

Performance and Robustness Analysis of Sequential Hypotheses Testing for Time Series with Trend

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Abstract

The problem of sequential testing of simple hypotheses for time series with a trend is considered. Analytic expressions and asymptotic expansions for error probabilities and expected numbers of observations are obtained. Robustness analysis is performed. Numerical results are given.

Keywords: sequential test, time series with trend, error probabilities, expected sample sizes, asymptotic expansions.

1. Introduction

The sequential approach to test parametric hypotheses proposed by Wald (see Wald (1947)) has been applied in many practical problems of computer data analysis. The sequential probability ratio test (SPRT) is proved to be optimal in terms of minimizing expected sample size under the assumption that type I and type II error probabilities do not exceed preassigned values (see Wald and Wolfowitz (1948)). The problem of sequential test performance characteristics (error probabilities and expected number of observations) evaluation is well studied for the case of identical distribution of observations (see Govindarajulu (2004), Kharin (2013), Kharin (2016)). In this paper, the model of non-identical distribution is considered for the problem of two simple hypotheses testing (see Kharin and Ton (2016)).

In practice, data does not often follow the hypothetical model exactly (see Huber (1981), Kharin (2005)), and the problem of robustness under distortions (see Kharin (1997), Maevskii and Kharin (2002)) is important for sequential testing (see Kharin (2011), Kharin and Kishylau (2015)). Here we consider the problem of robustness of sequential tests for time series with trend.

2. Mathematical model

Let x_1, x_2, \dots be time series with a trend:

$$x_t = \theta^T \psi(t) + \xi_t, \quad t = 1, 2, 3, \dots, \quad (1)$$

where $\psi(t) = (\psi_1(t), \psi_2(t), \dots, \psi_m(t))^T$, $t \geq 1$, are the vectors of basic functions of trend, $\theta = (\theta_1, \theta_2, \dots, \theta_m)^T \in \mathbb{R}^m$ is an unknown vector of coefficients, and $\{\xi_t, t \geq 1\}$ is the sequence of independent identically distributed random variables, $\xi_t \sim N(0, \sigma^2)$.

Consider two simple hypotheses ($\theta^0, \theta^1 \in \mathbb{R}^m$ are known vectors):

$$H_0 : \theta = \theta^0, H_1 : \theta = \theta^1. \quad (2)$$

Denote the accumulated log-likelihood ratio statistic:

$$\Lambda_n = \Lambda_n(x_1, x_2, \dots, x_n) = \sum_{t=1}^n \lambda_t, \quad (3)$$

where $\lambda_t = \ln \left(\frac{p_t(x_t, \theta^1)}{p_t(x_t, \theta^0)} \right)$ is the log-likelihood ratio calculated on observation x_t , and $p_t(x, \theta)$ is the probability density function of x_t provided the true parameter value is θ . To test hypotheses (2), after n observations one makes the decision:

$$d = \mathbf{1}_{[C_+, +\infty)}(\Lambda_n) + 2 \cdot \mathbf{1}_{(C_-, C_+)}(\Lambda_n). \quad (4)$$

Thresholds C_- and C_+ are the parameters of the test. Decisions $d = 0$ and $d = 1$ mean stopping of the observation process and acceptance of H_0 or H_1 correspondently. According to Wald (1947),

$$C_+ = \ln((1 - \beta_0)/\alpha_0), C_- = \ln(\beta_0/(1 - \alpha_0)),$$

where α_0, β_0 are given values for error probabilities of types I and II respectively.

3. Some auxiliary results

Let $\zeta_n, n \geq 1$ be a sequence of random variables satisfying the following conditions:

$$\text{i) } \zeta_n \in T = \{0, 1, 2, \dots, K, K + 1\}, n \in \mathbb{N}; \quad (5)$$

$$\text{ii) } P(\zeta_n = i_1 | \zeta_k = i_1) = 1, i_1 \in \{0, 1\}, n > k; \quad (6)$$

$$\text{iii) } P(\zeta_n = i_1 | \zeta_k = i_2, \zeta_l = i_3) = P(\zeta_n = i_1 | \zeta_k = i_2), n > k > l \geq 1, i_1, i_2, i_3 \in T. \quad (7)$$

Remark 1. A Markov chain with a finite state space T , in which the states 0, 1 are absorbing, satisfies conditions (5)-(7).

Introduce the notation:

$$\begin{aligned} T_1 &= \{0, 1\}, T_2 = \{2, 3, \dots, K, K + 1\}, T = T_1 \cup T_2, \\ P(k) &= \{p_{ij}(k)\}_{(K+2) \times (K+2)}, P(k, l) = \{p_{ij}(k, l)\}_{(K+2) \times (K+2)}, \\ p_{ij}(k) &= P(\zeta_k = j | \zeta_{k-1} = i), p_{ij}(k, l) = P(\zeta_{k+l} = j | \zeta_k = i). \end{aligned}$$

Since (6), matrices $P(k)$ and $P(1, k)$ can be expressed as follows:

$$P(k) = \left(\begin{array}{c|c} I_2 & O_{2 \times K} \\ \hline R_k & Q_k \end{array} \right), k \geq 2; \quad P(1, k) = \left(\begin{array}{c|c} I_2 & O_{2 \times K} \\ \hline \bar{R}_k & \bar{Q}_k \end{array} \right), k \geq 1, \quad (8)$$

where R_k, \bar{R}_k are some matrices of size $K \times 2$, I_k is the identity matrix of size k , $O_{2 \times K}$ is the $2 \times K$ -matrix with all elements equal to 0, and Q_k, \bar{Q}_k are some matrices of size $K \times K$. For $k < n < k + l$ we have:

$$p_{ij}(k, l) = \sum_{t \in T} P(\zeta_n = t | \zeta_k = i) P(\zeta_{k+l} = j | \zeta_n = t) = \sum_{t \in T} p_{it}(k, n - k) p_{tj}(n, k + l - n),$$

which implies that

$$P(k, l) = P(k, n - k) P(n, k + l - n), \quad k < n < k + l. \quad (9)$$

Therefore,

$$P(1, k) = P(2)P(3)\dots P(k + 1), \quad k \geq 1. \tag{10}$$

From (8) and (10), we get ($Q_1 = I_K$):

$$\bar{Q}_k = Q_1 Q_2 \dots Q_k Q_{k+1}. \tag{11}$$

Let t be the total number of time moments for which process ζ_n belongs to T_2 ; $n_j (j \in T_2)$ be the number of time moments for which $\zeta_n = j$; u_j^k be the function that is 1 if the process $\zeta_n = j$ after k steps, and is 0 otherwise; $E_i(\cdot)$ be the conditional expected value given $\zeta_1 = i$. Denote: $N = \sum_{k=0}^{+\infty} \bar{Q}_k$; $\tau = N \mathbf{1}_K$; $\mathbf{1}_K$ is the vector of size K with all elements equal to 1.

Theorem 1. *In the above notation, for sequence (5)-(7) the following equations are satisfied:*

$$\{E_i(n_j)\}_{K \times K} = N, \quad i, j \in T_2, \quad \{E_i(t)\}_{K \times 1} = \tau, \quad i \in T_2. \tag{12}$$

Proof. Consider representation $n_j = \sum_{k=0}^{+\infty} u_j^k$. Therefore,

$$\begin{aligned} \{E_i(n_j)\} &= \left\{ \sum_{k=0}^{+\infty} E_i(u_j^k) \right\} = \sum_{k=0}^{+\infty} \{p_{ij}(1, k)\} = \sum_{k=0}^{+\infty} \bar{Q}_k; \\ \{E_i(t)\} &= \left\{ \sum_{j \in T_2} E_i(n_j) \right\} = N \mathbf{1}_K = \tau. \end{aligned}$$

□

Let B be a matrix of size $K \times 2$, $B = \{b_{ij}\}_{K \times 2}$, where b_{ij} is the probability that the sequence ζ_n started in i is absorbed in j , $i \in T_2, j \in T_1$.

Theorem 2. *If conditions (5)-(7) hold for ζ_n , then*

$$B = \sum_{k=1}^{+\infty} \bar{Q}_{k-1} R_{k+1}. \tag{13}$$

Proof. Let $B(k), k \geq 1$, be matrix of size $K \times 2$, $B(k) = \{b_{ij}(k)\}_{K \times 2}$, where $b_{ij}(k)$ is the probability that the process ζ_n starting in i is absorbed in j after exactly k steps, $i \in T_2, j \in T_1$. We obtain (13) from the facts that $B(k) = \bar{Q}_{k-1} R_{k+1}$ and $B = \sum_{k=1}^{+\infty} B(k)$. □

Corollary 1. *If $\pi = (\pi_0, \pi_1, \dots, \pi_K, \pi_{K+1}), \pi_i = P(\zeta_1 = i), i \in T$ and $\pi' = (\pi_2, \dots, \pi_K, \pi_{K+1})$, then the total expected value $E(t)$ equals:*

$$E(t) = \pi' \tau. \tag{14}$$

Let $U, V, T \in \mathbb{R}$ be three independent random variables, $U \sim N(\mu_u, \sigma_u^2), V \sim N(\mu_v, \sigma_v^2), T \sim N(\mu_t, \sigma_t^2)$. From the properties of multivariate normal distributions (see [Bilodeau and Brenner \(1999\)](#)), we have:

$$f_{U+V|V}(x|y) = \frac{f_{U+V,V}(x, y)}{f_V(y)} = n_1(x; y + \mu_u, \sigma_u^2), \tag{15}$$

$$f_{U+V+T|V+T,T}(x|y, z) = f_{U+V+T|V+T}(x|y) = n_1(x; y + \mu_u, \sigma_u^2), \tag{16}$$

$$|f'_U(x)| \leq \frac{e^{-1/2}}{\sigma_u^2 \sqrt{2\pi}}, \quad \forall x, \tag{17}$$

$$|f'_{U+V|V}(x|y)| \leq \frac{e^{-1/2}}{\sigma_u^2 \sqrt{2\pi}}, \quad \forall x, y. \tag{18}$$

Lemma 1. (*Gut 2005*) If X is a non-negative, integer valued random variable, then

$$E(X) = \sum_{n=1}^{+\infty} P(X \geq n).$$

Lemma 2. (*Gut 2005*) Let $r > 0$, and suppose that X is a non-negative random variable. Then the following inequalities hold:

$$\sum_{n=1}^{+\infty} n^{r-1} P(X \geq n) \leq E(X^r) \leq 1 + \sum_{n=1}^{+\infty} n^{r-1} P(X \geq n),$$

and

$$E(|X|^r) < \infty \text{ if and only if } \sum_{n=1}^{+\infty} n^{r-1} P(X \geq n) < \infty.$$

Lemma 3. (*Coope 1996*) For positive semidefinite matrices A, B of the same order

$$0 \leq \text{tr}(AB) \leq \text{tr}(A)\text{tr}(B).$$

4. Performance and robustness analysis

4.1. Performance analysis for the hypothetical model

For model (1), (3) we have ($t \geq 1$):

$$x_t \sim N(\theta^T \psi(t); \sigma^2), \quad p_t(x, \theta) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{1}{2\sigma^2} (x - \theta^T \psi(t))^2 \right\};$$

$$\lambda_t = \lambda_t(x_t) = -\frac{1}{2\sigma^2} \{ 2x_t(\theta^0 - \theta^1)^T \psi(t) + (\theta^1)^T \psi(t) \psi^T(t) \theta^1 - (\theta^0)^T \psi(t) \psi^T(t) \theta^0 \}.$$

Due to the properties of the normal distribution, λ_t and Λ_n have also normal distributions with the following parameters:

$$E(\lambda_t) = -\frac{1}{2\sigma^2} \{ 2(\theta^0 - \theta^1)^T \psi(t) \psi^T(t) \theta + (\theta^1)^T \psi(t) \psi^T(t) \theta^1 - (\theta^0)^T \psi(t) \psi^T(t) \theta^0 \},$$

$$D(\lambda_t) = \frac{(\theta^0 - \theta^1)^T \psi(t) \psi^T(t) (\theta^0 - \theta^1)}{\sigma^2};$$

$$E(\Lambda_n) = -\frac{1}{2\sigma^2} \{ 2(\theta^0 - \theta^1)^T H_n \theta + (\theta^1)^T H_n \theta^1 - (\theta^0)^T H_n \theta^0 \},$$

$$D(\Lambda_n) = \frac{(\theta^0 - \theta^1)^T H_n (\theta^0 - \theta^1)}{\sigma^2},$$

where $H_n = \sum_{t=1}^n \psi(t) \psi^T(t)$.

Introduce the notation: $E^{(k)}(\cdot), D^{(k)}(\cdot)$ are conditional expected value and variance provided the hypothesis H_k is true ($k = 0, 1$), $\Phi(\cdot)$ is the cumulative distribution function of the standard normal law,

$$\sigma_n^2 = D^{(0)}(\lambda_n) = D^{(1)}(\lambda_n) = \frac{(\theta^0 - \theta^1)^T \psi(n) \psi^T(n) (\theta^0 - \theta^1)}{\sigma^2},$$

$$\mu_n^{(k)} = E^{(k)}(\lambda_n) = \frac{(-1)^{k+1} \sigma_n^2}{2}, \quad s_n^2 = \sum_{t=1}^n \sigma_t^2, \quad m_n^{(k)} = \sum_{t=1}^n \mu_t^{(k)} = \frac{(-1)^{k+1} s_n^2}{2};$$

$$A_n = \{a_{ij}\}_{n \times n}, \quad a_{ij} = \begin{cases} 1, & i \geq j, \\ 0, & \text{otherwise,} \end{cases}$$

$$X_n = (\lambda_1, \lambda_2, \dots, \lambda_n)^T, \quad T_n = (\Lambda_1, \Lambda_2, \dots, \Lambda_n)^T = A_n X_n;$$

$$\mu_{T_n}^{(k)} = E^{(k)}(T_n) = A_n E^{(k)}(X_n), \quad \Sigma_{T_n} = \text{Cov}(T_n, T_n) = A_n \text{Cov}(X_n, X_n) A_n^T;$$

$$N = \inf\{n \in \mathbb{N} : \Lambda_n \notin (C_-, C_+)\}, \quad \Gamma = (\theta^0 - \theta^1)(\theta^0 - \theta^1)^T.$$

Theorem 3. *If $\text{tr}(\Gamma H_n) \rightarrow +\infty$ as $n \rightarrow +\infty$, then test (3)-(4) terminates finitely with probability 1.*

Proof. We have:

$$\begin{aligned} s_n^2 &= \sum_{t=1}^n \sigma_t^2 = \frac{1}{\sigma^2} \sum_{t=1}^n (\theta^0 - \theta^1)^T \psi(t) \psi^T(t) (\theta^0 - \theta^1) \\ &= \frac{1}{\sigma^2} \sum_{t=1}^n \text{tr}\{(\theta^0 - \theta^1)^T \psi(t) \psi^T(t) (\theta^0 - \theta^1)\} = \frac{1}{\sigma^2} \text{tr}(\Gamma H_n). \end{aligned}$$

Under the theorem condition, we get $s_n^2 \rightarrow +\infty$ as $n \rightarrow +\infty$. Furthermore, we also have $P_k(N > n) = P_k(\Lambda_i \in (C_-, C_+), i = \overline{1, n}), k = 0, 1$, and

$$\begin{aligned} P_k(\Lambda_i \in (C_-, C_+), i = \overline{1, n}) &\leq P_k(\Lambda_n \in (C_-, C_+)) = \\ &= \Phi\left(\frac{C_+ - \mu_n^{(k)}}{s_n}\right) - \Phi\left(\frac{C_- - \mu_n^{(k)}}{s_n}\right) \\ &= \Phi\left(\frac{2C_+ - (-1)^{k+1} s_n^2}{2s_n}\right) - \Phi\left(\frac{2C_- - (-1)^{k+1} s_n^2}{2s_n}\right), \end{aligned}$$

which implies that $\lim_{n \rightarrow +\infty} P_k(\Lambda_i \in (C_-, C_+), i = \overline{1, n}) = 0$ or $\lim_{n \rightarrow +\infty} P_k(N > n) = 0$.

Therefore, $P_k(N < +\infty) = 1 - P_k(N = +\infty) = 1 - \lim_{n \rightarrow +\infty} P_k(N > n) = 1$. □

Corollary 2. *Under the conditions of Theorem 3 we get:*

$$\sum_{i=1}^n \sum_{j=1}^m \psi_j^2(i) \rightarrow +\infty \text{ as } n \rightarrow +\infty. \tag{19}$$

Additionally, if $(\theta_i^1 - \theta_i^0)^T \psi_i(t), i = \overline{1, m}$, are simultaneously nonnegative (or nonpositive) functions at t , then the result of Theorem 3 still holds if $\sum_{i=1}^n \psi_k^2(i) \rightarrow +\infty$, where k is such an index that $\theta_k^1 \neq \theta_k^0$.

Proof. Note that Γ and H_n are positive semi-definite matrices. The proof is derived directly from Lemma 3 and the facts that:

$$\text{tr}(H_n) = \sum_{i=1}^n \sum_{j=1}^m \psi_j^2(i), \text{tr}(\Gamma H_n) = \sum_{i=1}^n \left(\sum_{k=1}^m (\theta_k^0 - \theta_k^1) \psi_k(i)\right)^2.$$

□

Remark 2. *If $\sum_{i=1}^n \sum_{j=1}^m \psi_j^2(i)$ is bounded, then there exists a positive constant L such that $s_n^2 \rightarrow L$ as $n \rightarrow +\infty$. In this case, we get $\sigma_n \rightarrow 0$ and $\mu_n^{(k)} \rightarrow 0, k = 0, 1$. It means $\lambda_n \xrightarrow{P} 0$ as $n \rightarrow +\infty$ under hypothesis H_0 or H_1 . In addition, we also have:*

$$\lim_{n \rightarrow +\infty} P_k(\Lambda_n \in (C_-, C_+)) = \Phi\left(\frac{2C_+ - (-1)^{k+1} L}{2\sqrt{L}}\right) - \Phi\left(\frac{2C_- - (-1)^{k+1} L}{2\sqrt{L}}\right) > 0.$$

Theorem 4. *Under the conditions of Theorem 3 the following expressions are valid for the characteristics of test (3), (4):*

$$E^{(k)}(N) = 1 + \sum_{i=1}^{+\infty} \int_{C_-}^{C_+} ds_i \int_{C_-}^{C_+} ds_{i-1} \dots \int_{C_-}^{C_+} n_i(s; \mu_{T_i}^{(i)}, \Sigma_{T_i}) ds_1, k = 0, 1, \tag{20}$$

$$\alpha = \int_{C_+}^{+\infty} n_1(s_1; \mu_1^{(0)}, \sigma_1^2) ds_1 + \sum_{i=2}^{+\infty} \int_{C_+}^{+\infty} ds_i \int_{C_-}^{C_+} ds_{i-1} \dots \int_{C_-}^{C_+} n_i(s; \mu_{T_i}^{(0)}, \Sigma_{T_i}) ds_1, \tag{21}$$

$$\beta = \int_{-\infty}^{C_-} n_1(s_1; \mu_1^{(1)}, \sigma_1^2) ds_1 + \sum_{i=2}^{+\infty} \int_{-\infty}^{C_-} ds_i \int_{C_-}^{C_+} ds_{i-1} \dots \int_{C_-}^{C_+} n_i(s; \mu_{T_i}^{(1)}, \Sigma_{T_i}) ds_1. \tag{22}$$

Proof. Under the condition of Theorem 4 the test terminates finitely with probability 1. From Lemma 1, we have:

$$E^{(k)}(N) = 1 + \sum_{i=1}^{+\infty} P_k(N > i) = 1 + \sum_{i=1}^{+\infty} \int_{C_-}^{C_+} ds_i \int_{C_-}^{C_+} ds_{i-1} \dots \int_{C_-}^{C_+} n_i(s; \mu_{T_i}^{(k)}, \Sigma_{T_i}) ds_1.$$

For the error type I probability we get:

$$\begin{aligned} \alpha &= P_0(\Lambda_N \geq C_+) = \sum_{i=1}^{+\infty} P_0(N = i, \Lambda_N \geq C_+) \\ &= P_0(\Lambda_1 \geq C_+) + \sum_{i=2}^{+\infty} P_0(\Lambda_i \geq C_+, \Lambda_j \in (C_-, C_+), j = \overline{1, i-1}). \end{aligned} \quad (23)$$

From (23) we get (21). The expression of β in (22) is proved analogously. \square

In practice, it is difficult to use formulae (20)-(22) for computing the characteristics of the test: using numerical methods for approximating the multiple integration in the right hand sides of these equalities is unfeasible. To get upper bounds for these test characteristics, we can use the following estimate:

$$P(\Lambda_1 \in (a_1, b_1), \dots, \Lambda_n \in (a_n, b_n)) \leq P(\Lambda_i \in (a_i, b_i), \dots, \Lambda_n \in (a_n, b_n)), \quad (24)$$

where i is a fixed value in $\{1, 2, \dots, n\}$.

It is obvious that the smaller value i , the stricter the inequality (24). In particular, when $tr(\Gamma H_n)$ tends to $+\infty$ slowly, we should select the value i smaller to get better estimates.

Corollary 3. *Under the Theorem 4 condition, the following inequalities hold:*

$$\begin{aligned} E^{(k)}(N) &\leq \Phi\left(\frac{C_+ - \mu_1^{(k)}}{\sigma_1}\right) + \Phi\left(\frac{\mu_1^{(k)} - C_-}{\sigma_1}\right) \\ &\quad + \sum_{i=2}^{+\infty} \int_{C_-}^{C_+} \int_{C_-}^{C_+} n_1(x; m_{i-1}^{(k)}, s_{i-1}^2) n_1(y; x + \mu_i^{(k)}, \sigma_i^2) dx dy, k \in \{0, 1\}, \\ \alpha &\leq 1 - \Phi\left(\frac{C_+ - \mu_1^{(0)}}{\sigma_1}\right) + \sum_{i=2}^{+\infty} \int_{C_+}^{+\infty} \int_{C_-}^{C_+} n_1(x; m_{i-1}^{(0)}, s_{i-1}^2) n_1(y; x + \mu_i^{(0)}, \sigma_i^2) dx dy, \\ \beta &\leq \Phi\left(\frac{C_- - \mu_1^{(1)}}{\sigma_1}\right) + \sum_{i=2}^{+\infty} \int_{-\infty}^{C_-} \int_{C_-}^{C_+} n_1(x; m_{i-1}^{(1)}, s_{i-1}^2) n_1(y; x + \mu_i^{(1)}, \sigma_i^2) dx dy. \end{aligned}$$

Proof. The inequalities are resulting from (24), (15) with $i = n - 1$. \square

Time series data is usually collected at certain intervals. Sometimes, there are patterns that can repeat over fixed periods of time within the data set. Such patterns are known as periodic fluctuations or seasonality. In this case, the function of trend $g(t) = \theta^T \psi(t)$ will be periodic with some period $T > 0$. In particular, function $h(t) = (\theta^0 - \theta^1)^T \psi(t)$ will also be periodic.

Theorem 5. *If there exists an integer $T \geq 1$ such that $h(t+T) = h(t), \forall t = 1, 2, \dots$, then the stopping time N has finite moments of any order.*

Proof. Without loss of generality, assume that hypothesis H_0 is true. Due to the Theorem conditions, λ_t and λ_{t+kT} have the same distribution for all $t, k \in \mathbb{N}$.

On event $\{N > kT\}, k \geq 1, \Lambda_{kT} = \lambda_1 + \dots + \lambda_{kT} \in (C_-, C_+)$, which implies that $\Lambda_{kT}^2 \leq (C_+ - C_-)^2$. Since $D(\Lambda_{kT}) = \sum_{i=1}^{kT} D(\lambda_i) = kD(\Lambda_T) = \frac{k}{\sigma^2} tr(\Gamma H_T) \rightarrow +\infty$ as $k \rightarrow +\infty$, there exists $k_0 \geq 1$ such that on $\{N > k_0T\}$:

$$P_0(\Lambda_{k_0T}^2 < (C_+ - C_-)^2) = q \in (0, 1).$$

Put $n_0 = hk_0T, h \geq 1$. Note that from the fact $N > n_0$, we have

$$|\Lambda_{rk_0T} - \Lambda_{(r-1)k_0T}| < C_+ - C_-, \forall r = \overline{1, h}.$$

Because $\{\Lambda_{rk_0T} - \Lambda_{(r-1)k_0T}, r = \overline{1, h}\}$ is a sequence of independent identically distributed random variables and $\{N > n_0\} \subset \{|\Lambda_{rk_0T} - \Lambda_{(r-1)k_0T}| < C_+ - C_-, \forall r = \overline{1, h}\}$, we have

$$P_0(N > n_0) \leq P_0(|\Lambda_{rk_0T} - \Lambda_{(r-1)k_0T}| < C_+ - C_-, \forall r = \overline{1, h}) = q^h = q^{\frac{n_0}{k_0T}}.$$

Therefore, for any $n > k_0T$

$$P_0(N > n) \leq q^{\lceil \frac{n}{k_0T} \rceil} \leq q^{\frac{n}{k_0T} - 1},$$

which implies $P_0(N > n) = o(n^{-r})$ as $n \rightarrow +\infty$ for any finite r .

The rest part of proof is derived from Lemma 2. □

Corollary 4. *If the basic vector function of trend $\psi(t)$ is periodic on the set \mathbb{N} , then the stopping time N has finite moments of any order.*

4.2. Special case

Assume that there exists a constant $a \neq 0$ such that $h(t) = a, \forall t \geq 1$, and H_0 is the true hypothesis.

In this case, $\{\lambda_t, t \geq 1\}$ becomes the sequence of independent and identically distributed random variables from $N(\mu, \sigma_0^2)$, where

$$\mu = \frac{-a^2}{2\sigma^2}, \quad \sigma_0^2 = \frac{a^2}{\sigma^2}.$$

Let x be a fixed value and put $\Lambda_n^x = x + \Lambda_n$. The new test based on Λ_n^x is equivalent to the original SPRT whose region of indifference is the interval $(C_- - x, C_+ - x)$. Let $\beta_\theta(x)$ and $N_\theta(x)$ be the operating characteristic and average sample size functions of this SPRT respectively. From the Markov property of log-likelihood ratio statistic Λ_n and when H_0 is true, $\beta_{\theta^0}(x)$ and $N_{\theta^0}(x)$ are known to satisfy the Fredholm integral equations (see [Basseville and Nikiforov \(1993\)](#), [Cox and Miller \(1965\)](#)):

$$\beta_{\theta^0}(x) = F_{\theta^0}(C_- - x) + \int_{C_-}^{C_+} K_{\theta^0}(x, y)\beta_{\theta^0}(y)dy,$$

$$N_{\theta^0}(x) = 1 + \int_{C_-}^{C_+} K_{\theta^0}(x, y)N_{\theta^0}(y)dy,$$

where $F_{\theta^0}(x) = P_{\theta^0}(\lambda_1 < x) = \Phi\left(\frac{x-\mu}{\sigma_0}\right)$ and $K_{\theta^0}(x, y) = \frac{\partial}{\partial y}F_{\theta^0}(y-x) = \frac{1}{\sigma_0}\varphi\left(\frac{y-x-\mu}{\sigma_0}\right); \varphi(z) = n_1(z; 0, 1), z \in \mathbb{R}$.

A numerical method is used for solving these equations. Let m_0 be a positive integer and $\{y_i, i = \overline{1, m_0}\}, C_- = y_1 < y_2 < \dots < y_{m_0} = C_+$, be a partition of the interval $[C_-, C_+]$, in which y_i having the smallest absolute value is set to be 0. Note that $\alpha = 1 - \beta_{\theta^0}(0)$ and $E(N|H_0) = N_{\theta^0}(0)$. Let $\tilde{\beta}_{\theta^0}(y_i)$ be approximations of $\beta_{\theta^0}(y)$ at $y = y_i, i = \overline{1, m_0}$. Using the trapezoid formula (see [Hoffman \(2001\)](#)) we have:

$$\begin{aligned} \int_{C_-}^{C_+} K_{\theta^0}(x, y)\beta_{\theta^0}(y)dy &\approx \sum_{i=1}^{m_0-1} \frac{y_{i+1} - y_i}{2\sigma_0} \left(\varphi\left(\frac{y_i - x - \mu}{\sigma_0}\right) \tilde{\beta}_{\theta^0}(y_i) + \right. \\ &\quad \left. + \varphi\left(\frac{y_{i+1} - x - \mu}{\sigma_0}\right) \tilde{\beta}_{\theta^0}(y_{i+1}) \right) \\ &\approx \sum_{i=1}^{m_0} \rho_i \varphi\left(\frac{y_i - x - \mu}{\sigma_0}\right) \tilde{\beta}_{\theta^0}(y_i), \end{aligned}$$

Proof. From the Lemma condition and (17)-(18), there exist two positive constants C_1, C_2 such that:

$$|f'_{\Lambda_n}(x)| \leq C_1, \forall n \geq 1, |f'_{\Lambda_n|\Lambda_k}(x|y)| \leq C_2, \forall k < n, \forall x, y \in \mathbb{R}.$$

The rest part of proof is similar to the proof of Theorem 1 in Kharin (2008). □

Denote $f_{C_-}^{C_+}(x) = \left(\left\lceil \frac{x-C_-}{h} \right\rceil + 1 \right) \cdot \mathbf{1}_{(C_-, C_+)}(x) + (K + 1) \cdot \mathbf{1}_{[C_+, +\infty)}(x)$.

For the random sequence Λ_n introduce the discrete random sequence Z_n with the finite state space $V = \{0, 1, \dots, K + 1\}$. Put $Z_1 = f_{C_-}^{C_+}(\Lambda_1)$. For $n \geq 2$:

$$Z_n = \begin{cases} 0, & \text{if } Z_{n-1} = 0, \\ K + 1, & \text{if } Z_{n-1} = K + 1, \\ f_{C_-}^{C_+}(\Lambda_n), & \text{otherwise.} \end{cases}$$

To simplify the notation, let us renumerate the states space of Z_n :

$$V = \{\{0\}, \{K + 1\}, \{1\}, \dots, \{K\}\}.$$

Denote:

$$\begin{aligned} P^{(n)}(\theta^k) &= \left(\begin{array}{c|c} I_2 & \mathbf{O}_{2 \times K} \\ \hline R_n(\theta^k) & Q_n(\theta^k) \end{array} \right), k \in \{0, 1\}, \\ Q_n(\theta^k) &= \{q_{ij}^{(n)}(\theta^k)\}_{K \times K}, R_n(\theta^k) = \{r_{ij}^{(n)}(\theta^k)\}_{K \times 2}, \\ q_{ij}^{(n)}(\theta^k) &= \frac{\int_{A_i} n_1(y; m_{n-1}^{(i)}, s_{n-1}^2) \int_{A_j} n_1(x; y + \mu_n^{(k)}, \sigma_n^2) dx dy}{\int_{A_i} n_1(y; m_{n-1}^{(k)}, s_{n-1}^2) dy}, 1 \leq i, j \leq K, \\ r_{i1}^{(n)}(\theta^k) &= \frac{\int_{A_i} n_1(y; m_{n-1}^{(k)}, s_{n-1}^2) \int_{A_0} n_1(x; y + \mu_n^{(k)}, \sigma_n^2) dx dy}{\int_{A_i} n_1(y; m_{n-1}^{(k)}, s_{n-1}^2) dy}, 1 \leq i \leq K, \\ r_{i2}^{(n)}(\theta^k) &= \frac{\int_{A_i} n_1(y; m_{n-1}^{(k)}, s_{n-1}^2) \int_{A_{K+1}} n_1(x; y + \mu_n^{(k)}, \sigma_n^2) dx dy}{\int_{A_i} n_1(y; m_{n-1}^{(k)}, s_{n-1}^2) dy}, 1 \leq i \leq K, \\ S(\theta^k) &= I_K + \sum_{i=1}^{+\infty} \prod_{j=1}^{i+1} Q_j(\theta^k), \quad B(\theta^k) = R_2(\theta^k) + \sum_{i=2}^{+\infty} \prod_{j=1}^i Q_j(\theta^k) R_{i+1}(\theta^k), \end{aligned}$$

$B_{(j)}(\cdot)$ is the j^{th} column of matrix $B(\cdot)$, $\pi(\theta^k)$ is the probability distribution of $Z_1, k \in \{0, 1\}$.

Theorem 6. *If $\exists C = const > 0, \inf_n tr(\Gamma\psi(n)\psi^T(n)) \geq C$, then the characteristics of the test (3),(4) satisfy the following expansions:*

$$E^{(k)}(N) = 1 + (\pi(\theta^k))' S(\theta^k) \cdot \mathbf{1}_K + O(h), k \in \{0, 1\}, \tag{28}$$

$$\alpha = (\pi(\theta^0))' B_{(2)}(\theta^0) + \pi_{K+1}(\theta^0) + O(h), \tag{29}$$

$$\beta = (\pi(\theta^1))' B_{(1)}(\theta^1) + \pi_0(\theta^1) + O(h), \tag{30}$$

where $(\pi(\theta^k))' = (\pi_1(\theta^k), \dots, \pi_K(\theta^k))$.

Proof. Under the conditions of Theorem 6 sequence Z_n satisfies the conditions (5)-(7) asymptotically as $h \rightarrow 0$. The results of this theorem are derived from the Lemma 4, Theorems 1 and 2. □

4.4. Robustness evaluation

In practice the observed data can often come from more complicated sources than hypothetical ones because of the distortion. Some contamination models (see Huber (1981)) can be used

to analyze the robustness of statistical procedures. However, in this paper we consider only the case where the noise components ξ_t in model (1) are distorted by another noises in the Gaussian distribution family.

Suppose that the observed data come from the following mixed model:

$$\bar{x}_t = \theta^T \psi(t) + (1 - \varepsilon)\xi_t + \varepsilon\tilde{\xi}_t, \quad (31)$$

where $\{\tilde{\xi}_t, t \geq 1\}$ is a sequence of independent identically distributed random variables, $\tilde{\xi}_t \sim N(0, \tilde{\sigma}^2)$, $\tilde{\sigma}$ is a given positive constant, ξ_t and $\tilde{\xi}_t$ are independent for all t , and $\varepsilon \in [0, 1/2)$ is a level of distortion.

Introduce the notation: $\bar{\lambda}_t = \lambda_t(\bar{x}_t)$, $\bar{\mu}_t^{(k)} = E^{(k)}(\bar{\lambda}_t)$, $\bar{\sigma}_t^2 = D(\bar{\lambda}_t)$, $\bar{m}_n^{(k)} = \sum_{i=1}^n \bar{\mu}_i^{(k)}$, $\bar{s}_n^2 = \sum_{i=1}^n \bar{\sigma}_i^2$, and $\bar{P}^{(n)}(\theta^k)$, $\bar{R}_n(\theta^k)$, $\bar{Q}_n(\theta^k)$, $\bar{\pi}(\theta^k)$, $\bar{\alpha}$, $\bar{\beta}$, $\bar{t}(\theta^k)$ are calculated analogously by replacing x_t with \bar{x}_t .

For $t, n \geq 1, k \in \{0, 1\}$, we get:

$$\bar{\mu}_t^{(k)} = \mu_t^{(k)}, \bar{m}_n^{(k)} = m_n^{(k)}, \bar{\sigma}_t^2 = \frac{\sigma_t^2}{\sigma^2} [(1 - \varepsilon)^2 \sigma^2 + \varepsilon^2 \tilde{\sigma}^2], \bar{s}_n^2 = \frac{s_n^2}{\sigma^2} [(1 - \varepsilon)^2 \sigma^2 + \varepsilon^2 \tilde{\sigma}^2]. \quad (32)$$

Theorem 7. *For the contaminated model (31), under the conditions of Theorem 6 the following asymptotic expansions hold as $\varepsilon \rightarrow 0$ and $h \rightarrow 0$:*

$$\bar{\alpha} = \alpha + O(\varepsilon) + O(h), \quad (33)$$

$$\bar{\beta} = \beta + O(\varepsilon) + O(h), \quad (34)$$

$$\bar{t}(\theta^k) = t(\theta^k) + O(\varepsilon) + O(h), k \in \{0, 1\}. \quad (35)$$

Proof. From (32), as $\varepsilon \rightarrow 0$ we have:

$$\frac{1}{\bar{\sigma}_t^2} = \frac{1}{\sigma_t^2} (1 + O(\varepsilon)), \quad \frac{1}{\bar{s}_n^2} = \frac{1}{s_n^2} (1 + O(\varepsilon)),$$

$$\begin{aligned} n_1(x; \bar{\mu}_1^{(k)}, \bar{\sigma}_1^2) &= \frac{1 + O(\varepsilon)}{\sigma_1 \sqrt{2\pi}} \exp \left[-\frac{(x - \mu_1^{(k)})^2}{2\sigma_1^2} (1 + O(\varepsilon)) \right] \\ &= (1 + O(\varepsilon)) n_1(x; \mu_1^{(k)}, \sigma_1^2), \forall x \in \mathbb{R}, \end{aligned}$$

which implies that

$$\bar{\pi}_i(\theta^k) = \pi_i(\theta^k) + O(\varepsilon), i = \overline{0, K+1}, k \in \{0, 1\}. \quad (36)$$

Similarly, we also have as $\varepsilon \rightarrow 0$

$$n_1(y; \bar{m}_{n-1}^{(k)}, \bar{s}_{n-1}^2) n_1(x; y + \bar{\mu}_n^{(k)}, \bar{\sigma}_n^2) = (1 + O(\varepsilon))^2 n_1(y; m_{n-1}^{(k)}, s_{n-1}^2) n_1(x; y + \mu_n^{(k)}, \sigma_n^2),$$

and obtain

$$\bar{r}_{ij}^{(n)}(\theta^k) = r_{ij}^{(n)}(\theta^k) + O(\varepsilon), 1 \leq i \leq K, j = 1, 2, \quad (37)$$

$$\bar{q}_{ij}^{(n)}(\theta^k) = q_{ij}^{(n)}(\theta^k) + O(\varepsilon), 1 \leq i, j \leq K. \quad (38)$$

Combining (36)-(38) and using Theorem 6 we get (33)-(35). \square

5. Numerical examples

First consider the probability model (1) for the special case from Section 4.2 with the following parameters values: $m = 4, \sigma = 10, \psi(t) = (1 + 1/t, 4 - t/10, t/10, t^2/10)^T, \theta^0 =$

$(1, 2, 2, 1)^T, \theta^1 = (1, 1, 1, 1)^T$; hypotheses (2) were tested. Denote the sample estimate of a characteristic γ with Monte-Carlo method by $\hat{\gamma}$. The number of simulation runs used in this method was 100 000. Denote $t_k = E^{(k)}(N), k = \overline{0, 1}$.

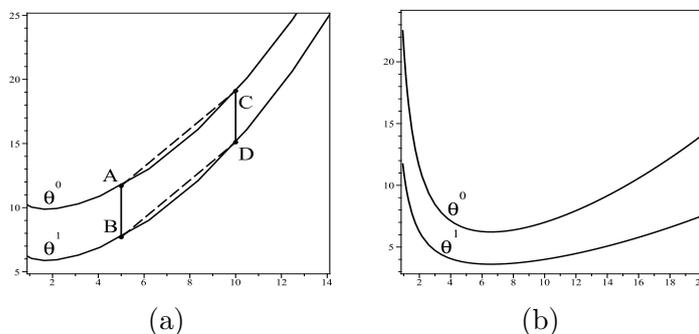


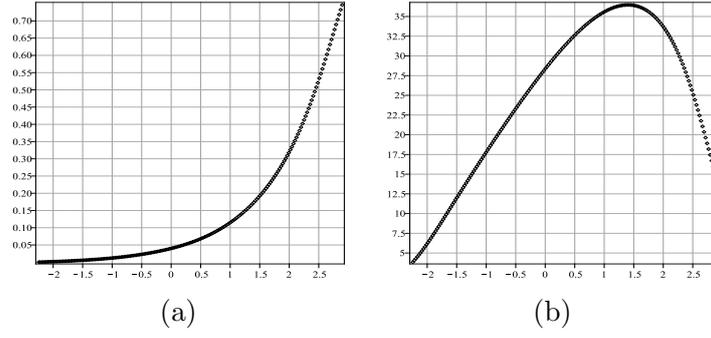
Figure 1: Trend functions

Suppose that hypothesis H_0 is true. Then we have $a = (\theta^0 - \theta^1)^T \psi(t) = 4, \forall t$. This means that the lengths of segments AB and CD are always the same in all positions such that they are parallel to the vertical axis (figure 1a). In addition, $\lambda_t, t \geq 1$, are independent identically distributed random variables, $\lambda_t \sim N(\mu, \sigma_0^2)$, where $\mu = -50, \sigma_0 = 0.4$. Monte-Carlo estimates ($\hat{\alpha}$ and \hat{t}_0) and approximate values ($\tilde{\alpha} = 1 - \beta_{\theta^0}(0)$ and $\tilde{t}_0 = \tilde{N}_{\theta^0}(0)$) calculated according to Section 4.2 for Type I error probability α and conditional average number of observations t_0 respectively are presented in Table 1. When the value of m_0 increases, the approximate values of test characteristics tend to their corresponding Monte-Carlo estimates.

Table 1: Performance characteristics estimates

| α_0 | β_0 | $\hat{\alpha}$ | \hat{t}_0 | m_0 | $\tilde{\alpha}$ | \tilde{t}_0 |
|------------|-----------|----------------|-------------|-------|------------------|---------------|
| 0.1 | 0.1 | 0.08034 | 25.73666 | 200 | 0.08236 | 25.70478 |
| | | | | 500 | 0.08076 | 25.72103 |
| 0.05 | 0.1 | 0.04072 | 28.46724 | 200 | 0.04128 | 28.46644 |
| | | | | 500 | 0.04080 | 28.47073 |
| 0.05 | 0.05 | 0.03990 | 36.83060 | 200 | 0.03816 | 36.83469 |
| | | | | 500 | 0.03974 | 36.80202 |

The dependence of the operating characteristic and the average sample size functions on the initial value x in the modified test is presented in figure 2 for the case of $m_0 = 200, \alpha_0 = 0.05, \beta_0 = 0.1$. Under hypothesis H_0 , $\beta_{\theta^0}(x)$ is a decreasing function with respect to x (figure 2a). This fact is easily understood because when x increases, the probability that $x + \Lambda_n$ comes out of the interval $(C_- - x, C_+ - x)$ through the upper boundary $C_+ - x$ also increases. However, function $N_{\theta^0}(x)$ increases to the maximum value in the interval (C_-, C_+) before dropping (figure 2b).

Figure 2: Plots of functions $1 - \beta_{\theta^0}(x)$ and $N_{\theta^0}(x)$

In the next examples, the following values of parameters are used for calculating: $m = 4, \sigma = 10, \psi(t) = (1, t/10, t^2/100, 10/t)^T, \theta^0 = (1, 2, 2, 2)^T, \theta^1 = (1, 1, 1, 1)^T$. In this case, $P_k(\Lambda_{50} \in (C_-, C_+)) \leq 10^{-6}, k \in \{0, 1\}$. All infinite sums were replaced by the sums of the first 50 summands, this provides the accuracy of the order 0.00001. Figure 1b shows the plots of trend functions. The upper bounds for the test performance characteristics constructed in Corollary 3 are given in Tables 2 and 3.

Table 2: The upper bounds for error probabilities

| α_0 | β_0 | $\hat{\alpha}$ | $\alpha \leq$ | $\hat{\beta}$ | $\beta \leq$ |
|------------|-----------|----------------|---------------|---------------|--------------|
| 0.1 | 0.1 | 0.07166 | 0.12155 | 0.07244 | 0.12155 |
| 0.05 | 0.1 | 0.03392 | 0.05307 | 0.07006 | 0.11402 |
| 0.01 | 0.05 | 0.00534 | 0.00761 | 0.03216 | 0.04734 |

Table 3: The upper bounds for the average number of observations

| α_0 | β_0 | \hat{t}_0 | $E^{(0)}(N) \leq$ | \hat{t}_1 | $E^{(1)}(N) \leq$ |
|------------|-----------|-------------|-------------------|-------------|-------------------|
| 0.1 | 0.1 | 16.07760 | 18.09357 | 16.03320 | 18.09357 |
| 0.05 | 0.1 | 17.03316 | 18.68805 | 19.70408 | 21.30107 |
| 0.01 | 0.05 | 21.13740 | 22.13213 | 25.61644 | 26.41215 |

Denote the main terms of asymptotic expansions of $\alpha, \beta, E^{(k)}(N)$ respectively by $\bar{\alpha}_{ASYM}, \bar{\beta}_{ASYM}, \bar{E}_{ASYM}^{(k)}(N)$. The numerical results for these main terms are presented in Table 4.

Table 4: The main terms of asymptotic expansions of the test characteristics

| α_0 | β_0 | K | $\bar{\alpha}_{ASYM}$ | $\bar{\beta}_{ASYM}$ | $\bar{E}_{ASYM}^{(0)}(N)$ | $\bar{E}_{ASYM}^{(1)}(N)$ |
|------------|-----------|-----|-----------------------|----------------------|---------------------------|---------------------------|
| 0.1 | 0.1 | 10 | 0.07449 | 0.07449 | 15.85942 | 15.85942 |
| | | 20 | 0.07273 | 0.07273 | 16.03470 | 16.03470 |
| | | 40 | 0.07225 | 0.07225 | 16.08029 | 16.08029 |
| 0.05 | 0.1 | 10 | 0.03501 | 0.07367 | 16.76240 | 19.49727 |
| | | 20 | 0.03433 | 0.07104 | 16.97260 | 19.65483 |
| | | 40 | 0.03414 | 0.07033 | 17.02777 | 19.69613 |
| 0.01 | 0.05 | 10 | 0.00586 | 0.03466 | 20.83295 | 25.57559 |
| | | 20 | 0.00579 | 0.03291 | 21.06344 | 25.65018 |
| | | 40 | 0.00576 | 0.0324 | 21.12768 | 25.67067 |

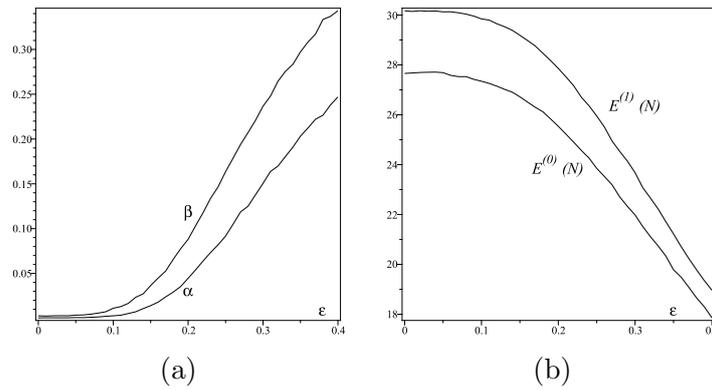


Figure 3: Dependence of performance characteristics on probability of contamination ε

Due to symmetric property, we have $\bar{\alpha}_{ASYM} = \bar{\beta}_{ASYM}$, $\bar{E}_{ASYM}^{(0)}(N) = \bar{E}_{ASYM}^{(1)}(N)$ provided $\alpha_0 = \beta_0$. When the value K increases, the main terms of asymptotic expansions of the test characteristics become closer to their corresponding Monte-Carlo estimates (see Tables 2 and 3). The orders of approximation in (28)-(30) are only $O(h)$. Therefore, if we want to make the main terms of asymptotic expansions better, the value K must be larger. However, with the large value K the computation on infinite sums (in practice, they can be reasonably replaced with finite ones because of the termination of the test) of matrices with high dimensions $S(\theta^i)$ and $B(\theta^i)$ will cost much time.

Figure 3 shows the dependence of the error probabilities and average number of observations on the probability of contamination ε in the model (31), when $\tilde{\sigma}^2 = 50\sigma^2$, $\alpha_0 = 0.001$, $\beta_0 = 0.005$. When the contamination probability ε increases, both error probabilities increase. For both conditional average numbers of observations, there are opposite pictures.

6. Conclusion

The problem of sequential testing for time series with trend is studied. The sufficient condition of termination of the test is given. Beside the explicit (but not useful for further analysis) formulae for the test characteristics, an approach to approximate test characteristics is also constructed. This approach allows us not only to estimate the test characteristics, but also to analyze the robustness of the test.

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Austrian Journal of Statistics
 published by the Austrian Society of Statistics

<http://www.ajs.or.at/>
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Volume 46
 April 2017

Submitted: 2016-11-15
 Accepted: 2017-02-02