ALGORITHMIC AND TOPOLOGICAL ASPECTS OF SEMI-ALGEBRAIC SETS DEFINED BY QUADRATIC POLYNOMIALS

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ALGORITHMIC AND TOPOLOGICAL ASPECTS OF SEMI-ALGEBRAIC SETS DEFINED BY QUADRATIC POLYNOMIALS

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For my Mum

and the memory of my Dad

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LIST OF SYMBOLS OR ABBREVIATIONS

$[m]$	the set $\{1, \ldots, m\}$ with $m \in \mathbb{N}$.
$\mathbf{B}_k(x, r)$	the closed ball centered at the point $\mathbf{x}$ and of radius r in R^k.
$b(j, k)$	as $b(j, k, \bar{d})$ with $\bar{d} = (2, \ldots, 2)$.
$b(j, k, \bar{d})$	the sum of the Betti numbers with $\mathbb{Z}_2$ coefficients of a complex projective variety $\mathrm{Zer}(\{Q_1, \ldots, Q_j\}, \mathbb{P}^k_{\mathbb{C}})$ of codimension j which is a non-singular complete intersection such that the degree of Q_i is d_i and $\bar{d} = (d_1, \ldots, d_j)$.
$b(S, \mathbb{Z}_2)$	the sum of the Betti numbers $b_p(S, \mathbb{Z}_2)$ of a semi-algebraic set S with coefficients in $\mathbb{Z}_2$.
$b_p(S, \mathbb{Z}_2)$	the p-th Betti number of a semi-algebraic set S with coefficients in $\mathbb{Z}_2$.
$\mathrm{cut}(P_1, P_i)$	the resultant $\mathrm{Res}(P_1, P_i)$ with respect to X_3 for $P_1, P_i \in \mathrm{R}[X_1, X_2, X_3]$.
C	an algebraic closed field containg R such that $\mathrm{C} = \mathrm{R}[i]$.
$\mathrm{Ext}(S, \mathrm{R}')$	the semi-algebraic subset of R'^k defined by the same quantifier free formula that defines $S \subset \mathrm{R}^k$, where $\mathrm{R} \subset \mathrm{R}'$ are real closed fields.
$\lambda_i(Q)$	the eigenvalues of a quadratic form $Q \in \mathrm{R}[Y_0, \ldots, Y_\ell]$ in non-decreasing order.
$\mathrm{hocolim}(\mathcal{A})$	the homotopy colimit of a finite set $\mathcal{A}$ of finite CW-complexes.
$\mathrm{index}(Q)$	the number of negative eigenvalues of the symmetric matrix of the corresponding bilinear form of a quadratic form $Q \in \mathrm{R}[Y_0, \ldots, Y_\ell]$, that is of the matrix M_Q such that $Q(\mathbf{y}) = \langle M_Q\mathbf{y}, \mathbf{y}\rangle$ for all $\mathbf{y} \in \mathrm{R}^{\ell+1}$.
P^h	the homogenization of P.
$\mathrm{Res}(P, Q)$	the resultant of P and Q.
R	a real closed field.
$\mathrm{R}\langle\zeta\rangle$	the real closed field of algebraic Puiseux series in ζ with coefficients in R.
$\mathcal{R}(\sigma)$	the realization of the sign condition σ.
$\mathrm{sResP}_j(P, Q)$	the j-th subresultant of P and Q.
$\mathrm{sRes}_j(P, Q)$	the j-th signed subresultant coefficient of P and Q.
$\mathrm{SyHa}_j(P, Q)$	the j-th Sylvester-Habicht matrix of P and Q.
$\mathrm{Sil}(P_1)$	the resultant $\mathrm{Res}(P_1, \partial P_1/\partial X_3)$ for $P_1 \in \mathrm{R}[X_1, X_2, X_3]$.
$\mathbf{S}^{k-1}(x, r)$	the sphere centered at the point $\mathbf{x}$ and of radius r in R^k.
$\mathbf{S}(X)$	the suspension of a finite CW-complex X.

$S_{\mathbf{x}}$	the fiber $\pi^{-1}(\mathbf{x}) \cap S$ where $\pi : \mathrm{R}^{\ell+k} \to \mathrm{R}^k$ is the projection map on the last k co-ordinates, $S \subset \mathrm{R}^{\ell+k}$ and $\mathbf{x} \in \mathrm{R}^k$.
$\mathrm{Sign}(\mathcal{P})$	the set of realizable sign conditions on $\mathcal{P}$.
$\mathrm{Sign}_p(\mathcal{P})$	the subset of $\mathrm{Sign}(\mathcal{P})$ of elements of level p.
$\mathrm{Zer}(\mathcal{P}, S)$	the set of common zeros of $\mathcal{P}$ in $S \subset \mathrm{R}^k$.
$\mathrm{Zer}(\mathcal{Q}, \mathbb{P}^k_{\mathrm{C}})$	the set of common zeros of $\mathcal{Q}$ in the complex projective space $\mathbb{P}^k_{\mathrm{C}}$ of dimension k.
$\mathrm{Zer}(\mathcal{Q}, \mathbb{P}^k_{\mathrm{R}})$	the set of common zeros of $\mathcal{Q}$ in the real projective space $\mathbb{P}^k_{\mathrm{R}}$ of dimension k.

SUMMARY

In this thesis, we consider semi-algebraic sets over a real closed field R defined by quadratic polynomials. Semi-algebraic sets of R^k are defined as the smallest family of sets in R^k that contains the algebraic sets as well as the sets defined by polynomial inequalities, and which is also closed under the boolean operations (complementation, finite unions and finite intersections). We prove the following new bounds on the topological complexity of semi-algebraic sets over a real closed field R defined by quadratic polynomials, in terms of the parameters of the system of polynomials defining them, which improve the known results.

1. Let $S \subset \mathrm{R}^k$ be defined by $P_1 \geq 0, \ldots, P_m \geq 0$ with $P_i \in \mathrm{R}[X_1, \ldots, X_k]$, $m < k$, and $\deg(P_i) \leq 2$, for $1 \leq i \leq m$. We prove that $b_i(S) \leq \frac{3}{2} \cdot \left(\frac{6ek}{m}\right)^m + k$, $0 \leq i \leq k-1$.

2. Let $\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[Y_1, \ldots, Y_\ell, X_1, \ldots, X_k]$, with $\deg_Y(P_i) \leq 2$, $\deg_X(P_i) \leq d$, $1 \leq i \leq m$. Let $S \subset \mathrm{R}^{\ell+k}$ be a semi-algebaic set, defined by a Boolean formula without negations, whose atoms are of the form, $P \geq 0, P \leq 0, P \in \mathcal{P}$. Let $\pi : \mathrm{R}^{\ell+k} \rightarrow \mathrm{R}^k$ be the projection on the last k co-ordinates. We prove that the number of stable homotopy types amongst the fibers $\pi^{-1}(\mathbf{x}) \cap S$ is bounded by $(2^m \ell k d)^{O(mk)}$.

We conclude the thesis with presenting two new algorithms along with their implementations. The first algorithm computes the number of connected components and the first Betti number of a semi-algebraic set defined by compact objects in $\mathbb{R}^k$ which are simply connected. This algorithm improves the well-know method using a triangulation of the semi-algebraic set. Moreover, the algorithm has been efficiently implemented which was not possible before. The second algorithm computes efficiently the real intersection of three quadratic surfaces in $\mathbb{R}^3$ using a semi-numerical approach.

CHAPTER I

INTRODUCTION

1.1 Real Algebraic Geometry

In classical algebraic geometry, the main objects of interest are complex algebraic sets, i.e. the zero set of a finite family of polynomials over the field $\mathbb{C}$ of complex numbers, meaning the set of all points that simultaneously satisfy one or more polynomial equations. But in many applications in computer-aided geometric design, computational geometry, robotics or computer graphics one is interested in the solutions over the field $\mathbb{R}$ of real numbers. Moreover, they also deal with the real solutions of finite systems of inequalities which are the main objects of real algebraic geometry. Unfortunately, real algebraic sets have a very different behavior than their complex counterparts. For example, an irreducible algebraic subset of $\mathbb{C}^k$ having complex dimension n, considered as an algebraic subset of $\mathbb{R}^{2k}$ is connected, not bounded (unless it is a point) and has local real dimension $2n$ at every point (see, for instance, [29]). But this is no longer true for real algebraic sets (see Example 2.38).

In 1926, Emil Artin and Otto Schreier [7, 6] introduced the notion of a real closed field. Artin [5, 6] used this new theory for solving the 17th problem of Hilbert which asks whether a polynomial which is nonnegative on $\mathbb{R}^n$ is a sum of squares of rational functions. A real closed field R is an ordered field whose positive cone is the set of squares $\mathrm{R}^{(2)}$ and such that every polynomial in $\mathrm{R}[X]$ of odd degree has a root in R. Notice that real closed fields need not be complete nor archimedean (see Chapter 2.1.2).

In this thesis, we consider semi-algebraic sets over a real closed field R defined by quadratic polynomials in k variables. Semi-algebraic sets of R^k are defined as the smallest family of sets in R^k that contains the algebraic sets as well as the sets defined by polynomial inequalities, and which is also closed under the boolean operations (complementation, finite unions and finite intersections). Furthermore, unlike algebraic sets (over R), the projection of a semi-algebraic set is again semi-algebraic, this was proved by Tarski [91] and

Seidenberg [84].

It is worthwhile to mention that in many applications in computer-aided geometric design or computational geometry one deals with arrangements of many geometric objects having a similar simple description [55]. For instance, each object is a semi-algebraic set defined by few polynomials of fixed degree. Thus, understanding the properties of semi-algebraic sets and designing algorithms are important topics in real algebraic geometry.

The class of semi-algebraic set defined by quadratic polynomials is of particular interest for several reasons. First, any semi-algebraic set can be defined by (quantified) formulas involving only quadratic polynomials (at the cost of increasing the number of variables and the size of the formula). Secondly, they are distinguished from arbitrary semi-algebraic sets since one can obtain better results from an algorithmic standpoint, as well as from the point of view of topological complexity (as we will see later). Moreover, they can be much more complicated topologically than semi-algebraic sets defined by only linear polynomials. Thirdly, quadratic surfaces are widely used in computer-aided geometric design, computational geometry [82] and computer graphics as well as in robotics ([79]) and computational physics ([69, 75]).

One basic ingredient in most algorithms for computing topological properties of semi-algebraic sets is an algorithm due to Collins [38], called cylindrical decomposition (see Chapter 2.1.4) which decomposes a given semi-algebraic set into topological balls. Cylindrical decomposition can be used to compute a semi-algebraic triangulation of a semi-algebraic set (see Chapter 2.1.5), and from this triangulation one can compute the homology groups, Betti numbers, et cetera. One disadvantage of the cylindrical decomposition is that it uses iterated projections (reducing the dimension by one in each step) and the number of polynomials (as well as the degrees) is squared in each step of the process. Thus, the complexity of performing cylindrical decomposition is double exponential in the number of variables which makes it impractical in most cases for computing topological information. Nevertheless, we will see in Chapters 2.1.4.2 and 5 that it can be used quite efficiently for several important problems in low dimensions.

1.2 Betti numbers

Important topological invariants of a semi-algebraic sets are the Betti numbers b_i (see Chapter 2.2.1 for a precise definition) which, roughly speaking, measure the number of i-dimensional holes of a semi-algebriac set. The zero-th Betti number b_0 is the number of connected components.

The initial result on bounding the Betti numbers of semi-algebraic sets defined by polynomial inequalities was proved independently by Oleinik and Petrovskii [76], Thom [92] and Milnor [74]. They proved (see Theorem 2.34) that the sum of the Betti numbers of a semi-algebraic set in R^k defined by m polynomial inequalities of degree at most d has a bound of the form $O(md)^k$. Notice that this bound is exponential in k and this exponential dependence is unavoidable (see Example 2.35). Recently, the above bound was extended to more general classes of semi-algebraic sets. For example, Basu [11] improved the bound of the individual Betti numbers of $\mathcal{P}$-closed semi-algebraic sets (which are defined by a Boolean formula with atoms of the form $P = 0$, $P < 0$ or $P > 0$, where $P \in \mathcal{P}$), while Gabrielov and Vorobjov [51] extended the above bound to any $\mathcal{P}$-semi-algebraic set (which is defined by a Boolean formula with atoms of the form $P = 0$, $P \leq 0$ or $P \geq 0$, where $P \in \mathcal{P}$). They proved a bound of $O(m^2d)^k$. Moreover, Basu, Pollack and Roy [19] proved a similar bound for the individual Betti numbers of the realizations of sign conditions.

However, it turns out that for a semi-algebraic set $S \subset \mathrm{R}^k$ defined by m quadratic inequalities, it is possible to obtain upper bounds on the sum of Betti numbers of S which are polynomial in k and exponential only in m. The first such result was proved by Barvinok [9] who proved a bound of $k^{O(m)}$ (see Theorem 2.36). The exponential dependence on m is unavoidable as already remarked by Barvinok, but the implied constant (which is at least two) in the exponent of Barvinok's bound is not optimal.

Using Barvinok's result, as well as inequalities derived from the Mayer-Vietoris sequence (see Chapter 2.2.2), Basu [11] proved a polynomial bound (polynomial both in k and m) on the top few Betti numbers of a set defined by quadratic inequalities (see Theorem 2.37). Very recently, Basu, Pasechnik and Roy [18] extended these bounds to arbitrary $\mathcal{P}$-closed (not just basic closed) semi-algebraic sets defined in terms of quadratic inequalities.

Apart from their intrinsic mathematical interest, for example in distinguishing the semi-algebraic sets defined by quadratic inequalities from general semi-algebraic sets, the bounds proved by Barvinok and Basu respectively have motivated recent work on designing polynomial time algorithms for computing topological invariants of semi-algebraic sets defined by quadratic inequalities. For instance, Grigoriev and Pasechnik [54] presented a polynomial time algorithm (in k) for computing sampling points meeting each connected component of a real algebraic set defined over a quadratic map. Their result improves a result of Barvinok [8] about the the feasibility of systems of real quadratic equations. Basu [14, 13] gave polynomial time algorithms for computing the Euler characteristic and the higher Betti numbers of semi-algebraic sets defined by quadratic inequalities. Furthermore, Basu and Zell [23] gave a polynomial time algorithm for computing the lower Betti numbers of projections defined by such semi-algebraic sets. For details, we refer the reader to the papers mentioned above.

Traditionally an important goal in algorithmic semi-algebraic geometry has been to design algorithms for computing topological invariants of semi-algebraic sets, whose worst-case complexity matches the best upper bounds known for the quantity being computed. It is thus of interest to tighten the bounds on the Betti numbers of semi-algebraic sets defined by quadratic inequalities, as it has been done recently in the case of general semi-algebraic sets (see for example [51, 11, 19, 18]). Notice that the problem of computing the Betti numbers of semi-algebraic sets in single exponential time is considered to be a very important open problem in algorithmic semi-algebraic geometry. Recent progress has been made in several special cases (see [21, 12, 14]).

In another direction, the bounds of the Betti numbers are used to produce lower bounds for complexity decision problems. For instance, Steele and Yao [89] recognized that the bounds for the sum of the Betti numbers can be applied to obtain non-trivial lower bounds in terms of the number of connected components for the model of algebraic decision trees. This was extended to algebraic computation trees by Ben-Or [25].

1.3 Homotopy Types

A fundamental theorem in semi-algebraic geometry is Hardt's Theorem (see Theorem 2.15) which is a corollary of the existence of the cylindrical decomposition. For a projection map $\pi : \mathrm{R}^{\ell+k} \to \mathrm{R}^k$ on the last k co-ordinates and semi-algebraic subset S of R^k, it implies that there is a semi-algebraic partition of R^k, $\{T_i\}_{i\in I}$, such that for each $i \in I$ and any point $\mathbf{y} \in T_i$, the pre-image $\pi^{-1}(T_i) \cap S$ is semi-algebraically homeomorphic to $(\pi^{-1}(\mathbf{y}) \cap S) \times T_i$ by a fiber preserving homeomorphism. In particular, for each $i \in I$, all fibers $\pi^{-1}(\mathbf{y}) \cap S$, $\mathbf{y} \in T_i$, are semi-algebraically homeomorphic. Unfortunately, the cylindrical decomposition algorithm implies a double exponential (in k and ℓ) upper bound on the cardinality of I and, hence, on the number of homeomorphism types of the fibers of the map $\pi|_S$. No better bounds than the double exponential bound are known, even though it seems reasonable to conjecture a single exponential upper bound on the number of homeomorphism types of the fibers of the map π_S.

Basu and Vorobjov [22] considered the weaker problem of bounding the number of distinct homotopy types, occurring amongst the set of all fibers of $\pi|_S$, and a single exponential upper bound was proved on the number of homotopy types of such fibers (see Theorem 2.42). They proved in the same paper a similar result for semi-Pfaffian sets as well, and Basu [14] extended it to arbitrary o-minimal structures. Both these bounds on the number of homotopy types are exponential in ℓ as well as k. As already pointed out in [22], in this generality the single exponential dependence on ℓ is unavoidable (see Example 2.43).

Since sets defined by quadratic equalities and inequalities are the simplest class of topologically non-trivial semi-algebraic sets, the problem of classifying such sets topologically has attracted the attention of many researchers. Motivated by problems related to stability of maps, Wall [97] considered the special case of real algebraic sets defined by two simultaneously diagonalizable quadratic forms in ℓ variables. He obtained a full topological classification of such varieties making use of Gale diagrams (from the theory of convex polytopes). To be more precise, letting

$$Q_1 = \sum_{i=1}^{\ell} X_i Y_i^2,$$

$$Q_2 = \sum_{i=1}^{\ell} X_{i+\ell} Y_i^2,$$

and

$$S = \{(\mathbf{y}, \mathbf{x}) \in \mathrm{R}^{3\ell} \mid \quad \| \mathbf{y} \| = 1, \quad Q_1(\mathbf{y}, \mathbf{x}) = Q_2(\mathbf{y}, \mathbf{x}) = 0\},$$

Wall obtains as a consequence of his classification theorem, that the number of different topological types of fibers $\pi^{-1}(\mathbf{x}) \cap S$ is bounded by $2^{\ell-1}$. Similar results were also obtained by López [70] using different techniques. Much more recently Briand [31] has obtained explicit characterization of the isotopy classes of real varieties defined by two general conics in two dimensional real projective space $\mathbb{P}^2_{\mathrm{R}}$ in terms of the coefficients of the polynomials. His method also gives a decision algorithm for testing whether two such given varieties are isotopic.

In another direction Agrachev [1] studied the topology of semi-algebraic sets defined by quadratic inequalities, and he defined a certain spectral sequence converging to the homology groups of such sets. We will give a parametrized version of Agrachev's construction in Chapter 4.3 which is due to Basu.

In view of the topological simplicity of semi-algebraic sets defined by few quadratic inequalities as opposed to general semi-algebraic sets, one might expect a much tighter bound on the number of topological types compared to the general case. However one should be cautious, since a tight bound on the Betti numbers of a class of semi-algebraic sets does not automatically imply a similar bound on the number of topological or even homotopy types occurring in that class. We refer the reader to [24] for an explicit example of the large number of possible homotopy types amongst finite cell complexes having very small Betti numbers.

1.4 Arrangements

Arrangements of geometric objects in fixed dimensional Euclidean space are fundamental objects in computational geometry and computer-aided geometric design (for instance, see [55]). As already mentioned before, usually it is assumed that each individual object in such an arrangement has a simple description – for instance, they are semi-algebraic sets defined by few polynomials of fixed degree.

Arrangements of quadratic surfaces, or quadrics, in three dimensional space are of particular interest since they are widely used in CAD/CAM and computer graphics as well as in robotics ([79]) and computational physics ([69, 75]). Therefore, it is often necessary to compute or characterize the intersection of quadratic surfaces and many approaches have already been proposed (see [66, 67, 99, 98, 100, 46, 44, 43, 93]). In particular, computing the real intersection of three quadrics is an important subject in computational geometry and computer-aided geometric design (for instance, see [37, 102, 101, 82]).

Chionh, Goldman and Miller [37] used Macaulay's multivariate resultant to solve the problem in the case of finitely many intersection points. But, as pointed out by Xu, Wang, Chen and Sun [102], one can produce quite general examples where the real intersection cannot be computed using this approach. In [102], the computation of the real intersection of three quadrics is reduced to computing the real intersection of two planar curves obtained by Levin's method. Though useful for curve tracing, Levin's method ([66, 67]) and its improvement by Wang, Goldman and Tu [98] has serious limitations. First of all, it produces a parameterization of the real intersection curve of two quadrics with a square-root function but does not yield information about reducibility or singularity of the real intersection. Secondly, Levin's method and similar methods ([44, 65]) for computing parameterization for the intersection set are restricted to quadratic surfaces since higher degree intersection curves cannot be parameterized easily.

In another direction, Chazelle, Edelsbrunner, Guibas and Sharir [36] showed how to decompose an arrangement of m objects in R^k into $O^*(m^{2k-3})$ simple pieces. This was further improved by Koltun in the case $k = 4$ [63]. However, these decompositions while suitable for many applications, are not useful for computing topological properties of the arrangements, since they fail to produce a cell complex. Furthermore, arrangements of finitely many balls in $\mathbb{R}^3$ have been studied by Edelsbrunner [45] from both combinatorial and topological viewpoint, motivated by applications in molecular biology. But these techniques use special properties of the objects, such as convexity, and are not applicable to general semi-algebraic sets.

1.5 *Review of the Results*

We review the main results of this thesis.

1.5.1 Bounding the Betti Numbers

In Chapter 3 we consider the problem of bounding the Betti numbers, $b_i(S)$, of a semi-algebraic set $S \subset \mathrm{R}^k$ defined by polynomial inequalities

$$P_1 \geq 0, \ldots, P_m \geq 0,$$

where $P_i \in \mathrm{R}[X_1, \ldots, X_k]$, $m < k$, and $\deg(P_i) \leq 2$, for $1 \leq i \leq m$.

We prove (see Theorem 3.1) that for $0 \leq i \leq k-1$,

$$\begin{aligned} b_i(S) &\leq \frac{1}{2} + (k-m) + \frac{1}{2} \cdot \sum_{j=0}^{min\{m+1,k-i\}} 2^j \binom{m+1}{j} \binom{k}{j-1} \\ &\leq \frac{3}{2} \cdot \left(\frac{6ek}{m}\right)^m + k. \end{aligned}$$

We first bound the Betti numbers of non-singular complete intersections of complex projective varieties defined by generic quadratic forms, and use this bound to obtain bounds in the real semi-algebraic case. Because of this new approach we are able to remove the constant in the exponent in the bounds proved in [9, 11] and this constitutes the main contribution which appears in [17].

1.5.2 Bounding the Stable Homotopy Types of a Parameterized Family

In Chapter 4 we consider the following problem. Let

$$\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[Y_1, \ldots, Y_\ell, X_1, \ldots, X_k],$$

with $\deg_Y(P_i) \leq 2$, $\deg_X(P_i) \leq d$, $1 \leq i \leq m$. Let $S \subset \mathrm{R}^{\ell+k}$ be a semi-algebaic set, defined by a Boolean formula without negations, whose atoms are of the form, $P \geq 0, P \leq 0, P \in \mathcal{P}$. Let $\pi : \mathrm{R}^{\ell+k} \to \mathrm{R}^k$ be the projection on the last k co-ordinates. Then the number of stable homotopy types (see Definition 2.28) amongst the fibers $\pi^{-1}(\mathbf{x}) \cap S$ is bounded by

$$(2^m \ell k d)^{O(mk)}$$

(see Theorem 4.1).

Our result can be seen as a follow-up to the recent work by Basu and Vorobjov [22] on bounding the number of homotopy types of fibers of general semi-algebraic maps (see Theorem 2.42). However, our bound (unlike the one proven in [22]) is polynomial in ℓ for fixed m and k, which constitutes the main contribution and appears in [16]. Unfortunately, the exponential dependence on m is unavoidable (see Remark 4.2).

Due to technical reasons, we only obtain a bound on the number of stable homotopy types, rather than homotopy types. But note that the notions of homeomorphism type, homotopy type and stable homotopy type are each strictly weaker than the previous one, since two semi-algebraic sets might be stable homotopy equivalent, without being homotopy equivalent (see [88], p. 462), and also homotopy equivalent without being homeomorphic. However, two closed and bounded semi-algebraic sets which are stable homotopy equivalent have isomorphic homology groups.

1.5.3 Algorithms and Their Implementations

In Chapter 5 we consider the problem of computing the first Betti Numbers of arrangements of compact objects in $\mathbb{R}^k$ as well as computing the intersection of three quadratic surfaces in three dimensional space $\mathbb{R}^3$.

1.5.3.1 Computing the Betti Numbers of Arrangements

In Chapter 5.1 we consider arrangements of compact objects in $\mathbb{R}^k$ which are simply connected. This implies, in particular, that their first Betti number is zero. We describe an algorithm (see Algorithm 5.2) for computing the number of connected components and the first Betti number of such an arrangement, along with its implementation. For the implementation, we restrict our attention to arrangements in $\mathbb{R}^3$ and take for our objects the simplest possible semi-algebraic sets in $\mathbb{R}^3$ which are topologically non-trivial – namely, each object is an ellipsoid defined by a single quadratic equation. Ellipsoids are simply connected, but with non-zero second Betti number. We also allow solid ellipsoids defined by a single quadratic inequality. This algorithm appears in [15].

1.5.3.2 *Computing the Real Intersection of Quadratic Surfaces*

In Chapter 5.2 we consider the problem of computing the real intersection of three quadratic surfaces, or quadrics, defined by the quadratic polynomials P_1, P_2 and P_3 in $\mathbb{R}^3$. We describe an algorithm for computing the isolated points and a linear graph embedded into $\mathbb{R}^3$ (if the real intersection form a curve) representing the real intersection of the three quadrics defined by the three polynomials P_i, along with its prototypical implementation into the computer algebra system `Maple` (Version 9.5). For our implementation, we restrict our attention to quadrics with defining equation having rational coefficients. This algorithm appears in [60].

CHAPTER II

MATHEMATICAL PRELIMINARIES

2.1 Real Algebraic Geometry

2.1.1 Some Notations

Let R be a real closed field and let C be an algebraic closed field containing R such that $\mathrm{C} = \mathrm{R}[i]$. For each $m \in \mathbb{N}$ we will denote by $[m]$ the set $\{1, \ldots, m\}$.

For $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_k) \in \mathrm{R}^k$ and $r \in \mathrm{R}$, $r > 0$, we denote

$$\begin{aligned}
||\mathbf{x}|| &= \sqrt{\mathbf{x}_1^2 + \cdots + \mathbf{x}_k^2}, \\
\mathbf{B}_k(x, r) &= \{\mathbf{y} \in \mathrm{R}^k \mid ||\mathbf{y} - \mathbf{x}||^2 \leq r^2\} \quad \text{(the } \textbf{closed ball}\text{)}, \\
\mathbf{S}^{k-1}(x, r) &= \{\mathbf{y} \in \mathrm{R}^k \mid ||\mathbf{y} - \mathbf{x}||^2 = r^2\} \quad \text{(the } \mathbf{(k-1)}\textbf{-sphere}\text{)}.
\end{aligned}$$

We omit both x and r from the notation for the unit sphere centered at the origin. For any polynomial $P \in \mathrm{R}[X_1, \ldots, X_k]$, let

$$P^h(X_0, \ldots, X_k) = X_0^d P(\frac{X_1}{X_0}, \ldots, \frac{X_k}{X_0}),$$

where d is the total degree of P, the **homogenization** of P with respect to X_0. The polynomial P is X_i**-regular** if $\deg_{X_i}(P) = \deg P$, i.e., if the polynomial P has a non-vanishing constant leading coefficient in the variable X_i. The **gcd-free part** of a polynomial P with respect to another polynomial Q is the polynomial $\bar{P} = P/\gcd(P, Q)$. A polynomial $P \in \mathrm{R}[X]$ is **square-free** if there is no non-constant polynomial $A \in \mathrm{R}[X]$ such that A^2 divides P. Equivalently, the polynomial P is square-free if and only if P is equal (up to a constant) to the gcd-free part of P and $\partial P/\partial X$.

For any family of polynomials $\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[X_1, \ldots, X_k]$, and $S \subset \mathrm{R}^k$, we denote by $\mathrm{Zer}(\mathcal{P}, S)$ the set of common zeros of $\mathcal{P}$ in S, i.e.,

$$\mathrm{Zer}(\mathcal{P}, S) := \Big\{\mathbf{x} \in S \mid \bigwedge_{i=1}^{m} P_i(\mathbf{x}) = 0\Big\}.$$

Let ϕ be a Boolean formula with atoms of the form $P = 0$, $P > 0$, or $P < 0$, where

$P \in \mathcal{P}$. We call ϕ a **$\mathcal{P}$-formula**, and the semi-algebraic set $S \subset \mathrm{R}^k$ defined by ϕ, a **$\mathcal{P}$-semi-algebraic set**.

If the Boolean formula ϕ contains no negations, and its atoms are of the form $P = 0$, $P \geq 0$, or $P \leq 0$, with $P \in \mathcal{P}$, then we call ϕ a **$\mathcal{P}$-closed formula**, and the semi-algebraic set $S \subset \mathrm{R}^k$ defined by ϕ, a **$\mathcal{P}$-closed semi-algebraic set**.

For an element $a \in \mathrm{R}$ introduce

$$\mathrm{sign}(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a > 0, \\ \text{-1} & \text{if } a < 0. \end{cases}$$

A **sign condition** σ on $\mathcal{P}$ is an element of $\{0, 1, -1\}^{\mathcal{P}}$. The **realization of the sign condition** σ is the basic semi-algebraic set

$$\mathcal{R}(\sigma) := \Big\{ \mathbf{x} \in \mathrm{R}^k \ \mid \ \bigwedge_{P \in \mathcal{P}} \mathrm{sign}(P(\mathbf{x})) = \sigma(P) \Big\}.$$

A sign condition σ is **realizable** if $\mathcal{R}(\sigma) \neq \emptyset$. We denote by $\mathrm{Sign}(\mathcal{P})$ the set of realizable sign conditions on $\mathcal{P}$. For $\sigma \in \mathrm{Sign}(\mathcal{P})$ we define the **level of** σ as the cardinality

$$\#\{P \in \mathcal{P} | \sigma(P) = 0\}.$$

For each level p, $0 \leq p \leq \#\mathcal{P}$, we denote by $\mathrm{Sign}_p(\mathcal{P})$ the subset of $\mathrm{Sign}(\mathcal{P})$ of elements of level p. Furthermore, for a sign condition σ let

$$\mathcal{Z}(\sigma) := \Big\{ \mathbf{x} \in \mathrm{R}^k \ \mid \ \bigwedge_{P \in \mathcal{P},\ \sigma(P)=0} P(\mathbf{x}) = 0 \Big\}.$$

Finally, for any family of homogeneous polynomials $\mathcal{Q} = \{Q_1, \ldots, Q_m\} \subset \mathrm{R}[X_0, \ldots, X_k]$, we denote by $\mathrm{Zer}(\mathcal{Q}, \mathbb{P}^k_{\mathrm{R}})$ (resp., $\mathrm{Zer}(\mathcal{Q}, \mathbb{P}^k_{\mathrm{C}})$) the set of common zeros of $\mathcal{Q}$ in the real (resp., complex) projective space $\mathbb{P}^k_{\mathrm{R}}$ (resp., $\mathbb{P}^k_{\mathrm{C}}$) of dimension k.

2.1.2 Infinitesimals

In Chapter 3 and 4 we will extend the ground field R by infinitesimal elements which are smaller than any positive element of R. The infinitesimals are used to deform our semi-algebraic sets such that we get very similar semi-algebraic sets having some additional properties.

We denote by $R\langle\zeta\rangle$ the real closed field of algebraic Puiseux series in ζ with coefficients in R (see [20] for more details). The sign of a Puiseux series in $R\langle\zeta\rangle$ agrees with the sign of the coefficient of the lowest degree term in ζ. This induces a unique order on $R\langle\zeta\rangle$ which makes ζ infinitesimal, i.e., ζ is positive and smaller than any positive element of R. Given a semi-algebraic set S in R^k, the **extension** of S to $R\langle\zeta\rangle$, denoted $\mathrm{Ext}(S, R\langle\zeta\rangle)$, is the semi-algebraic subset of $R\langle\zeta\rangle^k$ defined by the same quantifier free formula that defines S. The set $\mathrm{Ext}(S, R\langle\zeta\rangle)$ is well defined (i.e., it only depends on the set S and not on the quantifier free formula chosen to describe it). This is an easy consequence of the Tarski-Seidenberg principle (see for instance [20]).

We will also need the following remark about extensions which is again a consequence of the Tarski-Seidenberg transfer principle.

Remark 2.1. Let S, T be two closed and bounded semi-algebraic subsets of R^k, and let R' be a real closed extension of R. Then S and T are semi-algebraically homotopy equivalent if and only if $\mathrm{Ext}(S, R')$ and $\mathrm{Ext}(T, R')$ are semi-algebraically homotopy equivalent.

2.1.3 Resultants and Subresultants

We recall next the notion of resultant and subresultant which will play an important role in the cylindrical decomposition and its applications (see Chapter 2.1.4). We will define them and recall some of their properties which will be very helpful in our settings. But we will omit the details on how to compute them. We refer to [20] for more details on the algorithm. Nevertheless, it is worthwhile to mention that subresultants can be computed very efficiently in practice.

Let $\mathbb{K}$ be a field. Let $P(X)$ and $Q(X)$ be two polynomials in $\mathbb{K}[X]$ of positive degree p and q, $p > q$[1],

$$P = a_p X^p + \cdots + a_0, \quad Q = b_q X^q + \cdots + b_0$$

Next, we introduce the well-known Sylvester-Habicht matrix.

Definition 2.2 (Sylvester-Habicht matrix)**.** For $0 \le j \le q$, the ***j*-th Sylvester-Habicht matrix of *P* and *Q***, denoted by $\mathrm{SyHa}_j(P, Q)$, is the matrix whose rows are

[1] in the case $p = q$, we replace Q by $a_p Q - b_q P$

$X^{q-j-1}P, \ldots, P, Q, \ldots, X^{p-j-1}Q$ considered as vectors in the basis $X^{p+q-j-1}, \ldots, X, 1$:

$$\begin{bmatrix} a_p & \cdots & \cdots & \cdots & \cdots & a_0 & 0 & 0 \\ 0 & \ddots & & & & & \ddots & 0 \\ \vdots & \ddots & a_p & \cdots & \cdots & \cdots & \cdots & a_0 \\ \vdots & & 0 & b_q & \cdots & \cdots & \cdots & b_0 \\ \vdots & \iddots & \iddots & & & & \iddots & 0 \\ 0 & \iddots & & & & \iddots & \iddots & \vdots \\ b_q & \cdots & \cdots & \cdots & b_0 & 0 & \cdots & 0 \end{bmatrix}$$

Under these conditions, the resultant of two polynomials P and Q is defined as follows.

Definition 2.3 (Resultant)**.** The **(univariate) resultant of** P **and** Q, denoted by $\mathrm{Res}(P, Q)$, is $\det(\mathrm{SyHa}_0(P, Q))$.

The signed subresultants of P and Q will play a key role in what follows. For any $j \in \{0, 1, \ldots, p\}$, the signed subresultant of P and Q of index j is the polynomial

$$\mathrm{sResP}_j(P, Q) = \mathrm{sRes}_j X^j + \cdots + \mathrm{sRes}_{j,1} X + \mathrm{sRes}_{j,0}$$

where sRes_j and each $\mathrm{sRes}_{j,k}$ are elements of $\mathbb{K}$ defined as determinants of submatrices coming from $\mathrm{SyHa}_j(P, Q)$ (see [20] for a precise definition). Note that $\mathrm{Res}(P, Q) = \mathrm{sRes}_0$.

We write $\mathrm{sResP}_j(P, Q)$ (resp., $\mathrm{Res}(P, Q)$) for the j-th subresultant (resp., resultant) of the polynomials P, $Q \in \mathbb{K}[X_1, \ldots, X_k]$ with respect to X_k. The **j-th signed subresultant coefficient of** P **and** Q, denoted by $\mathrm{sRes}_j(P, Q)$ or sRes_j, is the coefficient of X^j in $\mathrm{sResP}_j(P, Q)$.

Next, we notice that one of the main characteristics of subresultants is that they provide a very easy to use characterization of the greatest common divisor of two polynomials (see [20] for a proof).

Theorem 2.4. *Let P, $Q \in \mathrm{R}[X]$ be two polynomials of degree p and q. Then the following are equivalent:*

1. *P and Q have a gcd of degree j*

2. $\mathrm{sRes}_0(P, Q) = \ldots = \mathrm{sRes}_{j-1}(P, Q) = 0$*,* $\mathrm{sRes}_j(P, Q) \neq 0$

In this case, $\mathrm{sResP}_j(P, Q)$ *is the greatest common divisor of* P *and* Q*.*

The following well-known theorem is very helpful.

Theorem 2.5 (The Extension Theorem)**.** *Let* $P, Q \in \mathrm{C}[X_1, \ldots, X_{k-1}][X_k]$*,*

$$P = a_p(X_1, \ldots, X_{k-1})X_k^p + \cdots + a_0(X_1, \ldots, X_{k-1})$$

$$Q = b_q(X_1, \ldots, X_{k-1})X_k^q + \cdots + b_0(X_1, \ldots, X_{k-1}).$$

Let $(\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}) \in \mathrm{C}^{k-1}$ *and assume that* $\mathrm{Res}(P, Q)(\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}) = 0$*, then either*

1. a_p *or* b_q *vanish at* $(\mathbf{x}_1, \ldots, \mathbf{x}_{k-1})$*, or*

2. *there is a number* $\mathbf{x}_k \in \mathrm{C}$ *such that* P *and* Q *vanish at* $(\mathbf{x}_1, \ldots, \mathbf{x}_k) \in \mathrm{C}^k$*.*

Proof. See [40]. □

In other words, if we assume that a_p and b_q are in C, i.e., P and Q are X_k-regular, and that P and Q do not have a common factor, then any solution $(\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}) \in \mathrm{C}^{k-1}$ of the equation $\mathrm{Res}(P, Q) = 0$ can be extended to a solution $(\mathbf{x}_1, \ldots, \mathbf{x}_k) \in \mathrm{C}^k$ of the polynomials P and Q. Note that we always can ensure that the polynomials are X_k-regular by a change of coordinates. (see [101] for details). Moreover, the common factor can be detected a priori by computing the greatest common divisor of P and Q.

The following proposition shows why resultants are very useful in our setting (see, for instance, Chapter 2.1.4 and 5.2).

Proposition 2.6. *Let* P_1*,* P_2 *and* P_3 *be three square-free and* X_3*-regular polynomials in* $\mathrm{C}[X_1, X_2, X_3]$ *such that two of them do not have a common factor. Moreover, assume that the polynomials* $\mathrm{Res}(P_1, P_2)$ *and* $\mathrm{Res}(P_1, P_3)$ *do not have a common factor, i.e.* $\gcd(\mathrm{Res}(P_1, P_2), \mathrm{Res}(P_1, P_3)) = 1$*. Then the number of distinct roots of the system*

$$P_1(X_1, X_2, X_3) = 0, P_2(X_1, X_2, X_3) = 0, P_3(X_1, X_2, X_3) = 0$$

is finite.

Proof. By [40], Chapter 3.6., Proposition 1, we know that $\mathrm{Res}(P_1, P_i)$ is in the elimination ideal $\langle P_1, P_i\rangle \cap \mathrm{C}[X_1, X_2]$. Therefore, by Proposition 2.5, only the solutions of the system

$$\mathrm{Res}(P_1, P_2) = \mathrm{Res}(P_1, P_3) = 0 \tag{2.1}$$

can be extended to a solution of the equations (2.6). But there are only finitely many such solutions since $\gcd(\mathrm{Res}(P_1, P_2), \mathrm{Res}(P_1, P_3)) = 1$.

Hence, let $(\mathbf{x}, \mathbf{y})$ be a solution of the equations (2.1). Then every $P_i(\mathbf{x}, \mathbf{y}, X_3)$ is not identically zero, as all of them are X_3-regular. In particular, they only have finitely many solutions. Now, the claim follows. □

2.1.4 The Cylindrical Decomposition

2.1.4.1 Definition

One basic ingredient in most algorithms for computing topological properties of semi-algebraic sets is an algorithm due to Collins [38], called cylindrical decomposition, which decomposes a given semi-algebraic set into topological balls. In this chapter, we recall some facts about the cylindrical decomposition which can be turned into an algorithm for solving several important problems. For instance, computing the topology of planar curves (see Chapter 2.1.4.2), computing the (real) intersection of quadratic surfaces (see Chapter 5.2), the general decision problem or the quantifier elimination problem (see [20]). Moreover, cylindrical decomposition can be used to compute a semi-algebraic triangulation of a semi-algebraic set (see Chapter 2.1.5). For more details on the algorithm in the general case we refer to [38, 2, 3, 4, 20].

Definition 2.7. A **Cylindrical Decomposition** of R^k is a sequence $\mathcal{S}_1, \ldots, \mathcal{S}_k$, where, for each $1 \leq i \leq k$, $\mathcal{S}_i$ is a finite partition of R^i into semi-algebraic subsets (**cells of level i**), which satisfy the following properties:

– Each cell $S \in \mathcal{S}_1$ is either a point or an open interval.

– For every $1 \leq i < k$ and every $S \in \mathcal{S}_i$ there are finitely many continuous semi-algebraic functions

$$\xi_{S,1} < \cdots < \xi_{S,n_S} : S \to \mathrm{R}$$

such that the **cylinder** $S \times \mathrm{R} \subset \mathrm{R}^{i+1}$ (also called a **stack over the cell S**) is a disjoint union of cells of $\mathcal{S}_{i+1}$ which are:

– either the graph of one of the functions $\xi_{S,j}$, for $j = 1, \ldots, n_S$:

$$\{(\mathbf{x}', \mathbf{x}_{j+1}) \in S \times \mathrm{R} \mid \mathbf{x}_{j+1} = \xi_{S,j}(\mathbf{x}')\},$$

– or a band of the cylinder bounded from below and above by the graphs of the functions $\xi_{S,j}$ and $\xi_{S,j+1}$, for $j = 0, \ldots, n_S$, where we take $\xi_{S,0} = -\infty$ and $\xi_{S,\ell_S+1} = +\infty$.

Note that a cylindrical decomposition has a recursive structure, i.e., the decomposition of R^i induces a decomposition of R^{i+1} and vice-versa.

Definition 2.8. Given a finite set $\mathcal{P}$ of polynomials in $\mathrm{R}[X_1, \ldots, X_k]$, a subset S of R^k is **$\mathcal{P}$-invariant** if every polynomial P in $\mathcal{P}$ has constant sign on S. A **cylindrical decomposition of R^k adapted to $\mathcal{P}$** is a cylindrical decomposition for which each cell in $\mathcal{S}_k$ is $\mathcal{P}$-invariant.

The following example illustrate the above definitions.

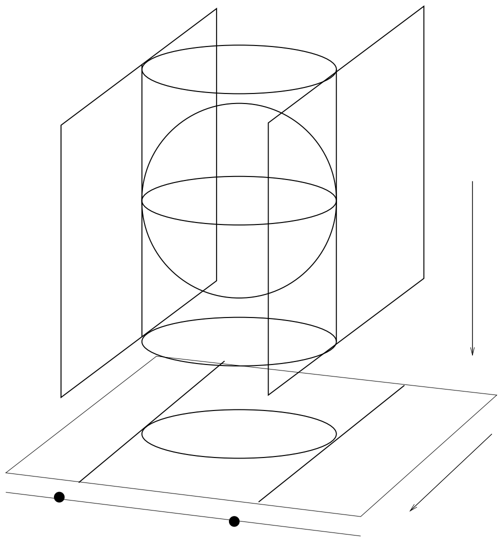

Figure 1: A cylindrical decomposition adapted to the unit sphere in R^3

Example 2.9 (Decomposition adapted to the unit sphere)**.** Let

$$S = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathrm{R}^3 \mid \mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 - 1 = 0\}$$

(see Figure 1). The decomposition of R (i.e., the line) consists of five cells of level 1 corresponding to the points -1 and 1 and the three intervals they define. The decomposition of R^2 (i.e., the plane) consists of 13 cells of level 2. For instance, the two bands to the left and right of the circle, the two cells corresponding to the points $(-1, 0)$ and $(1, 0)$ and the cell that corresponds to the set $S_{3,2} = \{(\mathbf{x}, \mathbf{y}) \in \mathrm{R}^2 \mid 1 < \mathbf{x} < 1, \mathbf{y} = -\sqrt{1 - \mathbf{x}^2}\}$. The decomposition of R^3 consists of 25 cells of level 3. For instance, the two cells corresponding to the points $(-1, 0, 0)$ and $(1, 0, 0)$ and the cell that corresponds to the set $S_{3,2,2} = S_{3,2} \times \{0\}$. For a more detailed description of this example see [20], Shapter 5.1.

The Cylindrical Decomposition Algorithm ([38, 20]) consists of two phases: the projection and the lifting phase. During the projection phase one eliminates the variables $X_k, \ldots, X_2$ by iterative use of (sub)-resultant computations. In the lifting phase the cells defined by these (sub)-resultants are used to define inductively, starting with $i = 1$, the cylindrical decomposition.

One disadvantage of the Cylindrical Decomposition Algorithm is that it uses iterated projections (reducing the dimension by one in each step) and the number of polynomials (as well as the degrees) square in each step of the process. Thus, the complexity of performing cylindrical decomposition is double-exponential in the number of variables which makes it impractical in most cases for computing topological information.

Nevertheless, we will see in the next chapters that it can be used quite efficiently for several important problems in low dimensions.

2.1.4.2 Computing the Topology of Planer Curves

The simplest situation where the cylindrical decomposition method can be performed is the case of one single non-zero bivariate polynomial $P \in \mathrm{R}[X_1, X_2]$ or a set of bivariate polynomials $\mathcal{P} \subset \mathrm{R}[X_1, X_2]$. In particular, we are interested in the topology of the curve $\mathrm{Zer}(P, \mathrm{R}^2)$ (resp., of $\mathrm{Zer}(\mathcal{P}, \mathrm{R}^2)$), i.e., to determine a planar graph homeomorphic to $\mathrm{Zer}(P, \mathrm{R}^2)$ (resp., $\mathrm{Zer}(\mathcal{P}, \mathrm{R}^2)$).

We consider planar algebraic curves being in generic position which we define next.

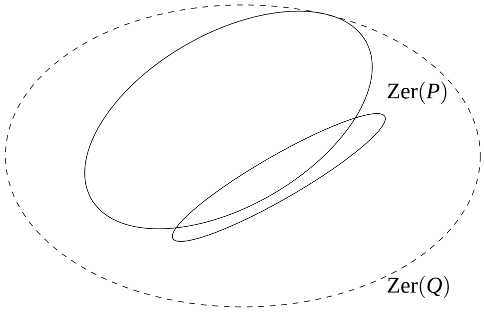

Figure 2: The polynomial P is in generic position with respect to Q

Definition 2.10. Two square-free bi-variate polynomials P_1 and P_2 are in **generic position** with respect to the projection on the X_1-axis if the following conditions hold.

1. $\deg(P_i) = \deg_{X_2}(P_i)$ (X_2-regular),
2. $\gcd(P_1, P_2) = 1$,
3. for all $x \in \mathrm{R}$ the number of distinct (complex) roots of

$$P_1(\mathbf{x}, X_2) = 0, \quad P_2(\mathbf{x}, X_2) = 0$$

is 0 or 1.

In particular, a single bi-variate polynomial P_1 is called in **generic position with respect to** P_2 (resp., **generic position**) if P_1 and $\partial P_1/\partial X_2 \cdot P_2$ are in generic position and, for $0 \neq \lambda \in \mathrm{R}$, $P_2 \neq \lambda \cdot \partial P_1/\partial X_2$ (resp., $P_2 = 1$).

It is worthwhile to mention that it is always possible to put a set of planar algebraic curves in generic position by a linear change of coordinates and computing the gcd-free part of each polynomial. Furthermore, two plane curves in generic position behalf nicely, i.e., their intersection points can be described using signed subresultant computations. The following proposition makes this precise.

Proposition 2.11. *Let P, $Q \in \mathrm{R}[X_1, X_2]$ be two square-free polynomials in generic position. If $(\mathbf{x}, \mathbf{y})$ is an intersection point of $\mathrm{Zer}(P, \mathrm{R}^2)$ and $\mathrm{Zer}(Q, \mathrm{R}^2)$, then there exists a*

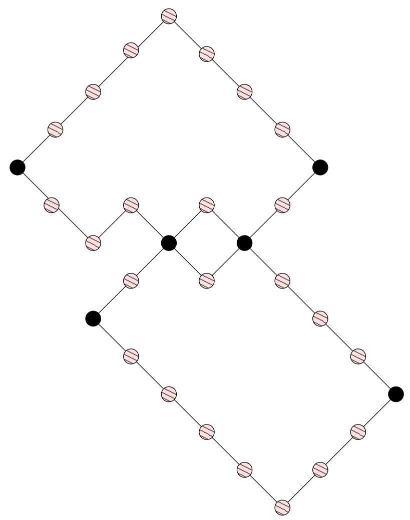

Figure 3: The topology of $\mathrm{Zer}(P, \mathrm{R}^2)$

unique j *such that*

$$\mathrm{sRes}_0(\mathbf{x}) = \cdots = \mathrm{sRes}_{j-1}(\mathbf{x}) = 0, \quad \mathrm{sRes}_j(\mathbf{x}) \neq 0$$

$$\mathbf{y} = -\frac{1}{j} \cdot \frac{\mathrm{sRes}_{j,j-1}(\mathbf{x})}{\mathrm{sRes}_j(\mathbf{x})}$$

Proof. Let j be the unique integer such that $\mathrm{sRes}_0(\mathbf{x}) = \cdots = \mathrm{sRes}_{j-1}(\mathbf{x}) = 0$ and $\mathrm{sRes}_j(\mathbf{x}) \neq 0$. Then $\mathrm{sResP}_j(P, Q)(\mathbf{x}, X_2)$ is the greatest common divisor of the polynomials $P(\mathbf{x}, X_2)$ and $Q(\mathbf{x}, X_2)$ by Theorem 2.4. Since P and Q are in generic position, there is only one intersection point of P and Q with X_1-coordinate equal to $\mathbf{x}$. In particular, $\mathbf{y}$ is the only root of $\mathrm{sResP}_j(P, Q)(\mathbf{x}, X_2)$ and hence $\mathbf{y} = -(j \cdot \mathrm{sRes}_j(\mathbf{x}))^{-1}\mathrm{sRes}_{j,j-1}(\mathbf{x})$. $\square$

González-Vega and Necula presented an algorithm TOP [52] which computes the topology of a plane curve. The TOP-algorithm takes a single bi-variate polynomial P as an input. While computing, it checks if the polynomial P is in generic position and performs a change of coordinates until the polynomial is in generic position. The TOP-algorithm outputs the topology of $\mathrm{Zer}(P, \mathrm{R}^2)$ as described below (see Algorithm 2.12).

For example, consider the curves given in Figure 2. The polynomial P (defining the two ellipses) is in generic position with respect to the polynomial Q (defining the dotted ellipse). The output of the TOP-algorithm is as in Figure 3.

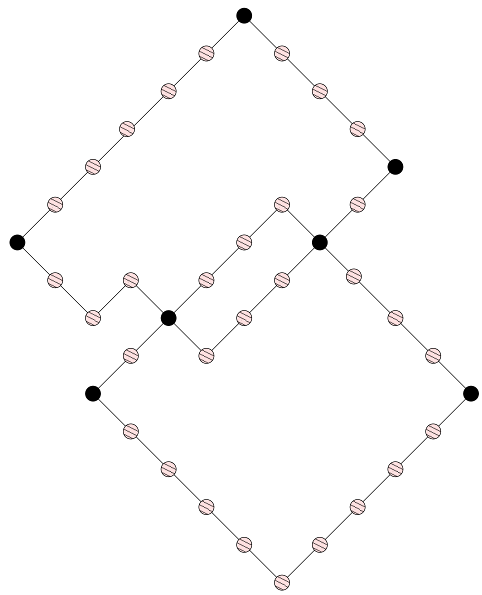

Figure 4: The topology of $\mathrm{Zer}(P, \mathrm{R}^2)$ with respect to $\mathrm{Zer}(Q, \mathrm{R}^2)$

After some slight modifications one can use this algorithm for the following two problems, which might occur simultaneously.

1. Computing the topology of a plane curve $\mathrm{Zer}(P_1, \mathrm{R}^2)$ with respect to another plane curve $\mathrm{Zer}(P_2, \mathrm{R}^2)$, and

2. computing the common roots of two plane curves.

Note that the proof presented in [52] can easily be adapted to those two problems, but the modified algorithm detects for the first problem whether or not the polynomial P_1 is in generic position with respect to P_2 and for the second one if P_1 and P_2 are in generic position.

For our example considered above the modified TOP-algorithm output is as in Figure 4. Note that 8 additional points are computed.

Finally, we simply recall the in- and output of the TOP-algorithm which we later will use as a black-box in Chapter 5.2, and we refer the reader to [52, 20] for more details.

Algorithm 2.12 (TOP)**.**

Input: a square-free polynomial $P \in \mathrm{R}[X_1, X_2]$.

Output: the topology of the curve $\mathrm{Zer}(P, \mathrm{R}^2)$, described by

- The real roots $\mathbf{x}_1, \ldots, \mathbf{x}_r$ of $\mathrm{Res}(P, \partial P/\partial X_2)(X_1)$. We set by $\mathbf{x}_0 = -\infty$, $\mathbf{x}_{r+1} = \infty$.

- The number m_i of roots of $P(\mathbf{x}, X_2)$ in R when $\mathbf{x}$ varies on $(\mathbf{x}_i, \mathbf{x}_{i+1})$.

- The number n_i of roots of $P(\mathbf{x}_i, X_2)$ in R. We denote these roots by $\mathbf{y}_{i,1}, \ldots, \mathbf{y}_{i,n_i}$.

- A number $c_i \leq n_i$ such that if $(\mathbf{x}_i, \mathbf{z}_i)$ is the unique critical point of the projection of $\mathrm{Zer}(P, \mathrm{C}^2)$ on the X_1-axis above $\mathbf{x}_i$, $\mathbf{z}_i = \mathbf{y}_{i,c_i}$.

2.1.4.3 Cell Adjacency

An important piece of information that we require from the cylindrical decomposition algorithm is that of cell adjacency. In other words, we need to know given two cells in a set $\mathcal{S}_i$, whether the closure of one intersects the other. In Example 2.9, for instance, we have that the cell corresponding to the point $(-1, 0, 0)$ is adjacent to the cell $C_{3,2,2}$.

We need the following notation. We distinguish between the **inter-stack cell adjacency of level** i, which is the adjacency of cells of level i in two different stacks, and the **intra-stack cell adjacency of level** i, which is the adjacency of cells of level i within the same stack.

Moreover, we use the following intuitive labeling of cells.

A cell in R, i.e., a cell in the induced decomposition (line) of the induced decomposition (plane), is denoted by (i), where the i ranges over the number of cells in the induced decomposition of R. Note that $i_1 < i_2$ if and only if the cell (i_1) "occurs to the left" of the cell (i_2).

A cell in R^2, i.e., a cell in the induced decomposition of the plane, is denoted by (i, j), where i ranges over the number of cells in the line and the j ranges over the number of cells in the stack over the cell (i). Note that $j_1 < j_2$ if and only if the cell (i, j_1) "occurs lower in the plane" than the cell (i, j_2).

A cell in R^3 is denoted by (i, j, k), where (i, j) is a cell in the induced decomposition of the plane and the k ranges over the number of cells in the stack over the cell (i, j). Note that $k_1 < k_2$ if and only if the cell (i, j, k_1) "occurs lower" than the cell (i, j, k_2).

Furthermore, we distinguish among **0-cells, 1-cells, 2-cells** and **3-cells** of the cylindrical decomposition, that are points, graphs and cylinders bounded below and above by graphs. The adjacency between a ℓ-cell and k-cell will be denoted by $\{\ell, \mathbf{k}\}$**-adjacency**.

We illustrate the above notation on Example 2.9 (Decomposition adapted to the unit sphere).

Example 2.13 (cont.)**.** For instance, the cell (2) and (4) correspond to the points -1 and 1 (in the line), whereas the cells $(2, 2)$ and $(3, 2)$ correspond the point $(-1, 0)$ and the set

$$S_{3,2} = \{(\mathbf{x}, \mathbf{y}) \in \mathrm{R}^2 \mid -1 < \mathbf{x} < 1, \mathbf{y} = -\sqrt{1 - \mathbf{x}^2}\}.$$

Moreover, the cell $(2, 2, 2)$ corresponds to the point $(-1, 0, 0)$ and the cell $(3, 2, 2)$ corresponds to the set $S_{3,2,2} = S_{3,2} \times \{0\}$.

While there are algorithms known for computing the cell adjacencies of a cylindrical decomposition of R^k (see ([3, 4]), we will only be interested in the cell adjacencies for a cylindrical decomposition adapted to family $\mathcal{P} \subset \mathrm{R}[X_1, X_2, X_3]$ such that $\deg(P) \leq 2$ and P is X_3-regular for every polynomial $P \in \mathcal{P}$.

It is worthwhile to mention that we do not need to compute all cell adjacencies. In our applications (see Chapter 5) it suffices to compute the $\{0, 1\}$-inter-stack adjacencies which we can do by a simple combinatorial type approach. In other words, we determine the full adjacency information for the boundary of the semi-algebraic set by using the simpler structure induced by the quadratic polynomials which we describe next.

Assume that the 0-cell (i, j_1) and the 1-cell $(i + 1, j_2)$ are adjacent in the induced decomposition of the plane. To be more precise, the 0-cell (i, j_1) and the 1-cell $(i + 1, j_2)$ correspond to a point and a curve segment of $\mathrm{Zer}(\mathrm{Res}(P_m, P_t, X_3), \mathrm{R}^2)$ where P_m and P_t are two input quadratic polynomials that are X_3-regular. We have the following two cases:

Case 1: The stack over the 0-cell (i, j_1) contains exactly one 0-cell (i, j_1, k). Note, that the stack over 1-cell $(i + 1, j_2)$ must contain two 1-cells $(i + 1, j_2, l_1)$ and $(i + 1, j_2, l_2)$ (corresponding to graphs), since the polynomial P_m is of degree equal to 2 in the variable X_3. Therefore, the 0-cell (i, j_1, k) must be adjacent to both cells $(i + 1, j_2, l_1)$ and $(i + 1, j_2, l_2)$, since the semi-algebraic set S_i is closed.

Case 2: The stack over the 0-cell (i, j_1) contains two 0-cells (i, j_1, k_1) and (i, j_1, k_2). As above, the stack over the 1-cell $(i+1, j_2)$ must contain two 1-cells $(i+1, j_2, \ell_1)$ and $(i+1, j_2, \ell_2)$. Remember that both stacks are ordered from the bottom to the top. Hence, the cells (i, j_1, k_1) and $(i+1, j_2, \ell_1)$ as well as the cells (i, j_1, k_2) and $(i+1, j_2, \ell_2)$ must be adjacent for the same reason as above. It is worthwhile to mention that is not possible to have just one 1-cell above $(i+1, j_2)$, i.e., $\ell_1 = \ell_2$, by the properties of the cylindrical decomposition.

2.1.5 Triangulation of Semi-algebraic Sets

Another important property of closed and bounded semi-algebraic sets is that they are homeomorphic to a simplicial complex. The following makes this statement precise.

Let $a_0, \ldots, a_p$ be points of R^k that are affinely independent. The **p-simplex** with vertices $a_0, \ldots, a_p$ is

$$[a_0, \ldots, a_p] = \{\lambda_0 a_0 + \cdots + \lambda_p a_p \mid \sum_{i=0}^{p} \lambda_i = 1 \text{ and } \lambda_0, \ldots, \lambda_p \geq 0\}$$

Note that the dimension of $[a_0, \ldots, a_p]$ is p.

An **q-face** of the p-simplex $s = [a_0, \ldots, a_p]$ is any simplex $s' = [b_0, \ldots, b_q]$ such that

$$\{b_0, \ldots, b_q\} \subset \{a_0, \ldots, a_p\}$$

The open simplex, denoted by s^o, corresponding to a simplex s consists of all points of s which do not belong to any proper face of s:

$$s^o = (a_0, \ldots, a_p) = \{\lambda_0 a_0 + \cdots + \lambda_p a_p \mid \sum_{i=0}^{p} \lambda_i = 1 \text{ and } \lambda_0 > 0, \ldots, \lambda_p > 0\}$$

A **simplicial complex** K in R^k is a finite set of simplices in R^k such that $s, s' \in \mathrm{K}$ implies

– every face of s is in K,

– $s \cap s'$ is a common face of both s and s'.

A **triangulation** of a semi-algebraic set S is a simplicial complex K together with a semi-algebraic homeomorphism $h : |\mathrm{K}| \to S$, where the set $|\mathrm{K}| = \bigcup_{s \in K} s$ is the **realization of** K.

A triangulation of S **respecting a finite family of semi-algebraic sets** $S_1, \dots, S_n$ contained in S is a triangulation (K, h) such that each S_j is the union of images by h of open simplices of K.

We have the following theorem.

Theorem 2.14. *Let $S \subset \mathrm{R}^k$ be a closed and bounded semi-algebraic set, and let $S_1, \dots, S_n$ be semi-algebraic subsets of S. There exists a triangulation of S respecting $S_1, \dots, S_n$. Moreover, the vertices of* K *can be chosen with rational coefficients.*

Proof. See [20] □

For example, let S be a closed and bounded subset of R^k such that $S = \bigcup_{i=1}^n S_i \subset \mathrm{R}^k$. Then Theorem 2.14 implies that there is a triangulation (K, h) of S such that for every simplex $s \in \mathrm{K}$ and $1 \le i \le n$ either $h(s) \cap S_i = h(s)$ or $h(s) \cap S_i = \emptyset$.

Finally, note that one can compute a triangulation of a closed and bounded semi-algebraic set using the cylindrical decomposition which decomposes a given semi-algebraic set into double exponential number (in the dimension) of topological balls.

2.1.6 Triviality of Semi-algebraic Mappings

The finiteness of the topological types of algebraic subsets of R^k defined by polynomials of fixed degree is an easy consequence of Hardt's triviality theorem, which we recall next.

Theorem 2.15 (Hardt's triviality theorem [56, 20])**.** *Let $S \subset \mathrm{R}^n$ and $T \subset \mathrm{R}^k$ be semi-algebraic sets. Given a continuos semi-algebraic function $f : S \to T$, there exists a finite partition of T into semi-algebraic sets $T = \bigcup_{i \in I} T_i$, so that for each i and any $\mathbf{x}_i \in T_i$, $T_i \times f^{-1}(\mathbf{x}_i)$ is semi-algebraically homeomorphic to $f^{-1}(T_i)$.*

Hardt's theorem is a corollary of the existence of cylindrical decompositions (see Chapter 2.1.4), which implies a double exponential (in n) upper bound on the cardinality of the set I. Moreover, it follows that one can always retract a closed semi-algebraic set to a closed and bounded set. The following proposition makes this precise.

Proposition 2.16 (Conic structure at infinity)**.** *Let $S \subset \mathrm{R}^k$ be a closed semi-algebraic set. There exists $r \in \mathrm{R}$, $r > 0$, such that for every r', $r' \ge r$, there is a semi-algebraic* *deformation retraction from S to $S_{r'} = S \cap \mathbf{B}_k(0, r')$ and a semi-algebraic deformation retraction from $S_{r'}$ to S_r.*

Proof. See [20], Proposition 5.49. $\square$

2.2 Algebraic Topology

2.2.1 Some Notations

In this chapter we recall the basic objects from algebraic topology like homology and cohomology theory. Unless otherwise noted, we will consider vector spaces over $\mathbb{Q}$ in what follows next.

Given a simplicial complex K, we denote by $\mathrm{C}_p(\mathrm{K})$ the vector space generated by the p-dimensional oriented simplices of K. The elements of $\mathrm{C}_p(\mathrm{K})$ are called the **p-chains** of K. For $p < 0$, we define $\mathrm{C}_p(\mathrm{K}) = 0$.

Given an oriented p-simplex $s = [a_0, \ldots, a_p]$, $p > 0$, the boundary of s is the $(p-1)$-chain

$$\partial_p(s) = \sum_{0 \le i \le p} (-1)^i [a_0, \ldots, a_{i-1}, \hat{a}_i, a_{i+1}, \ldots, a_p],$$

where $\hat{a}_i$ means that the a_i is omitted. For $p \le 0$, we define $\partial_p = 0$. The map ∂_p extends linearly to a homomorphism

$$\partial_p : \mathrm{C}_p(\mathrm{K}) \to \mathrm{C}_{p-1}(\mathrm{K}).$$

Thus, we have the following sequence of vector space homomorphism with $\partial_{p-1} \circ \partial_p = 0$,

$$\cdots \longrightarrow \mathrm{C}_p(\mathrm{K}) \xrightarrow{\partial_p} \mathrm{C}_{p-1}(\mathrm{K}) \xrightarrow{\partial_{p-1}} \mathrm{C}_{p-2}(\mathrm{K}) \xrightarrow{\partial_{p-2}} \cdots \xrightarrow{\partial_1} \mathrm{C}_0(\mathrm{K}) \xrightarrow{\partial_0} 0$$

The sequence of pairs $\{(\mathrm{C}_p(\mathrm{K}), \partial_p)\}_{p \in \mathbb{N}}$, denoted by $\mathrm{C}_\bullet(\mathrm{K})$, is called the **simplicial chain complex**.

We denote by $\mathrm{H}_p(\mathrm{K})$ the **p-th simplicial homology group** of K, that is

$$\mathrm{H}_p(\mathrm{C}_\bullet(\mathrm{K})) = \mathrm{Z}_p(\mathrm{C}_\bullet(\mathrm{K}))/\mathrm{B}_p(\mathrm{C}_\bullet(\mathrm{K})),$$

where $\mathrm{Z}_p(\mathrm{C}_\bullet(\mathrm{K})) = \mathrm{Ker}(\partial_p)$ is the subspace of **p-cycles**, and $\mathrm{B}_p(\mathrm{C}_\bullet(\mathrm{K})) = \mathrm{Im}(\partial_{p+1})$ is the subspace of **p-boundaries**.

Note that $\mathrm{H}_p(\mathrm{K})$ is a finite dimensional vector space. The dimension of $\mathrm{H}_p(\mathrm{K})$ as a vector space is called the **p-th Betti number** of K and denoted by $b_p(\mathrm{K})$. We will denote by $b(\mathrm{K})$ the sum $\sum_{p\geq 0} b_p(\mathrm{K})$.

Next, we define the dual notion of cohomology groups.

We denote by $\mathrm{C}^p(K) = \mathrm{Hom}(C_p(K), \mathbb{Q})$ the vector space dual to $\mathrm{C}_p(K)$, and by δ^p the co-boundary map $\delta^p : \mathrm{C}^p(\mathrm{K}) \to \mathrm{C}^{p+1}(\mathrm{K})$ which is the homomorphism dual to ∂_{p+1} in the simplicial chain complex $\mathrm{C}_\bullet(K)$. More precisely, given $\omega \in \mathrm{C}^p(\mathrm{K})$, and a $p+1$-simplex $[a_0, \ldots, a_{p+1}]$ of K, then

$$\delta\omega([a_0, \ldots, a_{p+1}]) = \sum_{0\leq i\leq p+1} (-1)^i \omega([a_0, \ldots, a_{i-1}, \hat{a}_i, a_{i+1}, \ldots, a_{p+1}])$$

Thus, we have the following sequence of (dual) vector space homomorphism,

$$0 \to \mathrm{C}^0(\mathrm{K}) \xrightarrow{\delta^0} \mathrm{C}^1(\mathrm{K}) \xrightarrow{\delta^1} \mathrm{C}^2(\mathrm{K}) \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^{p-1}} \mathrm{C}^p(\mathrm{K}) \xrightarrow{\delta^p} \mathrm{C}^{p+1}(\mathrm{K}) \xrightarrow{\delta^{p+1}} \cdots,$$

with $\delta^{p+1} \circ \delta^p = 0$. The sequence of pairs $\{(\mathrm{C}^p(\mathrm{K}), \delta^p)\}_{p\in\mathbb{N}}$, denoted by $\mathrm{C}^\bullet(K)$, is called the **simplicial cochain complex**.

We denote by $\mathrm{H}^p(\mathrm{K})$ the **p-th simplicial cohomology group** of K, that is

$$\mathrm{H}^p(\mathrm{C}^\bullet(\mathrm{K})) = \mathrm{Z}^p(\mathrm{C}^\bullet(\mathrm{K}))/\mathrm{B}^p(\mathrm{C}^\bullet(\mathrm{K})),$$

where $\mathrm{Z}^p(\mathrm{C}^\bullet(\mathrm{K})) = \mathrm{Ker}(\partial^{p-1})$ is the subspace of **p-cocycles**, and $\mathrm{B}^p(\mathrm{C}^\bullet(\mathrm{K})) = \mathrm{Im}(\partial_p)$ is the subspace of **p-coboundaries**.

Note that $\mathrm{H}^p(\mathrm{K})$ is a finite dimensional vector space and its dimension as a vector space is equal to $b_p(\mathrm{K})$. To be more precise, we have by the Universal Coefficient Theorem for cohomology (see [58], Theorem 3.2, page 195) that $\mathrm{H}^p(\mathrm{C}^\bullet(\mathrm{K}))$ and $\mathrm{H}_p(\mathrm{C}_\bullet(\mathrm{K}))$ are isomorphic for every $p \geq 0$. Moreover, the cohomology group $\mathrm{H}^0(\mathrm{K})$ can be identified with the vector space of locally constant functions on $|\mathrm{K}|$ (see [20], Proposition 6.5).

Next, we define simplicial (co)-homology groups for a closed semi-algebraic set. Let $S \subset \mathrm{R}^k$ be a closed semi-algebraic set. By Proposition 2.16 (Conic structure at infinity), there exists $r \in \mathrm{R}$, $r > 0$, such that for every r', $r' \geq r$, there is a semi-algebraic deformation from S to $S_{r'} = S \cap \mathbf{B}_k(0, r')$ and a semi-algebraic deformation from $S_{r'}$ to S_r. Note that the set S_r is closed and bounded. By Theorem 2.14, the set S_r can be triangulated by a simplicial complex K with rational coordinates. Choose a semi-algebraic triangulation $f : |\mathrm{K}| \longrightarrow S_r$, then for $p \geq 0$ the **homology groups** $\mathbf{H_p(S)}$ are $\mathrm{H}_p(\mathrm{K})$ (resp., **cohomology groups** $\mathbf{H^p(S)}$ are $\mathrm{H}^p(\mathrm{K})$). Note that the (co)-homology groups do not depend on the particular triangulation. The dimension of $\mathrm{H}_p(S)$ as a vector space is called the **p-th Betti number** of S and denoted by $b_p(S)$. We will denote by $b(S)$ the sum $\sum_{p\geq 0} b_p(S)$.

For completeness we now consider a basic locally closed semi-algebraic set S which is, by definition, the intersection of a closed semi-algebraic set with a basic open one. Let $\dot{S}$ be the (one point) Alexandroff compactification of S. Then the dimension of $\mathrm{H}_p(\dot{S})$ as a vector space is called the **p-th Betti number** of S and denoted by $b_p(S)$. This definition is well-defined since the Alexandroff compactification $\dot{S}$ of S is closed, bounded, unique (up to semi-algebraic homeomorphism) and semi-algebraically homeomorphic to S. We will denote by $b(S)$ the sum $\sum_{p\geq 0} b_p(S)$. Note that the homology groups of a semi-algebraic set $S \subset \mathrm{R}^k$ are finitely generated. Hence, the Betti numbers $b_i(S)$ are finite.

We illustrate Betti numbers with the following example.

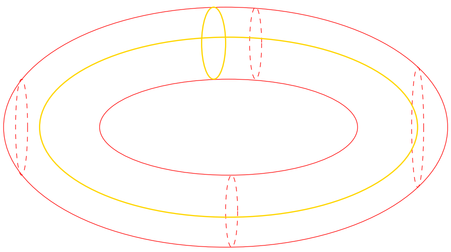

Figure 5: The hollow torus

Example 2.17. Let S be the hollow torus in R^3 (see Figure 5), then

$$b_0(S) = 1, \quad b_1(S) = 2, \quad b_2(S) = 1 \text{ and } b_p(S) = 0, p > 2.$$

Intuitively, $b_p(S)$ measures the number of p-dimensional holes in the set S. The zero-th Betti number, $b_0(S)$, is the number of connected components.

Similarly, one can define $b_p(S, \mathbb{Z}_2)$, the p-th Betti number with coefficients in $\mathbb{Z}_2$, as the $\mathbb{Z}_2$-vector space dimension of $\mathrm{H}_p(S, \mathbb{Z}_2)$. We denote by $b(S, \mathbb{Z}_2)$ the sum $\sum_{p\geq 0} b_p(S, \mathbb{Z}_2)$.

It follows from the Universal Coefficients Theorem, that

$$b_i(S, \mathbb{Z}_2) \geq b_i(S)$$

(see [58], Corollary 3.A6 (b)).

Hence, any bounds proved for Betti numbers with $\mathbb{Z}_2$-coefficients also apply to the ordinary Betti numbers (with coefficients in $\mathbb{Q}$).

2.2.2 The Mayer-Vietoris Theorem

We have seen in Chapter 2.1.4 that we can use the cylindrical decomposition in order to decompose a semi-algebraic set into smaller pieces. The Mayer-Vietoris inequalities (see Proposition 2.19) bound the Betti numbers of the union (resp., intersection) of semi-algebraic sets in terms of intersections (resp., unions) of fewer semi-algebraic sets. This will be very useful in Chapter 3 and Chapter 4. We first recall a semi-algebraic version of the Mayer-Vietoris theorem.

Theorem 2.18 (Semi-algebraic Mayer-Vietoris)**.** *Let S_1 and S_2 be two closed and bounded semi-algebraic subsets of R^k. Then there is a long exact sequence.*

$$\cdots \to \mathrm{H}_p(S_1 \cap S_2) \to \mathrm{H}_p(S_1) \oplus \mathrm{H}_p(S_2) \to \mathrm{H}_p(S_1 \cup S_2) \to \mathrm{H}_{p-1}(S_1 \cap S_2) \to \cdots$$

Proof. By Theorem 2.14 there is a triangulation of $S_1 \cup S_2$ that is simultaneously a triangulation of S_1, S_2, and $S_1 \cap S_2$. Let K_i be the simplicial complex corresponding to S_i. Then there is a a short exact sequence of simplicial chain complexes,

$$0 \to \mathrm{C}_\bullet(K_1 \cap K_2) \to \mathrm{C}_\bullet(K_1) \oplus \mathrm{C}_\bullet(K_2) \to \mathrm{C}_\bullet(K_1 \cup K_2) \to 0$$

The claim follows by a standard argument about short and long exact sequences (see [20], Lemma 6.10). □

From the exactness of the Mayer-Vieotoris sequence, we have the following proposition.

Proposition 2.19 (Mayer-Vietoris inequalities)**.** *Let be $S_1, \ldots, S_n$ subsets of R^k be all open or all closed. Then for each $i \geq 0$ we have,*

$$b_i\left(\bigcup_{1 \leq j \leq n} S_j\right) \leq \sum_{J \subset [n]} b_{i-(\#J)+1}\left(\bigcap_{j \in J} S_j\right) \tag{2.2}$$

and

$$b_i\left(\bigcap_{1\le j\le n} S_j\right) \le \sum_{J\subset[n]} b_{i+(\#J)-1}\left(\bigcup_{j\in J} S_j\right). \tag{2.3}$$

Proof. Follows from [20], Proposition 7.33. □

The following proposition characterizes b_0 and b_1 in a special case of unions of simplicial complexes. It is a slightly strengthened version of a similar proposition appearing in [21, 20]. We do not require that the complexes A_i be acyclic, but only that their first co-homology group vanishes. We need the following notations.

Let $A_1, \ldots, A_n$ be sub-complexes of a finite simplicial complex A such that

- each A_i is connected, i.e., $\mathrm{H}^0(A_i) = \mathbb{Q}$,
- $A = \bigcup_{i=1}^n A_i$, and
- $\mathrm{H}^1(A_i) = 0$, $1 \le i \le n$.

Note that the intersections of any number of the sub-complexes, A_i, is again a sub-complex of A. We will denote by $A_{i,j}$ the sub-complex $A_i \cap A_j$, and by $A_{i,j,\ell}$ the sub-complex $A_i \cap A_j \cap A_\ell$.

Recall that $\mathrm{H}^0(\mathrm{K})$ can be identified as the vector space of locally constant functions on the simplicial complex K. Hence, we can define the following sequence of generalized restriction homomorphisms.

Let $\phi \in \bigoplus_{1\le i\le n} \mathrm{H}^0(A_i)$, define

$$(\delta_0\phi)_{i,j} = \phi_i|_{A_{i,j}} - \phi_j|_{A_{i,j}}$$

and let $\psi \in \bigoplus_{1\le i<j\le n} \mathrm{H}^0(A_{i,j})$, define

$$(\delta_1\psi)_{i,j,\ell} = \psi_{i,j}|_{A_{i,j,\ell}} - \psi_{i,\ell}|_{A_{i,j,\ell}} + \psi_{j,\ell}|_{A_{i,j,\ell}}.$$

We now are able to state our proposition.

Proposition 2.20. *Let $A_1, \ldots, A_n$ be sub-complexes of a finite simplicial complex A such that $A = \bigcup_{i=1}^n A_i$ and for each $i, 1 \le i \le n$,* *1.* $\mathrm{H}^0(A_i) = \mathbb{Q}$, *and*

2. $\mathrm{H}^1(A_i) = 0$.

Let the homomorphisms δ_0 *and* δ_1 *in the following sequence be defined as above.*

$$\prod_i \mathrm{H}^0(A_i) \xrightarrow{\delta_0} \prod_{i<j} \mathrm{H}^0(A_{i,j}) \xrightarrow{\delta_1} \prod_{i<j<\ell} \mathrm{H}^0(A_{i,j,\ell})$$

Then,

1. $b_0(A) = \dim(\mathrm{Ker}(\delta_0))$,

2. $b_1(A) = \dim(\mathrm{Ker}(\delta_1)) - \dim(\mathrm{Im}(\delta_0))$.

Proof. Follows from [20], Theorem 6.9. □

Remark 2.21. One could use the so-called generalized Mayer-Vietoris sequence and some spectral sequence argument in order to prove Proposition 2.20. We refer to [10, 15] for more details.

2.2.3 Smith Theory

In Chapter 3 we will reduce the problem of bounding the Betti numbers of a semi-algebraic set to the problem of bounding the Betti numbers of some real projective algebraic sets. Using the Smith inequality (see Theorem 2.22 below) allows us to relate the Betti numbers of these real projective algebraic sets to the corresponding complex projective algebraic sets. As we will see in Chapter 2.3.2, we have precise information about the corresponding complex projective algebraic set. Before we recall a version of the Smith inequality, we need the following.

Let X be a compact topological space and $c : X \to X$ an involution. We regard X as a G-space, where $G = \{id, c\} \cong \mathbb{Z}_2$. We denote by $X' = X/c$ the orbit space, and by $F = \mathrm{Fix}\ c$, the fixed point set of the involution c. Moreover, we identify F with its image in X'.

Then there are two exact sequences, called **(homology and cohomology) Smith sequences of** (X, c):

$$\cdots \to \mathrm{H}_{p+1}(X', F; \mathbb{Z}_2) \to \mathrm{H}_p(X', F; \mathbb{Z}_2) \oplus \mathrm{H}_p(F; \mathbb{Z}_2) \to \mathrm{H}_p(X; \mathbb{Z}_2) \to \mathrm{H}_p(X', F; \mathbb{Z}_2) \to \cdots,$$

$$\cdots \to \mathrm{H}_p(X', F; \mathbb{Z}_2) \to \mathrm{H}^p(X; \mathbb{Z}_2) \to \mathrm{H}^p(X', F; \mathbb{Z}_2) \oplus \mathrm{H}^p(F; \mathbb{Z}_2) \to \mathrm{H}^{p+1}(X', F; \mathbb{Z}_2) \to \cdots.$$

We refer the reader to [95], p. 131, for more details.

Next, we state a version of the Smith inequality which follows from the exactness of the Smith sequence. We consider the special case where X is a complex projective algebraic set defined by real forms, with the involution taken to be complex conjugation. Then we have the following theorem.

Theorem 2.22 (Smith inequality)**.** *Let $\mathcal{Q} \subset \mathrm{R}[X_0, \ldots, X_k]$ be a family of homogeneous polynomials. Then,*

$$b(\mathrm{Zer}(\mathcal{Q}, \mathbb{P}^k_{\mathrm{R}}), \mathbb{Z}_2) \leq b(\mathrm{Zer}(\mathcal{Q}, \mathbb{P}^k_{\mathrm{C}}), \mathbb{Z}_2).$$

2.2.4 Alexander Duality

In Chapter 3, we also use the well-known Alexander duality theorem which relates the Betti numbers of a compact subset of a sphere to those of its complement.

Theorem 2.23 (Alexander Duality)**.** *Let $r > 0$. For any closed subset $A \subset S^k(0, r)$,*

$$\mathrm{H}_i(S^k(0, r) \setminus A) \approx \tilde{\mathrm{H}}^{k-i-1}(A),$$

where $\tilde{\mathrm{H}}^i(A)$, $0 \leq i \leq k-1$, denotes the reduced cohomology group of A.

Proof. See [73], Theorem 6.6. □

2.2.5 The Betti Numbers of a Double Cover

Let X be a topological space. A **covering space** of X is a space $\tilde{X}$ together with a continuous surjective map $f : \tilde{X} \to X$, such that for every $\mathbf{x} \in X$ there exists an open neighborhood U of $\mathbf{x}$ such that $f^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$ each of which is mapped homeomorphically onto U by f. In particular, if for every $\mathbf{x} \in X$ the fiber $f^{-1}(\mathbf{x})$ has two elements, we speak of a **double cover**.

The following proposition relates the Betti numbers (with $\mathbb{Z}_2$ coefficients) of a finite simplicial complex to its double cover. Note that the proposition is no longer true for Betti numbers (with $\mathbb{Q}$-coefficients). A simple counterexample is provided by the 2-torus which is a double cover of the Klein bottle, for which the stated inequality is not true for $i = 2$ for Betti numbers (with $\mathbb{Q}$-coefficients).

Proposition 2.24. *Let X be a finite simplicial complex and $f : \tilde{X} \to X$ a double cover of X. Then for each $i \geq 0$,*

$$b_i(\tilde{X}, \mathbb{Z}_2) \leq 2b_i(X, \mathbb{Z}_2).$$

Proof. Let

$$\phi_\bullet : \mathrm{C}_\bullet(X, \mathbb{Z}_2) \longrightarrow \mathrm{C}_\bullet(\tilde{X}, \mathbb{Z}_2)$$

denote the chain map sending each simplex of X to the sum of its two preimages in $\tilde{X}$. Let

$$\psi_\bullet : \mathrm{C}_\bullet(\tilde{X}, \mathbb{Z}_2) \longrightarrow \mathrm{C}_\bullet(X, \mathbb{Z}_2)$$

be the chain map induced by the covering map f.

It is an easy exercise to check that the following sequence is exact,

$$0 \longrightarrow \mathrm{C}_\bullet(X, \mathbb{Z}_2) \xrightarrow{\phi_\bullet} \mathrm{C}_\bullet(\tilde{X}, \mathbb{Z}_2) \xrightarrow{\psi_\bullet} \mathrm{C}_\bullet(X, \mathbb{Z}_2) \longrightarrow 0.$$

The corresponding long exact sequence in homology,

$$\cdots \longrightarrow \mathrm{H}_i(X, \mathbb{Z}_2) \longrightarrow \mathrm{H}_i(\tilde{X}, \mathbb{Z}_2) \longrightarrow \mathrm{H}_i(X, \mathbb{Z}_2) \longrightarrow \cdots$$

gives the required inequality. □

Remark 2.25. The above proof is due to Michel Coste.

2.2.6 The Betti Numbers of a Projection

The following proposition gives a bound on the Betti numbers of the projection $\pi(S)$ of a closed and bounded semi-algebraic set S in terms of the number and degrees of polynomials defining S.

Proposition 2.26 ([50])**.** *Let R be a real closed field and let $\pi : \mathrm{R}^{m+k} \to \mathrm{R}^k$ be the projection map on to last k co-ordinates. Let $S \subset \mathrm{R}^{m+k}$ be a closed and bounded semi-algebraic set defined by a Boolean formula with s distinct polynomials of degrees not exceeding d. Then the n-th Betti number of the projection*

$$\mathrm{b}_n(\pi(S)) \leq (mnd)^{O(k+nm)}.$$

Proof. See [50]. □

2.2.7 The Smale-Vietoris Theorem

In Chapter 4 we also need the following version of the well-known Smale-Vietoris Theorem [86].

Theorem 2.27 ([86])**.** *Let S and T be closed and bounded semi-algebraic sets, and $f : S \to T$ a continuous semi-algebraic map such that $f^{-1}(\mathbf{y})$ is contractible for every $\mathbf{y} \in T$. Then the map f is a homotopy equivalence.*

2.2.8 Stable homotopy equivalence and Spanier-Whitehead duality

For any finite CW-complex X we will denote by $\mathbf{S}(X)$ the suspension of X, which is the quotient of $X \times [0,1]$ by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.

Recall from [87] that for two finite CW-complexes X and Y, an element of

$$\{X;Y\} = \varinjlim_{i} [\mathbf{S}^i(X), \mathbf{S}^i(Y)] \tag{2.4}$$

is called an *S-map* (or map in the *suspension category*). (When the context is clear we will sometime denote an S-map $f \in \{X;Y\}$ by $f : X \to Y$).

Definition 2.28. An S-map $f \in \{X;Y\}$ is an **S-equivalence** (also called a **stable homotopy equivalence**) if it admits an inverse $f^{-1} \in \{Y;X\}$. In this case we say that X and Y are **stable homotopy equivalent**.

If $f \in \{X;Y\}$ is an S-map, then f induces a homomorphism,

$$f_* : \mathrm{H}_*(X) \to \mathrm{H}_*(Y).$$

The following theorem characterizes stable homotopy equivalence in terms of homology.

Theorem 2.29. *[88] Let X and Y be two finite CW-complexes. Then X and Y are stable homotopy equivalent if and only if there exists an S-map $f \in \{X;Y\}$ which induces isomorphisms $f_* : \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ (see [42], pp. 604) for all $i \geq 0$.*

In order to compare the complements of closed and bounded semi-algebraic sets which are homotopy equivalent, we will use the duality theory due to Spanier and Whitehead [87]. We will need the following facts about Spanier-Whitehead duality (see [42], pp. 603 for more details). Let $X \subset \mathbf{S}^n$ be a finite CW-complex. Then there exists a dual complex, denoted $D_n X \subset \mathbf{S}^n \setminus X$. The dual complex $D_n X$ is defined only upto S-equivalence. In particular, any deformation retract of $\mathbf{S}^n \setminus X$ represents $D_n X$. Moreover, the functor D_n has the following property. If $Y \subset \mathbf{S}^n$ is another finite CW-complex, and the S-map represented by $\phi : X \to Y$ is a stable homotopy equivalence, then there exists a stable homotopy equivalence $D_n \phi$. Moreover, if the map $\phi : X \to Y$ is an inclusion, then the dual S-map $D_n \phi$ is also represented by a corresponding inclusion.

Remark 2.30. Note that, since Spanier-Whitehead duality theory deals only with finite polyhedra over $\mathbb{R}$, it extends without difficulty to general real closed fields using the Tarski-Seidenberg transfer principle.

2.2.9 Homotopy colimits

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$, where each A_i is a sub-complex of a finite CW-complex.

Let $\Delta_{[n]}$ denote the standard simplex of dimension $n-1$ with vertices in $[n]$. For $I \subset [n]$, we denote by Δ_I the $(\#I - 1)$-dimensional face of $\Delta_{[n]}$ corresponding to I, and by A_I the CW-complex $\bigcap_{i \in I} A_i$.

The homotopy colimit, hocolim$(\mathcal{A})$, is a CW-complex defined as follows.

Definition 2.31.

$$\mathrm{hocolim}(\mathcal{A}) = \bigcupdot_{I \subset [n]} \Delta_I \times A_I / \sim$$

where the equivalence relation $\sim$ is defined as follows.

For $I \subset J \subset [n]$, let $s_{I,J} : \Delta_I \hookrightarrow \Delta_J$ denote the inclusion map of the face Δ_I in Δ_J, and let $i_{I,J} : A_J \hookrightarrow A_I$ denote the inclusion map of A_J in A_I.

Given $(\mathbf{s}, \mathbf{x}) \in \Delta_I \times A_I$ and $(\mathbf{t}, \mathbf{y}) \in \Delta_J \times A_J$ with $I \subset J$, then $(\mathbf{s}, \mathbf{x}) \sim (\mathbf{t}, \mathbf{y})$ if and only if $\mathbf{t} = s_{I,J}(\mathbf{s})$ and $\mathbf{x} = i_{I,J}(\mathbf{y})$.

We have a obvious map

$$f : \mathrm{hocolim}(\mathcal{A}) \longrightarrow \mathrm{colim}(\mathcal{A}) = \bigcup_{i \in [n]} A_i$$

sending $(\mathbf{s}, \mathbf{x}) \mapsto \mathbf{x}$. It is a consequence of the Smale-Vietoris theorem (see Theorem **??**) that

Lemma 2.32. *The map*

$$f : \mathrm{hocolim}(\mathcal{A}) \longrightarrow \mathrm{colim}(\mathcal{A}) = \bigcup_{i \in [n]} A_i$$

is a homotopy equivalence.

Now let $\mathcal{A} = \{A_1, \ldots, A_n\}$ (resp. $\mathcal{B} = \{B_1, \ldots, B_n\}$) be a set of sub-complexes of a finite CW-complex. For each $I \subset [n]$ let $f_I \in \{A_I; B_I\}$ be a stable homotopy equivalence, having the property that for each $I \subset J \subset [n]$, $f_J = f_I|_{A_J}$. Then we have an induced S-map, $f \in \{\mathrm{hocolim}(\mathcal{A}); \mathrm{hocolim}(\mathcal{B})\}$, and we have that

Lemma 2.33. *The induced S-map* $f \in \{\mathrm{hocolim}(\mathcal{A}); \mathrm{hocolim}(\mathcal{B})\}$ *is a stable homotopy equivalence.*

Proof. Using the Mayer-Vietoris exact sequence it is easy to see that if the f_I's induce isomorphisms in homology, so does the map f. Now apply Theorem 2.29. □

2.3 The Topology of Algebraic and Semi-Algbraic Sets

2.3.1 Bounds on the Topology of Semi-Algebraic Sets

The initial result on bounding the Betti numbers of semi-algebraic sets defined by polynomial inequalities was proved independently by Oleinik and Petrovskii [76], Thom [92] and Milnor [74]. They proved:

Theorem 2.34. *[76, 92, 74] Let*

$$\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[X_1, \ldots, X_k]$$

with $\deg(P_i) \leq d$*,* $1 \leq i \leq m$ *and let* $S \subset \mathrm{R}^k$ *be the set defined by*

$$P_1 \geq 0, \ldots, P_m \geq 0.$$

Then

$$b(S) = O(md)^k.$$

Notice that the theorem includes the case where the set S is a real algebraic set. Moreover, the above bound is exponential in k and this exponential dependence is unavoidable (see Example 2.35 below). Recently, the above bound was extended to more general classes of semi-algebraic sets. For example, Basu [11] improved the bound of the individual Betti numbers of $\mathcal{P}$-closed semi-algebraic sets while Gabrielov and Vorobjov [51] extended the above bound to any $\mathcal{P}$-semi-algebraic set. They proved a bound of $O(m^2d)^k$. Moreover, Basu, Pollack and Roy [19] proved a similar bound for the individual Betti numbers of the realizations of sign conditions.

Example 2.35. The set $S \subset \mathrm{R}^k$ defined by

$$X_1(X_1 - 1) \geq 0, \ldots, X_k(X_k - 1) \geq 0,$$

has $b_0(S) = 2^k$.

However, it turns out that for a semi-algebraic set $S \subset \mathrm{R}^k$ defined by m quadratic inequalities, it is possible to obtain upper bounds on the Betti numbers of S which are polynomial in k and exponential only in m. The first such result was proved by Barvinok who proved the following theorem.

Theorem 2.36. *[9] Let $S \subset \mathrm{R}^k$ be defined by*

$$P_1 \geq 0, \ldots, P_m \geq 0,$$

with $\deg(P_i) \leq 2, 1 \leq i \leq m$. *Then,* $b(S) \leq k^{O(m)}$.

Theorem 2.36 is proved using a duality argument that interchanges the roles of k and m, and reduces the original problem to that of bounding the Betti numbers of a semi-algebraic set in R^s defined by $k^{O(1)}$ polynomials of degree at most k. One can then use Theorem 2.34 to obtain a bound of $k^{O(m)}$. The constant hidden in the exponent of the above bound is at least two. Also, the bound in Theorem 2.36 is polynomial in k but exponential in m. The exponential dependence on m is unavoidable as remarked in [9], but the implied constant (which is at least two) in the exponent of Barvinok's bound is not optimal.

Using Barvinok's result, as well as inequalities derived from the Mayer-Vietoris sequence, Basu proved a polynomial bound (polynomial both in k and m) on the top few Betti numbers of a set defined by quadratic inequalities. More precisely, he proved the following theorem.

Theorem 2.37. *[11] Let $\ell > 0$ and let $S \subset \mathrm{R}^k$ be defined by*

$$P_1 \geq 0, \ldots, P_m \geq 0,$$

with $\deg(P_i) \leq 2$. *Then*

$$b_{k-\ell}(S) \leq \binom{m}{\ell} k^{O(\ell)}.$$

Notice that for fixed ℓ, the bound in Theorem 2.37 is polynomial in both m and k.

2.3.2 Bounds on the Topology of Complex Algebraic Sets

By separating the real and imaginary parts one can consider a complex algebraic set $X \subset \mathrm{C}^k$ as a real algebraic subset of R^{2k}. Unfortunately, real and complex algebraic sets do not have the same properties. To be more precise, an irreducible algebraic subset of C^k having complex dimension n, considered as an algebraic subset of R^{2k} is connected, not bounded (unless it is a point) and has local real dimension $2n$ at every point (see, for instance, [29]). But this is no longer true for real algebraic sets as we will see in the following examples.

Example 2.38 ([29]).

1. The circle $\{(\mathbf{x}, \mathbf{y}) \in \mathrm{R}^2 \mid \mathbf{x}^2 + \mathbf{y}^2 = 1\}$ is bounded.
2. The cubic curve $\{(\mathbf{x}, \mathbf{y}) \in \mathrm{R}^2 \mid \mathbf{x}^2 + \mathbf{y}^2 - \mathbf{x}^3 = 0\}$ has an isolated point at the origin.

However, in Chapter 3 we will show how to reduce the problem of bounding the Betti numbers of a real algebraic set to the problem of bounding the Betti numbers of a complex projective algebraic set involving the same polynomials. Moreover, this complex projective algebraic set will have the property that is a non-singular complete intersection, which we define next.

Definition 2.39. A projective algebraic set $X \subset \mathbb{P}^k_{\mathrm{C}}$ of codimension n is a **non-singular complete intersection** if it is the intersection of n non-singular hypersurfaces in $\mathbb{P}^k_{\mathrm{C}}$ that meet transversally at each point of the intersection.

Next, we recall some results about the Betti numbers of a complex projective algebraic set which is a non-singular complete intersection. We need the following notation.

Fix a j-tuple of natural numbers $\bar{d} = (d_1, \ldots, d_j)$. Let $X_{\mathrm{C}} = \mathrm{Zer}(\{Q_1, \ldots, Q_j\}, \mathbb{P}^k_{\mathrm{C}})$, such that the degree of Q_i is d_i, denote a complex projective algebraic set of codimension j which is a non-singular complete intersection.

Let $b(j, k, \bar{d})$ denote the sum of the Betti numbers with $\mathbb{Z}_2$ coefficients of X_{C}. This is well defined since the Betti numbers only depend only on the degree sequence and not on the specific X_{C} (see, for instance, [47]).

The function $b(j, k, \bar{d})$ satisfies the following (see [26]):

$$b(j,k,\bar{d}) = \begin{cases} c(j,k,\bar{d}) & \text{if } k-j \text{ is even,} \\ 2(k-j+1) - c(j,k,\bar{d}) & \text{if } k-j \text{ is odd,} \end{cases}$$

where

$$c(j,k,\bar{d}) = \begin{cases} k+1 & \text{if } j = 0, \\ d_1 \ldots d_j & \text{if } j = k, \\ d_k c(j-1, k-1, (d_1, \ldots, d_{k-1})) - (d_k - 1) c(j, k-1, \bar{d}) & \text{if } 0 < j < k. \end{cases}$$

In the special case when each $d_i = 2$, we denote by $b(j,k) = b(j,k,(2,\ldots,2))$. We then have the following recurrence for $b(j,k)$.

$$b(j,k) = \begin{cases} q(j,k) & \text{if } k-j \text{ is even,} \\ 2(k-j+1) - q(j,k) & \text{if } k-j \text{ is odd,} \end{cases}$$

where

$$q(j,k) = \begin{cases} k+1 & \text{if } j = 0, \\ 2^j & \text{if } j = k, \\ 2q(j-1,k-1) - q(j,k-1) & \text{if } 0 < j < k. \end{cases}$$

Next, we show some properties of $q(j,k)$.

Lemma 2.40. *1. $q(1,k) = k + 1/2(1-(-1)^k)$ and $q(2,k) = (-1)^k k + k$.*

2. *For* $2 \le j \le k$, $|q(j,k)| \le 2^{j-1}\binom{k}{j-1}$.

3. *For* $2 \le j \le k$ *and* $k-j$ *odd,* $2(k-j+1) - q(j,k) \le 2^{j-1}\binom{k}{j-1}$.

Proof. The first part is shown by two easy computations and noting that

$$2(k-2+1) - q(2,k) = 2k-2 \text{ if } k-2 \text{ is odd.}$$

Hence, we can assume that the statements are true for $k-1$ and that $3 \le j < k$. Note that for the special case $j = k-1$, we have that $2^{k-1} \le 2^{k-2}\binom{k-1}{k-2}$ since $k > 2$. Then

$$\begin{aligned}
|q(j,k)| &= |2q(j-1,k-1) - q(j,k-1)| \\
&\le 2|q(j-1,k-1)| + |q(j,k-1)| \\
&\le 2 \cdot 2^{j-2}\binom{k-1}{j-2} + 2^{j-1}\binom{k-1}{j-1} \\
&= 2^{j-1}\binom{k}{j-1}.
\end{aligned}$$

and, for $k-j$ odd,

$$\begin{aligned}
2(k-j+1) - q(j,k) &= 2(k-j+1) - 2q(j-1,k-1) + q(j,k-1) \\
&\le |2((k-1)-(j-1)+1) - q(j-1,k-1)| \\
&\quad +|q(j-1,k-1)| + |q(j,k-1)| \\
&\le 2^{j-2}\binom{k-1}{j-2} + 2^{j-2}\binom{k-1}{j-2} + 2^{j-1}\binom{k-1}{j-1} \\
&\le 2^{j-1}\left(\binom{k-1}{j-2} + \binom{k-1}{j-1}\right) = 2^{j-1}\binom{k}{j-1}.
\end{aligned}$$

□

Hence, we get the following bound for $b(j,k)$.

Theorem 2.41.

1. $b(1,k) = \begin{cases} q(0,k-1) & \textit{if } k \textit{ is even,} \\ q(0,k) & \textit{if } k \textit{ is odd,} \end{cases}$

2. $b(j,k) \le 2^{j-1}\binom{k}{j-1}$, *for* $2 \le j \le k$.

Proof. Follows from Lemma 2.40. □

2.3.3 Bounds on the Topology of Parametrized Semi-algebraic Sets

Let $\pi : \mathrm{R}^{\ell+k} \to \mathrm{R}^k$ be the projection map on the last k co-ordinates, and for any $S \subset \mathrm{R}^{\ell+k}$ we will denote by π_S the restriction of π to S. Moreover, when the map π is clear from context, for any $\mathbf{x} \in \mathrm{R}^k$ we will denote by $S_{\mathbf{x}}$ the fiber $\pi^{-1}(\mathbf{x}) \cap S$. One way to interpret this setting is that the set S depends on k parameters and π is the projection onto the parameter space.

Hardt's triviality theorem (see Theorem 2.15) implies that there exists a semi-algebraic partition $\{T_i\}_{i \in I}$ of R^k having the following property. For each $i \in I$ and any point $\mathbf{x} \in T_i$, the pre-image $\pi^{-1}(T_i) \cap S$ is semi-algebraically homeomorphic to $S_{\mathbf{x}} \times T_i$ by a fiber preserving homeomorphism. In particular, for each $i \in I$, all fibers $S_{\mathbf{x}}$, $\mathbf{x} \in T_i$ are semi-algebraically homeomorphic.

As mentioned in Chapter 2.1.6 the existence of cylindrical decompositions implies a double exponential (in k and ℓ) upper bound on the cardinality of I and, hence, on the number of homeomorphism types of the fibers of the map π_S. No better bounds than the double exponential bound are known, even though it seems reasonable to conjecture a single exponential upper bound on the number of homeomorphism types of the fibers of the map π_S.

In [22], Basu and Vorobjov considered the weaker problem of bounding the number of distinct homotopy types occurring amongst the set of all fibers of $S_{\mathbf{x}}$, and they proved a single exponential upper bound (in k and ℓ) on the number of homotopy types of such fibers.

They proved the following theorem.

Theorem 2.42. *[22] Let $\mathcal{P} \subset \mathrm{R}[Y_1, \ldots, Y_\ell, X_1, \ldots, X_k]$, with $\deg(P) \le d$ for each $P \in \mathcal{P}$ and cardinality $\#\mathcal{P} = m$. Then there exists a finite set $A \subset \mathrm{R}^k$ with*

$$\#A \le (2^\ell mkd)^{O(k\ell)}$$

such that for every $\mathbf{x} \in \mathrm{R}^k$, there exists $\mathbf{z} \in A$ such that for every $\mathcal{P}$-semi-algebraic set $S \subset \mathrm{R}^{\ell+k}$, the set $S_{\mathbf{x}}$ is semi-algebraically homotopy equivalent to $S_{\mathbf{z}}$. In particular, for any fixed $\mathcal{P}$-semi-algebraic set S, the number of different homotopy types of fibers $S_{\mathbf{x}}$ for *various* $\mathbf{x} \in \pi(S)$ *is also bounded by*

$$(2^\ell mkd)^{O(k\ell)}.$$

Notice that the bound in Theorem 2.42 is single exponential in $k\ell$. The following example, which also appears in [22], shows that the single exponential dependence on ℓ is unavoidable.

Example 2.43. Let $P \in \mathrm{R}[Y_1, \ldots, Y_\ell] \hookrightarrow \mathrm{R}[Y_1, \ldots, Y_\ell, X]$ be the polynomial defined by

$$P = \sum_{i=1}^{\ell} \prod_{j=0}^{d-1} (Y_i - j)^2.$$

The algebraic set defined by $P = 0$ in $\mathrm{R}^{\ell+1}$ with coordinates $Y_1, \ldots, Y_\ell, X$, consists of d^ℓ lines all parallel to the X axis. Consider now the semi-algebraic set $S \subset \mathrm{R}^{\ell+1}$ defined by

$$(P = 0) \ \wedge \ (0 \leq X \leq Y_1 + dY_2 + d^2Y_3 + \cdots + d^{\ell-1}Y_\ell).$$

It is easy to verify that, if $\pi : \mathrm{R}^{\ell+1} \to \mathrm{R}$ is the projection map on the X co-ordinate, then the fibers $S_{\mathbf{x}}$, for $\mathbf{x} \in \{0, 1, 2, \ldots, d^\ell - 1\} \subset \mathrm{R}$ are 0-dimensional and of different cardinality, and hence have different homotopy types.

2.3.4 Some Useful Constructions

In this chapter, we recall some very useful constructions for semi-algebraic subsets of R^k which are well-known in real algebraic geometry.

Let $\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[X_1, \ldots, X_k]$ with $\deg(P_i) \leq 2$, $1 \leq i \leq m$. Let $S \subset \mathrm{R}^k$ be the basic semi-algebraic set defined by

$$S = \{\mathbf{x} \in \mathrm{R}^k \mid P_1(\mathbf{x}) \geq 0, \ldots, P_m(\mathbf{x}) \geq 0\}.$$

Let $1 \gg \varepsilon > 0$ be an infinitesimal, and let

$$P_{m+1} = 1 - \varepsilon^2 \sum_{i=1}^{k} X_i^2.$$

Let $S_b \subset \mathrm{R}\langle\varepsilon\rangle^k$ be the basic semi-algebraic set defined by

$$S_b = \{\mathbf{x} \in \mathrm{R}\langle\varepsilon\rangle^k \mid P_1(\mathbf{x}) \geq 0, \ldots, P_m(\mathbf{x}) \geq 0, P_{m+1}(\mathbf{x}) \geq 0\}.$$

Proposition 2.44. *The bounded set S_b and the set* $\mathrm{Ext}(S, \mathrm{R}\langle\varepsilon\rangle)$ *are homotopy equivalent. Moreover, the homology groups of the S_b and S are isomorphic.*

Proof. It follows from Proposition 2.16 (Conic structure at infinity) that the semi-algebraic set S_b has the same homotopy type as $\mathrm{Ext}(S, \mathrm{R}\langle\varepsilon\rangle)$. The claim now follows since one can extend any triangulation over R to a triangulation over $\mathrm{R}\langle\varepsilon\rangle$. □

Let $S^h \subset \mathbf{S}^k$ be the basic semi-algebraic set defined by

$$S^h = \{\mathbf{x} \in \mathrm{R}\langle\varepsilon\rangle^{k+1} \mid ||\mathbf{x}|| = 1,\ P_1^h(\mathbf{x}) \geq 0, \ldots, P_m^h(\mathbf{x}) \geq 0, P_{m+1}^h(\mathbf{x}) \geq 0\}.$$

Lemma 2.45. *For $0 \leq i \leq k$, we have*

$$b_i(S_b) = \frac{1}{2} b_i(S^h).$$

Proof. Note that S_b is bounded by Proposition 2.44 and S^h is the projection from the origin of the set $\{1\} \times S_b \subset \{1\} \times \mathrm{R}\langle\varepsilon\rangle^k$ onto the unit sphere $\mathbf{S}^k$ in $\mathrm{R}\langle\varepsilon\rangle^{k+1}$. Since S_b is bounded, the projection does not intersect the equator and consists of two disjoint copies (each homeomorphic to the set S_b) in the upper and lower hemispheres. □

CHAPTER III

BOUNDING THE BETTI NUMBERS

3.1 Results

We prove the following theorem.

Theorem 3.1. *[17] Let* $\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[X_1, \ldots, X_k]$, $m < k$. *Let* $S \subset \mathrm{R}^k$ *be defined by*

$$P_1 \geq 0, \ldots, P_m \geq 0$$

with $\deg(P_i) \leq 2$. *Then, for* $0 \leq i \leq k-1$,

$$b_i(S) \leq \frac{1}{2} + (k-m) + \frac{1}{2} \cdot \sum_{j=0}^{min\{m+1,k-i\}} 2^j \binom{m+1}{j} \binom{k}{j-1} \leq \frac{3}{2} \cdot \left(\frac{6ek}{m}\right)^m + k.$$

As a consequence of Theorem 3.1 we get a new bound on the sum of the Betti numbers, which we state for the sake of completeness.

Corollary 3.2. *Let* $\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[X_1, \ldots, X_k]$, $m < k$. *Let* $S \subset \mathrm{R}^k$ *be defined by*

$$P_1 \geq 0, \ldots, P_m \geq 0$$

with $\deg(P_i) \leq 2$. *Then*

$$b(S) \leq k \left(\frac{1}{2} + (k-m) + \frac{1}{2} \cdot \sum_{j=0}^{min\{m+1,k-i\}} 2^j \binom{m+1}{j} \binom{k}{j-1} \right).$$

Remark 3.3. The technique used in this chapter was proposed as a possible alternative method by Barvinok in [9], who did not pursue this further in that paper. Also, Benedetti, Loeser, and Risler [26] used a similar technique for proving upper bounds on the number of connected components of real algebraic sets in R^k defined by polynomials of degrees bounded by d. However, these bounds (unlike the bounds we obtain) are exponential in k. Finally, there exists another possible method for bounding the Betti numbers of semi-algebraic sets defined by quadratic inequalities, using a spectral sequence argument due to

Agrachev [1]. However, this method also produces a non-optimal bound of the form $k^{O(m)}$ (similar to Barvinok's bound) where the constant in the exponent is at least two. We omit the details of this argument referring the reader to [13] for an indication of the proof (where the case of computing, and as a result, bounding the Euler-Poincaré characteristics of such sets is worked out in full details).

3.2 Proof Strategy

Our strategy for proving Theorem 3.1 is as follows. Using certain infinitesimal deformations we first reduce the problem to bounding the Betti numbers of another closed and bounded semi-algebraic set defined by a new family of quadratic polynomials. We then use inequalities obtained from the Mayer-Vietoris exact sequence to further reduce the problem of bounding the Betti numbers of this new semi-algebraic set to the problem of bounding the Betti numbers of the real projective algebraic sets defined by each ℓ-tuple, $\ell \leq m$, of the new polynomials. The new family of polynomials also has the property that the complex projective algebraic set defined by each ℓ-tuple, $\ell \leq k$, of these polynomials is a non-singular complete intersection. According to Theorem 2.41 we have precise information about the Betti numbers of these complex complete intersections. An application of the Smith inequality (see Theorem 2.22) then allows us to obtain bounds on the Betti numbers of the real parts of these algebraic sets and, as a result, on the Betti numbers of the original semi-algebraic set.

3.3 Constructing Non-singular Complete Intersections

In Chapter 2.3.2 we introduced the notion of a projective complex algebraic set which is a non-singular complete intersection (see Definition 2.39). Next, we show the existence of such a set and how to obtain a non-singular complete intersection from a given algebraic set in complex projective space.

Proposition 3.4. *There exists a family* $\mathcal{H} = \{H_1, \ldots, H_m\} \subset \mathrm{R}[X_0, \ldots, X_k]$ *of positive definite quadratic forms such that* $\mathrm{Zer}(\mathcal{H}_J, \mathbb{P}^k_{\mathrm{C}})$ *is a non-singular complete intersection for every* $J \subset \{1, \ldots, m\}$.

Proof. Recall that the set of positive definite quadratic forms is open in the set of quadratic forms over R. Moreover, any real closed field contains the real closure of $\mathbb{Q}$. Thus, we can choose a family $\mathcal{H} = \{H_1, \ldots, H_m\} \subset \mathrm{R}[X_0, \ldots, X_k]$ of positive definite quadratic forms such that their coefficients are algebraically independent over $\mathbb{Q}$. It follows by Bertini's Theorem (see [57], Theorem 17.16) that $\mathrm{Zer}(\mathcal{H}_J, \mathbb{P}^{k+1}_{\mathrm{C}})$, $J \subset \{1, \ldots, m\}$, is a non-singular complete intersection. $\square$

The following proposition allows us to replace a family of real quadratic forms by another family obtained by infinitesimal perturbations of the original family and whose zero sets are non-singular complete intersections in complex projective space.

Proposition 3.5. *Let*

$$\mathcal{Q} = \{Q_1, \ldots, Q_m\} \subset \mathrm{R}[X_0, \ldots, X_k]$$

be a set of quadratic forms and let

$$\mathcal{H} = \{H_1, \ldots, H_m\} \subset \mathrm{R}[X_0, \ldots, X_k]$$

be a family of positive definite quadratic forms such that $\mathrm{Zer}(\mathcal{H}, \mathbb{P}^k_{\mathrm{C}})$ *is a non-singular complete intersection for every* $J \subset \{1, \ldots, m\}$.

Let $1 \gg \delta > 0$ *be infinitesimals, and let*

$$\tilde{\mathcal{Q}} = \{\tilde{Q}_1, \ldots, \tilde{Q}_m\} \textit{ with}$$
$$\tilde{Q}_i = (1 - \delta)Q_i + \delta H_i.$$

Then for any $J \subset \{1, \ldots, m\}$,

$$\mathrm{Zer}(\tilde{\mathcal{Q}}_J, \mathbb{P}^k_{\mathrm{C}\langle\delta\rangle})$$

is a non-singular complete intersection.

Proof. Consider

$$\tilde{\mathcal{Q}}_t = \{\tilde{Q}_{t,1}, \ldots, \tilde{Q}_{t,m}\} \text{ with}$$
$$\tilde{Q}_{t,i} = (1 - t)Q_i + tH_i.$$

Let $J \subset \{1, \dots, m\}$, and let $T_J \subset \mathrm{C}$ be defined by,

$$T_J = \{t \in \mathrm{C} \mid \mathrm{Zer}(\tilde{\mathcal{Q}}_{t,J}, \mathbb{P}^k_{\mathrm{C}}) \text{ is a non-singular complete intersection } \}.$$

Clearly, T_J contains 1. Moreover, since being a non-singular complete intersection is a stable condition, T_J must contain an open neighborhood of 1 in C and so must $T = \cap_{J \subset \{1,\dots,m\}} T_J$. Finally, the set T is constructible, since it can be defined by a first order formula. Since a constructible subset of C is either finite or the complement of a finite set (see for instance, [19], Corollary 1.25), T must contain an interval $(0, t_0), t_0 > 0$. Hence, its extension to $\mathrm{C}\langle\delta\rangle$ contains δ. □

3.4 Proof of Theorem 3.1

Before we prove Theorem 3.1, we need what follows next:
Let $\mathcal{P} = \{P_1, \dots, P_m\} \subset \mathrm{R}[X_1, \dots, X_k]$, $m < k$, with $\deg(P_i) \le 2$, $1 \le i \le m$. Let $S \subset \mathrm{R}^k$ be the basic semi-algebraic set defined by

$$S = \{\mathbf{x} \in \mathrm{R}^k \mid P_1(\mathbf{x}) \ge 0, \dots, P_m(\mathbf{x}) \ge 0\}.$$

Let $1 \gg \varepsilon \gg \delta > 0$ be infinitesimals, and let

$$P_{m+1} = 1 - \varepsilon^2 \sum_{i=1}^{k} X_i^2.$$

Let $S_b \subset \mathrm{R}\langle\varepsilon\rangle^k$ be the basic semi-algebraic set defined by

$$S_b = \{\mathbf{x} \in \mathrm{R}\langle\varepsilon\rangle^k \mid P_1(\mathbf{x}) \ge 0, \dots, P_m(\mathbf{x}) \ge 0, P_{m+1}(\mathbf{x}) \ge 0\}.$$

The homology groups of S and S_b are isomorphic by Proposition 2.44. Moreover, the set S_b is bounded.

Let $S^h \subset \mathbf{S}^k$ be the basic semi-algebraic set defined by

$$S^h = \{\mathbf{x} \in \mathrm{R}\langle\varepsilon\rangle^{k+1} \mid |\mathbf{x}| = 1,\ P_1^h(\mathbf{x}) \ge 0, \dots, P_m^h(\mathbf{x}) \ge 0, P_{m+1}^h(\mathbf{x}) \ge 0\}.$$

Then, for $0 \le i \le k$, we have

$$b_i(S_b, \mathbb{Z}_2) = \frac{1}{2} b_i(S^h, \mathbb{Z}_2).$$

by Lemma 2.45.

We now fix a family of polynomials that will be useful in what follows. By Proposition 3.4 we can choose a family $\mathcal{H} = \{H_1, \ldots, H_{m+1}\} \subset \mathrm{R}[X_0, \ldots, X_k]$ of positive definite quadratic forms such that $\mathrm{Zer}(\mathcal{H}_J, \mathbb{P}^k_{\mathrm{C}\langle\varepsilon\rangle})$ is a non-singular complete intersection for every $J \subset \{1, \ldots, m+1\}$.

Let $\tilde{P}_i = (1-\delta)P_i^h + \delta H_i$, $1 \le i \le m+1$. Let T (resp., $\bar{T}$) be the basic semi-algebraic set defined by

$$T = \{\mathbf{x} \in \mathrm{R}\langle\varepsilon,\delta\rangle^{k+1} \mid \ ||\mathbf{x}|| = 1,\ \tilde{P}_1(\mathbf{x}) > 0, \ldots, \tilde{P}_m(\mathbf{x}) > 0, , \tilde{P}_{m+1}(\mathbf{x}) > 0\}$$

and

$$\bar{T} = \{\mathbf{x} \in \mathrm{R}\langle\varepsilon,\delta\rangle^{k+1} \mid \ ||\mathbf{x}|| = 1,\ \tilde{P}_1(\mathbf{x}) \ge 0, \ldots, \tilde{P}_m(\mathbf{x}) \ge 0, \tilde{P}_{m+1}(\mathbf{x}) \ge 0\},$$

respectively.

Also, let

$$\tilde{\mathcal{P}} = \{\tilde{P}_1, \ldots, \tilde{P}_m, \tilde{P}_{m+1}\}.$$

Lemma 3.6. *We have,*

1. *the homology groups of S^h and $\bar{T}$ are isomorphic,*

2. *the homology groups of T and $\bar{T}$ are isomorphic,*

3. *for all $J \subset \{1, \ldots, m+1\}$,*
   $\mathrm{Zer}(\tilde{P}_J, \mathbb{P}^k_{\mathrm{C}\langle\varepsilon,\delta\rangle})$ *is a non-singular complete intersection, and*

4. *for all $J \subset \{1, \ldots, m+1\}$,*
   $b_i\left(\mathrm{Zer}(\tilde{\mathcal{P}}_J, \mathrm{Ext}(\mathbf{S}^k, \mathrm{R}\langle\varepsilon,\delta\rangle), \mathbb{Z}_2\right) \le 2b_i\left(\mathrm{Zer}(\tilde{\mathcal{P}}_J, \mathbb{P}^k_{\mathrm{R}\langle\varepsilon,\delta\rangle}), \mathbb{Z}_2\right).$

Proof. For the first part note that the sets $\mathrm{Ext}(S^h, \mathrm{R}\langle\varepsilon,\delta\rangle)$ and $\bar{T}$ have the same homotopy type using Lemma 16.17 in [20].

The second part is clear since we have a retraction from T to $\bar{T}$.

The third part follows from Proposition 3.5.

For the last part, let $\pi : \mathrm{Ext}(\mathbf{S}^k, \mathrm{R}\langle\varepsilon,\delta\rangle) \to \mathbb{P}^k_{\mathrm{R}\langle\varepsilon,\delta\rangle}$ be the double cover obtained by identifying antipodal points. Then the restriction of π to $\mathrm{Zer}(\tilde{\mathcal{P}}_J, \mathrm{Ext}(\mathbf{S}^k, \mathrm{R}\langle\varepsilon,\delta\rangle))$ gives a double cover,

$$\pi : \mathrm{Zer}(\tilde{\mathcal{P}}_J, \mathrm{Ext}(\mathbf{S}^k, \mathrm{R}\langle\varepsilon,\delta\rangle)) \to \mathrm{Zer}(\tilde{P}_J, \mathbb{P}^k_{\mathrm{R}\langle\varepsilon,\delta\rangle}).$$

Now apply Proposition 2.24. □

Proposition 3.7. *For $0 \le i \le k-1$, we have*

$$b_i(T, \mathbb{Z}_2) \le 1 + 2(k-m) + \sum_{j=0}^{min\{m+1,k-i\}} 2^j \binom{m+1}{j}\binom{k}{j-1}.$$

Proof. First note that by Lemma 3.6 (3) $\mathrm{Zer}(\tilde{\mathcal{P}}_J, \mathbb{P}^k_{\mathrm{C}\langle\varepsilon,\delta\rangle})$ is a complete intersection for all $J \subset \{1,\ldots,m+1\}$. For $0 \le i \le k-1$, we have

$$\begin{aligned} b_i(T,\mathbb{Z}_2) &\le b_i\left(\mathrm{Ext}(\mathbf{S}^k, \mathrm{R}\langle\varepsilon,\delta\rangle) \setminus \bigcup_{i=1}^{m+1} \mathrm{Zer}(\tilde{P}_i, \mathrm{Ext}(\mathbf{S}^k, \mathrm{R}\langle\varepsilon,\delta\rangle)), \mathbb{Z}_2\right) \\ &\le 1 + b_{k-1-i}\left(\bigcup_{i=1}^{m+1} \mathrm{Zer}(\tilde{P}_i, \mathrm{Ext}(\mathbf{S}^k, \mathrm{R}\langle\varepsilon,\delta\rangle)), \mathbb{Z}_2\right), \end{aligned}$$

where the first inequality is a consequence of the fact that, T is an open subset of

$$\mathrm{Ext}(\mathbf{S}^k, \mathrm{R}\langle\varepsilon,\delta\rangle) \setminus \bigcup_{i=1}^{m+1} \mathrm{Zer}(\tilde{P}_i, \mathrm{Ext}(\mathbf{S}^k, \mathrm{R}\langle\varepsilon,\delta\rangle))$$

and disconnected from its complement in $\mathrm{Ext}(\mathbf{S}^k, \mathrm{R}\langle\varepsilon,\delta\rangle) \setminus \bigcup_{i=1}^{m+1} \mathrm{Zer}(\tilde{P}_i, \mathrm{Ext}(\mathbf{S}^k, \mathrm{R}\langle\varepsilon,\delta\rangle))$, and the last inequality follows from Theorem 2.23 (Alexander Duality).

It follows from Proposition 2.19 (2.2), Lemma 3.6 (4) and Theorem 2.22 (Smith inequality) that

$$\begin{aligned} b_i(T,\mathbb{Z}_2) &\le 1 + \sum_{j=1}^{k-i} \sum_{|J|=j} b_{k-i-j}\left(\mathrm{Zer}(\tilde{\mathcal{P}}_J, \mathrm{Ext}(\mathbf{S}^k, \mathrm{R}\langle\varepsilon,\delta\rangle)), \mathbb{Z}_2\right) \\ &\le 1 + 2\cdot \sum_{j=1}^{k-i} \sum_{|J|=j} b_{k-i-j}\left(\mathrm{Zer}(\tilde{\mathcal{P}}_J, \mathbb{P}^k_{\mathrm{R}\langle\varepsilon,\delta\rangle}), \mathbb{Z}_2\right) \\ &\le 1 + 2\cdot \sum_{j=1}^{min\{m+1,k-i\}} \sum_{|J|=j} b\left(\mathrm{Zer}(\tilde{\mathcal{P}}_J, \mathbb{P}^k_{\mathrm{C}\langle\varepsilon,\delta\rangle}), \mathbb{Z}_2\right). \end{aligned}$$

Note that for $j \le m+1$ the number of possible j-ary intersections is equal to $\binom{m+1}{j}$ and using Theorem 2.41, we conclude

$$\begin{aligned}
b_i(T,\mathbb{Z}_2) &\leq 1 + 2 \cdot \sum_{j=1}^{min\{m+1,k-i\}} \binom{m+1}{j} b(j,k) \\
&\leq 1 + 2(k+1) + 2 \cdot \sum_{j=2}^{min\{m+1,k-i\}} \binom{m+1}{j} 2^{j-1} \binom{k}{j-1} \\
&= 1 + 2(k+1) + \sum_{j=2}^{min\{m+1,k-i\}} 2^j \binom{m+1}{j} \binom{k}{j-1} \\
&= 1 + 2(k+1) - 2(m+1) + \sum_{j=0}^{min\{m+1,k-i\}} 2^j \binom{m+1}{j} \binom{k}{j-1} \\
&= 1 + 2(k-m) + \sum_{j=0}^{min\{m+1,k-i\}} 2^j \binom{m+1}{j} \binom{k}{j-1}.
\end{aligned}$$

□

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. It follows from the Universal Coefficients Theorem (see [58], Corollary 3.A6 (b)), that $b_i(S) \leq b_i(S,\mathbb{Z}_2)$. We have by Lemma 3.6 that the homology groups (with $\mathbb{Z}_2$ coefficients) of S^h and T are isomorphic. Moreover $b_i(S,\mathbb{Z}_2) = \frac{1}{2} b_i(S^h,\mathbb{Z}_2)$, for $0 \leq i \leq k-1$, by Proposition 2.44 and Lemma 2.45. Hence, the first inequality follows from Proposition 3.7.

The second inequality follows from an easy computation. □

CHAPTER IV

BOUNDING THE NUMBER OF HOMOTOPY TYPES

4.1 Result

We prove the following theorem.

Theorem 4.1. *[16] Let* R *be a real closed field and let*

$$\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[Y_1, \ldots, Y_\ell, X_1, \ldots, X_k],$$

with $\deg_Y(P_i) \leq 2, \deg_X(P_i) \leq d, 1 \leq i \leq m$. *Let* $\pi : \mathrm{R}^{\ell+k} \to \mathrm{R}^k$ *be the projection on the last* k *co-ordinates. Then for any* $\mathcal{P}$*-closed semi-algebraic set* $S \subset \mathrm{R}^{\ell+k}$*, the number of stable homotopy types (see Definition 2.28) amongst the fibers,* $S_{\mathbf{x}}$*, is bounded by* $(2^m \ell k d)^{O(mk)}$.

Remark 4.2.

1. The bound in Theorem 4.1 (unlike the one in Theorem 2.42) is polynomial in ℓ for fixed m and k. The exponential dependence on m is unavoidable, as can be seen from a slight modification of Example 2.43. Consider the semi-algebaic set $S \subset \mathrm{R}^{\ell+1}$ defined by

$$Y_i(Y_i - 1) = 0, \ 1 \leq i \leq m \leq \ell,$$
$$0 \leq X \leq Y_1 + 2 \cdot Y_2 + \ldots + 2^{m-1} \cdot Y_m.$$

   Let $\pi : \mathrm{R}^{\ell+1} \to \mathrm{R}$ be the projection on the X-coordinate. Then, the sets $S_{\mathbf{x}}$, $\mathbf{x} \in \{0, 1 \ldots, 2^{m-1}\}$, have different number of connected components, and hence have distinct (stable) homotopy types.

2. The technique used to prove Theorem 2.42 in [22] does not directly produce better bounds in the quadratic case, and hence we need a new approach to prove a substantially better bound in this case. For technical reasons, we only obtain a bound on the number of stable homotopy types, rather than homotopy types. But note that the notions of homeomorphism type, homotopy type and stable homotopy type are each

strictly weaker than the previous one, since two semi-algebraic sets might be stable homotopy equivalent, without being homotopy equivalent (see [88], p. 462), and also homotopy equivalent without being homeomorphic. However, two closed and bounded semi-algebraic sets which are stable homotopy equivalent have isomorphic homology groups.

4.2 Proof Strategy

The strategy underlying our proof of Theorem 4.1 is as follows. We first consider the special case of a semi-algebraic subset, $A \subset \mathbf{S}^{\ell}$, defined by a disjunction of m homogeneous quadratic inequalities restricted to the unit sphere in $\mathrm{R}^{\ell+1}$. We then show that there exists a closed and bounded semi-algebraic set C' (see (4.14) below for the precise definition of the semi-algebraic set C'), consisting of certain sphere bundles, glued along certain sub-sphere bundles, which is homotopy equivalent to A. The number of these sphere bundles, as well descriptions of their bases, are bounded polynomially in ℓ (for fixed m).

In the presence of parameters $X_1, \ldots, X_k$, the set A, as well as C', will depend on the values of the parameters. However, using some basic homotopy properties of bundles, we show that the homotopy type of the set C' stays invariant under continuous deformation of the bases of the different sphere bundles which constitute C'. These bases also depend on the parameters, $X_1, \ldots, X_k$, but the degrees in $X_1, \ldots, X_k$ of the polynomials defining them are bounded by $O(\ell d)$. Now, using techniques similar to those used in [22], we are able to control the number of isotopy types of the bases which occur as the parameters vary over R^k. The bound on the number of isotopy types, also gives a bound on the number of possible homotopy types of the set C', and hence of A, for different values of the parameter.

In order to prove the results for semi-algebraic sets defined by more general formulas than disjunctions of weak inequalities, we first use Spanier-Whitehead duality to obtain a bound in the case of conjunctions, and then use the construction of homotopy colimits to prove the theorem for general $\mathcal{P}$-closed sets. Because of the use of Spanier-Whitehead duality we get bounds on the number of stable homotopy types, rather than homotopy types.

4.3 Topology of Sets Defined by Quadratic Constraints

One of the main ideas behind our proof of Theorem 4.1 is to parametrize a construction introduced by Agrachev in [1] while studying the topology of sets defined by (purely) quadratic inequalities (that is without the parameters $X_1, \ldots, X_k$ in our notation). However, we avoid construction of Leray spectral sequences as was done in [1]. For the rest of this section, we fix a set of polynomials

$$\mathcal{Q} = \{Q_1, \ldots, Q_m\} \subset \mathrm{R}[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k]$$

which are homogeneous of degree 2 in $Y_0, \ldots, Y_\ell$, and of degree at most d in $X_1, \ldots, X_k$.

We will denote by

$$Q = (Q_1, \ldots, Q_m) : \mathrm{R}^{\ell+1} \times \mathrm{R}^k \to \mathrm{R}^m,$$

the map defined by the polynomials $Q_1, \ldots, Q_m$, and generally, for $I \subset \{1, \ldots, m\}$, we denote by $Q_I : \mathrm{R}^{\ell+1} \times \mathrm{R}^k \to \mathrm{R}^I$, the map whose co-ordinates are given by Q_i, $i \in I$. When $I = [m]$, we will often drop the subscript I from our notation.

For any subset $I \subset [m]$, let $A_I \subset \mathbf{S}^\ell \times \mathrm{R}^k$ be the semi-algebraic set defined by

$$A_I = \bigcup_{i \in I} \{(\mathbf{y}, \mathbf{x}) \mid |\mathbf{y}| = 1 \ \wedge \ Q_i(\mathbf{y}, \mathbf{x}) \leq 0\}, \tag{4.1}$$

and let

$$\Omega_I = \{\omega \in \mathrm{R}^m \mid |\omega| = 1, \omega_i = 0, i \notin I, \omega_i \leq 0, i \in I\}. \tag{4.2}$$

For $\omega \in \Omega_I$ we denote by $\omega Q \in \mathrm{R}[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k]$ the polynomial defined by

$$\omega Q = \sum_{i=0}^{m} \omega_i Q_i. \tag{4.3}$$

For $(\omega, \mathbf{x}) \in F_I = \Omega_I \times \mathrm{R}^k$, we will denote by $\omega Q(\cdot, \mathbf{x})$ the quadratic form in $Y_0, \ldots, Y_\ell$ obtained from ωQ by specializing $X_i = \mathbf{x}_i, 1 \leq i \leq k$.

Let $B_I \subset \Omega_I \times \mathbf{S}^\ell \times \mathrm{R}^k$ be the semi-algebraic set defined by

$$B_I = \{(\omega, \mathbf{y}, \mathbf{x}) \mid \omega \in \Omega_I, \mathbf{y} \in \mathbf{S}^\ell, \mathbf{x} \in \mathrm{R}^k, \ \omega Q(\mathbf{y}, \mathbf{x}) \geq 0\}. \tag{4.4}$$

We denote by $\phi_1 : B_I \to F_I$ and $\phi_2 : B_I \to \mathbf{S}^\ell \times \mathrm{R}^k$ the two projection maps (see diagram below).

$$\begin{tikzcd} & B_I \arrow[dl, "\phi_{I,1}"'] \arrow[d, dotted] \arrow[dr, "\phi_{I,2}"] & \\ F_I = \Omega_I \times \mathrm{R}^k \arrow[r] & \mathrm{R}^k & \mathbf{S}^\ell \times \mathrm{R}^k \arrow[l] \end{tikzcd} \tag{4.5}$$

The following key proposition was proved by Agrachev [1] in the unparametrized situation, but as we see below it works in the parametrized case as well.

Proposition 4.3. *The map ϕ_2 gives a homotopy equivalence between B_I and $\phi_2(B_I) = A_I$.*

Proof. In order to simplify notation we prove it in the case $I = [m]$, and the case for any other I would follow immediately. We first prove that $\phi_2(B) = A$. If $(\mathbf{y}, \mathbf{x}) \in A$, then there exists some $i, 1 \leq i \leq m$, such that $Q_i(\mathbf{y}, \mathbf{x}) \leq 0$. Then for $\omega = (-\delta_{1,i}, \ldots, -\delta_{m,i})$ (where $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise), we see that $(\omega, \mathbf{y}, \mathbf{x}) \in B$. Conversely, if $(\mathbf{y}, \mathbf{x}) \in \phi_2(B)$, then there exists $\omega = (\omega_1, \ldots, \omega_m) \in \Omega$ such that, $\sum_{i=1}^m \omega_i Q_i(\mathbf{y}, \mathbf{x}) \geq 0$. Since $\omega_i \leq 0, 1 \leq i \leq m$, and not all $\omega_i = 0$, this implies that $Q_i(\mathbf{y}, \mathbf{x}) \leq 0$ for some $i, 1 \leq i \leq m$. This shows that $(\mathbf{y}, \mathbf{x}) \in A$.

For $(\mathbf{y}, \mathbf{x}) \in \phi_2(B)$, the fiber

$$\phi_2^{-1}(\mathbf{y}, \mathbf{x}) = \{(\omega, \mathbf{y}, \mathbf{x}) \mid \omega \in \Omega \text{ such that } \omega Q(\mathbf{y}, \mathbf{x}) \geq 0\}$$

is a non-empty subset of Ω defined by a single linear inequality. Thus each non-empty fiber is an intersection of a convex cone with $\mathbf{S}^{m-1}$, and hence contractible.

The proposition now follows from the well-known Vietoris-Smale theorem (see Theorem 2.27). □

We will use the following notation.

Notation 4.4. For any quadratic form $Q \in \mathrm{R}[Y_0, \ldots, Y_\ell]$, we will denote by index(Q) the number of negative eigenvalues of the symmetric matrix of the corresponding bilinear form, that is of the matrix M_Q such that, $Q(\mathbf{y}) = \langle M_Q \mathbf{y}, \mathbf{y} \rangle$ for all $\mathbf{y} \in \mathrm{R}^{\ell+1}$ (here $\langle \cdot, \cdot \rangle$ denotes the usual inner product). We will also denote by $\lambda_i(Q)$, $0 \leq i \leq \ell$, the eigenvalues of Q in non-decreasing order, i.e.,

$$\lambda_0(Q) \leq \lambda_1(Q) \leq \cdots \leq \lambda_\ell(Q).$$

For $I \subset [m]$, let

$$F_{I,j} = \{(\omega, \mathbf{x}) \in \Omega_I \times \mathrm{R}^k \mid \operatorname{index}(\omega Q(\cdot, \mathbf{x})) \leq j\}. \tag{4.6}$$

It is clear that each $F_{I,j}$ is a closed semi-algebraic subset of F_I and that they induce a filtration of the space F_I given by

$$F_{I,0} \subset F_{I,1} \subset \cdots \subset F_{I,\ell+1} = F_I.$$

Lemma 4.5. *The fiber of the map $\phi_{I,1}$ over a point $(\omega, \mathbf{x}) \in F_{I,j} \setminus F_{I,j-1}$ has the homotopy type of a sphere of dimension $\ell - j$.*

Proof. As before, we prove the lemma only for $I = [m]$. The proof for a general I is identical. First notice that for $(\omega, \mathbf{x}) \in F_j \setminus F_{j-1}$, the first j eigenvalues of $\omega Q(\cdot, \mathbf{x})$

$$\lambda_0(\omega Q(\cdot, \mathbf{x})), \ldots, \lambda_{j-1}(\omega Q(\cdot, \mathbf{x})) < 0.$$

Moreover, letting $W_0(\omega Q(\cdot, \mathbf{x})), \ldots, W_\ell(\omega Q(\cdot, \mathbf{x}))$ be the co-ordinates with respect to an orthonormal basis $e_0(\omega Q(\cdot, \mathbf{x})), \ldots, e_\ell(\omega Q(\cdot, \mathbf{x}))$, consisting of eigenvectors of $\omega Q(\cdot, \mathbf{x})$, we have that $\phi_1^{-1}(\omega, \mathbf{x})$ is the subset of $\mathbf{S}^\ell = \{\omega\} \times \mathbf{S}^\ell \times \{\mathbf{x}\}$ defined by

$$\sum_{i=0}^{\ell} \lambda_i(\omega Q(\cdot, \mathbf{x})) W_i(\omega Q(\cdot, \mathbf{x}))^2 \geq 0,$$
$$\sum_{i=0}^{\ell} W_i(\omega Q(\cdot, \mathbf{x}))^2 = 1.$$

Since, $\lambda_i(\omega Q(\cdot, \mathbf{x})) < 0, 0 \leq i < j$, it follows that for $(\omega, \mathbf{x}) \in F_j \setminus F_{j-1}$, the fiber $\phi_1^{-1}(\omega, \mathbf{x})$ is homotopy equivalent to the $(\ell - j)$-dimensional sphere defined by setting

$$W_0(\omega Q(\cdot, \mathbf{x})) = \cdots = W_{j-1}(\omega Q(\cdot, \mathbf{x})) = 0$$

on the sphere defined by $\sum_{i=0}^{\ell} W_i(\omega Q(\cdot, \mathbf{x}))^2 = 1$. $\square$

For each $(\omega, \mathbf{x}) \in F_{I,j} \setminus F_{I,j-1}$, let $L_j^+(\omega, \mathbf{x}) \subset \mathrm{R}^{\ell+1}$ denote the sum of the non-negative eigenspaces of $\omega Q(\cdot, \mathbf{x})$ (i.e., $L_j^+(\omega, \mathbf{x})$ is the largest linear subspace of $\mathrm{R}^{\ell+1}$ on which $\omega Q(\cdot, \mathbf{x})$ is positive semi-definite). Since $\mathrm{index}(\omega Q(\cdot, \mathbf{x})) = j$ stays invariant as $(\omega, \mathbf{x})$ varies over $F_{I,j} \setminus F_{I,j-1}$, $L_j^+(\omega, \mathbf{x})$ varies continuously with $(\omega, \mathbf{x})$.

We will denote by C_I the semi-algebraic set defined by

$$C_I = \bigcup_{j=0}^{\ell+1} \{(\omega, \mathbf{y}, \mathbf{x}) \mid (\omega, \mathbf{x}) \in F_{I,j} \setminus F_{I,j-1}, \mathbf{y} \in L_j^+(\omega, \mathbf{x}), |\mathbf{y}| = 1\}. \tag{4.7}$$

The following proposition relates the homotopy type of B_I to that of C_I.

Proposition 4.6. *The semi-algebraic set C_I defined above is homotopy equivalent to B_I (see (4.4) for the definition of B_I).*

Proof. We give a deformation retraction of B_I to C_I constructed as follows. For each $(\omega, x) \in F_{I,\ell} \setminus F_{I,\ell-1}$, we can retract the fiber $\phi_1^{-1}(\omega, x)$ to the zero-dimensional sphere, $L_\ell^+(\omega, x) \cap \mathbf{S}^\ell$ by the following retraction. Let

$$W_0(\omega Q_I(\cdot, x)), \ldots, W_\ell(\omega Q_I(\cdot, x))$$

be the co-ordinates with respect to an orthonormal basis $e_0(\omega Q(\cdot, \mathbf{x})), \ldots, e_\ell(\omega Q(\cdot, \mathbf{x}))$, consisting of eigenvectors of $\omega Q_I(\cdot, x)$ corresponding to non-decreasing order of the eigenvalues of $\omega Q(\cdot, \mathbf{x})$. Then, $\phi_1^{-1}(\omega, x)$ is the subset of $\mathbf{S}^\ell$ defined by

$$\sum_{i=0}^{\ell} \lambda_i(\omega Q_I(\cdot, x)) W_i(\omega Q_I(\cdot, x))^2 \geq 0,$$
$$\sum_{i=0}^{\ell} W_i(\omega Q_I(\cdot, x))^2 = 1.$$

and $L_\ell^+(\omega, x)$ is defined by $W_0(\omega Q_I(\cdot, x)) = \cdots = W_{\ell-1}(\omega Q_I(\cdot, x)) = 0$. We retract $\phi_1^{-1}(\omega, x)$ to the zero-dimensional sphere, $L_\ell^+(\omega, x) \cap \mathbf{S}^\ell$ by the retraction sending,

$$(w_0, \ldots, w_\ell) \in \phi_1^{-1}(\omega, x),$$

at time t to

$$((1-t)w_0, \ldots, (1-t)w_{\ell-1}, t' w_\ell),$$

where $0 \leq t \leq 1$, and

$$t' = \left(\frac{1 - (1-t)^2 \sum_{i=0}^{\ell-1} w_i^2}{w_\ell^2} \right)^{1/2}.$$

Notice that even though the local co-ordinates $(W_0, \ldots, W_\ell)$ in $\mathrm{R}^{\ell+1}$ with respect to the orthonormal basis $(e_0, \ldots, e_\ell)$ may not be uniquely defined at the point (ω, x) (for instance, if the quadratic form $\omega Q_I(\cdot, x)$ has multiple eigenvalues), the retraction is still well-defined since it only depends on the decomposition of $R^{\ell+1}$ into orthogonal complements $\mathrm{span}(e_0, \ldots, e_{\ell-1})$ and $\mathrm{span}(e_\ell)$. We can thus retract simultaneously all fibers over $F_{I\ell} \setminus F_{I,\ell-1}$ continuously, to obtain a semi-algebraic set $B_{I,\ell} \subset B_I$, which is moreover homotopy equivalent to B_I.

This retraction is schematically shown in Figure 6, where $F_{I,\ell}$ is the closed segment, and $F_{I,\ell-1}$ are its end points.

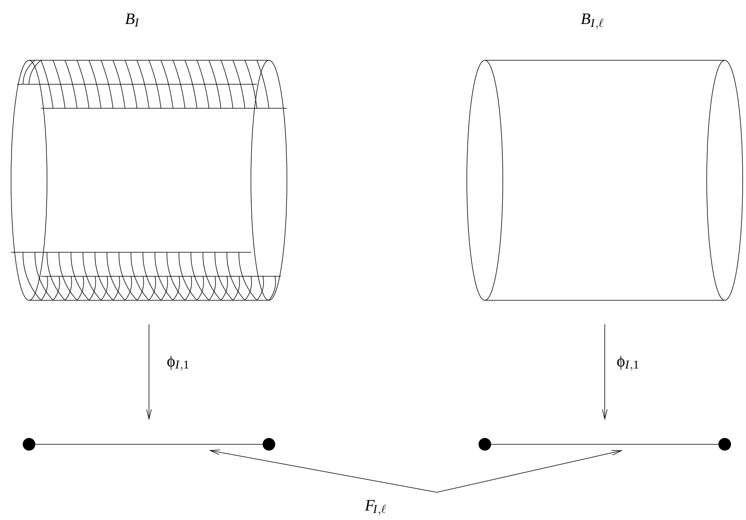

Figure 6: Schematic picture of the retraction of B_I to $B_{I,\ell}$.

Now starting from $B_{I,\ell}$, retract all fibers over $F_{I,\ell-1} \setminus F_{I,\ell-2}$ to the corresponding one dimensional spheres, by the retraction sending

$$(w_0, \ldots, w_\ell) \in \phi_1^{-1}(\omega, x),$$

at time t to

$$((1-t)w_0, \ldots, (1-t)w_{\ell-2}, t'w_{\ell-1}, t'w_\ell),$$

where $0 \leq t \leq 1$, and

$$t' = \left(\frac{1 - (1-t)^2 \sum_{i=0}^{\ell-2} w_i^2}{\sum_{i=\ell-1}^{\ell} w_i^2} \right)^{1/2}$$

to obtain $B_{I,\ell-1}$, which is homotopy equivalent to $B_{I,\ell}$. Continuing this process we finally obtain $B_{I,0} = C_I$, which is clearly homotopy equivalent to B_I by construction. $\square$

Notice that the semi-algebraic set $\phi_1^{-1}(F_{I,j} \setminus F_{I,j-1}) \cap C_I$ is a $\mathbf{S}^{\ell-j}$-bundle over $F_{I,j} \setminus F_{I,j-1}$ under the map ϕ_1, and C_I is a union of these sphere bundles. We have good control over the bases, $F_{I,j} \setminus F_{I,j-1}$, of these bundles, that is we have good bounds on the number as well as the degrees of polynomials used to define them. However, these bundles could be possibly glued to each other in complicated ways, and it is not immediate how to control this glueing data, since different types of glueing could give rise to different homotopy types of the underlying space. In order to get around this difficulty, we consider certain closed subsets, $F'_{I,j}$ of F_I, where each $F'_{I,j}$ is an infinitesimal deformation of $F_{I,j} \setminus F_{I,j-1}$, and form the base of a $\mathbf{S}^{\ell-j}$-bundle. Moreover, these new sphere bundles are glued to each other along sphere bundles over $F'_{I,j} \cap F'_{I,j-1}$, and their union, C'_I, is homotopy equivalent to C_I. Finally, the polynomials defining the sets $F'_{I,j}$ are in general position in a very strong sense, and this property is used later to bound the number of isotopy classes of the sets $F'_{I,j}$ in the parametrized situation.

We now make precise the argument outlined above. Let Λ_I be the polynomial in $\mathrm{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k, T]$ defined by

$$\begin{aligned} \Lambda_I &= \det(M_{Z_I \cdot Q} + T \,\mathrm{Id}_{\ell+1}), \\ &= T^{\ell+1} + H_{I,\ell} T^{\ell} + \cdots + H_{I,0}, \end{aligned}$$

where $Z_I \cdot Q = \sum_{i \in I} Z_i Q_i$, and each $H_{I,j} \in \mathrm{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k]$.

Notice, that $H_{I,j}$ is obtained from $H_j = H_{[m],j}$ by setting the variable Z_i to 0 in the polynomial H_j for each $i \notin I$.

Note also that for $(\mathbf{z}, \mathbf{x}) \in \mathrm{R}^m \times \mathrm{R}^k$, the polynomial $\Lambda_I(\mathbf{z}, \mathbf{x}, T)$ being the characteristic polynomial of a real symmetric matrix has all its roots real. It then follows from Descartes' rule of signs (see for instance [20]), that for each $(\mathbf{z}, \mathbf{x}) \in \mathrm{R}^m \times \mathrm{R}^k$, where $\mathbf{z}_i = 0$ for all

$i \notin I$, index($\mathbf{z}Q(\cdot, \mathbf{x})$) is determined by the sign vector

$$(\mathrm{sign}(H_{I,\ell}(\mathbf{z}, \mathbf{x})), \ldots, \mathrm{sign}(H_{I,0}(\mathbf{z}, \mathbf{x}))).$$

Hence, denoting by

$$\mathcal{H}_I = \{H_{I,0}, \ldots, H_{I,\ell}\} \subset \mathrm{R}[Z_1, \ldots, Z_m, X_1, \ldots, X_k], \tag{4.8}$$

we have

Lemma 4.7. *For each $j, 0 \leq j \leq \ell + 1$, $F_{I,j}$ is the intersection of F_I with a $\mathcal{H}_I$-closed semi-algebraic set $D_{I,j} \subset \mathrm{R}^{m+k}$.*

Notation 4.8. Let $D_{I,j}$ be defined by the formula

$$D_{I,j} = \bigcup_{\sigma \in \Sigma_{I,j}} \mathcal{R}(\sigma), \tag{4.9}$$

for some $\Sigma_{I,j} \subset \mathrm{Sign}(\mathcal{H}_I)$. Note that, $\mathrm{Sign}(\mathcal{H}_I) \subset \mathrm{Sign}(\mathcal{H})$ and $\Sigma_{I,j} \subset \Sigma_j$ for all $I \subset [m]$.

Now, let $\bar{\delta} = (\delta_\ell, \ldots, \delta_0)$ and $\bar{\varepsilon} = (\varepsilon_{\ell+1}, \ldots, \varepsilon_0)$ be infinitesimals such that

$$0 < \delta_0 \ll \cdots \ll \delta_\ell \ll \varepsilon_0 \ll \cdots \ll \varepsilon_{\ell+1} \ll 1,$$

and let

$$\mathrm{R}' = \mathrm{R}\langle \bar{\varepsilon}, \bar{\delta} \rangle \tag{4.10}$$

Given $\sigma \in \mathrm{Sign}(\mathcal{H}_I)$, and $0 \leq j \leq \ell + 1$, we denote by $\mathcal{R}(\sigma_j^c) \subset \mathrm{R}'^{m+k}$ the set defined by the formula σ_j^c obtained by taking the conjunction of

$$\begin{aligned}
&-\varepsilon_j - \delta_i \leq H_{I,i} \leq \varepsilon_j + \delta_i \text{ for each } H_{I,i} \in \mathcal{H}_I \text{ such that } \sigma(H_{I,i}) = 0,\\
&H_{I,i} \geq -\varepsilon_j - \delta_i, \text{ for each } H_{I,i} \in \mathcal{H}_I \text{ such that } \sigma(H_{I,i}) = 1,\\
&H_{I,i} \leq \varepsilon_j + \delta_i, \text{ for each } H_{I,i} \in \mathcal{H}_I \text{ such that } \sigma(H_{I,i}) = -1.
\end{aligned}$$

Similarly, we denote by $\mathcal{R}(\sigma_j^o) \subset \mathrm{R}'^{m+k}$ the set defined by the formula σ^o obtained by taking the conjunction of

$$\begin{aligned}
&-\varepsilon_j - \delta_i < H_{I,i} < \varepsilon_j + \delta_i \text{ for each } H_{i,I} \in \mathcal{H}_I \text{ such that } \sigma(H_{I,i}) = 0,\\
&H_{I,i} > -\varepsilon_j - \delta_i, \text{ for each } H_{I,i} \in \mathcal{H}_I \text{ such that } \sigma(H_{I,i}) = 1,\\
&H_{I,i} < \varepsilon_j + \delta_i, \text{ for each } H_{I,i} \in \mathcal{H}_I \text{ such that } \sigma(H_{I,i}) = -1.
\end{aligned}$$

For each $j, 0 \leq j \leq \ell + 1$, let

$$\begin{aligned}
D^o_{I,j} &= \bigcup_{\sigma \in \Sigma_{I,j}} \mathcal{R}(\sigma^o_j), \\
D^c_{I,j} &= \bigcup_{\sigma \in \Sigma_{I,j}} \mathcal{R}(\sigma^c_j), \\
D'_{I,j} &= D^c_{I,j} \setminus D^o_{I,j-1}, \\
F'_{I,j} &= \mathrm{Ext}(F_I, \mathrm{R}') \cap D'_{I,j}. \qquad (4.11)
\end{aligned}$$

where we denote by $D^o_{I,-1} = \emptyset$. We also denote by $F'_I = \mathrm{Ext}(F_I, \mathrm{R}')$.

We now note some extra properties of the sets $D'_{I,j}$'s.

Lemma 4.9. *For each $j, 0 \leq j \leq \ell + 1$, $D'_{I,j}$ is a $\mathcal{H}'_I$-closed semi-algebraic set, where*

$$\mathcal{H}'_I = \bigcup_{i=0}^{\ell} \bigcup_{j=0}^{\ell+1} \{H_{I,i} + \varepsilon_j + \delta_i, H_{I,i} - \varepsilon_j - \delta_i\}. \qquad (4.12)$$

Proof. Follows from the definition of the sets $D'_{I,j}$. □

Lemma 4.10. *For $0 \leq j + 1 < i \leq \ell + 1$,*

$$D'_{I,i} \cap D'_{I,j} = \emptyset.$$

Proof. In order to keep notation simple we prove the proposition only for $I = [m]$. The proof for a general I is identical. The inclusions,

$$D_{j-1} \subset D_j \subset D_{i-1} \subset D_i,$$

$$D^o_{j-1} \subset D^c_j \subset D^o_{i-1} \subset D^c_i.$$

follow directly from the definitions of the sets

$$D_i, D_j, D_{j-1}, D^c_i, D^c_j, D^o_{i-1}, D^o_{j-1},$$

and the fact that,

$$\varepsilon_{j-1} \ll \varepsilon_j \ll \varepsilon_{i-1} \ll \varepsilon_i.$$

It follows immediately that,

$$D'_i = D^c_i \setminus D^o_{i-1}$$

is disjoint from D^c_j, and hence from D'_j. □

We now associate to each $F'_{I,j}$ a $(\ell - j)$-dimensional sphere bundle as follows. For each $(\omega, \mathbf{x}) \in F''_{I,j} = F_{I,j} \setminus F'_{I,j-1}$, let $L_j^+(\omega, \mathbf{x}) \subset \mathrm{R}^{\ell+1}$ denote the sum of the non-negative eigenspaces of $\omega Q(\cdot, \mathbf{x})$ (i.e., $L_j^+(\omega, \mathbf{x})$ is the largest linear subspace of $\mathrm{R}^{\ell+1}$ on which $\omega Q(\cdot, \mathbf{x})$ is positive semi-definite). Since $\mathrm{index}(\omega Q(\cdot, \mathbf{x})) = j$ stays invariant as $(\omega, \mathbf{x})$ varies over $F''_{I,j}$, $L_j^+(\omega, \mathbf{x})$ varies continuously with $(\omega, \mathbf{x})$.

Let,

$$\lambda_0(\omega, \mathbf{x}) \leq \cdots \leq \lambda_{j-1}(\omega, \mathbf{x}) < 0 \leq \lambda_j(\omega, \mathbf{x}) \leq \cdots \leq \lambda_\ell(\omega, \mathbf{x}),$$

be the eigenvalues of $\omega Q(\cdot, \mathbf{x})$ for $(\omega, \mathbf{x}) \in F''_{I,j}$. There is a continuous extension of the map sending $(\omega, \mathbf{x}) \mapsto L_j^+(\omega, \mathbf{x})$ to $(\omega, \mathbf{x}) \in F'_{I,j}$.

To see this observe that for $(\omega, \mathbf{x}) \in F''_{I,j}$ the block of the first j (negative) eigenvalues, $\lambda_0(\omega, \mathbf{x}) \leq \cdots \leq \lambda_{j-1}(\omega, \mathbf{x})$, and hence the sum of the eigenspaces corresponding to them can be extended continuously to any infinitesimal neighborhood of $F''_{I,j}$, and in particular to $F'_{I,j}$. Now $L_j^+(\omega, \mathbf{x})$ is the orthogonal complement of the sum of the eigenspaces corresponding to the block of negative eigenvalues, $\lambda_0(\omega, \mathbf{x}) \leq \cdots \leq \lambda_{j-1}(\omega, \mathbf{x})$.

We will denote by $C'_{I,j} \subset F'_{I,j} \times \mathrm{R}'^{\ell+1}$ the semi-algebraic set defined by

$$C'_{I,j} = \{(\omega, \mathbf{y}, \mathbf{x}) \mid (\omega, \mathbf{x}) \in F'_{I,j}, \mathbf{y} \in L_j^+(\omega, \mathbf{x}), |\mathbf{y}| = 1\}. \tag{4.13}$$

Note that the projection $\pi_{I,j} : C'_{I,j} \to F'_{I,j}$, makes $C'_{I,j}$ the total space of a $(\ell - j)$-dimensional sphere bundle over $F'_{I,j}$.

Now observe that

$$C'_{I,j-1} \cap C'_{I,j} = \pi_{I,j}^{-1}(F'_{I,j} \cap F'_{I,j-1}),$$

and

$$\pi_{I,j}|_{C'_{I,j-1} \cap C'_{I,j}} : C'_{I,j-1} \cap C'_{I,j} \to F'_{I,j} \cap F'_{I,j-1}$$

is also a $(\ell - j)$ dimensional sphere bundle over $F'_{I,j} \cap F'_{I,j-1}$.

Let

$$C'_I = \bigcup_{j=0}^{\ell+1} C'_{I,j}. \tag{4.14}$$

We have that

Proposition 4.11. *C'_I is homotopy equivalent to* $\mathrm{Ext}(C_I, \mathrm{R}')$*, where C_I and* R' *are defined in (4.7) and (4.10) respectively.*

Proof. Let $\bar{\varepsilon} = (\varepsilon_{\ell+1}, \ldots, \varepsilon_0)$ and let

$$R_i = \begin{cases} R\langle \bar{\varepsilon}, \delta_\ell, \ldots, \delta_i \rangle, \ 0 \leq i \leq \ell, \\ R\langle \varepsilon_{\ell+1}, \ldots, \varepsilon_{i-\ell-1} \rangle, \ \ell+1 \leq i \leq 2\ell+2, \\ R, \ i = 2\ell+3. \end{cases}$$

First observe that $C_I = \lim_{\varepsilon_{\ell+1}} C'_I$ where C_I is the semi-algebraic set defined in (4.7) above.

Now let,

$$\begin{aligned} C_{I,-1} &= C'_I, \\ C_{I,0} &= \lim_{\delta_0} C'_I, \\ C_{I,i} &= \lim_{\delta_i} C_{I,i-1}, 1 \leq i \leq \ell, \\ C_{I,\ell+1} &= \lim_{\varepsilon_0} C_{I,\ell}, \\ C_{I,i} &= \lim_{\varepsilon_{i-\ell-2}} C_{I,i-1}, \ell+2 \leq i \leq 2\ell+3. \end{aligned}$$

Notice that each $C_{I,i}$ is a closed and bounded semi-algebraic set. Also, for $i \geq 0$, let $C_{I,i-1,t} \subset \mathrm{R}_i^{m+\ell+k}$ be the semi-algebraic set obtained by replacing δ_i (resp., ε_i) in the definition of $C_{I,i-1}$ by the variable t. Then, there exists $t_0 > 0$, such that for all $0 < t_1 < t_2 \leq t_0$, $C_{I,i-1,t_1} \subset C_{I,i-1,t_2}$.

It follows (see Lemma 16.17 in [20]) that for each i, $0 \leq i \leq 2\ell+3$, $\mathrm{Ext}(C_{I,i}, \mathrm{R}_i)$ is homotopy equivalent to $C_{I,i-1}$. □

4.4 Partitioning the Parameter Space

The goal of this section is to prove the following proposition (Proposition 4.12). The techniques used in the proof are similar to those used in [22] for proving a similar result. We go through the proof in detail in order to extract the right bound in terms of the parameters d, k, ℓ and m.

Proposition 4.12. *There exists a finite set of points $T \subset \mathrm{R}^k$ with*

$$\#T \leq (2^m \ell k d)^{O(mk)}$$

such that for any $\mathbf{x} \in \mathrm{R}^k$*, there exists* $\mathbf{z} \in T$*, with the following property.*

There is a semi-algebraic path, $\gamma : [0,1] \to \mathrm{R}'^k$ *and a continuous semi-algebraic map,* $\phi : \Omega \times [0,1] \to \Omega$ *(see (4.2) and (4.10) for the definition of* Ω *and* R'*),*

with $\gamma(0) = \mathbf{x}$*,* $\gamma(1) = \mathbf{z}$*, and for each* $I \subset [m]$*,*

$$\phi(\cdot, t)|_{F'_{I,j,\mathbf{x}}} : F'_{I,j,\mathbf{x}} \to F'_{I,j,\gamma(t)},$$

is a homeomorphism for each $0 \leq t \leq 1$*.*

Before proving Proposition 4.12 we need a few preliminary results. Let

$$\mathcal{H}'' = \mathcal{H}' \cup \{Z_1, \ldots, Z_m, Z_1^2 + \cdots + Z_m^2 - 1\}, \tag{4.15}$$

where $\mathcal{H}' = \mathcal{H}'_{[m]}$ is defined in (4.12) above.

Note that for each j, $0 \leq j \leq \ell + 1$, $F'_{I,j}$ is a $\mathcal{H}''$-closed semi-algebraic set. Moreover, let $\psi : \mathrm{R}'^{m+k} \to \mathrm{R}'^k$ be the projection onto the last k co-ordinates.

Notation 4.13. We fix a finite set of points $T \subset \mathrm{R}^k$ such that for every $\mathbf{x} \in \mathrm{R}^k$ there exists $\mathbf{z} \in T$ such that for every $\mathcal{H}''$-semi-algebraic set V, the set $\psi^{-1}(\mathbf{x}) \cap V$ is homeomorphic to $\psi^{-1}(\mathbf{z}) \cap V$.

The existence of a finite set T with this property follows from Hardt's triviality theorem (Theorem 2.15) and the Tarski-Seidenberg transfer principle, as well as the fact that the number of $\mathcal{H}''$-semi-algebraic sets is finite.

Now, we note some extra properties of the family $\mathcal{H}''$. The notations Sign_p and $\mathcal{R}(\sigma)$ were introduced in Chapter 2.1.1.

Lemma 4.14. *If* $\sigma \in \mathrm{Sign}_p(\mathcal{H}'')$*, then* $p \leq k + m$ *and* $\mathcal{R}(\sigma) \subset \mathrm{R}'^{m+k}$ *is a non-singular* $(m + k - p)$*-dimensional manifold such that at every point* $(\mathbf{z}, \mathbf{x}) \in \mathcal{R}(\sigma)$*, the* $(p \times (m + k))$*-Jacobi matrix,*

$$\left(\frac{\partial P}{\partial Z_i}, \frac{\partial P}{\partial Y_j} \right)_{P \in \mathcal{H}'',\ \sigma(P)=0,\ 1 \leq i \leq m,\ 1 \leq j \leq k}$$

has maximal rank p*.*

Proof. Let $\mathrm{Ext}(\mathbf{S}^{m-1}, \mathrm{R}')$ be the unit sphere in R'^m. Suppose without loss of generality that

$$\{P \in \mathcal{H}'' |\, \sigma(P) = 0\} = \{H_{i_1} - \varepsilon_{j_1} - \delta_{i_1}, \ldots, H_{i_{p-1}} - \varepsilon_{j_{p-1}} - \delta_{i_{p-1}}, \sum_{i=1}^{m} Z_i^2 - 1\}$$

since the equation $Z_i = 0$ eliminates the variable Z_i from the polynomials. It follows that it suffices to show that the algebraic set

$$V = \bigcap_{r=1}^{p-1} \{(\mathbf{z}, \mathbf{x}) \in \mathrm{Ext}(\mathbf{S}^{m-1}, \mathrm{R}') \times \mathrm{R}'^k \mid H_{i_r}(\mathbf{z}, \mathbf{x}) = \varepsilon_{j_r} + \delta_{i_r}\} \tag{4.16}$$

is a smooth $((m-1) + k - (p-1))$-dimensional manifold such that at every point on it the $(p \times (m+k))$-Jacobi matrix,

$$\left(\frac{\partial P}{\partial Z_i}, \frac{\partial P}{\partial Y_j}\right)_{P \in \mathcal{H}'',\ \sigma(P)=0,\ 1 \le i \le m,\ 1 \le j \le k}$$

has maximal rank p.

Let $p \le m + k$. Consider the semi-algebraic map $P_{i_1,\ldots,i_{p-1}} : \mathbf{S}^{m-1} \times \mathrm{R}^k \to \mathrm{R}^{p-1}$ defined by

$$(\mathbf{z}, \mathbf{x}) \mapsto (H_{i_1}(\mathbf{z}, \mathbf{x}), \ldots, H_{i_{p-1}}(\mathbf{z}, \mathbf{x})).$$

By the semi-algebraic version of Sard's theorem (see [29]), the set of critical values of $P_{i_1,\ldots,i_{p-1}}$ is a semi-algebraic subset C of R^{p-1} of dimension strictly less than $p-1$. Since $\bar{\delta}$ and $\bar{\varepsilon}$ are infinitesimals, it follows that

$$(\varepsilon_{j_1} + \delta_{i_1}, \ldots, \varepsilon_{j_{p-1}} + \delta_{i_{p-1}}) \notin \mathrm{Ext}(C, R').$$

Hence, the algebraic set V defined in (4.16) has the desired properties, and the same is true for the basic semi-algebraic set $\mathcal{R}(\sigma)$.

We now prove that $p \le m + k$. Suppose that $p > m + k$. As we have just proved,

$$\{H_{i_1}(\mathbf{z}, \mathbf{x}) = \varepsilon_{j_1} + \delta_{i_1}, \ldots, H_{i_{m+k-1}}(\mathbf{z}, \mathbf{x}) = \varepsilon_{j_{m+k-1}} + \delta_{i_{m+k-1}}\}$$

is a finite set of points. But the polynomial $H_{i_{p-1}} - \varepsilon_{j_{p-1}} - \delta_{i_{p-1}}$ cannot vanish on each of these points as $\bar{\delta}$ and $\bar{\varepsilon}$ are infinitesimals. $\square$

Lemma 4.15. *For every* $\mathbf{x} \in \mathrm{R}^k$*, and* $\sigma \in \mathrm{Sign}_p(\mathcal{H}''_{\mathbf{x}})$*, where*

$$\mathcal{H}''_{\mathbf{x}} = \{P(Z_1, \ldots, Z_m, \mathbf{x}) |\ P \in \mathcal{H}''\},$$

the following holds.

1. $0 \leq p \leq m$*, and*

2. $\mathcal{R}(\sigma) \cap \psi^{-1}(\mathbf{x})$ *is a non-singular* $(m-p)$*-dimensional manifold such that at every point* $(\mathbf{z}, \mathbf{x}) \in \mathcal{R}(\sigma) \cap \psi^{-1}(\mathbf{x})$*, the* $(p \times m)$*-Jacobi matrix,*

$$\left(\frac{\partial P}{\partial Z_i}\right)_{P \in \mathcal{H}''_{\mathbf{x}}, \sigma(P)=0, 1 \leq i \leq m}$$

   has maximal rank p*.*

Proof. Note that $P_{\mathbf{x}} = P(Z_1, \ldots, Z_m, \mathbf{x}) \in \mathrm{R}'[Z_1, \ldots, Z_m]$ for each $P \in \mathcal{H}''$ and $\mathbf{x} \in \mathrm{R}^k$. The proof is now identical to the proof of Lemma 4.14. □

Lemma 4.16. *For any bounded* $\mathcal{H}''$*-semi-algebraic set* V *defined by*

$$V = \bigcup_{\sigma \in \Sigma_V \subset \mathrm{Sign}(\mathcal{H}'')} \mathcal{R}(\sigma),$$

the partitions

$$\begin{aligned} \mathrm{R}'^{m+k} &= \bigcup_{\sigma \in \mathrm{Sign}(\mathcal{H}'')} \mathcal{R}(\sigma), \\ V &= \bigcup_{\sigma \in \Sigma_V} \mathcal{R}(\sigma), \end{aligned}$$

are compatible Whitney stratifications of R'^{m+k} *and* V *respectively.*

Proof. Follows directly from the definition of Whitney stratification (see [53, 39]), and Lemma 4.14. □

Fix some sign condition $\sigma \in \mathrm{Sign}(\mathcal{H}'')$. Recall that $(\mathbf{z}, \mathbf{x}) \in \mathcal{R}(\sigma)$ is a *critical point* of the map $\psi_{\mathcal{R}(\sigma)}$ if the Jacobi matrix,

$$\left(\frac{\partial P}{\partial Z_i}\right)_{P \in \mathcal{H}'', \sigma(P)=0,\ 1 \leq i \leq m}$$

at $(\mathbf{z}, \mathbf{x})$ is not of the maximal possible rank. The projection $\psi(\mathbf{z}, \mathbf{x})$ of a critical point is a *critical value* of $\psi_{\mathcal{R}(\sigma)}$.

Let $C_1 \subset \mathrm{R}'^{m+k}$ be the set of critical points of $\psi_{\mathcal{R}(\sigma)}$ over all sign conditions

$$\sigma \in \bigcup_{p \le m} \mathrm{Sign}_p(\mathcal{H}''),$$

(i.e., over all $\sigma \in \mathrm{Sign}_p(\mathcal{H}'')$ with $\dim(\mathcal{R}(\sigma)) \ge k$). For a bounded $\mathcal{H}''$-semi-algebraic set V, let $C_1(V) \subset V$ be the set of critical points of $\psi_{\mathcal{R}(\sigma)}$ over all sign conditions

$$\sigma \in \bigcup_{p \le m} \mathrm{Sign}_p(\mathcal{H}'') \cap \Sigma_V$$

(i.e., over all $\sigma \in \Sigma_V$ with $\dim(\mathcal{R}(\sigma)) \ge k$).

Let $C_2 \subset \mathrm{R}'^{m+k}$ be the union of $\mathcal{R}(\sigma)$ over all

$$\sigma \in \bigcup_{p > m} \mathrm{Sign}_p(\mathcal{H}'')$$

(i.e., over all $\sigma \in \mathrm{Sign}_p(\mathcal{H}'')$ with $\dim(\mathcal{R}(\sigma)) < k$). For a bounded $\mathcal{H}''$-semi-algebraic set V, let $C_2(V) \subset V$ be the union of $\mathcal{R}(\sigma)$ over all

$$\sigma \in \bigcup_{p > m} \mathrm{Sign}_p(\mathcal{H}'') \cap \Sigma_V$$

(i.e., over all $\sigma \in \Sigma_V$ with $\dim(\mathcal{R}(\sigma)) < k$).

Denote $C = C_1 \cup C_2$, and $C(V) = C_1(V) \cup C_2(V)$.

Lemma 4.17. *For each bounded $\mathcal{H}''$-semi-algebraic V, the set $C(V)$ is closed and bounded.*

Proof. The set $C(V)$ is bounded since V is bounded. The union $C_2(V)$ of strata of dimensions less than k is closed since V is closed.

Let $\sigma_1 \in \mathrm{Sign}_{p_1}(\mathcal{H}'') \cap \Sigma_V$, $\sigma_2 \in \mathrm{Sign}_{p_2}(\mathcal{H}'') \cap \Sigma_V$, where $p_1 \le m$, $p_1 < p_2$, and if $\sigma_1(P) = 0$, then $\sigma_2(P) = 0$ for any $P \in \mathcal{H}''$. It follows that stratum $\mathcal{R}(\sigma_2)$ lies in the closure of the stratum $\mathcal{R}(\sigma_1)$. Let $\mathcal{J}$ be the finite family of $(p_1 \times p_1)$-minors such that $\mathrm{Zer}(\mathcal{J}, \mathrm{R}') \cap \mathcal{R}(\sigma_1)$ is the set of all critical points of $\pi_{\mathcal{R}(\sigma_1)}$. Then $\mathrm{Zer}(\mathcal{J}, \mathrm{R}') \cap \mathcal{R}(\sigma_2)$ is either contained in $C_2(V)$ (when $\dim(\mathcal{R}(\sigma_2)) < k$), or is contained in the set of all critical points of $\pi_{\mathcal{R}(\sigma_2)}$ (when $\dim(\mathcal{R}(\sigma_2)) \ge k$). It follows that the closure of $\mathrm{Zer}(\mathcal{J}, \mathrm{R}') \cap \mathcal{R}(\sigma_1)$ lies in the union of the following sets:

1. $\mathrm{Zer}(\mathcal{J}, \mathrm{R}') \cap \mathcal{R}(\sigma_1)$,

2. sets of critical points of some strata of dimensions less than $m + k - p_1$,

3. some strata of dimension less than k.

Using induction on descending dimensions in case (2), we conclude that the closure of $\mathrm{Zer}(\mathcal{J}, \mathrm{R}') \cap \mathcal{R}(\sigma_1)$ is contained in $C(V)$. Hence, $C(V)$ is closed. □

Definition 4.18. We denote by $G_i = \psi(C_i), i = 1, 2$, and $G = G_1 \cup G_2$. Similarly, for each bounded $\mathcal{H}''$-semi-algebraic set V, we denote by $G_i(V) = \psi(C_i(V))$, $i = 1, 2$, and $G(V) = G_1(V) \cup G_2(V)$.

Lemma 4.19. *We have $T \cap G = \emptyset$. In particular, $T \cap G(V) = \emptyset$ for every bounded $\mathcal{H}''$-semi-algebraic set V.*

Proof. By Lemma 4.15, for all $\mathbf{x} \in T$, and $\sigma \in \mathrm{Sign}_p(\mathcal{H}''_{\mathbf{x}})$,

1. $0 \leq p \leq m$, and

2. $\mathcal{R}(\sigma) \cap \psi^{-1}(\mathbf{x})$ is a non-singular $(m-p)$-dimensional manifold such that at every point $(\mathbf{z}, \mathbf{x}) \in \mathcal{R}(\sigma) \cap \psi^{-1}(\mathbf{x})$, the $(p \times m)$-Jacobi matrix,

$$\left(\frac{\partial P}{\partial Z_i}\right)_{P \in \mathcal{H}''_{\mathbf{x}}, \sigma(P)=0, 1 \leq i \leq m}$$

has the maximal rank p.

If a point $\mathbf{x} \in T \cap G_1 = T \cap \psi(C_1)$, then there exists $\mathbf{z} \in \mathrm{R}'^m$ such that $(\mathbf{z}, \mathbf{x})$ is a critical point of $\psi_{\mathcal{R}(\sigma)}$ for some $\sigma \in \bigcup_{p \leq m} \mathrm{Sign}_p(\mathcal{H}'')$, and this is impossible by (2).

Similarly, $\mathbf{x} \in T \cap G_2 = T \cap \psi(C_2)$, implies that there exists $\mathbf{z} \in \mathrm{R}'^m$ such that $(\mathbf{z}, \mathbf{x}) \in \mathcal{R}(\sigma)$ for some $\sigma \in \bigcup_{p > m} \mathrm{Sign}_p(\mathcal{H}'')$, and this is impossible by (1). □

Let D be a connected component of $\mathrm{R}'^k \setminus G$, and for a bounded $\mathcal{H}''$-semi-algebraic set V, let $D(V)$ be a connected component of $\psi(V) \setminus G(V)$.

Lemma 4.20. *For every bounded $\mathcal{H}''$-semi-algebraic set V, all fibers $\psi^{-1}(\mathbf{x}) \cap V$, $\mathbf{x} \in D$ are homeomorphic.*

Proof. Lemma 4.15 and Lemma 4.16 imply that $\widehat{V} = \psi^{-1}(\psi(V) \setminus G(V)) \cap V$ is a Whitney stratified set having strata of dimensions at least k. Moreover, $\psi|_{\widehat{V}}$ is a proper stratified submersion. By Thom's first isotopy lemma (in the semi-algebraic version, over real closed fields [39]) the map $\psi|_{\widehat{V}}$ is a locally trivial fibration. In particular, all fibers $\psi^{-1}(\mathbf{x}) \cap V$, $\mathbf{x} \in D(V)$ are homeomorphic for every connected component $D(V)$. The lemma follows, since the inclusion $G(V) \subset G$ implies that either $D \subset D(V)$ for some connected component $D(V)$, or $D \cap \psi(V) = \emptyset$. $\square$

Lemma 4.21. *For each $\mathbf{x} \in T$, there exists a connected component D of $\mathrm{R}'^k \setminus G$, such that $\psi^{-1}(\mathbf{x}) \cap V$ is homeomorphic to $\psi^{-1}(\mathbf{x}_1) \cap V$ for every bounded $\mathcal{H}''$-semi-algebraic set V and for every $\mathbf{x}_1 \in D$.*

Proof. Let V be a bounded $\mathcal{H}''$-semi-algebraic set and $\mathbf{x} \in T$. By Lemma 4.19, $\mathbf{x}$ belongs to some connected component D of $\mathrm{R}'^k \setminus G$. Lemma 4.20 implies that $\psi^{-1}(\mathbf{x}) \cap V$ is homeomorphic to $\psi^{-1}(\mathbf{x}_1) \cap V$ for every $\mathbf{x}_1 \in D$. $\square$

We now are able to prove Proposition 4.12.

Proof of Proposition 4.12. Recall that $G = G_1 \cup G_2$, where G_1 is the union of sets of critical values of $\psi_{\mathcal{R}(\sigma)}$ over all strata $\mathcal{R}(\sigma)$ of dimensions at least k, and G_2 is the union of projections of all strata of dimensions less than k.

By Lemma 4.21 it suffices to bound the number of connected components of the set $\mathrm{R}'^k \setminus G$. Denote by $\mathcal{E}_1$ the family of closed sets of critical points of $\psi_{\mathcal{Z}(\sigma)}$, over all sign conditions σ such that strata $\mathcal{R}(\sigma)$ have dimensions at least k (the notation $\mathcal{Z}(\sigma)$ was introduced in Chapter 2.1.1). Let $\mathcal{E}_2$ be the family of closed sets $\mathcal{Z}(\sigma)$, over all sign conditions σ such that strata $\mathcal{R}(\sigma)$ have dimensions equal to $k-1$. Let $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$. Denote by E the image under the projection ψ of the union of all sets in the family $\mathcal{E}$.

Because of the transversality condition, every stratum of the stratification of V, having the dimension less than $m+k$, lies in the closure of a stratum having the next higher dimension. In particular, this is true for strata of dimensions less than $k-1$. It follows that $G \subset E$, and thus every connected component of the complement $\mathrm{R}'^k \setminus E$ is contained in a connected component of $\mathrm{R}'^k \setminus G$. Since $\dim(E) < k$, every connected component of $\mathrm{R}'^k \setminus G$ contains a connected component of $\mathrm{R}'^k \setminus E$. Therefore, it is sufficient to estimate from above the Betti number $\mathrm{b}_0(\mathrm{R}'^k \setminus E)$ which is equal to $\mathrm{b}_{k-1}(E)$ by the Alexander's duality.

The total number of sets $\mathcal{Z}(\sigma)$, such that $\sigma \in \mathrm{Sign}(\mathcal{H}'')$ and $\dim(\mathcal{Z}(\sigma)) \geq k-1$, is $O(\ell^{2(m+1)})$ because each $\mathcal{Z}(\sigma)$ is defined by a conjunction of at most $m+1$ of possible $O(\ell^2+m)$ polynomial equations.

Thus, the cardinality $\#\mathcal{E}$, as well as the number of images under the projection π of sets in $\mathcal{E}$ is $O(\ell^{2(m+1)})$. According to (2.2) in Proposition 2.19, $\mathrm{b}_{k-1}(E)$ does not exceed the sum of certain Betti numbers of sets of the type

$$\Phi = \bigcap_{1 \leq i \leq p} \pi(U_i),$$

where every $U_i \in \mathcal{E}$ and $1 \leq p \leq k$. More precisely, we have

$$\mathrm{b}_{k-1}(E) \leq \sum_{1 \leq p \leq k} \quad \sum_{\{U_1,\ldots,U_p\} \subset \mathcal{E}} \mathrm{b}_{k-p}\left(\bigcap_{1 \leq i \leq p} \pi(U_i)\right).$$

Obviously, there are $O(\ell^{2(m+1)k})$ sets of the kind Φ.

Using inequality (2.3) in Proposition 2.19, we have that for each Φ as above, the Betti number $\mathrm{b}_{k-p}(\Phi)$ does not exceed the sum of certain Betti numbers of unions of the kind,

$$\Psi = \bigcup_{1 \leq j \leq q} \pi(U_{i_j}) = \pi\left(\bigcup_{1 \leq j \leq q} U_{i_j}\right),$$

with $1 \leq q \leq p$. More precisely,

$$\mathrm{b}_{k-p}(\Phi) \quad \leq \quad \sum_{1 \leq q \leq p} \quad \sum_{1 \leq i_1 < \cdots < i_q \leq p} \mathrm{b}_{k-p+q-1}\left(\pi\left(\bigcup_{1 \leq j \leq q} U_{i_j}\right)\right).$$

It is clear that there are at most $2^p \leq 2^k$ sets of the kind Ψ.

If a set $U \in \mathcal{E}_1$, then it is defined by m polynomials of degrees at most $O(\ell d)$. If a set $U \in \mathcal{E}_2$, then it is defined by $O(2^m)$ polynomials of degrees $O(m\ell d)$, since the critical points on strata of dimensions at least k are defined by $O(2^m)$ determinantal equations, the corresponding matrices have orders $O(m)$, and the entries of these matrices are polynomials of degrees at most $O(\ell d)$.

It follows that the closed and bounded set

$$\bigcup_{1\le j\le q} U_{i_j}$$

is defined by $O(k2^m))$ polynomials of degrees $O(\ell d)$.

By Proposition 2.26, $\mathrm{b}_{k-p+q-1}(\Psi) \le (2^m k\ell d)^{O(mk)}$ for all $1 \le p \le k$, $1 \le q \le p$. Then $\mathrm{b}_{k-p}(\Phi) \le (2^m k\ell d)^{O(mk)}$ for every $1 \le p \le k$. Since there are $O(\ell^{2(m+1)k})$ sets of the kind Φ, we get the claimed bound

$$\mathrm{b}_{k-1}(E) \le (2^m k\ell d)^{O(mk)}.$$

The rest of the proof follows from Proposition 4.21. □

4.5 Proof of the Result

4.5.1 The Homogeneous Case

We first consider the case where all the polynomials in $\mathcal{Q}$ are homogeneous in variables $Y_0,\ldots,Y_\ell$ and we bound the number of homotopy types among the fibers $S_{\mathbf{x}}$, defined by the $\mathcal{Q}$-closed semi-algebraic subsets S of $\mathbf{S}^\ell \times \mathrm{R}^k$. We first the prove the following theorems for the special cases of unions and intersections.

Theorem 4.22. *Let* R *be a real closed field and let*

$$\mathcal{Q} = \{Q_1,\ldots,Q_m\} \subset \mathrm{R}[Y_0,\ldots,Y_\ell,X_1,\ldots,X_k],$$

where each Q_i is homogeneous of degree 2 in the variables $Y_0,\ldots,Y_\ell$, and of degree at most d in $X_1,\ldots,X_k$.

For $i \in [m]$, let $A_i \subset \mathbf{S}^\ell \times \mathrm{R}^k$ be semi-algebraic sets defined by

$$A_i = \{(\mathbf{y},\mathbf{x}) \mid |\mathbf{y}| = 1 \wedge Q_i(\mathbf{y},\mathbf{x}) \le 0)\},$$

Let $\pi : \mathbf{S}^\ell \times \mathrm{R}^k \to \mathrm{R}^k$ be the projection on the last k co-ordinates.

Then, the number of homotopy types amongst the fibers $\bigcup_{i=1}^m A_{i,\mathbf{x}}$ is bounded by

$$(2^m \ell k d)^{O(mk)}.$$

With the same assumptions as in Theorem 4.22 we have

Theorem 4.23. *The number of stable homotopy types amongst the fibers* $\bigcap_{i=1}^{m} A_{i,\mathbf{x}}$ *is bounded by*

$$(2^m \ell k d)^{O(mk)}.$$

Before proving Theorems 4.22 and 4.23 we first prove two preliminary lemmas.

Lemma 4.24. *There exists a finite set* $T \subset \mathrm{R}^k$*, with*

$$\#T \leq (2^m \ell k d)^{O(mk)},$$

such that for every $\mathbf{x} \in \mathrm{R}^k$ *there exists* $\mathbf{z} \in T$*, a semi-algebraic set* $D_{\mathbf{x},\mathbf{z}} \subset \mathrm{R}'^{m+\ell}$*, and semi-algebraic maps* $f_{\mathbf{x}}, f_{\mathbf{z}}$*, as shown in the diagram below, such that* $f_{\mathbf{x}}, f_{\mathbf{z}}$ *are both homotopy equivalences.*

$$\begin{tikzcd} & D_{\mathbf{x},\mathbf{z}} \arrow[dl, "f_{\mathbf{x}}"', "\sim"] \arrow[dr, "f_{\mathbf{z}}", "\sim"'] & \\ \mathrm{Ext}(\bigcup_{i\in[m]} A_{i,\mathbf{x}}, \mathrm{R}') & & \mathrm{Ext}(\bigcup_{i\in[m]} A_{i,\mathbf{z}}, \mathrm{R}') \end{tikzcd} \tag{4.17}$$

Moreover, for each $I \subset [m]$*, there exists a subset* $D_{I,\mathbf{x},\mathbf{z}} \subset D_{\mathbf{x},\mathbf{z}}$*, such that the restrictions,* $f_{I,\mathbf{x}}, f_{I,\mathbf{z}}$*, of* $f_{\mathbf{x}}, f_{\mathbf{z}}$ *to* $D_{I,\mathbf{x},\mathbf{z}}$ *give rise to the following diagram in which all maps are again homotopy equivalences.*

$$\begin{tikzcd} & D_{I,\mathbf{x},\mathbf{z}} \arrow[dl, "f_{I,\mathbf{x}}"', "\sim"] \arrow[dr, "f_{I,\mathbf{z}}", "\sim"'] & \\ \mathrm{Ext}(\bigcup_{i\in I} A_{i,\mathbf{x}}, \mathrm{R}') & & \mathrm{Ext}(\bigcup_{i\in I} A_{i,\mathbf{z}}, \mathrm{R}') \end{tikzcd} \tag{4.18}$$

For each $I \subset J \subset [m]$*,* $D_{I,\mathbf{x},\mathbf{z}} \subset D_{J,\mathbf{x},\mathbf{z}}$ *and the maps* $f_{I,\mathbf{x}}, f_{I,\mathbf{z}}$ *are restrictions of* $f_{J,\mathbf{x}}, f_{J,\mathbf{z}}$.

Proof of Lemma 4.24. By Proposition 4.12, there exists $T \subset \mathrm{R}^k$ with

$$\#T \leq (2^m \ell k d)^{O(mk)},$$

such that for every $\mathbf{x} \in \mathrm{R}^k$, there exists $\mathbf{z} \in T$, with the following property.

There is a semi-algebraic path, $\gamma : [0,1] \to \mathrm{R}'^k$ and a continuous semi-algebraic map, $\phi : \Omega \times [0,1] \to \Omega$, with $\gamma(0) = \mathbf{x}$, $\gamma(1) = \mathbf{z}$, and for each $I \subset [m]$,

$$\phi(\cdot, t)|_{F'_{I,j,\mathbf{x}}} : F'_{I,j,\mathbf{x}} \to F'_{I,j,\gamma(t)},$$

is a homeomorphism for each $0 \leq t \leq 1$ (see (4.2), (4.10) and (4.11) for the definition of Ω, R' and $F'_{I,j}$).

Now, observe that $C'_{I,j,\mathbf{x}}$ (resp. $C'_{I,j,\mathbf{z}}$) is a sphere bundle over $F'_{I,j,\mathbf{x}}$ (resp. $F'_{I,j,\mathbf{z}}$). Moreover

$$C'_{I,j,\mathbf{x}} = \{(\omega, \mathbf{y}) \mid \omega \in F'_{I,j,\mathbf{x}}, \mathbf{y} \in L_j^+(\omega, \mathbf{x}), |\mathbf{y}| = 1\},$$

and, for $\omega \in F'_{I,j,\mathbf{x}} \cap F'_{I,j-1,\mathbf{x}}$, we have $L_j^+(\omega, \mathbf{x}) \subset L_{j-1}^+(\omega, \mathbf{x})$.

We now prove that the map ϕ induces a homeomorphism $\tilde{\phi} : C'_{\mathbf{x}} \to C'_{\mathbf{z}}$, which for each $I \subset [m]$ and $0 \leq j \leq \ell$ restricts to a homeomorphism $\tilde{\phi}_{I,j} : C'_{I,j,\mathbf{x}} \to C'_{I,j,\mathbf{z}}$.

First recall that by a standard result in the theory of bundles (see for instance, [48], p. 313, Lemma 5), the isomorphism class of the sphere bundle $C'_{I,j,\mathbf{x}} \to F'_{I,j,\mathbf{x}}$, is determined by the homotopy class of the map,

$$\begin{array}{rcl} F'_{I,j,\mathbf{x}} & \to & Gr(\ell+1-j, \ell+1) \\ \omega & \mapsto & L_j^+(\omega, \mathbf{x}), \end{array}$$

where $Gr(m, n)$ denotes the Grassmannian variety of m dimensional subspaces of R'^n.

The map ϕ induces for each $j, 0 \leq j \leq \ell$, a homotopy between the maps

$$\begin{array}{rcl} f_0 : F'_{I,j,\mathbf{x}} & \to & Gr(\ell+1-j, \ell+1) \\ \omega & \mapsto & L_j^+(\omega, \mathbf{x}) \end{array}$$

and

$$\begin{array}{rcl} f_1 : F'_{I,j,\mathbf{z}} & \to & Gr(\ell+1-j, \ell+1) \\ \omega & \mapsto & L_j^+(\omega, \mathbf{z}) \end{array}$$

(after indentifying the sets $F'_{I,j,\mathbf{x}}$ and $F'_{I,j,\mathbf{z}}$ since they are homeomorphic) which respects the inclusions $L_j^+(\omega, \mathbf{x}) \subset L_{j-1}^+(\omega, \mathbf{x})$, and $L_j^+(\omega, \mathbf{z}) \subset L_{j-1}^+(\omega, \mathbf{z})$.

The above observation in conjunction with Lemma 5 in [48] is sufficient to prove the equivalence of the sphere bundles $C'_{I,j,\mathbf{x}}$ and $C'_{I,j,\mathbf{z}}$. But we need to prove a more general equivalence, involving all the sphere bundles $C'_{I,j,\mathbf{x}}$ simultaneously, for $0 \leq j \leq \ell$.

However, note that the proof of Lemma 5 in [48] proceeds by induction on the skeleton of the CW-complex of the base of the bundle. After choosing a sufficiently fine triangulation of the set $F'_{I,\mathbf{x}} \cong F'_{I,\mathbf{z}}$ compatible with the closed subsets $F'_{I,j,\mathbf{x}} \cong F'_{I,j,\mathbf{z}}$, the same proof extends without difficulty to this slightly more general situation to give a fiber preserving homeomorphism, $\tilde{\phi} : C'_{\mathbf{x}} \to C'_{\mathbf{z}}$, which restricts to an isomorphism of sphere bundles, $\tilde{\phi}_{I,j} : C'_{I,j,\mathbf{x}} \to C'_{I,j,\mathbf{z}}$, for each $I \subset [m]$ and $0 \le j \le \ell$.

We have the following maps.

$$\begin{tikzcd}
\mathrm{Ext}(A_{\mathbf{x}}, \mathrm{R}') & \mathrm{Ext}(B_{\mathbf{x}}, \mathrm{R}') \arrow[l, "\phi_2"'] & \mathrm{Ext}(C_{\mathbf{x}}, \mathrm{R}') \arrow[l, "i"'] & C'_{\mathbf{x}} \arrow[l, "r"'] \arrow[d, "\tilde{\phi}"] \\
\mathrm{Ext}(A_{\mathbf{z}}, \mathrm{R}') & \mathrm{Ext}(B_{\mathbf{z}}, \mathrm{R}') \arrow[l, "\phi_2"'] & \mathrm{Ext}(C_{\mathbf{z}}, \mathrm{R}') \arrow[l, "i"'] & C'_{\mathbf{z}} \arrow[l, "r"']
\end{tikzcd} \tag{4.19}$$

The map i is the inclusion map, and r is a retraction shown to exist by Proposition 4.11.

Since all the maps ϕ_2, i, r have been shown to be homotopy equivalences, by Propositions 4.6, 4.3, and 4.11, their composition is also a homotopy equivalence.

Moreover, for each $I \subset [m]$, the maps in the above diagram restrict properly to give a corresponding diagram:

$$\begin{tikzcd}
\mathrm{Ext}(A_{I,\mathbf{x}}, \mathrm{R}') & \mathrm{Ext}(B_{I,\mathbf{x}}, \mathrm{R}') \arrow[l, "\phi_2"'] & \mathrm{Ext}(C_{I,\mathbf{x}}, \mathrm{R}') \arrow[l, "i"'] & C'_{I,\mathbf{x}} \arrow[l, "r"'] \arrow[d, "\tilde{\phi}"] \\
\mathrm{Ext}(A_{I,\mathbf{z}}, \mathrm{R}') & \mathrm{Ext}(B_{I,\mathbf{z}}, \mathrm{R}') \arrow[l, "\phi_2"'] & \mathrm{Ext}(C_{I,\mathbf{z}}, \mathrm{R}') \arrow[l, "i"'] & C'_{I,\mathbf{z}} \arrow[l, "r"']
\end{tikzcd} \tag{4.20}$$

Now let $D_{\mathbf{x},\mathbf{z}} = C'_{\mathbf{x}}$, and $f_{\mathbf{x}} = \phi_2 \circ i \circ r$ and $f_{\mathbf{z}} = \phi_2 \circ i \circ r \circ \tilde{\phi}$. Finally, for each $I \subset [m]$, let $D_{I,\mathbf{x},\mathbf{z}} = C'_{I,\mathbf{x}}$ and the maps $f_{I,\mathbf{x}}, f_{I,\mathbf{z}}$ the restrictions of $f_{\mathbf{x}}$ and $f_{\mathbf{z}}$ respectively to $D_{I,\mathbf{x},\mathbf{z}}$. The collection of sets $D_{I,\mathbf{x},\mathbf{z}}$ and the maps $f_{I,\mathbf{x}}, f_{I,\mathbf{z}}$ clearly satisfy the conditions of the lemma. This completes the proof of the lemma. □

Remark 4.25. Note that if R_1 is a real closed sub-field of R, then Lemma 4.24 continues to hold after we substitute "$T \subset \mathrm{R}_1^k$" and "for all $\mathbf{x} \in \mathrm{R}_1^k$" in place of "$T \subset \mathrm{R}^k$" and "for all $\mathbf{x} \in \mathrm{R}^k$" in the statement of the lemma. This is a consequence of the Tarski-Seidenberg transfer principle.

With the same hypothesis as in Lemma 4.24 we also have,

Lemma 4.26. *There exists a finite set $T \subset \mathrm{R}^k$ with*

$$\#T \leq (2^m \ell k d)^{O(mk)}$$

such that for every $\mathbf{x} \in \mathrm{R}^k$, there exists $\mathbf{z} \in T$, for each $I \subset [m]$, a semi-algebraic set $E_{I,\mathbf{x},\mathbf{z}}$ defined over R'', where $\mathrm{R}'' = \mathrm{R}\langle \varepsilon, \bar{\varepsilon}, \bar{\delta}\rangle$ (see (4.10 for the definition of $\bar{\varepsilon}$ and $\bar{\delta}$), and S-maps $g_{I,\mathbf{x}}, g_{I,\mathbf{z}}$ as shown in the diagram below such that $g_{I,\mathbf{x}}, g_{I,\mathbf{z}}$ are both stable homotopy equivalences.

$$\begin{tikzcd} & E_{I,\mathbf{x},\mathbf{z}} & \\ \mathrm{Ext}(\bigcap_{i\in I} A_{i,\mathbf{x}}, \mathrm{R}'') \arrow[ur, "g_{\mathbf{x}}", "\sim"'] & & \mathrm{Ext}(\bigcap_{i\in I} A_{i,\mathbf{z}}, \mathrm{R}'') \arrow[ul, "g_{\mathbf{z}}"', "\sim"] \end{tikzcd} \tag{4.21}$$

For each $I \subset J \subset [m]$, $E_{J,\mathbf{x},\mathbf{z}} \subset E_{I,\mathbf{x},\mathbf{z}}$ and the maps $g_{J,\mathbf{x}}, g_{J,\mathbf{z}}$ are restrictions of of $g_{I,\mathbf{x}}, g_{I,\mathbf{z}}$.

Proof. Let $1 \gg \varepsilon > 0$ be an infinitesimal. For $1 \leq i \leq m$, we define

$$\tilde{Q}_i = Q_i + \varepsilon(Y_0^2 + \cdots + Y_\ell^2),$$

$$\tilde{A}_i = \{(\mathbf{y}, \mathbf{x}) \mid |\mathbf{y}| = 1 \wedge \tilde{Q}_i(\mathbf{y}, \mathbf{x}) \leq 0)\}.$$

Note that the set $\bigcap_{i\in I} \tilde{A}_{i,\mathbf{x}}$ is homotopy equivalent to $\mathrm{Ext}(\bigcap_{i\in I} A_{i,\mathbf{x}}, \mathrm{R}\langle\varepsilon\rangle)$ for each $I \subset [m]$ and $\mathbf{x} \in \mathrm{R}^k$. Applying Lemma 4.24 (see Remark 4.25) to the family $\tilde{\mathcal{Q}} = \{-\tilde{Q}_1, \ldots, -\tilde{Q}_m\}$, we have that there exists a finite set $T \subset \mathrm{R}^k$ with

$$\#T \leq (2^m \ell k d)^{O(mk)}$$

such that for every $\mathbf{x} \in \mathrm{R}^k$, there exists $\mathbf{z} \in T$ such that for each $I \subset [m]$, the following diagram

$$\begin{tikzcd} & \tilde{D}_{I,\mathbf{x},\mathbf{z}} \arrow[dl, "\tilde{f}_{I,\mathbf{x}}"', "\sim"] \arrow[dr, "\tilde{f}_{I,\mathbf{z}}", "\sim"'] & \\ \mathrm{Ext}(\bigcup_{i\in I} \tilde{A}_{i,\mathbf{x}}, \mathrm{R}'') & & \mathrm{Ext}(\bigcup_{i\in I} \tilde{A}_{i,\mathbf{z}}, \mathrm{R}'') \end{tikzcd} \tag{4.22}$$

where for each $\mathbf{x} \in \mathrm{R}^k$ we denote

$$\tilde{A}_{i,\mathbf{x}} = \{(\mathbf{y},\mathbf{x}) \mid |\mathbf{y}| = 1 \wedge -\tilde{Q}_i(\mathbf{y},\mathbf{x}) \leq 0)\},$$

$\tilde{f}_{I,\mathbf{x}}, \tilde{f}_{I,\mathbf{z}}$ are homotopy equivalences.

Note that for each $\mathbf{x} \in \mathrm{R}^k$, the set $\mathrm{Ext}(\bigcap_{i\in I} A_{i,\mathbf{x}}, \mathrm{R}'')$ is a deformation retract of the complement of $\mathrm{Ext}(\bigcup_{i\in I} \tilde{A}_{i,\mathbf{x}}, \mathrm{R}'')$ and hence is Spanier-Whitehead dual to $\mathrm{Ext}(\bigcup_{i\in I} \tilde{A}_{i,\mathbf{x}}, \mathrm{R}'')$. The lemma now follows by taking the Spanier-Whitehead dual of diagram (4.22) above for each $I \subset [m]$. □

Proof of Theorem 4.22. Follows directly from Lemma 4.24. □

Proof of Theorem 4.23. Follows directly from Lemma 4.26. □

We now prove a homogenous version of Theorem 4.1

Theorem 4.27. *Let* R *be a real closed field and let*

$$\mathcal{Q} = \{Q_1, \ldots, Q_m\} \subset \mathrm{R}[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k],$$

where each Q_i is homogeneous of degree 2 in the variables $Y_0, \ldots, Y_\ell$, and of degree at most d in $X_1, \ldots, X_k$.

Let $\pi : \mathbf{S}^\ell \times \mathrm{R}^k \to \mathrm{R}^k$ be the projection on the last k co-ordinates. Then, for any $\mathcal{Q}$-closed semi-algebraic set $S \subset \mathbf{S}^\ell \times \mathrm{R}^k$, the number of stable homotopy types amongst the fibers $S_\mathbf{x}$ is bounded by

$$(2^m \ell k d)^{O(mk)}.$$

Proof. We first replace the family $\mathcal{Q}$ by the family,

$$\mathcal{Q}' = \{Q_1, \ldots, Q_{2m}\} = \{Q, -Q \mid Q \in \mathcal{Q}\}.$$

Note that the cardinality of $\mathcal{Q}'$ is $2m$. Let

$$A_i = \{(\mathbf{y},\mathbf{x}) \mid |\mathbf{y}| = 1 \wedge Q_i(\mathbf{y},\mathbf{x}) \leq 0)\}.$$

It follows from Lemma 4.26 that there exists a set $T \subset \mathrm{R}^k$ with

$$\#T \leq (2^m \ell k d)^{O(mk)}$$

such that for every $I \subset [2m]$ and $\mathbf{x} \in \mathrm{R}^k$, there exists $\mathbf{z} \in T$ and a semi-algebraic set $E_{I,\mathbf{x},\mathbf{z}}$ defined over $\mathrm{R}'' = \mathrm{R}\langle \varepsilon, \bar{\varepsilon}, \bar{\delta} \rangle$ and S-maps $g_{I,\mathbf{x}}, g_{I,\mathbf{z}}$ as shown in the diagram below such that $g_{I,\mathbf{x}}, g_{I,\mathbf{z}}$ are both stable homotopy equivalences.

$$\begin{tikzcd} & E_{I,\mathbf{x},\mathbf{z}} & \\ \mathrm{Ext}(\bigcap_{i \in I} A_{i,\mathbf{x}}, \mathrm{R}'') \arrow[ur, "g_{I,\mathbf{x}}", "\sim"'] & & \mathrm{Ext}(\bigcap_{i \in I} A_{i,\mathbf{z}}, \mathrm{R}'') \arrow[ul, "g_{I,\mathbf{z}}"', "\sim"] \end{tikzcd} \tag{4.23}$$

Now notice that each $\mathcal{Q}$-closed set S is a union of sets of the form $\bigcap_{i \in I} A_i$ with $I \subset [2m]$. Let

$$S = \bigcup_{I \in \Sigma \subset 2^{[2m]}} \bigcap_{i \in I} A_i.$$

Moreover, the intersection of any sub-collection of sets of the kind, $\bigcap_{i \in I} A_i$ with $I \subset [2m]$, is also a set of the same kind. More precisely, for any $\Sigma' \subset \Sigma$ there exists $I_{\Sigma'} \in 2^{[2m]}$ such that

$$\bigcap_{I \in \Sigma'} \bigcap_{i \in I} A_i = \bigcap_{i \in I_{\Sigma'}} A_i.$$

We are not able to show directly a stable homotopy equivalence between $S_{\mathbf{x}}$ and $S_{\mathbf{z}}$. Instead, we note that the S-maps $g_{I,\mathbf{x}}$ and $g_{I,\mathbf{z}}$ induce S-maps (cf. Definition 2.31)

$$\tilde{g}_{\mathbf{x}} : \mathrm{hocolim}(\{\mathrm{Ext}(\bigcap_{i \in I} A_{i,\mathbf{x}}, \mathrm{R}'') \mid I \in \Sigma\}) \longrightarrow \mathrm{hocolim}(\{E_{I,\mathbf{x},\mathbf{z}} \mid I \in \Sigma\})$$
$$\tilde{g}_{\mathbf{z}} : \mathrm{hocolim}(\{\mathrm{Ext}(\bigcap_{i \in I} A_{i,\mathbf{z}}, \mathrm{R}'') \mid I \in \Sigma\}) \longrightarrow \mathrm{hocolim}(\{E_{I,\mathbf{x},\mathbf{z}} \mid I \in \Sigma\})$$

which are stable homotopy equivalences by Lemma 2.33 since each $g_{I,\mathbf{x}}$ and $g_{I,\mathbf{z}}$ is a stable homotopy equivalence.

Since $\mathrm{hocolim}(\{\bigcap_{i \in I} A_{i,\mathbf{x}} \mid I \in \Sigma\})$ (resp. $\mathrm{hocolim}(\{\bigcap_{i \in I} A_{i,\mathbf{z}} \mid I \in \Sigma\})$) is homotopy equivalent by Lemma 2.32 to $\bigcup_{I \in \Sigma} \bigcap_{i \in I} A_{i,\mathbf{x}}$ (resp. $\bigcup_{I \in \Sigma} \bigcap_{i \in I} A_{i,\mathbf{z}}$), it follows (see Remark 2.1) that $S_{\mathbf{x}} = \bigcup_{I \in \Sigma} \bigcap_{i \in I} A_{i,\mathbf{x}}$ is stable homotopy equivalent to $S_{\mathbf{z}} = \bigcup_{I \in \Sigma} \bigcap_{i \in I} A_{i,\mathbf{z}}$. This proves the theorem. □

4.5.2 Inhomogeneous case

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. Let ϕ be a $\mathcal{P}$-closed formula defining the $\mathcal{P}$-closed semi-algebraic set $S \subset \mathrm{R}^{\ell+k}$. Let $1 \gg \varepsilon > 0$ be an infinitesimal, and let

$$P_0 = \varepsilon^2 \left(\sum_{i=1}^{\ell} Y_i^2 + \sum_{i=1}^{k} X_i^2 \right) - 1.$$

Let $\tilde{\mathcal{P}} = \mathcal{P} \cup \{P_0\}$, and let $\tilde{\phi}$ be the $\tilde{\mathcal{P}}$-closed formula defined by

$$\tilde{\phi} = \phi \wedge \{P_0 \leq 0\},$$

defining the $\tilde{\mathcal{P}}$-closed semi-algebraic set $S_b \subset \mathrm{R}\langle\varepsilon\rangle^{\ell+k}$. Note that the set S_b is bounded.

It follows from the local conical structure of semi-algebraic sets at infinity [29] that the semi-algebraic set S_b has the same homotopy type as $\mathrm{Ext}(S, \mathrm{R}\langle\varepsilon\rangle)$.

Considering each P_i as a polynomial in the variables $Y_1, \ldots, Y_\ell$ with coefficients in $\mathrm{R}[X_1, \ldots, X_k]$, and let P_i^h denote the homogenization of P_i. Thus the polynomials $P_i^h \in \mathrm{R}[Y_0, \ldots, Y_\ell, X_1, \ldots, X_k]$ and are homogeneous of degree 2 in the variables $Y_0, \ldots, Y_\ell$.

Let $S_b^h \subset \mathbf{S}^\ell \times \mathrm{R}\langle\varepsilon\rangle^k$ be the semi-algebraic set defined by the $\tilde{\mathcal{P}}^h$-closed formula $\tilde{\phi}^h$ (replacing P_i by P_i^h in $\tilde{\phi}$). It is clear that S_b^h is a union of two disjoint, closed and bounded semi-algebraic sets each homeomorphic to S_b, which has the same homotopy type as $\mathrm{Ext}(S, \mathrm{R}\langle\varepsilon\rangle)$.

The theorem is now proven by applying Theorem 4.27 to the family $\tilde{\mathcal{P}}^h$ and the semi-algebraic set S_b^h. Note that two fibers $S_{\mathbf{x}}$ and $S_{\mathbf{y}}$ are stable homotopy equivalent if and only if $\mathrm{Ext}(S_{\mathbf{x}}, \mathrm{R}\langle\varepsilon\rangle)$ and $\mathrm{Ext}(S_{\mathbf{y}}, \mathrm{R}\langle\varepsilon\rangle)$ are stable homotopy equivalent (see Remark 2.1). $\square$

4.6 Metric upper bounds

In [22] certain metric upper bounds related to homotopy types were proven as applications of the main result. Similar results hold in the quadratic case, except now the bounds have a better dependence on ℓ. We state these results without proof.

We first recall the following results from [22]. Let $V \subset \mathrm{R}^\ell$ be a $\mathcal{P}$-semi-algebraic set, where $\mathcal{P} \subset \mathbb{Z}[Y_1, \ldots, Y_\ell]$. Suppose for each $P \in \mathcal{P}$, $\deg(P) < d$, and the maximum of the absolute values of coefficients in P is less than some constant M, $0 < M \in \mathbb{Z}$.

Theorem 4.28. *There exists a constant $c > 0$, such that for any $r_1 > r_2 > M^{d^{c\ell}}$ we have*

1. $V \cap B_\ell(0, r_1)$ *and* $V \cap B_\ell(0, r_2)$ *are homotopy equivalent, and*

2. $V \setminus B_\ell(0, r_1)$ *and* $V \setminus B_\ell(0, r_2)$ *are homotopy equivalent.*

In the special case of quadratic polynomials we get the following improvement of Theorem 4.28.

Theorem 4.29. *Let* R *be a real closed field. Let* $V \subset \mathrm{R}^\ell$ *be a* $\mathcal{P}$*-semi-algebraic set, where*

$$\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[Y_1, \ldots, Y_\ell],$$

with $\deg(P_i) \leq 2$, $1 \leq i \leq m$ *and the maximum of the absolute values of coefficients in* $\mathcal{P}$ *is less than some constant* M, $0 < M \in \mathbb{Z}$.

There exists a constant $c > 0$, *such that for any* $r_1 > r_2 > M^{\ell^{cm}}$ *we have,*

1. $V \cap B_\ell(0, r_1)$ *and* $V \cap B_\ell(0, r_2)$ *are stable homotopy equivalent, and*

2. $V \setminus B_\ell(0, r_1)$ *and* $V \setminus B_\ell(0, r_2)$ *are stable homotopy equivalent.*

CHAPTER V

ALGORITHMS AND THEIR IMPLEMENTATION

5.1 Computing the Betti Numbers of Arrangements

In this chapter, we consider arrangements of compact objects in $\mathbb{R}^k$ which are simply connected. This implies, in particular, that their first Betti number is zero. We describe an algorithm for computing the zero-th and the first Betti number of such an arrangement, along with its implementation [15]. For the implementation, we restrict our attention to arrangements in $\mathbb{R}^3$ and take for our objects the simplest possible semi-algebraic sets in $\mathbb{R}^3$ which are topologically non-trivial – namely, each object is an ellipsoid defined by a single quadratic equation. Ellipsoids are simply connected, but with non-vanishing second co-homology groups. We also allow solid ellipsoids defined by a single quadratic inequality. Computing the Betti numbers of an arrangement of ellipsoids in $\mathbb{R}^3$ is already a challenging computational problem in practice and to our knowledge no existing software can effectively deal with this case. Note that arrangements of ellipsoids are topologically quite different from arrangements of balls. For instance, the union of two ellipsoids can have non-zero first Betti number, unlike in the case of balls.

5.1.1 Outline of the Method

The following corollary follows immediately from Proposition 2.20.

Corollary 5.1. *Let be* $S = \bigcup_{i=1}^m S_i \subset \mathbb{R}^k$ *such that* $S_1, \ldots, S_m$ *are compact semi-algebraic sets with*

1. $\mathrm{H}^0(S_i) = \mathbb{Q}$, *and*

2. $\mathrm{H}^1(S_i) = 0$, $1 \leq i \leq m$.

Let the homomorphisms δ_0 *and* δ_1 *in the following sequence be defined as in Chapter 2.2.2 (identifying* $\mathrm{H}^0(\mathrm{K})$ *with the* $\mathbb{Q}$*-vector space of locally constant functions on a simplicial* *complex* K*).*

$$\bigoplus_i \mathrm{H}^0(S_i) \xrightarrow{\delta_0} \bigoplus_{i<j} \mathrm{H}^0(S_i \cap S_j) \xrightarrow{\delta_1} \bigoplus_{i<j<\ell} \mathrm{H}^0(S_i \cap S_j \cap S_\ell).$$

Then,

$$\begin{aligned} b_0(S) &= \dim(\mathrm{Ker}(\delta_0)), \\ b_1(S) &= \dim(\mathrm{Ker}(\delta_1)) - \dim(\mathrm{Im}(\delta_0)). \end{aligned}$$

The importance of Corollary 5.1 lies in the following observation. Given an arrangement, $\{S_1, \ldots, S_m\}$, of m simply connected objects in $\mathbb{R}^k$, suppose we are able to identify the connected components of all pairwise and triple-wise intersections of these objects and their incidences (that is, which connected component of $S_i \cap S_j \cap S_\ell$ is contained in which connected component of $S_i \cap S_j$). Then this information is sufficient to compute the zero-th and the first Betti number of the arrangement. We only have to look at the objects of the arrangement at most three at a time. Thus, the cost of computing the connected components and incidences is $O(m^3)$. This is to be compared with having to compute a global triangulation of the whole arrangement using cylindrical algebraic decomposition which would have entailed a cost of $O(m^{2^k})$.

Recall that a cylindrical decomposition (see Chapter 2.1.4) adapted to a finite set $\mathcal{P}$ of polynomials in $\mathbb{R}[X_1, \ldots, X_k]$ produces a graph where the vertices correspond to cells in $\mathcal{S}_k$ and edges correspond to adjacencies. Moreover, each cell in $\mathcal{S}_k$ is $\mathcal{P}$-invariant and we know the sign for each P in $\mathcal{P}$ on each such cell. Hence, given an arrangement, $\{S_1, \ldots, S_m\}$, of m semi-algebraic sets in $\mathbb{R}^k$, we are able to identify the connected components of all pairwise and triple-wise intersections of these objects and their incidences by computing a cylindrical decomposition adapted to the families $\mathcal{P}_{i,j,\ell}$, $1 \le i < j < \ell \le m$, where $\mathcal{P}_{i,j,\ell}$ is the set of polynomials used in the definition of S_i, S_j, and S_ℓ and by performing a graph transversal algorithm on the graph described above.

To sum up, we now formally describe our algorithm for computing the zero-th and the first Betti numbers of an arrangement of m simply connected compact objects in $\mathbb{R}^k$.

Algorithm 5.2 (Computing the zero-th and the first Betti number)**.**

Input: compact sets $S_i \subset \mathbb{R}^k$, $1 \le i \le m$, with $b_0(S_i) = 1$ and $b_1(S_i) = 0$.

Output: $b_0(S)$ and $b_1(S)$.

Procedure:

- For each triple (i, j, ℓ), $1 \leq i < j < \ell \leq m$, do the following:

  Compute a cylindrical decomposition adapted to the set $\{S_i, S_j, S_\ell\}$.

  Identify the connected components of all pairwise and triple-wise intersections and their incidences.

- Compute the matrices A and B corresponding to the sequence of homomorphisms:

$$\bigoplus_i \mathrm{H}^0(S_i) \xrightarrow{\delta_0} \bigoplus_{i<j} \mathrm{H}^0(S_i \cap S_j) \xrightarrow{\delta_1} \bigoplus_{i<j<\ell} \mathrm{H}^0(S_i \cap S_j \cap S_\ell).$$

- Compute

$$\begin{aligned} b_0(S) &= d_0 - \mathrm{rk}(A), \text{ and} \\ b_1(S) &= d_1 - \mathrm{rk}(B) - \mathrm{rk}(A), \end{aligned}$$

  where d_0 is the dimension of $\bigoplus_{1 \leq i \leq m} \mathrm{H}^0(S_i)$, d_1 is the dimension of $\bigoplus_{1 \leq i < j \leq m} \mathrm{H}^0(S_i \cap S_j)$, and the rank of a matrix is denoted by $\mathrm{rk}(\cdot)$.

5.1.2 The Implementation

The algorithm has been prototypically implemented using QEPCAD B (Version 1.27) [77] and Magma [71] for compact sets $S_i \subset \mathbb{R}^3$. We use the package QEPCAD B for computing the cylindrical decompositions, in Step 1 of Algorithm 5.2. There are several other packages available for computing cylindrical decompositions, for instance REDLOG [78]. The main reason for using QEPCAD B is that it provides some important information regarding cell adjacency, that is not provided by the other systems.

In Figure 7, which shows the QEPCAD B output for a cylindrical decomposition adapted to the unit sphere, the first (resp., second and third) column corresponds to the cylindrical decomposition of the line (resp. plane and $\mathbb{R}^3$). Note that the signs accompanying the cells give the signs of projection factors computed by QEPCAD B and the letter "T" and "F" corresponds to true and false value of the cells, i.e., depending upon whether our input formula is true or false on this cell.

```
()---(1)p1(-,-)---(1,1)p1(+)---(1,1,1)p1(+) F
  ---(2)p1(0,-)---(2,1)p1(+)---(2,1,1)p1(+) F
                ---(2,2)p1(0)---(2,2,1)p1(+) F
                               ---(2,2,2)p1(0) T
                               ---(2,2,3)p1(+) F
                ---(2,3)p1(+)---(2,3,1)p1(+) F
  ---(3)p1(+,-)---(3,1)p1(+)---(3,1,1)p1(+) F
                ---(3,2)p1(0)---(3,2,1)p1(+) F
                               ---(3,2,2)p1(0) T
                               ---(3,2,3)p1(+) F
                ---(3,3)p1(-)---(3,3,1)p1(+) F
                               ---(3,3,2)p2(0) T
                               ---(3,3,3)p1(-) F
                               ---(3,3,4)p2(0) T
                               ---(3,3,5)p1(+) F
                ---(3,4)p1(0)---(3,4,1)p1(+) F
                               ---(3,4,2)p1(0) T
                               ---(3,4,3)p1(+) F
                ---(3,5)p1(+)---(3,5,1)p1(+) F
  ---(4)p1(+,0)---(4,1)p1(+)---(4,1,1)p1(+) F
                ---(4,2)p1(0)---(4,2,1)p1(+) F
                               ---(4,2,2)p1(0) T
                               ---(4,2,3)p1(+) F
                ---(4,3)p1(+)---(4,3,1)p1(+) F
  ---(5)p1(+,+)---(5,1)p1(+)---(5,1,1)p1(+) F
```

Figure 7: Output of a cylindrical decomposition using QEPCAD B

Even though QEPCAD B does not provide full information regarding cell adjacencies in dimension three, we are still able to deduce all the needed cell adjacencies as described in Chapter 2.1.4.3, making use of the fact that input polynomials are quadratic.

We use Magma for post-processing of the information output by QEPCAD B, in Steps 2 and 3 of the algorithm. Note that all computations performed are exact with no possibility of numerical errors.

To illustrate our implementation, we consider four examples where the ellipsoids

$$S_i = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbb{R}^3 \mid P_i(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0\},$$

$1 \leq i \leq 27$, are defined by the following list of polynomials (see Table 1) We denote by A and B the matrices of the homomorphisms δ_1 and δ_2 with respect to the obvious basis. The columns (resp., the rows) of the matrix A are labeled by e_i (resp., $e^p_{i,j}$), while the columns (resp., the rows) of the matrix B are labeled by $e^p_{i,j}$ (resp., $e^p_{i,j,k}$), where e_i corresponds to S_i, $e^p_{i,j}$ corresponds to the p-th connected component of $S_i \cap S_j$ and $e^p_{i,j,\ell}$ corresponds to the p-th connected component of $S_i \cap S_j \cap S_\ell$.

Remark 5.3. In the examples described below, we have modified the matrix A as follows.

Table 1: Input polynomials defining the different arrangements

$$
\begin{aligned}
P_1 &= 8/9X_1^2 + 1/64X_2^2 + 1/6X_3^2 - 1\\
P_2 &= 1/64X_1^2 + 8/9X_2^2 + 8/9X_3^2 - 1\\
P_3 &= 8/9X_1^2 + 8/9X_2^2 + 1/64X_3^2 - 1\\
P_4 &= 8/9(X_1-4)^2 + 1/64(X_2-4)^2 + 1/6X_3^2 - 1\\
P_5 &= 1/64(X_1-4)^2 + 8/9(X_2-4)^2 + 8/9X_3^2 - 1\\
P_6 &= 8/9(X_1-4)^2 + 8/9(X_2-4)^2 + 1/64X_3^2 - 1\\
P_7 &= (X_1-1)^2 + (X_2-2)^2 + X_3^2 - 3\\
P_8 &= 5X_1^2 + 1/9X_2^2 + 2X_3^2 - 1\\
P_9 &= 1/9X_1^2 + 5X_2^2 + 5X_3^2 - 1\\
P_{10} &= 5X_1^2 + 5X_2^2 + 1/9X_3^2 - 1\\
P_{11} &= 5(X_1-1)^2 + 1/9(X_2-1)^2 + 2X_3^2 - 1\\
P_{12} &= 1/9(X_1-1)^2 + 5(X_2-1)^2 + 5X_3^2 - 1\\
P_{13} &= 5(X_1-1)^2 + 5(X_2-1)^2 + 1/9X_3^2 - 1\\
P_{14} &= 5(X_1+1)^2 + 1/9(X_2-1)^2 + 2X_3^2 - 1\\
P_{15} &= 1/9(X_1+1)^2 + 5(X_2-1)^2 + 5X_3^2 - 1\\
P_{16} &= 5(X_1+1)^2 + 5(X_2-1)^2 + 1/9X_3^2 - 1\\
P_{17} &= 5(X_1-1)^2 + 1/9(X_2+1)^2 + 2X_3^2 - 1\\
P_{18} &= 1/9(X_1-1)^2 + 5(X_2+1)^2 + 5X_3^2 - 1\\
P_{19} &= 5(X_1-1)^2 + 5(X_2+1)^2 + 1/9X_3^2 - 1\\
P_{20} &= 5(X_1+1)^2 + 1/9(X_2+1)^2 + 2X_3^2 - 1\\
P_{21} &= 1/9(X_1+1)^2 + 5(X_2+1)^2 + 5X_3^2 - 1\\
P_{22} &= 5(X_1+1)^2 + 5(X_2+1)^2 + 1/9X_3^2 - 1\\
P_{23} &= 6(X_1-1/2)^2 + 6X_2^2 + 1/6X_3^2 - 1\\
P_{24} &= 4X_1^2 + 4(X_2-1/2)^2 + 1/6X_3^2 - 1\\
P_{25} &= 5(X_1+2)^2 + 5X_2^2 + 1/6X_3^2 - 1\\
P_{26} &= 1/6(X_1+2)^2 + 5(X_2-2)^2 + 5X_3^2 - 1\\
P_{27} &= 5(X_1+2)^2 + 1/6(X_2-2)^2 + 5X_3^2 - 1
\end{aligned}
$$

Since we know that each input set S_i has exactly one connected component, we can simplify the computation. We only need to check whether or not the intersection $S_i \cap S_j$ is empty. Therefore, we have exactly one row for each intersection instead of one row for each connected component of each intersection $S_i \cap S_j$, and this reduces the size of the matrix A without changing its rank. For the matrix B we delete all rows containing only zeros which correspond to empty triple intersections $S_i \cap S_j \cap S_\ell$.

Example 5.4 (Three ellipsoids)**.** Let S be the union of the first three ellipsoids, i.e.,

Figure 8: Three ellipsoids

$S = \bigcup_{i=1}^{3} S_i$ (see Figure 8). Then

$$A = \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \left(\begin{array}{ccc} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{array}\right) \end{array} \begin{array}{c} \\ e_{1,2} \\ e_{1,3} \\ e_{2,3} \end{array}$$

$$B = \begin{array}{c} \begin{array}{cccc} e_{1,2}^1 & e_{1,2}^2 & e_{1,3} & e_{2,3} \end{array} \\ \left(\begin{array}{cccc} 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right) \end{array} \begin{array}{c} \\ e_{1,2,3}^1 \\ e_{1,2,3}^2 \\ e_{1,2,3}^3 \\ e_{1,2,3}^4 \\ e_{1,2,3}^5 \\ e_{1,2,3}^6 \\ e_{1,2,3}^7 \\ e_{1,2,3}^8 \end{array}$$

In this case,

$$\begin{aligned} b_0(S) &= d_0 - \mathrm{rk}(A) = 3 - 2 = 1 \\ b_1(S) &= d_1 - \mathrm{rk}(B) - \mathrm{rk}(A) = (4 - 2) - 2 = 0 \end{aligned}$$

Example 5.5 (Six ellipsoids)**.** Let the set S be the union of the first six ellipsoids S_i, $1 \leq i \leq 6$, i.e., $S = \bigcup_{i=1}^{6} S_i$ (see Figure 9). Then

Figure 9: Six ellipsoids

$$A = \begin{array}{cc} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} & \\ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} & \begin{array}{c} e_{1,2} \\ e_{1,3} \\ e_{1,4} \\ e_{1,5} \\ e_{1,6} \\ e_{2,3} \\ e_{2,4} \\ e_{2,5} \\ e_{2,6} \\ e_{3,4} \\ e_{3,5} \\ e_{3,6} \\ e_{4,5} \\ e_{4,6} \\ e_{5,6} \end{array} \end{array}$$

and

$$B = \begin{array}{cc} \begin{array}{cccccccccccc} e_{1,2}^1 & e_{1,2}^2 & e_{1,3}^1 & e_{1,5}^1 & e_{1,5}^2 & e_{2,3}^1 & e_{2,4}^1 & e_{2,4}^2 & e_{4,5}^1 & e_{4,5}^2 & e_{4,6}^1 & e_{5,6}^1 \end{array} & \\ \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix} & \begin{array}{c} e_{1,2,3}^1 \\ e_{1,2,3}^2 \\ e_{1,2,3}^3 \\ e_{1,2,3}^4 \\ e_{1,2,3}^5 \\ e_{1,2,3}^6 \\ e_{1,2,3}^7 \\ e_{1,2,3}^8 \\ e_{4,5,6}^1 \\ e_{4,5,6}^2 \\ e_{4,5,6}^3 \\ e_{4,5,6}^4 \\ e_{4,5,6}^5 \\ e_{4,5,6}^6 \\ e_{4,5,6}^7 \\ e_{4,5,6}^8 \end{array} \end{array}$$

In this case,

$$\begin{aligned} b_0(S) &= d_0 - \mathrm{rk}(A) = 6 - 5 = 1 \\ b_1(S) &= d_1 - \mathrm{rk}(B) - \mathrm{rk}(A) = (12 - 4) - 5 = 3 \end{aligned}$$

Figure 10: Seven ellipsoids

Example 5.6 (Seven ellipsoids)**.** Let the set S be the union of the first seven ellipsoids S_i, $1 \leq i \leq 7$, i.e., $S = \bigcup_{i=1}^{7} S_i$ (see Figure 10). Then

$$A = \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \left(\begin{array}{ccccccc}
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \end{array}
\begin{array}{c} \\ e_{1,2} \\ e_{1,3} \\ e_{1,4} \\ e_{1,5} \\ e_{1,6} \\ e_{1,7} \\ e_{2,3} \\ e_{2,4} \\ e_{2,5} \\ e_{2,6} \\ e_{2,7} \\ e_{3,4} \\ e_{3,5} \\ e_{3,6} \\ e_{3,7} \\ e_{4,5} \\ e_{4,6} \\ e_{4,7} \\ e_{5,6} \\ e_{5,7} \\ e_{6,7} \end{array}$$

and

$$
B = \begin{array}{cc}
\begin{matrix} e^1_{1,2} & e^2_{1,2} & e^1_{1,3} & e^1_{1,5} & e^2_{1,5} & e^1_{1,7} & e^1_{2,3} & e^1_{2,4} & e^2_{2,4} & e^1_{2,7} & e^1_{3,7} & e^1_{4,5} & e^2_{4,5} & e^1_{4,6} & e^1_{5,6} & e^1_{5,7} \end{matrix} & \\
\begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0
\end{pmatrix} &
\begin{matrix} e^1_{1,2,3} \\ e^2_{1,2,3} \\ e^3_{1,2,3} \\ e^4_{1,2,3} \\ e^5_{1,2,3} \\ e^6_{1,2,3} \\ e^7_{1,2,3} \\ e^8_{1,2,3} \\ e^1_{1,2,7} \\ e^2_{1,2,7} \\ e^1_{1,5,7} \\ e^2_{1,5,7} \\ e^1_{2,3,7} \\ e^2_{2,3,7} \\ e^3_{2,3,7} \\ e^4_{2,3,7} \\ e^1_{4,5,6} \\ e^2_{4,5,6} \\ e^3_{4,5,6} \\ e^4_{4,5,6} \\ e^5_{4,5,6} \\ e^6_{4,5,6} \\ e^7_{4,5,6} \\ e^8_{4,5,6} \end{matrix}
\end{array}
$$

In this case,

$$
\begin{aligned}
b_0(S) &= d_0 - \mathrm{rk}(A) = 7 - 6 = 1 \\
b_1(S) &= d_1 - \mathrm{rk}(B) - \mathrm{rk}(A) = (16 - 7) - 6 = 3
\end{aligned}
$$

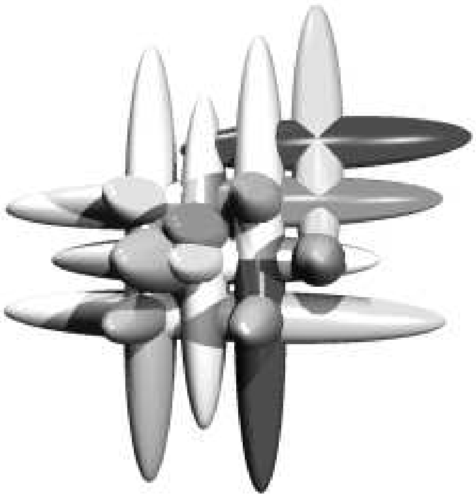

Figure 11: Twenty ellipsoids

Example 5.7 (20 ellipsoids)**.** Let the set S be the union of the last 20 ellipsoids S_i, $8 \le i \le 27$, i.e., $S = \bigcup_{i=8}^{27} S_i$ (see Figure 11). Thus, we get a 190×20-matrix A of rank equal to 19, a 190×107-matrix B of rank equal to 55, and the dimension of $\bigoplus_{i<j} H^0(S_i \cap S_j)$

is equal to 107. In this case,

$$\begin{aligned} b_0(S) &= d_0 - \mathrm{rk}(A) = 20 - 19 = 1 \\ b_1(S) &= d_1 - \mathrm{rk}(B) - \mathrm{rk}(A) = 107 - 55 - 19 = 33 \end{aligned}$$

5.2 Computing the Real Intersection of Quadratic Surfaces

In this chapter, we consider the problem of computing the real intersection of three quadratic surfaces, or quadrics, defined by the quadratic polynomials P_1, P_2 and P_3 in $\mathbb{R}^3$. We describe an algorithm for computing the isolated points and a linear graph embedded into $\mathbb{R}^3$ (if the real intersection form a curve) representing the real intersection of the three quadrics defined by the three polynomials P_i, along with its implementation [60]. For the implementation, we restrict our attention to quadrics with defining equation having rational coefficients.

Before outlining our method, we define the silhouette curve and cut curve, which can be interpreted in our setting as the projection of one quadric and the projection of intersection curve of two quadrics into the $X_1 - X_2$-plane, respectively.

Definition 5.8. Let P, $Q \in \mathrm{R}[X_1, X_2, X_3]$. The algebraic curves with defining polynomials

$$\mathrm{Sil}(P) := \mathrm{Res}(P, \partial P/\partial X_3), \quad \mathrm{cut}(P,Q) := \mathrm{Res}(P,Q)$$

are called **silhouette curve** and **cut curve** respectively.

Another geometric interpretation of the silhouette curve in our setting is the following. The silhouette curve defined by $\mathrm{Sil}(P_1)$ contains all points $(\mathbf{x}, \mathbf{y})$ such that the polynomial $P_1(\mathbf{x}, \mathbf{y}, X_3)$ has exactly one root $\mathbf{z}$ of multiplicity 2.

5.2.1 Outline of the Method

The basic idea of computing the intersection of three quadrics is based on the cylindrical decomposition (see Chapter 2.1.4). As the algorithm of Schömer and Wolpert [82, 101], our approach can be summarized by several phases:

preparation, projection, planar arrangement analysis and **lifting phase**.

But our analysis of the planar arrangement and lifting phase differs from the methods presented in [82, 101].

First, we project one input quadric and the resulting space intersection curves of the pairwise intersections by computing (univariate) resultants onto the plane assuming that we have a "good" coordinate system by using the *Brown-McCallum projection operation* (see [34, 33]). The *Brown-McCallum projection operation* produces, based on the current literature, the smallest projection set in our setting. Then we analyze the planar arrangement of curves before we lift our solution into space (if possible). In other words, we compute the defining polynomial $\mathrm{Sil}(P_1)$ of the silhouette curve of the input quadric P_1, and the defining polynomial $\mathrm{cut}(P_1, P_i)$ of the corresponding cut curves (see Definition 5.8). Then we identify the common factor G of $\mathrm{cut}(P_1, P_2)$ and $\mathrm{cut}(P_1, P_3)$ and the corresponding gcd-free parts $H_i = \mathrm{cut}(P_1, P_i)/G$.

While Schömer and Wolpert [82] use resultants for computing the candidates of the X_1 and X_2-coordinates and analyze the resulting grid (see [82] for more details) afterwards, our planar analysis is based on the TOP algorithm (see Algorithm 2.12). To be more precise, we use the TOP-algorithm in order to obtain the topology of the curve defined by the common factor G (including some relevant points on the curve). In addition, we generalize the idea of [52] of using subresultants to two planar curves. We perform a linear change of coordinates (if it is needed) in order to have two planar curves of the arrangement in generic position (see Definition 2.10). Then we compute the X_1-coordinates of the (planar) intersection points of two curves using resultants as well as the X_2-coordinates of those points that can be described rationally in terms of X_1 via subresultant computations.

Furthermore, when the intersection points form a curve, the set of solutions is described topologically via a linear graph embedded in $\mathbb{R}^3$. The computed graph provides all the information for tracing this curve numerically since we know exactly how to proceed when we are close to a complicated point. Nevertheless, all computed points lie in the real intersection set of the three quadric surfaces defined by the three polynomials P_i.

Next we describe all necessary steps, but we omit the details on the change of coordinates to which we refer to several times. For more details on the change of coordinates see [52].

5.2.2 Details on the Preparation Phase

Before starting the real computation, we test the input quadrics for degeneracy and if they behave well under resultant computations. We want to make sure that all quadrics are of degree equal to two, X_3-regular, square-free and pairwise do not have a common factor of degree equal to one. The absence of the latter two conditions can easily be detected and solved as it simplifies the considered problem. For example, one quadric describes a single plane if it is not square-free, whereas two quadrics define three planes, with one of them in common, if they have a common factor of degree equal to one. Therefore, we omit the details on these cases, though we can detect and solve them easily. In the case that one of the other conditions is violated, we make a change of coordinates and start the computation again.

Finally, from now on we use the following assumption for the input polynomials P_i.

Assumption 5.9. The trivariate polynomials P_1, P_2 and P_3 with coefficients in $\mathbb{R}$ are all of degree equal to two, square-free, X_3-regular and pairwise do not have a common factor of degree equal to one.

5.2.3 Details on the Projection Phase

After a suitable preparation of our input quadrics, we assume throughout this and the following sections that the quadratic input polynomials have the properties of Assumption 5.9. It is worthwhile to mention that Assumption 5.9 is necessary in order to interpret correctly the projection onto the X_1-X_2 plane via resultant computation. Our projection method is based on the so-called restricted equational version of the *Brown-McCallum projection operation* (see [34],[33]) where we use the polynomial P_1 as the pivot constraint. The *Brown-McCallum projection operation* produces, based on the current literature, the smallest projection set $\tilde{\mathcal{P}}$ and consists of the following polynomials in our setting,

$$\tilde{\mathcal{P}} = \{\mathrm{Sil}(P_1), \mathrm{cut}(P_1, P_2), \mathrm{cut}(P_1, P_3)\}.$$

As in the beginning of our computation, we need to test the polynomials contained in the set $\tilde{\mathcal{P}}$ for degeneracy in order to interpret correctly the following resultant computations.

Thus, we simplify the set $\tilde{\mathcal{P}}$ further and we obtain the set $\mathcal{P}$ containing the following polynomials,

$$\mathcal{P} = \{\mathrm{Sil}(P_1), H_2, H_3, G\},$$

such that $H_i = \mathrm{cut}(P_1, P_i)/G$, where G is the greatest common divisor of $\mathrm{cut}(P_1, P_2)$ and $\mathrm{cut}(P_1, P_3)$. Moreover, we decompose the polynomial G further. We write $G = \widetilde{G} \cdot \widetilde{\mathrm{Sil}(P_1)}$ where $\widetilde{G}$ (resp., $\widetilde{\mathrm{Sil}(P_1)}$) is the gcd-free part (resp., greatest common divisor) of G and $\mathrm{Sil}(P_1)$. Note, that the decomposition of the polynomial G into $\widetilde{\mathrm{Sil}(P_1)}$ and $\widetilde{G}$ will be very useful for the lifting phase (see Chapter 5.2.5). Finally, we can summarize the projection phase as follows.

Algorithm 5.10 (Projection)**.**

Input: three polynomials P_1, P_2 and P_3 in $\mathbb{R}[X_1, X_2, X_3]$ with the properties of Assumption 5.9.

Output: $\mathcal{P} = \{\mathrm{Sil}(P_1), H_2, H_3, G\}$

such that H_i is the square-free part of $\mathrm{cut}(P_1, P_i)$ with respect to G,

where $G = \gcd(\mathrm{cut}(P_1, P_2), \mathrm{cut}(P_1, P_3))$. Moreover, we decompose the polynomial G into $G = \widetilde{G} \cdot \widetilde{\mathrm{Sil}(P_1)}$ where $\widetilde{G}$ (resp., $\widetilde{\mathrm{Sil}(P_1)}$) is the gcd-free part (resp., greatest common factor) of G and $\mathrm{Sil}(P_1)$.

5.2.4 Details on the Analysis of the Planar Arrangement

In this section, we describe how we analyze the planar arrangement. We assume from now on that the set $\mathcal{P}$ computed before is of the following form:

$$\mathcal{P} = \{\mathrm{Sil}(P_1), H_2, H_3, G\},$$

such that $H_i = \mathrm{cut}(P_1, P_i)/G$, where G is the square-free part of the common factor of $\mathrm{cut}(P_1, P_2)$ and $\mathrm{cut}(P_1, P_3)$). Moreover, we decompose the polynomial G further. We write $G = \widetilde{G} \cdot \widetilde{\mathrm{Sil}(P_1)}$ where $\widetilde{G}$ (resp., $\widetilde{\mathrm{Sil}(P_1)}$) is the gcd-free part (resp., greatest common factor) of G and $\mathrm{Sil}(P_1)$.

The problem, which might occur, is that the planar curves might not be in generic position which would ensure that we can use subresultants in order to compute the critical points (including intersection points with another curve) of the planar curves in our arrangement. In this case, we start the computation again after a change of coordinates if the planar curves are not in generic position. Hence, we assume throughout this and the following sections that the set $\mathcal{P}$ has the following properties.

Assumption 5.11. Let $\mathcal{P} = \{\mathrm{Sil}(P_1), H_2, H_3, G\}$ as computed in Algorithm 5.10 such that all polynomials are X_2-regular. The polynomials H_2 and H_3 as well as G are in generic position. Moreover, $\widetilde{G}$ is in generic position with respect to $\mathrm{Sil}(P_1)$.

By using the Brown-McCallum projection operation for eliminating the variable X_3, it follows that the (possible) intersection points of all three quadrics lie on the cut curves defined by $\mathrm{cut}(P_1, P_2)$ and $\mathrm{cut}(P_1, P_3)$, i.e., on the intersection of $\mathrm{Zer}(H_2, \mathbb{R}^2)$ and $\mathrm{Zer}(H_3, \mathbb{R}^2)$, or on $\mathrm{Zer}(G, \mathbb{R}^2)$. In addition, we need to identify the common points of those curves with the silhouette curve $\mathrm{Zer}(\mathrm{Sil}(P_1), \mathbb{R}^2)$ since the number and type of points above a point on $\mathrm{Zer}(\mathrm{Sil}(P_1), \mathbb{R}^2)$ might be different than for points which do not lie on $\mathrm{Zer}(\mathrm{Sil}(P_1), \mathbb{R}^2)$. But observe that the curve $\mathrm{Zer}(\mathrm{Sil}(P_1), \mathbb{R}^2)$ contains all points $(\mathbf{x}, \mathbf{y})$ such that $P_1(\mathbf{x}, \mathbf{y}, X_3)$ has exactly one root z of multiplicity 2. To sum up, we need to compute the following:

1. the intersection points of $\mathrm{Zer}(H_2, \mathbb{R}^2)$ and $\mathrm{Zer}(H_3, \mathbb{R}^2)$ and whether or not they lie on the curve $\mathrm{Zer}(\mathrm{Sil}(P_1), \mathbb{R}^2)$, and

2. the topology of $\mathrm{Zer}(G, \mathbb{R}^2)$ including the common points with $\mathrm{Zer}(\mathrm{Sil}(P_1), \mathbb{R}^2)$ which could be finitely or infinitely many.

By decomposing the polynomial $G = \widetilde{G} \cdot \widetilde{\mathrm{Sil}(P_1)}$ we simplify the second problem further since we just need to compute the following:

1. the topology of $\mathrm{Zer}(\tilde{G}, \mathbb{R}^2)$ including the common points with $\mathrm{Sil}(P_1)$,
2. the topology of $\mathrm{Zer}(\widetilde{\mathrm{Sil}(P_1)})$ including the common points with $\mathrm{Zer}(\tilde{G}, \mathbb{R}^2)$.

It is worthwhile to mention that we can not decide without further computation whether or not a planar point can be lifted to a solution of all three quadrics. This comes from the fact that two different (space) points in $\mathbb{R}^3$ might get projected to the same (planar) point. Nevertheless, this problem can be solved easily as we will see in Chapter 5.2.5. We summarize the above discussion in the following algorithm.

Algorithm 5.12 (Planar Arrangement Analysis)**.**

Input: the set of polynomials

$$\mathcal{P} = \{\mathrm{Sil}(P_1), H_2, H_3, G\},$$

with the properties of Assumption 5.11

Output:

- the common points of $\mathrm{Zer}(H_2, \mathbb{R}^2)$ and $\mathrm{Zer}(H_3, \mathbb{R}^2)$,
- the topology of the curve $\mathrm{Zer}(G, \mathbb{R}^2)$, described by
  - the real roots $\mathbf{x}_1, \ldots, \mathbf{x}_r$ of $\mathrm{Res}(\widetilde{G}, \partial\widetilde{G}/\partial X_2)$, $\mathrm{Res}(\widetilde{\mathrm{Sil}(P_1)}, \partial\widetilde{\mathrm{Sil}(P_1)}/\partial X_2)$ and $\mathrm{Res}(\widetilde{G}, \mathrm{Sil}(P_1))$. We denote by $\mathbf{x}_0 = -\infty$, $\mathbf{x}_{r+1} = \infty$.
  - The number m_i of roots of $G(\mathbf{x}, X_2)$ in $\mathbb{R}$ when $\mathbf{x}$ varies on $(\mathbf{x}_i, \mathbf{x}_{i+1})$. We denote this root by $\mathbf{x}_{i,1}, \ldots, \mathbf{x}_{i,m_i}$.
  - The number n_i of roots of $G(\mathbf{x}_i, X_2)$ in $\mathbb{R}$. We denote these roots by $\mathbf{y}_{i,1}, \ldots, \mathbf{y}_{i,n_i}$.
  - A number $c_i \leq n_i$ such that if $(\mathbf{x}_i, \mathbf{z}_i)$ is the unique critical point of the projection of $\mathrm{Zer}(G, \mathbb{C}^2)$ on the X_1-axis or an intersection point of $\mathrm{Zer}(\widetilde{G}, \mathbb{R}^2)$ and $\mathrm{Zer}(\mathrm{Sil}(P_1), \mathbb{R}^2)$ above $\mathbf{x}_i$, $\mathbf{z}_i = \mathbf{y}_{i,c_i}$.

Procedure:

– Compute the common points of $\mathrm{Zer}(H_2, \mathbb{R}^2)$ and $\mathrm{Zer}(H_3, \mathbb{R}^2)$ using Algorithm 2.12 (TOP) as a black-box.

– Compute the topology of $\mathrm{Zer}(G, \mathbb{R}^2)$ including the common points with $\mathrm{Zer}(\mathrm{Sil}(P_1), \mathbb{R}^2)$ by using Algorithm 2.12 (TOP) as a black-box.

5.2.5 Details on the Lifting Phase

5.2.5.1 Lifting of Single Points

We recall some well-known facts about the real roots of a quadratic polynomial in one variable. Note that we know a priori what case we do have to consider in our setting. For example, a candidate $(\mathbf{x}, \mathbf{y}) \in \mathrm{Zer}(G, \mathbb{R}^2)$ which also lie on $\mathrm{Zer}(\mathrm{Sil}(P_1), \mathbb{R}^2)$ corresponds to the case $D = 0$ (see Proposition 5.13), i.e., the polynomial $P_1(\mathbf{x}, \mathbf{y}, X_3)$ has exactly one real root.

Proposition 5.13. *Let $P = aX^2 + bX + c$ with $a, b, c \in \mathbb{R}$ and let $D = b^2 - 4ac$. Then we get the following cases.*

1. *If $D = 0$, then the polynomial P has exactly one solution $\mathbf{x} = -\frac{b}{2a}$.*
2. *If $D > 0$, then the polynomial P has two real solution $\mathbf{x}_1$ and $\mathbf{x}_2$. In this case, $\mathbf{x}_1 = \frac{1}{2a}\left(-b - \sqrt{D}\right)$ and $\mathbf{x}_2 = \frac{1}{2a}\left(-b + \sqrt{D}\right)$.*
3. *If $D < 0$, then the polynomial P has only two complex conjugated roots.*

By using the information computed by Algorithm 5.12 we can now easily determine the solutions $\mathbf{z}_1, \ldots, \mathbf{z}_i$, $i \leq 2$, of the polynomial $P_1(\mathbf{x}, \mathbf{y}, X_3)$ where $(\mathbf{x}, \mathbf{y})$ is a possible candidate in the plane.

5.2.5.2 Lifting of a Curve

Our approach for lifting a curve is similar to lifting a single point as described in the chapter before. By computing some extra points on $\mathrm{Zer}(P_1, \mathbb{R}^3)$ as described in the previous section, we can determine easily the adjacency of the (possible) space curve which is induced by the plane curve $\mathrm{Zer}(G, \mathbb{R}^2)$.

Assume that we computed the topology of $\mathrm{Zer}(G, \mathbb{R}^2)$ as described in Algorithm 5.12. First, we lift all points (if possible) onto $\mathrm{Zer}(P_1, \mathbb{R}^3)$. Note that we can easily determine the missing adjacencies as described in Chapter 2.1.4.3, since there are only one or two points above. Then we just need to test whether or not our candidates lie on $\mathrm{Zer}(P_2, \mathbb{R}^3)$ and $\mathrm{Zer}(P_3, \mathbb{R}^3)$ as well. It is worthwhile to mention that not all components of $\mathrm{Zer}(G, \mathbb{R}^2)$ might get lifted even though they can be lifted to a solution on $\mathrm{Zer}(P_1, \mathbb{R}^3)$.

5.2.6 The Implementation

The algorithm has been prototypically implemented in the Computer Algebra System `Maple` (version 9.5) [72] and it follows the approach outlined closely. It starts always with three quadratic polynomials P_1, P_2 and P_3 in $\mathbb{Q}[X_1, X_2, X_3]$ and, due to efficiency reasons, it performs most of the computations by using floating point arithmetic. The latter one comes from the fact that we extended Laureano Gonzalez-Vega and Ioana Necula's TOP algorithm code ([52]). Hence, the only computations that are performed symbolically are:

1. the computation of the projection set $\mathcal{P} = \{\mathrm{Sil}(P_1), H_2, H_3, G\}$.

2. the computations of the different signed subresultant sequences and their coefficients for the projection set $\mathcal{P}$.

3. the computation of the square-free part of the resultant of two polynomials P_1 and P_2 in $\mathbb{Q}[X_1, X_2]$ and its decomposition with respect to the signed subresultant coefficients.

The remaining computations consist in solving numerically different polynomial equations (without multiple roots) or evaluating at these roots some of the polynomials symbolically computed. Initially the chosen precision is 15 digits, but one can choose any other starting precision t_1. As in the implementation of the TOP-algorithm, we choose a threshold ε that depends on the chosen precision in order to decide whether or not a polynomial is zero at a given point.

Once the planar arrangement $\mathcal{P}$ is computed, we analyze the size t_2 of the input polynomials P_i and the set $\mathcal{P}$. Afterwards, we update the precision to t digits, where $t = max\{t_1, t_2 + 10, 15\}$. Furthermore, the `Maple` function `fsolve` is used to solve the square-free univariate polynomial equations before mentioned. If `fsolve` does not return the correct number of roots (which are known in advance) or some numerical evaluation returns some non guaranteed value, the precision is increased by 10 digits and those computations are performed again. Moreover, we output the coordinates of the isolated points and a three dimensional linear graph if the intersection points form a curve.

We end this section by giving some examples, which illustrate our approach. The experimentations were performed on a PowerPC G4 1GHz. The following example is taken from [82].

Example 5.14 (two isolated points, [82])**.** Let be

$$
\begin{aligned}
P_1 &= 7216X_1^2 - 11022X_1X_2 - 12220X_1X_3 + 15624X_2^2 + 15168X_2X_3 + 11186X_3^2 - 1000 \\
P_2 &= 4854X_1^2 - 3560X_1X_2 + 4468X_1X_3 + 658X_1 + 5040X_2^2 + 32X_2X_3 + 1914X_2 + \\
&\quad 10244X_3^2 + 3242X_3 - 536 \\
P_3 &= 8877X_1^2 - 10488X_1X_2 + 9754X_1X_3 + 1280X_1 + 16219X_2^2 - 16282X_2X_3 - 808X_2 + \\
&\quad 10152X_3^2 - 1118X_3 - 796
\end{aligned}
$$

Then the projection set $\mathcal{P}$ contains of

$$
\begin{aligned}
\mathrm{Sil}(P_1) &= 10846519X_1^2 - 7653903X_1X_2 - 2796500 + 29313252X_2^2 \\
H_2 &= -56556109351696X_1 + 61135807177688X_2 - 6220192626724 + \\
&\quad 203315497528241X_1X_2 - 56404750618857X_1^2 - 55861103592035X_2^2 \\
&\quad -910824371936818X_2X_1^2 + 972629091137652X_1X_2^2 - 659086885094112X_2^3 + \\
&\quad 533601199106972X_1^3 - 2885241224346328X_1^3X_2 + 4223689039107028X_1^2X_2^2 \\
&\quad -3571456229045952X_1X_2^3 + 1026392565603269X_1^4 + 1407622740362496X_2^4 \\
H_3 &= 2872582087600X_1 - 4005061111776X_2 + 69677228486124X_1X_2 \\
&\quad -23228971077672X_1^2 - 49611754602456X_2^2 - 5464061993528X_2X_1^2 \\
&\quad -17976875889356X_1X_2^2 + 40411859296976X_2^3 + 1462282618132X_1^3 \\
&\quad -926282674085672X_1^3X_2 + 1733300718748310X_1^2X_2^2 - 1854003852157600X_1X_2^3 + \\
&\quad 225439274765947X_1^4 + 897407958763127X_2^4 - 66086625728 \\
G &= 1
\end{aligned}
$$

Our computations end with a precision of 26 digits. The real intersection is computed in 0.572 seconds and consists of two isolated points, namely,

$$
p_1 = \begin{pmatrix} -0.47111071472741316264056772 \\ -0.19897789206886601999604553 \\ 0.18592931583225857372754588 \end{pmatrix}
$$

and

$$p_2 = \begin{pmatrix} -0.16627634657169906116678201 \\ 0.10827914469994312737865267 \\ -0.011248383019525287650192532 \end{pmatrix}$$

Table 2: Experimental results for Example 5.14

Size of Input	Size of $\mathcal{P}$	Changes	Precision	Time
5	16	1	26	0.572
8	32	0	42	0.466
12	46	0	56	1.120
15	54	0	64	4.453
20	75	0	85	4.662
23	90	0	100	7.361
28	106	0	116	6.479
32	122	0	132	6.665
36	137	0	147	8.077
40	147	0	157	7.609

Moreover, Table 2 and Table 3 present a comparison between the computing times (in seconds) obtained by our approach and the prototypically and improved implementation of [82][1] using different numbers of decimal digits for the three input quadrics. Moreover, Table 2 contains the following additional information:

size of Input (resp., $\mathcal{P}$) – number of decimal digits of the Input (resp., the projection set $\mathcal{P}$.

Changes – number of linear changes of variables

Precision – used precision for obtaining the result

Table 3: Experimental results of Schömer and Wolpert [82]

Number of digits	5	10	15	20	25	30
Running time 1	18	33	56	92	126	186
Running time 2	1.1	2.7	5.0	7.8	12.1	16.1

It is worthwhile to mention that we obtain similar running times for all our experiments. Additionally, the improvement of the running times do not only depend on the newer computer.

[1]running times are measured on a Intel Pentium 700 and Pentium III Mobile 800

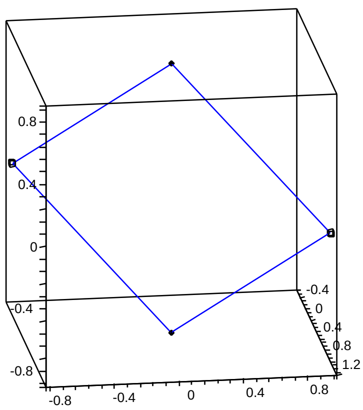

Figure 12: The intersection of three linearly independent quadrics

Example 5.15 (closed curve)**.** Let be

$$
\begin{aligned}
P_1 &= (X_1 - X_2)^2 + X_2^2 + X_3^2 - 1 \\
P_2 &= (X_1 - X_2 - 1)^2 + X_2^2 + X_3^2 - 1 \\
P_3 &= 4X_2^2 + 4X_3^2 - 3
\end{aligned}
$$

Note, that the three quadrics are linearly independent and the projection set $\mathcal{P}$ contains of

$$
\begin{aligned}
\mathrm{Sil}(P_1) &= X_1^2 - 2X_1X_2 + 2X_2^2 - 1 \\
H_2 &= 1 \\
H_3 &= -1 - 2X_1 + 2X_2 \\
\widetilde{\mathrm{Sil}(P_1)} &= 1 \\
\widetilde{G} &= 1 - 2X_1 + 2X_2
\end{aligned}
$$

Then the real intersection of the three quadrics defined by P_1, P_2 and P_3 consists of infinitely many points. Figure 12 shows the linear three dimensional graph computed by our implementation. The computations start and end with a precision of 15 digits and is computed in 0.101 seconds. For representing the linear graph we computed the following four points. The points

$$
\begin{aligned}
p_1 &= (-0.366025403784439, -0.866025403784439, 0) \\
p_2 &= (1.36602540378444, 0.866025403784440, 0)
\end{aligned}
$$

which correspond to the lift of the intersection points of the two plane curves $\mathrm{Zer}(\mathrm{Sil}(P_1), \mathbb{R}^2)$ and $\mathrm{Zer}(\widetilde{G}, \mathbb{R}^2)$, and

$$(0.500000000000000, 0, -0.866025403784440),$$
$$0.500000000000000, 0, 0.866025403784440)$$

which are two sample points for the two curve segments between the critical points p_1 and p_2.

Table 4: Experimental results for Example 5.15

Size of Input	Size of $\mathcal{P}$	Changes	Precision	Time
0.101	1	1	0	15
0.185	4	8	0	18
0.257	8	15	0	25
0.196	12	20	0	30
0.307	16	30	0	40
0.323	20	38	0	48
0.498	25	47	2	57
0.520	28	53	2	64
0.591	33	62	2	82
0.368	36	66	0	76

Example 5.16 (2 isolated points, $\widetilde{G} \neq 1$)**.** Let be

$$\begin{aligned}
P_1 &= 27X_1^2 + 62X_2^2 + 249X_3^2 - 10 \\
P_2 &= 88X_1^2 + 45X_2^2 + 67X_3^2 - 66X_1X_2 - 25X_1X_3 + 12X_2X_3 - 24X_1 + 2X_2 + 29X_3 - 5 \\
P_3 &= 88X_1^2 + 45X_2^2 + 67X_3^2 - 66X_1X_2 + 25X_1X_3 - 12X_2X_3 - 24X_1 + 2X_2 - 29X_3 - 5.
\end{aligned}$$

Note, that $P_3(X_1, X_2, X_3) = P_2(X_1, X_2, -X_3)$. Then the projection set $\mathcal{P}$ contains of

$$\begin{aligned}
\mathrm{Sil}(P_1) &= 27X_1^2 + 62X_2^2 - 10 \\
H_2 &= H_3 = \widetilde{\mathrm{Sil}(P_1)} = 1 \\
\widetilde{G} &= -1763465 + 408332484X_1^4 + 51939673X_2^4 + 10482900X_1 - 2305740X_2 \\
&\quad -123026916X_1X_2^2 + 221120964X_1^2X_2 + 4764152X_2^2 + 17767644X_2^3 + \\
&\quad 14441004X_1X_2 - 250019406X_1^3 + 16691919X_1^2 - 664779204X_1^3X_2 + \\
&\quad 564185724X_1^2X_2^2 - 241015068X_1X_2^3
\end{aligned}$$

Our computations end with a precision of 19 digits. The real intersection consists of two isolated points

$$(0.06676451891748808143, 0.3991856119605212449, 0),$$

$$(0.4954772252006942431, 0.2331952878577051550, 0)$$

and is computed in 0.490 seconds.

Table 5: Experimental results for Example 5.16

Size of Input	Size of $\mathcal{P}$	Changes	Precision	Time
2	9	0	19	0.490
6	22	0	32	0.355
10	37	0	47	2.374
14	46	0	56	4.939
18	67	0	77	5.018
21	82	0	92	6.362
26	98	0	108	6.515
30	113	0	123	7.109
34	129	0	139	7.694
38	138	0	148	9.671
41	158	0	168	9.056

Example 5.17 (empty intersection)**.** Let be

$$\begin{aligned} P_1 &= X_2 + X_1^2 + 2X_1X_2 + 2X_1X_3 + X_2^2 + 2X_2X_3 + X_3^2 \\ P_2 &= X_3^2 + 1 - X_2 \\ P_3 &= 2X_3^2 + 2 - 2X_2 \end{aligned}$$

Then the projection set $\mathcal{P}$ contains of

$$\begin{aligned} \mathrm{Sil}(P_1) &= X_2 \\ H_2 &= H_3 = \widetilde{\mathrm{Sil}(P_1)} = 1 \\ \widetilde{G} &= 1 + X_1^4 + X_2^4 + 4X_1^3X_2 + 6X_1^2X_2^2 + 4X_1X_2^3 - 4X_2 + 6X_2^2 + 4X_1X_2 + 2X_1^2 \end{aligned}$$

Our computations start and end with precision of 15 digits. The real intersection is empty and computed in 0.182 s.

Table 6: Experimental results for Example 5.17

Size of Input	Size of $\mathcal{P}$	Changes	Precision	Time
1	1	0	15	0.182
4	15	0	25	0.191
8	30	0	40	0.187
12	39	0	49	0.274
16	59	0	69	1.025
20	74	0	84	0.978
24	92	2	121	2.345
28	105	1	126	1.863
32	122	1	142	1.821
36	133	1	153	2.090
40	152	2	182	2.740

Example 5.18 (a curve and an isolated point)**.** Let be

$$\begin{aligned} P_1 &= X_2 + X_1^2 + 2X_1X_2 + 2X_1X_3 + X_2^2 + 2X_2X_3 + X_3^2 \\ P_2 &= X_3^2 - X_2 + X_1X_2 + X_2^2 + X_2X_3 \\ P_3 &= 2X_3^2 - 2X_2 + 2X_1X_2 + 2X_2^2 + 2X_2X_3 \end{aligned}$$

Note, that $P_3 = 2P_2$. Then the projection set $\mathcal{P}$ contains of

$$\begin{aligned} \mathrm{Sil}(P_1) &= X_2 \\ H_2 &= H_3 = \widetilde{\mathrm{Sil}(P_1)} = 1 \\ \widetilde{G} &= 4X_2^2 + X_1^4 + 6X_1^2X_2^2 - 3X_2^3 + 4X_1X_2^3 - 4X_1X_2^2 + 4X_1^3X_2 + X_2^4 \end{aligned}$$

Note, that $\mathrm{Zer}(\widetilde{G}, \mathbb{R}^2)$ consists of two isolated points and and open curve. Our computations start and end with precision of 15 digits. The real intersection is computed in 0.305 seconds and consists of the isolated point $(0, 0, 0)$ and an open curve (see Figure 13). For the curve we computed the following three points.

$$p = (1.91241422362700, -1.06499480841233, 0.184566441477331)$$

which corresponds to the lift of the (non-isolated) critical point of $\mathrm{Zer}(\widetilde{G}, \mathbb{R}^2)$, and

$$(2.91241422362700, -2.54899069044757, 1.23313235073054),$$
$$(2.91241422362700, -1.32006472767900, -0.443408797879122)$$

which are sample points for the two branches ending and starting of p.

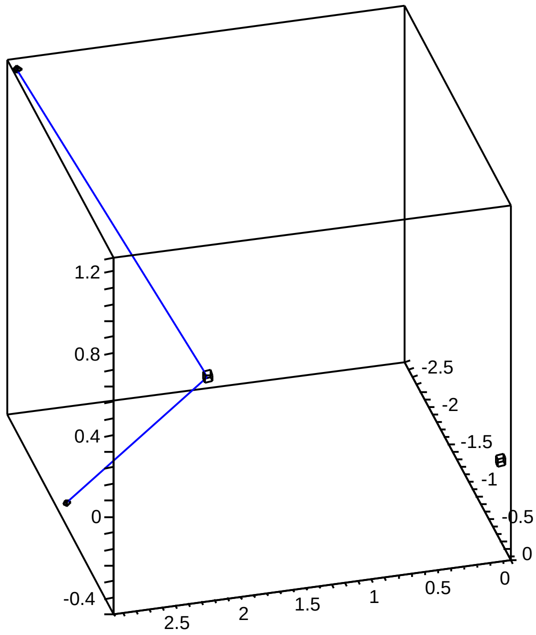

Figure 13: A curve and an isolated point

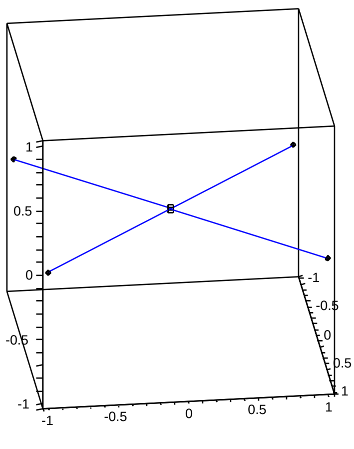

Figure 14: Two intersecting lines with $\widetilde{\mathrm{Sil}(P_1)} \neq 1$

Table 7: Experimental results for Example 5.18

Size of Input	Size of $\mathcal{P}$	Changes	Precision	Time
1	1	0	15	0.305
4	16	0	26	0.272
8	30	0	40	0.421
12	38	0	48	0.437
17	60	7	80	6.119
20	72	0	82	1.215
25	92	7	112	4.600
28	104	1	115	1.900
32	118	0	128	1.716
38	134	7	154	6.131
41	151	6	161	5.303
45	165	14	194	10.003

Example 5.19 (a curve, $\widetilde{\mathrm{Sil}(P_1)} \neq 1$)**.** Let be

$$\begin{aligned} P_1 &= X_3^2 + X_1^2 - X_2^2 \\ P_2 &= X_3^2 + X_1X_3 + X_2X_3 - X_3 + X_1^2 - X_2^2 \\ P_3 &= X_3^2 + X_1X_3 + X_2X_3 + X_3 + X_1^2 - X_2^2 \end{aligned}$$

Then the projection set $\mathcal{P}$ contains of

$$\begin{aligned} \mathrm{Sil}(P_1) &= X_1^2 - X_2^2 \\ H_2 &= -1 + X_1 + X_2 \\ H_3 &= 1 - X_1 + X_2 \\ \widetilde{\mathrm{Sil}(P_1)} &= X_1^2 - X_2^2 \\ \widetilde{G} &= 1 \end{aligned}$$

Our computations start and end with precision of 15 digits. The real intersection is computed in 0.152 seconds and consists of two intersecting lines. We computed the following five points. The point

$$p = (0, 0, 0)$$

which corresponds to the lift of the critical point of $\widetilde{\mathrm{Sil}(P_1)}$, and

$$(-1, -1, 0), (-1, 1, 0) \text{ and}$$
$$(1, -1, 0), (1, -1, 0)$$

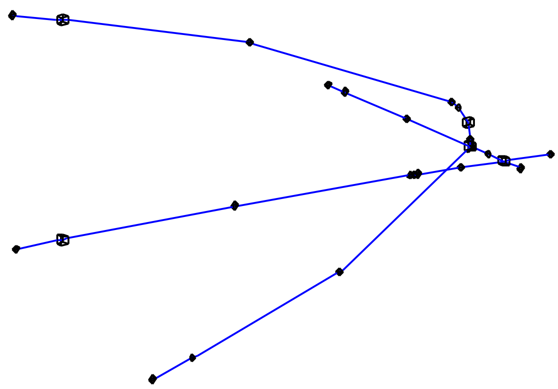

Figure 15: One connected component

which are sample points for the two branches attached to the left and to the right of p.

Table 8: Experimental results for Example 5.19

Size of Input	Size of $\mathcal{P}$	Changes	Precision	Time
1	1	0	15	0.152
4	8	0	18	0.139
8	15	0	25	0.117
11	19	0	29	0.183
15	29	0	39	0.137
19	37	0	47	0.244
23	45	0	55	0.187
27	53	0	63	0.273
31	61	0	71	0.212
35	66	0	76	0.274
39	75	0	85	0.283
43	82	0	92	0.233

Example 5.20 (one connected component)**.** Let be

$$\begin{aligned}
P_1 &= X_2 - X_3 + X_1X_3 + 5X_2X_3 + 2X_3^2 \\
P_2 &= 6X_2^2 - 5X_2X_3 - X_3^2 + X_1X_2 - X_1X_3 + X_3 \\
P_3 &= 6X_2^2 - 5X_2X_3 - X_3^2 + X_1X_2 - X_1X_3 + X_3
\end{aligned}$$

Note, that $P_2 = P_3$. Then the projection set $\mathcal{P}$ contains of

$$\begin{aligned}
\mathrm{Sil}(P_1) &= 18X_2 - 1 + 2X_1 - X_1^2 - 10X_2X_1 - 25X_2^2 \\
H_2 &= H_3 = \widetilde{\mathrm{Sil}(P_1)} = 1 \\
\widetilde{G} &= -3X_2^2 - 8X_2^2X_1 - 11X_2^3 + 20X_2^2X_1^2 + 133X_2^3X_1 \\
&\quad +294X_2^4 - X_2X_1^2 + X_1^3X_2 + X_2 - X_2X_1
\end{aligned}$$

Our computations start and end with precision of 15 digits. The real intersection is computed in 0.529 seconds and consists of one connected component (see Figure 15).

5.2.7 Remark on Cubic Surfaces

We would like to remark that the algorithm presented in Chapter 5.2 has been extended to three cubic surfaces defined by the polynomials C_1, C_2 and C_3 in $\mathbb{R}[X_1, X_2, X_3]$. Note that in this case the silhouette curve $\mathrm{Zer}(\mathrm{Sil}(C_1), \mathbb{R}^2)$ contains all points $(\mathbf{x}, \mathbf{y})$ such that the polynomial $C_1(\mathbf{x}, \mathbf{y}, X_3)$ has a root $\mathbf{z}$ of multiplicity 2 or 3. Theorem 2.4 implies that in the first case the polynomial $\mathrm{sRes}_1(C_1, \partial C_1/\partial X_3)(\mathbf{x}, \mathbf{y}) \neq 0$ whereas in the latter one $\mathrm{sRes}_1(C_1, \partial C_1/\partial X_3)(\mathbf{x}, \mathbf{y}) = 0$. Moreover, one can also use a solution formula for cubic polynomials in one variable in order to lift a single point.

Like in the case for quadrics, we can easily determine the missing adjacency information while lifting the curve $\mathrm{Zer}(G, \mathbb{R}^2)$ using a simple combinatorial type approach.

Finally, this new algorithm has similarly implemented in the Computer Algebra System `Maple` (version 9.5) as well. The experimental results archived show a very good performance. We refer to [61] for more details.

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