Artin L-functions for abelian extensions of imaginary quadratic fields.

Thesis by

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In memory of my grandfather

Lyle Albert Bean

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It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.

–Karl Friedrich Gauss

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Abstract

Let F be an abelian extension of an imaginary quadratic field K with Galois group G. We form the Galois-equivariant L-function of the motive $M = h^0(\operatorname{Spec} F)(j)$ where the Tate twists j are negative integers. The leading term in Taylor expansion at $s = 0$ decomposes over the group algebra $\mathbb{Q}[G]$ into a product of Artin L-functions indexed by the characters of G. We construct a motivic element ξ via the Eisenstein symbol and relate the L-value to periods of ξ via regulator maps. Working toward the equivariant Tamagawa number conjecture, we prove that the L-value gives a basis in étale cohomology which coincides with the basis given by the p-adic L-function according to the main conjecture of Iwasawa theory.

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Chapter 1

Introduction

Number theorists are like lotus-eaters – having once tasted of this food they can never give it up.

–Leopold Kronecker

1.1 History

Euler's eighteenth century solution to the Basel problem on finding a closed form for the infinite sum

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$$

led to his interest in the zeta function and subsequent discovery of the Euler product

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}$$

and sowed the seed for the study of special values of L-functions. This seed germinated in the mid-nineteenth century with the work of Dirichlet, Riemann and Dedekind who recognized relationships between zeta functions and the structure of integers. In order to study the density of the prime numbers, Riemann considered $\zeta(s)$ as a function on the entire complex plane with a simple pole at $s = 1$. While Dedekind, after formulating the theory of ideals, was able to define for any number field F the zeta function

$$\zeta_F(s) = \sum_{\mathfrak{a}\subseteq\mathcal{O}_F} \mathcal{N}\mathfrak{a}^{-s} = \prod_{\mathfrak{p}} (1-\mathcal{N}\mathfrak{p}^{-s})^{-1}.$$

Much of modern number theory is rooted in these discoveries, though for the present investigation we shall focus on the following pair of results.

Theorem 1.1.1. Unit Theorem *(Dirichlet)*

Let S be a finite set of places of F containing the infinite ones, $Y_{F,S} = \bigoplus_{v\in S}\mathbb{Z} = \{\sum_{v\in S} n_v \cdot v : n_v \in \mathbb{Z}\}$, and $\mathcal{O}_{F,S}$ denote the ring of S-integers of F. The regulator is the map

$$\begin{aligned} \lambda_{F,S} :&\mathcal{O}_{F,S}^\times \to Y_{F,S} \otimes_{\mathbb{Z}} \mathbb{R} \\ &u \mapsto \sum_{v\in S} \log|u|_v \cdot v. \end{aligned}$$

Setting $X_{F,S} := \ker\left(Y_{F,S} \xrightarrow{\Sigma} \mathbb{Z}\right)$, the following properties hold:

a) $\ker(\lambda_{F,S})$ *is a finite group.*

b) $\operatorname{im}(\lambda_{F,S})$ *is a discrete lattice in $X_{F,S} \otimes_{\mathbb{Z}} \mathbb{R}$.*

c) $\lambda_{F,S}$ induces an isomorphism $\mathcal{O}_{F,S}^\times \otimes_{\mathbb{Z}} \mathbb{R} \simeq X_{F,S} \otimes_{\mathbb{Z}} \mathbb{R}$. Therefore,

$$\mathcal{O}_{F,S}^\times \simeq W_{F,S} \times \mathbb{Z}^{|S|-1}$$

where $W_{F,S}$ is a finite group.

We formulate the class number formula in the case that S is comprised only of the inifinite places.

Theorem 1.1.2. Analytic Class Number Formula *(Dedekind)*

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1+r_2}\pi^{r_2}}{w_F\sqrt{|d_F|}} R_F h_F$$

where r_1 (r_2) is the number of real (complex) places of F, h_F is the class number, d_F is the discriminant, w_F is the order of the group of roots of unity and the regulator R_F is the covolume of the lattice determined by the units.

In fact, fixing a place v_0, determines a pairing

$$\mathcal{O}_{F,S}^{\times} \times \mathrm{Hom}_{\mathbb{Z}}(X_{F,S}, \mathbb{Z}) \to \mathbb{R},$$

and the regulator $R_F = |\det(\log |u_i|_{v_j})|$ is independent of the choice of basis u_i and place v_0. Using the functional equation of the zeta function, this can be reformulated to a statement at $s = 0$

$$\lim_{s \to 0} s^{1-r_1-r_2}\zeta_F(s) = -h_F R_F / w_F,$$

giving a direct method for computing the class number and tantalizing mathematicians with the promise of a new window into the structure of integers.

1.2 Modern directions

The philosophy that locally defined complex analytic functions can encode global algebraic information motivated a large body of number theory in the twentieth century. With the development of algebraic geometry by Weil and Grothendieck number theorists acquired new tools of investigation. The notion of L-function generalized that of the zeta function, and conjectures proliferated. Again, we focus on a specific generalization which is due to Stark.

Let F/K be any finite Galois extension of number fields with group $G := \mathrm{Gal}(F/K)$. For a complex representation V of G, we can attach an Artin L-function which depends only on the character (or trace) χ of the representation

$$L(\chi, s) = \prod_{\mathfrak{p}} (\det(1 - \mathrm{Fr}_{\mathfrak{P}} \mathcal{N}\mathfrak{p}^{-s} | V^{I_{\mathfrak{P}}})^{-1}.$$

Here $I_{\mathfrak{P}}$ denotes the inertia group of the prime $\mathfrak{P} \mid \mathfrak{p}$ and $\mathrm{Fr}_{\mathfrak{P}}$ is a lift of the Frobenius at $\mathfrak{P}$. The regulator $\lambda_{F,S}$ is G-equivariant with the action $|x|_{\sigma v} = |\sigma^{-1} x|_v$. Stark fixes a $\mathbb{Q}[G]$ isomorphism

$$f : X_{F,S} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\simeq} \mathcal{O}_{F,S}^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The composition $(\lambda_{F,S} \circ f)_V \in \mathrm{Aut}_{\mathbb{C}}(\mathrm{Hom}_G(V^*, X_{F,S} \otimes_{\mathbb{Z}} \mathbb{C}))$ and we define the Stark regulator to be

$$R(\chi, f) = \det((\lambda_{F,S} \circ f)_V).$$

Conjecture 1. *(Stark) Let $L^*(\chi, 0)$ denote the leading term in the Taylor expansion at $s = 0$ of $L(\chi, s)$, and define $A(\chi, f) := R(\chi, f)/L^*(\chi, 0)$. Then $A(\chi, f) \in \mathbb{Q}(\chi)$, the field of values of χ, and*

$$A(\chi, f)^{\alpha} = A(\chi^{\alpha}, f)$$

for all $\alpha \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$.

Thus Stark conjectured that the L-function not only encodes data about the number field but also retains information about the group action. This statement is proved in a very limited number of cases, for example $\mathbb{Q}$-valued characters.

Generalizing from the unit theorem in a different direction than Stark, Borel's work [Bor74] on the K-theory of fields gives a suitable alternative to units of a number field when $s \neq 0$ and defines a regulator map on these K-groups. Gross then formulated a version of Stark's conjecture for negative values of the L-functions in terms of Borel's regulator map [Neu88].

The equivariant Tamagawa number conjecture for an extension of number fields generalizes the analytic class number formula in the same way that Gross's conjecture generalizes Dirichlet's unit theorem. The Tamagawa number conjecture is originally due to Bloch and Kato [BK90], and the formulation used in this work is due to Fontaine and Perrin-Riou [FPR94]. Many of these conjectures have been made for algebraic varieties in general. In fact our investigation is a special case of the equivariant Tamagawa number conjecture of Burns and Flach [BF01] for motives over $\mathbb{Q}$ with (non-commutative) coefficients which is summarized in section 2.3.

1.3 Current Work

In this work, we take K to be an imaginary quadratic field and G an abelian group. We study the Artin L-functions of the representations of G at negative integral values and describe them in terms of the ℓ-adic cohomology of the number field. By taking all representations of the group, we prove the equivariant Tamagawa number conjecture for the field F with the action of G modulo our formulation of the main conjecture. The next chapter is devoted to the precise formulation of the conjecture, and an exposition of the main theorem.

The proof relies heavily on an auxiliary object. Namely, we choose an elliptic curve E over F with complex multiplication by the ring of integers $\mathcal{O}_K$. We can then express the L-values, K-theory elements and ℓ-adic cohomology classes in terms of torsion points on the elliptic curve. Section 2.2 outlines the strategy of the proof.

As G is abelian, all representations of G are linear. Indeed, if χ is a representation of G, then we can also consider χ to be a character on the ideals of K where $\chi(\mathfrak{p}) := 0$ if $\mathfrak{p}$ is ramified in F. Thus, if the conductor of χ is a proper divisor of the conductor of F, the Artin L-function of χ differs from its Dirichlet L-function by a finite number of Euler factors. These Euler factors play a nontrivial role in our investigation.

1.4 Related Results

The only completely proven case of the equivairant Tamagawa number conjecture is the proof of Burns and Greither for abelian extensions of $\mathbb{Q}$ [BG03]. Huber and Kings proved independently a weaker version of this cyclotomic case [HK03].

Bley has also been considering the case of abelian extensions of imaginary quadratic fields, but at the point $s = 0$. Recall that this thesis looks at all $s = j$ where j is a negative integer. His recent preprint [Ble05] gives the ℓ-part of the conjecture at $s = 0$ for split $\ell \nmid h_K$ for abelian extensions of imaginary quadratic fields. The argument follows a similar argument to ours, but the difficulties lie in different steps.

For non-abelian extensions of $\mathbb{Q}$, the results are even more sparse. Burns and Flach give a proof for an infinite family of quaternion extensions [BF03]. Breuning tackles a family of dihedral extensions [Bre04], and Navilarekallu gives a method of proof for A_4 extensions which he employs for a specific case [Nav04].

There are also several theorems that are not equivariant. Gealy recently proved a weakened version of the Tamagawa number conjecture for modular forms of weight greater than 1 [Gea05]. Kings also proved a weakened version for elliptic curves with CM by an imaginary quadratic field of class number 1 [Kin01]. Bars builds on work of Kings to give some non-equivariant results for Hecke characters of imaginary quadratic fields [Bar03]

The survey paper of Flach [Fla04] includes a nice formulation of the general version of the equivariant Tamagawa number conjecture and discusses the proven cases.

1.5 A word on motives

By a motive we simply mean a pure Chow motive $M = h^i(X)(j)$ of a smooth projective variety $X/\operatorname{Spec}\mathbb{Q}$ with a Tate twist j. Indeed our concern is with the motive associated to the spectrum of a number field, and we will thus speak almost exclusively about $h^0(X)(j)$. In the context of Chow motives, the Tate twist is achieved by tensoring with the Tate motive. For a nice introductory treatment of correspondences and motives, see the notes by Murre [Mur04]. M will be endowed with the action of a semisimple $\mathbb{Q}$-algebra A. We study M via its realizations and the action of A on these spaces, focusing on the Betti realization $M_B := H^i(X(\mathbb{C}), \mathbb{Q}(j))$ which carries an action of complex conjugation, the étale or ℓ-adic realization $M_\ell := H^i_{et}(X \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_\ell(j))$ which is a continuous representation of the Galois group $G_{\mathbb{Q}}$, and the de Rham realization $M_{dR} := H^i_{dR}(X/\mathbb{Q})(j)$ with its Hodge filtration.

Chapter 2

The Main Theorem

The equivariant Tamagawa number conjecture has been stated very generally by Burns and Flach [BF01, BF03] for any motive over $\mathbb{Q}$ with the action of a semisimple, finite-dimensional $\mathbb{Q}$-algebra. For the present work we are considering the motive $M = h^0(\mathrm{Spec}(F))(j)$, where F/K is an abelian extension of an imaginary quadratic field and j is a negative integer. M carries an action of the semisimple $\mathbb{Q}$-algebra $A = \mathbb{Q}[\mathrm{Gal}(F/K)]$, so we formulate an equivariant Tamagawa number conjecture for the pair (M, A).

The A-equivariant L-function is constructed by taking an Euler product over the rational primes,

$$L({}_AM, s) = \prod_p \mathrm{Det}_A(1 - \mathrm{Fr}_p^{-1} \cdot p^{-s} | M_\ell^{I_p})^{-1}$$

where I_p is the inertia group at p and M_ℓ is the ℓ-adic realization of the motive for some fixed, auxiliary prime ℓ. Twisting by the Tate motive $\mathbb{Q}(j) = \mathbb{Q}(1)^{\otimes j}$ for $j \in \mathbb{Z}$ effects a shift in the Galois action on the ℓ-adic realization of the motive. We are concerned with the leading term of the Taylor expansion about $s = 0$, denoted by $L^*({}_AM)$ when $j < 0$.

We first formulate the conjecture for the case in question in section 2.1; we then state the main theorem and explain the method of proof in section 2.2. Section 2.3 briefly describes the general formulation.

2.1 Statement of the conjecture

A decomposes into a product of number fields indexed by the rational characters of $G := \mathrm{Gal}(F/K)$

$$A = \prod_{\chi \in \hat{G}^{\mathbb{Q}}} \mathbb{Q}(\chi),$$

where by a rational character, we mean an $\mathrm{Aut}(\mathbb{C})$ orbit $(\eta)_{\eta\in\chi}$ of complex characters. The A-equivariant L-function of M also decomposes over $\hat{G}^{\mathbb{Q}}$ so that the leading term

$$L^*({}_AM, s) = (L'(\chi, j))_{\chi\in\hat{G}^{\mathbb{Q}}} = (L'(\eta, j))_{\eta\in\hat{G}} \in A \otimes_{\mathbb{Q}} \mathbb{C}$$

exists by meromorphic continuation of the Artin L-function and lies in $(A \otimes_{\mathbb{Q}} \mathbb{R})^\times$. Note that $L(\eta, s)$ has a simple zero at every negative integer.

The dual of the Borel regulator gives an A-equivariant isomorphism

$$K_{1-2j}(\mathcal{O}_F)^* \otimes_{\mathbb{Z}} \mathbb{R} \xleftarrow{\rho_\infty^\vee} H^0_B(\mathrm{Spec}(F)(\mathbb{C}), \mathbb{Q}(j))^+ \otimes_{\mathbb{Q}} \mathbb{R},$$

where $H^\bullet_B$ denotes the Betti cohomology of the complex space, and $K_{1-2j}(\mathcal{O}_F)^*$ denotes the dual of the algebraic K-group $K_{1-2j}(\mathcal{O}_F) = K_{1-2j}(F)$ which is finite-dimensional over $\mathbb{Q}$ [Bor74].

Thus, defining

$$\Xi({}_AM) := \mathrm{Det}_A(K_{1-2j}(\mathcal{O}_F)^* \otimes \mathbb{Q}) \otimes_A \mathrm{Det}_A^{-1}(H^0_B(\mathrm{Spec}(F)(\mathbb{C}), \mathbb{Q}(j))^+),$$

we obtain an isomorphism

$${}_A\vartheta_\infty : A \otimes_{\mathbb{Q}} \mathbb{R} \to \Xi({}_AM) \otimes_{\mathbb{Q}} \mathbb{R}.$$

Gross conjectured that the image of $L^*({}_AM, 0)$ in fact lies in the rational space $\Xi({}_AM) \otimes_{\mathbb{Q}} 1$ [Neu88]. This can be understood as a Stark-type conjecture for negative integers. The Gross conjecture for abelian extensions of K was proved by Deninger [Den90] in his work on the Beilinson conjectures for Hecke characters of imaginary quadratic fields. We recall his result in chapter 3 and make some modifications to apply his construction to our situation.

Fix a prime number ℓ and put $A_\ell := A \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. We now construct an isomorphism which will compute the L-value in terms of compact support cohomology. First, for every finite prime p, define a complex of the ℓ-adic realization.

$$R\Gamma_f(\mathbb{Q}_p, M_\ell) = \begin{cases} M_\ell^{I_p} \xrightarrow{1-\mathrm{Fr}_p} M_\ell^{I_p} & \ell \neq p \\ D_{cris}(M_\ell) \xrightarrow{(1-\mathrm{Fr}_p, \pi)} D_{cris}(M_\ell) \oplus (D_{dR}(M_\ell)/\mathrm{Fil}^0 D_{dR}(M_\ell)) & \ell = p. \end{cases}$$

The modules D_{cris} and D_{dR} come from Fontaine's p-adic Hodge theory. We will distill the essential facts for our work below, but a nice introduction to the subject may be found in Berger's paper [Ber04].

There is a map $R\Gamma_f(\mathbb{Q}_p, M_\ell) \to R\Gamma(\mathbb{Q}_p, M_\ell)$, and we define $R\Gamma_{/f}(\mathbb{Q}_p, M_\ell)$ to be the mapping cone so that we have a distinguished triangle

$$R\Gamma_f(\mathbb{Q}_p, M_\ell) \to R\Gamma(\mathbb{Q}_p, M_\ell) \to R\Gamma_{/f}(\mathbb{Q}_p, M_\ell)$$

in the derived category of $\mathbb{Q}_\ell$ vector spaces. We also define $R\Gamma_{/f}(\mathbb{R}, M_\ell)$ to be trivial.

Let S be a finite set of primes containing ℓ, ∞, and the ramified primes. Then the definition of cohomology with compact supports gives the triangle

$$R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_\ell) \to R\Gamma(\mathbb{Z}[\frac{1}{S}], M_\ell) \to \bigoplus_{p \in S} R\Gamma(\mathbb{Q}_p, M_\ell).$$

The third distinguished triangle is the definition of $R\Gamma_f(\mathbb{Q}, M_\ell)$ as the shifted mapping cone in

$$R\Gamma_f(\mathbb{Q}, M_\ell) \to R\Gamma(\mathbb{Z}[\frac{1}{S}], M_\ell) \to \bigoplus_{p \in S} R\Gamma_{/f}(\mathbb{Q}_p, M_\ell).$$

Note that the definition is independent of S.

From the octahedral axiom, we have the triangle

$$R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_\ell) \to R\Gamma_f(\mathbb{Q}, M_\ell) \to \bigoplus_{p \in S} R\Gamma_f(\mathbb{Q}_p, M_\ell) \tag{2.1.1}$$

which induces an isomorphism

$$\mathrm{Det}_{A_\ell}(R\Gamma_f(\mathbb{Q}, M_\ell)) \otimes \mathrm{Det}^{-1}_{A_\ell}(\bigoplus_{p\in S} R\Gamma_f(\mathbb{Q}_p, M_\ell)) \simeq \mathrm{Det}_{A_\ell} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_\ell).$$

Since $j < 0$, we have $M_{dR}/\mathrm{Fil^0M_{dR}} = 0$ and isomorphisms

$$\mathrm{Det}_{A_\ell}(R\Gamma_f(\mathbb{Q}_p, M_\ell)) \simeq A_\ell,$$

induced by the identity map on $M_\ell^{I_p}$ (resp. $D_{cris}(M_\ell)$) for $\ell \neq p$ (resp. $\ell = p$). For $p = \infty$,

$$\mathrm{Det}_{A_\ell}(R\Gamma_f(\mathbb{R}, M_\ell)) := \mathrm{Det}_{A_\ell}(R\Gamma(\mathbb{R}, M_\ell)) \simeq \mathrm{Det}_{A_\ell}((M_B^+) \otimes_{\mathbb{Q}} A_\ell).$$

The cohomology of $R\Gamma_f(\mathbb{Q}, M_\ell)$ is computed in all degrees in two steps. First, there is an isomorphism with motivic cohomology: $H^0_f(M)_{\mathbb{Q}_\ell} \simeq H^0_f(\mathbb{Q}, M_\ell)$ via the cycle class map, and $H^1_f(M)_{\mathbb{Q}_\ell} \simeq H^1_f(\mathbb{Q}, M_\ell)$ via the Chern class map. Note that the motivic cohomology groups are rationally given by algebraic K-theory; $H^0_f(M) = 0$ for weight reasons, and $H^1_f(M) = K_{1-2j}(F) \otimes \mathbb{Q}$. Second we have Artin-Verdier duality $H^i_f(\mathbb{Q}, M_\ell) \simeq H^{3-i}_f(\mathbb{Q}, M^*_\ell(1))^*$. Thus the exact triangle (2.1.1) induces an isomorphism

$$_A\vartheta_\ell : \Xi(_AM) \otimes_{\mathbb{Q}} A_\ell \simeq \mathrm{Det}_{A_\ell} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_\ell). \qquad (2.1.2)$$

Choosing the order $\mathbb{Z}[G]$ in A and a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-stable projective $\mathbb{Z}_\ell[G]$-lattice $T_\ell = H^0_{et}(\mathrm{Spec}(F \otimes_K \bar{K}), \mathbb{Z}_\ell(j))$ in M_ℓ, the equivariant Tamagawa number conjecture can be stated.

Conjecture 2. *(ETNC) For every prime number ℓ,*

$$_A\vartheta_\ell \circ {_A\vartheta_\infty}(L^*(_AM)^{-1}) \cdot \mathbb{Z}_\ell[G] = \mathrm{Det}_{\mathbb{Z}_\ell[G]} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_\ell),$$

inside of $\mathrm{Det}_{A_\ell} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_\ell)$.

Remarks: i) The equivariant Tamagawa number conjecture computes the tuple

$L^*({}_AM)$ up to a unit in $\mathbb{Z}[G]$.

ii) The statement is independent of the choice of S and T_ℓ [Fla00], but depends on the choice of order in A_ℓ. Indeed, if there are 2 orders $\mathfrak{A}_\ell \subseteq \mathfrak{A}'_\ell$ of A then conjecture 2 for $\mathfrak{A}_\ell$ implies conjecture 2 for $\mathfrak{A}'_\ell$ but not vice versa. Thus, the conjecture as stated gives the equivariant Tamagawa number conjecture for any order in A_ℓ.

iii) The composition ${}_A\vartheta_l \circ {}_A\vartheta_\infty$ is only well defined for elements of $A \otimes_{\mathbb{Q}} \mathbb{R}$ whose image under ${}_A\vartheta_\infty$ have rational coefficients. Hence, Gross's conjecture for the extension F/K is assumed in the formulation of conjecture 2.

iv) The conjecture can be made similarly for $j \geq 0$. The precise formulation follows from the discussion in section 2.3.

2.2 Main theorem

This thesis concerns the equivariant Tamagawa number conjecture for abelian extensions of imaginary quadratic fields at negative integral values of the L-function (Conjecture 2). We make significant progress, though we do not prove the conjecture in full. In the cases for which we have Rubin's 2-variable main conjecture, $\ell \nmid [F : K]$, it remains to reconcile Rubin's module of elliptic units with that constructed in chapter 4. For the remaining primes $\ell \neq 2$, the main conjecture is still the conditional part of the proof, but proving it requires the vanishing of certain Iwasawa μ-invariants. For the prime 2, some of the complexes are no longer perfect and different methods are needed.

Having explicated what is not done, we now explain the main theorem.

Theorem 2.2.1. *The ℓ-part of the equivariant Tamagawa number conjecture for the motive $h^0(\operatorname{Spec} F)(j)$ and the order $\mathbb{Z}[\mathrm{Gal}(F/K)]$ in the group algebra $\mathbb{Q}[\mathrm{Gal}(F/K)]$ holds for $j < 0$ whenever $\ell \neq 2$ and we have the 2-variable Iwasawa main conjecture for imaginary quadratic fields (Conjecture 3).*

Proof Strategy: The remainder of the text is devoted to the proof of this theorem, but we will summarize the method here.

To begin, we make some general reductions. Burns and Flach prove a general functoriality statement for the ETNC ([BF01] Prop. 4.1b) which implies that if F'/F is also an abelian extension of K then the conjecture for F' implies the conjecture for F. Thus by class field theory, it suffices to give a proof for $F = K(\mathfrak{m})$ where $\mathfrak{m} \subset \mathcal{O}_K$ and $K(\mathfrak{m})$ is the ray class field modulo $\mathfrak{m}$. We can also assume that $w_{\mathfrak{m}} = 1$ where $w_{\mathfrak{m}}$ denotes the number of roots of unity of K which are congruent to 1 modulo $\mathfrak{m}$.

Let $G_{\mathfrak{m}}$ be the Galois group of the extension $K(\mathfrak{m})/K$. The conjecture asserts an equality of rank one $\mathbb{Z}_\ell[G_{\mathfrak{m}}]$-modules inside of $\mathrm{Det}_{A_\ell} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_\ell)$, the bases of which can be computed over A_ℓ. Recall that the ring A is a semisimple $\mathbb{Q}$-algebra, so it splits as a product of number fields according the rational characters of $G_{\mathfrak{m}}$, and hence so does A_ℓ. Thus after finding a canonical global basis of $\mathrm{Det}_{\mathbb{Z}_\ell[G_{\mathfrak{m}}]} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_\ell)$, denoted by $\mathcal{L}_{\mathfrak{m},j}$, by descending from the main conjecture we can compare $\mathcal{L}_{\mathfrak{m},j}$ to the image of the L-value under ${}_A\vartheta_\ell \circ {}_A\vartheta_\infty$ character by character. Theorem 2.2.1 reduces to the following result.

Theorem 2.2.2. *For every rational character χ of $G_{\mathfrak{m}}$,*

$$({}_A\vartheta_\ell \circ {}_A\vartheta_\infty(L^*({}_AM, 0))_\chi = (\mathcal{L}_{\mathfrak{m},j})_\chi$$

where the subscript χ denotes the projection of the element to the χ-isotypical component

$$\mathrm{Det}_{\mathbb{Q}_\ell(\chi)}(R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_\ell) \otimes_{A_\ell} \mathbb{Q}_\ell(\chi)).$$

We construct in chapter 3 elements $\xi_{\mathfrak{f}}(j) \in K_{1-2j}(\mathcal{O}_{K(\mathfrak{m})})$ such that whenever χ has conductor $\mathfrak{f}$, the Artin L-function

$$L'(\bar{\chi}, j) \sim_{\mathbb{Q}} e_\chi(\rho_\infty(\xi_{\mathfrak{f}}(j))).$$

Computing the image under the étale Chern class map $\rho_\ell(\xi_{\mathfrak{f}}(j))$ is sufficient to determine for each χ the element

$$({}_A\vartheta_\ell \circ {}_A\vartheta_\infty(L^*({}_AM, 0))_\chi.$$

The proof is completed by comparing this element with $(\mathcal{L}_{\mathfrak{m},j})_\chi$ in chapter 5.

2.3 A conjecture for any smooth projective variety over $\mathbb{Q}$

Given a smooth projective variety

$$X \to \operatorname{Spec} \mathbb{Q}$$

we consider the pure motive of weight $i-2j$ denoted $M = h^i(X)(j)$ with the action of a semisimple, possibly non-commutative, $\mathbb{Q}$-algebra A. To formulate the general conjecture we proceed much in the same fashion as above, though we must assume quite a few "nice properties" that are far from being proved. First of all, the motivic cohomology spaces $H^0_f(M)$ and $H^1_f(M)$ are defined via algebraic K-theory. We must assume that these K-groups are finitely generated over $\mathbb{Q}$ in order to take determinants (compare bases). Notice that there is a well-defined theory of non-commutative determinants due to Burns and Flach [BF03].

Instead of merely a regulator isomorphism, we now have a six term exact sequence.

$$0 \to H^0_f(M)_{\mathbb{R}} \xrightarrow{c} \ker(\alpha_M) \to H^1_f(M^*(1))^*_{\mathbb{R}} \xrightarrow{h} H^1_f(M)_{\mathbb{R}} \xrightarrow{r} \operatorname{coker}(\alpha_M) \to H^0_f(M^*(1))^*_{\mathbb{R}} \to 0$$

where c is the cycle class map, h is the height pairing, r is the Beilinson regulator, and

$$\alpha_M : M_B^+ \to M_{dR}/\mathrm{Fil}^0(\mathrm{M_{dR}})$$

is induced from the period isomorphism. In fact we use the same sequence in the formulation of the conjecture above, but all spaces vanish save $\ker(\alpha_M)$ and $H^1_f(M^*(1))^*$. The exactness of this sequence is not known in general.

The L-function $L({}_AM, s)$ is defined at the beginning of this chapter. Write $L^*({}_AM, 0) \in (A \otimes \mathbb{R})^\times$ for the leading term in the Taylor expansion about $s = 0$ and $r({}_AM) \in H^0(\operatorname{Spec}(A \otimes \mathbb{R}), \mathbb{Z})$ for the order vanishing. We take as part of our framework that $L({}_AM, s)$ has meromorphic continuation so that this makes sense. Meromorphic continuation of motivic L-functions is an open question in general, though it is known for automorphic L-functions. The vanishing order conjecture states that

$$r({}_AM) = \dim_A H^1_f(M^*(1)) - \dim_A H^0_f(M^*(1)).$$

With these definitions, the general version of conjecture 2 has precisely the same form.

This seems to be an apt language for the phrasing of conjectures about special values of L-functions. Indeed, when considering X to be the spectrum of a number field, we have the strong Stark conjecture; while taking X to be an elliptic curve, gives a version of the Birch and Swinnerton-Dyer conjecture. A good reference for the general formualtion and all work done toward the conjecture is the survey paper of Flach [Fla04].

Chapter 3

Formulas for L-values

Recall that the special value of the L-function $L^*(M,0)$ decomposes over the rational characters of the group $\mathrm{Gal}(F/K)$. We can consider χ as a representation of $G_{\mathfrak{m}}$, but if the conductor $\mathfrak{f}_\chi \neq \mathfrak{m}$, then χ is induced from a character of $G_{\mathfrak{f}_\chi}$. The Artin L-function of the $G_{\mathfrak{m}}$-representation differs from the Dirichlet L-function of χ by a finite number of Euler factors. Indeed,

$$L(\chi,s)_{Dir} = \prod_{\mathfrak{p}|\mathfrak{m},\mathfrak{p}\nmid\mathfrak{f}_\chi} (1-\chi(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-s})L(\chi,s).$$

As mentioned in section 2.2, to prove Conjecture 2 for all abelian extensions of imaginary quadratic fields, it suffices to consider all ray class fields of imaginary quadratic fields. Moreover, the modulus $\mathfrak{m}$ can be enlarged as necessary without loss of generality.

In this chapter, we will consider the special value of the (primitive) Artin L-function $L(\chi,s)$ at negative integers j, and we will denote the conductor of χ by $\mathfrak{f}$. Section 3.1 follows the work of Deninger [Den89, Den90] to give a proof of Gross's conjecture and discusses the modifications to Deninger's construction which are necessary for our situation. In Section 3.2, we compute the ℓ-adic realization of the motivic cohomology classes constructed in Section 3.1, building primarily on work of Huber and Kings [HK99, Kin01]. The two main theorems of this chapter are the construction of canonical motivic elements in Theorem 3.1.1 and the computation of the étale chern class of these elements in Theorem 3.2.1.

3.1 Analytic Computation

We devote this section to the proof of the following theorem which is a modification of a result of Deninger [Den90] (see proposition 3.1.4 below).

Theorem 3.1.1. *For every ideal $1 \neq \mathfrak{f} \mid \mathfrak{m}$, there are motivic elements*

$$\xi_{\mathfrak{f}}(j) \in H^1_{\mathcal{M}}(K(\mathfrak{m}), 1-j)$$

with the property that if χ is a rational character of $G_{\mathfrak{m}}$ of conductor $\mathfrak{f}$, then

$$e_\chi(\rho_\infty(\xi_{\mathfrak{f}}(j))) = \frac{\mathcal{N}\mathfrak{f}^{-1-j}2^{-1-j}\Phi(\mathfrak{m})}{(-1)^{1+j}(-2j)!\Phi(\mathfrak{f})}L'(\chi, j)\eta_{\mathbb{Q}}.$$

where $\eta_{\mathbb{Q}}$ is a basis of the χ-component of $(H^0_B(\mathrm{Spec}(F)(\mathbb{C}), \mathbb{Q}(j))^+)^*$, and

$$\Phi(\mathfrak{m}) = |(\mathcal{O}_K/\mathfrak{m})^\times| = \mathcal{N}\mathfrak{m}\prod_{\mathfrak{p}|\mathfrak{m}}(1-\mathcal{N}\mathfrak{p}^{-1})$$

is Euler's totient function. Moreover, these elements form a norm compatible system, and for $\mathfrak{f} = 1$ we have a family of elements, ${}_{\mathfrak{q}}\xi_1(j)$, indexed by the primes of K which are defined via the norm map and satisfying the above formula for any choice of $\mathfrak{q}$ with $w_{\mathfrak{q}} = 1$.

Remark: Combining this result with theorem 3.2.1 gives an analog of the theorem of Huber and Wildeshaus over $\mathbb{Q}$ [HW98] (9.6, 9.7).

We begin by recalling some results of Deninger [Den89, Den90]. Notice that taking $F = K(\mathfrak{m})$ and using functoriality to increase $\mathfrak{m}$ as necessary, the formulas become somewhat simpler.

We first establish some notation. Let E be an elliptic curve defined over $K(\mathfrak{m})$ with complex multiplication by $\mathcal{O}_K$ where E has the additional property that the CM character factors through the norm map from $K(\mathfrak{m})$ to K. Shimura [Shi71] showed that this is equivalent to the condition that the torsion points of the elliptic curve E generate an abelian extension of K.

Fix an isomorphism

$$\theta_E : \mathcal{O}_K \simeq \mathrm{End}_{K(\mathfrak{m})}(E)$$

such that $\theta_E^*(\alpha)\omega = \alpha\omega$ for all $\omega \in H^0(E, \Omega^1_{E/K(\mathfrak{m})})$ and an embedding, τ_0 of $K(\mathfrak{m})$ in $\mathbb{C}$ such that $j(E) = j(\mathcal{O}_K)$. Then, over the complex numbers, $E \simeq \mathbb{C}/\Gamma$ where $\Gamma = \Omega\mathcal{O}_K$ for some $\Omega \in \mathbb{C}$. Notice that this choice is entirely non-canonical. It determines a class in the Betti cohomology of E.

In order to distinguish an $\mathfrak{f}$-torsion point on E, we let $\rho_\mathfrak{f} \in \mathbb{A}_K^*$ be an idèle with ideal $\mathfrak{f}$, and choose $f_\mathfrak{f} \in K^*$ with

$$v_\mathfrak{p}(f_\mathfrak{f}) \leq 0 \text{ if } \mathfrak{p} \nmid \mathfrak{f} \text{ and } v_\mathfrak{p}(f_\mathfrak{f}^{-1} - (\rho_\mathfrak{f})_\mathfrak{p}^{-1}) \geq 0 \text{ if } \mathfrak{p} \mid \mathfrak{f}. \tag{3.1.1}$$

Let ${}_\mathfrak{f}\beta = ([\Omega f_\mathfrak{f}^{-1}])$. Then ${}_\mathfrak{f}\beta$ is an $\mathfrak{f}$-torsion point on E which is rational over $K(\mathfrak{f})$. For $g \in G_\mathfrak{m}$, let ${}^gE$ be the curve obtained by base change according to the diagram

$$\begin{tikzcd} {}^gE \arrow[r] \arrow[d] & E \arrow[d] \\ F \arrow[r, "g"] & F \end{tikzcd}$$

Let $\mathcal{A} = R_{K(\mathfrak{m})/K}E$ be the Weil restriction of the elliptic curve. $\mathcal{A}$ is an abelian variety over K with CM by a semisimple K-algebra T and Serre-Tate character $\varphi_\mathcal{A}$. Deninger proves that any Hecke character, φ of weight $w > 0$ is of the form $\prod_{i=1}^w \varphi_{\lambda_i}$ where $\lambda_i \in \mathrm{Hom}(T, \mathbb{C})$ and $\varphi_{\lambda_i} = \lambda_i \circ \varphi_\mathcal{A}$ [Den89] (Proposition 1.3.1).

Twisting a complex character $\eta \in \chi$ by the norm character gives a Hecke character of K of weight 2

$$\varphi_\eta = \eta N_{K/\mathbb{Q}} = \varphi_{\lambda_1}\varphi_{\lambda_2}.$$

Once again by functoriality, we can increase $\mathfrak{m}$ so that it is a multiple of the conductors of φ_{λ_1} and φ_{λ_2}. Notice that this can be done once and for all by choosing a type $(1,0)$ character φ with $N_{K/\mathbb{Q}} = \varphi\bar{\varphi}$ and taking $\mathfrak{m}$ to be a multiple of the conductor of φ. Fix a set of ideals $\{\mathfrak{b}_g \subseteq \mathcal{O}_K\}_{g \in G_\mathfrak{m}}$ with Artin symbol $(\mathfrak{b}_g, K(\mathfrak{m})/K) = g \in G_\mathfrak{m}$. For integral ideals $\mathfrak{a}$ of K prime to the conductors of φ_{λ_1} and φ_{λ_2}, define $\Lambda(\mathfrak{a}) \in K(\mathfrak{m})^\times$ by

$$\varphi_\mathcal{A}(\mathfrak{a})^* \omega^{\sigma_\mathfrak{a}} = \Lambda(\mathfrak{a})\omega$$

where $\omega \in H^0(E, \Omega^1)$ has period lattice Γ, $\varphi_\mathcal{A}(\mathfrak{a}) \in T^\times$ is viewed as an isogeny $E \to {}^{\sigma_\mathfrak{a}}E$, and $\sigma_\mathfrak{a}$ is the Artin automorphism of $\mathfrak{a}$. Now, for all $g \in G_\mathfrak{m}$, ω^g has period lattice Γ_g, and we can identify ${}^gE(\mathbb{C})$ with $\mathbb{C}/\Gamma_g$ via the Abel-Jacobi map to obtain a divisor

$${}_\mathfrak{f}\beta_g = ([\Lambda(\mathfrak{b}_g)\Omega f_\mathfrak{f}^{-1}])$$

on ${}^gE_\mathfrak{f}(K(\mathfrak{m}))$ with $G_\mathfrak{m}$ action given by ${}^h{}_\mathfrak{f}\beta_g = {}_\mathfrak{f}\beta_{hg}$. Notice that if $\mathfrak{f} = 1$, ${}_\mathfrak{f}\beta$ is just the identity on E.

We return to the consideration of rational characters with an formula for the special values of these Artin L-functions at negative integers.

Proposition 3.1.2. *(Deninger [Den90] (3.4))*
The L-series, $L(\chi, s)$ has a first order zero for every $s = j < 0$, and the special value is given by the formula

$$L'(\chi, j) = (-1)^{-j} \frac{\Phi(\mathfrak{f})(-j)!^2}{\Phi(\mathfrak{m})} \left(\frac{\sqrt{d_K}\mathcal{N}\mathfrak{f}}{(2\pi i)} \right)^{-j} \chi(\rho_\mathfrak{f}) \sum_{g \in G_\mathfrak{m}} \chi(g) A(\Gamma_g)^{1-j} \mathcal{M}_j({}_\mathfrak{f}\beta_g),$$

where Φ is the totient function, and d_K is the discriminant of K. For any $\mathbb{Z}$-basis of Γ_g with $Im(v/u) > 0$,

$$A(\Gamma_g) = (\bar{u}v - \bar{v}u)/2\pi i,$$

and for divisors on the $\mathfrak{f}$-torsion of E, $\mathcal{M}_j$ is defined by linearity from

$${}_j\mathcal{M}_{\log}(x) = \sum_{\gamma \in \Gamma_g}{}' \frac{(x, \gamma)_g}{|\gamma|^{2(1-j)}}, \; x \in E_\mathfrak{f}$$

with the Pontrjagin pairing $(\,,\,)_g : \mathbb{C}/\Gamma_g \times \Gamma_g \to U(1)$ *given by* $(z,\gamma)_g = \exp(A(\Gamma_g)^{-1}(z\bar{\gamma} - \bar{z}\gamma))$.

Deninger constructs motivic elements from the divisors ${}_{\mathfrak{f}}\beta$ using an early variation of the Eisenstein symbol

$$\mathcal{E}^k_{\mathcal{M}} : \mathbb{Q}[E]^0 \to H^{k+1}_{\mathcal{M}}(E^k, k+1)$$

which is defined only for divisors of degree 0. Lemma 3.1.7 demonstrates the relationship between Deninger's Eisenstein symbol and the one that used in the later work of Huber, Kings, and Scholl [HK99, Kin01, Sch98], which we will also need. Choose an integer $N \geq 2$ and define a degree 0 divisor on ${}^gE(\mathbb{C})$

$${}_N\alpha_g = N^2(0) - \sum_{p \in {}^gE(\mathbb{C})_N} (p).$$

Deninger shows that

Proposition 3.1.3. *(Deninger [Den90] (2.6))*

$${}_{\mathfrak{f}}\beta'_g = {}_{\mathfrak{f}}\beta_g - (\deg {}_{\mathfrak{f}}\beta_g)(0) + \frac{\deg {}_{\mathfrak{f}}\beta_g}{N^2}\left(1 - \frac{1}{N^{4-2j}}\right)^{-1} {}_N\alpha_g$$

is a degree 0 divisor on ${}^gE(K(\mathfrak{m}))$ *with* ${}_{\mathfrak{f}}{}^h\beta'_g = {}_{\mathfrak{f}}\beta'_{hg}$ *and*

$$\mathcal{M}_j({}_{\mathfrak{f}}\beta'_g) = \mathcal{M}_j({}_{\mathfrak{f}}\beta_g).$$

His notation does not distinguish between the group $G_{\mathfrak{m}}$ and the embeddings $\mathrm{Hom}_K(K(\mathfrak{m}), \mathbb{C})$. In Lemma 3.2, he computes that for an embedding τ of F into $\mathbb{C}$,

$$\rho_\infty(\mathcal{K}_{\mathcal{M}}\mathcal{E}^{-2j}_{\mathcal{M}}({}_{\mathfrak{f}}\beta'))_\tau = -\frac{|E_{N\mathfrak{f}}(\mathbb{C})|^{-2j} A(\Gamma_\tau)^{1-j}(-j)!^2}{2(-2j)!}(2\sqrt{d_K})^{-j}\mathcal{M}_j({}_{\mathfrak{f}}\beta'_\tau)$$

The Kronecker map is a projector given by the composition

$$\begin{tikzcd}
H^{1-2j}_{\mathcal{M}}(E^{-2j}, 1-2j) \arrow[r, "(id,\theta_E(\sqrt{d_K})^{-j,*}"] \arrow[dr, "\mathcal{K}_{\mathcal{M}}"'] & H^{1-2j}_{\mathcal{M}}(E^{-j}, 1-2j) \arrow[d, "\pi_{-j,*}"] \\
& H^{1}_{\mathcal{M}}(\mathrm{Spec}(K(\mathfrak{m})), 1-j),
\end{tikzcd}$$

where the map π_{-j*} is a proper push forward.

Recall that the regulator, ρ_∞ is an isomorphism

$$K_{1-2j}(\mathcal{O}_{K(\mathfrak{m})}) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\rho_\infty} \left(\bigoplus_{\sigma \in \mathcal{T}} \mathbb{C}/\mathbb{R} \cdot (2\pi i)^{1-j} \cdot \sigma \right)^+$$

where $\mathcal{T} = \mathrm{Hom}(K(\mathfrak{m}), \mathbb{C})$. Since $j < 0$, $K_{1-2j}(\mathcal{O}_{K(\mathfrak{m})}) \simeq K_{1-2j}(K(\mathfrak{m}))$, and the $\mathbb{R}$-dual of this last space is identified with $M^+_{B,\mathbb{R}}$ by taking invariants in the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant perfect pairing

$$\bigoplus_{\sigma \in \mathcal{T}} \mathbb{R} \cdot (2\pi i)^j \times \bigoplus_{\sigma \in \mathcal{T}} \mathbb{C}/\mathbb{R} \cdot (2\pi i)^{1-j} \to \bigoplus_{\sigma \in \mathcal{T}} \mathbb{C}/2\pi i \cdot \mathbb{R} \xrightarrow{\Sigma} \mathbb{R}$$

induced by multiplication.

To prove ETNC, it is essential to distinguish between the group $G_{\mathfrak{m}}$ and the group $\mathcal{T}$. The reader should note that there are two commuting left actions: that of the Galois group $G_{\mathfrak{m}}$, and the group of embeddings, $\mathcal{T}$. For an element, $\sum_{\sigma \in \mathcal{T}} x_\sigma \cdot \sigma$, the Galois group acts via

$$g \cdot \left(\sum_{\sigma \in \mathcal{T}} x \cdot \sigma \right) = \sum_{\sigma \in \mathcal{T}} x \cdot g^{-1}\sigma.$$

With this action, ρ_∞ is A-equivariant just as in the case of the Dirichlet regulator.

Therefore by [Den90] (Lemma 3.2),

$$\rho_\infty(\mathcal{K}_{\mathcal{M}}\mathcal{E}_{\mathcal{M}}^{-2j}({}_{\mathfrak{f}}\beta')) = \sum_{\tau\in\mathcal{T}}(2\pi i)^j\left(-\frac{|E_{N\mathfrak{f}}(\mathbb{C})|^{-2j}A(\Gamma_\tau)^{1-j}(-j)!^2}{2(-2j)!}(2\sqrt{d_K})^{-j}\mathcal{M}_j({}_{\mathfrak{f}}\beta_\tau)\right)\cdot\tau \tag{3.1.2}$$

$$= \sum_{g\in G_{\mathfrak{m}}}(2\pi i)^j\left(-\frac{|E_{N\mathfrak{f}}(\mathbb{C})|^{-2j}A(\Gamma_g)^{1-j}(-j)!^2}{2(-2j)!}(2\sqrt{d_K})^{-j}\mathcal{M}_j({}_{\mathfrak{f}}\beta_g)\right)\cdot g\tau_0$$

$$= \sum_{g\in G_{\mathfrak{m}}}g^{-1}\cdot(2\pi i)^j\left(-\frac{|E_{N\mathfrak{f}}(\mathbb{C})|^{-2j}A(\Gamma_g)^{1-j}(-j)!^2}{2(-2j)!}(2\sqrt{d_K})^{-j}\mathcal{M}_j({}_{\mathfrak{f}}\beta_g)\right)\cdot\tau_0.$$

Again, the analysis over $\mathbb{Q}[G_{\mathfrak{m}}]$ is done character by character, so one projects to the χ-isotypical component

$$e_\chi(\rho_\infty(\mathcal{K}_{\mathcal{M}}\mathcal{E}_{\mathcal{M}}^{-2j}({}_{\mathfrak{f}}\beta'))) = \left(\sum_{g\in G_{\mathfrak{m}}}-\frac{|E_{N\mathfrak{f}(\mathbb{C})}|^{-2j}A(\Gamma_g)^{1-j}(-j)!^2}{2(-2j)!}(2\sqrt{d_K})^{-j}\mathcal{M}_j({}_{\mathfrak{f}}\beta_g)\bar{\chi}(g)\right)\cdot\eta_{\mathbb{Q}}$$

where $\eta_{\mathbb{Q}} = e_\chi\cdot(2\pi i)^j\tau_0$ is a basis of $e_\chi\left(M_B^{+*}\right)$. This settles the proof of Beilinson's conjecture for a rational character χ of $G_{\mathfrak{m}}$ which is summarized in the following proposition.

Proposition 3.1.4. *(Deninger [Den90] (3.1))*

$$e_\chi(\rho_\infty(\mathcal{K}_{\mathcal{M}}\mathcal{E}_{\mathcal{M}}^{-2j}({}_{\mathfrak{f}}\beta'))) = \frac{(-2)^{-1-j}N^{-4j)}\mathcal{N}\mathfrak{f}^{-j}}{(-2j)!\chi(\rho_{\mathfrak{f}})}\frac{\Phi(\mathfrak{m})}{\Phi(\mathfrak{f})}L'(\bar{\chi},-j)$$

Remark: Notice that this differs from theorem 3.1.1 by a factor of $\chi(\rho_{\mathfrak{f}})$.

Corollary 3.1.5. *The Gross conjecture holds for the extension $K(\mathfrak{m})/K$.*

Proof of 3.1.5: The group algebra $A = \mathbb{Q}[G_{\mathfrak{m}}]$ is semisimple and thus decomposes as a product of fields indexed by the rational characters of $G_{\mathfrak{m}}$. Hence, $\Xi({}_AM)$ also decomposes

$$\Xi({}_AM) = \prod_{\chi\in\hat{G}_{\mathfrak{m}}^{\mathbb{Q}}}\mathrm{Det}_{\mathbb{Q}(\chi)}(K_{1-2j}(\mathcal{O}_{K(\mathfrak{m})})^*\otimes_{\mathbb{Z}}\mathbb{Q}(\chi))\otimes\mathrm{Det}^{-1}_{\mathbb{Q}_\ell(\chi)}(H_B^0(\mathrm{Spec}(K(\mathfrak{m}))(\mathbb{C}),\mathbb{Q}(j))^+\otimes_{\mathbb{Q}}\mathbb{Q}(\chi)).$$

The corollary follows from the Beilinson conjecture at each χ component. $\square$

In order to make the necessary modifications to Deninger's elements, we digress on the theory of complex multiplication for elliptic curves. For an idèle s of K, we can define multiplication by s componentwise.

Theorem 3.1.6. Main theorem of complex multiplication*(Shimura [Shi71] (5.3))*
Let X be an elliptic curve with CM by $\mathcal{O}_K$. Given a map $r : K/\mathcal{O}_K \to X$ and an idèle $s \in \mathbb{A}_K^\times$ with ideal (s) and Artin symbol $\psi(s)$ we have a unique map $r' : K/(s) \to {}^{\psi(s)}X$ such that the following diagram commutes.

$$\begin{tikzcd} K/\mathcal{O}_K \arrow[r, "s"] \arrow[d, "r"'] & K/(s) \arrow[d, "r'"] \\ X \arrow[r, "\psi(s)^{-1}"] & {}^{\psi(s)}X. \end{tikzcd}$$

By a *CM* pair of modulus $\mathfrak{f}$ over F, we mean a pair (X, α) where X is an elliptic curve over F with complex multiplication by $\mathcal{O}_K$ and such that the inculsion of $\mathcal{O}_K$ into F factors through $\mathrm{End}(X)$. By [Kat04] (15.3.1), there is a CM pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})$ which is isomorphic to $(\mathbb{C}/\mathfrak{f}, 1 \bmod \mathfrak{f})$ over $\mathbb{C}$. This pair is unique up to isomorphism, and whenever $\mathcal{O}_K^\times \to (\mathcal{O}_K/\mathfrak{f})^\times$ is injective the isomorphism is unique. We call this pair the canonical CM pair. There is also a good account of the theory of complex multiplication in the book by Silverman [Sil94]

Proof of 3.1.1: Continuing with the previous notation, we have fixed a choice of an embedding $\tau_0 : K \hookrightarrow \mathbb{C}$ and of a uniformization $E \simeq \mathbb{C}/\Omega\mathcal{O}_K$. The torsion point ${}_{\mathfrak{f}}\beta \in E_{\mathfrak{f}}$ is dependent on the choice of idèle $\rho_{\mathfrak{f}}$. In fact, by theorem 3.1.6

$$\psi(\rho_{\mathfrak{f}})^{-1} : E \to {}^{\psi(\rho_{\mathfrak{f}})}E$$

maps the pair $(E(\mathbb{C}), {}_{\mathfrak{f}}\beta)$ to $(\mathbb{C}/\Omega\mathfrak{f}, 1 \bmod \Omega\mathfrak{f})$ since ${}_{\mathfrak{f}}\beta = \Omega f_{\mathfrak{f}}^{-1}$

$$\rho_{\mathfrak{f}} \cdot f_{\mathfrak{f}}^{-1} \equiv 1 \bmod \mathfrak{f}.$$

Indeed, the restrictions on the valuation of $f_{\mathfrak{f}}$ at each prime $\mathfrak{p} \mid \mathfrak{f}$ in (3.1.1), give that

$$\rho_{\mathfrak{f},\mathfrak{p}}/f_{\mathfrak{f}} \in 1 + \mathfrak{m}_{\mathfrak{p}}^{\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{f}}$$

where $\mathfrak{m}_{\mathfrak{p}}$ is the maximal ideal in the local ring $\mathcal{O}_{K_{\mathfrak{p}}}$. Moreover, one may choose the idèles ρ to be multiplicative in the sense that $\rho_{\mathfrak{f}\mathfrak{p}} = \rho_{\mathfrak{f}}\rho_{\mathfrak{p}}$. We require the following lemma in order to compute the Eisenstein symbol of individual torsion points.

Lemma 3.1.7. *For $k > 0$, there is a variation of the Eisenstein symbol $\mathcal{E}is^k_{\mathcal{M}}$: $\mathbb{Q}[E[\mathfrak{f}] \setminus 0] \to H^{k+1}_{\mathcal{M}}(E^k, k+1)$ which is defined for divisors of any degree. Moreover,*

$$\mathcal{E}is({}_{\mathfrak{f}}\beta') = \mathcal{E}is({}_{\mathfrak{f}}\beta).$$

Proof of 3.1.7:

For $N = \mathcal{N}\mathfrak{f} \geq 3$, let M be the modular curve parameterizing elliptic curves with full level N structure, and let $\mathfrak{E}$ be the universal elliptic curve over M. Choose a level N structure on E, $\alpha : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N]$, rational over some extension K' of $K(\mathfrak{m})$. By the universality of $\mathfrak{E}$, we have the following diagram depending on the choice of level-N structure.

$$\begin{tikzcd} E \arrow[r, "\alpha^*"] \arrow[d] & \mathfrak{E} \arrow[d] \\ \mathrm{Spec}(K') \arrow[r] & M \end{tikzcd}$$

Denote by $\tilde{\mathfrak{E}}^0$ the fiber over the cusps of the connected component of the generalized elliptic curve over the compactification of M. Then we can define $\mathrm{Isom} = \mathrm{Isom}(\mathbb{G}_m, \tilde{\mathfrak{E}}^0_{\mathrm{Cusp}})$. Isom is a μ_2 torsor over the subscheme of cusps, and we consider the subset $\mathbb{Q}[\mathrm{Isom}]^{(k)} \subseteq \mathbb{Q}[\mathrm{Isom}]$ where μ_2 acts by $(-1)^k$. Define the horospherical map $\varrho^k : \mathbb{Q}[\mathfrak{E}[N]]^0 \to \mathbb{Q}[\mathrm{Isom}]^{(k)}$ explicitly by

$$\varrho^k(\psi)(g) = \frac{N^k}{k!(k+2)} \sum_t \psi(g^{-1}t) B_{k+2}(\frac{t_2}{N}),$$

where $t = (t_1, t_2) \in (\mathbb{Z}/N\mathbb{Z})^2$ and $B_k(x)$ is the kth Bernoulli polynomial. Moreover, when $k > 0$, ϱ is well-defined for divisors of any degree.

For an elliptic curve over any base, Beilinson [Bei86] constructs an Eisenstein symbol $\mathcal{E}is^k : \mathbb{Q}[E[N]]^0 \to H^{k+1}_{\mathcal{M}}(E^k, k+1)$ which is preserved under base change. For the universal elliptic curve $\mathfrak{E}^0$, we also have a boundary map

$$\mathrm{res^k : H^{k+1}_{\mathcal{M}}(\mathfrak{E}^k, k+1) \to \mathbb{Q}[Isom]^{(k)}}$$

coming from the long exact cohomology sequence, and another Eisenstein symbol

$$\mathrm{Eis^k : \mathbb{Q}[Isom]^{(k)} \to H^{k+1}_{\mathcal{M}}(\mathfrak{E}^k, k+1)}$$

with $\mathrm{res^k \circ Eis^k = id}$. The following diagram commutes when restricting to degree zero divisors.

$$\begin{tikzcd}
\mathbb{Q}[\mathfrak{E}[N]] \arrow[r, "\varrho"] & \mathbb{Q}[\mathrm{Isom}]^{(k)} \arrow[r, "\mathrm{Eis}"] & H^{k+1}_{\mathcal{M}}(\mathfrak{E}^k, k+1) \arrow[d, "\alpha^*"'] \\
\mathbb{Q}[E[N]]^0 \arrow[u, "\alpha"] \arrow[rr, "\mathcal{E}is^k"] & & H^{k+1}_{\mathcal{M}}(E^k, k+1)
\end{tikzcd}$$

Indeed, he horospherical map above was computed by Schappacher and Scholl to be the composition $\mathcal{E}is^k \circ \mathrm{res^k}$ [SS91]. Combining this fact with base change, the diagram commutes, and we can compute the Eisenstein symbol at torsion points on the elliptic curve. Moreover, this computation does not depend on the choice of full level structure since the assignment of Eisenstein symbols commutes with the GL_2 action on the torsion sections and is thus invariant under the trace $Y(N) \to Y_1(N)$ [Kin99] (Lemma 3.1.2).

To show that $\mathcal{E}is({}_{\mathfrak{f}}\beta') = \mathcal{E}is({}_{\mathfrak{f}}\beta)$, it suffices to show that

$$ {}_{\mathfrak{f}}\beta' - {}_{\mathfrak{f}}\beta = \frac{1}{\tilde{N}^{4-2j}-1}(0) - \frac{\tilde{N}^{2-2j}}{\tilde{N}^{4-2j}-1} \sum_{p \in E(\mathbb{C})[\tilde{N}]} (p) \in \ker \varrho. $$

Here, $\tilde{N}$ denotes the auxilliary integer defined by Deninger as discussed in section 3.1. To see that this lies in the kernel of ϱ, we first note that the action of $\mathbb{G}_m$ preserves the identity section on the curve. So,

$$ \varrho^{-2j}(0)(g) = \frac{N^{-2j}}{(-2j)!(2-2j)} B_{2-2j}(0). $$

We compute

$$ \begin{aligned} \varrho^{-2j}\left(\sum_{p \in E(\mathbb{C})[\tilde{N}]} (p)\right)(g) &= \frac{\tilde{N}^{-2j}}{(-2j)!(2-2j)} \sum_{(p)=(t_1,t_2)\in(\mathbb{Z}/\tilde{N}\mathbb{Z})^2} B_{2-2j}\left(\frac{t_2}{\tilde{N}}\right) \\ &= \frac{\tilde{N}^{1-2j}}{(-2j)!(2-2j)} \sum_{a=0}^{\tilde{N}-1} B_{2-2j}\left(\frac{a}{\tilde{N}}\right). \end{aligned} $$

Moreover, we have the distribution relation

$$ B_k(X) = \tilde{N}^{k-1} \sum_{a=0}^{\tilde{N}-1} B_k\left(\frac{X+a}{\tilde{N}}\right) $$

which implies that

$$ \varrho^{-2j}\left(\sum_{p \in E(\mathbb{C})[\tilde{N}]} (p)\right)(g) = \frac{1}{\tilde{N}^{2-2j}} \varrho^{-2j}(0)(g). $$

$\square$

Remark: This lemma allows us to extend Deninger's formula [Den89] (Theorem 10.9) for computing the Beilinson regulator of the Eisenstein symbol of a divisor to our torsion point ${}_{\mathfrak{f}}\beta$ without the modification to degree 0 in the sequel [Den90] (Proposition 2.6). This is necessary in order to have the desired shape in the ℓ-adic computation. It is used incorrectly in [Kin01]. We also note that $\mathcal{E}_{\mathcal{M}} = N\mathcal{E}is$ due to an error in the normalization of the residue map in Deninger's work (one can find this error in for example [Den94] formula 3.7).

Thus we define

$$\xi_{\mathfrak{f}}(j) := \mathcal{K}_{\mathcal{M}}\mathcal{E}is^{-2j}(\rho_{\mathfrak{f}} \cdot {}_{\mathfrak{f}}\beta)$$

One will notice that we have not constructed motivic elements for small $\mathfrak{f}$. We atone for this omission with the following lemma.

Lemma 3.1.8.

$$w_{\mathfrak{f}}/w_{\mathfrak{p}\mathfrak{f}} \operatorname{Tr}_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})} \xi_{\mathfrak{f}\mathfrak{p}}(j) = \begin{cases} \xi_{\mathfrak{f}}(j) & \mathfrak{p} \mid \mathfrak{f} \neq 1 \\ (1 - \mathrm{Fr}_{\mathfrak{p}}^{-1})\xi_{\mathfrak{f}}(j) & \mathfrak{p} \nmid \mathfrak{f} \neq 1. \end{cases}$$

Proof of 3.1.8: Given an isogeny of CM elliptic curves of order $\mathfrak{p}$ over the field $K(\mathfrak{f})$ and an $\mathfrak{f}$-torsion section β according to the diagram,

$$\begin{tikzcd} E \arrow[r, "\mathrm{Fr}_{\mathfrak{p}}"] \arrow[dr] & E \arrow[d, shift right] \\ & K(\mathfrak{f}) \arrow[u, bend right, "\beta"'] \end{tikzcd}$$

there is an $\mathfrak{f}\mathfrak{p}$-torsion section β' pullback diagram

$$\begin{tikzcd} E \arrow[r, "\mathrm{Fr}_{\mathfrak{p}}"] & E \\ K(\mathfrak{f}\mathfrak{p}) \arrow[u, "\beta'"] \arrow[r] & K(\mathfrak{f}) \arrow[u, "\beta"'] \end{tikzcd}$$

when $\mathfrak{p} \mid \mathfrak{f}$ and by

$$\begin{tikzcd} E \arrow[r, "\mathrm{Fr}_{\mathfrak{p}}"] & E \\ K(\mathfrak{f}) \sqcup K(\mathfrak{f}\mathfrak{p}) \arrow[u, "\beta'"] \arrow[r] & K(\mathfrak{f}) \arrow[u, "\beta"'] \end{tikzcd}$$

when $\mathfrak{p} \nmid \mathfrak{f}$. So we see that the trace on fields gives a sum on torsion sections, though when $\mathfrak{p} \nmid \mathfrak{f}$ we must account for the fact that we only have $\mathcal{N}\mathfrak{p} - 1$ primitive roots.

Scholl gives these relations for the Eisenstein symbol on the universal elliptic curve in [Sch98] (A.2.2, A.2.3). Pulling back to our elliptic curve over $K(\mathfrak{m})$, we deduce them for $\mathcal{E}is(\rho_{\mathfrak{f}} \cdot {}_{\mathfrak{f}}\beta)$. Finally, applying the Kronecker map, we have the lemma. $\square$.

With this lemma we can define motivic elements for small ideals via the trace map. In particular, for any prime $\mathfrak{q}$ of K with $w_{\mathfrak{q}} = 1$, we define

$$ {}_{\mathfrak{q}}\xi_1(j) := (1 - \mathrm{Fr}_{\mathfrak{q}}^{-1})^{-1} w_K \operatorname{Tr}_{K(\mathfrak{q})/K(1)} \xi_{\mathfrak{q}}(j), $$

for a family of motivic elements at level 1. To complete the proof of the theorem, we compute

$$ \begin{aligned} \rho_\infty(w_{\mathfrak{f}}/w_{\mathfrak{p}\mathfrak{f}} \operatorname{Tr}_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})} \xi_{\mathfrak{f}\mathfrak{p}}(j)) =& w_{\mathfrak{f}}/w_{\mathfrak{p}/\mathfrak{f}} \operatorname{Tr}_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})} \rho_\infty \mathcal{K}_{\mathcal{M}} \mathcal{E}is^{-2j}(\rho_{\mathfrak{f}\mathfrak{p}} \cdot {}_{\mathfrak{f}\mathfrak{p}}\beta) \\ =& \psi(\rho_{\mathfrak{f}\mathfrak{p}})^{-1} \cdot w_{\mathfrak{f}}/w_{\mathfrak{p}/\mathfrak{f}} \mathcal{N}\mathfrak{f}\mathfrak{p}^{-1} \operatorname{Tr}_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})} \rho_\infty \mathcal{K}_{\mathcal{M}} \mathcal{E}_{\mathcal{M}}^{-2j}({}_{\mathfrak{f}\mathfrak{p}}\beta) \end{aligned} $$

By the computation in 3.1.2 we have that

$$ \begin{aligned} &\operatorname{Tr}_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})} \rho_\infty \mathcal{K}_{\mathcal{M}} \mathcal{E}_{\mathcal{M}}^{-2j}({}_{\mathfrak{f}\mathfrak{p}}\beta) \\ &\quad = \operatorname{Tr}_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})} \sum_{g \in G_{\mathfrak{m}}} -g^{-1}(2\pi i)^j \frac{\mathcal{N}\mathfrak{f}\mathfrak{p}^{-2j} A(\Gamma_g)^{1-j}(-j)!^2}{2(-2j)!} (2\sqrt{d_K})^{-j} \mathcal{M}_j({}_{\mathfrak{f}\mathfrak{p}}\beta_g) \cdot \tau_0 \\ &\quad = \sum_{g \in G_{\mathfrak{m}}} -g^{-1}(2\pi i)^j \frac{\mathcal{N}\mathfrak{f}\mathfrak{p}^{-2j} A(\Gamma_g)^{1-j}(-j)!^2}{2(-2j)!} (2\sqrt{d_K})^{-j} \operatorname{Tr}_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})} \mathcal{M}_j({}_{\mathfrak{f}\mathfrak{p}}\beta_g) \cdot \tau_0 \end{aligned} $$

Focusing on $\mathcal{M}_j$, which should probably be called an Eisenstein number, we proceed as in the proof of 3.1.8 taking first the case of $\mathfrak{p} \mid \mathfrak{f}$.

$$\begin{aligned}\psi(\rho_{\mathfrak{f}\mathfrak{p}})^{-1}\operatorname{Tr}_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})}\mathcal{M}_j({}_{\mathfrak{f}}\beta_g) =& \psi(\rho_{\mathfrak{f}})^{-1}\mathcal{N}\mathfrak{p}\operatorname{Tr}_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})}\mathcal{M}_j(\rho_{\mathfrak{p}}\Lambda(g)\Omega f_{\mathfrak{f}}^{-1}f_{\mathfrak{p}}^{-1})\\ =&\psi(\rho_{\mathfrak{f}})^{-1}\mathcal{N}\mathfrak{p}\operatorname{Tr}_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})}\mathcal{M}_j(\Lambda(g)\Omega f_{\mathfrak{f}}^{-1})\\ =&\psi(\rho_{\mathfrak{f}})^{-1}\mathcal{N}\mathfrak{p}w_{\mathfrak{p}\mathfrak{f}}/w_{\mathfrak{f}}\sum_{u\in\mathfrak{p}^*{}_{\mathfrak{f}}\beta_g}\mathcal{M}_j({}_{\mathfrak{f}}\beta_g+u)\\ =&w_{\mathfrak{p}\mathfrak{f}}/w_{\mathfrak{f}}\psi(\rho_{\mathfrak{f}})^{-1}\mathcal{N}\mathfrak{p}^{2j+1}\mathcal{M}_j({}_{\mathfrak{f}}\beta_g) \qquad (3.1.3)\end{aligned}$$

Here the u are the primitive $\mathfrak{p}$th roots of ${}_{\mathfrak{f}}\beta_g$ resulting from pulling back by the isogeny $E \xrightarrow{\mathfrak{p}} {}^{\mathrm{Fr}_{\mathfrak{p}}}E$, and the equality in 3.1.3 follows from a formula in the proof of [Den90] (Proposition 2.6). Now in the case that $\mathfrak{p} \nmid \mathfrak{f}$, there is a unique point $u_0 \in \{u : \mathfrak{p}u = {}_{\mathfrak{f}}\beta_g\}$ with $u_0 \notin \mathfrak{p}^*{}_{\mathfrak{f}}\beta_g$. Adding and subtracting this point from the sum, we conclude that

$$\psi(\rho_{\mathfrak{f}\mathfrak{p}})^{-1}\operatorname{Tr}_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})}\mathcal{M}_j({}_{\mathfrak{f}}\beta_g) = (1-\mathrm{Fr}_{\mathfrak{p}}^{-1})w_{\mathfrak{p}\mathfrak{f}}/w_{\mathfrak{f}}\psi(\rho_{\mathfrak{f}})^{-1}\mathcal{N}\mathfrak{p}^{2j+1}\mathcal{M}_j({}_{\mathfrak{f}}\beta_g).$$

Compare with the formulas in 3.1.2 to prove that for a character χ of conductor $\mathfrak{f}$

$$e_\chi(\rho_\infty(w_{\mathfrak{f}}/w_{\mathfrak{p}\mathfrak{f}}\operatorname{Tr}_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})}\xi_{\mathfrak{f}\mathfrak{p}}(j)) = \begin{cases}\frac{\mathcal{N}\mathfrak{f}^{-1-j}2^{-1-j}\Phi(\mathfrak{m})}{(-1)^{1+j}(-2j)!\Phi(\mathfrak{f})}L'(\bar{\chi},j)\eta_{\mathbb{Q}} & \mathfrak{p}\mid\mathfrak{f}\\ (1-\bar{\chi}(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-j})\frac{\mathcal{N}\mathfrak{f}^{-1-j}2^{-1-j}\Phi(\mathfrak{m})}{(-1)^{1+j}(-2j)!\Phi(\mathfrak{f})}L'(\bar{\chi},j)\eta_{\mathbb{Q}} & \mathfrak{p}\nmid\mathfrak{f}.\end{cases}$$

□

3.2 ℓ-adic Computation

The second main result of this chapter is the computation of the ℓ-adic regulator of the motivic elements constructed in the proof of proposition 3.1.1.

Theorem 3.2.1. *For all $1 \neq \mathfrak{f} \mid \mathfrak{m}$, we have that*

$$\rho_{et}(\xi_{\mathfrak{f}}(j)) = \frac{\mathcal{N}\mathfrak{f}^{-1-j}w_{\mathfrak{f}}}{(\mathcal{N}\mathfrak{a}-\sigma(\mathfrak{a}))\prod_{\mathfrak{l}\mid\ell}(1-\mathrm{Fr}_{\mathfrak{l}}^{-1})(-2j)!}\cdot\left(\operatorname{Tr}_{K(\ell^n\mathfrak{f})/K(\mathfrak{f})}{}_{\mathfrak{a}}z_{\ell^n\mathfrak{f}}\zeta_{\ell^n}^{-j}\right)_n$$

up to a sign, where $\mathfrak{a} \nmid 6\ell\mathfrak{f}$ *is an auxilliary ideal and the* ${}_{\mathfrak{a}}z_{\ell^n\mathfrak{f}}$ *are elliptic units.*

Following the treatment in de Shalit's book [dS87](Ch.II) except for minor improvements involving the canonical choice of various 12-th roots, we review the construction of elliptic units. We first introduce certain very classical functions associated to lattices in the complex plane, and then indicate the connection to elliptic curves. The elliptic curves in fact play an auxiliary role.

Let $L = \mathbb{Z} \cdot w_1 + \mathbb{Z} \cdot w_2$ be a lattice in $\mathbb{C}$ with oriented basis w_1, w_2, i.e. so that $\tau := w_1/w_2$ has positive imaginary part. The Dedekind Eta-function is defined as

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - q_\tau^n); \quad q_\tau := e^{2\pi i \tau}$$

and we put

$$\eta^{(2)}(w_1, w_2) = w_2^{-1} 2\pi \eta(w_1/w_2)^2.$$

This function depends on the choice of basis but

$$\Delta(L) = \Delta(\tau) = \eta^{(2)}(w_1, w_2)^{12}$$

does not. Define a Theta-function

$$\phi(z, \tau) = i e^{\frac{\pi i z}{2} \frac{z - \bar{z}}{\tau - \bar{\tau}}} q_\tau^{1/12} q_z^{-1/2} (1 - q_z) \prod_{n=1}^{\infty} (1 - q_z q_\tau^n)(1 - q_z^{-1} q_\tau^n)$$

where $q_z = e^{2\pi i z}$ and

$$\phi(z; w_1, w_2) = \phi(z/w_2, w_1/w_2).$$

The function ϕ is holomorphic in z and τ and has a simple zero at each lattice point $z \in \mathbb{Z} \cdot w_1 + \mathbb{Z} \cdot w_2$. For any pair of lattices $L \subseteq L'$ of index prime to 6 with oriented bases $\omega := (w_1, w_2)$ and $\omega' := (w_1', w_2')$ it is shown by Robert in [Rob92](Thms. 1,2) that there exists a unique choice of 12-th root of unity $C(\omega, \omega')$ so that the functions

$$\delta(L, L') := C(\omega; \omega')\eta^{(2)}(\omega)^{[L':L]}/\eta^{(2)}(\omega')$$

and

$$\psi(z; L, L') = C(\omega; \omega')\phi(z; \omega)^{[L':L]}/\phi(z; \omega') = \delta(L, L') \prod_{u \in T} (\wp(z; L) - \wp(u; L))^{-1}$$

only depend on the lattices L, L' and so that ψ satisfies the distribution relation

$$\psi(z; K, K') = \prod_{i=1}^{[L:K]} \psi(z + t_i; L, L') \tag{3.2.1}$$

for any lattice $L \subseteq K$ so that $K \cap L' = L$ (and where $K' = K + L'$). The $t_i \in K$ are a set of representatives of K/L. The set T is any set of representatives of $(L' \setminus \{0\})/(\pm 1 \ltimes L)$ and $\wp$ is the Weierstrass $\wp$-function associated to L. In particular we see that $\psi(z; L, L')$ is an elliptic function, i.e. a rational function on the elliptic curve $E = \mathbb{C}/L$ with divisor $[L' : L](O) - \sum_{P \in L'/L}(P)$.

Kato reproves Robert's result in a scheme theoretic context. Again the key insight is that the distribution relation (or norm compatibility) suffices to canonically normalize the 12-th root.

Lemma 3.2.2. *(Kato [Kat04] (15.5.4))*
Let E be an elliptic curve over a field F with $\mathcal{O}_K \cong \operatorname{End}_F(E)$ and $\mathfrak{a}$ an ideal in $\mathcal{O}_K$ prime to 6. Then there is a unique function

$${}_{\mathfrak{a}}\Theta_E \in \Gamma(E \setminus {}_{\mathfrak{a}}E, \mathcal{O}^\times)$$

satisfying

(i) $div({}_{\mathfrak{a}}\Theta_E) = N\mathfrak{a} \cdot (0) - {}_{\mathfrak{a}}E$

(ii) For any $b \in \mathbb{Z}$ prime to $\mathfrak{a}$ we have $N_b({}_{\mathfrak{a}}\Theta_E) = {}_{\mathfrak{a}}\Theta_E$ where N_b is the norm map

associated to the finite flat morphism $E \setminus {}_{b\mathfrak{a}}E \to E \setminus {}_{\mathfrak{a}}E$ given by multiplication with b.

Moreover, for any isogeny $\phi : E \to E'$ (where $\mathrm{End}_F(E') = \mathcal{O}_K$) we have $\phi_*({}_{\mathfrak{a}}\Theta_E) = {}_{\mathfrak{a}}\Theta_{E'}$, in particular property (ii) also holds with $b \in \mathcal{O}_K$ prime to $\mathfrak{a}$. For $F = \mathbb{C}$ we have

$${}_{\mathfrak{a}}\Theta_E(z) = \psi(z; L, \mathfrak{a}^{-1}L).$$

Given $\mathfrak{f} \neq 1$ and any (auxiliary) $\mathfrak{a}$ which is prime to $6\mathfrak{f}$ we define an analog of the cyclotomic unit $1 - \zeta_f$ by

$${}_{\mathfrak{a}}z_{\mathfrak{f}} = \psi(1; \mathfrak{f}, \mathfrak{a}^{-1}\mathfrak{f})$$

and for $\mathfrak{f} = 1$ we define a family of elements indexed by *all* ideals $\mathfrak{a}$ of K by

$$u(\mathfrak{a}) = \frac{\Delta(\mathcal{O}_K)}{\Delta(\mathfrak{a}^{-1})}.$$

Lemma 3.2.3. *The complex numbers ${}_{\mathfrak{a}}z_{\mathfrak{f}}$ and $u(\mathfrak{a})$ satisfy the following properties*

a) (Rationality) ${}_{\mathfrak{a}}z_{\mathfrak{f}} \in K(\mathfrak{f})$, $u(\mathfrak{a}) \in K(1)$

b) (Integrality)

$${}_{\mathfrak{a}}z_{\mathfrak{f}} \in \begin{cases} \mathcal{O}^{\times}_{K(\mathfrak{f})} & \mathfrak{f} \textit{ divisible by primes } \mathfrak{p} \neq \mathfrak{q} \\ \mathcal{O}^{\times}_{K(\mathfrak{f}),\{v|\mathfrak{f}\}} & \mathfrak{f} = \mathfrak{p}^n \textit{ for some prime } \mathfrak{p} \end{cases}$$

$$u(\mathfrak{a}) \cdot \mathcal{O}_{K(1)} = \mathfrak{a}^{-12}\mathcal{O}_{K(1)}$$

c) (Galois action) For $(\mathfrak{c}, \mathfrak{f}\mathfrak{a}) = 1$ with Artin symbol $\sigma(\mathfrak{c}) \in \mathrm{Gal}(K(\mathfrak{f})/K)$ we have

$${}_{\mathfrak{a}}z_{\mathfrak{f}}^{\sigma(\mathfrak{c})} = \psi(1; \mathfrak{c}^{-1}\mathfrak{f}, \mathfrak{c}^{-1}\mathfrak{a}^{-1}\mathfrak{f}); \quad u(\mathfrak{a})^{\sigma(\mathfrak{c})} = u(\mathfrak{a}\mathfrak{c})/u(\mathfrak{c}).$$

This implies (see also [Kat04](15.4.4))

$$ {}_{\mathfrak{a}}z_{\mathfrak{f}}^{N\mathfrak{c}-\sigma(\mathfrak{c})} = {}_{\mathfrak{c}}z_{\mathfrak{f}}^{N\mathfrak{a}-\sigma(\mathfrak{a})}; \quad u(\mathfrak{a})^{1-\sigma(\mathfrak{c})} = u(\mathfrak{c})^{1-\sigma(\mathfrak{a})}. $$

d) (Norm compatibility) For a prime ideal $\mathfrak{p}$ *one has*

$$ N_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})}({}_{\mathfrak{a}}z_{\mathfrak{p}\mathfrak{f}})^{w_{\mathfrak{f}}/w_{\mathfrak{p}\mathfrak{f}}} = \begin{cases} {}_{\mathfrak{a}}z_{\mathfrak{f}} & \mathfrak{p} \mid \mathfrak{f} \neq 1 \\ {}_{\mathfrak{a}}z_{\mathfrak{f}}^{1-\sigma(\mathfrak{p})^{-1}} & \mathfrak{p} \nmid \mathfrak{f} \neq 1 \\ u(\mathfrak{p})^{(\sigma(\mathfrak{a})-N\mathfrak{a})/12} & \mathfrak{f} = 1 \end{cases} $$

e) (Kronecker limit formula). Let η *be a complex character of* $G_{\mathfrak{f}}$*. If* $\mathfrak{f} = 1$ *and* $\eta \neq 1$ *choose any ideal* $\mathfrak{a}$ *so that* $\eta(\mathfrak{a}) \neq 1$*. Then*

$$ \begin{aligned} L(\eta, 0) = \zeta_K(0) &= -\frac{h}{w_1}R && \eta = 1 \\ \frac{d}{ds}L(s,\eta)|_{s=0} &= -\frac{1}{1-\eta(\mathfrak{a})}\frac{1}{12w_1}\sum_{\sigma\in G_1}\log|\sigma(u(\mathfrak{a}))|\eta(\sigma) && \eta \neq 1, \quad \mathfrak{f} = 1 \\ \frac{d}{ds}L(s,\eta)|_{s=0} &= -\frac{1}{N\mathfrak{a}-\eta(\mathfrak{a})}\frac{1}{w_{\mathfrak{f}}}\sum_{\sigma\in G_{\mathfrak{f}}}\log|\sigma({}_{\mathfrak{a}}z_{\mathfrak{f}})|\eta(\sigma) && \mathfrak{f} \neq 1. \end{aligned} $$

Proof of 3.2.3: See [dS87] (Chapter II) and [Sie61] (Chapter II Section 2). □

Remarks: i) The relations in c) show the auxiliary nature of $\mathfrak{a}$. In $\mathcal{O}_{K(\mathfrak{f})}^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$ we can invert the element $N\mathfrak{a} - \sigma(\mathfrak{a}) \in \mathbb{Q}[G_{\mathfrak{f}}]$ and obtain an element $z_{\mathfrak{f}} = (N\mathfrak{a} - \sigma(\mathfrak{a}))^{-1}{}_{\mathfrak{a}}z_{\mathfrak{f}}$ independent of $\mathfrak{a}$. The last item in b) shows that $u(\mathfrak{c})^{1-\sigma(\mathfrak{a})} \in \mathcal{O}_{K(1)}^{\times}$ is a unit. However, here we cannot invert $1 - \sigma(\mathfrak{a})$ in $\mathbb{Q}[G_1]$ and obtain a unit independent of $\mathfrak{a}$, only in eigenspaces where $\eta(\mathfrak{a}) \neq 1$.

ii) The Galois action in c) together with the relation

$$ \psi(\lambda z; \lambda L, \lambda L') = \psi(z; L, L') $$

for any $\lambda \in \mathbb{C}$ shows that the Galois conjugates of ${}_{\mathfrak{a}}z_{\mathfrak{f}}$ are the numbers ${}_{\mathfrak{a}}\Theta_E(\alpha)$ where (E, α) runs through all pairs with $E/\mathbb{C}$ an elliptic curve and $\alpha \in E(\mathbb{C})$ a primitive $\mathfrak{f}$-division point. In fact ${}_{\mathfrak{a}}z_{\mathfrak{f}}$ is the value of ${}_{\mathfrak{a}}\Theta_E$ at a single closed point with residue field $K(\mathfrak{f})$ on a scheme E (with constant field $K(1)$).

Proof of 3.2.1: We compute the image of $\xi_{\mathfrak{f}}(j)$ under the étale Chern class map ρ_{et}. The following diagram commutes.

$$\begin{tikzcd}
H^{1-2j}_{\mathcal{M}}(E^{-2j}, 1-2j) \arrow[r, "\rho_{et}"] \arrow[d, "\mathcal{K}_{\mathcal{M}}"'] & H^{1-2j}(E^{-2j}, \mathbb{Q}_\ell(1-2j)) \arrow[d, "\mathcal{K}_\ell"] \\
H^1_{\mathcal{M}}(\mathrm{Spec}(K(\mathfrak{m})), 1-j) \arrow[r, "\rho_{et}"'] & H^1(K(\mathfrak{m}), \mathbb{Q}_\ell(1-j))
\end{tikzcd}$$

By Theorem 2.2.4 of [HK99], the étale realization of the Eisenstein symbol can be computed in terms of the pullback of the elliptic polylogarithm along torsion sections. Thus,

$$\rho_{et}(\xi_{\mathfrak{f}}(j)) = \mathcal{K}_\ell(\rho_{et}(\mathcal{E}is^{-2j}(\rho_{\mathfrak{f}} \cdot {}_{\mathfrak{f}}\beta))) = \psi(\rho_{\mathfrak{f}})^{-1} \cdot \mathcal{N}\mathfrak{f}^{-2j-1}\mathcal{K}_\ell({}_{\mathfrak{f}}\beta^*\mathcal{P}\mathrm{ol}_{\mathbb{Q}_\ell})^{-2\mathrm{j}}.$$

This computation has been done for an elliptic curve over any base by Kings [Kin01] (Theorem 4.2.9) using the geometric elliptic polylogarithm under the assumption that $\ell \nmid \mathfrak{f}$.

$$({}_{\mathfrak{f}}\beta^*\mathcal{P}\mathrm{ol}_{\mathbb{Q}_\ell})^{-2j} = \frac{\pm}{\mathcal{N}\mathfrak{a}([\mathfrak{a}]^{-2j}\mathcal{N}\mathfrak{a} - 1)(-2j)!}\left(\delta \sum_{[\ell^n]t_n = {}_{\mathfrak{f}}\beta} {}_{\mathfrak{a}}\Theta_E(-t_n)\tilde{t}_n^{\otimes -2j}\right)_n$$

where δ is the connecting homomorphism in a Kummer sequence which we hereafter omit, the $\Theta_{\mathfrak{a}}(-t_n)$ are the elliptic units defined in 3.2.2, $\tilde{t}_n$ is the projection of t_n to $E[\ell^n]$, and $\mathfrak{a} \subset \mathcal{O}_K$ is chosen prime to $\ell\mathfrak{f}$. Gealy [Gea05] uses a more direct method to prove a slightly weaker statement for the universal elliptic curve over the modular curve with full level N structure. What's more he fixes the sign by computing the residue of both sides. Strengthening Gealy's result is plausible and would yield a more straightforward solution. However, without this we must make some adjustments to

King's result as follows.

We are considering the elliptic curve E over $K(\mathfrak{m})$ with a uniformization $\mathbb{C}/\Gamma$. We can define a multiplication by $\rho_{\mathfrak{f}}$ on the elliptic curve componentwise. Notice that this is not the "map of multiplication by $\rho_{\mathfrak{f}}$" described in the main theorem of complex multiplication. By the choice of torsion point ${}_{\mathfrak{f}}\beta$, we see that $\rho_{\mathfrak{f}} t_n \in E[\ell^n]$. Taking this as our $\tilde{t_n}$ we must then multiply Kings' result by a factor of $\rho_{\mathfrak{f}}^{2j}$.

For a point $t \in E[\ell^n]$, we define $\gamma(t)^k :=< t, \sqrt{d_K} t >^{\otimes k}$ where $<,>$ is the Weil pairing. Then, the Kronecker map acts on the Tate module via

$$\mathcal{K}_\ell(\tilde{t}_n^{\otimes -2j}) = \gamma(\tilde{t}_n)^{-j} = \zeta_{\ell^n}^{\otimes -j},$$

on the integral isogenies $[\mathfrak{a}]$ by

$$\mathcal{K}_\ell([\mathfrak{a}]^{-2j}) = \mathcal{N}\mathfrak{a}^{-j}$$

and on the idele $\rho_{\mathfrak{f}}$ by $\mathcal{K}_\ell(\rho_{\mathfrak{f}}^{2j}) = \mathcal{N}(\rho_{\mathfrak{f}})^j = \mathcal{N}\mathfrak{f}^j$. Note also that the Artin automorphism $\sigma(\mathfrak{a})$ acts on the space $H^1(K(\mathfrak{m}), \mathbb{Q}_\ell)$ by $\mathcal{N}\mathfrak{a}$, and thus on the space $H^1(K(\mathfrak{m}), \mathbb{Q}_\ell(1-j))$ by $\mathcal{N}\mathfrak{a}^{2-j}$. We conclude that

$$\rho_{et}(\xi_{\mathfrak{f}}(j)) = \frac{\psi(\rho_{\mathfrak{f}})^{-1}}{\mathcal{N}\mathfrak{a} - \sigma(\mathfrak{a})} \cdot \mathcal{N}\mathfrak{f}^{-1-j} \left(\delta \sum_{[\ell^n] t_n = {}_{\mathfrak{f}}\beta} {}_{\mathfrak{a}}\Theta_E(-t_n) \zeta_{\ell^n}^{\otimes -j} \right)_n .$$

Lemma 3.2.4. *For any rational prime ℓ,*

$$\prod_{\mathfrak{l}|\ell} (1 - \mathrm{Fr}_{\mathfrak{l}}^{-1})^{-1} \left(\sum_{[\ell^n] t_n = \Omega f_{\mathfrak{f}}^{-1}} {}_{\mathfrak{a}}\Theta_E(-t_n) \zeta_{\ell^n}^{\otimes -j} \right)_n = w_{\mathfrak{f}} \left(\mathrm{Tr}_{K(\ell^n \mathfrak{f})/K(\mathfrak{f})}\, {}_{\mathfrak{a}}\Theta_E(-s_n) \zeta_{\ell^n}^{-j} \right)_n ,$$

where s_n is a primitive ℓ^nth root of ${}_{\mathfrak{f}}\beta$.

Notice that this lemma is similar to Kings [Kin01] (5.1.2).

Proof of 3.2.4: Let $\mathfrak{l}$ be a prime of K and $\nu = \mathrm{ord}_{\mathfrak{l}}(\mathfrak{f})$. For $\mathfrak{l}^r t_r = \Omega f_{\mathfrak{f}}^{-1}$ write

$t_r = (\tilde{t}_r, t_{r,0}) \in E[\mathfrak{l}^{r+\nu}] \oplus E[\mathfrak{f}_0] = E[\mathfrak{l}^r\mathfrak{f}_0]$.Define a filtration $F^\bullet$ on the set $H_r^{\mathfrak{l}} = \{\mathfrak{l}^r t_r = \Omega f_{\mathfrak{f}}^{-1}\}$ by

$$F_r^i := \{t_r = (\tilde{t}_r, t_{r,0}) \in H_{r,t}^{\mathfrak{l}} : \mathfrak{l}^{r+\nu-i}\tilde{t}_r = 0\}.$$

The Frobenius at $\mathfrak{l}$ acts via $(\mathrm{Fr}_{\mathfrak{l}}^{-1})\zeta_{\ell^r}^{\otimes k} = \zeta_{\ell^{r-1}}^{\otimes k}$ Thus, we compute

$$\begin{aligned} \mathrm{Fr}_{\mathfrak{l}}^{-i}\,\mathrm{Tr}_{K(\mathfrak{l}^r\mathfrak{f})/K(\mathfrak{l}^{r-i}\mathfrak{f})}\,{}_{\mathfrak{a}}\Theta_E(-s_r)\otimes\zeta_{\ell^r}^{\otimes -j} &= \mathrm{Tr}_{K(\mathfrak{l}^r\mathfrak{f})/K(\mathfrak{l}^{r-i}\mathfrak{f})}\,{}_{\mathfrak{a}}\Theta_E(-(\tilde{s}_r, s_{r,0}))\otimes\zeta_{\ell^{r-i}}^{\otimes -j} \\ &= {}_{\mathfrak{a}}\Theta_E(-(\tilde{s}_{r-i}, s_{r-i,0}))\otimes\zeta_{\ell^{r-i}}^{\otimes -j}. \end{aligned}$$

The second equality follows from the distribution relation for elliptic units in lemma 3.2.3. Notice that the elliptic function ${}_{\mathfrak{a}}\Theta_E$ does not change in the distribution relation even though the curve does because the lattices are homothetic.

The Galois group $\mathrm{Gal}(K(\mathfrak{l}^{r-i}\mathfrak{f})/K(\mathfrak{f}))$ acts transitively on $F_r^i \setminus F_r^{i+1}$ with each conjugate appearing $w_{\mathfrak{f}}$ times. Hence we can write

$$\mathrm{Fr}_{\mathfrak{l}}^{-i}\,\mathrm{Tr}_{K(\mathfrak{l}^r\mathfrak{f})/K(\mathfrak{f})}\,{}_{\mathfrak{a}}\Theta_E(-s_r)\otimes\zeta_{\ell^r}^{\otimes -j} = \frac{1}{w_{\mathfrak{f}}}\sum_{t_{r-i}\in F_r^i\setminus F_r^{i+1}} {}_{\mathfrak{a}}\Theta_E(-(\tilde{t}_{r-i}, t_{r-i,0}))\otimes\zeta_{\ell^{r-i}}^{\otimes -j}$$

These elements are annihilated by $\mathfrak{l}^r$, so summing over i we can take the limit as $r \to \infty$ to get

$$\begin{aligned} \left(\sum_{\mathfrak{l}^r t_r = _{\mathfrak{f}}\beta} {}_{\mathfrak{a}}\Theta_E(-t_r)\otimes\zeta_{\ell^r}^{\otimes -j}\right)_r &= w_{\mathfrak{f}}\left(\sum_{i=1}^{r}(\mathrm{Fr}_{\mathfrak{l}}^{-1})^i\,\mathrm{Tr}_{K(\mathfrak{l}^r\mathfrak{f})/K(\mathfrak{f})}\,{}_{\mathfrak{a}}\Theta_E(-s_r)\otimes\zeta_{\ell^r}^{\otimes -j}\right)_r \\ &= w_{\mathfrak{f}}(1-\mathrm{Fr}_{\mathfrak{l}}^{-1})^{-1}\left(\mathrm{Tr}_{K(\mathfrak{l}^r\mathfrak{f})/K(\mathfrak{f})}\,{}_{\mathfrak{a}}\Theta_E(-s_r)\otimes\zeta_{\ell^r}^{\otimes -j}\right)_r. \end{aligned}$$

For ℓ inert in K, the lemma is proved, and for ℓ split or ramified in K we apply the results to $\mathrm{Tr}_{K(\ell^n\mathfrak{f})/K(\mathfrak{f})} = \mathrm{Tr}_{K\ell^n\mathfrak{f})/K(\mathfrak{l}^n\mathfrak{f})}\,\mathrm{Tr}_{K(\mathfrak{l}^n\mathfrak{f})/K(\mathfrak{f})}$. □

By the main theorem of complex multiplication (3.1.6), $\rho_{\mathfrak{f}}\cdot s_n$ gives a primitive torsion point of 1 mod $\mathfrak{f}$ on the curve ${}^{\sigma}E$ with $\mathbb{C}/\Omega\mathfrak{f} \simeq {}^{\sigma}E(\mathbb{C})$. Therefore, we effectively "undo" our choice of ${}_{\mathfrak{f}}\beta$ via the identity

$$\psi(\rho_{\mathfrak{f}})^{-1}{}_{a}\Theta_{E}(-s_n) = {}_{\mathfrak{a}}z_{\mathfrak{f}\ell^n}.$$

In particular, we have shown that the χ component is given by

$$e_\chi \cdot \rho_{et}(\xi_{\mathfrak{f}}(j)) = \prod_{\mathfrak{l}|\ell}(1-\chi(\mathfrak{l})\mathcal{N}\mathfrak{l}^{-j})^{-1}\mathcal{N}\mathfrak{f}^{-1-j}\frac{w_{\mathfrak{f}}}{(-2j)!}\left(\mathrm{Tr}_{K(\ell^n\mathfrak{f})/K(\mathfrak{f})}\, z_{\ell^n\mathfrak{f}}\zeta_{\ell^n}^{-j}\right)_n. \quad (3.2.2)$$

where we follow Kato to set $z_{\ell^n\mathfrak{f}} = (\mathcal{N}\mathfrak{a}-\sigma(\mathfrak{a}))^{-1}{}_{\mathfrak{a}}z_{\ell^n\mathfrak{f}}$. This completes the proof of theorem 3.2.1. □

Chapter 4

Iwasawa Main Conjecture

In Chapter 2 we stated the equivariant Tamagawa number conjecture in terms of an equality of lattices for each prime ℓ of $\mathbb{Z}$. This formulation allows for a direct comparison to the main conjecture of Iwasawa theory by taking inverse limits up the tower of fields unramified outside of ℓ. In section 4.1 we construct a perfect complex of modules over the group ring $\mathbb{Z}_\ell[G]$ and compute their cohomology. Taking the inverse limit, we obtain a perfect complex of Iwasawa modules. The main conjecture can then be formulated as an analog of the ETNC for the Iwasawa modules. We give a precise statement in Conjecture 3.

Unfortunately, there is not a complete proof of this version of the main conjecture in this 2-variable case. Rubin's original work [Rub91] involved a somewhat different formulation. He defines a different module of elliptic units and shows an equality of characteristic ideals whenever $\ell \nmid [K(\mathfrak{m}) : K]$. In Bley's work on the ETNC at $j = 0$, he formulates a main conjecture in the same spirit as our conjecture 3 and gives a proof for split $\ell \nmid h_K$. However, his result does not apply to our situation. We formulate the 2-variable main conjecture in section 4.2 and discuss Bley's theorem in section 4.3.

4.1 Preliminaries

For an integral ideal $\mathfrak{f}$ of $\mathcal{O}_K$, let $G_\mathfrak{f}$ denote the Galois group of the extentsion $K(\mathfrak{f})$ over K. Fix an integral ideal $\mathfrak{m}$. Let $\mu = \mathrm{ord}_\ell \mathfrak{m}$ be compound notation.

$$\ell^\mu = \begin{cases} \mathfrak{l}_1^{\mu_1}\mathfrak{l}_2^{\mu_2} & \ell = \mathfrak{l}_1\mathfrak{l}_2 \text{ split} \\ \mathfrak{l}_1^{\mu_1} & \ell = \mathfrak{l}_1^2 \text{ ramified} \\ \ell^\mu & \ell \text{ inert,} \end{cases}$$

where $\mu_1, \mu_2 \in \mathbb{Z}$, and we write $\mathfrak{m} = \mathfrak{m}_0\ell^\mu$. For any finite set of places S of $K(\mathfrak{m})$, the $\mathbb{Z}[G_\mathfrak{m}]$ module X_S is defined to be the kernel of the sum map

$$0 \to X_S(K(\mathfrak{m})) \to Y_S(K(\mathfrak{m})) \to \mathbb{Z} \to 0$$

where $Y_S(K(\mathfrak{m})) := \bigoplus_{v\in S}\mathbb{Z}$. When there is no confusion we will suppress the field. We choose a projective $G_\mathbb{Q}$-stable $\mathbb{Z}_\ell[G_\mathfrak{m}]$ lattice

$$T'_\ell = H^0_{et}(\mathrm{Spec}(K(\mathfrak{m}) \otimes_K \overline{K}), \mathbb{Z}_\ell) = T_\ell(-j)$$

in the ℓ-adic realization,

$$M_\ell(-j) = H^0_{et}(\mathrm{Spec}(K(\mathfrak{m}) \otimes_K \overline{K}), \mathbb{Q}_\ell).$$

Then define a perfect complex of $\mathbb{Z}_\ell[G_\mathfrak{m}]$-modules

$$\Delta(K(\mathfrak{m})) := R\,\mathrm{Hom}_{\mathbb{Z}_\ell}(R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{m}\ell}], T'_\ell), \mathbb{Z}_\ell)[-3].$$

Lemma 4.1.1. *The cohomology of of $\Delta(K(\mathfrak{m}))$ is given by a canonical isomorphism,*

$$H^1(\Delta(K(\mathfrak{m})) \simeq H^1(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}_\ell(1)) \simeq \mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}]^\times \otimes_\mathbb{Z} \mathbb{Z}_\ell,$$

a short exact sequence,

$$0 \to \operatorname{Pic}(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}]) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to H^2(\Delta(K(\mathfrak{m}))) \to X_{\{v|\mathfrak{m}\ell\infty\}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to 0,$$

and $H^i(\Delta(K(\mathfrak{m}))) = 0$ *for* $i \neq 1, 2$*.*

Proof of 4.1.1 By Shapiro's lemma,

$$R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{m}\ell}], T'_\ell) \simeq R\Gamma_c(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}_\ell),$$

and by Artin-Verdier duality,

$$R\operatorname{Hom}_{\mathbb{Z}_\ell}(\tilde{R\Gamma}_c(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}_\ell), \mathbb{Z}_\ell)[-3] \simeq R\Gamma(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}_\ell(1)).$$

Here the tilde denotes that this complex differs from the usual definition of cohomology with compact supports (see Chapter 2) by substituting Tate cohomology for the purpose of dualizing. More specifically, the complex is defined as the shifted mapping cone

$$\tilde{R\Gamma}_c(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}_\ell) \to R\Gamma(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}_\ell) \to \bigoplus_{\mathfrak{p}|\mathfrak{m}\ell} R\Gamma_T(K(\mathfrak{m})_{\mathfrak{p}}, \mathbb{Z}_\ell),$$

where the infinite places are excluded from the direct sum as they are complex, and their Tate cohomology vanishes. The Kummer sequence

$$0 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m \to 0$$

induces the long exact cohomology sequence

$$\xrightarrow{\ell^n} H^i(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{G}_m) \to H^{i+1}(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mu_{\ell^n}) \to H^{i+1}(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{G}_m) \xrightarrow{\ell^n} .$$

The Galois cohomology is then computed by the short exact sequences

$$0 \to H^0(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{G}_m)/\ell^n \to H^1(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mu_{\ell^n}) \to H^1(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{G}_m)[\ell^n] \to 0$$
$$0 \to H^1(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{G}_m)/\ell^n \to H^2(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mu_{\ell^n}) \to H^2(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{G}_m)[\ell^n] \to 0$$

and the canonical isomorphism

$$H^0(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mu_{\ell^n}) \simeq H^0(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{G}_m)[\ell^n].$$

The cohomology of $\mathbb{G}_m$ on a curve is given by [Mil80]

$$H^i(X, \mathbb{G}_m) = \begin{cases} \Gamma(X, \mathcal{O}_X^\times) & i = 0 \\ \mathrm{Pic}(X) & i = 1 \\ 0 & i > 1. \end{cases}$$

Hence, taking inverse limits we compute that

$$H^i(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}_\ell(1)) = \begin{cases} \mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}]^\times \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & i = 1 \\ \mathrm{Pic}(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}]) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Now we need to account for the use of Tate cohomology. By the octahedral axiom for triangulated categories, there is an exact triangle

$$R\Gamma_c(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}_\ell), \mathbb{Z}_\ell) \to \widetilde{R\Gamma}_c (\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}_\ell), \mathbb{Z}_\ell) \to \bigoplus_{\mathfrak{p} | \mathfrak{m}\ell\infty} R\Gamma_\Delta(K(\mathfrak{m})$$

which induces the triangle

$$\bigoplus_{\mathfrak{p} | \mathfrak{m}\ell\infty} R\,\mathrm{Hom}(R\Gamma_\Delta(K(\mathfrak{m})_\mathfrak{p})[-3] \to R\Gamma(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}_\ell(1)) \to \Delta(K(\mathfrak{m}))$$

where $R\Gamma_\Delta$ is defined by the exact triangle

$$R\Gamma_\Delta(K(\mathfrak{m})_\mathfrak{p}) \to R\Gamma(K(\mathfrak{m})_\mathfrak{p}, \mathbb{Z}_\ell) \to R\Gamma_T(K(\mathfrak{m})_\mathfrak{p}, \mathbb{Z}_\ell).$$

Finally, computing the cohomology of the $R\Gamma_\Delta(K(\mathfrak{m})_\mathfrak{p})$ we have the lemma. $\square$

Remark: This computation is given in [BF98](Prop 3.3) under the condition that the S-restricted class group is trivial.

For invertible $\mathbb{Z}_\ell[G_\mathfrak{m}]$-modules, the dual of the inverse module (or vice versa) is isomorphic to the original module with the action of $G_\mathfrak{m}$ twisted by the automorphism $g \mapsto g^{-1}$. We denote the twisted action with a #. Hence, there is a natural isomorphism of determinants,

$$\mathrm{Det}_{\mathbb{Z}_\ell[G_\mathfrak{m}]} \Delta(K(\mathfrak{m})) \simeq \mathrm{Det}_{\mathbb{Z}_\ell[G_\mathfrak{m}]} R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{m}\ell}], T'_\ell)^{\#}.$$

For a complete treatment of Artin-Verdier duality, see Milne [Mil86].

4.2 Iwasawa theory

We first formulate the 2-variable main conjecture by considering the tower of ray class fields over $K(\mathfrak{m})$ unramified outside of the primes above ℓ. The Iwasawa algebra

$$\Lambda := \varprojlim_n \mathbb{Z}_\ell[G_{\mathfrak{m}\ell^n}] \simeq \mathbb{Z}_\ell[G^{tor}_{\mathfrak{m}\ell^\infty}][[S,T]]$$

is a finite product of complete local 3-dimensional Cohen-Macaulay rings, where $G^{tor}_{\mathfrak{m}\ell^\infty}$ is the torsion subgroup of $G_{\mathfrak{m}\ell^\infty} = \varprojlim_n G_{\mathfrak{m}\ell^n}$. Λ is regular if and only if $\ell \nmid \# G^{tor}_{\mathfrak{m}\ell^\infty}$. In general, this torsion subgroup is not $G_{\mathfrak{m}_0\ell}$ where $\mathfrak{m}_0$ is the prime to ℓ part of $\mathfrak{m}$. (Consider that case that $\ell \mid h_K$.)

The elements $S, T \in \Lambda$ depend on the choice of a complement $F \simeq \mathbb{Z}_\ell^2$ of the torsion subgroup in $G_{\mathfrak{m}\ell^\infty}$ as well as the choice topological generators γ_1, γ_2 of F. The cohomology of the perfect complex of Λ modules,

$$\Delta^\infty = \varprojlim_n \Delta(K(\mathfrak{m}\ell^n))$$

is computed by functoriality. By Lemma 4.1.1, $H^i(\Delta^\infty) = 0$ for $i \neq 1, 2$, and we have a canonical isomorphism,

$$H^1(\Delta^\infty) \simeq U^\infty_{\{v|\mathfrak{m}\ell\}} := \varprojlim_n \mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[\frac{1}{\mathfrak{m}\ell}]^\times \otimes_{\mathbb{Z}} \mathbb{Z}_\ell,$$

and a short exact sequence,

$$0 \to P^\infty_{\{v|\mathfrak{m}\ell\}} \to H^2(\Delta^\infty) \to X^\infty_{\{v|\mathfrak{m}\ell\infty\}} \to 0,$$

where

$$P^\infty_{\{v|\mathfrak{m}\ell\}} := \varprojlim_n \mathrm{Pic}(\mathcal{O}_{K(\mathfrak{m}\ell^n)}[\frac{1}{\mathfrak{m}\ell}]) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$$
$$X^\infty_{\{v|\mathfrak{m}\ell\infty\}} := \varprojlim_n X_{\{v|\mathfrak{m}\ell\infty\}}(K(\mathfrak{m}\ell^n)) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell.$$

The limits are taken with respect to the Norm maps, which on the module Y_S is the map sending a place to its restriction. We also consider $K(\mathfrak{m}\ell^n)$ as a subfield of $\mathbb{C}$ and denote the corresponding archimedean place by $\sigma_{\mathfrak{m}\ell^n}$. Notice that for $\mathfrak{f}_0 \mid \mathfrak{m}_0$, the elliptic units ${}_{\mathfrak{a}}z_{\mathfrak{f}_0\ell^n}$ discussed in section 3.2 form a Norm-compatible system of units. We set

$${}_{\mathfrak{a}}\eta_{\mathfrak{f}_0} := ({}_{\mathfrak{a}}z_{\mathfrak{f}_0\ell^n})_{n>>0} \in U^\infty_{\{v|\mathfrak{m}\ell\}}$$
$$\sigma := (\sigma_{\mathfrak{m}\ell^n})_{n>>0} \in Y^\infty_{\{v|\mathfrak{m}\ell\infty\}}$$

We fix an embedding $\bar{\mathbb{Q}}_\ell \to \mathbb{C}$ and identify $\hat{G}$ with the set of $\bar{\mathbb{Q}}_\ell$-valued characters.

The total ring of fractions

$$Q(\Lambda) \cong \prod_{\psi \in (\hat{G}^{tor}_{\mathfrak{m}\ell^\infty})^{\mathbb{Q}_\ell}} Q(\psi) \tag{4.2.1}$$

of Λ is a product of fields indexed by the $\mathbb{Q}_\ell$-rational characters of $G^{tor}_{\mathfrak{m}\ell^\infty}$. Since for any place w of K, the $\mathbb{Z}[G_{\mathfrak{m}\ell^n}]$-module $Y_{\{v|w\}}(K(\mathfrak{m}\ell^n))$ is induced from the trivial module $\mathbb{Z}$ on the decomposition group $D_w \subseteq G_{\mathfrak{m}\ell^n}$, and for $w = \infty$ (resp. nonarchimedean w) we have $[G_{\mathfrak{m}\ell^n} : D_w] = [K(\mathfrak{m}\ell^n) : K]$ (resp. the index $[G_{\mathfrak{m}\ell^n} : D_w]$ is bounded as $n \to \infty$), one computes easily

$$\dim_{Q(\psi)}(Y^\infty_{\{v|m\ell\infty\}} \otimes_\Lambda Q(\psi)) = 1 \tag{4.2.2}$$

for all characters ψ. Note that the inclusion $X^\infty_{\{v|\mathfrak{m}\ell\infty\}} \subseteq Y^\infty_{\{v|\mathfrak{m}\ell\infty\}}$ becomes an isomorphism after tensoring with $Q(\psi)$, and thus by the unit theorem

$$\dim_{Q(\psi)}(U^\infty_{\{v|\mathfrak{m}\ell\}} \otimes_\Lambda Q(\psi)) = 1. \tag{4.2.3}$$

So we have that $e_\psi({}_\mathfrak{a}\eta^{-1}_{\mathfrak{m}_0} \otimes \sigma)$ is a $Q(\psi)$-basis of

$$\begin{aligned} &\mathrm{Det}^{-1}_{Q(\psi)}(U^\infty_{\{v|\mathfrak{m}\ell\}} \otimes_\Lambda Q(\psi)) \otimes \mathrm{Det}_{Q(\psi)}(X^\infty_{\{v|\mathfrak{m}\ell\infty\}} \otimes_\Lambda Q(\psi) \\ \cong &\mathrm{Det}_{Q(\psi)}\left(\Delta^\infty \otimes_\Lambda Q(\psi)\right). \end{aligned}$$

The last isomorphism follows from the fact that the class group, $P^\infty_{\{v|\mathfrak{m}\ell\}}$ is a torsion Λ-module. Hence we obtain an element

$$\mathcal{L} := (\mathcal{N}\mathfrak{a} - \sigma(\mathfrak{a}))_\mathfrak{a}\eta^{-1}_{\mathfrak{m}_0} \otimes \sigma \in \mathrm{Det}_{Q(\Lambda)}\left(\Delta^\infty \otimes_\Lambda Q(\Lambda)\right).$$

Conjecture 3. *There is an equality of invertible Λ-submodules*

$$\Lambda \cdot \mathcal{L} = Det_\Lambda \Delta^\infty$$

of $Det_{Q(\Lambda)}(\Delta^\infty \otimes_\Lambda Q(\Lambda))$.

Remark: One proves this by localizing at all height 1 primes of Λ, [Fla04] (Lemma 5.3). We note the similarities to Rubin's main conjecture. For a height 1 prime $\mathfrak{q}$ of Λ, conjecture 3 is equivalent to

$$\mathrm{Fit}_{\Lambda_\mathfrak{q}}(U^\infty_\mathfrak{q}/\Lambda_\mathfrak{q}\cdot(\mathcal{N}\mathfrak{a}-\sigma(\mathfrak{a}))^{-1}{}_\mathfrak{a}\eta_{\mathfrak{m}_0}) = \mathrm{Fit}_{\Lambda_\mathfrak{q}}(P^\infty_\mathfrak{q})\cdot \mathrm{Fit}_{\Lambda_\mathfrak{q}}(X^\infty_{\{v|\mathfrak{m}_0\},\mathfrak{q}}).$$

Thus, if we suppose that Rubins module of elliptic units is the same as the module of elliptic units generated by the $(\mathcal{N}\mathfrak{a}-\sigma(\mathfrak{a}))^{-1}{}_\mathfrak{a}\eta_{\mathfrak{f}_0}$ (this should be true) we have the conjecture after accounting for the Euler factors in $X^\infty_{\{v|\mathfrak{m}_0\},\mathfrak{q}}$ whenever $\ell \nmid [K(\mathfrak{m}):K]$. For the other primes, the argument is more delicate.

4.3 A Theorem of Bley

When $\ell = \mathfrak{l}\bar{\mathfrak{l}}$ is split in K, the inverse limit

$$\Delta^\infty_\mathfrak{l} := \varprojlim_n \Delta(K(\mathfrak{m}_0\mathfrak{l}^n))$$

forms a perfect complex over the 1-variable Iwasawa algebra

$$\Lambda_\mathfrak{l} := \varprojlim_n \mathbb{Z}_\ell[G_{\mathfrak{m}\mathfrak{l}^n}] \simeq \mathbb{Z}_\ell[G_{\mathfrak{m}_0\mathfrak{l}}][[T]],$$

where we have changed notation and $\mathfrak{m} = \mathfrak{m}_0\mathfrak{l}^n$, so in particular it is possible that $(\mathfrak{m}_0,\ell)\neq 1$. Arguing as above, we deduce that $\mathcal{L}$ is in fact an element of $\mathrm{Det}_{Q(\Lambda_\mathfrak{l})}(\Delta^\infty_\mathfrak{l}\otimes_{\Lambda_\mathfrak{l}} Q(\Lambda_\mathfrak{l}))$. Bley proves the following 1-variable main conjecture in his preprint treating the case of $j=0$.

Theorem 4.3.1. *([Ble05] (Theorem 5.1)) If* $\mathfrak{l}\nmid h_K$*, then there is an equality of invertible* $\Lambda_\mathfrak{l}$*-submodules*

$$\Lambda_\mathfrak{l}\cdot\mathcal{L} = Det_{\Lambda_\mathfrak{l}}\Delta^\infty_\mathfrak{l}$$

of $Det_{Q(\Lambda_{\mathfrak{l}})}\,(\Delta_{\mathfrak{l}}^{\infty} \otimes_{\Lambda_{\mathfrak{l}}} Q(\Lambda_{\mathfrak{l}}))$.

Since we are taking negative Tate twists, we must descend from a tower of fields which contains the cyclotomic tower for the prime ℓ over $\mathbb{Q}$. $\Lambda_{\mathfrak{l}}$ does not factor over any such tower, so we are not able to use Bley's theorem to make the descent argument in the next chapter.

Chapter 5

Comparison of Integral Lattices

We continue with the notation of the previous chapters. In particular, $G_{\mathfrak{f}} = \mathrm{Gal}(K(\mathfrak{f})/K)$, $G = G_{\mathfrak{m}}$, $A = \mathbb{Q}[G]$, and $T_\ell = H^0(\mathrm{Spec}(K(\mathfrak{m}) \otimes_K \overline{K}), \mathbb{Z}_\ell(j))$.

In Chapter 3 we constructed elements in K-theory attached to the L-values of the motive and computed their realization in étale cohomology. Using this data, the L-value gives an element in the space $\mathrm{Det}_{A_\ell} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_\ell)$ as shown in theorem 5.1.1. We will show that the $\mathbb{Z}_\ell[G]$ lattice spanned by this element is the same as the one given by the ℓ-adic L-function of conjecture 3 via a descent argument in section 5.2.

5.1 The image of the L-value in $\mathrm{Det}_{A_\ell} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_\ell)$

The result in Theorem 5.1.1 is the upshot of the calculations in Chapter 3. The twist by $g \mapsto g^{-1}$ in the Galois action, denoted by # is necessary to make the regulator map equivariant for the action of the Galois group on the group of embeddings. Moreover, since $j < 0$ we must reinterpret in L-value in terms of the dual of the regulator map.

Theorem 5.1.1. *The element ${}_A\vartheta_\ell({}_A\vartheta_\infty(L^*({}_AM, 0)^{-1}))^{\#}$ of*

$$\mathrm{Det}_{A_\ell} \Delta(K(\mathfrak{m})) = \prod_{\chi \in \hat{G}} (\mathrm{Det}_{\mathbb{Q}_\ell(\chi)} \Delta(K(\mathfrak{m})) \otimes \mathbb{Q}_\ell(\chi))$$

has χ component given by

$$\prod_{\mathfrak{p}|\mathfrak{m}_0}(1-\bar{\chi}(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-j})^{-1}\frac{[K(\mathfrak{m}):K(\mathfrak{f}_\chi)]}{(-2)^{1+j}}(\mathcal{N}\mathfrak{a}-\chi(\mathfrak{a})\mathcal{N}\mathfrak{a}^{-j})\left(\mathrm{Tr}_{K(\mathfrak{f}_{\chi,0}\ell^n)/K(\mathfrak{f}_\chi)}({}_{\mathfrak{a}}z_{\mathfrak{f}_{\chi,0}\ell^n}\zeta_{\ell^n}^{-j})\right)_n^{-1}\otimes\zeta_{\ell^\infty}^{-j}\cdot e_\chi\tau_0.$$

Proof of 5.1.1:

The dual of the regulator isomorphism

$$\rho_\infty^\vee : H^0_B(K(\mathfrak{m})(\mathbb{C}),\mathbb{Q}(j))^+\otimes_{\mathbb{Q}}\mathbb{R}\xrightarrow{\sim}K_{1-2j}(\mathcal{O}_{K(\mathfrak{m})})^*\otimes_{\mathbb{Z}}\mathbb{R}$$

induces an isomorphism of rank 1 $A\otimes\mathbb{R}$-modules ${}_A\vartheta_\infty : A\otimes_{\mathbb{Q}}\mathbb{R}\to\Xi({}_AM)\otimes_{\mathbb{Q}}\mathbb{R}$, where we recall that

$$\Xi({}_AM)=(K_{1-2j}(\mathcal{O}_{K(\mathfrak{m})})^*\otimes_{\mathbb{Z}}\mathbb{Q})\otimes H^0_B(K(\mathfrak{m})(\mathbb{C}),\mathbb{Q}(j))^+.$$

In Theorem 3.1.1 we proved that for $\mathfrak{f}_\chi\neq 1$,

$$e_\chi(\rho_\infty(\xi_{\mathfrak{f}_\chi}(j)))=\frac{\mathcal{N}\mathfrak{f}_\chi^{-1-j}2^{-1-j}\Phi(\mathfrak{m})}{(-1)^{j+1}(-2j)!\Phi(\mathfrak{f}_\chi)}L'(\chi,j)\eta_{\mathbb{Q}},$$

where $\eta_{\mathbb{Q}}$ is a basis of $e_\chi(M_B^{+*})$ and we refer to Chapter 3 for the remaining notation. Moreover, for $\mathfrak{f}_\chi=1$, section 3.1 gives the formula in terms of trace maps

$$e_\chi\cdot\rho_\infty({}_{\mathfrak{q}}\xi_1(j))=e_\chi\cdot\rho_\infty(w_K(1-\mathrm{Fr}_{\mathfrak{q}}^{-1})^{-1}\,\mathrm{Tr}_{K(\mathfrak{q})/K(1)}\,\xi_{\mathfrak{q}}(j))=\frac{\Phi(\mathfrak{m})2^{-j-1}}{(-1)^{1+j}(-2j)!}L'(\chi,j)\eta_{\mathbb{Q}},$$

where we take the primitive L-function for χ. We will sometimes abuse notation at write $\xi_1(j)$ for a choice of ${}_{\mathfrak{q}}\xi_1(j)$.

Since both $H^0_B(K(\mathfrak{m})(\mathbb{C}),\mathbb{Q}(j))^+$ and $K_{1-2j}(\mathcal{O}_{K(\mathfrak{m})})\otimes_{\mathbb{Z}}\mathbb{Q}$ are invertible A-modules duality manifests in terms of the twist $g\mapsto g^{-1}$ according to the computation

$$\begin{aligned}\Xi({}_AM)^\# &= (K_{1-2j}(\mathcal{O}_{K(\mathfrak{m})})^*\otimes_{\mathbb{Z}}\mathbb{Q})^\#\otimes(H^0_B(K(\mathfrak{m})(\mathbb{C}),\mathbb{Q}(j))^{+,-1})^\#\\ &= (K_{1-2j}(\mathcal{O}_{K(\mathfrak{m})})\otimes_{\mathbb{Z}}\mathbb{Q})^{-1}\otimes(H^0_B(K(\mathfrak{m})(\mathbb{C}),\mathbb{Q}(j))^*)^+\\ &= (K_{1-2j}(\mathcal{O}_{K(\mathfrak{m})})\otimes_{\mathbb{Z}}\mathbb{Q})^{-1}\otimes Y(-j),\end{aligned}\tag{5.1.1}$$

where for v a place of $K(\mathfrak{m})$

$$Y(-j) := \bigoplus_{v|\infty} \mathbb{Q} \cdot (2\pi i)^{-j}.$$

The $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant perfect pairing

$$\bigoplus_{\tau\in\mathcal{T}} \mathbb{R}\cdot(2\pi i)^{j} \times \bigoplus_{\tau\in\mathcal{T}} \mathbb{C}/\mathbb{R}\cdot(2\pi i)^{1-j} \to \bigoplus_{\tau\in\mathcal{T}} \mathbb{C}/2\pi i\cdot\mathbb{R} \xrightarrow{\Sigma} \mathbb{R}$$

for $\mathcal{T} = \mathrm{Hom}(K(\mathfrak{m}),\mathbb{C})$ identifies the $\mathbb{Q}$-dual of $H^0_B(K(\mathfrak{m})(\mathbb{C}),\mathbb{Q}(j))$ with $\bigoplus_{\tau\in\mathcal{T}}\mathbb{Q}\cdot(2\pi i)^{-j}$. Taking invariants under complex conjugation gives the equality in 5.1.1.

We compute that the χ components of ${}_A\vartheta^{\#}_{\infty}(L^*({}_AM,0)^{-1}) = (L^*({}_AM,0)^{-1})^{\#}{}_A\vartheta_\infty(1)$ are given by

$$({}_A\vartheta^{\#}_{\infty}(L^*({}_AM,0)^{-1}))_\chi = \frac{\mathcal{N}\mathfrak{f}_\chi^{-1-j}2^{-1-j}\Phi(\mathfrak{m})}{(-1)^{j+1}(-2j)!\Phi(\mathfrak{f}_\chi)}[\xi_{\mathfrak{f}_\chi}(j)]^{-1}\otimes(2\pi i)^{-j}e_\chi\tau_0.$$

In Chapter 4 we defined $\Delta(K(\mathfrak{m}))$ and computed its cohomology (Lemma 4.1.1). Denote by $\Delta(K(\mathfrak{m}))_j$ the "twist" of $\Delta(K(\mathfrak{m}))$. Namely,

$$\Delta(K(\mathfrak{m}))_j := R\,\mathrm{Hom}_{\mathbb{Z}_\ell}(R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{m}\ell}],T_\ell),\mathbb{Z}_\ell)[-3].$$

The natural isomorphism

$$\begin{aligned}\mathrm{Det}_{\mathbb{Z}_\ell[G]}\,\Delta(K(\mathfrak{m})) &= (\mathrm{Det}_{\mathbb{Z}_\ell[G]}\,R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{m}\ell}],T'_\ell)^*)^{-1}\\ &\simeq \mathrm{Det}_{\mathbb{Z}_\ell[G]}\,R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{m}\ell}],T'_\ell)^{\#}\end{aligned}$$

induces

$$\mathrm{Det}_{\mathbb{Z}_\ell[G]}\,\Delta(K(\mathfrak{m}))_j \simeq \mathrm{Det}_{\mathbb{Z}_\ell[G]}\,R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{m}\ell}],T_\ell)^{\#},$$

and there are isomorphisms in cohomology

$$H^1(\Delta(K(\mathfrak{m}))_j) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \simeq H^1(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Q}_\ell(1-j))$$
$$H^2(\Delta(K(\mathfrak{m}))_j) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \simeq \left(\bigoplus_{\tau\in\mathcal{T}} \mathbb{Q}_\ell(-j)\right)^+,$$

with $H^i(\Delta(K(\mathfrak{m}))_j) = 0$ for $i \neq 1, 2$.

Thus, ${}_A\vartheta_\ell$ is given by the composite

$$\begin{aligned}\Xi({}_AM)^\# \otimes \mathbb{Q}_\ell &\simeq \mathrm{Det}_{A_\ell}^{-1}(K_{1-2j}(\mathcal{O}_{K(\mathfrak{m})}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell) \otimes \mathrm{Det}_{A_\ell}(Y(-j) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \\ &\xrightarrow{\sim} \mathrm{Det}_{A_\ell}^{-1}(H^1(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Q}_\ell(1-j))) \otimes \mathrm{Det}_{A_\ell}(\bigoplus_{\tau\in\mathcal{T}} \mathbb{Q}_\ell(-j))^+ \qquad (5.1.2) \\ &\xrightarrow{\sim} \mathrm{Det}_{A_\ell}^{-1}(H^1(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Q}_\ell(1-j))) \otimes \mathrm{Det}_{A_\ell}(\bigoplus_{\tau\in\mathcal{T}} \mathbb{Q}_\ell(-j))^+ \qquad (5.1.3) \\ &\xrightarrow{\sim} \mathrm{Det}_{A_\ell} \Delta(K(\mathfrak{m})),\end{aligned}$$

where the map (5.1.3) is multiplication with the Euler factors [BF98](Lemma 2) $\prod_{\mathfrak{p}|\mathfrak{m}\ell} \mathcal{E}_\mathfrak{p}^\# \in \mathfrak{A}^\times$ and $\mathcal{E}_\mathfrak{p} = (1 - \mathrm{Fr}_\mathfrak{p}^{-1})^{-1}$. These factors measure the difference in the two trivializations of the complex $R\Gamma_f(K_\mathfrak{p}, M_\ell)$ for each $\mathfrak{p} \mid \mathfrak{m}\ell$. The map (5.1.2) is induced by the isomorphism

$$K_{1-2j}(\mathcal{O}_{K(\mathfrak{m})}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \xrightarrow{\rho_{et}} H^1(\mathcal{O}_{K(\mathfrak{m})}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Q}_\ell(1-j)).$$

Thus far, we have shown for $\mathfrak{f}_\chi \neq 1$,

$$({}_A\vartheta_\ell \circ {}_A\vartheta_\infty(L^*({}_AM, 0)^{-1}))_\chi = \prod_{\mathfrak{p}|\mathfrak{m}\ell} (1 - \bar{\chi}(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-j})^{-1} \frac{\mathcal{N}\mathfrak{f}_\chi^{-1-j} 2^{-1-j} \Phi(\mathfrak{m})}{(-1)^{j+1}(-2j)!\Phi(\mathfrak{f}_\chi)} \rho_{et}(\xi_{\mathfrak{f}_\chi}(j))^{-1} \otimes \zeta_{\ell^\infty}^{-j} \cdot \sigma,$$

and for $\mathfrak{f}_\chi = 1$, we choose a $\mathfrak{q} \mid \mathfrak{m}$ to show

$$({}_A\vartheta_\ell \circ {}_A\vartheta_\infty(L^*({}_AM, 0)^{-1}))_\chi = \prod_{\mathfrak{q}\neq\mathfrak{p}|\mathfrak{m}\ell} (1 - \bar{\chi}(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-j})^{-1} \frac{\Phi(\mathfrak{m})}{(-2)^{1+j}} \rho_{et}(w_K \operatorname{Tr}_{K(\mathfrak{q})/K(1)} \xi_\mathfrak{q}(j))^{-1} \otimes \zeta_{\ell^\infty}^{-j} \cdot \sigma,$$

Theorem 3.2.1 states that for any $1 \neq \mathfrak{f} \mid \mathfrak{m}$,

$$\rho_{et}(\xi_{\mathfrak{f}}(j)) = \frac{\mathcal{N}\mathfrak{f}^{-1-j} w_{\mathfrak{f}}}{(\mathcal{N}\mathfrak{a} - \sigma(\mathfrak{a})) \prod_{\mathfrak{l}|\ell}(1 - \mathrm{Fr}_{\mathfrak{l}}^{-1})(-2j)!} \cdot \left(\mathrm{Tr}_{K(\ell^n\mathfrak{f})/K(\mathfrak{f})}\, {}_{\mathfrak{a}}z_{\ell^n\mathfrak{f}} \zeta_{\ell^n}^{-j}\right)_n.$$

We recall that $[K(\mathfrak{f}) : K(1)] = \Phi(\mathfrak{f}) w_{\mathfrak{f}}/w_K$ where $w_K \in \{2, 4, 6\}$ is the number of roots of unity in the imaginary quadratic field K, and $w_{\mathfrak{f}}$ is the number of roots of unity in K which are congruent to 1 modulo $\mathfrak{f}$. For $\mathfrak{f}$ large enough (at least bigger than 2) this number is 1. Thus, we can choose $\mathfrak{m}$ so that $w_{\mathfrak{m}} = 1$ and we have that $\Phi(\mathfrak{m})/\Phi(\mathfrak{f}_\chi) = [K(\mathfrak{m}) : K(\mathfrak{f}_\chi)] w_{\mathfrak{f}_\chi}$. What's more, if $(\ell, \mathfrak{f}) \neq 1$, then

$$\begin{aligned}\left(\mathrm{Tr}_{K(\ell^n\mathfrak{f})/K(\mathfrak{f})}\, {}_{\mathfrak{a}}z_{\ell^n\mathfrak{f}} \zeta_{\ell^n}^{-j}\right)_n &= \left(\mathrm{Tr}_{K(\ell^n\mathfrak{f}_0)/K(\mathfrak{f})} \left(\mathrm{Tr}_{K(\ell^{n+\mu}\mathfrak{f}_0)/K(\ell^n\mathfrak{f}_0)}\, {}_{\mathfrak{a}}z_{\ell^n\mathfrak{f}}\right) \zeta_{\ell^n}^{-j}\right)_n \\ &= \left(\mathrm{Tr}_{K(\ell^n\mathfrak{f}_0)/K(\mathfrak{f})}\, {}_{\mathfrak{a}}z_{\ell^n\mathfrak{f}_0} \zeta_{\ell^n}^{-j}\right)_n\end{aligned}$$

by lemma 3.2.3, where μ denotes the compound notation discussed in chapter 4. Thus for $\mathfrak{f}_\chi \neq 1$ we have the computed the component of the theorem.

When $\mathfrak{f}_\chi = 1$, choose $\mathfrak{q}$ so that $w_{\mathfrak{q}} = 1$ and compute

$$\begin{aligned}\rho_{et}(w_K \,\mathrm{Tr}_{K(\mathfrak{q})/K(1)}\, \xi_{\mathfrak{q}}(j)) =& w_K \frac{1}{(\mathcal{N}\mathfrak{a} - \sigma(\mathfrak{a})) \prod_{\mathfrak{l}|\ell}(1 - \mathrm{Fr}_{\mathfrak{l}}^{-1})} \cdot (\mathrm{Tr}_{K(\ell^n\mathfrak{q})/K(1)}\, {}_{\mathfrak{a}}z_{\ell^n\mathfrak{q}} \zeta_{\ell^n}^{-j})_n \\ =& \frac{(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1})}{(\mathcal{N}\mathfrak{a} - \sigma(\mathfrak{a})) \prod_{\mathfrak{l}|\ell}(1 - \mathrm{Fr}_{\mathfrak{l}}^{-1})} \cdot (\mathrm{Tr}_{K(\ell^n)/K(1)}\, {}_{\mathfrak{a}}z_{\ell^n} \zeta_{\ell^n}^{-j})_n.\end{aligned}$$

Substituting the formulas for ρ_{et} completes the proof of the theorem. □

5.2 Descent from the 2-variable main conjecture

In this section, we will prove that conjecture 3 implies ETNC for all $\ell \neq 2$. We expect the case of $\ell = 2$ to hold and that the treatment will be similar to that in [Fla04]. Recall that according to conjecture 3,

$$\mathcal{L} \cdot \Lambda = \mathrm{Det}_\Lambda(\Delta^\infty) \tag{5.2.1}$$

in $\mathrm{Det}_{Q(\Lambda)}(\Delta^\infty \otimes_\Lambda Q(\Lambda))$ where $\mathcal{L} = (\mathcal{N}\mathfrak{a} - \sigma(\mathfrak{a}))_{\mathfrak{a}}\eta_{\mathfrak{m}_0}^{-1} \otimes \sigma$. We refer to chapter 4 for the remaining notation. In this section we will justify the term "ℓ-adic L-function" by relating $\mathcal{L}$ to the special values of $L({}_AM, s)$. More precisely, we show that from the identity (5.2.1) we can deduce the identity

$$ {}_A\vartheta_\ell \circ {}_A\vartheta_\infty^{\#}(L^*({}_AM, 0)^{-1}) \cdot \mathbb{Z}_\ell[G_{\mathfrak{m}}] = \mathrm{Det}_{\mathbb{Z}_\ell[G_{\mathfrak{m}}]} \Delta(K(\mathfrak{m})) $$

in $\mathrm{Det}_{\mathbb{Q}_\ell[G_{\mathfrak{m}}]}(\Delta(K(\mathfrak{m})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$ to complete the proof that conjecture 4 implies conjecture 2.

We begin by proving a twisting lemma. For $j \in \mathbb{Z}$ we denote by $\kappa^j : G_{\mathfrak{m}\ell^\infty} \longrightarrow \Lambda^\times$ the character $g \mapsto \chi_{\mathrm{cyclo}}(g)^j g$ as well as the induced ring automorphism $\kappa^j : \Lambda \to \Lambda$. If there is no risk of confusion we also denote by $\kappa^j : \Lambda \to \mathbb{Z}_\ell[G_{\mathfrak{m}}] \subseteq A_\ell$ the composite of κ^j and the natural projection to $\mathbb{Z}_\ell[G_{\mathfrak{m}}]$ or A_ℓ.

Lemma 5.2.1. *a) For $j \in \mathbb{Z}$ there is a natural isomorphism*

$$ \Delta^\infty \otimes^{\mathbb{L}}_{\Lambda,\kappa^j} \mathbb{Z}_\ell[G_{\mathfrak{m}}] \xrightarrow{\simeq} \Delta(K(\mathfrak{m}))_j. $$

b) On the cohomology groups, the map $H^i(\Delta^\infty) \to H^i(\Delta^\infty_j)$ induces

$$ u \mapsto (u_n \cup \zeta_{\ell^n}^{\otimes -j})_{n>>0} \text{ and } s \mapsto (s_n \cup \zeta_{\ell^n}^{\otimes -j})_{n\geq 0} $$

where

$$ u = (u_n)_{n\geq 0} \in \varprojlim_n H^1(\mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}/\ell^n\mathbb{Z}(1)) \simeq U^\infty_{\{v|\mathfrak{m}\ell\}} = H^1(\Delta^\infty) $$

and

$$ s = (s_n)_{n\geq 0} \in \varprojlim \mathbb{Z}/\ell^n\mathbb{Z}[G_{\mathfrak{m}_0\ell^n}] \cdot \sigma = Y^\infty_{\{v|\infty\}} $$

Proof of 5.2.1 (As in [Fla04] (Lemma 5.13)) The automorphism κ^j can be viewed as the inverse limit of similarly defined automorphisms κ^j of the rings $\Lambda_n := \mathbb{Z}/\ell^n\mathbb{Z}[G_{\mathfrak{m}_0\ell^n}]$. Let $f_n : \mathrm{Spec}(\mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[\frac{1}{\mathfrak{m}\ell}]) \to \mathrm{Spec}(\mathcal{O}_{K(\mathfrak{m})})$ be the natural map. The sheaf $\mathcal{F}_n := f_{n,*}f_n^*\mathbb{Z}/\ell^n\mathbb{Z}$ is free of rank one over Λ_n with $\pi_1(\mathrm{Spec}(\mathcal{O}_{K(\mathfrak{m})}))$-action given by the natural projection $G_{\mathbb{Q}} \to G_{\mathfrak{m}_0\ell^n}$, twisted by the automorphism $g \mapsto g^{-1}$. There is a Λ_n-κ^{-j}-semilinear isomorphism $\mathrm{tw}^j : \mathcal{F}_n \to \mathcal{F}_n(j)$ so that Shapiro's lemma gives a commutative diagram of isomorphisms

$$\begin{array}{ccc} R\Gamma_c(\mathcal{O}_{K(\mathfrak{m})}, \mathcal{F}_n) & \xrightarrow{\mathrm{tw}^j} & R\Gamma_c(\mathcal{O}_{K(\mathfrak{m})}, \mathcal{F}_n(j)) \\ \downarrow & & \downarrow \\ R\Gamma_c(\mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}/\ell^n\mathbb{Z}) & \xrightarrow{\cup\zeta_{\ell^n}^{\otimes j}} & R\Gamma_c(\mathcal{O}_{K_{\mathfrak{m}_0\ell^n}}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}/\ell^n\mathbb{Z}(j)), \end{array} \tag{5.2.2}$$

with the horizontal arrows Λ_n-κ^{-j}-semilinear. Taking the $\mathbb{Z}/\ell^n\mathbb{Z}$-dual of the lower row (with contragredient $G_{\mathfrak{m}_0\ell^n}$-action), we obtain a $\# \circ \kappa^{-j} \circ \# = \kappa^j$-semilinear isomorphism

$$R\Gamma_c(\mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}/\ell^n\mathbb{Z}(j))^*[-3] \to R\Gamma_c(\mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}/\ell^n\mathbb{Z})^*[-3].$$

After passage to the limit this gives a κ^j-semilinear isomorphism $\Delta^\infty \simeq \Delta_j^\infty$, i.e. a Λ-linear isomorphism $\Delta^\infty \otimes_{\Lambda,\kappa^j} \Lambda \simeq \Delta_j^\infty$. The part a) follows by tensoring over Λ with $\mathbb{Z}_\ell[G_{\mathfrak{m}}]$. For b), consider the inverse map of the lower row of 5.2.2 on the degree two cohomology given by

$$H_c^2(\mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}/\ell^n\mathbb{Z}) \overset{\cup\zeta_{\ell^n}^{\otimes j}}{\leftarrow} H_c^2(\mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}/\ell^n\mathbb{Z}(j)).$$

Artin-Verdier duality says that

$$H_c^i(\mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}/\ell^n\mathbb{Z}(j))^\vee = H^{3-i}(\mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}/\ell^n\mathbb{Z}(1-j)).$$

Thus we have a dual map which is a κ^j semi-linear isomorphism.

$$H^1(\mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}/\ell^n\mathbb{Z}(1)) \overset{\cup\zeta_{\ell^n}^{\otimes j}}{\to} H^1(\mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[\frac{1}{\mathfrak{m}\ell}], \mathbb{Z}/\ell^n\mathbb{Z}(1-j)).$$

Moreover, we have a similar diagram to 5.2.2 on the level of sheaves where c denotes complex conjugation

$$\begin{array}{ccc} \mathcal{F}_n & \xrightarrow{\mathrm{tw}^j} & \mathcal{F}_n(j) \\ \downarrow & & \downarrow \\ \mathcal{F}_n^{c=1} = H^0(K(\mathfrak{m}_0\ell^n)\otimes\mathbb{R}, \mathbb{Z}/\ell^n\mathbb{Z}) & \xrightarrow{\cup\zeta_{\ell^n}^{\otimes j}} & H^0(K(\mathfrak{m}_0\ell^n)\otimes\mathbb{R}, \mathbb{Z}/\ell^n\mathbb{Z}(j)) = \mathcal{F}_n(j)^{c=1}. \end{array} \tag{5.2.3}$$

Again using the inverse map and taking the $\mathbb{Z}/\ell^n\mathbb{Z}$ dual, we again have a κ^j-semilinear isomorphism given by the cup product with $\zeta_{\ell^n}^{\otimes -j}$

$$\Lambda_n \cdot \sigma \mapsto \Lambda_n \cdot \sigma \cup \zeta_{\ell^n}^{\otimes -j}.$$

Taking inverse limits, we have part b). $\square$

As $\Delta(K(\mathfrak{m}))_j$ is a rank 1 $\mathbb{Z}_\ell[G_\mathfrak{m}]$-module, the image of $\mathcal{L}\otimes 1$ is a basis of the lattice. So it remains to compare this image with ${}_A\vartheta_\ell \circ {}_A\vartheta_\infty^{\#}(L^*({}_AM,0)^{-1})$ inside of the rational space $\Delta(K(\mathfrak{m}))\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ which is a rank one module over A_ℓ. Recall that in section 2.2 we showed that in order to prove theorem 2.2.1 it suffices to prove theorem 2.2.2 which states

$$({}_A\vartheta_\ell \circ {}_A\vartheta_\infty(L^*({}_AM,0))_\chi = (\mathcal{L}_{\mathfrak{m},j})_\chi.$$

The object of the descent computation is to make precise the notation $(\mathcal{L}_{\mathfrak{m},j})_\chi$ by computing the image of $\mathcal{L}$ in $\Delta(K(\mathfrak{m}))_j\otimes\mathbb{Q}_\ell(\chi)$. To this end, let $\mathfrak{q} = \mathfrak{q}_{\chi,j}$ be the height 2 prime of Λ given by the kernel of the composite ring homomorphism

$$\chi\kappa^j : \Lambda \xrightarrow{\kappa^j} \Lambda \to \mathbb{Z}_\ell[G(\mathfrak{m})] \subseteq A_\ell \to \mathbb{Q}_\ell(\chi).$$

$R := \Lambda_\mathfrak{q}$ is a regular local ring of dimension 2 with residue field $k := \mathbb{Q}_\ell(\chi)$. Let Δ be the module $\Delta_\mathfrak{q}^\infty$ over the localized ring R. To indicate the ℓ-divisibility of $\mathfrak{m}$ and

$\mathfrak{f}_\chi$, we continue with the compound notation of the previous chapter. Thus,

$$\mathfrak{m} = \mathfrak{m}_0 \ell^\mu \text{ and } \mathfrak{f}_\chi = \mathfrak{f}_{\chi,0} \ell^{\mu'},$$

where $(\mathfrak{m}_0, \ell) = (\mathfrak{f}_{\chi,0}, \ell) = 1$. For $\ell = \mathfrak{l}_1 \mathfrak{l}_2$ split, $\ell^\mu = \mathfrak{l}_1^{\mu_1} \mathfrak{l}_2^{\mu_2}$, and for $\ell = \mathfrak{l}_1^2$ ramified, $\ell^\mu = \mathfrak{l}_1^{\mu_1}$ where μ_1 and μ_2 are integers. Assuming conjecture 3, we can consider $\mathcal{L}$ to be a basis of the R-module $(\operatorname{Det}_\Lambda \Delta^\infty)_\mathfrak{q}$ which is isomorphic to $\operatorname{Det}_R \Delta$ since localization is exact and hence commutes with the determinant functor. Lemma 5.2.1 gives the following isomorphism of complexes of R-modules,

$$\Delta \otimes_R^{\mathbb{L}} k \xrightarrow{\simeq} \Delta(K(\mathfrak{m}))_j \otimes_{\mathbb{Z}_\ell[G_\mathfrak{m}]} k.$$

Lemma 5.2.2. *For $i = 1, 2$*

$$H^i(\Delta \otimes_R^{\mathbb{L}} k) \simeq H^i(\Delta) \otimes_R k.$$

Remark: Since the cohomology of Δ vanishes for $i \neq 1, 2$ this statement holds for all i.

Proof: Indeed, if (x, y) is a regular sequence for R, then the Koszul complex is the resolution

$$0 \to R \xrightarrow{\binom{x}{y}} R \oplus R \xrightarrow{(y,-x)} R \to k \to 0.$$

Thus, the homological spectral sequence for Tor degenerates to give an isomorphism $H^2(\Delta \otimes_R^{\mathbb{L}} k) \simeq H^2(\Delta) \otimes k$ and in degree 1 an exact sequence

$$0 \to \operatorname{Tor}_2(H^2(\Delta), k) \to H^1(\Delta) \otimes k \to H^1(\Delta \otimes_R^{\mathbb{L}} k) \to \operatorname{Tor}_1(H^2(\Delta), k) \to 0.$$

Now, the second degree cohomology is given by an exact sequence where the quotient is a free module (lemma 4.1.1)

$$0 \to P^\infty_{\{v|\mathfrak{m}\ell\}} \to H^2(\Delta^\infty) \to X^\infty_{\{v|\mathfrak{m}\ell\infty\}} \to 0.$$

Again, localization is exact, so we must show that the higher torsion groups of the localized class groups are zero. As R is a 2-dimensional local ring, the localization R_π at a height 1 prime is a DVR, and the image of ${}_{\mathfrak{a}}\eta_{\mathfrak{f}_{\chi,0}}$ in $H^1(\Delta)_\pi$ is non-zero because of its relationship to the non-vanishing L-value. Then, by Rubin's main conjecture, the fitting ideal of the $(P_{\mathfrak{q}}^{\infty})_\pi$ vanishes, and so by Nakayama's lemma does $P_{\mathfrak{q}}^{\infty}$. $\square$

By lemma 5.2.2 the isomorphism of determinants

$$\phi : \mathrm{Det}_k(\Delta \otimes_R^{\mathbb{L}} k) \xrightarrow{\simeq} \mathrm{Det}_k(\Delta(K(\mathfrak{m}))_j \otimes_{\mathbb{Z}_\ell[G_{\mathfrak{m}}]} k)$$

can be computed as a map on the cohomology groups

$$\begin{aligned}\phi : \bigotimes_{i=1}^{2} H^i(\Delta) \otimes k &\xrightarrow{\simeq} \bigotimes_{i=1}^{2} H^i(\Delta \otimes_R^{\mathbb{L}} k)\\ &\xrightarrow{\simeq} \bigotimes_{i=1}^{2} H^i(\Delta(K(\mathfrak{m}))_j) \otimes_{\mathbb{Q}_\ell[G]} k.\end{aligned}$$

Our descent computation amounts to computing $\phi(\mathcal{L} \otimes 1)$. We will consider the elements ${}_{\mathfrak{a}}\eta_{\mathfrak{m}_0}$ and σ independently. First, we recall that for an ideal $\mathfrak{d} \mid \mathfrak{m}_0$

$$N_{\mathfrak{d}} := \sum_{\tau \in \mathrm{Gal}(K(\mathfrak{m}_0)/K(\mathfrak{d}))} \tau.$$

When $\mathfrak{f}_{\chi,0} \mid \mathfrak{d}$, $N_{\mathfrak{d}}$ is invertible is the ring R since $\chi(N_{\mathfrak{d}}) = [K(\mathfrak{m}_0) : K(\mathfrak{d})]$. Thus, in the localized module Δ, the norm compatibility properties of the elliptic units give the equality

$$
\begin{aligned}
{}_{\mathfrak a}\eta_{\mathfrak m_0} =& N_{\mathfrak f_{\chi,0}}^{-1} N_{\mathfrak f_{\chi,0}}\,{}_{\mathfrak a}\eta_{\mathfrak m_0} \\
=& N_{\mathfrak f_{\chi,0}}^{-1} \prod_{\mathfrak p \mid \mathfrak m_0, \mathfrak p \nmid \mathfrak f_{\chi,0}} (1 - \mathrm{Fr}_{\mathfrak p}^{-1})(w_{\mathfrak m_0}/w_{\mathfrak f_{\chi,0}})\,{}_{\mathfrak a}\eta_{\mathfrak f_{\chi,0}} \\
=& (w_{\mathfrak m_0}/w_{\mathfrak f_{\chi,0}}) \left(\sum_{\tau \in \mathrm{Gal}(K(\mathfrak m)/K(\mathfrak m_0 \ell^{\mu'}))} \tau \right) \left(\sum_{\tau \in \mathrm{Gal}(K(\mathfrak m)/K(\mathfrak m_0 \ell^{\mu'}))} \tau \right)^{-1} \\
& N_{\mathfrak f_{\chi,0}}^{-1} \prod_{\mathfrak p \mid \mathfrak m_0, \mathfrak p \nmid \mathfrak f_{\chi,0}} (1 - \mathrm{Fr}_{\mathfrak p}^{-1})\,{}_{\mathfrak a}\eta_{\mathfrak f_{\chi,0}} \\
=& (w_{\mathfrak m_0 \ell^{\mu'}}/w_{\mathfrak f_\chi})[K(\mathfrak m) : K(\mathfrak f_\chi)]^{-1} \mathrm{Tr}_{K(\mathfrak m)/K(\mathfrak m_0 \ell^{\mu'})} \prod_{\mathfrak p \mid \mathfrak m_0, \mathfrak p \nmid \mathfrak f_{\chi,0}} (1 - \mathrm{Fr}_{\mathfrak p}^{-1})\,{}_{\mathfrak a}\eta_{\mathfrak f_{\chi,0}}.
\end{aligned} \tag{5.2.4}
$$

The last equality in 5.2.4 can be deduced from the diagram of fields below.

$$
\begin{tikzcd}
& K(\mathfrak m_0 \ell^{\mu'}) \arrow[d, dash, "\frac{w_{\mathfrak m_0 \ell^{\mu'}} w_{\mathfrak f_{\chi,0}}}{w_{\mathfrak m_0} w_{\mathfrak f_\chi}}"] & \\
& K(\mathfrak m_0)K(\mathfrak f_\chi) \arrow[dl, dash] \arrow[dr, dash] & \\
K(\mathfrak m_0) \arrow[dr, dash] & & K(\mathfrak f_\chi) \arrow[dl, dash] \\
& K(\mathfrak f_{\chi,0}) &
\end{tikzcd}
$$

Thus, by Lemma 5.2.1

$$
\begin{aligned}
\phi({}_{\mathfrak a}\eta_{\mathfrak m_0}) =& (w_{\mathfrak m_0 \ell^{\mu'}}/w_{\mathfrak f_\chi})[K(\mathfrak m) : K(\mathfrak f_\chi)]^{-1} \prod_{\mathfrak p \mid \mathfrak m_0, \mathfrak p \nmid \mathfrak f_{\chi,0}} (1 - \bar\chi(\mathfrak p)\mathcal N \mathfrak p^{-j}) \\
& \cdot \mathrm{Tr}_{K(\mathfrak m)/K(\mathfrak m_0 \ell^{\mu'})} (\mathrm{Tr}_{K(\mathfrak m_0 \ell^n)/K(\mathfrak m)}\ {}_{\mathfrak a}z_{\mathfrak f_{\chi,0}\ell^n} \otimes \zeta_{\ell^n}^{\otimes -j})_n \\
=& [K(\mathfrak m) : K(\mathfrak f_\chi)]^{-1} \prod_{\mathfrak p \mid \mathfrak m_0, \mathfrak p \nmid \mathfrak f_{\chi,0}} (1 - \bar\chi(\mathfrak p)\mathcal N \mathfrak p^{-j}) (\mathrm{Tr}_{K(\mathfrak f_{\chi,0}\ell^n)/K(\mathfrak f_\chi)}\ {}_{\mathfrak a}z_{\mathfrak f_{\chi,0}\ell^n} \otimes \zeta_{\ell^n}^{\otimes -j})_n.
\end{aligned}
$$

The second equality follows from a similar diagram of fields

$$\begin{tikzcd}
& K(\mathfrak{m}_0\ell^n) \arrow[d, dash, "\frac{w_{\mathfrak{f}_\chi}}{w_{\mathfrak{m}_0\ell^{\mu'}}}"] & \\
& K(\mathfrak{m}_0\ell^{\mu'})K(\mathfrak{f}_{\chi,0}\ell^n) \arrow[dl, dash] \arrow[dr, dash] & \\
K(\mathfrak{m}_0\ell^{\mu'}) \arrow[dr, dash] & & K(\mathfrak{f}_{\chi,0}\ell^n) \arrow[dl, dash] \\
& K(\mathfrak{f}_\chi) &
\end{tikzcd}$$

where we recall that we take $\mathfrak{m}$ and n to be large enough that $w_{\mathfrak{m}_0} = 1$ and $w_{\mathfrak{f}_{\chi,0}\ell^n} = 1$.

For the second degree cohomology, the situation is somewhat more simple. Indeed, by lemma 5.2.1,

$$\begin{aligned}
\phi(\sigma) =& e_\chi(\sigma_\mathfrak{m} \otimes \zeta_{\ell^n}^{-j})_n \\
=& e_\chi \sigma_\mathfrak{m} \otimes \zeta_{\ell^\infty}^{-j}.
\end{aligned}$$

Recalling that $\sigma_\mathfrak{m}$ was our fixed choice of embedding τ_0 from section 3.1 and multiplying by $\mathcal{N}\mathfrak{a} - \sigma(\mathfrak{a})$, we see that in fact

$$\phi(\mathcal{L}) = [K(\mathfrak{m}) : K(\mathfrak{f}_\chi)] \prod_{\mathfrak{p}|\mathfrak{m}_0, \mathfrak{p}\nmid\mathfrak{f}_{\chi,0}} (1-\bar{\chi}(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-j})^{-1}(\mathrm{Tr}_{K(\mathfrak{f}_{\chi,0}\ell^n)/K(\mathfrak{f}_\chi)}\, z_{\mathfrak{f}_{\chi,0}\ell^n} \otimes \zeta_{\ell^n}^{\otimes -j})_n^{-1} \otimes \zeta_{\ell^\infty}^{-j} \cdot e_\chi \tau_0.$$

Since $\chi(\mathfrak{p}) = 0$ for $\mathfrak{p} \mid \mathfrak{f}_\chi$ and 2 is a unit $\Lambda_\mathfrak{q}$ we have proved that $\phi(\mathcal{L} \otimes 1) = ({}_A\vartheta_\ell \circ {}_A\vartheta_\infty(L^*({}_AM, 0)^{-1}))^\#$. By its relation to the L-value established in theorem 5.1.1, the image of $(\mathcal{N}\mathfrak{a} - \sigma(\mathfrak{a}))^{-1}{}_\mathfrak{a}\eta_{\mathfrak{m}_0}$ does not vanish and thus is a basis of $H^1(\Delta(K(\mathfrak{m}))_j) \otimes \mathbb{Q}_\ell(\chi)$, making the image of σ is a basis of $H^2(\Delta(K(\mathfrak{m}))_j) \otimes \mathbb{Q}_\ell(\chi)$. This completes the proof of theorem 2.2.2. $\square$

Reason's last step is the recognition that there are an infinite number of things which are beyond it.

–Blaise Pascal

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