MULTILINEAR DYADIC OPERATORS AND THEIR COMMUTATORS

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MULTILINEAR DYADIC OPERATORS AND THEIR COMMUTATORS

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Dyadic operators have attracted a lot of attention in the recent years as the dyadic techniques have been established as fundamental tools in harmonic analysis. The theory of linear dyadic operators has been proved extremely useful in the advancement of the theory of linear Calderón-Zygmund operators and their commutarors with locally integrable functions. The proof of so-called A_2 theorem (see [1]) consisted in representing a linear Calderón-Zygmund operator as an average of dyadic operators, and then verifying some testing conditions for those simpler dyadic operators. It is now well-known that the linear Calderón-Zygmund operators, as well as their commutators with locally integrable functions can be dominated pointwise by sparse dyadic operators. These results have been particularly helpful in obtaining quantitative weighted inequalities for these operators, including the Bloom's inequality for the commutators in two-weight setting. The objects, statements, and often proofs are simpler in the dyadic world, but yet illuminating enough to guarantee that one can translate them into the non-dyadic world.

As in the linear case, we can expect a similar connection between the multilinear dyadic and non-dyadic worlds. The main goal of this dissertation is to develop a detailed theory of multilinear dyadic operators (paraproducts and Haar multipliers) and their commutators with locally integrable functions. These multilinear operators can be thought of as discrete dyadic models of multilinear Calderón-Zygmund operators introduced in [2], and we can expect that the results obtained in the multilinear dyadic setting will eventually imply corresponding results in the continuous setting.

We introduce multilinear dyadic paraproducts and Haar multipliers in Chapter 2, where we motivate their definitions by obtaining a generalized paraproduct decomposition of the pointwise product of two or more functions. These operators and their commutators with locally integrable functions are the main objects of our study.

In Chapter 3, we investigate the boundedness properties of multilinear dyadic paraproducts and Haar multipliers in the unweighted setting. The corresponding theory of linear dyadic operators, which we will be using very often, can be found in [3]. In [4], the authors have studied boundedness properties of bilinear paraproducts defined in terms of so-called "smooth molecules". The paraproduct operators we study are general multilinear operators defined in terms of indicators and Haar functions of dyadic intervals. In [5] Coifman, Rochberg and Weiss proved that the commutator of a BMO function with a singular integral operator is bounded in L^p, $1 < p < \infty$. The necessity of BMO condition for the boundedness of the commutator was also established for certain singular integral operators, such as the Hilbert transform. S. Janson [6] later studied its analogue for linear martingale transforms. In Chapter 3, we study commutators of multilinear dyadic operators, and characterize dyadic BMO functions via the boundedness of these commutators. The corresponding theory for general multilinear Calderón-Zygmund operators can be found in [2].

Using the unweighted theory from Chapter 3, and exploring some additional properties of these multilinear dyadic operators and their commutators, we obtain weighted estimates for them in Chapter 4. These results are the dyadic analogs of the corresponding results for multilinear Calderón-Zygmund operators obtained in [7], and are included in [8]. In this chapter, we also characterize dyadic BMO functions via the boundedness of the commutators of multilinear dyadic paraproducts in the weighted setting. Such characterization of BMO functions in the continuous case is yet to be known.

Domination by sparse operators has become a very useful idea in better understanding the weighted estimates for various linear and multilinear operators. It is now well known that a linear Calderón-Zygmund operator can be dominated pointwise by a finite number of sparse operators. Such domination results are particularly helpful in obtaining sharp weighted bounds for these operators. M. Lacey [9] showed that given a martingale transform T and a locally supported integrable function f, there exists a sparse operator S (depending on T and f) such that $|T(f)| \lesssim S(|f|)$. Using this result, the author then established a number of sharp weighted inequalities for martingale transforms, and gave an elementary proof of the A_2 bounds in the continuous setting.

A. K. Lerner, S. Ombrosi and I. P. Rivera-Ríos [10] obtained a sparse domination result for the commutator $[b, T]$ of a linear Calderón-Zygmund operator T with a locally integrable function b, and derived several weighted inequalities for the commutators. In particular, they obtained quantitative norm inequalities for $[b, T]$ in two-weight setting. Study of commutators in two-weight setting was initiated by Bloom [11] who, in 1985, characterized the boundedness of the commutator of the Hilbert transform H:

$$[b, H] : L^p(\lambda) \to L^p(\mu), \quad \lambda, \mu \in A_2,\ 0 < p < \infty,$$

by a BMO condition on b adapted to the weights λ and μ; namely

$$\|b\|_{BMO_\nu} := \sup_I \frac{1}{\nu(I)} \int_I |b(x) - \langle b \rangle_I| dx < \infty,$$

where $\nu = \left(\frac{\mu}{\lambda}\right)^{1/p}$. A modern proof of the same result was given in [12] for $p = 2$ by I. Holmes, M. Lacey and B. Wick, who in a subsequent paper [13], generalized the result for the commutators of Riesz transforms for $1 < p < \infty$, and also obtained the upper bound for the commutators of linear Calderón-Zygmund operators in the two-weight setting.

In Chapter 5, we show that the multilinear dyadic paraproducts and Haar multipliers can be dominated pointwise by multilinear sparse operators, and also obtain similar pointwise estimates for their commutators with locally integrable functions. We then introduce and prove the multilinear Bloom's inequality for the commutators of multilinear Haar multipliers. These results regarding commutators are new in the multilinear setting, and can be expected to hold also for the commutators of multilinear Calderón-Zygmund operators.

1.2 Preliminaries

1.2.1 The Haar System

Let $\mathcal{D}$ denote the standard dyadic grid on $\mathbb{R}$,

$$\mathcal{D} = \{[m2^{-k}, (m+1)2^{-k}) : m, k \in \mathbb{Z}\}.$$

Associated to each dyadic interval I, there is a Haar function h_I defined by

$$h_I(x) = \frac{1}{|I|^{1/2}} \left(\mathbb{1}_{I_+} - \mathbb{1}_{I_-}\right),$$

where I_- and I_+ are the left and right halves of I.

The collection of all Haar functions $\{h_I : I \in \mathcal{D}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, and an unconditional basis of L^p for $1 < p < \infty$. In fact, if a sequence $\epsilon = \{\epsilon_I\}_{I\in\mathcal{D}}$ is bounded, the operator T_ϵ defined by

$$T_\epsilon f(x) = \sum_{I\in\mathcal{D}} \epsilon_I \langle f, h_I\rangle h_I$$

is bounded in L^p for all $1 < p < \infty$. The converse also holds. The operator T_ϵ is called the Haar multiplier with symbol ϵ.

1.2.2 Weights and A_p Classes

A weight w is a non-negative locally integrable function on $\mathbb{R}$ such that $0 < w(x) < \infty$ for almost every x. Given a weight w and a measurable set $E \subseteq \mathbb{R}$, the w-measure of E is defined by

$$w(E) = \int_E w(x)dx.$$

We say that a weight w belongs to the class A_p for $1 < p < \infty$ if it satisfies the Muckenhoupt condition:

$$\sup_I \left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}\right)^{p-1} < \infty,$$

where the supremum is taken over all intervals. The expression on the left is called the A_p (Muckenhoupt) characteristic constant of w, and is denoted by $[w]_{A_p}$. Note that if p' is the conjugate index of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$, then $1 - p' = -\frac{1}{p-1} = -\frac{p'}{p}$. So,

$$\begin{aligned} [w]_{A_p} &= \sup_I \left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{1-p'}\right)^{1/p'} \\ &= \sup_I \left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{-\frac{p'}{p}}\right)^{\frac{p}{p'}}. \end{aligned}$$

It can be shown that $\lim_{p\to 1}\left(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}\right)^{p-1} = \|w^{-1}\|_{L^\infty(I)}$. This leads to the following definition of A_1 class:

A weight w is called an A_1 weight if

$$[w]_{A_1} := \sup_I \left(\frac{1}{|I|}\int_I w\right)\|w^{-1}\|_{L^\infty(I)} < \infty.$$

Thus $[w]_{A_1}$ is the infimum of all constants C such that for all intervals I,

$$\frac{1}{|I|}\int_I w \le Cw(x) \quad \text{for a.e. } x \in I.$$

The A_p classes are increasing with respect to p, i.e. for $1 \le p_1 < p_2 < \infty$,

$$[w]_{A_{p_2}} \le [w]_{A_{p_1}}.$$

It is natural to define the A_∞ class of weights by

$$A_\infty = \bigcup_{p>1} A_p,$$

with $[w]_{A_\infty} = \inf\{[w]_{A_p} : w \in A_p\}$.

For $1 \le p < \infty$, the dyadic A_p^d classes are defined by the same inequalities restricted to the dyadic intervals. Moreover, $A_\infty^d = \bigcup\limits_{p>1} A_p^d$.

1.2.3 Multilinear $A_{\vec{P}}$ Condition

We state the multilinear $A_{\vec{P}}$ condition introduced by Lerner et al. [7].

Let $\vec{P} = (p_1, \dots, p_m)$ and $\vec{w} = (w_1, \dots, w_m)$, where $1 \le p_1, \dots, p_m < \infty$ and $w_1, \dots, w_m$ are non-negative measurable functions. Let $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$.

We say that $\vec{w}$ satisfies the multilinear $A_{\vec{P}}$ condition and we write $\vec{w} \in A_{\vec{P}}$ if

$$\sup_I \left(\frac{1}{|I|} \int_I \nu_{\vec{w}} \right)^{\frac{1}{p}} \prod_{j=1}^m \left(\frac{1}{|I|} \int_I w_j^{1-p_j'} \right)^{\frac{1}{p_j'}} < \infty,$$

where $\nu_{\vec{w}} := \prod\limits_{j=1}^m w_j^{p/p_j}$, and $\left(\frac{1}{|I|} \int_I w_j^{1-p_j'} \right)^{\frac{1}{p_j'}}$ is understood as $\|w_j^{-1}\|_{L^\infty(I)}$ when $p_j = 1$.

Using Hölder's inequality, it is easy to see that

$$\prod_{j=1}^m A_{p_j} \subset A_{\vec{P}}.$$

Moreover, if $\vec{w} \in A_{\vec{P}}$, $\nu_{\vec{w}} \in A_{mp}$. We will denote the *dyadic multiliner $A_{\vec{P}}$ class* by $A_{\vec{P}}^d$.

1.2.4 Lebesgue Spaces

Given $0 < p < \infty$, the Lebesgue space $L^p(\mathbb{R})$ is defined by

$$L^p(\mathbb{R}) := \{f : \|f\|_p < \infty\},$$

where, $\|f\|_p = \|f\|_{L^p} := \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p}$. The Weak L^p space, also denoted by $L^{p,\infty}(\mathbb{R})$, is the space of all functions f such that

$$\|f\|_{L^{p,\infty}(\mathbb{R})} := \sup_{t>0} t \left|\{x \in \mathbb{R} : f(x) > t\}\right|^{1/p} < \infty.$$

For convenience, we will denote $L^p(\mathbb{R})$ and $L^{p,\infty}(\mathbb{R})$ by L^p and $L^{p,\infty}$ respectively.

Given a weight w, the weighted Lebesgue space $L^p(\mathbb{R}, w)$ is defined by

$$L^p(\mathbb{R}, w) := \left\{f : \|f\|_{L^p(\mathbb{R},w)} < \infty\right\},$$

where, $\|f\|_{L^p(\mathbb{R},w)} := \left(\int_R |f(x)|^p w(x) dx\right)^{1/p}$. Moreover, the weak space $L^{p,\infty}(\mathbb{R}, w)$ is the space of all functions f such that

$$\|f\|_{L^{p,\infty}(\mathbb{R},w)} := \sup_{t>0} t\, w\left(\{x \in \mathbb{R} : f(x) > t\}\right)^{1/p} < \infty.$$

For convenience, we will denote $L^p(\mathbb{R}, w)$ and $L^{p,\infty}(\mathbb{R})$ by $L^p(w)$ and $L^{p,\infty}(w)$ respectively.

1.2.5 Maximal Operators

Given a function f, the maximal function Mf is defined by

$$Mf(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(t)|\, dt,$$

where the supremum is taken over all intervals I in $\mathbb{R}$ that contain x.

For $\delta > 0$, the maximal operator M_δ is defined by

$$M_\delta f(x) := M(|f|^\delta)^{1/\delta}(x) = \left(\sup_{I \ni x} \frac{1}{|I|} \int_I |f(t)|^\delta \, dt \right)^{1/\delta}.$$

The sharp maximal function $M^\#$ is given by

$$M^\# f(x) := \sup_{I \ni x} \inf_c \frac{1}{|I|} \int_I |f(t) - c| \, dt.$$

In fact,

$$M^\# f(x) \approx \sup_{I \ni x} \frac{1}{|I|} \int_I |f(t) - \langle f \rangle_I| \, dt,$$

where $\langle f \rangle_I := \dfrac{1}{|I|} \displaystyle\int_I f(t)dt$ is the average of f over I.

Given $\vec{f} = (f_1, \ldots, f_m)$, the maximal operators $\mathcal{M}$ and $\mathcal{M}_r$ with $r > 0$ are defined by

$$\mathcal{M}(\vec{f})(x) = \sup_{I \ni x} \prod_{i=1}^m \frac{1}{|I|} \int_I |f_i(y_i)| dy_i$$

and

$$\mathcal{M}_r(\vec{f})(x) = \sup_{I \ni x} \prod_{i=1}^m \left(\frac{1}{|I|} \int_I |f_i(y_i)|^r dy_i \right)^{1/r}.$$

We will be using dyadic versions of the above maximal operators which are defined by taking supremum over all dyadic intervals $I \ni x$, instead of all intervals $I \ni x$. For convenience, we will use the same notation to denote the dyadic counterparts.

We will use the following results regarding maximal functions. The dyadic analogs of these statements are also true.

- For any locally integrable function f, $|f(x)| \leq Mf(x)$ almost everywhere. This inequality is a consequence of Lebesgue differentiation theorem and can be found in any standard Fourier Analysis textbooks, see for example [14] or [15]. In fact, for any $\delta > 0$, if $f \in L^{\delta}_{loc}(\mathbb{R})$, then $|f(x)| \leq M_{\delta} f(x)$ almost everywhere.

- For $0 < \delta_1 < \delta_2 < \infty$, $M_{\delta_1} f(x) \leq M_{\delta_2} f(x)$. This simple inequality can be verified just by using Hölder's inequality.

- For $w \in A_p$ with $1 < p < \infty$ there exists a constant C such that

$$\|Mf\|_{L^p(w)} \leq C\|f\|_{L^p(w)}. \qquad \text{(See [3], [14])}$$

- Fefferman-Stein's inequalities (see [16]): Let $w \in A_\infty$ and $0 < \delta, p < \infty$. Then there exists a constant C_1 such that

$$\|M_\delta f\|_{L^p(w)} \leq C_1 \|M^{\#}_\delta f\|_{L^p(w)} \tag{1.2.1}$$

for all functions f for which the left-hand side is finite.

Similarly, there exists a constant C_2 such that

$$\|M_\delta f\|_{L^{p,\infty}(w)} \leq C_2 \|M^{\#}_\delta f\|_{L^{p,\infty}(w)} \tag{1.2.2}$$

for all functions f for which the left-hand side is finite.

- Let $\vec{P} = (p_1, \ldots, p_m)$ and $\vec{w} = (w_1, \ldots, w_m)$, where $1 < p_1, \ldots, p_m < \infty$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$, and $w_1, \ldots, w_m$ are weights. Then the inequality

$$\|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)} \tag{1.2.3}$$

holds for every $\vec{f} = (f_1, \ldots, f_m)$ if and only if $\vec{w} \in A_{\vec{P}}$. For $1 \leq p_1, \ldots, p_m < \infty$, the same statement is true with the inequality

$$\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}. \tag{1.2.4}$$

These estimates and the one below have been obtained in [7].

- If $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$, for $\vec{P} = (p_1, \ldots, p_m)$ with $1 < p_1, \ldots, p_m < \infty$ and $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$, then there exists an $r > 1$ such that $\vec{w} \in A_{\vec{P}/r}$, and that

$$\|\mathcal{M}_r(\vec{f})\|_{L^{p}(\nu_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}. \tag{1.2.5}$$

1.2.6 The Dyadic Square Function

The dyadic Littlewood-Paley square function of a function f is defined by

$$Sf(x) := \left(\sum_{I \in \mathcal{D}} \frac{|\langle f, h_I \rangle|^2}{|I|} 1_I(x) \right)^{1/2}.$$

For $f \in L^p$ with $1 < p < \infty$, we have $\|Sf\|_p \approx \|f\|_p$ with equality when $p = 2$.

1.2.7 BMO Spaces

A locally integrable function b is said to be of bounded mean oscillation if

$$\|b\|_{BMO} := \sup_I \frac{1}{|I|} \int_I |b(x) - \langle b \rangle_I| \, dx < \infty,$$

where the supremum is taken over all intervals in $\mathbb{R}$. The space of all functions of bounded mean oscillation is denoted by BMO.

If we take the supremum over all dyadic intervals in $\mathbb{R}$, we get a larger space of dyadic BMO functions which we denote by BMO^d.

For $0 < r < \infty$, define

$$BMO_r = \{b \in L^r_{loc}(\mathbb{R}) : \|b\|_{BMO_r} < \infty\},$$

where, $\|b\|_{BMO_r} := \left(\sup_I \frac{1}{|I|}\int_I |b(x) - \langle b\rangle_I|^r\, dx\right)^{1/r}.$

For any $0 < r < \infty$, the norms $\|b\|_{BMO_r}$ and $\|b\|_{BMO}$ are equivalent. The equivalence of norms for $r > 1$ is well-known and follows from John-Nirenberg's lemma (see [17]), while the equivalence for $0 < r < 1$ has been proved by Hanks in [18]. (See also [19], page 179.)

For $r = 2$, it follows from the orthogonality of Haar system that

$$\|b\|_{BMO_2^d} = \left(\sup_{I\in\mathcal{D}} \frac{1}{|I|}\sum_{J\subseteq I} |\widehat{b}(J)|^2\right)^{1/2}.$$

Given a weight w on R, we define the weighted BMO space $BMO(w)$ to be the space of all locally integrable functions b that satisfy

$$\|b\|_{BMO(w)} \equiv \sup_I \frac{1}{w(I)}\int_I |b(x) - \langle b\rangle_I|\, dx < \infty,$$

where the supremum is taken over all intervals in $\mathbb{R}$. The dyadic counterpart $BMO^d(w)$ is defined by taking the suprimum over the dyadic intervals in $\mathbb{R}$.

1.2.8 The Linear and Bilinear Paraproducts

Given two functions f_1 and f_2, the point-wise product $f_1 f_2$ can be decomposed into the sum of bilinear paraproducts:

$$f_1 f_2 = P^{(0,0)}(f_1, f_2) + P^{(0,1)}(f_1, f_2) + P^{(1,0)}(f_1, f_2),$$

where for $\vec{\alpha} = (\alpha_1, \alpha_2) \in \{0,1\}^2$,

$$P^{\vec{\alpha}}(f_1, f_2) = \sum_{I \in \mathcal{D}} f_1(I, \alpha_1) f_2(I, \alpha_2) h_I^{\sigma(\vec{\alpha})}$$

with $f_i(I,0) = \langle f_i, h_I \rangle, \quad f_i(I,1) = \langle f_i \rangle_I, \ \sigma(\vec{\alpha}) = \#\{i : \alpha_i = 0\}$, and $h_I^{\sigma(\vec{\alpha})}$ being the pointwise product $h_I h_I \dots h_I$ of $\sigma(\vec{\alpha})$ factors.

The paraproduct $P^{(0,1)}(f_1, f_2)$ is also denoted by $\pi_{f_1}(f_2)$, i.e.,

$$\pi_{f_1}(f_2) = \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle f_2 \rangle_I h_I.$$

Observe that

$$\langle \pi_{f_1}(f_2), g \rangle = \left\langle \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle f_2 \rangle_I h_I, g \right\rangle = \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle f_2 \rangle_I \langle g, h_I \rangle$$

which is equal to

$$\begin{aligned} \left\langle f_2, P^{(0,0)}(f_1, g) \right\rangle &= \left\langle f_2, \ \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle g, h_I \rangle h_I^2 \right\rangle \\ &= \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle g, h_I \rangle \langle f_2, h_I^2 \rangle \\ &= \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle f_2 \rangle_I \langle g, h_I \rangle. \end{aligned}$$

This shows that $\pi_{f_1}^* = P^{(0,0)}(f_1, \cdot) = P^{(0,0)}(\cdot, f_1)$.

The ordinary multiplication operator $M_b : f \to bf$ can therefore be given by:

$$M_b(f) = bf = P^{(0,0)}(b,f) + P^{(0,1)}(b,f) + P^{(1,0)}(b,f) = \pi_b^*(f) + \pi_b(f) + \pi_f(b).$$

The function b is required to be in L^∞ for the boundedness of M_b in L^p. However, the paraproduct operator π_b is bounded in L^p for every $1 < p < \infty$ if $b \in BMO^d$. Note that

BMO^d properly contains L^∞. Detailed information on the operator π_b can be found in [3] or [20].

1.2.9 Commutators of Haar Multipliers

The commutator of T_ϵ with a locally integrable function b is defined by

$$[b, T_\epsilon](f)(x) := T_\epsilon(bf)(x) - M_b(T_\epsilon(f))(x).$$

It is well-known that for a bounded sequence ϵ and $1 < p < \infty$, the commutator $[b, T_\epsilon]$ is bounded in L^p for all $p \in (1, \infty)$ if $b \in BMO^d$. These commutators have been studied in [21] in non-homogeneous martingale settings.

1.2.10 Sparse Operators

A collection $\mathcal{S}$ of dyadic intervals is said to be sparse if for each $I \in \mathcal{S}$,

$$\sum_{J \in Ch_{\mathcal{S}}(I)} |J| \leq \frac{1}{2}|I|,$$

where $Ch_{\mathcal{S}}(I)$ is the collection of maximal dyadic intervals in $\mathcal{S}$ which are strictly contained in I.

Given a sparse collection $\mathcal{S}$ of dyadic intervals, the multilinear sparse operator $\mathcal{A}_{\mathcal{S}}$ is defined as follows:

$$\mathcal{A}_S(f_1, \ldots, f_m) = \sum_{I \in \mathcal{S}} \left(\prod_{i=1}^{m} \frac{1}{|I|} \int_I f_i \right) 1_I.$$

In [22], the authors have proved that if $\vec{P} = (p_1, \cdots, p_m)$ with $1 < p_1, \cdots, p_m < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$, then for $\vec{w} = (w_1, \cdots, w_m) \in A_{\vec{P}}$ and $\vec{f} = (f_1, \cdots, f_m)$, we have

$$\|\mathcal{A}_{\mathcal{S}}(\vec{f})\|_{L^p(v_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\left\{1, \frac{p_1'}{p}, \cdots, \frac{p_m'}{p}\right\}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}. \tag{1.2.6}$$

CHAPTER 2

MULTILINEAR DYADIC OPERATORS: MOTIVATION AND DEFINITIONS

In this chapter, we introduce multilinear dyadic paraproducts and Haar multipliers. Their definitions are motivated by the generalized paraproduct decomposition of the pointwise product of two or more functions. These operators and their commutators with locally integrable functions are the main objects of our study.

2.1 Decomposition of the Pointwise Product $\prod_{j=1}^{m} f_j$

.

In this section, we obtain a decomposition of the pointwise product $\prod_{j=1}^{m} f_j$ of m functions that generalizes the following paraproduct decomposition :

$$f_1 f_2 = P^{(0,0)}(f_1, f_2) + P^{(0,1)}(f_1, f_2) + P^{(1,0)}(f_1, f_2).$$

The decomposition of $\prod_{j=1}^{m} f_j$ will be the basis for defining *multi-linear paraproducts* and *m-linear Haar multipliers*, and will also be very useful in proving boundedness properties of multilinear commutators.

We first introduce the following notation:

- $f(I,0) := \widehat{f}(I) = \langle f, h_I \rangle = \displaystyle\int_{\mathbb{R}} f(x) h_I(x) dx.$
- $f(I,1) := \langle f \rangle_I = \frac{1}{|I|} \displaystyle\int_I f(x) dx.$
- $U_m := \{(\alpha_1, \alpha_2, \dots, \alpha_m) \in \{0,1\}^m : (\alpha_1, \alpha_2, \dots, \alpha_m) \neq (1, 1, \dots, 1)\}.$
- $\sigma(\vec{\alpha}) = \#\{i : \alpha_i = 0\}$ for $\vec{\alpha} = (\alpha_1, \dots, \alpha_m) \in \{0,1\}^m$.

- $(\vec{\alpha}, i) = (\alpha_1, \dots, \alpha_m, i), \ (i, \vec{\alpha}) = (i, \alpha_1, \dots, \alpha_m)$ for $\vec{\alpha} = (\alpha_1, \dots, \alpha_m) \in \{0,1\}^m$.
- $P_I^{\vec{\alpha}}(f_1, \dots, f_m) = \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}$ for $\vec{\alpha} \in U_m$ and $I \in \mathcal{D}$.
- $P^{\vec{\alpha}}(f_1, \dots, f_m) = \displaystyle\sum_{I \in \mathcal{D}} P_I^{\vec{\alpha}}(f_1, \dots, f_m) = \sum_{I \in \mathcal{D}} \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}$ for $\vec{\alpha} \in U_m$.

With this notation, the paraproduct decomposition of $f_1 f_2$ takes the following form:

$$f_1 f_2 = P^{(0,0)}(f_1, f_2) + P^{(0,1)}(f_1, f_2) + P^{(1,0)}(f_1, f_2) = \sum_{\vec{\alpha} \in U_2} P^{\vec{\alpha}}(f_1, f_2).$$

Note that

$$U_m = \{(\alpha, 1) : \vec{\alpha} \in U_{m-1}\} \cup \{(\vec{\alpha}, 0) : \vec{\alpha} \in U_{m-1}\} \cup \{(1, \dots, 1, 0)\}. \tag{2.1.1}$$

To obtain an analogous decomposition of $\displaystyle\prod_{j=1}^m f_j$, we need the following crucial lemma:

Lemma 2.1.1. *Given $m \geq 2$ and functions $f_1, f_2, \dots, f_m$, with $f_i \in L^{p_i}, 1 < p_i < \infty$,we have*

$$\prod_{j=1}^m \langle f_j \rangle_J \mathbb{1}_J = \sum_{\vec{\alpha} \in U_m} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_m) \, \mathbb{1}_J,$$

for all $J \in \mathcal{D}$.

Proof. We prove the lemma by induction on m.

First assume that $m = 2$. We want to prove the following:

$$\begin{aligned}
\langle f_1 \rangle_J \langle f_2 \rangle_J \mathbb{1}_J &= \sum_{\vec{\alpha} \in U_2} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2) \, \mathbb{1}_J \\
&= \left(\sum_{J \subsetneq I} P_I^{(0,1)}(f_1, f_2) + \sum_{J \subsetneq I} P_I^{(1,0)}(f_1, f_2) + \sum_{J \subsetneq I} P_I^{(0,0)}(f_1, f_2) \right) \mathbb{1}_J \\
&= \left(\sum_{J \subsetneq I} \widehat{f_1}(I) \langle f_2 \rangle_I h_I + \sum_{J \subsetneq I} \langle f_1 \rangle_I \widehat{f_2}(I) h_I + \sum_{J \subsetneq I} \widehat{f_1}(I) \widehat{f_2}(I) h_I^2 \right) \mathbb{1}_J.
\end{aligned}$$

For $1 < p_i < \infty$, $\langle f_i \rangle_J \mathsf{1}_J = \left(\sum_{J \subsetneq I} \widehat{f_i}(I) h_I \right) \mathsf{1}_J$. So,

$$\begin{aligned}
& \langle f_1 \rangle_J \langle f_2 \rangle_J \mathsf{1}_J \\
= & \left(\sum_{J \subsetneq I} \widehat{f_1}(I) h_I \right) \left(\sum_{J \subsetneq K} \widehat{f_2}(K) h_K \right) \mathsf{1}_J \\
= & \sum_{J \subsetneq I} \widehat{f_1}(I) h_I \left(\sum_{I \subsetneq K} \widehat{f_2}(K) h_K + \widehat{f_2}(I) h_I + \sum_{J \subsetneq K \subsetneq I} \widehat{f_2}(K) h_K \right) \mathsf{1}_J \\
= & \left\{ \sum_{J \subsetneq I} \widehat{f_1}(I) \langle f_2 \rangle_I h_I + \sum_{J \subsetneq I} \widehat{f_1}(I) \widehat{f_2}(I) h_I^2 + \sum_{J \subsetneq I} \widehat{f_1}(I) h_I \left(\sum_{J \subsetneq K \subsetneq I} \widehat{f_2}(K) h_K \right) \right\} \mathsf{1}_J \\
= & \left\{ \sum_{J \subsetneq I} \widehat{f_1}(I) \langle f_2 \rangle_I h_I + \sum_{J \subsetneq I} \widehat{f_1}(I) \widehat{f_2}(I) h_I^2 + \sum_{J \subsetneq K} \widehat{f_2}(K) h_K \left(\sum_{K \subsetneq I} \widehat{f_1}(I) h_I \right) \right\} \mathsf{1}_J \\
= & \left\{ \sum_{J \subsetneq I} \widehat{f_1}(I) \langle f_2 \rangle_I h_I + \sum_{J \subsetneq I} \widehat{f_1}(I) \widehat{f_2}(I) h_I^2 + \sum_{J \subsetneq K} \widehat{f_2}(K) \langle f_1 \rangle_K h_K \right\} \mathsf{1}_J \\
= & \left\{ \sum_{J \subsetneq I} \widehat{f_1}(I) \langle f_2 \rangle_I h_I + \sum_{J \subsetneq I} \widehat{f_1}(I) \widehat{f_2}(I) h_I^2 + \sum_{J \subsetneq I} \widehat{f_2}(I) \langle f_1 \rangle_I h_I \right\} \mathsf{1}_J \\
= & \left(\sum_{J \subsetneq I} \widehat{f_1}(I) \langle f_2 \rangle_I h_I + \sum_{J \subsetneq I} \langle f_1 \rangle_I \widehat{f_2}(I) h_I + \sum_{J \subsetneq I} \widehat{f_1}(I) \widehat{f_2}(I) h_I^2 \right) \mathsf{1}_J.
\end{aligned}$$

Now assume $m > 2$ and that $\prod_{j=1}^{m-1} \langle f_j \rangle_J \mathsf{1}_J = \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \ldots, f_{m-1}) \mathsf{1}_J$. Then,

$$\begin{aligned}
& \prod_{j=1}^{m} \langle f_j \rangle_J \mathsf{1}_J \\
= & \left(\prod_{j=1}^{m-1} \langle f_j \rangle_J \mathsf{1}_J \right) \langle f_m \rangle_J \mathsf{1}_J \\
= & \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \ldots, f_{m-1}) \left(\sum_{J \subsetneq K} \widehat{f_m}(K) h_K \right) \mathsf{1}_J \\
= & \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \ldots, f_{m-1}) \left(\sum_{I \subsetneq K} \widehat{f_m}(K) h_K + \widehat{f_m}(I) h_I + \sum_{J \subsetneq K \subsetneq I} \widehat{f_m}(K) h_K \right) \mathsf{1}_J
\end{aligned}$$

This gives

$$
\begin{aligned}
&\prod_{j=1}^{m}\langle f_j\rangle_J 1_J \\
= \quad & \sum_{\vec{\alpha}\in U_{m-1}}\sum_{J\subsetneq I} P_I^{\vec{\alpha}}(f_1,f_2,\dots,f_{m-1})\langle f_m\rangle_I 1_J + \sum_{\vec{\alpha}\in U_{m-1}}\sum_{J\subsetneq I} P_I^{\vec{\alpha}}(f_1,\dots,f_{m-1})\widehat{f_m}(I)h_I 1_J \\
& \quad + \sum_{\vec{\alpha}\in U_{m-1}}\sum_{J\subsetneq I} P_I^{\vec{\alpha}}(f_1,f_2,\dots,f_{m-1})\left(\sum_{J\subsetneq K\subsetneq I}\widehat{f_m}(K)h_K\right)1_J \\
= \quad & \sum_{\vec{\alpha}\in U_{m-1}}\sum_{J\subsetneq I} P_I^{(\vec{\alpha},1)}(f_1,f_2,\dots,f_m)1_J + \sum_{\vec{\alpha}\in U_{m-1}}\sum_{J\subsetneq I} P_I^{(\vec{\alpha},0)}(f_1,f_2,\dots,f_m)1_J \\
& \quad + \sum_{J\subsetneq K}\widehat{f_m}(K)h_K\left(\sum_{\vec{\alpha}\in U_{m-1}}\sum_{K\subsetneq I} P_I^{\vec{\alpha}}(f_1,f_2,\dots,f_{m-1})\right)1_J \\
= \quad & \sum_{\vec{\alpha}\in U_{m-1}}\sum_{J\subsetneq I} P_I^{(\vec{\alpha},1)}(f_1,f_2,\dots,f_m)1_J + \sum_{\vec{\alpha}\in U_{m-1}}\sum_{J\subsetneq I} P_I^{(\vec{\alpha},0)}(f_1,f_2,\dots,f_m)1_J \\
& \quad + \sum_{J\subsetneq K}\widehat{f_m}(K)h_K\langle f_1\rangle_K\dots\langle f_{m-1}\rangle_K 1_J \\
= \quad & \sum_{\vec{\alpha}\in U_{m-1}}\sum_{J\subsetneq I} P_I^{(\vec{\alpha},1)}(f_1,f_2,\dots,f_m)1_J + \sum_{\vec{\alpha}\in U_{m-1}}\sum_{J\subsetneq I} P_I^{(\vec{\alpha},0)}(f_1,f_2,\dots,f_m)1_J \\
& \quad + \sum_{J\subsetneq I} P_I^{(1,\dots,1,0)}(f_1,f_2,\dots,f_m)1_J \\
= \quad & \sum_{\vec{\alpha}\in U_m}\sum_{J\subsetneq I} P_I^{\vec{\alpha}}(f_1,f_2,\dots,f_m)1_J.
\end{aligned}
$$

The last equality follows from (2.1.1). $\square$

Lemma 2.1.2. *Given $m \geq 2$ and functions $f_1, f_2, \dots, f_m$, with $f_i \in L^{p_i}, 1 < p_i < \infty$,we have*

$$\prod_{j=1}^{m} f_j = \sum_{\vec{\alpha}\in U_m} P^{\vec{\alpha}}(f_1,f_2,\dots,f_m).$$

Proof. We have already seen that it is true for $m = 2$. By induction, assume that

$$
\begin{aligned}
\prod_{j=1}^{m-1} f_j &= \sum_{\vec{\alpha}\in U_{m-1}} P^{\vec{\alpha}}(f_1,f_2,\dots,f_{m-1}) \\
&= \sum_{\vec{\alpha}\in U_{m-1}}\sum_{I\in\mathcal{D}} P_I^{\vec{\alpha}}(f_1,f_2,\dots,f_{m-1})
\end{aligned}
$$

Then,

$$
\begin{aligned}
\prod_{j=1}^{m} f_j &= \left(\prod_{j=1}^{m-1} f_j\right) f_m \\
&= \sum_{\vec{\alpha}\in U_{m-1}} \sum_{I\in\mathcal{D}} P_I^{\vec{\alpha}}(f_1, f_2, \ldots, f_{m-1}) \left(\sum_{J\in\mathcal{D}} \widehat{f_m}(J) h_J\right) \\
&= \sum_{\vec{\alpha}\in U_{m-1}} \sum_{I\in\mathcal{D}} P_I^{\vec{\alpha}}(f_1, f_2, \ldots, f_{m-1}) \left(\sum_{I\subsetneq J} \widehat{f_m}(J) h_J + \widehat{f_m}(I) h_I + \sum_{J\subsetneq I} \widehat{f_m}(J) h_J\right) \\
&= \sum_{\vec{\alpha}\in U_{m-1}} \sum_{I\in\mathcal{D}} P_I^{\vec{\alpha}}(f_1, \ldots, f_{m-1}) \langle f_m \rangle_I + \sum_{\vec{\alpha}\in U_{m-1}} \sum_{I\in\mathcal{D}} P_I^{\vec{\alpha}}(f_1, \ldots, f_{m-1}) \widehat{f_m}(I) h_I \\
&\quad + \sum_{\vec{\alpha}\in U_{m-1}} \sum_{I\in\mathcal{D}} P_I^{\vec{\alpha}}(f_1, f_2, \ldots, f_{m-1}) \left(\sum_{J\subsetneq I} \widehat{f_m}(J) h_J\right) \\
&= \sum_{\vec{\alpha}\in U_{m-1}} \sum_{I\in\mathcal{D}} P_I^{(\vec{\alpha},1)}(f_1, f_2, \ldots, f_m) + \sum_{\vec{\alpha}\in U_{m-1}} \sum_{I\in\mathcal{D}} P_I^{(\vec{\alpha},0)}(f_1, f_2, \ldots, f_m) \\
&\quad + \sum_{J} \widehat{f_m}(J) h_J \left(\sum_{\vec{\alpha}\in U_{m-1}} \sum_{J\subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \ldots, f_{m-1})\right) \\
&= \sum_{\vec{\alpha}\in U_{m-1}} \sum_{I\in\mathcal{D}} P_I^{(\vec{\alpha},1)}(f_1, f_2, \ldots, f_m) + \sum_{\vec{\alpha}\in U_{m-1}} \sum_{I\in\mathcal{D}} P_I^{(\vec{\alpha},0)}(f_1, f_2, \ldots, f_m) \\
&\quad + \sum_{J} \widehat{f_m}(J) h_J \langle f_1 \rangle_J \ldots \langle f_{m-1} \rangle_J \\
&= \sum_{\vec{\alpha}\in U_{m-1}} \sum_{I\in\mathcal{D}} P_I^{(\vec{\alpha},1)}(f_1, f_2, \ldots, f_m) + \sum_{\vec{\alpha}\in U_{m-1}} \sum_{I\in\mathcal{D}} P_I^{(\vec{\alpha},0)}(f_1, f_2, \ldots, f_m) \\
&\quad + P^{(1,\ldots,1,0)}(f_1, f_2, \ldots, f_m) \\
&= \sum_{\vec{\alpha}\in U_m} P^{\vec{\alpha}}(f_1, f_2, \ldots, f_m).
\end{aligned}
$$

Here the last equality follows from $(2.1.1)$. $\square$

2.2 Multilinear Dyadic Paraproducts and Haar Multipliers

On the basis of the decomposition of the pointwise product $\prod_{j=1}^{m} f_j$, we now introduce multi-linear dyadic paraproducts and Haar-multipliers. These operators and their commutators with locally integrable functions are the main objects of our study.

Definition 2.2.1. *For $m \geq 2$ and $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \{0,1\}^m$, we define multi-linear dyadic paraproduct operators by*

$$P^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) = \sum_{I \in \mathcal{D}} \prod_{j=1}^{m} f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}$$

where $f_i(I,0) = \langle f_i, h_I \rangle$, $f_i(I,1) = \langle f_i \rangle_I$ and $\sigma(\vec{\alpha}) = \#\{i : \alpha_i = 0\}$.

Observe that if $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)$ is some permutation of $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $(g_1, g_2, \ldots, g_m)$ is the corresponding permutation of $(f_1, f_2, \ldots, f_m)$, then

$$P^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) = P^{\vec{\beta}}(g_1, g_2, \ldots, g_m).$$

Also note that $P^{(1,0)}$ and $P^{(0,1)}$ are the standard bilinear paraproduct operators:

$$P^{(0,1)}(f_1, f_2) = \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle f_2 \rangle_I h_I = P(f_1, f_2)$$

$$P^{(1,0)}(f_1, f_2) = \sum_{I \in \mathcal{D}} \langle f_1 \rangle_I \langle f_2, h_I \rangle h_I = P(f_1, f_2).$$

In terms of paraproducts, the decomposition of point-wise product $\prod_{j=1}^{m} f_j$, we obtained in the previous section takes the form

$$\prod_{j=1}^{m} f_j = \sum_{\substack{\vec{\alpha} \in \{0,1\}^m \\ \vec{\alpha} \neq (1,1,\ldots,1)}} P^{\vec{\alpha}}(f_1, f_2, \ldots, f_m).$$

Definition 2.2.2. *For a given function b and $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \{0,1\}^m$, we define the paraproduct operators $\pi_b^{\vec{\alpha}}$ by*

$$\pi_b^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) = P^{(0,\vec{\alpha})}(b, f_1, f_2, \ldots, f_m) = \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \prod_{j=1}^{m} f_j(I, \alpha_j)\ h_I^{1+\sigma(\vec{\alpha})}$$

where $(0, \vec{\alpha}) = (0, \alpha_1, \ldots, \alpha_m) \in \{0,1\}^{m+1}$.

Note that

$$\pi_b^1(f) = P^{(0,1)}(b,f) = \sum_{I\in\mathcal{D}} b(I,0)f(I,1)h_I = \sum_{I\in\mathcal{D}} \langle b, h_I\rangle\langle f\rangle_I h_I = \pi_b(f).$$

Definition 2.2.3. *Given* $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \{0,1\}^m$*, and a symbol sequence* $\epsilon = \{\epsilon_I\}_{I\in\mathcal{D}}$*, we define m-linear Haar multipliers by*

$$T_\epsilon^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) \equiv \sum_{I\in\mathcal{D}} \epsilon_I \prod_{j=1}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})}.$$

Note that for $\epsilon_I = 1$ for all $I \in \mathcal{D}$, $T_\epsilon^{\vec{\alpha}} = P^{\vec{\alpha}}$.

Besides, multilinear dyadic paraproducts and Haar multipliers, we are interested in studying the commutators of multilinear Haar multipliers with locally integrable functions, which are defined as follows:

$$[b, T_\epsilon^{\vec{\alpha}}]_i(f_1, f_2, \ldots, f_m)(x) \equiv (T_\epsilon^{\vec{\alpha}}(f_1, \ldots, bf_i, \ldots, f_m) - bT_\epsilon^{\vec{\alpha}}(f_1, f_2, \ldots, f_m))(x)$$

where $1 \leq i \leq m$.

CHAPTER 3

ESTIMATES FOR MULTILINEAR DYADIC OPERATORS: UNWEIGHTED SETTING

In this chapter, we study the boundedness properties of multilinear dyadic paraproducts and Haar multipliers, as well as their commutators with dyadic *BMO* functions. We also characterize the dyadic BMO functions via the boundedness of (a) certain paraproducts, and (b) the commutators of multilinear Haar multipliers and paraproduct operators.

3.1 Main Results

Following are the main results of this chapter.

Theorem: Let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \{0,1\}^m$ and $1 < p_1, p_2, \ldots, p_m < \infty$ with $\displaystyle\sum_{j=1}^{m} \frac{1}{p_j} = \frac{1}{r}$. Then

(a) For $\vec{\alpha} \neq (1,1,\ldots,1)$, $\left\| P^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) \right\|_r \lesssim \displaystyle\prod_{j=1}^{m} \|f_j\|_{p_j}$.

(b) For $\sigma(\vec{\alpha}) \leq 1$, $\left\| \pi_b^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) \right\|_r \lesssim \|b\|_{BMO^d} \displaystyle\prod_{j=1}^{m} \|f_j\|_{p_j}$, if and only if $b \in BMO^d$.

(c) For $\sigma(\vec{\alpha}) > 1$, $\left\| \pi_b^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) \right\|_r \leq C_b \displaystyle\prod_{j=1}^{m} \|f_j\|_{p_j}$, if and only if $\displaystyle\sup_{I \in \mathcal{D}} \frac{|\langle b, h_I \rangle|}{\sqrt{|I|}} < \infty$.

In each case, the paraproducts are weakly bounded if $1 \leq p_1, p_2, \ldots, p_m < \infty$.

Theorem: Let $\epsilon = \{\epsilon_I\}_{I \in \mathcal{D}}$ be a given sequence and let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in U_m$. Let $1 < p_1, p_2, \ldots, p_m < \infty$ with $\displaystyle\sum_{j=1}^{m} \frac{1}{p_j} = \frac{1}{r}$. Then $T_\epsilon^{\vec{\alpha}}$ is bounded from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m}$

to L^r if and only if $\|\epsilon\|_\infty := \sup\limits_{I\in\mathcal{D}} |\epsilon_I| < \infty$.
Moreover, $T_\epsilon^{\vec{\alpha}}$ has the corresponding weak-type boundedness if $1 \le p_1, p_2, \ldots, p_m < \infty$.

Theorem: Let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in U_m$, $1 \le i \le m$, and $1 < p_1, p_2, \ldots, p_m, r < \infty$ with $\displaystyle\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$. Suppose $b \in L^p$ for some $p \in (1, \infty)$. Then the following two statements are equivalent.

(a) $b \in BMO^d$.

(b) $[b, T_\epsilon^{\vec{\alpha}}]_i : L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m} \to L^r$ is bounded for every bounded sequence $\epsilon = \{\epsilon_I\}_{I\in\mathcal{D}}$.

In particular, $b \in BMO^d$ if and only if $[b, P^{\vec{\alpha}}]_i : L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m} \to L^r$ is bounded.

3.2 Multilinear Dyadic Paraproducts

This section is devoted to the boundedness properties of the multilinear paraproduct operators $P^{\vec{\alpha}}$ and $\pi_b^{\vec{\alpha}}$.

Lemma 3.2.1. *Let $1 < p_1, p_2, \ldots, p_m, r < \infty$ and $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$. Then for $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$ in U_m, the operators $P^{\vec{\alpha}}$ map $L^{p_1} \times \cdots \times L^{p_m} \to L^r$ with estimates of the form:*

$$\|P^{\vec{\alpha}}(f_1, f_2, \ldots, f_m)\|_r \lesssim \prod_{j=1}^m \|f_j\|_{p_j}$$

Proof. First we observe that, if $x \in I \in \mathcal{D}$, then $|\langle f\rangle_I| \le \langle |f|\rangle_I \le Mf(x)$, and that

$$\begin{aligned} \frac{|\langle f, h_I\rangle|}{\sqrt{|I|}} &= \frac{1}{\sqrt{|I|}} \left| \int_{\mathbb{R}} f h_I \right| \\ &= \frac{1}{|I|} \left| \int_{\mathbb{R}} f \mathbb{1}_{I_+} - \int_{\mathbb{R}} f \mathbb{1}_{I_-} \right| \\ &= \frac{1}{|I|} \left(\int_{I_+} |f| + \int_{I_-} |f| \right) \end{aligned}$$

$$\text{So,}\quad \frac{|\langle f, h_I\rangle|}{\sqrt{|I|}} \le \frac{1}{|I|}\int_I |f| \le Mf(x).$$

Case I: $\sigma(\vec{\alpha}) = 1$.

Let $\alpha_{j_0} = 0$. Then

$$\begin{aligned} P^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) &= \sum_{I\in\mathcal{D}} \prod_{j=1}^{m} f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \\ &= \sum_{I\in\mathcal{D}} \left(\prod_{\substack{j=1 \\ j\neq j_0}}^{m} \langle f_j\rangle_I \right) \langle f_{j_0}, h_I\rangle h_I. \end{aligned}$$

Using square function estimates, we obtain

$$\begin{aligned} \left\| P^{\vec{\alpha}}(f_1, f_2, \ldots, f_m)\right\|_r &\lesssim \left\| \left(\sum_{I\in\mathcal{D}} \prod_{\substack{j=1 \\ j\neq j_0}}^{m} |\langle f_j\rangle_I|^2 \, |\langle f_{j_0}, h_I\rangle|^2 \frac{\mathbb{1}_I}{|I|} \right)^{1/2} \right\|_r \\ &\le \left\| \left(\prod_{\substack{j=1 \\ j\neq j_0}}^{m} Mf_j \right) \left(\sum_{I\in\mathcal{D}} |\langle f_{j_0}, h_I\rangle|^2 \frac{\mathbb{1}_I}{|I|} \right)^{1/2} \right\|_r \\ &= \left\| \left(\prod_{\substack{j=1 \\ j\neq j_0}}^{m} Mf_j \right) (Sf_{j_0}) \right\|_r \\ &\le \prod_{\substack{j=1 \\ j\neq j_0}}^{m} \|Mf_j\|_{p_j} \|Sf_{j_0}\|_{p_{j_0}} \\ &\lesssim \prod_{j=1}^{m} \|f_j\|_{p_j}, \end{aligned}$$

where we have used Hölder inequality, and the boundedness of maximal and square function operators to obtain the last two inequalities.

Case II: $\sigma(\vec{\alpha}) > 1$.

Choose j' and j'' such that $\alpha_{j'} = \alpha_{j''} = 0$. Then

$$\begin{aligned}
&\left|P^{\vec{\alpha}}(f_1, f_2, \dots, f_m)(x)\right| \\
= &\left|\sum_{I\in\mathcal{D}} \left(\prod_{j:\alpha_j=1} \langle f_j\rangle_I\right) \left(\prod_{\substack{j:\alpha_j=0\\ j\neq j',\, j''}} \frac{\langle f_j, h_I\rangle}{\sqrt{|I|}}\right) \langle f_{j'}, h_I\rangle \langle f_{j''}, h_I\rangle \frac{\mathbf{1}_I(x)}{|I|}\right| \\
\leq &\left(\prod_{j:j\neq j',\, j''} Mf_j(x)\right) \left(\sum_{I\in\mathcal{D}} |\langle f_{j'}, h_I\rangle| |\langle f_{j''}, h_I\rangle| \frac{\mathbf{1}_I(x)}{|I|}\right).
\end{aligned}$$

By Cauchy-Schwarz inequality

$$\begin{aligned}
&\sum_{I\in\mathcal{D}} |\langle f_{j'}, h_I\rangle|\ |\langle f_{j''}, h_I\rangle|\ \frac{\mathbf{1}_I(x)}{|I|} \\
\leq &\left(\sum_{I\in\mathcal{D}} |\langle f_{j'}, h_I\rangle|^2 \frac{\mathbf{1}_I(x)}{|I|}\right)^{\frac{1}{2}} \left(\sum_{I\in\mathcal{D}} |\langle f_{j''}, h_I\rangle|^2 \frac{\mathbf{1}_I(x)}{|I|}\right)^{\frac{1}{2}} \\
= &\ Sf_{j'}(x)\, Sf_{j''}(x).
\end{aligned} \tag{3.2.1}$$

Therefore,

$$\left|P^{\vec{\alpha}}(f_1, f_2, \dots, f_m)(x)\right| \leq \left(\prod_{j:j\neq j',\, j''} Mf_j(x)\right) Sf_{j'}(x)\, Sf_{j''}(x).$$

Now using generalized Hölder's inequality and the boundedness properties of the maximal and square functions, we get

$$\begin{aligned}
\left\|P^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\right\|_r &\leq \left(\prod_{j:j\neq j',\, j''} \|Mf_j\|_{p_j}\right) \|Sf_{j'}\|_{p_{j'}}\, \|Sf_{j''}\|_{p_{j''}} \\
&\lesssim \prod_{j=1}^{m} \|f_j\|_{p_j}.
\end{aligned}$$

$\square$

Lemma 3.2.2. *Let* $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \{0,1\}^m$ *and* $1 < p_1, \ldots, p_m, r < \infty$ *with* $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$.

(a) *For* $\sigma(\vec{\alpha}) \leq 1$, $\pi_b^{\vec{\alpha}}$ *is a bounded operator from* $L^{p_1} \times \cdots \times L^{p_m}$ *to* L^r *if and only if* $b \in BMO^d$.

(b) *For* $\sigma(\vec{\alpha}) > 1$, $\pi_b^{\vec{\alpha}}$ *is a bounded operator from* $L^{p_1} \times \cdots \times L^{p_m}$ *to* L^r *if and only if* $\sup_{I \in \mathcal{D}} \dfrac{|\langle b, h_I \rangle|}{\sqrt{|I|}} < \infty$.

Proof. (a) We prove this part first for $\sigma(\vec{\alpha}) = 0$, that is, for $\alpha_1 = \cdots = \alpha_m = 1$.

Assume that $b \in BMO^d$. Then for $(f_1, \ldots, f_m) \in L^{p_1} \times \cdots \times L^{p_m}$, we have

$$\begin{aligned}
\pi_b^{\vec{\alpha}}(f_1, \ldots, f_m) &= P^{(0,\vec{\alpha})}(b, f_1, \ldots, f_m) \\
&= \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \prod_{j=1}^m \langle f_j \rangle_I h_I \\
&= \sum_{I \in \mathcal{D}} \langle \pi_b(f_1), h_I \rangle \prod_{j=2}^m \langle f_j \rangle_I h_I \\
&= P^{(0,\alpha_2,\ldots,\alpha_m)}\left(\pi_b(f_1), f_2, \ldots, f_m\right).
\end{aligned}$$

Since $b \in BMO^d$ and $f_1 \in L^{p_1}$ with $p_1 > 1$, we have $\|\pi_b(f_1)\|_{p_1} \lesssim \|b\|_{BMO^d} \|f_1\|_{p_1}$. So,

$$\begin{aligned}
\|\pi_b^{\vec{\alpha}}(f_1, \ldots, f_m)\|_r &= \|P^{(0,\alpha_2,\ldots,\alpha_m)}\left(\pi_b(f_1), f_2, \ldots, f_m\right)\|_r \\
&\lesssim \|\pi_b(f_1)\|_{p_1} \prod_{j=2}^m \|f_j\|_{p_j} \\
&\lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j},
\end{aligned}$$

where the first inequality follows from Lemma 3.2.1.

Conversely, assume that $\pi_b^{(1,\ldots,1)} : L^{p_1} \times \cdots \times L^{p_m} \to L^r$ is bounded. Then for $f_i =$

$|J|^{-\frac{1}{p_i}}\mathbb{1}_J(x)$ with $J \in \mathcal{D}$,

$$\left\|\pi_b^{(1,1,\ldots,1)}(f_1, f_2, \ldots, f_m)\right\|_r \leq \left\|\pi_b^{(1,1,\ldots,1)}\right\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^r},$$

since $\|f_i\|_{p_i} = 1$ for all $1 \leq i \leq m$. For such f_i,

$$\begin{aligned}\left\|\pi_b^{(1,1,\ldots,1)}(f_1, f_2, \ldots, f_m)\right\|_r &= \left\||J|^{-\left(\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}\right)}\,\pi_b^{(1,1,\ldots,1)}(\mathbb{1}_J, \mathbb{1}_J, \ldots, \mathbb{1}_J)\right\|_r \\ &= |J|^{-\frac{1}{r}}\left\|\sum_{I\in\mathcal{D}}\widehat{b}(I)\langle\mathbb{1}_J\rangle_I^m h_I\right\|_r .\end{aligned}$$

Taking $\epsilon_I = 1$ if $I \subseteq J$ and $\epsilon_I = 0$ otherwise, we observe that

$$\begin{aligned}\left\|\sum_{J\supseteq I\in\mathcal{D}}\widehat{b}(I)h_I\right\|_r &= \left\|\sum_{J\supseteq I\in\mathcal{D}}\widehat{b}(I)\langle\mathbb{1}_J\rangle_I^m h_I\right\|_r \\ &= \left\|\sum_{I\in\mathcal{D}}\epsilon_I\widehat{b}(I)\langle\mathbb{1}_J\rangle_I^m h_I\right\|_r \\ &\lesssim \left\|\sum_{I\in\mathcal{D}}\widehat{b}(I)\langle\mathbb{1}_J\rangle_I^m h_I\right\|_r ,\end{aligned}$$

where the last inequality follows from the boundedness of Haar multiplier T_ϵ on L^r. Thus, we have

$$\begin{aligned}\sup_{J\in\mathcal{D}}|J|^{-1/r}\left\|\sum_{J\supseteq I\in\mathcal{D}}\widehat{b}(I)h_I\right\|_r &\lesssim \sup_{J\in\mathcal{D}}|J|^{-1/r}\left\|\sum_{I\in\mathcal{D}}\widehat{b}(I)\langle\mathbb{1}_J\rangle_I^m h_I\right\|_r \\ &\lesssim \left\|\pi_b^{(1,1,\ldots,1)}\right\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^r},\end{aligned}$$

proving that $b \in BMO^d$.

Now the proof for $\sigma(\vec{\alpha}) = 1$ follows from the simple observation that $\pi_b^{\vec{\alpha}}$ is a transpose of $\pi_b^{(1,\ldots,1)}$. For example, if $\sigma(\vec{\alpha}) = 1$ with $\alpha_1 = 0$ and $\alpha_2 = \cdots = \alpha_m = 1$ and if r' is the conjugate exponent of r, then for $g \in L^{r'}$

$$\begin{aligned}
\left\langle \pi_b^{\vec{\alpha}}(f_1,\ldots,f_m), g\right\rangle &= \left\langle \sum_{I\in\mathcal{D}} \langle b,h_I\rangle\langle f_1,h_I\rangle \prod_{j=2}^{m}\langle f_j\rangle_I h_I^2, g\right\rangle \\
&= \sum_{I\in\mathcal{D}} \langle b,h_I\rangle\langle f_1,h_I\rangle \prod_{j=2}^{m}\langle f_j\rangle_I \langle g, h_I^2\rangle \\
&= \sum_{I\in\mathcal{D}} \langle b,h_I\rangle\langle f_1,h_I\rangle \prod_{j=1}^{m}\langle f_j\rangle_I \langle g\rangle_I \\
&= \left\langle \sum_{I\in\mathcal{D}} \langle b,h_I\rangle\langle g\rangle_I \prod_{j=1}^{m}\langle f_j\rangle_I h_I, f_1\right\rangle \\
&= \left\langle \pi_b^{(1,\ldots,1)}(g,f_2,\ldots,f_m), f_1\right\rangle.
\end{aligned}$$

(b) Assume that $\|b\|_* \equiv \sup_{I\in\mathcal{D}} \dfrac{|\langle b,h_I\rangle|}{\sqrt{|I|}} < \infty$. For $m=2$ we have

$$\begin{aligned}
\int_{\mathbb{R}} \left|\pi_b^{(0,0)}(f_1,f_2)\right|^r dx &= \int_{\mathbb{R}} \left|\sum_{I\in\mathcal{D}} \langle b,h_I\rangle\langle f_1,h_I\rangle\langle f_2,h_I\rangle h_I^3(x)\right|^r dx \\
&\le \int_{\mathbb{R}} \left(\sum_{I\in\mathcal{D}} |\langle b,h_I\rangle|\,|\langle f_1,h_I\rangle|\,|\langle f_2,h_I\rangle| \frac{\mathbf{1}_I(x)}{|I|^{3/2}}\right)^r dx \\
&\le \int_{\mathbb{R}} \left(\sup_{I\in\mathcal{D}} \frac{|\langle b,h_I\rangle|}{\sqrt{|I|}} \sum_{I\in\mathcal{D}} |\langle f_1,h_I\rangle|\,|\langle f_2,h_I\rangle| \frac{\mathbf{1}_I(x)}{|I|}\right)^r dx \\
&= \|b\|_*^r \int_{\mathbb{R}} \left(\sum_{I\in\mathcal{D}} |\langle f_1,h_I\rangle|\,|\langle f_2,h_I\rangle| \frac{\mathbf{1}_I(x)}{|I|}\right)^r dx.
\end{aligned}$$

Using (3.2.1) and Hölder's inequality we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \left|\pi_b^{(0,0)}(f_1,f_2)\right|^r dx &\le \|b\|_*^r \int_{\mathbb{R}} (Sf_1)^r(x)\,(Sf_2)^r(x)\,dx \\
&\le \|b\|_*^r \left(\int_{\mathbb{R}} \{(Sf_1)^r(x)\}^{p_1/r}\,dx\right)^{r/p_1} \left(\int_{\mathbb{R}} \{(Sf_2)^r(x)\}^{p_2/r}\,dx\right)^{r/p_2} \\
&\le \|b\|_*^r \|Sf_1\|_{p_1}^r \|Sf_2\|_{p_2}^r \\
&\lesssim \|b\|_*^r \|f_1\|_{p_1}^r \|f_2\|_{p_2}^r.
\end{aligned}$$

Thus we have,

$$\|\pi_b^{(0,0)}(f_1, f_2)\|_r \lesssim \|b\|_* \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

Observe that

$$\pi_b^{(0,0)}(f_1, f_2)(I, 0) = \langle \pi_b^{(0,0)}(f_1, f_2), h_I\rangle = \frac{1}{|I|}\langle b, h_I\rangle\langle f_1, h_I\rangle\langle f_2, h_I\rangle.$$

Now consider $m > 2$ and let $\sigma(\vec{\alpha}) > 1$. Without loss of generality we may assume that $\alpha_1 = \alpha_2 = 0$. Then,

$$\begin{aligned}
\|\pi_b^{\vec{\alpha}}(f_1, f_2, \ldots, f_m)\|_r &= \left\| \sum_{I\in\mathcal{D}} \langle b, h_I\rangle\langle f_1, h_I\rangle\langle f_2, h_I\rangle \prod_{j=3}^{m} f_j(I, \alpha_j) h_I^{1+\sigma(\vec{\alpha})} \right\|_r \\
&= \left\| \sum_{I\in\mathcal{D}} \frac{1}{|I|}\langle b, h_I\rangle\langle f_1, h_I\rangle\langle f_2, h_I\rangle \prod_{j=3}^{m} f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})-1} \right\|_r \\
&= \left\| \sum_{I\in\mathcal{D}} \langle \pi_b^{(0,0)}(f_1, f_2), h_I\rangle \prod_{j=3}^{m} f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})-1} \right\|_r \\
&= \left\| P^{\vec{\beta}}(\pi_b^{(0,0)}(f_1, f_2), f_3, \ldots, f_m) \right\|_r \\
&\lesssim \|\pi_b^{(0,0)}(f_1, f_2)\|_q \prod_{j=3}^{m} \|f_j\|_{p_j} \\
&\lesssim \|b\|_* \prod_{j=1}^{m} \|f_j\|_{p_j}
\end{aligned}$$

where $\vec{\beta} = (0, \alpha_3, \ldots, \alpha_m) \in \{0,1\}^{m-1}$ and $\pi_b^{(0,0)}(f_1, f_2) \in L^q$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q}, q > r > 1$.

Conversely, assume that $\pi_b^{\vec{\alpha}} : L^{p_1} \times \cdots \times L^{p_m} \to L^r$ is bounded and that $\sigma(\vec{\alpha}) > 1$. Choose any $J \in \mathcal{D}$, and take $f_j = |J|^{\frac{1}{2}-\frac{1}{p_j}} h_J$ if $\alpha_j = 0$, and $f_j = |J|^{-\frac{1}{p_j}} \mathbb{1}_J$ if $\alpha_j = 1$ so that $\|f_j\|_{p_j} = 1$. Then,

$$\left\|\pi_b^{\vec{\alpha}}(f_1, \ldots, f_m)\right\|_r \leq \left\|\pi_b^{\vec{\alpha}}\right\|_{L^{p_1}\times\cdots\times L^{p_m}}.$$

We also have

$$\begin{aligned}
\left\|\pi_b^{\vec{\alpha}}(f_1,\ldots,f_m)\right\|_r &= \left\| |J|^{\frac{\sigma(\vec{\alpha})}{2}-\sum_{j=1}^m \frac{1}{p_j}} \langle b,h_J\rangle h_J^{1+\sigma(\vec{\alpha})} \right\|_r \\
&= |J|^{\frac{\sigma(\vec{\alpha})}{2}-\frac{1}{r}} |\langle b,h_J\rangle| \left\| h_J^{1+\sigma(\vec{\alpha})} \right\|_r \\
&= |J|^{\frac{\sigma(\vec{\alpha})}{2}-\frac{1}{r}} |\langle b,h_J\rangle| |J|^{-\frac{1+\sigma(\vec{\alpha})}{2}} \left\| \mathsf{1}_J \right\|_r \\
&= |J|^{\frac{\sigma(\vec{\alpha})}{2}-\frac{1}{r}} |\langle b,h_J\rangle| |J|^{-\frac{1+\sigma(\vec{\alpha})}{2}} |J|^{\frac{1}{r}} \\
&= \frac{|\langle b,h_J\rangle|}{\sqrt{|J|}}.
\end{aligned}$$

Thus $\frac{|\langle b,h_J\rangle|}{\sqrt{|J|}} \leq \left\|\pi_b^{\vec{\alpha}}\right\|_{L^{p_1}\times\cdots\times L^{p_m}}$. Since it is true for any $J \in D$, we have $\sup_{J\in\mathcal{D}} \frac{|\langle b,h_J\rangle|}{\sqrt{|J|}} \leq \left\|\pi_b^{\vec{\alpha}}\right\|_{L^{p_1}\times\cdots\times L^{p_m}} < \infty,$ as desired. □

Now that we have obtained strong type $L^{p_1} \times \cdots \times L^{p_m} \to L^r$ boundedness estimates for the paraproduct operators $P^{\vec{\alpha}}$ with $\vec{\alpha} \in U_m$ and $\pi_b^{\vec{\alpha}}$ with $\vec{\alpha} \in \{0,1\}^m$ in the case when $1 < p_1, p_2, \ldots, p_m, r < \infty$ and $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$, we are interested to investigate estimates corresponding to $\frac{1}{m} \leq r < \infty$. We will prove in Lemma 3.2.4 that we obtain weak type estimates if one or more p_i's are equal to 1. In particular, we obtain $L^1 \times \cdots \times L^1 \to L^{\frac{1}{m},\infty}$ estimates for those operators. Then it follows from multilinear interpolation that the paraproduct operators are strongly bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^r for $1 < p_1, p_2, \ldots, p_m < \infty$ and $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$, even if $\frac{1}{m} < r \leq 1$.

We first prove the following general lemma, which when applied to the operators $P^{\vec{\alpha}}$ and $\pi_b^{\vec{\alpha}}$ gives aforementioned weak type estimates.

Lemma 3.2.3. *Let T be a multi-sublinear operator that is bounded from the product of Lebesgue spaces $L^{p_1} \times \cdots \times L^{p_m}$ to $L^{r,\infty}$ for some $1 < p_1, p_2, \ldots, p_m < \infty$ with*

$$\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}.$$

Suppose that for every $I \in \mathcal{D}$, $T(f_1, \dots, f_m)$ is supported in I if $f_i = h_I$ for some $i \in \{1, 2, \dots, m\}$. Then T is bounded from $L^1 \times \cdots \times L^1 \times L^{p_{k+1}} \times \cdots \times L^{p_m} \to L^{\frac{q_k}{q_k+1}, \infty}$ for each $k = 1, 2, \dots, m$, where q_k is given by

$$\frac{1}{q_k} = (k-1) + \frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m}.$$

In particular, T is bounded from $L^1 \times \cdots \times L^1$ to $L^{\frac{1}{m}, \infty}$.

Proof. We first prove that T is bounded from $L^1 \times L^{p_2} \times \cdots \times L^{p_m}$ to $L^{\frac{q_1}{q_1+1}, \infty}$.

Let $\lambda > 0$ be given. We have to show that

$$|\{x : |T(f_1, f_2, \dots, f_m)(x)| > \lambda\}| \lesssim \left(\frac{\|f_1\|_1 \prod_{j=2}^m \|f_j\|_{p_j}}{\lambda} \right)^{\frac{q_1}{1+q_1}}$$

for all $(f_1, f_2, \dots, f_m) \in L^1 \times L^{p_2} \cdots \times L^{p_m}$.

Without loss of generality, we assume $\|f_1\|_1 = \|f_2\|_{p_2} = \cdots = \|f_m\|_{p_m} = 1$, and prove that

$$|\{x : |T(f_1, f_2, \dots, f_m)(x)| > \lambda\}| \lesssim \lambda^{-\frac{q_1}{1+q_1}}.$$

For this, we apply Calderón-Zygmund decomposition to the function f_1 at height $\lambda^{\frac{q_1}{q_1+1}}$ to obtain 'good' and 'bad' functions g_1 and b_1, and a sequence $\{I_{1,j}\}$ of disjoint dyadic intervals such that

$$f_1 = g_1 + b_1;$$

$$b_1 = \sum_j b_{1,j} \text{ with } \operatorname{supp}(b_{1,j}) \subseteq I_{1,j} \text{ and } \int_{I_{1,j}} b_{1,j} dx = 0;$$

$$\text{and } \sum_j |I_{1,j}| \leq \lambda^{-\frac{q_1}{q_1+1}} \|f_1\|_1 = \lambda^{-\frac{q_1}{q_1+1}}. \text{ (Recall that we have assumed } \|f_1\|_1 = 1.)$$

Moreover, since $1 < p_1 < \infty$, the good function $g_1 \in L^{p_1}$ with

$$\|g_1\|_{p_1} \leq \left(2\lambda^{\frac{q_1}{q_1+1}}\right)^{\frac{1}{p_1'}} \|f_1\|_1^{1/p_1} = \left(2\lambda^{\frac{q_1}{q_1+1}}\right)^{\frac{p_1-1}{p_1}},$$

where p_1' is the Hölder conjugate of p_1.

Since T is multi-sublinear,

$$|\{x : |T(f_1, \ldots, f_m)(x)| > \lambda\}|$$

$$\leq \left|\left\{x : |T(g_1, f_2, \ldots, f_m)(x)| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x : |T(b_1, f_2, \ldots, f_m)(x)| > \frac{\lambda}{2}\right\}\right|.$$

Since $g_1 \in L^{p_1}$ and T is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to $L^{r,\infty}$, we have

$$\begin{aligned} |\{x : |T(g_1, f_2, \ldots, f_m)(x)| > \lambda/2\}| &\lesssim \left(\frac{2\|g_1\|_{p_1} \prod_{j=2}^{m} f_j(J, \alpha_j)}{\lambda}\right)^r \\ &\leq \left(\frac{2\left(2\lambda^{\frac{q_1}{q_1+1}}\right)^{\frac{p_1-1}{p_1}}}{\lambda}\right)^r \\ &\lesssim \lambda^{r\left(\frac{q_1(p_1-1)}{p_1(q_1+1)}-1\right)}. \end{aligned}$$

Now, $\frac{1}{r} = \sum_{j=1}^{m} \frac{1}{p_j} = \frac{1}{p_1} + \frac{1}{q_1}$ implies that $r = \frac{p_1 q_1}{p_1+q_1}$. So,

$$\begin{aligned} r\left(\frac{q_1(p_1-1)}{p_1(q_1+1)} - 1\right) &= \frac{p_1 q_1}{(p_1+q_1)}\left(\frac{p_1q_1 - q_1 - p_1q_1 - p_1}{p_1(q_1+1)}\right) \\ &= \frac{p_1 q_1}{(p_1+q_1)} \frac{(-p_1-q_1)}{p_1(q_1+1)} \\ &= -\frac{q_1}{q_1+1}. \end{aligned}$$

Thus we have: $\quad |\{x : |T(g_1, f_2, \ldots, f_m)(x)| > \lambda/2\}| \lesssim \lambda^{-\frac{q_1}{1+q_1}}$.

From the properties of 'bad' function b_1 we deduce that $\langle b_1, h_I\rangle \neq 0$ only if $I \subseteq I_{1,j}$ for some j. The hypothesis of the lemma on the support of $T(f_1, \ldots, f_m)$ then implies that

$$\text{supp}\,(T(b_1, f_2, \ldots, f_m)) \subseteq \cup_j I_{1,j}.$$

Thus,

$$\left|\left\{x : |T(b_1, f_2, \ldots, f_m)(x)| > \frac{\lambda}{2}\right\}\right| \leq |\cup_j I_{1,j}| \leq \lambda^{-\frac{q_1}{1+q_1}}.$$

Combining these estimates corresponding to g_1 and b_1, we have the desired estimate

$$|\{x : |T(f_1, f_2, \ldots, f_m)(x)| > \lambda\}| \lesssim \lambda^{-\frac{q_1}{1+q_1}}.$$

Now beginning with the $L^1 \times L^{p_2} \times \cdots \times L^{p_m} \to L^{\frac{q_1}{q_1+1},\infty}$ estimate, we use the same argument to lower the second exponent to 1 proving that T is bounded from $L^1 \times L^1 \times L^{p_3} \times \cdots \times L^{p_m}$ to $L^{\frac{q_2}{q_2+1},\infty}$, where q_2 is given by $\frac{1}{q_2} = 1 + \frac{1}{p_3} + \cdots + \frac{1}{p_m}$.

We continue the same process until we obtain $L^1 \times L^1 \times \cdots \times L^1 \to L^{\frac{q_m}{q_m+1},\infty}$ boundedness of T with $\frac{1}{q_m} = 1 + 1 + \cdots + 1$ $(m-1$ terms$) = m-1$. This completes the proof since $\frac{q_m}{q_m+1} = \frac{1}{m}$. $\square$

Lemma 3.2.4. *Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \{0,1\}^m, 1 \leq p_1, \ldots, p_m < \infty$ and $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$. Then*

(a) *For $\vec{\alpha} \neq (1, 1, \ldots, 1)$, $P^{\vec{\alpha}}$ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to $L^{r,\infty}$.*

(b) *If $b \in BMO^d$ and $\sigma(\vec{\alpha}) \leq 1$, $\pi_b^{\vec{\alpha}}$ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to $L^{r,\infty}$.*

(c) *If $\sup_{I \in \mathcal{D}} \dfrac{|\langle b, h_I\rangle|}{\sqrt{|I|}} < \infty$ and $\sigma(\vec{\alpha}) > 1$, $\pi_b^{\vec{\alpha}}$ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to $L^{r,\infty}$.*

Proof. By orthogonality of Haar functions, $h_I(J, 0) = \langle h_I, h_J\rangle = 0$ for any two distinct dyadic intervals I and J. The Haar functions have mean value 0, so it is easy to see that

$$\langle h_I\rangle_J \neq 0 \text{ only if } J \subsetneq I$$

since any two dyadic intervals are either disjoint or one is contained in the other.

Consequently, if some $f_i = h_I$, then

$$P^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) = \sum_{J \subseteq I} \prod_{j=1}^{m} f_j(J, \alpha_j) h_J^{\sigma(\vec{\alpha})}$$

and,

$$\pi_b^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) = \sum_{J \subseteq I} \langle b, h_J \rangle \prod_{j=1}^{m} f_j(J, \alpha_j) h_J^{1+\sigma(\vec{\alpha})},$$

which are both supported in I. Since the paraproducts are strongly (and hence weakly) bounded from $L^{p_1} \times \cdots \times L^{p_m} \to L^r$, the proof follows immediately from Lemma 3.2.3.

□

Combining the results of Lemmas 3.2.1, 3.2.2 and 3.2.4, and using multilinear interpolation (see [23]), we have the following theorem:

Theorem 3.2.5. *Let* $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \{0, 1\}^m$ *and* $1 < p_1, \ldots, p_m < \infty$ *with* $\sum_{j=1}^{m} \frac{1}{p_j} = \frac{1}{r}$. *Then*

(a) *For* $\vec{\alpha} \neq (1, 1, \ldots, 1)$, $\left\| P^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) \right\|_r \lesssim \prod_{j=1}^{m} \|f_j\|_{p_j}$.

(b) *For* $\sigma(\vec{\alpha}) \leq 1$, $\left\| \pi_b^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) \right\|_r \lesssim \|b\|_{BMO^d} \prod_{j=1}^{m} \|f_j\|_{p_j}$, *if and only if* $b \in BMO^d$.

(c) *For* $\sigma(\vec{\alpha}) > 1$, $\left\| \pi_b^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) \right\|_r \leq C_b \prod_{j=1}^{m} \|f_j\|_{p_j}$, *if and only if* $\sup_{I \in \mathcal{D}} \frac{|\langle b, h_I \rangle|}{\sqrt{|I|}} < \infty$.

In each of the above cases, the paraproducts are weakly bounded if $1 \leq p_1, p_2, \ldots, p_m < \infty$.

3.3 Multilinear Haar Multipliers

In this section, we present the boundedness properties of multilinear Haar multipliers.

Theorem 3.3.1. *Let $\epsilon = \{\epsilon_I\}_{I\in\mathcal{D}}$ be a given sequence and let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in U_m$. Let $1 < p_1, p_2, \ldots, p_m < \infty$ with $\displaystyle\sum_{j=1}^{m} \frac{1}{p_j} = \frac{1}{r}$. Then $T_\epsilon^{\vec{\alpha}}$ is bounded from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m}$ to L^r if and only if $\|\epsilon\|_\infty := \sup_{I\in\mathcal{D}} |\epsilon_I| < \infty$.*
Moreover, $T_\epsilon^{\vec{\alpha}}$ has the corresponding weak-type boundedness if $1 \leq p_1, p_2, \ldots, p_m < \infty$.

Proof. To prove this lemma we use the fact that the linear Haar multiplier

$$T_\epsilon(f) = \sum_{I\in\mathcal{D}} \epsilon_I \langle f, h_I\rangle h_I$$

is bounded on L^p for all $1 < p < \infty$ if $\|\epsilon\|_\infty := \sup_{I\in\mathcal{D}} |\epsilon_I| < \infty$, and that $\langle T_\epsilon(f), h_I\rangle = \epsilon_I\langle f, h_I\rangle$.

By assumption $\sigma(\vec{\alpha}) \geq 1$. Without loss of generality we may assume that $\alpha_i = 0$ if $1 \leq i \leq \sigma(\vec{\alpha})$ and $\alpha_i = 1$ if $\sigma(\vec{\alpha}) < i \leq m$. In particular, we have $\alpha_1 = 0$. Then

$$\epsilon_I f_1(I, \alpha_1) = \epsilon_I\langle f_1, h_I\rangle = \langle T_\epsilon(f_1), h_I\rangle = T_\epsilon(f_1)(I, \alpha_1).$$

First assume that $\|\epsilon\|_\infty := \sup_{I\in\mathcal{D}} |\epsilon_I| < \infty$.

Then,

$$\begin{aligned}
\|T_\epsilon^{\vec{\alpha}}(f_1, f_2, \ldots, f_m)\|_r &= \left\|\sum_{I\in\mathcal{D}} \epsilon_I \prod_{j=1}^{m} f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}\right\|_r \\
&= \left\|\sum_{I\in\mathcal{D}} T_\epsilon(f_1)(I, \alpha_1) \prod_{j=2}^{m} f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}\right\|_r \\
&= \|P^{\vec{\alpha}}(T_\epsilon(f_1), f_2, \ldots, f_m)\|_r \\
&\lesssim \|T_\epsilon(f_1)\|_{p_1} \prod_{j=2}^{m} \|f_j\|_{p_j} \\
&\lesssim \prod_{j=1}^{m} \|f_j\|_{p_j}.
\end{aligned}$$

Conversely, assume that $T_\epsilon^{\vec{\alpha}} : L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m} \to L^r$ is bounded, and let $\sigma(\vec{\alpha}) = k$. Recall that $\alpha_i = 0$ if $1 \le i \le \sigma(\vec{\alpha}) = k$ and $\alpha_i = 1$ if $k = \sigma(\vec{\alpha}) < i \le m$. Taking $f_i = h_I$ if $1 \le i \le k$ and $f_i = \mathbb{1}_I$ if $k < i \le m$, we observe that

$$\|T_\epsilon^{\vec{\alpha}}(f_1, f_2, \ldots, f_m)\|_r = \left(\int_{\mathbb{R}} |\epsilon_I h_I^k(x)|^r dx\right)^{1/r} = \left(\frac{|\epsilon_I|^r}{|I|^{kr/2}} \int_{\mathbb{R}} \mathbb{1}_I(x) dx\right)^{1/r} = \frac{|\epsilon_I|}{|I|^{k/2}} |I|^{1/r},$$

and

$$\begin{aligned}
\prod_{j=1}^m \|f_j\|_{p_j} &= \prod_{i=1}^k \left(\int_{\mathbb{R}} |h_I(x)|^{p_i} dx\right)^{1/p_i} \prod_{j=k+1}^m \left(\int_{\mathbb{R}} |\mathbb{1}_I(x)|^{p_j} dx\right)^{1/p_j} \\
&= \prod_{i=1}^k \left(\frac{1}{|I|^{p_i/2}} \int_{\mathbb{R}} \mathbb{1}_I(x) dx\right)^{1/p_i} \prod_{j=k+1}^m \left(\int_{\mathbb{R}} \mathbb{1}_I(x) dx\right)^{1/p_j} \\
&= \prod_{i=1}^k \left(\frac{1}{|I|^{1/2}} |I|^{1/p_i}\right) \prod_{j=k+1}^m |I|^{1/p_j} \\
&= \frac{|I|^{1/r}}{|I|^{k/2}}
\end{aligned}$$

Since $(f_1, f_2, \ldots, f_m) \in L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m}$, the boundedness of T_ϵ implies that

$$\|T_\epsilon^{\vec{\alpha}}(f_1, f_2, \ldots, f_m)\|_r \le \|T_\epsilon^{\vec{\alpha}}\|_{L^{p_1} \times \cdots \times L^{p_m} \to L^r} \prod_{j=1}^m \|f_j\|_{p_j}.$$

That is, $\dfrac{|\epsilon_I|}{|I|^{k/2}} |I|^{1/r} \le \|T_\epsilon^{\vec{\alpha}}\|_{L^{p_1} \times \cdots \times L^{p_m}} \dfrac{|I|^{1/r}}{|I|^{k/2}}$, for all $I \in \mathcal{D}$. Consequently, $\|\epsilon\|_\infty = \sup_{I \in \mathcal{D}} |\epsilon_I| \le \|T_\epsilon^{\vec{\alpha}}\|_{L^{p_1} \times \cdots \times L^{p_m}} < \infty$, as desired.
If $1 \le p_1, p_2, \ldots, p_m < \infty$, the weak-type boundedness of $T_\epsilon^{\vec{\alpha}}$ follows from Lemma 3.2.3.

□

3.4 Commutators of Multilinear Haar Multipliers.

In this section, we study boundedness properties of the commutators of $T_\epsilon^{\vec{\alpha}}$ with the multiplication operator M_b when $b \in BMO^d$. For convenience we denote the operator M_b by b

itself. We are interested in the following commutators:

$$[b, T_\epsilon^{\vec{\alpha}}]_i(f_1, f_2, \dots, f_m)(x) \equiv (T_\epsilon^{\vec{\alpha}}(f_1, \dots, bf_i, \dots, f_m) - bT_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m))(x)$$

where $1 \le i \le m$.

Note that if b is a constant function, $[b, T_\epsilon^{\vec{\alpha}}]_i(f_1, f_2, \dots, f_m)(x) = 0$ for all x. Our approach to study the boundedness properties of $[b, T_\epsilon^{\vec{\alpha}}]_i : L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m} \to L^r$ with $1 < p_1, p_2, \dots, p_m < \infty$ and $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$ for non-constant b requires us to assume that $b \in L^p$ for some $p \in (1, \infty)$, and that $r > 1$. However, this restricted unweighted theory turns out to be sufficient to obtain a weighted theory, which in turn implies the unrestricted unweighted theory of these multilinear commutators. We will present the weighted theory of these commutators in the next chapter.

Theorem 3.4.1. *Let $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in U_m$. If $b \in BMO^d \cap L^p$ for some $1 < p < \infty$ and $\|\epsilon\|_\infty := \sup_{I \in \mathcal{D}} |\epsilon_I| < \infty$, then each commutator $[b, T_\epsilon^{\vec{\alpha}}]_i$ is bounded from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m} \to L^r$ for all $1 < p_1, p_2, \dots, p_m, r < \infty$ with $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$, with estimates of the form:* $\|[b, T_\epsilon^{\vec{\alpha}}]_i(f_1, f_2, \dots, f_m)\|_r \lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}$.

Proof. It suffices to prove boundedness of $[b, T_\epsilon^{\vec{\alpha}}]_1$, as the others are identical. Moreover, we may assume that each f_i is bounded and has compact support, since such functions are dense in the L^p spaces.

Writing $bf_1 = \pi_b(f_1) + \pi_b^*(f_1) + \pi_{f_1}(b)$ and using multilinearity of $T_\epsilon^{\vec{\alpha}}$, we have

$$\begin{aligned} & T_\epsilon^{\vec{\alpha}}(bf_1, f_2, \dots, f_m) \\ = \;& T_\epsilon^{\vec{\alpha}}(\pi_b(f_1), f_2, \dots, f_m) + T_\epsilon^{\vec{\alpha}}(\pi_b^*(f_1), f_2, \dots, f_m) + T_\epsilon^{\vec{\alpha}}(\pi_{f_1}(b), f_2, \dots, f_m). \end{aligned}$$

On the other hand,

$$\begin{aligned}
bT_\epsilon^{\vec{\alpha}}(f_1, f_2, \ldots, f_m) &= \sum_{I\in\mathcal{D}} \epsilon_I \prod_{j=1}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \left(\sum_{J\in\mathcal{D}} \widehat{b}(J) h_J\right) \\
&= \sum_{I\in\mathcal{D}} \epsilon_I \widehat{b}(I) \prod_{j=1}^m f_j(I,\alpha_j) h_I^{1+\sigma(\vec{\alpha})} \\
&\quad + \sum_{I\in\mathcal{D}} \epsilon_I \prod_{j=1}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \left(\sum_{I\subsetneq J} \widehat{b}(J) h_J\right) \\
&\quad + \sum_{I\in\mathcal{D}} \epsilon_I \prod_{j=1}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \left(\sum_{J\subsetneq I} \widehat{b}(J) h_J\right) \\
&= \pi_b^{\vec{\alpha}}(f_1, \ldots, T_\epsilon(f_i), \ldots, f_m) \\
&\quad + \sum_{I\in\mathcal{D}} \epsilon_I \langle b\rangle_I \prod_{j=1}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&\quad + \sum_{J\in\mathcal{D}} \widehat{b}(J) h_J \left(\sum_{J\subsetneq I} \epsilon_I \prod_{j=1}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})}\right)
\end{aligned}$$

for some i with $\alpha_i = 0$. Indeed, some α_i equals 0 by assumption, and for such i, we have

$$T_\epsilon(f_i)(I,\alpha_i) = \widehat{T_\epsilon(f_i)}(I) = \epsilon_I \widehat{f_i}(I) = \epsilon_I f_i(I,\alpha_i).$$

For $(f_1, f_2, \ldots, f_m) \in L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m}$, we have

$$\begin{aligned}
\|T_\epsilon^{\vec{\alpha}}(\pi_b(f_1), f_2, \ldots, f_m)\|_r &\lesssim \|\pi_b(f_1)\|_{p_1} \prod_{j=2}^m \|f_j\|_{p_j} \\
&\lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}
\end{aligned}$$

$$\begin{aligned}
\|T_\epsilon^{\vec{\alpha}}(\pi_b^*(f_1), f_2, \ldots, f_m)\|_r &\lesssim \|\pi_b^*(f_1)\|_{p_1} \prod_{j=2}^m \|f_j\|_{p_j} \\
&\lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}.
\end{aligned}$$

and,

$$\begin{aligned}\|\pi_b^{\vec{\alpha}}(f_1,\ldots,T_\epsilon(f_i),\ldots,f_m)\|_r &\lesssim \|b\|_{BMO^d}\|f_1\|_{p_1}\cdots\|T_\epsilon(f_i)\|_{p_i}\cdots\|f_m\|_{p_m}\\ &\lesssim \|b\|_{BMO^d}\prod_{j=1}^m\|f_j\|_{p_j}.\end{aligned}$$

So, to prove boundedness of $[b,T_\epsilon^{\vec{\alpha}}]_1$, is suffices to show similar control over the terms:

$$\left\|\sum_{J\in\mathcal{D}}\widehat{b}(J)h_J\left(\sum_{J\subsetneq I}\epsilon_I\prod_{j=1}^m f_j(I,\alpha_j)h_I^{\sigma(\vec{\alpha})}\right)\right\|_r \tag{3.4.1}$$

and,

$$\left\|T_\epsilon^{\vec{\alpha}}(\pi_{f_1}(b),f_2,\ldots,f_m)-\sum_{I\in\mathcal{D}}\epsilon_I\langle b\rangle_I\prod_{j=1}^m f_j(I,\alpha_j)h_I^{\sigma(\vec{\alpha})}\right\|_r. \tag{3.4.2}$$

Estimation of $(3.4.1)$:

Case I: $\sigma(\vec{\alpha})$ odd. In this case,

$$T_\epsilon^{\vec{\alpha}}(f_1,f_2,\ldots,f_m)=\sum_{I\in\mathcal{D}}\epsilon_I\prod_{j=1}^m f_j(I,\alpha_j)h_I^{\sigma(\vec{\alpha})}=\sum_{I\in\mathcal{D}}\epsilon_I|I|^{\frac{1-\sigma(\vec{\alpha})}{2}}\prod_{j=1}^m f_j(I,\alpha_j)h_I.$$

So, $\quad \langle T_\epsilon^{\vec{\alpha}}(f_1,f_2,\ldots,f_m),h_I\rangle h_I=\epsilon_I|I|^{\frac{1-\sigma(\vec{\alpha})}{2}}\prod_{j=1}^m f_j(I,\alpha_j)h_I=\epsilon_I\prod_{j=1}^m f_j(I,\alpha_j)h_I^{\sigma(\vec{\alpha})}.$

This implies that

$$\begin{aligned}(3.4.1) &= \left\|\sum_{J\in\mathcal{D}}\widehat{b}(J)h_J\left(\sum_{J\subsetneq I}\langle T_\epsilon^{\vec{\alpha}}(f_1,f_2,\ldots,f_m),h_I\rangle h_I\right)\right\|_r\\ &= \left\|\sum_{J\in\mathcal{D}}\widehat{b}(J)\langle T_\epsilon^{\vec{\alpha}}(f_1,f_2,\ldots,f_m)\rangle_J h_J\right\|_r\\ &= \left\|\pi_b\left(T_\epsilon^{\vec{\alpha}}(f_1,f_2,\ldots,f_m)\right)\right\|_r\\ &\lesssim \|b\|_{BMO^d}\left\|T_\epsilon^{\vec{\alpha}}(f_1,f_2,\ldots,f_m)\right\|_r\\ &\lesssim \|b\|_{BMO^d}\prod_{j=1}^m\|f_j\|_{p_j}.\end{aligned}$$

Case II: $\sigma(\vec{\alpha})$ even.

In this case at least two $\alpha_i's$ are equal to 0. Without loss of generality we may assume that $\alpha_1 = 0$. Then denoting $T_\epsilon(f_1)$ by g_1, $P^{(\alpha_2,\dots,\alpha_m)}(f_2,\dots,f_m)$ by g_2, and using the fact that

$$\langle g_1\rangle_J\langle g_2\rangle_J 1_J = \left(\sum_{J\subsetneq I}\widehat{g_1}(I)\langle g_2\rangle_I h_I + \sum_{J\subsetneq I}\langle g_1\rangle_I\widehat{g_2}(I)h_I + \sum_{J\subsetneq I}\widehat{g_1}(I)\widehat{g_2}(I)h_I^2\right)1_J,$$

we have

$$\begin{aligned}
&\left\|\sum_{J\in\mathcal{D}}\widehat{b}(J)h_J\left(\sum_{J\subsetneq I}\epsilon_I\prod_{j=1}^m f_j(I,\alpha_j)h_I^{\sigma(\vec{\alpha})}\right)\right\|_r\\
= &\left\|\sum_{J\in\mathcal{D}}\widehat{b}(J)h_J\left(\sum_{J\subsetneq I}\widehat{g_1}(I)\widehat{g_2}(I)h_I^2\right)\right\|_r\\
= &\left\|\sum_{J\in\mathcal{D}}\widehat{b}(J)h_J\left(\langle g_1\rangle_J\langle g_2\rangle_J 1_J - \sum_{J\subsetneq I}\widehat{g_1}(I)\langle g_2\rangle_I h_I - \sum_{J\subsetneq I}\langle g_1\rangle_I\widehat{g_2}(I)h_I\right)\right\|_r\\
\leq &\left\|\sum_{J\in\mathcal{D}}\widehat{b}(J)\langle g_1\rangle_J\langle g_2\rangle_J h_J\right\|_r + \left\|\sum_{J\in\mathcal{D}}\widehat{b}(J)\langle P^{(0,1)}(g_1,g_2)\rangle_J h_J\right\|_r\\
&\qquad + \left\|\sum_{J\in\mathcal{D}}\widehat{b}(J)\langle P^{(1,0)}(g_1,g_2)\rangle_J h_J\right\|_r\\
\lesssim &\|b\|_{BMO^d}\|g_1\|_{p_1}\|g_2\|_q + \|b\|_{BMO^d}\|P^{(0,1)}(g_1,g_2)\|_r + \|b\|_{BMO^d}\|P^{(1,0)}(g_1,g_2)\|_r\\
\lesssim &\|b\|_{BMO^d}\|g_1\|_{p_1}\|g_2\|_q\\
\lesssim &\|b\|_{BMO^d}\prod_{j=1}^m\|f_j\|_{p_j}.
\end{aligned}$$

where, q is given by $\dfrac{1}{q} = \displaystyle\sum_{j=2}^m \frac{1}{p_j}$. Here the last three inequalities follow from Lemmas 3.2.1 and 3.2.2, and the fact that $\|g_1\|_{p_1} = \|T_\epsilon(f_1)\|_{p_1} \lesssim \|f_1\|_{p_1}$.

Estimation of (3.4.2) :

Case I: $\alpha_1 = 0$.

This case is easy as we observe that

$$\begin{aligned}
& T_\epsilon^{\vec{\alpha}}(\pi_{f_1}(b), f_2, \ldots, f_m) - \sum_{I\in\mathcal{D}} \epsilon_I \langle b\rangle_I \prod_{j=1}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \\
= & \sum_{I\in\mathcal{D}} \epsilon_I \widehat{\pi_{f_1}(b)}(I) \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} - \sum_{I\in\mathcal{D}} \epsilon_I \langle b\rangle_I \widehat{f_1}(I) \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \\
= & \sum_{I\in\mathcal{D}} \epsilon_I \langle b\rangle_I \widehat{f_1}(I) \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} - \sum_{I\in\mathcal{D}} \epsilon_I \langle b\rangle_I \widehat{f_1}(I) \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \\
= & \ 0.
\end{aligned}$$

So there is nothing to estimate.

Case II: $\alpha_1 = 1$.

In this case,

$$\begin{aligned}
& T_\epsilon^{\vec{\alpha}}(\pi_{f_1}(b), f_2, \ldots, f_m) - \sum_{I\in\mathcal{D}} \epsilon_I \langle b\rangle_I \prod_{j=1}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \\
= & \sum_{I\in\mathcal{D}} \epsilon_I \langle \pi_{f_1}(b)\rangle_I \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} - \sum_{I\in\mathcal{D}} \epsilon_I \langle b\rangle_I \langle f_1\rangle_I \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \\
= & \sum_{I\in\mathcal{D}} \epsilon_I \left(\langle \pi_{f_1}(b)\rangle_I - \langle b\rangle_I \langle f_1\rangle_I \right) \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})}
\end{aligned}$$

We have assumed that $b \in L^p$ for some $p \in (1,\infty)$. So, using Lemma 2.1.1, we have

$$\begin{aligned}
\langle b\rangle_I \langle f_1\rangle_I \mathbf{1}_I &= \sum_{I\subsetneq J} \widehat{b}(J) \langle f_1\rangle_J h_J \mathbf{1}_I + \sum_{I\subsetneq J} \langle b\rangle_J \widehat{f_1}(J) h_J \mathbf{1}_I + \sum_{I\subsetneq J} \widehat{b}(J) \widehat{f_1}(J) h_J^2 \mathbf{1}_I \\
&= \langle \pi_b(f_1)\rangle_I \mathbf{1}_I + \langle \pi_{f_1}(b)\rangle_I \mathbf{1}_I + \sum_{I\subsetneq J} \widehat{b}(J) \widehat{f_1}(J) h_J^2 \mathbf{1}_I.
\end{aligned}$$

Hence, $\langle b\rangle_I \langle f_1\rangle_I \mathbf{1}_I - \langle \pi_{f_1}(b)\rangle_I \mathbf{1}_I = \langle \pi_b(f_1)\rangle_I \mathbf{1}_I + \sum_{I\subsetneq J} \widehat{b}(J) \widehat{f_1}(J) h_J^2 \mathbf{1}_I$.

So we have

$$
\begin{aligned}
&T_\epsilon^{\vec{\alpha}}(\pi_{f_1}(b), f_2, \dots, f_m) - \sum_{I\in\mathcal{D}} \epsilon_I \langle b\rangle_I \prod_{j=1}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&= -\sum_{I\in\mathcal{D}} \epsilon_I \left(\langle \pi_b(f_1)\rangle_I 1_I + \sum_{I\subsetneq J} \widehat{b}(J)\widehat{f}_1(J) h_J^2 \right) \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&= -\sum_{I\in\mathcal{D}} \epsilon_I \langle \pi_b(f_1)\rangle_I \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&\qquad\qquad - \sum_{I\in\mathcal{D}} \epsilon_I \left(\sum_{I\subsetneq J} \widehat{b}(J)\widehat{f}_1(J) h_J^2 \right) \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&= -T_\epsilon(\pi_b(f_1), f_2, \dots, f_m) - \sum_{J\in\mathcal{D}} \widehat{b}(J)\widehat{f}_1(J) h_J^2 \left(\sum_{I\subsetneq J} \epsilon_I \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \right).
\end{aligned}
$$

Since

$$
\|T_\epsilon(\pi_b(f_1), f_2, \dots, f_m)\|_r \lesssim \|\pi_b(f_1)\|_{p_1} \prod_{j=2}^m f_j(J,\alpha_j) \lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j},
$$

we are left with controlling

$$
\left\| \sum_{J\in\mathcal{D}} \widehat{b}(J)\widehat{f}_1(J) h_J^2 \left(\sum_{I\subsetneq J} \epsilon_I \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \right) \right\|_r .
$$

For this we observe that $\left\|T_\epsilon^{(\alpha_2,\dots,\alpha_m)}(f_2,\dots,f_m)\right\|_q \lesssim \prod_{j=2}^m \|f_j\|_{p_j}$, and that

$$
\begin{aligned}
\pi_b^*(f_1)\, T_\epsilon^{(\alpha_2,\dots,\alpha_m)}(f_2,\dots,f_m) \;=\;& \sum_{J\in\mathcal{D}} \widehat{b}(J)\widehat{f}_1(J) h_J^2 \left(\sum_{I\subsetneq J} \epsilon_I \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \right) \\
&+ \sum_{J\in\mathcal{D}} \epsilon_J \widehat{b}(J)\widehat{f}_1(J) \prod_{j=2}^m f_j(J,\alpha_j) h_J^{2+\sigma(\vec{\alpha})} \\
&+ \sum_{J\in\mathcal{D}} \widehat{b}(J)\widehat{f}_1(J) h_J^2 \left(\sum_{J\subsetneq I} \epsilon_I \prod_{j=2}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})} \right)
\end{aligned}
$$

Now, following the same technique we used to control $(3.4.1)$, we obtain

$$\left\|\sum_{J\in\mathcal{D}}\widehat{b}(J)\widehat{f}_1(J)h_J^2\left(\sum_{J\subsetneq I}\epsilon_I\prod_{j=2}^{m}f_j(I,\alpha_j)h_I^{\sigma(\vec{\alpha})}\right)\right\|_r \lesssim \|b\|_{BMO^d}\prod_{j=1}^{m}\|f_j\|_{p_j}.$$

We also have

$$\begin{aligned}\left\|\pi_b^*(f_1)\,T_\epsilon^{(\alpha_2,\dots,\alpha_m)}(f_2,\dots,f_m)\right\|_r &\leq \|\pi_b^*(f_1)\|_{p_1}\left\|T_\epsilon^{(\alpha_2,\dots,\alpha_m)}(f_2,\dots,f_m)\right\|_q\\ &\lesssim \|b\|_{BMO^d}\prod_{j=1}^{m}\|f_j\|_{p_j}\end{aligned}$$

and,

$$\left\|\sum_{J\in\mathcal{D}}\epsilon_J\widehat{b}(J)\widehat{f}_1(J)\prod_{j=2}^{m}f_j(J,\alpha_j)h_J^{2+\sigma(\vec{\alpha})}\right\|_r \lesssim \|b\|_{BMO^d}\prod_{j=1}^{m}\|f_j\|_{p_j}.$$

.

So we conclude that

$$\left\|\sum_{J\in\mathcal{D}}\widehat{b}(J)\widehat{f}_1(J)h_J^2\left(\sum_{I\subsetneq J}\epsilon_I\prod_{j=2}^{m}f_j(I,\alpha_j)h_I^{\sigma(\vec{\alpha})}\right)\right\|_r \lesssim \|b\|_{BMO^d}\prod_{j=1}^{m}\|f_j\|_{p_j}.$$

Thus we have strong type boundedness of

$$[b,T_\epsilon^{\vec{\alpha}}]_1 \to L^{p_1}\times L^{p_2}\times\cdots\times L^{p_m}\to L^r$$

for all $1<p_1,p_2,\dots,p_m,r<\infty$ with

$$\sum_{j=1}^{m}\frac{1}{p_j}=\frac{1}{r}.$$

$\square$

Note that $T_\epsilon^{\vec{\alpha}}=P^{\vec{\alpha}}$ if $\epsilon_I=1$ for all $I\in\mathcal{D}$. The following theorem shows that the BMO condition on b is *necessary* for the boundedness of the commutator $[b,P^{\vec{\alpha}}]_i$.

Theorem 3.4.2. *Let* $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in U_m$*, and* $1 < p_1, p_2, \ldots, p_m, r < \infty$ *with*

$$\sum_{j=1}^{m} \frac{1}{p_j} = \frac{1}{r}.$$

Assume that for given b *and* i*,*

$$\|[b, P^{\vec{\alpha}}]_i(f_1, f_2, \ldots, f_m)\|_r \leq C \prod_{j=1}^{m} \|f_j\|_{p_j}, \tag{3.4.3}$$

for all $f_j \in L^{p_j}$*. Then* $b \in BMO^d$*.*

Proof. Without loss of generality we may assume that $i = 1$. Fix $I_0 \in \mathcal{D}$.

Case I: $\alpha_1 = 0, \sigma(\vec{\alpha}) = 1$.

Take $f_1 = \mathbb{1}_{I_0}$ and $f_i = h_{I_0^{(1)}}$ for $i > 1$, where $I_0^{(1)}$ is the parent of I_0. Then,

$$P^{\vec{\alpha}}(f_1, f_2, \ldots, f_m)) = \sum_{I \in \mathcal{D}} \langle \mathbb{1}_{I_0}, h_I \rangle \langle h_{I_0^{(1)}} \rangle_I^{m-1} h_I = 0,$$

and,

$$\begin{aligned} P^{\vec{\alpha}}(bf_1, f_2, \ldots, \ldots, f_m) &= \sum_{I \in \mathcal{D}} \langle b\mathbb{1}_{I_0}, h_I \rangle \langle h_{I_0^{(1)}} \rangle_I^{m-1} h_I \\ &= \sum_{I \subseteq I_0} \langle b\mathbb{1}_{I_0}, h_I \rangle \left(\frac{K(I_0, I_0^{(1)})}{\sqrt{\left|I_0^{(1)}\right|}} \right)^{m-1} h_I \\ &= \left(\frac{K(I_0, I_0^{(1)})}{\sqrt{\left|I_0^{(1)}\right|}} \right)^{m-1} \sum_{I \subseteq I_0} \langle b, h_I \rangle h_I, \end{aligned}$$

where $K(I_0, I_0^{(1)})$ is either 1 or -1 depending on whether I_0 is the right or left half of $I_0^{(1)}$. For the second to last equality we observe that, if I is not a proper subset of $I_0^{(1)}$, $\langle h_{I_0^{(1)}} \rangle_I = 0$, and that if I is a proper subset of $I_0^{(1)}$ but is not a subset of I_0, then $\langle b\mathbb{1}_{I_0}, h_I \rangle = 0$.

Moreover, for $I \subseteq I_0$, $\langle b\mathbb{1}_{I_0}, h_I\rangle = \int_{\mathbb{R}} b\mathbb{1}_{I_0} h_I = \int_{\mathbb{R}} bh_I = \langle b, h_I\rangle$.

Now from inequality (3.4.3), we get

$$\left\| \left(\frac{K(I_0, I_0^{(1)})}{\sqrt{\left|I_0^{(1)}\right|}} \right)^{m-1} \sum_{I\subseteq I_0} \langle b, h_I\rangle h_I \right\|_r \leq C|I_0|^{\frac{1}{p_1}} \prod_{i=2}^{m} \frac{|I_0^{(1)}|^{\frac{1}{p_i}}}{\sqrt{|I_0^{(1)}|}}$$

$$i.e. \quad \left\| \sum_{I\subseteq I_0} \langle b, h_I\rangle h_I \right\|_r \leq 2^{\frac{1}{p_2}+\cdots+\frac{1}{p_m}} C|I_0|^{\frac{1}{r}}.$$

Thus for every $I_0 \in \mathcal{D}$,

$$\frac{1}{|I_0|^{\frac{1}{r}}} \left\| \sum_{I\subseteq I_0} \langle b, h_I\rangle h_I \right\|_r \leq 2^{\frac{1}{p_2}+\cdots+\frac{1}{p_m}} C,$$

and hence $b \in BMO^d$.

Case II: $\alpha_1 \neq 0$ or $\sigma(\vec{\alpha}) > 1$.

Taking $f_i = \begin{cases} h_{I_0}, & \text{if } \alpha_i = 0 \\ \mathbb{1}_{I_0}, & \text{if } \alpha_i = 1, \end{cases}$ we observe that

$$P^{\vec{\alpha}}(f_1, f_2, \ldots, f_m)) = h_{I_0}^{\sigma(\vec{\alpha})} \quad \text{and} \quad P^{\vec{\alpha}}(bf_1, f_2, \ldots, \ldots, f_m) = (bf_1)(I_0, \alpha_1) h_{I_0}^{\sigma(\vec{\alpha})}.$$

If $\alpha_1 = 0$,

$$(bf_1)(I_0, \alpha_1) = bh_{I_0}(I_0, 0) = \widehat{bh_{I_0}}(I_0) = \int_{\mathbb{R}} bh_{I_0} h_{I_0} = \frac{1}{|I_0|} \int_{\mathbb{R}} b\mathbb{1}_{I_0} = \langle b\rangle_{I_0}.$$

If $\alpha_1 = 1$,

$$(bf_1)(I_0, \alpha_1) = b\mathbb{1}_{I_0}(I_0, 1) = \langle b\mathbb{1}_{I_0}\rangle_{I_0} = \langle b\rangle_{I_0}.$$

So in each case,

$$\begin{aligned}
\|[b,P^{\vec{\alpha}}]_1(f_1,f_2,\dots,f_m)\|_r &= \left\|bP^{\vec{\alpha}}(f_1,f_2,\dots,f_m) - P^{\vec{\alpha}}(bf_1,f_2,\dots,\dots,f_m)\right\|_r \\
&= \left\|bh_{I_0}^{\sigma(\vec{\alpha})} - \langle b\rangle_{I_0} h_{I_0}^{\sigma(\vec{\alpha})}\right\|_r \\
&= \left\|(b-\langle b\rangle_{I_0}) h_{I_0}^{\sigma(\vec{\alpha})}\right\|_r \\
&= \frac{1}{(\sqrt{|I_0|})^{\sigma(\vec{\alpha})}} \|(b-\langle b\rangle_{I_0})\mathbb{1}_{I_0}\|_r.
\end{aligned}$$

On the other hand,

$$\prod_{j=1}^m \|f_j\|_{p_j} = \frac{1}{(\sqrt{|I_0|})^{\sigma(\vec{\alpha})}} |I_0|^{\frac{1}{p_1}+\cdots+\frac{1}{p_m}} = \frac{1}{(\sqrt{|I_0|})^{\sigma(\vec{\alpha})}} |I_0|^{\frac{1}{r}}.$$

Inequality (3.4.3) then gives

$$\frac{1}{(\sqrt{|I_0|})^{\sigma(\vec{\alpha})}} \|(b-\langle b\rangle_{I_0})\mathbb{1}_{I_0}\|_r \le C\frac{1}{(\sqrt{|I_0|})^{\sigma(\vec{\alpha})}} |I_0|^{\frac{1}{r}}$$

$$\text{i.e.} \quad \frac{1}{|I_0|^{\frac{1}{r}}} \|(b-\langle b\rangle_{I_0})\mathbb{1}_{I_0}\|_r \le C.$$

Since this is true for any $I_0 \in \mathcal{D}$, we have $b \in BMO^d$. □

Combining the results from Theorems 3.4.1 and 3.4.2, we have the following characterization of the dyadic BMO functions.

Theorem 3.4.3. *Let* $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in U_m$, $1 \le i \le m$, *and* $1 < p_1, p_2, \dots, p_m, r < \infty$ *with* $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$. *Suppose* $b \in L^p$ *for some* $p \in (1,\infty)$. *Then the following two statements are equivalent.*

(a) $b \in BMO^d$.

(b) $[b, T_\epsilon^{\vec{\alpha}}]_i : L^{p_1} \times \cdots \times L^{p_m} \to L^r$ *is bounded for every bounded sequence* $\epsilon = \{\epsilon_I\}_{I \in \mathcal{D}}$.

In particular, $b \in BMO^d$ *if and only if* $[b, P^{\vec{\alpha}}]_i : L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m} \to L^r$ *is bounded.*

CHAPTER 4

ESTIMATES FOR MULTILINEAR DYADIC OPERATORS: WEIGHTED SETTING

In this chapter, we investigate the boundedness properties of the multilinear dyadic paraproduct operators in the weighted setting. We also obtain weighted estimates for the multilinear Haar multipliers and their commutators with dyadic BMO functions, and characterize dyadic BMO functions by the boundedness of the commutators of multilinear dyadic paraproducts.

4.1 Main Results

The main results of this chapter are as follows:

Theorem: Let $b \in BMO^d$, and $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ be bounded. Suppose $T \in \left\{P^{\vec{\alpha}}, T_\epsilon^{\vec{\alpha}}\right\}$ with $\vec{\alpha} \in U_m$, or $T = \pi_b^{\vec{\alpha}}$ with $\vec{\alpha} \in \{0,1\}^m$. Let $\vec{w} = (w_1, \dots, w_m) \in A^d_{\vec{P}}$ for $\vec{P} = (p_1, \dots, p_m)$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$.

(a) If $1 < p_1, \dots, p_m < \infty$, then $\|T(f_1, \dots, f_m)\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}$.

(b) If $1 \leq p_1, \dots, p_m < \infty$, then $\|T(f_1, \dots, f_m)\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}$.

Theorem: Let $\vec{\alpha} \in U_m$ and $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ be bounded. Suppose $b \in BMO^d$ and $\vec{w} = (w_1, \dots, w_m) \in A^d_{\vec{P}}$ for $\vec{P} = (p_1, \dots, p_m)$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$ and $1 < p_1, \dots, p_m < \infty$. Then there exists a constant C such that

$$\left\|[b, T_\epsilon^{\vec{\alpha}}]_i(f_1, \dots, f_m)\right\|_{L^p(\nu_{\vec{w}})} \leq C\|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

Theorem: Assume $\sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{p}$ with $1 < p_i < \infty$, and let $\vec{w} = (w_1, \ldots, w_m)$ with $w_i \in A_{p_i}$. Then for $j \in \{1, \ldots, m\}$ and $\vec{\alpha} \in U_m$, the following two statements are equivalent.

1. $b \in BMO$.

2. $[b, P^{\vec{\alpha}}]_j : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \to L^p(\nu_{\vec{w}})$ is bounded.

4.2 Multilinear Dyadic Paraproducts and Haar Multipliers

We first present the following property of the multilinear dyadic operators, which will be very useful for our purpose.

Lemma 4.2.1. *Let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \{0,1\}^m$, and let T be any of the $m-$linear operators $P^{\vec{\alpha}}, \pi_b^{\vec{\alpha}}$ or $T_\epsilon^{\vec{\alpha}}$. Then for a given function g and $J \in \mathcal{D}$, the function*

$$T\left(M_g^i(f_1, f_2, \ldots, f_m)\right) - T\left(M_g^i(f_1 \mathbf{1}_J, f_2 \mathbf{1}_J, \ldots, f_m \mathbf{1}_J)\right)$$

is constant on J. In particular, $T(f_1, f_2, \ldots, f_m) - T(f_1 \mathbf{1}_J, f_2 \mathbf{1}_J, \ldots, f_m \mathbf{1}_J)$ is constant on J.

Proof. Fix $J \in \mathcal{D}$. Let $f_i \mathbf{1}_J = f_i^0$ and $f_i - f_i \mathbf{1}_J = f_i^\infty$.
Since $T(M_g^i)$ is multilinear,

$$\begin{aligned} T\left(M_g^i(f_1, f_2, \ldots, f_m)\right) &= T\left(M_g^i(f_1^0 + f_1^\infty, f_2^0 + f_2^\infty, \ldots, f_m^0 + f_m^\infty)\right) \\ &= T\left(M_g^i(f_1^0, \ldots, f_m^0)\right) + \sum_{\substack{\vec{\beta} \in \{0,\infty\}^m \\ \vec{\beta} \neq \vec{0}}} T\left(M_g^i(f_1^{\beta_1}, \ldots, f_m^{\beta_m})\right), \end{aligned}$$

where $\vec{\beta} = (\beta_1, \ldots, \beta_m)$.
Observe that if $I \subseteq J$, $\widehat{f_j^\infty}(I) = \widehat{g f_j^\infty}(I) = \langle f_j^\infty \rangle_I = \langle g f_j^\infty \rangle_I = 0$, since each of the

functions f_j^∞, gf_j^∞ is identically 0 on J. So for $\vec{\beta} \neq \vec{0}$,

$$T\left(M_g^i(f_1^{\beta_1}, \ldots, f_m^{\beta_m})\right) = \sum_{I \in \mathcal{D}} \delta_J^T \prod_{j=1}^m F_j^{\beta_j}(I, \alpha_j) h_I^{\sigma(\vec{\alpha},T)} = \sum_{I: I \not\subseteq J} \delta_J^T \prod_{j=1}^m F_j^{\beta_j}(I, \alpha_j) h_I^{\sigma(\vec{\alpha},T)},$$

where

$$\delta_J^T = \begin{cases} 1, & \text{if } T = P^{\vec{\alpha}} \\ \widehat{b}(J), & \text{if } T = \pi_b^{\vec{\alpha}} \\ \epsilon_J & \text{if } T = T_\epsilon^{\vec{\alpha}} \end{cases},$$

$$F_j^{\beta_j} = \begin{cases} f_j^{\beta_j}, & \text{if } j \neq i \\ gf_j^{\beta_j}, & \text{if } j = i \end{cases},$$

and

$$\sigma(\vec{\alpha}, T) = \begin{cases} \sigma(\vec{\alpha}), & \text{if } T = P^{\vec{\alpha}} \text{ or } T_\epsilon^{\vec{\alpha}} \\ \sigma(\vec{\alpha}) + 1, & \text{if } T = \pi_b^{\vec{\alpha}} \end{cases}.$$

Since each h_I with $I \not\subseteq J$ is constant on J, so is $T\left(M_g^i(f_1^{\beta_1}, f_2^{\beta_2}, \ldots, f_m^{\beta_m})\right)$ for $\vec{\beta} \neq \vec{0}$. Consequently, $\sum_{\substack{\vec{\beta} \in \{0,\infty\}^m \\ \vec{\beta} \neq \vec{0}}} T\left(M_g^i(f_1^{\beta_1}, f_2^{\beta_2}, \ldots, f_m^{\beta_m})\right)$ is constant on J, say C_J. Then for every $x \in J$,

$$T\left(M_g^i(f_1, f_2, \ldots, f_m)\right)(x) - T\left(M_g^i(T)(f_1 \mathbf{1}_J, f_2 \mathbf{1}_J, \ldots, f_m \mathbf{1}_J)\right)(x) = c_J.$$

Taking $g = 1$, we see that $T(f_1, f_2, \ldots, f_m) - T(f_1 \mathbf{1}_J, f_2 \mathbf{1}_J, \ldots, f_m \mathbf{1}_J)$ is constant on J. □

Lemma 4.2.2. *Let $b \in BMO^d$, and $\epsilon = (\epsilon_I)_{I \in \mathcal{D}}$ be bounded. Let $T \in \left\{P^{\vec{\alpha}}, T_\epsilon^{\vec{\alpha}}\right\}$ with $\vec{\alpha} \in U_m$, or $T = \pi_b^{\vec{\alpha}}$ with $\vec{\alpha} \in \{0,1\}^m$. Then for $0 < \delta < \frac{1}{m}$, and $\vec{f} = (f_1, f_2, \ldots, f_m) \in$*

$L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m}$ *with* $1 \leq p_i < \infty$, *we have*

$$M_\delta^{\#}\left(T(\vec{f}\,)\right)(x) \lesssim \mathcal{M}(\vec{f}\,)(x).$$

Proof. Fix a point x. We will show that for every dyadic interval I containing x, there exists a constant c_I such that

$$\left(\frac{1}{|I|}\int_I \left|\left|T(\vec{f})(y)\right|^\delta - |c_I|^\delta\right| dy\right)^{1/\delta} \lesssim \mathcal{M}(\vec{f}\,)(x),$$

from which the assertion follows. In fact, since $\left|\left|T(\vec{f})(y)\right|^\delta - |c_I|^\delta\right| \leq \left|T(\vec{f})(y) - c_I\right|^\delta$ for $0 < \delta < 1$, it suffices to show that

$$\left(\frac{1}{|I|}\int_I \left|T(\vec{f})(y) - c_I\right|^\delta\right)^{1/\delta} \lesssim \mathcal{M}(\vec{f}\,)(x).$$

Fix a dyadic interval I that contains x, and let $f_i^0 = f1_I, f_i^\infty = f_i - f_i^0$.

Writing $\vec{f^0} = (f_i^0, \ldots, f_m^0)$, Lemma 4.2.1 says that $T(\vec{f})(y) - T(\vec{f^0})(y)$ is constant for all y in I, say c_I. We then have $T(\vec{f})(y) - c_I = T(\vec{f^0})(y)$ for all $y \in I$. So,

$$\left(\frac{1}{|I|}\int_I \left|T(\vec{f})(y) - c_I\right|^\delta\right)^{1/\delta} = \left(\frac{1}{|I|}\int_I \left|T(\vec{f^0})(y)\right|^\delta\right)^{1/\delta}.$$

We can estimate this using the following form of Kolmogorov inequality:
If $0 < p < q < \infty$, then for any measurable function f, there exists a constant $C = C(p,q)$ such that

$$\|f\|_{L^p\left(I,\frac{dy}{|I|}\right)} \leq C\,\|f\|_{L^{q,\infty}\left(I,\frac{dy}{|I|}\right)}. \tag{4.2.1}$$

For $p = \delta, q = 1/m$ and $f = T(\vec{f^0})$, (4.2.1) becomes

$$\left(\frac{1}{|I|}\int_I \left|T(\vec{f^0})(y)\right|^\delta dy\right)^{1/\delta} \leq C\left\|T(\vec{f^0})(y)\right\|_{L^{1/m,\infty}\left(I,\frac{dy}{|I|}\right)}.$$

Now,

$$\begin{aligned}
\left\|T(\vec{f}^0)(y)\right\|_{L^{1/m,\infty}\left(I,\frac{dy}{|I|}\right)} &= \sup_{t>0} t\left(\frac{1}{|I|}\left|\left\{y \in I : \left|T(\vec{f}^0)(y)\right| > t\right\}\right|\right)^m \\
&\leq \sup_{t>0} \frac{t}{|I|^m}\left|\left\{y : \frac{1}{|I|^m}\left|T(\vec{f}^0)(y)\right| > \frac{t}{|I|^m}\right\}\right|^m \\
&= \sup_{t>0} \frac{t}{|I|^m}\left|\left\{y : \left|T\left(\frac{f_1^0}{|I|}, \dots, \frac{f_m^0}{|I|}\right)(y)\right| > \frac{t}{|I|^m}\right\}\right|^m \\
&= \left\|T\left(\frac{f_1^0}{|I|}, \dots, \frac{f_m^0}{|I|}\right)(y)\right\|_{L^{1/m,\infty}}.
\end{aligned}$$

Since $\frac{f_i^0}{|I|} \in L^1$ for all $1 \leq i \leq m$, it follows from the boundedness of $T : L^1 \times \cdots \times L^1 \to L^{1/m,\infty}$ that

$$\begin{aligned}
\left\|T\left(\frac{f_1^0}{|I|}, \dots, \frac{f_m^0}{|I|}\right)(y)\right\|_{L^{1/m,\infty}} &\lesssim \prod_{i=1}^m \left\|\frac{f_i^0}{|I|}\right\|_{L^1} \\
&= \prod_{i=1}^m \int \frac{|f_i^0|}{|I|} \\
&= \prod_{i=1}^m \frac{1}{|I|}\int_I |f_i| \\
&\leq \mathcal{M}(\vec{f}\,)(x).
\end{aligned}$$

This completes the proof. □

The following lemma gives us the finiteness condition needed to apply Fefferman-Stein inequalities 1.2.1 and 1.2.2 for the multilinear dyadic operators.

Lemma 4.2.3. *Let $w \in A_\infty^d$ and $\vec{f} = (f_1, \dots, f_m)$ where each f_i is bounded and has compact support. If $\left\|\mathcal{M}(\vec{f}\,)\right\|_{L^p(w)} < \infty$ for some $p > 0$, then there exists a $\delta \in (0, 1/m)$ such that $\left\|M_\delta\left(T(\vec{f}\,)\right)\right\|_{L^p(w)} < \infty$. Similarly, if $\left\|\mathcal{M}(\vec{f}\,)\right\|_{L^{p,\infty}(w)} < \infty$ for some $p > 0$, then there exists a $\delta \in (0, 1/m)$ such that $\left\|M_\delta\left(T(\vec{f}\,)\right)\right\|_{L^{p,\infty}(w)} < \infty$.*

Proof. We prove the first assertion, the second one follows from similar arguments.

Since $w \in A_\infty^d$, it is in $A_{p_0}^d$ for some $p_0 > \max(1, pm)$. Then for any δ with $0 < \delta < p/p_0 < 1/m$, we have

$$\begin{aligned}
\left\| M_\delta \left(T(\vec{f}) \right) \right\|_{L^p(w)} &\leq \left\| M_{p/p_0} \left(T(\vec{f}) \right) \right\|_{L^p(w)} \\
&= \left[\int_{\mathbb{R}} \left\{ \left(\sup_{I \ni x} \frac{1}{|I|} \int_I |T(\vec{f})|^{p/p_0} \, dt \right)^{p_0/p} \right\}^p dw(x) \right]^{1/p} \\
&= \left[\int_{\mathbb{R}} M \left(T(\vec{f})^{p/p_0} \right)^{p_0} dw \right]^{\frac{1}{p_0} \times \frac{p_0}{p}} \\
&= \left\| M \left(T(\vec{f})^{p/p_0} \right) \right\|_{L^{p_0}(w)}^{p_0/p},
\end{aligned}$$

The boundedness of $M : L^{p_0}(w) \to L^{p_0}(w)$ for $w \in A_{p_0}^d$ gives

$$\left\| M \left(T(\vec{f})^{p/p_0} \right) \right\|_{L^{p_0}(w)} \lesssim \left\| T(\vec{f})^{p/p_0} \right\|_{L^{p_0}(w)}.$$

Consequently,

$$\begin{aligned}
\left\| M_\delta \left(T(\vec{f}) \right) \right\|_{L^p(w)} &\lesssim \left\| T(\vec{f})^{p/p_0} \right\|_{L^{p_0}(w)}^{p_0/p} \\
&= \left(\int_{\mathbb{R}} \left| T(\vec{f})^{p/p_0} \right|^{p_0} dw \right)^{\frac{1}{p_0} \times \frac{p_0}{p}} \\
&= \left(\int_{\mathbb{R}} \left| T(\vec{f}) \right|^p dw \right)^{1/p} \\
&= \left\| T(\vec{f}) \right\|_{L^p(w)},
\end{aligned}$$

So, it suffices to prove that $\left\| T(\vec{f}) \right\|_{L^p(w)} < \infty$.

Since each f_i has compact support, there exist dyadic intervals $S' = [0, 2^{-k})$ and $S'' = [-2^{-k}, 0)$ such that the support of every f_i is contained in $S = S' \cup S''$.

To prove the assertion, it suffices to show that

$$\left\|T(\vec{f}\,)\right\|_{L^p(S,w)} < \infty \ \text{ and } \ \left\|T(\vec{f}\,)\right\|_{L^p(\mathbb{R}\backslash S,w)} < \infty.$$

Since $w \in A^d_\infty$, $w^{1+\gamma} \in L^1_{loc}$ for sufficiently small γ, (see [3] or [24]). In particular, $w \in L^q(S)$ for $q := 1+\gamma$. We can choose γ small enough so that $w \in L^q(S)$ and $q'p > \frac{1}{m}$. Then by Hölder's inequality, we have

$$\begin{aligned} \left\|T(\vec{f}\,)\right\|_{L^p(S,w)} &= \left(\int_S \left|T(\vec{f}\,)\right|^p w dx\right)^{1/p} \\ &\leq \left(\left(\int_S \left|T(\vec{f}\,)\right|^{pq'} dx\right)^{1/q'} \left(\int_S w^q dx\right)^{1/q}\right)^{1/p} \\ &< \infty. \end{aligned}$$

Here, the finiteness of $\displaystyle\int_S \left|T(\vec{f}\,)\right|^{pq'} dx$ follows from the boundedness of $T : L^{mpq'} \times \cdots \times L^{mpq'} \to L^{pq'}$, and the fact that each f_i (being bounded with compact support) is in $L^{mpq'}$. We refer to [25] for the unweighted theory of multilinear dyadic operators.

To prove $\left\|T(\vec{f}\,)\right\|_{L^p(\mathbb{R}\backslash S,w)} < \infty$, it suffices to show that

$$\left|T(\vec{f}\,)(x)\right| \leq C\mathcal{M}(\vec{f}\,)(x) \quad \text{ for every } x \in \mathbb{R}\backslash S.$$

We prove this for $T = \pi_b^{\vec{\alpha}}$. Proofs for $P^{\vec{\alpha}}$ and $T_\epsilon^{\vec{\alpha}}$ follow similarly.

Fix $x \in \mathbb{R}\backslash S$. Let I_x be the smallest dyadic interval that contains x and one of the intervals S' and S''.

For definiteness, assume $x > 0$. In this case I_x is the smallest dyadic interval containing x and S'. Note that if $x \notin I$, $h_I(x) = 0$ and, if $x \in I$ with $I \cap S' = \emptyset$, $f_j(I, \alpha_j) = 0$ for each j. So,

$$\begin{aligned}
\left|\pi_b^{\vec{\alpha}}(\vec{f}\,)(x)\right| &= \left|\sum_{I\in\mathcal{D}} \widehat{b}(I) \prod_{j=1}^{m} f_j(I,\alpha_j)\; h_I^{1+\sigma(\vec{\alpha})}(x)\right| \\
&= \left|\sum_{I\supseteq I_x} \widehat{b}(I) \prod_{j=1}^{m} f_j(I,\alpha_j)\; h_I^{1+\sigma(\vec{\alpha})}(x)\right| \\
&\leq \sum_{I\supseteq I_x} \frac{\left|\widehat{b}(I)\right|}{\sqrt{|I|}} \left(\prod_{j:\alpha_j=0} \frac{\left|\widehat{f_j}(I)\right|}{\sqrt{|I|}}\right) \left(\prod_{j:\alpha_j=1} |\langle f_j\rangle_I|\right)\; \mathsf{1}_I(x) \\
&\leq \|b\|_{BMO^d} \sum_{I\supseteq I_x} \left(\prod_{j:\alpha_j=0} \frac{\left|\widehat{f_j}(I)\right|}{\sqrt{|I|}}\right) \left(\prod_{j:\alpha_j=1} |\langle f_j\rangle_I|\right),
\end{aligned}$$

where the last inequality follows from the fact that for $b \in BMO^d$,

$$\frac{\left|\widehat{b}(I)\right|}{\sqrt{|I|}} \leq \left(\frac{1}{|I|}\sum_{J\subseteq I} \left|\widehat{b}(I)\right|^2\right)^{1/2} \leq \|b\|_{BMO^d}\,.$$

Note that $\dfrac{\left|\widehat{f_j}(I)\right|}{\sqrt{|I|}} = \dfrac{1}{\sqrt{|I|}}\left|\displaystyle\int f_j h_I\right| \leq \dfrac{1}{\sqrt{|I|}}\displaystyle\int |f_j|\,\dfrac{\mathsf{1}_I}{\sqrt{|I|}} = \dfrac{1}{|I|}\displaystyle\int_I |f_j| = \langle |f_j|\rangle_I$, and since f_j is 0 on $\mathbb{R}\backslash S$, we have $\langle |f_j|\rangle_{I^1} = \dfrac{\langle |f_j|\rangle_I}{2}$ whenever I^1 is the parent of I with $I_x \subseteq I$. So, we have

$$\begin{aligned}
\left|\pi_b^{\vec{\alpha}}(\vec{f}\,)(x)\right| &\leq \|b\|_{BMO^d} \sum_{I\supseteq I_x} \prod_{j=1}^{m} \langle |f_j|\rangle_I \\
&= \|b\|_{BMO^d} \left(\prod_{j=1}^{m}\langle |f_j|\rangle_{I_x} + \frac{1}{2^m}\prod_{j=1}^{m}\langle |f_j|\rangle_{I_x} + \frac{1}{2^{2m}}\prod_{j=1}^{m}\langle |f_j|\rangle_{I_x} + \cdots\right) \\
&= \frac{2^m}{(2^m-1)}\,\|b\|_{BMO^d} \prod_{j=1}^{m}\langle |f_j|\rangle_{I_x} \\
&\leq \frac{2^m}{(2^m-1)}\,\|b\|_{BMO^d}\, \mathcal{M}(\vec{f}\,)(x).
\end{aligned}$$

The same proof works for $x < 0$ too. This completes the proof. □

Theorem 4.2.4. *Let $b \in BMO^d$, and $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ be bounded. Let $T \in \left\{P^{\vec{\alpha}}, T_\epsilon^{\vec{\alpha}}\right\}$ with $\vec{\alpha} \in U_m$, or $T = \pi_b^{\vec{\alpha}}$ with $\vec{\alpha} \in \{0,1\}^m$. Then for $w \in A_\infty^d$ and $p > 0$,*

$$\|T(\vec{f})\|_{L^p(w)} \lesssim \|\mathcal{M}(\vec{f})\|_{L^p(w)}$$

and

$$\|T(\vec{f})\|_{L^{p,\infty}(w)} \lesssim \|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(w)}$$

for all m-tuples $\vec{f} = (f_1, \ldots, f_m)$ of bounded functions with compact support.

Proof. To prove the first inequality, assume that $\|\mathcal{M}(\vec{f})\|_{L^p(w)} < \infty$, otherwise there is nothing to prove. Then by Lemma 4.2.3, there exists a $\delta \in (0, 1/m)$ such that

$$\left\|M_\delta\left(T(\vec{f})\right)\right\|_{L^p(w)} < \infty.$$

For such δ, we have

$$\left\|T(\vec{f})\right\|_{L^p(w)} \le \left\|M_\delta\left(T(\vec{f})\right)\right\|_{L^p(w)} \le C\left\|M_\delta^{\#}\left(T(\vec{f})\right)\right\|_{L^p(w)} \le C\left\|\mathcal{M}(\vec{f})\right\|_{L^p(w)},$$

where the first and last inequalities follow from pointwise control and the second inequality is the Fefferman-Stein's inequality (1.2.1).

Proof of the second inequality follows similarly, by applying Lemma 4.2.3 and using the Fefferman-Stein's inequality (1.2.2) for weak-type estimates. $\square$

Theorem 4.2.5. *Let $b \in BMO^d$, and $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ be bounded. Suppose $T \in \left\{P^{\vec{\alpha}}, T_\epsilon^{\vec{\alpha}}\right\}$ with $\vec{\alpha} \in U_m$, or $T = \pi_b^{\vec{\alpha}}$ with $\vec{\alpha} \in \{0,1\}^m$. Let $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}^d$ for $\vec{P} = (p_1, \ldots, p_m)$ with*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}.$$

(a) *If* $1 < p_1, \ldots, p_m < \infty$, *then*

$$\|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}. \tag{4.2.2}$$

(b) *If* $1 \leq p_1, \ldots, p_m < \infty$, *then*

$$\|T(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}. \tag{4.2.3}$$

Proof. Since the simple functions in $L^p(w)$ are dense in $L^p(w)$ for any weight w (see [26]), it suffices to prove the estimates for $\vec{f} = (f_1, f_2, \ldots, f_m)$ with $f_i \in L^{p_i}(w_i)$ simple. Note that $\vec{w} = (w_1, \ldots, w_m) \in A^d_{\vec{P}}$ implies that $\nu_{\vec{w}} \in A^d_\infty$. So, by Theorem 4.2.4 and the boundedness properties of the multilinear maximal function $\mathcal{M}$, we have

$$\|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)},$$

and

$$\|T(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \lesssim \|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}.$$

□

4.3 Commutators of Multilinear Haar Multipliers

Definition 4.3.1. *Let $\vec{\alpha} \in U_m$ and $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ be bounded. Given a locally integrable function b, we define the commutator $[b, T_\epsilon^{\vec{\alpha}}]_i, 1 \leq i \leq m$, by*

$$[b, T_\epsilon^{\vec{\alpha}}]_i(f_1, f_2, \ldots, f_m)(x) := b(x)T_\epsilon^{\vec{\alpha}}(f_1, f_2, \ldots, f_m)(x) - T_\epsilon^{\vec{\alpha}}(f_1, \ldots, bf_i, \ldots, f_m)(x).$$

$$i.e. \quad [b, T_\epsilon^{\vec{\alpha}}]_i = M_b \circ T_\epsilon^{\vec{\alpha}} - T_\epsilon^{\vec{\alpha}} \circ M_b^i.$$

Theorem 4.3.1. *Let $\vec{\alpha} \in U_m$ and $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ be bounded. Let $\delta \in (0, 1/m)$ and $\gamma > \delta$.*

Then for any $r > 1$,

$$M_\delta^\# \left([b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})\right)(x) \lesssim \|b\|_{BMO^d} \left(\mathcal{M}_r(\vec{f})(x) + M_\gamma \left(T_\epsilon^{\vec{\alpha}}(\vec{f})\right)(x)\right) \tag{4.3.1}$$

for all m*-tuples* $\vec{f} = (f_1, f_2, \ldots, f_m)$ *of bounded measurable functions with compact support.*

Proof. Fix $x \in \mathbb{R}$. As in the proof of Lemma 4.2.2, it suffices to show that for every $I \in \mathcal{D}$ containing x, there exists a constant C_I such that

$$\left(\frac{1}{|I|}\int_I \left|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(t) - C_I\right|^\delta dt\right)^{1/\delta} \lesssim \|b\|_{BMO^d} \left(\mathcal{M}_r(\vec{f})(x) + M_\gamma \left(T_\epsilon^{\vec{\alpha}}(\vec{f})\right)(x)\right).$$

Fix $I \in \mathcal{D}$ containing x, and take $C_I = T_\epsilon^{\vec{\alpha}}\left(M_g^i(\vec{f^0})\right)(t) - T_\epsilon^{\vec{\alpha}}\left(M_g^i(\vec{f})\right)(t)$, where $g = b - \langle b\rangle_I$ and $\vec{f^0} = (f_1^0, \ldots, f_m^0)$ with $f_i^0 = f_i 1_I$. Lemma 4.2.1 shows that this is indeed a constant on I. Since $T_\epsilon^{\vec{\alpha}}$ is multilinear,

$$\begin{aligned}
[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(t) &= b(t)\, T_\epsilon^{\vec{\alpha}}(\vec{f})(t) - T_\epsilon^{\vec{\alpha}}(f_1, \ldots, bf_i, \ldots, f_m)(t) \\
&= (b(t) - \langle b\rangle_I)\ T_\epsilon^{\vec{\alpha}}(\vec{f})(t) - T_\epsilon^{\vec{\alpha}}(f_1, \ldots, (b - \langle b\rangle_I)f_i, \ldots, f_m)(t) \\
&= (b(t) - \langle b\rangle_I)\ T_\epsilon^{\vec{\alpha}}(\vec{f})(t) - T_\epsilon^{\vec{\alpha}}\left(M_g^i(\vec{f})\right)(t).
\end{aligned}$$

So,

$$\begin{aligned}
&\left(\frac{1}{|I|}\int_I \left|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(t) - C_I\right|^\delta dt\right)^{1/\delta} \\
=\ &\left(\frac{1}{|I|}\int_I \left|(b(t) - \langle b\rangle_I)\ T_\epsilon^{\vec{\alpha}}(\vec{f})(t) - T_\epsilon^{\vec{\alpha}}\left(M_g^i(\vec{f})\right)(t) - C_I\right|^\delta dt\right)^{1/\delta} \\
=\ &\left(\frac{1}{|I|}\int_I \left|(b(t) - \langle b\rangle_I)\ T_\epsilon^{\vec{\alpha}}(\vec{f})(t) - T_\epsilon^{\vec{\alpha}}\left(M_g^i(\vec{f^0})\right)(t)\right|^\delta dt\right)^{1/\delta} \\
\lesssim\ &\left(\frac{1}{|I|}\int_I \left|(b(t) - \langle b\rangle_I)\ T_\epsilon^{\vec{\alpha}}(\vec{f})(t)\right|^\delta dt\right)^{1/\delta} + \left(\frac{1}{|I|}\int_I \left|T_\epsilon^{\vec{\alpha}}\left(M_g^i(\vec{f^0})\right)(t)\right|^\delta dt\right)^{1/\delta}.
\end{aligned}$$

Note that $\gamma/\delta > 1$. For any $q \in (1, \gamma/\delta)$, Hölder's inequlity gives

$$\begin{aligned}
&\left(\frac{1}{|I|}\int_I \left|(b(t)-\langle b\rangle_I)\; T_\epsilon^{\vec{\alpha}}(\vec{f})(t)\right|^\delta dt\right)^{1/\delta} \\
\leq\;& \left(\frac{1}{|I|}\int_I |(b(t)-\langle b\rangle_I)|^{\delta q'} dt\right)^{1/\delta q'} \left(\frac{1}{|I|}\int_I \left|T_\epsilon^{\vec{\alpha}}(\vec{f})(t)\right|^{\delta q} dt\right)^{1/\delta q} \\
\lesssim\;& \|b\|_{BMO^d}\, M_{\delta q}\left(T_\epsilon^{\vec{\alpha}}(\vec{f})\right)(x) \\
\leq\;& \|b\|_{BMO^d}\, M_{\gamma}\left(T_\epsilon^{\vec{\alpha}}(\vec{f})\right)(x).
\end{aligned}$$

As in the proof of Lemma 4.2.2, we can apply Kolmogorov's inequality to obtain

$$\begin{aligned}
&\left(\frac{1}{|I|}\int_I \left|T_\epsilon^{\vec{\alpha}}(f_1^0,\ldots,(b-\langle b\rangle_I)f_i^0,\ldots,f_m^0)(t)\right|^\delta dt\right)^{1/\delta} \\
\leq\;& \left\|T_\epsilon^{\vec{\alpha}}(f_1^0,\ldots,(b-\langle b\rangle_I)f_i^0,\ldots,f_m^0)(t)\right\|_{L^{\frac{1}{m},\infty}\left(I,\frac{dt}{|I|}\right)} \\
\leq\;& \frac{1}{|I|}\int_I \left|(b(t)-\langle b\rangle_I)\; f_i^0(t)\right| dt \prod_{j=1,j\neq i}^m \frac{1}{|I|}\int_I \left|f_j^0(t)\right| dt \\
\leq\;& \left(\frac{1}{|I|}\int_I |b(t)-\langle b\rangle_I|^{r'} dt\right)^{1/r'} \left(\frac{1}{|I|}\int_I \left|f_i^0(t)\right|^r dt\right)^{1/r} \prod_{j=1,j\neq i}^m \left(\frac{1}{|I|}\int_I \left|f_j^0(t)\right|^r dt\right)^{1/r} \\
\lesssim\;& \|b\|_{BMO^d} \prod_{j=1}^m \left(\frac{1}{|I|}\int_I |f_j(t)|^r dt\right)^{1/r} \\
\leq\;& \|b\|_{BMO^d}\, \mathcal{M}_r(\vec{f})(x).
\end{aligned}$$

We thus have

$$M_\delta^{\#}\left([b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})\right)(x) \lesssim \|b\|_{BMO^d}\left(\mathcal{M}_r(\vec{f})(x) + M_\gamma\left(T_\epsilon^{\vec{\alpha}}(\vec{f})\right)(x)\right).$$

□

Lemma 4.2.3 is also true for the commutators of the multilinear Haar multipliers with a bounded function b.

Lemma 4.3.2. *Let $w \in A_\infty^d$ and $\vec{f} = (f_1, \ldots, f_m)$ where each f_i is bounded and has* *compact support. If* $\left\|\mathcal{M}(\vec{f}\,)\right\|_{L^p(w)} < \infty$ *for some* $p > 0$, *and* b *bounded, then there exists a* $\delta \in (0, 1/m)$ *such that* $\left\|M_\delta\left([b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}\,)\right)\right\|_{L^p(w)} < \infty$.

Proof. Since each f_i has compact support, there exist dyadic intervals $S' = [0, 2^{-k})$ and $S'' = [-2^{-k}, 0)$ such that the support of every f_i is contained in $S = S' \cup S''$.

Following the arguments used in the proof of Lemma 4.2.3, we get

$$\left\|M_\delta\left([b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}\,)\right)\right\|_{L^p(w)} \leq \left\|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}\,)\right\|_{L^p(w)}.$$

So, it suffices to prove that

$$\left\|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}\,)\right\|_{L^p(S,w)} < \infty \text{ and } \left\|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}\,)\right\|_{L^p(\mathbb{R}\setminus S,w)} < \infty.$$

Since $w \in A_\infty^d$, $w^{1+\gamma} \in L^1_{loc}$ for sufficiently small γ, (see [3] or [24]). In particular, $w \in L^q(S)$ for $q := 1 + \gamma$. We can choose γ small enough so that $w \in L^q(S)$ and $q'p > 1$. Then by Hölder's inequality, we have

$$\begin{aligned}
\left\|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})\right\|_{L^p(S,w)} &= \left(\int_S \left|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})\right|^p w dx\right)^{1/p} \\
&\leq \left(\left(\int_S \left|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})\right|^{pq'} dx\right)^{1/q'} \left(\int_S w^q dx\right)^{1/q}\right)^{1/p} \\
&< \infty.
\end{aligned}$$

Here, $\int_S w^q dx < \infty$ because $w \in L^q_{loc}$, and the finiteness of $\int_S \left|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})\right|^{pq'} dx$ follows from boundedness of $[b, T_\epsilon^{\vec{\alpha}}]_i : L^{mpq'} \times \cdots \times L^{mpq'} \to L^{pq'}$, and the fact that each f_i (being bounded with compact support) is in $L^{mpq'}$. For the unweighted theory of the commutators of multilinear Haar multipliers we refer to [25]. Note that to prove finiteness of $\left\|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})\right\|_{L^p(S,w)}$ we may assume that the BMO function b is in some L^p space with $1 < p < \infty$. Indeed, for all $x \in S$,

$$[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x) = [b\mathbf{1}_S, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x),$$

for all $\vec{f} = (f_1, \dots, f_m)$ with f_i supported in S.

Now to prove $\left\| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}) \right\|_{L^p(\mathbb{R}\setminus S, w)} < \infty$, it suffices to show that for every $x \in \mathbb{R}\setminus S$,

$$\left| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x) \right| \le \mathcal{M}(\vec{f})(x).$$

Fix $x \in \mathbb{R}\setminus S$. For definiteness, assume that $x > 0$, and let I_x be the smallest dyadic interval that contains x and the interval S'. Note that if $x \notin I$, $h_I(x) = 0$ and, if $x \in I$ with $I \cap S' = \emptyset$, $f_j(I, \alpha_j) = 0$ for each j. So,

$$\begin{aligned}
& \left| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x) \right| \\
\le \;& |b(x)| \left| T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m)(x) \right| + \left| T_\epsilon^{\vec{\alpha}}(f_1, \dots, bf_i, \dots, f_m)(x) \right| \\
= \;& |b(x)| \left| \sum_{I \supseteq I_x} \epsilon_I \prod_{j=1}^{m} f_j(I, \alpha_j)\, h_I^{\sigma(\vec{\alpha})}(x) \right| + \left| \sum_{I \supseteq I_x} \epsilon_I (bf_i)(I, \alpha_i) \prod_{\substack{j=1 \\ j \ne i}}^{m} f_j(I, \alpha_j)\, h_I^{\sigma(\vec{\alpha})}(x) \right| \\
\le \;& |b(x)| \sum_{I \supseteq I_x} |\epsilon_I| \left(\prod_{j:\alpha_j = 0} \frac{\left| \widehat{f_j}(I) \right|}{\sqrt{|I|}} \right) \left(\prod_{j:\alpha_j = 1} |\langle f_j \rangle_I| \right) \mathbf{1}_I(x) \\
& + |b(x)| \sum_{I \supseteq I_x} |\epsilon_I| |(bf_i)(I, \alpha_i)| \left(\prod_{\substack{j:\alpha_j = 0 \\ j \ne i}} \frac{\left| \widehat{f_j}(I) \right|}{\sqrt{|I|}} \right) \left(\prod_{\substack{j:\alpha_j = 1 \\ j \ne i}} |\langle f_j \rangle_I| \right) \mathbf{1}_I(x)
\end{aligned}$$

We have $\dfrac{\left| \widehat{f_j}(I) \right|}{\sqrt{|I|}} = \dfrac{1}{\sqrt{|I|}} \left| \displaystyle\int f_j h_I \right| \le \dfrac{1}{\sqrt{|I|}} \displaystyle\int |f_j| \, \dfrac{\mathbf{1}_I}{\sqrt{|I|}} = \dfrac{1}{|I|} \displaystyle\int_I |f_j| = \langle |f_j| \rangle_I$. Since f_j is 0 on $\mathbb{R}\setminus S$, $\langle |f_j| \rangle_{I^1} = \dfrac{\langle |f_j| \rangle_I}{2}$ whenever I^1 is the parent of I with $I_x \subseteq I$. Moreover, $|(bf_i)(I, \alpha_i)| \le |b| \langle |f_i| \rangle_I$. So,

$$
\begin{aligned}
&\left|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x)\right| \\
\leq\ & 2\left(\sup_{x\in\mathbb{R}}|b(x)|\right)\left(\sup_{I\in\mathcal{D}}|\epsilon_I|\right)\sum_{I\supseteq I_x}\left(\prod_{j:\alpha_j=0}\frac{\left|\widehat{f}_j(I)\right|}{\sqrt{|I|}}\right)\left(\prod_{j:\alpha_j=1}|\langle f_j\rangle_I|\right). \\
\leq\ & 2\left(\sup_{x\in\mathbb{R}}|b(x)|\right)\left(\sup_{I\in\mathcal{D}}|\epsilon_I|\right)\sum_{I\supseteq I_x}\prod_{j=1}^{m}\langle|f_j|\rangle_I \\
=\ & 2\left(\sup_{x\in\mathbb{R}}|b(x)|\right)\left(\sup_{I\in\mathcal{D}}|\epsilon_I|\right)\left(\prod_{j=1}^{m}\langle|f_j|\rangle_{I_x}+\frac{1}{2^m}\prod_{j=1}^{m}\langle|f_j|\rangle_{I_x}+\frac{1}{2^{2m}}\prod_{j=1}^{m}\langle|f_j|\rangle_{I_x}+\cdots\right) \\
=\ & 2\left(\frac{2^m}{2^m-1}\right)\left(\sup_{x\in\mathbb{R}}|b(x)|\right)\left(\sup_{I\in\mathcal{D}}|\epsilon_I|\right)\prod_{j=1}^{m}\langle|f_j|\rangle_{I_x} \\
\leq\ & \frac{2^{m+1}}{(2^m-1)}\left(\sup_{x\in\mathbb{R}}|b(x)|\right)\left(\sup_{I\in\mathcal{D}}|\epsilon_I|\right)\mathcal{M}(\vec{f}\,)(x).
\end{aligned}
$$

The same proof works for $x < 0$ with I_x the smallest dyadic interval that contains both x and the interval S''.

□

Theorem 4.3.3. *Let $\vec{\alpha} \in U_m$ and $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ be bounded. Suppose $b \in BMO^d$ and $\vec{w} = (w_1, \ldots, w_m) \in A^d_{\vec{P}}$ for $\vec{P} = (p_1, \ldots, p_m)$ with $\frac{1}{p_1}+\cdots+\frac{1}{p_m} = \frac{1}{p}$ and $1 < p_1, \ldots, p_m < \infty$. Then there exists a constant C such that*

$$
\left\|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} \leq C\|b\|_{BMO^d}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(w_j)}. \tag{4.3.2}
$$

Proof. First assume that b is bounded.

Since the simple functions in $L^p(\nu_{\vec{w}})$ are dense in $L^p(\nu_{\vec{w}})$, it suffices to prove (4.3.2) for $\vec{f} = (f_1, f_2, \ldots, f_m)$ with $f_i \in L^{p_i}(w_i)$ simple. For all such $\vec{f}$, there exists, by Lemma 4.3.2, a $\delta \in (0, 1/m)$ such that $\left\|M_\delta\left([b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}\,)\right)\right\|_{L^p(w)} < \infty$. So, for any $r > 1$ and

$\gamma > \delta$ we have

$$\begin{aligned}
\left\|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} &\leq \left\|M_\delta[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} \\
&\lesssim \left\|M_\delta^{\#}[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} \\
&\lesssim \|b\|_{BMO^d}\left(\left\|\mathcal{M}_r(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} + \left\|M_\gamma\left(T_\epsilon^{\vec{\alpha}}(\vec{f})\right)\right\|_{L^p(\nu_{\vec{w}})}\right),
\end{aligned}$$

where the first inequality follows from the pointwise control, the second one is the Fefferman-Stein inequality (1.2.1) and the last inequality follows from Theorem 4.3.1.

Now we can choose $\gamma \in (\delta, 1/m)$ such that $\left\|M_\gamma\left(T_\epsilon^{\vec{\alpha}}(\vec{f})\right)\right\|_{L^p(\nu_{\vec{w}})} < \infty$. In fact, looking at the proofs of Lemmas 4.2.3 and 4.3.2, any $\gamma \in (\delta, p/p_0)$ would work. For such γ, we have

$$\begin{aligned}
\left\|M_\gamma\left(T_\epsilon^{\vec{\alpha}}(\vec{f})\right)\right\|_{L^p(\nu_{\vec{w}})} &\lesssim \left\|M_\gamma^{\#}\left(T_\epsilon^{\vec{\alpha}}(\vec{f})\right)\right\|_{L^p(\nu_{\vec{w}})} \\
&\leq \left\|\mathcal{M}(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} \\
&\leq \left\|\mathcal{M}_r(\vec{f})\right\|_{L^p(\nu_{\vec{w}})}
\end{aligned}$$

We thus have

$$\left\|[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} \lesssim \|b\|_{BMO^d}\left\|\mathcal{M}_r(\vec{f})\right\|_{L^p(\nu_{\vec{w}})}$$

for all $r > 1$.

Finally, we can choose $r > 1$ such that the inequality (1.2.5) holds, i.e.

$$\left\|\mathcal{M}_r(\vec{f})\right\|_{L^p(\nu_{\vec{w}})} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

This completes the proof when b is bounded.

Now following [7], we use a limiting argument to prove the theorem for general $b \in BMO^d$.

Let $\{b_j\}$ be the sequence of functions defined by

$$b_j(x) = \begin{cases} j, & \text{if } b(x) > j, \\ b(x), & \text{if } |b(x)| \leq j, \\ -j & \text{if } b(x) < -j. \end{cases}$$

Clearly, $b_j \to b$ pointwise, and we have $\|b_j\|_{BMO^d} \leq c\|b\|_{BMO^d}$ for all j. In fact, c = 9/4 works (see [24], page 129).

For any $q \in (1, \infty)$,

$$T_\epsilon^{\vec{\alpha}}(f_1, \ldots, b_j f_i, \ldots, f_m) \to T_\epsilon^{\vec{\alpha}}(f_1, \ldots, bf_i, \ldots, f_m) \quad \text{in } L^q \text{ as } j \to \infty$$

due to boundedness of $T_\epsilon^{\vec{\alpha}} : L^{mq} \times \cdots \times L^{mq} \to L^q$ and the fact that bounded functions $f_1, \ldots, f_m$ with compact support are all in L^{mq}. Note that since $b_j, b \in BMO^d$ and bounded function f_i has compact support $b_j f_i \to bf_i$ in L^{mq} as $j \to \infty$. Then there exists a subsequence $\{b_{j_k}\}$ such that

$$T_\epsilon^{\vec{\alpha}}(f_1, \ldots, b_{j_k} f_i, \ldots, f_m)(x) \to T_\epsilon^{\vec{\alpha}}(f_1, \ldots, bf_i, \ldots, f_m)(x) \quad \text{for almost every } x.$$

For such x, we have $[b_{j_k}, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x) \to [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x)$. Now,

$$\begin{aligned} \left\| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}) \right\|_{L^p(\nu_{\vec{w}})} &= \left(\int_{\mathbb{R}} \left| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x) \right|^p dx \right)^{1/p} \\ &\leq \liminf_{k \to \infty} \left(\int_{\mathbb{R}} \left| [b_{j_k}, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x) \right|^p dx \right)^{1/p} \\ &\leq C' \liminf_{k \to \infty} \|b_{j_k}\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)} \\ &\leq C\|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}, \end{aligned}$$

where we have used Fatou's lemma to obtain the first inequality, and the second inequality follows from the result already proved for bounded function b. $\square$

The following theorem characterizes dyadic BMO functions via the boundedness of the commutators of multilinear dyadic paraproducts.

Theorem 4.3.4. *Assume* $\sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{p}$ *with* $1 < p_i < \infty$, *and let* $\vec{w} = (w_1, \ldots, w_m)$ *with* $w_i \in A_{p_i}$. *Then for* $j \in \{1, \ldots, m\}$ *and* $\vec{\alpha} \in U_m$, *the following two statements are equivalent.*

(a) $b \in BMO$.

(b) $[b, P^{\vec{\alpha}}]_j : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \to L^p(\nu_{\vec{w}})$ *is bounded.*

Proof. It suffices to prove the theorem for $j = 1$.

"$(a) \Rightarrow (b)$" follows from Theorem 4.3.3, since $T_{\epsilon}^{\vec{\alpha}} = P^{\vec{\alpha}}$ when $\epsilon_I = 1$ for all $I \in \mathcal{D}$, and that $BMO(\nu_1) = BMO$ for $\nu_1 = 1$.

To prove the converse, assume that $[b, P^{\vec{\alpha}}]_1 : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \to L^p(\nu_{\vec{w}})$ is bounded, and fix $J \in \mathcal{D}$.

Case I: $\vec{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_m) = (0, 1, ..., 1)$.

Let J' be the parent of J. Take $f_1 = 1_J$ and $f_2 = \cdots = f_m = \sqrt{|J'|}h_{J'}$. Then for $\vec{f} = (f_1, \ldots, f_m)$,

$$\begin{aligned}
|[b, P^{\vec{\alpha}}]_1(\vec{f})| &= \left| bP^{\vec{\alpha}}(\vec{f}) - P^{\vec{\alpha}}(bf_1, f_2, \ldots, f_m) \right| \\
&= \left| 0 - \sum_{I \in \mathcal{D}} \langle b 1_J, h_I \rangle \langle \sqrt{|J'|} h_{J'} \rangle_I^{m-1} h_I \right| \\
&= \left| \sum_{I \subseteq J} \langle b 1_J, h_I \rangle h_I \right| \\
&= |b - \langle b \rangle_J| \, 1_J.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_J |b-\langle b\rangle_J|^{\frac{1}{m}}\, dx &= \int_J |b-\langle b\rangle_J|^{\frac{1}{m}}\, \nu_{\vec{w}}^{\frac{1}{mp}} \nu_{\vec{w}}^{-\frac{1}{mp}} dx \\
&\leq \left(\int_J |b-\langle b\rangle_J|^p\, \nu_{\vec{w}} dx\right)^{\frac{1}{mp}} \left(\int_J \nu_{\vec{w}}^{-\frac{(mp)'}{mp}}\, dx\right)^{\frac{1}{(mp)'}} \\
&= \left\| [b,P^{\vec{\alpha}}]_1(\vec{f}) \right\|_{L^p(\nu_{\vec{w}})}^{1/m} \left(\int_J \nu_{\vec{w}}^{\frac{1}{1-mp}} dx\right)^{\frac{mp-1}{mp}} \\
&\lesssim \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}\right)^{1/m} \left(\int_J \nu_{\vec{w}}^{\frac{1}{1-mp}} dx\right)^{\frac{mp-1}{mp}} \\
&= \left(\int_J \nu_{\vec{w}}^{\frac{1}{1-mp}} dx\right)^{\frac{mp-1}{mp}} \left\{ \left(\int_J w_1 dx\right)^{1/p_1} \prod_{i=2}^m \left(\int_{J'} w_i dx\right)^{1/p_i} \right\}^{1/m}.
\end{aligned}$$

So,

$$\int_J |b-\langle b\rangle_J|^{\frac{1}{m}}\, dx \lesssim \left(\int_{J'} \nu_{\vec{w}}^{\frac{1}{1-mp}} dx\right)^{\frac{mp-1}{mp}} \prod_{i=1}^m \left(\int_{J'} w_i dx\right)^{\frac{1}{mp_i}}. \tag{4.3.3}$$

Let $w_i' := w_i^{1-p_i'} = w_i^{-\frac{p_i'}{p_i}}$, $\vec{w}' := (w_1', \ldots, w_m')$, and $\vec{P}' := (p_1', \ldots, p_m')$. Since $w_i' \in A_{p_i'}$, $\vec{w}'$ satisfies the multilinear $A_{\vec{P}'}$ condition. Observing that $\sum_{i=1}^m \frac{1}{p_i'} = m - \sum_{i=1}^m \frac{1}{p_i} = m - \frac{1}{p} = \frac{mp-1}{p}$, we therefore have

$$\left(\frac{1}{|J'|}\int_{J'} \nu_{\vec{w}'}\right) \prod_{i=1}^m \left(\frac{1}{|J'|}\int_{J'} (w_i')^{1-p_i}\right)^{\frac{p/(mp-1)}{p_i}} \leq [\vec{w}']_{A_{\vec{P}'}} < \infty. \tag{4.3.4}$$

Note that

$$\nu_{\vec{w}'} = \prod_{i=1}^m \left(w_i^{-\frac{p_i'}{p_i}}\right)^{\frac{p/(mp-1)}{p_i'}} = \prod_{i=1}^m \left(w_i^{\frac{p}{p_i}}\right)^{\frac{1}{1-mp}} = \nu_{\vec{w}}^{\frac{1}{1-mp}},$$

and

$$(w_i')^{1-p_i} = \left(w_i^{-\frac{p_i'}{p_i}}\right)^{-\frac{p_i}{p_i'}} = w_i.$$

So, from (4.3.4), we get

$$\left(\frac{1}{|J'|}\int_{J'} \nu_{\vec{w}}^{\frac{1}{1-mp}}\right) \prod_{i=1}^m \left(\frac{1}{|J'|}\int_{J'} w_i\right)^{\frac{p/(mp-1)}{p_i}} \leq [\vec{w}']_{A_{\vec{P}'}}.$$

This implies that

$$\begin{aligned}
\left(\int_{J'} \nu_{\vec{w}}^{\frac{1}{1-mp}}\right) \prod_{i=1}^{m} \left(\int_{J'} w_i\right)^{\frac{p/(mp-1)}{p_i}} &\leq [\vec{w}']_{A_{\vec{P}'}} |J'|^{1+\frac{p}{mp-1}(\frac{1}{p_1}+\cdots+\frac{1}{p_m})} \\
&= [\vec{w}']_{A_{\vec{P}'}} |J'|^{1+\frac{1}{mp-1}} \\
&= [\vec{w}']_{A_{\vec{P}'}} |J'|^{\frac{mp}{mp-1}}.
\end{aligned}$$

Consequently,

$$\left(\int_{J'} \nu_{\vec{w}}^{\frac{1}{1-mp}} dx\right)^{\frac{mp-1}{mp}} \prod_{i=1}^{m} \left(\int_{J'} w_i dx\right)^{\frac{1}{mp_i}} \leq [\vec{w}']_{A_{\vec{P}'}}^{\frac{mp-1}{mp}} |J'| = 2[\vec{w}']_{A_{\vec{P}'}}^{\frac{mp-1}{mp}} |J|.$$

Using this in (4.3.3), we get

$$\int_J |b - \langle b \rangle_J|^{\frac{1}{m}} \, dx \lesssim [\vec{w}']_{A_{\vec{P}'}}^{\frac{mp-1}{mp}} |J|.$$

i.e.

$$\left(\frac{1}{|J|} \int_J |b - \langle b \rangle_J|^{\frac{1}{m}} \, dx\right)^m \lesssim [\vec{w}']_{A_{\vec{P}'}}^{\frac{mp-1}{p}}.$$

Since $J \in \mathcal{D}$ is arbitrary, this proves that $b \in BMO$.

Case II: $\vec{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_m) \neq (0, 1, ..., 1)$.

In this case, $\alpha_i = 0$ for some $i > 1$. Let

$$f_i = \begin{cases} \sqrt{|J|} h_J, & \text{if } \alpha_i = 0 \\ \mathsf{1}_J, & \text{if } \alpha_i = 1 \end{cases}.$$

Note that if $\alpha_i = 0$, $f_i(J, \alpha_i) = \langle \sqrt{|J|} h_J, h_J \rangle = \sqrt{|J|}$, and if $\alpha_i = 1$, $f_i(J, \alpha_i) = \langle \mathsf{1}_J \rangle_J = 1$.

Also, if $\alpha_1 = 0$,

$$(bf_1)(J,\alpha_1) = \sqrt{|J|}\int_{\mathbb{R}} bh_J h_J = \sqrt{|J|}\frac{1}{|J|}\int_{\mathbb{R}} b\mathbb{1}_J = \sqrt{|J|}\langle b\rangle_J,$$

and if $\alpha_1 = 1$,

$$(bf_1)(J,\alpha_1) = \langle b\mathbb{1}_J\rangle_J = \langle b\rangle_J.$$

So we have,

$$\begin{aligned}
|[b, P^{\vec{\alpha}}]_1(\vec{f})| &= \left|bP^{\vec{\alpha}}(\vec{f}) - P^{\vec{\alpha}}(bf_1, f_2, \ldots, f_m)\right| \\
&= \left|b(\sqrt{|J|}h_J)^{\sigma(\vec{\alpha})} - (bf_1)(J,\alpha_1)\left(\prod_{i=2}^m f_i(J,\alpha_i)\right) h_J^{\sigma(\vec{\alpha})}\right| \\
&= \left|b(\sqrt{|J|}h_J)^{\sigma(\vec{\alpha})} - \langle b\rangle_J(\sqrt{|J|}h_J)^{\sigma(\vec{\alpha})}\right| \\
&= |b - \langle b\rangle_J|\,\mathbb{1}_J.
\end{aligned}$$

Proceeding as in the first case, we get

$$\begin{aligned}
\int_J |b - \langle b\rangle_J|^{\frac{1}{m}}\,dx &\lesssim \left(\int_J \nu_{\vec{w}}^{\frac{1}{1-mp}}\,dx\right)^{\frac{mp-1}{mp}} \prod_{i=1}^m \left(\int_J w_i dx\right)^{\frac{1}{mp_i}} \\
&\lesssim [\vec{w}']_{A_{\vec{P}'}}^{\frac{mp-1}{mp}}|J|,
\end{aligned}$$

which implies that $b \in BMO$. This completes the proof. $\square$

Some Remarks:

1. In the previous chapter, we presented the unweighted theory of the multilinear commutators with some restrictions, where we required that $b \in L^q$ for some $q \in (1,\infty)$ and that $p > 1$. As we have seen, this restricted unweighted theory was sufficient to obtain the weighted theory presented in this chapter. Taking $w_i = 1$ for all $1 \leq i \leq m$, we see that the weighted theory implies the unweighted theory for all $b \in BMO^d$ and $1/m < p < \infty$.

2. With the results obtained in this chapter, it is easy to see that the end-point results obtained in [7] for the commutators of the multilinear Calderón-Zygmund operators also hold for the commutators of the multilinear Haar multipliers.

CHAPTER 5

SPARSE DOMINATION THEOREMS AND MULTILINEAR BLOOM'S INEQUALITY

In this chapter, we show that multilinear dyadic paraproducts and Haar multipliers can be pointwise dominated by multilinear sparse operators. We also obtain similar pointwise estimates for their commutators with locally integrable functions. As a consequence, we obtain various quantitative weighted norm inequalities for these operators. In particular, we introduce multilinear analog of Bloom's inequality, and prove it for the commutators of the multilinear Haar multipliers.

5.1 Main Results

Theorem: Let $b \in BMO^d$, and $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ be bounded. Let $T \in \left\{P^{\vec{\alpha}}, T_\epsilon^{\vec{\alpha}}\right\}$ with $\vec{\alpha} \in U_m$, or $T = \pi_b^{\vec{\alpha}}$ with $\vec{\alpha} \in \{0,1\}^m$. There exists a constant C so that for every compactly supported $\vec{f} = (f_1, \ldots, f_m) \in L^1 \times \cdots \times L^1$, there is a sparse collection $\mathcal{S}$ of dyadic intervals (depending on T and $\vec{f}$) such that

$$\left|T(\vec{f})\right| \leq C\mathcal{A}_{\mathcal{S}}(|\vec{f}|).$$

Theorem: Let $T \in \left\{P^{\vec{\alpha}}, T_\epsilon^{\vec{\alpha}}\right\}$ with $\vec{\alpha} \in U_m$ and $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ bounded, or $T = \pi_b^{\vec{\alpha}}$ with $\vec{\alpha} \in \{0,1\}^m$ and $b \in BMO^d$. Let $\vec{P} = (p_1, \cdots, p_m)$ with $1 < p_1, \cdots, p_m < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$. Then for $\vec{w} = (w_1, \cdots, w_m) \in A_{\vec{P},}$ we have

$$\|T(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C_{m,\vec{P},T}\, [\vec{w}]_{A_{\vec{P}}}^{\max\left\{1, \frac{p_1'}{p}, \cdots, \frac{p_m'}{p}\right\}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}.$$

Theorem: Let $T := T_\epsilon^{\vec{\alpha}}$ for some $\vec{\alpha} \in U_m$, and bounded sequence $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$, and let

b be a locally integrable function. There exists a constant C so that for every bounded $\vec{f} = (f_1, \dots, f_m)$ with compact support, there is a sparse collection $\mathcal{S}$ of dyadic intervals (depending on T, $\vec{f}$, and b) such that

$$\left|[b,T]_i(\vec{f})\right| \le C\left(\sum_{I\in\mathcal{S}} |b - \langle b\rangle_I| \prod_{j=1}^{m} \langle |f_j|\rangle_I \mathbb{1}_I + \sum_{I\in\mathcal{S}} \langle |(b-\langle b\rangle_I)f_i|\rangle_I \prod_{\substack{j=1\\ j\neq i}}^{m} \langle |f_j|\rangle_I \mathbb{1}_I\right).$$

Theorem: Let $\vec{P} = (p_1, \cdots, p_m)$ with $1 < p_1, \cdots, p_m < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$, and let $\vec{w} = (w_1, \dots, w_m)$ where w_i's are weights. Assume that $w_1, \lambda_1 \in A_{p_1}$, and that $\vec{w}^1 = (w_2, \dots, w_m)$ satisfies the $A_{\vec{P}^1}$ condition, where $\vec{P}^1 = (p_2, \dots, p_m)$ with $\sum_{i=2}^{m} \frac{1}{p_i} = \frac{1}{q}$. Let $\vec{\alpha} \in U_m$, $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ be bounded, and b be locally integrable. Then for $\vec{\mu}_1 = (\lambda_1, w_2, \dots, w_m)$ and $\nu_1 = \left(\frac{w_1}{\lambda_1}\right)^{\frac{1}{p_1}}$, we have

$$\left\|[b, T_\epsilon^{\vec{\alpha}}]_1(\vec{f})\right\|_{L^p(\nu_{\vec{\mu}_1})} \lesssim C(\lambda_1, \vec{w}, \vec{P})\, \|b\|_{BMO_{\nu_1}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)},$$

where $C(\lambda_1, \vec{w}, \vec{P}) = [w_1]_{A_{p_1}}^{\frac{1}{p_1}\max\left\{p_1, p_1'\right\}} [\lambda_1]_{A_{p_1}}^{\frac{1}{p_1}\max\left\{p_1, p_1', \cdots, p_m'\right\}} [\vec{w}^1]_{A_{\vec{P}^1}}^{\frac{1}{q}\max\left\{q, p_1', \cdots, p_m'\right\}}$. Similar estimates also hold for the commutators $[b, T_\epsilon^{\vec{\alpha}}]_j$ with $j \in \{2, \dots, m\}$.

5.2 Domination by Sparse Operators

In this section, we first obtain weak type endpoint estimates for the maximal trunctions of the multiliner dyadic paraproducts and Haar multipliers. Using this result, we will then obtain pointwise estimates of multilinear dyadic operators and their commutators.

Lemma 5.2.1. *Let $b \in BMO^d$, and $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ be bounded. Let $T \in \left\{P^{\vec{\alpha}}, T_\epsilon^{\vec{\alpha}}\right\}$ with $\vec{\alpha} \in U_m$, or $T = \pi_b^{\vec{\alpha}}$ with $\vec{\alpha} \in \{0,1\}^m$. Then*

$$\left\|T_\sharp(\vec{f})\right\|_{L^{\frac{1}{m},\infty}} \lesssim \prod_{i=1}^{m} \|f_i\|_{L^1}, \tag{5.2.1}$$

where $T_\sharp(\vec{f})$ is the maximal truncation given by

$$T_\sharp(\vec{f}) := \sup_{J\in\mathcal{D}} \left| \sum_{\substack{I\in\mathcal{D} \\ I\supsetneq J}} \delta_I^T \prod_{i=1}^m f_i(I,\alpha_i) h_I^{\sigma(\vec{\alpha},T)} \right|, \tag{5.2.2}$$

with

$$\delta_I^T = \begin{cases} 1, & \textit{if } T = P^{\vec{\alpha}} \\ \widehat{b}(I), & \textit{if } T = \pi_b^{\vec{\alpha}} \\ \epsilon_I & \textit{if } T = T_\epsilon^{\vec{\alpha}} \end{cases},$$

and

$$\sigma(\vec{\alpha},T) = \begin{cases} \sigma(\vec{\alpha}), & \textit{if } T = P^{\vec{\alpha}} \textit{ or } T_\epsilon^{\vec{\alpha}} \\ \sigma(\vec{\alpha})+1, & \textit{if } T = \pi_b^{\vec{\alpha}} \end{cases}.$$

Proof. It is easy to see that $T_\sharp$ is multi-sublinear, and that $T_\sharp(\vec{f})$ is supported on $I\in\mathcal{D}$ if $f_i = h_I$ for some i. So, by Lemma 3.2.3, it suffices to prove that

$$\left\| T_\sharp(\vec{f}) \right\|_{L^p} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}},$$

for $1 < p_i, p < \infty$ with $\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p}$.

For $J \in D$, define

$$T^J(\vec{f}) := \sum_{\substack{I\in\mathcal{D} \\ I\supsetneq J}} \delta_I^T \prod_{i=1}^m f_i(I,\alpha_i) h_I^{\sigma(\vec{\alpha},T)}.$$

Note that this is the expansion of $T^J(\vec{f})$ in terms of the Haar basis $\{h_I\}_{I\in\mathcal{D}}$ if and only if $\sigma(\vec{\alpha},T)$ is odd.

Case I: $\sigma(\vec{\alpha},T)$ odd.

In this case, we show that

$$T_\sharp(\vec{f})(x) \leq M(T(\vec{f}))(x) \quad \text{for every } x \in \mathbb{R}.$$

Fix $J \in D$ and $x \in \mathbb{R}$. Suppose there exists a dyadic interval that properly contains J, and also contains the point x. Let J' be the smallest of such intervals, and let J'' be the child of J' that contains x. Then,

$$\begin{aligned}
T^J(\vec{f})(x) &= \sum_{\substack{I\in\mathcal{D}\\ I\supsetneq J}} \delta_I^T \prod_{i=1}^m f_i(I,\alpha_i) h_I^{\sigma(\vec{\alpha},T)}(x) \\
&= \sum_{\substack{I\in\mathcal{D}\\ I\supseteq J'}} \delta_I^T \prod_{i=1}^m f_i(I,\alpha_i) h_I^{\sigma(\vec{\alpha},T)}(x) \\
&= \left(\sum_{\substack{I\in\mathcal{D}\\ I\supsetneq J''}} \delta_I^T \prod_{i=1}^m f_i(I,\alpha_i) h_I^{\sigma(\vec{\alpha},T)}(x) \right) \mathbf{1}_{J''}(x), \\
&= \langle T(\vec{f}) \rangle_{J''} \mathbf{1}_{J''}(x).
\end{aligned}$$

We then have, $\left| T^J(\vec{f})(x) \right| \leq \langle |T(\vec{f})| \rangle_{J''} \mathbf{1}_{J''}(x) \leq M(T(\vec{f}))(x)$.
If no dyadic interval containing x properly contains J, we have

$$\left| T^J(\vec{f})(x) \right| = 0 \leq M(T(\vec{f}))(x).$$

Thus for each $J \in \mathcal{D}$ and all $x \in \mathbb{R}$,

$$\left| T^J(\vec{f})(x) \right| \leq M(T(\vec{f}))(x),$$

which implies that

$$\sup_{J\in\mathcal{D}} \left| T^J(\vec{f})(x) \right| \leq M(T(\vec{f}))(x)$$

$$\text{i.e.} \quad T_\sharp(\vec{f})(x) \leq M(T(\vec{f}))(x).$$

Case II: $\sigma(\vec{\alpha}, T)$ even.

Assume without loss of generality that $\alpha_1 = 0$, and define

$$T_1(f_2, \ldots, f_m) := \sum_{I \in \mathcal{D}} \delta_I^T \prod_{i=2}^{m} f_i(I, \alpha_i) h_I^{\sigma(\vec{\alpha}, T)-1}.$$

Note that T_1 is an $(m-1)$-linear paraproduct or Haar multiplier which is bounded from $L^{p_2} \times \cdots \times L^{p_m} \to L^{q_1}$ for q_1 given by $\sum_{i=2}^{m} \frac{1}{p_i} = \frac{1}{q_1}$. Moreover, $\sigma(\vec{\alpha}, T) - 1$ being odd, $T_1(f_2, \ldots, f_m)(I, 0) = \dfrac{\delta_I^T \prod_{i=2}^m f_i(I, \alpha_i)}{(\sqrt{|I|})^{\sigma(\vec{\alpha}, T)-2}}$. Writing $T_1(f_2, \ldots, f_m) = g$, we have

$$\begin{aligned}
T^J(\vec{f})(x) &= \sum_{\substack{I \in \mathcal{D} \\ I \supsetneq J}} \delta_I^T \prod_{i=1}^{m} f_i(I, \alpha_i) h_I^{\sigma(\vec{\alpha}, T)}(x) \\
&= \sum_{\substack{I \in \mathcal{D} \\ I \supsetneq J}} \widehat{f}_1(I) \frac{\delta_I^T \prod_{i=2}^m f_i(I, \alpha_i)}{(\sqrt{|I|})^{\sigma(\vec{\alpha}, T)-2}} h_I^2(x) \\
&= \sum_{\substack{I \in \mathcal{D} \\ I \supsetneq J}} \widehat{f}_1(I) \widehat{g}(I) h_I^2(x).
\end{aligned}$$

With J'' as in case *I*, we have

$$\begin{aligned}
&T^J(\vec{f})(x) \\
=& \left(\sum_{\substack{I \in \mathcal{D} \\ I \supsetneq J''}} \widehat{f}_1(I) \widehat{g}(I) h_I^2(x) \right) \mathbb{1}_{J''}(x). \\
=& \left(\langle f_1 \rangle_{J''} \langle g \rangle_{J''} - \sum_{\substack{I \in \mathcal{D} \\ I \supsetneq J''}} \widehat{f}_1(I) \langle g \rangle_I h_I(x) - \sum_{\substack{I \in \mathcal{D} \\ I \supsetneq J''}} \langle f_1 \rangle_I \widehat{g}(I) h_I(x) \right) \mathbb{1}_{J''}(x).
\end{aligned}$$

The last equality follows from Lemma 2.1.1. We then have,

$$
\begin{aligned}
& \left|T^J(\vec{f})(x)\right| \\
\leq\ & \langle|f_1|\rangle_{J''}\langle|g|\rangle_{J''}1_{J''}(x) + \left|\sum_{\substack{I\in\mathcal{D}\\ I\supsetneq J''}} \widehat{f}_1(I)\langle g\rangle_I h_I(x)\right| 1_{J''}(x) + \left|\sum_{\substack{I\in\mathcal{D}\\ I\supsetneq J''}} \langle f_1\rangle_I \widehat{g}(I) h_I(x)\right| 1_{J''}(x) \\
\leq\ & \mathcal{M}(f_1,g)(x) + M(P^{(0,1)}(f_1,g))(x) + M(P^{(1,0)}(f_1,g))(x).
\end{aligned}
$$

The definition of the multilinear maximal function, and the result from case *I* above imply the last inequality. If J' does not exist,

$$
\left|T^J(\vec{f})(x)\right| = 0 \leq \mathcal{M}(f_1,g)(x) + M(P^{(0,1)}(f_1,g))(x) + M(P^{(1,0)}(f_1,g))(x).
$$

Thus, for all $x\in\mathbb{R}$,

$$
T_\sharp(\vec{f})(x) \leq \mathcal{M}(f_1,g)(x) + M(P^{(0,1)}(f_1,g))(x) + M(P^{(1,0)}(f_1,g))(x).
$$

Using the boundedness properties of the linear/multilinear maximal functions as well as the multilinear paraproducts and Haar multipliers, we observe that

$$
\left\|M(T(\vec{f}))\right\|_{L^p} \lesssim \left\|T(\vec{f})\right\|_{L^p} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}},
$$

$$
\|\mathcal{M}(f_1,g)\|_{L^p} \lesssim \|f_1\|_{L^{p_1}} \|g\|_{L^{q_1}} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}},
$$

$$
\left\|M(P^{(0,1)}(f_1,g))\right\|_{L^p} \lesssim \left\|P^{(0,1)}(f_1,g)\right\|_{L^p} \lesssim \|f_1\|_{L^{p_1}} \|g\|_{L^{q_1}} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}},
$$

and,

$$
\left\|M(P^{(1,0)}(f_1,g))\right\|_{L^p} \lesssim \left\|P^{(1,0)}(f_1,g)\right\|_{L^p} \lesssim \|f_1\|_{L^{p_1}} \|g\|_{L^{q_1}} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}}.
$$

By the domination of maximal truncation $T_\sharp(\vec{f})$ obtained above, we then have

$$\left\|T_\sharp(\vec{f})\right\|_{L^p} \lesssim \prod_{i=1}^{m} \|f_i\|_{L^{p_i}}.$$

This completes the proof. $\square$

Theorem 5.2.2. *Let $b \in BMO^d$, and $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ be bounded. Let $T \in \left\{P^{\vec{\alpha}}, T_\epsilon^{\vec{\alpha}}\right\}$ with $\vec{\alpha} \in U_m$, or $T = \pi_b^{\vec{\alpha}}$ with $\vec{\alpha} \in \{0,1\}^m$. There exists a constant C so that for every compactly supported $\vec{f} = (f_1, \dots, f_m) \in L^1 \times \cdots \times L^1$, there is a sparse collection $\mathcal{S}$ of dyadic intervals $($depending on T and $\vec{f})$ such that*

$$\left|T(\vec{f})\right| \le C\mathcal{A}_{\mathcal{S}}(|\vec{f}|). \tag{5.2.3}$$

Proof. It suffices to prove the theorem for $\vec{f} = (f_1, \dots, f_m)$ supported on a dyadic interval $I_0 = [0, 2^k)$. We first obtain a sparse collection $\mathcal{S}'$ such that the corresponding sparse operator $\mathcal{A}_{\mathcal{S}'}$ satisfies

$$\left|T(\vec{f})\right| 1_{I_0} \le C\,\mathcal{A}_{\mathcal{S}'}(|\vec{f}|). \tag{5.2.4}$$

We use the weak-type estimates for the multilinear maximal function $\mathcal{M}$ and the maximal truncation $T_\sharp$, namely

$$\left|\{x : \mathcal{M}(\vec{f})(x) > \lambda\}\right| \le \frac{C_1}{\lambda^{1/m}} \prod_{i=1}^{m} \|f_i\|_{L^1}^{1/m}, \qquad \left|\{x : T_\sharp(\vec{f})(x) > \lambda\}\right| \le \frac{C_2}{\lambda^{1/m}} \prod_{i=1}^{m} \|f_i\|_{L^1}^{1/m}.$$

Let $E = \left\{x \in I_0 : \max\{\mathcal{M}(\vec{f})(x), T_\sharp(\vec{f})(x)\} > \frac{1}{2}C_0 \prod_{i=1}^{m} \langle |f_i| \rangle_{I_0}\right\}$.

Since each f_i is supported on I_0, $\|f_i\|_{L^1} = \int_{\mathbb{R}} |f_i| = |I_0| \left(\frac{1}{|I_0|}\int_{I_0} |f_i|\right) = |I_0|\langle |f_i|\rangle_{I_0}$. So,

$$\begin{aligned}
|E| &\le \left|\left\{x : \mathcal{M}(\vec{f})(x) > \frac{1}{2}C_0\prod_{i=1}^{m}\langle|f_i|\rangle_{I_0}\right\}\right| + \left|\left\{x : T_\sharp(\vec{f})(x) > \frac{1}{2}C_0\prod_{i=1}^{m}\langle|f_i|\rangle_{I_0}\right\}\right| \\
&\le \frac{2^{1/m}C_1}{C_0^{1/m}}|I_0| + \frac{2^{1/m}C_2}{C_0^{1/m}}|I_0| \\
&= \frac{2^{1/m}(C_1+C_2)}{C_0^{1/m}}|I_0|
\end{aligned}$$

We choose C_0 so large that $|E| \le \dfrac{1}{2}|I_0|$.

Let $\mathcal{E}$ be the collection of maximal dyadic intervals contained in E. We claim that

$$\left|T(\vec{f})(x)\right| 1_{I_0}(x) \le C\prod_{i=1}^{m}\langle|f_i|\rangle_{I_0} + \sum_{J\in\mathcal{E}}|T_J(\vec{f})(x)|, \tag{5.2.5}$$

where $T_J(\vec{f}) := \sum_{\substack{I\in\mathcal{D}\\ I\subseteq J}}\delta_I^T\prod_{i=1}^{m}f_i(I,\alpha_i)h_I^{\sigma(\vec{\alpha},T)}$, and $C = \max\{C_0, C'C_0\}$ with

$$C' = \begin{cases} 1, & \text{if } T = P^{\vec{\alpha}} \\ \|b\|_{BMO^d}, & \text{if } T = \pi_b^{\vec{\alpha}} \\ \sup_I|\epsilon_I| & \text{if } T = T_\epsilon^{\vec{\alpha}} \end{cases}.$$

If $x \in E$, there is a unique $K \in \mathcal{E}$ that contains x. If K' is the parent of K, then

$$\left|T(\vec{f})(x)\right| 1_{I_0}(x) \;\le\; \left|T^{K'}(\vec{f})(x)\right| + \left|\delta_{K'}^T\prod_{i=1}^{m}f_i(K',\alpha_i)h_{K'}^{\sigma(\vec{\alpha},T)}(x)\right| + \left|T_K(\vec{f})(x)\right|.$$

Note that $\dfrac{|\widehat{b}(K')|}{\sqrt{|K'|}} \le \|b\|_{BMO^d},\ \dfrac{|\widehat{f}_i(K')|}{\sqrt{|K'|}} \le \langle|f_i|\rangle_{K'},$ and $|\langle f_i\rangle_{K'}| \le \langle|f_i|\rangle_{K'}$. So,

$$\left|\delta_{K'}^T \prod_{i=1}^m f_i(K',\alpha_i) h_{K'}^{\sigma(\vec{\alpha},T)}(x)\right| \leq \left|\eta_{K'}^T \left(\prod_{i:\alpha_i=0} \frac{\widehat{f}_i(K')}{\sqrt{|K'|}}\right)\left(\prod_{i:\alpha_i=1} \langle f_i\rangle_{K'}\right)\right| 1_{K'}(x)$$
$$\leq C' \prod_{i=1}^m \langle |f_i|\rangle_{K'},$$

where $\eta_{K'}^T = \begin{cases} 1, & \text{if } T = P^{\vec{\alpha}} \\ \frac{\widehat{b}(K')}{\sqrt{|K'|}}, & \text{if } T = \pi_b^{\vec{\alpha}} \\ \epsilon_I & \text{if } T = T_\epsilon^{\vec{\alpha}} \end{cases}$.

We thus have

$$\left|T(\vec{f})(x)\right| 1_{I_0}(x) \leq \left|T^{K'}(\vec{f})(x)\right| + C' \prod_{i=1}^m \langle |f_i|\rangle_{K'} + \sum_{J\in\mathcal{E}} \left|T_J(\vec{f})(x)\right|.$$

By the maximality of K, $\left|T^{K'}(\vec{f})(x)\right| \leq \dfrac{1}{2}C_0 \displaystyle\prod_{i=1}^m \langle |f_i|\rangle_{I_0}$, and $\displaystyle\prod_{i=1}^m \langle |f_i|\rangle_{K'} \leq \dfrac{1}{2}C_0 \prod_{i=1}^m \langle |f_i|\rangle_{I_0}$.

In fact, if $|T^{K'}(\vec{f})(x)| > \frac{1}{2}C_0 \prod_{i=1}^m \langle |f_i|\rangle_{I_0}$, then $T^{K'}(\vec{f})$ being constant on K', we have

$$T_\sharp(\vec{f})(y) \geq |T^{K'}(\vec{f})(y)| = |T^{K'}(\vec{f})(x)| > \frac{1}{2}C_0 \prod_{i=1}^m \langle |f_i|\rangle_{I_0}$$

for every $y \in K'$, which implies that $K' \in \mathcal{E}$ contradicting the maximality of K. On the other hand, if $\displaystyle\prod_{i=1}^m \langle |f_i|\rangle_{K'} > \frac{1}{2}C_0 \prod_{i=1}^m \langle |f_i|\rangle_{I_0}$, then for every $y \in K'$,

$$\mathcal{M}(\vec{f})(y) > \frac{1}{2}C_0 \prod_{i=1}^m \langle |f_i|\rangle_{I_0}$$

which also contradicts the maximality of K. So, for all $x \in E$ we have

$$\begin{aligned}\left|T(\vec{f})(x)\right| 1_{I_0}(x) &\leq \frac{1}{2}C_0\prod_{i=1}^{m}\langle|f_i|\rangle_{I_0} + \frac{1}{2}C'C_0\prod_{i=1}^{m}\langle|f_i|\rangle_{I_0} + \sum_{J\in\mathcal{E}}\left|T_J(\vec{f})(x)\right| \\ &\leq C\prod_{i=1}^{m}\langle|f_i|\rangle_{I_0} + \sum_{J\in\mathcal{E}}|T_J(\vec{f})(x)|.\end{aligned}$$

For $x \in I_0\backslash E$, (5.2.5) is obviously true, since $T^J(\vec{f})(x) \leq \frac{1}{2}C_0\prod_{i=1}^{m}\langle|f_i|\rangle_{I_0}$ for all $J \in \mathcal{D}$, and $T(\vec{f})(x) = \lim_{k\to\infty} T^{J_x^k}(x)$, where J_x^k is the dyadic interval of length 2^{-k} that contains x.

As the inequality (5.2.5) suggests, we include I_0 in $\mathcal{S}'$. Now we recurse on $T_J(\vec{f})$, $J \in \mathcal{E}$. At this stage, we add each member of $\mathcal{E}$ to $\mathcal{S}'$ as the $\mathcal{S}'$-children of I_0. The sparseness condition is satisfied since $\sum_{J\in\mathcal{E}}|J| \leq |E| \leq \frac{1}{2}|I_0|$. Continuing the recursion, we get the sparse operator satisfying (5.2.4).

For $x \notin I_0$,

$$\begin{aligned}\left|T(\vec{f})(x)\right| &= \left|\sum_{I\in\mathcal{D}}\delta_I^T\prod_{i=1}^{m}f_i(I,\alpha_i)h_I^{\sigma(\vec{\alpha},T)}(x)\right| \\ &= \left|\sum_{I\supsetneq I_0}\delta_I^T\prod_{i=1}^{m}f_i(I,\alpha_i)h_I^{\sigma(\vec{\alpha},T)}(x)\right| \\ &\leq C'\sum_{I\supsetneq I_0}\prod_{i=0}^{m}\langle|f_i|\rangle_I.\end{aligned}$$

Clearly, the sparse operator $A_{\mathcal{S}}$ corresponding to the sparse collection

$$\mathcal{S} = \mathcal{S}' \cup \{I \in \mathcal{D} : I \supsetneq I_0\}$$

satisfies (5.2.3) with $C = \max\{C_0, C', C'C_0\}$. $\square$

As an immediate consequence of this theorem and (1.2.6), we have the following weighted estimate for the multilinear dyadic paraproducts and Haar multilpliers.

Theorem 5.2.3. *Let $T \in \left\{P^{\vec{\alpha}}, T_\epsilon^{\vec{\alpha}}\right\}$ with $\vec{\alpha} \in U_m$ and $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$ bounded, or $T = \pi_b^{\vec{\alpha}}$ with $\vec{\alpha} \in \{0,1\}^m$ and $b \in BMO^d$. Let $\vec{P} = (p_1, \cdots, p_m)$ with $1 < p_1, \cdots, p_m < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$. Then for $\vec{w} = (w_1, \cdots, w_m) \in A_{\vec{P},}$ we have*

$$\|T(\vec{f})\|_{L^p(v_{\vec{w}})} \le C_{m,\vec{P},T}\, [\vec{w}]_{A_{\vec{P}}}^{\max\left\{1, \frac{p_1'}{p}, \cdots, \frac{p_m'}{p}\right\}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}. \tag{5.2.6}$$

Theorem 5.2.4. *Let $T := T_\epsilon^{\vec{\alpha}}$ for some $\vec{\alpha} \in U_m$, and bounded sequence $\epsilon = (\epsilon_I)_{I\in\mathcal{D}}$, and let b be a locally integrable function. There exists a constant C so that for every bounded $\vec{f} = (f_1, \ldots, f_m)$ with compact support, there is a sparse collection $\mathcal{S}$ of dyadic intervals (depending on T, $\vec{f}$, and b) such that*

$$\left|[b,T]_i(\vec{f})\right| \le C \left(\sum_{I\in\mathcal{S}} |b - \langle b\rangle_I| \prod_{j=1}^m \langle |f_j|\rangle_I \mathbb{1}_I + \sum_{I\in\mathcal{S}} \langle |(b-\langle b\rangle_I) f_i|\rangle_I \prod_{\substack{j=1\\ j\neq i}}^m \langle |f_j|\rangle_I \mathbb{1}_I \right). \tag{5.2.7}$$

Proof. It suffices to prove (5.2.7) for $i = 1$ and for $\vec{f}$ supported in a dyadic interval $I_0 = [0, 2^k)$. We first find a sparse collection $\mathcal{S}'$ of dyadic intervals such that

$$\left|[b,T]_1(\vec{f})\right| \mathbb{1}_{I_0} \le C' \left(\sum_{I\in\mathcal{S}'} |b - \langle b\rangle_I| \prod_{j=1}^m \langle |f_j|\rangle_I \mathbb{1}_I + \sum_{I\in\mathcal{S}'} \langle |(b-\langle b\rangle_I) f_1|\rangle_I \prod_{j=2}^m \langle |f_j|\rangle_I \mathbb{1}_I \right). \tag{5.2.8}$$

Let $\vec{g} = (g_1, g_2, \ldots, g_m) = ((b - \langle b\rangle_{I_0}) f_1, f_2, \ldots, f_m)$, and $E = E_1 \cup E_2$, where

$$E_1 = \left\{ x \in I_0 : \max\{\mathcal{M}(\vec{f})(x), T_\sharp(\vec{f})(x)\} > C_0 \prod_{j=1}^m \langle |f_j|\rangle_{I_0} \right\}$$

and

$$E_2 = \left\{ x \in I_0 : \max\{\mathcal{M}(\vec{g})(x), T_\sharp(\vec{g})(x)\} > C_0 \prod_{j=1}^m \langle |g_j|\rangle_{I_0} \right\}.$$

Due to weak-type boundedness of $\mathcal{M}$ and $T_\sharp$, we can choose C_0 so large that $|E| \leq \frac{1}{2}|I_0|$. Let $\mathcal{E}$ be the collection of maximal dyadic intervals contained in E. It suffices to prove the recursive claim:

$$\begin{aligned}\left|[b,T]_1(\vec{f})(x)\right| \mathbb{1}_{I_0}(x) \;\; &\leq \;\; C'\left(|b(x)-\langle b\rangle_{I_0}|\prod_{j=1}^m \langle |f_j|\rangle_{I_0} + \langle |(b-\langle b\rangle_{I_0})f_1|\rangle_{I_0}\prod_{j=2}^m \langle |f_j|\rangle_{I_0}\right) \\ &\qquad + \sum_{J\in\mathcal{E}}\left|[b,T_J](\vec{f})(x)\right| \mathbb{1}_J(x),\end{aligned}$$

where $T_J(\vec{f}) := \sum_{\substack{I\in\mathcal{D}\\ I\subseteq J}} \epsilon_I \prod_{j=1}^m f_j(I,\alpha_j) h_I^{\sigma(\vec{\alpha})}$.

If $x \in E$, there is a unique $K \in \mathcal{E}$ that contains x. If K' is the parent of K, then

$$\begin{aligned}[b,T](\vec{f})(x) \;\; &= \;\; [b,T^K](\vec{f})(x) + [b,T_K](\vec{f})(x) \\ &= \;\; [b-\langle b\rangle_{I_0},T^K](\vec{f})(x) + [b,T_K](\vec{f})(x) \\ &= \;\; (b(x)-\langle b\rangle_{I_0})\,T^K(\vec{f})(x) - T^K((b-\langle b\rangle_{I_0})f_1, f_2, \ldots, f_m)(x) \\ &\qquad\qquad + [b,T_K](\vec{f})(x) \\ &= \;\; (b(x)-\langle b\rangle_{I_0})\,T^K(\vec{f})(x) - T^K(\vec{g})(x) + [b,T_K](\vec{f})(x) \\ &= \;\; (b(x)-\langle b\rangle_{I_0})\,T^{K'}(\vec{f})(x) + (b(x)-\langle b\rangle_{I_0})\,\epsilon_{K'}\prod_{j=1}^m f_j(K',\alpha_j)h_{K'}^{\sigma(\vec{\alpha})}(x) \\ &\qquad\qquad - T^{K'}(\vec{g})(x) - \epsilon_{K'}\prod_{j=1}^m g_j(K',\alpha_j)h_{K'}^{\sigma(\vec{\alpha})}(x) + [b,T_K](\vec{f})(x).\end{aligned}$$

As argued in the proof of Theorem 5.2.2, the maximality of K implies that

$$\left|T^{K'}(\vec{f})(x)\right| \leq C_0\prod_{j=1}^m \langle |f_j|\rangle_{I_0}, \quad \left|\prod_{j=1}^m f_j(K',\alpha_j)h_{K'}^{\sigma(\vec{\alpha})}(x)\right| \leq C_0\prod_{j=1}^m \langle |f_j|\rangle_{I_0},$$

$$\left|T^{K'}(\vec{g})(x)\right| \leq C_0\prod_{j=1}^m \langle |g_j|\rangle_{I_0}, \quad \left|\prod_{j=1}^m g_j(K',\alpha_j)h_{K'}^{\sigma(\vec{\alpha})}(x)\right| \leq C_0\prod_{j=1}^m \langle |g_j|\rangle_{I_0}.$$

So, for $x \in E$, we have

$$
\begin{aligned}
\left|[b,T](\vec{f})(x)\right| &\leq |b(x)-\langle b\rangle_{I_0}|\, C_0 \prod_{j=1}^m \langle |f_j|\rangle_{I_0} + |b(x)-\langle b\rangle_{I_0}| \sup_I |\epsilon_I|\, C_0 \prod_{j=1}^m \langle |f_j|\rangle_{I_0} \\
&\quad + C_0 \prod_{j=1}^m \langle |g_j|\rangle_{I_0} + \left(\sup_I |\epsilon_I|\right) C_0 \prod_{j=1}^m \langle |g_j|\rangle_{I_0} + \left|[b,T_K](\vec{f})(x)\right| \\
&= C' \left(|b(x)-\langle b\rangle_{I_0}| \prod_{j=1}^m \langle |f_j|\rangle_{I_0} + \langle |(b-\langle b\rangle_{I_0}) f_1|\rangle_{I_0} \prod_{j=2}^m \langle |f_j|\rangle_{I_0} \right) \\
&\qquad + \left|[b,T_K](\vec{f})(x)\right|,
\end{aligned}
$$

where $C' = C_0 + (\sup_I |\epsilon_I|)\, C_0$. Now for $x \in I_0 \backslash E$, we have

$$
\begin{aligned}
\left|[b,T](\vec{f})(x)\right| &= \left|[b-\langle b\rangle_{I_0}, T](\vec{f})(x)\right| \\
&\leq |b(x)-\langle b\rangle_{I_0}| \left|T(\vec{f})(x)\right| + |T(\vec{g})(x)| \\
&\leq |b(x)-\langle b\rangle_{I_0}| \left|T_\sharp(\vec{f})(x)\right| + |T_\sharp(\vec{g})(x)| \\
&\leq C_0 \left(|b(x)-\langle b\rangle_{I_0}| \prod_{j=1}^m \langle |f_j|\rangle_{I_0} + \langle |(b-\langle b\rangle_{I_0}) f_1|\rangle_{I_0} \prod_{j=2}^m \langle |f_j|\rangle_{I_0} \right).
\end{aligned}
$$

Thus the recursive claim is true for all x, and by iterating this estimate, we see that (5.2.8) holds for the sparse collection $\mathcal{S}'$ that contains I_0 and all the dyadic intervals that are contained in $\mathcal{E}$ and those arising from the iteration.

Now observe that if $x \notin I_0$, and I_x is the smallest dyadic interval containing I_0 and x, then as in the proof of Lemma 4.2.3, we get

$$
\begin{aligned}
&\left|[b,T](\vec{f})(x)\right| \\
= &\left|[b-\langle b\rangle_{I_x}, T](\vec{f})(x)\right| \\
\leq &\frac{2^m}{2^m-1}\left(\sup_I |\epsilon_I|\right) \left(|b(x)-\langle b\rangle_{I_x}| \prod_{j=1}^m \langle |f_j|\rangle_{I_x} + \langle |(b-\langle b\rangle_{I_x}) f_1|\rangle_{I_x} \prod_{j=2}^m \langle |f_j|\rangle_{I_x} \right).
\end{aligned}
$$

So, (5.2.7) holds for $\mathcal{S} = \mathcal{S}' \cup \{I \in \mathcal{D} : I_0 \subsetneq I\}$ and $C = \max\left\{C', \frac{2^m}{2^m-1}\left(\sup_I |\epsilon_I|\right)\right\}$. $\square$

5.3 Multilinear Bloom's Inequality

Theorem 5.3.1. *Let $\vec{P} = (p_1, \cdots, p_m)$ with $1 < p_1, \cdots, p_m < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$, and let $\vec{w} = (w_1, \ldots, w_m)$ where w_i's are weights. Assume that $w_1, \lambda_1 \in A_{p_1}$, and that $\vec{w}^1 = (w_2, \ldots, w_m)$ satisfies the $A_{\vec{P}^1}$ condition, where $\vec{P}^1 = (p_2, \ldots, p_m)$ with $\sum_{i=2}^m \frac{1}{p_i} = \frac{1}{q}$. Let $\vec{\alpha} \in U_m$, $\epsilon = (\epsilon_I)_{I \in \mathcal{D}}$ be bounded, and b be locally integrable. Then for $\vec{\mu}_1 = (\lambda_1, w_2, \ldots, w_m)$ and $\nu_1 = \left(\frac{w_1}{\lambda_1}\right)^{\frac{1}{p_1}}$, we have*

$$\left\| [b, T_\epsilon^{\vec{\alpha}}]_1(\vec{f}) \right\|_{L^p(\nu_{\vec{\mu}_1})} \lesssim C(\lambda_1, \vec{w}, \vec{P}) \, \|b\|_{BMO_{\nu_1}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}, \tag{5.3.1}$$

where $C(\lambda_1, \vec{w}, \vec{P}) = [w_1]_{A_{p_1}}^{\frac{1}{p_1}\max\left\{p_1, p_1'\right\}} [\lambda_1]_{A_{p_1}}^{\frac{1}{p_1}\max\left\{p_1, p_1', \cdots, p_m'\right\}} [\vec{w}^1]_{A_{\vec{P}^1}}^{\frac{1}{q}\max\left\{q, p_1', \cdots, p_m'\right\}}$.
Similar estimates also hold for the commutators $[b, T_\epsilon^{\vec{\alpha}}]_j$ with $j \in \{2, \ldots, m\}$.

Proof. Due to Theorem 5.2.4, it suffices to prove the above estimate for

$$\mathcal{A}_{\mathcal{S},b}(\vec{f}) := \sum_{I \in \mathcal{S}} |b - \langle b \rangle_I| \prod_{i=1}^m \langle |f_i| \rangle_I 1_I$$

and,

$$\mathcal{A}_{\mathcal{S},b,1}(\vec{f}) := \sum_{I \in \mathcal{S}} \langle |(b - \langle b \rangle_I) f_1| \rangle_I \prod_{i=2}^m \langle |f_i| \rangle_I 1_I.$$

By Lemma (5.1) in [10], there exists a sparse collection $\tilde{\mathcal{S}}$ of dyadic intervals such that $\mathcal{S} \subset \tilde{\mathcal{S}}$, and for a.e. $x \in I \in \mathcal{S}$,

$$|b(x) - \langle b \rangle_I| \leq C_1 \sum_{J \in \tilde{\mathcal{S}}, J \subseteq I} \langle |b - \langle b \rangle_J| \rangle_J 1_J.$$

So,

$$\begin{aligned}
\langle|(b-\langle b\rangle_I)f_1|\rangle_I &= \frac{1}{|I|}\int_I |(b-\langle b\rangle_I)f_1| \\
&\lesssim \frac{1}{|I|}\int_I \sum_{J\in\tilde{\mathcal{S}},J\subseteq I}\langle|b-\langle b\rangle_J|\rangle_J |f_1|1_J \\
&\leq \frac{1}{|I|}\|b\|_{BMO_{\nu_1}}\sum_{J\in\tilde{\mathcal{S}},J\subseteq I}\frac{\nu_1(J)}{|J|}\int_I |f_1|1_J \\
&= \frac{1}{|I|}\|b\|_{BMO_{\nu_1}}\sum_{J\in\tilde{\mathcal{S}},J\subseteq I}\langle|f_1|\rangle_J\nu_1(J) \\
&\leq \frac{1}{|I|}\|b\|_{BMO_{\nu_1}}\sum_{J\in\tilde{\mathcal{S}}}\langle|f_1|\rangle_J\int_I \nu_1 1_J \\
&= \frac{1}{|I|}\|b\|_{BMO_{\nu_1}}\int_I\left(\sum_{J\in\tilde{\mathcal{S}}}\langle|f_1|\rangle_J 1_J\right)\nu_1 \\
&= \frac{1}{|I|}\|b\|_{BMO_{\nu_1}}\int_I \mathcal{A}_{\tilde{\mathcal{S}}}(f_1)\nu_1 \\
&= \|b\|_{BMO_{\nu_1}}\langle\mathcal{A}_{\tilde{\mathcal{S}}}(f_1)\nu_1\rangle_I.
\end{aligned}$$

This implies that,

$$\begin{aligned}
\left\|\mathcal{A}_{\mathcal{S},b,1}(\vec{f})\right\|_{L^p(\nu_{\vec{\mu}_1})} &\lesssim \|b\|_{BMO_{\nu_1}}\left\|\sum_{I\in\mathcal{S}}\langle\mathcal{A}_{\tilde{\mathcal{S}}}(f_1)\nu_1\rangle_I\prod_{i=2}^m\langle|f_i|\rangle_I 1_I\right\|_{L^p(\nu_{\vec{\mu}_1})} \\
&\lesssim \|b\|_{BMO_{\nu_1}}[\vec{\mu}_1]_{A_{\vec{P}}}^{\frac{1}{p}\max\left\{p,p_1',\cdots,p_m'\right\}}\|\mathcal{A}_{\tilde{\mathcal{S}}}(f_j)\nu_1\|_{L^{p_1}(\lambda_1)}\prod_{i=2}^m\|f_i\|_{L^{p_i}(w_i)}.
\end{aligned}$$

Note that $\|\mathcal{A}_{\tilde{\mathcal{S}}}(f_1)\nu_1\|_{L^{p_1}(\lambda_1)} = \|\mathcal{A}_{\tilde{\mathcal{S}}}(f_1)\|_{L^{p_1}(w_1)} \lesssim [w_1]_{A_{p_1}}^{\frac{1}{p_1}\max\left\{p_1,p_1'\right\}}\|f_1\|_{L^{p_1}(w_1)}$.

So,

$$\left\|\mathcal{A}_{\mathcal{S},b,1}(\vec{f})\right\|_{L^p(\nu_{\vec{\mu}_1})} \lesssim [w_1]_{A_{p_1}}^{\frac{1}{p_1}\max\left\{p_1,p_1'\right\}}[\vec{\mu}_1]_{A_{\vec{P}}}^{\frac{1}{p}\max\left\{p,p_1',\cdots,p_m'\right\}}\|b\|_{BMO_{\nu_1}}\prod_{i=1}^m\|f_i\|_{L^{p_i}(w_i)}. \tag{5.3.2}$$

Now to obtain an estimate for the norm of $\mathcal{A}_{\mathcal{S},b}(\vec{f})$, we observe that

$$\begin{aligned}
\mathcal{A}_{\mathcal{S},b}(\vec{f}) &= \sum_{I\in\mathcal{S}} |b-\langle b\rangle_I| \prod_{i=1}^{m} \langle |f_i|\rangle_I 1_I \\
&= \sum_{I\in\mathcal{S}} |b-\langle b\rangle_I| \, \langle |f_1|\rangle_I \prod_{i=2}^{m} \langle |f_i|\rangle_I 1_I \\
&\le \sum_{I\in\mathcal{S}} \left(\sum_{J\in\mathcal{S}} |b-\langle b\rangle_J| \, \langle |f_1|\rangle_J 1_J \right) \prod_{i=2}^{m} \langle |f_i|\rangle_I 1_I \\
&= \left(\sum_{J\in\mathcal{S}} |b-\langle b\rangle_J| \, \langle |f_1|\rangle_J 1_J \right) \left(\sum_{I\in\mathcal{S}} \prod_{i=2}^{m} \langle |f_i|\rangle_I 1_I \right) \\
&= T_1(f_1) T_2(\vec{f}^1) \quad \text{(say)}.
\end{aligned}$$

Since $\nu_{\vec{\mu}_1} = \lambda_1^{p/p_1} w_2^{p/p_2} \dots w_m^{p/p_m} = \lambda_1^{p/p_1} \left(w_2^{q/p_2} \dots w_m^{q/p_m} \right)^{p/q} = \lambda_1^{p/p_1} \nu_{\vec{w}^1}^{p/q}$,

$$\begin{aligned}
\left\| \mathcal{A}_{\mathcal{S},b}(\vec{f}) \right\|_{L^p(\nu_{\vec{\mu}_1})} &= \left(\int \left(\mathcal{A}_{\mathcal{S},b}(\vec{f}) \right)^p \nu_{\vec{\mu}_1} dx \right)^{1/p} \\
&= \left(\int T_1(f_1)^p T_2(\vec{f}^1)^p \lambda_1^{p/p_1} \nu_{\vec{w}^1}^{p/q} dx \right)^{1/p} \\
&\le \left(\int T_1(f_1)^{p_1} \lambda_1 dx \right)^{1/p_1} \left(\int T_2(\vec{f}^1)^q \nu_{\vec{w}^1} dx \right)^{1/q} \\
&= \|T_1(f_1)\|_{L^{p_1}(\lambda_1)} \left\| T_2(\vec{f}^1) \right\|_{L^q(\nu_{\vec{w}^1})}
\end{aligned}$$

Now, as shown in [10],

$$\|T_1(f_1)\|_{L^{p_1}(\lambda_1)} \lesssim \left\{ [w_1]_{A_{p_1}} [\lambda_1]_{A_{p_1}} \right\}^{\frac{1}{p_1}\max\left\{p_1, p_1'\right\}} \|b\|_{BMO_{\nu_1}} \|f_1\|_{L^{p_1}(\lambda_1)}$$

and,

$$\left\| T_2(\vec{f}^1) \right\|_{L^q(\nu_{\vec{w}^1})} \lesssim [\vec{w}^1]_{A_{\vec{P}^1}}^{\frac{1}{q}\max\left\{q, p_2', \cdots, p_m'\right\}} \prod_{i=2}^{m} \|f_i\|_{L^{p_i}(w_i)}.$$

So,

$$\begin{aligned}
&\left\|\mathcal{A}_{\mathcal{S},b}(\vec{f})\right\|_{L^p(\nu_{\vec{\mu}_1})} \\
\lesssim\;& \left\{[w_1]_{A_{p_1}}[\lambda_1]_{A_{p_1}}\right\}^{\frac{1}{p_1}\max\left\{p_1,p_1'\right\}}[\vec{w}^1]_{A_{\vec{P}^1}}^{\frac{1}{q}\max\left\{q,p_2',\cdots,p_m'\right\}}\|b\|_{BMO_{\nu_1}}\prod_{i=1}^m\|f_i\|_{L^{p_i}(w_i)}.
\end{aligned}$$

Using the above estimates for $\left\|\mathcal{A}_{\mathcal{S},b,1}(\vec{f})\right\|_{L^p(\nu_{\vec{\mu}_1})}$ and $\left\|\mathcal{A}_{\mathcal{S},b}(\vec{f})\right\|_{L^p(\nu_{\vec{\mu}_1})}$, it follows from Theorem 5.2.4 that

$$\left\|[b,T_\epsilon^{\vec{\alpha}}]_1(\vec{f})\right\|_{L^p(\nu_{\vec{\mu}_1})}\lesssim C(\lambda_1,\vec{w},\vec{P})\,\|b\|_{BMO_{\nu_1}}\prod_{i=1}^m\|f_i\|_{L^{p_i}(w_i)}.$$

To see that the estimate holds for

$$C(\lambda_1,\vec{w},\vec{P})=[w_1]_{A_{p_1}}^{\frac{1}{p_1}\max\left\{p_1,p_1'\right\}}[\lambda_1]_{A_{p_1}}^{\frac{1}{p_1}\max\left\{p_1,p_1',\cdots,p_m'\right\}}[\vec{w}^1]_{A_{\vec{P}^1}}^{\frac{1}{q}\max\left\{q,p_1',\cdots,p_m'\right\}},$$

it suffices to prove

$$[\vec{\mu}_1]_{A_{\vec{P}}}^{\frac{1}{p}\max\left\{p,p_1',\cdots,p_m'\right\}}\le[\lambda_1]_{A_{p_1}}^{\frac{1}{p_1}\max\left\{p,p_1',\cdots,p_m'\right\}}[\vec{w}^1]_{A_{\vec{P}^1}}^{\frac{1}{q}\max\left\{q,p_1',\cdots,p_m'\right\}}. \tag{5.3.3}$$

Observe that $\int_I\nu_{\vec{\mu}_1}=\int_I\lambda_1^{p/p_1}\nu_{\vec{w}^1}^{p/q}\le\left(\int_I\lambda_1\right)^{p/p_1}\left(\int_I\nu_{\vec{w}^1}\right)^{p/q}$. So,

$$\begin{aligned}
&[\vec{\mu}_1]_{A_{\vec{P}}}\\
=\;&\sup_I\left(\frac{1}{|I|}\int_I\nu_{\vec{\mu}_1}\right)\left(\frac{1}{|I|}\int_I\lambda_1^{1-p_1'}\right)^{p/p_1'}\prod_{i=2}^m\left(\frac{1}{|I|}\int_I w_i^{1-p_i'}\right)^{p/p_i'}\\
\le\;&\sup_I\left(\frac{1}{|I|}\int_I\lambda_1\right)^{p/p_1}\left(\frac{1}{|I|}\int_I\nu_{\vec{w}^1}\right)^{p/q}\left(\frac{1}{|I|}\int_I\lambda_1^{1-p_1'}\right)^{p/p_1'}\prod_{i=2}^m\left(\frac{1}{|I|}\int_I w_i^{1-p_i'}\right)^{p/p_i'}\\
=\;&\sup_I\left\{\left(\frac{1}{|I|}\int_I\lambda_1\right)\left(\frac{1}{|I|}\int_I\lambda_1^{1-p_1'}\right)^{\frac{p_1}{p_1'}}\right\}^{\frac{p}{p_1}}\left\{\left(\frac{1}{|I|}\int_I\nu_{\vec{w}^1}\right)\prod_{i=2}^m\left(\frac{1}{|I|}\int_I w_i^{1-p_i'}\right)^{\frac{q}{p_i'}}\right\}^{\frac{p}{q}}\\
=\;&[\lambda_1]_{A_{p_1}}^{\frac{p}{p_1}}[\vec{w}^1]_{A_{\vec{P}^1}}^{\frac{p}{q}}.
\end{aligned}$$

This gives

$$[\vec{\mu}_1]_{A_{\vec{P}}}^{\frac{1}{p}\max\left\{p,p_1',\cdots,p_m'\right\}} \leq [\lambda_1]_{A_{p_1}}^{\frac{1}{p_1}\max\left\{p,p_1',\cdots,p_m'\right\}} [\vec{w}^1]_{A_{\vec{P}1}}^{\frac{1}{q}\max\left\{p,p_1',\cdots,p_m'\right\}}.$$

Since $q > p$, (5.3.3) follows. This completes the proof. $\square$

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