경제학석사 학위논문

# The Shapley value for two-source minimum cost spanning tree problems

## 소스가 두 개인 최소신장가지문제에서의 샤플리 밸류

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# Abstract

We study two-source minimum cost spanning tree problem. Agents need to connect to the sources either directly or through other agents. For each connection there is an associated cost, and the total cost of connecting all agents must be shared among them. We introduce a cost allocation rule that is defined based on the Boruvka algorithm and show that this rule coincides with a widely used rule, the Shapley value, in the irreducible form of the problem.

**Keywords : Minimum Cost Spanning Tree Problems, Boruvka Algorithm, Irreducible Form, Cost Allocation, Shapley Value**

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# 1 Introduction

In this paper, we study *minimum cost spanning tree problems* (*mcstp*). Consider a situation where a group of agents located at geographically different places need a certain service that can only be delivered by a common supplier. They have to be connected to a set of sources that the supplier provides, either directly or through other agents, and there exists a corresponding cost for each possible connection (*link*) between two *nodes* (agents or sources).

There are many real life examples of this context such as networks for power supply cables, water pipes, and internet connection among several cities.

In most of these situations, agents may benefit from cooperating. Thus, the problem consists of two parts. One is to find a way to minimize the total cost while connecting all agents to the source, i.e. to find a *minimum cost spanning tree* (*mcst)*. The other is to decide how to allocate the cost of the minimum cost spanning tree found.

There are a few papers in the literature that study the first part. For instance, Boruvka (1926), Kruskal (1956), and Prim (1957) showed algorithms for constructing the *mcst* in any *mcstp*. In this paper, we use the Boruvka algorithm.

For the cost allocation aspect, Bird (1976) represented a cooperative game based on the *mcstp* and came up with a cost allocation rule, called the Bird rule. Granot and Huberman (1981, 1984) have studied the core and the nucleolus in the cooperative game of the *mcstp*. Kar (2002) studied the Shapley value in this context, and Dutta and Kar (2004) came up with another rule, called the Dutta-Kar rule. Bergantiños and Vidal-Puga (2007) studied the Kar rule in the irreducible form, which was later called the Folk solution.

The Bird rule, the Dutta-Kar rule, and the Folk solution are all based on the Prim algorithm.

Feltkamp et al. (1994) proposed a rule based on the Kruskal algorithm, which is shown to be the same as the Folk solution (Bergantiños and Lorenzo-Freire, 2008).

Bergantiños and Vidal-Puga (2011) proposed still another rule based on the Boruvka algorithm and showed that it coincides with the Folk solution as well.

The literature on *mcstp* has mainly focused on the case with one source only, and the papers mentioned above are of no exception. Hence, not much work has been done nor has many of the rules been defined in the case with two or more sources.

As a first step in the study into the multiple source context in the *mcstp*, we study and modify the rule proposed by Bergantiños and Vidal-Puga (2011) for the two-source case and show that this rule is the same as the Shapley value in the irreducible form of the two-source minimum spanning tree problem (*2s-mcstp*).

In section 2, we introduce the two-source minimum cost spanning tree problem. The Boruvka algorithm is presented in section 3. Section 4 explains the irreducible form and the rule based on the Boruvka algorithm. In section 5, we show and prove the main result. And, section 6 concludes.

## 2 Preliminaries

Let $\mathbf{N} = \{1, 2, \ldots\}$ be the set of all possible agents. A subset $N = \{1, \ldots, n\} \in \mathbf{N}$ represents a (typically selected) set of agents, and $O = \{o_1, o_2\}$ is the set of *sources*. Given $N$ and $O$, let $N_O = N \cup O$ denote the set of nodes and also define $N_{o_h} = N \cup \{o_h\}, o_h \in O$.

Given $N_O$, a cost matrix $C = (c_{ij})_{i,j \in N_O}$ represents the cost of the (direct) *link* $(i, j)$ between $i$ and $j$. For all $i, j \in N_0, c_{ij} = c_{ji}, c_{ij} \geq 0$, and $c_{ii} = 0$. That is, the links are undirected and nonnegative. Let $\mathcal{C}^{N_O}$ be the set of all cost matrices over $N_O$.

A *two-source minimum cost spanning tree problem, 2s-mcstp,* is a pair $(N_O, C)$ where $N \subset \mathbf{N}$ is a set of agents, $o_1$ and $o_2$ are the sources, and $C \in \mathcal{C}^{N_O}$ is the cost matrix.

Let $\mathbf{L} = \{(i, j) | i, j \in N_O, i \neq j\}$ denote the set of all possible links.

A network over $N_O$ is denoted $g \subset \mathbf{L}$.

A *path* from $i$ to $j$ in $g$ is a sequence of distinct links $\{(i_{k-1}, i_k)\}_{k=1}^{K}$ such that $(i_{k-1}, i_k) \in g$ for all $k \in \{1, \ldots, K\}$ with $i_0 = i$ and $i_K = j$. Two nodes $i$ and $j$ $(i \neq j)$ are *connected* in $g$ if there exists a path from $i$ to $j$ in $g$.

A subset of nodes $D \subset N_O$ is called a *component* in $g$ if all $i, j \in D$ are connected in $g$. Consider isolated nodes as components. Let $\mathbf{D}(g)$ be the set of components in $g$, $D \in \mathbf{D}(g)$.

A network $g$ is connected if all pairs of nodes in $N_O$ are connected in $g$.

A *tree* $t$ is a connected network with exactly $|N_O| - 1$ links, in which there exists a unique path from $i$ to $j$, for all $i, j \in N_O$. Denote this path as $t_{ij} \subset t$.

Let $\mathbf{G}^{N_O}$ denote the set of all networks over $N_O$ and $\hat{\mathbf{G}}^{N_O}$ the set of all connected networks over $N_O$.

The *cost* of $g \in \mathbf{G}^{N_O}$ in a *2s-mcstp* $(N_O, C)$ is

$$c(g) = c(N_O, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

A *minimum cost spanning tree, mcst,* for $(N_O, C)$ is a tree $t \in \hat{\mathbf{G}}^{N_O}$ such that

$$c(t) = \min_{g \in \hat{\mathbf{G}}^{N_O}} c(N_O, C, g).$$

Denote the cost of the *mcst* $t$ in $(N_O, C)$ as $m(N_O, C)$.

A *rule*, or *cost allocation rule*, is a function $\psi$ where $\psi(N_O, C) \in \mathbb{R}^n$ for each *2s-mcstp* $(N_O, C)$ such that $\sum_{i \in N} \psi_i(N_O, C) = m(N_O, C)$.

Given *2s-mcstp* $(N_O, C)$, let $(S_O, C)$ denote the *2s-mcstp* induced by $C$ and $S \subset N$.

Let $\sigma$ be an ordering over $N$ and $\Sigma_N$ the set of all such orderings. Also, let $\sigma_S$ be an ordering over $S \subset N$ and $\sigma_S \in \Sigma_S$.

Given $\sigma \in \Sigma_N$, denote $P^\sigma(i)$ as the set of agents in $N$ that stand before $i$ according to $\sigma$,

$$P^\sigma(i) = \{j \in N | \sigma(j) < \sigma(i)\}.$$

The *Shapley value* $\phi$ for *2s-mcstp* $(N_O, C)$ is defined as

$$\phi_i(N_O, C) = \frac{1}{n!} \sum_{\sigma \in \Sigma_N} [m((P^\sigma(i) \cup \{i\})_O, C) - m(P^\sigma(i)_O, C)].$$

## 3 The Boruvka Algorithm

Since we are interested in the Shapley value in the irreducible form game, we need to find an *mcst* for any given *2s-mcstp* $(N_O, C)$. We introduce the Boruvka algorithm that is modified for the two-source case from Bergantinos and Vidal-Puga (2011).

**The Boruvka algorithm**

Let $\pi$ be an ordering over the set of all possible links over $N_O$,

$$\pi : \{(i,j) : i,j \in N_O, i \neq j\} \to \left\{1, 2, \ldots, \frac{(n+2)(n+1)}{2}\right\}.$$

Let $g^{\pi,0} = \emptyset$, the initial network. Then, the set of components is $\mathbf{D}(g^{\pi,0}) = \{\{o_1\}, \{o_2\}, \{1\}, \{2\}, \ldots, \{n\}\}$.

**Step 1**. For each component $D \in \mathbf{D}(g^{\pi,0})$, where $D \neq \{o_1\}, \{o_2\}$, choose a link $(i,j)^{\pi,D} \in D \times (N_O \setminus D)$ that is the cheapest to connect $D$ and $N_O \setminus D$. We pick the one in the front according to the ordering $\pi$ in the case of a tie. This link is added to the network. Formally,

$$g^{\pi,1} = g^{\pi,0} \cup \{(i,j)^{\pi,D} : D \in \mathbf{D}(g^{\pi,0}), D \neq \{o_1\}, \{o_2\}\}.$$

Then, we have $\mathbf{D}(g^{\pi,1})$ from $g^{\pi,1}$.

Assume we have reached Step $s$ $(s = 1, 2, \ldots)$ and we have defined $g^{\pi,s-1}$.

**Step s**. For each component $D \in \mathbf{D}(g^{\pi,s-1})$, where $D \neq \{o_1\}, \{o_2\}$, let $(i,j)^{\pi,D} \in D \times (N_O \setminus D)$ be the cheapest link to connect $D$ and $(N_O \setminus D)$. Formally,

$$g^{\pi,s} = g^{\pi,s-1} \cup \{(i,j)^{\pi,D} : D \in \mathbf{D}(g^{\pi,s-1}), D \neq \{o_1\}, \{o_2\}\}.$$

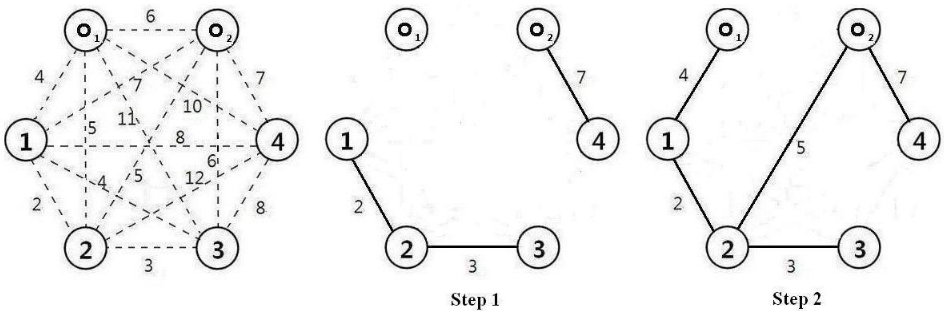


Figure 1: A *2s-mcstp* and its *mcst* following the Boruvka algorithm

When all nodes in $N_O$ are connected, the process ends.

Let $\tau$ be the final step of the process. Then, $g^{\pi,\tau}$ has no cycle and is an *mcst* of $(N_O, C)$, $t^{\pi}$.

For two different orderings over links $\pi$ and $\pi'$, it might be that $t^{\pi} = t^{\pi'}$. In particular, it is always the case when the costs of all links are different. When there is no ambiguity, we leave out $\pi$ from the notation (for example, $g^s$ instead of $g^{\pi,s}$).

**Example 1.** Consider a *2s-mcstp* $(N_O, C)$ with $N = \{1, 2, 3, 4\}$ as shown in Figure 1.

Set $g^0 = \emptyset$. In step 1, $\{1\}$ and $\{2\}$ select $(1, 2)$, $\{3\}$ selects $(2, 3)$, and $\{4\}$ selects $(o_2, 4)$. $g^1 = \{(1, 2), (2, 3), (o_2, 4)\}$. In step 2, $\{1, 2, 3\}$ selects $(o_1, 1)$ and $\{o_2, 4\}$ selects $(o_2, 2)$. $g^2 = g^1 \cup \{(o_1, 1), (o_2, 4)\}$. The process ends.

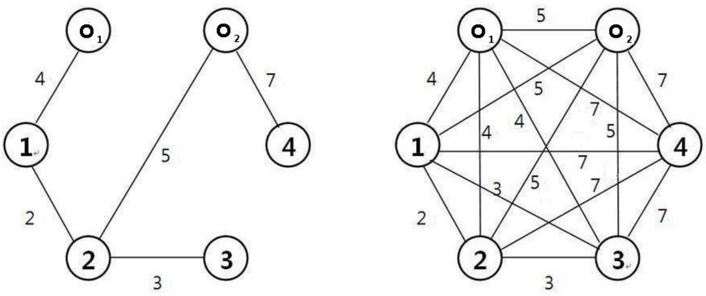


Figure 2: An *mcst* following the Boruvka algorithm and the corresponding irreducible form

## 4 The Irreducible Form and the Rule $\beta^{\pi}$

Now that we have an *mcst* $t$ from the Boruvka algorithm, we can define the irreducible form of the problem. Given a *2s-mcstp* $(N_O, C)$ and an *mcst* $t$, define $C^* = (c^*_{ij})_{i,j \in N_O}$ as the irreducible cost matrix associated with $C$ , where $c^*_{ij} = \max_{(kl)\in t_{ij}}\{c_{kl}\}$ for each pair $(i, j)$. Recall that $t_{ij}$ is the unique path from $i$ to $j$ in $t$.

Let $(N_O, C^*)$ be an irreducible form of a *2s-mcstp* $(N_O, C)$. Note that $(N_O, C^*)$ is an irreducible form if and only if reducing the cost of a link always reduces the cost of connecting all agents to the sources.

In Figure 2, the network on the right represents the irreducible form generated based on the *mcst* on the left.

For a given a *2s-mcstp* $(N_O, C)$, we can now compute the Shapley value in the irreducible form,

$$\phi(N_O, C^*) = \frac{1}{n!}\sum_{\sigma\in\Sigma_N}[m((P^{\sigma}(i)\cup\{i\})_O, C^*) - m(P^{\sigma}(i)_O, C^*)].$$

Note that $\phi(N_O, C^*) = (\frac{49}{12}, \frac{49}{12}, \frac{55}{12}, \frac{33}{4})$ for the problem in Example 1.

We define the rule $\beta^\pi$ following the process below, which is a modification for two-source case from Bergantinos and Vidal-Puga (2011).

We first define some notation.

Denote: $\ell_i^{s,\pi}$ as the link in $t$ that agent $i$ is assigned to (pays in part) in Step $s$; $p^{s,\pi}$ as the proportion of the cost of the link that each agent has to pay in Step $s$; $\bar{p}_{ij}^{s,\pi}$ as the proportion of the cost of link $(i,j)$ already paid up to Step $s$; $L^{s,\pi}$ as the set of non-completely paid links in Step $s$, $L^{s,\pi} = \{(i,j) \in t : \bar{p}_{ij}^{s,\pi} < 1\}$; $f_i^{s,\pi}$ as the cost that agent $i$ must pay in Step $s$, $f_i^{s,\pi} = p^{s,\pi} c_{\ell_i^{s,\pi}}$; $\bar{L}^{s,\pi} = t \backslash L^{s,\pi} = \{(i,j) \in t : \bar{p}_{ij}^{s,\pi} = 1\}$; and $\mathbf{D}^{s,\pi}$ as the set of components of $N_O$ associated to $\bar{L}^s$.

For simplicity, we omit $\pi$ from the notation.

Let $\ell_i^0 = \emptyset$ for all $i \in N$, $p^0 = 0$, $\bar{p}_{ij}^0 = 0$ for all $(i,j) \in t$, $L^0 = t$, $\bar{L}^0 = \emptyset$, $\mathbf{D}^0 = \mathbf{D}(\emptyset)$, and $f_i^0 = 0$ for all $i \in N$. Assume we have reached Step $s$ $(s = 1, 2, \ldots)$.

**Step s**. Given a component $D \in \mathbf{D}^0$, $D \neq \{o_1\}, \{o_2\}$, select a link $(i,j)^D \in t$ as in the Boruvka algorithm. If $(i,j)^D$ has been selected by $D$ in Step $s-1$ but has not been fully paid, then $D$ must select $(i,j)^D$ again in Step $s$.

For all $d \in D \in \mathbf{D}^{s-1}$, set $\ell_d^s = (i,j)^D$. That is, each agent pays the cost of the link that is selected by the component he/she is in. For each link $(i,j) \in L^{s-1}$, let $N_{ij}^s = \{k \in N : \ell_k^s = (i,j)\}$ be the set of agents that pay the cost of link $(i,j)$. Then, define

$$p^s = \min\left\{\frac{1 - \bar{p}_{ij}^{s-1}}{|N_{ij}^s|} : (i,j) \in L^{s-1}, N_{ij}^s \neq \emptyset\right\}.$$

For each $(i,j) \in L^{s-1}$, define $\bar{p}_{ij}^s = \bar{p}_{ij}^{s-1} + |N_{ij}^s| p^s$.

Notice that $\bar{p}_{ij}^s \leq 1$ for all $(i,j) \in L^{s-1}$ and $\bar{p}_{ij}^s = 1$ for at least one $(i,j) \in L^{s-1}$, which leads to $L^s \subsetneq L^{s-1}$ and $\bar{L}^{s-1} \subsetneq \bar{L}^s$. That is, there are more completely paid links in the latter step. The process ends when $\bar{L}^s = t$, and from $\bar{L}^{s-1} \subsetneq \bar{L}^s$ we know that the process ends in a finite number of steps, $\tau$. Finally, the process ends when $\sum_{s=1}^{\tau} p^s = \frac{n+1}{n}$.

**Definition 1.** Given a *2s-mcstp* $(N_O, C)$ and an ordering $\pi$ over the set of all possible links, the rule $\beta^\pi$ is defined as

$$\beta_i^\pi(N_O, C) = \sum_{s=1}^{\tau} f_i^s$$

for each $i \in N$.

**Example 2.** Consider the same problem as in Example 1.

Following the Boruvka algorithm, the *mcst* $t = \{(o_1, 1), (1, 2), (2, 3), (2, o_2), (o_2, 4)\}$ as shown in Figure 3.

In step 1, $\{1\}$ and $\{2\}$ selects $(1, 2)$ and agent 1 and 2 share the cost equally, $\{3\}$ selects $(2, 3)$ and agent 3 pays the half of its cost, and $\{4\}$ selects $(o_2, 4)$ and agent 4 pays the half of its cost. In step 2, $\{1, 2\}$ and $\{3\}$ selects $(2, 3)$ and agent 1,2, and 3 each pay $\frac{1}{6}$ of its cost while $\{4\}$ selects $(o_2, 4)$ and agent 4 pays $\frac{1}{6}$ of its cost. In step 3, $\{1, 2, 3\}$ selects $(o_1, 1)$ and agent 1,2, and 3 each pay $\frac{1}{3}$ of its cost while $\{4\}$ still selects $(o_2, 4)$ and agent 4 pays the remaining $\frac{1}{3}$ of its cost. In step 4, $\{1, 2, 3\}$ and $\{4\}$ select $(2, o_2)$ and all four agents share its cost evenly by $\frac{1}{4}$.

This process is presented in Table 1. We can see that the final allocation is the same as the Shapley value of the irreducible form obtained above. Notice in the first row of the table the agents are partitioned into two groups, namely a group with agent

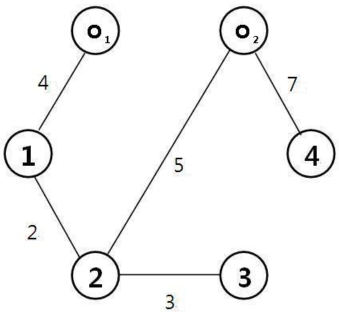


Figure 3: An *mcst* following the Boruvka algorithm

| | { 1 | 2 | 3 } | { 4 } | *p* | ∑*p* |
|---|---|---|---|---|---|---|
| **S1** | (1/2)*2=1 | (1/2)*2=1 | (1/2)*3=3/2 | (1/2)*7=7/2 | 1/2 | 1/2 |
| **S2** | (1/6)*3=1/2 | (1/6)*3=1/2 | (1/6)*3=1/2 | (1/6)*7=7/6 | 1/6 | 2/3 |
| **S3** | (1/3)*4=4/3 | (1/3)*4=4/3 | (1/3)*4=4/3 | (1/3)*7=7/3 | 1/3 | **1** |
| **S4** | (1/4)***5**=5/4 | (1/4)***5**=5/4 | (1/4)***5**=5/4 | (1/4)***5**=5/4 | 1/4 | 5/4 |
| | 49/12 | + 49/12 | + 55/12 | + 33/4 | = 21 | |

Table 1: The process of computing the rule $\beta^{\pi}$

1, 2, and 3 and another group with agent 4. These two groups are never connected until they are connected by the last link, for which all agents pays the same amount at the final step. In this case the link is $\{2, o_2\}$ with the cost 5.

In other words, these two groups of agents never share the cost of any links until the very last step of the process. This idea is a key to the proof of our main result that we now present.

## 5 Main Result

The main result shows that the Shapley value in the irreducible form of a *2s-mcstp* is the same as the rule $\beta^\pi$.

**Theorem 1.** Given a *2s-mcstp* $(N_O, C)$ and an ordering $\pi$ over the set of all possible links,

$$\beta^\pi(N_O, C) = \phi(N_O, C^*).$$

*Proof of Theorem 1*

First, we define the problem with one source only.

A *one-source minimum cost spanning tree problem, 1s-mcstp,* is a pair $(N_o, C)$ where $N \subset \mathbf{N}$ is a set of agents, $o$ is the source, and $C$ is the cost matrix ($N_o = N \cup \{o\}$).

Assume that we have an *mcst* $t$ following the Boruvka algorithm. Let $M$ be the link with the highest cost along the path between $o_1$ and $o_2$ in $t$. Removing $M$ breaks $t$ into two trees containing one source each.

Without loss of generality, denote $t^1$ as the tree containing $o_1$ and $t^2$ the one with $o_2$. Also, Denote $S^1$ as the set of agents in $t^1$ and $S^2$ as the set of agents in $t^2$.

Then, we prove the following two lemmas:

**Lemma 1.** For all $i \in N$,

$$\beta_i^\pi(N_O, C) = \beta_i^\pi(S^h_{o_h}, C) + \frac{c_M}{|N|}, \tag{1}$$

where $h$ indicates the index for the tree that $i$ belongs to.

**Lemma 2.** For all $i \in N$,

$$\phi_i(N_O, C^*) = \phi_i(S^h_{o_h}, C^*) + \frac{c_M}{|N|}, \tag{2}$$

where $h$ indicates the index for the tree that $i$ belongs to.

Bergantinos and Vidal-Puga (2011) showed that $\beta^\pi = \phi$ for each $\pi$ in any *1s-mcstp* $(N_o, C)$. Thus,

$$\beta^\pi_i(S^h_{o_h}, C) = \phi_i(S^h_{o_h}, C^*). \tag{3}$$

Putting Equation (1), (2), and (3) together yields,

$$\beta^\pi_i(N_O, C) = \beta^\pi_i(S^h_{o_h}, C) + \frac{c_M}{|N|} = \phi_i(S^h_{o_h}, C^*) + \frac{c_M}{|N|} = \phi_i(N_O, C^*). \tag{4}$$

Without loss of generality, we only analyze $S^1$.

*Proof of Lemma 1* $\beta^\pi_i(N_O, C) = \beta^\pi_i(S^1_{o_1}, C) + \frac{c_M}{|N|}$.

We prove this by showing that all agents are assigned to link $M$ in the Step $\tau$ following the Boruvka algorithm, which also guarantees the cost within $t^1$ is paid only by agents in $S^1$. Then, by definition of the rule $\beta^\pi$, Lemma 1 holds.

When the process ends, $\sum_{s=1}^{\tau} p^s = \frac{n+1}{n}$. If link $M$ is, in fact, the link being selected in Step $\tau$ by all agent, then all links $t^1$ must have been paid already in Step $\tau - 1$.

Denote $\hat{S}^1$ as the set of agents directly connected to $o_1$ in $t^1$. For each $i \in \hat{S}^1$, let $F^i$ be the set of agents that need agent $i$ in order to be (indirectly) connected to $o_1$ in $t^1$, including $i$ himself/herself. Then, $\{F^i\}_{i \in \hat{S}^1}$ forms a partition of $S^1$.

Define $t_{F^i} = \{(i,j) : (i,j) \in t^1; i,j \in F^i \cup \{o_1\}\}$. Clearly, $t_{F^i}$ is an *mcst* in $(F^i_{o_1}, C)$. Also, $\cup(t_{F^i})_{i \in \hat{S}^1} = t^1$ and $\sum_{i \in \hat{S}^1} m(F^i_{o_1}, C) = m(S^1_{o_1}, C)$.

We need to show that: **(i)** no agent in any $F^i$ is assigned to a link $(k,l) \in F^i \times S^1 \setminus F^i$ before Step $\tau$, $\sum_{s=1}^{\tau-1} p^s = \frac{|F^i|}{|F^i|} = 1$. That is, no agent crosses over to another "branch" that he/she does not belong to.

In order to show this, we use induction. If $|F^i| = 1$, then condition **(i)** is satisfied by definition. Assume that we have verified that condition **(i)** holds for less than $|F^i|$ agents.

Denote $M_{F^i}$ as the link with the highest cost in $t_{F^i}$. Removing it from $t_{F^i}$ yields a tree containing $\alpha < |F^i|$ agents ($\alpha$ is a positive integer) and $o_1$ and another tree with $|F^i| - \alpha$ agents. By induction hypothesis, condition **(i)** is satisfied in the first tree. The agents in the second tree pays for the links within it up to some Step $r$, where $r = \tau - 2$ with $\sum_{s=1}^{\tau-2} p^s = \frac{|F^i|-\alpha-1}{|F^i|-\alpha}$. Then they pay for $M_{F^i}$ with $p^{\tau-1} = \frac{1}{|F^i|-\alpha}$, making $\sum_{s=1}^{\tau-1} p^s = 1$. Thus, condition **(i)** is satisfied for $|F^i|$ agents, and Equation (1) holds.

*Proof of Lemma 2* $\phi_i(N_O, C^*) = \phi_i(S^1_{o_1}, C^*) + \frac{c_M}{|N|}$

By definition, we know that

$$\phi(N_O, C^*) = \frac{1}{n!} \sum_{\sigma \in \Sigma_N} [m((P^\sigma(i) \cup \{i\})_O, C^*) - m(P^\sigma(i)_O, C^*)]$$

Setting $mc^{i,\sigma} = m((P^\sigma(i) \cup \{i\})_O, C^*) - m(P^\sigma(i)_O, C^*)$,

$$\phi_i(N_O, C^*) = \frac{1}{n!} \sum_{\sigma \in \Sigma_N} mc^{i,\sigma}.$$

Also, we already know under the irreducible cost matrix,

$$c^*_{ij} = max_{(k,l) \in t_{ij}} \{c_{kl}\}. \tag{5}$$

Since the *2s-mcstp* requires agents to be connected to both sources, for any given ordering over agents, the first agent in the order connects to both sources and pays the corresponding costs. Starting from the second agent in the order, they can connect to either one of their predecessors or the (closest) source that minimizes their marginal costs.

Formally, given an ordering $\sigma \in \Sigma_N$, agent $i \in S^1$ who is not the first agent in $\sigma$ chooses a node $j \in P^\sigma(j) \cup O$, $j \neq i$ that minimizes $c^*_{ij}$, which $i$ pays under $\sigma$. By (5), it must be that $j \in (P^\sigma(j) \cap S^1) \cup \{o_1\}$. Denote this cost as $c^*_{i,\sigma}$. Thus, for $i \in S^1$, $i \neq \sigma(1)$,

$$mc^{i,\sigma} = c^*_{i,\sigma} \tag{6}$$

If $i \in S^1$ is the first agent in $\sigma$, he/she not only pays $c^*_{i,\sigma}$ but also the cost of connecting to $o_2$. Because of (5), $i$ does not connect directly to $o_2$ but through $o_1$, so the cost is $c^*_{o_1 o_2}$. Again by (5), $c^*_{o_1 o_2} = c^*_M$, since we know $M$ is the link with the highest cost along the path between $o_1$ and $o_2$ in $t$. Thus, for $i \in S^1$, $i = \sigma(1)$,

$$mc^{i,\sigma} = c^*_{i,\sigma} + c^*_M \tag{7}$$

Now, let $\sigma^i \in \Sigma_N$ be the ordering over $N$ that starts with agent $i$. Then, we have

$$mc^{i,\sigma^i} = c^*_{i,\sigma} + c^*_M. \tag{8}$$

Recall that

$$\phi_i(N_O, C^*) = \frac{1}{n!} \sum_{\sigma \in \Sigma_N} mc^{i,\sigma}.$$

Using (6) and (8), we have

$$\phi_i(N_O, C^*) = \frac{1}{n!} \sum_{\sigma \in \Sigma_N} c^*_{i,\sigma} + \frac{1}{n!} \sum_{\sigma^i \in \Sigma_N} c^*_M$$

It is clear that the number of $\sigma^i$ in $\Sigma_N$ is the same for all $i \in N$. Then,

$$\phi_i(N_O, C^*) = \frac{1}{n!} \sum_{\sigma \in \Sigma_N} c^*_{i,\sigma} + \frac{1}{n!} \cdot \frac{n!}{n} c^*_M.$$

$$\phi_i(N_O, C^*) = \frac{1}{n!} \sum_{\sigma \in \Sigma_N} c^*_{i,\sigma} + \frac{1}{n} c^*_M. \tag{9}$$

Now, for the *1s-mcstp* $(S^1_{o_1}, C)$,

$$\phi(S^1_{o_1}, C^*) = \frac{1}{s^1!} \sum_{\sigma_{S^1} \in \Sigma_{S^1}} [m((P^{\sigma^{S^1}}(i) \cup \{i\})_{o_1}, C^*) - m(P^{\sigma^{S^1}}(i)_{o_1}, C^*)]$$

We follow the same logic and get

$$\phi(S^1_{o_1}, C^*) = \frac{1}{s^1!} \sum_{\sigma_{S^1} \in \Sigma_{S^1}} mc^{i,\sigma_{S^1}},$$

$$\phi(S^1_{o_1}, C^*) = \frac{1}{s^1!} \sum_{\sigma_{S^1} \in \Sigma_{S^1}} c^*_{i,\sigma_{S^1}}. \tag{10}$$

Given an ordering $\sigma_{S^1} \in \Sigma_{S^1}$, there exist $\frac{n!}{s^1!}$ orderings in $\Sigma_N$ such that removing $N \setminus S^1$ from those orderings yields $\sigma_{S^1}$. In other words, for each ordering $\sigma_{S^1} \in \Sigma_{S^1}$, there are $\frac{n!}{s^1!}$ orderings in $\Sigma_N$ that give the same marginal cost to the agents in $S^1$. Thus, from (9) we obtain

$$\phi_i(N_O, C^*) = \frac{1}{n!} \cdot \frac{n!}{s^1!} \sum_{\sigma_{S^1} \in \Sigma_{S^1}} c^*_{i,\sigma_{S^1}} + \frac{1}{n} c^*_M.$$

$$\phi_i(N_O, C^*) = \frac{1}{s^1!} \sum_{\sigma_{S^1} \in \Sigma_{S^1}} c^*_{i,\sigma_{S^1}} + \frac{1}{n} c^*_M. \tag{11}$$

Combining (10) and (11) yields (2). □

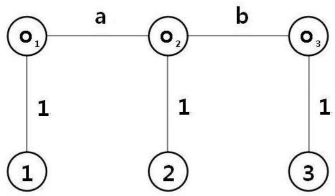


Figure 4: An *mcst* with three sources

## 6 Concluding Remarks

As mentioned in the introduction, this paper is merely a small step to analyzing the *2s-mcstp*. Defining other rules such as the Bird rule and the Dutta-Kar rule for *2s-mcstp* and analyzing them may be of interest. Extending the study to three or more sources can be another area for future research. We conclude the paper with a counter example that Theorem 1 does not hold in general for *mcstp* with multiple sources.

**Example 3** In Figure 4, we have an *mcst* of an *mcstp* with three sources and three agents. Given $a < b$, we can see that Theorem 1 does not hold as shown below.

$$\beta^{\pi}(N_O, C) = (1 + \frac{1}{2}a + \frac{1}{6}b, 1 + \frac{1}{2}a + \frac{1}{6}b, 1 + \frac{2}{3}b).$$

$$\phi(N_O, C^*) = (1 + \frac{1}{3}(a + b), 1 + \frac{1}{3}(a + b), 1 + \frac{1}{3}(a + b)).$$

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# 국문초록

# 소스가 두 개인 최소신장가지문제에서의 샤플리 밸류

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소스가 두 개인 최소신장가지문제(Minimim Cost Spanning Tree Problem)에 대해 알아본다. 경기자들은 두 개의 소스에 직접 연결되거나 다른 경기자들을 통해 연결되어야 한다. 각 연결에는 비용이 발생하고 전체의 비용은 경기자들 사이에서 모두 지불되어야한다. 이를 위해 보르부카 알고리듬에 기반을 둔 비용배분 규칙을 정의하고 축약불가문제에서 이 비용배분이 이미 잘 알려진 샤플리 밸류와 동일함을 밝힌다.

주요어 : 최소신장가지문제, 보르부카 알고리듬, 축약불가문제,
비용배분, 샤플리 밸류

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