SUPERSONIC FLUTTER OF A PLATE CONTAINING MANY BAYS IN THE SPANWISE DIRECTION

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ABSTRACT

A special problem of panel flutter is considered by the method of Galerkin. The solution to the supersonic, non-steady potential equation is obtained by the approach of Garrick and Rubinow. Plate equations of motion are obtained by use of the Lagrange equations. Numerical solutions to the flutter determinant are presented for two cases.

TABLE OF CONTENTS

1. INTRODUCTION 1

2. PROBLEM STATEMENT. 2

3. AERODYNAMIC SOLUTION 3

4. PLATE EQUATIONS OF MOTION. 4

5. THE FLUTTER DETERMINANT. 13

6. NUMERICAL CALCULATIONS 14

7. DISCUSSION OF RESULTS. 15

8. REFERENCES 17

9. APPENDIX 18

FIGURES. 23

TABLES 26

LIST OF SYMBOLS

x, y, z	Cartesian coordinates
L_1	Streamwise length of the plate (x-direction)
L_2	Bay length of the plate (z-direction)
M	Mach number ($M = \frac{U}{a}$)
U	Free stream velocity
a	Free stream velocity of sound
$\beta = \sqrt{M^2 - 1}$	
t	Time variable
φ	Supersonic potential function
$\Phi(x,y)$	Plate mode shape
A_{mn}	Plate mode amplitude factor
m	Mode number in x-direction
n	Mode number in z-direction
ω	Angular frequency associated with flutter
$\omega_{\ell s}$	Natural angular frequency associated with the plate
$k = \frac{\omega L_1}{U}$	Reduced frequency
W, w	Upwash
$f_{\ell s} = A_{\ell s} e^{i\omega t}$	Generalized coordinates
T	Kinetic energy
V	Strain energy
ρ_s	Plate density
h	Plate thickness
ℓ, s	Summation indices

$D = \dfrac{Eh^3}{12(1 - \nu^2)}$ Plate flexural rigidity

E Modules of elasticity

ν Poisson's ratio

$\left.\begin{array}{l} \Lambda_{1,2,3} \\ \varepsilon_i \\ \alpha_i \\ \Delta(\ell,m) \end{array}\right\}$ Quantities used to evaluate $Q'_{mn\ell s}$

$Q'_{mn\ell s}$ Reduced generalized forces

$\mu = \dfrac{\rho_a L_1}{\rho_s h}$ Mass ratio

$\eta = \eta_o \cos\theta + y$

$\eta_o = \dfrac{x - \xi}{\beta}$

$\theta = \cos^{-1}\left(\dfrac{\eta - y}{x - \xi}\right)\beta$

ρ_a Air density

ξ Dummy variable in x-direction

$\tau_1 = \dfrac{x - \xi}{\alpha\beta^2}(M - \sin\theta)$

$\tau_2 = \dfrac{x - \xi}{\alpha\beta^2}(M + \sin\theta)$

$\psi = \tan^{-1}\left(\dfrac{ik}{m}\right)$ Phase shift in upwash

$i = \sqrt{-1}$

I_o, I_1, I_c Specified integrals

J_o Bessel's function of order zero

J_1 Bessel's function of order one

$$\Omega = \frac{kM^2}{\beta^2}$$

$$\Gamma = \frac{1}{L_1}\sqrt{\left(\frac{kM}{\beta}\right)^2 + \left(\frac{n\pi}{AR}\right)^2}$$

N_x, N_y Inplane plate stresses

$AR = \frac{L_2}{L_1}$ Aspect ratio

$$\lambda_r = \frac{\Gamma}{\beta}\cos\left[\pi\left(\frac{2r-1}{4q}\right)\right]$$

r Summation index

q Summation limit

1. INTRODUCTION

The problem of flutter of a thin elastic plate subjected to a supersonic stream on one surface has received considerable study. It seems that in most such considerations some approximation in addition to those of linearizing the aerodynamic and plate equations is used. In his Ph. D. thesis (1), Eisley carried out in detail the case of "strip theory". In the same work Eisley suggests a method for finding the exact solution by supposition of two separate problems. The first of these problems is that of a large number of bays side by side (and running at right angles to the flow direction) fluttering in phase with each other (see fig. 1). The second is the same except that the bays now flutter out of phase with their neighboring bays by π radians (see fig. 2). The two solutions would then be added to give the solution for discrete bays surrounded by non-fluttering plates.

The second of the problems has interest of itself since it seems likely to be at least a first approximation to the way in which the skin on wing or fuselage shell might flutter if the constraints at the edge of each bay are close to the case of simple supports. And in addition it lends itself to a simple model analysis which may yield an exact solution to the linearized, non-steady, supersonic potential equation.

It is the purpose of this work to pursue the second problem mentioned above, i.e. the supersonic flutter of a plate divided into many bays (at right angles to the flow direction) fluttering out of phase with its neighboring bays by π radians.

2. PROBLEM STATEMENT

The configuration to be studied here is depicted in fig. 2. It is composed of a uniform elastic plate of thickness h mounted on rigid simple supports at leading and trailing edges, and at a number of lines spaced uniformly in the y-direction. The individual bays are of length L_1 in the x or streamwise direction and L_2 in the y or spanwise direction. The upper surface of the plate is subjected to a supersonic stream at Mach number M and density ρ_a.

The plate has a flexural rigidity D and is subjected to inplane stresses N_x in the x-direction and N_y in the y-direction.

In the following work a solution for the aerodynamic potential equation is found and from it generalized forces are calculated. These generalized forces are then used with the Lagrange equations to determine the flutter determinate (2).

3. AERODYNAMIC SOLUTION

The linearized, non-steady, supersonic potential equation is

$$\beta^2 \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial z^2} + \frac{2M}{a^2}\frac{\partial^2\varphi}{\partial x\partial t} + \frac{\partial^2\varphi}{a^2\partial t^2} = 0 \qquad (1)$$

where $\beta^2 = M^2 - 1$, a is the velocity of sound, and M is the Mach number in the free stream. The three-dimensional solution of this equation due to Garrick and Rubinow (3) is briefly outlined below and is then applied to the problem under consideration.

The Garrick and Rubinow solution to equation 1 is obtained from the solution of the simple wave equation by using a Lorentz transformation and a Galilean transformation of the solution to the wave equation. The boundary condition that the airflow must be tangent to the air wetted surface is used to evaluate the potential at the surface $z = 0^+$ in the form

$$\varphi(x,y,0^+,t) = -\int_0^x \int_{\eta_1}^{\eta_2} W(\xi,\eta)\frac{w(t-\tau_1) + w(t-\tau_2)}{\sqrt{(\eta-\eta_1)(\eta_2-\eta)}}\, d\eta d\xi \qquad (2)$$

where

$$\tau_1 = \frac{M(x-\xi)}{a\beta^2} - \frac{\sqrt{(\eta-\eta_1)(\eta_2-\eta)}}{a\beta}$$

$$\tau_2 = \frac{M(x-\xi)}{a\beta^2} + \frac{\sqrt{(\eta-\eta_1)(\eta_2-\eta)}}{a\beta}$$

$$\eta_1 = y - \eta_o \quad , \quad \eta_2 = y + \eta_o \quad , \quad \eta_o = \frac{x - \xi}{\beta} \; .$$

Garrick and Rubinow then use the substitution

$$\eta = \eta_o \cos\theta + y \quad , \quad \text{where } \theta = \cos^{-1} \beta \, \frac{\eta - y}{x - \xi}$$

to obtain the potential equation in the form

$$\varphi(x,y,0^+,t) = -\frac{1}{2\pi\beta}\int_0^x \int_0^\pi W[\xi,\eta(\theta)][w(t-\tau_1)+w(t-\tau_2)]d\theta d\xi \qquad (3)$$

where

$$\tau_1 = \frac{(x-\xi)}{a\beta^2}(M - \sin\theta)$$

$$\tau_2 = \frac{(x-\xi)}{a\beta^2}(M + \sin\theta)$$

It should be noted that this is an integral over a Mach cone.

By the nature of the problem to be examined, the mode shape to be used is suggested as

$$\Phi(x,y,t) = \Phi'(x,y)\, e^{i\omega t}$$

$$= A_{mn} \sin\left(\frac{m\pi x}{L_1}\right) \sin\left(\frac{m\pi y}{L_2}\right) e^{i\omega t} \qquad (4)$$

From this the upwash is found to be

$$W(x,y,0^+,t) = [\, U \frac{\partial\Phi'}{\partial x} + i\omega\Phi']\, e^{i\omega t} \qquad (5)$$

the first term being due to the plate slope and the second due to the plate motion. In more detail equation 5 becomes

$$W(x,y,0^+,t) = A_{mn}e^{i\omega t}\sin(\frac{n\pi y}{L_2})[\frac{Um\pi}{L_1}\cos(\frac{m\pi x}{L_1}) + i\omega\sin(\frac{m\pi x}{L_1})] , \qquad (6)$$

which may be written

$$W(x,y,0^+,t) = A_{mn}(\frac{U}{L_1})\sqrt{m^2\pi^2 - k^2}\ \sin(\frac{n\pi y}{L_2})\ \cos\ (\frac{m\pi x}{L_1} - \psi)e^{i\omega t} \qquad (7)$$

where

$$\psi = \tan^{-1}(\frac{ik}{m\pi}) \quad , \quad k = \frac{\omega L_1}{U} \quad .$$

When equations 3 and 7 are used together the expression for the potential function becomes

$$\varphi(x,y,0^+,t) = -\frac{A_{mn}U\sqrt{m^2\pi^2 - k^2}\ e^{i\omega t}}{2\pi L_1\beta}\int_0^x\int_0^\pi \sin(\frac{n\pi\eta}{L_2})\cos(\frac{m\pi\xi}{L_1} - \psi)[e^{-i\omega\tau_1} + e^{-i\omega\tau_2}]d\theta d\xi \qquad (8)$$

Replacing η, τ_1 and τ_2 with the expressions following equation 2 and simplifying, the expression for φ becomes

$$\varphi(x,y,0^+,t) = -\frac{A_{mn}U\ \sqrt{m^2\pi^2 - k^2}}{\pi L_1\beta}\ e^{i\omega t}\int_0^x e^{-i(\Omega/L_1)(x-\xi)}\cos(\frac{m\pi\xi}{L_1} - \psi)\ I_o d\xi , \qquad (9)$$

where

$$I_o = \int_0^\pi \cos[\frac{\omega\eta_o}{a\beta}\sin\theta]\ \sin\ [\frac{n\pi}{L_1}(y + \eta_o\cos\theta)]\ d\theta \ .$$

To evaluate I_o it is written in the following way:

$$\sin\left(\frac{n\pi y}{L_2}\right)\int_0^{\pi}\cos\left[\frac{\omega\eta_o}{a\beta}\sin\theta\right]\cos\left[\frac{n\pi}{L_2}\eta_o\cos\theta\right]d\theta$$

$$+\cos\left(\frac{n\pi y}{L_2}\right)\int_0^{\pi}\cos\left[\frac{\omega\eta_o}{a\beta}\sin\theta\right]\sin\left[\frac{n\pi}{L_2}\eta_o\cos\theta\right]d\theta \tag{10}$$

As pointed out in reference 1 the second integral is identically zero. To evaluate the first integral the relation

$$J_o\left\{\sqrt{\gamma^2+\alpha^2}\right\} = \frac{1}{\pi}\int_0^{\pi}\cos[\alpha\cos\theta]\cos[\gamma\sin\theta]\,d\theta\,, \tag{11}$$

which follows from page 21 of reference 5, is used to arrive at the expression

$$I_o = \pi\sin\left(\frac{n\pi y}{L_2}\right)J_o\left\{\frac{\eta_o}{L_1}\sqrt{\left(\frac{kM}{\beta}\right)^2+\left(\frac{n\pi}{AR}\right)^2}\right\}. \tag{12}$$

To determine the generalized forces the pressure distribution must be calculated from the supersonic potential by means of the momentum equation

$$\frac{D}{Dt}(\vec{q}) = -\frac{1}{\rho_a}\,\mathrm{GRAD}\,(P)\,,$$

where

$$\vec{q} = \mathrm{GRAD}\,\varphi$$

then

$$\frac{D}{Dt}(\varphi) = -\frac{1}{\rho_a}P\,, \tag{13}$$

or

$$P = -\rho_a(U \frac{\partial}{\partial x} \varphi + \frac{\partial}{\partial t} \varphi) . \tag{13a}$$

Introducing equations 9 and 12 into 13a yields

$$P = - \rho_a H[U\left\{I_c + \cos(\frac{m\pi x}{L_1} - \psi)\right\} + i\omega I_1] \tag{14}$$

where

$$I_c = - \int_0^x e^{-i(\Omega/L_1)(x-\xi)} \cos(\frac{m\pi \xi}{L_1} - \psi)\left\{i \frac{\Omega}{L_1} J_o[\frac{\Gamma}{\beta} (x - \xi)]\right.$$

$$\left. + \frac{\Gamma}{\beta} J_1[\frac{\Gamma}{\beta} (x - \xi)]\right\} d\xi$$

$$I_1 = \int_0^x e^{-i(\Omega/L_1)(x-\xi)} \cos(\frac{m\pi \xi}{L_1} - \psi) J_o[\frac{\Gamma}{\beta} (x - \xi)] \, d\xi$$

$$H = - \frac{A_{mn} U \sqrt{m^2\pi^2 - k^2}}{L_1\beta} \sin (\frac{n\pi y}{L_2}) e^{i\omega t}$$

$$\Gamma = \frac{1}{L_1} \sqrt{(\frac{kM}{\beta})^2 + (\frac{n\pi}{AR})^2}$$

The Bessel functions in these expressions are replaced by the following approximate expressions:

$$J_o[\frac{\Gamma}{\beta} (x - \xi)]= \frac{1}{q} \sum_{r=1}^{q} \cos[\lambda_r(x - \xi)] + 2J_{4q}[\frac{\Gamma}{\beta} (x - \xi)]$$

$$J_1[\frac{\Gamma}{\beta} (x - \xi)]= \frac{1}{q} \sum_{r=1}^{q} \cos \pi(\frac{2r-1}{4q}) \sin [\lambda_r(x - \xi)]$$

$$- J_{4q-1} [\frac{\Gamma}{\beta} (x - \xi)] + J_{4q+1}[\frac{\Gamma}{\beta} (x - \xi)]$$

$$\lambda_r = \frac{\Gamma}{\beta} \cos \pi (\frac{2r - 1}{4q}) \tag{15}$$

in which the trailing terms are estimates of the error in taking q terms of the series as an approximation to the Bessel function.

The expression for the pressure distribution associated with the mn^{th} mode is the rather lengthy expression:

$$P_{mn} = \rho_a \left\{ \frac{A_{mn}}{L_1 \beta} U^2 \sqrt{m^2\pi^2 - k^2} \sin\left(\frac{n\pi y}{L_2} \right) e^{i\omega t} \right\} \left\{ \cos\left(\frac{m\pi x}{L_1} - \psi \right) \right.$$

$$+ i\left(\frac{k}{L_1} - \frac{\Omega}{L_1} \right) \frac{1}{q} \sum_{r=1}^{q} \int_0^x e^{-i(\Omega/L_1)(x-\xi)} \cos\left(\frac{m\pi\xi}{L_1} - \psi \right) \cos\left[\lambda_r (x - \xi)\right] d\xi$$

$$\left. - \frac{\Gamma}{\beta} \frac{1}{q} \sum_{r=1}^{q} \int_0^x e^{-i(\Omega/L_1)(x-\xi)} \cos\left(\frac{m\pi\xi}{L_1} - \psi \right) \cos \pi \left(\frac{2r-1}{4q}\right) \sin\left[\lambda_r (x-\xi)\right] d\xi \right\} \tag{16}$$

4. PLATE EQUATIONS OF MOTION

The equations of motion for a single bay subjected to the pressure distribution calculated from the full group of bays are to be derived. This is a correct procedure since the assumed mode shapes for the adjacent bays only differ in sense.

The assumed mode shape is

$$\Phi(x,y,t) = \sum_{\ell} \sum_{s} \quad f_{\ell s} \sin(\frac{\ell \pi x}{L_1}) \sin(\frac{s \pi y}{L_2}) \tag{17}$$

$$f_{\ell s} = A_{\ell s} e^{i\omega t}$$

while the generalized forces are given by

$$Q_{\ell s} \delta A_{\ell s} = \sum_{m} \sum_{n} \int_0^{L_1}\int_0^{L_2} P_{mn} \delta A_{\ell s} \sin(\frac{\ell \pi x}{L_1}) \sin(\frac{s \pi y}{L_2}) dydx \tag{18}$$

and the Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{f}_{\ell s}} \right) + \frac{\partial V}{\partial f_{\ell s}} = Q_{\ell s} \tag{19}$$

The kinetic energy for a plate is given by

$$T = \frac{\rho_s h}{2} \int_0^{L_1}\int_0^{L_2} (\dot{\Phi})^2 \, dxdy$$

$$= \frac{\rho_s h}{2} \sum_{\ell} \sum_{s} \int_0^{L_1}\int_0^{L_2} (\dot{f}_{\ell s})^2 \sin^2(\frac{\ell \pi x}{L_1}) \sin^2(\frac{s \pi y}{L_2}) dydx \tag{20}$$

which yields

$$T = \left(\frac{\rho_s h}{2}\right)\left(\frac{L_1 L_2}{4}\right) \sum_{\ell} \sum_{s} (\dot{f}_{\ell s})^2 \tag{20a}$$

The strain energy, V, calculated from the linearized plate theory has two contributions: one from bending, V_B, and one from the in plane stresses, V_T.

The strain energy due to bending is

$$V_B = \frac{D}{2}\int_0^{L_1}\int_0^{L_2}\left\{\left(\frac{\partial^2\Phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\Phi}{\partial y^2}\right)^2 + 2\nu \frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2\Phi}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2\Phi}{\partial x\partial y}\right)^2\right\}dydx \tag{21}$$

or, performing the indicated operations,

$$V_B = \frac{DL_1L_2}{8} \sum_{\ell} \sum_{s} (f_{\ell s})^2\left[\left(\frac{\ell\pi}{L_1}\right)^2 + \left(\frac{s\pi}{L_2}\right)^2\right]^2 \tag{21a}$$

and

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

ν being Poisson's ratio and E, Young's modulus.

If N_x and N_y are the in plane stresses in the x- and y-directions respectively the strain energy due to them may be expressed as

$$V_T = -\frac{1}{2}\int_0^{L_1}\int_0^{L_2}\left[N_x\left(\frac{\partial\Phi}{\partial x}\right)^2 + N_y\left(\frac{\partial\Phi}{\partial y}\right)^2\right] dydx \tag{22}$$

Again performing the indicated operations

$$V_T = - \frac{\pi^2 L_1 L_2}{8} \sum_{\ell} \sum_{s} (f_{\ell s})^2 [N_x (\frac{\ell}{L_1})^2 + N_y (\frac{s}{L_2})^2] \tag{22a}$$

If the relations for strain energy and kinetic energy are substituted in equation 19 the equations of motion are found to be

$$(\ddot{f}_{\ell s} + \omega_{\ell s}^2 f_{\ell s}) \frac{\rho_s L_1 L_2}{4} = Q_{\ell s} \tag{23}$$

with $\omega_{\ell s}$ being the natural plate frequency defined by

$$(\omega_{\ell s})^2 = \frac{\pi^2}{\rho_s h} \left\{ D\pi^2 [(\frac{\ell}{L_1})^2 + (\frac{s}{L_2})^2]^2 - [N_x (\frac{\ell}{L_1})^2 + N_y (\frac{s}{L_2})^2] \right\}$$

An equivalent form of equation 23 which will be more useful in calculating the flutter determinate is

$$\frac{Q_{\ell s}}{e^{i\omega t}} = \rho_a \sum_{m} \sum_{n} A_{mn} Q_{mn\ell s} = A_{\ell s} \frac{\rho_a L_1 L_2}{4} [\omega_{\ell s}^2 - \omega^2] \tag{23a}$$

Representation of the Generalized Forces

At this point the principal remaining difficulty is to find a usable representation for the generalized forces. It will be recognized that substitution of equation 16 into equation 18 yields a rather cumbersome relation containing both double and triple integrations. The exponential is replaced by use of Euler's formula while the products of trigonometric functions are converted to sums of trigonometric functions. In this form the integration over ξ can be carried out conveniently.

After considerable manipulation there are left some thirty integrals which must be carried out over x and y. These integrations yield results of the following form:

$$\frac{\cos\psi + \sin\psi}{\left(\frac{\Omega}{L_1} - \lambda_r + \frac{m\pi}{L_1}\right)}\left[\frac{\sin\left\{L_1\left(\frac{\Omega}{L_1} - \lambda_r - \frac{\ell\pi}{L_1}\right)\right\}}{2\left(\frac{\Omega}{L_1} - \lambda_r - \frac{\ell\pi}{L_1}\right)} - \frac{\sin\left\{L_1\left(\frac{\Omega}{L_1} - \lambda_r + \frac{\ell\pi}{L_1}\right)\right\}}{2\left(\frac{\Omega}{L_1} - \lambda_r + \frac{\ell\pi}{L_1}\right)}\right]$$

$$\text{if}\quad \left(\frac{\Omega}{L_1} - \lambda_r - \frac{\ell\pi}{L_1}\right) \neq 0 \tag{24}$$

and

$$\frac{\frac{L_1}{2}\cos\psi}{\left(\frac{\Omega}{L_1} - \lambda_r + \frac{m\pi}{L_1}\right)} \qquad \text{if}\ \left(\frac{\Omega}{L_1} - \lambda_r - \frac{\ell\pi}{L_1}\right) = 0$$

When all these terms are combined and simplified to a certain extent, the representation for $Q_{\ell s}$ is

$$\frac{Q_{\ell s}}{e^{i\omega t}} = \rho_a \sum_m \sum_n A_{mn} Q_{mn\ell s}$$

$$Q_{mn\ell s} = \left[\frac{U^2\sqrt{m^2\pi^2 - k^2}}{L_1\beta}\right]\left[\Lambda_1 + i\left(\frac{k-\Omega}{L_1}\right)\frac{1}{q}\sum_{r=1}^{q}\Lambda_2 - \frac{\Gamma}{\beta}\cdot\frac{1}{q}\sum_{r=1}^{q}\Lambda_3\right] \tag{25}$$

The representations for Λ_1, Λ_2 and Λ_3 are given in Appendix A.

5. The Flutter Determinant

The condition that the flutter represented thus should exist is that the amplitudes, A_{mn}, have a non-vanishing value. This means that the determinant formed from the coefficients of A_{mn} in equation 23a should vanish identically. This condition may be written as

$$\left| \delta_{mn\ell s} \left[\left(\frac{k_{\ell s}}{k_{11}} \right)^2 - \left(\frac{k}{k_{11}} \right)^2 \right] - \frac{4\mu}{k_{11}^2} Q'_{mn\ell s} \right| = 0 \qquad (26)$$

where

$$Q'_{mn\ell s} = \frac{Q_{mn\ell s}}{U^2 L_2} \quad ; \quad \mu = \frac{\rho_a L_1}{\rho_a h}$$

$$\delta_{mn\ell s} = \begin{cases} 1 & \text{if } m = \ell \text{ and } n = s \\ 0 & \text{if } m \neq \ell \text{ or } n \neq s \end{cases}$$

The term containing $\delta_{mn\ell s}$ comes in because $A_{\ell s} = A_{mn}$ at $\ell = m$ and $s = n$.

6. NUMERICAL CALCULATIONS

Numerical calculations have been carried out for a two mode analysis of two cases: $AR = 1$, $M = 2$ and $AR = 1$, $M = \sqrt{2}$.

The first stage of these calculations was to determine the values of $Q'_{\ell m}$* from equation 25 where q is taken as three. This value for q was chosen since it will allow a good representation for the Bessel's function approximations without making the computational routine excessively lengthy. The range of the reduced frequency parameter, k, was from 0.1000 to 0.5000 in this calculation. For the values of $Q'_{\ell m}$ thus obtained see tables I and II.

Stage number two in the calculation was solution of the two mode flutter determinant using the results of the first stage. As a matter of convenience the notation $\kappa = \frac{k_2^2}{k_{11}^2}$ and $\gamma = \frac{\mu}{k_{11}^2}$ is introduced and the solution of the flutter determinant is carried out in terms of these parameters. Since μ is a ratio of real masses only those values of κ and γ which lead to a real value of μ correspond to the existance of flutter. Solutions of the flutter determinants are presented in tables III and IV.

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* $Q'_{m\ell}$ is written in place of $\frac{Q'_{m_1 \ell_1}}{U^2 L_2}$.

7. DISCUSSION OF RESULTS

An interesting limit case can be obtained from the expression for $Q'_{\ell m}$ if the aspect ratio is allowed to approach infinity. First let equation 23a be divided by $\frac{L_2}{2}$. Then we see that what concerns us is the limit of $\frac{2}{L_2} Q_{mn\ell s}$ as the aspect ratio (or L_2) tends to infinity. Then if equation 25 and expressions for Λ_i (which are given in the appendix) are considered it is seen that a factor $\frac{L_2}{2}$ is present in each Λ_i to cancel the $\frac{2}{L_2}$ factor we have in front of $Q_{mn\ell s}$. Examination of the remaining expression shows that $\frac{2}{L_2}\Lambda_1$ is now independent of aspect ratio, while $\frac{2}{L_2}\Lambda_2$ and $\frac{2}{L_2}\Lambda_3$ depend on aspect ratio through the parameter

$$\Gamma/\beta = \frac{1}{L_1\beta}\sqrt{\left(\frac{kM}{\beta}\right)^2 + \left(\frac{n\pi}{AR}\right)^2}\,.$$

In the limiting case of infinite aspect ratio then Γ/β is replaced by $\Gamma'/\beta = \frac{kM}{L_1\beta^2}$ (which is equal to the parameter σ used by Shen (8) and Eisley (1)). Thus the limiting case is achieved by replacing Γ by Γ' in the equations for $\frac{2}{L_2}\Lambda_i$. This causes a change in the terms Γ, λ_r and $\alpha_{1,2}$. But none of these terms is a function of ℓ or m so that the form of the expression for $Q'_{\ell m}$ is not altered.

It is also to be noticed that this form for $Q'_{\ell m}$ is such that $Q'_{\ell m} \neq -Q'_{m\ell}$, a result which does not agree with Shen and Eisley.

Phase angle relations of $Q'_{\ell m}$ in the case $AR = 1$, $M = 2$ are shown in figure 3. It is seen that Q'_{11} and Q'_{22} have a stronger phase dependence on k than do either Q'_{12} or Q'_{21}. This difference of dependence on k can be seen by examining the expressions for Λ_i in the two cases $m = \ell$ and $m \neq \ell$ as shown in the appendix.

The results of the numerical calculations indicate that for the range of k considered, no flutter exists at $M = 2$, $AR = 1$; while for $M = \sqrt{2}$, $AR = 1$ flutter does not exist except in the neighborhood of $k = 0.250$. These results show the need for further calculations; first in the neighborhood of $k = 0.250$ for $AR = 1$, $M = \sqrt{2}$, and secondly extending the range of k for $AR = 1$, $M = 2$. These calculations have not been carried out to date because of their rather extensive nature.

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APPENDIX A

In the representation for $Q'_{mn\,s}$, three terms Λ_1, Λ_2, Λ_3 are used. In each of these the terms sin ψ and cos ψ appear. Since these functions are given by

$$\sin\psi = \frac{ik}{\sqrt{m^2\pi^2 - k^2}} \quad ; \quad \cos\psi = \frac{m\pi}{\sqrt{m^2\pi^2 - k^2}},$$

it will be convenient to give the representations for $\sqrt{m^2\pi^2 - k^2}\ \Lambda_i$ rather than for Λ_i.

It should also be noted that the expressions Λ_i are composed of terms which have different values under the conditions α_i equal or not equal to $(\frac{\ell\pi}{L_1})$, where

$$\alpha_1 = \frac{\Omega}{L_1} + \lambda_r \quad , \quad \alpha_2 = \frac{\Omega}{L_1} - \lambda_r \quad .$$

Then the expressions for $\sqrt{m^2\pi^2 - k^2}\ \Lambda_i$ are given below with the condition that L_1 be set equal to unity which is really the same as using a nondimensional length system $L_1' = \frac{L_1}{L_1}$, $L_2' = \frac{L_2}{L_1}$.

$$\Lambda_1 = \Lambda_2 = \Lambda_3 = 0 \qquad \text{if} \qquad s \neq n$$

$$\sqrt{m^2\pi^2 - k^2}\ \Lambda_1 = \frac{L_2}{2}(ik) \qquad \text{if} \qquad m = \ell$$

$$= \frac{L_2}{2}\left[\frac{\ell m}{\ell^2 - m^2}\right]\left[1 - (1)^{\ell+m}\right] \qquad \text{if } m \neq \ell$$

$$\sqrt{m^2\pi^2 - k^2}\,\Lambda_2 = \frac{L_2}{16}\left\{\left[\frac{(-1)^\ell 4\pi^2 \ell m(\alpha_1 + ik)\sin\alpha_1}{(\alpha_1^2 - \ell^2\pi^2)(\alpha_1^2 - m^2\pi^2)}\right](1 - i)\right.$$

$$+\left[\frac{(-1)^\ell 4\pi^2 \ell m[\alpha_2(1 - \frac{k}{m\pi}) + i(m\pi + k)]\sin\alpha_2}{(\alpha_2^2 - \ell^2\pi^2)(\alpha_2^2 - m^2\pi^2)}\right]$$

$$\left.-8i\left[\frac{\ell m}{\ell^2 - m^2}\right]\left[1 - (-1)^{\ell+m}\right]\left[\frac{\Omega^2(k+\lambda_r)+\lambda_r^2(k-\lambda_r)-(m\pi)^2(k-\lambda_r)}{(\alpha_1^2 - m^2\pi^2)(\alpha_2^2 - m^2\pi^2)}\right]\right\}$$

If $\alpha_1 = \ell\pi$, replace the first term by

$$\frac{m\pi(\alpha_2 - k)}{\alpha_1^2 - m^2\pi^2}$$

If $\alpha_2 = \ell\pi$, replace the second term by

$$\frac{\alpha_2(m\pi - k)}{\alpha_2^2 - m^2\pi^2}$$

If $m = \ell$, replace the third term by

$$4\left[\frac{k\lambda_r(\alpha_1\alpha_2 - m^2\pi^2) - \frac{1}{2}m^2\pi^2(\alpha_1^2 + \alpha_2^2 - 2m^2\pi^2)}{(\alpha_1^2 - m^2\pi^2)(\alpha_2^2 - m^2\pi^2)}\right]$$

$$\sqrt{m^2\pi^2 - k^2}\,\Lambda_3 = \frac{L_2}{16}\cos\left[\pi\left(\frac{2r-1}{4q}\right)\right]\Bigg\{\left[\frac{(-1)\;4\pi^2\ell m\;\sin\alpha_2}{\alpha_2^{\,2} - \ell^2\pi^2}\right]\left[\frac{(m\pi - k) + i\alpha_2\left(\frac{k}{m\pi} - 1\right)}{\alpha_2^{\,2} - m^2\pi^2}\right]$$

$$+\left[\frac{(-1)^{\ell}4\pi^2\ell m\;\sin\alpha_1}{\alpha_1^{\,2} - \ell^2\pi^2}\right]\left[\frac{\left(\frac{\alpha_1}{m\pi} - 1\right)(1 - i)}{\alpha_1^{\,2} - m^2\pi^2}\right]$$

$$+ 8\left[\frac{\ell m}{\ell^2 - m^2}\right]\left[1 - (-1)^{\ell+m}\right]\left[\frac{\Omega(\alpha_1\alpha_2 - k\lambda_r - m^2\pi^2)}{(\alpha_1^{\,2} - m^2\pi^2)(\alpha_2^{\,2} - m^2\pi^2)}\right]\Bigg\}$$

If $\alpha_2 = \ell\pi$, replace the first term by

$$\frac{i2m\pi\,(k - \alpha_2)}{\alpha_2^{\,2} - m^2\pi^2}$$

If $\alpha_1 = \ell\pi$, replace the second term by

$$\frac{i2\alpha_1(k + m\pi)}{\alpha_1^{\,2} - m^2\pi^2}$$

If $m = \ell$, replace the third term by

$$\frac{i4\lambda_r(\Omega m\pi - \alpha_1\alpha_2 - m^2\pi^2)}{(\alpha_1^{\,2} - m^2\pi^2)(\alpha_2^{\,2} - m^2\pi^2)}$$

The techniques used to determine these representations for Λ_i will be illustrated by a specific example which is typical. To establish a useful representation for Λ_3 the expression

$$\int_0^{L_2}\int_0^{L_1}\sin(\frac{\ell\pi x}{L_1})\sin(\frac{s\pi y}{L_2})\sin(\frac{n\pi y}{L_2})\int_0^x e^{-i(\Omega/L_1)(x-\xi)}\cos(\frac{m\pi\xi}{L_1}-\psi)$$

$$\sin[\lambda_r(x-\xi)]\cos[\pi(\frac{2r-1}{4q})]\ d\xi\ dx\ dy$$

must be carried out. To do this we first consider the last integral which we write as:

$$(\varepsilon_1 - i\varepsilon_2)\cos[\pi(\frac{2r-1}{4q})]$$

$$= \Big(\int_0^x \cos[\frac{\Omega}{L_1}(x-\xi)]\cos(\frac{m\pi\xi}{L_1}-\psi)\sin[\lambda_r(x-\xi)]\,d\xi$$

$$-i\int_0^x \sin[\frac{\Omega}{L_1}(x-\xi)]\cos(\frac{m\pi\xi}{L_1}-\psi)\sin[\lambda_r(x-\xi)]d\xi\Big)\cos[\pi(\frac{2r-1}{4q})]$$

And then let us consider ε_1. The triple product of trigonometric functions can be expanded and recombined to give the following relation

$$\varepsilon_1 = \frac{1}{4}\sin(\alpha_1 x)\int_0^x\Big\{\cos[\xi(\alpha_1+\frac{m\pi}{L_1})-\psi] + \cos[\xi(\alpha_1-\frac{m\pi}{L_1})+\psi]\Big\}d\xi$$

$$-\frac{1}{4}\cos(\alpha_1 x)\int_0^x\Big\{\sin[\xi(\alpha_1+\frac{m\pi}{L_1})-\psi] + \sin[\xi(\alpha_1-\frac{m\pi}{L_1})+\psi]\Big\}d\xi$$

$$+\frac{1}{4}\cos(\alpha_2 x)\int_0^x\Big\{\sin[\xi(\alpha_2+\frac{m\pi}{L_1})-\psi] + \sin[\xi(\alpha_2-\frac{m\pi}{L_1})+\psi]\Big\}d\xi$$

$$-\frac{1}{4}\sin(\alpha_2 x)\int_0^x\Big\{\cos[\xi(\alpha_2+\frac{m\pi}{L_1})-\psi] + \cos[\xi(\alpha_2-\frac{m\pi}{L_1})+\psi]\Big\}d\xi$$

These integrals are easily carried out and after some manipulation there results

$$\varepsilon_1 = \frac{1}{4}\left\{ \frac{\lambda_r[\lambda_r^2-(\frac{m\pi}{L_1})^2-3(\frac{\Omega}{L_1})^2]\cos(\frac{m\pi x}{L_1}-\psi)}{[\alpha_1^2-(\frac{m\pi}{L_1})^2][\alpha_2^2-(\frac{m\pi}{L_1})^2]} + \frac{\cos(\alpha_2 x-\psi)}{(\alpha_2-\frac{m\pi}{L_1})}\right.$$

$$\left. + \frac{\cos(\alpha_2 x+\psi)}{(\alpha_2+\frac{m\pi}{L_1})} - \frac{\cos(\alpha_1 x+\psi)}{(\alpha_1+\frac{m\pi}{L_1})} - \frac{\cos(\alpha_1 x-\psi)}{(\alpha_1-\frac{m\pi}{L_1})} \right\}$$

To achieve the final representation for Λ_3 integrals of the sort

$$\int_0^{L_2}\int_0^{L_1} \varepsilon_1(x)\sin(\frac{\ell\pi x}{L_1})\sin(\frac{s\pi y}{L_2})\sin(\frac{n\pi y}{L_2})\,dx\,dy$$

are carried out to give the results which have already been stated.

The expressions for Λ_1 and Λ_2 are obtained by a procedure similar to the one outlined above for determining Λ_3.

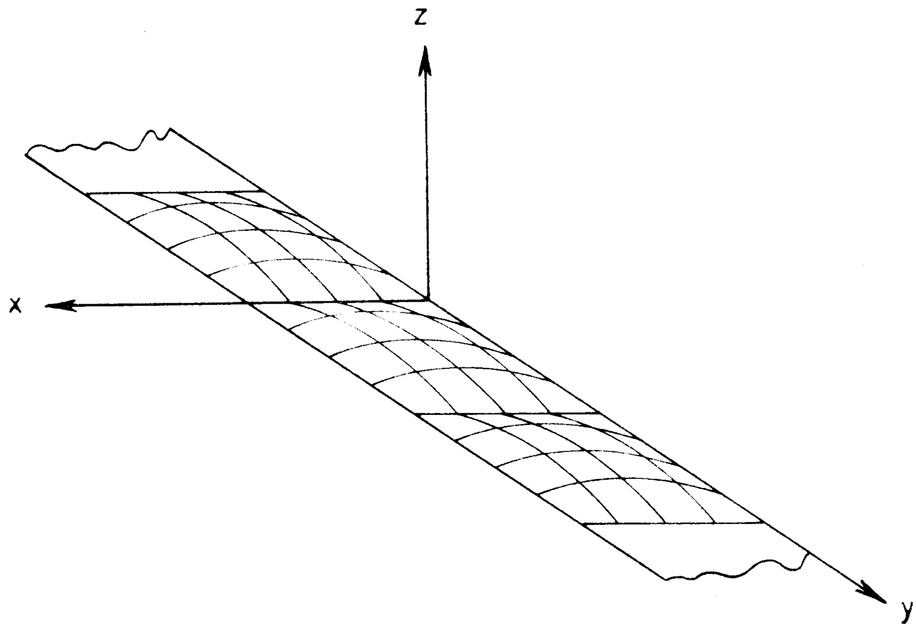

FIG. 1

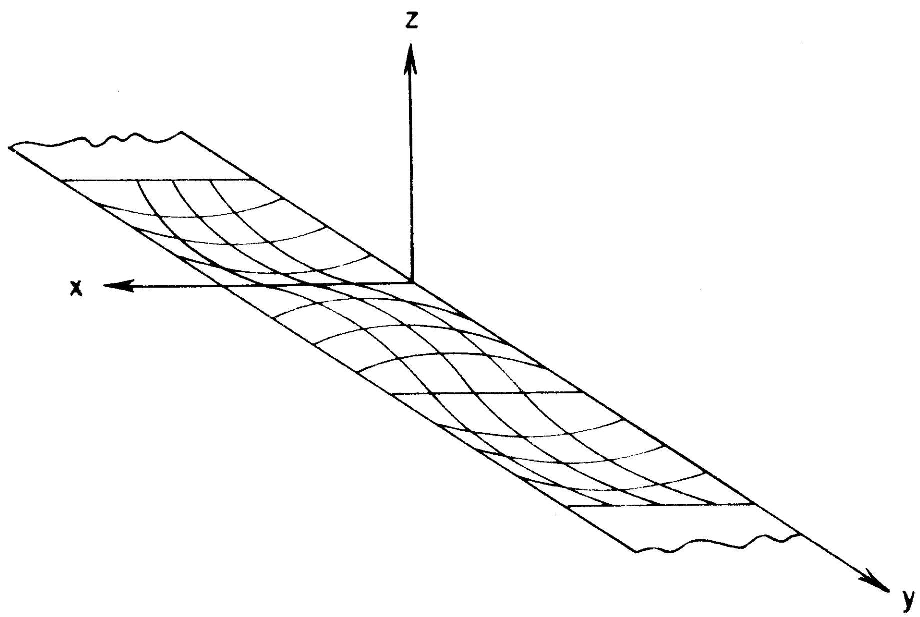

FIG. 2

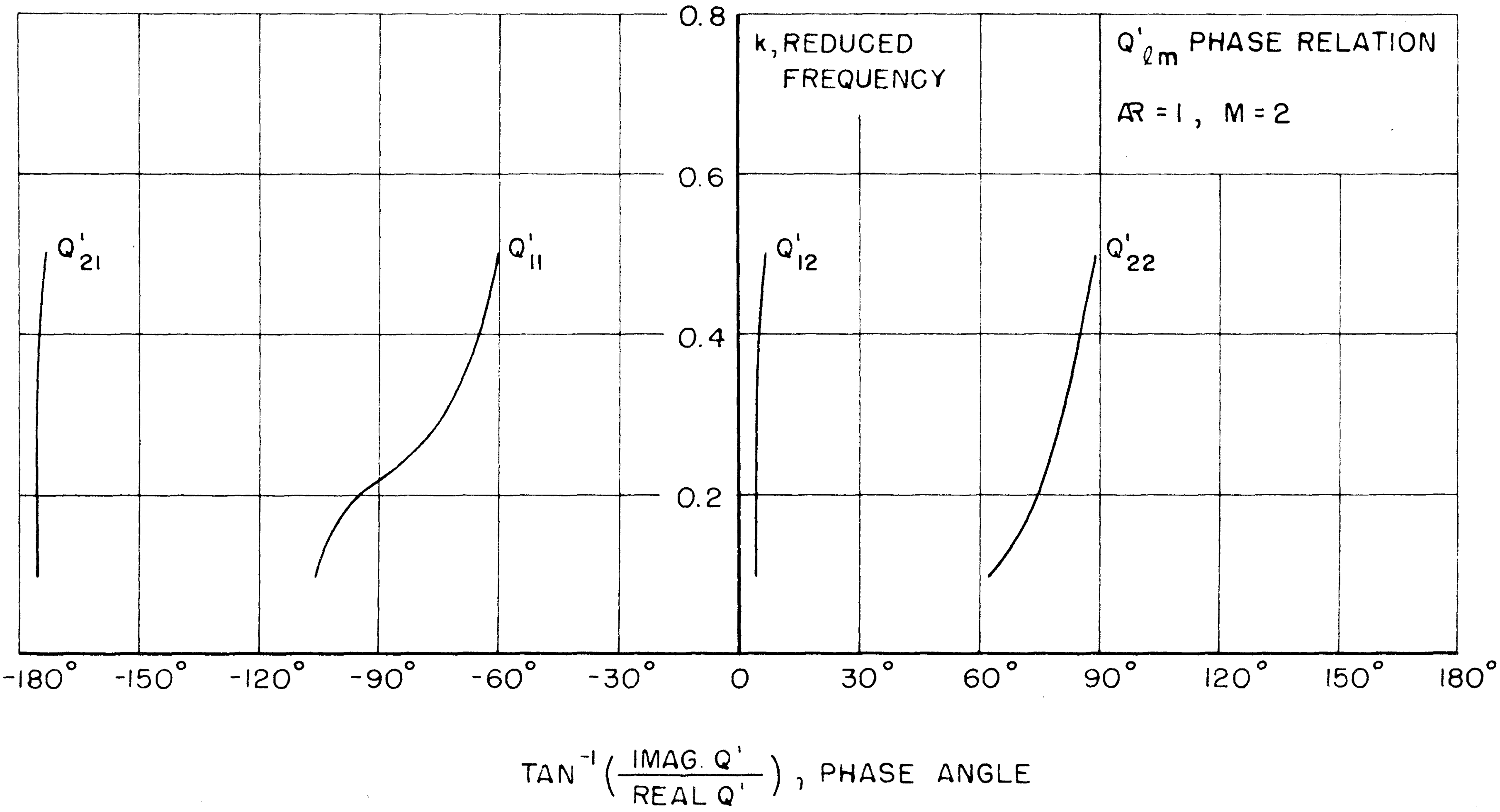
k, REDUCED
FREQUENCY
Q'ℓm PHASE RELATION
ÆR = 1 , M = 2
Q'21
Q'11
Q'12
Q'22
0.8
0.6
0.4
0.2
-180°
-150°
-120°
-90°
-60°
-30°
0
30°
60°
90°
120°
150°
180°
TAN⁻¹ (IMAG. Q' / REAL Q'), PHASE ANGLE

FIG. 3

TABLE I

GENERALIZED FORCES @ $M = 2, \quad = 1$

k	Q'_{11}	Q'_{12}	Q'_{21}	Q'_{22}
0.1000	-0.023911 -0.068941i	0.40824 0.034359i	-0.42902 -0.034625i	0.018476 0.037010i
0.1667	-0.011462 -0.061320i	0.41283 0.035627i	-0.42416 -0.036082i	0.016834 0.052970i
0.2500	0.003678 -0.051217i	0.41976 0.037817i	-0.41843 -0.038552i	0.014785 0.072896i
0.3333	0.018689 -0.055691i	0.42849 0.040660i	-0.41311 -0.041659i	0.012784 0.092746i
0.5000	0.050360 -0.077861i	0.45452 0.048176i	-0.40380 -0.049903i	0.0091158 0.13193i

TABLE II

GENERALIZED FORCES @ $M = \sqrt{2}, \quad = 1$

k	Q'_{11}	Q'_{12}	Q'_{21}	Q'_{22}
0.1000	-2.60936 2.70814i	-1.60305 0.24492i	-0.81356 -0.24685i	0.047273 0.042873i
0.1667	-1.13992 0.84016i	-0.36107 0.24925i	-0.79826 -0.25411i	0.057302 0.046122i
0.2500	-0.54580 0.16111i	0.13062 0.24716i	-0.78195 -0.26563i	0.069655 0.04358i
0.3333	-0.12151 -0.51170i	0.50912 0.26758i	-0.76859 -0.28034i	0.082915 0.03175i
0.5000	-1.21504 3.19141i	-0.81620 0.29459i	-0.75334 -0.31717i	0.05449 0.05022i

TABLE III

RESULTS OF THE FLUTTER DETERMINANT

@ M = 2, = 1

k	κ	γ
0.1000	5.30832 2.43099i	-2.09951 0.72636i
0.1667	4.18323 3.40455i	-2.60242 0.25052i
0.2500	3.84935 -46.08146i	-2.62275 7.9242i
0.3333	2.00135 -3.18392i	-2.30937 0.82689i
0.5000	1.76031 -3.66303i	-2.33974 1.03011i

TABLE IV

RESULTS OF THE FLUTTER DETERMINANT

@ M = $\sqrt{2}$, = 1

k	κ	γ
0.1000	6.18286 -0.028502i	0.055945 0.10872i
0.1667	5.96745 0.24015i	0.020169 0.51767i
0.2500	2.40872	0.62436
0.3333	6.27609	-0.12757
0.5000	6.16529 0.015593i	0.036006 0.16634i