

Inventory Control and Demand Distribution Characterization

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Inventory Control and Demand Distribution Characterization

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To my parents

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SUMMARY

This thesis studies three problems related to inventory control. The first problem is motivated by the need to eliminate the bullwhip effect in a supply chain. An important source of this effect is the inventory control policy, which is originally designed to smooth production in response to demand variation along the supply chain arising from the customers. To address this issue, we propose an estimation method based on the control variate technique. A byproduct of this approach is a stabilizing inventory control policy. We evaluate the effectiveness of the proposed method using the models from the literature.

Generally, the derivation of the inventory policies requires the knowledge of the specific demand distribution. Unfortunately, in several cases the demand is not observable in a direct way. The second problem is motivated by a practical application where only partial demand information is observable. Towards this end we derive estimators of the first two moments of the (daily) demand by means of the renewal theoretical concepts. We also propose a regression-based approximation to improve the quality of the estimators. A series of numerical studies are carried out to evaluate the accuracy and precision of the estimators and to investigate the impact of the estimation on the optimality of the inventory policies.

The last part of this dissertation studies a periodic-review inventory system with regular and emergency orders. Emergency orders, characterized by shorter lead-time, higher ordering cost and higher setup cost, are placed when the inventory level becomes critically low. Based on our assumptions, we formulate a dynamic programming model and prove the optimality of state-dependent (s, S) type policies for both emergency and regular orders. We also derive analytic properties of the optimal policies. We gain some managerial insights into the optimal policies and cost performance from numerical studies.

CHAPTER I

INTRODUCTION

1.1 Introduction

Inventory is a key component in the logistical behavior of essentially all manufacturing systems. The classical inventory results are core to modern techniques of manufacturing management such as material requirement planning, just-in-time, and time-based competition. Inventory consists of goods and materials used in the production and distribution process such as raw materials, component parts, subassemblies, and finished goods, as well as the various products and supplies required in the production and distribution process.

Uncertainty is the main motivation for a retailer to store inventory. One of the most important factors for uncertainty is external demand; while other sources of uncertainty include the variation of delivery time (lead-time), fluctuations of price, and uncertain production schedules. Another main reason to hold inventory is to take advantage of volume discounts that amortize the fixed setup cost over a large number of units.

1.1.1 Characteristics of Inventory Control

Through empirical studies and deductive mathematical modeling, a number of factors have been identified that are important for inventory control.

Demand is the most important factor in inventory systems; this in turn determines the complexity of the resulting models. Generally speaking, the demand process could be categorized as follows: deterministic vs. random, known vs. unknown distribution, and independent vs. dependent. Lead-time, the time an order is placed until it arrives, is another important aspect of inventory systems that can be either constant or random. A third important aspect of inventory systems is the reaction to excess demand. Most often, unsatisfied demand is either backordered in full or is entirely lost.

A number of cost factors commonly involved in an inventory system will be briefly described here. The holding cost is the sum of all costs that are associated the amount of inventory physically on hand; these include opportunity costs of the more invested, expenses incurred in running a warehouse, handling and counting costs, costs of special storage requirements as well as costs related to deterioration of stock, damage, theft, obsolescence, insurance, and taxes. Fixed setup costs are only associated with replenishments (independently of the size of a replenishment). They consist of the costs for order forms, postage, telephone calls, authorization, typing of orders, receiving, inspection, following up on unexpected situations, and handling of vendor invoices. Backorder costs are associated with stockouts.

Besides the above cost factors, another key concern in inventory models is the service level. In many real systems, managers often have a difficult time determining backorder costs. Since these costs include intangible components such as loss of goodwill and potential delays to other parts of the system. A common substitute for a backorder cost is service level. Two types of service level are considered. In the first case, the service level is specified by the probability of not stocking out during a lead-time. The second type of service level measures the proportion of demands that are met from stock.

1.1.2 Review Strategies and Optimal Policies

Inventory is both an asset and a liability. Too much inventory consumes physical space, creates a financial burden, and increases the possibility of damage, spoilage and loss. Further, excessive inventory frequently compensates for sloppy and inefficient management, poor forecasting, haphazard scheduling, and inadequate attention to process and procedures. On the other hand, too little inventory often disrupts manufacturing operations, and increases the likelihood of poor customer service. In many cases good customers may become irate and take their business elsewhere if the desired product is not immediately available.

Therefore, the three key questions inventory control attempts to answer are:

1. How often should the inventory status be determined?
2. When should a replenishment order be placed?

3. How large should a replenishment order be?

The first question involves the review interval, T , elapsing between consecutive moments at which we inspect the stock level. An extreme case is the continuous review ($T = 0$), that is, the stock status is known at any time. With periodic review, as the name implies, the stock status is determined only every $T > 0$ time units; between the review epochs there may be considerable uncertainty as to the value of the stock level. Detailed discussion of their advantage and disadvantages will be in Chapter 5.

There is an enormous variety of inventory management models of potential interest. Here we briefly describe some milestone work in the development of inventory control, for detail review, see Silver (1981). First and foremost, we consider the EOQ (economic order quantity) model with deterministic demand. A second all-important inventory policy is the (s, S) policy under stochastic demand and positive setup cost.

One of the simplest and the most useful models is the classical EOQ model, which assumes that demand process is deterministic with rate μ , no shortage and lead-time. The costs include the fixed setup cost K , unit order cost c and holding cost h per unit per unit of time. Minimization of the total cost in a single period yields the optimal order quantity $Q = \sqrt{2K\mu/h}$. The EOQ address the tradeoff between order quantity and inventory. More discussion of EOQ model and its extensions can be seen in Nahmias (1997) and in Silver et al. (1998).

Generally speaking, most previous work assumes knowledge of the demand distribution. Under this assumption, one of the most fundamental results in inventory control is the optimality of the (s, S) policy by Scarf (1960) for fixed setup cost, linear holding cost and linear backorder cost. Under this policy, if the inventory position during an inspection epoch is found to be below s , an order is placed to raise the inventory position up to S . Scarf's proof used the concept of K -convexity. Other work related to the optimality of (s, S) policies (e.g., existence and uniqueness), can be found in Arrow et al. (1951), Bellman et al. (1955), Dvoretzky et al. (1953), Karlin (1958), Iglehart (1963), and Veinott (1966). Even though the optimal policy has an attractive form, it is often hard to compute the optimal values of s and S . Some elegant algorithms were developed by Veinott and Wagner

(1965), Johnson (1968), Bell (1970), Archibald and Silver (1978), and Federgruen and Zheng (1991). Some approximation methods have been developed to calculate the optimal values of s and S ; see Robert (1962), Wagner et al. (1965), Naddor (1975), Ehrhardt (1979, 1984, 1985), Porteus (1979), Freeland and Porteus (1980), and Federgruen and Zipkin (1984, 1985). These algorithms are greatly helpful and useful, especially when we do not know the demand distribution. We will focus on methods that rely only on the first two moments of the demand distribution and the lead-time distribution.

Similar results and policies can easily be extended to scenarios with periodic review. Especially if the lead-time is a multiple of the review period T , the model with periodic review can be transformed to an equivalent continuous-review model by assuming that the original demand process is transferred to T periodic demand and suitably scaling the cost. A good reference studying the periodic review problem is by Hadley and Whitin (1963).

1.2 Outline of Thesis

Chapter 2 focuses on the variation reduction problems under an order-up-to policy and a single retailer facing a demand process that can be modeled as an order-1 autoregressive AR(1) sequence. ¹Chapter 3 investigates the impact of demand estimation on the system total cost when the demand is not transparent. Chapter 4 evaluates the accuracy of the demand estimation technique in Chapter 3. Chapter 5 studies an inventory control model with regular and emergency orders. We can place emergency orders during a period.

¹Joint work with , Hancong Liu, K. Tsui and Hengqing Ye

CHAPTER II

REDUCING BULLWHIP EFFECT IN SUPPLY CHAINS USING CONTROL VARIATES

2.1 Introduction

The motivation behind this chapter is the recent studies on the bullwhip effect in supply chain management. The bullwhip effect refers to the phenomenon where orders to the supplier tend to have larger variance than sales to the buyer (i.e., demand distortion), and the distortion propagates as one moves upstream in an amplified form (i.e., variance amplification). An interesting example can be found in Lee et al. (1997b), which is concerned with the demand amplification of a diaper product from babies and customers to upstream manufacturers.

It is known that the bullwhip effect may lead to tremendous inefficiency in supply chains. For instance, manufacturers in a supply chain may have to encounter extra production costs due to high demand fluctuation, such as the expenses for hiring, layoffs, overtime, excessive inventory investment, additional transportation (e.g., due to premium shipping rates) and lost revenues (e.g., due to unmet demand). A company's reputation may also be damaged by its inability to meet the customer demand due to the variance amplification. Therefore, a fundamental question in the management of supply chains is raised: How to control the bullwhip effect so that the system-wide performance of the supply chain is improved? This chapter is concerned with the reduction of the bullwhip effect, or more generally, the variance reduction of the demand processes in supply chains via the design of inventory control policies.

Some major causes of bullwhip effect, as identified in Lee et al. (1997ab), include demand forecasting, lead-times, batch orders, supply shortages, and price variations. In response to these causes, various methods have been proposed to reduce this effect, such as reducing

uncertainty by centralized information, reducing demand variability, shortening lead-time and building more decisions strategic partnerships. The readers are referred to, e.g., Lee et al. (1997ab), Lee and Tang (1997, 1998), Chen (1999), Cachon (1999), Cachon and Zipkin (1999), Tsay (1999), Li and Kouvelis (1999), Moinszadeh and Nahmias (2000).

An important source of bullwhip effect, which has been partly reflected in the above-mentioned causes, is the inventory control policy. A fundamental function of inventory control is to smooth production in response to demand variation along the supply chain arising from the customers. However, as shown in Baganha and Cohen (1998), many widely used inventory control policies actually enlarge such variation when one moves up a supply chain, and thus the stabilizing function of inventory is not realized. This argument is also reinforced by studies on some specific supply chain models and inventory control policies. For instance, Caplin (1985) and Kelle and Milne (1999) point out that if n retailers use (s, S) policies simultaneously, the aggregate order variance exceeds the variance of sales. Cachon (1999) shows in his supply chain model with multiple retailers and a supplier that the demand variation faced by the supplier may be reduced by increasing the retailers' order intervals. Furthermore, Chen et al. (2000ab) show that the order variance is greater than the demand variance when various forecasting strategies are used.

Some techniques on inventory control to reduce demand variability within supply chains have been studied in various contexts. Henig et al. (1997) derive an optimal control policy for an inventory system given a supply contract that specifies the available amount of supply. By such a policy, that specifies a range of inventory level within which the fixed contract supply volume is ordered, the variability of order quantity can be reduced and excessive premium transportation costs can be avoided. Cachon (1999), as mentioned above, claims that the supplier's demand variance can be reduced through carefully scheduling the retailers' orders, and such reduction may also lead to the reduction of system cost for the model used in his paper. Motivated by engineering process control methods, Liu (2001) and Liu et al. (2001) propose a class of order-up-to level policies and a nearly optimal inventory control policy to reduce the bullwhip effect. This nearly optimal policy significantly reduces the order variance while the inventory cost is only increased slightly.

These recent studies on reducing bullwhip effect due to inventory control policy are stimulating, and have provided insight into this managerial issue. They are also important steps towards more practical solutions to such bullwhip effect issues in management. However, there are some difficulties in using these approaches. For instance, (1) The techniques mentioned above and the respective models employ various system-wide optimization approaches, for which experts are needed to formulate and solve the complex mathematical models even for supply chains with only slightly complex configurations. (2) Great effort is needed if those techniques are to be adapted to different models and situations. This is almost impossible for most companies unless experts are approached, who would not always be available due to the high variety of practical situations and models. (3) The consequences of bullwhip effect are usually not considered in inventory control policies that are easy to implement and are widely used; in practical work these policies are often the first choice for managers in managing their supply systems. Even when the bullwhip has a detrimental impact on the performance of the supply chain, it would often be difficult to implement a totally new and optimal (but complicated) inventory control policy due to some managerial issues such as the resistance to change in the company.

In this chapter, we propose a control variate technique for reducing the bullwhip effect, or more generally the demand variability in supply chains; this technique is effective and easy to implement in the practical management of supply systems. The main idea behind such a technique is as follows: When an order amount derived from an (original) inventory control policy leads to detrimentally large variance in the demand for upstream supplier, then we impose to the order a (correlated) control variate in suitable form to the original order quantity. Such a technique may lead in some cases to an optimal system-wide solution.

We organize our presentation as follows. We outline the key ideas of the control variate technique and then use a simple example to illustrate the technique in Section 2.2. In Section 2.3, we examine in detail the application of the control variate technique to variations of the inventory model studied in Lee et al. (1997a) and Chen et al. (2000a). From this application, we gain more insight into the technique. We conclude in Section 2.4 with a few remarks.

2.2 The Control Variate Technique

To motivate our study of the control variate technique for dampening the bullwhip effect in supply chains, we briefly review the use of this technique for reducing variance of simulation output. It is known that such techniques have been widely used in the variance reduction in simulation and statistical process control. When properly applied, it may transform an impossibly expensive simulation project into a feasible one.

Suppose that one aims to estimate a performance measure $\phi = E[Z]$ of a system using an output Z that is generated from a simulation of the system. In many situations, for example, in simulating the steady-state expected queue length evolution of a heavily loaded queueing system, the variance of the estimator Z , $\text{Var}(Z)$, would become unacceptably large, which often makes it impossible to obtain reasonably good point estimate for ϕ . Then some modification of the original estimator Z would be necessary. For this purpose, one may identify some other output variable V generated from the simulation, with known expectation $E[V] = \mu_v$. The random variable

$$Z + \alpha(\mu_v - V),$$

where α is a constant, can be used as an alternative estimator of ϕ with variance

$$\text{Var}(Z + \alpha(\mu_v - V)) = \text{Var}(Z) + \alpha^2 \text{Var}(V) - 2\alpha \text{Cov}(Z, V).$$

It is direct to show that one can minimize the above variance at

$$\alpha^* = \frac{\text{Cov}(Z, V)}{\text{Var}(V)}$$

with the minimal variance being

$$\text{Var}(Z + \alpha(\mu_v - V)) = \text{Var}(Z) - \frac{[\text{Cov}(Z, V)]^2}{\text{Var}(V)}.$$

We call the variable V a *control variate* with respect to the simulation estimator Z . To gain more perspective on how control variate leads to an estimator with less variance, consider the following scenario. Assume Z and V to be highly positively correlated, so that α^* is a positive. Therefore, when a large value of the original estimator Z (or $Z > \phi$) is observed

from a simulation run, the observed value of V would also tend to be larger than its mean μ_v . As α^* is positive, appending the term $\alpha^*(\mu_v - V)$ to the original estimator Z helps to cancel out the deviation of Z from its mean ϕ . A similar observation applies if Z and V are negatively correlated.

Inspired by the above example on the control variate technique for variance reduction in simulation, we propose a variance reduction method that is easy to implement. Suppose that an inventory system (which is a stage of a supply chain), the inventory control policy (called the original policy below) is derived using a conventional method without taking into account the bullwhip effect on the entire supply chain. When it is necessary to avoid the system-wide performance degradation caused by the bullwhip effect, a new inventory control policy (called a stabilizing policy below) that leads to less variance in the demand process for its upstream supplier in the supply chain can be obtained in three steps.

1. *Generic stabilizing policy.* Identify a control variate that is (positively or negatively) correlated to the order quantity given by the original inventory control policy, and use this control variate to construct a generic control variable for the original policy. Then, a generic stabilizing inventory control policy is formed by appending the generic control variable to the original order in a suitable format. The generic policy may not be a feasible inventory control policy. For instance, it may violate the service level requirement. It may also have side effects on other performance measures, e.g., the holding cost, of the supply chain.
2. *Feasible stabilizing policy.* Modify the generic stabilizing inventory control policy (or the generic control variable) according to some specified managerial requirement, so that it becomes a feasible one.
3. *Fine-tuning.* Make some necessary adjustments and further modifications on the stabilizing policy so that a better tradeoff among various performance measures of the supply chain and thus a better system-wide performance can be achieved. In practice, such fine-tuning of the stabilizing policy could be carried out by trial and error based on the manager's insight into the supply chain system. An appropriate

use of simulation and analytical analysis would be helpful in the fine-tuning step.

We illustrate the control variate technique using the following simple example. Consider a supply chain with a retailer and a manufacturer, both belonging to the same organization or company. The demands for an item to the retailer at each period, D_t ($t = 1, 2, \dots$), are independent $N(d, \sigma^2)$ random variables. (For simplicity, the demand distribution is truncated so that no negative demands would appear. The normal distribution assumption remains a good approximation if σ/d is small.) A service requirement is specified as the service level, λ , the probability of not stocking out in each period. An order from the retailer to the manufacturer is placed and filled at the beginning of each period. The holding cost is h per item per period. Assume that the backorder and ordering costs are negligible. Then the optimal order-up-to-level for each period in this model is

$$S_t \equiv S := d + \Phi^{-1}(\lambda)\sigma,$$

the inventory level at the beginning of the next period is

$$I_t = \max(S_t - D_t, 0), \tag{1}$$

and the order quantity placed at the end of each period is

$$Z_{t+1} = S_{t+1} - I_t. \tag{2}$$

Here, $\Phi(\cdot)$ denotes the standard normal distribution function.

Now suppose that the normal production capacity of the manufacturer is P units and, when the order quantity Z_t is greater than this production capacity, a penalty cost of p is incurred for producing each extra unit. In practical situations, such a cost would probably be caused by, e.g., hiring new workers or outsourcing, and would be very high sometimes in order to meet some pre-determined service requirements. Then, if the variance of the order quantity Z_t is high and leads to significant production penalty costs, the system-wide performance of the whole supply chain would be seriously degraded, as shown in the numerical example below. In this case, it becomes necessary to derive an inventory control policy with less variance in the order quantity so that a better tradeoff between the inventory

cost and the extra production cost is achieved, and thus a better system-wide performance of the whole supply chain is obtained. This can be done heuristically following the above three-step procedure.

First, we formulate a generic stabilizing inventory control policy as follows. The key in this step is to identify a control variate and form an auxiliary variable that is negatively correlated with the original order quantity. To this end, we note that, heuristically by the original policy,

$$Z_{t+1} = S_{t+1} - I_t = S_{t+1} - \max(S_t - D_t, 0) \approx D_t$$

if the service level λ is close to 1. Thus, an obvious choice of control variate that is (positively) correlated to the original order quantity Z_{t+1} is D_t , and hence a simple choice of a generic control variable could be a linear function of D_t ,

$$V'_t = \theta(d - D_t) + \gamma,$$

where θ and γ are parameters. A generic stabilizing policy can be formulated as an order-up-to-level policy with order quantity

$$Z_{t+1} = S_t - I_t + V'_t = S_t - I_t + \theta(d - D_t) + \gamma$$

and the order-up-to-level

$$S_{t+1} = Z_{t+1} + I_t = S_t + \theta(d - D_t) + \gamma, \tag{3}$$

where the dynamics of the inventory level I_t still follows from equation (1). Compared to the original policy given in equation (2), the addition of the control variable counterfeits the variability of the original order quantity if the parameters θ and γ are carefully chosen. Intuitively, the parameter θ , chosen between zero and one, is used to adjust the variability of the order quantity, and the parameter γ , which should be a positive number, provides additional safety stock to make the adjustment. However, it can be seen that the service level requirement may be violated if we use the generic stabilizing policy directly, since the V'_t may be less than zero and hence the order-up-to-level at the next period, S_{t+1} , may fall below S .

Next, we formulate a feasible stabilizing inventory control policy by appropriately revising the above generic stabilizing policy. A simple truncation on the order-up-to-level in the generic stabilizing policy would work in this case. That is, to avoid the violation of the service level requirement, we modify the order-up-to-level of the generic stabilizing policy in equation (3) as

$$S_{t+1} = \max(S_t + \theta(d - D_t) + \gamma, S). \quad (4)$$

Finally, we fine-tune the feasible stabilizing inventory control policy to achieve the best possible system-wide performance. For our simple example, we may resort to simulation to search for a set of parameters, θ and γ , to determine a stabilizing inventory control policy that leads to a satisfactory system-wide performance. As a numerical example, the parameters of the model are given as

$$d = 10, \sigma = 4, \lambda = 0.95, h = 2, P = 12, p = 100.$$

Hence

$$\Phi^{-1}(0.95) = 1.64, S = d + \Phi^{-1}(\lambda)\sigma = 16.56.$$

The high penalty cost ($p = 100$) here is consistent with many practical situations, where the production cost incurred by the outsourcing, hiring new workers, or using overtime is much higher than the cost from routine production. By trial and error, it is not difficult to find some set of parameters (θ, γ) to achieve a satisfactory policy (which should be near the optimal one in certain sense) from the class of feasible stabilizing policies. (There is no doubt that analytical tools, such as using sophisticated searching algorithms work better.) For instance, we would suggest that the parameters can be chosen as $\theta = 0.6$ and $\gamma = 0.8$. Some performance measures, including the average inventory cost, the average extra production cost and the average total system cost (which is the sum of the former two costs), are shown in Table 1.

To gain more perspective on how to fine-tune the feasible stabilizing inventory control policy in different situations, we investigate a case motivated by choosing the parameters

Table 1: Stabilizing policy with $\theta = 0.6$ and $\gamma = 0.8$

	Inventory Cost	Production Cost	Total Cost
Original policy	13.41	68.17	81.59
Stabilizing policy	19.15	29.89	49.04

Table 2: Stabilizing policy with $\theta = 1.0$ and $\gamma = 0$

	Inventory Cost	Production Cost	Total Cost
Original policy	13.25	74.73	87.98
Stabilizing policy	168.15	3.66	171.81

$\theta = 1.0$ and $\gamma = 0$ in the above stabilizing policy. An appealing feasible (and possibly stabilizing) inventory control policy would specify an order-up-to-level as

$$S_{t+1} = \max(S_t + d - D_t, S) = \max(I_t + d, S), \quad (5)$$

where the second equality is due to the identity $S_t - D_t = I_t$ when there is no stockout at period t . With such a policy, the order quantity tends to be a constant. However this policy does not lead to a better system-wide performance for the supply chain, as shown in Table 2. From Table 2 we notice that the average inventory cost turns out to be significantly higher than the corresponding parts in the previous simulation (see for example Table 1). Further investigation into the inventory process I_t , as shown in Figure 1, reveals that the inventory level may become very high from time to time. Thus, from this observation, a refinement of the inventory control policy given in equation (5) can be made by introducing an upper bound B on the order-up-to-level. The resulting stabilizing inventory control policy is given by

$$S_{t+1} = \begin{cases} S, & I_t + d < S \\ I_t + d, & S \leq I_t + d \leq B \\ B, & B < I_t + d. \end{cases} \quad (6)$$

Again, the bound B can be chosen by trial and error. Our simulation shows that $B = 20$ does give a satisfactory system-wide performance for our simple example, compared to the

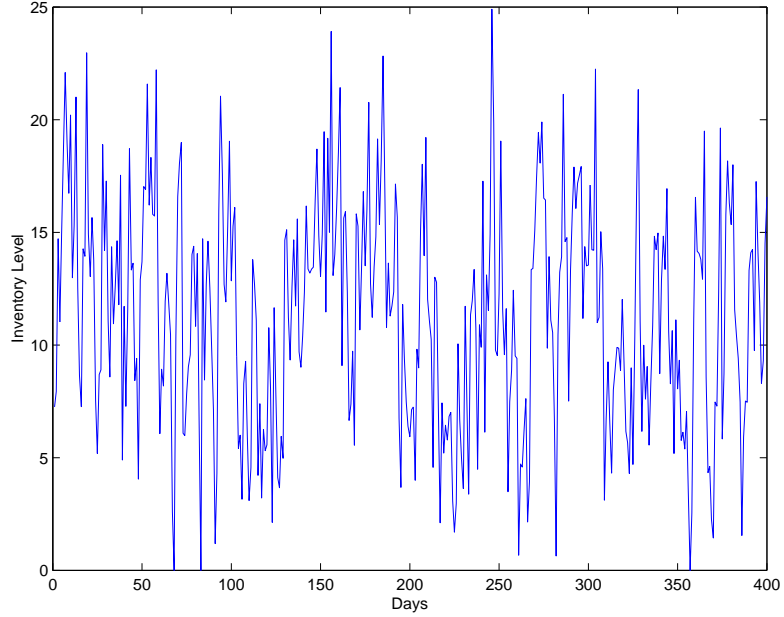


Figure 1: Inventory level vs. days

Table 3: Stabilizing policy with $\theta = 1.0$, $\gamma = 0$, and $B = 20$

	Inventory Cost	Production Cost	Total Cost
Original policy	13.37	68.24	81.61
Stabilizing policy	16.72	39.43	56.16

above stabilizing inventory control policy we have examined. The numerical results are shown in Table 3.

The practical implementation of the technique is an art, as shown in the above instance. However, it would be interesting to gain more understanding from a theoretical perspective. Below, we apply the technique to (variations of) the model studied in Lee, et al. (1997) and Chen et al. (2001ab). The model is used to illustrate the bullwhip effect in these papers. The optimal policy (within a wide class of feasible inventory control policies) is obtained, which indicates that the control variate technique is indeed effective. The analysis of such a model also provides more insight into the practical implementation of the variance reduction technique.

2.3 Variance Reduction for a Supply Chain with AR(1) Demand

In this section, we apply the control variate technique to a supply chain with AR(1) demands and with a retailer and a manufacturer, and illustrate how the analytical analysis helps in fine-tuning the stabilizing policy and gaining more insight into the variance reduction technique. In next subsection, we describe the supply chain with a first-order autoregressive (AR(1)) demand that is studied in Lee et al. (1997) and Chen et al. (2001ab). An AR(1) process is a first-order process, meaning that only the immediately previous value has a direct effect on the current value. We explicitly consider the performance degradation caused by bullwhip effect. More specifically in our example, such a bullwhip effect causes difficulty in arranging production for upstream suppliers (or manufacturers) and leads to significant extra production cost. A class of feasible stabilizing inventory control policies are then proposed for this model on heuristic base. Before fine-tuning the policy for a more-satisfactory policy, we also observe how the stabilizing policy affects the variability of the demand from the retailer to the manufacturer without considering the impact of extra production cost (caused by bullwhip effect) on the system-wide performance of the supply chain. In particular, we provide more intuition on how the order quantity variance is reduced at the expense of additional inventory cost. In Section 2.3.2, we derive the optimal stabilizing policy within a wide class of inventory control policies for the supply chain.

2.3.1 The Model and a Class of Stabilizing Policy

Consider a supply chain with a retailer and a manufacturer. The retailer is faced with single-item multi-period inventory problem in which the retailer orders a single item from a supplier each period. There is a delay of ν periods between ordering and receiving the goods. To simplify the analysis, excess inventory is assumed to be returned without cost and there is not a service level requirement. (We comment on this later.) We consider the AR(1) demand model

$$D_t = d + \rho D_{t-1} + u_t, \tag{7}$$

where D_t is the demand in period t , $-1 < \rho < 1$, and the u_t 's are independent $N(0, \sigma^2)$ random variables, and are uncorrelated with anything known at time $t-1$. It is assumed that σ is significantly smaller than d , so that the probability of a negative demand is negligible. In this model, we assume that orders are placed and filled at the beginning of each period and that demands are realized at the end of each period.

Momentarily, we consider the inventory management for the retailer without considering its impact on the other components of the supply chain, i.e., the manufacturer. Then, the cost-minimization problem for the retailer in an arbitrary period (normalized at 1) is formulated as

$$\min_{S_t} \sum_{t=1}^{\infty} \beta^{t-1} E_1 \left[cZ_t + \beta^\nu g \left(S_t, \sum_{i=t}^{t+\nu} D_i \right) \right], \quad (8)$$

with

$$g \left(S_t, \sum_{i=t}^{t+\nu} D_i \right) = h \left(S_t - \sum_{i=t}^{t+\nu} D_i \right)^+ + \pi \left(\sum_{i=t}^{t+\nu} D_i - S_t \right)^+.$$

(See for example Lee et al. (1997).) Here h , π , and c denote the unit holding cost, the unit shortage penalty cost, and the unit ordering cost, respectively. The random variable Z_t is the quantity ordered at the beginning of period t , and the order-up-to-level S_t is the amount in stock plus on order (ordered goods in transit) after decision Z_t has been made in period t . Let β be the cost discount factor per period, and ν be the delivery time. The notation x^+ denotes $\max(0, x)$. We assume that the demands D_t , for periods $t = 0, -1, -2$ are realized and known. It can be shown (Lee et al. (1997)) that the optimal order-up-to-level for period t ($t \geq 0$) is

$$S_t^* = M + \rho^* D_{t-1}, \quad (9)$$

where

$$M = d \sum_{k=1}^{\nu+1} \frac{1 - \rho^k}{1 - \rho} + \Phi^{-1} \left(\frac{\pi - c(1 - \beta)/\beta^\nu}{h + \pi} \right) \sigma \sqrt{\sum_{k=1}^{\nu+1} \left(\frac{1 - \rho^{\nu+2-k}}{1 - \rho} \right)^2}$$

$$\rho^* = \frac{\rho(1 - \rho^{\nu+1})}{1 - \rho},$$

and $\Phi(\cdot)$ denotes the standard normal distribution function. It follows that the optimal order amount is given by

$$Z_t^* = S_t^* - S_{t-1}^* + D_{t-1} = (1 + \rho^*) D_{t-1} - \rho^* D_{t-2}. \quad (10)$$

Thus, the variance of order can also be obtained:

$$\text{Var}(Z_1^*) = \text{Var}(D_0) + \frac{2\rho(1 - \rho^{\nu+1})(1 - \rho^{\nu+2})}{(1 + \rho)(1 - \rho)^2} \sigma^2.$$

Theorem 1 of Lee et al. (1997) says that “the variance amplification takes place when the retailer adjusts the order-up-to level based on the positive correlated demand signals. ” Also, the order variance of the optimal policy is amplified by ρ and ν .

Next, we consider the supply chain including both the retailer and the manufacturer as a whole, and investigate the impact of bullwhip effect and its variance reduction using the control variate technique. We model the impact of the bullwhip effect on the manufacturer by introducing a production capacity. As in the simple example in Section 2.2, we suppose that the usual production capacity of the manufacturer is P units and, when the order quantity Z_t is greater than this production capacity, a penalty cost of p is incurred for producing each extra unit. Now the inventory control problem for the supply chain is formulated as a minimization problem

$$\min_{S_t} \sum_{t=1}^{\infty} \beta^{t-1} \mathbf{E} \left[cZ_t + \beta^\nu g \left(S_t, \sum_{i=t}^{t+\nu} D_i \right) + p \left(Z_t - P \right)^+ \right]. \quad (11)$$

With the control variate technique in mind, instead of solving the above problem for the optimal inventory control policy, we would rather construct a satisfactory solution based on the original optimal inventory control policy for problem in equation (8), where the impact of bullwhip effect is not yet explicitly modeled.

As the first step of the control variate technique, we construct a class of generic stabilizing inventory control policies. To this end, inspired by the format of the order-up-to-level policy in equation (9), we examine a class of generic order-up-to-level policies specified as

$$S_{t+1} = m + GD_t, \quad (12)$$

where parameters m and G are to be determined. To see how a control variable comes into place, notice that, by this policy, the order quantity is

$$\begin{aligned} Z_{t+1} &= S_{t+1} - (S_t - D_t) = D_t + \rho^*(D_t - D_{t-1}) + \theta(D_t - D_{t-1}) \\ &= Z_{t+1}^* + \theta(D_t - D_{t-1}), \end{aligned}$$

where $\theta = G - \rho^*$. As compared to the order quantity Z_{t+1}^* , the control variable $\theta(D_t - D_{t-1})$ is implicitly enclosed in the generic policy in equation (12). Since the variable $D_t - D_{t-1}$ is correlated to the order quantity Z_{t+1}^* given in equation (10), it can be viewed as a control variate and, by suitably choosing the parameter θ or equivalently the parameter G , the order-up-to-level policy given in equation (12) is indeed a (generic) stabilizing inventory control policy. Moreover, it can be seen from a simple Markovian argument that, given the current state of the inventory level, the optimal ordering policy depends only on the latest demand information. Based on this understanding, the order-up-to-level policy given in equation (12) is a natural class of candidate policies that could lead to a satisfactory solution to the optimization problem in equation (11).

In fact, the generic policy given in equation (12) is feasible. Furthermore, we can directly fine-tune the generic policy to obtain a satisfactory one, as suggested in the final step of the control variable technique. Before we perform the fine-tuning (see Subsection 2.3.2), we briefly investigate how the stabilizing policy given in equation (12) (with different choice of parameters m and G) affects the variability of the demand from the retailer to the manufacturer as well as the other performance measures of the supply chain.

Stabilizing Effect of the Generic Feasible Policy. The order variance is

$$\text{Var}(Z_t) = [1 + 2G(1 + G)(1 - \rho)]\text{Var}(D_t). \quad (13)$$

As $\text{Var}(D_t)$ is a constant, the order variance will depend on G only. It can be seen that if $G < -1$ or $G > 0$, $\text{Var}(Z_t)$ is larger than $\text{Var}(D_t)$. That is, for any policy with $G < -1$ or $G > 0$ within the class, the order variance is always larger than that of the demand, thus the bullwhip effect occurs. From equation (13), we can see that the order variance is a quadratic function in G . Their relation can be further revealed by taking the first derivative of the order variance with respect to G :

$$\frac{d}{dG}\text{Var}(Z_t) = 2(1 - \rho)(1 + 2G)\text{Var}(D_t).$$

The derivative is negative for $G < -1/2$, and positive for $G > -1/2$. This implies that the order variance is a decreasing function when $G > -1/2$ and an increasing function when $G < -1/2$. For a smaller positive ρ , the order variance changes more quickly when

G varies, while for a larger positive ρ , the order variance changes more slowly under the same situation. Also, the order variance attains the minimum when $G = -1/2$, where the minimal order variance is $(1 + \rho)\text{Var}(D_t)/2$, i.e., the minimal order variance in the class of order-up-to policies. When $G = \rho^*$, the modified order-up-to level is just the optimal order-up-to level. Consequently, tuning the parameter G allows us to dampen the level of order variance. Additional characterizations of the stabilizing effect of this type of generic inventory control policy can be found in Liu (2001).

2.3.2 Fine-tuning the Generic Policy

The following theorem facilitates the search for the optimal policy among the class of generic policy given in equation (12).

PROPOSITION 2.1. *Given the parameter G , the optimal solution m to problem (11) is*

$$m_G = \frac{(\nu + 1 - G)d}{1 - \rho} + \Phi^{-1}\left(\frac{\pi}{h + \pi}\right)\sigma\sqrt{\frac{(\rho^* - G)^2}{1 - \rho^2} + \sum_{k=1}^{\nu+1}\left(\frac{1 - \rho^{\nu+2-k}}{1 - \rho}\right)^2},$$

where $\rho^* = \frac{\rho(1-\rho^{\nu+1})}{1-\rho}$. The respective optimal expected cost is

$$\begin{aligned} F(m_G, G) &= \frac{cd}{(1 - \beta)(1 - \rho)} \\ &\quad + \frac{1}{1 - \beta}\sigma\sqrt{\frac{2G^2(1 - \rho) + 2G(1 - \rho) + 1}{1 - \rho^2}}\left(pr_2[\Phi(r_2) - 1] + p\frac{1}{\sqrt{2\pi}}e^{-\frac{r_2^2}{2}}\right) \\ &\quad + \frac{\beta^\nu}{1 - \beta}\sigma\sqrt{\frac{(\rho^* - G)^2}{1 - \rho^2} + \sum_{k=1}^{\nu+1}\left(\frac{1 - \rho^{\nu+2-k}}{1 - \rho}\right)^2}\left((h + \pi)\frac{1}{\sqrt{2\pi}}e^{-\frac{r_3^2}{2}}\right), \end{aligned}$$

where

$$r_2 = \left(P - \frac{d}{1 - \rho}\right) / \left(\sigma\sqrt{\frac{2G^2(1 - \rho) + 2G(1 - \rho) + 1}{1 - \rho^2}}\right)$$

and $r_3 = \Phi^{-1}\left(\frac{\pi}{h + \pi}\right)$.

The proof of the theorem is listed in Appendix A. With this theorem in hand, what is left is to determine the optimal parameter G for the generic policy, so that a satisfactory inventory control policy for the retailer is obtained. This can be accomplished by using various search methods.

From the above equation of the total cost, we can see that the production capacity P , the lead-time ν , the penalty cost p and the standard deviation σ of the demand play crucial roles in the expected total cost $F(m_G, G)$ and affect the choice of parameter G . In the rest of this subsection, we conduct numerical experiments and analyze implications. In all these numerical experiments, we consider stationary systems and assume the following parameters: $h = 2$, $\pi = 10$, $d = 10$, $\rho = 0.3$ and $\beta = 0.9$.

Impact of Production Capacity. Assuming $\nu = 0$, $p = 100$ and $\sigma = 2$, the numerical results concerning the impact of production capacity on the total cost and the choice of G are shown in Figure 2. First, we remind that the reader the order variability is reduced when the parameter G for the inventory control policy (13) is chosen from $(-\rho - 1, \rho)$. From the figure, we can see that the total costs decrease with the capacity P , which is consistent with the intuition that the more the capacity, the less chance to incur the penalty. However, it would be more interesting to note that, when the production capacity P is moderate, the control variate technique is effective in dampening the bullwhip effect and thus avoids a high penalty cost due to insufficient production capacity. In particular, when the capacity P is between 10 and 20 under the current parameter settings, the chosen G is equal or close to $-1/2$, which achieves the minimal order variability and expected cost. In contrast, when the production capacity is very low or very high (e.g., $P \leq 5$ or $P \geq 25$ in our example), the control variate technique becomes less effective. In these cases, the optimal parameter is $G = \rho$, which implies that the policy (13) induced by control variate technique coincides with the optimal policy (9) for the original inventory system without a production capacity limit (or $P = \infty$). This is due to the fact that, when the production capacity P is very low, the penalty cost due to insufficient capacity is always incurred; and thus the penalty cost in effect can be modelled as part of the normal production cost. In this case, the system is reduced to the original one with un-capacitated normal production. On the other hand, when the production capacity P is very high, there would be little chance to incur the penalty cost, and again the system with production capacity P is reduced to the original one.

Impact of Lead-Time. The lead-time is an important cause for the bullwhip effect, which

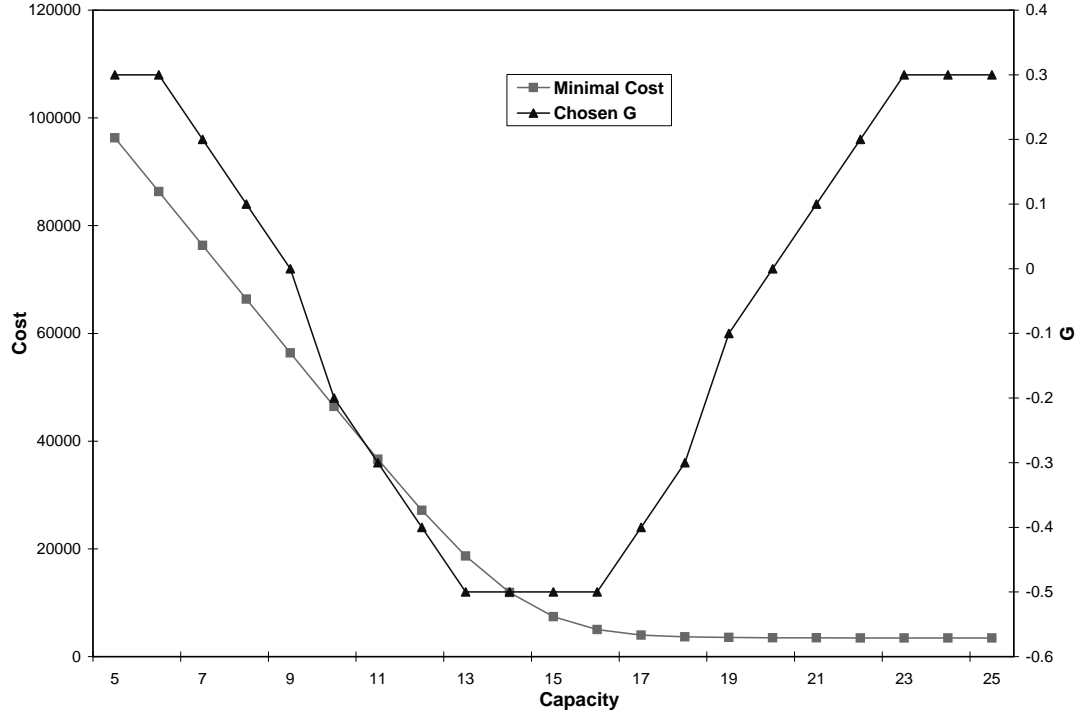


Figure 2: Optimal expected cost and chosen G vs. production capacity P

has been extensively studied in literature. In this numerical experiment, we show the impact of lead-time on the parameter G , i.e., the optimal inventory control policy (see Figure 3. Here, we let the capacity $P = 20$, penalty cost $p = 100$, and $\sigma = 2$ respectively. It is shown in the figure that the parameter G approaches $-1/2$ as the lead-time increases. In other words, when the order variance significantly increases as the lead-time increases, the control variate technique play a more significant role.

Impact of Penalty Cost. Here we intend to study the impact of unit penalty cost p on the optimal total cost and the parameter G . In this case, we assume the production capacity $P = 20$, the standard deviation of demand $\sigma = 2$ and the lead-time $\nu = 6$, respectively. The numerical results displayed in Figure 3 indicate that the minimal total cost of the

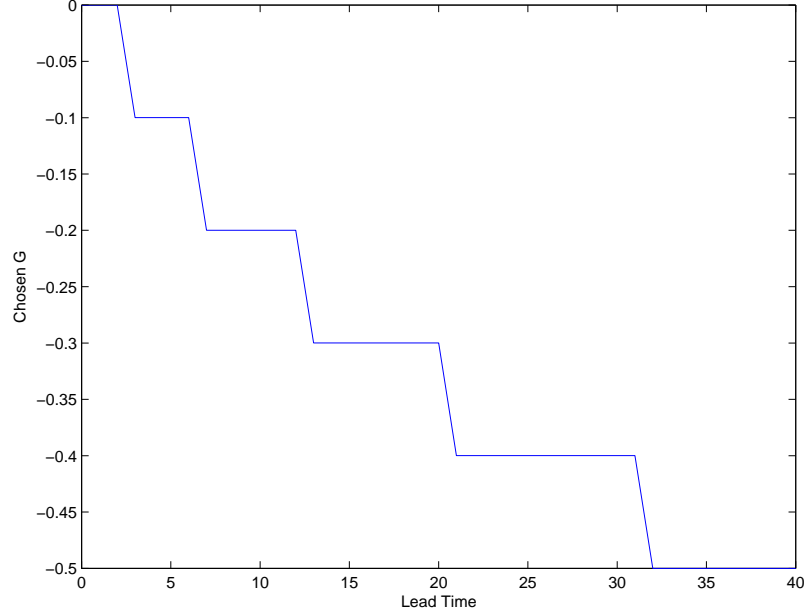


Figure 3: Parameter G vs. lead-time ν

system is an increasing concave function of the unit penalty cost p . In particular, when $p = 0$, the inventory systems with and without production capacity limit coincide and the optimal parameter G is equal to $\rho = 0.3$. When the unit penalty cost p increases, it becomes more important to dampen the bullwhip effect to avoid the heavy penalty cost incurred by extra production and eventually the optimal parameter G approaches $-1/2$. Furthermore, when p exceeds a certain level (e.g., $p > 200$) much larger than the unit holding cost h and backorder cost π , the optimal total cost increases almost linearly. This is expected if we note that, when $G \approx -1/2$, there is little room left for dampening the bullwhip effect and hence reducing the possibility of extra production. The total penalty cost due to extra production is then linearly increasing as the unit penalty cost p increases, which leads to the linear increase of the total cost.

Impact of Demand Standard Deviation. We let the production capacity $P = 20$, the unit penalty cost $p = 100$, and the lead-time $\nu = 6$, and examine the impact of demand standard deviation on the optimal choice of the parameter G (see Figure 5). The results confirm our

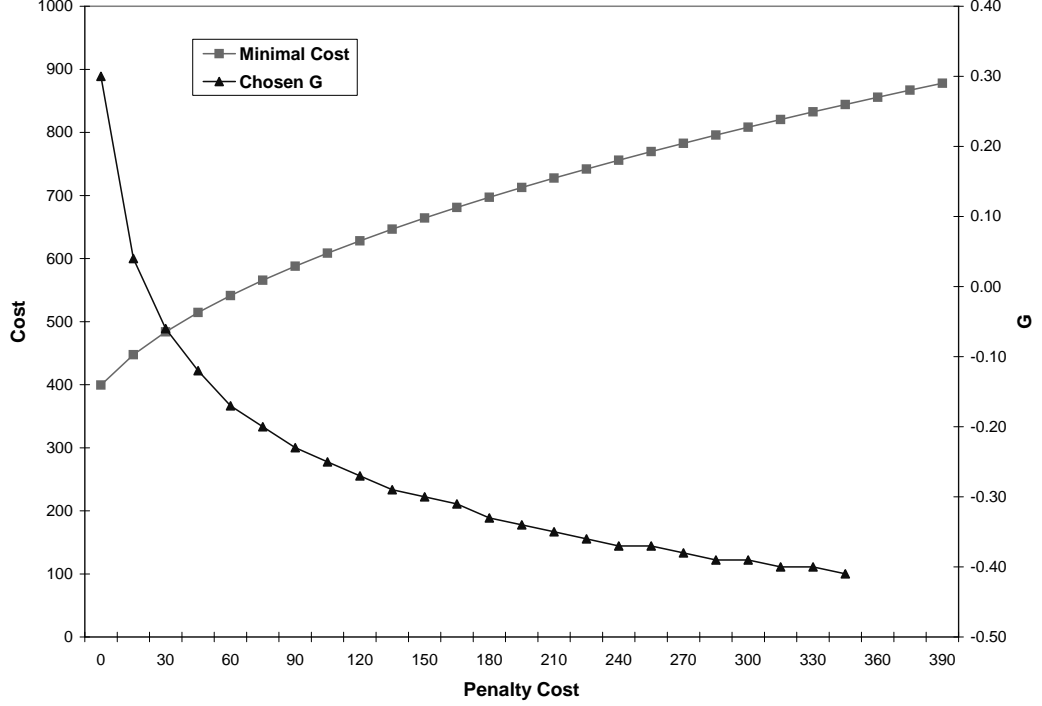


Figure 4: Expected total cost and parameter G vs. unit penalty cost p

belief that the expected total cost is strongly affected by the demand deviation. In our example, the minimal expected total cost is increased by nearly 50% when the demand deviation is doubled from $\sigma = 2$ to $\sigma = 4$. As the standard deviation of demand increases, the optimal choice of parameter G (which should be between $-(1 + \rho)$ and ρ) gradually approaches $-1/2$, which indicates that it becomes more important to dampen the bullwhip effect to a greater extent. Finally, we remark that multiple demand forecasting with input from their downstream members in a supply chain that enlarge the deviation of demand process can also create the bullwhip effect. Therefore, though the control variates technique is an effective tool for dampening the bullwhip effect, it is important to apply the forecasting technique appropriately. The relationship between the forecasting and the bullwhip effect is

thus an interesting topic that deserves further investigation. Interested readers are referred to Liu (2001) and references there.

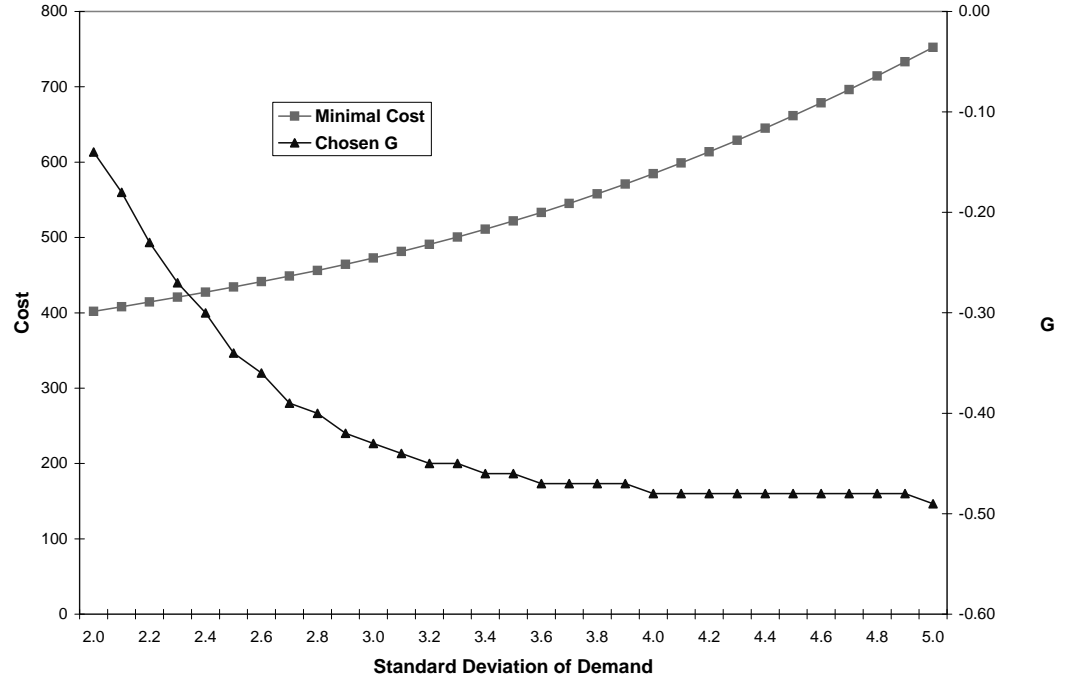


Figure 5: Expected total cost and parameter G vs. demand standard deviation σ

2.4 Concluding Remarks

In this chapter, we proposed a three-step guideline for a variance reduction technique for bullwhip effect of supply chains. For the guideline, we should emphasize that the choice of the control variate(s) is not unique and should be based on the understanding of the underlying structure of the supply chain. We also demonstrated through some examples that such technique is effective and easy to use.

We only considered two-stage supply chains as examples, and showed that the proposed

variance reduction technique leads to better performance and is easy to implement. When there are more stages in a supply chain, the impact of bullwhip effect would become more significant and our effective and easy-to-use technique to reduce the bullwhip effect would be an appealing tool to improve the system wide performance of the supply chain. We should note that our technique applies when the supply chain can be coordinated centrally, e.g., in the situation where there is a single owner of the chain or VMI (Vender Managed Inventory) is implemented. When the supply chain is managed in a decentralized manner, the investigation of the applicability of our technique (with the necessary adaptations) is an interesting problem. In this regard, games among entities in the supply chain would be involved. To this end, our results provide a benchmark for designing the game mechanism for the supply chain to achieve the system wide performance and to share the surplus. We also note that the effectiveness of our control variate technique is only demonstrated for stationary demand. The application of our technique to demand processes with seasonality is another problem worth future investigation.

CHAPTER III

DEMAND ESTIMATION AND INVENTORY CONTROL UNDER IMPERFECT INFORMATION IN SUPPLY CHAIN

3.1 Motivation

Inventory theory has been developed for more than 50 years—there are a lot of advanced theoretical results available in the literature. However, the application of these theoretical results in real-world applications is often challenging. In particular, some theoretical results cannot be applied since the assumptions in the underlying models do not actually hold. Some efficient and easy-to-use techniques should be created to solve the actual problems; this is one of main motivations of this research. The practical problems under study involve a large consulting company.

The company manages the inventory of aircraft spare parts. The inventory consists of about 60,000 items, ranging from small/inexpensive to large/expensive. Parts are stored in bins which, in turn, are arranged in bin stock locations. This study will focus on inventory management policies for the small and inexpensive parts (e.g., bolts). Since these parts are small, mechanics do not record the number of parts they retrieve from the bins. The demand process is unknown. This chapter focuses on a single item.

The company currently uses the following simplistic policy: each bin is divided into two halves (by a demarkating line). An inspector reviews the bins periodically (say, every 4 days). If a bin is more than one half full, no action is taken; otherwise, the inspector places an order for an Authorized Quantity (AQ) defined by the company based on past history. Orders are not delivered instantaneously; the lead-times are i.i.d. from an unknown distribution that depends on the item and its manufacturer or supplier.

Under their current operating policy, when an order is placed, the order quantity, and the

time it took place, and the arrival time of the order is recorded in their database. Also, the order arrival time is placed in the database. Nevertheless, the exact inventory level cannot be tracked (the parts are usually small) and the direct demand information is unknown. However, modern inventory theory implies that a one-parameter inventory policy such as AQ is not as efficient and cost-effective as other policy choices. These practical constraints along with imperfect information result in the sheer difficulties of solving the problems and improving the system performance.

3.2 Optimal Inventory Policies and Power Approximation

Let $E(L)$ and $\text{Var}(L)$ denote the mean and variance of the lead-time distribution. Replenishment costs consist of a setup cost K and a unit cost c . At the end of each review period a cost of h or p is incurred for each unit on hand or backlogged, respectively. We will assume that the demands for a specific item during different days are i.i.d. from an unspecified distribution with mean μ and variance σ^2 . The objective is the minimization of the long-run un-discounted total expected cost per day.

Under these assumptions, it is known that an (s, S) policy is optimal. That is, if during the periodic inspection the inventory position (inventory on hand plus on order), say x , is less than s , an order of $S - x$ units is placed. The computation of the reorder point s and the value S requires the complete specification of the demand distribution, and is difficult to carry out for practical implementation. For deterministic lead-times and large K and p , Roberts (1962) derives approximations that are easy to compute, but still require the knowledge of the demand distribution. In fact, Roberts showed via a renewal theoretic approach that the difference between the optimal parameters s^* and S^* approaches the well-known EOQ, that is, $D^* \equiv S^* - s^* = \sqrt{2K\mu/h} + o(D^*)$, as $D^* \rightarrow \infty$. Unfortunately, the demand distribution is rarely known and the lead-times are frequently random; in fact, managers are fortunate if they know the first two moments of these random variables. To address these issues, Ehrhardt (1979) proposes the Power Approximation (PA) method. For fixed lead-times, the PA assumes that the mean total cost per day, T , can be approximated

by a function of the form

$$T/h = cf_1(L, \theta)f_2(\mu, \theta)f_3(\pi, \theta)f_4(\kappa, \theta), \quad (14)$$

where $\theta = \sigma^2/\mu$, $\kappa = K/h$, $\pi = p/h$, $f_i(x, \theta) = x^{\gamma_i(x, \theta)} \exp[\delta_i(x, \theta)]$ ($i = 1, \dots, 4$), and $\gamma_i(x, \theta)$, $\delta_i(x, \theta)$ are linear combinations of variables from the set

$$\{1, x, 1/x, \theta, 1/\theta, x^2, 1/x^2, \theta^2, 1/\theta^2, x\theta, x/\theta, \theta/x, 1/(x\theta)\}.$$

The model (14) is fitted via regression based on optimal values obtained for a grid of 288 models based on Poisson or negative binomial demand distributions (with variance-to-mean ratios θ equal to 3 or 9), three lead-times (0, 2 and 4 days), two values for κ (32 and 64), and four values for π (4, 4, 24, and 99). In addition to the parameters s and S , the PA method provides easy-to-compute formulas for the mean holding cost per day, the mean replenishment cost per day, the mean backlog cost per day, and the long-run backlog protection (defined as the probability that a stockout does not occur during a day); see Ehrhardt (1985) for details.

The following formulas for computing the optimal values of s and S are from Ehrhardt and Mosier (1984). $\mu_L = [E(L) + 1]\mu$ is the mean demand during a lead-time and $\sigma_L^2 = [E(L) + 1]\sigma^2 + \mu^2\text{Var}(L)$ is the variance of the demand during a lead-time. The subscript “ p ” stands for “power”.

$$D_p = 1.30\mu^{0.494}\kappa^{0.506}(1 + \sigma_L^2/\mu^2)^{0.116},$$

$$z = \sqrt{D_p/(\sigma_L\pi)},$$

$$s_p = 0.973\mu_L + \sigma_L(0.183/z + 1.063 - 2.192z).$$

If $D_p/\mu > 1.5$, we let $s = s_p$ and $S = s_p + D_p$. Otherwise, we set $s = \min\{s_p, S_0\}$ and $S = \min\{s_p + D_p, S_0\}$, where $S_0 = \mu_L + \Phi^{-1}(p/(p+h))$ and $\Phi^{-1}(\cdot)$ is the inverse c.d.f. of the standard normal distribution. Since we are dealing with discrete values, s_p , D_p and S_0 must be rounded to the nearest integer.

The PA method has been used successfully in a variety of settings. It owns its popularity to its simplicity and the surprisingly good fit of the regression model.

Clearly, (s, S) policies are appealing for the problem facing by company the company. However, difficulties arise as the inventory position is not known with certainty and the daily demands are not recorded. Further, the suppliers of the company prefer to ship fixed quantities.

We plan to address the estimation of the first two moments of the daily demand by means of renewal theoretic approach so that the (s, S) policies applies approximately.

3.3 Objectives of the Proposed Research

The derivation of the optimal (s, S) policies require the complete knowledge of the specific demand distribution, which is practically unknown and needs to be estimated. It is important to investigate the impact of the estimation of the distribution to the optimality of the policy under two situations, observable demand and unobservable demand.

In the case of unknown but observable demand process, a commonly used approach is to employ replenishment formulas that are derived assuming a completely specified demand distribution, and to substitute statistical estimates for the demand distribution parameters. Limited historical data can be used to estimate the parameters of the demand distribution. An excellent paper by Jacobs and Wagner (1989) investigates how the choices of statistical estimators affect the system total cost. Their findings show that when demand variability is large, exponentially smoothed estimators can substantially outperform sample means and sample variances. When demand variability is relatively small, the cost of demand uncertainty is negligible, and the choice of statistical estimators is not critical.

In the case of unobservable (partially observable) demand distribution, the problem becomes a lot more challenging. There is little research in the literature to study estimation of this distribution and on the impact of estimation to the optimal policy under unobservable demand. This happens when the inventory position cannot be observed, for example, the personnel at the Logistics Centers of the company cannot record the quantity of each part used per day. Meanwhile, imperfect demand information makes tracking of inventory levels very difficult. Unfortunately, this inability arises when one has to manage many small and inexpensive parts.

The first objective of our research is to develop estimation methods for the demand parameters based on imperfect information of demand and inventory level. Then we will derive approximate optimal inventory policies based on the PA method. We will also investigate the impact of the estimation of distribution to the optimality of some commonly used policies.

3.4 Formulation of the Demand Estimation Problem

As we described in the previous section, the Power Approximation can be obtained based on estimators of the mean and variance of the demand and lead-time distribution. Below we will discuss how we can derive good estimation based on imperfect inventory information.

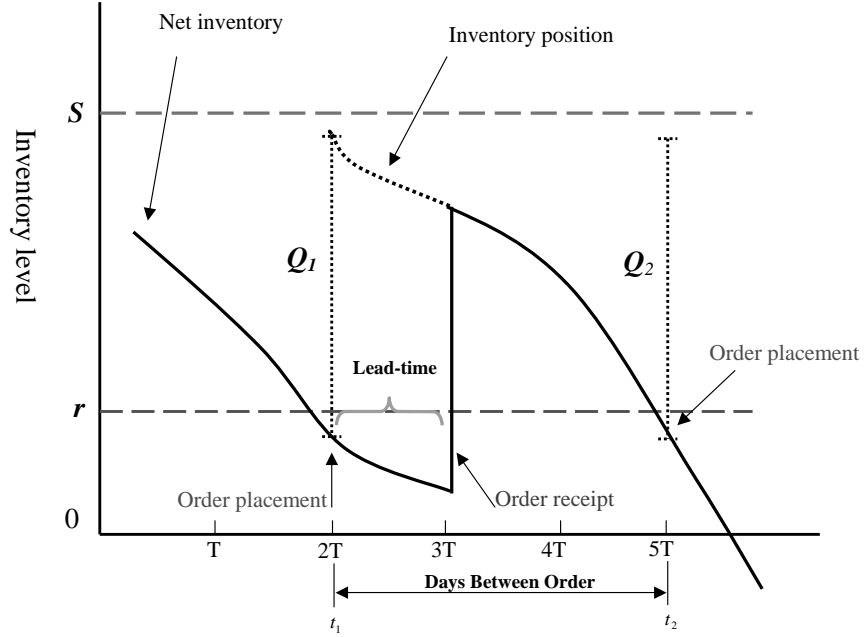


Figure 6: Illustration of order Process under periodic review

Our imperfect historical data are from the company. As we discussed earlier, their current inventory system uses a non-optimal periodic review (r, Q) inventory policy with $r = AQ/2$ and $Q = AQ$. The ordering process of the (r, Q) policy under periodic review is depicted in Figure 10. As shown in the figure, the inventory status is checked every T days. If the inventory level is less than r , a quantity Q is ordered. Even though they ideally want to order a fixed quantity each time, due to uncertainties and complexities of reality the order quantity is frequently altered. Let t_i and Q_i denote the time and order quantity that the i th order is placed. These pairs of values are recorded in a database. When an order arrives, its arrival time R_i is also recorded in the database. Only these three pieces of information are available.

Obviously, one-parameter control policies sometimes create serious problems by resulting in excessive amounts of inventory for some parts or frequent backorders for others.

The mean and variance of lead-time are also the input of the Power Approximation. However, it is trivial to obtain the historical lead-times by $R_i - t_i$, so we can easily obtain their sample mean and variance. Let $\tau_i = t_i - t_{i-1}$, denote the days between two successive orders (DBO) as shown in Figure 10 with $t_0 = 0$. As a random variable, DBO carries some information that we plan to use towards the estimation of the demand process. The next section describes our approach.

3.5 Estimation of Mean and Variance under Imperfect Information

In this section, two scenarios including constant and random order quantity are considered in Sections 3.5.2 and 3.5.3, respectively. For each scenario, the methods to approximately and asymptotically estimate the mean μ and variance σ^2 of daily demand are provided.

3.5.1 Preliminary Analysis

In order to simplify the model, we first assume that the order quantity Q_i is fixed, say Q . Meanwhile, we assume that the daily demands X_i are independent with the same mean and variance, but may have a different type of distribution. Moreover, assume that we review the inventory at the start of each day ($T = 1$) and that delivery is instantaneous.

Based on our inventory control policy, once the approximate inventory position I_i is below the reorder point r , then a order quantity of size Q will be ordered at t_i . On the other hand, during the time interval $[t_{i-1}, t_i)$ with length $\tau_i = t_i - t_{i-1}$, the relationship between accumulated demand and inventory position can be expressed as

$$I_i = I_{i-1} + Q - \sum_{j=1}^{\tau_i} X_j. \quad (15)$$

Then the accumulated demand during time interval τ_i is

$$\sum_{j=1}^{\tau_i} X_j = I_{i-1} + Q - I_i.$$

Therefore, an interval length τ can be defined by the minimal value of n for which the n th accumulated demand comes be greater than Q , that is,

$$\tau = N(Q) + 1 \equiv \min\{n : S_n = X_1 + X_2 + \cdots + X_n > Q\}, \quad (16)$$

where

$$N(Q) \equiv \max\{n : S_n = X_1 + X_2 + \cdots + X_n \leq Q\}. \quad (17)$$

The renewal function $M(Q) = E[N(Q)]$ satisfies the renewal equation

$$M(Q) = F(Q) + \sum_{y=0}^Q M(Q-y)f(y), \quad (18)$$

where $F(x)$ and $f(x)$ are the c.d.f. and p.d.f. of the random variable X . When $F(\cdot)$ is absolutely continuous, we can differentiate $M(\cdot)$ and obtain $m(\cdot)$, the renewal density. This density satisfies

$$m(Q) = f(Q) + \sum_{y=0}^Q m(Q-y)f(y). \quad (19)$$

These renewal results will be used in later section to derive the modified (r, Q) policies.

3.5.2 Constant Order Quantity

The following lemma from Feller (1949) provides the reasoning basis of the first two moments of the demand distribution for deriving the estimates.

LEMMA 3.1. *If the random variables X_1, X_2, \dots have finite mean $E[X_i] = \mu$ and variance $\text{Var}[X_i] = \sigma^2$, and τ is defined by (16), then $E[\tau]$ and $\text{Var}[\tau]$ are given by*

$$E[\tau] = \frac{Q}{\mu} + \frac{\sigma^2 + \mu^2}{2\mu^2} + o(1) \quad \text{as } Q \rightarrow \infty \quad (20)$$

and

$$\text{Var}[\tau] = \frac{Q\sigma^2}{\mu^3} + o(1) \quad \text{as } Q \rightarrow \infty, \quad (21)$$

respectively.

The next theorem provides the asymptotic distribution of τ . Its proof is a trivial extension to Theorem 3.3.5 in Ross (1996).

THEOREM 3.1. *Under the assumptions of Lemma 3.1, τ has the asymptotic normal distribution with mean Q/μ and variance $Q\sigma^2/\mu^3$:*

$$\tau \xrightarrow{d} N(Q/\mu, \sqrt{Q\sigma^2/\mu^3}) \quad \text{as } Q \rightarrow \infty.$$

According to Theorem 2.7.1 of Lehmann (1990), the theorem still holds even when the daily demands are not identically distributed, but are independent with finite third moments.

The next theorem provides the estimates of mean and variance followed from the lemma above. We start with an auxiliary lemma, which follows from the continuous mapping theorem and Slutsky's theorem, see Lehmann (1990).

LEMMA 3.2. *If X_n and Y_n are two sequences of random variables satisfying $X_n \xrightarrow{P} c$ and $Y_n \xrightarrow{P} d$, respectively, then*

$$\frac{X_n}{Y_n} \xrightarrow{P} \frac{c}{d} \quad \text{provided } d \neq 0. \quad (22)$$

THEOREM 3.2. *Assume that τ_1, \dots, τ_m is an independent sample of times between orders and that the daily demand are i.i.d. Using the first term of the r.h.s.'s of (20) and (21) we propose the following estimators for μ and σ^2 :*

$$\hat{\mu} = Q/\bar{\tau}(n) \quad (23)$$

and

$$\widehat{\sigma^2} = S_\tau^2(n)Q^2/\bar{\tau}^3(n), \quad (24)$$

where $\bar{\tau}(n)$ and $S_\tau^2(n)$ are the sample mean and sample variance, respectively, of the τ_i .

Furthermore, as $Q \rightarrow \infty$ and $n \rightarrow \infty$

$$\hat{\mu} = \frac{Q}{\bar{\tau}(n)} \xrightarrow{P} \mu \quad \text{and} \quad \widehat{\sigma^2} \xrightarrow{P} \sigma^2. \quad (25)$$

Proof. From Theorem 3.1, we have known that τ has an asymptotic normal distribution with mean Q/μ and variance $Q\sigma^2/\mu^3$ as $Q \rightarrow \infty$. From the weak law of large numbers, the sample mean $\bar{\tau}(n)$ and the sample variance $S_\tau^2(n)$ converge in probability to the respective mean and variance as the sample size increases. Using Lemma 3.2, it is easy to prove that

$$\hat{\mu} = \frac{Q}{\bar{\tau}(n)} \xrightarrow{P} \mu \quad \text{and} \quad \widehat{\sigma^2} \xrightarrow{P} \sigma^2. \quad (26)$$

□

3.5.3 Random Order Quantity

In reality, as the price of materials, labor and many other factors in market change, the inventory manager may alter the order quantity. Therefore the order quantity Q can somehow exhibit unpredictable behavior. So in this section, we consider asymptotic estimators by assuming that the order quantity Q_i at order placement epochs form an i.i.d. sequence.

3.5.3.1 Method I

From Lemma 3.1, we already have known the conditional mean and variance of τ when a given Q is large. We can use this information to derive the unconditional mean and variance. Hence, we have

$$\mathbb{E}[\tau] = \mathbb{E}_Q[\mathbb{E}[\tau|Q]] \approx \frac{\mathbb{E}[Q]}{\mu} \quad (27)$$

and

$$\begin{aligned} \text{Var}[\tau] &= \mathbb{E}_Q[\text{Var}(\tau|Q)] + \text{Var}_Q[\mathbb{E}(\tau|Q)] \\ &\approx \mathbb{E}\left[\frac{Q\sigma^2}{\mu^3}\right] + \text{Var}\left[\frac{Q}{\mu}\right] \\ &\approx \frac{\sigma^2\mathbb{E}[Q]}{\mu^3} + \frac{\text{Var}[Q]}{\mu^2}. \end{aligned} \quad (28)$$

By replacing the mean and variance by the sample mean and variance, we can easily obtain the estimators of the mean and the variance of the daily demand.

3.5.3.2 Method II

Before we provide the other approximate method, we first show a lemma from Janson (1983).

LEMMA 3.3. *Let X_1, X_2, \dots be i.i.d. with $X \geq 0$ a.s. and $E[X^r] < \infty$ for $1 \leq r < \infty$. Then as $Q \rightarrow \infty$ we have $E[(S_\tau - Q)^{r-1}] = O(1)$ where τ and S_τ are defined in (16).*

Lemma 3.3 indicates that as Q becomes large, the difference between Q and S_τ becomes smaller. Since only Q is observable, we can replace S_τ with Q to approximately estimate the mean and variance of daily demand.

DEFINITION 3.1. *An integer valued random variable N is said to be a stopping time for the sequence X_1, X_2, \dots if for each n the event $\{N = n\}$ depends only on X_1, X_2, \dots, X_n .*

First we derive the first moment of S_τ based on the Wald's equation, see Ross (1996).

LEMMA 3.4. *Suppose that*

- (a) X_1, X_2, \dots are i.i.d. random variables having expectation μ , and
- (b) there is a finite A such that $E[|X_i|] \leq A$ for all i .

If N is a stopping time for X_1, X_2, \dots with mean $E[N]$, then the mean of $S_N = \sum_{i=1}^N X_i$ is given by

$$E[S_N] = E\left[\sum_{i=1}^N X_i\right] = \mu E[N]. \quad (29)$$

Next we turn to the derivation of the second moment of S_τ . We start with the following lemma from Johnson (1959).

LEMMA 3.5. *In addition to assumptions (a) and (b) in Lemma 3.4, assume that*

- (c) $\text{Var}[X_i] = \sigma^2 < \infty$,
- (d) there is a constant $B < \infty$ such that $E[(X_j - \mu)^2 | N \geq i] \leq B$ for all $j < i$, and

(e) $E[N^2] < \infty$.

Then the second moment of $Z_N \equiv S_N - N\mu$ is given by

$$E[Z_N^2] = \sigma^2 E[N]. \quad (30)$$

The following theorem captures the covariance between the stopping time and the corresponding random sums.

THEOREM 3.3. *Under the conditions in Lemmas 3.4 and 3.5, we have*

$$\sigma^2 E[N] = \text{Var}[S_N] - 2\mu \text{Cov}[N, S_N] + \mu^2 \text{Var}[N]. \quad (31)$$

Proof. From Lemmas 3.4 and 3.5, we have

$$\begin{aligned} \sigma^2 E[N] &= E[Z_N^2] = E[(S_N - N\mu)^2] \\ &= E[S_N^2] - 2\mu E[NS_N] + \mu^2 E[N^2] \\ &= E[S_N^2] - 2\mu(E[NS_N] - E[N]E[S_N]) + \mu^2 E[N^2] - 2\mu E[N]E[S_N] \\ &= \text{Var}[S_N] - 2\mu \text{Cov}[N, S_N] + \mu^2 \text{Var}[N]. \end{aligned}$$

□

Based on assumptions of our model and historical data, we have the sequence of order quantities and corresponding stopping times Q_1, Q_2, \dots, Q_m and $\tau_1, \tau_2, \dots, \tau_m$, respectively, where τ_i is defined in (16) based on the order quantity Q_i . Obviously, Q_i and τ_i are dependent. Recall that the Q_i 's were assumed to be independent.

Based on Lemma 3.4 and Theorem 3.3, one can estimate the mean daily demand by

$$\hat{\mu} = \frac{\bar{Q}(n)}{\bar{\tau}(n)} \quad (32)$$

and the variance of the daily demand by

$$\widehat{\sigma^2} = \frac{1}{\bar{\tau}(n)} S_Q^2(n) - \frac{2\bar{Q}(n)}{\bar{\tau}^2(n)} S_{Q,\tau}(n) + \frac{\bar{Q}^2(n)}{\bar{\tau}^3(n)} S_\tau^2(n), \quad (33)$$

where $S_{Q,\tau}(n)$ is the sample covariance between the Q_i 's and the τ_i 's.

3.6 *Implementation of the Power Approximation Policy*

Once the estimates of mean and variance of daily demand and lead-time are obtained, they can be applied to the Power Approximation formulas. The approximate optimal reorder point s and order-up-to level S are then available to be applied. Since the current inventory position is not observable, such an order-up-to level policy cannot be applied directly. Instead, we need to implement an (r, Q) policy with fixed order quantity Q . In this section, we will discuss how to obtain an (r, Q) policy based on the derived (s, S) policy so that the total cost of the (r, Q) policy is close to that of the (s, S) policy.

We choose the same numerical examples from Veinott and Wanger (1965) with linear holding and backlogging cost, zero lead-time, and Poisson-distributed daily demand. In their paper, they provide optimal values for s and S , and the total cost based on the known information on the demand distribution. The system parameters and optimal values are presented in Table 4, with the mean demand listed in column 1.

3.6.1 **Direct (r, Q) Policy**

An intuitive and direct transformation of an (s, S) policy to an (r, Q) policy is to use $r = s$ as the reorder point and $Q = S - s$ as the order quantity. The direct (r, Q) policy and the respective cost are shown in the columns 4 and 5 of Table 4. Unfortunately, the inventory position prior to the placement of an order will typically be lower than r and the fixed order quantity Q will bring it under S .

3.6.2 **Modified (r, Q) Policy**

In this section we will compute an approximation for the inventory position prior to order placements.

Define the excess random variable as $B(Q) = S_{N(Q)+1} - Q$. We denote the distribution function of $B(Q)$ by $H(Q, z) = \Pr\{B(Q) = z\}$. It is well-known that $H(Q, z)$ satisfies following integral equation

$$H(Q, z) = F(Q + z) - F(Q) + \sum_{y=0}^Q H(Q - y, z)f(y) \quad (34)$$

and its mean is given by

$$E[B(Q)] = \mu(1 + M(Q)) - Q. \quad (35)$$

See Ross (1996) for details. From Lemma 3.1 we have

$$M(Q) = \frac{Q}{\mu} + \frac{\sigma^2 + \mu^2}{2\mu^2} - 1 + o(1) \quad \text{as } Q \rightarrow \infty. \quad (36)$$

We proceed with an easy-to-use and effective adjusted (r, Q) policy that is based on the above renewal results.

Assume that the daily demands X_i are i.i.d. with mean μ and variance σ^2 , deliveries are instantaneous, and suppose that optimal values s and S have been computed. Under an (s, S) policy, Karlin (1958) has shown that the limiting distribution of the inventory position has probability function

$$g(x) = \begin{cases} \frac{m(S-x)}{1+M(Q)}, & s < x \leq S \\ \frac{h(Q, s-x)}{1+M(Q)}, & x \leq s, \end{cases} \quad (37)$$

where $Q = S - s$, and $M(\cdot)$ and $m(\cdot)$ are given in equations (18) and (19). $h(Q, x)$ is the density of $H(Q, x)$ defined in equation (34). Using the stationary distribution in equation (37), one can compute the asymptotic expected inventory position when an order is placed:

$$\begin{aligned} \lim_{Q \rightarrow \infty} E[xI(x \leq s)] &= \lim_{Q \rightarrow \infty} \sum_{x=-\infty}^s xh(Q, s-x) \\ &= \lim_{Q \rightarrow \infty} \left\{ \sum_{x=0}^{\infty} sh(Q, x) - \sum_{x=0}^{\infty} xh(Q, x) \right\} \\ &= \lim_{Q \rightarrow \infty} \{s - \mu[1 + M(Q)] + Q\} \\ &= s - \frac{\sigma^2 + \mu^2}{2\mu}. \end{aligned}$$

For simplicity we disregard the term $\sigma^2/(2\mu)$ and use the adjusted order quantity $Q = S - s + \mu/2$.

This approximation is an asymptotic result when the direct order quantity ($Q = S - s$) is sufficiently large. However, in the case of a small order quantity, such a modified (r, Q) policy may deviate from the optimal policy significantly. Further adjustment and improvement should be done to deal with such cases.

Table 4: Comparison of optimal (s, S) policy and adjusted (r, Q) policy based on the experiments in Veinott and Wagner (1965)

μ	Optimal (s, S)		Direct (r, Q)		Improved (r, Q)		Optimal (r, Q)	
	(s, S)	Cost	(r, Q)	Cost	(r, Q)	Cost	(r, Q)	Cost
21	(15, 65)	50.410	(15, 50)	51.411	(15, 61)	51.157	(15, 56)	50.992
22	(16, 68)	51.630	(16, 52)	52.508	(16, 63)	52.416	(16, 57)	52.193
23	(17, 52)	52.757	(17, 35)	61.532	(17, 55)	53.504	(17, 59)	53.354
24	(18, 54)	53.514	(18, 36)	62.540	(18, 56)	54.659	(18, 60)	54.516
51	(43, 110)	71.612	(43, 67)	82.510	(43, 93)	79.709	(43, 87)	79.495
52	(44, 112)	72.249	(44, 68)	83.180	(44, 94)	80.489	(43, 89)	80.236
55	(47, 118)	74.165	(47, 71)	85.185	(44, 99)	82.995	(44, 90)	82.582
59	(51, 126)	76.679	(51, 75)	87.770	(51, 105)	86.238	(51, 92)	85.566
61	(52, 131)	77.933	(52, 79)	88.655	(52, 110)	87.884	(52, 97)	86.928
63	(54, 73)	78.290	(54, 19)	∞	(54, 90)	88.802	(54, 99)	88.376
64	(55, 74)	78.414	(55, 19)	∞	(54, 91)	89.263	(54, 97)	89.047

3.6.3 Improved (r, Q) Policy

Based on these optimal values and previous results, we further adjust the order quantity to

$$Q = \max\{S - s + \mu/2, EOQ = \sqrt{2K\mu/h}\}. \quad (38)$$

This empirical adjustment is motivated from the argument that optimal order quantity should be around EOQ to allow for trading off between setup cost and holding cost. The respective (r, Q) policy and cost are shown in the columns 6 and 7 of Table 4.

To evaluate the performance of the further improved (r, Q) policy, we search for an optimal (r, Q) policy and the corresponding cost by running 100 replications of a simulation model over 10 years. The values of r and Q were chosen from the neighborhoods of s^* and $S^* - s^*$, where s^* and S^* are given in the column 2. The simulated results are shown in the last two columns. These numerical results show that the adjusted based on equation (38) works much better than the (r, Q) policies with $r = s$ and $Q = S - s$, and is close to an optimal (r, Q) policy.

CHAPTER IV

NUMERICAL STUDIES OF ESTIMATION ACCURACY AND POLICY PERFORMANCE

4.1 Introduction

In Chapter 3, we derived asymptotic estimators for the first two moments of the daily demand by assuming that partial demand information is known. In this chapter, we consider the same scenarios as the previous chapter. However, our objective is to evaluate the estimation of the first two moments of the daily demand and study the cost effectiveness by applying these estimates based on the Power Approximation (PA) and simulation.

The chapter is organized as follows. First we setup an experimental design based on demand distribution, lead-time, cost structure and constant order quantity; this design grid will be used for the simulation studying and is analogous to those used in Ehrhardt (1979). Then we collect the sample of observed Days-Between-Orders (DBO) from the simulation to estimate the mean and variance of the daily demand by the algorithm derived in Chapter 3. We evaluate the accuracy of the estimates by means of several statistical measures.

The PA method uses these estimates to derive a near-optimal inventory policy, i.e., a reorder point and an order-up-to level. By applying the nearly optimal inventory policy for each case, it is possible to estimate the average total cost. We benchmark the performance of our algorithm in terms of average total cost by means of a case in which the first two moments of the daily demand is fully known. Thus in the following numerical studies, the numerical results under partial information relate to the practical use of our model, while those under full information is used for benchmarking purpose.

Based on the estimates presented in Chapter 3, we also propose a regression model to improve the accuracy of variance estimate. The exact same procedures above are applied to evaluate the accuracy of the adjusted estimate and the cost effectiveness of this method

on the average total cost. We also test its performance by considering extrapolation of parameter values beyond the ranges used in the derivation of the regression model.

All the detailed numerical results are presented in the Appendix B.

4.2 *Experiment Design and Simulation*

In this section, we present the system parameters for the simulation model.

We remind the reader that μ and σ^2 denote the mean and variance of the underlying daily demand, respectively. Further, K , p and h are as the replenishment setup cost, unit penalty cost and unit holding cost, respectively. In addition, the replenishment lead-time is denoted by L and the constant order quantity is denoted by Q .

4.2.1 **Experiment Design**

Before discussing the performance of estimation of the daily demand, we present the parameters used in this study.

A grid of 216 inventory cases has been specified to generate data for the analysis; Table 5 lists the parameter assignments. Three types of demand distribution are used: Poisson, and negative binomial with variance-to-mean ratios of 3 and 5. Research has indicated that the negative binomial distribution fits closely the observed distribution of consumer purchases in a single day, and purchases of a given consumer in successive days will follow the Poisson distribution (Chatfield et al. 1966). As a result, these distributions are thought to more closely model the demand of most inventory systems. Each demand distribution is given two mean values, 8 and 16. Two values, 2 and 4, are assigned to lead-time. Since the cost function is linear in the parameters K , p , and h , the value of the unit holding cost is a redundant parameter which is set at unity. The unit penalty costs are 4, 24 or 99, and the setup costs are 32 or 64. The unit replenishment cost is unspecified because it does not effect the computation of an optimal policy for undiscounted and infinite horizon models. Based on the system parameters above and the well-known EOQ model, we calculate the minimal and maximal order quantity of all combinations with the minimum being around 25 and the maximum being around 46. Therefore, the constant order quantity is specified

Table 5: System parameters

Factor	Levels	Number of Levels
Demand distribution	Poisson Negative Binomial ($\sigma^2 = 3\mu$) Negative Binomial ($\sigma^2 = 5\mu$)	3
Mean demand	8, 16	2
Replenishment lead-time (L)	2, 4	2
Replenishment setup cost (K)	32, 64	2
Unit penalty cost (p)	4, 24, 99	3
Unit holding cost (h)	1	1
Order quantity (Q)	20, 40, 80	3

at three values: 20, 40, and 80. All combinations of these parameters settings are included in the grid, which yields 216 cases.

4.2.2 Description of Simulation

The observed daily demand, observed DBO and average total cost for each of the 216 cases of the inventory system are obtained through simulation. Since all the observed data are simulated, we use the terms observed and simulated interchangeably.

First, the system parameters for each case are input by calling the initialization function and the simulation clock is started. Aside from the system parameters in Section 4.2.1, we assume that an (r, Q) policy is applied with a reorder point equal to 15 for all of the 216 cases. The simulation clock is used to determine the next event and the respective event type including demand arriving, order placement, order receiving, and periodic inventory reviewing. The simulation is running until the clock reaches the simulation end time, which is long enough to guarantee the inventory system will be stable. To this end, we specify a simulation horizon of 2 years.

The PA function is called to calculate the reorder point and order quantity, which are then used to replace the original reorder point and order quantity for each case. With these two new values along with the existing system parameters, we will estimate the average total cost over 5 years for each case including holding, penalty and setup cost.

It is worth noting that for each case, the same random number generator seed is always

used in each simulation to generate the sequences of demands needed allowing us to obtain the historical daily demand and historical DBO.

The flow chart in Figure 7 depicts the evaluation process within one simulation. In order to reduce the variation coming from the random generator, we repeat the simulation 100 times.

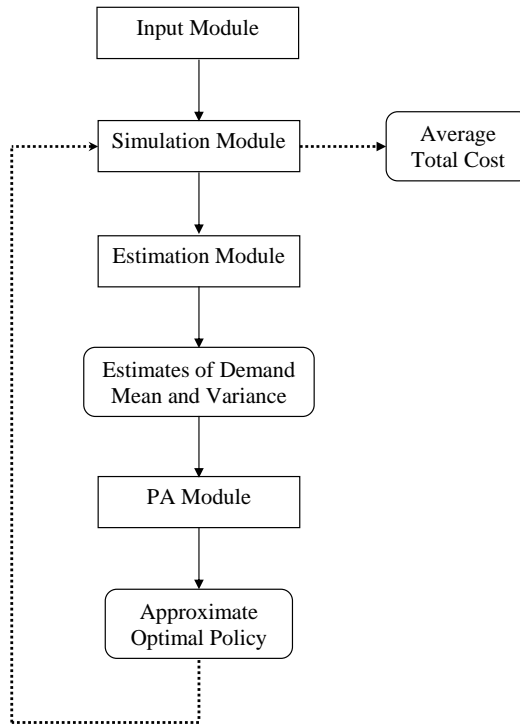


Figure 7: Flow chart of the evaluation process

4.3 Performance Evaluation

In this section, we proceed with a thorough evaluation of the performance of our estimates and the cost effectiveness by applying the PA and our algorithm.

We first examine the statistical properties of the estimates. Several standard statistical measurements are applied to verify the accuracy and precision of the demand estimators. We use following notation. Let n denote the number of observations within one simulation run, and let r denote the number of independent replications. As we mentioned before, we set $r = 100$. Moreover, since we fix the simulation time, the number of observed DBO from one simulation is random but the number of observations for daily demands is equal to 730.

4.3.1 Evaluation of Estimation

As described in the previous chapter, the objective of this research is to study the case in which the full demand information is unknown, with the days between two successive orders being the only known information. Then based on this partial information, we will derive the estimates of the first two moments of the daily demand. In this section, we propose some standard statistical measurements to evaluate the estimate error.

First, we define some notations needed. Let $\tau_{i,j}$, $j = 1, \dots, n$, denote the observed DBO during the i th replication. The estimated mean within the i th replication is $\hat{\mu}_i = Q/\bar{\tau}_i$, where $\bar{\tau}_i = \sum_{j=1}^n \tau_{i,j}/n$. The estimate of daily demand mean over all samples is $\hat{\mu} = Q/\bar{\tau}$, where $\bar{\tau} = \sum_{i=1}^r \bar{\tau}_i/r$.

Let $S_{\tau,i}^2$ denote the sample variance of DBO within the i th replication. Then by Theorem 3.2, the estimate of the daily demand standard deviation is given by

$$\hat{\sigma}_i = \sqrt{S_{\tau,i}^2 Q^2 / \bar{\tau}_i^3}. \quad (39)$$

The overall estimate of the demand standard deviation obtained by combining all replications is

$$\hat{\sigma} = \frac{1}{r} \sum_{i=1}^r \hat{\sigma}_i. \quad (40)$$

In addition, we define the overall sample variance of historical DBO by $S_{\tau}^2 = \sum_{i=1}^r S_{\tau,i}^2/r$.

In order to evaluate the accuracy and precision of the estimates, we use the following measures. The first measure is the Relative Root Mean Squared Error (RRMSE) of the mean estimate, which is denoted by

$$\text{RRMSE}_\mu = \sqrt{\frac{\sum_{i=1}^r (\hat{\mu}_i - \mu)^2}{r}} / \mu. \quad (41)$$

RRMSE is also called the standard error of the estimate and measures the spread of the estimate from the true value. The second measure is called the Relative-Standard-Deviation (RSD) of the mean estimate, which is denoted by

$$\text{RSD}_\mu = \sqrt{\frac{\sum_{i=1}^r (\hat{\mu}_i - \hat{\mu})^2}{r - 1}} / \mu. \quad (42)$$

Moreover, we define the Relative Bias (RBias) by

$$\text{RBias}_\mu = (\hat{\mu} - \mu) / \mu, \quad (43)$$

which measures the relative precision of the estimate. The detailed numerical results are labeled and presented in Tables in Appendix B.

We calculate the average of absolute RBias, RSD and RRMSE according to the level of coefficient of variation σ/μ and Q with other parameters being fixed. We also calculate the maximum, minimum, and overall average of absolute RBias, RSD and RRMSE; we present the summary in Table 6.

From Table 6, we can clearly draw the conclusion that the estimate of demand mean under partial information works as well as that under full information since the differences of all three measurements are small. For example, the average of RRMSE under full information is 0.0186 and that under partial information is 0.0187. Furthermore, the difference of the average of absolute RBias between partial information and full information is only 0.0001.

The constant order quantity has negligible effect on the estimate of demand mean. The demand parameters slightly affect on the performance of mean estimate. It is interesting to see that as the ratio σ/μ increases, the average of absolute RBias and RSD slightly increase as well; this behavior is expected.

Table 6: Summary of mean estimates under full and partial information

Parameter		Full Information			Partial Information		
		RBias	RSD	RRMSE	RBias	RSD	RRMSE
σ/μ	0.250	0.0007	0.0092	0.0092	0.0007	0.0092	0.0093
	0.354	0.0011	0.0132	0.0133	0.0011	0.0132	0.0133
	0.433	0.0015	0.0161	0.0162	0.0015	0.0162	0.0163
	0.559	0.0018	0.0208	0.0209	0.0020	0.0208	0.0210
	0.612	0.0019	0.0227	0.0228	0.0021	0.0227	0.0229
	0.791	0.0021	0.0291	0.0292	0.0023	0.0294	0.0295
Q	20	0.0014	0.0188	0.0188	0.0016	0.0188	0.0189
	40	0.0014	0.0184	0.0185	0.0016	0.0185	0.0186
	80	0.0016	0.0183	0.0184	0.0017	0.0185	0.0186
Max		0.0067	0.0342	0.0342	0.0074	0.0342	0.0344
Min		-0.0041	0.0079	0.0079	-0.0034	0.0080	0.0081
Average		0.0015	0.0185	0.0186	0.0016	0.0186	0.0187

Similarly, we define the error measures for standard deviation estimate as follows:

$$\text{RRMSE}_\sigma = \sqrt{\frac{\sum_{i=1}^r (\hat{\sigma}_i - \sigma)^2}{r}} / \sigma, \quad (44)$$

$$\text{RSD}_\sigma = \sqrt{\frac{\sum_{i=1}^r (\hat{\sigma}_i - \hat{\sigma})^2}{r-1}} / \sigma, \quad (45)$$

and $\text{RBias}_\sigma = (\hat{\sigma} - \sigma) / \sigma$. The numerical comparisons between full and partial information are displayed in Tables in Appendix B. Similarly, we summarize the numerical results and present them in Table 7.

First we observe that the estimate of the standard deviation based on partial information performs worse than that under full information. Since the average of RSD under partial information is close to that under full information, the error mainly comes from RBias. The overall average RBias from partial information almost 50 times larger than that from full information.

Next we focus our efforts on analyzing the performance of the estimate from partial information. Based on the relationship among the RBias, RSD and RRMSE, we note that RSD does not vary significantly and that the RRMSE is primarily determined by RBias. Therefore we can focus our analysis on RBias as it is the dominant contributor. From Table 7, we can conclude that both the constant order quantity Q and demand parameters

Table 7: Summary of the standard deviation estimate with full and partial information

Parameter		Full Information			Partial Information		
		RBias	RSD	RRMSE	RBias	RSD	RRMSE
σ/μ	0.250	0.0023	0.0266	0.0267	0.4764	0.0580	0.4817
	0.354	0.0038	0.0277	0.0280	0.1471	0.0709	0.1699
	0.433	0.0023	0.0307	0.0309	0.1834	0.0553	0.1950
	0.559	0.0031	0.0350	0.0352	0.1066	0.0537	0.1243
	0.612	0.0042	0.0348	0.0353	0.0492	0.0671	0.0889
	0.791	0.0043	0.0411	0.0415	0.0253	0.0703	0.0765
Q	20	0.0034	0.0329	0.0332	0.2638	0.0493	0.2734
	40	0.0033	0.0326	0.0329	0.1517	0.0593	0.1723
	80	0.0033	0.0324	0.0327	0.0784	0.0791	0.1224
Max		0.0178	0.0491	0.0498	0.7476	0.0998	0.7490
Min		-0.0067	0.0214	0.0214	0.0035	0.0362	0.0594
Average		0.0033	0.0327	0.0329	0.1647	0.0626	0.1894

characterized by σ/μ significantly affect the performance of the estimate of the standard deviation. As the constant order quantity Q increases, RBias, RSD and RRMSE decrease quickly. When the order quantity Q is large enough, our estimate of standard deviation works well with the average RBias being only 0.0784. The impact of the demand coefficient of variation σ/μ on the estimate is more complex. Overall, the performance for the negative binomial model is better than for the Poisson demand model. The maximal RBias is 0.7476 when the demand is Poisson distributed and the order quantity is 20. For the same type of demand distribution, when the mean of demand is smaller, RBias is smaller. Especially for Poisson demand, when the demand mean is 8 and 16, RBias is 0.1471 and 0.4764, respectively. In general, as the ratio σ/μ goes up, the RBias exhibits downward trend even though this trend is not monotone. This suggests that there is much room to improve the estimation of the demand variation σ^2 for our problem; topic is worth additional research.

4.3.2 Evaluation of Cost Performance

Next we examine the impact of demand estimation to cost estimates by comparing the cases in which true and full demand information are known.

Let C^* denote the average total cost for a given case when controlled using the PA method with known mean and variance of daily demand. Here we need to point out that

the term optimal PA policy refers to the (s, S) policy, not our suboptimal (r, Q) policy.

Additionally, let $C_{p_0,i}$ denote the average total cost for a specific case from the i th replication that uses the estimated mean and variance of demand under full information. The average from all replications is denoted by $C_{p_0}^* = \sum_{i=1}^r C_{p_0,i}/r$. Similarly, $C_{p_1,i}$ denotes the respective average total cost from the i th replication based on the estimated mean and variance under partial information. The overall average is denoted by $C_{p_1}^* = \sum_{i=1}^r C_{p_1,i}/r$.

First we examine the policy performance based on the known mean and variance of the daily demand. Similar to the Section 4.3.1, we also use these the measures: RRMSE, RSD and RBias to evaluate the policy performance in terms of average total cost by simply replacing $\hat{\mu}_i$, $\hat{\mu}$ and μ by $C_{p_0,i}$, $C_{p_1,i}$, $C_{p_0}^*$, $C_{p_1}^*$ and C^* , respectively. The numerical results are presented in Tables 34 to 39 (Appendix B), and the summary is presented in Table 8. When the distribution of the daily demand is known, the average total cost is very close to the nearly optimal average total cost. The average RBias from all 216 cases is very small, only 0.0009. Since in most of the 216 cases the demand variance under partial information is overestimated, most of these 216 cases are associated with higher average total cost than the nearly optimal cost. The average of RBias under partial information is 0.047. As the constant order quantity increases, the RBias under partial information decreases. Overall, the cost performance under a negative binomial demand is better than under a Poisson demand. In the worst case, the RBias is 0.3762.

In practice, one uses demand history and then estimates the mean and variance of the demand. Ehrhart (1979) tested the cost performance of the PA method by substituting estimates of the demand mean and variance in place of the actual mean and variance by simulating 72 systems, each having a negative binomial demand distribution with $\sigma^2/\mu = 9$. He found that using classical estimates with a year's worth of weekly demand history resulted in an aggregated cost being only 6% above the actual optimal cost for known demand parameters.

Therefore, we propose the following performance measure for each simulation run

$$\Delta_{p_1,i} = \frac{(C_{p_1,i} - C_{p_0,i}) \times 100}{C_{p_0,i}}, \quad i = 1, \dots, r, \quad (46)$$

Table 8: Summary of the average total cost under full and partial information

Parameter		Full Information			Partial Information		
		RBias	RSD	RRMSE	RBias	RSD	RRMSE
σ/μ	0.250	-0.0011	0.0144	0.0156	0.1043	0.0213	0.1095
	0.354	-0.0007	0.0167	0.019	0.0275	0.0226	0.0404
	0.433	0.0011	0.0226	0.0236	0.0690	0.0310	0.0780
	0.559	0.0037	0.029	0.0298	0.0523	0.0369	0.0671
	0.612	0.0067	0.0257	0.0278	0.0233	0.0341	0.043
	0.791	-0.0046	0.0326	0.0336	0.0059	0.0429	0.0443
Q	20	0.0009	0.0239	0.0253	0.0766	0.029	0.0873
	40	0.0009	0.0235	0.0248	0.0429	0.0306	0.0584
	80	0.0008	0.0231	0.0246	0.0217	0.0348	0.0454
Max		0.0244	0.05	0.05	0.3762	0.0734	0.378
Min		-0.0177	0.0098	0.0103	-0.0135	0.0084	0.0085
Average		0.0009	0.0235	0.0249	0.0471	0.0315	0.0637

Table 9: Frequencies of average $\Delta_{p_1,i}$ under partial information

Range for $\Delta_{p_1,i}$	Number of Cases	Percentage of Cases
$[-\infty, -1.5\%)$	0	0.0%
$[-1.5\%, 1.5\%)$	74	34.3%
$[1.5\%, \infty)$	142	65.7%

namely, the percentage by which the average total cost under partial information exceeds that under full information. The average and standard deviation of $\Delta_{p_1,i}$ over the r replications are displayed in Tables 40 to 42 (Appendix B). Our results for the 216 cases are summarized in Table 9, which lists the number of cases in the system having values of average $\Delta_{p_1,i}$ in various ranges. If the cases are in the range of $[-1.5\%, 1.5\%)$, then we assume that the optimization under partial information are as good as the optimization under full information. There are 74 cases which are in the range $[-1.5\%, 1.5\%)$, but 66% of the 216 cases are outside the range.

4.4 Regression Based Approximation (RBA)

In this section, we propose an adaptive method to improve the estimate of the demand variance by using a least square regression model. Afterwards, additional testing procures

are presented as well.

Based on the analysis in Chapter 3, equation (39) holds only as $Q \rightarrow \infty$. This fact is supported by the numerical results. When Q is small with respect to the mean of the demand, the estimate of demand variance does not work well. We propose a regression method to improve our estimate of demand variance based on equation (40). Towards to this goal, we generalize expression (40) to the following multiplicative form

$$\sigma^2 = C (S_\tau^2)^\alpha Q^\beta / (\bar{\tau})^\gamma, \quad (47)$$

where C , α , β , and γ are constants to be fitted. We form a linear model by taking the logarithm of equation (47) and use least squares regression to fit the model to our 216 values for σ^2 . The independent variables are the sample average of DBO, the sample variance of DBO, and the fixed order quantity Q . The true variance of the daily demand is used as the dependent variable. The regression model generates the following estimator for σ^2 :

$$\sigma_p^2 = 0.7418 (S_\tau^2)^{1.2685} Q^{2.0012} / (\bar{\tau})^{3.0060}. \quad (48)$$

Note that the exponent of the order quantity Q is close to 2 and the exponent of the overall average of DBO is close to -3 . However, the exponent of the sample variance of DBO is slightly larger than 1, and the model includes a multiplier that is less than 1. The regression model in equation (48) has a coefficient of determination $R^2 = 0.9633$.

4.4.1 Evaluation of Estimation and Cost Performance

In this section, we will evaluate the RBA estimate. Applying equation (48), we can obtain the estimated variance; then it is possible to estimate the average total cost. As in Section 4.3, we use three measures, RBias, RSD and RRMSE, to evaluate the performance of the variance estimate and the average total cost. The numerical results are displayed in the last three columns of Tables 28 to 39 (Appendix B), respectively. We summarize the performance of the (demand) standard deviation estimate and the average total cost in Tables 10 and 11, respectively.

From Table 10, we can observe that RBA significantly improves the performance of the standard deviation estimate. The average of RBias under partial information decreases

Table 10: Summary of the standard deviation estimate under partial information and RBA

Parameter		Partial Information			RBA		
		RBias	RSD	RRMSE	RBias	RSD	RRMSE
σ/μ	0.250	0.4764	0.0580	0.4817	0.1165	0.0545	0.1388
	0.354	0.1471	0.0709	0.1699	0.0377	0.0759	0.0858
	0.433	0.1834	0.0553	0.1950	0.0362	0.0558	0.0686
	0.559	0.1066	0.0537	0.1243	0.0507	0.0559	0.0781
	0.612	0.0492	0.0671	0.0889	0.0380	0.0798	0.0911
	0.791	0.0253	0.0703	0.0765	0.0633	0.0874	0.1095
Q	20	0.2638	0.0493	0.2734	0.0702	0.0477	0.0934
	40	0.1517	0.0593	0.1723	0.0481	0.0636	0.0834
	80	0.0784	0.0791	0.1224	0.0528	0.0933	0.1091
Max		0.7476	0.0998	0.7490	0.2481	0.1351	0.2522
Min		0.0035	0.0362	0.0594	-0.0722	0.0336	0.0370
Average		0.1647	0.0626	0.1894	0.0570	0.0682	0.0953

from 0.1647 to 0.057. Most importantly, RSD does not vary much over the 216 cases in the system. The regression method “flattens” the bias and makes the estimate less sensitive to the demand parameters and the order quantity. For example, the average absolute RBias is 0.07, 0.05 and 0.05 when Q is at 20, 40 and 80, respectively.

Figure 8 plots the differences of the absolute RBias, RSD and RRMSE based on partial information and RBA. The figure indicates that the differences have certain patterns based on the demand distribution. The differences of RSD vary slightly around zero. When the mean demand is 16, many RSDs under partial information are less than those by RBA. The differences of absolute RBias and RRMSE is obviously larger when the demand mean is 16.

Since the differences of RRMSE have the same pattern as those of RBias, we plot the RBias both under partial information and under RBA based on the demand parameter σ/μ in Figure 9. From this figure, we note as the ratio σ/μ increases, the estimate by RBA becomes larger than that under partial information.

Based on Table 11, RBA significantly reduces the average total cost to values that are close to the nearly optimal cost. The average of RBias over the 216 cases is 0.0002, which is less than the average of RBias under full information. This implies that the total cost induced by RBA is less than the cost under full information. This happens because we

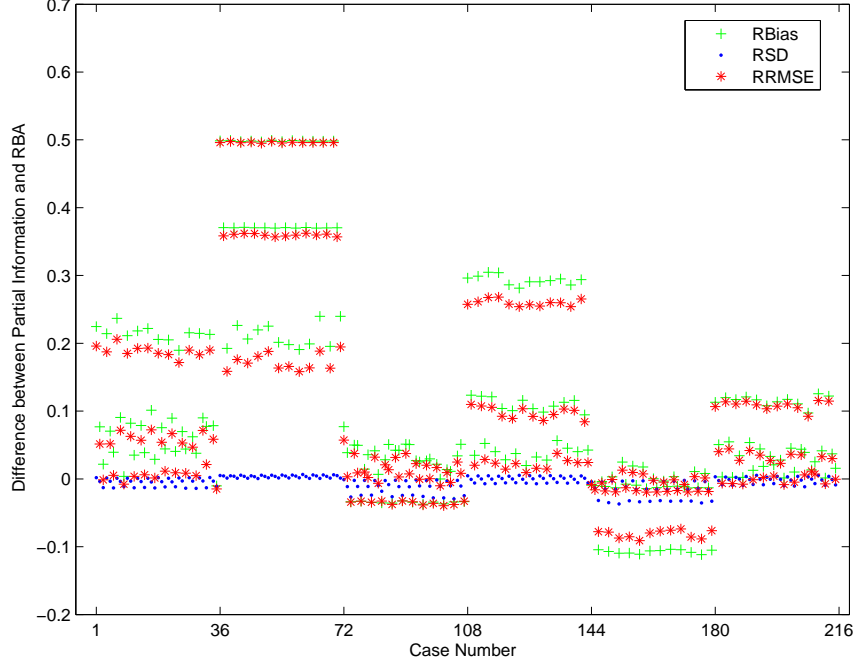


Figure 8: Difference of the absolute RBias, RSD and RMSE based on partial information and RBA

apply an (r, Q) policy by using the PA method to derive the reorder point and the order quantity. The average total cost for each case is less sensitive to the demand parameters and the order quantity. The range of RBias on the average total cost is from 0.1170 to -0.0443 , which is much less than the range under partial information.

Similarly, let $C_{p_2,i}$ denote the average total cost from the i th replication based on the demand estimate from RBA. We define the percentage differences

$$\Delta_{p_2,i} = \frac{(C_{p_2,i} - C_{p_0,i}) \times 100}{C_{p_0,i}}, \quad i = 1, \dots, r. \quad (49)$$

The average and standard deviation of $\Delta_{p_2,i}$ over the r replications are listed in Tables 40 to 42 (Appendix B). Similarly, we list the number of cases in various ranges in Table 12. Nearly 60% of the 216 cases in the system are the range of $[-1.5\%, 1.5\%)$, one half of the remaining cases are in the higher range, and one half are in the lower range. Therefore it seems that RBA works as well as the scenario under full information in terms of the total

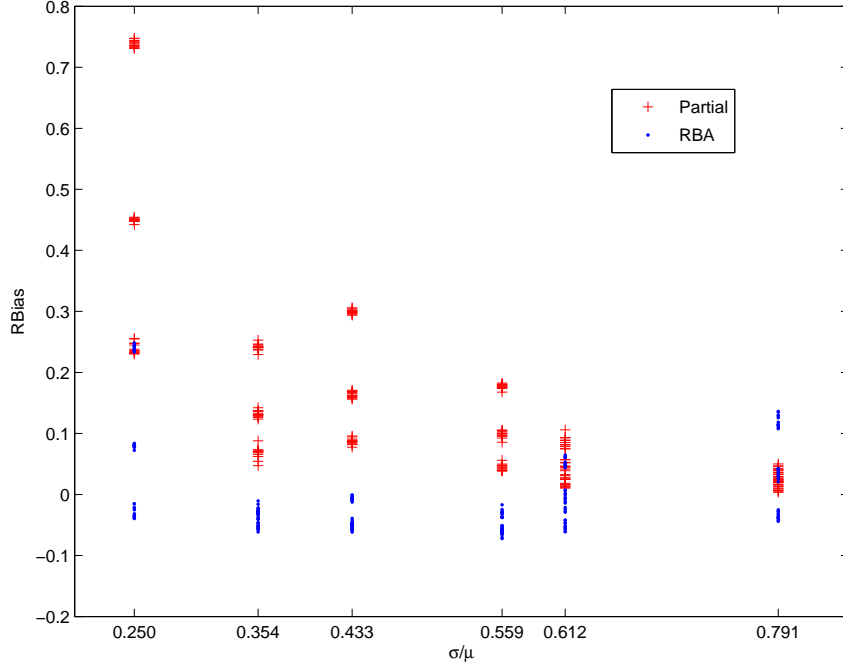


Figure 9: Comparison of RBias under partial information to RBias under RBA based on variable coefficient of variation σ/μ

cost.

4.4.2 Extrapolation

Not only do we evaluate the performance of RBA within the range of the parameters used in our experiments, but also we consider extreme extrapolations of parameter values. A single case with interpolated parameter settings is used as a base case: negative binomial demand, $\sigma^2 = 4\mu$, $\mu = 12$, $L = 3$, $h = 1$, $p = 49$, $K = 48$, and $Q = 60$. Then in each case, we change the value of one parameter while fixing other parameters at the base value. The parameter values and cost performance are presented in Table 13 for each of the 16 cases under consideration.

Based on Table 13, the average RBias induced by RBA and under partial information is -0.001 and 0.06 , respectively. This suggests that RBA works well with regard to the aggregated average total cost even though the parameter values are beyond the range. Based

Table 11: Summary of the average total cost under partial information and RBA

Parameter		Partial Information			RBA		
		RBias	RSD	RRMSE	RBias	RSD	RRMSE
σ/μ	0.250	0.1043	0.0213	0.1095	0.0191	0.0189	0.0347
	0.354	0.0275	0.0226	0.0404	-0.0076	0.0225	0.0264
	0.433	0.069	0.031	0.078	-0.0109	0.0283	0.0319
	0.559	0.0523	0.0369	0.0671	-0.0179	0.0352	0.0408
	0.612	0.0233	0.0341	0.043	0.0057	0.0370	0.0415
	0.791	0.0059	0.0429	0.0443	0.0130	0.049	0.0571
Q	20	0.0766	0.029	0.0873	-0.0019	0.0266	0.0368
	40	0.0429	0.0306	0.0584	-0.0032	0.0302	0.0351
	80	0.0217	0.0348	0.0454	0.0059	0.0387	0.0443
Max		0.3762	0.0734	0.3780	0.1170	0.0994	0.1297
Min		-0.0135	0.0084	0.0085	-0.0443	0.0102	0.0111
Average		0.0471	0.0315	0.0637	0.0002	0.0318	0.0387

Table 12: Frequencies of average $\Delta_{p_2,i}$ under RBA

Range for $\Delta_{p_2,i}$	Number of Cases	Percentage of Cases
$[-\infty, -1.5\%)$	54	25%
$[-1.5\%, 1.5\%)$	121	56%
$[1.5\%, \infty)$	41	19%

on the average RBias of 0.006, the regression fit under imperfect information rivals the benchmark under full information. As discussed before, two reasons cause the aggregated cost of the 16 cases induced by RBA to be slightly less than that under full information.

Table 13: Single parameter extrapolations

Parameter	Value	Full Information			Partial Information			RBA		
		RBias	RSD	RRMSE	RBias	RSD	RRMSE	RBias	RSD	RRMSE
$\sigma^2/\mu \sigma^2/\mu$	7	0.003	0.039	0.039	0.014	0.049	0.051	0.015	0.055	0.057
	24	0.014	0.026	0.030	0.112	0.036	0.118	-0.013	0.033	0.035
	30	0.002	0.026	0.027	0.151	0.038	0.156	-0.007	0.034	0.034
	36	-0.002	0.021	0.021	0.213	0.033	0.215	0.019	0.028	0.033
K	16	-0.004	0.038	0.039	0.031	0.059	0.067	-0.018	0.065	0.067
	72	0.008	0.025	0.026	0.026	0.034	0.043	-0.002	0.038	0.038
p	2	0.005	0.017	0.018	0.003	0.017	0.018	0.008	0.018	0.020
	132	0.008	0.030	0.031	0.036	0.045	0.058	-0.006	0.049	0.049
	199	0.006	0.033	0.034	0.043	0.051	0.066	-0.003	0.055	0.055
L	0	0.000	0.016	0.016	0.014	0.023	0.027	-0.004	0.025	0.025
	5	-0.003	0.034	0.034	0.028	0.045	0.053	-0.013	0.048	0.050
Q	10	0.011	0.027	0.029	0.119	0.031	0.123	-0.027	0.026	0.037
	15	0.009	0.030	0.032	0.097	0.037	0.104	-0.023	0.032	0.039
	85	0.013	0.028	0.031	0.029	0.041	0.050	0.014	0.047	0.050
	90	0.012	0.028	0.030	0.032	0.048	0.058	0.025	0.056	0.062
	100	0.007	0.032	0.033	0.021	0.046	0.050	0.019	0.055	0.058

CHAPTER V

INVENTORY POLICIES WITH REGULAR AND EMERGENCY ORDERS

5.1 Introduction

An important problem for an inventory system is how often the inventory status should be determined. A common approach is to specify the review interval T , which is the time that elapses between two consecutive inspection epochs. An extreme case is continuous review. In reality, the continuous review is usually unnecessary; instead, each transaction (shipment, receipt, demand, etc.) triggers an immediate update of the stock level; this type of control is often called transactions reporting. With periodic review, the stock level is inspected only every T time units. Between the points of review, there may be considerable uncertainty as to the value of the stock level.

There are several reasons for companies to review their stocks periodically rather than continuously. Chiang and Gutierrez (1996) mention the following: (1) avoidance of large review cost; (2) savings on ordering and transportation costs by coordinating orders for different items; (3) practical and organizational considerations; and (4) compliance with the supplier's Just-In-Time systems.

The main disadvantage of a periodic review system, pointed out by Tagaras and Vlachos (2001), is that protection against stockouts is required over a longer period than under continuous review. Thus, the necessary safety stock to provide a given service rate is higher under a periodic review system. When moderate safety stock levels are used, stockouts are likely to occur during a review cycle. For this reason, an emergency order can be an attractive option, especially for inventory systems with long review periods. To alleviate this problem while maintaining the attractive characteristics of periodic review, we analyze a periodic review inventory management system with the additional possibility of emergency

orders. Such two supply modes are commonly used by many companies. For example, a retailer could choose to replenish the inventory of an item under review by a fast supply mode (by air) if the inventory of the item is dangerously low. More detailed discussion can be found in Chiang and Gutierrez (1998).

The model in this chapter is particularly applicable in the case of demand seasonalities. Under periodic review, when there is a sudden surge in demand, the inventory status will be checked to see whether there is enough inventory to meet the demand and whether there is a need to place a new order after fulfilling the immediate demand requirement although it is not at review epoch.

Generally, an emergency order has shorter lead-time than a regular order, but it normally incurs a higher setup cost. We assume that the unit ordering cost of an emergency order is at least equal to that of a regular order. Furthermore, the fixed positive setup cost for the regular order is usually small since a regular order for an item is part of a joint order that includes a mix of products. The higher fixed setup cost for emergency order represents the extra expense of making a special arrangement with the supplier such as using a express media to deliver the order.

To the best of our knowledge, there are no theoretical research and results on the mixed order policy in the framework of a periodic review system with regular and emergency orders and positive setup costs. The remainder of the chapter is organized as follows. Section 5.2 contains a review of relevant literature. Section 5.3 describes the main assumptions, defines the necessary notation, and formulates a stochastic dynamic programming model. Section 5.4 establishes a state-dependent optimal policy, which is a natural extension of optimal (s, S) policy. The optimal policy for emergency orders during a review cycle depends on the amount of inventory on hand and the amount of inventory on order. Section 5.5 establishes some monotone properties of the optimal policies. Section 5.6 reports numerical results to illustrate some properties of the optimal order policies, the effect of using emergency order policies for various parameter values, and the benefits from the use of optimal policies instead of stationary policies.

5.2 *Literature Review*

A periodic review inventory system with emergency orders and without considering setup cost dates back to Barankin (1961). He develops a one-product single period inventory model where the lead-time of a regular order is one period and an emergency order has zero lead-time. Daniel (1962) studies an extension of this model to multiple planning periods and derives an optimal policy by assuming that the emergency order is bound by a given constant.

Bulinskaya (1964), Fukuda (1964) and Veinott (1966) extend Daniel's model to allow both emergency and regular orders to be placed simultaneously and to have longer lead-times, but always differing by one period. Wright (1968) further extends the analysis to the inventory system with multiple products.

Whittemore and Saunders (1977) consider a more general case in which the two lead-times can take any multiples of review periods. Unfortunately, the form of the optimal policy they derive is extremely complex, relying on the use and solution of a multidimensional dynamic program. They were able to obtain explicit results only for the case where two lead-times differ by one period. Furthermore, they do not consider the setup cost for emergency and regular orders.

Rosenshine and Obee (1976) examine a standing order inventory system where a constant regular order quantity is received every period and a constant emergency order quantity may be placed once per period. They assume that an emergency order can be placed at most once in each period for immediate delivery. The minimum system cost is obtained by formulating the standing order system as a Markov chain.

Blumenfeld et al. (1985) introduces a simple model with emergency orders, where it is assumed that an emergency order is sufficiently large to completely avoid stockouts.

Chiang and Gutierrez (1996) consider a different inventory model where each review epoch the inventory manager must decide which of the two supply modes to use, and then order enough units to raise the inventory position to a given level. They show that given any positive order-up-to level, either the regular supply mode is used alone, or there exists an indifference inventory level such that if the inventory at the review epoch is below the

level, the emergency order is used.

Chiang and Gutierrez (1998) allow multiple emergency orders to be placed at any discrete time within a review period including at the regular review epoch. Thus, the proposed policy is essentially a mixture of periodic review (for regular orders) and continuous review (for emergency orders). They analyze the problem within the framework of a stochastic dynamic program and derive an optimal control policy which is quite complex, especially if the two lead-times differ by more than one period. They also derive a stopping rule to end the computation and obtain optimal parameters. Computational results are included that support the contention that easily implemented policies can be computed with reasonable effort.

Tagaras and Vlachos (2001) propose and analyze a periodic review inventory system with two replenishment modes. Regular orders are placed periodically following a base stock policy with a deterministic lead-time. The manager also has the option of placing emergency orders, characterized by a shorter fixed lead-time but higher acquisition cost. One crucial assumption is that only one emergency order is placed per cycle as late as possible so that it arrives before the end of review period. An approximate cost model is developed which can be optimized easily with respect to the order-up-to parameters.

All of this previous work ignores setup costs, so the inventory policy is a simple order-up-to policy. If we consider the different setup costs for emergency orders and regular orders, the problem becomes more challenging. To our best knowledge, there is no such work in the literature.

Emergency order models under continuous review have also been proposed. The model with the most general assumptions is that of Moinzadeh and Nahmias (1988). They develop a heuristic policy that places a regular order for Q_1 units when the inventory on hand reaches R_1 and an emergency order for Q_2 if the inventory on hand reaches R_2 . To derive cost expressions, they assume that there is never more than one outstanding order of each type.

Moinzadeh and Schmidt (1991) develop an approximate model of an inventory system in which there exist two options for supply, with one having a shorter lead-time. Then

they assume that the demand and fixed ordering cost are small relative to the holding cost so that a one-for-one ordering policy is appropriate. The policy they consider for placing emergency orders uses information about the age of outstanding orders. They derive a steady state behavior and present some computational results. The proposed policy is to cancel the normal order if it has not been delivered by a certain time and issue an emergency order instead.

Inspired by Moynzadeh and Nahmias (1988), Johansen and Thorstenson (1998) explicitly consider the opportunity to use an emergency supply mode to hedge against demand uncertainty when replenishing a single item inventory. Normal orders with a relative long and constant lead-time are controlled by a standard (r, Q) policy. These orders can only be issued when no other orders are outstanding. When a normal order is outstanding, emergency orders are controlled by a reorder point $s(j)$ and an order-up-to level $S(j)$, where j is a measure of the time remaining until the normal order is delivered. The emergency orders have a short lead-time, and may also have different ordering costs compared to normal orders. They formulate a long run average cost model that includes ordering costs for the two types of orders, backordering costs, and holding costs.

Although our model is closely related to the one in Chiang and Gutierrez (1998), there are important differences between these two models, primarily because of the setup costs.

5.3 Model Formulation and Cost Functions

In this section, we make some assumptions and define some notations in order to formulate a dynamic inventory model. To begin with, we first make some assumptions for model formulation.

We consider a dynamic periodic review inventory model in which demands for a single product in each day are independent and identically distributed random variables. Generally, the inventory is reviewed every m days, with the interval between reviews called a review cycle. As in much of the periodic review literature, we assume that the manager has already decided the length of a review cycle. The ordering policies in our model are mixed strategies including emergency orders and regular orders. At a review epoch, we have

two options to place order; either a regular order or an emergency order. We first identify whether we need to place an emergency order and then decide a regular order. Between the review epochs only an emergency order can be placed to have the inventory replenished.

The order placement incurs the unit ordering costs and fixed setup costs for emergency and regular orders. Meanwhile we assume that the lead-times for one emergency order and a regular order are 1 and τ days respectively, where $\tau < m$. In addition, there is a linear holding and backorder cost that is incurred based on the net inventory at the end of every day. The unfilled demand will be fully backordered. An example of the inventory process from a periodic review system is plotted in Figure 5.3, in which we assume that $m = 10$ days and $\tau = 6$ days.

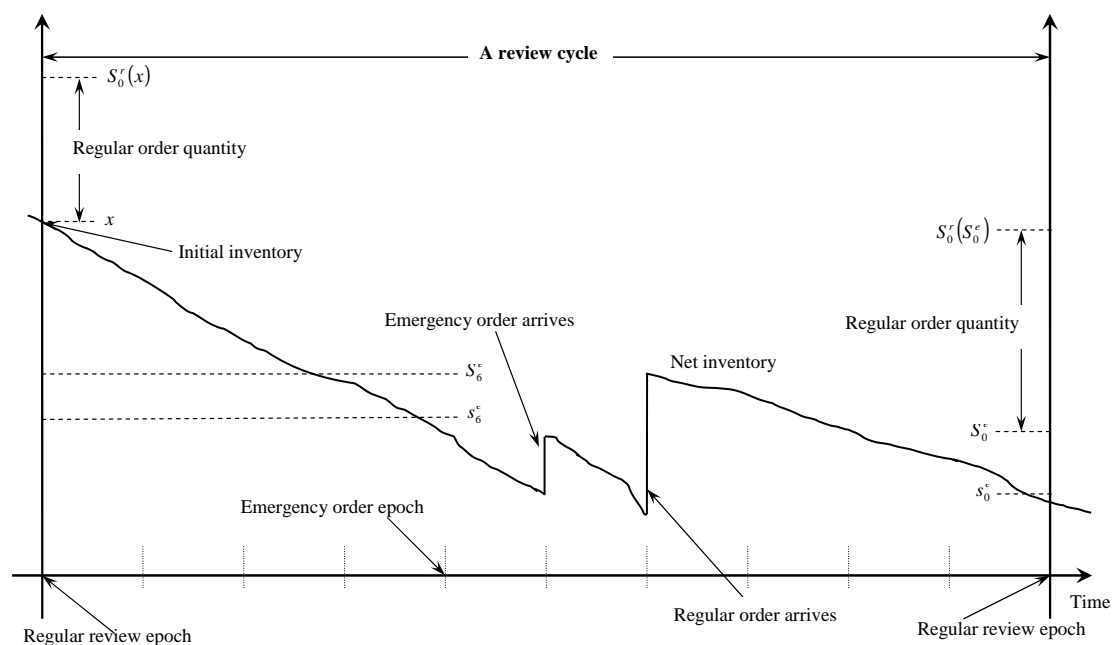


Figure 10: An example of the inventory process for the periodic system with $m = 10$ days and $\tau = 6$ days

If the net inventory is x at the beginning of a day, then the expected holding and shortage costs to be charged during that day will be given by

$$L(x) = \begin{cases} \int_0^x h(x - \xi) f(\xi) d\xi + \int_x^\infty p(t - x) f(\xi) d\xi, & x \geq 0 \\ \int_0^\infty p(\xi - x) f(\xi) d\xi, & x < 0, \end{cases} \quad (50)$$

where $f(\cdot)$ is the density of the demand distribution, the constants h and p are the unit holding and shortage costs per unit and per day, respectively. Therefore, $L(x)$ is a convex and differential function with $\lim_{|x| \rightarrow \infty} L(x) = \infty$.

Let us assume that the inventory problem has a horizon of n review cycles, and we start with an initial inventory of x units. Let $C_{i,j}(x, z)$ represent the expected value of the discounted cost during these i review cycles and j days remaining until the end of the i th review cycle when the starting net inventory is x and the inventory on order is z , where $0 \leq j \leq m - 1$. Here we notice that the inventory on order is zero after the regular order arrives. In other words, when the remaining days are less than $m - \tau + 1$, $C_{i,j}(x, z) = C_{i,j}(x, 0)$, for $j = 0, 1, \dots, m - \tau$.

Let $K_{i,j}^e$ and $K_{i,j}^r$ be the setup cost with i review cycles and j days remaining for emergency and regular order, respectively. The superscripts e and r denote the corresponding notation for emergency and regular order, respectively. Based on our assumptions, only at the review epoch can the regular order be placed, i.e., the fixed setup cost for regular orders $K_{i,j}^r$ is positive only if $j = 0$. Meanwhile, we assume that $K_{i,m-1}^e \geq K_{i,m-2}^e \geq \dots \geq K_{i,0}^e \geq K_{i,0}^r \geq K_{i-1,m-1}^e \geq K_{i-1,m-2}^e \geq \dots \geq K_{i-1,0}^e \geq K_{i-1,0}^r$. This condition of “decreasing” setup cost is needed for establishing the optimality of (s, S) type policies. This case may arise due to the learning curve effect associated with fixed ordering cost over time.

Moreover, we define the delta function by

$$\delta(z) = \begin{cases} 1, & z > 0 \\ 0, & \text{else.} \end{cases} \quad (51)$$

So the corresponding ordering cost function including the setup cost can be defined by

$$c_{i,j}^k(z) = K_{i,j}^k \delta(z) + c_{i,j}^k z, \quad k = e, r. \quad (52)$$

Here the unit ordering cost is assumed to be fixed for emergency and regular orders over time, but the unit ordering cost of an emergency order is at least equal to that of regular order, i.e., $c^e \geq c^r$. Therefore, the above equation can be simplified as follows:

$$c_{i,j}^k(z) = K_{i,j}^k \delta(z) + c^k z, \quad k = e, r. \quad (53)$$

In addition, we let y^e and y^r be the inventory position after a possible emergency order and regular order, respectively. Let $\alpha \in (0, 1)$ be the discount factor per day.

5.3.1 Cost Functions

By definition, $C_{i,j}(x, z)$ satisfies the following functional equation

$$C_{i,0}(x, 0) = \min_{y^e \geq x, y^r \geq y^e} \{c_{i,0}^e(y^e - x) + c_{i,0}^r(y^r - y^e) + \alpha E_\xi L(y^e - \xi) + \alpha E_\xi C_{i-1,m-1}(y^e - \xi, y^r - y^e)\}, \quad (54)$$

$$C_{i,j}(x, z) = \min_{y^e \geq x} \{c_{i,j}^e(y^e - x) + \alpha E_\xi L(y^e - \xi) + \alpha E_\xi C_{i,j-1}(y^e - \xi, z)\}, \quad (55)$$

$$j = 1, \dots, m-1 \text{ and } j \neq m - \tau + 1,$$

$$C_{i,m-\tau+1}(x, z) = \min_{y^e \geq x} \{c_{i,m-\tau+1}^e(y^e - x) + \alpha E_\xi L(y^e + z - \xi) + \alpha E_\xi C_{i,m-\tau}(y^e + z - \xi, 0)\}, \quad (56)$$

where $C_{0,0}(x, 0) \equiv 0$. At a review epoch, $y^r - y^e$ is the order quantity via the regular mode which becomes inventory on order afterward.

To simplify the expressions of the cost functions above, we define the auxiliary functions

$$J_{i,0}(y^e, y^r - y^e) = c^r y^r + \alpha E_\xi C_{i-1,m-1}(y^e - \xi, y^r - y^e), \quad (57)$$

and

$$J_i(y^e) = \min_{y^r \geq y^e} \{J_{i,0}(y^e, y^r - y^e) + K_{i,0}^r \delta(y^r - y^e)\} - c^r y^e. \quad (58)$$

Hence equation (54) can be written as

$$\begin{aligned} C_{i,0}(x, 0) &= \min_{y^e \geq x} \{c_{i,0}^e(y^e - x) + \alpha E_\xi L(y^e - \xi) + \min_{y^r \geq y^e} \{J_{i,0}(y^e, y^r - y^e) + K_{i,0}^r \delta(y^r - y^e)\}\} \\ &= \min_{y^e \geq x} \{c_{i,0}^e(y^e - x) + \alpha E_\xi L(y^e - \xi) + J_i(y^e)\}. \end{aligned} \quad (59)$$

In addition, we define

$$G_{i,0}(y^e) = c^e y^e + \alpha E_\xi L(y^e - \xi) + J_i(y^e), \quad (60)$$

$$G_{i,j}(y^e, z) = c^e y^e + \alpha E_\xi L(y^e - \xi) + \alpha E_\xi C_{i,j-1}(y^e - \xi, z),$$

$$j = 1, \dots, m-1 \text{ and } j \neq m-\tau+1, \quad (61)$$

$$G_{i,m-\tau+1}(y^e, z) = c^e y^e + \alpha E_\xi L(y^e + z - \xi) + \alpha E_\xi C_{i,m-\tau}(y^e + z - \xi, 0). \quad (62)$$

Finally, we can rewrite equations (54)-(56) as follows:

$$C_{i,0}(x, 0) = \min_{y^e \geq x} \{G_{i,0}(y^e) + K_{i,0}^e \delta(y^e - x)\} - c^e x, \quad (63)$$

$$C_{i,j}(x, z) = \min_{y^e \geq x} \{G_{i,j}(y^e, z) + K_{i,j}^e \delta(y^e - x)\} - c^e x, \quad \text{for } j = 1, \dots, m-1. \quad (64)$$

As we mentioned before, there is no inventory on order after the regular order arrives. Then $G_{i,j}(y^e, z)$ simplifies to $G_{i,j}(y^e, 0)$ as $j = 1, \dots, m-\tau$. In addition, we assume that

$$\lim_{|y^e| \rightarrow \infty} [c^e y^e + \alpha E_\xi L(y^e - \xi)] = \infty. \quad (65)$$

5.4 Optimal Policies

In this section, we establish the optimality of inventory order policies based on the property of K -convexity introduced by Scarf (1963).

5.4.1 Definition and Properties of K -Convexity

We now relate our problem to the stochastic inventory control problem with positive setup cost discussed by Scarf (1963). For the classical stochastic inventory problem, Scarf shows that an (s, S) policy is optimal. Under this policy, the optimal decision is characterized by two parameters; a reorder point, s , and an order-up-to level, S . If the initial inventory level is smaller than the reorder point, then order up to level S . Otherwise, no order is placed.

To prove that an (s, S) policy is optimal, Scarf (1963) uses the concept of K -convexity. We start with two equivalent definitions of K -convexity.

DEFINITION 5.1. Let $K \geq 0$, and let $f(x)$ be a differentiable function. We say that $f(x)$ is K -convex if

$$K + f(a+x) - f(x) - af'(x) \geq 0 \quad \text{for all } a \geq 0 \text{ and } x. \quad (66)$$

If differentiability is not assumed, we say that $f(x)$ is K -convex if

$$K + f(a + x) - f(x) - a \left[\frac{f(x) - f(x - b)}{b} \right] \geq 0 \quad \text{for all } a \geq 0, b > 0 \text{ and } x. \quad (67)$$

DEFINITION 5.2. Let $K \geq 0$, and let $f(x)$ be a real-valued function. We say that $f(x)$ is K -convex if for any $x_1 \leq x_2$ and $\lambda \in [0, 1]$,

$$(1 - \lambda)f(x_1) + \lambda f(x_2) + \lambda K \geq f((1 - \lambda)x_1 + \lambda x_2). \quad (68)$$

Conditions (67) and (68) are equivalent. This is evident by substituting $x = (1 - \lambda)x_1 + \lambda x_2$, $a = x_2 - x$ and $b = x - x_1$ into the right side of (67).

Definition 5.2 emphasizes the difference between K -convexity and traditional convexity.

Some of the following properties of K -convex functions are used in our proofs. See Bertsekas (2000) for details.

LEMMA 5.1.

- (a) A real-valued convex function is K -convex for all $K \geq 0$.
- (b) If $f(x)$ is K -convex, then $f(x + h)$ is K -convex for all h .
- (c) If f is K -convex, then f is K' -convex for all $K' \geq K$.
- (d) If f and g are K -convex and M -convex, respectively, then $\alpha f + \beta g$ is $(\alpha K + \beta M)$ -convex for any positive constants α and β .
- (e) If f is K -convex and ξ is a random variable, then for any $x \in R$, $E_\xi\{f(x - \xi)\}$ is also K -convex, provided $E_\xi|f(x - \xi)| < \infty$.

LEMMA 5.2. If f is a continuous K -convex function and $f(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, then there are exist values s and S with $s \leq S$ such that

- (a) $f(S) \leq f(x)$, for all x .
- (b) $f(S) + K = f(s) < f(x)$, for all $x < s$.
- (c) $f(x)$ is a decreasing function on $(-\infty, s)$.

(d) $f(x) \leq f(z) + K$ for all x, z with $s \leq x \leq z$.

(e) The function

$$h(x) = \min_{y \geq x} \{K\delta(y - x) + f(y)\} = \begin{cases} K + f(S), & x < s \\ f(x), & x \geq s \end{cases}$$

is K -convex.

Proof. Parts (a)-(d) are proved in Bertsekas (2000). To show that $h(x)$ is K -convex, we need to consider the following four cases:

Case 1: $s \leq x - b < x \leq x + a$.

$$K + h(a + x) - h(x) - a \left[\frac{h(x) - h(x - b)}{b} \right] = K + f(a + x) - f(x) - a \left[\frac{f(x) - f(x - b)}{b} \right] \geq 0.$$

Case 2: $x - b < s \leq x \leq x + a$.

$$K + h(a + x) - h(x) - a \left[\frac{h(x) - h(x - b)}{b} \right] = K + f(a + x) - f(x) - a \left[\frac{f(x) - K - f(S)}{b} \right].$$

Here we shall discuss two subcases: If $s < S \leq x$, then the last equation can be written as

$$\begin{aligned} & K + f(a + x) - f(x) - a \left[\frac{f(x) - K - f(S)}{b} \right] \\ &= K + f(a + x) - f(x) - a \left[\frac{f(x) - f(S)}{b} \right] + \frac{a}{b} K \\ &\geq a \left[\frac{f(x) - f(S)}{x - S} \right] - a \left[\frac{f(x) - f(S)}{b} \right] + \frac{a}{b} K \\ &\geq a[f(x) - f(S)] \left[\frac{1}{x - S} - \frac{1}{b} \right] + \frac{a}{b} K > 0. \end{aligned}$$

Alternatively, if $s \leq x \leq S$, then we have

$$K + f(a + x) - f(x) - a \left[\frac{f(x) - K - f(S)}{b} \right] = K + f(a + x) - f(x) - a \left[\frac{f(x) - f(s)}{b} \right].$$

From part (d), it is known that $K + f(x + a) \geq f(x)$ since $s \leq x \leq x + a$. If $s \leq x \leq S$, by K -convexity,

$$f(s) = K + f(S) \geq f(x) + (S - x) \left[\frac{f(x) - f(s)}{x - s} \right],$$

from which we deduce

$$\left(1 + \frac{S - x}{x - s} \right) f(s) \leq \left(1 + \frac{S - x}{x - s} \right) f(x),$$

so that $f(x) \leq f(s)$ and

$$K + f(a+x) - f(x) - a \left[\frac{f(x) - f(s)}{b} \right] \geq 0.$$

Case 3: $x - b < x < s \leq x + a$.

$$\begin{aligned} & K + h(a+x) - h(x) - a \left[\frac{h(x) - h(x-b)}{b} \right] \\ &= K + f(a+x) - K - f(S) - a \left[\frac{K + f(S) - K - f(S)}{b} \right] \\ &= f(a+x) - f(S) \geq 0. \end{aligned}$$

Case 4: $x - b < x \leq x + a < s$.

$$\begin{aligned} & K + h(a+x) - h(x) - a \left[\frac{h(x) - h(x-b)}{b} \right] \\ &= K + K + f(S) - K - f(S) - a \left[\frac{K + f(S) - K - f(S)}{b} \right] \\ &= K \geq 0. \end{aligned}$$

This completes the proof. □

5.4.2 Optimality of State-Dependent (s, S) Type Policies

In this section, we prove that the state-dependent (s, S) type policies are optimal in our model. We start with the following theorem.

THEOREM 5.1. *For each pair (i, j) , the function $C_{i,j}(x, z)$ is $K_{i,j}^e$ -convex.*

Proof. We first show that $C_{0,j}(x, z)$ is $K_{0,j}^e$ -convex by induction on j .

We start with $j = 1$. Since $C_{0,0}(x, 0) \equiv 0$, $G_{0,1}(y^e, 0) = c^e y^e + \alpha E_\xi L(y^e - \xi)$ is a convex function in y^e ; hence $K_{0,1}^e$ -convex for any $K_{0,1}^e \geq 0$. From assumption (65), we know that

$$\lim_{|y^e| \rightarrow \infty} G_{0,1}(y^e, 0) = \lim_{|y^e| \rightarrow \infty} [c^e y^e + \alpha E_\xi L(y^e - \xi)] = \infty. \quad (69)$$

It follows from Lemma 5.2 (e) that there are constants $s_{0,1}^e$ and $S_{0,1}^e$, independent of x , with $s_{0,1}^e \leq S_{0,1}^e < \infty$, such that

$$\min_{y^e \geq x} \{K_{0,1}^e \delta(y^e - x) + G_{0,1}(y^e, 0)\} = \begin{cases} K_{0,1}^e + G_{0,1}(S_{0,1}^e, 0), & x \leq s_{0,1}^e \\ G_{0,1}(x, 0), & x > s_{0,1}^e. \end{cases} \quad (70)$$

From equations (64) and (70), we have

$$\begin{aligned} C_{0,1}(x, 0) &= \begin{cases} K_{0,1}^e + c^e S_{0,1}^e + \alpha E_\xi L(S_{0,1}^e - \xi) - c^e x, & x \leq s_{0,1}^e \\ \alpha E_\xi L(x - \xi), & x > s_{0,1}^e \end{cases} \\ &\geq \alpha E_\xi [\delta(x - s_{0,1}^e) L(x - \xi)]. \end{aligned} \quad (71)$$

Hence, Lemma 5.2 implies that $C_{0,1}(x, 0)$ is a nonnegative, continuous $K_{0,1}^e$ -convex function of x .

Now assume that $C_{0,\nu}(x, 0)$ is a nonnegative, continuous and $K_{0,\nu}^e$ -convex function for $\nu = 1, \dots, T - \tau - 1$. We shall prove that $C_{0,\nu+1}(x, 0)$ is a nonnegative, continuous and $K_{0,\nu+1}^e$ -convex function.

Since $C_{0,\nu}(x, 0)$ is a continuous $K_{0,\nu}^e$ -convex function with

$$\lim_{|y^e| \rightarrow \infty} G_{0,\nu}(y^e, 0) = \infty,$$

and $G_{0,\nu+1}(y^e, 0) = c^e y^e + \alpha E_\xi L(y^e - \xi) + \alpha E_\xi C_{0,\nu}(y^e - \xi, 0)$, Lemma 5.2 implies that $G_{0,\nu+1}(y^e, 0)$ is a $K_{0,\nu+1}^e$ -convex function. It follows that there exist values $s_{0,\nu+1}^e$ and $S_{0,\nu+1}^e$ independent of x , with $s_{0,\nu+1}^e \leq S_{0,\nu+1}^e < \infty$, such that

$$\begin{aligned} C_{0,\nu+1}(x, 0) &= \min_{y^e \geq x} \{K_{0,\nu+1}^e \delta(y^e - x) + G_{0,\nu}(y^e, 0)\} - c^e x \\ &= \begin{cases} K_{0,\nu+1}^e + c^e S_{0,\nu+1}^e + \alpha E_\xi L(S_{0,\nu+1}^e - \xi) \\ \quad + \alpha E_\xi C_{0,\nu}(S_{0,\nu+1}^e - \xi, 0) - c^e x, & x \leq s_{0,\nu+1}^e \\ \alpha E_\xi L(x - \xi) + \alpha E_\xi C_{0,\nu}(x - \xi, 0), & x > s_{0,\nu+1}^e. \end{cases} \end{aligned}$$

Using the assumption $K_{0,\nu+1}^e \geq K_{0,\nu}^e$, we can show that $C_{0,\nu+1}(x, 0)$ is a nonnegative, continuous and $K_{0,\nu+1}^e$ -convex function.

Therefore, we have proved that $C_{0,j}(x, 0)$ is a nonnegative, continuous and $K_{0,j}^e$ -convex function for $j = 1, \dots, m - \tau$. There also exists an optimal $(s_{0,j}^e, S_{0,j}^e)$ policy.

Now we consider the cost functions associated with the days before regular orders could arrive, starting with $C_{0,m-\tau+1}(x, z)$ for any given inventory on order z , $0 \leq z < \infty$. We just proved that $C_{0,m-\tau}(x, 0)$ is a nonnegative, continuous and $K_{0,m-\tau}^e$ -convex function. For a

given z , we choose values y_1^e and y_2^e , such that $y^e = (1 - \lambda)y_1^e + \lambda y_2^e$ for some $\lambda \in [0, 1]$.

Based on equation (62), it follows that

$$\begin{aligned}
& (1 - \lambda)G_{0,m-\tau+1}(y_1^e, z) + \lambda G_{0,m-\tau+1}(y_2^e, z) + \lambda K_{0,m-\tau}^e \\
&= (1 - \lambda)[c^e y_1^e + E_\xi L(y_1^e + z - \xi) + E_\xi C_{0,m-\tau}(y_1^e + z - \xi, 0)] \\
&\quad + \lambda[c^e y_2^e + E_\xi L(y_2^e + z - \xi) + E_\xi C_{0,m-\tau}(y_2^e + z - \xi, 0)] + \lambda K_{0,m-\tau}^e \\
&\geq c^e y^e + E_\xi L(y^e + z - \xi) + E_\xi C_{0,m-\tau}(y^e + z - \xi, 0) = G_{0,m-\tau+1}(y^e, z).
\end{aligned}$$

Consequently, $G_{0,m-\tau+1}(y^e, z)$ is a continuous $K_{0,m-\tau}^e$ -convex function. From (69), for any given finite z we know that

$$\lim_{|y^e| \rightarrow \infty} [c^e y^e + \alpha E_\xi L(y^e + z - \xi)] = \lim_{|y^e| \rightarrow \infty} [c^e (y^e + z) + \alpha E_\xi L(y^e + z - \xi)] - c^e z = \infty.$$

Thus, there exist values $s_{0,m-\tau+1}^e(z)$ and $S_{0,m-\tau+1}^e(z)$, independent of x with $s_{0,m-\tau+1}^e(z) \leq S_{0,m-\tau+1}^e(z) < \infty$, such that

$$\begin{aligned}
& C_{0,m-\tau+1}(x, z) \\
&= \min_{y^e \geq x} \{K_{0,m-\tau+1} \delta(y^e - x) + G_{0,m-\tau+1}(y^e, z)\} - c^e x \\
&= \begin{cases} K_{0,m-\tau+1}^e + c^e S_{0,m-\tau+1}^e(z) + \alpha E_\xi L(S_{0,m-\tau+1}^e(z) + z - \xi) \\ \quad + \alpha E_\xi C_{0,m-\tau}(S_{0,m-\tau+1}^e(z) + z - \xi, 0) - c^e x, & x \leq s_{0,m-\tau+1}^e(z) \\ \alpha E_\xi L(x + z - \xi) + \alpha E_\xi C_{0,m-\tau}(x + z - \xi, 0), & x > s_{0,m-\tau+1}^e(z). \end{cases} \tag{72}
\end{aligned}$$

Based on Lemma 5.2 and the assumption $K_{0,m-\tau+1}^e \geq K_{0,m-\tau}^e$ we have that $C_{0,m-\tau+1}(x, z)$ is a $K_{0,m-\tau+1}^e$ -convex function for given z . Choosing any x_1 and x_2 , such that $x = (1 - \lambda)x_1 + \lambda x_2$ for some $\lambda \in [0, 1]$, it is easy to prove that $C_{0,m-\tau+1}(x, z)$ is $K_{0,m-\tau+1}^e$ -convex if either both x_1 and x_2 are larger than $s_{0,m-\tau+1}^e$ or both are smaller than $s_{0,m-\tau+1}^e$. So here we only provide the proof for the remaining two cases. Without loss of generality, we assume that $x_1 \leq x_2$.

Case 1: $x_1 \leq x \leq s_{0,m-\tau+1}^e \leq x_2$. Using equation (72), we have

$$\begin{aligned}
& (1-\lambda)C_{0,m-\tau+1}(x_1, z) + \lambda C_{0,m-\tau+1}(x_2, z) + \lambda K_{0,m-\tau+1}^e \\
&= (1-\lambda)[K_{0,m-\tau+1}^e + G_{0,m-\tau+1}(S_{0,m-\tau+1}^e(z), z) - c^e x_1] \\
&\quad + \lambda[G_{0,m-\tau+1}(x_2, z) - c^e x_2] + \lambda K_{0,m-\tau+1}^e \\
&\geq (1-\lambda)[K_{0,m-\tau+1}^e + G_{0,m-\tau+1}(S_{0,m-\tau+1}^e(z), z) - c^e x_1] \\
&\quad + \lambda[K_{0,m-\tau+1}^e + G_{0,m-\tau+1}(S_{0,m-\tau+1}^e(z), z) - c^e x_2] \\
&= C_{0,m-\tau+1}(x, z).
\end{aligned}$$

Case 2: $x_1 \leq s_{0,m-\tau+1}^e \leq x \leq x_2$. We have

$$\begin{aligned}
& (1-\lambda)C_{0,m-\tau+1}(x_1, z) + \lambda C_{0,m-\tau+1}(x_2, z) + \lambda K_{0,m-\tau+1}^e \\
&= (1-\lambda)[K_{0,m-\tau+1}^e + G_{0,m-\tau+1}(S_{0,m-\tau+1}^e(z), z) - c^e x_1] \\
&\quad + \lambda[G_{0,m-\tau+1}(x_2, z) - c^e x_2] + \lambda K_{0,m-\tau+1}^e \\
&= (1-\lambda)[G_{0,m-\tau+1}(s_{0,m-\tau+1}^e(z), z) - c^e x_1] \\
&\quad + \lambda[K_{0,m-\tau+1}^e + G_{0,m-\tau+1}(S_{0,m-\tau+1}^e(z), z) - c^e x_2] \\
&\geq (1-\lambda)[G_{0,m-\tau+1}(x_2, z) - c^e x_1] + \lambda[G_{0,m-\tau+1}(x_2, z) - c^e x_2] \\
&= C_{0,m-\tau+1}(x, z).
\end{aligned}$$

Assuming that $C_{0,\nu}(x, z)$ is a nonnegative, continuous and $K_{0,\nu}^e$ -convex function for $\nu = m - \tau + 1, \dots, m - 2$, we shall prove that $C_{0,\nu+1}(x, z)$ is a nonnegative, continuous and $K_{0,\nu+1}^e$ -convex.

Since $C_{0,\nu}(x, z)$ is a nonnegative, continuous $K_{0,\nu}^e$ -convex function with

$$\lim_{|y^e| \rightarrow \infty} G_{0,\nu+1}(y^e, 0) = \infty,$$

and $G_{0,\nu+1}(y^e, z) = c^e y^e + E_\xi L(y^e - \xi) + E_\xi C_{0,\nu}(y^e - \xi, z)$, Lemma 5.2 implies that $G_{0,\nu+1}(y^e, z)$ is a continuous $K_{0,\nu}^e$ -convex function. It follows that there are values $s_{0,\nu+1}^e(z)$ and $S_{0,\nu+1}^e(z)$,

independent of x with $s_{0,\nu+1}^e(z) \leq S_{0,\nu+1}^e(z) < \infty$, such that

$$\begin{aligned} C_{0,\nu+1}(x, z) &= \min_{y^e \geq x} \{K_{0,\nu+1}\delta(y^e - x) + G_{0,\nu+1}(y^e, z)\} - c^e x \\ &= \begin{cases} K_{0,\nu+1}^e + c^e S_{0,\nu+1}^e(z) + \alpha E_\xi L(S_{0,\nu+1}^e(z) - \xi) \\ \quad + \alpha E_\xi C_{0,\nu}(S_{0,\nu+1}^e(z) - \xi, z) - c^e x, & x \leq s_{0,\nu+1}^e(z) \\ \alpha E_\xi L(x - \xi) + \alpha E_\xi C_{0,\nu}(x - \xi, z), & x > s_{0,\nu+1}^e(z). \end{cases} \end{aligned} \quad (73)$$

By Lemma 5.2 and the inequality $K_{0,\nu+1}^e \geq K_{0,\nu}^e$, we have that $C_{0,\nu+1}(x, z)$ is a nonnegative, continuous and $K_{0,\nu+1}^e$ -convex function.

We have proved that $C_{0,j}(x, z)$ is a nonnegative, continuous and $K_{0,j}^e$ -convex for $j = m - \tau + 1, \dots, m - 1$. Hence there exists an optimal $(s_{0,j}^e(z), S_{0,j}^e(z))$ policy for given z .

Lastly, consider the last review epoch and the associated cost function $C_{1,0}(x, 0)$. Since $C_{0,m-1}(x, z)$ is $K_{0,m-1}^e$ -convex, it follows that $J_{1,0}(y^e, y^r - y^e) = c^r y^r + \alpha E_\xi C_{0,m-1}(y^e - \xi, y^r - y^e)$ is a continuous $K_{0,m-1}^e$ -convex for given y^e . Now we shall show that $\lim_{|y^r| \rightarrow \infty} J_{1,0}(y^e, y^r - y^e) = \infty$ for given y^e .

From equations (72) and (73), we have

$$C_{0,j}(x, z) \geq \alpha E_{\xi_{0,j}}(\delta(x - s_{0,j}^e)C_{0,j-1}(x - \xi_{0,j}, z)), \quad \text{for } j = m - \tau + 2, \dots, m - 1$$

$$C_{0,m-\tau+1}(x, z) \geq \alpha E_{\xi_{0,m-\tau+1}}(\delta(x - s_{0,m-\tau+1}^e)C_{0,m-\tau}(x + z - \xi_{0,m-\tau}, 0)),$$

$$C_{0,m-\tau}(x, 0) \geq \alpha E_{\xi_{0,m-\tau}}(\delta(x - s_{0,m-\tau}^e)L(x - \xi_{0,m-\tau})).$$

Hence

$$\begin{aligned} &\lim_{|y^r| \rightarrow \infty} J_{1,0}(y^e, y^r - y^e) \\ &\geq \lim_{|y^r| \rightarrow \infty} \left\{ c^r y^r + \alpha^2 E_{\xi_{1,0}}[E_{\xi_{0,m-1}}(\delta(y^e - \xi_{1,0})C_{0,m-2}(y^e - \xi_{1,0} - \xi_{0,m-1}, y^r - y^e)) \right\} \\ &\quad \vdots \\ &\geq \lim_{|y^r| \rightarrow \infty} \left\{ c^r y^r \right. \\ &\quad \left. + \alpha^{\tau+1} E_{\xi_{1,0}} E_{\xi_{0,m-1}} \cdots E_{\xi_{0,m-\tau-1}} \left[\prod_{\kappa=m-1}^{m-\tau+2} \delta\left(y^e - \xi_{1,0} - \sum_{j=m-1}^{\kappa} \xi_{0,j} - s_{0,\kappa}^e(y^r - y^e)\right) \right] \right. \\ &\quad \left. \times \delta\left(y^r - \xi_{1,0} - \sum_{j=m-1}^{m-\tau+1} \xi_{0,j} - s_{0,m-\tau}^e\right) L\left(y^r - \xi_{1,0} - \sum_{j=m-1}^{m-\tau} \xi_{0,j}\right) \right\} = \infty, \end{aligned}$$

and there exists an $(s_{1,0}^r(y^e), S_{1,0}^r(y^e))$ optimal policy such that

$$J_1(y^e) = \begin{cases} K_{1,0}^r + c^r S_{1,0}^r(y^e) + \alpha E_\xi C_{0,m-1}(S_{1,0}^r(y^e) - \xi, S_{1,0}^r(y^e) - y^e) - c^r y^e, & y^e \leq s_{1,0}^r(y^e) \\ \alpha E_\xi C_{0,m-1}(y^e - \xi, 0), & y^e > s_{1,0}^r(y^e). \end{cases}$$

This function is nonnegative, continuous and $K_{1,0}^r$ -convex since $K_{1,0}^r \geq K_{0,m-1}^e$. Therefore, it is easy to prove that $G_{1,0}(y^e)$ is $K_{1,0}^r$ -convex and $\lim_{|y^e| \rightarrow \infty} G_{1,0}(y^e) = \infty$. It follows that there exist values $s_{1,0}^e$ and $S_{1,0}^e$ such that

$$\begin{aligned} C_{1,0}(x, 0) &= \min_{y^e \geq x} \{G_{1,0}(y^e) + K_{1,0}^e(y^e - x)\} - c^e x \\ &= \begin{cases} K_{1,0}^e + c^e S_{1,0}^e + \alpha E_\xi L(S_{1,0}^e - \xi) + J_1(S_{1,0}^e) - c^e x, & x \leq s_{1,0}^e \\ \alpha E_\xi L(x - \xi) + J_1(x), & x > s_{1,0}^e, \end{cases} \end{aligned}$$

and $C_{1,0}(x, 0)$ is a nonnegative, continuous and $K_{1,0}^e$ -convex function since $K_{1,0}^e \geq K_{1,0}^r$.

Now assume that $C_{i-1,j}(x, z)$ is a nonnegative, continuous and $K_{i-1,j}^e$ -convex function, and that $C_{i,0}(x, 0)$ is a nonnegative, continuous and $K_{i,0}^e$ -convex function for $i = 1, \dots, n-1$ and $j = 1, \dots, m-1$. We shall prove that $C_{i,j}(x, z)$ is a nonnegative, continuous and $K_{i,j}^e$ -convex function for $j = 1, \dots, m-1$, and that $C_{i+1,0}(x, 0)$ is a nonnegative, continuous and $K_{i+1,0}^e$ -convex function.

Let us start with $j = 1$. Since $C_{i,0}(x, 0)$ is a nonnegative, continuous and $K_{i,0}^e$ -convex function with

$$\lim_{|y^e| \rightarrow \infty} G_{i,1}(y^e, 0) = \infty,$$

and $G_{i,1}(y^e, 0) = c^e y^e + \alpha E_\xi L(y^e - \xi) + \alpha E_\xi C_{i,0}(y^e - \xi, 0)$, Lemma 5.2 implies that $G_{i,1}(y^e, 0)$ is $K_{i,0}^e$ -convex and that there are constants $s_{i,1}^e$ and $S_{i,1}^e$, independent of x , with $s_{i,1}^e \leq S_{i,1}^e < \infty$, such that

$$\begin{aligned} C_{i,1}(x, 0) &= \min_{y^e \geq x} \{K_{i,1}^e \delta(y^e - x) + G_{i,1}(y^e, 0)\} - c^e x \\ &= \begin{cases} K_{i,1}^e + c^e S_{i,1}^e + \alpha E_\xi L(S_{i,1}^e - \xi) - c^e x, & x \leq s_{i,1}^e \\ \alpha E_\xi L(x - \xi), & x > s_{i,1}^e. \end{cases} \end{aligned}$$

Furthermore, Lemma 5.2 implies that $C_{i,1}(x, 0)$ is a nonnegative, continuous and $K_{i,1}^e$ -convex function.

Now assume that $C_{i,\nu}(x, 0)$ is a nonnegative, continuous and $K_{i,\nu}^e$ -convex function, where $\nu = 1, \dots, m - \tau - 1$. We shall prove that $C_{i,\nu+1}(x, 0)$ is a nonnegative, continuous and $K_{i,\nu+1}^e$ -convex function.

Since $C_{i,\nu}(x, 0)$ is a nonnegative, continuous and $K_{i,\nu}^e$ -convex function with

$$\lim_{|y^e| \rightarrow \infty} G_{i,\nu+1}(y^e, 0) = \infty,$$

and $G_{i,\nu+1}(y^e, 0) = c^e y^e + \alpha E_\xi L(y^e - \xi) + \alpha E_\xi C_{i,\nu}(y^e - \xi, 0)$, Lemma 5.2 implies that $G_{i,\nu+1}(y^e, 0)$ is a $K_{i,\nu+1}^e$ -convex function. Furthermore, there are values $s_{i,\nu+1}^e$ and $S_{i,\nu+1}^e$ independent of x , with $s_{i,\nu+1}^e \leq S_{i,\nu+1}^e < \infty$, such that

$$\begin{aligned} C_{i,\nu+1}(x, 0) &= \min_{y^e \geq x} \{K_{i,\nu+1}^e \delta(y^e - x) + G_{i,\nu+1}(y^e, 0)\} - c^e x \\ &= \begin{cases} K_{i,\nu+1}^e + c^e S_{i,\nu+1}^e + \alpha E_\xi L(S_{i,\nu+1}^e - \xi) \\ \quad + \alpha E_\xi C_{i,\nu}(S_{i,\nu+1}^e - \xi, 0) - c^e x, & x \leq s_{i,\nu+1}^e \\ \alpha E_\xi L(x - \xi) + \alpha E_\xi C_{i,\nu}(x - \xi, 0), & x > s_{i,\nu+1}^e. \end{cases} \end{aligned}$$

Using the assumption $K_{i,\nu+1}^e \geq K_{i,\nu}^e$, we can now show that $C_{i,\nu+1}(x, 0)$ is a nonnegative, continuous and $K_{i,\nu+1}^e$ -convex function.

Therefore, we proved that $C_{i,j}(x, 0)$ is a nonnegative, continuous and $K_{i,j}^e$ -convex function for $j = 1, \dots, m - \tau$. Also, there exists an optimal $(s_{i,j}^e, S_{i,j}^e)$ policy.

Now we consider the costs associated with the days before the regular order arrives, starting with $C_{i,m-\tau+1}(x, z)$ for any given inventory on order z , $0 \leq z < \infty$. We just proved that $C_{i,m-\tau}(x, 0)$ is a nonnegative, continuous and $K_{i,m-\tau}^e$ -convex function. For given inventory on order z , $G_{i,m-\tau+1}(y^e, z) = c^e y^e + E_\xi L(y^e + z - \xi) + E_\xi C_{i,m-\tau}(y^e + z - \xi, 0)$ is a continuous $K_{i,m-\tau}^e$ -convex. From equation (69), for any given finite z we know that

$$\lim_{|y^e| \rightarrow \infty} [c^e y^e + \alpha E_\xi L(y^e + z - \xi)] = \lim_{|y^e| \rightarrow \infty} [c^e (y^e + z) + \alpha E_\xi L(y^e + z - \xi)] - c^e z = \infty.$$

Thus, there exist values $s_{i,m-\tau+1}^e(z)$ and $S_{i,m-\tau+1}^e(z)$, independent of x with $s_{i,m-\tau+1}^e(z) \leq$

$S_{i,m-\tau+1}^e(z) < \infty$, such that

$$\begin{aligned}
C_{i,m-\tau+1}(x, z) &= \min_{y^e \geq x} \{K_{i,m-\tau+1} \delta(y^e - x) + G_{i,m-\tau+1}(y^e, z)\} - c^e x \\
&= \begin{cases} K_{i,m-\tau+1}^e + c^e S_{i,m-\tau+1}^e(z) + \alpha E_\xi L(S_{i,m-\tau+1}^e(z) + z - \xi) \\ \quad + \alpha E_\xi C_{i,m-\tau}(S_{i,m-\tau+1}^e(z) + z - \xi, 0) - c^e x, & x \leq s_{i,m-\tau+1}^e(z) \\ \alpha E_\xi L(x + z - \xi) + \alpha E_\xi C_{i,m-\tau}(x + z - \xi, 0), & x > s_{i,m-\tau+1}^e(z). \end{cases}
\end{aligned} \tag{74}$$

Based on Lemma 5.2 and the assumption $K_{i,m-\tau+1}^e \geq K_{i,m-\tau}^e$ we have that $C_{i,m-\tau+1}(x, z)$ is a nonnegative, continuous and $K_{i,m-\tau+1}^e$ -convex function for given z .

Assuming that $C_{i,\nu}(x, z)$ is a nonnegative, continuous and $K_{i,\nu}^e$ -convex function for $\nu = m - \tau + 1, \dots, m - 2$, we shall prove that $C_{i,\nu+1}(x, z)$ is a nonnegative, continuous and $K_{i,\nu+1}^e$ -convex function.

Since $C_{i,\nu}(x, z)$ is a nonnegative, continuous and $K_{i,\nu}^e$ -convex function and $G_{i,\nu+1}(y^e, z) = c^e y^e + E_\xi L(y^e - \xi) + E_\xi C_{i,\nu}(y^e - \xi, z)$, Lemma 5.2 implies that $G_{i,\nu+1}(y^e, z)$ is a continuous $K_{i,\nu}^e$ -convex function. It follows that there are values $s_{i,\nu+1}^e(z)$ and $S_{i,\nu+1}^e(z)$, independent of x with $s_{i,\nu+1}^e(z) \leq S_{i,\nu+1}^e(z) < \infty$, such that

$$\begin{aligned}
C_{i,\nu+1}(x, z) &= \min_{y^e \geq x} \{K_{i,\nu+1} \delta(y^e - x) + G_{i,\nu+1}(y^e, z)\} - c^e x \\
&= \begin{cases} K_{i,\nu+1}^e + c^e S_{i,\nu+1}^e(z) + \alpha E_\xi L(S_{i,\nu+1}^e(z) - \xi) \\ \quad + \alpha E_\xi C_{i,\nu}(S_{i,\nu+1}^e(z) + z - \xi, 0) - c^e x, & x \leq s_{i,\nu+1}^e(z) \\ \alpha E_\xi L(x - \xi) + \alpha E_\xi C_{i,\nu}(x + z - \xi, 0), & x > s_{i,\nu+1}^e(z). \end{cases}
\end{aligned} \tag{75}$$

Based on Lemmas and $K_{i,\nu+1}^e \geq K_{i,\nu}^e$, we have $C_{i,\nu+1}(x, z)$ is a nonnegative, continuous and $K_{i,\nu+1}^e$ -convex function.

Therefore, we proved that $C_{i,j}(x, z)$ is a nonnegative, continuous and $K_{i,j}^e$ -convex function for $j = m - \tau + 1, \dots, m - 1$. This implies that there are exists an optimal $(s_{i,j}^e(z), S_{i,j}^e(z))$ policy for given finite z .

Finally, consider the review epoch for review cycle $i + 1$ and the associated cost function $C_{i+1,0}(x, 0)$. Since $C_{i,m-1}(x, z)$ is $K_{i,m-1}^e$ -convex, it follows that $J_{i+1,0}(y^e, y^r - y^e) = c^r y^r +$

$\alpha E_\xi C_{i,m-1}(y^e - \xi, y^r - y^e)$ is a nonnegative, continuous and $K_{i,m-1}^e$ -convex function for any y^e . Now we shall show that $\lim_{|y^r| \rightarrow \infty} J_{i+1,0}(y^e, y^r - y^e) = \infty$ for given y^e .

Equation (74) and (75) implying

$$\begin{aligned}
& \lim_{|y^r| \rightarrow \infty} J_{i+1,0}(y^e, y^r - y^e) \\
& \geq \lim_{|y^r| \rightarrow \infty} \left\{ c^r y^r + \alpha^2 E_{\xi_{i+1,0}} [E_{\xi_{i,m-1}} (\delta(y^e - \xi_{i+1,0}) C_{i,m-2}(y^e - \xi_{i+1,0} - \xi_{i,m-1}, y^r - y^e)) \right\} \\
& \quad \vdots \\
& \geq \lim_{|y^r| \rightarrow \infty} \left\{ c^r y^r \right. \\
& \quad \left. + \alpha^{\tau+1} E_{\xi_{i+1,0}} E_{\xi_{i,m-1}} \cdots E_{\xi_{i,m-\tau-1}} \left[\prod_{\kappa=m-1}^{m-\tau+2} \delta \left(y^e - \xi_{i+1,0} - \sum_{j=m-1}^{\kappa} \xi_{i,j} - s_{i,\kappa}^e (y^r - y^e) \right) \right] \right. \\
& \quad \left. \times \delta \left(y^r - \xi_{i+1,0} - \sum_{j=m-1}^{m-\tau+1} \xi_{i,j} - s_{i,m-\tau}^e \right) L \left(y^r - \xi_{i+1,0} - \sum_{j=m-1}^{m-\tau} \xi_{i,j} \right) \right\} = \infty.
\end{aligned}$$

Hence there exists an $(s_{i+1,0}^r(y^e), S_{i+1,0}^T(y^e))$ optimal policy such that

$$J_{i+1}(y^e) = \begin{cases} K_{i+1,0}^r + c^r S_{i+1,0}^r(y^e) \\ \quad + \alpha E_\xi C_{i,m-1}(S_{i+1,0}^r(y^e) - \xi, S_{i+1,0}^r(y^e) - y^e) - c^r y^e, & y^e \leq s_{i+1,0}^r(y^e) \\ \alpha E_\xi C_{i,m-1}(y^e - \xi, 0), & y^e > s_{i+1,0}^r(y^e). \end{cases}$$

This function is nonnegative, continuous and $K_{i+1,0}^r$ -convex since $K_{i+1,0}^r \geq K_{i,m-1}^e$. Therefore, it is easy to prove that $G_{i+1,0}(y^e)$ is $K_{i+1,0}^r$ -convex and $\lim_{|y^e| \rightarrow \infty} G_{i+1,0}(y^e) = \infty$; hence there exist values $s_{i+1,0}^e$ and $S_{i+1,0}^e$ such that

$$\begin{aligned}
C_{i+1,0}(x, 0) &= \min_{y^e \geq x} \{ G_{i+1,0}(y^e) + K_{i+1,0}^e(y^e - x) \} - c^e x \\
&= \begin{cases} K_{i+1,0}^e + c^e S_{i+1,0}^e + \alpha E_\xi L(S_{i+1,0}^e - \xi) + J_{i+1}(S_{i+1,0}^e) - c^e x, & x \leq s_{i+1,0}^e \\ \alpha E_\xi L(x - \xi) + J_{i+1}(x), & x > s_{i+1,0}^e, \end{cases}
\end{aligned}$$

and $C_{i+1,0}(x, 0)$ is a nonnegative, continuous and $K_{i+1,0}^e$ -convex function since $K_{i+1,0}^e \geq K_{i+1,0}^r$.

This completes the proof. □

5.5 Properties of Optimal Policies

While the state-dependent (s, S) type policies in Section 5.4 are optimal, it is very difficult to determine the optimal values of the parameters. Below we present some analytical properties of these optimal policies. Section 5.6 will attempt to some properties based on numerical studies. We start with the following auxiliary result.

LEMMA 5.3. *[Sethi et al. 2003] Let W_1 and W_2 be continuous and almost everywhere differentiable K -convex functions with $\lim_{u \rightarrow \infty} W_i(u) = \infty, i = 1, 2$. Assume that*

$$dW_1(u)/du \geq dW_2(u)/du, \quad (76)$$

whenever both derivatives exist. Let s_i and S_i be such that for $i = 1, 2$

$$W_i(S_i) = \min_u \{W_i(u)\},$$

$$s_i = \min_u \{u : W_i(u) = K + W_i(S_i), u \leq S_i\}.$$

Then $s_1 \leq s_2$ and $S_1 \leq S_2$.

Based on the above lemma, we obtain the following results.

PROPOSITION 5.1. *For each i and $\Delta \geq 0$, $s_{i,m-\tau+1}^e(z + \Delta) = s_{i,m-\tau+1}^e(z) - \Delta$ and $S_{i,m-\tau+1}^e(z + \Delta) = S_{i,m-\tau+1}^e(z) - \Delta$.*

Proof. Taking the first derivative on equation (62), we obtain

$$\frac{d}{dy^e} G_{i,m-\tau+1}(y^e, z) = c^e + E_\xi \left[\frac{d}{dy^e} L(y^e + z - \xi) \right] + \alpha E_\xi \left[\frac{d}{dy^e} C_{i,m-\tau}(y^e + z - \xi, 0) \right]. \quad (77)$$

Notice that $S_{i,m-\tau+1}^e(z)$ satisfies $\frac{d}{dy^e} G_{i,m-\tau+1}(S_{i,m-\tau+1}^e(z), z) = 0$. If we increase z by Δ and replace y^e by $S_{i,m-\tau+1}^0(z) - \Delta$, we find that $\frac{d}{dy^e} G_{i,m-\tau+1}(S_{i,m-\tau+1}^0(z) - \Delta, z + \Delta) = 0$. This implies that $S_{i,m-\tau+1}^0(z + \Delta) = S_{i,m-\tau+1}^0(z) - \Delta$.

Next we substitute $S_{i,m-\tau+1}^0(z)$ and $S_{i,m-\tau+1}^0(z + \Delta)$ into equation (62) with the inventory on order being z and $z + \Delta$, respectively. We have

$$G_{i,m-\tau+1}(S_{i,m-\tau+1}^0(z + \Delta), z + \Delta) = G_{i,m-\tau+1}(S_{i,m-\tau+1}^0(z), z) - c^e \Delta. \quad (78)$$

Adding $K_{i,m-\tau+1}^e$ to both side of equation (78), we get

$$G_{i,m-\tau+1}(s_{i,m-\tau+1}^0(z+\Delta), z+\Delta) = G_{i,m-\tau+1}(s_{i,m-\tau+1}^0(z), z) - c^e \Delta.$$

Therefore, we can prove that $s_{i,m-\tau+1}^0(z+\Delta) = s_{i,m-\tau+1}^0(z) - \Delta$. \square

COROLLARY 5.1. *For each i and $\Delta \geq 0$, $\frac{d}{dy^e} G_{i,m-\tau+1}(y^e, z+\Delta) = \frac{d}{dy^e} G_{i,m-\tau+1}(y^e + \Delta, z)$ and $\frac{d}{dy^e} G_{i,m-\tau+1}(y^e, z) = \frac{d}{dy^e} G_{i,m-\tau+1}(y^e - \Delta, z + \Delta)$.*

Proof. The result follows from equation (77). \square

COROLLARY 5.2. *For all $\Delta \geq 0$, $s_{i,m-\tau+2}^e(z) \geq s_{i,m-\tau+2}^e(z+\Delta)$ and $S_{i,m-\tau+2}^e(z) \geq S_{i,m-\tau+2}^e(z+\Delta)$.*

Proof. Based on Lemma 5.3, it suffices to show

$$\frac{d}{dy^e} G_{i,m-\tau+2}(y^e, z) \leq \frac{d}{dy^e} G_{i,m-\tau+2}(y^e, z+\Delta).$$

From equation (61) we have

$$\begin{aligned} & \frac{d}{dy^e} G_{i,m-\tau+2}(y^e, z) \\ &= c^e + \mathbb{E}_\xi \left[\frac{d}{dy^e} L(y^e - \xi) \right] + \mathbb{E}_\xi \left[\frac{d}{dy^e} C_{i,m-\tau+1}(y^e - \xi, z) \right] \\ &= c^e + \mathbb{E}_\xi \left[\frac{d}{dy^e} L(y^e - \xi) \right] + \int_{y^e - \xi \leq s_{i,m-\tau+1}^e(z)} -c^e f(\xi) d\xi \\ & \quad + \int_{s_{i,m-\tau+1}^e(z) \leq y^e - \xi \leq S_{i,m-\tau+1}^e(z)} \left[-c^e + \frac{d}{dy^e} G_{i,m-\tau+1}(y^e - \xi, z) \right] f(\xi) d\xi \\ &= c^e + \mathbb{E}_\xi \left[\frac{d}{dy^e} L(y^e - \xi) \right] - c^e [1 - F(y^e - S_{i,m-\tau+1}^e(z))] \\ & \quad + \int_{s_{i,m-\tau+1}^e(z) \leq y^e - \xi \leq S_{i,m-\tau+1}^e(z)} \frac{d}{dy^e} G_{i,m-\tau+1}(y^e - \xi, z) f(\xi) d\xi. \end{aligned}$$

Similarly,

$$\begin{aligned}
& \frac{d}{dy^e} G_{i,m-\tau+2}(y^e, z + \Delta) \\
&= c^e + E_\xi \left[\frac{d}{dy^e} L(y^e - \xi) \right] + E_\xi \left[\frac{d}{dy^e} C_{i,m-\tau+1}(y^e - \xi, z + \Delta) \right] \\
&= c^e + E_\xi \left[\frac{d}{dy^e} L(y^e - \xi) \right] + \int_{y^e - \xi \leq s_{0,m-\tau+1}^e(z+\Delta)} -c^e f(\xi) d\xi \\
&\quad + \int_{s_{0,m-\tau+1}^e(z+\Delta) \leq y^e - \xi \leq S_{0,m-\tau+1}^e(z+\Delta)} \left[-c^e + \frac{d}{dy^e} G_{i,m-\tau+1}(y^e - \xi, z + \Delta) \right] f(\xi) d\xi \\
&= c^e + E_\xi \left[\frac{d}{dy^e} L(y^e - \xi) \right] - c^e [1 - F(y^e - S_{i,m-\tau+1}^e(z + \Delta))] \\
&\quad + \int_{s_{i,m-\tau+1}^e(z+\Delta) \leq y^e - \xi \leq S_{i,m-\tau+1}^e(z+\Delta)} \frac{d}{dy^e} G_{i,m-\tau+1}(y^e - \xi, z + \Delta) f(\xi) d\xi \\
&= c^e + E_\xi \left[\frac{d}{dy^e} L(y^e - \xi) \right] - c^e [1 - F(y^e - S_{i,m-\tau+1}^e(z + \Delta))] \\
&\quad + \int_{s_{i,m-\tau+1}^e(z) - \Delta \leq y^e - \xi \leq S_{i,m-\tau+1}^e(z) - \Delta} \frac{d}{dy^e} G_{i,m-\tau+1}(y^e + \Delta - \xi, z) f(\xi) d\xi \\
&= c^e + E_\xi \left[\frac{d}{dy^e} L(y^e - \xi) \right] - c^e [1 - F(y^e - S_{i,m-\tau+1}^e(z + \Delta))] \\
&\quad + \int_{s_{i,m-\tau+1}^e(z) \leq y^e + \Delta - \xi \leq S_{i,m-\tau+1}^e(y)} \frac{d}{dy^e} G_{i,m-\tau+1}(y^e + \Delta - \xi, z) f(\xi) d\xi \\
&= c^e + E_\xi \left[\frac{d}{dy^e} L(y^e - \xi) \right] - c^e [1 - F(y^e - S_{i,m-\tau+1}^e(z + \Delta))] \\
&\quad + \int_{s_{i,m-\tau+1}^e(z) \leq y^{e'} - \xi \leq S_{i,m-\tau+1}^e(z)} \frac{d}{dy^e} G_{i,m-\tau+1}(y^{e'} - \xi, z) f(\xi) d\xi \\
&\geq \frac{d}{dy^e} G_{i,m-\tau+2}(y^e, z).
\end{aligned}$$

The last inequality follows from Theorem 5.1 and $F(y^e - S_{i,m-\tau+1}^e(z + \Delta)) \geq F(y^e - S_{i,m-\tau+1}^e(z))$. This completes the proof. \square

5.6 Numerical Results

In this section we attempt to obtain an insight on some properties of optimal policies based on numerical studies. To carry out these numerical studies, we consider a baseline inventory system with the following parameters, that are analogous to those used in Chiang and Gutierrez (1998): length of a review cycle $m = 10$ days, lead-time of an emergency order $\tau = 6$ days, unit ordering cost of emergency order $c^e = 15$, unit ordering cost of regular order $c^r = 10$, Poisson daily demand with mean $\mu = 2$, unit holding cost per day

$h = 0.01$, unit backorder cost per day $p = 20$, and discount factor $\alpha = 0.999$. The setup cost for emergency and regular order is set at 30 for all the days in the planning horizon of 40 review cycles.

Once all the system parameters are specified, we can compute numerically the expected discounted cost by using the backward dynamic programming approach based on the cost functions in Section 5.3.1. In Section 5.4 we proved that the state-dependent (s, S) type policies with varying reorder points and order-up-to levels are optimal. Hence, the optimal values of inventory policy and the optimal expected discounted cost can be obtained.

5.6.1 Properties of Optimal Policies

As we derived in Section 5.4, the regular ordering decision depends on the inventory position after a possible emergency order. Therefore, we first investigate the relationship between the size of the optimal regular order and the inventory position after a possible emergency order at the review epoch. The optimal regular order quantity z as a function of the inventory position y^e after a possible emergency order is shown in Table 14. The first row of the table is the remaining number of review cycles.

Two conclusions can be drawn from the numerical results in this table. For a given value of the inventory position after a possible emergency order, the optimal regular order quantity z appears to converge to a certain value, which we call a *stationary optimal regular order quantity*. For example, this optimal value is stabilized at 92 for $y^e = 10$ if $i \geq 30$. As shown in the table, the stationary regular order quantity appears to be a non-increasing function of the inventory position y^e .

Second, we will study the effect of the regular order quantity z on the decision for the emergency order before the regular order arrives (i.e., $5 \leq j \leq 9$) within each review cycle. By fixing the regular order quantity z at the review epoch, we search for the optimal reorder points and order-up-to levels for emergency orders and present them in Table 15. Based on these numerical results, the optimal policy for emergency orders converges to a stationary policy after certain review cycles. However, it converges slower than the case where the setup cost is zero presented in Chiang and Gutierrez (1998). At the epoch just before the

Table 14: The optimal regular order quantity z as a function of the inventory position y^e after a possible emergency order at a regular review epoch i

$y^e \setminus i$	1	2	3	4	5	6	7	8	9	10	15	20	25	≥ 30
10	10	29	49	67	84	100	88	90	92	94	91	91	92	92
11	10	30	49	67	84	100	88	90	92	94	91	91	92	92
12	10	29	48	67	84	100	88	90	92	94	91	91	92	92
13	9	29	48	66	83	99	88	89	92	93	91	90	91	92
14	9	28	47	65	83	99	87	89	91	93	90	90	91	91
15	8	27	46	64	82	98	86	88	90	92	89	89	90	90
16	7	27	46	64	81	97	85	87	89	91	88	88	89	89
17	6	26	45	63	80	96	84	86	88	90	87	87	88	88
18	0	25	44	62	79	95	83	85	87	89	86	86	87	87
19	0	24	43	61	78	94	82	84	86	88	85	85	86	86
20	0	23	42	60	77	93	81	83	85	87	84	84	85	85
21	0	22	41	59	76	92	80	82	84	86	83	83	84	84
22	0	21	40	58	75	91	79	81	83	85	82	82	83	83
23	0	20	39	57	74	90	78	80	82	84	81	81	82	82
24	0	19	38	56	73	89	77	79	81	83	80	80	81	81
25	0	18	37	55	72	88	76	78	80	82	79	79	80	80
26	0	17	36	54	71	87	75	77	79	81	78	78	79	79
27	0	16	35	53	70	86	74	76	78	80	77	77	78	78
28	0	15	34	52	69	85	73	75	77	79	76	76	77	77
29	0	14	33	51	68	84	72	74	76	78	75	75	76	76
30	0	13	32	50	67	83	71	73	75	77	74	74	75	75
31	0	12	31	49	66	82	70	72	74	76	73	73	74	74
32	0	11	30	48	65	81	69	71	73	75	72	72	73	73
33	0	10	29	47	64	80	68	70	72	74	71	71	72	72
34	0	9	28	46	63	79	67	69	71	73	70	70	71	71
35	0	0	27	45	62	78	66	68	70	72	69	69	70	70

regular order arrives, i.e., $j = 5$, it appears that the stationary values of s^e and S^e change by the same quantity but in the opposite direction compared to the regular order quantity. For example, as the regular order quantity increases from 0 to 1, then the stationary reorder point decreases from 7 to 6. However, two days before the regular order arrives, i.e., $j = 6$, the stationary values of s^e and S^e change less than the regular order quantity. Moreover, it is worth mentioning that the optimal reorder point and the order-up-to level for emergency orders do not decrease within each review cycle for $j = 6, \dots, 9$. Unfortunately, we were not able to obtain a theoretical proof.

5.6.2 Effect of Emergency Orders

In this section, we investigate the impact of emergency orders on the system cost by comparing our model with a case in which emergency orders can not be placed within a review cycle. For simplicity, we use the fixed regular order policies without reorder points. This heuristics policy induces larger costs than the optimal policy.

The optimal expected discounted costs at review epochs, denoted by C_0^* and C_1^* for the cases with and without emergency order, respectively, are displayed in Tables 16 to 18. Here Δ_C is the percentage difference of which C_1^* exceeds C_0^* , i.e., $\Delta_C = (C_1^* - C_0^*) \times 100 / C_0^* \%$. For simplification, we consider the regular order quantity z at three fixed values. The setup costs for both regular and emergency orders are still fixed at 30. To see how the effect depends on some parameters, we consider different unit holding costs h and unit backorder costs p , as shown in the tables.

From the numerical results, we can clearly see that the performance of our model with the option of emergency orders is better than that without emergency orders within a review cycle. The performance deteriorates as the review cycle becomes longer. Furthermore, for fixed review cycle length even though C_0^* and C_1^* keep increasing as the model has more review cycles, the percentage difference Δ_C seems to converge to a certain value. As the unit holding or the backorder cost increase, the “stationary” value of Δ_C increases as well. Since the unit holding cost is much smaller than the unit backorder cost, the stationary value of Δ_C becomes smaller while the regular order quantity z increases.

Table 15: Optimal emergency order policy (s^e, S^e) as a function of the regular order quantity z

z	j	$i = 0$	$i = 10$	$i = 20$	$i = 30$	$i \geq 40$
0	5	(6,12)	(7,75)	(7,75)	(7,75)	(7,75)
	6	(7,14)	(7,75)	(7,74)	(7,75)	(7,75)
	7	(7,15)	(7,76)	(7,74)	(7,75)	(7,75)
	8	(7,17)	(7,76)	(7,75)	(7,75)	(7,75)
	9	(7,19)	(7,77)	(7,75)	(7,75)	(7,75)
1	5	(5,12)	(6,72)	(6,73)	(6,74)	(6,74)
	6	(6,14)	(7,73)	(7,73)	(7,73)	(7,74)
	7	(7,15)	(7,73)	(7,73)	(7,75)	(7,74)
	8	(7,17)	(7,74)	(7,73)	(7,75)	(7,74)
	9	(7,19)	(7,74)	(7,73)	(7,75)	(7,75)
2	5	(4,12)	(6,70)	(5,73)	(5,73)	(5,73)
	6	(6,14)	(7,70)	(7,73)	(7,73)	(7,73)
	7	(7,15)	(8,71)	(7,73)	(7,75)	(7,73)
	8	(7,17)	(8,72)	(8,73)	(7,74)	(7,74)
	9	(7,19)	(8,72)	(8,73)	(8,74)	(8,74)
3	5	(3,12)	(5,68)	(4,73)	(4,72)	(4,72)
	6	(6,14)	(7,68)	(7,73)	(7,72)	(7,72)
	7	(7,15)	(8,69)	(7,73)	(7,72)	(7,72)
	8	(7,17)	(8,70)	(8,73)	(8,73)	(8,73)
	9	(7,19)	(8,70)	(8,73)	(8,73)	(8,73)
4	5	(2,12)	(4,66)	(4,71)	(3,71)	(3,71)
	6	(5,14)	(7,67)	(7,72)	(7,70)	(7,70)
	7	(6,16)	(8,67)	(7,72)	(7,71)	(7,71)
	8	(7,17)	(8,68)	(8,72)	(8,71)	(8,71)
	9	(7,19)	(8,69)	(8,73)	(8,72)	(8,72)
5	5	(1,12)	(3,67)	(2,70)	(2,70)	(2,70)
	6	(5,14)	(7,67)	(7,69)	(7,69)	(7,69)
	7	(6,16)	(7,67)	(7,70)	(7,70)	(7,70)
	8	(7,18)	(8,67)	(8,70)	(8,70)	(8,70)
	9	(7,19)	(8,68)	(8,71)	(8,71)	(8,71)

Table 16: Comparison of system costs with and without emergency order ($h = 0.01$)

z	i	$p = 20$			$p = 50$			$p = 100$		
		C_0^*	C_1^*	Δ_C	C_0^*	C_1^*	Δ_C	C_0^*	C_1^*	Δ_C
3	1	395.0	411.1	4.1	418.8	448.5	7.1	433.2	473.3	9.3
	5	1682.1	1709.5	1.6	1705.6	1748.1	2.5	1720.1	1773.8	3.1
	10	3223.1	3260.9	1.2	3246.4	3300.1	1.7	3260.9	3326.4	2.0
	15	4687.4	4733.3	1.0	4710.8	4772.9	1.3	4725.2	4799.4	1.6
	20	6081.3	6134.1	0.9	6104.6	6174.1	1.1	6119.1	6201.0	1.3
	25	7407.1	7466.4	0.8	7430.5	7506.9	1.0	7444.9	7534.2	1.2
	30	8668.0	8733.0	0.7	8691.3	8774.0	1.0	8705.8	8801.6	1.1
	35	9867.5	9937.2	0.7	9890.8	9978.5	0.9	9905.3	10006.5	1.0
	40	11008.5	11082.2	0.7	11031.8	11123.9	0.8	11046.2	11152.2	1.0
5	1	386.9	402.3	4.0	410.0	438.3	6.9	424.3	463.4	9.2
	5	1632.3	1658.7	1.6	1655.3	1696.6	2.5	1669.6	1722.0	3.1
	10	3123.4	3157.8	1.1	3146.5	3196.3	1.6	3160.8	3222.2	1.9
	15	4540.4	4582.5	0.9	4563.5	4621.4	1.3	4577.9	4647.5	1.5
	20	5888.3	5936.6	0.8	5911.3	5975.9	1.1	5925.7	6002.8	1.3
	25	7170.7	7224.3	0.7	7193.8	7264.1	1.0	7208.2	7291.3	1.2
	30	8390.6	8448.8	0.7	8413.7	8489.0	0.9	8428.2	8516.6	1.0
	35	9550.9	9613.0	0.7	9574.0	9653.6	0.8	9588.5	9681.7	1.0
	40	10654.6	10720.0	0.6	10677.6	10761.0	0.8	10692.2	10789.4	0.9
10	1	380.3	390.8	2.8	401.7	422.8	5.3	415.4	444.8	7.1
	5	1524.6	1545.7	1.4	1545.8	1578.5	2.1	1559.4	1601.2	2.7
	10	2887.6	2913.4	0.9	2908.9	2946.8	1.3	2922.7	2970.0	1.6
	15	4184.0	4213.9	0.7	4205.5	4247.9	1.0	4219.4	4271.4	1.2
	20	5417.1	5450.3	0.6	5438.7	5484.8	0.8	5452.8	5508.8	1.0
	25	6590.0	6626.0	0.5	6611.7	6661.0	0.7	6625.8	6685.3	0.9
	30	7705.7	7743.8	0.5	7727.5	7779.3	0.7	7741.8	7804.0	0.8
	35	8766.9	8806.6	0.5	8788.8	8842.6	0.6	8803.3	8867.6	0.7
	40	9776.4	9817.2	0.4	9798.3	9853.5	0.6	9812.9	9878.9	0.7

Table 17: Comparison of system costs with and without emergency order ($h = 0.1$)

z	i	$p = 20$			$p = 50$			$p = 100$		
		C_0^*	C_1^*	Δ_C	C_0^*	C_1^*	Δ_C	C_0^*	C_1^*	Δ_C
3	1	401.6	419.0	4.3	426.3	458.9	7.6	441.6	485.4	9.9
	5	1770.9	1821.1	2.8	1799.2	1869.9	3.9	1817.2	1902.6	4.7
	10	3409.2	3499.4	2.6	3441.6	3558.0	3.4	3462.6	3597.5	3.9
	15	4967.8	5093.6	2.5	5004.0	5161.0	3.1	5028.0	5206.8	3.6
	20	6450.4	6610.2	2.5	6490.2	6686.1	3.0	6517.0	6738.0	3.4
	25	7860.6	8051.8	2.4	7903.9	8135.7	2.9	7933.3	8193.4	3.3
	30	9202.0	9422.7	2.4	9248.6	9514.2	2.9	9280.6	9577.3	3.2
	35	10477.9	10726.0	2.4	10527.7	10824.9	2.8	10562.0	10893.2	3.1
	40	11691.6	11965.3	2.3	11744.3	12071.1	2.8	11781.0	12144.3	3.1
5	1	392.7	409.4	4.2	417.3	448.4	7.4	432.3	474.7	9.8
	5	1716.3	1762.5	2.7	1744.2	1810.3	3.8	1762.1	1842.8	4.6
	10	3300.1	3382.4	2.5	3332.1	3439.8	3.2	3353.3	3479.1	3.8
	15	4806.9	4921.3	2.4	4842.7	4987.6	3.0	4867.0	5033.3	3.4
	20	6240.1	6385.1	2.3	6279.6	6459.8	2.9	6306.9	6511.5	3.2
	25	7603.4	7776.6	2.3	7646.4	7859.4	2.8	7676.5	7916.9	3.1
	30	8900.2	9099.8	2.2	8946.5	9190.2	2.7	8979.3	9253.2	3.1
	35	10133.6	10357.9	2.2	10183.2	10455.5	2.7	10218.5	10523.8	3.0
	40	11307.0	11554.0	2.2	11359.5	11658.6	2.6	11397.3	11731.8	2.9
10	1	385.6	397.2	3.0	408.6	430.9	5.5	423.1	454.7	7.5
	5	1596.0	1629.3	2.1	1622.4	1671.0	3.0	1639.5	1700.3	3.7
	10	3043.8	3099.6	1.8	3074.7	3149.9	2.4	3095.0	3185.5	2.9
	15	4421.0	4497.8	1.7	4456.2	4556.2	2.2	4479.5	4597.9	2.6
	20	5731.1	5827.2	1.7	5770.3	5893.3	2.1	5796.4	5940.8	2.5
	25	6977.2	7091.2	1.6	7020.2	7164.7	2.1	7049.1	7217.7	2.4
	30	8162.5	8293.0	1.6	8209.2	8373.5	2.0	8240.7	8431.7	2.3
	35	9289.9	9435.7	1.6	9340.2	9522.9	2.0	9374.1	9586.1	2.3
	40	10362.4	10522.2	1.5	10415.9	10615.7	1.9	10452.2	10683.6	2.2

Table 18: Comparison of system costs with and without emergency order ($h = 1.0$)

z	i	$p = 20$			$p = 50$			$p = 100$		
		C_0^*	C_1^*	Δ_C	C_0^*	C_1^*	Δ_C	C_0^*	C_1^*	Δ_C
3	1	455.5	492.6	8.2	492.1	557.9	13.4	515.1	601.5	16.8
	5	2148.7	2348.8	9.3	2227.8	2502.0	12.3	2279.3	2604.0	14.2
	10	4172.1	4566.2	9.4	4301.9	4824.3	12.1	4387.8	4996.0	13.9
	15	6096.9	6674.6	9.5	6274.9	7032.5	12.1	6393.3	7270.4	13.7
	20	7927.7	8679.2	9.5	8151.6	9132.0	12.0	8301.0	9432.9	13.6
	25	9669.2	10585.2	9.5	9936.7	11128.3	12.0	10115.5	11489.1	13.6
	30	11325.7	12397.5	9.5	11634.8	13026.3	12.0	11841.6	13444.1	13.5
	35	12901.4	14120.7	9.5	13250.0	14831.0	11.9	13483.4	15302.9	13.5
	40	14400.2	15759.0	9.4	14786.3	16546.9	11.9	15045.1	17070.3	13.5
5	1	442.6	475.2	7.4	478.9	538.9	12.5	501.8	582.1	16.0
	5	2078.6	2254.8	8.5	2158.0	2405.7	11.5	2210.1	2506.9	13.4
	10	4034.2	4380.9	8.6	4164.8	4635.9	11.3	4251.9	4806.3	13.0
	15	5894.3	6402.4	8.6	6073.7	6756.4	11.2	6194.0	6992.6	12.9
	20	7663.7	8324.5	8.6	7889.4	8772.5	11.2	8041.4	9071.3	12.8
	25	9346.7	10152.0	8.6	9616.6	10689.6	11.2	9798.6	11047.9	12.7
	30	10947.6	11889.6	8.6	11259.5	12512.3	11.1	11470.1	12927.1	12.7
	35	12470.4	13541.8	8.6	12822.2	14245.3	11.1	13060.0	14714.0	12.7
	40	13918.9	15112.7	8.6	14308.7	15893.1	11.1	14572.3	16412.9	12.6
10	1	433.6	454.7	4.9	468.3	508.2	8.5	491.0	546.4	11.3
	5	1949.9	2064.0	5.9	2029.0	2194.4	8.2	2081.2	2285.8	9.8
	10	3763.5	3988.1	6.0	3895.1	4209.7	8.1	3982.7	4364.7	9.6
	15	5488.6	5817.5	6.0	5670.1	6125.9	8.0	5791.5	6341.3	9.5
	20	7129.6	7556.9	6.0	7358.5	7947.8	8.0	7511.9	8220.6	9.4
	25	8690.4	9210.7	6.0	8964.6	9680.1	8.0	9148.5	10007.5	9.4
	30	10175.1	10783.2	6.0	10492.2	11327.2	8.0	10705.1	11706.5	9.4
	35	11587.4	12278.4	6.0	11945.3	12893.2	7.9	12185.8	13322.0	9.3
	40	12930.8	13700.0	5.9	13327.6	14382.2	7.9	13594.3	14857.9	9.3

Table 19: Optimal expected discount cost with different setup costs

Scenario	$i = 1$	$i = 5$	$i = 10$	$i = 15$	$i = 20$	$i = 25$	$i = 30$
1	379.1	1229.6	2249.9	3215.8	4135.1	5010.1	5841.4
2	379.3	1255.5	2336.4	3389.8	4422.1	5436.3	6432.6

5.6.3 Effect of Optimal Policy

In Section 5.4, we already proved that the optimal inventory policy for emergency orders and regular orders is state-dependent. Then it is natural to investigate how much the optimal policies improve the system performance over a stationary policy. In this section, numerical studies are carried out to address this question.

First, consider our state-dependent inventory control policies. The numerical studies are composed of two scenarios with different setup cost structures. In the first scenario, we assume that the setup cost of both emergency and regular order is fixed at 30 for the entire planning horizon. In the second scenario, the setup costs of emergency and regular order $K_{i,j}^e$ and $K_{i,j}^r$ are given by the following equations: $K_{i,j}^e = 30 + 5i$ and $K_{i,j}^r = K_{i-1,m-1}^e + 2$. The optimal expected discount costs under our state-dependent inventory control policy at some review epochs are reported in Table 19.

Next, to compare our optimal dynamic control policy with the stationary policy to illustrate the impact of our optimal policy on the expected discount cost, we choose several stationary policies on the basis of previous numerical results. Stationarity refers to the fact that the regular order quantity and the order-up-to level of emergency order are fixed instead of dynamically changing for all the decision epochs as shown in Tables 20 and 21. The reorder points for emergency orders can be determined by the property of K -convexity and the order-up-to level. Afterwards, the expected discounted cost can be calculated for each stationary policy. The results for two scenarios are respectively reported Tables 20 and 21.

Apparently, the stationary policies incur significantly larger costs than the dynamic optimal policies. Moreover, the stationary policies worsen when the setup costs change as in scenario 2. Therefore, in practice the inventory manager has to consider the trade-offs

Table 20: Expected discount cost with different stationary policies for scenario 1

z	$S_{i,j}^e$	$i = 1$	$i = 5$	$i = 10$	$i = 15$	$i = 20$	$i = 25$	$i = 30$
5	30	532.1	1695.3	3214.6	4658.2	6030.8	7335.8	8576.6
	35	607.4	1697.8	3201.3	4633.5	5995.0	7289.5	8520.4
	40	682.9	1698.6	3196.5	4620.3	5974.5	7262.2	8486.5
	45	758.4	1698.2	3194.4	4614.7	5964.3	7247.4	8467.3
	50	833.9	1711.5	3204.6	4617.1	5963.4	7243.5	8460.2
	55	909.4	1724.3	3205.4	4620.9	5963.8	7241.9	8456.7
	60	984.9	1734.6	3208.7	4624.8	5967.2	7242.7	8457.1
	65	1060.4	1751.6	3227.7	4633.3	5976.8	7253.8	8465.9
	70	1135.9	1760.3	3239.3	4645.5	5983.7	7258.5	8472.0
	75	1211.4	1765.3	3246.3	4656.4	5996.5	7270.6	8482.3
	80	1286.9	1761.9	2956.5	4665.1	6007.7	7283.9	8497.1
10	30	582.2	1580.0	2955.8	4263.3	5506.3	6688.3	7812.1
	35	657.7	1586.2	2954.6	4254.7	5491.0	6666.6	7784.2
	40	733.2	1604.5	2961.8	4258.5	5490.2	6661.5	7775.1
	45	808.6	1616.3	2969.6	4261.8	5491.4	6660.1	7771.3
	50	884.2	1629.3	2985.0	4274.2	5500.7	6667.4	7777.0
	55	959.7	1634.0	2992.6	4282.7	5509.2	6675.3	7784.1
	60	1035.2	1638.4	3000.7	4292.3	5519.8	6686.3	7795.1

between benefits (ease of use) and disadvantages (additional cost).

Here is another important observation: If we specify a high order-up-to level and a large regular order quantity, the cost will be large when the model has a short planning horizon as in the case of a fashion product. When the planning horizon is long, the cost becomes smaller than that with a small order-up-level and regular order quantity. Therefore, properly choosing a stationary policy based on the type of product can outweigh their disadvantage of the complexity of optimal policies and will be our future work.

Table 21: Expected discount cost with different stationary policies for scenario 2

z	$S_{i,j}^e$	$i = 1$	$i = 5$	$i = 10$	$i = 15$	$i = 20$	$i = 25$	$i = 30$
5	30	534.1	1775.5	3561.4	5450.1	7436.9	9515.7	11681.6
	35	609.4	1777.8	3530.3	5381.6	7322.1	9346.9	11451.2
	40	684.9	1773.1	3510.5	5334.4	7241.4	9227.1	11287.0
	45	760.4	1768.3	3496.4	5302.8	7185.2	9141.3	11167.5
	50	835.9	1780.9	3498.7	5283.7	7148.5	9081.8	11081.4
	55	911.4	1798.4	3492.5	5275.9	7121.7	9038.5	11016.6
	60	986.9	1808.8	3490.2	5264.0	7102.7	9001.5	10964.3
	65	1062.4	1826.8	3507.5	5261.0	7090.0	8982.1	10930.2
	70	1137.9	1838.1	3520.1	5270.0	7084.7	8962.8	10900.0
	75	1213.4	1838.0	3521.0	5271.7	7083.5	8953.8	10879.9
	80	1288.9	1824.8	3518.1	5268.4	7077.7	8943.7	10863.7
10	30	584.2	1652.7	3263.4	4962.1	6745.4	8608.3	10546.4
	35	659.7	1656.1	3248.3	4922.0	6674.8	8502.1	10399.6
	40	735.2	1678.0	3246.9	4905.0	6635.6	8436.8	10304.7
	45	810.7	1688.6	3248.5	4890.6	6605.8	8388.0	10233.8
	50	886.2	1697.6	3258.0	4890.4	6591.9	8358.7	10187.1
	55	961.7	1698.4	3260.5	4888.5	6582.0	8337.9	10153.2
	60	1037.2	1698.6	3263.0	4888.5	6576.8	8324.7	10129.4

CHAPTER VI

CONCLUSION AND FUTURE RESEARCH

6.1 Conclusion

This thesis studied three problems related to inventory control. The first problem relates to the bullwhip effect in supply chains, the second problem investigate the estimation of the demand distribution under partial information, and the third problem studies an inventory system with periodic reviews and emergency orders. In this thesis, we assume that the time sequence of events in a day is as follows. At the beginning of day, an ordering decision is made. Next, an order placed earlier is delivered. Lastly, customer demand is realized. The available inventory is used to fill the demand.

Inspired by the control variate technique for variance reduction in simulation, we proposed a three-step variance reduction method that is easy to implement for dampening the bullwhip effect in supply chains. First, we identified a control variate that is correlated to the original order quantity to establish a generic stabilizing policy. Next we modified the generic stabilizing policy according to some specific managerial requirements to be a feasible policy. The last step of our technique adjusts the feasible policy to achieve better system-wide performance. Two examples including both time-independent and time-dependent demand process were considered to illustrate how to apply the technique step by step and analyze its effectiveness. The numerical and analytical results indicate that the technique could effectively reduce the bullwhip effect while it may achieve a system-wide optimal performance within a class of stabilizing policies. We also conducted a series of numerical experiments to study how the system parameters, such as production capacity, penalty cost, lead-time, and demand standard deviation, affect the decision of fine-tuning policy and the impact of the technique.

The second part of this thesis studied the estimation of the first two moments of the distribution of the daily demand in a practical case where the demand is not monitored.

Based on the estimator of demand variance, we proposed a regression-based approach to improve the estimator. We then used a simulation model on a grid of 216 cases, and applied three statistical measures, RBias, RSD and RRMSE, to evaluate the performance of the estimation. The numerical experiments showed that the estimator of the mean demand under partial information works as well as in the case under full demand information. We also observed that the estimate of the demand variance works well when the order quantity is large and the daily demand follows the negative binomial distribution. On the whole, the adjusted estimate of demand variance by RBA is more stable and robust. In terms of the aggregated cost from 216 cases, the RBA approach may rival the case under full demand information even though the case under partial information is worse.

The third part of this dissertation introduced a periodic inventory model that includes two supply modes. We generalized the assumptions on lead-time by two exist modes and included the setup cost for two modes as well. On the analytical front, we formulated the recursive cost functions and proved the optimality of state-dependent (s, S) type inventory policies for both modes. A few interesting properties of the optimal inventory policies are analytically characterized. Letting daily Poisson demands, we conducted various numerical studies to illustrate the relationship between the optimal regular order quantity and the inventory position after a possible emergency order, and the relation between optimal emergency order and regular order quantity. The numerical results confirm that when a periodic inventory system has longer review cycle or larger unit penalty cost, the emergency orders can significantly reduce the system cost. While the implementation of state-dependent optimal inventory policies is more complex than a stationary policy, it can significantly reduce the system cost. Moreover, we showed that a proper choice of a stationary policy based on the type of product can outweigh the disadvantage of the complexity of optimal policies.

6.2 *Future Research*

There are several points to be explored in the future research.

1. We demonstrated our control variate technique for stationary demand in a centralized manner in Chapter 2. The investigation of the applicability of our technique is an

interesting problem for future research. The application of our technique to demand processes with seasonality is another problem worth future investigation.

2. Based on the assumption that the daily demands are independent, we derived the estimators of the first two moments of the daily demand in Chapter 3. Therefore, it is an interesting and challenging research topic to derive these estimators if the daily demands are correlated.
3. In Chapter 4, we use simulation to evaluate the performance of our estimation by assuming that the order quantity is constant. Testing the estimation of varied order quantity is another research topic.
4. We assume that the lead-times of emergency and regular order are shorter than the length of a review cycle in Chapter 5. If relaxing this assumption, we need to track the arrival time of each order placed so that it increases the dimensions of the state space. Moreover, we did not theoretically obtain the monotone properties of the optimal reorder point and order-up-to level for emergency orders before regular orders could arrive. These topics will be for future research.

APPENDIX A

THEORETICAL PROOF

We start with a few auxiliary lemmas.

LEMMA A.1. *Let u be an standard normal random variable with c.d.f. $\Phi(\cdot)$ and let r be a real constant number. Define*

$$g(r, u) = h(r - u)^+ + \pi(u - r)^+, \quad (79)$$

where constants h and π are unit holding and penalty costs. The value of r that minimizes $E[g(r, u)]$ satisfies $\Phi(r) = \pi/(h + \pi)$.

LEMMA A.2. *Let u be an standard normal random variable, let k be a real constant, and let σ be a positive constant. Then the expectation $E[g(r, u)]$ defined in equation (79) satisfies*

$$E[g(\sigma k, \sigma u)] = \sigma E[g(k, u)]. \quad (80)$$

Proof.

$$E(g(\sigma k, \sigma u)) = h \int_{-\infty}^{\sigma k} (\sigma k - x) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx + \pi \int_{\sigma k}^{\infty} (x - \sigma k) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx.$$

Application of the transformation $y = x/\sigma$ yields $E[g(\sigma k, \sigma u)] = \sigma E[g(k, u)]$. □

LEMMA A.3. *Consider an AR(1) demand process $D_t = d + \rho D_{t-1} + u_t$ with the u_t 's are independent $N(0, \sigma^2)$, let r be a real constant number, and define*

$$v_t = \frac{(G - \rho^*)d}{1 - \rho} + (\rho^* - G)D_{t-1} + \sum_{k=t}^{t+\nu} \sum_{i=t}^k \rho^{k-i} u_i.$$

Then

$$E_D[E_1(g(r, v_t))] = E\left\{g\left[r, \left(\sigma \sqrt{\frac{(\rho^* - G)^2}{1 - \rho^2} + \sum_{k=1}^{\nu+1} \left(\frac{1 - \rho^{\nu+2-k}}{1 - \rho}\right)^2}\right)u\right]\right\}, \quad (81)$$

where u is a standard normal random variable.

Proof. Let

$$x = (\rho^* - G)D_{t-1} - \frac{(\rho^* - G)d}{1 - \rho},$$

$$y = \sum_{k=t}^{t+\nu} \sum_{i=t}^k \rho^{k-i} u_i,$$

$$\sigma_1 = \sqrt{\frac{(\rho^* - G)^2 \sigma^2}{1 - \rho^2}},$$

and

$$\sigma_2 = \sqrt{\sum_{k=1}^{\nu+1} \left(\frac{1 - \rho^{\nu+2-k}}{1 - \rho} \right)^2 \sigma^2}.$$

Since $D_{-1} \sim N(\frac{d}{1-\rho}, \frac{\sigma^2}{1-\rho^2})$ and $D_k = d + \rho D_{k-1} + u_k$ with $u_k \sim N(0, \sigma^2)$, $k \geq 0$, then we have $x \sim N(0, \frac{(\rho^* - G)^2 \sigma^2}{1 - \rho^2})$, $y \sim N(0, \sum_{k=1}^{\nu+1} \left(\frac{1 - \rho^{\nu+2-k}}{1 - \rho} \right)^2 \sigma^2)$; moreover x and y are independent.

Thus,

$$\begin{aligned} E_D[E_1(g(r, v_t))] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{x^2}{2\sigma_1^2}} \left\{ h \int_{-\infty}^{r-x} (r - (x + y)) \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{y^2}{2\sigma_2^2}} dy \right. \\ &\quad \left. + \pi \int_{r-x}^{\infty} ((x + y) - r) \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{y^2}{2\sigma_2^2}} dy \right\} dx. \end{aligned}$$

Letting $z = x + y$, we have

$$\begin{aligned} E_D[E_1(g(r, v_t))] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{x^2}{2\sigma_1^2}} \left\{ h \int_{-\infty}^r (r - z) \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(z-x)^2}{2\sigma_2^2}} dz \right. \\ &\quad \left. + \pi \int_r^{\infty} (z - r) \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(z-x)^2}{2\sigma_2^2}} dz \right\} dx \\ &= h \int_{-\infty}^r \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{x^2}{2\sigma_1^2}} (r - z) \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(z-x)^2}{2\sigma_2^2}} dx dz \\ &\quad + \pi \int_r^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{x^2}{2\sigma_1^2}} (z - r) \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(z-x)^2}{2\sigma_2^2}} dx dz. \quad (82) \end{aligned}$$

We also have

$$\frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} e^{-\frac{z^2}{2(\sigma_1^2 + \sigma_2^2)}} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{x^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(z-x)^2}{2\sigma_2^2}} dx. \quad (83)$$

The proof follows from equation (82) and (83). \square

Derivation of Expected Cost

Under our inventory policy, we set the order-up-to level at period t as

$$S_t = m + GD_{t-1}.$$

To derive the expected cost, the demand during lead time $\sum_{i=t}^{t+\nu} D_i$ can be written as

$$\sum_{i=t}^{t+\nu} D_i = d \sum_{k=1}^{\nu+1} \frac{1-\rho^k}{1-\rho} + GD_{t-1} + (\rho^* - G)D_{t-1} + \sum_{k=4}^{t+\nu} \sum_{i=t}^k \rho^{k-i} u_i. \quad (84)$$

From equations (81) and (84), we have

$$\begin{aligned} & g\left(m + GD_{t-1}, \sum_{i=t}^{t+\nu} D_i\right) \\ &= g\left(m - d \sum_{k=1}^{\nu+1} \frac{1-\rho^k}{1-\rho} - \frac{(\rho^* - G)d}{1-\rho}, v_t\right) \\ &= g\left(m - \frac{(\nu+1-G)d}{1-\rho}, \left(\sigma \sqrt{\frac{(\rho^* - G)^2}{1-\rho^2} + \sum_{k=1}^{\nu+1} \left(\frac{1-\rho^{\nu+2-k}}{1-\rho}\right)^2} u\right)\right). \end{aligned}$$

Then,

$$\begin{aligned} & F(m, G) \\ &= E_D \left[\sum_{t=1}^{\infty} \beta^{t-1} E_1 \left(cZ_t + \beta^\nu g\left(m + GD_{t-1}, \sum_{i=t}^{t+\nu} D_i\right) \right) \right] \\ &= c \left\{ m + GE_D[E_1(D_0)] - n_1 + \sum_{t=2}^{\infty} \beta^{t-1} E_D \left[E_1 \left((1+G)D_{t-1} - GD_{t-2} \right) \right] \right. \\ &\quad \left. + \sum_{t=1}^{\infty} \beta^{t+\nu-1} E_D \left[E_1 \left(g\left(m - \frac{(\nu+1-G)d}{1-\rho}, v_t\right) \right) \right] \right\} \\ &= c \left\{ m + GE_D[E_1(D_0)] - n_1 + \sum_{t=2}^{\infty} \beta^{t-1} E_D[E_1((1+G)D_{t-1} - GD_{t-2})] \right. \\ &\quad \left. + \frac{\beta^\nu E \left[g\left(m - \frac{(\nu+1-G)d}{1-\rho}, \left(\sigma \sqrt{\frac{(\rho^* - G)^2}{1-\rho^2} + \sum_{k=1}^{\nu+1} \left(\frac{1-\rho^{\nu+2-k}}{1-\rho}\right)^2} u\right)\right) \right]}{1-\beta} \right\}, \end{aligned}$$

where $u \sim N(0, 1)$ and

$$E_D[E_1(D_0)] = E_D[E_1(D_{t-1})] = \frac{d}{1-\rho} \quad t \geq 1.$$

Since $\varphi(x)$ is the p.d.f. of the standard normal distribution, we have

$$\begin{aligned} E[g(r, u)] &= E[h(r-u)^+ + \pi(u-r)^+] \\ &= h \int_{-\infty}^r (r-x)\varphi(x) dx + \pi \int_r^{\infty} (x-r)\varphi(x) dx \\ &= (h+\pi)r\Phi(r) - \pi r + (h+\pi) \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}}. \end{aligned} \quad (85)$$

Thus, from equation (80), we have

$$\begin{aligned}
& F(m, G) \\
&= c \left(m - n_1 + \frac{Gd}{1-\rho} + \frac{\beta d}{(1-\beta)(1-\rho)} \right) \\
&\quad + \frac{\beta^\nu \sigma \sqrt{\frac{(\rho^*-G)^2}{1-\rho^2} + \sum_{k=1}^{\nu+1} \left(\frac{1-\rho^{\nu+2-k}}{1-\rho} \right)^2} \left((h+\pi)r\Phi(r) - \pi r + (h+\pi)\frac{1}{\sqrt{2\pi}}e^{-\frac{r^2}{2}} \right)}{1-\beta},
\end{aligned}$$

where

$$r = \frac{m - \frac{(\nu+1-G)d}{1-\rho}}{\sigma \sqrt{\frac{(\rho^*-G)^2}{1-\rho^2} + \sum_{k=1}^{\nu+1} \left(\frac{1-\rho^{\nu+2-k}}{1-\rho} \right)^2}},$$

and n_1 denotes the initial inventory level at period 1.

Proof of Proposition 1.

First, we define following notation for convenience:

$$g_1(Z_t, P) = p(Z_t - P)^+ = p((1+G)D_{t-1} - GD_{t-2} - P)^+, \quad (86)$$

and

$$g_2\left(S_t, \sum_{i=t}^{t+\nu} D_i\right) = h\left(m + GD_{t-1} - \sum_{i=t}^{t+\nu} D_i\right)^+ + \pi\left(\sum_{i=t}^{t+\nu} D_i - m - GD_{t-1}\right)^+. \quad (87)$$

Since Z_t follows the normal distribution $N(\frac{d}{1-\rho}, \frac{\sigma^2}{1-\rho^2}[2G^2(1-\rho) + 2G(1-\rho) + 1])$, we can write as

$$Z_t = \frac{d}{1-\rho} + \sqrt{\frac{\sigma^2}{1-\rho^2}[2G^2(1-\rho) + 2G(1-\rho) + 1]}u, \quad (88)$$

where u is a standard normal random variable.

Substitution of Z_t to equation (86) gives

$$\begin{aligned}
& g_1(Z_t, P) \\
&= g_1\left(P, \frac{d}{1-\rho} + \sqrt{\frac{\sigma^2}{1-\rho^2}[2G^2(1-\rho) + 2G(1-\rho) + 1]}u\right) \\
&= g_1\left(P - \frac{d}{1-\rho}, \sqrt{\frac{\sigma^2}{1-\rho^2}[2G^2(1-\rho) + 2G(1-\rho) + 1]}u\right) \\
&= \sqrt{\frac{\sigma^2}{1-\rho^2}[2G^2(1-\rho) + 2G(1-\rho) + 1]}g_1\left(\frac{P - \frac{d}{1-\rho}}{\sqrt{\frac{\sigma^2}{1-\rho^2}[2G^2(1-\rho) + 2G(1-\rho) + 1]}}, u\right).
\end{aligned}$$

Moreover, we rewrite $\sum_{i=t}^{t+\nu} D_i$ as in equation (84) and define

$$v_t = \frac{(G - \rho^*)d}{1 - \rho} + (\rho^* - G)D_{t-1} + \sum_{k=4}^{t+\nu} \sum_{i=t}^k \rho^{k-i} u_i. \quad (89)$$

Obviously, v_t has the a normal distribution with mean and variance given by

$$\mathbb{E}[v_t] = 0,$$

and

$$\text{Var}[v_t] = \text{Var}\left[(\rho^* - G)D_{t-1} + \sum_{k=4}^{t+\nu} \sum_{i=t}^k \rho^{k-i} u_i\right] = \frac{(\rho^* - G)\sigma^2}{1 - \rho^2} + \sum_{k=1}^{\nu+1} \left(\frac{1 - \rho^{\nu+2-k}}{1 - \rho}\right)^2 \sigma^2.$$

Next, we substitute equations (84) and (89) into equation (87) to get

$$\begin{aligned} & g_2\left(S_t, \sum_{i=t}^{t+\nu} D_i\right) \\ &= g_2\left(m + GD_{t-1}, \sum_{i=t}^{t+\nu} D_i\right) \\ &= g_2\left(m, \sum_{i=t}^{t+\nu} D_i - GD_{t-1}\right) \\ &= g_2\left(m - \frac{(\nu+1-G)d}{1-\rho}, \sigma \sqrt{\frac{(\rho^* - G)^2}{1 - \rho^2} + \sum_{k=1}^{\nu+1} \left(\frac{1 - \rho^{\nu+2-k}}{1 - \rho}\right)^2} u\right) \\ &= \sigma \sqrt{\frac{(\rho^* - G)^2}{1 - \rho^2} + \sum_{k=1}^{\nu+1} \left(\frac{1 - \rho^{\nu+2-k}}{1 - \rho}\right)^2} g_2\left(\frac{m - \frac{(\nu+1-G)d}{1-\rho}}{\sigma \sqrt{\frac{(\rho^* - G)^2}{1 - \rho^2} + \sum_{k=1}^{\nu+1} \left(\frac{1 - \rho^{\nu+2-k}}{1 - \rho}\right)^2}}, u\right). \end{aligned}$$

Replace all the equations into the expected cost, we obtain

$$\begin{aligned} & F(m, G) \\ &= \frac{cd}{(1-\beta)(1-\rho)} \\ & \quad + \frac{\beta\sigma}{1-\beta} \sqrt{\frac{2G^2(1-\rho) + 2G(1-\rho) + 1}{1-\rho^2}} \left(pr_2[\Phi(r_2) - 1] + p \frac{1}{\sqrt{2\pi}} e^{-\frac{r_2^2}{2}} \right) \\ & \quad + \frac{\beta^\nu \sigma}{1-\beta} \sqrt{\frac{(\rho^* - G)^2}{1 - \rho^2} + \sum_{k=1}^{\nu+1} \left(\frac{1 - \rho^{\nu+2-k}}{1 - \rho}\right)^2} \left((h + \pi)r_3\Phi(r_3) - \pi r_3 \right. \\ & \quad \left. + (h + \pi) \frac{1}{\sqrt{2\pi}} e^{-\frac{r_3^2}{2}} \right), \end{aligned}$$

where

$$r_2 = \frac{P - \frac{d}{1-\rho}}{\sigma \sqrt{\frac{[2G^2(1-\rho)+2G(1-\rho)+1]}{1-\rho^2}}},$$

and

$$r_3 = \frac{m - \frac{(\nu+1-G)d}{1-\rho}}{\sigma \sqrt{\frac{(\rho^*-G)^2}{1-\rho^2} + \sum_{k=1}^{\nu+1} \left(\frac{1-\rho^{\nu+2-k}}{1-\rho} \right)^2}}.$$

Differentiating the discounted expected cost $F(m, G)$ with respect to m and then equate it to zero, we obtain the equation

$$(h + \pi)\Phi(r_3) - \pi = 0. \quad (90)$$

whose solution is

$$r_3 = \Phi^{-1}\left(\frac{\pi}{h + \pi}\right). \quad (91)$$

Finally, the optimal m is obtained:

$$m^*(G) = \frac{(\nu + 1 - G)d}{1 - \rho} + \Phi^{-1}\left(\frac{\pi}{h + \pi}\right) \sigma \sqrt{\frac{(\rho^* - G)^2}{1 - \rho^2} + \sum_{k=1}^{\nu+1} \left(\frac{1 - \rho^{\nu+2-k}}{1 - \rho} \right)^2}. \quad (92)$$

The expression for the expected discount cost follows from substitution of the optimal $m^*(G)$.

APPENDIX B

NUMERICAL RESULTS

Table 22: Comparison of the mean estimates (Poisson demand with $\mu = 8$)

No.	Full Information			Partial Information		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE
1	-0.001	0.012	0.012	-0.001	0.012	0.012
2	0.000	0.013	0.013	0.001	0.013	0.013
3	-0.001	0.013	0.013	-0.001	0.013	0.013
4	0.001	0.013	0.013	0.002	0.013	0.013
5	0.001	0.013	0.013	0.001	0.013	0.013
6	0.000	0.012	0.012	0.000	0.013	0.013
7	0.000	0.014	0.014	0.000	0.014	0.014
8	0.001	0.013	0.013	0.001	0.013	0.013
9	0.001	0.012	0.012	0.002	0.012	0.012
10	0.001	0.013	0.013	0.001	0.013	0.013
11	0.002	0.012	0.012	0.002	0.012	0.013
12	-0.001	0.014	0.014	-0.001	0.014	0.014
13	0.000	0.012	0.012	0.000	0.012	0.012
14	0.001	0.013	0.013	0.001	0.014	0.014
15	-0.001	0.013	0.013	0.000	0.013	0.013
16	0.000	0.013	0.013	0.000	0.013	0.013
17	-0.001	0.014	0.014	0.000	0.014	0.014
18	0.000	0.013	0.013	0.000	0.013	0.013
19	0.001	0.014	0.014	0.001	0.014	0.014
20	0.000	0.013	0.013	0.001	0.013	0.013
21	-0.001	0.013	0.013	-0.001	0.014	0.014
22	-0.001	0.014	0.014	0.000	0.015	0.015
23	0.001	0.012	0.012	0.001	0.012	0.012
24	0.000	0.012	0.012	0.000	0.011	0.011
25	0.000	0.015	0.015	0.001	0.015	0.015
26	-0.003	0.014	0.014	-0.003	0.014	0.014
27	0.001	0.015	0.015	0.001	0.015	0.015
28	0.000	0.014	0.014	0.000	0.014	0.014
29	0.001	0.013	0.013	0.002	0.013	0.013
30	-0.001	0.015	0.015	-0.001	0.015	0.015
31	-0.003	0.015	0.015	-0.003	0.015	0.015
32	-0.003	0.012	0.012	-0.002	0.012	0.012
33	-0.002	0.013	0.013	-0.002	0.013	0.013
34	-0.002	0.012	0.012	-0.001	0.012	0.012
35	-0.004	0.014	0.014	-0.003	0.014	0.014
36	-0.002	0.012	0.012	-0.002	0.012	0.012

Table 23: Comparison of the mean estimates (Poisson demand with $\mu = 16$)

No.	Full Information			Partial Information		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE
37	0.001	0.009	0.009	0.001	0.009	0.009
38	0.000	0.010	0.010	0.000	0.010	0.010
39	0.000	0.009	0.009	0.000	0.009	0.009
40	0.001	0.010	0.010	0.002	0.010	0.010
41	0.000	0.009	0.009	0.000	0.009	0.009
42	0.000	0.009	0.010	0.000	0.009	0.009
43	-0.001	0.009	0.010	0.000	0.010	0.010
44	0.001	0.009	0.009	0.001	0.009	0.009
45	0.000	0.010	0.010	0.000	0.010	0.010
46	0.001	0.009	0.009	0.001	0.009	0.009
47	0.000	0.009	0.009	0.000	0.009	0.009
48	0.001	0.008	0.008	0.001	0.008	0.008
49	-0.001	0.010	0.010	-0.001	0.010	0.010
50	0.000	0.010	0.010	0.000	0.010	0.010
51	0.001	0.008	0.008	0.001	0.008	0.009
52	0.001	0.009	0.009	0.001	0.009	0.009
53	0.000	0.009	0.009	0.000	0.009	0.010
54	0.002	0.008	0.009	0.002	0.008	0.009
55	0.000	0.009	0.009	0.000	0.009	0.009
56	0.000	0.009	0.009	0.001	0.009	0.009
57	-0.001	0.009	0.009	-0.001	0.009	0.009
58	0.000	0.010	0.010	0.000	0.010	0.010
59	-0.001	0.009	0.009	0.000	0.009	0.009
60	0.000	0.008	0.008	0.000	0.009	0.009
61	-0.001	0.010	0.010	0.000	0.010	0.010
62	0.000	0.009	0.009	0.000	0.009	0.009
63	0.002	0.009	0.009	0.002	0.009	0.009
64	0.000	0.010	0.010	0.001	0.010	0.010
65	-0.001	0.009	0.009	-0.001	0.009	0.009
66	0.001	0.009	0.009	0.001	0.009	0.009
67	-0.001	0.009	0.009	0.000	0.009	0.009
68	-0.001	0.009	0.009	-0.001	0.009	0.009
69	0.001	0.010	0.010	0.001	0.011	0.011
70	0.000	0.009	0.009	0.000	0.009	0.009
71	0.000	0.010	0.010	0.000	0.009	0.009
72	0.003	0.009	0.010	0.003	0.009	0.009

Table 24: Comparison of the mean estimates (negative binomial demand with $\mu = 8$ and $\sigma^2 = 3\mu$)

No.	Full Information			Partial Information		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE
73	0.006	0.027	0.028	0.006	0.028	0.028
74	0.001	0.024	0.024	0.002	0.024	0.024
75	-0.002	0.024	0.024	-0.002	0.024	0.024
76	0.003	0.025	0.025	0.003	0.025	0.026
77	0.000	0.021	0.021	0.000	0.021	0.021
78	0.004	0.023	0.023	0.005	0.022	0.023
79	0.000	0.025	0.025	0.001	0.024	0.024
80	0.000	0.022	0.022	0.000	0.022	0.022
81	0.002	0.025	0.025	0.002	0.026	0.026
82	0.001	0.021	0.021	0.002	0.021	0.021
83	0.000	0.023	0.023	0.001	0.023	0.023
84	0.003	0.025	0.025	0.003	0.026	0.026
85	0.000	0.021	0.021	0.001	0.021	0.021
86	0.003	0.025	0.025	0.003	0.026	0.026
87	0.004	0.023	0.024	0.004	0.024	0.024
88	0.004	0.022	0.022	0.004	0.021	0.022
89	0.000	0.023	0.023	0.000	0.023	0.023
90	0.002	0.021	0.021	0.003	0.021	0.021
91	0.003	0.023	0.023	0.004	0.023	0.024
92	0.000	0.020	0.020	0.001	0.021	0.021
93	-0.001	0.021	0.021	0.000	0.021	0.021
94	-0.001	0.021	0.021	0.000	0.021	0.021
95	-0.001	0.024	0.024	-0.001	0.024	0.024
96	0.001	0.023	0.023	0.002	0.023	0.023
97	0.004	0.021	0.022	0.005	0.021	0.022
98	0.002	0.022	0.022	0.002	0.022	0.022
99	0.000	0.022	0.022	0.000	0.021	0.021
100	-0.001	0.022	0.022	-0.001	0.021	0.021
101	-0.003	0.023	0.023	-0.002	0.023	0.023
102	0.002	0.022	0.022	0.002	0.022	0.022
103	-0.001	0.021	0.021	-0.001	0.022	0.022
104	-0.002	0.024	0.024	-0.001	0.024	0.024
105	0.002	0.023	0.023	0.003	0.023	0.023
106	0.003	0.023	0.023	0.004	0.023	0.023
107	0.004	0.020	0.020	0.004	0.020	0.020
108	-0.002	0.021	0.022	-0.002	0.022	0.022

Table 25: Comparison of the mean estimates (negative binomial demand with $\mu = 16$ and $\sigma^2 = 3\mu$)

No.	Full Information			Partial Information		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE
109	0.000	0.018	0.018	0.000	0.017	0.017
110	0.003	0.017	0.017	0.004	0.017	0.017
111	0.001	0.016	0.016	0.001	0.016	0.016
112	0.000	0.015	0.015	0.001	0.015	0.015
113	0.001	0.015	0.015	0.001	0.015	0.015
114	0.005	0.015	0.016	0.005	0.016	0.017
115	0.001	0.016	0.016	0.001	0.016	0.016
116	-0.002	0.015	0.015	-0.002	0.015	0.015
117	0.000	0.014	0.014	0.000	0.014	0.014
118	0.002	0.017	0.017	0.002	0.017	0.017
119	0.001	0.017	0.017	0.001	0.017	0.017
120	-0.002	0.016	0.016	-0.002	0.015	0.016
121	0.001	0.018	0.018	0.002	0.018	0.018
122	-0.001	0.014	0.015	-0.001	0.015	0.015
123	0.002	0.017	0.017	0.002	0.017	0.017
124	0.000	0.017	0.017	0.001	0.017	0.017
125	0.002	0.017	0.017	0.002	0.018	0.018
126	-0.003	0.015	0.016	-0.002	0.016	0.016
127	0.000	0.016	0.016	0.001	0.016	0.017
128	0.002	0.017	0.017	0.003	0.017	0.017
129	0.000	0.014	0.014	0.000	0.014	0.014
130	-0.003	0.016	0.016	-0.002	0.016	0.016
131	-0.001	0.016	0.016	-0.001	0.016	0.016
132	0.000	0.015	0.015	0.001	0.015	0.015
133	0.002	0.018	0.018	0.002	0.018	0.018
134	-0.001	0.017	0.017	-0.001	0.017	0.017
135	0.001	0.017	0.017	0.001	0.018	0.018
136	0.001	0.016	0.016	0.002	0.016	0.016
137	-0.001	0.015	0.015	-0.001	0.015	0.015
138	0.003	0.016	0.016	0.003	0.016	0.016
139	-0.002	0.017	0.017	-0.002	0.017	0.017
140	0.000	0.014	0.014	0.000	0.014	0.014
141	-0.003	0.018	0.018	-0.002	0.018	0.018
142	-0.002	0.015	0.015	-0.001	0.015	0.015
143	0.001	0.017	0.017	0.001	0.017	0.017
144	0.000	0.018	0.018	0.000	0.018	0.018

Table 26: Comparison of the mean estimates (negative binomial demand with $\mu = 8$ and $\sigma^2 = 5\mu$)

No.	Full Information			Partial Information		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE
145	-0.002	0.032	0.032	-0.001	0.032	0.032
146	0.000	0.027	0.027	0.001	0.027	0.027
147	0.004	0.028	0.028	0.004	0.028	0.028
148	-0.004	0.028	0.029	-0.003	0.029	0.029
149	0.002	0.030	0.030	0.004	0.029	0.030
150	-0.003	0.028	0.028	-0.002	0.029	0.029
151	0.003	0.032	0.032	0.004	0.032	0.032
152	-0.001	0.027	0.027	0.000	0.028	0.028
153	0.000	0.027	0.027	0.000	0.028	0.028
154	0.003	0.026	0.026	0.004	0.026	0.026
155	0.003	0.031	0.031	0.004	0.031	0.032
156	0.001	0.030	0.030	0.003	0.030	0.030
157	0.000	0.027	0.027	0.001	0.027	0.027
158	0.005	0.029	0.030	0.006	0.029	0.030
159	0.002	0.028	0.028	0.004	0.028	0.029
160	0.000	0.032	0.032	0.001	0.032	0.032
161	-0.003	0.031	0.031	-0.002	0.031	0.031
162	0.002	0.027	0.027	0.004	0.027	0.027
163	0.002	0.030	0.030	0.003	0.030	0.030
164	0.002	0.030	0.030	0.003	0.030	0.030
165	-0.002	0.027	0.027	-0.001	0.027	0.027
166	-0.003	0.029	0.029	-0.002	0.029	0.029
167	0.000	0.023	0.023	0.001	0.023	0.023
168	0.002	0.029	0.029	0.003	0.029	0.029
169	0.002	0.034	0.034	0.003	0.034	0.034
170	-0.003	0.027	0.028	-0.001	0.028	0.028
171	-0.001	0.027	0.027	-0.001	0.027	0.027
172	-0.001	0.032	0.032	0.000	0.032	0.032
173	-0.003	0.027	0.027	-0.002	0.027	0.027
174	0.006	0.030	0.031	0.006	0.030	0.031
175	-0.001	0.032	0.032	0.000	0.032	0.032
176	0.001	0.030	0.030	0.002	0.030	0.030
177	0.000	0.029	0.029	0.001	0.029	0.029
178	-0.001	0.031	0.031	0.000	0.031	0.031
179	-0.004	0.030	0.030	-0.003	0.030	0.030
180	0.003	0.032	0.032	0.003	0.033	0.034

Table 27: Comparison of the mean estimates (negative binomial demand with $\mu = 16$ and $\sigma^2 = 5\mu$)

No.	Full Information			Partial Information		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE
181	0.000	0.019	0.019	0.001	0.019	0.019
182	0.002	0.018	0.018	0.002	0.018	0.019
183	0.004	0.021	0.021	0.005	0.021	0.021
184	0.007	0.019	0.020	0.007	0.019	0.020
185	0.003	0.021	0.021	0.003	0.021	0.021
186	0.001	0.020	0.020	0.001	0.020	0.020
187	-0.001	0.020	0.020	-0.001	0.020	0.020
188	0.004	0.023	0.023	0.005	0.023	0.024
189	0.001	0.023	0.023	0.001	0.023	0.023
190	0.002	0.019	0.019	0.002	0.019	0.019
191	0.001	0.023	0.023	0.002	0.023	0.023
192	0.000	0.022	0.022	0.001	0.022	0.022
193	0.000	0.023	0.023	0.001	0.023	0.023
194	0.000	0.023	0.023	0.000	0.023	0.023
195	0.003	0.019	0.019	0.003	0.019	0.019
196	0.003	0.018	0.019	0.003	0.018	0.019
197	0.002	0.020	0.020	0.003	0.020	0.020
198	-0.001	0.024	0.024	-0.001	0.024	0.024
199	0.001	0.019	0.019	0.002	0.019	0.019
200	0.001	0.021	0.021	0.001	0.021	0.021
201	-0.002	0.023	0.023	-0.002	0.024	0.024
202	0.002	0.020	0.020	0.003	0.020	0.020
203	-0.002	0.020	0.020	-0.001	0.020	0.020
204	-0.001	0.020	0.020	0.000	0.020	0.020
205	0.000	0.021	0.021	0.001	0.021	0.021
206	0.002	0.020	0.021	0.003	0.020	0.021
207	-0.001	0.019	0.019	-0.001	0.019	0.019
208	-0.001	0.018	0.018	0.000	0.017	0.017
209	0.001	0.019	0.019	0.001	0.019	0.019
210	-0.001	0.019	0.019	-0.001	0.019	0.019
211	0.003	0.025	0.025	0.004	0.025	0.025
212	0.002	0.021	0.021	0.002	0.022	0.022
213	-0.003	0.021	0.021	-0.003	0.021	0.021
214	0.002	0.023	0.023	0.002	0.023	0.023
215	0.002	0.023	0.024	0.003	0.024	0.024
216	-0.002	0.022	0.022	-0.001	0.021	0.021

Table 28: Comparison of the standard deviation estimates (Poisson demand with $\mu = 8$)

No.	Full Information			Partial Information			RBA		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE	RBias	RSD	RRMSE
1	0.003	0.032	0.032	0.247	0.054	0.252	-0.022	0.052	0.056
2	0.004	0.027	0.028	0.130	0.087	0.156	-0.053	0.090	0.105
3	0.002	0.031	0.031	0.062	0.092	0.111	-0.041	0.105	0.113
4	0.001	0.026	0.026	0.242	0.056	0.248	-0.028	0.054	0.061
5	0.004	0.032	0.032	0.127	0.073	0.147	-0.057	0.076	0.095
6	0.003	0.024	0.024	0.071	0.099	0.121	-0.031	0.112	0.116
7	0.005	0.029	0.029	0.253	0.050	0.258	-0.016	0.049	0.052
8	0.005	0.029	0.029	0.137	0.058	0.149	-0.047	0.062	0.077
9	0.003	0.033	0.033	0.054	0.080	0.097	-0.051	0.092	0.105
10	0.006	0.026	0.026	0.240	0.055	0.247	-0.029	0.054	0.061
11	0.004	0.025	0.026	0.133	0.066	0.148	-0.051	0.069	0.086
12	0.002	0.028	0.028	0.068	0.097	0.119	-0.033	0.110	0.115
13	0.006	0.027	0.027	0.244	0.051	0.249	-0.025	0.050	0.056
14	0.003	0.029	0.029	0.131	0.074	0.151	-0.053	0.077	0.094
15	0.004	0.029	0.029	0.070	0.093	0.117	-0.031	0.106	0.110
16	0.003	0.025	0.026	0.245	0.054	0.251	-0.023	0.054	0.058
17	0.005	0.025	0.026	0.142	0.073	0.160	-0.041	0.077	0.087
18	0.006	0.027	0.028	0.066	0.093	0.114	-0.037	0.106	0.112
19	-0.003	0.026	0.026	0.237	0.049	0.242	-0.032	0.048	0.057
20	0.001	0.027	0.027	0.130	0.075	0.150	-0.054	0.079	0.096
21	0.006	0.026	0.027	0.073	0.083	0.110	-0.029	0.095	0.099
22	0.004	0.027	0.027	0.237	0.051	0.242	-0.032	0.050	0.059
23	0.008	0.029	0.030	0.137	0.067	0.152	-0.047	0.071	0.085
24	0.003	0.028	0.028	0.071	0.088	0.113	-0.031	0.099	0.104
25	0.006	0.028	0.028	0.229	0.051	0.235	-0.040	0.049	0.063
26	0.003	0.028	0.029	0.126	0.067	0.143	-0.057	0.070	0.090
27	0.001	0.027	0.027	0.073	0.089	0.115	-0.029	0.103	0.107
28	0.004	0.029	0.029	0.242	0.053	0.248	-0.027	0.052	0.058
29	0.005	0.029	0.030	0.123	0.068	0.141	-0.061	0.071	0.094
30	0.002	0.030	0.030	0.070	0.093	0.116	-0.032	0.106	0.111
31	0.003	0.028	0.028	0.241	0.061	0.249	-0.026	0.060	0.066
32	0.003	0.026	0.026	0.136	0.058	0.148	-0.046	0.061	0.076
33	0.010	0.025	0.027	0.088	0.094	0.128	-0.010	0.107	0.107
34	-0.001	0.026	0.026	0.241	0.051	0.246	-0.027	0.049	0.056
35	-0.003	0.025	0.025	0.130	0.069	0.147	-0.052	0.072	0.089
36	0.000	0.030	0.030	0.047	0.085	0.097	-0.058	0.096	0.112

Table 29: Comparison of the standard deviation estimates (Poisson demand with $\mu = 16$)

No.	Full Information			Partial Information			RBA		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE	RBias	RSD	RRMSE
37	0.003	0.026	0.026	0.736	0.050	0.738	0.238	0.045	0.242
38	0.003	0.026	0.026	0.453	0.058	0.456	0.082	0.053	0.098
39	-0.001	0.026	0.026	0.231	0.077	0.243	-0.038	0.076	0.085
40	0.004	0.026	0.026	0.743	0.044	0.744	0.243	0.039	0.246
41	0.000	0.025	0.025	0.451	0.050	0.454	0.081	0.046	0.093
42	0.002	0.024	0.024	0.248	0.084	0.262	-0.021	0.083	0.085
43	0.004	0.028	0.029	0.734	0.044	0.735	0.236	0.038	0.239
44	-0.002	0.026	0.026	0.453	0.049	0.456	0.082	0.045	0.094
45	-0.003	0.026	0.026	0.238	0.072	0.249	-0.032	0.071	0.078
46	0.004	0.031	0.031	0.736	0.042	0.737	0.238	0.037	0.241
47	-0.002	0.029	0.029	0.450	0.047	0.453	0.080	0.043	0.091
48	0.003	0.026	0.027	0.245	0.070	0.255	-0.025	0.070	0.074
49	0.000	0.025	0.025	0.732	0.047	0.734	0.235	0.041	0.239
50	0.005	0.025	0.025	0.450	0.053	0.453	0.080	0.049	0.094
51	0.001	0.028	0.028	0.248	0.066	0.256	-0.022	0.064	0.068
52	0.001	0.027	0.027	0.748	0.045	0.749	0.248	0.039	0.251
53	0.001	0.027	0.027	0.442	0.055	0.446	0.072	0.051	0.089
54	0.002	0.026	0.026	0.236	0.080	0.249	-0.034	0.078	0.085
55	-0.002	0.026	0.026	0.731	0.047	0.732	0.233	0.041	0.237
56	0.005	0.027	0.027	0.450	0.057	0.453	0.079	0.053	0.095
57	0.000	0.028	0.028	0.234	0.071	0.244	-0.036	0.070	0.078
58	-0.001	0.026	0.026	0.747	0.053	0.749	0.248	0.047	0.252
59	-0.002	0.027	0.027	0.447	0.052	0.450	0.077	0.047	0.091
60	-0.004	0.029	0.029	0.230	0.076	0.242	-0.039	0.074	0.084
61	0.002	0.029	0.030	0.744	0.052	0.746	0.245	0.045	0.249
62	-0.003	0.023	0.023	0.454	0.048	0.457	0.084	0.044	0.095
63	0.000	0.021	0.021	0.235	0.076	0.247	-0.036	0.075	0.083
64	-0.002	0.026	0.026	0.736	0.048	0.737	0.238	0.043	0.241
65	0.003	0.030	0.030	0.448	0.052	0.451	0.078	0.047	0.091
66	0.003	0.028	0.028	0.255	0.077	0.266	-0.015	0.076	0.078
67	0.002	0.024	0.024	0.739	0.048	0.741	0.241	0.042	0.245
68	0.006	0.028	0.029	0.450	0.048	0.452	0.079	0.045	0.091
69	0.000	0.027	0.027	0.233	0.073	0.244	-0.037	0.071	0.081
70	-0.001	0.025	0.025	0.741	0.050	0.743	0.243	0.044	0.246
71	0.001	0.030	0.030	0.450	0.061	0.454	0.079	0.056	0.097
72	0.004	0.023	0.023	0.255	0.069	0.265	-0.016	0.068	0.070

Table 30: Comparison of the standard deviation estimates (negative binomial demand with $\mu = 8$ and $\sigma^2 = 3\mu$)

No.	Full Information			Partial Information			RBA		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE	RBias	RSD	RRMSE
73	0.018	0.039	0.043	0.106	0.053	0.118	-0.029	0.054	0.061
74	0.005	0.030	0.031	0.046	0.063	0.077	-0.007	0.074	0.074
75	0.001	0.039	0.039	0.011	0.092	0.092	0.045	0.118	0.126
76	0.004	0.038	0.038	0.093	0.051	0.106	-0.043	0.053	0.068
77	0.010	0.033	0.034	0.057	0.070	0.090	0.007	0.081	0.081
78	0.010	0.036	0.037	0.017	0.089	0.091	0.049	0.113	0.124
79	-0.002	0.040	0.040	0.075	0.050	0.091	-0.061	0.053	0.080
80	0.002	0.035	0.035	0.044	0.065	0.078	-0.008	0.076	0.076
81	0.006	0.033	0.034	0.018	0.081	0.083	0.053	0.105	0.117
82	0.005	0.038	0.038	0.088	0.046	0.100	-0.047	0.049	0.068
83	0.001	0.035	0.035	0.031	0.064	0.071	-0.024	0.074	0.078
84	0.002	0.033	0.033	0.025	0.070	0.074	0.061	0.088	0.107
85	0.002	0.036	0.036	0.082	0.050	0.096	-0.054	0.053	0.075
86	0.007	0.038	0.039	0.057	0.063	0.085	0.006	0.071	0.071
87	0.009	0.032	0.033	0.025	0.093	0.097	0.060	0.120	0.134
88	0.009	0.040	0.041	0.089	0.052	0.103	-0.047	0.054	0.072
89	0.000	0.035	0.035	0.047	0.068	0.082	-0.005	0.079	0.079
90	0.001	0.036	0.036	0.013	0.084	0.085	0.045	0.108	0.117
91	0.003	0.036	0.036	0.093	0.053	0.107	-0.042	0.056	0.070
92	0.006	0.031	0.031	0.053	0.064	0.082	0.002	0.075	0.075
93	-0.003	0.031	0.031	0.016	0.081	0.082	0.050	0.105	0.116
94	0.005	0.032	0.033	0.081	0.039	0.090	-0.054	0.040	0.067
95	-0.004	0.035	0.035	0.039	0.062	0.073	-0.014	0.072	0.074
96	-0.004	0.032	0.032	0.027	0.091	0.095	0.065	0.117	0.133
97	0.002	0.033	0.033	0.082	0.049	0.095	-0.055	0.051	0.075
98	0.001	0.033	0.033	0.041	0.065	0.077	-0.012	0.076	0.077
99	-0.001	0.034	0.034	0.014	0.095	0.096	0.048	0.123	0.132
100	-0.002	0.031	0.031	0.078	0.047	0.091	-0.057	0.048	0.075
101	-0.001	0.035	0.035	0.027	0.064	0.070	-0.027	0.075	0.080
102	0.005	0.034	0.035	0.025	0.092	0.095	0.061	0.120	0.135
103	0.002	0.034	0.034	0.075	0.049	0.090	-0.061	0.052	0.080
104	-0.004	0.033	0.033	0.033	0.064	0.072	-0.021	0.074	0.077
105	0.004	0.035	0.035	0.016	0.099	0.100	0.050	0.128	0.138
106	0.003	0.036	0.036	0.085	0.054	0.101	-0.051	0.056	0.076
107	0.004	0.036	0.036	0.052	0.061	0.080	-0.001	0.072	0.072
108	0.003	0.035	0.036	0.011	0.084	0.085	0.044	0.109	0.118

Table 31: Comparison of the standard deviation estimates (negative binomial demand with $\mu = 16$ and $\sigma^2 = 3\mu$)

No.	Full Information			Partial Information			RBA		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE	RBias	RSD	RRMSE
109	0.000	0.033	0.033	0.301	0.053	0.306	-0.005	0.048	0.048
110	0.005	0.030	0.030	0.171	0.047	0.177	-0.047	0.048	0.067
111	-0.001	0.030	0.030	0.085	0.065	0.107	-0.050	0.071	0.087
112	-0.001	0.035	0.035	0.303	0.050	0.307	-0.004	0.045	0.045
113	0.002	0.028	0.028	0.170	0.050	0.177	-0.048	0.051	0.069
114	0.010	0.030	0.031	0.094	0.074	0.120	-0.042	0.081	0.091
115	0.000	0.032	0.032	0.306	0.046	0.309	-0.001	0.042	0.042
116	0.002	0.034	0.034	0.169	0.052	0.176	-0.047	0.052	0.071
117	-0.002	0.031	0.031	0.088	0.065	0.109	-0.048	0.071	0.086
118	0.003	0.034	0.034	0.305	0.044	0.309	-0.001	0.040	0.040
119	0.001	0.027	0.027	0.161	0.051	0.169	-0.057	0.051	0.076
120	-0.003	0.032	0.032	0.082	0.073	0.110	-0.053	0.079	0.095
121	0.000	0.029	0.029	0.296	0.045	0.300	-0.010	0.041	0.042
122	-0.001	0.030	0.030	0.159	0.052	0.167	-0.058	0.052	0.078
123	0.004	0.031	0.032	0.087	0.064	0.108	-0.049	0.069	0.085
124	0.000	0.030	0.030	0.294	0.047	0.297	-0.012	0.041	0.043
125	0.003	0.029	0.029	0.167	0.047	0.173	-0.051	0.048	0.070
126	0.001	0.033	0.033	0.078	0.066	0.102	-0.058	0.073	0.093
127	0.005	0.029	0.029	0.298	0.050	0.302	-0.008	0.045	0.046
128	-0.003	0.032	0.032	0.161	0.051	0.169	-0.057	0.051	0.076
129	0.003	0.030	0.030	0.084	0.075	0.112	-0.052	0.082	0.097
130	-0.002	0.033	0.033	0.298	0.051	0.302	-0.007	0.047	0.047
131	-0.003	0.034	0.034	0.158	0.057	0.168	-0.059	0.056	0.081
132	-0.002	0.033	0.033	0.082	0.068	0.107	-0.054	0.075	0.092
133	0.002	0.028	0.028	0.300	0.047	0.303	-0.007	0.043	0.043
134	-0.007	0.030	0.031	0.162	0.052	0.170	-0.055	0.052	0.075
135	0.000	0.031	0.031	0.096	0.058	0.112	-0.039	0.063	0.074
136	-0.001	0.031	0.031	0.301	0.049	0.305	-0.006	0.044	0.045
137	-0.001	0.028	0.028	0.165	0.045	0.171	-0.052	0.044	0.068
138	0.004	0.028	0.028	0.091	0.065	0.112	-0.046	0.071	0.085
139	-0.001	0.030	0.030	0.295	0.048	0.299	-0.009	0.044	0.045
140	0.000	0.030	0.030	0.166	0.053	0.174	-0.051	0.053	0.073
141	-0.004	0.032	0.032	0.087	0.065	0.109	-0.047	0.071	0.085
142	0.000	0.029	0.029	0.300	0.041	0.302	-0.006	0.036	0.037
143	0.002	0.031	0.031	0.156	0.053	0.165	-0.062	0.052	0.080
144	0.002	0.030	0.031	0.089	0.070	0.113	-0.046	0.076	0.089

Table 32: Comparison of the standard deviation estimates (negative binomial demand with $\mu = 8$ and $\sigma^2 = 5\mu$)

No.	Full Information			Partial Information			RBA		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE	RBias	RSD	RRMSE
145	-0.004	0.044	0.045	0.033	0.060	0.068	-0.043	0.064	0.077
146	-0.001	0.040	0.040	0.017	0.057	0.059	0.027	0.071	0.075
147	0.012	0.039	0.041	0.009	0.083	0.084	0.114	0.115	0.162
148	-0.007	0.041	0.041	0.036	0.055	0.065	-0.040	0.060	0.071
149	-0.001	0.044	0.044	0.024	0.072	0.076	0.035	0.087	0.094
150	-0.001	0.038	0.039	0.007	0.092	0.092	0.114	0.127	0.170
151	0.003	0.043	0.043	0.040	0.058	0.071	-0.037	0.062	0.072
152	0.008	0.044	0.044	0.027	0.068	0.073	0.040	0.083	0.092
153	0.002	0.038	0.038	0.016	0.090	0.091	0.126	0.127	0.179
154	0.011	0.040	0.042	0.050	0.044	0.067	-0.025	0.048	0.054
155	0.002	0.039	0.039	0.013	0.063	0.064	0.021	0.074	0.077
156	0.007	0.040	0.041	0.019	0.091	0.093	0.129	0.123	0.178
157	0.002	0.036	0.036	0.047	0.045	0.065	-0.028	0.048	0.056
158	0.009	0.049	0.050	0.021	0.073	0.076	0.030	0.089	0.094
159	0.005	0.038	0.038	0.025	0.088	0.091	0.136	0.122	0.182
160	-0.002	0.038	0.039	0.046	0.055	0.072	-0.029	0.058	0.065
161	0.005	0.041	0.041	0.027	0.067	0.073	0.042	0.082	0.092
162	0.001	0.044	0.044	0.012	0.090	0.091	0.118	0.124	0.171
163	0.001	0.041	0.041	0.039	0.055	0.067	-0.038	0.057	0.069
164	0.007	0.038	0.039	0.030	0.067	0.074	0.042	0.083	0.093
165	-0.006	0.041	0.041	0.007	0.085	0.085	0.112	0.117	0.162
166	0.001	0.041	0.041	0.036	0.055	0.066	-0.039	0.058	0.070
167	0.001	0.038	0.038	0.021	0.066	0.069	0.032	0.082	0.088
168	0.002	0.042	0.042	0.005	0.085	0.085	0.109	0.118	0.161
169	0.007	0.047	0.048	0.040	0.064	0.075	-0.037	0.067	0.077
170	-0.002	0.041	0.042	0.015	0.063	0.065	0.026	0.078	0.082
171	-0.001	0.036	0.036	0.003	0.088	0.088	0.108	0.120	0.161
172	0.002	0.040	0.040	0.033	0.056	0.065	-0.044	0.058	0.073
173	0.003	0.042	0.042	0.024	0.065	0.069	0.037	0.080	0.089
174	0.010	0.042	0.043	0.021	0.091	0.093	0.130	0.123	0.179
175	0.011	0.045	0.046	0.043	0.057	0.071	-0.032	0.060	0.068
176	0.001	0.036	0.036	0.029	0.065	0.071	0.042	0.079	0.089
177	0.001	0.045	0.045	0.023	0.100	0.102	0.135	0.135	0.191
178	0.004	0.038	0.038	0.042	0.052	0.066	-0.034	0.056	0.065
179	0.004	0.046	0.046	0.021	0.073	0.076	0.034	0.088	0.095
180	0.005	0.047	0.047	0.009	0.094	0.095	0.114	0.127	0.171

Table 33: Comparison of the standard deviation estimates (negative binomial demand with $\mu = 16$ and $\sigma^2 = 5\mu$)

No.	Full Information			Partial Information			RBA		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE	RBias	RSD	RRMSE
181	0.003	0.035	0.035	0.176	0.042	0.181	-0.063	0.039	0.074
182	0.004	0.034	0.034	0.104	0.051	0.115	-0.053	0.053	0.075
183	0.004	0.037	0.037	0.040	0.068	0.079	-0.038	0.077	0.086
184	0.008	0.036	0.037	0.181	0.040	0.185	-0.061	0.038	0.071
185	0.003	0.033	0.033	0.106	0.052	0.118	-0.051	0.053	0.074
186	0.001	0.033	0.033	0.040	0.060	0.072	-0.037	0.069	0.078
187	0.001	0.039	0.039	0.178	0.041	0.182	-0.060	0.039	0.072
188	0.005	0.037	0.037	0.096	0.058	0.112	-0.062	0.058	0.085
189	0.004	0.032	0.032	0.040	0.074	0.084	-0.037	0.084	0.092
190	0.001	0.030	0.030	0.180	0.041	0.185	-0.059	0.038	0.070
191	0.007	0.036	0.036	0.105	0.057	0.119	-0.051	0.058	0.077
192	-0.002	0.034	0.034	0.044	0.065	0.078	-0.032	0.073	0.079
193	0.003	0.038	0.038	0.178	0.049	0.184	-0.061	0.044	0.075
194	0.005	0.039	0.039	0.100	0.057	0.115	-0.056	0.057	0.080
195	0.001	0.033	0.033	0.047	0.059	0.076	-0.029	0.068	0.074
196	0.003	0.029	0.029	0.174	0.036	0.177	-0.066	0.034	0.074
197	0.000	0.038	0.038	0.095	0.053	0.109	-0.062	0.053	0.082
198	-0.001	0.036	0.036	0.048	0.062	0.078	-0.027	0.069	0.074
199	-0.002	0.036	0.036	0.176	0.044	0.182	-0.063	0.040	0.074
200	-0.003	0.035	0.035	0.092	0.050	0.105	-0.065	0.051	0.082
201	0.000	0.034	0.034	0.039	0.068	0.079	-0.037	0.079	0.087
202	0.003	0.036	0.036	0.178	0.043	0.184	-0.061	0.040	0.073
203	0.000	0.036	0.036	0.100	0.051	0.113	-0.055	0.052	0.076
204	-0.004	0.036	0.036	0.043	0.071	0.083	-0.033	0.082	0.088
205	-0.005	0.036	0.036	0.175	0.042	0.180	-0.064	0.039	0.075
206	0.004	0.035	0.035	0.100	0.049	0.112	-0.057	0.051	0.076
207	-0.001	0.032	0.032	0.056	0.072	0.092	-0.017	0.084	0.086
208	-0.003	0.034	0.035	0.168	0.042	0.173	-0.071	0.039	0.081
209	0.001	0.033	0.033	0.086	0.045	0.097	-0.072	0.046	0.085
210	-0.002	0.036	0.036	0.050	0.058	0.076	-0.025	0.064	0.069
211	0.007	0.037	0.038	0.183	0.054	0.190	-0.057	0.048	0.075
212	0.005	0.033	0.033	0.099	0.054	0.113	-0.058	0.056	0.080
213	-0.005	0.035	0.035	0.038	0.060	0.071	-0.037	0.069	0.078
214	0.003	0.038	0.038	0.181	0.045	0.186	-0.059	0.041	0.071
215	0.005	0.034	0.034	0.097	0.052	0.110	-0.060	0.053	0.080
216	-0.004	0.036	0.036	0.045	0.067	0.081	-0.030	0.076	0.082

Table 34: Comparison of the average total costs (Poisson demand with $\mu = 8$)

No.	Full Information			Partial Information			RBA		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE	RBias	RSD	RRMSE
1	-0.015	0.018	0.024	-0.010	0.014	0.017	-0.016	0.018	0.024
2	-0.013	0.018	0.022	-0.012	0.017	0.021	-0.013	0.018	0.023
3	-0.013	0.018	0.023	-0.014	0.018	0.022	-0.015	0.019	0.024
4	-0.017	0.018	0.025	0.039	0.023	0.045	-0.022	0.020	0.030
5	-0.016	0.020	0.025	0.011	0.023	0.026	-0.029	0.024	0.038
6	-0.018	0.018	0.025	-0.001	0.031	0.031	-0.024	0.029	0.037
7	0.003	0.018	0.018	0.091	0.024	0.094	-0.003	0.023	0.023
8	0.005	0.017	0.017	0.051	0.026	0.057	-0.013	0.026	0.029
9	0.002	0.018	0.018	0.022	0.031	0.038	-0.014	0.033	0.036
10	0.006	0.013	0.014	-0.002	0.008	0.009	0.009	0.014	0.016
11	0.006	0.013	0.014	0.001	0.011	0.011	0.008	0.013	0.015
12	0.004	0.011	0.012	0.004	0.011	0.011	0.008	0.013	0.015
13	0.007	0.013	0.015	0.045	0.015	0.047	0.003	0.013	0.014
14	0.007	0.013	0.015	0.030	0.018	0.035	0.004	0.018	0.018
15	0.008	0.013	0.015	0.017	0.019	0.026	0.006	0.020	0.020
16	-0.002	0.010	0.011	0.057	0.018	0.060	-0.008	0.017	0.019
17	0.000	0.013	0.013	0.032	0.022	0.038	-0.012	0.021	0.024
18	-0.001	0.012	0.012	0.014	0.026	0.029	-0.011	0.027	0.030
19	0.017	0.020	0.026	0.033	0.022	0.039	0.016	0.020	0.025
20	0.016	0.020	0.026	0.025	0.024	0.035	0.016	0.018	0.025
21	0.017	0.020	0.026	0.023	0.021	0.031	0.015	0.021	0.026
22	-0.004	0.022	0.022	0.069	0.029	0.075	-0.014	0.024	0.028
23	-0.002	0.021	0.021	0.038	0.029	0.048	-0.020	0.026	0.033
24	-0.002	0.021	0.021	0.017	0.034	0.038	-0.013	0.034	0.037
25	-0.006	0.023	0.023	0.097	0.029	0.101	-0.026	0.028	0.038
26	-0.011	0.023	0.026	0.045	0.036	0.058	-0.037	0.036	0.052
27	-0.011	0.020	0.023	0.022	0.043	0.048	-0.020	0.047	0.051
28	-0.005	0.014	0.015	-0.006	0.014	0.015	-0.005	0.014	0.015
29	-0.008	0.014	0.016	-0.007	0.013	0.015	-0.007	0.013	0.015
30	-0.010	0.016	0.019	-0.009	0.016	0.018	-0.008	0.016	0.018
31	0.006	0.016	0.017	0.057	0.022	0.061	0.002	0.020	0.020
32	0.009	0.015	0.018	0.038	0.020	0.043	-0.001	0.019	0.019
33	0.011	0.016	0.019	0.027	0.023	0.036	0.008	0.026	0.027
34	0.002	0.014	0.014	0.083	0.023	0.086	-0.007	0.020	0.022
35	-0.001	0.017	0.017	0.045	0.029	0.053	-0.015	0.028	0.032
36	0.002	0.017	0.017	0.017	0.032	0.037	-0.014	0.033	0.036

Table 35: Comparison of the average total costs (Poisson demand with $\mu = 16$)

No.	Full Information			Partial Information			RBA		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE	RBias	RSD	RRMSE
37	0.002	0.012	0.012	0.041	0.015	0.044	0.007	0.013	0.015
38	0.000	0.010	0.010	0.024	0.014	0.027	0.002	0.012	0.013
39	-0.001	0.011	0.011	0.008	0.014	0.016	-0.001	0.012	0.012
40	0.005	0.014	0.015	0.199	0.018	0.200	0.063	0.017	0.065
41	0.002	0.015	0.015	0.118	0.021	0.120	0.021	0.018	0.028
42	0.002	0.014	0.015	0.063	0.027	0.069	-0.004	0.023	0.023
43	0.012	0.015	0.019	0.301	0.023	0.302	0.100	0.020	0.102
44	0.011	0.016	0.019	0.185	0.024	0.187	0.042	0.022	0.047
45	0.010	0.016	0.019	0.100	0.032	0.105	-0.001	0.028	0.028
46	-0.007	0.010	0.012	-0.002	0.010	0.010	-0.009	0.010	0.014
47	-0.006	0.010	0.012	-0.007	0.010	0.013	-0.007	0.011	0.013
48	-0.006	0.010	0.011	-0.008	0.010	0.013	-0.004	0.010	0.011
49	0.006	0.012	0.013	0.129	0.015	0.130	0.044	0.014	0.046
50	0.007	0.010	0.012	0.079	0.015	0.080	0.018	0.012	0.022
51	0.008	0.011	0.014	0.045	0.015	0.048	0.005	0.016	0.017
52	-0.003	0.012	0.012	0.197	0.016	0.198	0.060	0.013	0.061
53	-0.002	0.011	0.011	0.111	0.019	0.113	0.016	0.016	0.022
54	-0.001	0.011	0.011	0.058	0.024	0.063	-0.009	0.021	0.023
55	-0.013	0.018	0.022	0.065	0.022	0.069	0.007	0.018	0.019
56	-0.012	0.018	0.022	0.032	0.021	0.038	-0.005	0.019	0.020
57	-0.013	0.019	0.023	0.005	0.019	0.019	-0.016	0.018	0.024
58	-0.004	0.018	0.019	0.259	0.030	0.261	0.075	0.026	0.079
59	-0.007	0.021	0.022	0.143	0.029	0.146	0.018	0.024	0.030
60	-0.006	0.022	0.022	0.068	0.032	0.075	-0.016	0.029	0.033
61	0.000	0.023	0.023	0.376	0.036	0.378	0.117	0.030	0.121
62	0.001	0.018	0.018	0.224	0.031	0.227	0.040	0.028	0.049
63	0.003	0.017	0.018	0.114	0.039	0.120	-0.013	0.038	0.040
64	-0.002	0.014	0.014	0.019	0.014	0.024	0.000	0.014	0.014
65	-0.005	0.012	0.013	0.006	0.014	0.015	-0.004	0.013	0.014
66	-0.003	0.013	0.014	0.001	0.014	0.014	-0.003	0.013	0.013
67	-0.006	0.013	0.014	0.166	0.021	0.168	0.046	0.017	0.049
68	-0.006	0.014	0.015	0.090	0.019	0.092	0.009	0.017	0.019
69	-0.004	0.015	0.015	0.044	0.023	0.050	-0.012	0.021	0.024
70	-0.002	0.014	0.014	0.260	0.025	0.261	0.078	0.020	0.081
71	-0.001	0.016	0.016	0.153	0.029	0.155	0.026	0.024	0.035
72	0.003	0.014	0.014	0.088	0.028	0.093	-0.002	0.025	0.026

Table 36: Comparison of the average total costs (negative binomial demand with $\mu = 8$ and $\sigma^2 = 3\mu$)

No.	Full Information			Partial Information			RBA		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE	RBias	RSD	RRMSE
73	0.024	0.027	0.036	0.037	0.027	0.045	0.020	0.026	0.032
74	0.021	0.022	0.031	0.027	0.024	0.036	0.019	0.025	0.031
75	0.019	0.025	0.031	0.021	0.026	0.033	0.026	0.028	0.038
76	0.013	0.028	0.030	0.049	0.031	0.058	-0.006	0.031	0.032
77	0.013	0.025	0.028	0.032	0.038	0.050	0.014	0.040	0.042
78	0.016	0.023	0.029	0.018	0.042	0.046	0.033	0.051	0.061
79	0.005	0.029	0.029	0.048	0.035	0.059	-0.026	0.034	0.043
80	0.006	0.026	0.027	0.031	0.041	0.051	0.002	0.044	0.044
81	0.010	0.030	0.031	0.018	0.049	0.053	0.037	0.062	0.072
82	0.009	0.016	0.018	0.012	0.016	0.020	0.009	0.016	0.018
83	0.008	0.017	0.019	0.009	0.018	0.020	0.007	0.018	0.019
84	0.008	0.016	0.018	0.009	0.017	0.019	0.011	0.018	0.021
85	0.009	0.021	0.023	0.031	0.024	0.040	-0.007	0.023	0.024
86	0.012	0.022	0.025	0.026	0.025	0.036	0.012	0.028	0.031
87	0.013	0.019	0.023	0.018	0.033	0.038	0.027	0.041	0.049
88	0.007	0.025	0.026	0.043	0.028	0.051	-0.015	0.028	0.031
89	0.004	0.023	0.024	0.021	0.033	0.039	0.001	0.037	0.037
90	0.004	0.023	0.023	0.011	0.037	0.038	0.022	0.047	0.052
91	0.006	0.032	0.033	0.027	0.032	0.042	-0.002	0.034	0.034
92	0.007	0.027	0.028	0.015	0.032	0.035	0.007	0.032	0.033
93	0.006	0.027	0.028	0.010	0.030	0.032	0.016	0.031	0.035
94	-0.013	0.031	0.034	0.026	0.033	0.042	-0.040	0.028	0.049
95	-0.015	0.033	0.037	0.008	0.042	0.043	-0.020	0.043	0.047
96	-0.014	0.032	0.035	0.004	0.055	0.055	0.023	0.067	0.071
97	0.004	0.031	0.032	0.061	0.041	0.074	-0.031	0.039	0.050
98	0.002	0.032	0.032	0.031	0.052	0.060	-0.004	0.056	0.056
99	0.002	0.033	0.033	0.013	0.069	0.070	0.037	0.089	0.096
100	0.008	0.023	0.025	0.014	0.023	0.027	0.003	0.022	0.023
101	0.005	0.022	0.022	0.010	0.022	0.024	0.005	0.022	0.023
102	0.012	0.024	0.026	0.012	0.024	0.027	0.015	0.025	0.029
103	0.003	0.025	0.025	0.029	0.028	0.041	-0.018	0.026	0.032
104	-0.001	0.027	0.027	0.013	0.033	0.035	-0.007	0.033	0.034
105	0.006	0.026	0.027	0.010	0.042	0.043	0.024	0.052	0.057
106	0.008	0.029	0.030	0.052	0.038	0.064	-0.018	0.036	0.041
107	0.010	0.026	0.028	0.034	0.038	0.051	0.008	0.041	0.041
108	0.004	0.027	0.027	0.009	0.048	0.049	0.026	0.060	0.065

Table 37: Comparison of the average total costs (negative binomial demand with $\mu = 16$ and $\sigma^2 = 3\mu$)

No.	Full Information			Partial Information			RBA		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE	RBias	RSD	RRMSE
109	-0.012	0.019	0.022	0.033	0.023	0.041	-0.011	0.019	0.022
110	-0.007	0.018	0.020	0.017	0.020	0.026	-0.012	0.017	0.021
111	-0.012	0.019	0.023	0.001	0.023	0.023	-0.017	0.020	0.027
112	0.000	0.024	0.024	0.129	0.033	0.133	0.000	0.028	0.028
113	0.001	0.021	0.022	0.071	0.029	0.077	-0.019	0.027	0.033
114	0.008	0.021	0.022	0.043	0.037	0.057	-0.013	0.037	0.039
115	0.007	0.027	0.028	0.188	0.036	0.191	0.005	0.030	0.030
116	0.004	0.024	0.025	0.103	0.035	0.109	-0.022	0.034	0.040
117	0.004	0.022	0.022	0.056	0.038	0.068	-0.021	0.041	0.045
118	0.002	0.016	0.016	0.021	0.017	0.027	0.001	0.016	0.016
119	0.003	0.014	0.015	0.010	0.016	0.019	0.001	0.014	0.014
120	0.000	0.014	0.014	0.004	0.016	0.017	-0.001	0.014	0.014
121	0.006	0.018	0.019	0.094	0.023	0.097	0.003	0.019	0.020
122	0.004	0.017	0.017	0.051	0.023	0.056	-0.011	0.019	0.022
123	0.007	0.018	0.019	0.032	0.025	0.040	-0.007	0.024	0.025
124	0.004	0.019	0.019	0.133	0.028	0.136	-0.001	0.023	0.023
125	0.006	0.019	0.020	0.077	0.025	0.081	-0.016	0.023	0.028
126	0.003	0.020	0.021	0.036	0.033	0.049	-0.020	0.033	0.039
127	0.010	0.030	0.032	0.082	0.036	0.089	0.009	0.031	0.032
128	0.013	0.026	0.029	0.046	0.032	0.056	0.001	0.029	0.029
129	0.010	0.024	0.026	0.027	0.031	0.041	0.000	0.029	0.029
130	-0.014	0.031	0.034	0.146	0.042	0.152	-0.015	0.035	0.038
131	-0.011	0.031	0.033	0.073	0.042	0.084	-0.038	0.038	0.054
132	-0.007	0.030	0.031	0.036	0.044	0.056	-0.031	0.044	0.054
133	-0.003	0.028	0.029	0.209	0.043	0.213	-0.009	0.038	0.039
134	-0.013	0.030	0.032	0.106	0.044	0.115	-0.044	0.041	0.060
135	-0.004	0.032	0.032	0.060	0.047	0.076	-0.033	0.048	0.059
136	0.004	0.020	0.020	0.039	0.024	0.046	0.004	0.021	0.021
137	0.002	0.018	0.018	0.019	0.022	0.029	-0.003	0.019	0.019
138	0.007	0.019	0.020	0.015	0.021	0.026	0.002	0.019	0.019
139	-0.001	0.024	0.024	0.114	0.030	0.118	-0.003	0.027	0.027
140	0.000	0.020	0.020	0.064	0.031	0.071	-0.017	0.027	0.032
141	-0.003	0.023	0.023	0.032	0.034	0.046	-0.017	0.033	0.036
142	0.005	0.024	0.024	0.172	0.032	0.175	0.003	0.027	0.027
143	0.009	0.027	0.028	0.093	0.037	0.100	-0.024	0.034	0.042
144	0.007	0.026	0.027	0.053	0.043	0.068	-0.017	0.044	0.047

Table 38: Comparison of the average total costs (negative binomial demand with $\mu = 8$ and $\sigma^2 = 5\mu$)

No.	Full Information			Partial Information			RBA		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE	RBias	RSD	RRMSE
145	-0.016	0.032	0.036	-0.011	0.034	0.035	-0.022	0.031	0.038
146	-0.017	0.025	0.031	-0.012	0.027	0.029	-0.010	0.027	0.029
147	-0.012	0.026	0.028	-0.012	0.028	0.030	0.010	0.034	0.036
148	-0.006	0.031	0.032	0.015	0.038	0.041	-0.020	0.037	0.042
149	0.000	0.033	0.033	0.013	0.045	0.047	0.019	0.051	0.054
150	-0.002	0.029	0.029	0.002	0.050	0.050	0.056	0.067	0.088
151	-0.008	0.037	0.038	0.017	0.047	0.050	-0.034	0.048	0.058
152	-0.005	0.037	0.038	0.007	0.052	0.052	0.018	0.060	0.062
153	-0.009	0.032	0.033	0.002	0.061	0.061	0.074	0.088	0.115
154	-0.006	0.017	0.018	-0.002	0.018	0.018	-0.007	0.017	0.018
155	-0.005	0.022	0.022	-0.005	0.023	0.023	-0.004	0.023	0.024
156	-0.006	0.022	0.023	-0.002	0.024	0.024	0.007	0.027	0.027
157	-0.011	0.024	0.026	0.005	0.028	0.028	-0.020	0.027	0.034
158	-0.004	0.029	0.030	0.001	0.036	0.036	0.003	0.039	0.039
159	-0.009	0.024	0.026	-0.001	0.037	0.037	0.039	0.050	0.063
160	-0.007	0.030	0.031	0.018	0.037	0.041	-0.019	0.036	0.041
161	-0.005	0.029	0.029	0.006	0.040	0.040	0.013	0.046	0.048
162	-0.006	0.030	0.031	0.001	0.050	0.050	0.056	0.067	0.087
163	-0.002	0.038	0.038	0.009	0.043	0.044	-0.010	0.040	0.041
164	0.002	0.038	0.038	0.007	0.046	0.047	0.012	0.048	0.050
165	-0.006	0.032	0.033	-0.001	0.041	0.041	0.032	0.048	0.057
166	-0.009	0.039	0.040	0.013	0.049	0.050	-0.031	0.048	0.058
167	-0.005	0.037	0.037	0.006	0.049	0.049	0.013	0.056	0.057
168	-0.004	0.041	0.041	0.000	0.059	0.059	0.064	0.078	0.101
169	0.002	0.050	0.050	0.029	0.062	0.068	-0.030	0.061	0.068
170	-0.006	0.041	0.042	0.010	0.057	0.058	0.017	0.066	0.068
171	-0.002	0.037	0.037	0.001	0.073	0.073	0.083	0.099	0.130
172	0.008	0.030	0.031	0.014	0.033	0.036	0.002	0.032	0.032
173	0.006	0.028	0.028	0.011	0.027	0.029	0.012	0.028	0.030
174	0.015	0.030	0.034	0.019	0.034	0.038	0.037	0.036	0.052
175	-0.005	0.037	0.038	0.010	0.042	0.043	-0.023	0.041	0.047
176	-0.010	0.034	0.036	0.003	0.041	0.041	0.011	0.046	0.047
177	-0.011	0.035	0.036	0.002	0.054	0.054	0.054	0.071	0.089
178	0.001	0.036	0.036	0.024	0.042	0.049	-0.022	0.042	0.048
179	-0.003	0.039	0.039	0.009	0.054	0.055	0.017	0.062	0.064
180	0.001	0.041	0.041	0.006	0.066	0.066	0.072	0.086	0.112

Table 39: Comparison of the average total costs (negative binomial demand with $\mu = 16$ and $\sigma^2 = 5\mu$)

No.	Full Information			Partial Information			RBA		
	RBias	RSD	RRMSE	RBias	RSD	RRMSE	RBias	RSD	RRMSE
181	0.008	0.024	0.025	0.045	0.025	0.052	-0.006	0.023	0.024
182	0.008	0.023	0.025	0.033	0.026	0.042	-0.002	0.024	0.025
183	0.011	0.024	0.026	0.017	0.028	0.033	0.003	0.029	0.029
184	0.017	0.028	0.033	0.108	0.032	0.112	-0.016	0.026	0.031
185	0.012	0.029	0.032	0.065	0.037	0.075	-0.015	0.035	0.038
186	0.011	0.026	0.028	0.031	0.036	0.048	-0.007	0.038	0.039
187	0.007	0.034	0.034	0.131	0.036	0.136	-0.033	0.031	0.045
188	0.014	0.033	0.036	0.077	0.049	0.091	-0.031	0.045	0.055
189	0.011	0.029	0.031	0.037	0.058	0.068	-0.016	0.061	0.063
190	0.001	0.016	0.016	0.019	0.020	0.027	-0.004	0.016	0.017
191	0.001	0.019	0.019	0.012	0.022	0.025	-0.004	0.020	0.020
192	0.001	0.018	0.018	0.006	0.020	0.021	-0.002	0.020	0.020
193	0.002	0.025	0.025	0.066	0.031	0.073	-0.020	0.026	0.033
194	0.001	0.025	0.025	0.037	0.033	0.049	-0.019	0.029	0.035
195	0.001	0.021	0.021	0.018	0.029	0.034	-0.009	0.029	0.030
196	0.007	0.022	0.023	0.098	0.027	0.102	-0.027	0.023	0.036
197	0.006	0.028	0.028	0.056	0.035	0.066	-0.025	0.033	0.042
198	0.005	0.027	0.028	0.031	0.040	0.050	-0.008	0.042	0.043
199	0.003	0.035	0.035	0.061	0.040	0.073	-0.013	0.035	0.037
200	0.009	0.038	0.039	0.040	0.040	0.056	-0.008	0.037	0.038
201	0.001	0.037	0.037	0.014	0.043	0.045	-0.009	0.042	0.043
202	0.008	0.037	0.037	0.119	0.042	0.126	-0.030	0.036	0.047
203	-0.001	0.036	0.036	0.065	0.043	0.078	-0.032	0.041	0.052
204	0.001	0.037	0.037	0.030	0.054	0.061	-0.016	0.056	0.058
205	0.001	0.038	0.038	0.149	0.046	0.156	-0.044	0.039	0.059
206	0.011	0.037	0.039	0.089	0.049	0.102	-0.035	0.046	0.058
207	0.005	0.036	0.036	0.050	0.063	0.081	-0.007	0.070	0.070
208	-0.004	0.023	0.024	0.026	0.025	0.036	-0.012	0.023	0.026
209	0.000	0.024	0.024	0.015	0.026	0.030	-0.010	0.025	0.027
210	-0.003	0.026	0.026	0.006	0.030	0.030	-0.006	0.029	0.030
211	0.002	0.034	0.034	0.085	0.042	0.095	-0.026	0.037	0.046
212	-0.001	0.029	0.029	0.043	0.038	0.057	-0.028	0.035	0.045
213	-0.011	0.028	0.030	0.010	0.036	0.038	-0.026	0.038	0.046
214	-0.001	0.035	0.035	0.113	0.040	0.120	-0.039	0.034	0.052
215	0.000	0.031	0.031	0.060	0.042	0.073	-0.039	0.039	0.055
216	-0.008	0.032	0.033	0.024	0.049	0.055	-0.023	0.052	0.057

Table 40: Pairwise comparison of the average total costs based on Poisson demand

No.	Partial vs. Full		RBA vs. Full		No.	Partial vs. Full		RBA vs. Full	
	Average	StDev.	Average	StDev.		Average	StDev.	Average	StDev.
1	0.54	1.75	-0.07	0.76	37	3.99	1.60	0.59	1.14
2	0.05	0.98	-0.04	0.70	38	2.35	1.17	0.14	0.67
3	-0.01	0.87	-0.13	0.84	39	0.89	1.35	0.03	0.61
4	5.77	2.27	-0.42	1.83	40	19.31	1.82	5.71	1.62
5	2.70	2.14	-1.37	2.10	41	11.55	1.83	1.91	1.53
6	1.68	2.78	-0.60	2.62	42	6.12	2.52	-0.60	2.16
7	8.77	2.34	-0.59	2.06	43	28.59	2.08	8.68	2.11
8	4.58	2.60	-1.75	2.61	44	17.21	2.11	2.99	1.86
9	2.01	2.87	-1.60	3.19	45	8.88	2.95	-1.04	2.58
10	-0.78	1.33	0.23	0.87	46	0.43	1.14	-0.29	0.65
11	-0.47	1.01	0.21	0.78	47	-0.14	0.72	-0.18	0.57
12	-0.06	0.63	0.34	0.98	48	-0.25	0.61	0.14	0.61
13	3.70	1.44	-0.47	1.30	49	12.24	1.38	3.73	1.30
14	2.29	1.53	-0.35	1.53	50	7.12	1.28	1.13	1.07
15	0.92	1.55	-0.23	1.62	51	3.74	1.27	-0.26	1.29
16	5.92	1.67	-0.63	1.58	52	20.06	1.81	6.32	1.55
17	3.21	1.90	-1.17	1.94	53	11.35	1.79	1.78	1.57
18	1.54	2.44	-0.95	2.59	54	5.95	2.17	-0.74	1.81
19	1.58	1.98	-0.13	0.97	55	7.94	1.45	2.04	1.21
20	0.89	1.71	0.02	1.16	56	4.42	1.58	0.73	1.27
21	0.64	1.48	-0.18	1.33	57	1.80	1.40	-0.26	1.17
22	7.33	2.06	-1.03	1.90	58	26.40	2.24	7.93	2.05
23	3.99	2.75	-1.85	2.41	59	15.18	2.28	2.58	1.71
24	1.92	3.02	-1.06	3.24	60	7.52	2.86	-0.97	2.42
25	10.36	2.52	-2.02	2.35	61	37.62	2.67	11.70	2.23
26	5.67	3.22	-2.63	3.29	62	22.33	2.80	3.91	2.42
27	3.35	3.99	-0.92	4.54	63	11.07	3.71	-1.59	3.70
28	-0.05	0.85	0.01	0.58	64	2.17	1.07	0.23	0.86
29	0.04	0.59	0.10	0.58	65	1.08	1.21	0.06	0.50
30	0.11	0.77	0.19	0.75	66	0.43	1.07	0.04	0.44
31	5.07	1.68	-0.44	1.47	67	17.30	1.60	5.20	1.38
32	2.81	1.64	-0.97	1.65	68	9.71	1.47	1.60	1.31
33	1.66	2.20	-0.23	2.53	69	4.83	1.81	-0.85	1.54
34	8.02	1.78	-0.91	1.63	70	26.26	2.07	8.06	1.68
35	4.61	2.48	-1.40	2.38	71	15.40	2.26	2.67	1.84
36	1.57	2.78	-1.53	2.84	72	8.46	2.27	-0.58	2.02

Table 41: Pairwise comparison of the average total costs based on negative binomial demand with $\sigma^2 = 3\mu$

No.	Partial vs. Full		RBA vs. Full		No.	Partial vs. Full		RBA vs. Full	
	Average	StDev.	Average	StDev.		Average	StDev.	Average	StDev.
73	1.23	1.91	-0.45	1.24	109	4.56	1.56	0.02	1.24
74	0.60	1.64	-0.21	1.75	110	2.37	1.08	-0.47	1.20
75	0.18	1.80	0.66	2.15	111	1.35	1.45	-0.52	1.25
76	3.58	1.77	-1.87	1.94	112	12.84	2.08	-0.05	1.63
77	1.88	2.63	0.12	2.86	113	6.98	2.31	-2.01	2.17
78	0.21	3.32	1.60	4.19	114	3.48	2.99	-2.09	3.11
79	4.36	2.12	-3.04	2.23	115	18.01	2.37	-0.15	2.07
80	2.48	3.07	-0.45	3.63	116	9.88	2.62	-2.56	2.61
81	0.82	4.12	2.71	5.42	117	5.11	3.58	-2.47	3.83
82	0.28	1.05	-0.01	0.65	118	1.95	0.85	0.00	0.63
83	0.09	0.83	-0.07	0.89	119	0.73	1.09	-0.19	0.79
84	0.05	0.69	0.29	1.05	120	0.45	0.92	-0.06	0.63
85	2.20	1.59	-1.59	1.56	121	8.78	1.50	-0.29	1.18
86	1.36	1.74	-0.04	1.96	122	4.67	1.74	-1.48	1.48
87	0.48	2.67	1.35	3.42	123	2.43	1.87	-1.44	1.86
88	3.58	2.02	-2.16	1.90	124	12.79	1.87	-0.55	1.59
89	1.73	2.39	-0.31	2.91	125	7.05	2.01	-2.17	1.90
90	0.72	3.15	1.89	4.28	126	3.30	2.67	-2.37	2.78
91	2.07	1.94	-0.87	1.89	127	7.06	1.82	-0.16	1.40
92	0.79	1.87	0.03	2.10	128	3.33	1.33	-1.11	1.34
93	0.39	2.02	1.03	2.42	129	1.65	1.96	-1.00	1.88
94	3.96	2.22	-2.75	2.20	130	16.25	2.35	-0.09	1.83
95	2.30	2.82	-0.52	3.27	131	8.56	2.73	-2.72	2.32
96	1.85	4.32	3.72	5.86	132	4.24	3.14	-2.49	3.26
97	5.66	2.81	-3.52	2.72	133	21.32	2.94	-0.52	2.72
98	2.92	4.16	-0.59	4.73	134	11.99	3.07	-3.22	2.87
99	1.12	5.62	3.47	7.70	135	6.46	3.69	-2.85	3.90
100	0.59	1.43	-0.46	1.24	136	3.41	1.09	0.01	0.79
101	0.49	1.49	-0.02	1.48	137	1.69	1.13	-0.53	0.94
102	0.06	1.46	0.28	1.66	138	0.83	1.11	-0.46	1.09
103	2.67	1.58	-2.06	1.69	139	11.50	1.78	-0.27	1.51
104	1.41	2.33	-0.60	2.72	140	6.36	2.01	-1.74	1.82
105	0.45	3.46	1.86	4.59	141	3.51	2.29	-1.36	2.31
106	4.32	2.58	-2.60	2.71	142	16.57	2.02	-0.23	1.73
107	2.38	2.62	-0.23	3.14	143	8.30	2.57	-3.29	2.33
108	0.51	4.02	2.22	5.34	144	4.54	3.27	-2.35	3.44

Table 42: Pairwise comparison of the average total costs for negative binomial demand with $\sigma^2 = 5\mu$

No.	Partial vs. Full		RBA vs. Full		No.	Partial vs. Full		RBA vs. Full	
	Average	StDev.	Average	StDev.		Average	StDev.	Average	StDev.
145	0.58	1.66	-0.57	1.85	181	3.69	1.38	-1.35	1.26
146	0.58	1.77	0.75	2.13	182	2.41	1.47	-1.03	1.52
147	0.11	2.32	2.32	3.18	183	0.59	1.46	-0.76	1.69
148	2.13	2.24	-1.42	2.21	184	8.94	2.03	-3.22	1.73
149	1.30	2.71	1.86	3.56	185	5.29	2.31	-2.66	2.25
150	0.39	4.16	5.81	5.92	186	2.00	2.69	-1.83	3.03
151	2.52	2.58	-2.63	2.87	187	12.41	2.15	-3.91	2.05
152	1.29	3.99	2.35	4.87	188	6.27	2.76	-4.36	2.82
153	1.13	6.15	8.41	8.89	189	2.46	4.27	-2.72	4.71
154	0.40	1.12	-0.10	0.79	190	1.71	1.32	-0.51	0.95
155	0.08	1.06	0.17	1.08	191	1.12	1.17	-0.47	0.93
156	0.39	1.47	1.34	1.84	192	0.49	1.13	-0.25	1.11
157	1.57	1.77	-0.95	1.85	193	6.38	1.77	-2.22	1.47
158	0.53	2.33	0.76	2.66	194	3.54	1.72	-2.05	1.55
159	0.84	2.97	4.84	4.31	195	1.64	1.98	-1.00	2.08
160	2.53	2.15	-1.19	2.38	196	9.08	1.85	-3.40	1.63
161	1.15	2.89	1.80	3.74	197	4.91	2.19	-3.15	2.16
162	0.70	4.30	6.23	6.01	198	2.59	2.72	-1.27	3.16
163	1.09	1.78	-0.85	1.65	199	5.73	1.64	-1.58	1.54
164	0.48	2.32	1.03	2.75	200	3.05	1.45	-1.68	1.29
165	0.58	2.63	3.83	3.57	201	1.39	2.33	-0.97	2.46
166	2.13	2.47	-2.30	2.61	202	10.97	2.17	-3.77	1.84
167	1.16	3.19	1.85	4.04	203	6.62	2.57	-3.11	2.48
168	0.42	4.90	6.83	7.15	204	2.95	3.87	-1.61	4.37
169	2.68	3.47	-3.23	3.73	205	14.83	2.91	-4.48	2.53
170	1.69	4.68	2.35	5.86	206	7.75	3.46	-4.55	3.39
171	0.31	5.91	8.55	8.53	207	4.50	4.98	-1.22	5.81
172	0.53	1.41	-0.60	1.42	208	3.10	1.30	-0.80	1.05
173	0.49	1.64	0.64	1.77	209	1.52	1.12	-0.99	1.07
174	0.32	1.66	2.15	2.30	210	0.93	1.12	-0.24	1.08
175	1.56	2.13	-1.73	2.53	211	8.32	1.84	-2.76	1.62
176	1.28	2.40	2.09	3.21	212	4.43	2.03	-2.66	2.12
177	1.30	4.16	6.51	5.90	213	2.08	2.48	-1.47	2.85
178	2.37	2.42	-2.26	2.85	214	11.42	2.23	-3.76	2.07
179	1.13	3.65	1.94	4.56	215	6.00	2.64	-3.88	2.55
180	0.46	5.18	7.03	7.36	216	3.26	3.63	-1.49	4.09

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