August 7, 1930

Receipt is hereby acknowledged of two copies of a thesis for the degree of Doctor of Philosophy conferred in Jule 1930 by

Benedict Cassen

THEORY OF GRAVITATIONAL SPIN.

Thesis by

Benedict Cassen

In Partial Fulfillment of the

Requirements for the degree of Doctor of

Philosophy

California Institute of Technology

Pasadena, California

1930

THEORY OF GRAVITATIONAL SPIN.

Benedict Cassen.

Dirac's well known Hamiltomian operator

$$p_0 + \frac{e}{c}A_0 + \sum_{i=1}^{3} \alpha_i \left(p_i + \frac{e}{c}A_i\right) + \alpha_4 mc$$

describing an electron in an electromagnetic field determined by the potentials A_0, A_1, A_2, A_3 , when multiplied by its conjugate operator, gives a quadratic Hamiltonian operator

$$-\left(p_0 + \frac{e}{c}A_0\right)^2 + \sum_{i=1}^{3}\left(p_i + \frac{e}{c}A_i\right)^2 + \frac{eh}{c}\sum_{i=1}^{3}\sigma_i H_i + \frac{ieh}{c}\sum_{i=1}^{3}\alpha_i E_i$$

The last term of this operator shows that the electron acts as if it had an imaginary spin energy in an electric field, or what amounts to the same thing, a real spin energy in an imaginary electric field. In what follows, it will be shown that an imaginary electric field has the same properties as a real acceleration or gravitational field and that we may either say that a gravitational field is an imaginary electric field or that an electric field is an imaginary gravitational field. Further , by the introduction of a gravitational vector potential as an imaginary electromagnetic vector potential, it is possible to describe electromagnetic and gravitational phenomena by a simple unified theory. This theory has as its basis the known theory of the electromagnetic field and not the Einsteinian geometrical picture of the gravitational field. When an imaginary electric field is interpreted as a gravitational field, then a Diracian electron, described by the above Hamiltonian operator, will act as if it had a gravitational spin moment when in a gravitational field. This seems to be the real physical significance of the imaginary spin terms which are necessary in the quadratic form of Dirac's Hamiltonian operator in order that it should be relativistically invariant.

Let A_0 be the scalar potential and A_1, A_2, A_3 the components of the vector potential of the electromagnetic field. If we define

$$I_i = \frac{1}{4\pi}\left(\frac{1}{c^2}\frac{\partial^2 A_i}{\partial t^2} - \nabla^2 A_i\right) \quad i = 0,1,2,3 \qquad (1)$$

and identify I_0 with electric charge density, I_1, I_2, I_3 with the current densities, then the equations (1) are the Maxwell-Poisson equations of the electromagnetic field.

Let V_0 be the scalar potential of the acceleration field. We introduce a gravitational vector potential with components V_1, V_2, V_3 . We define

$$C_i = \frac{1}{4\pi}\left(\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \nabla^2 V_i\right), \quad i = 0,1,2,3 \qquad (2)$$

and identify C_0 with the mass density or $\frac{1}{c^2}$ times the energy density, and C_1, C_2, C_3 with the components of momentum. We will then call equations (2) the Newton-Poisson equations of the gravitational field.

In the same way that the components of force on a unit electric charge are

$$E_i = -\frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x_i}$$

the components of gravitational force per unit mass are proportional to

$$-\frac{1}{c}\frac{\partial V_i}{\partial t} - \frac{\partial V_0}{\partial x_i}$$

It seems necessary to introduce this gravitational vector potential in order that gravitational waves might be quantized. Otherwise, there would be no gravitational Poynting's vector and there would be no action in the field that could be quantized .

Of course, equations (2) are equivalent to four of Einstein's ten equations for a weak gravitational field where V_i is a linear function of the g_{i4}'s. It would be possible to consider a space time manifold with a complex line element, that is, with complex g_{ik}'s and interpret the real part of the g_{ik}'s as gravitational potentials in the usual way and the imaginary parts as electromagnetic potentials. However, this proceedure would lead to the well known difficulties, that the true equations satisfied by the g_{ik}'s are non-linear and only take the Poisson form for weak fields. We know from experience that the electromagnetic equations are linear for strong fields as well as for weak fields. As we desire to consider the gravitational laws as being exactly analogous to the electromagnetic laws, instead of using the methods of the general theory of relativity, we will assume the gravitational equations to be linear and of the same form as the electromagnetic equations as given by equations (1) and (2).

By defining

$$\left.\begin{aligned} T_i &= I_i + i\gamma C_i \\ G_i &= A_i + i\gamma V_i \end{aligned}\right\} \qquad (3)$$

where γ is a constant, we can combine equations (1) and (2) in the form

$$T_\iota = \frac{1}{4\pi}\left(\frac{1}{c^2}\frac{\partial^2 G_\iota}{\partial t^2} - \nabla^2 G_\iota\right) \qquad \iota = 0,1,2,3 \qquad (4)$$

This combination can be made more than a merely formal one by giving γ a particular numerical value. We define γ as the square root of Newton's constant k which enters into the Newtonian law of gravitation. The accepted numerical value of k is

$$k = 6.664 \times 10^{-8} \ cm^3/gm\ sec^2$$

so that

$$\gamma = 2.58 \times 10^{-4} \ cm^{3/2}/gm^{1/2}\ sec \qquad (5)$$

The reason why we take this particular value of γ is illustrated in the simple limiting case of the repulsion of two particles . For a particle

$$T_0 = I_0 + \iota\gamma C_0 = e + \iota\gamma m$$

where e is the charge of the particle in elestrotatic units and m is its mass in grams. Then a particle of charge e and mass m in an electro-gravistatic field of potential

$$G_0 = A_0 + \iota\gamma V_0$$

is acted upon by a force

$$(e + \iota\gamma m)(\text{grad}\,A_0 + \iota\gamma\,\text{grad}\,V_0) =$$
$$-e\,\text{grad}\,A_0 + \gamma^2 m\,\text{grad}\,V_0 - \iota\gamma(m\,\text{grad}\,A_0 + e\,\text{grad}\,V_0)$$

The field of a particle at rest of mass m_2 and charge e_2 follows from (4) to be

$$G_0 = \frac{e_2 + \iota\gamma m_2}{r}, \quad G_1 = G_2 = G_3 = 0$$

Therefore the Coulomb-Newton repulsion of two particles is given by

$$f = \frac{(e_1 + i\gamma m_1)(e_2 + i\gamma m_2)}{r^2} = \frac{e_1 e_2}{r^2} - \frac{\gamma^2 m_1 m_2}{r^2} + \frac{i\gamma(e_1 m_2 + e_2 m_1)}{r^2} \qquad (6)$$

The real part of this force is the actual force to be expected with the correct relative signs for the electrical and the gravitational parts. That is, the fact that masses of like sign attract, follows from the fact that charges of like sign repel, if mass is considered to be an imaginary charge.

There does not seem to be any reason why negative mass should be excluded from existing. However, negative mass would have the property of collecting itself together and being repelled by positive masses. Therefore it would seem that it would have a tendency to remove itself to infinity with respect to the universe of positive mass. If any negative mass is observable by us, it will probably be found receeding away from our universe of positive mass at an accelerating speed!

It should be noted that two particles such as a nucleus and an electron, on the present theory, would have a complex interaction energy. The real part of the interaction is the combined electrostatic and gravitational interaction. The imaginary part has no meaning with respect to any previous theories.

If we are to consider complex Hamiltonian functions, we must be able to interpret the non-Hermitian Hamiltonian operators obtained by means of the correspondence principle, describing the corresponding quantized system.

The Hamiltonian operator must be of the type known as normal, that is, that if H is the operator and H^* is its conjugate transposed operator then

$$HH^* = H^*H$$

in order that it may be transformed to the diagonal canonical form. A non-Hermitian Hamiltonian operator, of the normal type, when transformed to the diagonal form, will in general, have complex component terms.

According to the parallelism between electromagnetic and gravitational phenomena which we are following, and which is exhibited in the following table, the real parts

real		imaginary	real		imaginary
electric charge	→	mass	electric field	→	gravit. field
electric current	→	momentum	magnetic field	→	grav. vector fie

of the eigenwerte of the Hamiltonian operator should constitute the spectrum of the ordinary energy, which energy is proportional to the total massof the system represented by the Hamiltonian. The imaginary parts should then constitute the spectrum of the imaginary energy, which energy must be proportional to the electric charge of the system.

If the total mass and charge of the system are to be simultaneously measurable, and if M represents the mass operator and E the charge operator we have

$$[M,E] = ME - EM = 0 \qquad (7)$$

Then M and E can be simultaneously transformed to the diagonal form and we must have

$$H = M - \frac{\iota}{\gamma} E \qquad (8)$$

where H is the total Hamiltonian operator. The wave transformation S that changes H to its diagonal form, therefore transforms both M and E to their diagonal forms simultaneously.

So that in the case that mass and charge are simultaneously measurable, the wave functions of the system can be obtained by just considering the real part of the Hamiltonian function, that is, the Hermitian part of the Hamiltonian operator. In this case, the electric charge is a constant of the motion. This corresponds to the facts in all systems of which we are familiar.

The potential energy in the real Hamiltonian function for an electron in a central field of force due to a positive nucleus of charge Ze and mass M is

$$R\left[\frac{(Ze+\iota\gamma M)(-e+\iota\gamma m)}{r}\right] = -\frac{Ze^2}{r} - \frac{\gamma^2 Mm}{r} \qquad (9)$$

as should be expected. Of course, the gravitational part of this interaction is numerically insignificant in comparison with the electrostatic part and has no effect in any ordinary measurements. It is however necessary for a complete theory to take into account and quantize the gravitational field.

We will quantize the gravitational waves by the method developed by Heisenberg and Pauli [1] for the

(1) Zeits. f. Physik 56, 1, (1929)

quantization of the electromaghetic field. Or, to be more precise, we will quantize the electromagnetic and gravitational fields together.

If we put $G_4 = \iota G_0$, $x_4 = \iota c t$ and let

$$F_{\alpha\beta} = \frac{\partial G_\beta}{\partial x_\alpha} - \frac{\partial G_\alpha}{\partial x_\beta}, \quad (\alpha, \beta = 1, 2, 3, 4) \qquad (10)$$

then

$$F_{4k} = \iota(E_k + \iota\gamma F_k), \quad (k = 1, 2, 3) \qquad (11)$$

and F_{23}, F_{31}, F_{12} are the components $H_k + k\gamma J_k$ of the complex magnetic field. Also $F_{\alpha\beta} = -F_{\beta\alpha}$. Then for a space free of charge and mass, equations (4) are equivalent to

$$\sum_\beta \frac{\partial F_{\alpha\beta}}{\partial x_\beta} = 0 \qquad (12)$$

These equations are derivable from the variation principle

$$\delta \int L \, dV dt = 0$$

where

$$L = -\frac{1}{4} \sum_{\alpha,\beta} F_{\alpha\beta} F_{\alpha\beta} \qquad (13)$$

where in this case L is no longer in general a real function. As Heisenberg and Pauli have shown for the real case, the variables $P_{\alpha 4}$ which are canonically conjugate to the G_α are given by

$$P_{\alpha 4} = \frac{\partial L}{\partial \frac{\partial G_\alpha}{\partial x_4}} \qquad (14)$$

so that

$$P_{k4} = -F_{4k} \quad (k = 1, 2, 3), \quad P_{44} = 0.$$

The commutation rules for quantizing the electromagnetic and gravitational waves jointly in vacuum are then

$$\left.\begin{aligned} &[G_\alpha, G_\beta'] = 0 \\ &[F_{4i}, F_{4k}'] = 0 \\ &[F_{4k}, G_\alpha'] = \delta_{k\alpha} \cdot \frac{-hc}{2\pi} \delta(\underline{r}, \underline{r}') \end{aligned}\right\} \quad (15)$$

where $[P, Q']$ is an abbreviation of $[P(x_1, x_2, x_3, x_4), Q(x_1', x_2', x_3', x_4')]$

According to the general formulation, the particular values of ~~the~~ any fields at different space- time points commute with each other. This is expressed by the fact that $\delta(\underline{r}, \underline{r}') = 0$ when $\underline{r}, \underline{r}'$ are different vectors, but $\delta(\underline{r}, \underline{r}) = 1$

From equations (15) follow

$$\left.\begin{aligned} &[(H_i + i\gamma J_i), (H_k' + i\gamma J_k')] = 0 \\ &[(E_i + i\gamma F_i), (E_k' + i\gamma F_k')] = 0 \\ &[(E_i + i\gamma F_i), (H_{i+1}' + i\gamma J_{i+1}')] = \\ &-[(E_{i+1} + i\gamma F_{i+1}), (H_i' + i\gamma J_i')] = \frac{hc}{2\pi i} \frac{\partial \delta(\underline{r}, \underline{r}')}{\partial x_{i+2}} \end{aligned}\right\} \quad (16)$$

where $\frac{\partial \delta(\underline{r}, \underline{r}')}{\partial x}$ is an operator such that

$$\int \left(\frac{\partial}{\partial x} \delta(\underline{r}, \underline{r}')\right) f(\underline{r}) d\underline{r} = \begin{cases} -\frac{\partial f}{\partial x}(0) & \text{when } \underline{r} = \underline{r}' \\ 0 & \text{otherwise} \end{cases}$$

The equations (16), on expansion give

$$\left.\begin{aligned} &[H_i, H_k'] - \gamma^2 [J_i, J_k'] + i\gamma([H_i, J_k'] + [J_i, H_k']) = 0 \\ &[E_i, E_k'] - \gamma^2 [F_i, F_k'] + i\gamma([E_i, F_k'] + [F_i, E_k']) = 0 \\ &[E_i, H_{i+1}'] - \gamma^2 [F_i, J_{i+1}'] + i\gamma([E_i, J_{i+1}'] + [F_i, H_{i+1}']) = \frac{hc}{2\pi i} \frac{\partial \delta}{\partial x_{i+2}} \end{aligned}\right\}$$

We no longer have simple commutation laws for the electromagnetic field alone or for the gravitational fiels alone. The laws given by Heisenberg and Pauli are a first approximation to these. But all the methods used by Heisenberg and Pauli, especially those used in their second paper[(2)]

(2) Zeits. f. Physik 59, 168 (1930)

are applicable in our case if we use complex potentials where they use real potentials. Our complex Lagrangian function $L^{(s)}$ for the electromagnetic and gravitational radiation is

$$L^{(s)} = -\frac{1}{4}\sum_{\alpha,\beta} F_{\alpha\beta} F_{\alpha\beta} \qquad (13)$$

The corresponding complex Hamiltonian for the gravi-electromagnetic radiation is

$$H^{(s)} = -\sum_{k=1}^{3} F_{4k}\frac{\partial G_k}{\partial x_4} - L^{(s)} = -\sum_{k=1}^{3} F_{4k}\frac{\partial G_4}{\partial x_k} - \frac{1}{2}\sum_{k=1}^{3} F_{4k}F_{4k} + \frac{1}{4}\sum_{i,k=1}^{3} F_{ik}F_{ik} \qquad (18)$$

For want of a better Hamiltonian for the material part we follow Heisenberg and Pauli in using Dirac's operator cited in the beginning of this thesis. The wave equation obtained by use of this operator and its adjoint are obtained by the usual variational principle from the Lagrangian function

$$L^{(m)} = -\sum_{\mu,\sigma,\rho}\left[(\alpha_\mu)_{\rho\sigma}\psi_\rho^{+}\left(\frac{hc}{2\pi i}\frac{\partial}{\partial x_\mu} + t\,G_\mu\right)\psi_\rho - i m c^2 \psi_\rho^{+}\psi_\rho\right] \qquad (19)$$

where we are now using the complex potentials G_μ instead of the ordinary potentials of the electromagnetic field and $t = e + i\gamma m$ is the complex charge of the particle. The ψ_ρ's are the operators corresponding to the wave functions for the particles and the ψ_ρ^{+}'s are the operators adjoint to these. The operators conjugate to the ψ_ρ's are

$$-\frac{hc}{2\pi}\psi_\sigma^{*} = \frac{\partial L}{\partial\frac{\partial\psi_\rho}{\partial x_4}} = -\frac{hc}{2\pi i}\sum_{\rho}(\alpha_4)_{\rho\sigma}\psi_\rho^{+}$$

or

$$\psi_\sigma^{*} = \frac{1}{i}\sum_{\rho}(\alpha_4)_{\rho\sigma}\psi_\rho^{+} \qquad (20)$$

The commutation laws for the ψ_ρ's are not in any way affected by the use of the complex G's so that we will not give them here.

A current vector with components s_μ satisfying

$$\sum_\mu \frac{\partial s_\mu}{\partial x_\mu} = 0 \tag{21}$$

is given by

$$s_\mu = -\frac{\partial L^{(m)}}{\partial G_\mu} = t\sum_{\rho,\sigma}(\alpha_\mu)_{\rho\sigma}\psi_\rho^{+}\psi_\sigma \tag{22}$$

As $t = e + i\eta m$ is complex, the components of the current vector are complex and represent momenta as well as current. Then the real part of (21) is the equation of continuity for electric flow while the imaginary part is the equation of continuity of momentum flow.

The Hamiltonian for material is then

$$\begin{aligned} H^{(m)} &= -\frac{hc}{2\pi}\psi_\sigma^{*}\frac{\partial\psi_\sigma}{\partial x_4} - L^{(m)} \\ &= \sum_{\rho=1}^{4}\sum_{k=1}^{3}(\alpha_k)_{\rho\sigma}\psi_\rho^{*}\left(\frac{hc}{2\pi i}\frac{\partial\psi_\rho}{\partial x_k} + t\psi_\sigma G_k\right) + mc^2\sum_{\rho,\sigma}(\alpha_4)_{\rho\sigma}\psi_\rho^{*}\psi_\rho \\ &\quad + ti\sum_\sigma \psi_\sigma^{*}\psi_\sigma G_4 \end{aligned} \tag{23}$$

and the complex Hamiltonian analogous to the real one used by Heisenberg and Pauli is the sum of this and that for the radiation given in (18).

The invariance of a Hamiltonian functional of the G's, ψ's and ψ^{*}'s under the transformation

$$G_\nu' = G_\mu + \frac{\partial\chi}{\partial x_\nu},\quad \psi_\rho' = e^{-\frac{8\pi^2 i t}{hc}\chi}\psi_\rho,\quad \psi_\rho^{*\prime} = \psi_\rho^{*}e^{\frac{8\pi^2 i t^{*}}{hc}\chi} \tag{24}$$

or the corresponding infinitesimal transformations, in the case when G_4 is taken as zero and χ is an arbitrary function of x_1, x_2, x_3

$$G_i' = G_i + \delta \frac{\partial \chi}{\partial x_i}, \quad \psi_\rho' = \psi_\rho - \frac{8\pi^2 i t}{hc} \delta \chi \cdot \psi_\rho \qquad (25)$$

require the existence of an operator commuting with the Hamiltonian operator, which for this particular transformation is

$$\int \chi \left(\sum_i \frac{\partial}{\partial x_i} \frac{\delta}{\delta G_i} + \frac{8\pi^2 i t}{hc} \psi_\rho \frac{\delta}{\delta \psi_\rho} \right) dV \qquad (26)$$

The operator conjugate to G_i is from (14)

$$-F_{4i} = -i\,(E_i + i\gamma F_i)$$

Therefore

$$\frac{\delta}{\delta G_i} = \frac{2\pi}{h} (E_i + i\gamma F_i) \qquad (27)$$

The operator conjugate to ψ_ρ is from (20)

$$-\frac{hc}{2\pi} \psi_\rho^*$$

Therfore

$$\frac{\delta}{\delta \psi_\rho} = -ic\,\psi_\sigma^* \qquad (28)$$

Then (26) becomes

$$\frac{2\pi}{h} \int \chi \left(\sum_i \frac{\partial E_i}{\partial x_i} + i\gamma \sum_i \frac{\partial F_i}{\partial x_i} + 4\pi t \sum_\rho \psi_\rho^* \psi_\rho \right) dV \qquad (29)$$

So that if the Hamiltonian is invariant in the above described manner, (29) is a constant of the motion. As it would be so for an arbitrary continuous function $\chi(x_1, x_2, x_3)$ then

$$C = \sum_i \frac{\partial E_i}{\partial x_i} + i\gamma \sum_i \frac{\partial F_i}{\partial x_i} + 4\pi (e + i\gamma m) \sum_\rho \psi_\rho^* \psi_\rho \qquad (30)$$

is a constant of the motion. If C is 0, it will then always be zero and in that case we can separate the real and imaginary parts of (30) obtaining

$$\left.\begin{aligned} \sum_i \frac{\partial E_i}{\partial x_i} + 4\pi e \sum_\rho \psi_\rho^* \psi_\rho = 0 \\ \sum_i \frac{\partial F_i}{\partial x_i} + 4\pi m \sum_\rho \psi_\rho^* \psi_\rho = 0 \end{aligned}\right\} \quad (31)$$

This separation is allowable as $\sum_\rho \psi_\rho^* \psi_\rho$ commutes with the $\frac{\partial E}{\partial x}$'s and the $\frac{\partial F}{\partial x}$'s

As the total Hamiltonian $H^{(s)} + H^{(m)}$, when $G_4 = 0$, is gauge invariant under the transformation (24) then the relations (31) hold in this case. If they hold in general then G_4 must enter into the Hamiltonian as a product with C, so that for the particular value $C = 0$ the Hamiltonian does not depend on G_4. That this is the case is shown by Heisenberg and Pauli for the Hamiltonians involving real potentials. It holds in an exactly similar way with the Hamiltonian involving complex potentials.

We have shown then, that the quantum theory of the electromagnetic field as developed by Heisenberg and Pauli can be used without difficulty when the potentials are complex and the imaginary parts are interpreted as gravitational potentials.

Returning to Dirac's Hamiltonian for a single particle in a gravielectromagnetic field we have:

for the linear operator

$$p_0 + \frac{e+irm}{c}(A_0+irV_0) + \sum_{i=1}^{3}\alpha_i\left(p_i + \frac{e+irm}{c}(A_i+irV_i)\right) + \alpha_4 mc \tag{32}$$

The spin part of the quadratic Hamiltonian is

$$\frac{h}{2\pi c}(e+irm)\sum_{i=1}^{3}\sigma_i(H_i+irJ_i) + \frac{ih}{2\pi c}(e+irm)\sum_{i=1}^{3}\alpha_i(E_i+irF_i) \tag{33}$$

The real part of this

$$\frac{he}{2\pi c}\sum_{i=1}^{3}\sigma_i H_i - \frac{hmr^2}{2\pi c}\sum_{i=1}^{3}\sigma_i J_i - \frac{her}{2\pi c}\sum_{i=1}^{3}\alpha_i F_i - \frac{hmr}{2\pi c}\sum_{i=1}^{3}\alpha_i E_i \tag{34}$$

The real magnetic spin moment is then for the usual h = 6.554×10^{-27}

$$\pm\frac{eh}{4\pi mc} \qquad \text{as usual,}$$

the real electric spin moment is

$$\pm\frac{hr}{4\pi c}$$

which is extremely small,

the real gravitational spin moment is

$$\pm\frac{her}{4\pi mc}$$

and the real spin moment in a gravitational field is

$$\pm\frac{hr^2}{4\pi c}$$

which is also extremely small.

The fact that a particle has a spin energy of $\pm\frac{her}{4\pi mc}F$ in a gravitational field F indicates that an intense gravitational field should cause a splitting of spectral terms. Due to the fact that the spin matrices for the gravitational spin are the α's and not the σ's the nature of the splitting will be entirely dissimilar to that due to the magnetic spin. The gravitational Stark and Zeeman effects would not be analogous to the corresponding electromagnetic effects.

The numerical value of $\frac{he\tau}{4\pi mc}$ is 2.14×10^{-23}
At the surface of the earth the value of F due to the normal gravity of the earth is 1.47x 10^{10} so that the maximum energy difference in ergs for an electron spinning with and against the earth's gravitational field is

$2 \times 3.13 \times 10^{-13}$ ergs

This is equivalent to 2x .196 electron volts which is very considerable . But the signs of the eigenwerte of the matrices e.g.

$$\alpha_1 \begin{cases} 0001 \\ 0010 \\ 0100 \\ 1000 \end{cases} \quad \alpha_2 \begin{cases} 000i \\ 00\text{-}i0 \\ 0i00 \\ \text{-}i000 \end{cases} \quad \alpha_3 \begin{cases} 0010 \\ 000\text{-}1 \\ 1000 \\ 0\text{-}100 \end{cases}$$

corresponding to the $+1$ and -1 eigenwerte of the matrices are both of the same sign. An opposite gravitational spin will therefore pertain to one of the four ψ's which is only very slightly excited under ordinary conditions.

If the gravitational shift of the energy levels of an atom are the same for all of them, then no shift will occur in the observed positions of spectral lines as the shift in the final level will be the same as the shift in the initial level . Gravitational shifts of spectral lines could be produced only when there is a coupling between the gravitational spin and the other quantum numbers of an electron in the atom.

Besides the well known commutation rules for the σ's and the α's

$$\left.\begin{aligned} \sigma_i \sigma_k &= -\sigma_k \sigma_i , & (i \neq k) \\ \alpha_i \alpha_k &= -\alpha_k \alpha_i , & (i \neq k) \end{aligned}\right\} \tag{35}$$

we can easily find those between the σ's and the α's as

$$\left.\begin{aligned} \sigma_i \alpha_i &= \alpha_i \sigma_i , & (i = 1, 2, 3) \\ \sigma_i \alpha_k &= -\alpha_k \sigma_i , & (i, k = 1, 2, 3,\ i \neq k) \\ \sigma_i \alpha_4 &= \alpha_4 \sigma_i , & (i = 1, 2, 3) \end{aligned}\right\} \tag{36}$$

The first set of relations in (36) show that the magnetic spin commutes with the gravitational spin in the same direction, so that it is possible to simultaneously specify the two spins in a given direction. If the exclusion principle, excluding states of negative energy, as recently suggested by Dirac, is in operation , then the relative probabilities of finding an electron with a given magnetic spin having gravitational spins with or against a gravitational field suddenly applied in some direction will be greatly modified from those expected by the second set of relations in (36).