

INFLATION OF A PLANAR DOMAIN

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For Faith.

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SUMMARY

Take a soap film or other capillary surface that spans a fixed boundary at one end of a container and bounds the volume inside. If you increase the pressure in the bounded volume the soap film will expand outwards into the unbounded volume. In the absence of gravitational effects, if the boundary of the soap film remains stationary then the film will be a constant mean curvature surface at each point in time during the expansion.

We model this process mathematically as an inflation, a one parameter family of constant mean curvature surfaces with the same boundary and with increasing bounded volume. Such families have been shown to exist as graphs over planar domains. However, this places an artificial restriction on an inflation as not all constant mean curvature surfaces can be represented as graphs over a plane.

We avoid these restrictions by using an alternative representation of the surfaces. Specifically, we consider surfaces as graphs, not over a planar domain, but over a known nearby constant mean curvature surface. In so doing we prove the existence of new constant mean curvature surfaces beyond the limits of previous approaches.

CHAPTER 1

BACKGROUND AND INTRODUCTION

1.1 Background

Soap films and soap bubbles have long been a topic of mathematical research [1, 2, 3, 4, 5]. These soap films tend to minimize total internal energy. When gravitational effects are negligible this is realized by minimizing surface area subject to their boundary and volume constraints.

Without volume/pressure constraints soap films form *minimal surfaces* with everywhere zero mean curvature. Finding such surfaces is the object of the Plateau problem and its variants. When volume and/or pressure constraints are present, however, variational arguments can show that to remain area minimizing soap films must have *constant mean curvature* [6, 7]. Models in physics also demonstrate that the mean curvature is proportional to the difference in pressure between the two separated volumes, as shown by Young and Laplace in 1805 [8, 9].

Soap films that span a given boundary and separate volumes of different pressures are realized mathematically as constant mean curvature surfaces with boundary. Such surfaces have often been studied in the context of graphs over a planar domain [10, 11, 12]. Existence and uniqueness results have been proven in this framework for a large class of domains, so long as the surfaces in questions can still be represented as graphs. However, constant mean curvature surfaces that bound larger volumes are typically not planar graphs. This leads to an artificial restriction to this approach as boundary gradient values of constant mean curvature graphs will approach infinity as the contained volume increases (See Appendix A of [12]).

Our approach to this problem is to represent the surfaces in question as graphs, not

over a planar domain, but over a nearby constant mean curvature surface with the same boundary. This allows a local reparameterization that eliminates boundary gradient blow-up. In this way we are able to ‘push-off’ from a surface close to a gradient blow-up found using previous approaches and demonstrate the existence of new constant mean curvature surfaces with boundary that are not planar graphs. This builds on the techniques of [13, 14, 15, 16].

As our primary application of this technique we consider the physical situation of a planar soap film spanning some fixed boundary, such as a bubble wand. Increasing the air pressure on one side of the film will cause the soap film to expand towards the other side. At each point of the expansion the soap film will remain a constant mean curvature surface with mean curvature proportional to the difference in pressure (Laplace pressure) across the soap film.

We can represent this situation mathematically as an *inflation*: a smooth, one parameter family of constant mean curvature surfaces with the same boundary with increasing bounded volume. Inflations of planar domains are known to exist so long as the constant mean curvature surfaces are planar graphs [12]. The techniques of this paper are able to extend inflations of planar domains beyond this limit.

1.2 Previous Results

In this section we review what previous results are known about inflations. These results are found in [12].

Starting with a smooth, connected, planar domain D we may consider the boundary value problem for graphs over D with constant mean curvature and satisfying a zero Dirichlet boundary value condition on ∂D . Such graphs are known to exist so long as the prescribed constant mean curvature is not too large. In addition, these graphs depend smoothly on the prescribed mean curvature $H \in [-H_{max}, H_{max}]$ and foliate the volume enclosed between the graphs of constant mean curvature $\pm H_{max}$.

One can obtain from among these graphs of constant mean curvature a unique smooth one parameter family of graphs with the mean curvature H as the parameter. There is in fact a unique smooth such family containing all constant mean curvature graphs over the domain D . Let V_H be the volume enclosed by the graph of constant mean curvature H and the planar domain D . Then V_H is monotone over $[-H_{max}, 0]$: if $-H_{max} \leq H_1 < H_2 \leq 0$ then $V_{H_2} \subsetneq V_{H_1}$ and $\overline{V_{H_2}} \subsetneq \overline{V_{H_1}}$.

For $|H| < H_{max}$ each graph solution has globally bounded gradient on \overline{D} . For $|H| = H_{max}$ the corresponding constant mean curvature graph has locally bounded gradient on the interior of D , but is unbounded on ∂D . This occurs when the inflating constant mean curvature surfaces become perpendicular to the plane of D at some point of ∂D .

We show in this thesis that if D is not a circular disk the smooth inflation family can be extended beyond H_{max} if additional surfaces which are not graphs over D are allowed. In the case of a circular disk, an explicit such family of surfaces is known (see Chapter 2). Additionally we show the same volume enclosing, nesting, and foliation properties hold for the extended inflation family of (non-graph) surfaces.

1.3 Statement of the Main Theorem

We now give an outline of our main results. The following hypotheses will be made rigorous in Chapter 3. Let M be an embedded constant mean curvature surface (cmc surface) of mean curvature H_M with boundary ∂M such that \overline{M} is compact. Let ν_M be the Gauss map of M and consider φ some scalar function on M . We let $N = \Gamma_\varphi(M)$, the graph of φ over M . Here $\Gamma_\varphi : M \rightarrow \mathbb{R}^3$ is given by

$$\Gamma_\varphi(\mathbf{x}) = \mathbf{x} + \nu_M(\mathbf{x})\varphi(\mathbf{x}).$$

Let \mathcal{H} be the mean curvature operator for the graph of φ . Our goal then is to solve the

Dirichlet boundary value problem

$$\begin{cases} 2\mathcal{H}[\varphi] = 2H_N \\ \varphi|_{\partial M} = 0 \end{cases}$$

Remark 1.3.1. Note that \mathcal{H} is the mean curvature equation for graphs over the constant mean curvature surface M . This is not the commonly seen mean curvature operator,

$$\mathcal{M}u = \operatorname{div} \left(\frac{Du}{\sqrt{1 + Du^2}} \right),$$

which only applies to graphs over planar domains.

Theorem 1.3.2 (Main Theorem). *Let M be a smooth constant mean curvature graph over a planar domain D where $\partial M = \partial D$. If M is not a half-sphere then for all $\epsilon > 0$, sufficiently small, the Dirichlet boundary value problems*

$$\begin{cases} 2\mathcal{H}[\varphi] = 2H_M \pm \epsilon \\ \varphi|_{\partial M} = 0 \end{cases}$$

both have unique solutions in $C^{p+2,\alpha}(\overline{M})$.

This result also applies to graphs with infinite boundary gradients, the limit case of the planar graph approach. In this case one of the two solutions will describe a constant mean curvature surface with the same boundary as M that is not a graph over a plane.

1.4 Outline of the Proof

We are interested in the boundary value problem

$$\begin{cases} 2\mathcal{H}[\varphi] = 2H_N \\ \varphi|_{\partial M} = 0 \end{cases}$$

for $\varphi \in C^{p+2,\alpha}(\overline{M})$, where H_N is a constant near H_M . The graph over M of a solution to this problem provides the constant mean curvature surface N as described above.

This differential equation is well known [15] to separate into zeroth, first and higher degree terms as

$$2H_N = 2H_M - (-\Delta_M - |A_M|^2)\varphi + Q[\varphi].$$

To solve this differential equation we use the Fredholm alternative to show that the linear operator $L = -\Delta_M - |A_M|^2$ is invertible, and that the inverse operator L^{-1} is continuous. But as $-L$ is the Fréchet derivative of $n\mathcal{H}$ at $\varphi = 0$, this allows us to apply the inverse mapping theorem to $n\mathcal{H}$ near $\varphi = 0$. We then find the solutions to the boundary value problem as

$$\varphi = (n\mathcal{H})^{-1}[nH_N]$$

for nH_N near nH_M .

1.5 Outline of the Paper

We begin by looking at the example of a circular planar domain in Chapter 2 and discuss the maximum Laplace pressure problem. Chapters 4 and 5 discuss the linear operator L and prove it's solvability under certain conditions. In Chapter 6 we complete the proof of a generalized form of our main theorem, while in Chapter 7 we demonstrate its application to cmc graphs over planar domains and get our first new cmc surfaces. Finally in Chapter 8 we discuss some results for inflations of general domains.

CHAPTER 2

MAXIMUM LAPLACE PRESSURE AND CIRCULAR DOMAINS

2.1 Laplace Pressure and Mean Curvature Extrema

Consider a planar wire-frame spanned by a flat soap film in the same plane. This surface has constant zero mean curvature corresponding to the zero Laplace pressure (pressure differential) across the film. Now consider what would happen as we begin to increase the volume bounded by the soap film, say by sealing one side and pumping air in. At first we expect to see the Laplace pressure and mean curvature to increase as the bounded volume increases. There will, however, be a point after which pumping in more air to the bounded side will actually decrease the Laplace pressure. When the bounded volume becomes truly large, the shape of the soap film will approach a large sphere attached to the wire-frame on one side. As the volume and radius of this ‘sphere’ increases, its mean curvature ($\approx \frac{1}{r}$) must decrease, asymptotically approaching zero at infinite volume.

For all soap films the Laplace pressure across the film is proportional to the (constant) mean curvature its surface [9]. Thus the soap film of maximum Laplace pressure coincides with the constant mean curvature surface of maximum mean curvature (or minimum depending on orientation) for its fixed boundary. Specifically, for each sufficiently smooth simple boundary curve we expect to find a one parameter family of constant mean curvature surfaces that include the surfaces of maximum, minimum, and zero curvature, corresponding to maximum, minimum, and zero Laplace pressure respectively. Chapter 8 contains results from the present work on this problem, and conjectures for further research.

2.2 A Look at Circular Domains

In the statement of our main theorem we have excluded the case of a half-sphere over a circular domain. This is because the constant mean curvature surface of maximum Laplace pressure with a circular boundary is the half-sphere of the same radius. (For non-circular domains the maximum Laplace pressure is beyond the gradient blow-up of the planar graph approach, as we show).

Consider, for example, a circular disk of \mathbb{R}^2 embedded in \mathbb{R}^3 ,

$$B_r = \{(x, y, 0) \in \mathbb{R}^3 | x^2 + y^2 < r^2\}.$$

Then an inflation of B_r is a one parameter family of spherical caps: For all $c \in \mathbb{R}$ there is a sphere S_c with center $(0, 0, c)$ of radius $\sqrt{c^2 + r^2}$ that contains ∂B_r . Taking the upper half space ($z > 0$) portion of these yields a smooth family, S_c^+ , of cmc surfaces with the same boundary ∂B_r . With the addition of B_r itself as $S_{-\infty}^+$, this family can be parameterized by $V \in [0, \infty)$, the volume enclosed by $S_c^+ - B_r$, given by

$$V_c = \frac{\pi}{3}(\sqrt{c^2 + r^2} + c)^2(2\sqrt{c^2 + r^2} - c).$$

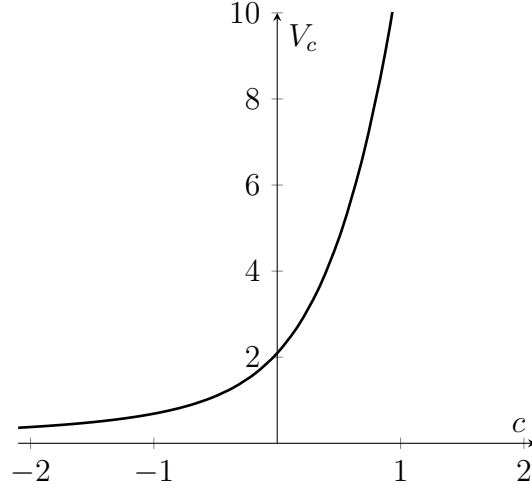


Figure 2.1: Volume as a function of c ($r = 1$)

At the same time, however, the mean curvature of S_c^+ is given by

$$2H_c = \frac{-2}{\sqrt{c^2 + r^2}}.$$

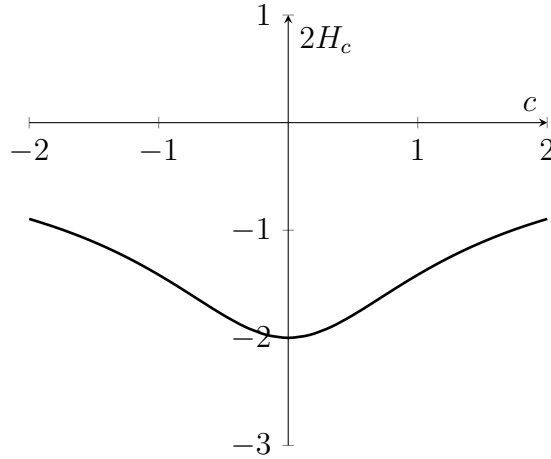


Figure 2.2: Mean curvature as a function of c ($r = 1$)

This has a minimum value of $-\frac{2}{r}$ at $c = 0$, $V = \frac{2}{3}\pi r^3$. Thus the maximum Laplace pressure in the inflation of B_r occurs at the half-sphere of the same radius as B_r . Both

increasing or decreasing the bounded volume at this critical point will lead to a decrease in the Laplace pressure. As the half-sphere can still be represented as a graph over the planar domain B_r , this special case must be excluded in the statement of Theorem 1.3.2.

It will be shown in Chapter 8 that this minimum of mean curvature corresponds to a zero eigenvalue for the stability operator L . Indeed, L on the surface S_C^+ is given by

$$\begin{aligned} L[u] &= \Delta_{S_C^+} u + |A_{S_C^+}|^2 u \\ &= -\Delta_{S_C^+} u - \frac{2}{c^2 + r^2} u \end{aligned}$$

and has as an eigenfunction

$$u = \frac{z - c}{\sqrt{x^2 + y^2 + (z - c)^2}}$$

with

$$\begin{aligned} L[u] &= -\Delta_{S_C^+} u - \frac{2}{\sqrt{c^2 + r^2}} u \\ &= + \frac{3(z - c)}{(x^2 + y^2 + (z - c)^2)^{3/2}} - \frac{3(x^2 + y^2 + (z - c)^2)(z - c)}{(x^2 + y^2 + (z - c)^2)^{5/2}} \\ &\quad + \frac{2(z - c)}{(x^2 + y^2 + (z - c)^2)^{3/2}} - \frac{2}{c^2 + r^2} \frac{z - c}{\sqrt{x^2 + y^2 + (z - c)^2}} \\ &= 0 \end{aligned}$$

since $u(\vec{x}) = u(\vec{x}/\|\vec{x}\|)$ and $x^2 + y^2 + (z - c)^2 = c^2 + r^2$ on S_C^+ . In particular if $c = 0$ this is a Dirichlet eigenfunction with zero boundary values. We will see in Chapter 5 that this implies the operator $L : C_0^{p+2}(\overline{S_0^+}) \rightarrow C^p(\overline{S_0^+})$ is not invertible, and our argument breaks down on S_0^+ the half-sphere. Inflatons can still be obtained around extrema of mean curvature in some cases (as we have for the half-sphere) but require conditions on the kernel of L which are beyond the scope of this work [15, 17].

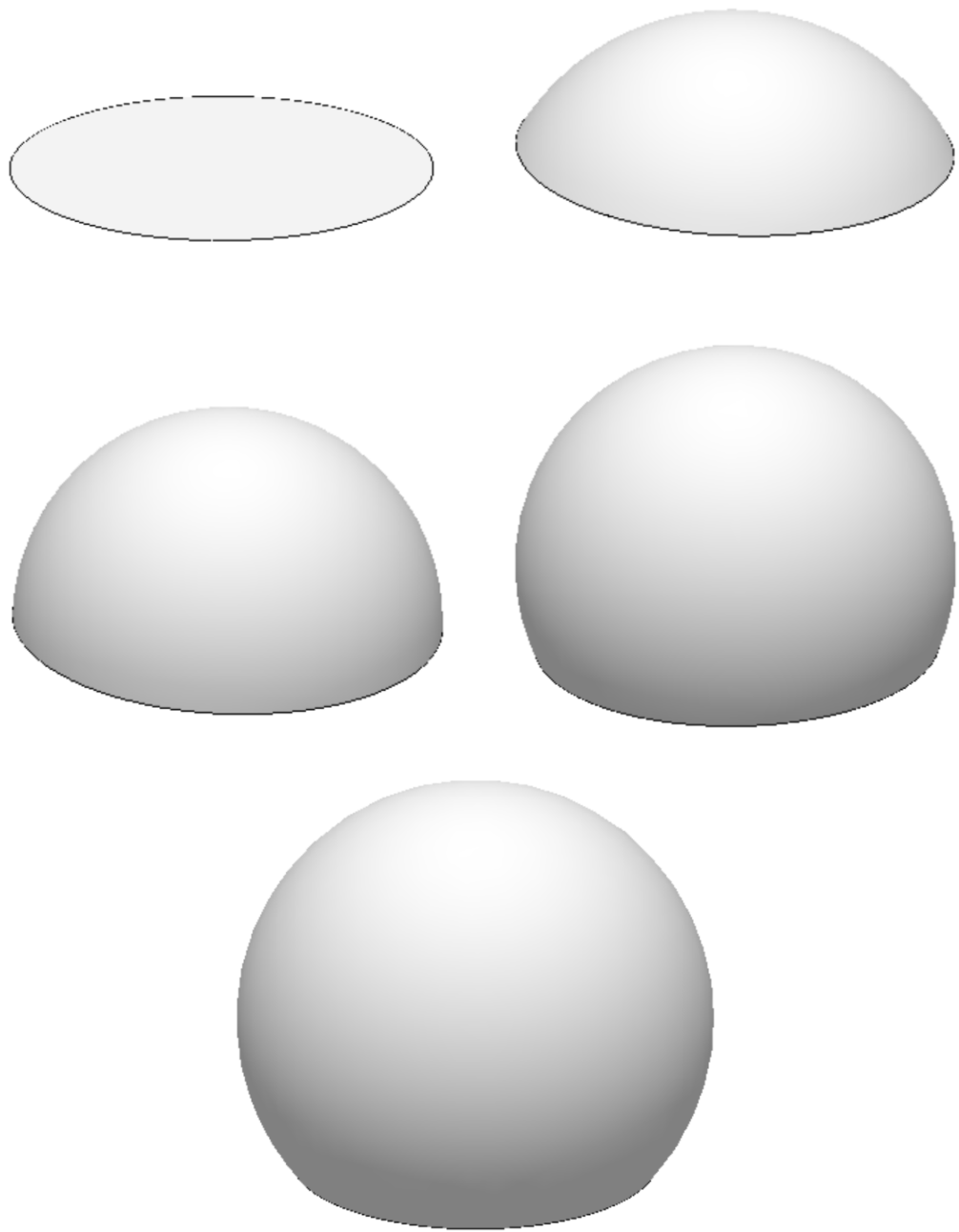


Figure 2.3: Inflation of a Circular Domain

CHAPTER 3

BASIC NOTIONS

While our main theorem is applicable to graphs over planar domains, most of our results apply to general constant mean curvature surfaces and we work in that framework. The following notations and assumptions will be used throughout the work.

3.1 Hypotheses

Let $M \subset \mathbb{R}^{n+1}$ be an n -manifold, a (hyper)-surface, with boundary ∂M . We suppose that $\overline{M} = M \cup \partial M$ bounded, compact, complete, connected, orientable, and embedded. We also suppose that M and ∂M are of class C^∞ .

Let $\nu_M(x)$ be the unit normal to M at $x \in \overline{M}$. Define $\Gamma : \overline{M} \times \mathbb{R} \rightarrow \mathbb{R}^n$ to be the map given by

$$(x, t) \mapsto x + t\nu_M(x).$$

Let $\text{dist}(x, y)$ be the geodesic distance between $x, y \in \overline{M}$. Let $\vec{n}(x)$ be the unit exterior boundary normal to ∂M at $x \in \partial M$. For all $x \in \partial M$ let $\gamma_x : [0, b] \rightarrow \overline{M}$ be the unit speed geodesic in \overline{M} of maximum domain such that $\gamma_x(0) = x$ and $\gamma'_x(0) = -\vec{n}$. We extend this map to $\gamma_x : [0, \infty) \rightarrow \overline{M}$ where $\gamma_x(t) = \gamma_x(b)$ for all $t > b$. Define $\gamma : \partial M \times [0, \infty) \rightarrow \overline{M}$ to be the map given by

$$(x, t) \mapsto \gamma_x(t).$$

These assumptions are well known to imply the following:

- The surface M satisfies an *interior sphere condition* at each point of ∂M : For all $x \in \partial M$ there exists $y_x \in M$ and $R_x > 0$ such that $B_{R_x}(y) = \{z \in \overline{M} \mid \text{dist}(y, z) < R_x\} \subset M$ and $x \in \partial B_{R_x}(y)$.

- The surface M has *positive reach*: There exists $R_M > 0$ such that Γ is one-to-one when restricted to the domain $M \times [-R_M, R_M]$. Since \overline{M} is C^∞ and compact, there is a minimum value for the radius of curvature for geodesics on \overline{M} . The bound R_M must be less than this minimum radius, but may need to be even smaller if $\Gamma(M, r)$ is not an embedding for some $r \neq 0$.
- The boundary of M has *positive reach in M* : There exists $R_{\partial M} > 0$ such that γ is one-to-one when restricted to the domain $\partial M \times [0, R_{\partial M}]$. Similar to the above, there is a minimum value for the radius of M -geodesic curvature for geodesics on ∂M . The bound $R_{\partial M}$ must be less than this minimum radius, but may need to be even smaller if $\gamma(\partial M, r)$ is not an embedding for some $r \neq 0$.

3.2 Differential Operators

Suppose as well that M has everywhere constant mean curvature H_M . We describe the differential operator that gives the mean curvature of graphs over M . To define a graph over M we let $\varphi \in C^{p+2}(\overline{M})$ be a scalar function on \overline{M} where $p \geq 0$. Define $\Gamma_\varphi : \overline{M} \rightarrow \mathbb{R}^{n+1}$ to be the map given by

$$x \mapsto \Gamma(x, \varphi(x)) = x + \varphi(x)\nu_M.$$

Then if $\varphi \equiv 0$ on ∂M the surface $N = \Gamma_\varphi(M)$ has the same boundary as M . For $\|\varphi\|_{p+2} < R_M$, Γ_φ is an injection and the mean curvature of N at $\Gamma_\varphi(x)$ is well defined for all $x \in \overline{M}$.

We define the operator

$$n\mathcal{H} : U \rightarrow C^p(\overline{M})$$

for

$$U = \{\varphi \in C^{p+2}(\overline{M}) \mid \varphi \equiv 0 \text{ on } \partial M, \|\varphi\|_{p+2} < R_M\}$$

by setting $\mathcal{H}[\varphi](x)$ equal to the mean curvature of N at $\Gamma_\varphi(x)$. This differential operator

separates into zeroth, first, and higher degree terms as

$$n\mathcal{H}[\varphi] = nH_M - (-\Delta_M - |A_M|^2)\varphi + Q[\varphi].$$

where H_M is the mean curvature of M , Δ_M is the Laplace-Beltrami operator on M , and A_M the second fundamental form of M (see Appendix A). Thus $|A_M|^2$ is also the sum of the squares of the principle curvatures of M . The linear operator $L = -\Delta_M - |A_M|^2$ is known as the stability operator for M [12, 17], see Chapter 8.

We are interested in the solutions of the Dirichlet boundary value problem

$$\begin{cases} 2\mathcal{H}[\varphi] = 2H_N \\ \varphi|_{\partial M} = 0. \end{cases}$$

Remark 3.2.1. This approach is a variation of the approach of Kapouleas. In [14, 15, 16] he looks for solutions to a similar problem without boundary constraints. However, where we have our base surface M as constant mean curvature and fixed, the base surfaces of Kapouleas are explicitly constructed to be only nearly-constant mean curvature and must be adjusted to arrive at the necessary inequalities.

We use the non-standard notation

$$C_0^{k,\alpha}(M) = \{\varphi \in C^{k,\alpha}(M) \cap C^0(\overline{M}) | \varphi \equiv 0 \text{ on } \partial M\}$$

and

$$H_0^k(M) = \{\varphi \in H^k(M) \cap C^0(\overline{M}) | \varphi \equiv 0 \text{ on } \partial M\}.$$

Under this notation $L : C_0^{p+2,\alpha}(M) \rightarrow C^{p,\alpha}(M)$. We can also define the standard bilinear form associated with L ,

$$B : C_0^{p+2,\alpha}(M) \times C_0^{p+2,\alpha}(M) \rightarrow \mathbb{R}.$$

For all $u, v \in C_0^{p+2,\alpha}(M)$, let

$$\begin{aligned}
B[u, v] &= \langle L[u], v \rangle_{L^2(M)} \\
&= \int_M L[u]v dx \\
&= \int_M -(\Delta_M u)v - |A|^2 uv dx \\
&= \int_M -(\nabla \cdot \nabla u)v - |A|^2 uv dx \\
&= \int_M \nabla u \cdot \nabla v - |A|^2 uv dx - \int_{\partial M} v \nabla u \cdot \vec{n} ds \\
&= \int_M \nabla u \cdot \nabla v - |A|^2 uv dx.
\end{aligned}$$

Using this last equation as a definition for all $u, v \in H_0^1(M)$, the bilinear form B can be extended to $B : H_0^1(M) \times H_0^1(M) \rightarrow \mathbb{R}$. Note also that B is symmetric over both $C_0^{p+2,\alpha}(M)$ and $H_0^1(M)$.

Since the second order terms of L are simply those from the symmetric Laplace-Beltrami operator, we note that L is uniformly elliptic. Since \overline{M} is C^∞ , the coefficients of L are also C^∞ and bounded on \overline{M} . The symmetry of B also implies that $L : C_0^{p+2,\alpha}(M) \rightarrow C^{p,\alpha}(M) \subset L^2(M)$ is formally self-adjoint:

$$\langle L[u], v \rangle_{L^2(M)} = B[u, v] = B[v, u] = \langle u, L[v] \rangle_{L^2(M)}$$

for all $u, v \in C_0^{p+2,\alpha}(M)$. Thus the eigenvalues of L are countably infinite, real, and bounded below. The first eigenvalue is also known to be simple and have a positive eigenfunction [11, Theorems 8.37-38].

3.3 Solutions and Regularity

Given $f \in L^2(M)$ we let $f^* \in L^2(M)^*$ denote its dual given by

$$f^*(v) = \int_M f v dx$$

for all $v \in L^2(M)$. Then we say $u \in H_0^1(M)$ is a *weak solution* to the Dirichlet boundary value problem

$$\begin{cases} L[u] = f \text{ on } M \\ u \equiv 0 \text{ on } \partial M \end{cases}$$

if for all $v \in H_0^1(M)$ we have

$$B[u, v] = f^*(v).$$

Because the coefficients of L are smooth we get greater regularity for weak solutions. We will use the following regularity theorem from [18, Section 6.3.2 Theorem 6]:

Theorem 3.3.1. *Let L' be a uniformly elliptic linear second order differential operator with $C^\infty(\overline{M})$ coefficients. Then if $f \in C^\infty(\overline{M})$, ∂M is C^∞ , and $u \in H_0^1(M)$ is a weak solution to the Dirichlet boundary value problem*

$$\begin{cases} L'[u] = f \text{ on } M \\ u \equiv 0 \text{ on } \partial M \end{cases}$$

then

$$u \in C_0^\infty(\overline{M}).$$

Since L satisfies these conditions we will use Theorem 3.3.1 to show that weak solutions are in fact C^∞ classical solutions. Specifically we note that Theorem 3.3.1 implies that the eigenfunctions of L are in $C^\infty(\overline{M})$.

We can also state a form of Theorem 3.3.1 with weaker hypotheses.

Theorem 3.3.2. *Let L' be a uniformly elliptic linear second order differential operator with $C^\infty(\overline{M})$ coefficients. Then if $f \in C^{p,\alpha}(\overline{M})$, ∂M is C^∞ , and $u \in H_0^1(M)$ is a weak solution to the Dirichlet boundary value problem*

$$\begin{cases} L'[u] = f \text{ on } M \\ u \equiv 0 \text{ on } \partial M \end{cases}$$

then

$$u \in H_0^{p+2}(M).$$

Also if $p > 2 + n/2$ then there exists $\alpha \in (0, 1)$ such that

$$u \in C_0^{p+2,\alpha}(\overline{M}).$$

Proof. The first part of this theorem is equivalent to [18, Section 6.3.2, Theorem 5]. If $p > 2 + n/2$, then the Sobolev inequalities imply that there exists α such that $H_0^p(M) \subset C_0^{2,\alpha}(\overline{M})$. Then u is also a classical solution and by standard regularity theorems $u \in C_0^{p+2,\alpha}(M)$, see [11, Theorem 6.19]. \square

Remark 3.3.3. In their original sources the two regularity theorems above are stated in the non-parametric realm of functions over a domain in \mathbb{R}^n . These results are, however, well known to extend to the parametric case of functions on a sufficiently smooth manifold. See for example [19].

CHAPTER 4

BOUNDS ON THE EIGENVALUES OF L

In this chapter we show when there exists a positive super-solution h to L the eigenvalues of L are non-negative. If h is also non-zero on a boundary point of M then we show that all eigenvalues of L are strictly positive. This result is an adaptation of the methods of [20] to include boundary conditions.

4.1 A Proposition about Super-Solutions of L

Since the boundary of M has positive reach in M there is some $R_{\partial M} > 0$ such that the map $\gamma : \partial M \times [0, R_{\partial M}] \rightarrow \overline{M}$ is a one-to-one diffeomorphism. Thus we can also define $\pi : \gamma(\partial M \times [0, R_{\partial M}]) \rightarrow \partial M$ to be the boundary projection $x \mapsto y$ where $\gamma_y(\text{dist}(x, \partial M)) = x$. For ease of notation we also define on the same domain the unit vector field

$$\vec{v}(x) = \gamma'_{\pi(x)}(\text{dist}(x, \partial M)) \in T_x M.$$

We first prove a proposition concerning super-solutions of L that will be used later. We will use the following version of the Hopf boundary point lemma [11, Lemma 3.4]:

Lemma 4.1.1 (Hopf Boundary Point Lemma). *Suppose that L' is a uniformly elliptic linear differential operator and that $L'[h] \geq 0$ on M . Let $x_0 \in \partial M$ be such that*

- *h is continuous at x_0 ,*
- *$h(x) > h(x_0) = 0$ for all $x \in M$,*
- *and ∂M satisfies an interior sphere condition at x_0 .*

Then if the exterior normal derivative of h at x_0 exists, it satisfies the strict inequality

$$\frac{\partial h}{\partial \vec{n}} = \langle \nabla h(x_0), \vec{n}(x_0) \rangle < 0.$$

Proposition 4.1.2. *If there exists a function $h : \overline{M} \rightarrow \mathbb{R}$ in $C^2(M) \cap C^1(\overline{M})$ such that*

$$\begin{cases} L[h] \geq 0 \\ h > 0 \text{ on } M \end{cases}$$

and $u \in C^1(\overline{M})$ such that

$$u = 0 \text{ on } \partial M$$

then $\frac{u}{h} \in L^\infty(\overline{M})$

Proof. Suppose instead that there exists $x_i \in M$ such that

$$\left| \frac{u(x_i)}{h(x_i)} \right| \rightarrow \infty.$$

Without loss of generality we can replace x_i with a subsequence convergent to some $x_0 \in \overline{M}$ since \overline{M} is compact. As u is bounded on \overline{M} , we must have $h(x_0) = 0$. This may occur only when $x_0 \in \partial M$.

Now ∂M satisfies an interior sphere condition at all of its points. As $L[h] \geq 0$ and $h(x) > h(x_0) = 0$ as well, h satisfies the conditions of the Hopt boundary point lemma at x_0 . Thus we have

$$\langle \nabla h(x_0), \vec{v}(x_0) \rangle = -\langle \nabla h(x_0), \vec{n}(x_0) \rangle > \epsilon > 0$$

for some ϵ . By the continuity of the first derivatives of h there is a neighborhood $U \subset \overline{M}$ of x_0 such that $\langle \nabla h(x), \vec{v}(x) \rangle > \epsilon/2 > 0$ for all $x \in U$. Note as well that since u is C^1 on the compact \overline{M} there exists δ such that $\|\nabla u\| \leq \delta < \infty$ everywhere in \overline{M} . Consider then

that for all $x \in U$

$$\begin{aligned}
\left| \frac{u(x)}{h(x)} \right| &= \left| \frac{u(\pi(x)) + \int_0^{\text{dist}(x, \partial M)} \langle \nabla u(x), \vec{v}(x) \rangle dt}{h(\pi(x)) + \int_0^{\text{dist}(x, \partial M)} \langle \nabla h(x), \vec{v}(x) \rangle dt} \right| \\
&\leq \left| \frac{\int_0^{\text{dist}(x, \partial M)} \langle \nabla u(x), \vec{v}(x) \rangle dt}{\int_0^{\text{dist}(x, \partial M)} \langle \nabla h(x), \vec{v}(x) \rangle dt} \right| \\
&\leq \left| \frac{\int_0^{\text{dist}(x, \partial M)} \delta dt}{\int_0^{\text{dist}(x, \partial M)} \epsilon/2 dt} \right| \\
&\leq \left| \frac{\text{dist}(x, \partial M) \delta}{\text{dist}(x, \partial M) \epsilon/2} \right| \\
&= \frac{2\delta}{\epsilon} < \infty
\end{aligned}$$

since $u(\pi(x)) = 0$. In particular this gives a contradiction to the first supposition as the sequence of x_i is eventually in U . Thus $\frac{u}{h}$ is bounded and in $L_\infty(\overline{M})$.

□

4.2 A Condition for Non-negative Eigenvalues

This allows us to use positive super-solutions to prove results about the eigenvalues of L .

Lemma 4.2.1. *Suppose that there exists a function $h : \overline{M} \rightarrow \mathbb{R}$ in $C^2(M) \cap C^1(\overline{M})$ such that*

$$\begin{cases} L[h] \geq 0 \\ h > 0 \text{ on } M \end{cases}$$

Then $\lambda \geq 0$ for all eigenvalues λ of L .

Proof. Letting $w = \log h$, we consider the following:

$$\begin{aligned}
L[h] &= -\Delta h - |A|^2 h \geq 0 \\
\Delta w &= \frac{\Delta h}{h} - \left| \frac{\nabla h}{h} \right|^2 \\
&\leq \frac{-|A|^2 h}{h} - |\nabla w|^2 \\
&= -|A|^2 - |\nabla w|^2
\end{aligned}$$

Let $M_\epsilon = \{x \in M \mid \text{dist}(x, \partial M) > \epsilon\}$. Also let $u \in C_0^\infty(M) \subset H_0^1(M) \subset L^2(M)$ be an arbitrary nontrivial eigenfunction such that $\|u\|_{L^2(M)} = 1$. Multiplying the above inequality by u^2 , integrating over M_ϵ , and rearranging terms gives us

$$\int_{M_\epsilon} |A|^2 u^2 + |\nabla w|^2 u^2 dx \leq - \int_{M_\epsilon} (\Delta w) u^2 dx$$

Then as h is C^2 and positive on $M_\epsilon \cup \partial M_\epsilon \subset M$ integration by parts also yields

$$\begin{aligned}
- \int_{M_\epsilon} (\Delta w) u^2 dx &= \int_{M_\epsilon} 2u \langle \nabla w, \nabla u \rangle dx - \int_{\partial M_\epsilon} u^2 \langle \nabla w, \vec{n} \rangle dx \\
&\leq \int_{M_\epsilon} 2|u| |\nabla w| |\nabla u| dx - \int_{\partial M_\epsilon} u^2 \langle \nabla w, \vec{n} \rangle dx \tag{1}
\end{aligned}$$

$$\leq \int_{M_\epsilon} u^2 |\nabla w|^2 + |\nabla u|^2 dx - \int_{\partial M_\epsilon} u^2 \langle \nabla w, \vec{n} \rangle dx \tag{2}$$

by application of the Cauchy-Schwartz inequality and the arithmetic mean-geometric mean inequality. Thus

$$\int_{M_\epsilon} |A|^2 u^2 + |\nabla w|^2 u^2 dx \leq \int_{M_\epsilon} u^2 |\nabla w|^2 + |\nabla u|^2 dx - \int_{\partial M_\epsilon} u^2 \langle \nabla w, \vec{n} \rangle dx$$

Canceling like terms and rearranging yields

$$\begin{aligned}
\int_{\partial M_\epsilon} u^2 \langle \nabla w, \vec{n} \rangle dx &\leq \int_{M_\epsilon} |\nabla u|^2 - |A|^2 u^2 dx \\
\lim_{\epsilon \rightarrow 0} \int_{\partial M_\epsilon} u^2 \langle \nabla w, \vec{n} \rangle dx &\leq \lim_{\epsilon \rightarrow 0} \int_{M_\epsilon} |\nabla u|^2 - |A|^2 u^2 dx \\
&= \int_M |\nabla u|^2 - |A|^2 u^2 dx \\
&= B[u, u]
\end{aligned}$$

To complete the proof we now need only to show that the limit on the left evaluates to zero. Let $\pi_\epsilon^{-1} : \partial M \rightarrow \partial M_\epsilon$ be the inverse of $\pi_\epsilon = \pi|_{\partial M_\epsilon}$. Consider then

$$\begin{aligned}
&\lim_{\epsilon \rightarrow 0} \int_{\partial M_\epsilon} u^2 \langle \nabla w, \vec{n} \rangle dx \\
&= \lim_{\epsilon \rightarrow 0} \int_{\partial M_\epsilon} u \frac{u}{h} \langle \nabla h, \vec{n} \rangle dx \\
&= \lim_{\epsilon \rightarrow 0} \int_{\partial M} u(\pi_\epsilon^{-1}) \frac{u(\pi_\epsilon^{-1})}{h(\pi_\epsilon^{-1})} \langle \nabla h(\pi_\epsilon^{-1}), -\vec{v}(\pi_\epsilon^{-1}) \rangle |\det(D\pi_\epsilon^{-1})| dx \\
&= \int_{\partial M} \lim_{\epsilon \rightarrow 0} u(\pi_\epsilon^{-1}) \frac{u(\pi_\epsilon^{-1})}{h(\pi_\epsilon^{-1})} \langle \nabla h(\pi_\epsilon^{-1}), -\vec{v}(\pi_\epsilon^{-1}) \rangle |\det(D\pi_\epsilon^{-1})| dx \\
&= \int_{\partial M} 0 dx = 0
\end{aligned}$$

as a result of the dominated convergence theorem. Specifically u , $\langle \nabla h, -\vec{v} \rangle$, and $|\det(D\pi_\epsilon^{-1})|$ are uniformly bounded from continuity on the compact domain \overline{M} . Proposition 4.1.2 shows that $\frac{u}{h}$ is also uniformly bounded since $u \in C^\infty(\overline{M})$ as an eigenfunction. The limit then follows by noting that

$$\lim_{\epsilon \rightarrow 0} u(\pi_\epsilon^{-1}(x)) = u(x) = 0$$

for all $x \in \partial M$. Thus

$$0 \leq B[u, u].$$

Let λ be the eigenvalue associated with the eigenfunction u . Then

$$\begin{aligned}
 B[u, u] &= \langle L[u], u \rangle_{L^2(M)} \\
 &= \langle \lambda u, u \rangle_{L^2(M)} \\
 &= \lambda \langle u, u \rangle_{L^2(M)} \\
 &= \lambda
 \end{aligned}$$

Since u was an arbitrary eigenfunction of L ,

$$0 \leq \lambda$$

for all eigenvalues of L .

□

4.3 A Condition for Positive Eigenvalues

In the following chapters we will apply the Fredholm alternative on the operator L . To do so we need the strict inequality $\lambda > 0$. We can show this by adding the condition that h is non-zero at some point on the boundary of M .

Corollary 4.3.1. *If there exists a function $h : \overline{M} \rightarrow \mathbb{R}$ in $C^2(M) \cap C^1(\overline{M})$ such that*

$$\begin{cases}
 L[h] \geq 0 \\
 h > 0 \text{ on } M \\
 h(x_0) > 0 \text{ for some } x_0 \in \partial M
 \end{cases}$$

Then $\lambda > 0$ for all Dirichlet eigenvalues λ of L .

Proof. Let λ be a Dirichlet eigenvalue of L with $u \in C^\infty(\overline{M})$ is associated eigenfunction such that $\|u\|_{L^2} = 1$. From Lemma 4.2.1 we have $\lambda \geq 0$. However, equality in the proof

of Lemma 4.2.1 requires equality in the Cauchy-Schwartz and arithmetic mean-geometric mean inequalities (1) and (2). This would imply that

$$f\nabla w = \nabla u$$

and

$$|u||\nabla w| = |\nabla u|$$

respectively, where f is a scalar function on M . Taken together these imply that $f \equiv \pm u$, so that

$$\begin{aligned}\nabla u &= \pm u \nabla w \\ &= \pm u \frac{\nabla h}{h} \\ 0 &= \nabla u \pm \frac{u}{h} \nabla h\end{aligned}$$

from which we obtain

$$\nabla \frac{u}{h} = 0 \text{ or } \nabla(uh) = 0.$$

But this has only the solution $u \equiv ch^{\pm 1}$. Since $0 > h > -\infty$ on M and u is nontrivial, $c \neq 0$. But as this would imply that $u(x_0) = ch(x_0)^{\pm 1} \neq 0$ this leads a contradiction to the Dirichlet conditions on u . Thus we can take these inequalities to be strict, giving us $\lambda > 0$. □

We also note that the same conclusion would result if h was a strict super-solution.

Corollary 4.3.2. *If there exists a function $h : \overline{M} \rightarrow \mathbb{R}$ in $C^2(M) \cap C^1(\overline{M})$ such that*

$$\begin{cases} L[h] > 0 \\ h > 0 \text{ on } M \end{cases}$$

Then $\lambda > 0$ for all Dirichlet eigenvalues λ of L .

CHAPTER 5

INVERTIBILITY OF THE LINEAR OPERATOR L

In this chapter we complete the proof of the invertibility of the linear operator L by using the Fredholm alternative to reduce this problem to the Dirichlet eigenvalues of L all being non-zero. This follows the standard approach for weak solutions to linear problems [11, 18].

5.1 The Eigenvalue Condition on the Invertibility of L

First, we note that regularity results for linear, uniformly elliptic operators on manifolds are well established and it is known that a weak solution, $u \in H^1(M)$, to the linear problem

$$\begin{cases} L[u] = f \\ u|_{\partial M} = 0 \end{cases}$$

is typically also a classical solution in $C^\infty(M)$ (See Theorem 3.3.1). Therefore in this chapter we will concentrate primarily on weak solutions. We add regularity arguments at the end of this chapter.

Lemma 5.1.1. *The following are equivalent:*

1. *The boundary value problem*

$$\begin{cases} L[u] = f \\ u|_{\partial M} = 0 \end{cases}$$

has a unique weak solution in $H_0^1(M)$ for every $f \in L^2(M)$, and the inverse function $L^{-1} : L^2(M) \rightarrow H_0^1(M)$ is continuous.

2. The homogeneous boundary value problem

$$\begin{cases} L[u] = 0 \\ u|_{\partial M} = 0 \end{cases}$$

has only the trivial weak solution in $H_0^1(M)$.

To prove Lemma 5.1.1 we will make use of the Lax-Milgram Theorem and the Fredholm alternative. We begin with the Lax-Milgram Theorem for bilinear forms:

Theorem 5.1.2 (The Lax-Milgram Theorem). *Suppose H is a real Hilbert space with norm $\|\cdot\|$, and that*

$$B : H \times H \rightarrow \mathbb{R}$$

is a bilinear form. If B is bounded,

$$|B[u, v]| \leq C_1 \|u\| \|v\| \text{ for all } u, v \in H,$$

and coercive,

$$B[u, u] \geq C_2 \|u\|^2 \text{ for all } u \in H,$$

then for all bounded linear functionals $F : H \rightarrow \mathbb{R}$ in H^ there exists a unique element $u_F \in H$ such that*

$$B[u_F, v] = F(v) \text{ for all } v \in H.$$

In subsequent proofs we wish the element u_F produced by the Lax-Milgram theorem to be a weak solution to the linear problem $L[u] = F$. Recall then the bilinear form B defined in Chapter 3 and extended to $u, v \in H_0^1(M)$,

$$B(u, v) = \int_M \nabla u \cdot \nabla v - |A|^2 u v dx.$$

This bilinear form is bounded, but not coercive. As an alternate approach we investigate instead the related linear operator and its associated bilinear form

$$\begin{aligned}
L_\sigma[u] &= L[u] + \sigma u \\
&= -\Delta_M u + (\sigma - |A|^2)u \\
B_\sigma(u, v) &= \int_M L_\sigma[u]v dx \\
&= \int_M \nabla u \cdot \nabla v + (\sigma - |A|^2)uv dx
\end{aligned}$$

where $\sigma = \sup_M |A|^2 + 1$. As before, using this last equation as the definition we may extend B_σ to

$$B_\sigma : H_0^1(M) \times H_0^1(M) \rightarrow \mathbb{R}.$$

Proposition 5.1.3. *The bilinear form $B_\sigma(u, v)$ is bounded and coercive.*

Proof. Now consider that

$$\begin{aligned}
|B_\sigma(u, v)| &\leq \int_M |\nabla u| |\nabla v| + (\sigma - |A|^2)|u||v| dx \\
&\leq \int_M \sigma |\nabla u| |\nabla v| + \sigma |u||v| dx \\
&\leq \sigma \|u\|_{H_0^1(M)} \|v\|_{H_0^1(M)}
\end{aligned}$$

by the Cauchy-Schwartz inequality, so B_σ is bounded on $H_0^1(M)$. But also

$$\begin{aligned}
|B_\sigma(u, u)| &= \int_M |\nabla u|^2 + (\sigma - |A|^2)|u|^2 dx \\
&\geq \int_M |\nabla u|^2 + |u|^2 dx \\
&= \|u\|_{H_0^1(M)}^2,
\end{aligned}$$

which proves coercivity.

□

Thus the bilinear form B_σ satisfies the hypotheses of the Lax-Milgram theorem. This allows us to show the solvability of problems of the form $L_\sigma u = f$.

Lemma 5.1.4. *For all $f \in L^2(M)$ there exists a unique weak solution to the linear problem*

$$L_\sigma[u] = f.$$

Proof. Now $u \in H_0^1(M)$ is a weak solution to $L_\sigma[u] = f$ if

$$B[u, v] = \int_M f v dx$$

for all $v \in H_0^1(M)$. Let $f^* \in H_0^1(M)^*$ be the linear functional

$$f^*(v) = \int_M f v dx.$$

Then since B_σ satisfies the hypotheses of the Lax-Milgram by Proposition 5.1.3 there exists a unique $u_f \in H_0^1(M)$ such that

$$B[u_f, v] = f^*(v) = \int_M f v dx \text{ for all } v \in H.$$

Thus u_f is the unique weak solution to $L_\sigma[u] = f$. □

In light of Lemma 5.1.4 we define the function

$$L_\sigma^{-1} : L^2(M) \rightarrow H_0^1(M)$$

given by $L_\sigma^{-1}[f] = u_f$, the unique weak solution to $L_\sigma[u] = f$.

To complete the proof of Lemma 5.1.1 we will require the following form of the Fredholm alternative [11, Theorem 5.3]:

Theorem 5.1.5 (The Fredholm Alternative). *Suppose H is a real Hilbert space and $K :$*

$H \rightarrow H$ is a compact linear operator. Then the following are equivalent:

- The equation $u - Ku = f$ has a unique solution for all $f \in H$ and the inverse operator $(I - K)^{-1}$ is bounded.
- The equation $u - Ku = 0$ has only the trivial solution in H .

We give a proposition to facilitate our use of the Fredholm alternative.

Proposition 5.1.6. *The operator $L_\sigma^{-1} : L^2(M) \rightarrow H_0^1(M)$ is continuous. When restricted to the domain $H_0^1(M) \subset L^2(M)$ the operator $L_\sigma^{-1} : H_0^1(M) \rightarrow H_0^1(M)$ is compact.*

Proof. Take any $f \in L^2(M)$ and let $u_f = L_\sigma^{-1}[f]$. As a consequence of the coercivity of B_σ we have

$$|B_\sigma[u_f, u_f]| \geq \|u_f\|_{H_0^1(M)}^2.$$

But we also have

$$\begin{aligned} |B_\sigma[u_f, u_f]| &= |f^*(u_f)| \\ &= \|f^*\|_{H_0^1(M)^*} \|u_f\|_{H_0^1(M)} \\ &\leq \|f^*\|_{L^2(M)^*} \|u_f\|_{H_0^1(M)} \\ &= \|f\|_{L^2(M)} \|u_f\|_{H_0^1(M)} \end{aligned}$$

by the isometry of $L^2(M)$ and $L^2(M)^*$. Together imply

$$\|L_\sigma^{-1}[f]\|_{H_0^1(M)} = \|u_f\|_{H_0^1(M)} \leq \|f\|_{L^2(M)}$$

which shows the boundedness, and thus continuity, of L_σ^{-1} .

Lastly consider that

$$L_\sigma^{-1}|_{H_0^1(M)} = L_\sigma^{-1} \circ I$$

where $I : H_0^1(M) \hookrightarrow L^2(M)$ is the inclusion map. But as the imbedding I is compact [11, Theorem 7.22], so is $L_\sigma^{-1}|_{H_0^1(M)}$, its composition with the continuous function L_σ^{-1} . \square

We can now use these results to prove Lemma 5.1.1.

Proof of Lemma 5.1.1. The operator $\sigma L_\sigma^{-1} : H_0^1(M) \rightarrow H_0^1(M)$ is compact by Proposition 5.1.6. Thus we may apply the Fredholm alternative to σL_σ^{-1} . Then the following are equivalent

(i) The equation

$$u - \sigma L_\sigma^{-1}[u] = \phi$$

has a unique solution for all $\phi \in H_0^1(M)$ and the operator $(I - \sigma L_\sigma^{-1})^{-1}$ is continuous.

(ii) The equation

$$u - \sigma L_\sigma^{-1}[u] = 0$$

has only the trivial solution in $H_0^1(M)$.

Consider (i). This equation has a unique solution for all $\phi \in H_0^1(M)$, so in particular has a unique solution for $\phi = L_\sigma^{-1}[f] \in H_0^1(M)$ for all $f \in L^2(M)$. But $u \in H_0^1(M)$ is a unique solution to

$$u - \sigma L_\sigma^{-1}[u] = L_\sigma^{-1}[f]$$

$$u = L_\sigma^{-1}[\sigma u + f]$$

if and only if it is also a unique weak solution to $L_\sigma[u] = \sigma u + f$, namely that for all

$v \in H_0^1(M)$ we have

$$\begin{aligned}
B_\sigma[u, v] &= \int_M (\sigma u + f) v dx \\
\int_M \nabla u \cdot \nabla v + (\sigma - |A|^2) u v dx &= \int_M (\sigma u + f) v dx \\
\int_M \nabla u \cdot \nabla v - |A|^2 u v dx &= \int_M f v dx \\
B[u, v] &= f^*(v)
\end{aligned}$$

which is the condition for u to be a weak solution to $L[u] = f$. Therefore if $L^{-1}[f]$ denotes the unique weak solution to $L[u] = f$ we have

$$L^{-1}[f] = (I - \sigma L_\sigma^{-1})^{-1} L_\sigma^{-1}[f]$$

where $L^{-1} : L^2(M) \rightarrow H_0^1(M)$. Now as L_σ and its inverse are continuous, L^{-1} is continuous if and only if $(I - \sigma L_\sigma^{-1})^{-1}$ is continuous. Thus (i) is equivalent to (1) in the statement of Lemma 5.1.1.

Consider (ii). A non-trivial solution to

$$\begin{aligned}
u - \sigma L_\sigma^{-1}[u] &= 0 \\
L_\sigma^{-1}[u] &= \frac{1}{\sigma} u
\end{aligned}$$

in $H_0^1(M)$ is equivalently a non-trivial weak solution to $L_\sigma \left[\frac{1}{\sigma} u \right] = u$, i.e. that for all

$v \in H_0^1(M)$ we have

$$\begin{aligned}
B_\sigma \left[\frac{1}{\sigma} u, v \right] &= \int_M u v dx \\
\frac{1}{\sigma} \int_M \nabla u \cdot \nabla v + (\sigma - |A|^2) u v dx &= \int_M u v dx \\
\frac{1}{\sigma} \int_M \nabla u \cdot \nabla v - |A|^2 u v dx &= 0 \\
\frac{1}{\sigma} B[u, v] &= 0 \\
B[u, v] &= 0
\end{aligned}$$

so that u is a weak solution of $L[u] = 0$. Therefore (ii) is equivalent to (2). Thus as (1) \equiv (i) \equiv (ii) \equiv (2), this completes the proof of Lemma 5.1.1. \square

5.2 Regularity of the Inverse L^{-1}

Based on this result we can turn the question of the invertibility of L into a question about the eigenvalues of L with respect to functions that vanish on ∂M . Combining this with the regularity theorems in Chapter 3 we have the following corollaries.

Corollary 5.2.1. *If $\lambda \neq 0$ for all Dirichlet eigenvalues of L then the boundary value problem*

$$\begin{cases} L[u] = f \\ u|_{\partial M} = 0 \end{cases}$$

has a unique solution in $C_0^\infty(\overline{M})$ for every $f \in C^\infty(\overline{M})$, and the inverse function $L^{-1} : C^\infty(\overline{M}) \rightarrow C_0^\infty(\overline{M})$ is continuous.

Proof. Suppose that $u \in H_0^1(M)$ is a nontrivial weak solution to the boundary value problem

$$\begin{cases} L[u] = 0 \\ u|_{\partial M} = 0 \end{cases}$$

must also be in $C_0^\infty(\overline{M})$ by the regularity theorem 3.3.1. Therefore u is a classical eigenfunction with eigenvalue 0, a contradiction. The homogeneous boundary value problem then has only the trivial weak solution in $H_0^1(M)$ and Lemma 5.1.1 implies that

$$\begin{cases} L[u] = f \\ u|_{\partial M} = 0 \end{cases}$$

has a unique weak solution in $H_0^1(M)$ for every $f \in C^\infty(\overline{M}) \subset L^2(M)$, and the inverse function $L^{-1} : C^\infty(\overline{M}) \rightarrow H_0^1(M)$ is continuous. Again by the regularity theorem 3.3.1 we see that any weak solution u must also be a classical solution in $C_0^\infty(\overline{M})$ and $L^{-1} : C^\infty(\overline{M}) \rightarrow C_0^\infty(\overline{M})$. \square

We can state a similar result with weaker hypotheses.

Corollary 5.2.2. *If $p > 2 + n/2$ and $\lambda \neq 0$ for all Dirichlet eigenvalues of L then the boundary value problem*

$$\begin{cases} L[u] = f \\ u|_{\partial M} = 0 \end{cases}$$

has a unique solution in $C_0^{p+2,\alpha}(\overline{M})$ for every $f \in C^{p,\alpha}(\overline{M})$, and the inverse function $L^{-1} : C^{p,\alpha}(\overline{M}) \rightarrow C_0^{p+2,\alpha}(\overline{M})$ is continuous.

Proof. Proof follows just as in 5.2.1 with the weaker regularity theorem 3.3.2 substituted for theorem 3.3.1. Specifically, under these hypotheses we have that for all $f \in C^{p,\alpha}(\overline{M})$ there exists a unique weak solution $u_f \in H_0^1(M)$ and the inverse function $L^{-1} : C^{p,\alpha}(\overline{M}) \rightarrow H_0^1(M)$ is continuous. But by Theorem 3.3.2 this implies that $u_f \in C_0^{p+2,\alpha}(\overline{M})$ and is a classical solution. Thus $L^{-1} : C^{p,\alpha}(\overline{M}) \rightarrow C_0^{p+2,\alpha}(\overline{M})$. \square

The invertibility of L allows us to complete the proof of our main theorem in the next chapter.

CHAPTER 6

PROOF OF THE MAIN THEOREM

Consider again the mean curvature operator

$$n\mathcal{H} : U \subset C_0^{p+2,\alpha}(\overline{M}) \rightarrow C^{p,\alpha}(\overline{M})$$

for

$$U = \{\varphi \in C_0^{p+2,\alpha}(\overline{M}) \mid \|\varphi\|_{p+2,\alpha} < R_M\}$$

which gives the mean curvature of $N = \Gamma_\varphi(M)$ as

$$n\mathcal{H}[\varphi] = nH_M - L[\varphi] + Q[\varphi].$$

To complete the proof of the main theorem we wish to show that this operator is invertible near 0 and is a local C^∞ -isomorphism. We do this via the inverse function theorem, following the approach of [17]. Throughout this chapter we assume $p > 2 + n/2$.

6.1 Bounds on the Operator Q

Consider first the operator Q . Since Q is of at least quadratic degree we can show that it is quadratically bounded near 0. For ease of use we denote

$$\|\varphi\|_{p,\alpha} = \|\varphi\|_{C^{p,\alpha}(M)} = \sum_{|\beta| \leq p} \|\varphi_\beta\|_{C^{0,\alpha}(M)}.$$

Lemma 6.1.1. *There exists $C_Q, r > 0$ such that $\|Q[\varphi]\|_{p,\alpha} \leq C_Q \|\varphi\|_{p+2,\alpha}^2$ for all $\varphi \in C^2(M)$ such that $\|\varphi\|_{p+2,\alpha} \leq r$.*

Proof. Suppose that $F : C^{p+2,\alpha}(\overline{M}) \rightarrow C^{p,\alpha}(\overline{M})$ is a polynomial of at least quadratic

degree in φ, φ_i and φ_{ij} . Then $F(0) = 0$ and there exist polynomials $P_{\beta\gamma}$ in φ, φ_i and φ_{ij} such that

$$\begin{aligned} F(\varphi) &= \sum_{|\beta|, |\gamma| \leq 2} P_{\beta\gamma}(\varphi) \varphi_\beta \varphi_\gamma \\ \|F(\varphi)\|_{p,\alpha} &\leq \sum_{|\beta|, |\gamma| \leq 2} \|P_{\beta\gamma}(\varphi)\|_{p,\alpha} \|\varphi_\beta\|_{p,\alpha} \|\varphi_\gamma\|_{p,\alpha} \\ &\leq \sum_{|\beta|, |\gamma| \leq 2} \|P_{\beta\gamma}(\varphi)\|_{p,\alpha} \|\varphi\|_{p+2,\alpha}^2 \end{aligned}$$

Now if $\|\varphi\|_{p,\alpha} < r$ for some $r > 0$, then by compactness

$$\sum_{|\beta|, |\gamma| \leq 2} \|P_{\beta\gamma}(\varphi)\|_{p,\alpha} < C_F$$

for some $C_F(r) > 0$. In this case we have

$$\|F(\varphi)\|_{p,\alpha} \leq C_F \|\varphi\|_{p+2,\alpha}^2.$$

It is shown in Appendix A that the operator Q is given by

$$Q[\varphi] = \frac{F_1}{(1 + F_3)^{1/2}} + \frac{F_2}{1 + F_3 + (1 + F_3)^{1/2}}$$

where F_1, F_2, F_3 are polynomials of at least quadratic degree in φ, φ_i and φ_{ij} . Therefore given $r > 0$ there exists $C_{F_k} > 0$ such that $\|F_k\|_{p,\alpha} \leq C_F \|\varphi\|_{p+2,\alpha}^2$ for all $\|\varphi\|_{p+2,\alpha}^2 < r$. Let r be sufficiently small to ensure that if $\|\varphi\|_{p+2,\alpha} \leq r$ then $\|F_3\|_{p,\alpha} \leq C_{F_3} \|\varphi\|_{p+2,\alpha}^2 \leq C_{F_3} r^2 < 1$ (this will also ensure that $\Gamma(M)$ is an immersion, see Appendix A). Under this

assumption we have

$$\begin{aligned}
\|Q[\varphi]\|_{p,\alpha} &= \left\| \frac{F_1}{(1+F_3)^{1/2}} + \frac{F_2}{1+F_3+(1+F_3)^{1/2}} \right\|_{p,\alpha} \\
&\leq \left\| \frac{F_1}{(1+F_3)^{1/2}} \right\|_{p,\alpha} + \left\| \frac{F_2}{1+F_3+(1+F_3)^{1/2}} \right\|_{p,\alpha} \\
&\leq \frac{\|F_1\|_{p,\alpha}}{(1-\|F_3\|_{p,\alpha})^{1/2}} + \frac{\|F_2\|_{p,\alpha}}{1-\|F_3\|_{p,\alpha}+(1-\|F_3\|_{p,\alpha})^{1/2}} \\
&\leq \left[\frac{C_{F_1}\|\varphi\|_{p+2,\alpha}^2}{(1-C_{F_3}r^2)^{1/2}} + \frac{C_{F_2}\|\varphi\|_{p+2,\alpha}^2}{1-C_{F_3}r^2+(1-C_{F_3}r^2)^{1/2}} \right] \\
&\leq \left[\frac{C_{F_1}}{(1-C_{F_3}r^2)^{1/2}} + \frac{C_{F_2}}{1-C_{F_3}r^2+(1-C_{F_3}r^2)^{1/2}} \right] \|\varphi\|_{p+2,\alpha}^2 \\
&= C_Q \|\varphi\|_{p+2,\alpha}^2
\end{aligned}$$

□

Using this result we note that the Fréchet derivative of $n\mathcal{H} : C_0^{p+2,\alpha}(M) \rightarrow C^{p,\alpha}(M)$ at 0 is the linear operator $-L$. Indeed,

$$\begin{aligned}
&\lim_{\|\varphi\|_{p+2,\alpha} \rightarrow 0} \frac{\|n\mathcal{H}[\varphi] - n\mathcal{H}[0] - (-L[\varphi])\|_{p,\alpha}}{\|\varphi\|_{p+2,\alpha}} \\
&= \lim_{\|\varphi\|_{p+2,\alpha} \rightarrow 0} \frac{\|nH_M - L[\varphi] + Q[\varphi] - nH_M + L[\varphi]\|_{p,\alpha}}{\|\varphi\|_{p+2,\alpha}} \\
&= \lim_{\|\varphi\|_{p+2,\alpha} \rightarrow 0} \frac{\|Q[\varphi]\|_{p,\alpha}}{\|\varphi\|_{p+2,\alpha}} \\
&= \lim_{\|\varphi\|_{p+2,\alpha} \rightarrow 0} \frac{C_Q \|\varphi\|_{p+2,\alpha}^2}{\|\varphi\|_{p+2,\alpha}} \\
&= \lim_{\|\varphi\|_{p+2,\alpha} \rightarrow 0} C_Q \|\varphi\|_{p+2,\alpha} \\
&= 0
\end{aligned}$$

6.2 Application of the Inverse Mapping Theorem

We now wish to use the inverse mapping theorem on $n\mathcal{H}$. The following two isomorphism results for Banach spaces are found in Section I.5, Theorem 5.2 and Proposition 5.3 of

[21]. Here and in the following we say a map between Banach spaces is of class C^q if it has Fréchet derivatives up to order q which are all continuous.

Lemma 6.2.1 (Inverse Function Theorem). *Let E, E' be Banach spaces with $F : U \rightarrow E'$ a C^q -morphism, $q \geq 1$, from the open set $U \subset E$ to E' . If $F'(x_0) : E \rightarrow E'$ is a bounded linear isomorphism for some $x_0 \in U$, then F is a C^1 -isomorphism in a neighborhood of x_0 .*

Lemma 6.2.2 (Inverse Function Theorem (Induction)). *Let E, E' be Banach spaces with $U \subset E, V \subset E'$ and let $F : U \rightarrow V$ be a C^q -morphism, $q \geq 1$. If F is a C^1 isomorphism, then F is a C^q -isomorphism.*

Remark 6.2.3. In their original statement in [21] the two lemmas above are stated for finite dimensional normed vector spaces. They are, however, well known to hold for Banach spaces, or any normed vector space where the contraction mapping principle holds. See for example [22].

To use these results we prove the following. Without loss of generality we may reduce R_M (the reach of M) or r (from Lemma 6.1.1) so that $r = R_M > 0$.

Proposition 6.2.4. *The map $n\mathcal{H} : U \rightarrow C^{p,\alpha}(\overline{M})$ is of class C^∞ .*

Proof. Since in the decomposition $n\mathcal{H}[\varphi] = nH_M - L[\varphi] + Q[\varphi]$ the constant and linear terms are C^∞ by default we need only show that $Q : U \rightarrow C^{p,\alpha}(\overline{M})$ is C^∞ . However it is shown in Appendix A that $Q[\varphi]$ is the sum of two rational functions in φ, φ_i , and φ_{ij} . Note though that the product and chain rules hold for Fréchet derivatives (see for example [23]). Thus rational functions are C^∞ as compositions of the C^∞ multiplication, derivative, and division (multiplicative inverse, see Appendix D) operations. We need only show that the denominators in Q are never 0. This is guaranteed by $\|\varphi\|_{p+2,\alpha} < R_M = r$ as in the proof of Lemma 6.1.1. \square

Proposition 6.2.5. *If $\lambda > 0$ for all Dirichlet eigenvalues of L , then the map $L : C_0^{p+2,\alpha}(\overline{M}) \rightarrow C^{p,\alpha}(\overline{M})$ is an isomorphism.*

Proof. By Corollary 5.2.2 for all $f \in C^{p,\alpha}(\overline{M})$ $L^{-1}[f] \in C_0^{p+2,\alpha}(\overline{M})$ gives the unique classical solution to

$$\begin{cases} L[u] = f \\ u|_{\partial M} = 0 \end{cases}$$

As suggested by our choice of notation, we need only show that L^{-1} (defined via weak solutions and added regularity) is the inverse of the operator L . Let $\varphi \in C_0^{p+2,\alpha}(\overline{M})$. Then φ is a solution of the boundary value problem

$$\begin{cases} L[u] = L[\varphi] \\ u|_{\partial M} = 0 \end{cases}$$

But $L^{-1}[L[\varphi]]$ is the unique solution to this problem, so $\varphi = L^{-1}[L[\varphi]]$. Suppose $f \in C^{p,\alpha}(\overline{M})$. Then $L^{-1}[f]$ is a classical solution to $L[u] = f$, i.e. $L[L^{-1}[f]] = f$. Thus $L^{-1} \circ L = I_{C_0^{p+2,\alpha}(\overline{M})}$ and $L \circ L^{-1} = I_{C^{p,\alpha}(\overline{M})}$. This completes the proof. \square

Using these we can prove the following theorem.

Theorem 6.2.6. *If $\lambda \neq 0$ for all Dirichlet eigenvalues of L , then there exists a neighborhood $U_p \subset U \subset C_0^{p+2,\alpha}(M)$ of 0 and its image $V_p = n\mathcal{H}(U_p) \subset C^{p,\alpha}(M)$ such that*

$$n\mathcal{H} : U_p \rightarrow V_p$$

is a C^∞ isomorphism.

Proof. By Proposition 6.2.4 the operator

$$n\mathcal{H} : U \rightarrow C^{p,\alpha}(M)$$

is a C^∞ -morphism. We noted above that the Fréchet derivative of $n\mathcal{H}$ at $\varphi = 0$ is the linear operator $-L$. But if $\lambda \neq 0$ for all Dirichlet eigenvalues of L we see from Proposition 6.2.5

that L , and thus $-L$, are linear isomorphisms. Lemma 6.2.1 then implies that $n\mathcal{H}$ must be a local C^1 isomorphism. Namely there exists a neighborhood $U_p \subset U$ of 0 and its image $V_p = n\mathcal{H}(U_p)$ such that

$$n\mathcal{H} : U_p \rightarrow V_p$$

is a C^1 isomorphism. But as $n\mathcal{H}$ is C^∞ as a differential operator, Lemma 6.2.2 implies that

$$n\mathcal{H} : U_p \rightarrow V_p$$

is a C^q isomorphism for all $q \geq 1$. Thus $n\mathcal{H}$ is also a C^∞ isomorphism on U_p . \square

Remark 6.2.7. The above proof does not apply to functions in $C_0^\infty(M)$ since it is not a normed space. The conclusions of Theorem 6.2.6 can, however, be improved to show that

$$n\mathcal{H} : U' \subset C_0^\infty(\overline{M}) \rightarrow C^\infty(\overline{M})$$

is a local C^∞ isomorphism near $\varphi \equiv 0$ by an appeal to the Nash-Moser theorem. This would require that the linear map $(n\mathcal{H})'$ remain a bijection in a neighborhood of $\varphi \equiv 0$ instead of at only $\varphi \equiv 0$ itself, which is beyond the scope of the present work.

This yields a version of our main theorem.

Corollary 6.2.8. *If the Dirichlet eigenvalues of $L = -\Delta_M - |A_M|^2$ are all non-zero then for all $\epsilon > 0$, sufficiently small, the Dirichlet boundary value problems*

$$\begin{cases} n\mathcal{H}[\varphi] = nH_M \pm \epsilon \\ \varphi|_{\partial M} = 0 \end{cases}$$

both have unique solutions in a neighborhood of 0 in $C_0^{p+2,\alpha}(M)$.

Proof. Since $0 \in U_p$, $n\mathcal{H}[0] = nH_M \in V_p$. But as V_p is open then there exists $\delta > 0$ such that if $\epsilon < \delta$ the constant functions $nH_M \pm \epsilon$ are both in V_p . Thus there exist unique

solutions $u_{\pm\epsilon} \in U_p$ of the Dirichlet boundary value problems given by

$$u_{\pm\epsilon} = (n\mathcal{H})^{-1}[nH_m \pm \epsilon].$$

□

Combining this result with Lemma 4.3.1 gives the following corollary.

Corollary 6.2.9. *If there exists a function $h : \overline{M} \rightarrow \mathbb{R}$ in $C^2(M) \cap C^1(\overline{M})$ such that*

$$\begin{cases} L[h] \geq 0 \\ h > 0 \text{ on } M \\ h(x_0) > 0 \text{ for some } x_0 \in \partial M \end{cases}$$

then for all $\epsilon > 0$, sufficiently small, the Dirichlet boundary value problems

$$\begin{cases} 2\mathcal{H}[\varphi] = 2H_M \pm \epsilon \\ \varphi|_{\partial M} = 0 \end{cases}$$

both have unique solutions in a neighborhood of 0 in $C_0^{p+2,\alpha}(M)$.

Remark 6.2.10. Instead of using the inverse mapping theorem we could have arrived at the same conclusions using the more classical approach of finding fixed points of an operator related to $n\mathcal{H}$. As this alternate method is both instructive and non-trivial, it has been included in Appendix B.

CHAPTER 7

EXISTENCE OF A SUPER-SOLUTION TO L

The methods of the preceding chapters apply to all compact constant mean curvature surfaces with boundary. Thus the existence of a nearby constant mean curvature surface N to any given cmc surface M with boundary can be guaranteed by the existence of the super-solution $h \in C^2(M) \cap C^1(\overline{M})$ from Chapter 4:

$$(*) \begin{cases} L[h] \leq 0 \\ h > 0 \text{ on } M \\ h(x_0) > 0 \text{ for some } x_0 \in \partial M \end{cases}$$

Such super-solutions are, however, somewhat elusive. For this reason we restrict M to be a cmc surface that is a graph over a convex planar domain. In this case we find the vertical component of the gauss map satisfies the above conditions.

Lemma 7.0.1. *Let M be a cmc graph over a (hyper-)planar domain D , with $\partial M = \partial D$. Let \vec{e}_0 be the unit normal to D and $h \in C^2(M)$ be given by*

$$x \mapsto \nu_M(x) \cdot \vec{e}_0.$$

If M is not a half-sphere, then h satisfies $()$.*

Proof. It is well known that the Gauss map ν_M of a cmc surface M satisfies the linear elliptic PDE

$$L[\nu_M] = \Delta_M \nu_M + |A_M|^2 \nu_M = \vec{0}$$

and in particular does so in each of its coordinates. (See for example Remark 7 in [12].)

Thus $\nu_M \cdot \vec{e}_0$ also satisfies $L[\nu_M \cdot \vec{e}_0] = 0$.

Since M is a planar graph of some function $f : D \rightarrow \mathbb{R}$,

$$h = \frac{1}{\sqrt{1 + \|\nabla f\|^2}} > 0$$

necessarily in M .

Suppose that $h \equiv 0$ on ∂M . Then $\|\nabla f\| = \infty$ on ∂D and in particular $\frac{\partial f}{\partial \vec{n}} = \langle \nabla f, \vec{n} \rangle = \infty$ on ∂D where \vec{n} is the outward unit normal to ∂D . Since f also satisfies $\mathcal{M}[f] = \operatorname{div} \left(\frac{\nabla f}{\sqrt{1 + \|\nabla f\|^2}} \right) = nH_m \neq 0$ and $f = 0$ on ∂D , we can use Serrin's application of Alexandrov reflection [24, Theorem 2] to show that D is a ball and f is radially symmetric. Now the only radially symmetric constant mean curvature graph over a ball with infinite boundary gradient is the half-sphere of the same radius. Thus if M is not a half-sphere there exists $x_0 \in \partial M$ such that $h(x_0) > 0$. \square

Lemma 7.0.1 and Corollary 6.2.9 together complete the proof of the main theorem 1.3.2 as stated in Chapter 1.3. This proves the existence of new cmc surfaces beyond the scope of planar graph methods.

CHAPTER 8

INFLATIONS OF GENERAL DOMAINS

8.1 Inflations

As mentioned in Chapter 2, we conjecture that there is a smooth, one parameter family of constant mean curvature surfaces for each boundary curve. We formalize that here as an inflation. This gives us a mathematical analog to a soap film inflating due to an increasing bounded volume.

Definition 8.1.1. Let $M \subset \mathbb{R}^{n+1}$ be a known smooth cmc (hyper-) surface with smooth boundary ∂M . A C^p -inflation of M is a C^p map

$$X : M \times (a, b) \rightarrow \mathbb{R}^{n+1}$$

such that $X_t(M) = X(M, t)$ is a cmc surface with $\partial X_t(M) = \partial M$. We also require the volume of the region bounded by $X_t(M) - M$ to be strictly increasing in t . Equivalently if $t_1 < t_2$ then the region bounded by $X_{t_2}(M) - X_{t_1}(M)$ has positive signed volume.

In this situation we may define H_t as the constant (in M) mean curvature of $X_t(M)$. We can also define a bounded volume parameter V_t as signed volume of the region bounded by $X_t(M) - M$. As seen in Chapter 2 we have the natural example of the inflation of the unit circular disk:

Example 8.1.2. Let $B_1 = \{(x, y, 0) \in \mathbb{R}^3 | x^2 + y^2 < 1\}$. Then an inflation of B_1 is given by

$$X(r, \theta, t) = \begin{pmatrix} \sqrt{t^2 + 1} \sin \left(r \sin^{-1} \left(\frac{1}{\sqrt{t^2 + 1}} \right) \right) \sin \theta \\ \sqrt{t^2 + 1} \sin \left(r \sin^{-1} \left(\frac{1}{\sqrt{t^2 + 1}} \right) \right) \cos \theta \\ \sqrt{t^2 + 1} \cos \left(r \sin^{-1} \left(\frac{1}{\sqrt{t^2 + 1}} \right) \right) - t \end{pmatrix}$$

Under this inflation we have

$$V_t = \frac{\pi}{3}(\sqrt{t^2 + 1^2} + t)^2(2\sqrt{t^2 + 1^2} - t)$$

$$2H_t = \frac{-2}{\sqrt{t^2 + 1^2}}$$

We also note that this is the unique inflation of B_1 up to reparameterization.

In [12], McCuan showed that a $C^{p,\alpha}$ domain in \mathbb{R}^n has $C^{p,\alpha}$ inflation parameterized by mean curvature up to the gradient blow up of planar graph methods.

We also see that the C^∞ regularity of $(n\mathcal{H})^{-1} : V_p \subset C^{p+2,\alpha}(M) \rightarrow U_p \subset C^{p,\alpha}(M)$ implied by Theorem 6.2.6 extends the results of our main theorem 1.3.2 to the existence of a $C^{p+2,\alpha}$ inflation.

Corollary 8.1.3. *Suppose that M is a smooth constant mean curvature surface that is a graph over a planar domain D such that $\partial M = \partial D$. If M is not a half-sphere then there exists a $C^{p+2,\alpha}$ inflation of M given by*

$$(x, \epsilon) \mapsto \Gamma_{(n\mathcal{H})^{-1}[nH_M + \epsilon]}(x)$$

for all $x \in M$ and $\epsilon \in [-\delta, \delta]$ for some $\delta > 0$ sufficiently small.

Remark 8.1.4. Strictly we also need an increasing volume condition for this map to be an inflation. This follows as a consequence of Proposition 8.2.1 below.

8.2 The Eigenvalues of L in an Inflation

Inflations also have many important properties, some of which we can describe here. In the following we let $X : M \times (a, b) \rightarrow \mathbb{R}^{n+1}$ be an inflation of M . There is a natural normal

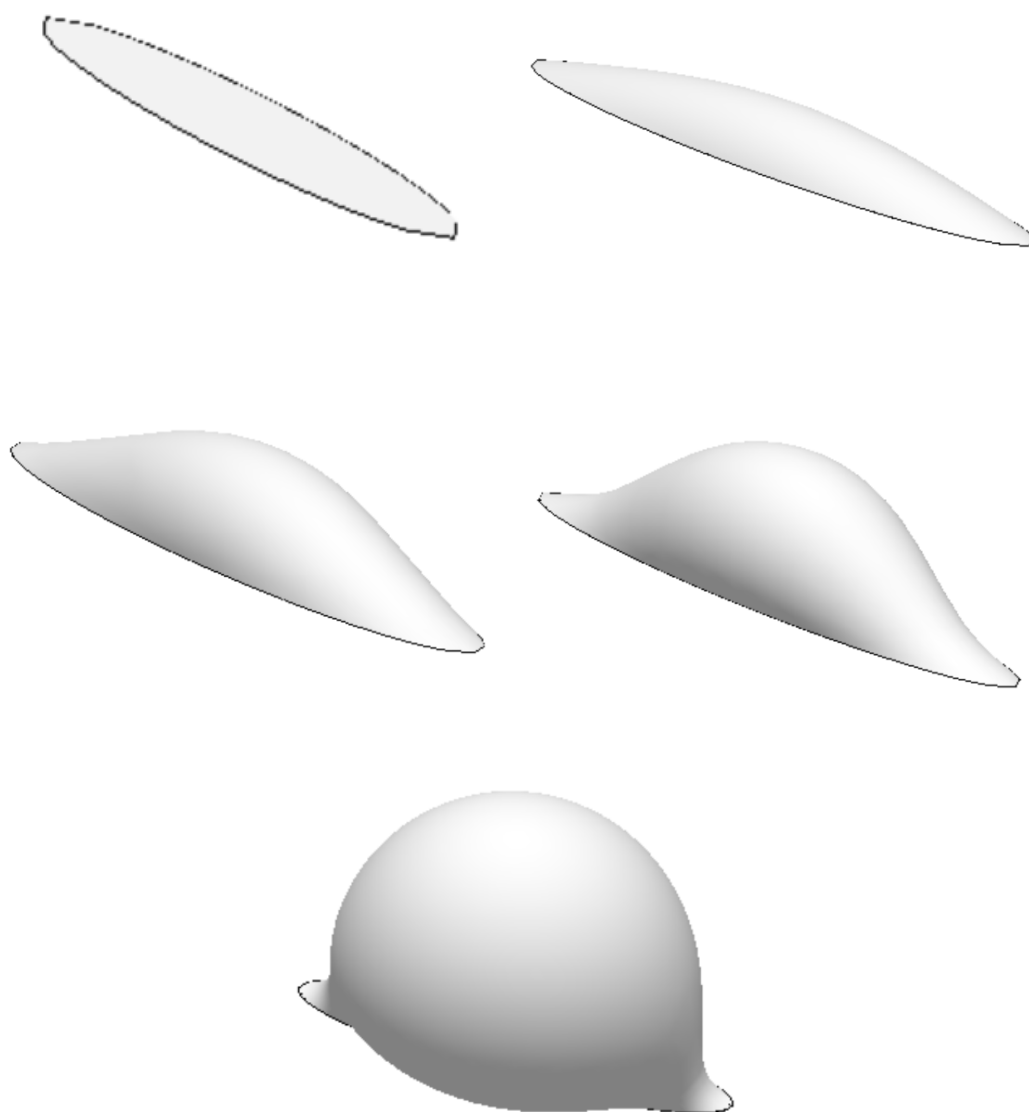


Figure 8.1: Inflation of a Non-Circular Domain

variation to $X_t(M)$ at each t given by

$$\left\langle \frac{\partial X_t}{\partial t}, \nu_{X_t(M)} \right\rangle = \varphi_t.$$

This normal variation is also related to the derivative of mean curvature (and thus Laplace pressure) in an inflation. We let L_t, Q_t denote the linear and higher degree terms respectively of the mean curvature operator $n\mathcal{H}_t$ for graphs over $X_t(M)$. Then

$$n\mathcal{H}_t, L_t, Q_t : C_0^{p+2,\alpha}(X_t(M)) \rightarrow C^{p,\alpha}(X_t(M)).$$

Proposition 8.2.1. *The derivative $\frac{d}{dt}n\mathcal{H}_t = -L_t[\varphi_t]$ on $X_t(M)$.*

Proof. For all t_1, t_2 with $|t_2 - t_1|$ sufficiently small, there exists $\varphi_{t_1 t_2}$ such that $\Gamma_{\varphi_{t_1 t_2}}[X_{t_1}(M)] = X_{t_2}(M)$. Here $\varphi_{t_1 t_2}$ is the height function that represents $X_{t_2}(M)$ as a graph over $X_{t_1}(M)$.

Consider then that by Lemma 6.1.1 we have at each point of $X_{t_0}(M)$

$$\begin{aligned} \lim_{t \rightarrow t_0} \frac{|Q_{t_0}[\varphi_{t_0 t}]|}{|t - t_0|} &\leq \lim_{t \rightarrow t_0} \frac{\|Q_{t_0}[\varphi_{t_0 t}]\|_{p,\alpha}}{|t - t_0|} \\ &\leq \lim_{t \rightarrow t_0} \frac{C_{Q_{t_0}} \|\varphi_{t_0 t}\|_{p+2,\alpha}^2}{|t - t_0|} \\ &\leq C_{Q_{t_0}} \|\varphi_{t_0}\|_{p+2,\alpha} \lim_{t \rightarrow t_0} \|\varphi_{t_0 t}\|_{p+2,\alpha} \\ &= 0 \end{aligned}$$

Then at each point of $X_{t_0}(M)$ we have

$$\begin{aligned}
\left. \frac{d}{dt} nH_t \right|_{t=t_0} &= \lim_{t \rightarrow t_0} \frac{n\mathcal{H}_{t_0}[\varphi_{t_0 t}] - nH_{t_0}}{t - t_0} \\
&= \lim_{t \rightarrow t_0} \frac{-L_{t_0}[\varphi_{t_0 t}] + Q_{t_0}[\varphi_{t_0 t}]}{t - t_0} \\
&= - \lim_{t \rightarrow t_0} \frac{L_{t_0}[\varphi_{t_0 t}]}{t - t_0} \\
&= - L_{t_0} \left[\lim_{t \rightarrow t_0} \frac{\varphi_{t_0 t}}{t - t_0} \right] \\
&= - L_{t_0}[\varphi_{t_0}]
\end{aligned}$$

which follows from the continuity of L_{t_0} . □

This shows that if $\frac{d}{dt} nH_t = 0$, then $L_t[\varphi_t] = 0$ and φ_t is a Dirichlet eigenfunction of L_t on $X_t(M)$ with an eigenvalue of zero. If in addition φ_t is positive on $X_t(M)$ then 0 is also the first eigenvalue. In fact, we get a similar result when $\frac{d}{dt} nH_t > 0$, i.e. in the realm of decreasing Laplace pressure for high $|V_t|$.

Proposition 8.2.2. *Let $\lambda_{1,t}$ be the first eigenvalue of L on the cmc surface $X_t(M)$. If the derivative $\frac{d}{dt} nH_t > 0$ then $\lambda_{1,t} < 0$.*

Proof. By the Rayleigh characterization of the eigenvalues

$$\begin{aligned}
\lambda_{1,t} &= \inf \frac{B_t[u, u]}{\|u\|_{L^2}^2} \\
&\leq \frac{B_t[\varphi_t, \varphi_t]}{\|\varphi_t\|_{L^2}^2} \\
&= \frac{\int_M L_t[\varphi_t] \varphi_t \, dx}{\|\varphi_t\|_{L^2}^2} \\
&= \frac{\int_M -\frac{d}{dt} nH_t \varphi_t \, dx}{\|\varphi_t\|_{L^2}^2} \\
&= - \frac{d}{dt} nH_t \frac{\int_M \varphi_t \, dx}{\|\varphi_t\|_{L^2}^2} \\
&< 0
\end{aligned}$$

since $\int_M \varphi_t dx = \frac{d}{dt} V_t > 0$. □

We get a similar result for $\frac{d}{dt} nH_t < 0$ in the increasing Laplace pressure regime with relatively weak additional assumptions.

Proposition 8.2.3. *Suppose*

- X_t in an embedding for all $t \in (a, b)$,
- $\frac{d}{dt} nH_t < 0$ over (a, b) ,
- and there exists $t_0 \in (a, b)$ such that $\varphi_{t_0} > 0$ on $X_{t_0}(M)$.

Then X is a foliation of its image. This implies that for every $x \in X(M \times (a, b))$ there exists a unique $t_x \in (a, b)$ such that $x \in X_{t_x}(M)$.

Proof. As X itself acts as a foliation cover we need only show that X is one-to-one. Since each X_t is an embedding, may complete the proof by showing that $X_t(M)$ is distinct for all $t \in (a, b)$.

Since $\frac{d}{dt} \varphi_{t_0 t} |_{t=t_0} = \varphi_{t_0} > 0$ on $X_{t_0}(M)$ then for all $x \in M$ $\varphi_{t_0 t}(x)$ must be strictly monotone in t for t near t_0 . Thus there exists some $\epsilon > 0$ such that $X_{t_1}(M)$ and $X_{t_2}(M)$ are distinct for all $t_1, t_2 \in (t_0 - \epsilon, t_0 + \epsilon)$.

Suppose that not all $X_t(M)$ are distinct. Let

$$t_1 = \inf \{t \in (t_0, b) | X_t(M) \cap X(M, [t_0, t)) \neq \emptyset\}.$$

Since $t_1 > t_0$ we may conclude that $\varphi_{t_1} \geq 0$ on $X_{t_1}(M)$ but there exists $x \in X_{t_1}(M)$ such that $\varphi_{t_1}(x) = 0$. Now

$$L_{t_1}[\varphi_{t_1}] = -\frac{d}{dt} nH_t |_{t=t_1} > 0$$

implies that

$$\begin{aligned}
0 &< L_{t_1}[\varphi_{t_1}](x) \\
&= -(\Delta\varphi_{t_1})(x) - |A|^2\varphi_{t_1}(x) \\
&= -(\Delta\varphi_{t_1})(x)
\end{aligned}$$

By continuity there must be a neighborhood $U \subset X_{t_1}(M)$ of x such that $\Delta\varphi_{t_1} > 0$. Thus the maximum principle on U implies that for the non-constant function φ_{t_1} we have

$$\varphi_{t_1}(p) > \min_U \varphi_{t_1} \geq 0,$$

a contradiction. Thus we must have $X_t(M)$ distinct for all $t \in [t_0, b)$. An equivalent argument extends this result to $t \in (a, b)$, which completes the proof. \square

Corollary 8.2.4. *If the hypotheses of Proposition 8.2.3 hold, then $\lambda > 0$ for all Dirichlet eigenvalues of L_t on $X_t(M)$ for all $t \in (a, b)$.*

Proof. As a consequence of the proof of Proposition 8.2.3, $\varphi_t > 0$ for all $t \in (a, b)$. But at the same time

$$L_t[\varphi_t] = -\frac{d}{dt}nH_t > 0.$$

Thus φ_t is a strict super-solution to L_t and satisfies the conditions of corollary 4.3.2. This allows us to conclude that $\lambda > 0$ for all Dirichlet eigenvalues of L_t on $X_t(M)$. \square

8.3 Stability in an Inflation

A related property of inflations is the stability of the cmc surfaces under volume preserving variations. Given a variation of a cmc surface M ,

$$Y : M \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{n+1},$$

we can define its normal variation

$$\varphi_Y = \left\langle \frac{\partial Y_t}{\partial t} \Big|_{t=0}, \nu_M \right\rangle.$$

We say that Y is volume preserving at $t = 0$ if

$$\int_M \varphi_Y dx = 0.$$

Since M is a cmc surface it is necessarily a critical point of the area functional with respect to volume preserving variations that fix the boundary. However, being a stationary point does not necessarily imply that M is stable. For M to be stable under volume preserving variations it must satisfy the positivity of the second variation

$$\int_M (-\Delta \varphi_Y - |A|^2 \varphi_Y) \varphi_Y dx = B[\varphi_Y, \varphi_Y] > 0$$

for all volume preserving variations Y that fix the boundary of M .

Conditions for the stability of cmc surfaces have are given in [17] in terms of the eigenvalues of L . We use the following results, translated into the notation of this work, from Theorem 1.3 and Corollary 1.1.

Theorem 8.3.1. *Given a cmc surface M let λ_1, λ_2 be the first and second eigenvalues of the operator L on M . Then*

- *If $\lambda_1 \geq 0$ then M is stable.*
- *If $\lambda_2 < 0$ then M is unstable.*
- *If $\lambda_1 < 0 < \lambda_2$, let $X : M \times (-a, a) \rightarrow \mathbb{R}^{n+1}$ be a $C^{3,\alpha}$ inflation of M such that $X_{t_0}(M) = M$ and $\frac{d}{dt} V_t|_{t=t_0} > 0$. Then*
 - *If $\frac{d}{dt} n H_t|_{t=t_0} > 0$ then M is stable.*

– If $\frac{d}{dt}nH_t|_{t=t_0} < 0$ then M is unstable.

Thus we can translate our above conditions on the eigenvalues of L_t into statements about the stability of $X_t(M)$.

Proposition 8.3.2. *The surface $X_t(M)$ is stable under volume preserving variations under any of the following conditions:*

- $\frac{d}{dt}nH_t > 0$ and $\lambda_{2,t} > 0$
- $\frac{d}{dt}nH_t = 0$ and $\varphi_t > 0$ on $X_t(M)$
- or $\frac{d}{dt}nH_t < 0$ and $\varphi_{t_0} > 0$ on $X_{t_0}(M)$ for some t_0 such that $\frac{d}{dt}nH_t < 0$ on the interval $[t, t_0]$.

Proof. This results from a combination of Theorem 8.3.1 with Proposition 8.2.2, Lemma 4.2.1 (with $h = \varphi_t$), and Corollary 8.2.4 respectively. \square

Lastly we note that in the $\frac{d}{dt}nH_t < 0$ case the additional condition $\varphi_{t_0} > 0$ holds for any $X_{t_0}(M)$ that is a graph over a planar domain.

Proposition 8.3.3. *If $X_t(M)$ is a graph over a planar domain D for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$ and $\frac{d}{dt}nH_t < 0$ over the same interval then $\varphi_{t_0} > 0$ and $\lambda > 0$ for all eigenvalues of L_t on $X_{t_0}(M)$.*

Proof. Let $f_t : D \rightarrow \mathbb{R}$ denote the function whose graph is $X_t(M)$. Since $\mathcal{M}[f_t] = H_t > H_{t'} = \mathcal{M}[f_{t'}]$ if $t < t'$ in $(t_0 - \epsilon, t_0 + \epsilon)$ the comparison principle implies that $f_t < f_{t'}$ in D . But this implies that $X_t(M)$ and $X_{t'}$ are distinct for all $t \neq t'$ in $(t_0 - \epsilon, t_0 + \epsilon)$. Thus X is a foliation over this domain and in particular $\varphi_{t_0} > 0$. The remainder follows from Corollary 4.3.2 with $h = \varphi_{t_0}$. \square

Remark 8.3.4. The condition that $\lambda_1 > 0$ for the first eigenvalue of L is termed *overstability* by [12], as it is a stronger condition than stability. It implies that $\delta^2 A(\varphi_Y) > 0$ for all

variations Y (not just volume preserving ones) where A is the area functional on $C_0^2(\overline{M})$. This holds even though $\delta A(\varphi_Y) \neq 0$ in general. Thus an inflation of a planar domain will remain overstable up to a critical point of mean curvature.

Additionally some further conditions for the positivity of λ_1, λ_2 are given in [25].

8.4 Conjectures about Inflations

These results suggest our conjecture from the introduction, that every planar domain has an inflation up to and beyond the point of maximum Laplace pressure.

Conjecture 8.4.1. *Suppose that D is a open pre-compact connected (hyper-) planar domain in $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ with $C^{p+1,\alpha}$ boundary. Then there exists a $C^{p,\alpha}$ inflation $X : D \times \mathbb{R} \rightarrow \mathbb{R}^{n+1}$ such that*

- $X_0(D) = D, nH_0 = 0,$
- $V_t = t$, i.e. the inflation is parameterized by the signed volume of the region bounded by $X_t(D) - D$,
- $\lim_{t \rightarrow \pm\infty} H_t = 0,$
- $H_t = -H_{-t}$
- there exists some $t_0 > 0$ such that
 - $\frac{d}{dt}nH_t < 0$ for $|t| < t_0$
 - $\frac{d}{dt}nH_t = 0$ for $|t| = t_0$
 - $\frac{d}{dt}nH_t > 0$ for $|t| > t_0$
- $X_t(D)$ is stable with respect to volume preserving variations for all $t \neq t_0$
- and all C^p inflations of D are equivalent to X for all $|t| \leq t_0$ up to reparameterization.

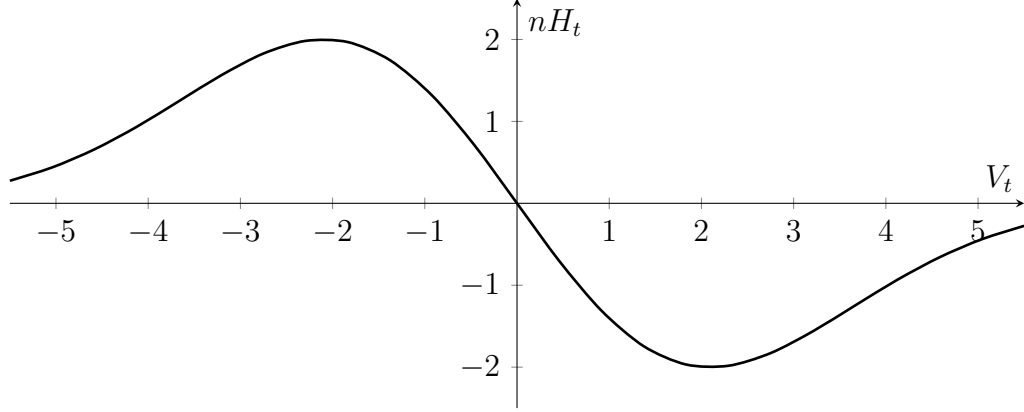


Figure 8.2: Example graph of nH_t in terms of V_t

Using this notation we conclude with an isoperimetric conjecture.

Conjecture 8.4.2. *For any open compact connected (hyper)-planar domain \overline{D} , $|nH_{t_0}| \leq \frac{n}{r}$ where r is the radius of the circular disk in $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ with the same boundary measure as D . Similarly $|V_{t_0}|$ is less than or equal to the volume of the half sphere of radius r . I.e. the minimum of maximum mean curvature for planar domains with the same boundary measure occurs only when D is a circular disk and the maximum volume of mean curvature maxima occurs at the same point.*

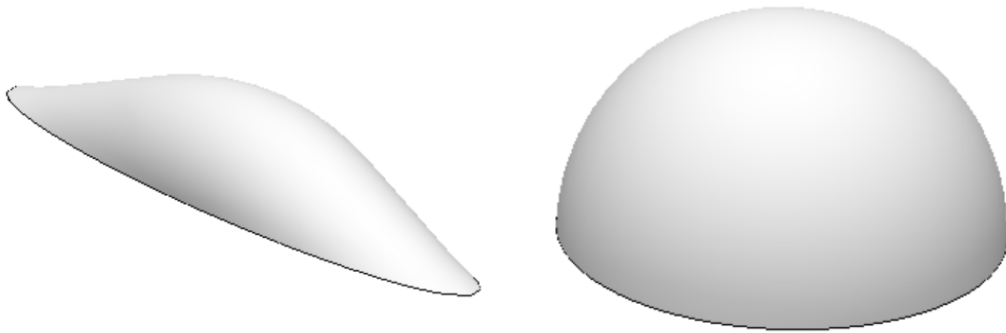


Figure 8.3: Two surfaces of mean curvature extrema with the same boundary length

Computational results give suggestive evidence for Conjecture 8.4.2. Take for example the two surfaces in Figure 8.4. The boundaries of both surfaces have the same length and both surfaces are mean curvature extrema for their respective boundaries. In this example however the half-sphere has a mean curvature of ≈ 0.3 while the surface on the left has a mean curvature of ≈ 0.57 . Similarly the half sphere bounds a volume over twice as large as the other. Should this conjecture prove true, it would add one more property to the long list of isoperimetric qualities enjoyed by the geodesic sphere.

Appendices

APPENDIX A

THE MEAN CURVATURE OPERATOR

In this appendix we derive the form of $n\mathcal{H}[\varphi]$ in arbitrary dimension. Let $X : M \rightarrow \mathbb{R}^{n+1}$ be an isometric embedding of the hypersurface M into \mathbb{R}^{n+1} of constant mean curvature H_M . Let $\varphi \in C^2(M)$ be a real valued function on M . Then $\mathcal{H}[\varphi]$ will be the mean curvature of the hypersurface $X_\varphi = X + \varphi\nu_M$ where ν_M is the unit normal (Gauss map) of $X(M)$. In this appendix, Einstein summation convention is used.

First, let e_0, e_1, \dots, e_n be a local orthonormal frame for \mathbb{R}^{n+1} where e_1, e_2, \dots, e_n are tangent to M , and thus $e_0 = \nu_M$. Let $\eta^0, \eta^1, \dots, \eta^n$ be the dual orthonormal coframe, and let ∇ be the connection on M induced by the embedding in \mathbb{R}^{n+1} . Letting indices range over $1, \dots, n$, we can find

$$\nabla X = \eta^i e_i$$

$$\nabla \varphi = \varphi_i \eta^i$$

$$\nabla e_0 = -A_j^i \eta^j e_i$$

$$\nabla e_j = A_{ij} \eta^i e_0$$

where $A_j^i = A_{ij}$ is the second fundamental form of M . (Typically $A_j^i = g^{ik} A_{kj}$ where g_{ij} is the first fundamental form or metric on M . In the given orthonormal frame, however, $g^{ij} = g_{ij}^{-1} = \delta_j^i$ is the identity matrix.)

In the following we denote by $\hat{e}_i, \hat{e}_0, \hat{g}$, and \hat{A} the tangent vectors, unit normal (Gauss map), and first and second fundamental forms of $N = X_\varphi(M)$, respectively. (Note that

$\hat{A}_j^i = \hat{g}^{ik} \hat{A}_{kj}$ and $n\mathcal{H}[\varphi] = \hat{A}_i^i$.) Thus,

$$\begin{aligned}
\hat{e}_i &= \nabla_{e_i} X_\varphi \\
&= \nabla_{e_i} (X + \varphi \nu_M) \\
&= \nabla_{e_i} X + (\nabla_{e_i} \varphi) \nu_M + \varphi \nabla_{e_i} \nu_M \\
&= e_i - \varphi A_i^j e_j + \varphi_i e_0
\end{aligned}$$

As $\hat{g}_{ij} = \hat{e}_i \cdot \hat{e}_j$, we have

$$\begin{aligned}
\hat{g}_{ij} &= \delta_i^j - 2\varphi A_j^i + \varphi_i \varphi_j + \varphi^2 \sum_k A_i^k A_j^k \\
&= \delta_i^j - 2\varphi A_j^i + G_{ij}
\end{aligned}$$

where the G_{ij} are of quadratic degree in φ, φ_i . From this we see that in matrix form,

$$\begin{aligned}
\hat{g} &= I - 2\varphi A + D\varphi D\varphi^T + \varphi^2 A^2 \\
&= I - (2\varphi A - G)
\end{aligned}$$

Setting $\|\varphi\|_2$ sufficiently small we can make $|2\varphi A - G| < 1$, and ensure that \hat{g} is non-singular (and that X_φ is an immersion). Given this we can see that

$$\begin{aligned}
\hat{g}^{-1} &= \frac{I + 2\varphi A + G'}{\det(I - (2\varphi A - G))} \\
&= \frac{I + 2\varphi A + G'}{1 + G''}
\end{aligned}$$

where G'' and the elements of G' are polynomials of at least of quadratic degree in φ, φ_i .

Consider now

$$\begin{aligned}
\bigwedge(\hat{e}_1, \dots, \hat{e}_n) &= \bigwedge(e_1 - \varphi A_1^j e_j + \varphi_1 e_0, \dots, e_n - \varphi A_n^k e_k + \varphi_n e_0) \\
&= \bigwedge(e_1, \dots, e_n) + \bigwedge(-\varphi A_1^j e_j + \varphi_1 e_0, e_2, \dots, e_n) \\
&\quad + \dots + \bigwedge(e_1, \dots, e_{n-1}, -\varphi A_n^k e_k + \varphi_n e_0) \\
&\quad + F^0 e_0 + F^i e_i \\
&= e_0 + (-\varphi A_1^1 e_0 - \varphi_1 e_1) \\
&\quad + \dots + (-\varphi A_n^n e_0 - \varphi_n e_n) \\
&\quad + F^0 e_0 + F^i e_i \\
&= (1 - \varphi A_i^i + F^0) e_0 + (-\varphi_i + F^i) e_i \\
&= (1 - \varphi n H_M + F^0) e_0 + (-\varphi_i + F^i) e_i
\end{aligned}$$

where F^0, F^i are polynomial of at least of quadratic degree in φ, φ_i .

Thus

$$\begin{aligned}
\|\bigwedge(\hat{e}_1, \dots, \hat{e}_n)\| &= \|(1 - \varphi n H_M + F^0) e_0 + (-\varphi_i + F^i) e_i\| \\
&= \left[(1 - \varphi n H_M + F^0)^2 + \sum_i (-\varphi_i + F^i)^2 \right]^{1/2} \\
&= \left[(1 - \varphi n H_M + F^0)^2 + \sum_i (-\varphi_i + F^i)^2 \right]^{1/2} \frac{1 + \varphi n H_M + F^0}{1 + \varphi n H_M + F^0} \\
&= \frac{[1 + F]^{1/2}}{1 + \varphi n H_M + F^0} \\
\|\bigwedge(\hat{e}_1, \dots, \hat{e}_n)\|^{-1} &= \frac{1 + \varphi n H_M + F^0}{[1 + F]^{1/2}}
\end{aligned}$$

where F is a polynomial of at least quadratic degree in φ, φ_i . From this we can find the

unit normal

$$\begin{aligned}
\hat{e}_0 &= \frac{\bigwedge(\hat{e}_1, \dots, \hat{e}_n)}{\|\bigwedge(\hat{e}_1, \dots, \hat{e}_n)\|} \\
&= ((1 - \varphi n H_M + F^0)e_0 + (-\varphi_i + F^i)e_i) \left(\frac{1 + \varphi n H_M + F^0}{[1 + F]^{1/2}} \right) \\
&= \frac{(1 + F'^0)e_0 + (-\varphi_i + F'^i)e_i}{[1 + F]^{1/2}}
\end{aligned}$$

where F'^0, F'^i are polynomials of at least of quadratic degree in φ, φ_i .

Consider also

$$\begin{aligned}
\nabla_{e_i e_j}^2 X_\varphi &= \nabla_{e_i} \hat{e}_j \\
&= \nabla_{e_i} (e_j + \varphi_j \nu_M - \varphi A_j^k e_k) \\
&= \nabla_{e_i} e_j + \nabla_{e_i} \varphi_j \nu_M + \varphi_j \nabla_{e_i} \nu_M - \nabla_{e_i} \varphi A_j^k e_k - \varphi \nabla_{e_i} A_j^k e_k - \varphi A_j^k \nabla_{e_i} e_k \\
&= A_{ij} \nu_M + \varphi_{ij} \nu_M - \varphi_j A_i^k e_k - \varphi_i A_j^k e_k - \varphi A_{ij}^k e_k - \varphi A_j^k A_{ik} \nu_M \\
&= (A_{ij} + \varphi_{ij} - \varphi A_j^k A_{ik}) \nu_M - (\varphi_j A_i^k + \varphi_i A_j^k + \varphi A_{ij}^k) e_k
\end{aligned}$$

where $\nabla A_j^i = A_{jk}^i \eta^k$.

Now,

$$\begin{aligned}
\hat{A}_{ij} &= \langle \nabla_{e_i e_j}^2 X_\varphi, \hat{e}_3 \rangle \\
&= \langle (A_{ij} + \varphi_{ij} - \varphi A_j^k A_{ik}) \nu_M - (\varphi_j A_i^k + \varphi_i A_j^k + \varphi A_{ij}^k) e_k, \\
&\quad \frac{(1 + F'^0)e_0 + (-\varphi_l + F'^l)e_l}{[1 + F]^{1/2}} \rangle \\
&= \frac{(A_{ij} + \varphi_{ij} - \varphi A_j^k A_{ik})(1 + F'^0) + \sum_k (\varphi_j A_i^k + \varphi_i A_j^k + \varphi A_{ij}^k)(-\varphi_k + F'^k)}{[1 + F]^{1/2}} \\
&= \frac{A_{ij} + \varphi_{ij} - \varphi A_j^k A_{ik} + E_{ij}}{[1 + F]^{1/2}} \\
&= \frac{1}{[1 + F]^{1/2}} (A + D^2 \varphi - \varphi A^2 + E)
\end{aligned}$$

where the E_{ij} are polynomials of at least of quadratic degree in φ, φ_i .

Lastly,

$$\begin{aligned}
\hat{A}_j^i &= \hat{g}^{ik} \hat{A}_{kj} \\
&= \left(\frac{I + 2\varphi A + G'}{1 + G''} \right) \frac{1}{[1 + F]^{1/2}} (A + D^2\varphi - \varphi A^2 + E) \\
&= \frac{1}{(1 + G'')(1 + F)^{1/2}} (A + 2\varphi A^2 + D^2\varphi - \varphi A^2 + E') \\
&= \frac{1}{(1 + F')^{1/2}} (A + D^2\varphi + \varphi A^2 + E')
\end{aligned}$$

where F' is a polynomial of at least quadratic degree in φ, φ_i , and elements of E' are polynomials of at least quadratic degree in φ, φ_i and φ_{ij} . Therefore,

$$\begin{aligned}
n\mathcal{H}[\varphi] &= \hat{A}_i^i \\
&= \frac{1}{(1 + F')^{1/2}} \text{tr}(A + D^2\varphi + \varphi A^2 + E') \\
&= \frac{1}{(1 + F')^{1/2}} (nH_M + \Delta\varphi + |A|^2\varphi + E_i'^i) \\
&= nH_M + \Delta\varphi + |A|^2\varphi + Q[\varphi]
\end{aligned}$$

where

$$\begin{aligned}
Q[\varphi] &= \frac{E_i'^i}{(1 + F')^{1/2}} - \frac{(1 + F')^{1/2} - 1}{(1 + F')^{1/2}} (nH_M + \Delta\varphi + |A|^2\varphi) \\
&= \frac{E_i'^i}{(1 + F')^{1/2}} - \frac{F'}{1 + F' + (1 + F')^{1/2}} (nH_M + \Delta\varphi + |A|^2\varphi) \\
&= \frac{E_1}{(1 + F')^{1/2}} + \frac{E_2}{1 + F' + (1 + F')^{1/2}}
\end{aligned}$$

where E_1, E_2 are polynomials of at least quadratic degree in φ, φ_i and φ_{ij} .

APPENDIX B

ALTERNATE APPROACH USING FIXED POINT THEOREMS

In this appendix we give an alternate approach to obtain the results of Chapter 6. While that chapter relies on the inverse function theorem for Banach spaces, here we use fixed point theorems to find a solution and subsequently use regularity estimates to prove uniqueness and continuity.

Specifically, we define the mapping $J : U \rightarrow C_0^{p+2,\alpha}(\overline{M})$ for

$$U = \{\varphi \in C_0^{p+2,\alpha}(\overline{M}) \mid \|\varphi\|_{p+2,\alpha} < R_M\}$$

by

$$J[\varphi] = -L^{-1}[2H_N - 2H_M - Q[\varphi]].$$

We show $J[\varphi]$ can be bounded in terms of $2H_N - 2H_M = \epsilon$ and $\|\varphi\|$ since Q is quadratic (and very small for $\|\varphi\|$ near 0). Using this we will show that J has a fixed point. This fixed point is the solution to our original differential equation and yields a new cmc surface $N = \Gamma(M)$.

B.1 Schauder Fixed Point Theorem Approach

We begin with the Schauder fixed point theorem. Here we can show the existence of a solution to $n\mathcal{H}[\varphi] = nH_N$ using no estimates stronger than those in chapter 6.

Theorem B.1.1 (Schauder Fixed Point Theorem). *Suppose K is a convex, compact subset of a Banach space. If $J : K \rightarrow K$ is a continuous map from K to itself then J has a fixed point in K .*

First we use the Arzelà-Ascoli Theorem to prove a result about Hölder spaces.

Theorem B.1.2 (Arzelà-Ascoli Theorem). *Let X is a compact Hausdorff space. If $\{f_i\}$ is equicontinuous and point-wise bounded then $\{f_i\}$ has a subsequence that converges uniformly to a continuous function f .*

Proposition B.1.3. *If $K \subset C^{p,\alpha'}(\overline{M})$ is closed and bounded by some $b < \infty$, then its embedding into $C^{p,\alpha}(\overline{M})$ for $0 < \alpha < \alpha'$ is compact.*

Proof. Since \overline{M} is Hausdorff as a manifold and compact by hypothesis the Arzelà-Ascoli Theorem applies. Take any sequence $\{f_i\}$ in $K \subset C^{p,\alpha}$. Then as $\|f_i\|_{p,\alpha} < b$, boundedness of derivatives implies that $(f_i)_\beta$ is equicontinuous for all derivatives $|\beta| < p$. Similarly the Hölder condition implies equicontinuity of all $(f_i)_\beta$, $|\beta| = p$. Thus by applying the Arzelà-Ascoli Theorem recursively to subsequences there exists a subsequence $\{f_j\}$ such that $\{(f_j)_\beta\}$ converges uniformly on \overline{M} to some continuous f^β for all $|\beta| \leq p$. Since this convergence of derivatives is uniform we have additionally that $f^\beta = f_\beta$ and $f \in C^p(\overline{M})$. Therefore $\{f_j\} \rightarrow f$ in $C^p(\overline{M})$, also by uniformity.

Lastly let $g_j = f - f_j$. We see that for all $|\beta| = p$ and all $x, y \in \overline{M}$

$$\begin{aligned}
\frac{|(g_j)_\beta(x) - (g_j)_\beta(y)|}{(\text{dist}(x, y))^\alpha} &\leq \left(\frac{|(g_j)_\beta(x) - (g_j)_\beta(y)|}{(\text{dist}(x, y))^{\alpha'}} \right)^{\alpha/\alpha'} |(g_j)_\beta(x) - (g_j)_\beta(y)|^{1-\alpha/\alpha'} \\
&\leq \left(\frac{|(f_j)_\beta(x) - (f_j)_\beta(y)|}{(\text{dist}(x, y))^{\alpha'}} \right)^{\alpha/\alpha'} |(g_j)_\beta(x) - (g_j)_\beta(y)|^{1-\alpha/\alpha'} \\
&\leq \|(f_j)_\beta\|_{0,\alpha}^{\alpha/\alpha'} (2\|(g_j)_\beta\|_0)^{1-\alpha/\alpha'} \\
&\leq 2^{1-\alpha'/\alpha} \|(f_j)_\beta\|_{0,\alpha}^{\alpha/\alpha'} \|(g_j)_\beta\|_0^{1-\alpha/\alpha'} \\
&\leq 2^{1-\alpha'/\alpha} \|f_j\|_{p,\alpha}^{\alpha/\alpha'} \|g_j\|_p^{1-\alpha/\alpha'} \\
&\leq 2^{1-\alpha'/\alpha} \|f_j\|_{p,\alpha'}^{\alpha/\alpha'} \|g_j\|_p^{1-\alpha/\alpha'} \\
&\leq 2^{1-\alpha'/\alpha} b^{\alpha/\alpha'} \|g_j\|_p^{1-\alpha/\alpha'} \rightarrow 0
\end{aligned}$$

as $j \rightarrow \infty$ since $\{g_j\} \rightarrow 0$ in $C^p(\overline{M})$. Thus $\{g_j\} \rightarrow 0$ in $C^{p,\alpha}(\overline{M})$ as well. Therefore $\{f_j\} \rightarrow f$ in the norm of $C^{p,\alpha}(\overline{M})$ and $f \in C^{p,\alpha}(\overline{M})$ since $C^{p,\alpha}(\overline{M})$ is a complete Banach space.

Since any sequence in K has a subsequence convergent in $C^{p,\alpha}(\overline{M})$, K is compact in $C^{p,\alpha}(\overline{M})$. □

We can now prove the following version of our main theorem.

Theorem B.1.4. *If $p > 2 + n/2$ and the Dirichlet eigenvalues of $L = -\Delta_M - |A_M|^2$ are all non-zero then for all $\epsilon \in C^p(\overline{M})$, $\|\epsilon\|$ sufficiently small, the Dirichlet boundary value problems*

$$\begin{cases} 2\mathcal{H}[\varphi] = 2H_M + \epsilon \\ \varphi|_{\partial M} = 0 \end{cases}$$

both have solutions in $C^{p+2,\alpha}(\overline{M})$.

Remark B.1.5. While our primary concern is with ϵ a constant, as with the inverse function theorem approach we get solutions for arbitrary variations $\epsilon \in C^{p,\alpha}(\overline{M})$ sufficiently small.

Proof. Consider again the PDE that φ must satisfy for $N = \Gamma_\varphi(M)$ to be a surface of constant mean curvature H_N with the same boundary as cmc surface M ,

$$2\mathcal{H}[\varphi] = 2H_N = 2H_M - L[\varphi] + Q[\varphi].$$

By Lemma 5.2.2 L is invertible and the operator

$$L^{-1} : C^{p,\alpha}(M) \rightarrow C_0^{p+2,\alpha}(M)$$

is continuous. Let $\epsilon = 2H_N - 2H_M$, the (assumed small) change in mean curvature between M and N , and define

$$J[\varphi] = L^{-1}[-\epsilon + Q[\varphi]].$$

Since L^{-1} was shown via Lemma 5.1.1 to be continuous, it is also bounded by its operator norm $\|L^{-1}\| < \infty$. Also we recall the constants C_Q and r from Theorem 6.1.1. Then we

can set

$$b = \min \left(\frac{1}{2C_Q \|L^{-1}\|}, r \right)$$

and let

$$K = \left\{ \varphi \in C_0^{2,\alpha'}(M) \mid \|\varphi\|_{p+2,\alpha'} \leq b \right\}$$

where $0 < \alpha < \alpha'$.

We note that for all $\varphi \in K$, $\|\varphi\|_{p+2,\alpha'} \leq r$ so that Lemma 6.1.1 applies for all $\varphi \in K$.

Let

$$\|\epsilon\|_p \leq \frac{b}{2\|L^{-1}\|}.$$

Then if $\varphi \in K$ we have

$$\begin{aligned} \|J[\varphi]\|_{p+2,\alpha} &= \|L^{-1}[-\epsilon + Q[\varphi]]\|_{p+2,\alpha} \\ &\leq \|L^{-1}[-\epsilon]\|_{p+2,\alpha} + \|L^{-1}[Q[\varphi]]\|_{p+2,\alpha} \\ &\leq \|L^{-1}\| \|\epsilon\| + \|L^{-1}\| \|Q(\varphi)\|_{p,\alpha} \\ &\leq \|L^{-1}\| \|\epsilon\| + C_Q \|L^{-1}\| \|\varphi\|_{p+2,\alpha}^2 \\ &\leq \|L^{-1}\| \frac{b}{2\|L^{-1}\|} + C_Q \|L^{-1}\| b^2 \\ &\leq \frac{b}{2} + C_Q \|L^{-1}\| \frac{b}{2C_Q \|L^{-1}\|} \\ &= b \end{aligned}$$

Thus for $J : K \subset C_0^{p+2,\alpha'}(\overline{M}) \subset C_0^{p+2,\alpha}(\overline{M}) \rightarrow C_0^{p+2,\alpha}(\overline{M})$ we have $J[K] \subset K$.

As K is closed and bounded in $C^{p+2,\alpha'}(M)$ the embedding of K in $C_0^{p+2,\alpha}(M)$ is compact via Proposition B.1.3. Since J is a continuous map from the convex compact set K to

itself, the Schauder fixed point theorem applies and J has a fixed point, φ . Thus

$$\begin{aligned}\varphi &= J[\varphi] = L^{-1}[-\epsilon + Q[\varphi]] \\ L[\varphi] &= -\epsilon + Q[\varphi] \\ -L[\varphi] + Q[\varphi] &= \epsilon \\ n\mathcal{H}[\varphi] &= nH_M - L[\varphi] + Q[\varphi] = nH_M + \epsilon\end{aligned}$$

and this fixed point is the solution to our initial differential equation. This yields a new cmc surface $N = \Gamma_\varphi(M)$ with mean curvature $nH_M + \epsilon$. \square

Combining this result with Lemma 4.3.1 gives the following corollary.

Corollary B.1.6. *If there exists a function $h : \overline{M} \rightarrow \mathbb{R}$ in $C^2(M) \cap C^1(\overline{M})$ such that*

$$\begin{cases} L[h] \geq 0 \\ h > 0 \text{ on } M \\ h(x_0) > 0 \text{ for some } x_0 \in \partial M \end{cases}$$

then for all $\epsilon > 0$, sufficiently small, the Dirichlet boundary value problems

$$\begin{cases} 2\mathcal{H}[\varphi] = 2H_M \pm \epsilon \\ \varphi|_{\partial M} = 0 \end{cases}$$

both have solutions in a neighborhood of 0 in $C_0^{p+2,\alpha}(\overline{M})$.

While the above result is essentially equivalent to Corollary 6.2.9, the approach of Chapter 6 can be extended to inflations almost immediately. This is because the inverse function theorem gives continuity/regularity of the inverse of $n\mathcal{H}$.

The Schauder fixed point theorem has no such standard result. In fact, the solutions found above are not even known to be unique near M . To achieve the same results then we

must also provide regularity results that lead to uniqueness and local continuity of solutions. One possible approach is to use the uniqueness result of [26]. However this would require a bound on the eigenvalues of $(n\mathcal{H})'$ in a neighborhood of $\varphi \equiv 0$, instead of only at zero. As mentioned in Remark 6.2.7, such a result is beyond the scope of this work.

Instead, by showing stronger conditions Q we can show that J is in fact a contraction near $\varphi \equiv 0$ and use the contraction mapping principle instead.

B.2 Contraction Mapping Principle Approach

In this section we are able to show both existence and uniqueness of fixed points of J for small(er) ϵ and $\|\varphi\|_{p+2}$ using the contraction mapping principle:

Theorem B.2.1 (Contraction Mapping Principle). *Suppose that B is a closed subset of the Banach space E . If $f : B \rightarrow B$ is a contraction, i.e there exists $C \in [0, 1)$ such that for all $x, y \in U$*

$$\|f(x) - f(y)\| \leq C\|x - y\|,$$

then f has a unique fixed point in U .

Remark B.2.2. While the contraction mapping principle requires only that E be a complete metric space, we use the present formulation as Banach spaces will be required in the next section for the continuity of fixed points.

To use this result we will need a stronger estimate on Q than that given in Lemma 6.1.1. Specifically we need to show that Q itself is a contraction near $\varphi \equiv 0$. While this result could be obtained directly, a simpler argument is given as follows from the mean value theorem for Fréchet derivatives.

Theorem B.2.3 (Mean Value Theorem). *Let $f : B \subset E \rightarrow F$ be a C^1 map between Banach spaces E, F . Let $Df(x)$ be the linear Fréchet derivative of f . Then if $\|Df(x)\| < C_f$ for all $x \in B$ then*

$$\|f(x) - f(y)\| \leq C_f\|x - y\|$$

for all $x, y \in B$.

Proposition B.2.4. For all $C > 0$ there exists $r_C > 0$ such that if $\varphi_1, \varphi_2 \in \overline{B_{r_C}(0)} \subset U \subset C_0^{p+2}(\overline{M})$ then

$$\|Q[\varphi_1] - Q[\varphi_2]\|_p < C\|\varphi_1 - \varphi_2\|_{p+2}.$$

Proof. Since Q is C^∞ (see Lemma 6.2.4) on $U \subset C_0^{p+2}(\overline{M})$ we let $DQ(\varphi_0)$ bet the Fréchet derivative at φ_0 . Thus $DQ(\varphi_0) : C_0^{p+2}(\overline{M}) \rightarrow C^p(\overline{M})$ is linear and

$$\lim_{\varphi \rightarrow \varphi_0} \frac{\|Q[\varphi] - Q[\varphi_0] - DQ(\varphi_0)[\varphi - \varphi_0]\|_p}{\|\varphi - \varphi_0\|_{p+2}} = 0.$$

Now by Lemma 6.1.1 we have

$$\begin{aligned} \lim_{\varphi \rightarrow 0} \frac{\|Q[\varphi]\|_p}{\|\varphi\|_{p+2}} &\leq \lim_{\varphi \rightarrow 0} \frac{C_Q \|\varphi\|_{p+2}^2}{\|\varphi\|_{p+2}} \\ &\leq \lim_{\varphi \rightarrow 0} C_Q \|\varphi\|_{p+2} = 0 \end{aligned}$$

Consider then that at $\varphi_0 \equiv 0$ we have

$$\begin{aligned} 0 &= \lim_{\varphi \rightarrow 0} \frac{\|Q[\varphi] - Q[0] - DQ(0)[\varphi - 0]\|_p}{\|\varphi - 0\|_{p+2}} \\ &= \lim_{\varphi \rightarrow 0} \frac{\|Q[\varphi] - DQ(0)[\varphi]\|_p}{\|\varphi\|_{p+2}} \\ &= \lim_{\varphi \rightarrow 0} \frac{\|Q[\varphi] - DQ(0)[\varphi]\|_p}{\|\varphi\|_{p+2}} + \frac{\|Q[\varphi]\|_p}{\|\varphi\|_{p+2}} \\ &\geq \lim_{\varphi \rightarrow 0} \frac{\|Q[\varphi] - DQ(0)[\varphi] - Q[\varphi]\|_p}{\|\varphi\|_{p+2}} \\ &= \lim_{\varphi \rightarrow 0} \frac{\|DQ(0)[\varphi]\|_p}{\|\varphi\|_{p+2}} \\ &\geq 0 \end{aligned}$$

In particular this implies that

$$\begin{aligned}
0 &= \lim_{t \rightarrow 0} \frac{\|DQ(0)[t\varphi]\|_p}{\|t\varphi\|_{p+2}} \\
&= \lim_{t \rightarrow 0} \frac{\|DQ(0)[\varphi]\|_p}{\|\varphi\|_{p+2}} \\
&= \frac{\|DQ(0)[\varphi]\|_p}{\|\varphi\|_{p+2}}
\end{aligned}$$

for all $\varphi \in C_0^{p+2}(\overline{M})$. Thus $DQ(0) = 0$.

Since Q is C^1 , $DQ(\varphi_0)$ is continuous in φ_0 and $DQ(\varphi_0)[\varphi]$ is continuous (and thus bounded) in φ . Thus the map $\varphi \mapsto \|DQ(\varphi)\|$ (in the operator norm for linear functions) is also continuous. Therefore as $\|DQ(0)\| = \|0\| = 0$ there exists $r_C > 0$ such that $\|DQ(\varphi_0)\| < C$ for all $\varphi_0 \in \overline{B_{r_C}(0)} \subset B_{2r_C}(0) \subset U$.

By the mean value theorem we may then conclude that for all $\varphi_1, \varphi_2 \in \overline{B_{r_C}(0)}$,

$$\|Q[\varphi_1] - Q[\varphi_2]\|_p \leq C\|\varphi_1 - \varphi_2\|_{p+2}.$$

□

Now we can conclude that J is a contraction.

Proposition B.2.5. *There exists $r_J > 0$ such that the map $J : B_{r_J}(0) \subset U \rightarrow C_0^{p+2}(\overline{M})$ is a contraction with factor $1/2$.*

Proof. Let $r_J > 0$ be given such that for all $\varphi_1, \varphi_2 \in B_{r_J}(0)$,

$$\|Q[\varphi_1] - Q[\varphi_2]\|_p \leq \frac{1}{2\|L^{-1}\|} \|\varphi_1 - \varphi_2\|_{p+2}.$$

Then for all $\varphi_1, \varphi_2 \in B_{r_J}(0)$,

$$\begin{aligned}
\|J[\varphi_1] - J[\varphi_2]\|_{p+2} &= \|L^{-1}[-\epsilon + Q[\varphi_1]] - L^{-1}[-\epsilon + Q[\varphi_2]]\|_{p+2} \\
&= \|L^{-1}[Q[\varphi_1] - Q[\varphi_2]]\|_{p+2} \\
&\leq \|L^{-1}\| \|Q[\varphi_1] - Q[\varphi_2]\|_p \\
&\leq \|L^{-1}\| \frac{1}{2\|L^{-1}\|} \|\varphi_1 - \varphi_2\|_{p+2} \\
&= \frac{1}{2} \|\varphi_1 - \varphi_2\|_{p+2}
\end{aligned}$$

□

We now give use the contraction mapping principle to find a unique solution.

Theorem B.2.6. *If $p > 2 + n/2$ and the Dirichlet eigenvalues of $L = -\Delta_M - |A_M|^2$ are all non-zero then for all ϵ , $\|\epsilon\|$ sufficiently small, the Dirichlet boundary value problems*

$$\begin{cases} 2\mathcal{H}[\varphi] = 2H_M + \epsilon \\ \varphi|_{\partial M} = 0 \end{cases}$$

both have unique solutions in $B_{r_J}(0) \subset C^{p+2}(\overline{M})$.

Proof. Let $\|\epsilon\|_p < \frac{r_J}{2\|L^{-1}\|}$. Then for all $\varphi \in B_{r_J}(0)$ we have

$$\begin{aligned}
\|J[\varphi]\|_{p+2} &\leq \|J[\varphi] - J[0]\|_{p+2} + \|J[0]\|_{p+2} \\
&\leq \frac{1}{2} \|\varphi - 0\|_{p+2} + \|L^{-1}[\epsilon]\|_{p+2} \\
&\leq \frac{1}{2} \|\varphi\|_{p+2} + \|L^{-1}\| \|\epsilon\|_p \\
&\leq \frac{1}{2} r_J + \|L^{-1}\| \frac{r_J}{2\|L^{-1}\|} \\
&= r_J
\end{aligned}$$

Thus $J : \overline{B_{r_J}(0)} \rightarrow \overline{B_{r_J}(0)}$ and J is a contraction, so by the contraction mapping principle

has a unique fixed point in $\overline{B_{r_J}(0)}$. As in the proof of Theorem B.1.4, this fixed point is a solution to our initial differential equation and yields a new cmc surface $N = \Gamma_\varphi(M)$ with mean curvature $nH_M + \epsilon$. Additionally however, this solution is unique in $\overline{B_{r_J}(0)}$. \square

Combined with Lemma 4.3.1 this gives an alternate proof of the Corollary 6.2.9. Now using a family of contractions we are in a position to show the continuity of this solution with respect to ϵ in the next section.

B.3 Regularity of the Fixed Point Operator

Using the above contraction principle approach to find unique solutions in $\overline{B_{r_J}(0)}$ we are able to define the map

$$X : B_{r_\epsilon}(0) \subset C^p(\overline{M}) \rightarrow \overline{B_{r_J}(0)} \subset C_0^{p+2}(\overline{M})$$

for $r_\epsilon = \frac{r_J}{2\|L^{-1}\|}$ where ϵ maps to φ_ϵ , the unique fixed point of the operator

$$J_\epsilon[\varphi] = L^{-1}[-\epsilon + Q[\varphi]].$$

We wish demonstrate that X is C^p . First we note that X is uniformly continuous in ϵ via the following recursive argument. In fact, we get Lipschitz continuity:

$$\begin{aligned} \|\varphi_{\epsilon_1} - \varphi_{\epsilon_2}\|_{p+2} &= \|J_{\epsilon_1}[\varphi_{\epsilon_1}] - J_{\epsilon_2}[\varphi_{\epsilon_2}]\|_{p+2} \\ &= \|L^{-1}[-\epsilon_1 + Q[\varphi_{\epsilon_1}]] - L^{-1}[-\epsilon_2 + Q[\varphi_{\epsilon_2}]]\|_{p+2} \\ &= \|L^{-1}[\epsilon_2 - \epsilon_1 + Q[\varphi_{\epsilon_1}] - Q[\varphi_{\epsilon_2}]]\|_{p+2} \\ &\leq \|L^{-1}\| \|\epsilon_2 - \epsilon_1\|_p + \|L^{-1}\| \|Q[\varphi_{\epsilon_1}] - Q[\varphi_{\epsilon_2}]\|_p \\ &\leq \|L^{-1}\| \|\epsilon_2 - \epsilon_1\|_p + \|L^{-1}\| \frac{1}{2\|L^{-1}\|} \|\varphi_1 - \varphi_2\|_{p+2} \\ \|X(\epsilon_1) - X(\epsilon_2)\|_{p+2} &= \|\varphi_{\epsilon_1} - \varphi_{\epsilon_2}\|_{p+2} \leq 2\|L^{-1}\| \|\epsilon_2 - \epsilon_1\|_p \end{aligned}$$

Next consider we consider the Fréchet derivative of X_ϵ with respect to ϵ . We see a similar recursive argument in an informal derivative argument.

$$\begin{aligned}
\frac{\partial}{\partial \epsilon} \varphi_\epsilon &= \frac{\partial}{\partial \epsilon} J_\epsilon[\varphi_\epsilon] \\
&= \frac{\partial}{\partial \epsilon} L^{-1}[-\epsilon + Q[\varphi_\epsilon]] \\
&= L^{-1} \left[\frac{\partial}{\partial \epsilon} (-\epsilon + Q[\varphi_\epsilon]) \right] \\
&= L^{-1} \left[-I + DQ(\varphi_\epsilon) \frac{\partial}{\partial \epsilon} \varphi_\epsilon \right] \\
L \left[\frac{\partial}{\partial \epsilon} \varphi_\epsilon \right] &= -I + DQ(\varphi_\epsilon) \frac{\partial}{\partial \epsilon} \varphi_\epsilon \\
\frac{\partial}{\partial \epsilon} \varphi_\epsilon &= (DQ(\varphi_\epsilon) - L)^{-1}
\end{aligned}$$

This suggests that the Fréchet derivative of X by ϵ ,

$$DX(\epsilon) : C^p(\overline{M}) \rightarrow C_0^{p+2}(\overline{M}),$$

is given by

$$DX(\epsilon) = (DQ(\varphi_\epsilon) - L)^{-1}.$$

This holds true formally as we show. Consider first the following relation.

$$\begin{aligned}
L[\varphi_\epsilon - \varphi_{\epsilon_0}] &= L[J_\epsilon[\varphi_\epsilon] - J_{\epsilon_0}[\varphi_{\epsilon_0}]] \\
&= L[L^{-1}[-\epsilon + Q[\varphi_\epsilon]] - L^{-1}[-\epsilon_0 + Q[\varphi_{\epsilon_0}]]] \\
&= Q[\varphi_\epsilon] - Q[\varphi_{\epsilon_0}] - (\epsilon - \epsilon_0) \\
&= DQ(\varphi_{\epsilon_0})[\varphi_\epsilon - \varphi_{\epsilon_0}] - (\epsilon - \epsilon_0) + Q[\varphi_\epsilon] - Q[\varphi_{\epsilon_0}] - DQ(\varphi_{\epsilon_0})[\varphi_\epsilon - \varphi_{\epsilon_0}] \\
(DQ(\varphi_{\epsilon_0}) - L)[\varphi_\epsilon - \varphi_{\epsilon_0}] &= (\epsilon - \epsilon_0) - [Q[\varphi_\epsilon] - Q[\varphi_{\epsilon_0}] - DQ(\varphi_{\epsilon_0})[\varphi_\epsilon - \varphi_{\epsilon_0}]] \\
\varphi_\epsilon - \varphi_{\epsilon_0} &= (DQ(\varphi_{\epsilon_0}) - L)^{-1}[(\epsilon - \epsilon_0) - [Q[\varphi_\epsilon] - Q[\varphi_{\epsilon_0}] - DQ(\varphi_{\epsilon_0})[\varphi_\epsilon - \varphi_{\epsilon_0}]]]
\end{aligned}$$

Therefore we see that

$$\begin{aligned}
& \lim_{\epsilon \rightarrow \epsilon_0} \frac{\|X(\epsilon) - X(\epsilon_0) - (DQ(\varphi_\epsilon) - L)^{-1}[\epsilon - \epsilon_0]\|_{p+2}}{\|\epsilon - \epsilon_0\|_p} \\
&= \lim_{\epsilon \rightarrow \epsilon_0} \frac{\|\varphi_\epsilon - \varphi_{\epsilon_0} - (DQ(\varphi_\epsilon) - L)^{-1}[\epsilon - \epsilon_0]\|_{p+2}}{\|\epsilon - \epsilon_0\|_p} \\
&\leq \lim_{\epsilon \rightarrow \epsilon_0} \frac{\|(DQ(\varphi_{\epsilon_0}) - L)^{-1}[(\epsilon - \epsilon_0)] - (DQ(\varphi_\epsilon) - L)^{-1}[\epsilon - \epsilon_0]\|_{p+2}}{\|\epsilon - \epsilon_0\|_p} \\
&\quad + \lim_{\epsilon \rightarrow \epsilon_0} \frac{\|-(DQ(\varphi_{\epsilon_0}) - L)^{-1}[Q[\varphi_\epsilon] - Q[\varphi_{\epsilon_0}] - DQ(\varphi_{\epsilon_0})[\varphi_\epsilon - \varphi_{\epsilon_0}]]\|_{p+2}}{\|\epsilon - \epsilon_0\|_p} \\
&\leq \|(DQ(\varphi_{\epsilon_0}) - L)^{-1}\| \lim_{\epsilon \rightarrow \epsilon_0} \frac{\|Q[\varphi_\epsilon] - Q[\varphi_{\epsilon_0}] - DQ(\varphi_{\epsilon_0})[\varphi_\epsilon - \varphi_{\epsilon_0}]\|_{p+2}}{\|\varphi_\epsilon - \varphi_{\epsilon_0}\|_{p+2}} \frac{\|\varphi_\epsilon - \varphi_{\epsilon_0}\|_{p+2}}{\|\epsilon - \epsilon_0\|_p} \\
&\leq 2\|L^{-1}\| \|(DQ(\varphi_{\epsilon_0}) - L)^{-1}\| \lim_{\varphi_\epsilon \rightarrow \varphi_{\epsilon_0}} \frac{\|Q[\varphi_\epsilon] - Q[\varphi_{\epsilon_0}] - DQ(\varphi_{\epsilon_0})[\varphi_\epsilon - \varphi_{\epsilon_0}]\|_{p+2}}{\|\varphi_\epsilon - \varphi_{\epsilon_0}\|_{p+2}} \\
&\leq 2\|L^{-1}\| \|(DQ(\varphi_{\epsilon_0}) - L)^{-1}\| \cdot 0 = 0
\end{aligned}$$

The above argument plus Lemma 5.2.1 proves the following:

Theorem B.3.1. *Suppose that $\lambda \neq 0$ for all Dirichlet eigenvalues of L . Then there X is C^1 and exists a C^1 inflation of M given by*

$$(x, \epsilon) \mapsto \Gamma_{X(\epsilon)}(x)$$

for all $x \in M$ and $\epsilon \in [-\delta, \delta]$ for some $\delta > 0$ sufficiently small.

Greater regularity of the fixed point operator X can be obtained by induction or by appealing to a version of the inverse function theorem. Though this would amount to using or reproving Lemma 6.2.2. This is leading us away from the stated fixed point approach. In fact, many of the arguments above have already been straying into the approach of Chapter 6. The proofs of Proposition B.2.5 and Theorem B.2.6 together contain many of the arguments used to prove the inverse function theorem (Lemma 6.2.1).

Considering the benefits of each approach, the inverse function theorem is very useful

in quickly arriving at strong results so long as the problem contains sufficient regularity. The fixed-point theorem approach is more useful in cases where very weak regularity is assumed. In such cases the Schauder, Contraction, and Leray-Schauder fixed point theorems may be able to guarantee the existence and/or uniqueness of a solution where the inverse function approach fails.

APPENDIX C

POSITIVITY OF THE FIRST EIGENFUNCTION

Given a uniformly elliptic operator with suitably smooth coefficients

$$L[u] = \sum_{ij} a_{ij} u_{x_i x_j} + \sum_i b_i u_{x_i} + cu$$

over a suitably smooth domain M it is well known that there exists a ‘first’ eigenvalue λ_1 such that λ_1 is real,

$$\operatorname{Re}(\lambda) < \lambda_1$$

for all other eigenvalues λ , and that λ_1 is a simple eigenfunction.

In addition, it is also known that if $c \leq 0$ then the corresponding eigenfunction is also known to be positive on the interior of the domain. We make the small extension to show the same result holds assuming only that c is bounded above.

Lemma C.0.1. *Let u be the eigenfunction of L associated with λ_1 . Suppose that c is bounded above by the constant $\sigma - 1$ everywhere in the domain M . Then $u > 0$ in $M - \partial M$.*

Proof. Let $L_\sigma = L - \sigma$. Consider that

$$L[w] = \lambda w$$

if and only if

$$L_\sigma[w] = (\lambda - \sigma)w.$$

Therefore the eigenfunctions of L_σ are the same as those for L , and in particular the first eigenvalue of L_σ is $\lambda_1 - \sigma$ and is associated with the eigenfunction u . But as $c_\sigma = c - \sigma < 0$ the standard approach shows that $u > 0$ on interior of the domain M . □

Since we are assuming a level of continuity for the function c , this result also holds whenever the domain M is compact.

APPENDIX D

FRÉCHET DIFFERENTIABILITY OF THE THE MULTIPLICATIVE INVERSE

Consider $g : W \subset C^p(\overline{M}) \rightarrow C^p(\overline{M})$ defined by

$$g[f] = \frac{1}{f}$$

for

$$W = \{f \in C^p(\overline{M}) \mid \|f(x)\| > w \text{ for all } x \in M\}.$$

Proposition D.0.1. *The operator g is C^1 in the Fréchet sense and*

$$g'[f] = -\frac{1}{f^2}I.$$

Proof. Consider

$$\begin{aligned} \lim_{f \rightarrow f_0} \frac{\|\frac{1}{f} - \frac{1}{f_0} - (-\frac{1}{f_0^2}I(f - f_0))\|_p}{\|f - f_0\|_p} &= \lim_{f \rightarrow f_0} \frac{\|\frac{1}{f} - \frac{1}{f_0} + \frac{1}{f_0^2}(f - f_0)\|_p}{\|f - f_0\|_p} \\ &= \lim_{f \rightarrow f_0} \frac{\|f_0^2 - f f_0 + f(f - f_0)\|_p}{\|f f_0^2\|_p \|f - f_0\|_p} \\ &= \lim_{f \rightarrow f_0} \frac{\|(f - f_0)^2\|_p}{\|f f_0^2\|_p \|f - f_0\|_p} \\ &= \lim_{f \rightarrow f_0} \frac{\|f - f_0\|_p}{\|f f_0^2\|_p} \\ &= \frac{0}{\|f_0^3\|_p} = 0 \end{aligned}$$

which holds for all $f_0 \in W$ since f, f_0 are never zero on M . □

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