Nodal sets and contact structures

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Nodal sets and contact structures

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I dedicate this thesis to my loving wife Aneta,

my parents, my sister, and my family in Poland.

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SUMMARY

In this work, I investigate the geometry of eigenfields of the curl operator in relation to contact topology, in dimension three. The initial observation, which has driven the rest of this study, was the fact that, under certain assumptions on the Riemannian metric, dividing curves become the zero set of an eigenfunction of the scalar Laplacian on a surface, i.e. the nodal set of an eigenfunction. In the context of Giroux's Theorem, which states how the dividing set controls the isotopy class of a contact structure, I prove that any closed orientable Riemannian surface admits a single contractible nodal curve. This result bears analogies to a conjecture stated by Payne in the setting of the Dirichlet problem. Further, it naturally leads to the proof of existence of an energy minimizing curl eigenfield which is orthogonal to an overtwisted contact structure - a fact conjectured to be false by J. Etnyre and R. Ghrist in their work on the hydrodynamics of contact structures. These results pave a way for further development. Using techniques developed by K. Uhlenbeck, J. Takahashi, and C. Anne, I prove that, for the first eigenfunction on a closed surface, all configurations of curves which divide the surface are nodal curves with respect to some Riemannian metric. Another question I address is whether we can characterize properties of metrics arising from tight or overtwisted contact structures. Extending the initial observation, I show a relation between characteristic surfaces of contact structures and zero sets of solutions to certain subelliptic PDEs. This relation makes it possible to derive, under a symmetry assumption, necessary and sufficient conditions for tightness of contact structures arising from a certain class of invariant curl eigenfields.

CHAPTER I

INTRODUCTION

There is an interesting connection between curl eigenfields and contact topology in dimension 3. This thesis is inspired by this connection and various unanswered questions which surround it. In the following paragraphs, I describe the connection and sketch ideas presented in this work.

From the perspective of functional analysis one studies the following equation on a Riemannian 3-manifold (M^3, g):

$$* \, d\,\alpha = \mu\,\alpha, \qquad \alpha \in \Omega^1(M), \quad \mu \in C^\infty(M), \tag{1}$$

where $\alpha = g(u, \,\cdot\,)$ is a 1-form dual to a vector field u which may be thought of as a velocity field of an inviscid, incompressible fluid on M, [5]. When $\mu = \text{const}$ we are clearly looking at eigenfields of the $*\,d$ operator. Because the $*\,d$ operator generalizes the classical curl operator $\nabla\times$ on $\mathbb{R}^3$, its eigenfields are known as CURL EIGENFIELDS. When μ is not a constant function we refer to these solutions as BELTRAMI FIELDS. Beltrami fields constitute an important class of time-independent STEADY EULER FLOWS, i.e. inviscid, incompressible fluid flows $u(t, \,\cdot\,)$ obeying the Euler equation:

$$u_t + \nabla_u u = -\nabla p, \quad \operatorname{div}(u) = 0, \quad u(0,\cdot) = u_0, \quad p \in C^\infty(\mathbb{R} \times M). \tag{2}$$

In fact one may argue that for a sufficiently complicated topology of the fluid domain M, this class of solutions is the only class of steady Euler flows which exist on M.

There is a natural variational problem associated to the Euler equation (2), and solutions to (1) are stationary points of the variational problem. Specifically, they extremize the kinetic energy of the fluid (i.e. the L^2-energy),

$$E(\alpha) = \int_M \alpha \wedge *\alpha, \tag{3}$$

$$\text{on } \Psi_\alpha = \{\beta : \beta = \varphi_*(\alpha),\ \varphi \in \mathrm{Diff}_0(M),\ \varphi_*(*1) = *1\}.$$

Energy relaxation and the topology of *minimizers* is of particular interest in ideal magnetohydrodynamics (MHD). In MHD one encounters a variational problem for plasmas and magnetic fields that is analogous to (3) (c.f. [5]). In this setting the role of u, $\alpha = g(u, \cdot)$, is played by the magnetic field $\mathbf{B}$, which is "frozen in" the fluid of infinite conductivity filling a star M, i.e. is transported by the velocity field of the plasma. The MHD equations indicate that during the evolution of the star, the kinetic energy $E(u)$ dissipates and the particle motion is ceased. Consequently, the "frozen in" magnetic field reaches a terminal position, and its energy $E(\mathbf{B})$ a minimum (c.f. [5]).

Classical examples of minimizers in the realm of closed 3-manifolds are Hopf fields on $\mathbb{S}^3$, and so called ABC-FIELDS on the flat 3-torus $T^3 \cong S^1 \times S^1 \times S^1$, [5], defined by the equations

$$\begin{aligned} \dot{x} &= A\sin(z) + C\cos(x), \\ \dot{y} &= B\sin(y) + A\cos(z), \\ \dot{z} &= C\sin(x) + B\cos(y). \end{aligned} \tag{4}$$

Connection of curl eigenfields to the world of contact topology comes from the following simple observation: if, in (1), the 1-form $\alpha = g(u, \cdot)$ is nonsingular, i.e. $\alpha \neq 0$ on M, one obtains

$$\alpha \wedge d\,\alpha = \mu \|\alpha\|^2 * 1 \neq 0.$$

This equation implies that the subbundle $\xi = \ker\alpha$ of the tangent bundle TM, i.e. the orthogonal plane distribution to the velocity field u, is an *anti-foliation.* Consequently, by definition, the subbundle ξ is a CONTACT STRUCTURE, i.e a nowhere integrable plane distribution on the manifold M. Contact structures have been an object of intensive study for topologists in recent years, [25, 27, 32, 24]. More specifically the question of the isotopy classification of these structures, in dimension 3, has been subjected to intensive scrutiny. Bennequin and Eliashberg [11], [25], have observed a dichotomy in the isotopy classes, namely two different classes of contact structures: *overtwisted* and *tight*. Eliashberg showed that the isotopy classification of overtwisted contact structures is equivalent to the classification by homotopy among plane distributions. In the case of tight contact structures the classification remains an unsolved problem.

The correspondence of nonsingular curl eigenfields to contact plane distributions raises interesting questions concerning the interplay between fluid dynamical properties of curl eigenfields and topological properties of contact structures. Specifically, one can investigate how the topological tight/overtwisted dichotomy relates to physical properties of a fluid such as helicity, energy minimization, periodic orbits, etc. These kind of questions where were first posed and investigated by Etnyre and Ghrist in series of papers: [29, 28, 37, 30]. Using the powerful tool of *contact homology*, introduced by Eliashberg, Givental, Hofer (c.f. [24]), they proved various theorems about existence of periodic orbits in the fluid flow defined by a nonsingular curl eigenfield. Recently, the same technique led them to a proof of generic instability of curl eigenfields [31].

Etnyre and Ghrist posed various questions and conjectures about the contact topology of curl eigenfields. One of the conjectures concerns the topology of minimizers for the variational principle (3). From [28], p. 17:

> It is very challenging to prove theorems about which smooth fields *minimize* the energy functional. It follows from remarks in Arnold [5] that the Reeb field[1] associated to the standard tight contact form on $\mathbb{S}^3$, as

[1]i.e. the Hopf field on $\mathbb{S}^3$

well as the ABC flows, each *minimize* energy. It thus follows that every known example of a smooth energy-minimizing field is the Reeb field for a *tight* contact structure. This leads to the conjecture that one can always reduce the energy of a Beltrami field associated to an overtwisted contact structure by a volume-preserving diffeomorphism: i.e., the minimal energy representative can only be smooth in the case of a tight fluid.

In Chapter 4 of this thesis, I show that this conjecture is false in full generality. I also identify sufficient conditions involving the topology of the manifold and the Riemannian metric, and prove the conjecture in these circumstances. An essential ingredient of the proof, the observation which motivated many of the ideas presented in this thesis, is that dividing curves, which appear in the convex decomposition theory of Honda, Kazez and Matić [48], become nodal sets of eigenfunctions of the Laplacian in a suitable Riemannian metric. Extending the initial observation in Chapter 3, I show a relation between characteristic surfaces of contact structures and zero sets of solutions to certain subelliptic PDEs. These observations lead into several natural questions, each of individual interest.

Problem 1.0.1. *What is the topology of nodal sets of eigenfunctions of the Laplacian, and how is it controlled by the geometry of the Riemannian metric? Similarly, what can we conclude about nodal sets of solutions to certain elliptic and subelliptic PDEs?*

Problem 1.0.2. *What relation exists between the geometric properties of nonsingular curl eigenfields and the geometry of underlying Riemannian metrics, and topological features of tightness/overtwistedness?*

Problem 1.0.3. *Is there any relation between tightness / overtwistedness and the variational principle* (3)*? Is there a special mechanism of energy relaxation for tight / overtwisted curl eigenfields?*

In this thesis, I address each of the above problems, and present several answers in specific situations. The approach presented in this thesis differs from that of Etnyre and Ghrist since I focus primarily on the geometry of the underlying Riemannian metric. I expect that further research on Riemannian geometric aspects of contact structures will provide more general answers. Each subsequent chapter of this thesis addresses one of the above problems. Parts of Chapter 2 and Chapter 4 resulted in two publications: [52], [38].

Chapter 2 is devoted to Problem 1.0.1 in the setting of an eigen-equation for the scalar Laplacian on a Riemannian surface. I show that one may prescribe arbitrary configurations for nodal curves of the first eigenfunction. I also prove that certain constraints on the scalar curvature and eigenvalues of the surface force nodal curves to be homotopically essential. These results are related to questions asked by Schoen and Yau in [63].

Chapter 3 is devoted to Problem 1.0.2. I investigate adapted metrics to contact structures and their relation to the topology of contact structures. In Chapter 3 I show the relation between characteristic surfaces of contact structures and zero sets of solutions to certain subelliptic PDEs, then I show necessary and sufficient conditions which force tightness of a certain class of invariant curl eigenfields. The only previously known result pertaining to Problem 1.0.2 is in the special case of K-CONTACT structures (c.f. [10]). In particular Belgun shows that all K-contact structures are tight. The ideas presented in [10], as well as conclusions of Chapter 1, are important ingredients in the main theorem of Chapter 3.

In Chapter 4 we present our construction of an overtwisted energy minimizer and, therefore a negative answer to the question of Etnyre and Ghrist. I also indicate examples of tight curl eigenfields minimizing the energy (3), which may be of some importance for further investigation of Problem 1.0.3.

CHAPTER II

ON THE TOPOLOGY OF NODAL SETS

If we think of a given Riemannian surface (Σ, g_Σ) as a vibrating membrane with $u(\mathbf{x}, t)$, $\mathbf{x} \in \Sigma$, the displacement u of the membrane from the original position at time t, u is a solution to the wave equation

$$\partial_{tt} u = \Delta_\Sigma u. \tag{5}$$

For a separable solution, i.e. $u(t, \mathbf{x}) = v(t)w(\mathbf{x})$, we obtain an equivalent system of equations $\partial_{tt} v = \lambda v$ and $\Delta_\Sigma w = \lambda w$, $(\lambda \in \mathbb{R})$. Therefore, the "stagnation points" on the membrane are exactly zeros of the eigenfunction w. This zero set, $\Xi(w) := \{\mathbf{x} \in \Sigma : w(\mathbf{x}) = 0\}$, is called the NODAL SET and forms interesting patterns, as originally studied by E. Chladni in the 18th century.

The goal of this chapter is to prove a variety of results regarding topology and geometry of nodal sets in dimension 2. We pursue questions in the spirit of an open problem #45, stated by Schoen and Yau in [63], p. 384:

> Melas ([55]), recently proved that the nodal line of any second eigenfunction cannot enclose a compact subregion of a bounded convex domain. Is there a similar conclusion for higher dimensional euclidean space? To what extend do these conclusions hold for compact manifolds with boundary? What is the topology of nodal sets of higher eigenvalues? For example, can one find an infinite sequence of eigenfunctions, which domains are disjoint union of cells?

We focus mostly on the case of generic metric on 2-dimensional manifolds, where the nodal set is a union of embedded circles. Our main technical result is the Gluing

Theorem 2.3.1, which says that given a fixed surface with the "prescribed" nodal curves one may form a connected sum by gluing a finite number of surfaces, and produce a metric on it with a nodal set isotopic to the original one. As a consequence of Gluing Theorem, we prove a version of Payne's conjecture for closed orientable surfaces, namely we prove that one may always construct a generic metric on any orientable surface such that the nodal set of the λ_1-eigenfunction is a circle bounding a disc. We have an independent interest in this problem due to connections with contact topological properties of eigenfields of curl eigenfields.

2.1 Outline and terminology

Here, all manifolds, unless stated otherwise, are equipped with a Riemannian metric, and are compact smooth and orientable with or without boundary. Throughout the article $C^j(M)$ stands for the set of j-differentiable functions on M, with $j = \infty$ smooth. Spaces $L^2(M)$, $H^j(M)$ are customary, square integrable real functions, and the Sobolev space of real valued functions with at least j bounded weak derivatives. The space $\Omega^k(M) = C^\infty(\Lambda^k M)$ is a set of smooth real valued k-differential forms on M making $\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M)$ a graded $C^\infty(M)$ module over $\mathbb{R}$, where $n = \dim(M)$. Here we denote by $L^2(\Lambda^k M)$ and $H^j(\Lambda^k M)$, respectively, the square integrable, and the Sobolev spaces of k-differential forms, where the measure is induced from the Riemannian metric. The Riemannian metric also induces an L^2-isometry: $* : \Omega^k(M) \to \Omega^{n-k}(M)$, namely the HODGE STAR OPERATOR. Consequently, we obtain de Rham graded complexes $(\Omega^*(M), d)$ and $(\Omega^*(M), \delta)$, where $d \equiv d^k : \Omega^k(M) \to \Omega^{k+1}(M)$ is an exterior derivative (also called a DIFFERENTIAL), and $\delta \equiv \delta^k : \Omega^{k+1}(M) \to \Omega^k(M)$ an adjoint of d (also called a CO-DIFFERENTIAL) given in terms of the Hodge star by $\delta^k = (-1)^{nk+1} * d^{n-k} *$ or equivalently as a formal adjoint of d,

$$(d^k\omega, \eta)_{L^2(\Lambda^{k+1}M)} = (\omega, \delta^k\eta)_{L^2(\Lambda^k M)}, \qquad \omega \in \Omega^k(M), \eta \in \Omega^{k+1}(M).$$

Most of the time we skip the superscripts in the notation for differentials and co-differentials and simply write d and δ. The Laplacian on k-forms is defined by $\Delta = \delta\, d + d\, \delta$, which in the case of functions reduces to $\Delta = \delta\, d$ (for further reference consult [61] or [7]). We also introduce the following notation for nodal sets. Let $\Xi(M, f) = \{x \in M : f(x) = 0\}$ stand for the zero set of the function f. In the case $f = f_k$, where f_k is a kth-eigenfunction of Δ_M, we write $\Xi(M, k) := \Xi(M, f_k)$. Here, the term EIGENFUNCTION refers to an eigenfunction of the scalar Laplacian, unless specified otherwise.

In the first section of this chapter we state several fundamental results concerning eigenvalues and topology of nodal domains. Section 2.3 is devoted to our main technical theorem, which we call the *Gluing Theorem.* As a consequence of this theorem we show, in Sections 2.4, 2.5, that all possible configurations of nodal curves for λ_1-eigenfunction may be archived on a surface of an arbitrary genus equipped with an appropriate generic metric.

2.2 Eigenvalues and Nodal sets

The Laplace-Beltrami operator (or simply the LAPLACIAN $\Delta_M = \delta\, d$) is a positive formally self-adjoint operator on any closed orientable Riemannian manifold (M, g). By the standard spectral theory of formally self-adjoint operators, the L^2-spectrum of Δ_M is real and countable,

$$0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \cdots \leq \lambda_k(M) \leq \cdots \to \infty, \tag{6}$$

and one can choose an orthonormal basis of eigenvectors $\{f_i\}_{i \in \mathbb{N} \cup \{0\}}$, $\|f_k\|_{L^2(M)} = 1$ in $L^2(M)$, smooth by regularity, c.f. [33], and satisfying (for $\lambda = \lambda_j(M)$)

$$\begin{cases} \Delta_M u = \lambda\, u, \\ u \restriction_{\partial M} = 0\,. \end{cases} \tag{7}$$

In case when the membrane M is a closed surface we drop the boundary condition and refer to the problem (7) as the FREE MEMBRANE PROBLEM. For surfaces with a boundary, and Dirichlet boundary conditions in (7), we refer to the problem as the FIXED MEMBRANE PROBLEM (see [20]).

Given a nodal set $\Xi(M,k)$ of a k-th eigenfunction, we refer to regions in $M \setminus \Xi(M,k)$ as NODAL DOMAINS. One of the fundamental results is the Courant's nodal domain theorem.

Theorem 2.2.1 (Courant's nodal domain theorem). *For the fixed membrane problem, the number of nodal domains of the i-th eigenvalue λ_i is not greater then i. For the free membrane problem the number of nodal domains of the i-th eigenvalue λ_i is not greater then $i+1$.*

Since $\int f_k = 0$ we conclude that $\Xi(M,k)$ is never empty and obtain the following:

Corollary 2.2.2. *In case of the fixed membrane problem a λ_2-eigenfunction has exactly two nodal domains, for the free membrane problem, λ_1-eigenfunction has exactly two nodal domains.*

First, we review Rayleigh's variational principle for eigenvalues. Let $B : H^1(M) \times H^1(M) \to \mathbb{R}$ be the bilinear form associated to the Laplacian Δ_{M_1}. Recall that for smooth functions u and w

$$B(u,w) = (\Delta_M u, w)_{L^2(M)} = (\delta\, d\, u, w)_{L^2(M)} = (d\, u, d\, w)_{L^2(\Lambda^1 M)}. \tag{8}$$

The last equality extends the definition of B to $H^1(M)$.

Theorem 2.2.3 (Rayleigh's Theorem). *Let $\{f_k\}$ be a complete orhonormal basis of $L^2(M)$ of eigenfunctions, then for $u \in H^1(M)$ satisfying*

$$(u, f_1)_{L^2} = (u, f_2)_{L^2} = \ldots = (u, f_{k-1})_{L^2} = 0, \tag{9}$$

i.e. $u \in \text{SPAN}(f_1, \ldots, f_{k-1})^\perp$, $u \in H^1(M)$, we have the RAYLEIGH'S QUOTIENT*:*

$$\lambda_k(M) \leq \frac{B(u,u)}{\|u\|_{L^2}}, \tag{10}$$

with equality iff u is a λ_k-eigenfunction.

Proof. For any $u \in H^1(M)$ let

$$\alpha_j = (u, f_j)_{L^2}$$

By (9): $\alpha_1 = \ldots = \alpha_{k-1} = 0$, let $r > k-1$, we obtain

$$\begin{aligned} 0 &\leq B\left[u - \sum_{j=k}^{r} \alpha_j\, f_j\,,\, u - \sum_{j=k}^{r} \alpha_j\, f_j\right] \\ &= B[u,u] - 2\sum_{j=k}^{r} \alpha_j\, B[u, f_j] + \sum_{j,l=k}^{r} \alpha_j \alpha_l\, B[f_j, f_l] \\ &= B[u,u] - 2\sum_{j=k}^{r} \alpha_j (u, \Delta f_j)_{L^2} + \sum_{j,\,l=k}^{r} \alpha_j \alpha_l\, (f_j, \Delta f_l)_{L^2} \\ &= B[u,u] - \sum_{j=k}^{r} \lambda_j\, \alpha_j^2, \end{aligned}$$

Since the above equality holds for any $r > 0$, we obtain

$$B[u,u] \geq \sum_{j=k}^{\infty} \lambda_j\, \alpha_j^2 \geq \lambda_k \sum_{j=k}^{\infty} \alpha_j^2 = \lambda_k \|u\|_{L^2}$$

where the last inequality holds due to (6) and Parseval's identity. □

Theorem 2.2.4 (Max-Min Principle [19]). *(1) Given $\phi_1, \ldots, \phi_{k-1} \in L^2(M)$, let*

$$\mu = \inf_u \frac{B(u,u)}{\|u\|_{L^2}}, \qquad u \in \mathrm{SPAN}(\phi_1, \ldots, \phi_{k-1})^{\perp}, \quad u \in H^1(M).$$

Then for eigenvalues in (6), *we have*

$$\mu \leq \lambda_k(M),$$

and we have equality iff each ϕ_i is a λ_i-eigenfunction.

(2) Similarly if $\{\phi_1 \ldots \phi_k\}$ span k-dimensional subspace and

$$\mu = \sup_u \frac{B(u,u)}{\|u\|_{L^2}}, \qquad u \in \mathrm{SPAN}(\phi_1, \ldots, \phi_k) \subset H^1(M).$$

Then for eigenvalues in (6), *we have*

$$\lambda_k(M) \leq \mu$$

Proof. Notice that one may choose a function f in the form,

$$u = \sum_{j=1}^{k} \alpha_j f_j \tag{11}$$

i.e. a linear combination of λ_j-eigenfunctions f_j, $j = 1, \dots k-1$, which is orthogonal to $\text{SPAN}(\phi_1, \dots, \phi_{k-1})$. Indeed it is equivalent to solving a system of $k-1$ equations: $(u, \phi_n)_{L^2} = \sum_{j=1}^{k} \alpha_j (f_j, \phi_n) = 0$ for k unknowns: α_j. We obtain then,

$$\mu \|u\|^2 \leq B[u,u] = \sum_{j=1}^{k} \lambda_j \alpha_j^2 \leq \lambda_k \|u\|^2,$$

which implies the claim. Analogously we show *(2)*, simply one shows that there exist $u = \sum_i \beta_i \phi_i$, which belongs to $\text{SPAN}(\phi_1, \dots, \phi_k)^{\perp}$. □

Proof of Courant's Theorem 2.2.1. assume by contradiction that $\Omega_1, \dots, \Omega_k, \Omega_{k+1}, \dots, \Omega_r$ are nodal domains of a λ_k-eigenfunction f_k. For each $j = 1, \dots, k$ we define

$$\phi_j = \begin{cases} f_k|_{\Omega_j}, & \text{on } \Omega_j; \\ 0, & \text{on } \bar{M} \setminus \Omega_j. \end{cases} \tag{12}$$

By *(1)* in the Min-Max Theorem 2.2.4 there is a function u in the form given by (11), orthogonal to $\text{SPAN}(\phi_1, \dots, \phi_{k-1})$, one verifies that $u \in H^1(M)$. Then by Theorem 2.2.3, and 2.2.4 we obtain

$$\lambda_k \leq \frac{B(u,u)}{\|u\|_{L^2}} \leq \lambda_k,$$

and therefore u is a λ_k-eigenfunction of Δ, vanishing identically on Ω_{k+1}, which contradicts the maximum principle [33]. □

As the corollary one derives

Theorem 2.2.5 (Domain of monotonicity of eigenvalues [19])**.** *Let $\Omega_1, \dots, \Omega_m$, be pairwise disjoint regular domains in M whose boundaries intersect ∂M transversally.*

We arrange all first eigenvalues (from (7)*) of* $\Omega_1, \ldots, \Omega_m$ *in the increasing order:*

$$0 \leq \mu_1 \leq \mu_2 \leq \ldots \mu_m$$

then for all $k = 1, \ldots m$ *we have*

$$\lambda_k \leq \mu_k.$$

For an arbitrary smooth Riemannian surface (Σ, g_Σ) nodal sets have been characterized, in [20], by S. Cheng, where it is proved that the nodal set is a collection of C^2-immersed closed curves in Σ.

Theorem 2.2.6 (Cheng [20], p. 49)**.** *Suppose* Σ *is a 2-dimensional manifold, then for any solution of the equation* $(\Delta_\Sigma + h)f = 0$*,* $h \in C^\infty(\Sigma)$*, the following are true:*
(1) The critical points in nodal curves are isolated.
(2) When the nodal curves meet, they form an equiangular system.
(3) The nodal curves consist of a number of C^2*-immersed one-dimensional closed submanifolds. Therefore, when* Σ *is compact they are number of* C^2*-immersed circles.*

For a generic metric K. Uhlenbeck showed, in [66], that these curves are embedded circles with no critical points. Let (M, g_0) be a Riemannian manifold with a fixed C^k-metric g_0, and let $\mathfrak{M}_k$ be a set of metrics which differ from g_0 on some open subset $U \subset M$, namely

$$\mathfrak{M}_{g_0,k} = \{g \in C^k(M, TM \otimes TM) : g \text{ is a metric tensor, } (g_0 - g) \restriction_{M \setminus U} = 0\}. \tag{13}$$

We call $\mathfrak{M}_{g_0,k}$ the SET OF PERTURBATIONS OF g_0.

Theorem 2.2.7 (Uhlenbeck [66], p. 1076)**.** *Let* Δ_g *be the Laplace operator on* (M^n, g) *for a metric* $g \in \mathfrak{M}_{g_0,k}$*. Then for* $k > n+3$*, the subset of metrics for which* Δ_g *satisfies properties (1)-(3), stated below, for non-constant eigenfunctions is residual in* $\mathfrak{M}_k$*.*
(1) Δ_g *has 1-dimensional eigenspaces.*

(2) Zero is not a critical value of the eigenfunction restricted to the interior of the domain of the operator.

(3) The eigenfunctions are Morse functions on the interior of the domain of the operator.

In the following sections we apply the term GENERIC METRIC to a metric which belongs to the residual set of metrics satisfying properties *(1) - (3)*. Recall that a subset $A \subset X$ is RESIDUAL in X iff it is a countable intersection of open dense subsets of X.

2.3 The Gluing Theorem

In this section I outline the proof of the following;

Theorem 2.3.1 (Gluing Theorem). *Let (Σ_0, g_0) be a closed Riemannian surface and g_0 be a smooth generic metric. Let f_k be $\lambda_k(\Sigma)$-eigenfunction on Σ_0 with exactly $k+1$ nodal domains. Choose m points: $\{x_1, x_2, \ldots, x_m\} \subset \Sigma_0 \setminus \Xi(\Sigma, f_k)$, and form an arbitrary connected sum: $\widetilde{\Sigma} = \Sigma_0 \# \Sigma_1 \# \ldots \# \Sigma_m$, by attaching m-surfaces: $\Sigma_1, \ldots, \Sigma_m$ called* HANDLES, *along small geodesic discs $D_i \subset \Sigma_0$ around points $\{x_i\}$.*

There exists a generic metric $\tilde{g}$ on $\widetilde{\Sigma}$ such that $g \restriction_{\Sigma \setminus \cup_i D_i} = \tilde{g} \restriction_{\widetilde{\Sigma} \setminus \cup_i D_i}$ and for which the nodal set $\Xi(\widetilde{\Sigma}, \tilde{f}_k)$, of the k-th eigenfunction $\tilde{f}_k$ is isotopic to $\Xi(\Sigma_0, f_k)$ in Σ_0.

The technique of the proof is based on the work of J. Takahashi, [65], about collapsing connected sums of surfaces, which is in turn based on work of C. Anné, [4]. The main idea of the proof is to start with a nodal set of k'th eigenfunction on the surface Σ_0, and "implant" m, ε-small handles $\{\Sigma_i\}$ of a given genus, forming a connected sum $\widetilde{\Sigma} = \Sigma_0 \# \Sigma_1 \# \ldots \# \Sigma_m$. Letting $\varepsilon \to 0$, i.e. collapsing the handles to centers of attaching discs we show that the nodal set $\Xi(\widetilde{\Sigma}, k)$ *converges* to the nodal set $\Xi(\Sigma_0, k)$.

First, observe the following elementary construction: If we choose an embedded contractible 2-disc D^2 in an orientable surface Σ_0 and define $\Sigma_0' \cong \Sigma_0 \setminus \mathrm{Int}(D^2)$,

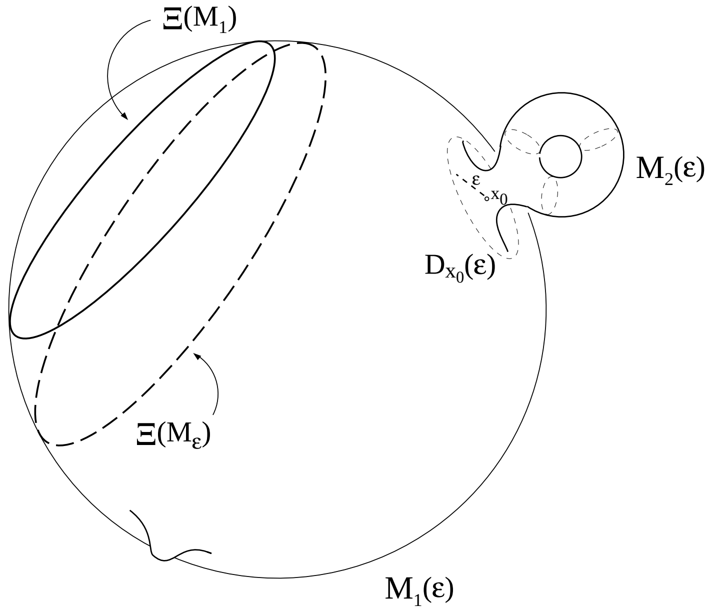

Figure 1: For small ε, nodal sets $\Xi(M_\varepsilon)$ and $\Xi(M_1)$ have to be "close" in $M_\varepsilon = (M_1(\varepsilon) \cup_{\Phi_\varepsilon} M_2(1), \tilde{g}_\varepsilon)$.

then we repeat this procedure for another orientable surface Σ_1 and define $\Sigma_1' \cong \Sigma_1 \setminus \mathrm{Int}(D^2)$. For an an orientation reversing diffeomorphism $\Phi : \partial\Sigma_0' \to \partial\Sigma_1'$ of the boundaries $\partial\Sigma_0'$, Σ_1' we form a topological manifold $\Sigma_\Phi = \Sigma_0' \cup_\Phi \Sigma_1'$, by gluing surfaces Σ_0', Σ_1' along Φ (c.f. [35]). Since Σ_Φ is homeomorphic to the connected sum $\Sigma_0 \# \Sigma_1$ we can make Σ_Φ into a smooth manifold by pulling back the differential structure from $\Sigma_0 \# \Sigma_1$ so that orientations of Σ_0' and Σ_1' induce the orientation of Σ_Φ. All Σ_Φ obtained this way are diffeomorphic. If we equip Σ_0' and Σ_1' with smooth Riemannian metrics g_0' and g_1' we can define a piecewise smooth metric g on Σ_Φ as follows

$$\widetilde{g} = \begin{cases} g_0' & \text{on } \Sigma_0', \\ g_1' & \text{on } \Sigma_1'. \end{cases}$$

Now, $\widetilde{g}$ is continuous on Σ_Φ if the gluing map Φ is an isometry of the boundaries $\partial\Sigma_0'$, $\partial\Sigma_1'$. In the case Φ admits an extension to the smooth isometry of tubular neighborhoods of boundaries $\partial\Sigma_i'$, $i = 0, 1$, the metric $\widetilde{g}$ is smooth as well. Clearly these considerations extend to an arbitrary finite connected sum: $\widetilde{\Sigma} = \Sigma_0 \# \Sigma_1 \# \ldots \# \Sigma_m$.

Now, we indicate an explicit case of the above construction, suitable for the proof of Theorem 2.3.1. For simplicity assume $m = 1$ (see Figure 1).

Let $M_1 = (\Sigma_0, g_0)$ be a an orientable surface equipped with a generic metric g_0. Consider a nodal set $\Xi(M_1, k)$ of a k-th eigenfunction f_k on M_1, and let $x_0 \notin \Xi(M_1, k)$. Let $D^2_{x_0}(\varepsilon) \subset U_{x_0}$ be a geodesic disc around x_0 of radius $\varepsilon \ll d$, smaller than a geodesic distance d between x_0 and $\Xi(M_1, k)$. We define $M_1(\varepsilon) = (M_1 \setminus \mathrm{Int}(D^2_{x_0}(\varepsilon)), g_0)$, which is diffeomorphic to Σ_0'.

(For simplicity, one may assume without loss of generality, [3], that for sufficiently small ε the metric g_0 is Euclidean on $D^2_{x_i}(\varepsilon)$, see [65], [3].)

In order to obtain a metric on Σ_1' we simply choose an arbitrary smooth metric g_1 on Σ_1, flat around a given point x_1', and a geodesic disc $D^2_{x_1}(r)$ of radius r which belongs to the flat neighborhood. Clearly, $\Sigma_1 \setminus D^2_{x_1}(r)$ is diffeomorphic to Σ_1'. A disc $D^2_{x_1}(r)$ of radius r in g_1 corresponds to a disc of radius 1: $D^2_{x_1}(1)$ in the rescaled metric $r^2\, g_1$. Define $M_2(1) = (\Sigma_1', \varepsilon^2 g_1)$.

For any $\varepsilon > 0$, choose local coordinates (x, y) such that the geodesic disc $D^2_{x_1}(\varepsilon)$ is an ε disc on $(\mathbb{R}^2, d^2s)$ and $D^2_{x_1}(1)$ is a unit disc on $(\mathbb{R}^2, \varepsilon^2 d^2 s)$, where $ds^2 = dx^2 + dy^2$. Observe that the boundaries $\partial M_1(\varepsilon)$, $\partial M_2(1)$ can be glued via an isometry Φ_ε of $(\mathbb{R}^2, d^2s)$ and $(\mathbb{R}^2, \varepsilon^2 d^2 s)$ restricted to a circle of radius ε in $(\mathbb{R}^2, d^2s)$. The isometry Φ_ε can be defined as

$$\Phi_\varepsilon : x \to -(1/\varepsilon) x. \tag{14}$$

By the discussion in the first paragraph of this section we can form a smooth manifold $M = M_1(\varepsilon) \cup_{\Phi_\varepsilon} M_2(1)$ with the orientation induced from $M_1(\varepsilon)$ and $M_2(1)$. We define a piecewise smooth and continuous metric on M

$$\widetilde{g}_\varepsilon = \begin{cases} g_0 & \text{on } M_1(\varepsilon), \\ \varepsilon^2 g_2 & \text{on } M_2(1), \end{cases} \tag{15}$$

see also [65]. Clearly, these considerations extend to an arbitrary finite connected sum i.e. when $m > 1$. Simply define: $M_1(\varepsilon) = (M_1 \setminus \bigcup_{i=1}^m D^2_{x_i}(\varepsilon), g_0)$, $M_2(1) = \bigsqcup_{i=1}^m (\Sigma_i \setminus D^2_{x_i'}(1), \varepsilon^2 g_i)$, and $M = M_1(\varepsilon) \cup_{\Phi_\varepsilon} M_2(1)$, where Φ_ε is defined in flat coordinates by (14) on each $\partial D^2_{x_i}(\varepsilon)$, and is gluing $\partial M_1(\varepsilon)$ to $\partial M_2(1)$. We summarize our notation below,

(a) $M = M_1(\varepsilon) \cup_{\Phi_\varepsilon} M_2(1)$,

(b) $M_1 = (\Sigma_0, g_0)$, $M_1(\varepsilon) = (\Sigma_0 \setminus \bigcup_{i=1}^m D^2_{x_i}(\varepsilon), g_0)$,

(c) $M_2(1) = \bigsqcup_{i=1}^m (\Sigma_i \setminus D^2_{x_i'}(1), \varepsilon^2 g_i)$,

(d) $M_\varepsilon = (M, \widetilde{g}_\varepsilon)$.

If we must specify a different metric on a manifold, we write e.g. $(M_2(\varepsilon), \hat{g})$.

Remark 2.3.2. In the above construction one obtains a piecewise smooth metric $\widetilde{g}_\varepsilon$. However, we may produce a smooth metric by a simple modification. Namely, cut out of the geodesic disc $D_{x_1}(\varepsilon) \subset M_1$, a smaller disc $D_{x_1}(\varepsilon/4)$, and attach Σ_2' to the annuli $A_\varepsilon = D_{x_1}(\varepsilon) \setminus D_{x_1}(\varepsilon/4)$ along the boundary $\partial D_{x_1}(\varepsilon/4)$, by an orientation reversing diffeomorphism $\Psi : \partial \Sigma_1' \to \partial D_{x_1}(\varepsilon/4)$. Now, extend smoothly the metric from A_ε to $\Sigma_1' \cong A_\varepsilon \cup_\Psi \Sigma_1'$, which results in a metric g_1 on Σ_1' which may be glued isometrically by the antipodal map.

2.3.1 Convergence of eigenvalues and eigenfunctions.

In [65], Takahashi shows the following convergence of eigenvalues for piecewise smooth metrics defined in (15).

Theorem 2.3.3 ([65])**.** *For all $k = 0, 1, 2, \ldots,$ we have*

$$\lim_{\varepsilon \to 0} \lambda_k(M_\varepsilon) = \lambda_k(M_1). \tag{16}$$

Because we essentially use parts of the proof in our later considerations we state the proof in the following paragraphs. Notice that one must define eigenvalues $\lambda_k(M_\varepsilon)$ in a piecewise smooth metric $\widetilde{g}_\varepsilon$ on M, we refer the reader to [65] for general definitions. Here we avoid these technicalities by assuming that $\widetilde{g}_\varepsilon$ is smooth, as explained in Remark 2.3.2.

One proves Theorem 2.3.3 in several steps, beginning with technical lemmas.

Lemma 2.3.4 ([65]). *For all $k = 0, 1, 2, \ldots,$ we have*

$$\limsup_{\varepsilon \to 0} \; \lambda_k(M_\varepsilon) \; \leq \lambda_k(M_1). \tag{17}$$

Proof. Let f_i be the i'th eigenfunction on M_1, with eigenvalue $\lambda_k(M_1)$, and $\{f_1, \ldots f_k\}$, an orthonormal set. Define the cut off function $\chi_\varepsilon : [0, \infty) \to [0, 1]$ as,

$$\chi_\varepsilon(r) = \begin{cases} 0, & (0 \leq r \leq \varepsilon), \\ -\frac{2}{\log \varepsilon} \log\left(\frac{r}{\varepsilon}\right), & (\varepsilon \leq r \leq \sqrt{\varepsilon}), \\ 1, & (r \geq \sqrt{\varepsilon}), \end{cases} \tag{18}$$

(see [4]). We set $\chi_\varepsilon(x) = \chi_\varepsilon(d_{g_0}(x_1, x))$, for $x \in M_1$, where d_{g_0} is the distance induced from g_0. Let $E_\varepsilon = \textsc{span}\{\chi_\varepsilon\, f_1, \ldots, \chi_\varepsilon\, f_k\}$ we consider E_ε as a subspace of $H^1(M_\varepsilon)$ by extending each $\chi_\varepsilon\, f_j$ to M by zero. Applying *(2)*, in Theorem 2.2.4, to E_ε we obtain

$$\lambda_k(M_\varepsilon) \leq \sup_{u \in E_\varepsilon} \left(\frac{B_{M_\varepsilon}[u, u]}{\|u\|^2_{L^2(M_\varepsilon)}} \right) \tag{19}$$

In $\dim M = 2$ we have

$$\begin{aligned} \|d\chi_\varepsilon\|^2_{L^2(D^2(x_1, \sqrt{\varepsilon}))} = (d\chi_\varepsilon, d\chi_\varepsilon)_{L^2(D^2(x_1, \sqrt{\varepsilon}))} &= \int_{D^2(x_1, \sqrt{\varepsilon})} d\chi_\varepsilon \wedge * d\chi_\varepsilon \\ &= \frac{4\, \mathrm{Vol}(S^1)}{(\ln \varepsilon)^2} \int_\varepsilon^{\sqrt{\varepsilon}} r^{-1} dr \to 0. \end{aligned} \tag{20}$$

as $\varepsilon \to 0$. Therefore

$$\begin{aligned} B_{M_\varepsilon}[\chi_\varepsilon\, f_i, \chi_\varepsilon\, f_i] &= \int_{M_\varepsilon} \langle d(\chi_\varepsilon\, f_i), d(\chi_\varepsilon\, f_i) \rangle = \int_{M_\varepsilon} \langle f_i d\chi_\varepsilon + \chi_\varepsilon df_i, f_j d\chi_\varepsilon + \chi_\varepsilon df_j) \rangle \\ &= \int_{M_\varepsilon} \langle f_i d\chi_\varepsilon, f_j d\chi_\varepsilon \rangle + \int_{M_\varepsilon} \chi_\varepsilon^2 \langle df_i, df_j \rangle + \int_{M_\varepsilon} f_i \langle d\chi_\varepsilon, df_j \rangle + \int_{M_\varepsilon} f_j \langle d\chi_\varepsilon, df_i \rangle, \end{aligned}$$

and we obtain

$$\begin{aligned}
\int_{M_\varepsilon} \langle f_i d\chi_\varepsilon, f_j d\chi_\varepsilon\rangle &\leq \|f_i f_j\|_{L^\infty(M_1)}\|d\chi_\varepsilon\|^2_{L^2(D^2(x_1,\sqrt{\varepsilon}))} \to 0, \text{ as } \varepsilon \to 0,\\
\int_{M_\varepsilon} \chi_\varepsilon^2\langle df_i, df_j\rangle &\leq (df_i, df_j)_{L^2(M_1)} = (\Delta_{M_1} f_i, f_j)_{L^2(M_1)} = \lambda_i(M_1)\delta^i_j,\\
\int_{M_\varepsilon} f_i\langle d\chi_\varepsilon, df_j\rangle &\leq \|f_i df_j\|_{L^2(M_1)}\|d\chi_\varepsilon\|_{L^2(M_1)} \leq C\|d\chi_\varepsilon\|_{L^2(D^2(x_1,\sqrt{\varepsilon}))} \to 0, \text{ as } \varepsilon \to 0.
\end{aligned}$$

By (19), we conclude

$$\lambda_k(M_\varepsilon) \leq \lambda_k(M_1) + \eta_\varepsilon, \qquad \eta_\varepsilon \to 0, \text{ as } \varepsilon \to 0.$$

$\square$

For the next step, in the proof of Theorem 2.3.3, we define

$$\alpha_k = \liminf_{\varepsilon\to 0}\ \lambda_k(M_\varepsilon). \tag{21}$$

In order to finish the proof we must show: $\lambda_k(M_1, g_1) \leq \alpha_k$. However, this requires several technical results.

Lemma 2.3.5 (Extension Lemma)**.** *Let g be an arbitrary smooth metric on M_1. Given $u \in C^\infty(M_1(\varepsilon)) \cap C^0(\overline{M_1(\varepsilon)})$, there exists a function $\bar{u} \in H^1(M_1, g)$, which is an extension of u, i.e. $\bar{u}|_{M_1(\varepsilon)} = u$ such that*

$$\|\bar{u}\|_{H^1(M_1,g)} \leq C\|u\|_{H^1(M_1(\varepsilon),g)}, \tag{22}$$

and C is independent of ε. For $l > 1$, and $u \in C^\infty(M_1(\varepsilon/2)) \cap C^0(\overline{M_1(\varepsilon/2)})$ we can find an extension $\bar{u} \in C^\infty(M_1)$, such that $\bar{u}|_{M_1(\varepsilon)} = u$ and

$$\|\bar{u}\|_{H^l(M_1,g)} \leq C'_{l,\varepsilon}\|u\|_{H^l(M_l(\varepsilon/2),g)}, \tag{23}$$

$$\|\Delta_{M_1}\bar{u}\|_{H^{l-2}(M_1,g)} \leq C''_{l,\varepsilon}\big(\|u\|_{H^{l-2}(M_l(\varepsilon/2),g)} + \|du\|_{H^{l-2}(\Lambda^1 M_l(\varepsilon/2),g)} + \|\Delta_{M_1}u\|_{H^{l-2}(M_l(\varepsilon/2),g)}\big) \tag{24}$$

where constants $C''_{l,\varepsilon}$, $C'_{l,\varepsilon}$ depend on l and ε.

Proof. The proof for $l = 1$ and dim=2 is given in [60] p. 40, where the authors show that for the unique harmonic extension the constant C is independent of ε. For the proof in the case $l > 1$ we follow the standard extension argument see ([39]). We define $\bar{u} = u\rho$, where $\rho \in C^\infty(M_1)$ is a "bump" function such that $\rho|_{M_1(\varepsilon)} = 1$, and $\mathrm{supp}(\rho) \subset M_1(\varepsilon/2)$. Inequality (23) follows immediately, and (24) is a consequence of the triangle inequality and the following product formula,

$$\Delta_{M_1}(\bar{u}) = \Delta_{M_1}(u\rho) = u\,\Delta_{M_1}\rho - 2g(\nabla u, \nabla\rho) + \rho\,\Delta_{M_1}u, \tag{25}$$

which holds pointwise (see e.g. [61]). □

For the given eigenfunction f_k^ε on M_ε we introduce the following notation:

$$f_k^\varepsilon = (f_k^{1,\varepsilon}, f_k^{2,\varepsilon}), \qquad \text{where} \qquad f_k^{1,\varepsilon} = f_k^\varepsilon|_{M_1(\varepsilon)},\ f_k^{2,\varepsilon} = f_k^\varepsilon|_{M_2(1)}. \tag{26}$$

In the following lemma the argument is essentially the same as in [65], p. 206.

Lemma 2.3.6 (L^2-convergence of eigenfunctions)**.** *For each k, one may choose a sequence of eigenfunctions $f_k^\varepsilon \in C^\infty(M_\varepsilon)$, $\|f_k^\varepsilon\|_{L^2(M_\varepsilon)} = 1$, with the following limit in $L^2(M_1)$*

$$\lim_{\varepsilon\to 0} f_k^{1,\varepsilon} = \hat{f}_k, \qquad in \quad L^2(M_1), \tag{27}$$

where $\hat{f}_k \in C^\infty(M_1)$ is a α_k-eigenfunction on M_1.

Proof. We prove that there is a family of extensions $\{\hat{f}_k^{1,\varepsilon}\}_\varepsilon$, $\hat{f}_k^{1,\varepsilon}|_{M_1(\varepsilon)} = f_k^{1,\varepsilon}$, convergent in $L^2(M_1)$ to $\hat{f}_k$. Choosing $\hat{f}_k^{1,\varepsilon}$ to be the H^1-extensions of $f_k^{1,\varepsilon} \in C^\infty(M_1(\varepsilon))$ given by Extension Lemma 2.3.5, we have the following;

$$\|\hat{f}_k^{1,\varepsilon}\|_{H^1(M_1)} \leq C\|f_k^{1,\varepsilon}\|_{H^1(M(\varepsilon),g_1)} \tag{28}$$

where C is independent of ε. From (28) we obtain

$$\begin{aligned}\|\hat{f}_k^{1,\varepsilon}\|_{H^1(M_1)} &\leq C\|f_k^\varepsilon\|_{H^1(M_1(\varepsilon),g_1)} \overset{(1)}{\leq} C\big(\|f_k^\varepsilon\|_{L^2(M_\varepsilon)} + \|df_k^\varepsilon\|_{L^2(\Lambda^1 M_\varepsilon)}\big) \\ &= C\big(1 + (\Delta_{M_\varepsilon} f_k^\varepsilon, f_k^\varepsilon)^{\frac{1}{2}}_{L^2(M_\varepsilon)}\big) \overset{(2)}{\leq} C(1 + \lambda_k^{\frac{1}{2}}(M_1) + \eta_\varepsilon),\end{aligned} \tag{29}$$

where $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$. Inequality (1) follows from the definition of the H^1-norm, (15) and the fact that $M_1(\varepsilon) \subset M_\varepsilon$; the equality (2) is the consequence of Lemma 2.3.4. We conclude that the family: $\{\hat{f}_k^{1,\varepsilon}\}_\varepsilon$ is bounded in $H^1(M_1)$, thus any sequence in the family contains a weakly convergent subsequence in $H^1(M_1)$. By Rellich's Theorem, the inclusion $H^1(M_1) \hookrightarrow L^2(M_1)$ is compact, thus any sequence in the family $\{\hat{f}_k^{1,\varepsilon}\}_\varepsilon$ contains a strongly convergent subsequence $\{\hat{f}_k^{1,\varepsilon_i}\}_{\varepsilon_i}$ in $L^2(M_1)$. We choose a subsequence in $\{\hat{f}_k^{1,\varepsilon}\}_\varepsilon$, such that $\lambda_k(M_\varepsilon) \to \alpha_k$ and denote a limit of the subsequence by $\hat{f}_k \in H^1(M_1)$. We wish to show that $\hat{f}_k$ is a smooth classical solution to $\Delta_{M_1} u = \alpha_k u$. Recall $B : H^1(M_1) \times H^1(M_1) \to \mathbb{R}$, the bilinear form associated to the Laplacian Δ_{M_1} defined in (8). Let $v \in C_c^\infty(M_1 \setminus \{x_1\})$ be a test function, we obtain

$$\begin{aligned}
B[\hat{f}_k, v] &= \int_{M_1} \langle d\hat{f}_k, dv\rangle_{g_0} dg_1 \overset{(1)}{=} \lim_{i\to\infty} \int_{M_1(\varepsilon_i)} \langle df_k^{1,\varepsilon_i}, dv\rangle_{g_0} dg_0 \\
&= \lim_{i\to\infty} \int_{M_1(\varepsilon_i)} \langle df_k^{1,\varepsilon_i}, dv\rangle_{g_0} dg_0 + \int_{M_2(1)} \langle df_k^{2,\varepsilon_i}, 0\rangle_{\widetilde{g}_{\varepsilon_i}} d\widetilde{g}_{\varepsilon_i} \\
&\overset{(2)}{=} \lim_{i\to\infty} (\Delta_{M_\varepsilon}(f_k^{1,\varepsilon_i}, f_k^{2,\varepsilon_i}), (v,0))_{L^2(M_{\varepsilon_i})} \\
&= \lim_{i\to\infty} \lambda_k(M_{\varepsilon_i})((f_k^{1,\varepsilon_i}, f_k^{2,\varepsilon_i}), (v,0))_{L^2(M_{\varepsilon_i})} \\
&\overset{(3)}{=} \alpha_k \lim_{i\to\infty} \int_{M_1(\varepsilon_i)} \hat{f}_k\, v\, dg_1 = \alpha_k(\hat{f}_k, v)_{L^2(M_1)}.
\end{aligned}$$

Equality (1) follows from the H^1-weak convergence of extensions $\hat{f}_k^{1,\varepsilon}$ and $\hat{f}_k^{1,\varepsilon}|_{M_1(\varepsilon)} = f_k^{1,\varepsilon}$. In equation (2) we used (26), equation (3) follows from the Lemma 2.3.4. Since $C_c^\infty(M_1 \setminus \{x_1\})$ is dense in $H^1(M_1)$, which holds in dimensions ≥ 2, (see [2], and Remark 2.3.7) the equality $B(\hat{f}_k, v) = \alpha_k(\hat{f}_k, v)_{L^2(M_1)}$ is valid for any $v \in H^1(M_1)$. Consequently, $\hat{f}_k$ is a weak solution to $\Delta_{M_1} u = \alpha_k u$, and by the regularity of weak solutions we conclude that $\hat{f}_k$ is a smooth classical solution. $\square$

Remark 2.3.7. To show density $C_c^\infty(M_1 \setminus \{x_1\})$ in $H^1(M_1)$, one may use cut-off functions defined in (18). Let $f \in C^\infty(M_1)$ we show that $\chi_\varepsilon f \in C_c^\infty(M_1 \setminus \{x_1\})$ is arbitrarily close to f in H^1-norm. Clearly, $\|f - \chi_\varepsilon f\|_{L^2(M_1)} \to 0$ as $\varepsilon \to 0$, and by (20) we obtain

$$\begin{aligned}\|df - d(\chi_\varepsilon f)\|_{L^2} &= \|df - \chi_\varepsilon df - f\, d\chi_\varepsilon\|_{L^2} \leq \|df - \chi_\varepsilon df\|_{L^2} + \|f\, d\chi_\varepsilon\|_{L^2} \\ &\leq \eta_\varepsilon + \|f\|_{L^\infty}\|d\chi_\varepsilon\|_{L^2} \to 0, \qquad \varepsilon \to 0.\end{aligned}$$

Since $C^\infty(M_1)$ is dense in $H^1(M_1)$, we conclude that $C_c^\infty(M_1 \setminus \{x_1\})$ is dense in $H^1(M_1)$.

Proof of Theorem 2.3.3. In order to finish the proof of Theorem 2.3.3 it suffices to show the following inequality;

$$\lambda_k(M_1) \leq \alpha_k. \tag{30}$$

From Lemma 2.3.6, we conclude that for each $j = 0, 1, \ldots, k$, there exists an α_j-eigenfunction on M_1: $\{\hat{f}_0, \ldots, \hat{f}_k\}$. If for $i \neq j$, $(\hat{f}_i, \hat{f}_j)_{L^2(M_1)} = 0$, i.e. $\hat{f}_i$ are orthogonal, then α_k is an l-eigenvalue for some $l \geq k$. Hence, we have $\lambda_k(M_1) \leq \alpha_k$. Therefore the only thing to show is the orthogonality of $\{\hat{f}_0, \ldots, \hat{f}_k\}$. □

Proof of orthogonality of $\{\hat{f}_0, \ldots, \hat{f}_k\}$. In the following we use notation from (26), we calculate

$$\begin{aligned}(\hat{f}_j, \hat{f}_l)_{L^2(M_1,g_1)} &= \lim_{i\to\infty}\{(\hat{f}_j^{1,\varepsilon_i}, \hat{f}_l^{1,\varepsilon_i})_{L^2(M_1(\varepsilon_i),g_1)} + (\hat{f}_j^{1,\varepsilon_i}, \hat{f}_l^{1,\varepsilon_i})_{L^2(B(x_1,\varepsilon_i),g_1)}\} \\ &= \lim_{i\to\infty}\{(f_j^{1,\varepsilon_i}, f_l^{1,\varepsilon_i})_{L^2(M_{\varepsilon_i})} - (f_j^{2,\varepsilon_i}, f_l^{2,\varepsilon_i})_{L^2(M_2(1),\varepsilon_i^2 g_2)}\} \\ &= \delta_l^j - \lim_{i\to\infty}(f_j^{2,\varepsilon_i}, f_l^{2,\varepsilon_i})_{L^2(M_2(1),\varepsilon_i^2 g_2)}.\end{aligned}$$

Hence, it suffices to prove the following;

$$\lim_{i\to\infty}(f_j^{2,\varepsilon_i}, f_l^{2,\varepsilon_i})_{L^2(M_2(1),\varepsilon_i^2 g_2)} = \lim_{i\to\infty}(\varepsilon_i\, f_j^{2,\varepsilon_i}, \varepsilon_i\, f_l^{2,\varepsilon_i})_{L^2(M_2(1),g_2)} = 0. \tag{31}$$

Define $\tilde{f}_j^{2,\varepsilon_i} = \varepsilon_i\, f_j^{2,\varepsilon_i}$, we will show that $\tilde{f}_j^{2,\varepsilon_i} \to 0$, as $i \to \infty$ in $L^2(M_2(1), g_2)$. By the second part of inequality (29) we conclude

$$\begin{aligned}&\|\tilde{f}_j^{2,\varepsilon_i}\|_{H^1(M_2(1),g_2)} = \|\varepsilon_i\, f_j^{2,\varepsilon_i}\|_{H^1(M_2(1),g_2)} = \|f_j^{2,\varepsilon_i}\|_{L^2(M_2(1),\varepsilon_i^2 g_2)} + \varepsilon_i\|df_j^{2,\varepsilon_i}\|_{H^1(M_2(1),g_2)} \\ &\leq \left(\|f_j^{\varepsilon_i}\|_{L^2(M_{\varepsilon_i})} + \varepsilon_i\|df_j^{\varepsilon_i}\|_{L^2(\Lambda^1 M_{\varepsilon_i})}\right) = \left(1 + \varepsilon_i(\Delta_{M_{\varepsilon_i}} f_j^{\varepsilon_i}, f_j^{\varepsilon_i})^{\frac{1}{2}}_{L^2(M_{\varepsilon_i})}\right) \leq (1 + \varepsilon_i\lambda_j^{\frac{1}{2}}(M_1)),\end{aligned}$$

where the last inequality follows from Lemma 2.3.4. Thus, $\{\tilde{f}_j^{2,\varepsilon_i}\}$ is bounded an therefore has a weakly convergent subsequence $\tilde{f}_j^{2,\varepsilon_i} \to \tilde{f}_j^2$, $i \to \infty$, in $H^1(M_2(1), g_2)$. This subsequence is strongly convergent in $L^2(M_2(1), g_2)$. It follows that

$$\|d\tilde{f}_j^2\|^2_{L^2(M_2(1),g_2)} \leq \liminf_{i\to\infty} \varepsilon_i^2(\Delta_{\varepsilon_i^2 g_2} f_j^{2,\varepsilon}, f_j^{2,\varepsilon})_{L^2(M_2(1),\varepsilon_i^2 g_2)} \leq \liminf_{i\to\infty} \varepsilon_i^2 \lambda_j(M_{\varepsilon_i}) = 0.$$

Consequently, $\tilde{f}_j^2 \equiv$ const and we obtain

$$\|\tilde{f}_j^{2,\varepsilon_i} - \tilde{f}_j^2\|^2_{H^1(M_2(1),g_2)} = \|\tilde{f}_j^{2,\varepsilon_i} - \tilde{f}_j^2\|^2_{L^2(M_2(1),g_2)} + \|d\tilde{f}_j^2\|^2_{L^2(M_2(1),g_2)} \to 0. \qquad (32)$$

Next, we show that $\tilde{f}_j^2 \restriction_{\partial M_2(1)} = 0$. Notice the following;

$$\begin{aligned} \|\tilde{f}_j^{2,\varepsilon_i} \restriction_{\partial M_2(1)} \|_{L^2(\partial M_2(1),\partial g_2)} &= \sqrt{\varepsilon_i}\|f_j^{2,\varepsilon_i} \restriction_{\partial M_2(1)} \|_{L^2(\partial M_2(1),\varepsilon_i^2 \partial g_2)} \\ = \sqrt{\varepsilon_i}\|f_j^{1,\varepsilon_i} \restriction_{\partial M_1(\varepsilon_i)} \|_{L^2(\partial M_1(\varepsilon_i),\partial g_1)} &\leq C\varepsilon_i\sqrt{|\ln \varepsilon_i|}\,\|f_j^{1,\varepsilon_i}\|_{H^1(M_1(\varepsilon_i),g_1)}, \end{aligned} \qquad (33)$$

where the last inequality was derived by Anné in [3]. Since $\|f_j^{1,\varepsilon_i}\|_{H^1(M_1(\varepsilon_i),g_1)}$ is bounded we get $\|\tilde{f}_j^{2,\varepsilon_i} \restriction_{\partial M_2(1)} \|_{L^2(\partial M_2(1),\partial g_2)} \to 0$, as $i \to \infty$. Hence, from the Trace Theorem and (32), we obtain

$$\begin{aligned} \|\tilde{f}_j^2 \restriction_{\partial M_2(1)} \|_{L^2(\partial M_2(1),\partial g_2)} &\leq \|\tilde{f}_j^{2,\varepsilon_i} \restriction_{\partial M_2(1)} \|_{L^2(\partial M_2(1),\partial g_2)} \\ +\|\tilde{f}_j^2 \restriction_{\partial M_2(1)} - \tilde{f}_j^{2,\varepsilon_i} \restriction_{\partial M_2(1)} \|_{L^2(\partial M_2(1),\partial g_2)} &\leq \|\tilde{f}_j^{2,\varepsilon_i} \restriction_{\partial M_2(1)} \|_{L^2(\partial M_2(1),\partial g_2)} \\ &+ C\,\|\tilde{f}_j^{2,\varepsilon_i} - \tilde{f}_j^2\|_{H^1(M_2(1),g_2)} \to 0, \end{aligned}$$

as $i \to \infty$. Since $\tilde{f}_j^2 \restriction_{\partial M_2(1)} = 0$, and $\tilde{f}_j^2$ is constant, we obtain $\tilde{f}_j^2 = 0$, which proves (31). □

2.3.2 Proof of Gluing Theorem.

For piecewise smooth metrics, eigenvalues of the Laplacian "vary" continuously with respect to the C^0-topology. It was derived in [8] as the following;

Theorem 2.3.8 ([8]). *Let $\mathfrak{M}$, be a set of metrics on M^n, the function $\mathfrak{M} \ni g \to \lambda_k(M) \in \mathbb{R}$, where λ_k is a k'th eigenvalue of Δ_g, is continuous w.r.t. C^0-topology on $\mathfrak{M}$.*

Therefore, for a given $\varepsilon > 0$ we can perturb the metric $\widetilde{g}_\varepsilon$ on M to a smooth generic metric g_ε so that eigenvalues are arbitrarily "close". By Theorem 2.2.7, we may assume that the support of the perturbation is contained in the complement $M_1(\varepsilon/2)^c$. Denote (M, g_ε) by M_ε. Hence, we can define a family of metrics $\{g_\varepsilon\}_\varepsilon$, satisfying the following requirements:

(i) g_ε are smooth and converge to $\widetilde{g}_\varepsilon$ in the C^0-topology of M, as $\varepsilon \to n0$;

(ii) $g_\varepsilon|_{M_1(\varepsilon/2)} = g_1$;

(iii) $\lambda_k(M_\varepsilon)$ are simple eigenvalues and nodal sets $\Xi(M_\varepsilon, k)$ are embedded circles;

(iv) $\lim_{\varepsilon\to 0} \lambda_k(M_\varepsilon) = \lambda_k(M_1)$.

We recall the notation *(a)* - *(d)* from page 16. We redefine: $M_\varepsilon = (M, g_\varepsilon)$. moreover, for convenience, we replace ε with factor $\varepsilon/4$ in *(a)* - *(d)* on page 16.

Comparing nodal sets $\Xi(M_\varepsilon, k)$ and $\Xi(M_1, k)$ can be a little bit subtle. Notice that for each $\varepsilon > 0$, M_ε is diffeomorphic to $\Sigma = \Sigma_0 \# \Sigma_1 \# \ldots \# \Sigma_m$ and $\{g_\varepsilon\}_\varepsilon$ is a family of metrics on Σ. In the limit (i.e. for $\varepsilon = 0$) the metric g_ε degenerates on $M_2(1)$, and $M_0 = (\Sigma, g_0)$ is not homeomorphic to $M_1 = (\Sigma_0, g_0)$. Rather, it inherits topology that is pulled back from M_1 under the quotient map, $\pi : \Sigma \to \Sigma/M_2(1)$. Thus we really have no control over a part of the nodal set in the "shrinking" portion: $M_2(1)$ of the manifold M_ε. Technically, we cannot compare eigenfunctions: f_k on M_1 to the eigenfunctions: $f_k^\varepsilon \in C^\infty(M_\varepsilon)$ on M_ε, we must restrict them to the common domain $M_1(\varepsilon_0)$ for a fixed $\varepsilon_0 > 0$. In order to prove the isotopy of nodal sets: $\Xi(M_1, k)$, and $\Xi(M_\varepsilon, k)$, for small ε, we must show C^1-convergence of eigenfunctions f_k^ε restricted to $M_1(\varepsilon_0)$. In this section we show a stronger result, namely, that any sequence $\{\varepsilon_j\}_j$; $\varepsilon_j \to 0$, $\{f_k^{\varepsilon_j}|_{M_1(\varepsilon_0)}\}_{\varepsilon_j}$ converges to $f_k|_{M_1(\varepsilon_0)} \in C^\infty(M_1(\varepsilon_0))$ in the C^∞-topology.

Lemma 2.3.9 (Gårding's inequality for differential forms, [61], p. 36). *For each $s \geq 0$, there exists a positive constant $C = C_s$ such that,*

$$\|\omega\|_{H^s(\Lambda^* M)} \leq C(\|\omega\|_{H^{s-1}(\Lambda^* M)} + \|(d+\delta)\omega\|_{H^{s-1}(\Lambda^* M)}) \tag{34}$$

where $\omega \in \Omega^*(M)$.

Lemma 2.3.10 (C^j-convergence of eigenfunctions). *For each k, and an arbitrary $j > 2$, the following C^j-convergence of eigenfunctions $f_k^\varepsilon \in C^\infty(M_\varepsilon)$, $\|f_k^\varepsilon\|_{L^2(M_\varepsilon)} = 1$ holds:*

$$\lim_{\varepsilon \to 0} f_k^\varepsilon = f_k \qquad \text{on compact subsets of} \quad M_1 \setminus \{x_1\} \tag{35}$$

where $f_k \in C^\infty(M_1)$ is a k'th-Δ_{M_1}-eigenfunction on M_1.

Proof. Recall the notation in (2.3). Lemma 2.3.6, implies that $\lim_{\varepsilon\to 0} f_k^{1,\varepsilon} = \hat{f}_k$ in $L^2(M_1)$. Since all the eigenvalues $\lambda_k(M_1)$ are simple in g_1 and, by Theorem 2.3.3, $\alpha_k = \lambda_k(M_1)$ we have

$$1 = \|f_k\|_{L^2(M_1,g_1)} = \lim_{i\to\infty} \|f_k\|_{L^2(M_1(\varepsilon_i),g_1)} = \lim_{i\to\infty} \|\hat{f}_k^{1,\varepsilon_i}\|_{L^2(M_1(\varepsilon_i),g_1)} = \|\hat{f}_k\|_{L^2(M_1,g_1)}.$$

We conclude that $\hat{f}_k = f_k$. In the next step, we argue C^j-convergence of f_k^{1,ε_i} on compact subsets of $M_1 \setminus \{x_1\}$.

Choose ε_0 such that $M_1(\varepsilon_0)$ contains a given compact subset and let $l > j + \frac{m}{2} + 1 = j+2$, where $m = 2$ is the dimension of M_1, (we assume $j > 2$ for convenience). Letting $\varepsilon \leq \varepsilon_0$, we apply Lemma 2.3.5 and consider a family of H^l-extensions $\bar{f}_k^{1,\varepsilon} \in H^l(M_1)$ for $f_k^{1,\varepsilon} \in C^\infty(M_1(\varepsilon_0/2))$. Lemma 2.3.9, with the constant D_l, implies the following;

$$\begin{aligned} \|\bar{f}_k^{1,\varepsilon}\|_{H^l(M_1)} &\leq D_l\big(\|\bar{f}_k^{1,\varepsilon}\|_{H^{l-1}(M_1)} + \|(d+\delta)\bar{f}_k^{1,\varepsilon}\|_{H^{l-1}(\Lambda^* M_1)}\big) \\ &= K_{l-1}\big(\|\bar{f}_k^{1,\varepsilon}\|_{H^{l-1}(M_1)} + \|d\bar{f}_k^{1,\varepsilon}\|_{H^{l-1}(\Lambda^1 M_1)}\big) \end{aligned} \tag{36}$$

(where $K_{l-1} = D_l$). Here $d + \delta$ is the Dirac operator (i.e. $(d+\delta)^2 = \Delta$) acting on forms of mixed degree. Applying Gårding's inequality again to each term of (36) and setting $D_{l,l-1} = D_l D_{l-1}$ results in

$$
\begin{aligned}
\text{(rhs of (36))} \quad &\leq \quad D_{l,l-1}\big(\|\bar{f}_k^{1,\varepsilon}\|_{H^{l-2}(M_1)} + 2\|d\bar{f}_k^{1,\varepsilon}\|_{H^{l-2}(\Lambda^1 M_1)} + \|\Delta_{M_1}\bar{f}_k^{1,\varepsilon}\|_{H^{l-2}(M_1)}\big) \\
&\overset{(1)}{\leq} \quad D_{l,l-1}\, C''_{l,\varepsilon_0}\big(\|\bar{f}_k^{1,\varepsilon}\|_{H^{l-2}(M_1)} + 2\|d\bar{f}_k^{1,\varepsilon}\|_{H^{l-2}(\Lambda^1 M_1)} + \|\Delta_{M_\varepsilon} f_k^{\varepsilon}\|_{H^{l-2}(M_1(\varepsilon_0/2),g_\varepsilon)}\big) \\
&\overset{(2)}{\leq} \quad 2D_{l,l-1}C''_{l-2,\varepsilon_0}(1+\lambda_k(M_\varepsilon))\big(\|\bar{f}_k^{1,\varepsilon}\|_{H^{l-2}(M_1)} + \|d\bar{f}_k^{1,\varepsilon}\|_{H^{l-2}(\Lambda^1 M_1)}\big) \\
&\overset{(3)}{\leq} \quad K_{l-2}\big(\|\bar{f}_k^{1,\varepsilon}\|_{H^{l-2}(M_1)} + \|d\bar{f}_k^{1,\varepsilon}\|_{H^{l-2}(\Lambda^1 M_1)}\big)
\end{aligned}
$$

Inequality (1) is a consequence of (24), whereas (2) follows from the fact that f_k^ε is a k'th-eigenfunction of the Laplacian on M_ε. In (3) we set $K_{l-2} > 2D_{l,l-1}C''_{l-2,\varepsilon_0}(1 + \lambda_k(M_\varepsilon))$ due to the requirement (iv) on page 23. Repeating the above steps finitely many times leads to

$$
\begin{aligned}
\|\bar{f}_k^{1,\varepsilon}\|_{H^l(M_1)} \quad &\leq \quad K_0\big(\|\bar{f}_k^{1,\varepsilon}\|_{L^2(M_1)} + \|d\bar{f}_k^{1,\varepsilon}\|_{L^2(\Lambda^1 M_1)}\big) = K_0\|\bar{f}_k^{1,\varepsilon}\|_{H^1(M_1)} \\
&\leq \quad K_0\, C'_{1,\varepsilon_0}\big(\|f_k^{1,\varepsilon}\|_{L^2(M_1(\varepsilon_0/2),g_1)} + \|df_k^{1,\varepsilon}\|_{L^2(\Lambda^1 M_1(\varepsilon_0/2),g_1)}\big) \\
&\leq \quad K_0\, C'_{1,\varepsilon_0}\big(1 + (\Delta_{M_\varepsilon} f_k^\varepsilon, f_k^\varepsilon)^{\frac{1}{2}}_{L^2(M_\varepsilon)}\big) \leq K_0\, C'_{1,\varepsilon_0}(1 + \lambda_k^{\frac{1}{2}}(M_1) + \eta'_\varepsilon),
\end{aligned}
$$

where $\eta'_\varepsilon \to 0$ as $\varepsilon \to 0$ and we applied Lemma 2.3.5 in the second inequality. Consequently, the family $\{\bar{f}_k^{1,\varepsilon}\}_\varepsilon$ is bounded in $H^l(M_1)$. By Rellich's Theorem we have a compact inclusion $H^l(M_1) \hookrightarrow H^{l-1}(M_1)$, and by Sobolev embedding theorem a bounded inclusion $H^{l-1}(M_1) \hookrightarrow C^j(M_1)$ (see [61]). Composition of these two gives us a compact inclusion $H^l(M_1) \hookrightarrow C^j(M_1)$. As a result, there exists a subsequence $\{\bar{f}_k^{1,\varepsilon_i}\}_i$ of any sequence in $\{\bar{f}_k^{1,\varepsilon}\}_\varepsilon$, convergent in the C^j topology of M_1. Denote a limit of this subsequence by $\bar{f}_k \in C^j(M_1)$.

Since $\hat{f}_k^{1,\varepsilon}|_{M_1(\varepsilon_0)} = f_k^{1,\varepsilon}|_{M_1(\varepsilon_0)} = \bar{f}_k^{1,\varepsilon}|_{M_1(\varepsilon_0)}$ for any $\varepsilon < \varepsilon_0$, the L^2-limits $\hat{f}_k$, $\bar{f}_k$ have to agree on $M_1(\varepsilon_0)$. Equality (35) holds, since for any sequence $\{\varepsilon_i\}_i$ converging to zero $\{f_k^{1,\varepsilon_i}|_{M_1(\varepsilon_0)}\}$ contains a convergent subsequence with a common limit. □

Corollary 2.3.11. *Convergence* (35) *holds on compact subsets of* $M_1 \setminus \{x_1\}$ *in the* C^∞*-topology of* M_1.

Proof of Gluing Theorem 2.3.1. We assume that the nodal set $\Xi(M_1) = \Xi(M_1, k)$ is at a geodesic distance $d > \varepsilon_0$ from gluing discs: $D_{x_i}(\varepsilon) \subset D_{x_i}(\varepsilon_0)$. The nodal sets $\Xi(M_1)$ and $\Xi(M_\varepsilon)$ can be compared only on the common subset $M_1(\varepsilon_0)$. What we really have to show is that for some $\varepsilon > 0$ the nodal set $\Xi(M_\varepsilon) = \Xi(M_\varepsilon, k)$ belongs entirely to the common domain $M_1(\varepsilon_0)$. First of all note the following pointwise convergence of nodal sets.

Lemma 2.3.12. *Consider a sequence of points $\{x_i\}_i$ such that for each i, $x_i \in \Xi(M_{\varepsilon_i}) \cap M_1(\varepsilon_0)$. If the limit x of $\{x_i\}_i$ exists, then $x \in \Xi(M_1)$.*

Proof. Applying the convergence $f_k^{\varepsilon_i}|_{M_1(\varepsilon_0)} \to f_k|_{M_1(\varepsilon_0)}$ in $C^0(M_1(\varepsilon_0))$ we obtain,

$$|f_k(x_i)| = |f_k(x_i) - f_k^{1,\varepsilon_i}(x_i)| \leq \|f_k - f_k^{1,\varepsilon_i}\|_{C^0(M_1(\varepsilon_0))} \overset{i\to\infty}{\longrightarrow} 0.$$

By continuity, of f_k and the assumption $x_i \to x \in M_1(\varepsilon_0)$, we have $0 = \lim_{i\to\infty} f_k(x_i) = f_k(x)$, and we conclude that $x \in \Xi(M_1)$. $\square$

The above Lemma implies that there exists an index n, such that for all $i \geq n$, $\Xi(M_{\varepsilon_i}) \cap \partial M_1(\varepsilon_0) = \varnothing$. Indeed, if $\Xi(M_{\varepsilon_i}) \cap \partial M_1(\varepsilon_0) \neq \varnothing$ for infinitely many i, we could find a convergent sequence of $x_i \in \Xi(M_{\varepsilon_i}) \subset \partial M_1(\varepsilon_0)$. Consequently, by Lemma 2.3.12 a limit $\hat{x}$ of $\{x_i\}$ would have to be in $\Xi(M_1)$, which contradicts the assumption that $\Xi(M_1)$ is away from gluing discs $\{D_{x_i}(\varepsilon_0)\}$.

In the following paragraphs we show that $\Xi'(M_{\varepsilon_i}) = M_1(\varepsilon_0) \cap \Xi(M_{\varepsilon_i})$ is isotopic to $\Xi(M_1)$ for sufficiently small ε_i. Define: $H : I \times M_1(\varepsilon_0) \to \mathbb{R}$, $I = (-\epsilon, 1 + \epsilon)$, as follows;

$$H(t, x) = t\, f_k(x) \restriction_{M_1(\varepsilon_0)} + (1 - t)\, f_k^{1,\varepsilon_i}(x). \tag{37}$$

Since $\Xi(M_1)$ is a regular level set of f_k we have the following bound:

$$\sup_{x\in\Xi(M_1)} \|df_k(x)\| \geq A > 0. \tag{38}$$

By Lemma 2.3.10, we may choose ε_i small enough so that

$$\| df_k \restriction_{M_1(\varepsilon_0)} - df_k^{1,\varepsilon_i} \|_{C^0} = \sup_{x \in M_1(\varepsilon_0)} \| df_k(x) \restriction_{M_1(\varepsilon_0)} - df_k^{1,\varepsilon_i}(x) \| < A/2 \qquad (39)$$

We claim that 0 is a regular value of H. Indeed, if $dH(t_0, x_0) = 0$, at some (t_0, x_0) then

$$dH(t,x) = (f_k(x) \restriction_{M_1(\varepsilon_0)} - f_k^{1,\varepsilon_i}(x))\, dt + t\, df_k(x) \restriction_{M_1(\varepsilon_0)} + (1-t)\, df_k^{1,\varepsilon_i}(x),$$

which implies $f_k(x_0) \restriction_{M_1(\varepsilon_0)} - f_k^{1,\varepsilon_i}(x_0) = 0$, and $t_0\, df_k(x_0) \restriction_{M_1(\varepsilon_0)} + (1-t_0)\, df_k^{1,\varepsilon_i}(x_0) = 0$. Consequently,

$$t_0\, df_k(x_0) \restriction_{M_1(\varepsilon_0)} = (t_0 - 1)\, df_k^{1,\varepsilon_i}(x_0). \qquad (40)$$

By (39),

$$A/2 > \| df_k \restriction_{M_1(\varepsilon_0)} - df_k^{1,\varepsilon_i} \|_{C^0} \geq \| df_k \restriction_{M_1(\varepsilon_0)} (x_0) - df_k^{1,\varepsilon_i}(x_0) \|. \qquad (41)$$

Since $0 < t_0 < 1$, we obtain

$$\begin{aligned} A/2 \geq (A/2)(1-t_0) &\geq \|(1-t_0)\, df_k \restriction_{M_1(\varepsilon_0)} (x_0) - (1-t_0) df_k^{1,\varepsilon_i}(x_0)\| \\ = \|(1-t_0)\, df_k \restriction_{M_1(\varepsilon_0)} (x_0) + t_0 df_k \restriction_{M_1(\varepsilon_0)} (x_0)\| &= \|df_k \restriction_{M_1(\varepsilon_0)} (x_0)\| \geq A. \qquad (42) \end{aligned}$$

Therefore we arrived to the contradiction with $d\,H(t_0, x_0) = 0$.

Consequently, 0 is a regular value of H and, as the above reasoning shows, it is also a regular value of $f_t = H(t, \cdot)$, $f_t \in C^\infty(M_1(\varepsilon_0))$, for all t. This implies that $f_t^{-1}(0)$, for each t, is a disjoint union of embedded circles. Consequently, the submanifold: $N = H^{-1}(0) \subset I \times M_1(\varepsilon_0)$, satisfies, $\partial N = \Xi(M_1) \sqcup \Xi'(M_{\varepsilon_i})$ and defines an isotopy between $\Xi(M_1)$ and $\Xi'(M_{\varepsilon_i})$. Since a number of nodal domains of f_k is $k+1$ Courant's Theorem 2.2.1 implies that $\Xi(M_{\varepsilon_i}) = \Xi'(M_{\varepsilon_i})$. $\square$

2.4 *Payne's conjecture for closed Riemannian surfaces.*

In [59], L. E. Payne conjectured that in case of the fixed membrane problem, for bounded domains in $\mathbb{R}^2$, the second eigenfunction of the Laplacian cannot possess a closed nodal curve.

Conjecture 2.4.1 (Payne(1967))**.** *The second eigenfunction of the Laplacian on a bounded region Ω in Euclidean $\mathbb{R}^2$ with the Dirichlet boundary conditions cannot have a closed curve in its nodal set.*

Since 1967, Payne's conjecture has been proved to be true in the case of convex domains (see [1] and [55]). Recently, it has been proved false by T. Hoffmann-Ostenhof and co-authors (see [46]), in the case of a non-simply connected domain (disc with slits on an inner circle removed). It is still not known, however, if Conjecture 2.4.1 is true for an arbitrary simply connected region in $\mathbb{R}^2$. In [34], P. Freitas has shown that Conjecture fails in case of $\Omega = D^2$ for a non-Euclidean metric.

We consider a more global version of Payne's conjecture:

Problem 2.4.2. *Does the first Δ_Σ-eigenfunction on a given closed surface Σ of genus ≥ 1 admit a contractible nodal curve in Σ?*

As a consequence of Gluing Theorem we derive the following;

Theorem 2.4.3 (Komendarczyk, [52])**.** *For an arbitrary closed compact orientable surface Σ, there always exists a smooth metric g_Σ such that $\Xi(\Sigma)$ is a single embedded circle which bounds a disc in Σ.*

Proof. First we observe that it is a straightforward corollary of Theorems 2.2.1, 2.2.7 in the case of $\Sigma = S^2$. Namely, it is enough to choose a generic metric and refer to Theorem 2.2.7 which states that $\Xi(S^2)$ has to be a one dimensional submanifold. By Courant's Theorem 2.2.1, $\Xi(S^2)$ splits S^2 into two open domains, thereby implying that $\Xi(S^2)$ must be a single embedded circle. If the surface Σ is of genus $g(\Sigma) \geq 1$, we produce a desired metric by gluing via boundary circles a "big" sphere $M_1 \cong S^2$ with an ε-disc removed, $M_1(\varepsilon) \cong S^2 \setminus \mathrm{Int}(D^2) \cong D^2$, to an ε-"small" surface $M_2(1)$, homeomorphic to $\Sigma \setminus \mathrm{Int}(D^2)$. The resulting manifold M_ε is homeomorphic to Σ and, as we have shown in Theorem 2.3.1, the nodal set $\Xi(M_\varepsilon)$ is isotopic to $\Xi(M_1)$ (see Figure 1). Thus for sufficiently small ε, $\Xi(M_\varepsilon)$ has to be a closed embedded circle that bounds a disc in M_ε. $\square$

2.5 Nodal curves of λ_1-eigenfunctions.

Below we generalize the proof of Theorem 2.4.3 and show that an arbitrary allowable configuration of nodal curves of λ_1-eigenfunction may be archived in a generic metric on Σ.

Theorem 2.5.1. *Let Σ be an orientable surface and S a collection of k disjoint circles in Σ, which divides the surface into two domains. Then there exists a generic metric on Σ such that $S = \Xi(\Sigma, 1)$ up to isotopy.*

Proof. Topologically: $\Sigma \setminus S = \Sigma_1 \cup \Sigma_2$, where each Σ_i is a genus $g(\Sigma_i)$ surface with k-boundary components. First, we consider a case when each Σ_i is a 2-sphere with k-disjoint discs removed, i.e. Σ is genus $g(\Sigma) = k - 1$ surface. Since Σ is a "double" one may find an appropriate metric by embedding Σ_1 in $\mathbb{R}^3$ so that $\partial \Sigma_1 = S$ is a collection of circles on xy-plane, and define g_1 on Σ_1 to be a metric induced from $\mathbb{R}^3$ (see Figure 2). We may also require that the "mirror image" Σ_1' of Σ_1 by reflection through the xy-plane, smoothly "fits" Σ_1. With such choices, we may define an embedding: $j : \Sigma \hookrightarrow \mathbb{R}^3$, such that $j(\Sigma_1) = \Sigma_1$, $j(\Sigma_2) = \Sigma_1'$, and equip Σ with a metric g induced from $\mathbb{R}^3$. Clearly, (Σ, g) admits a symmetry: namely a reflection about an xy-plane. Solving a fixed membrane problem on one half: Σ_1 results in λ_1-eigenfunction ϕ_1, $\phi_1 \restriction_{\partial\Sigma_1} = 0$. Now, the function ϕ defined as

$$\phi = \begin{cases} \phi_1(x, y, z), & \text{on } \Sigma_1,\ \{z \geq 0\}, \\ -\phi_1(x, y, -z), & \text{on } \Sigma_2,\ \{z \geq 0\} \end{cases}$$

is a λ_1-eigenfunction of (Σ, g), with the nodal set $\phi^{-1}(0) = S$. Perturbing if necessary, we obtain a desired metric g on Σ in the case: $g(\Sigma_1) = g(\Sigma_2) = 0$. When $g(\Sigma_1) > 0$ or

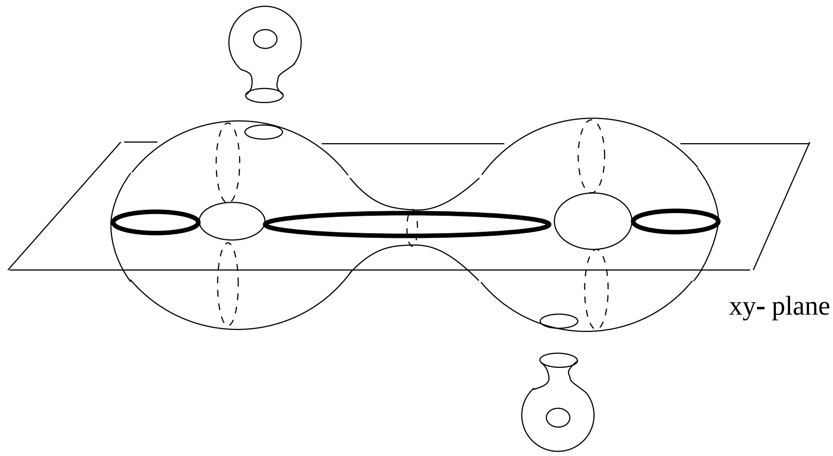

Figure 2: (Σ, g) is obtained by attaching higher genus surfaces on each side of $\Xi(\Sigma, k)$.

$g(\Sigma_2) > 0$ we may easily produce the metric, specified in the theorem, by attaching a genus $g(\Sigma_i)$ surfaces to Σ_i, $i = 1, 2$ to the previously constructed model surface (Σ, g) (see Figure 2). Now, the claim follows from Theorem 2.3.1. $\square$

Remark 2.5.2. One may certainly extend the proof of the Gluing Theorem to higher dimensions. Consequently, for a given topology of a nodal set $\Xi(M^n, k)$ one may change the topology of M by attaching finitely many manifolds via the connected sum. This may be helpful in addressing the following part of the question of Schoen and Yau:

> ... "Is there a similar conclusion for higher dimensional euclidean space? To what extend do these conclusions hold for compact manifolds with boundary?" ...

In this work we mostly considered the case of regular nodal sets on surfaces, i.e. nodal sets without singularities. What about immersed curves, then? In [66], Uhlenbeck proved that for a generic path of metrics g_t, $0 \leq t \leq 1$ on Σ, nodal sets of g_1 and g_0 are cobordant. Consequently, prescribing known configurations of nodal sets at the endpoints g_0, and g_1 one may obtain new immersed curves for metrics g_t, $0 < t < 1$. Clearly, such a procedure holds in higher dimensions as well.

2.6 Homotopically essential nodal curves.

In this section we investigate sufficient conditions which force homotopically essential nodal curves on a surface with a prescribed geometry. Our approach is guided by results of S. T. Dong, [23], and A. Savo in [62]. Throughout this section we use more classical notation stated in the following definitions.

Definition 2.6.1. *The* GRADIENT VECTOR FIELD ∇f *of* f *is defined through*

$$df(X) = \langle \nabla f, X \rangle, \qquad \text{for all } X \in C^\infty(M, TM). \tag{43}$$

Definition 2.6.2. *The* HESSIAN $\nabla^2 f$ *of a function* $f \in C^2(M)$ *on the Riemannian manifold* (M, g) *is a symmetric bilinear form defined as*

$$\nabla^2 f(X, Y) = (\nabla_X \, df)(Y) = (\nabla_Y \, df)(X), \qquad X, Y \in C^\infty(M, TM). \tag{44}$$

Let the covariant derivative along a vector field X on S be denoted by ∇^S_X or by ∇_X when S is known from the context.

Definition 2.6.3. *The* SECOND FUNDAMENTAL FORM *of a codimension 1 submanifold* M^m *in* N^{m+1} *is a symmetric bilinear form* $\ell : T_pM \times T_pM \to T_p^\perp M$ *defined as*

$$\nabla^N_X Y = \nabla^M_X Y + \ell(X, Y). \tag{45}$$

Choosing a unit normal vector $\nu \in T_p^\perp M$ *we define* $\ell_\nu : T_pM \times T_pM \to \mathbb{R}$ *as follows;*

$$\ell_\nu(X, Y) = \langle \ell(X, Y), \nu \rangle = -\langle \nabla^N_X \nu, Y \rangle = -\langle \nabla^N_Y \nu, X \rangle, \qquad X, Y \in T_pM. \tag{46}$$

Notice that choice of a normal vector ν affects the sign of ℓ_ν:

$$\ell_\nu(X, Y) = -\ell_{-\nu}(X, Y).$$

The following lemma shows that $\ell_\nu(X, Y)$ depends only on the value of ν at p.

Lemma 2.6.4 ([49]). *For any function f defined in a neighborhood of p we have*

$$\ell_{f\nu}(X,Y) = f\,\ell_\nu(X,Y). \tag{47}$$

Proof. Indeed, $(\nabla^N_X f\nu)^\top_p = ((Xf)\nu)^\top_p + f(\nabla^N_X\nu)^\top_p = f(\nabla^N_X\nu)^\top_p$. Consequently,

$$\begin{aligned}\ell_{f\nu}(X,Y) &= -\langle\nabla^N_X(f\nu),Y\rangle = -\langle(\nabla^N_X(f\nu))^\top,Y\rangle\\ &= -\langle f(\nabla^N_X\nu)^\top_p,Y\rangle = f\,\ell_\nu(X,Y).\end{aligned}$$

□

For an orthonormal frame $\{e_i\}\in T_pM$ the GAUSS-KRONECKER CURVATURE of M in a direction of a normal vector ν is given by

$$K_\nu = \det(\ell_\nu(e_i,e_j)), \tag{48}$$

and the MEAN CURVATURE by

$$H_\nu = \frac{1}{m}\mathrm{tr}(\ell_\nu(e_i,e_j)). \tag{49}$$

Clearly, K_ν, H_ν are independent of the choice of the frame and

$$K_\nu = (-1)^m K_{-\nu}, \qquad H_\nu = -H_{-\nu}. \tag{50}$$

The next step is to derive a formula for the mean curvature of a regular level set of a smooth function (c.f. [62]).

Proposition 2.6.5. *Let u be a smooth function on a manifold (N^m,g), and let $M = u^{-1}(c)$ be a c-regular level set of u. If $\nabla u(p)\neq 0$, $p\in M$, then*

$$m\,H_\nu = \frac{\Delta_N u}{\|\nabla u\|} + \frac{1}{\|\nabla u\|}\langle\nabla\|\nabla u\|,\nu\rangle = \frac{\Delta u}{\|\nabla u\|} + \frac{1}{2}\langle\nabla\ln\|\nabla u\|^2,\nu\rangle$$

where $\nu = \nabla u/\|\nabla u\|$ is pointing towards: $\{u>c\}$.

Proof. From [49], p. 139 we have the following formula for the Hessian $\nabla^2 u$:

$$\nabla^2 u(X, Y) = \langle \nabla_X \nabla u, Y \rangle. \tag{51}$$

For $\nu = \nabla u / \|\nabla u\|$ combining the above formula, Lemma 2.6.4, and the definition of ℓ yields

$$\ell_\nu(X, Y) = -\langle \nabla_X^N \frac{\nabla u}{\|\nabla u\|}, Y \rangle = -\frac{1}{\|\nabla u\|} \langle \nabla_X^N \nabla u, Y \rangle = -\frac{1}{\|\nabla u\|} \nabla^2 u(X, Y). \tag{52}$$

Hence: $\nabla^2 u|_{TM} = -\|\nabla u\| \ell$. Fixing an orhonormal basis $\{e_i, \nu\}$ of N at p we obtain

$$\begin{aligned} m\, H_\nu &= \operatorname{tr}(\ell_\nu(e_i, e_j)) = \sum_i \ell_\nu(e_i, e_i) + \nabla^2 u(\nu, \nu) - \nabla^2 u(\nu, \nu) \\ &= \frac{\Delta_N u}{\|\nabla u\|} - \frac{\nabla^2 u(\nu, \nu)}{\|\nabla u\|} = \frac{\Delta_N u}{\|\nabla u\|} - \frac{1}{\|\nabla u\|^3} \langle \nabla_{\nabla u} \nabla u, \nabla u \rangle \end{aligned}$$

where the second equation comes from (52), and $\Delta_N u = -\operatorname{tr}(\nabla^2 u)$. Using the formula: $X\langle X, X \rangle = 2\langle \nabla_X X, X \rangle$ we obtain

$$\begin{aligned} \frac{\langle \nabla_{\nabla u} \nabla u, \nabla u \rangle}{\|\nabla u\|^2} &= \frac{\langle \nabla \|\nabla u\|^2, \nabla u \rangle}{2\|\nabla u\|^2} = \frac{\langle \nabla \|\nabla u\|^2, \nu \rangle}{2\|\nabla u\|} \\ &= \frac{\langle 2\|\nabla u\|(\nabla \|\nabla u\|), \nu \rangle}{2\|\nabla u\|} = \langle \nabla \|\nabla u\|, \nu \rangle. \end{aligned}$$

Consequently,

$$\langle \nabla \ln(\|\nabla u\|^2), \nu \rangle = \frac{1}{\|\nabla u\|^2} \langle \nabla(\|\nabla u\|^2), \nu \rangle = \frac{2\langle \nabla \|\nabla u\|, \nu \rangle}{\|\nabla u\|}.$$

Combining the formulas leads to

$$m\, H_\nu = \frac{\Delta u}{\|\nabla u\|} + \frac{1}{\|\nabla u\|} \langle \nabla \|\nabla u\|, \nu \rangle = \frac{\Delta u}{\|\nabla u\|} + \frac{1}{2} \langle \nabla \ln \|\nabla u\|^2, \nu \rangle.$$

□

Given a frame of vectors $F = \{e_1, \ldots, e_n\}$ at $p \in M^n$, the linear isomorphism $F : \mathbb{R}^n \to T_p M$ defines a coordinate system on $T_p M$ in natural manner. Consequently, the diffeomorpism: $\exp \circ F : \mathbb{R}^n \to U_p$ given by the exponential map defines a local coordinate system on $U_p \subset M$. This coordinate system is called the NORMAL COORDINATE SYSTEM determined by the frame F. One of the crucial properties of a normal coordinate system $\{x_1, \ldots, x_n\}$ is that a geodesic determined by the initial condition (p, X), $X = a_i\, e_i$ is parametrized by ([51] p. 148)

$$x(t) = (a_1 t, \ldots, a_n t). \tag{53}$$

It yields the following fundamental result.

Proposition 2.6.6. *Let $\{x_1, \ldots, x_n\}$ be a normal coordinate system of M at p, and let $\partial_i = \frac{\partial}{\partial x_i}$. At p we have*

$$\omega_{ik}^j = \langle \nabla_{\partial_i} \partial_k, \partial_j \rangle = 0, \qquad \textit{for all } i, j, k. \tag{54}$$

(here ω_{ik}^j are Christoffel symbols at p of $\{\partial_i\}$).

Recall, [49], the following formulas in an orthonormal frame $\{e_i\}$ at p, and $\{\eta_i\}$ the dual coframe. Components of $\nabla^2 f$ in $\{e_i\}$ can be calculated as follows;

$$\begin{aligned} \nabla^2 f(e_i, e_j) &= \langle \nabla_{e_i} \nabla f, e_j \rangle = \langle \sum_k (\nabla_{e_i} \nabla_{e_k} f\, e_k + \nabla_{e_k} f\, \nabla_{e_i} e_k), e_j \rangle \\ &= \nabla_i \nabla_j f + \sum_k \nabla_k, f\, \omega_{i\,k}^{\,j} \end{aligned} \tag{55}$$

The Laplacian Δ_{M_1} is given by

$$\Delta_M f = -\sum_i \nabla_{e_i} \nabla_{e_i} f - \nabla_{\nabla_{e_i} e_i} f \tag{56}$$

(see [49]). In normal coordinates at p determined by an orthonormal frame $\{e_i\}$, $e_i = \partial_i$, these formulas simplify even further since $\omega_{i\,k}^{\,j} = 0$ and $[e_i, e_j] = [\nabla_i, \nabla_j] = 0$ at p. One derives

$$\nabla^2 f = \sum_{i,j} (\nabla_{i\,j} f)\, \eta_i \cdot \eta_j, \qquad \Delta_M f = -\sum_i \nabla_{i\,i} f = -\mathrm{Tr}(\nabla^2 f). \tag{57}$$

Proposition 2.6.7 ([63], p. 15)**.** *Let (M, g) be a Riemannian manifold and $f \in C^3(M)$. If $\{x_i\}$ is a normal coordinate system determined by $\{e_i\}$ at a point $p \in M$, then we have at p*

$$\begin{aligned}
\Delta\|\nabla f\|^2 &= 2\sum_{i,j}(\nabla_i\nabla_j f)^2 + 2\sum_{i,j} R_{ij}\nabla_i f\,\nabla_j f + 2\sum_i \nabla_i f\,\nabla_i(\Delta f) \\
&= 2\|\nabla^2 f\|^2 + \mathrm{Ric}(\nabla f, \nabla f) + 2\langle\nabla f, \nabla\Delta f\rangle, \qquad (58)
\end{aligned}$$

where R_{ij} are components of the Ricci tensor.

Results of this section are based on the following estimate proven by Dong in [23].

Lemma 2.6.8 (Dong's Lemma [23], p. 500)**.** *Let Σ be a surface, and f be a solution to (7) with $\lambda =$ const. Then the following estimate holds,*

$$\Delta \ln q \leq \lambda - 2\,K^-, \qquad K^- = \min(K_s, 0), \qquad (59)$$

where K_s is the scalar curvature of Σ and $q = \|\nabla f\|^2 + \frac{\lambda}{2} f^2$.

Proof. The proof is a calculation in normal coordinates. We obtain

$$\begin{aligned}
-\Delta \ln q &= \frac{1}{q} - \frac{1}{q^2}\|\nabla q\|^2 \\
&= \frac{1}{q}(2\,\|\nabla^2 f\|^2 + 2\langle\nabla f, \nabla\Delta\, f\rangle + (2\,K_s + \lambda)\|\nabla f\|^2 - \lambda^2\, f^2) \\
&- \frac{1}{q^2}\|2\langle\nabla^2 f, \nabla f\rangle + 2\,f\,\nabla f\|^2 \\
&= \frac{1}{q}(2\|\nabla^2\, f\|^2 - \lambda^2\, f^2 + (2\,K_s - \lambda)\|\nabla f\|^2) \\
&- \frac{1}{q^2}\|2\langle\nabla^2 f, \nabla f\rangle + 2\,f\,\nabla f\|^2,
\end{aligned}$$

where the second equality follows from the Bochner formula (58). For the remaining part of the calculation we choose normal coordinates which diagonalize $\nabla^2 f$ at a point p, (57) i.e. $\nabla^2 f = \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix}$, where $f_{ij} = \partial_i\partial_j f$. The equation $\Delta\, f = \lambda\, f$ implies that $f_{11}+f_{22}=-\lambda f$, substituting in the above identity and denoting $\nabla f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ at p, we have

$$\begin{aligned} -\Delta \ln q &= \frac{1}{q}\big(2(f_{11}^2+f_{22}^2)-(f_{11}+f_{22})^2+(2K_s-\lambda)\|\nabla f\|^2\big) \\ &- \frac{1}{q}\big((f_{11}-f_{22})^2u_1^2+(f_{11}-f_{22})^2u_2^2)\big) \\ &= \frac{1}{q^2}\big(q(f_{11}-f_{22})^2+q\,(2K_s-\lambda)\|\nabla f\|^2-(f_{11}-f_{22})^2\|\nabla f\|^2\big) \\ &= \frac{\lambda}{2}\,\frac{(f_{11}-f_{22})^2}{q^2}-(2K_s-\lambda)\frac{\|\nabla f\|^2}{q}. \end{aligned}$$

Notice that the first term is nonnegative and that $\|\nabla f\|^2/q \le 1$. Consequently,

$$-\Delta \ln q \ge -\lambda + 2\min(K_s, 0).$$

□

2.6.1 Contractible nodal domains.

Let $\gamma(s): I \to \Sigma$ be the arclength parametrization of a smooth curve γ. We define an adapted orthonormal frame $\{e_1,\nu\}$, by $e_1=\alpha'(s)$ and ν as a unique normal vector to e_1 making the frame into a positively oriented basis of $T_{\alpha(s)}\Sigma$. Following [22], we define the GEODESIC CURVATURE of the curve γ as

$$\kappa(s)=\langle\nabla_1 e_1,\nu\rangle=-\langle\nabla_1\nu,e_1\rangle.$$

Since γ is a 1-dimensional submanifold in Σ we immediately obtain (from (48), (49))

$$\kappa_\nu(s)=\ell_\nu(e_1,e_1)(\gamma(s))=K_\nu(\gamma(s))=H_\nu(\gamma(s)).$$

Clearly, κ_ν changes the sign depending on a choice of ν: $-\kappa_\nu=\kappa_{-\nu}$. Recall that for an orientable surface Σ the EULER CHARACTERISTIC OF Σ, $\chi(\Sigma)$, can be calculated as follows;

$$\chi(\Sigma)=2-2\,g(\Sigma)-k, \tag{60}$$

where $g(\Sigma)$ is the genus of Σ and k is a number of boundary components. Furthermore, we have the famous Gauss-Bonnet Theorem in our disposal.

Theorem 2.6.9 (Gauss-Bonnet Theorem)**.** *Let Σ be a Riemannian surface with or without boundary*

$$2\pi\chi(\Sigma) = \int_{\Sigma} K_s + \int_{\partial\Sigma} \kappa \tag{61}$$

where K_s is the sectional curvature of $T_p\Sigma$ (equal to the scalar curvature of Σ at p) and κ denotes the geodesic curvature of $\partial\Sigma$.

(Notice that the integral $\int_{\partial\Sigma} \kappa$ is independent on the choice of ν in $\kappa = \kappa_\nu$ since a different choice of ν changes the orientation of $\partial\,\Sigma$).

Now we state the main theorem of this section which is inspired by Lemma 11 in [62].

Theorem 2.6.10. *Let $\Omega \subset \Sigma$ be a nodal domain of the solution u to* (7) *with $\lambda =$* const*, and let $K^{\pm}$ be the positive (negative) part of the scalar curvature K_s of Σ. If Ω is diffeomorphic to a 2-disc D^2 with regular boundary then*

$$\frac{4\pi - 2\int_\Omega K^+}{\mathrm{Vol}(\Omega)} \le \lambda. \tag{62}$$

Proof. Based on facts from the beginning of this section, Equation (7) and Proposition 2.6.5, we have the following formula for the geodesic curvature of $\partial\Omega$:

$$\kappa_\nu = H_\nu = \frac{\Delta_\Sigma u}{\|\nabla u\|} + \frac{1}{2}\langle \nabla \ln\|\nabla u\|^2, \nu\rangle = \frac{1}{2}\langle \nabla \ln\|\nabla u\|^2, \nu\rangle, \tag{63}$$

where $\nu = \nabla u/\|\nabla u\|$ points towards $\{u > 0\}$, and the last equality is a consequence of (7). Assume that $u > 0$ on Ω so that $-\nu$ points outwards (it can be done without loss of generality since both u, and $-u$ satisfy (7)). Moreover the function q from Dong's Lemma 2.6.8 satisfies,

$$q \restriction_{\partial\Omega} = (\|\nabla u\|^2 + \frac{\lambda}{2}u^2) \restriction_{\partial\Omega} = \|\nabla u\|^2 \quad \Rightarrow \quad \kappa_\nu = \frac{1}{2}\langle \nabla \ln q, \nu\rangle.$$

By Green's formula (see e.g. [19], p. 7, noting that $\Delta = -\mathrm{div} \circ \nabla$, and the orientation e_1 of $\partial\Omega$ is chosen so that $\{-\nu, e_1\}$, agrees with the orientation of Ω, this is an opposite convention to (61) which justifies the minus sign on the right hand side) we obtain

$$\begin{aligned} -\frac{1}{2}\int_{\Omega} \Delta \ln q = -\int_{\partial\Omega} \frac{1}{2}\langle \nabla \ln q, -\nu \rangle, \qquad & \Rightarrow \\ \frac{1}{2}\int_{\Omega} \Delta \ln q = \int_{\partial\Omega} \kappa_{\nu} = \int_{\partial\Omega} \kappa \ , & \end{aligned}$$

in the last integral the orientation the same as in (61). Applying estimate (59) and Gauss-Bonnet Theorem we obtain

$$\begin{aligned} \int_{\partial\Omega} \kappa &\leq \frac{\lambda}{2}\,\mathrm{Vol}(\Omega) - \int_{\Omega} K^{-} \\ 2\pi\chi(\Omega) - \Big(\int_{\Omega} K^{+} + \int_{\Omega} K^{-}\Big) &\leq \frac{\lambda}{2}\,\mathrm{Vol}(\Omega) - \int_{\Omega} K^{-} \\ 2\pi\chi(\Omega) - \int_{\Omega} K^{+} &\leq \frac{\lambda}{2}\,\mathrm{Vol}(\Omega). \end{aligned}$$

Now, the claim follows from (60), since $\chi(D^2) = 1$. □

Corollary 2.6.11. *If Σ is a nonpositively curved surface (i.e. $K_s \leq 0$) of volume V then a necessary condition for $\Omega \cong D^2$ is*

$$\frac{4\pi}{V} \leq \lambda. \tag{64}$$

Proposition 2.6.12 ([19], p. 251)**.** *For any compact surface Σ with diameter $d(\Sigma)$, the k-th eigenvalue $\lambda_k(\Sigma)$ satisfies,*

$$\lambda_k(\Sigma) \leq \frac{1}{4} + \Big(\frac{2\pi k}{d(\Sigma)}\Big)^2. \tag{65}$$

Combining Proposition 2.6.12 with Theorem 2.6.10 leads to the following inequality for $\lambda = \lambda_k$,

$$\begin{aligned} 4\pi - 2\int_{\Omega} K^{+} &\leq \mathrm{Vol}(\Sigma)\Big(\frac{1}{4} + \Big(\frac{2\pi k}{d(\Sigma)}\Big)^2\Big), \quad \Rightarrow \\ d(\Sigma)\Big(\frac{16\pi - \mathrm{Vol}(\Sigma) - 8\int_{\Omega} K^{+}}{16\,\pi^2\,\mathrm{Vol}(\Sigma)}\Big)^{\frac{1}{2}} &\leq k. \end{aligned} \tag{66}$$

Therefore, for negatively curved surfaces one obtains

Proposition 2.6.13. *A necessary condition for the nodal domain of a k'th-eigenfunction to be a disc on a surface Σ of nonpositive curvature ($K_s \leq 0$) is*

$$d(\Sigma)\Big(\frac{1}{\pi\,\mathrm{Vol}(\Sigma)} - \frac{1}{16\pi^2}\Big)^{\frac{1}{2}} \;\leq\; k,$$
$$\textit{and if}\quad \mathrm{Vol}(\Sigma) = 1,\ \textit{then}\quad d(\Sigma) \;\leq\; \frac{89}{50}k. \tag{67}$$

It would be desirable to extend the above results to the case when λ (in (7)) is not constant, i.e. a smooth positive function ($\lambda > 0$, $\lambda \in C^\infty(\Sigma)$). This however requires an appropriate extension of Dong's Lemma which does not yet exist. At this point I may only offer the following remark.

Remark 2.6.14. If we allow λ (in (7)) to be a positive function and assume that $\|\nabla u\| = \text{const}$ along $\partial\Omega$, then $\kappa = \frac{1}{2}\langle\nabla \ln\|\nabla u\|^2, \nu\rangle = 0$ (by (63)). Consequently $\partial\,\Omega$ is geodesic. The Gauss-Bonnet Theorem then tells us the necessary and sufficient condition for $\Omega \cong D^2$ to be a disc, namely,

$$2\pi = \int_\Omega K_s. \tag{68}$$

2.6.2 Examples of surfaces with homotopically essential nodal curves.

We conclude this section by indicating a family of surfaces of constant negative curvature ($K_s = -1$) which have homotopically essential nodal curves for low (i.e. small k) eigenvalues λ_k. These examples have been built by Buser (see [19], p. 248 for an exposition), and they are intended to show existence of surfaces with arbitrarily small eigenvalues. In the context of Corollary 2.6.11, this is exactly what one must achieve to violate the necessary condition, in Theorem 2.6.10, for a contractible nodal domain. Here we briefly review Buser's construction and point out details that are relevant for us.

If Σ is a complete Riemannian surface with $K_s = -1$, then its universal cover is $\mathbb{H}^2$; and if Σ is compact, it must have genus $g(\Sigma) \leq 2$. Recall that along a given

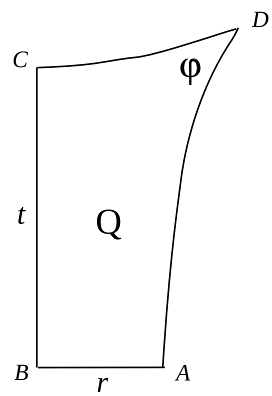

Figure 3: Quadrilateral Q on the hyperbolic plane $\mathbb{H}^2$.

geodesic γ parameterized by the arc-length t, we may express the metric in local coordinates via an exponential map (so called FERMI COORDINATES) as follows,

$$ds^2 = dt^2 + \eta^2(r,t)\,dr^2, \qquad dV = \eta(r,t)dt\,dr \tag{69}$$

where r parameterizes orbits of a unit vector field orthogonal to the geodesic γ. Note that when $K_s = -1$ we have $\eta(r,t) = \cosh t$.

Surfaces constructed by Buser are built from so called LÖBELL PIECES. Namely, one considers the quadrilateral Q of Figure 3, in the hyperbolic plane $\mathbb{H}^2$, where $\varphi = \pi/3$, and place six copies of Q centered at the vertex D, which results in a hexagon H, see Figure 4. We label geodesic segments bounding H: α_1, β_1, α_2, β_2, α_3, β_3 and denote their lengths by $L(\alpha_i) = 2\,R$, $L(\beta_j) = 2\,T$. The next step is to form the pair of pants Ω_L (i.e. S^2 with three discs removed), which is called the LÖBELL Y-PIECE [19], by gluing H to its copy $\tilde{H}$ along geodesics β_j and $\tilde{\beta}_j$. Let γ_j be the bounding geodesic obtained from α_j and $\tilde{\alpha}_j$, $j = 1, 2, 3$. Clearly each γ_j has length $4R$, and the area of Löbell Y-piece Ω_L equals (by (61), and (60)) to

$$\mathrm{Vol}(\Omega_L) = 2\,\pi. \tag{70}$$

Also notice the formula relating R and T in in the hyperbolic metric, [19]:

$$\frac{1}{2} = (\sinh R)(\sinh T). \tag{71}$$

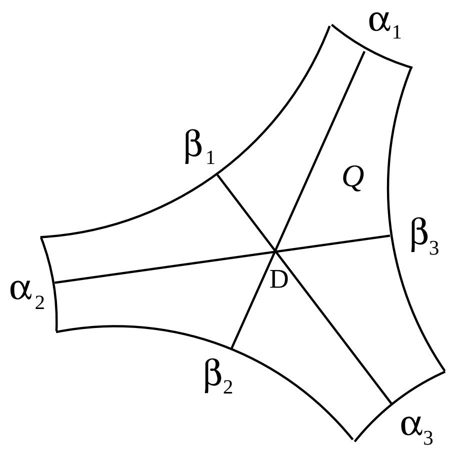

Figure 4: Hexagon H.

Therefore we may think about $\{\Omega_L\}$ as a family of surfaces parametrized by R.

Lemma 2.6.15 ([19] p. 249)**.** *For a Löbell Y-piece Ω_L satisfying $R < arcsinh(\frac{1}{2\sinh 1})$, i.e. $T > 1$, the lowest Dirichlet eigenvalue $\lambda(\Omega_L)$ of Ω_L satisfies*

$$\lambda(\Omega_L) \leq \frac{12\,R\,\sinh 1}{2\pi - 12\,R\,\sinh 1}\ . \tag{72}$$

Proof. Notice that the injectivity radius along γ_j is greater equal to T and $T > 1$. Consequently the collars:

$$D_j = \{x \in \Omega_L : \mathrm{dist}(x, \gamma_j) < 1\}, \qquad j = 1, 2, 3$$

are disjoint. Recalling the formula for the area element in the hyperbolic metric we compute the area of D_j's,

$$\mathrm{Vol}(D_j) = \int_0^1 \int_{\gamma_j} \cosh t\, dt\, dr = L(\gamma_j) \int_0^1 \cosh t\, dt = 4\,R\,\sinh 1.$$

Now, consider the test function ϕ on Ω_L:

$$\phi(x) = \begin{cases} \mathrm{dist}(x, \partial\Omega_L), & \mathrm{dist}(x, \partial\Omega_L) \leq 1; \\ 1, & \text{otherwise.} \end{cases}$$

Since $\mathrm{dist}(r,t,\partial\Omega_L) = \int_0^t d\tau = t$, we obtain,

$$\int_{\Omega_L} \|\nabla\phi\|^2\, dV = \sum_{j=1}^{3} \mathrm{Vol}(D_j) = 12\, R\, \sinh 1.$$

$$\int_{\Omega_L} \phi^2\, dV \geq \mathrm{Vol}(\Omega_L) - \sum_{j=1}^{3} \mathrm{Vol}(D_j) = 2\,\pi - 12\, R\, \sinh 1.$$

The estimate (72) follows from the Rayleigh's quotient (10). $\square$

Using Löbell pieces one can built closed orientable surfaces Σ of an arbitrary genus $g(\Sigma)$. We take $2\,g(\Sigma) - 2$ copies of Ω_L: Ω_1, Ω_2,..., $\Omega_{2g(\Sigma)-2}$, such that Ω_k has the boundary geodesic $\gamma_{k,j}$. Identify $\gamma_{k,2}$ with $\gamma_{k+1,3}^{-1}$ (reversed orientation) for all $k = 1, 2, \ldots, 2g(\Sigma) - 3$, and $\gamma_{2g(\Sigma)-2,2}$ with $\gamma_{1,3}^{-1}$, also identify $\gamma_{2l-1,1}$ with $\gamma_{2l,1}^{-1}$ for all $l = 1, \ldots, g(\Sigma) - 1$. The result is the Löbell surface Σ i.e. a compact orientable surface of genus $g(\Sigma)$ and $K_s = -1$. Since we used $2g(\Sigma) - 2$ copies of Ω_L to produce Σ, the Domain of Monotonicity Theorem 2.2.5 combined with the above estimate imply,

$$\lambda_{2g(\Sigma)-3}(\Sigma) \leq \lambda(\Omega_L) \leq \frac{12\, R\, \sinh 1}{2\pi - 12\, R\, \sinh 1}. \tag{73}$$

It is clear that the right hand side tends to zero as $R \to 0$, which implies the following

Theorem 2.6.16 (Buser [17]). *Given any ε and any positive integer $g \geq 2$, there exists a compact orientable surface Σ of genus $g(\Sigma)$ such that*

$$\lambda_{2g(\Sigma)-3}(\Sigma) < \varepsilon.$$

The important point for our purposes is that the area $\mathrm{Vol}(\Sigma)$ of the Löbell surface Σ is independent of R, and given by (from (71)),

$$\mathrm{Vol}(\Sigma) = 4(g(\Sigma) - 1)\pi. \tag{74}$$

As a consequence of Theorem 2.6.10, (73), and (74), we obtain the main theorem of this subsection.

Theorem 2.6.17. *Let Σ be a Löbell surface satisfying*

$$R < \frac{\pi}{6\, g(\Sigma)\, \sinh 1}. \tag{75}$$

Nodal domains of eigenfunctions on Σ corresponding to eigenvalues: $\lambda_1, \ldots, \lambda_{2g(\Sigma)-3}$, cannot be discs with smooth boundary.

Results of the previous sections tell us that perturbing the metric on the Löbell surface satisfying (75) assures homotopically essential nodal curves.

CHAPTER III

ON THE GEOMETRY OF CONTACT STRUCTURES

A CONTACT STRUCTURE ξ is a non-integrable distribution of planes on a 3-manifold (ie. ξ is a rank 2-subbundle of TM, which can be thought of as being as far from a 2-dimensional foliation as possible). One of the first results concerning contact structures comes from Martinet, [54], who shows that every orientable 3-manifold admits a contact structure. Investigation of these structures in the setting of Riemannian geometry was first probably first initiated by Chern and Hamilton, in [21], where the authors investigate a version of the Yamable problem in this setting. Their main theorem states that an arbitrary adapted metric can be conformally deformed to a metric of a constant Webster curvature (see [21]).

Recently, the 3-dimensional topology of contact structures has received a lot of attention from topologists, and has been a source of new invariants (e.g. contact homology [24], knot invariants [32]). One of the major open problems in the field is the classification of contact structures up to isotopy, which uncovered two different classes of contact structures: *overtwisted* and *tight*, introduced by Bennequin and Eliashberg ([11], [25]). In this chapter we relate tightness and overtwistedness to the geometry of adapted metrics, which was formulated as Problem 1.0.2 in Chapter 1.

In Section 3.1 of this chapter we describe adapted metrics in detail and address the problem of local compatibility of an arbitrary metric with a prescribed contact structure. We devote Section 3.2 to the introduction and more detailed investigation of the classes of tight and overtwisted contact structures, including the technique of dividing curves and characteristic surfaces. This technique provides, in certain circumstances, sufficient means to distinguish these classes. In Section 3.3 we show that characteristic surfaces are nodal sets of solutions to certain subelliptic PDEs. Section 3.2.1 is devoted to the proof of Giroux's Theorem, which classifies S^1-invariant contact structures on S^1-principal bundles. Giroux's Theorem is the main topological ingredient in the proof of the main result of this chapter, proven in Section 3.4.

3.1 Contact structures and adapted metrics.

Throughout this section and the rest of the chapter we work with orientable contact structures defined on 3-dimensional manifolds, a pair (M, ξ) is often called the CONTACT MANIFOLD.

Every orientable contact structure ξ is always defined by a kernel of a 1-form α, which is also called the CONTACT FORM. The nonintegrability of ξ may be expressed in terms of α as follows;

$$\alpha \wedge d\alpha \neq 0, \qquad \alpha \in \Omega^1(M^3). \tag{76}$$

Contact structures admit a local model, specifically we have the following;

Theorem 3.1.1 ([36]). *Any contact form α locally is diffeomorphic to the* STANDARD CONTACT FORM*: $\alpha_0 = dz + x\,dy$. Namely, for each neighborhood U of a point $p \in M$, there exists a diffeomorphism ϕ of U; $\phi : V \to U$, $V \subset M$ such that $\phi^*\alpha = \alpha_0$.*

Notice that Equation (76) simply says that the line field defined by $\ker(d\alpha) = \{X : d\alpha(X, \cdot) \equiv 0\}$ is transverse to $\xi = \ker\alpha$. This line field is spanned by a vector field which is of special importance.

Definition 3.1.2. *Given a contact form α, the* REEB FIELD *of α is the unique vector field X_α satisfying*

$$\alpha(X_\alpha) = 1, \qquad \mathcal{L}_{X_\alpha}\alpha = 0. \tag{77}$$

Cartan's formula yields

$$\iota(X_\alpha)\, d\alpha = 0, \qquad \text{so} \qquad X_\alpha \in \ker(d\alpha). \tag{78}$$

(Here $\iota(X)$ contracts $d\alpha$ by X i.e. $\iota(X)\, d\alpha = d\alpha(X, \cdot)$. The operation $\iota(X)$ is well defined for an arbitrary differential form.)

In Riemannian geometry, vector fields dual to contact forms are often called curl eigenfields or more generally Beltrami fields. This justifies the following;

Definition 3.1.3. *Let (M, g) be a Riemannian manifold. We call a 1-form α a μ-*BELTRAMI FORM*, or simply a Beltrami form iff it satisfies*

$$* \, d\alpha = \mu\, \alpha, \qquad \mu \neq 0, \quad \mu \in C^\infty(M), \tag{79}$$

where $*$ is the Hodge star operator and μ a nonzero smooth function. If $\mu \equiv const$ then α is an eigenform of the curl operator $*\,d$, i.e. the CURL EIGENFORM*, and the dual vector field is called the* CURL EIGENFIELD.

The theory of elliptic operators, [7], tells us that Beltrami forms exist on an arbitrary Riemannian manifold since the curl always has eigenforms. The relation of Beltrami forms to contact structures comes from the following;

Proposition 3.1.4. *A given Beltrami form α defines a contact structure on the complement of the zero set $\alpha^{-1}(0)$ of α.*

Proof. We check the nonintegrability condition (76),

$$\alpha \wedge d\alpha = \mu\, \alpha \wedge * \, \alpha = \mu\, \|\alpha\|^2 * 1.$$

Consequently, $\alpha \wedge d\alpha \neq 0$ iff $\alpha \neq 0$. □

In the rest of this chapter all Beltrami forms, unless stated otherwise, are assumed to be nonsingular (i.e. everywhere nonvanishing).

Definition 3.1.5. *Given a contact form α, we say that a Riemannian metric is* ADAPTED *to α if Equation* (79) *holds. We also refer to such metrics as* ADAPTED METRICS.

In Chapter 4 we discuss the importance of Beltrami forms in fluid mechanics and the physics of plasmas, but here we focus on their "contact geometric aspect". In the remaining part of this section we explore contact structures in the context of metric adaptation. The following result provides a useful characterization of adapted metrics;

Lemma 3.1.6. *Locally, for a given contact form α, a choice of a metric g adapted to α is equivalent to a choice of a local frame of vector-fields $\{e_1, e_2, e_3\}$ that satisfy:*
(1) $e_1 = v\, X_\alpha$, where X_α is a Reeb field of α and v a positive function, and
(2) $\xi = span\{e_2, e_3\}$.

Remark 3.1.7. *Moreover, one may define an almost complex structure $J : \xi \mapsto \xi$ on ξ in terms of the frame $\{e_2, e_3\}$ as follows;*
(3) $Je_2 = -e_3$, $Je_3 = e_2$.

Proof. We show both implications:

"$\Rightarrow$": Given an adapted metric g to a contact form α, we have the unique dual vector field X such that $\alpha(\,\cdot\,) = g(X, \,\cdot\,)$. We define $e_1 = X/\|X\|$ and choose arbitrary frame on ξ satisfying *(2)*. Let $\{\eta_i\}$ be the dual co-frame to $\{e_i\}$. Using Equation (79) we show *(1)*:

$$\iota(X)d\alpha = \iota(X)\mu * \alpha = \iota(e_1)\mu\, \|X\| * \eta_1 = \iota(e_1)\mu\, \|X\|\eta_2 \wedge \eta_3 = 0, \tag{80}$$

By (78), we conclude that $e_1 = v\, X_\alpha$ for some $v \neq 0$.

"$\Leftarrow$": Let $\{e_i\}$ be an adapted frame to α i.e. the frame satisfying *(1)* and *(2)*, and $\{\eta_i\}$ the co-frame. We show that $g = \sum_i \eta_i^2$ is adapted to α. By *(2)*: $e_1 \perp \xi$

therefore $\alpha(\,\cdot\,) = g(X,\,\cdot\,)$ for $X = h\,e_1 = h\,v\,X_\alpha$ and $\eta_1 = w\,\alpha$, for positive functions $v, h, w > 0$. Notice the following relations among v, h, w;

$$e_1 = v\,X_\alpha, \quad e_1 = \frac{X_\alpha}{\|X_\alpha\|} \quad \Rightarrow \quad v = \frac{1}{\|X_\alpha\|},$$
$$\eta_1(\,\cdot\,) = g(e_1,\,\cdot\,) = g(v\,X_\alpha,\,\cdot\,) = \frac{1}{h}\,g(X,\,\cdot\,) = \frac{1}{h}\alpha(\,\cdot\,) \quad \Rightarrow \quad w = h,$$
$$1 = \alpha(X_\alpha) = g(X, X_\alpha) = h\,v\,\|X_\alpha\|^2 \quad \Rightarrow \quad h = \frac{1}{\|X_\alpha\|}.$$

Therefore $v = w = h = 1/\|X_\alpha\|$. Since $\iota(e_1)d\,\alpha = v\iota(X_\alpha)d\,\alpha = 0$, one obtains,

$$0 = \iota(e_1)d\alpha = \iota(e_1)\,[a\eta_1\wedge\eta_2 + b\eta_1\wedge\eta_3 + c\eta_2\wedge\eta_3] = a\eta_2 + b\eta_3 \;\Rightarrow a = b = 0$$
$$\Rightarrow d\,\alpha = c\,\eta_2\wedge\eta_3 = c * \eta_1 = c\,v\, * \alpha.$$

We obtain (79) by defining $\mu = c\,v$. Notice that $\mu \neq 0$ since $\alpha\wedge d\alpha \neq 0$. □

Lemma 3.1.8. *Locally, let* $\{e_1, e_2, e_3\}$ *be the frame defined in Lemma 3.1.6 and* $\{\eta_1, \eta_2, \eta_3\}$ *the co-frame. We have the following formula for the adapted metric g:*

$$g(X,Y) = \sum_i \eta_i^2(X,Y) = \frac{1}{v^2}\,\alpha(X)\alpha(Y) + \frac{2\,v}{\mu}\,d\,\alpha(X, JY), \tag{81}$$

$$\textit{where} \quad \mu = v\,d\alpha(e_2, e_3) = v\,\alpha([e_2, e_3]) \;\textit{ and }\; v = \|X_\alpha\|.$$

Proof. Since $\eta_1 = \frac{1}{v}\,\alpha$ and

$$d\alpha(\,\cdot\,, J\,\cdot\,) = \frac{\mu}{v}\eta_2\wedge\eta_3(\,\cdot\,, J\,\cdot\,) = \frac{\mu}{2\,v}\,[\eta_2(\,\cdot\,)\otimes\eta_3(J\,\cdot\,) - \eta_3(\,\cdot\,)\otimes\eta_2(J\,\cdot\,)]$$
$$= \frac{\mu}{2\,v}\left(\eta_2^2(\,\cdot\,,\,\cdot\,) + \eta_3^2(\,\cdot\,,\,\cdot\,)\right),$$

where the last equality follows from *(3)*. We have,

$$g(\,\cdot\,,\,\cdot\,) = \sum_i \eta_i^2(\,\cdot\,,\,\cdot\,) = \frac{1}{v^2}\,\alpha^2(\,\cdot\,,\,\cdot\,) + \eta_2^2(\,\cdot\,,\,\cdot\,) + \eta_3^2(\,\cdot\,,\,\cdot\,)$$
$$= \frac{1}{v^2}\alpha^2(\,\cdot\,,\,\cdot\,) + \frac{2\,v}{\mu}d\alpha(\,\cdot\,, J\,\cdot\,).$$

□

Now we may conclude the global existence of adapted metrics in the following,

Theorem 3.1.9. *Given an arbitrary contact form α, one may always adapt the Riemannian metric g to α, such that Equation* (79) *is satisfied on* (M, g).

Proof. Indeed, by the formula (81) for g in Lemma 3.1.8, it is sufficient to choose a global almost complex structure $J : \xi \mapsto \xi$, $\xi = \ker\alpha$, and a vector $e_1 = v\,X_\alpha$, where v is an arbitrary positive function. Then the metric defined by (81) will be an adapted metric.

In order to define J globally, let g_ξ be an arbitrary metric on ξ (e.g. a metric induced from a global metric on M) and let J be the rotation by $\pi/2$ in (ξ, g_ξ). □

Remark 3.1.10. Formula (81) suggests that the moduli space of all metrics adapted to α can be locally parametrized by v and J.

A special case of adapted metric occurs when X_α is the unit vector field in the metric, this is the case introduced by Chern and Hamilton in [21]. We obtain their result as a consequence of Theorem 3.1.9 in the following;

Theorem 3.1.11 ([21])**.** *For every choice of a contact form α and an almost complex structure $J : \xi \mapsto \xi$, $\xi = \ker\alpha$. There exists a unique Riemanian metric adapted to α such that*

$$* \, d\alpha = 2\,\alpha, \qquad \alpha \wedge * \, \alpha = *1.$$

Adapted metrics with unit Reeb field X_α described by Chern and Hamilton in [21] admit local obstructions. Guildfoyle, in [43] shows:

Theorem 3.1.12 ([43])**.** *A hyperbolic metric cannot be the adapted metric for a contact form with a unit length Reeb field. Every smooth contact form on $\mathbb{R}^3$ adapted to a flat metric is contact isometric to $\mathbb{R}^3$ with the standard metric and adapted contact 1-form: $\beta_1 = \sin(\mu\, z)dx + \cos(\mu\, z)dy$. Every smooth contact form on $\mathbb{R}^3$ adapted to* *an elliptic metric is contact isometric to an open subset of* $\mathbb{S}^3$ *with the round metric and standard adapted contact 1-form* $\beta_2 = \frac{1}{\lambda} xdy - ydx + zdw - wdz$ *induced from* $\mathbb{R}^4$.

Remark 3.1.13. When $\|X_\alpha\| \neq 1$ one expects more flexibility in metrics adapted to α. Indeed, following examples in [16] (Example 3.7, p. 93) one shows that in the class of analytic metrics the equation

$$* \, d\alpha = \mu \, \alpha, \qquad \mu = \text{const},$$

may always be locally solved for a nonvanishing 1-form α, and an arbitrary choice of the constant μ. Consequently, e.g., the hyperbolic metric can be an adapted metric for some contact structure at least locally, which is not the case when $\|X_\alpha\| = 1$.

3.1.1 Local compatibility of contact forms and Riemannian metrics

Lemma 3.1.6 tells us that a metric adapted to α is essentially determined by choosing the normal vector to the contact planes to be proportional to the Reeb vector field X_α. Multiplication by a positive function u does not change the contact plane distribution determined by α, but the resulting 1-form $\alpha' = u \, \alpha$ has a different Reeb field $X_{\alpha'}$. The following natural question arises:

Problem 3.1.14. *Let* ξ *be a contact plane distribution defined on an open domain* (U, g)*, equipped with an arbitrary Riemannian metric* g*. Is it possible to find a contact form* α*,* $\xi = \ker \alpha$*, such that* α *satisfies Equation* (79) *on a possibly smaller open subset of* (U, g)*?*

In the remainder of this section we address this problem and give a negative answer. We also derive an obstruction or, in the language of [16], a compatibility condition.

We begin by investigating how the Reeb field changes under rescaling a contact form by a positive function. Let us choose a *random* contact form β such that $\xi = \ker \beta$. Let α be a different contact form which satisfies $\xi = \ker \alpha$. We assume that both α and β induce the same positive orientation on U, i.e. $\alpha \wedge d\alpha > 0$, $\beta \wedge d\beta > 0$. Then for some positive smooth function $v \in C^\infty(M)$, $v > 0$, the Reeb field X_α of α satisfies the following unique decomposition,

$$X_\alpha = v\, X_\beta + Y_\xi, \qquad Y_\xi \in \xi,$$

where X_β is the Reeb-field of β. We choose $\alpha = u\,\beta$, $u \in C^\infty(M)$, $u > 0$. Since $\beta(Y_\xi) = 0$, $\beta(X_\beta) = 1$ and $d\beta(X_\beta, \cdot\,) = 0$ we obtain

$$\iota(X_\alpha)\, d\alpha = 0;\ d\alpha = d\,u \wedge \beta + u\, d\beta \ \Rightarrow du(X_\alpha)\, \beta - \beta(X_\alpha)\, du + u\, d\beta(X_\alpha, \cdot) = 0 \ \Rightarrow$$
$$du(v\, X_\beta + Y_\xi)\, \beta - \beta(v\, X_\beta + Y_\xi)\, du + u\, d\beta(v\, X_\beta + Y_\xi, \cdot) = 0 \quad \Rightarrow$$
$$du(X_\alpha)\, \beta(\,\cdot\,) - v\, du(\,\cdot\,) + u\, d\beta(Y_\xi, \,\cdot\,) = 0. \tag{82}$$

Equation (82) must hold when restricted to ξ. Therefore

$$-v\, du(\,\cdot\,) + u\, d\beta(Y_\xi, \,\cdot\,) = 0, \quad \Rightarrow$$
$$d\beta(Y_u, \,\cdot\,) \restriction_\xi = du(\,\cdot\,) \restriction_\xi, \quad \text{where } Y_u = \frac{u}{v}\, Y_\xi. \tag{83}$$

Since $d\beta \restriction_\xi$ is a symplectic form on ξ, Equation (83) has a solution for any u in form of a Hamiltonian vector field Y_u. Then Y_ξ is determined uniquely in terms of Y_u, as in (83). Equation (83) also tells us

$$du(Y_u) = 0 \quad \Rightarrow \quad du(Y_\xi) = \mathcal{L}_{Y_\xi} u = 0, \tag{84}$$

that is, Y_ξ must be tangent to the level sets of u. The uniformization $\alpha(X_\alpha) = 1$ and $\beta(X_\beta) = 1$ also determines a relation between u and v:

$$1 = \alpha(X_\alpha) = u\beta(v\, X_\beta + Y_\xi) = u\, v, \quad u = \frac{1}{v}. \tag{85}$$

Based on these considerations we may now approach Problem 3.1.14. Lemma 3.1.6 implies that the metric is adapted to the contact form α iff:

(1) the orthogonal vector field to ξ is proportional to the Reeb field X_α;

(2) the metric on ξ is conformally equivalent to $d\alpha(\,\cdot\,, J\,\cdot\,)$, for some choice of J.

Given a contact structure ξ and an arbitrary Riemannian metric g, it is easy to find a contact form β that satisfies the condition *(2)*. Indeed, since $d\beta$ is a nondegenerate symplectic 2-form on ξ the condition *(2)* follows since it is sufficient to choose J which is a rotation by $\pi/2$ in terms of g restricted to ξ. However there is no reason for X_β to be orthogonal to ξ in g, i.e. the condition *(1)* does not need to hold.

Consequently, we seek a positive function u such that $\alpha = u\,\beta$ and $X_\alpha \perp \xi$. Choose a local orthonormal frame of vector fields adapted to ξ:

$$\{e_1, e_2, e_3\}, \quad \text{and} \quad e_1 \perp \xi.$$

Denote by $\{\eta_i\}$ the dual co-frame. In order to obtain *(1)* we need to find an appropriate u. Given the unique decomposition $e_1 = a\,X_\beta + e_\xi$, we require e_1 to be proportional to X_α i.e.

$$X_\alpha = w\,e_1 = w\,a\,X_\beta + w\,e_\xi, \quad \text{for some } w > 0.$$

By Equation (83), we must show that $w\,e_\xi$ satisfies,

$$\frac{u}{w\,a}\,d\beta(w\,e_\xi,\,\cdot\,)\restriction_\xi = du(\,\cdot\,)\restriction_\xi \quad \Leftrightarrow \quad d\beta(\frac{e_\xi}{a},\,\cdot\,)\restriction_\xi = d(\ln u)(\,\cdot\,)\restriction_\xi \;. \tag{86}$$

Notice that the rescailing factor w is irrelevant for this equation. Equation (86) may be expressed as follows;

$$\gamma + \psi\,\alpha = d\,h \qquad \text{where } \gamma = d\beta(e_\xi/a,\,\cdot\,),\ h = \ln u. \tag{87}$$

for some function ψ. By the Poincaré's Lemma (see e.g. [14]) the 1-form $\gamma + \psi\,\alpha$ is locally exact if and only if it is locally closed. Applying the exterior derivative to (87) we obtain

$$d\,\gamma + d\,\psi \wedge \alpha + \psi\,d\alpha \;=\; 0. \tag{88}$$

Choosing a smaller neighborhood U, if necessary, we may define the local coordinate system (x, y, z) on U such that $\beta = dz + x\,dy$ (see Theorem 3.1.1). Expressing

Equation (88) in coordinates (x,y,z) we obtain

$$(A + x\,\partial_x\psi + \psi)\,dx\wedge dy + (B + \partial_x\,\psi)\,dx\wedge dz + (C + \partial_y\psi - x\,\partial_z\psi)dy\wedge dz = 0,$$

where $d\gamma = A\,dx\wedge dy + B\,dx\wedge dz + C\,dy\wedge dz$. This leads to the system of equations for ψ,

$$\begin{cases} A + x\,\partial_x\psi + \psi = 0, \\ B + \partial_x\,\psi = 0, \\ C + \partial_y\psi - x\,\partial_z\psi = 0. \end{cases} \tag{89}$$

Now, we investigate these equations using the convenient language of exterior differential systems (EDS), [16], on 1-jet space: $J^1(\mathbb{R}^3,\mathbb{R}) \cong \mathbb{R}^3\times\mathbb{R}\times\mathbb{R}^3$. In coordinates: (x,y,z,ψ,p,q,r) on $J^1(\mathbb{R}^3,\mathbb{R})$, the system (89) is equivalent to the following EDS:

$$\begin{cases} \{F(x,y,z,\psi,p,q,r)=0\} \equiv \begin{cases} A + x\,p + \psi = 0, \\ B + p = 0, \\ C + q - x\,r = 0 \end{cases} \\ \varsigma = d\psi - p\,dx - q\,dy - r\,dz = 0, \\ d\varsigma = -dp\wedge dx - dq\wedge dy - dr\wedge dz = 0, \\ \{dF = 0\} \equiv \begin{cases} \theta_1 = dA + p\,dx + x\,dp + d\psi = 0, \\ \theta_2 = dB + dp = 0 \\ \theta_3 = dC + dq - r\,dx - x\,dr = 0 \end{cases} \end{cases} \tag{90}$$

By the Frobenius Theorem, [16], the solution to (90) exists iff the differential ideal $\mathcal{I}_{\text{diff}} = \langle F,\varsigma,dF,d\varsigma\rangle$ in $\Omega^*(U)$ is generated algebraically by $\mathcal{I}_{\text{alg}} = \langle F,\varsigma\rangle$. In the following, we uncover an obstruction to this condition.

Substitution in θ_1 gives

$$dA + x\,dB + 2\,B\,dx + q\,dy + r\,dz = 0. \tag{91}$$

Applying $dx\wedge dy$ to both sides of Equation (91) we obtain

$$dA\wedge dx\wedge dy + x\,dB\wedge dx\wedge dy + r\,dx\wedge dy\wedge dz = 0 \quad\Rightarrow$$
$$-\partial_z A - x\,\partial_z B = r. \tag{92}$$

Applying $dx \wedge dz$ to both sides of Equation (91) leads to

$$dA \wedge dx \wedge dz + x\, dB \wedge dx \wedge dz - q\, dx \wedge dy \wedge dz = 0 \quad \Rightarrow$$
$$-\partial_y A - x\, \partial_y B = q. \tag{93}$$

Substitution of (92), (93) into $d\varsigma$ yields

$$dB \wedge dx + d(\partial_y A + x\, \partial_y B) \wedge dy + d(\partial_z A + x\, \partial_z B) \wedge dz = 0.$$

This uncovers the following compatibility condition;

$$\begin{cases} \partial_x \partial_y A + \partial_x \partial_y B = 0, \\ \partial_x \partial_z A + \partial_z \partial_x B = 0, \\ \partial_z \partial_y A + x \partial_z \partial_y B - \partial_y \partial_z A - x \partial_y \partial_z B = 0. \end{cases} \tag{94}$$

Remark 3.1.15. Clearly, one may expect other compatibility conditions, e.g. involving the C component of $d\gamma$.

We summarize our considerations in the following;

Theorem 3.1.16. *Given a contact structure ξ, in a local neighborhood (U, g) (where g is a prescribed Riemannian metric), the condition defined in* (94) *is necessary to existence of a contact form α, $\ker \alpha = \xi$, which satisfies Equation* (79) *in g.*

Remark 3.1.17. [*A counterexample for the standard contact form* $\alpha_0 = dz + x\, dy$] Choose a Riemannian metric g on $\mathbb{R}^3$ such that the vector field $e_1 = \partial_z + x^3\, y\, \partial_x$ is unit normal to $\xi_0 = \ker \alpha_0$ in g. The following shows that g cannot be adapted to ξ_0. Clearly e_1 is transverse to ξ_0, since $\alpha_0(e_1) = 1$ and

$$\gamma(\,\cdot\,) = d\alpha_0(e_1, \,\cdot\,) = x^3\, y\, dy, \text{ and } d\gamma = 3\, x^2\, y\, dx \wedge dy,$$
$$A = 3\, x^2\, y,\ B = C = 0.$$

This contradicts the first equation in (94), since $\partial_x \partial_y A + \partial_x \partial_y B = 6\, x \neq 0$.

3.2 Tight and overtwisted contact structures.

It has been known since the work of Bennequin and Eliashberg [11, 26] that there are two fundamentally different classes of contact structures.

Definition 3.2.1. *A contact structure ξ is* OVERTWISTED *if and only if there exists an embedded disc $D^2 \subset M$ such that D is transverse to ξ near ∂D but ∂D is tangent to ξ. Any contact structure which is not overtwisted is called* TIGHT*. If all the covers of a structure are tight then we call it* UNIVERSALLY TIGHT.

These concepts arise in the context of the classification of contact structures up to isotopy. We say that two contact structures ξ_0 and ξ_1 are *isotopic* iff there exists a homotopy of plane fields ξ_t, $0 \leq t \leq 1$, such that ξ_t is a contact plane distribution for all t. Eliashberg, [25], showed the following fundamental theorem for the class of overtwisted structures.

Theorem 3.2.2 (Eliashberg, [25]). *Given a closed compact 3-manifold M, let $\pi_0(\Lambda^2(M))$ be the set of homotopy classes of oriented plane fields on M, and let $\mathcal{C}_0$ be the set of isotopy classes of oriented overtwisted contact structures on M. The natural inclusion map $\mathcal{C}_0 \hookrightarrow \pi_0(\Lambda^2(M))$ is a homotopy equivalence.*

An abundance of information concerning the isotopy classes of contact structures, comes from knots embedded in a contact manifold (M, ξ). One distinguishes TRANSVERSAL KNOTS (i.e. curves everywhere transverse to ξ) and LEGENDRIAN KNOTS (i.e. curves everywhere tangent to ξ). For a transversal null-homologous curve γ we may define the self linking number $l(\gamma) = \mathrm{lk}(\gamma, \gamma')$ as a linking number between γ and a curve γ' obtained by an ε-push off along a nonvanishing vector field X on Seifert surface Σ of γ (i.e. $\partial \Sigma = \gamma$), where X is everywhere tangent to ξ. It is shown in [11] that $l(\gamma)$ is independent of all choices, i.e. Σ and X.

One of the first techniques for detecting a class of contact structures involves Bennequin's Inequality.

Theorem 3.2.3 (Bennequin's Inequality, [11]). *If γ is a transverse null-homologus knot in a tight contact structure (M, ξ), then*

$$l(\gamma) \leq -\chi(\Sigma), \tag{95}$$

where Σ is a Seifert surface of γ.

It can be easily argued that if ξ is overtwisted than (95) is violated for some transverse null-homologus knot. In [11], Bennequin proves that any knot in $\mathbb{R}^3$ transverse to the standard contact structure $\xi_0 \equiv \{dz + x\,dy = 0\}$ satisfies (95) which implies the following;

Theorem 3.2.4 (Bennequin, [11]). *The standard contact structure $(\mathbb{R}^3, \xi_0)$ is tight.*

For closed embedded surfaces one has a version of inequality (95) due to Eliasherg

Theorem 3.2.5 (Eliashberg, [26]). *Let (M, ξ) be a tight contact manifold and Σ and embedded surface in M. If $e(\xi)$ denotes the Euler class of ξ, then*

$$|e(\xi)[\Sigma]| = |\int_\Sigma e(\xi)| \;\leq \begin{cases} -\chi(\Sigma), & \textit{if } \Sigma \neq S^2; \\ 0, & \Sigma = S^2. \end{cases} \tag{96}$$

More recent local technique of detecting tightness/overtwistedness of a contact structure involves the characteristic surfaces and dividing curves.

Definition 3.2.6. *Given a contact manifold (M, ξ) a vector field X is called contact iff the flow φ^t of X preserves ξ, i.e.*

$$\varphi^t_* \xi = \xi. \tag{97}$$

In terms of a contact form α defining ξ, (97) implies

$$\mathcal{L}_X \alpha = h\,\alpha, \qquad h \in C^\infty(M), \quad \tilde{h} \geq 0. \tag{98}$$

In (98), when $h = 0$ the contact field X also preserves the contact form α. This special case however implies restrictions on the topology of M, see Section 3.3. Obviously, the Reeb field X_α is an example of a contact field which is transverse to ξ. Contact fields which are not everywhere transverse to ξ are of special importance.

Definition 3.2.7. *Given a contact vector field on (M, ξ), the set of tangencies $\Gamma_X = \{p \in M : X_p \in \xi_p\}$ is called the* CHARACTERISTIC HYPERSURFACE *of X in ξ.*

Given an embedded surface Σ in the contact manifold (M, ξ), and a transverse contact vector field $X \pitchfork \Sigma$, the intersection of Γ_X with Σ is called the DIVIDING SET, $\Gamma_\Sigma = \Gamma_X \cap \Sigma$. The importance of this set comes from the following criteria due to Giroux [41];

Theorem 3.2.8 (Giroux, [41])**.** *Given an embedded orientable surface Σ in the contact manifold (M, ξ) and a contact vector field X transverse to Σ, we have the following:*

(i) Γ_Σ is a set of smooth curves, independent (up to isotopy) of the choice of the contact vector field X.

(ii) if $\Sigma \neq S^2$, then ξ has a tight tubular neighborhood iff none of the components in Γ_Σ bounds a disc.

(iii) if $\Sigma = S^2$, then ξ has a tight tubular neighborhood iff Γ_Σ is connected.

In a contact manifold (M, ξ), an embedded surface Σ admitting a transverse contact vector field is called CONVEX SURFACE. Convex surfaces exist in abundance, or more precisely are generic, as the following result states.

Theorem 3.2.9 (Honda [47])**.** *Given an arbitrary embedded surface Σ, which is either closed or has a Legendrian boundary, there exists a smooth perturbation Σ' of Σ which is convex.*

Theorem 3.2.10 (Honda [47])**.** *Two contact structures ξ, ξ' which have isotopic dividing sets on a convex surface Σ are isotopic as contact structures in a tubular neighborhood of Σ.*

Interpretation of the self-linking number $l(\gamma)$ of a transverse curve γ in (95) (and similarly of $e(\xi)[\Sigma]$) becomes more transparent in this setting. Specifically, given a convex surface Σ with a dividing set Γ_Σ we denote by Σ^+, Σ^- the components of $\Sigma \setminus \Gamma = \Sigma^+ \cup \Sigma^-$, where the orientation of Σ and ξ agree "+" and disagree "−", i.e. $\Sigma^+ = f^{-1}([0, +\infty))$, $\Sigma^- = f^{-1}((-\infty, 0])$, where $f = \alpha(X)$, $\xi = \ker \alpha$, and X is a contact vector field. The following formula holds:

$$e(\xi)[\Sigma] = \chi(\Sigma^-) - \chi(\Sigma^+). \tag{99}$$

Notice that components of Γ_Σ are always transverse curves to ξ. Let $\gamma \subset \Sigma$ be a closed curve transverse to Γ_Σ, and $\Sigma_\gamma \subset \Sigma$ be a Seifert surface for γ. Let $\Sigma^+_\gamma = \Sigma_\gamma \cap \Sigma^+$, $\Sigma^-_\gamma = \Sigma_\gamma \cap \Sigma^-$, we have the following

$$l(\gamma) = \chi(\Sigma^-_\gamma) - \chi(\Sigma^+_\gamma). \tag{100}$$

3.2.1 Contact structures on S^1-fibered spaces.

In this subsection we state and prove the result by Giroux concerning the classification of S^1-invariant contact structures on circle bundles over orientable surfaces. Our goal is to consider such structures in the Riemannian geometric setting later in this chapter.

In the case of a trivial fibration, Theorem 3.2.8 leads to the following;

Theorem 3.2.11 (Giroux, [42])**.** *Let ξ be an S^1-invariant contact structure on $S^1 \times \Sigma$, and let X be a contact vector field tangent to the S^1-fibers. Denote by Γ_{S^1} the characteristic surface Γ_X. Let $\Gamma = \pi(\Gamma_{S^1})$ be the projection of the characteristics surface Γ_{S^1} onto Σ then ξ is tight iff*

(I) $\Sigma \neq S^2$, none of the components of $\Sigma \setminus \Gamma$ is a disc.

(II) $\Sigma = S^2$, Γ is connected.

In the general case of circle bundles we have:

Theorem 3.2.12 (Giroux, [42]). *Let ξ be an S^1-invariant contact structure on the principal circle bundle $\pi : P \mapsto \Sigma$. Let $\Gamma = \pi(\Gamma_{S^1})$ be a projection of the characteristics surface Γ_{S^1} onto Σ. Denote by $e(P)$ the Euler number of P. The following holds.*

(a) *If ξ is tight and one of the connected components of $\Sigma \setminus \Gamma$ bounds a disc, then Γ has to be a single circle and $e(P)$ must satisfy*

$$\begin{cases} e(P) > 0, & \textit{if } \Sigma \neq S^2 \\ e(P) \geq 0, & \textit{if } \Sigma = S^2. \end{cases}$$

(b) *For ξ to be universally tight it is necessary and sufficient that one of the following holds,*

(b.1) $\Sigma \neq S^2$ *none of the connected components of $\Sigma \setminus \Gamma$ is a disc.*

(b.2) $\Sigma = S^2$, $e(P) < 0$ *and* $\Gamma = \emptyset$.

(b.3) $\Sigma = S^2$, $e(P) \geq 0$ *and Γ is connected.*

Below we present Giroux's proof of this theorem, which can be found in [42], p. 249 as Proposition 4.1.

Proof. The case of a trivial fibration ($e(P) = 0$) is already covered in Theorem 3.2.11. We prove **(a)** first. Assume that ξ is tight and the number of components in $\Sigma \setminus \Gamma$ is at least 3. Let H_0 be a component of $\Sigma \setminus \Gamma$ which is a disc, and H_1 be a neighboring component $H_1 \in \Sigma \setminus \Gamma$ such that $\partial H_0 = \partial H_1$. We can form a surface with a boundary $E = \overline{H_0 \cup H_1} \subset \Sigma$, ($\#\Sigma \setminus \Gamma \geq 3 \Rightarrow \partial E \neq \emptyset$). Now, one chooses a section $\hat{E}$ of P over E. Choosing an orientation of $\hat{E}$ in an appropriate way, one may satisfy:

(i) $\partial \hat{E}$ is positively transverse to ξ (i.e. the tangent vector to $\partial \hat{E}$ points in the direction of the Reeb field X_α for $\alpha \wedge d\alpha > 0$, $\xi = \ker \alpha$);

(ii) $H_0 = \hat{E}^-$, i.e. the orientations of ξ and TH_0 disagree.

Consequently, $\chi(\hat{E}^-) = 1$ and $l(\partial \hat{E}) = 1 - \chi(\hat{E}^+)$, which contradicts the inequality (95) which implies $l(\partial \hat{E}) = 1 - \chi(\hat{E}^+) \leq -1 - \chi(\hat{E}^+)$. Therefore, ξ is overtwisted if $\#\Gamma \geq 2$ and one of the components in $\Sigma \setminus \Gamma$ is a disc. Now, we establish the inequality for the Euler number. Let $e(P) < 0$ then there exist a sufficiently small disc $Q \subset \Sigma \setminus H_0$, and a section $\hat{E}$ over $\Sigma \setminus Q$ such that $\hat{E}$ has properties (i), (ii), which contradicts tightness.

Next, we show (b.1). Observe that if $\Sigma \neq S^2$, the inverse image $\tilde{\Gamma} = \pi^{-1}(\Gamma)$ under a nontrivial finite cover $\pi : \tilde{\Sigma} \to \Sigma$ has more then one component. Indeed, $\tilde{\Gamma}$ is a projection of the characteristic surface of $\tilde{\xi} = \pi_*(\xi)$, (i.e. $\tilde{\xi}$ is obtained via a pullback of ξ under the cover $\pi : \tilde{P} \to P$). Consequently, if one of the components of Γ bounds a disc the disc must lift to $\tilde{\Sigma}$. Therefore $\tilde{\xi}$ must be overtwisted and ξ virtually overtwisted by similar reasoning to the proof in **(a)**.

In the case $\Sigma = S^2$, Lutz's classification, [53], of universal covers $(\tilde{P}, \tilde{\xi})$ of (P, ξ) distinguishes, up to contactomorphism, two cases:

- if $e(P) = 0$ then $\tilde{P} = S^2 \times \mathbb{R} = \mathbb{R}^3 \setminus \{0\}$, and $\tilde{\xi}$ is defined via $dz + x\,dy - y\,dx = 0$ and is invariant under the flow: $(x, y, z) \to (e^t x, e^t y, e^{2t} z)$.

- if $e(P) = \pm 1$ then $\tilde{P} = S^3$ is a unit sphere in $\mathbb{C}^2$ and $\tilde{\xi}$ is defined through $\bar{z}\,dz + \bar{w}\,dw = 0$, $(z, w) \in \mathbb{C}^2$, which is invariant under the flow of the vector field $(z, w) \to (e^{\imath t} z, e^{\pm \imath t} w)$.

In both of the cases Bennequin's Inequality may be used to show tightness of $\tilde{\xi}$, as proven in [11].

When $\Sigma \neq S^2$ the universal cover of Σ is $\mathbb{R}^2$ and it suffices to show that the structure $\tilde{\xi}$ on $\tilde{P} = \mathbb{R}^2 \times S^1$ is tight. Since all the components of Γ are essential in Σ, the components of $\tilde{\Gamma}$ must be properly embedded lines in $\mathbb{R}^2$. Before the next step we note the following lemma.

Lemma 3.2.13. *Let Σ be a compact orientable surface, $R \subset \Sigma$ a compact subsurface and $v : \Sigma \to \mathbb{R}$ a function such that v, $v \restriction_{\partial S}$ and $v \restriction_{\partial R}$ have 0 as a regular value. Let $\Gamma = \{v = 0\}$. If $v \neq 0$ on each component of $\Sigma \setminus \{R \cup \Gamma\}$. Then any 1-form λ on R satisfying*

$$v\, d\lambda + \lambda \wedge dv > 0 \tag{101}$$

can be extended to Σ so that the above inequality is satisfied.

Proof. At a point of Γ, the inequality (101) implies that λ is transversal to dv. Therefore we can easily extend λ to a tubular neighborhood U of Γ (we may think about a vector field dual X to λ via a fixed volume form on Σ, transverse to Γ). On the other hand at every point of $\Sigma \setminus \Gamma$ we have $v\, d\lambda + \lambda \wedge dv = v^2\, d(\lambda/v)$. By the assumption each component D of $\Sigma \setminus (R \cup \Gamma)$ is such that ∂D contains at least one arc in Γ. Shrinking the neighborhood U, if necessary, we can extend λ/v to a 1-form γ on D in such a manner that $d\gamma$ is positive (equivalently we may think of extending X to the entire D so that divergence of X is positive). $\square$

Coming back to the main proof, the next step is to choose a sequence of discs D_n, the boundaries of which meet $\tilde{\Gamma}$ transversally and exhaust $\mathbb{R}^2$ as $n \to \infty$. We will show that for every n: $(D_n \times S^1, \tilde{\xi})$ is tight by embedding domains $D_n \times S^1$, into $(S^2 \times S^1, \varsigma)$, where ς is an S^1-invariant contact structure with connected Γ_ς (which is tight by Theorem 3.2.11).

One may choose 1-forms defining $\tilde{\xi}$ and ς by $\beta + u\, dt = 0$ and $\lambda + v\, dt = 0$, where $t \in S^1$ and β, u (resp. λ, v) are a 1-form and a function on $\mathbb{R}^2$ (resp. S^2). Collections of curves $\tilde{\Gamma}$ and Γ_ς are defined by $u = 0$ and $v = 0$. For each $n \geq 0$, we define an embedding $\phi_n : D_n \to S^2$ which sends $\tilde{\Gamma} \cap D_n$ to $\Gamma_\varsigma \cap \phi_n(D_n)$ and respects orientations induced by u and v. Therefore, there exist functions $h_n : D_n \to (0, +\infty)$ such that $v \circ \phi_n = h_n\, u$ and we define $\beta_n = h_n\, \beta$. By Lemma 3.2.13, we may extend forms $(\phi_n)_* \beta_n$ to the entire S^2 by a 1-form λ_n satisfying $v\, d\lambda_n + \lambda_n \wedge dv > 0$. This is equivalent to the contact form $v\,dt + \lambda_n$ defining a positive invariant contact structure ς_n on $S^2 \times S^1$. By the construction

$$\phi_n \times \mathrm{id} : (D_n \times S^1, \tilde{\xi}) \to (S^2 \times S^1, \varsigma_n),$$

defines a contactomorphic embedding. Since $\Gamma_{\varsigma_n} = \{v = 0\} = \Gamma_{\varsigma}$, ς_n is isotopic to ς by Theorem 3.2.10. Consequently, $\tilde{\xi}$ is tight and ξ is universally tight. $\square$

3.3 Geometry of the characteristic surface.

Results of previous sections indicate that the topology of characteristic surfaces is an indicator of tightness/overtwistedness for contact structures both on a local and a global level. The goal of this section is to interpret the characteristic surface in the Riemannian geometric setting of adapted metrics.

Since characteristic surfaces are defined via contact fields, we begin by providing a useful characterization of contact vector fields (see also [41] and [13] p. 57).

Theorem 3.3.1. *There is a one to one and onto linear correspondence $T_\alpha : \mathcal{F}_\xi \to C^\infty(M)$ between contact vector fields on (M, ξ):*

$$\mathcal{F}_\xi = \{X \in C^\infty(M, TM) : X \text{ is contact for } \xi\}$$

and the set of smooth functions $C^\infty(M)$ on M. This correspondence is not canonical and depends on a choice of a contact form α defining ξ. Moreover;

(i) Given contact forms α and α' ($\xi = \ker(\alpha) = \ker(\alpha')$). There exist a smooth positive function $\tilde{f}$ such that $T_{\alpha'} = \tilde{f}\,T_\alpha$.

(ii) $\mathcal{F}_\xi$ is a vector subspace of $C^\infty(M, TM)$.

Proof. Given a contact form α, $\xi = \ker\alpha$, we define T_α as follows;

$$\mathcal{F}_\xi \ni X \xrightarrow{T_\alpha} f = \alpha(X) \in C^\infty(M).$$

Notice that T_α is a bijection since any contact field X has a unique decomposition

$$X = f\, X_\alpha + Y_f, \qquad \alpha(Y_f) = 0, \quad f = \alpha(X). \tag{102}$$

where X_α is the Reeb field of α and $Y_f \in C^\infty(M, \xi)$. In order to show that Y_f is uniquely determined we apply Cartan's formula for the Lie derivative and obtain from Equations (77) and (78):

$$\mathcal{L}_X \alpha = d\alpha(X) + \iota(X)\, d\alpha = df + \iota(Y_f)\, d\alpha.$$

Therefore, by (98)

$$\iota(Y_f)\, d\alpha = \tilde{h}\, \alpha - df. \tag{103}$$

Since the left-hand side of (103) is a 1-form and $d\alpha \restriction_\xi \neq 0$, the vector field Y_f is uniquely determined.

Remark 3.3.2. Equation (103) also allows us to determine the relationship between $\tilde{h}$ and f. Simply applying $\iota(X_\alpha)$ to both sides of Equation (103) we obtain

$$\tilde{h} = df(X_\alpha). \tag{104}$$

Concluding, a function $f \in C^\infty(M)$ defines the unique contact field $X = f\, X_\alpha + Y$. Especially $f \equiv 0$ results in $X \equiv 0$.

For a different contact form α' there always exists a positive function $\tilde{f}$ such that $\alpha = \tilde{f}\, \alpha'$. Let X_α and $X_{\alpha'}$ be Reeb fields of α and α'. Since both are contact fields we obtain

$$X_\alpha = \tilde{f}\, X_{\alpha'} + Y_{\tilde{f}} \qquad \alpha'(Y_{\tilde{f}}) = \alpha(Y_{\tilde{f}}) = 0. \tag{105}$$

For any $X \in \mathcal{F}_\xi$ we have

$$T_\alpha(X) = T_\alpha(f\, X_\alpha + Y_f) = f = T_\alpha(f\, \tilde{f}\, X_{\alpha'} + (f\, Y_{\tilde{f}} + Y_f)). \tag{106}$$

Since $\alpha(f\,Y_{\tilde f} + Y_f) = 0$, we obtain $T_{\alpha'}(X) = f\,\tilde f$ and

$$T_{\alpha'}(X) = \tilde f\,T_\alpha(X). \tag{107}$$

It follows from the linearity of the Lie derivative that $\mathcal{F}_\xi$ is a vector subspace of $C^\infty(M, TM)$. The linearity of T_α is a direct consequence of Equation (103). Indeed, given $X_1, X_2 \in \mathcal{F}_\xi$, and $f_i = T_\alpha(X_i)$, $i = 1, 2$, we see that in the decomposition (102) of $a\,X_1 + b\,X_2$ we have $Y_{a\,f_1 + b\,f_2} = a\,Y_{f_1} + b\,Y_{f_2}$ which follows from Equation (103). □

Corollary 3.3.3. *Any local contact field can be extended to a global contact field.*

Corollary 3.3.4. *If $\mathcal{L}_X\alpha = 0$ for a contact field X then $f = T_\alpha(X)$ is an integral of the Reeb field X_α (i.e. $\mathcal{L}_{X_\alpha} f = 0$). Consequently, X_α is tangent to the level sets of f and M is foliated by $T^2 \cong S^1 \times S^1$ except possibly the singular level sets of f.*

Proof. The statement is clear from Equation (104) and the fact that T^2 is the only orientable surface admitting a nonvanishing vector field. □

Now, we may better justify the statement of Definition 3.2.7. Clearly, given a contact field X the characteristic surface Γ_X equals to $T_\alpha(X)^{-1}(0) = f^{-1}(0)$. For a generic X the function $T_\alpha(X)$ has 0 as a regular value. Therefore, Γ_X is a codimension one submanifold.

The next step in our investigation is to place Γ_X in the setting of adapted metrics. It is an essential part of our methodology and it allows us to approach Problem 1.0.2 specified in the introduction. The following result is a generalization of Lemma 2.7 from [52];

Theorem 3.3.5. *Assume that X is a global contact vector field on the Riemannian manifold (M, g) which preserves a Beltrami form α (i.e. $\mathcal{L}_X\alpha = 0$). Let $f = T_\alpha(X) = \alpha(X)$. Denote by $\{e_1 = \frac{X}{\|X\|}, e_2, e_3\}$ an adapted orthonormal frame and $\{\eta_1, \eta_2, \eta_3\}$ the dual coframe.*

Then coefficients of $\alpha = a_k\,\eta_k = \frac{f}{v}\eta_1 + a_2\eta_2 + a_3\eta_3$ *satisfy the following first order system,*

$$\begin{cases} \nabla_1 f = 0 \\ \nabla_2 f = -\mu\, v\, a_3 \\ \nabla_3 f = \mu\, v\, a_2 \end{cases} \tag{108}$$

where $v = \|X\| = \sqrt{\langle X, X\rangle}$*.*

Moreover, f *satisfies the following subelliptic equation;*

$$\Delta_E f - \langle \nabla \ln h, \nabla f\rangle + \mu(\mathcal{E} - \mu) f = 0 \tag{109}$$

where $\mathcal{E} = (*d\,\eta_1)(e_1)$*,* $h = 1/\mu v$*, and* Δ_E *is the Laplacian on the subbundle* $E = \ker \eta_1$*. One may express Equation* (109) *in terms of the Laplacian* Δ_M *as follows;*

$$\Delta_M f + \frac{1}{v^2}\nabla^2 f(X,X) - \langle \nabla \ln\Big(\frac{1}{\mu\|X\|}\Big), \nabla f\rangle + \mu(\mathcal{E} - \mu) f = 0. \tag{110}$$

Corollary 3.3.6. *If the contact field* X *is a unit field in the metric* g *(i.e.* $v = 1$*) and* $\mu \equiv$ *const,* (109) *becomes*

$$\Delta_E\, f + \mu(\mathcal{E} - \mu)\, f = 0. \tag{111}$$

Proof of Theorem 3.3.5. The proof is a calculation in the adapted co-frame $\{\eta_i\}_i$. We denote $\nabla_i \equiv \nabla_{e_i}$. Using Cartan's formula and Equation (79) we obtain

$$\begin{aligned} 0 &= \mathcal{L}_X \alpha = \iota(X) d\,\alpha + d\,f = \mu\,\iota(X) * \alpha + \nabla_i f\,\eta_i \;\Rightarrow \\ -\nabla_i f\,\eta_i &= \mu\, v\,\iota(X_1) * \alpha \;\Rightarrow \\ -\nabla_1 f\,\eta_1 - \nabla_2 f\,\eta_2 - \nabla_3 f\,\eta_3 &= \mu\, v(-a_2\,\eta_3 + a_3\,\eta_2), \qquad \text{where } v = \|X\|. \end{aligned}$$

The above expression leads to the following equations;

$$\begin{cases} \nabla_1 f = 0, \\ \nabla_2 f = -\mu\, v\, a_3, \\ \nabla_3 f = \mu\, v\, a_2 \end{cases} \tag{112}$$

Recall the following definitions (see [49])

$$\nabla_i e_j = \omega_{ij}^k e_k, \quad \omega_i^k = \nabla_i \eta_k = -\omega_{ij}^k \eta_j, \quad \omega_{ij}^k = -\omega_{ik}^j, \tag{113}$$

$$d\,\alpha = \eta_i \wedge \nabla_i \alpha, \tag{114}$$

$$\nabla\alpha = da_k \otimes \eta_k + a_k \nabla \eta_k = da_k \otimes \eta_k - a_k \omega_j^k \otimes \eta_j, \tag{115}$$

$$\Delta_M^0 = -\nabla_i \nabla_i + \omega_{i\,i}^j \nabla_j. \tag{116}$$

We obtain the following;

$$d\,\alpha = \sum_{i<j} a_{ij} \eta_i \wedge \eta_j = \eta_i \wedge \nabla_i \alpha = \eta_i \wedge (\nabla_i a_k \eta_k - a_k \omega_{ij}^k \eta_j) = \nabla_i a_k \eta_i \wedge \eta_k - a_k \omega_{ij}^k \eta_i \wedge \eta_j$$

(the summation is assumed over the repeating indices). Collecting terms in front of $*\eta_1 = \eta_2 \wedge \eta_3$ we obtain

$$a_{23} = \nabla_2 a_3 - \nabla_3 a_2 + a_k(\omega_{32}^k - \omega_{23}^k) = \nabla_2 a_3 - \nabla_3 a_2 + a_1(\omega_{32}^1 - \omega_{23}^1) - a_2 \omega_{23}^2 + a_3 \omega_{32}^3$$

From Equation (79) and (112):

$$a_{23} = \frac{\mu}{v} f = -\nabla_2(\frac{1}{\mu\, v} \nabla_2 f) - \nabla_3(\frac{1}{\mu\, v} \nabla_3 f) + \frac{f}{v}(\omega_{32}^1 - \omega_{23}^1) - \frac{1}{\mu\, v} \nabla_3 f \omega_{23}^2 - \frac{1}{\mu\, v} \nabla_2 f \omega_{32}^3.$$

Let $h = 1/(\mu\, v)$, distributing the terms we obtain

$$\begin{aligned} \mu^2 h\, f = h(-\nabla_2 \nabla_2\, f - \nabla_3 \nabla_3\, f + \omega_{22}^3 \nabla_3\, f + \omega_{33}^2 \nabla_2\, f) \\ +\mu h\, f(\omega_{32}^1 - \omega_{23}^1) - \nabla_2 h\, \nabla_2 f - \nabla_3 h \nabla_3 f\,. \end{aligned}$$

Dividing ($h \neq 0$) the above equation by h yields

$$\begin{aligned} (\Delta_E + L + \nu) f &= 0 \qquad \text{where,} \\ \Delta_E &= -\nabla_2 \nabla_2 - \nabla_3 \nabla_3 + \omega_{22}^3 \nabla_3 + \omega_{33}^2 \nabla_2, \\ L &= -\frac{1}{h}(\nabla_2 h\, \nabla_2 + \nabla_3 h \nabla_3) = -\langle \nabla \ln h, \nabla f \rangle \\ \nu &= \mu(\omega_{32}^1 - \omega_{23}^1 - \mu) = \mu(\mathcal{E} - \mu), \\ \mathcal{E} &= \iota(e_1) * d\eta_1 \end{aligned} \tag{117}$$

Recall Equations (55) and (56), applying $\nabla_1 f = 0$, and $\nabla_1 e_1 = \omega_{11}^k e_k$, we express (117) in terms of the Laplacian Δ_M below.

$$\langle \nabla f, \nabla_1 e_1 \rangle + \langle \nabla_1 \nabla f, e_1 \rangle = \nabla_1 \langle \nabla f, e_1 \rangle = 0, \quad \Rightarrow \quad \nabla^2 f(e_1, e_1) = -\langle \nabla f, \nabla_1 e_1 \rangle.$$

Consequently,

$$\Delta_E f = \Delta_M f - \langle \nabla f, \nabla_1 e_1 \rangle = \Delta_M f + \nabla^2 f(e_1, e_1) = \Delta_M f + \frac{1}{v^2} \nabla^2 f(X, X)$$

It yields

$$\Delta_M f + \frac{1}{v^2} \nabla^2 f(X, X) - \langle \nabla \ln \left(\frac{1}{\mu \|X\|} \right), \nabla f \rangle + \mu(\mathcal{E} - \mu) f = 0. \tag{118}$$

Equation (111) follows. □

The geometric interpretation of the characteristic surface Γ_X may be now stated as the following,

Theorem 3.3.7. *The characteristic surface $\Gamma_X = f^{-1}(0)$ is the zero set (i.e the the nodal set) of the solution f to the subelliptic equation* (110) *and consists of a disjoint union of smooth 2-tori: $\Gamma_X \cong \bigsqcup_i T_i^2$, $T_i^2 \cong S^1 \times S^1$.*

Proof. Clearly, Γ_X cannot contain a singular point p, since it would imply: $\alpha(p) = 0$, which contradicts the contact condition (76). Now the claim follows from Theorem 3.3.5 and Corollary 3.3.4. □

Remark 3.3.8. Equation (109) may be modified to become an elliptic equation by adding the term $\nabla_1 \nabla_1 f$. A global structure of nodal sets of solutions to elliptic equations has been studied e.g. in [9]. It follows from the methodology of [9] that given any solution f to (109) or (111) a nodal set $N = f^{-1}(0)$ is a union $N = N_{\text{sing}} \cup N_{\text{reg}}$ of the singular part N_{sing} of codimension at least 2 and the regular part N_{reg} which is a codimension 1 submanifold. We remark that in general the nodal set of the solution to an elliptic PDE can be very irregular. For example, it can be shown that any closed subset S in $\mathbb{R}^n$ can be realized as a nodal set of the solution to an elliptic PDE defined on $\mathbb{R}^{n+1}$ so that $S \in \mathbb{R}^n \times \{0\}$.

The following is a standard result from the theory of elliptic partial differential equations (see e.g. [39], [33]).

Proposition 3.3.9. *For $\mu > 0$ the solution f to* (110) *is nontrivial if the contact field X in Theorem 3.3.5 satisfies*

$$\mathcal{E} = * \, d\,\eta_1(X_1) \leq \mu. \tag{119}$$

Moreover, if the above inequality is strict then f cannot be a locally constant function.

Proof. Assume the setup of Theorem 3.3.5. In local coordinates (x_1, x_2, x_3), $\partial_i = \frac{\partial}{\partial x_i}$, such that the vector field $e_1 = \partial_1$, where $e_1 = X/\|X\|$, and X is the contact field, the subelliptic equation (110) reads

$$-\sum_{i,j} a_{ij}\,\partial_i\partial_j f + \sum_i b_i\,\partial_i f + c\,f = 0, \tag{120}$$

where $c = \mu(\mathcal{E} - \mu)$. Since $\partial_1 f = 0$ we may add the term $-\partial_1^2 f$ to the equation and assume that $L = \sum_{i,j} a_{ij}\,\partial_i\partial_j f + \sum_i b_i\,\partial_i$ is an elliptic operator (Remark 3.3.8). By contradiction to (119) let us assume that $\mathcal{E} > \mu$, and consequently: $c > 0$. By elliptic regularity, [39, 33], f must be a C^2-function and if $f \neq \text{const}$ then there exists a regular value y_0 for f. Let $D_1 = f^{-1}(\{y \leq y_0\})$, and $D_2 = f^{-1}(\{y \geq y_0\})$. Clearly, f is a solution to the following boundary value problems;

$$L f + c\,f = 0, \quad f\restriction_{\partial D_k} = y_0, \quad f \in C^2(D_k), \qquad k = 1, 2. \tag{121}$$

Recall the following corollary of the weak maximum principle ([33], p. 329):

If f is a solution to one of the boundary value problems in (121)*, then*

$$\max_{D_k} |f| = \max_{\partial D_k} |f|.$$

Consequently, $f\restriction_{D_j} = y_0 = \text{const}$ on D_j for either $j = 1$ or $j = 2$. Now, Equation (121) implies that $c\,f = 0$ on D_j. Consequently, $f\restriction_{D_j} = 0$, and $y_0 = 0$. Applying the maximum principle again we conclude that $f = 0$, which proves the first claim by contradiction. For the second claim ($c < 0$), if f were locally constant it would imply that it vanishes at some point to infinite order. But contradicts Aronszajn's Unique Continuation Principle (see [6], p. 235). $\square$

As a consequence of Proposition 3.3.9 and Corollary 3.3.4 we obtain.

Corollary 3.3.10. *If $\mathcal{E} < \mu$ then M is fillable almost everywhere by tori.*

Proof. Since f is invariant under a nonsingular vector field X, regular level sets of f must be 2-dimensional tori. Because f cannot be a locally constant function, regular level sets are dense in M. $\square$

Remark 3.3.11. Corollary 3.3.10 raises natural questions about the topology of manifolds "fillable almost everywhere by tori". Since f is X-invariant the set of singular points must be a set of disjoint circles in M. If we make additional assumptions on f, e.g. that the singular set has a nondegenerate Hessian in the transverse direction, one may define a nonsingular Morse-Smale vector field by $Y = X + \varepsilon\,\nabla f$ (for small ε). Results of Morgan, in [56], prove that manifolds admitting such a flow decompose as unions of Seifert fibered manifolds (see Section 3.4).

Summarizing the results obtained so far; we have interpreted the characteristic surface Γ_X of a contact vector field X, preserving the contact form α, as a nodal set of a solution to the subelliptic Equation (109). In Chapter 2 we saw, in the case of an eigen-equation, how the nodal sets are "controlled" by the geometry of an underlying manifold in dimension 2. On the other hand topological results of Giroux, Honda and others, presented in Section 3.2, describe how the topology of Γ_X influences the tightness/overtwistedness of $\xi = \ker\alpha$. We conclude that the topology of nodal sets influences the tightness/overtwistedness of a contact structures in certain specific situations. As we saw in Chapter 2, controlling the topology of nodal sets even in dimension 2 may not be an easy task. Facing serious difficulties in dimension 3 we seek situations where we may essentially reduce the problem to the setting of a surface, so that theorems of Chapter 2 would apply.

3.3.1 Geometry of the dividing set.

At the beginning we revisit some of techniques from Chapter 2 and draw conclusions about nodal sets of solutions to Equation (109). We consider an embedded convex surface Σ in (M, g), and assume that the contact vector field X is orthogonal to Σ. In the following, we investigate Equation (109) restricted to such a surface.

Proposition 3.3.12. *Let Σ be a surface embedded in M. If the contact field X is orthogonal to Σ and $\mu = const$, Equation* (117) *simplifies as follows;*

$$\Delta_\Sigma f + \frac{\langle \nabla \|X\|, \nabla f\rangle}{\|X\|} - \mu^2 f = 0. \tag{122}$$

Moreover, if $\|X\| = const$ we obtain the eigen-equation:

$$\Delta_\Sigma f = \mu^2 f. \tag{123}$$

Proof. Assume the setup of Theorem 3.3.5. First we show that $\Delta_E = \Delta_\Sigma$ in the frame $\{e_1 = X/\|X\|, e_2, e_3\}$, where $\{e_2, e_3\}$ span $T\Sigma$. Recall from (117) (here $E = T\Sigma$)

$$\Delta_E = -\nabla_2\nabla_2 - \nabla_3\nabla_3 + \omega^3_{22}\nabla_3 + \omega^2_{33}\nabla_2.$$

Since $\{e_2, e_3\}$ are tangent to the surface the bracket $[e_2, e_3]$ satisfies $[e_2, e_3] \in T\Sigma$. The following is the general formula for Christoffel symbols in a frame ([49])

$$\omega^k_{ij} = \frac{1}{2}\{\langle [e_i, e_j], e_k\rangle - \langle [e_j, e_k], e_i\rangle + \langle [e_k, e_i], e_j\rangle\}. \tag{124}$$

Consequently, the formula $\Delta_\Sigma = -\nabla_i\nabla_i + \omega^j_{ii}\nabla_j$ implies $\Delta_E = \Delta_\Sigma$ on Σ. Moreover

$$\langle [e_2, e_3], e_1\rangle = \eta_1([e_2, e_3]) = 0 \quad \Rightarrow \quad d\eta_1(e_2, e_3) = 0 \quad \Rightarrow \quad \mathcal{E} = (*d\eta_1)(e_1) = 0.$$

Secondly, for $h = 1/(\mu\|X\|)$, we have

$$-\langle \nabla \ln h, \nabla f\rangle = -\mu\|X\|\nabla\langle\Big(\frac{1}{\mu\|X\|}\Big), \nabla f\rangle = \frac{\|X\|}{\|X\|^2}\langle\nabla\|X\|, \nabla f\rangle = \frac{\langle\nabla\|X\|, \nabla f\rangle}{\|X\|}.$$

$\square$

In Theorem 2.6.10 of Chapter 2, we have obtained a condition for homotopically essential nodal curves $\Gamma_\Sigma = \Gamma_X \cap \Sigma$ in the case of Equation (123) ($\|X\| = const$). We revisit our technique in the following;

Proposition 3.3.13. *Let X be a contact vector field $X \perp \Sigma$, and α a contact form. Let Ω be a nodal domain of the solution f to Equation* (122). *We have the following*

$$2\pi\chi(\Omega) = \int_\Omega K_s + \int_\Omega \Delta_\Sigma \ln\|\alpha\|\,. \tag{125}$$

Proof. We need to calculate the geodesic curvature κ of $\partial\Omega$. Recall the formula from Proposition 2.6.5,

$$\kappa = H_\nu = \frac{\Delta_\Sigma f}{\|\nabla f\|} + \frac{1}{2}\langle\nabla \ln\|\nabla f\|^2, \nu\rangle.$$

Equation (122) yields

$$\kappa = -\frac{\langle\nabla\,\|X\|, \nabla f\rangle}{\|X\|\|\nabla f\|} + \frac{\mu^2\, f}{\|\nabla f\|} + \frac{1}{\|\nabla f\|}\langle\nabla\|\nabla f\|, \nu\rangle. \tag{126}$$

By Theorem 3.3.5 and $\alpha = a_i\eta_i$, $\nabla_1 f = 0$, $v = \|X\|$ we obtain

$$\begin{aligned}
\|\alpha\|^2 &= \sum_i a_i^2 = \Big(\frac{f}{v}\Big)^2 + \Big(\frac{\nabla_2 f}{\mu v}\Big)^2 + \Big(\frac{\nabla_3 f}{\mu v}\Big)^2 \quad \Rightarrow \\
(\mu\, v\|\alpha\|)^2 &= (\mu f)^2 + \|\nabla f\|^2 \quad \Rightarrow \quad \|X\|^2 = \frac{1}{(\mu\|\alpha\|)^2}(\mu^2 f^2 + \|\nabla f\|^2).
\end{aligned}$$

Since $f\restriction_{\partial\Omega} = 0$, we derive

$$\begin{gathered}
\|X\|\restriction_{\partial\Omega} = \frac{\|\nabla\, f\|}{\mu\,\|\alpha\|} \quad \Rightarrow \\
\nabla\|X\| = -\frac{1}{\mu\,\|\alpha\|^2}(\nabla\mu\|\alpha\|)\|\nabla f\| + \frac{1}{\mu\|\alpha\|}\nabla\|\nabla f\| \quad \Rightarrow \\
\frac{\langle\nabla\,\|X\|, \nabla f\rangle}{\|X\|\|\nabla f\|} = \frac{\langle\nabla\,\|X\|, \nu\rangle}{\|X\|} = -\frac{1}{\mu\|\alpha\|}\langle\nabla(\mu\|\alpha\|), \nu\rangle + \frac{\langle\nabla\|\nabla f\|, \nu\rangle}{\|\nabla f\|}.
\end{gathered}$$

Consequently,

$$\kappa = \frac{1}{\mu\|\alpha\|}\langle\nabla(\mu\|\alpha\|),\nu\rangle - \frac{\langle\nabla\|\nabla f\|,\nu\rangle}{\|\nabla f\|} + \frac{\mu^2 f}{\|\nabla f\|} + \frac{\langle\nabla\|\nabla f\|,\nu\rangle}{\|\nabla f\|} = \langle\nabla\ln(\|\alpha\|),\nu\rangle,$$

and the claim follows from the Gauss-Bonnet Theorem 2.6.9. □

If, for a given convex surface Σ, one finds an orthogonal contact vector field X (such that $\mathcal{L}_X\alpha = 0$), Proposition 3.3.13 provides a condition for a tight tubular neighborhood of Σ. For instance, under these assumptions, if $K_s \leq 0$, the sufficient condition for tight tubular neighborhood is:

$$\max_{p\in\Sigma}\Big(\Delta_\Sigma \ln\|\alpha\|(p)\Big) < \frac{2\pi}{\mathrm{Vol}(\Sigma)}, \tag{127}$$

(compare with Theorem 2.6.10 of Chapter 2). However, contrary to a method general convex surfaces, convex surfaces which admit an orthogonal contact field are special (as we indicate in Remark 3.3.15). Observe that, by definition, along the dividing set Γ_X we have $X \in \xi$. Orthogonality $X \perp \Sigma$ implies that X_α is tangent to the surface Σ since $X_\alpha \perp \xi$. From Equations (108) we conclude that the Reeb field X_α is tangent to the dividing curves Γ_Σ. Consequently, curves in Γ_Σ consist of periodic orbits of X_α, and we have proved the following;

Proposition 3.3.14. *For an embedded surface Σ in the contact manifold (M,ξ), $\xi = \ker\alpha$, if there exist a contact vector field X such that $\mathcal{L}_X\alpha = 0$, and $X \perp \Sigma$, then the dividing set Γ_X is a set of periodic orbits of the Reeb field X_α.*

Remark 3.3.15. The following example, [36], demonstrates that the dynamics of the Reeb field X_α may change drastically depending on a choice of a contact form α defining ξ. Consider the following family of contact forms on $S^3 \subset \mathbb{R}^4$:

$$\alpha_t = (x_1\,dy_1 - y_1\,dx_1) + (1+t)(x_2\,dy_2 - y_2\,dx_2), \qquad t \geq 0$$
$$X_{\alpha_t} = (x_1\,\partial_{y_1} - y_1\,\partial_{x_1}) + \frac{1}{1+t}(x_2\,\partial_{y_2} - y_2\,\partial_{x_2}).$$

If $t = 0$, X_{α_0} defines a Hopf fibration on S^3, in particular all the orbits of X_{α_0} are closed. For $t \in \mathbb{R}\backslash\mathbb{Q}^+$, X_{α_t} defines an irrational flow on tori of the Hopf fibration and has just two periodic orbits (at $x_1 = y_1 = 0$, and $x_2 = y_2 = 0$).

It demonstrates that in the irrational case any embedded surface away from the periodic orbits cannot admit the contact vector required in Proposition 3.3.14. (It also demonstrates that contact forms are not stable, i.e. in the above example there exist no family of diffeomorphisms ψ_t such that $\psi_{t*}\alpha_t = \alpha_0$, as otherwise the flows of X_{α_t} would have to be conjugate).

3.4 Tight Beltrami forms with symmetry.

In this section we prove the main theorem of this chaper. The theorem offers insight into Problem 3.1.14. Specifically, it describes conditions for an adapted metric which imply tightness of a certain class of the invariant curl eigenforms. The theorem is rather restrictive since we make strong assumptions of symmetry, both in the underlying Riemannian metric and in the curl eigenforms. These assumptions force M to a be Seifert fibered manifold which is covered by a principal S^1-bundle. Consequently we work in the topological setting of Giroux's Theorem 3.2.12, where tightness is completely characterized by the topology of the characteristic surface, and the techniques we developed for nodal sets can be applied. In the first part of this section we provide auxiliary lemmas which are essential for the main proof and also state necessary facts about Seifert fibered manifolds.

3.4.1 About Riemannian submersions.

In Proposition 3.3.12 we encountered situations where the operator Δ_E, from Equation (117), becomes the Laplacian on a surface. We begin by proving that a similar statement is true in the setting of a Riemannian submersion. One may consult [40] for the general treatment of this question for the Hodge Laplacian on forms.

Definition 3.4.1. *Let (M, g_M), and (N, g_N) be Riemannian manifolds. A submersion $\pi : M \to N$ is Riemannian iff $\pi^* : T_pM \supset \ker(\pi^*)_p^{\perp} \to T_{\pi(p)}N$ determines a linear isometry. In other words, for $V, W \in TM$ which are perpendicular to the kernel of $D\pi = \pi^*$, we have $g_M(V, W) = g_N(\pi^* V, \pi^* V)$.*

The Riemannian submersion π determines an orthogonal decomposition $TM = V \oplus H$ of the tangent bundle into the vertical subbundle $V = \mathrm{Ker}(\pi^*)$, and the horizontal subbundle $H = V^{\perp}$ (where $\pi^* : TM \to TN$ is a tangent map). The main feature of π is the possibility of lifting orthogonal frames on N to horizontal vectors on M which stay mutually orthogonal. Consequently, we may complete a lifted frame to an orthogonal frame on M. We introduce the following notation; vectors on the base N will be denoted with capital letters E, F and lifted vectors on M by small letters e, f. The HORIZONAL LIFT operation $HL : T_{\pi(p)}N \to T_pM$ has the following natural properties, [40];

(a) Lifted $f_p = HL(F_{\pi(p)})$ is horizontal i.e. $f_p \in H_p$.

(b) For any point $p \in M$ and a vector $F_{\pi(p)} \in T_{\pi(p)}N$, $\pi^* H(F_{\pi(p)}) = F_{\pi(p)}$.

We have the following standard result;

Lemma 3.4.2 ([40])**.** *Let $\pi : M \to N$ be a Riemannian submersion.*

(1) Let $f_i = HL(F_i)$, then $\pi^*([f_1, f_2]) = [F_1, F_2]$.

(2) Let $\nabla_i^M e_j = \omega_{ij}^k e_k$, and $\nabla_a^N E_b = \Omega_{ab}^c E_c$. Christoffel symbols satisfy

$$\omega_{ab}^c = \Omega_{ab}^c \circ \pi. \tag{128}$$

Proof. Let $\psi_i(t)$ and $\Psi_i(t)$ define flows of vector fields f_i and F_i respectively. By property (b) of the horizontal lift we have

$$\pi \circ \psi_i(t) = \Psi_i(t) \circ \pi. \tag{129}$$

Using the flows we compute,

$$a(t) = \psi_1(-\sqrt{t})\psi_2(-\sqrt{t})\psi_1(\sqrt{t})\psi_2(\sqrt{t}),$$

$$A(t) = \pi \circ a(t) = \Psi_1(-\sqrt{t})\Psi_2(-\sqrt{t})\Psi_1(\sqrt{t})\Psi_2(\sqrt{t}),$$

$$a'(0) = [f_1, f_2](p_0), \quad A'(0) = [F_1, F_2](\pi(p_0)), \qquad p_0 \in M.$$

Hence (129) proves the claim *(1)* of the lemma since $A'(0) = \pi^* a'(0)$. Now, the claim *(2)* follows directly from Equation (124);

$$\begin{aligned}
\omega_{ab}^c &= \frac{1}{2}\{\langle[e_a, e_b], e_c\rangle_M - \langle[e_b, e_c], e_a\rangle_M + \langle[e_c, e_a], e_b\rangle_M\} \\
&= \frac{1}{2}\{\langle\pi^*[e_a, e_b], \pi^* e_c\rangle_N - \langle\pi^*[e_b, e_c], \pi^* e_a\rangle_N + \langle\pi^*[e_c, e_a], \pi^* e_b\rangle_N\} \\
&= \frac{1}{2}\{\langle[E_a, E_b], E_c\rangle_N - \langle[E_b, E_c], E_a\rangle_N + \langle[E_c, E_a], E_b\rangle_N\} \\
&= \Omega_{ab}^c
\end{aligned}$$

□

We focus on the special case of the Riemannian submersion $\pi : M \to N$, namely we assume that $M = P$ is a total space of a principal S^1-bundle over an orientable surface $\Sigma = N$. We also consider Seifert fibered manifolds which may be defined as quotients $G\backslash P$ where G is a discrete group acting properly and discontinuously on P by isometries.

Lemma 3.4.3. *Suppose $\pi : P \to \Sigma$ is a projection of an S^1-bundle P, equipped with a Riemannian metric g_P which admits a vertical unit Killing vector field X.*

Then π defines the Riemannian submersion with an appropriate choice of the metric on Σ. Moreover, the following formulas hold, in a local orthogonal frame of vector fields $\{e_1, e_2, e_3\}$ where $e_1 = X$ and $\{e_2, e_3\}$ is the horizontal lift of a frame $\{E_2, E_3\}$ from Σ;

$$[e_1, e_k] = 0, \qquad k = 1, 2, 3, \tag{130}$$

$$\pi \circ \Delta_E = \Delta_\Sigma \circ \pi, \qquad E = \mathrm{SPAN}\{e_2, e_3\}. \tag{131}$$

Proof. Since X is a unit Killing vector field, its flow: ϕ^t is a flow of isometries on P. Therefore, in a local trivialization: $(t, \mathbf{x}) \in V \cong S^1 \times U$, $\mathbf{x} \in U \subset \Sigma$ of P, where $X = \partial_t$ and the flow ϕ^t acts by translations in the t-direction. One may choose a t-invariant frame $\{e_1, e_2, e_3\}$, $e_1 = \partial_t = X$ on V which satisfies

$$[e_1, e_k] = [\partial_t, e_k] = 0. \tag{132}$$

Clearly, any local vector-field Y on U lifts to $\tilde{Y}$ on $V \cong S^1 \times U$. One may define a metric g_Σ on $U \subset \Sigma$ by $g_\Sigma(Y, Y') = g_P(\tilde{Y}, \tilde{Y}')$ which turns π into the Riemannian submersion on V. Consequently, π is the global Riemannian submersion on P for the metric g_Σ extended globally. As a direct corollary of Lemma 3.4.2 we obtain (131). Indeed, Christoffel symbols project under Riemannian submersions (Lemma 3.4.2). For $u \in C^2(\Sigma)$, and a local frame $\{E_2, E_3\}$ on Σ, we obtain

$$\begin{aligned}
(\Delta_\Sigma u) \circ \pi &= (-\nabla_{E_2}\nabla_{E_2} u - \nabla_{E_3}\nabla_{E_3} u + \Omega^3_{22}\nabla_{E_3} u + \Omega^2_{33}\nabla_{E_2} u) \circ \pi \\
&= -\nabla_{e_2}\nabla_{e_2}(u \circ \pi) - \nabla_{e_3}\nabla_{e_3} + \omega^3_{22}\nabla_{e_3}(u \circ \pi) + \omega^2_{33}\nabla_{e_2}(u \circ \pi) \\
&= \Delta_E(u \circ \pi),
\end{aligned}$$

where $\{e_2, e_3\}$ is the lift of $\{E_2, E_3\}$. □

3.4.2 About Seifert fibered manifolds.

Next, we discuss the necessary facts concerning Seifert fibered manifolds. A Seifert fibering, [64], of a 3-manifold M is a decomposition of M into disjoint circles, the fibers, such that each fiber has a neighborhood diffeomorphic, preserving fibers, to a neighborhood of a fiber in some model Seifert fibering of $S^1 \times D^2$. A model Seifert fibering of $S^1 \times D^2$ is a decomposition of $S^1 \times D^2$ into disjoint circles, constructed as follows: Starting with $[0, 1] \times D^2$ decomposed into the segments $[0, 1] \times \{x\}$, identify the discs $\{0\} \times D^2$ and $\{1\} \times D^2$ via a $2\pi m/n$ rotation, for $m/n \in \mathbb{Q}$ with m and n relatively prime. The segment $[0 \times 1] \times \{0\}$ then becomes a fiber $S^1 \times \{0\}$, while every other fiber in $S^1 \times D^2$ is made from n segments $[0, 1] \times \{x\}$, we denote such a decomposition by $T(m,n)$. A SEIFERT FIBERED manifold is one which possesses a Seifert fibering. Each fiber C in a Seifert fibering of M has well-defined multiplicity, i.e. the number of times a small disc transverse to C meets each nearby fiber (which is equal to n in the model fibering $T(m,n)$). If a fiber has multiplicity > 1 it is called MULTIPLE, if the multiplicity $= 1$ the fibre is called REGULAR. Multiple fibers are isolated and lie in the interior of M.

The quotient space Σ of M obtained by identifying each fiber to a point is an example of an orientable 2-ORBIFOLD, i.e. a surface locally diffeomorphic to $\mathbb{R}^2/\mathbb{Z}_n$, $n = 1, 2, 3 \ldots$, where $\mathbb{Z}_n$ is the cyclic group of order n acting by rotations. The local structure is then a *cone point* with cone angle $2\pi/n$. Clearly, the singular fibers of M project to the cone points of Σ under the quotient projection $\pi : M \mapsto \Sigma$.

The term orbifold refers to the differential structure (i.e. atlas) on Σ, and every orientable orbifold is homeomorphic to an orientable surface. In the following we consider only so called *good* orbifolds i.e. orbifolds which are quotients S/G of a smooth surface S and a finite group G acting on it properly and discontinuously. It can be shown, [64] that every orbifold homeomorphic to a surface of nonzero genus is *good*.

Theorem 3.4.4 (Gauss-Bonnet (for orbifolds), see [57])**.** *If Σ is a closed orbifold of genus g and s cone points with cone angles: $2\,\pi/p_1$, $2\,\pi/p_2, \ldots,$ $2\,\pi/p_s$, equipped with a smooth Riemannian metric everywhere except at the cone points. Then,*

$$\int_\Sigma K_s = 2\pi\chi(\Sigma), \quad \textit{and} \quad \chi(\Sigma) = 2 - 2\,g - \sum_{i=1}^{s}\Big(1 - \frac{1}{p_i}\Big), \tag{133}$$

where K_s is a scalar curvature of Σ.

Proof. Subdivide Σ into the geodesic triangles Δ_i. Since $\int_{\Delta_i} K_s = \alpha_i + \beta_i + \gamma_i - \pi$, the claim follows if we sum up over all Δ_i. $\square$

The number $\chi(\Sigma)$ in (133) is called an Euler number of the orbifold Σ, and if Σ has no cone points it is simply the Euler characteristic of the surface.

Theorem 3.4.5 ([64] p. 425). *Every good closed 2-orbifold Σ without boundary is a quotient by a discrete group G of isometries of S^2, $\mathbb{R}^2$ or $\mathbb{H}^2$, depending whether: $\chi(\Sigma) > 0$, $\chi(\Sigma) = 0$, $\chi(\Sigma) < 0$.*

Theorem 3.4.6 ([64] p. 425). *Every good closed 2-orbifold is a quotient of a closed surface. Any orbifold Σ with $\chi(\Sigma) \leq 0$ is good.*

The following lemma, see [15], p. 148 Lemma 2.4.22, is of importance for our further considerations.

Lemma 3.4.7 ([15]). *Every closed, compact Seifert fibered 3-manifold M with the base* good *orbifold Σ is covered by a total space of a circle bundle P. We have the following diagram;*

$$\begin{array}{ccc} P & \xrightarrow{p} & M \\ \Big\downarrow \Pi & & \Big\downarrow \pi \\ \tilde{\Sigma} & \xrightarrow{r} & \Sigma \end{array} \tag{134}$$

where p is the covering map, r is the orbifold covering and π, Π are fibrations.

Proof. Let S be a model space for Σ, i.e. $S = S^2, \mathbb{R}^2$, or $\mathbb{H}^2$. Let G be a discrete subgroup of S such that $\Sigma = S/G$, we define $\tilde{\Sigma} = S/G'$ and $r : \tilde{\Sigma} \mapsto \Sigma$ to be a quotient map (notice that r is in general not a cover in the usual sense, see [44]). Theorem 3.4.6 is equivalent to asserting that any finitely generated discrete subgroup G of isometries of S with compact quotient space has a torsion free subgroup G' of finite index (i.e. the fundamental group of the closed surface $\tilde{\Sigma}$).

Let $h \in \pi_1(M)$ represent a regular fiber of M, a subgroup $\langle h \rangle$ of $\pi_1(M)$ is infinite cyclic and $\pi_1(M)/\langle h \rangle = \pi_*(\pi_1(M)) = G$. Denote by K the pre-image in $\pi_1(M)$ of a torsion free subgroup G' under the induced group homomorphism π_*. Denote by P the covering space of M corresponding to K if $\tilde{h}$ in $\pi_1(P) = K \subset \pi_1(M)$ is represented by a regular fiber then $\langle \tilde{h} \rangle = \langle h \rangle$. Since $K/\langle h \rangle = \pi_1(P)/\langle h \rangle = G'$ is torsion free P has no singular fibers and consequently must be an S^1-bundle over $\tilde{\Sigma}$. The diagram 134 follows accordingly. $\square$

3.4.3 Proof of the Main Theorem.

At this point we turn to the proof of the main theorem of this chapter.

Theorem 3.4.8. *Let (M, g_M) be a Riemannian manifold equipped with a contact structure ξ defined by a curl eigenform α (i.e. $*\,d\alpha = \mu\,\alpha$, $\mu \equiv$ const). Assume that α admits a contact vector field X, (i.e. $\mathcal{L}_X\alpha = 0$) with circular orbits, which is also a unit Killing vector field for g_M. Let $l_{\min}$ be a lower bound for the lengths of the orbits of X and denote by $\eta = g_M(X, \cdot\,)$ the 1-form dual to X.*

The form α defines a universally tight contact structure on M if the following conditions are satisfied:

(i) $\mathcal{E} = *\,d\eta(X) < \mu$; (ii) the sectional curvature κ_E of planes E orthogonal to the fibres satisfies: $\kappa_E \leq -\frac{3}{4}\,\mathcal{E}^2$;

(iii) $\mathcal{E}$ is constant and $\mu(\mu - \mathcal{E})\,\frac{k}{l_{\min}}(\mathrm{Vol}(M)) < 4\pi$; for a natural number k, which depends only on M.

If $\mathcal{E}$ is not a constant function we replace condition (iii) by the following;

(iv) $\frac{k}{l_{\min}} \int_M |\,\Delta_E \ln \|\alpha\|\,| < 2\pi$ where, Δ_E is defined in (117).

Proof. Since X has circular orbits, M is a Seifert fibered manifold and X induces an S^1-action by isometries on M. Consequently, we obtain an orbifold bundle: $\pi : M \mapsto M/S^1 \simeq \Sigma$. Let $C = \{x_1, \ldots, x_k\}$ be the cone points of Σ and $S = \pi^{-1}(C)$ the set of singular fibres in M. Since $M \setminus S \simeq S^1 \times \Sigma \setminus C$, [45], by Lemma (3.4.3) we may define a metric g_Σ on $\Sigma \setminus C$ so that $\pi : M \setminus S \mapsto \Sigma \setminus C$ is a Riemannian submersion. The metric g_Σ is smooth and extends continuously to Σ. In the first step of the proof we show that the scalar curvature of $(\Sigma \setminus C, g_\Sigma)$ is nonpositive which, by Theorem 3.4.4, implies that $\chi(\Sigma) \leq 0$ and consequently Σ will be a *good* orbifold (see Theorem 3.4.6).

Let us fix a local frame of vector fields $\{e_1 = X, e_2, e_3\}$, and the dual coframe $\{\eta = \eta_1, \eta_2, \eta_3\}$. Since X is the Killing vector field (i.e. $\mathcal{L}_X g_M = 0$), for any pair of vector fields V, W we have

$$\langle \nabla_V X, W\rangle = -\langle V, \nabla_W X\rangle.$$

Consequently, we obtain the following identities for Christoffel symbols in the frame $\{e_i\}$:

$$\omega_{1\,1}^{2} = \omega_{1\,1}^{3} = \omega_{2\,1}^{2} = \omega_{3\,1}^{3} = 0, \qquad \omega_{i\,j}^{k} = -\omega_{i\,k}^{j}, \tag{135}$$

$$-\omega_{3\,1}^{2} = \omega_{2\,1}^{3} = \frac{\mathcal{E}}{2}. \tag{136}$$

where $\nabla_i e_j = \omega_{i\,j}^{k} e_k$, and $\mathcal{E} = \iota(X) * d\eta$ is the parameter we have encountered in Equation (109). Cartan's structure equations imply that the 1-form $\eta = g_M(X, \cdot)$ satisfies

$$* \, d\,\eta = \mathcal{E}\,\eta. \tag{137}$$

Since $0 = d\,\mathcal{E} * \eta$, we have $d\,\mathcal{E}(X) = X\,\mathcal{E} = 0$, and consequently $\mathcal{E}$ is S^1-invariant. Using (135), we may compute the sectional curvature κ_E as follows (see also [58]);

$$\begin{aligned}
\nabla_2\nabla_3 e_3 &= \nabla_2(\omega_{3\,3}^{k}\, e_k) = (\nabla_2\, \omega_{3\,3}^{2}) e_2 + \omega_{3\,3}^{2}\nabla_2 e_2 \\
&= (\nabla_2\, \omega_{3\,3}^{2}) e_2 + \omega_{3\,3}^{2}\omega_{2\,2}^{3} e_3, \\
\nabla_3\nabla_2 e_3 &= \nabla_3(\omega_{2\,3}^{k}\, e_k) = (-\nabla_3\, \lambda)\, e_1 - \lambda\nabla_3 e_1 + (\nabla_3\, \omega_{2\,3}^{2})\, e_2 + \omega_{2\,3}^{2}\nabla_3 e_2 \\
&= -\nabla_3\, \lambda\, e_1 + \lambda^2 e_2 + (\nabla_3\, \omega_{2\,3}^{2}) e_2 + \lambda\omega_{2\,2}^{3} e_1 + \omega_{2\,2}^{3}\omega_{3\,3}^{2} e_3,
\end{aligned}$$

$$\begin{aligned}
[e_2, e_3] &= \nabla_2 e_3 - \nabla_3 e_2 = (\omega_{2\,3}^{k} - \omega_{3\,2}^{k})\, e_k = -2\,\lambda e_1 + \omega_{2\,3}^{2} e_2 - \omega_{3\,2}^{3} e_3, \\
\nabla_{[e_2,\, e_3]} e_3 &= -2\,\lambda\nabla_1 e_3 + \omega_{2\,3}^{2}\nabla_2 e_3 - \omega_{3\,2}^{3}\nabla_3 e_3 \\
&= -2\,\lambda^2\varphi e_2 + \omega_{2\,3}^{2}\lambda e_1 + \left((\omega_{2\,2}^{3})^2 + (\omega_{3\,3}^{2})^2\right) e_2
\end{aligned}$$

$$\begin{aligned}
\kappa_E &= \langle R(e_2, e_3) e_3, e_2\rangle \\
&= \nabla_2\, \omega_{3\,3}^{2} + \nabla_3\, \omega_{2\,3}^{2} - \lambda^2 + 2\lambda\varphi - \left((\omega_{2\,2}^{3})^2 + (\omega_{3\,2}^{3})^2\right) \\
&= \sigma - \lambda^2 + 2\,\lambda\varphi
\end{aligned} \tag{138}$$

$$\text{where:} \qquad \sigma = \nabla_2\, \omega_{3\,3}^{2} + \nabla_3\, \omega_{2\,3}^{2} - \left((\omega_{2\,2}^{3})^2 + (\omega_{3\,2}^{3})^2\right), \quad \varphi = \omega_{1\,3}^{2}, \quad \lambda = \frac{\mathcal{E}}{2}.$$

Notice that

$$\nabla_1 e_2 = \omega^3_{1\,2}\, e_3 = -\varphi\, e_3, \qquad \nabla_1 e_3 = \omega^2_{1\,3}\, e_2 = \varphi\, e_2\,.$$

Therefore φ measures a rotation of the frame in E, when parallel transported along orbits of X. Moreover by Lemma 3.4.2 the Christoffel symbols project under the Riemannian submersion: $\pi : (M \setminus S, g_M) \mapsto (\Sigma \setminus C, g_\Sigma)$ and the scalar curvature K_s of Σ satisfies

$$K_s \circ \pi(x) = \sigma(x), \qquad x \in M, \tag{139}$$

where σ is defined in (138). From Equation (130) we obtain

$$0 = [e_2, e_1] = \nabla_1 e_2 - \nabla_2 e_1 \quad \Rightarrow \quad \varphi = -\lambda = -\frac{\mathcal{E}}{2}.$$

Since $\kappa_E = \sigma - \lambda^2 + 2\lambda\,\varphi = \sigma - \frac{3}{4}\,\mathcal{E}^2$ by the assumption (ii) we conclude:

$$K_s \circ \pi = \kappa_E + \frac{3}{4}\,\mathcal{E} \leq 0. \tag{140}$$

Consequently, the Gauss-Bonnet Theorem 3.4.4 for orbifolds implies that $\chi(\Sigma) \leq 0$ and, by Theorem 3.4.5, Σ must be orbifold covered $r : \tilde{\Sigma} \mapsto \Sigma$ by a closed surface $\tilde{\Sigma}$ of nonzero genus (i.e. Σ is a *good* orbifold). Subsequently by Lemma 3.4.7 we may choose a principal bundle $\Pi : P \mapsto \tilde{\Sigma}$ such that the total space P is a covering space for M, the diagram (134) commutes and $p : P \mapsto M$ is a fiber preserving covering map. We define a metric g_P on P by pulling back the metric g_M from M, by p. This makes $p : (P, g_P) \mapsto (M, g_M)$ into a local isometry, and $\Pi : P \mapsto \tilde{\Sigma}$ into a Riemannian submersion, by Lemma 3.4.3. Let $\tilde{X}$ be the unique lift of X. Since p respects the fibers, which are orbits of the flow ϕ_X of X, we have

$$\phi_X(t, \cdot) \circ p = p \circ \phi_{\tilde{X}}(t, \cdot),$$

Consequently, the lift $\tilde{X}$ of X must also be a regular Killing vector field on P.

At this point we turn to the proof of the statement (iii) (the case of constant $\mathcal{E}$). Notice that it suffices to prove the claim for an S^1-invariant contact structure $\tilde{\xi}$ lifted to P (i.e. $\tilde{\xi} = \ker\tilde{\alpha}$, $\tilde{\alpha} = p_*\,\alpha$). Indeed, if $\tilde{\xi}$ is universally tight then so must be ξ, otherwise we could lift an overtwisted disc by the fiber-preserving cover $p : P \mapsto M$ to ξ. Theorem 3.2.12 states the necessary and sufficient condition for universal tightness of $(P, \tilde{\xi})$ in terms of the characteristic surface $\Gamma_{\tilde{X}}$, since $\chi(\tilde{\Sigma}) \leq 0$ the condition reads: $(*)$ The projection $\Gamma_{\tilde{\Sigma}} = \tilde{\pi}(\Gamma_{\tilde{X}})$ has to be either empty or a set of homotopically essential curves on $\tilde{\Sigma}$.

Applying techniques developed in Chapter 2: $\tilde{X}$ satisfies $\mathcal{L}_{\tilde{X}}\tilde{\alpha} = 0$ and $*\,d\,\tilde{\alpha} = \mu\,\tilde{\alpha}$, Theorem 3.3.5 and Theorem 3.3.7 imply that $\Gamma_{\tilde{X}} = f^{-1}(0)$ where $f = \alpha(X)\circ p$ is an S^1- invariant function which satisfies the following subelliptic equation on P:

$$\Delta_{\tilde{E}}\, f + \mu(\tilde{\mathcal{E}} - \mu)\, f = 0, \qquad (\|\tilde{X}\| = 1) \tag{141}$$

where $\tilde{\mathcal{E}} = \mathcal{E}\circ p$. By Proposition 3.3.9 f cannot be a locally constant function. As a consequence of Lemma 3.4.3, Equation (141) projects onto the surface $\tilde{\Sigma}$ as follows:

$$\Delta_{\tilde{\Sigma}}\, f + \mu(\tilde{\mathcal{E}} - \mu)\, f = 0, \tag{142}$$

(we may treat functions f, $\tilde{\mathcal{E}}$ as functions on $\tilde{\Sigma}$). Since f must change the sign on $\tilde{\Sigma}$ the dividing set $\Gamma_{\tilde{\Sigma}} = \tilde{\pi}(\Gamma_{\tilde{X}}) \neq \varnothing$ must be nonempty. We must show in $(*)$ that $\Gamma_{\tilde{X}}$ is a set of homotopically essential curves on $\tilde{\Sigma}$. (Notice that Theorem 3.3.7 implies that these curves cannot have self-intersections.) Assume by contradiction that one of the domains $\tilde{\Omega} \in \tilde{\Sigma}\setminus\Gamma_{\tilde{\Sigma}}$ is a disc $\tilde{\Omega} \cong D^2$. By (140), $\tilde{\Sigma}$ has a nonpositive scalar curvature. Since Equation (142) is an eigen-equation ($\mathcal{E} = \tilde{\mathcal{E}} = \text{const}$) by Theorem 2.6.10 of Chapter 2 we obtain

$$4\,\pi \leq \mu(\mu - \mathcal{E})\mathrm{Vol}(\tilde{\Omega}) \tag{143}$$

In the next step we bound the area: $\mathrm{Vol}(\tilde{\Omega})$. Since $r : \tilde{\Sigma}\setminus r^{-1}(C) \mapsto \Sigma\setminus C$ is a k-sheeted cover, and Σ is a quotient of $\tilde{\Sigma}$ by a group of isometries (see [64]) we have the following,

$$\mathrm{Vol}(\tilde{\Omega}) \leq \mathrm{Vol}(\tilde{\Sigma}) = k\,\mathrm{Vol}(\Sigma). \tag{144}$$

But since

$$\mathrm{Vol}(M) = \int_{\Sigma} l(x) \geq l_{\min}\,\mathrm{Vol}(\Sigma), \qquad l_{\min} = \min_{x\in\Sigma}\ l(x) \tag{145}$$

where $l : \Sigma \mapsto \mathbb{R}$ is the *length of the fibre function*, we obtain

$$4\pi \leq \frac{k}{l_{\min}}\mu(\mu - \mathcal{E})\mathrm{Vol}(M), \tag{146}$$

which contradicts the assumption *(iii)*.

In the case $\mathcal{E}$ is a nonconstant function we apply the analogous reasoning. By Lemma 3.4.3, Proposition 3.3.12 can be adapted to the setting of the Riemannian submersion Π. The function $\|\tilde{\alpha}\|$, $\tilde{\alpha} = p_*\,\alpha$, is S^1- invariant. From Proposition 3.3.12, and considerations for the area $\mathrm{Vol}(\tilde{\Omega})$ we derive

$$\begin{aligned} 2\pi = 2\pi\chi(\tilde{\Omega}) &= \int_{\tilde{\Omega}} K_s + \int_{\tilde{\Omega}} \Delta_{\tilde{\Sigma}}\ln\|\tilde{\alpha}\| \leq \int_{\tilde{\Omega}} |\Delta_{\tilde{\Sigma}}\ln\|\tilde{\alpha}\|| \ \Rightarrow \\ 2\pi &\leq k\int_{r(\tilde{\Omega})} |\Delta_{\Sigma}\ln\|\alpha\|| \leq \frac{k}{l_{\min}}\int_M |\Delta_E\ln\|\alpha\||. \end{aligned}$$

Consequently, we derive a contradiction to the assumption *(iv)*. $\square$

Remark 3.4.9. It seems to be feasible, in Theorem 3.4.8, to drop the assumption of regularity for the Killing field X (i.e. the assumption of circular orbits). By the compactness, [50], of the group of isometries of (M, g), one easily shows that there exists a regular Killing vector field X_ε arbitrarily close to X. One may expect that Equation (109) will hold for $f_\varepsilon = \alpha(X_\varepsilon)$ with possibly an error term. Consequently, one could imagine an approximation argument, with $\varepsilon \to 0$, that would show that the limit function f; $f_\varepsilon \to f$, is a solution to (109). Applying the reasoning presented in the proof should yield a similar conclusion.

Symmetry assumptions, namely the assumption of the Killing contact vector-field for α in Theorem 3.4.8, imply rather severe restrictions on the topology of the domain M. An ultimate goal of techniques presented in this chapter would be to prove a result similar to Theorem 3.4.8 without these assumptions. A "glimpse" of hope is offered by Proposition 3.3.13, if one could prove existence of convenient embedded surfaces. A more obvious extension may be sought in dropping the global Killing field assumption but keeping the property $\mathcal{L}_X\alpha = 0$. Then, one may conveniently average the metric over an S^1-action and try to relate the parameters of the original metric to the averaged parameters. This approach encounters one serious obstruction, though, namely the equation $* \, d\alpha = \mu \, \alpha$ does not seem to survive the averaging process.

As another conclusion of Theorem 3.4.8 one may give examples of dichotomous metrics, i.e. metrics which admit both tight and overtwisted Beltrami forms. For instance, we may prescribe on the base Σ of a principal S^1-bundle P the metric g_Σ constructed in Theorem 2.4.3 of Chapter 2. Then P would certainly admit both tight and overtwisted Beltrami forms. Consequently, tightness/overtwistedness may not be solely forced by assumptions on the geometry, but also on other parameters e.g. associated with the dynamics of a contact vector field.

CHAPTER IV

CURL EIGENFIELDS AND ENERGY RELAXATION

This chapter is devoted to the following conjecture of Etnyre and Ghrist ([28], p. 17) concerning the topology of smooth minimizers, for the variational principle (3).

Conjecture 4.0.10. *A smooth nonvanishing curl eigenfield which minimizes the L^2-energy in* (3) *defines a tight contact structure.*

We construct a counterexample by "building" a special S^1-invariant curl eigenfield on a product manifold $M = (S^1 \times \Sigma, 1 \oplus g_\Sigma)$, where Σ is a closed orientable surface of genus $g(\Sigma) > 0$. The crucial observation, subjected to scrutiny in Chapter 3, is that in the product metric an S^1-invariant curl eigenfield has the dividing set on Σ defined by a nodal set of a Δ_Σ-eigenfunction. Consequently, we may force these S^1-invariant curl eigenfields to be overtwisted by prescribing the metric on Σ constructed in Theorem 2.4.3. However, we must also assure that such curl eigenfields may minimize the L^2-energy (3) which may be achieved e.g. a choice of the principal eigenfields (i.e. the fields which correspond to the first eigenvalue of $*\, d$).

In the second part of this chapter, we demonstrate that under additional symmetry conditions, analogous to those in Theorem 3.4.8, the energy-relaxation leads to a tight curl-eigenfield which minimizes the energy.

4.1 Energy and eigenvalues

This section is devoted to the basic spectral analysis of the curl operator $*\, d$ on closed compact 3-manifolds. We establish the relation between the curl operator $*\, d$ and the

Laplace-Beltrami operator Δ acting on 1-forms. As a consequence the formula which relates eigenbasis of $*\,d$ to the the eigenbasis of Δ is proven, which may be thought of as an extension of the method by Chandrasekhar and Kendall presented in [18]. We start by proving, following Arnold (see e.g. [5]), that the principal curl-eigenfields are always energy minimizing. In the later construction of the energy minimizing overtwisted curl eigenfield we choose it to be a principal eigenfield to assure the *minimizing* property.

Recall from Chapter 1 that any curl eigenform α extremizes the L^2-energy defined as,

$$E(\alpha) = \|\alpha\|^2_{L^2} = \int_M \alpha \wedge *\alpha, \tag{147}$$

among all 1-forms obtained from α by pullbacks through volume preserving diffeomorphisms. The set of such forms is the COADJOINT ORBIT of α, for the action of the volume preserving diffeomorphisms group, defined as (c.f. [5]):

$$\Psi_\alpha = \{\beta : \beta = \varphi_*(\alpha),\ \varphi \in \mathrm{Diff}_0(M),\ \varphi_*(*1) = *1\}. \tag{148}$$

The question of energy minimization on the coadjoint orbit is closely related to spectral data. The following result is one of the few general results available (see e.g. [5]),

Proposition 4.1.1 ([5])**.** *A curl eigenform α_1, (i.e. an eigenform of the curl operator $*\,d : \mathcal{H} \to \mathcal{H}$, $\mathcal{H} = \{\alpha : \delta\alpha = 0\} = \{$ "divergence free" 1-forms$\}$) which corresponds to the first eigenvalue $\mu_1 \neq 0$ is a minimizer of the energy E on Ψ_{α_1}.*

Proof. The operator $*\,d$ is elliptic and consequently its analytic realization is unbounded on $L^2_{\mathcal{H}}(M, \Lambda^1 T^*M)$ (an L^2-completion of $\mathcal{H}$, closed and self-adjoint; it has a compact inverse $*\,d^{-1}$ defined on the orthogonal complement of its kernel (see [67]). We can also choose an orthonormal basis of eigenforms $\{\alpha_i\}$ in $(L^2_{\mathcal{H}}(M, \Lambda^1 T^*M), (\cdot,\cdot)_{L^2})$

such that,

$$* \, d^{-1}\alpha_i = \frac{1}{\mu_i}\,\alpha_i, \qquad 0 < \mu_1^2 \leq \mu_2^2 \leq \ldots \leq \mu_i^2 \leq \ldots \tag{149}$$

For an arbitrary L^2 1-form $\beta \in \mathrm{Im}(\delta)$ we have

$$|(* \, d^{-1}\beta, \beta)_{L^2}| = |\sum_i \frac{1}{\mu_i}(\alpha_i, \beta)^2_{L^2}| \leq \frac{1}{|\mu_1|}(\beta, \beta)_{L^2} = \frac{1}{|\mu_1|}E(\beta). \tag{150}$$

One obtains a lower bound for the energy $E(\beta)$,

$$E(\beta) \geq |\mu_1||(* \, d^{-1}\beta, \beta)_{L^2}|.$$

The above inequality becomes the equality if and only if β is a μ_1-eigenform of $* \, d$. The claim follows from the fact that the HELICITY, $(* \, d^{-1}\beta, \beta)_{L^2}$, is invariant under volume preserving transformations see [5]. □

From now on we do not distinguish between operators defined on various spaces of smooth differential forms and their analytic realizations defined on L^2-completions of those spaces.

The curl operator $* \, d : \mathcal{H} \to \mathcal{H}$ is a self-adjoint first-order elliptic operator, and the principal eigenvalue μ_1 enjoys a variational characterization through the Rellich's quotient. By Lemma 4.1.2 we have,

$$\mu_1 = \inf_{\alpha \in \mathcal{H}_0^\perp} \frac{|(* \, d\alpha, \alpha)_{L^2}|}{\|\alpha\|^2_{L^2}} \quad \Leftrightarrow \quad \mu_1^2 = \inf_{\alpha \in \mathcal{H}_0^\perp} \frac{(\Delta^1_M \alpha, \alpha)_{L^2}}{\|\alpha\|^2_{L^2}}, \tag{151}$$

$$\mathcal{H}_0^\perp = \mathrm{Ker}(* \, d)^\perp = \{\alpha \in \Omega^1(M) : \alpha = \delta\beta, \text{ for some } \beta \in \Omega^2(M)\},$$

Observe that the curl squared is equal to the Hodge Laplacian , $(* \, d)^2 = \delta \, d$, on $\mathcal{H}$. Therefore any curl eigenform α, (i.e. $* \, d\alpha = \pm\mu\,\alpha$) is automatically a co-closed μ^2-eigenform of the Hodge Laplacian $\Delta^1_M = d\,\delta + \delta\, d$, namely

$$\Delta^1_M \alpha = \delta \, d \, \alpha = * \, d * d\alpha = \mu^2\,\alpha. \tag{152}$$

The curl $*\,d$ commutes with Δ^1_M, therefore both of these operators are simultaneously diagonizable on $\mathcal{H}$ in a suitable orhonormal basis of curl eigenforms;

$$\mathcal{H} = \bigoplus_{i=1}^{\infty} E^{\Delta}(\mu_i^2), \quad E^{\Delta}(\mu_i^2) \perp E^{\Delta}(\mu_j^2),\ i \neq j \qquad 0 < \mu_1^2 \leq \mu_2^2 \leq \ldots \leq \mu_i^2 \leq \ldots$$

where $E^{\Delta}(\mu_i^2)$ stands for the μ_i^2-eigenspace of Δ^1_M, and

$$E^{\Delta}(\mu_i^2) = E^{*\,d}(\mu_i) \oplus E^{*\,d}(-\mu_i).$$

(We allow one of $E^{*\,d}(\mu_i)$, $E^{*\,d}(-\mu_i)$ to be trivial). We may conclude further that there exist two positive operators $\sqrt{\Delta_+}$, $\sqrt{\Delta_-}$, such that $*\,d = \sqrt{\Delta_+} - \sqrt{\Delta_-}$.

The following useful fact, which can be traced back to work in [18], tells us how to effectively find a basis of curl eigenforms from a basis of co-closed Δ^1_M-eigenforms.

Lemma 4.1.2. *Any curl μ-eigenform is automatically a co-closed μ^2-eigenform of the Laplacian Δ^1_M. Conversely, given a co-closed μ^2-eigenform $\alpha \in \Omega^1(M)$ of Δ^1_M there exists a corresponding $\pm\mu$-curl eigenform $\beta_\pm \in \Omega^1(M)$ given by*

$$\beta_\pm = \mu\,\alpha \pm *\,d\alpha \tag{153}$$

Proof. The first claim follows from (152). The second claim we verify by a direct calculation. Let $\beta_\pm$ be defined by (153), we will show: $*d\,\beta_\pm = \pm\,\mu\,\beta_\pm$. Since $*d*d = \delta\,d = \Delta^1_M \restriction_{\mathcal{H}}$, and $\delta\alpha = 0$ we obtain

$$*\,d\beta_\pm = \mu * d\alpha \pm \Delta^1_M \alpha\,.$$

Secondly, $\Delta^1_M \alpha = \mu^2\alpha$, therefore

$$*\,d\beta_\pm = \mu * d\alpha \pm \mu^2\alpha = \pm\mu\,\beta_\pm.$$

□

4.2 Overtwisted principal eigenfields

Based on Lemma 4.1.2 we may "produce" a curl eigenfield from a divergence free eigenform of the Laplace-Beltrami operator. For that reason, in the first part of this section, we fully characterize eigenvalues and eigenforms of the Laplace-Beltrami operator on $\Omega^1(S^1 \times \Sigma)$, where Σ is an orientable surface. In the construction of an overtwisted minimizer we use Proposition 4.1.1, which implies that it is sufficient to construct an overtwisted principal curl eigenfield. In the following paragraphs we elaborate on the construction.

We start with a full characterization of eigenforms on M which we assume to be a trivial bundle $P = S^1 \times \Sigma$ and the metric g on P is a product metric $g = 1 \oplus g_\Sigma$ with constant length l fibres. Consequently, the space of smooth 1-forms $\Omega^1(P)$ decomposes with respect to the L^2-inner product induced by the metric g as

$$\Omega^1(P) = \Omega^1_N(P) \oplus \Omega^1_T(P), \tag{154}$$

where,

$$\begin{aligned} \Omega^1_N(P) &= \{\alpha \in \Omega^1(P) : \alpha = f\eta,\ f \in C^\infty(P)\}, \\ \Omega^1_T(P) &= \{\alpha \in \Omega^1(P) : \alpha(X_\eta) = 0\}, \\ \Omega^1_T(P) &= \Omega^1_N(P)^\perp \cap \Omega^1(P). \end{aligned}$$

with η and X_η being the tangent 1-form and vector field (resp). to the S^1-fibres of unit magnitude ($\|X_\eta\|_g = 1$). In this setting we prove the following;

Lemma 4.2.1. *The Laplacian Δ^1_P preserves $\Omega^1_N(P)$, $\Omega^1_T(P)$ and for $\alpha = f\eta + \beta$, $f\eta \in \Omega^1_N(P)$, $\beta \in \Omega^1_T(P)$ we have the following formula for the Laplacian at a point $(t, q) \in S^1 \times \Sigma$,*

$$\Delta^1_P \alpha = (-\mathcal{L}^2_\eta f + \Delta^0_\Sigma f_t)\eta + (-\mathcal{L}^2_\eta \beta + \Delta^1_\Sigma \beta_t), \quad at \quad (t, q), \tag{155}$$

where $f_t = f \restriction_{\{t\}\times\Sigma} \in C^\infty(\Sigma)$ and Δ^0_Σ is the scalar Laplacian on Σ. Similarly $\beta_t = \beta \restriction_{\{t\}\times\Sigma}$ and Δ^1_Σ is the 1-form Laplacian on Σ.

Proof. The first claim follows immediately from the formula (155). We justify (155) by a direct calculation in the X_η-invariant frame $\{e_1, e_2, e_3\}$, $e_1 = X_\eta$, (denote the co-frame by $\{\eta_i\}$, $\eta_1 = \eta$) on P, where e_2, e_3 are tangent to the Σ fibers. Denote: $\nabla_i = \nabla_{e_i}$ and recall the following formulas (see e.g. [49])

$$\nabla_i e_j = \omega_{ij}^k e_k, \quad \nabla_i \eta_k = -\omega_{ij}^k \eta_j, \quad \omega_{ij}^k = -\omega_{ik}^j, \tag{156}$$

$$\nabla_{ii}^2 = -\nabla_i \nabla_i + \omega_{ii}^j \nabla_j, \quad \Delta^0 = -\nabla_{ii}^2. \tag{157}$$

The well known Weitzenböck formula (see [49] p. 138) for the k-form Laplacian $\Delta^k = d\,\delta + \delta\, d$ tells us ($k = 1$):

$$\begin{aligned} \Delta^1 \alpha &= -\nabla_{ii}^2 \alpha - \eta_i \wedge (\iota(e_j)\ R(e_i, e_j)\alpha), \quad \alpha \in \Omega^1(M), \qquad (158)\\ R(e_i, e_j)\alpha &= \nabla_i \nabla_j \alpha - \nabla_j \nabla_i \alpha - \nabla_{[e_i, e_j]}\alpha. \end{aligned}$$

In the product metric we may choose locally an X_η-invariant frame $\{e_i\}$, meaning:

$$[e_1, e_j] = -[e_j, e_1] = 0, \quad [e_2, e_3] \in T\Sigma.$$

Consequently (by $\omega_{i\,j}^k = \frac{1}{2}\langle [e_i, e_j], e_k\rangle - \langle [e_j, e_k], e_i\rangle + \langle [e_k, e_i], e_j\rangle$):

$$\omega_{i\,j}^k = 0, \quad \text{if one of the indices } i, j, k = 1, \tag{159}$$

$$\nabla_1 \omega_{i\,j}^k = 0, \quad \text{for all } i, j, k \tag{160}$$

$$R(e_i, e_j)\eta_r = 0, \quad \text{if one of the indices } i, j, k, r = 1, \tag{161}$$

where (161) is a consequence of the following;

$$\iota(e_k)\, R(e_i, e_j)\eta_r = \nabla_j \omega_{i\,k}^r - \nabla_i \omega_{j\,k}^r + \omega_{j\,n}^r\, \omega_{i\,k}^n - \omega_{i\,n}^r \omega_{j\,k}^n + (\omega_{i\,j}^n - \omega_{j\,i}^n)\omega_{n\,k}^r.$$

In turn we obtain,

$$\nabla \eta_1 = 0, \quad \nabla_1 \eta_k = 0, \quad \nabla_2 \eta_i = -\omega_{i\,3}^2\, \eta_3, \quad \nabla_3 \eta_i = -\omega_{i\,2}^3\, \eta_2\ . \tag{162}$$

Let $\alpha = f\eta + \beta$, then $\Delta^1_P \alpha = \Delta^1_P(f\eta_1) + \Delta^1_P \beta$, we obtain from (158), (161),

$$\begin{aligned}
\Delta^1_P(f\eta_1) &= -\nabla^2_{i\,i}(f\eta_1) = -\nabla_i\nabla_i(f\eta_1) + \omega^{\,j}_{i\,i}\nabla_j(f\eta_1) \\
(162): &= (-\nabla_i\nabla_i f + \omega^{\,j}_{i\,i}\nabla_j f)\eta_1 \\
(159)\text{ - }(161): &= (-\nabla_1\nabla_1 f - \nabla_2\nabla_2 f + \omega^{3}_{2\,2}\nabla_3 f - \nabla_3\nabla_3 f + \omega^{2}_{3\,3}\nabla_2 f)\eta_1
\end{aligned}$$

Treating $f = f_t$ as a family of functions $f_t \in C^\infty(\Sigma)$ dependent on $t \in S^1$, Equation (157) implies

$$-\nabla_2\nabla_2 f_t + \omega^{3}_{2\,2}\nabla_3 f_t - \nabla_3\nabla_3 f_t + \omega^{2}_{3\,3}\nabla_2 f_t = \Delta^0_\Sigma f_t, \quad \text{and}$$
$$\Delta^1_P(f\eta) = (-\nabla_1\nabla_1 f + \Delta^0_\Sigma f_t)\eta = (-\mathcal{L}^2_\eta f + \Delta^0_\Sigma f_t)\eta.$$

Similar reasoning can be applied to $\beta = a_2\,\eta_2 + a_3\,\eta_3$, by (162) and (161) we have

$$\begin{aligned}
(-\nabla_1\nabla_1 + \omega^{\,j}_{1\,1}\nabla_j)\beta - \eta_1 \wedge (\iota(e_j)\ R(e_1, e_j)\beta) &= -\nabla_1\nabla_1(a_2\eta_2 + a_3\eta_3) \\
&= -(\nabla_1\nabla_1 a_2)\eta_2 - (\nabla_1\nabla_1 a_3)\eta_3 = -\mathcal{L}^2_\eta \beta.
\end{aligned}$$

Treating $\beta = \beta_t$ as a family of 1-forms $\beta_t \in \Omega^1(\Sigma)$ and using (162), (161) and (158) one shows

$$\sum_{i=2}^{3}\{-\nabla_i\nabla_i\beta + \omega^{\,j}_{i\,i}\nabla_j)\beta - \eta_i \wedge (\iota(e_j)\,R(e_i, e_j)\beta\} = \Delta^1_\Sigma \beta_t\,.$$

□

Lemma 4.2.2. *On the product manifold $P = S^1 \times \Sigma$, $g = 1 \oplus g_\Sigma$ with constant length l fibers. The first eigenvalue μ_1 of the curl operator satisfies*

$$\mu_1^2 = \min\left\{\nu_1, \left(\frac{2\pi}{l}\right)^2\right\}, \quad \textit{where} \quad \nu_1 = \inf_{f \in L^2(\Sigma),\ f \neq const}\left\{\frac{(\Delta^0_\Sigma f, f)_{L^2}}{\|f\|^2_{L^2}}\right\}.$$

Proof. From the decomposition (154) and the fact that Δ^1_P preserves $\Omega^1_T(P)$ and $\Omega^1_N(P)$ (see Lemma 4.2.1) we have

$$\mu_1^2 = \min\left\{\mu^2_{1,T}, \mu^2_{1,N}\right\}; \quad \mu^2_{1,r} = \inf_{\alpha \in \mathcal{H}_0^\perp \cap \Omega^1_r(P)}\left\{\frac{(\Delta^1_P\alpha, \alpha)_{L^2}}{\|\alpha\|^2_{L^2}}\right\} \quad r = T, N. \tag{163}$$

In order to calculate $\mu_{1,N}^2$, notice that for any $\alpha \in \mathcal{H} \cap \Omega_N^1(P)$, $\alpha = f\,\eta$, the function f is constant on the fibers; hence $f \in C^\infty(\Sigma)$. Indeed, $\delta\alpha = 0$, and, since $\nabla\eta = 0$ in the adapted frame $\{e_1, e_2, e_3\}$ with $e_1 = X_\eta$, we obtain

$$0 = \delta\alpha = \iota(e_i)\nabla_i\alpha = \iota(e_i)(\nabla_i f\eta + f\nabla_i\eta) = \nabla_1 f = X_\eta f.$$

From Equation (155), we conclude that for any $\alpha = f\,\eta \in \mathcal{H} \cap \Omega_N^1(P)$ we have: $\Delta_\Sigma^1\alpha = (\Delta_\Sigma^0 f)\eta$ and consequently

$$\begin{aligned}
(\Delta_P^1\alpha, \alpha)_{L^2} &= (\Delta_\Sigma^0 f\eta, f\eta)_{L^2} = \int_{S^1\times\Sigma}(f\,\Delta_\Sigma^0 f)\eta \wedge *\eta \\
&= \int_{S^1}\int_\Sigma f\,\Delta_\Sigma^0 f = l\int_\Sigma f\,\Delta_\Sigma^0 f = l\,(\Delta_\Sigma^0 f, f)_{L^2}, \\
\text{and} \quad \|\alpha\|_{L^2}^2 &= (\alpha,\alpha)_{L^2} = \int_{S^1\times\Sigma} f^2\eta\wedge *\eta = l\,\|f\|_{L^2}^2 \quad \text{where } \eta\wedge*\eta = *1.
\end{aligned}$$

As a result, (151),

$$\mu_{1,N}^2 = \nu_1, \qquad \nu_1 = \inf_{f\in L^2(\Sigma),\ f\neq\text{const}}\left\{\frac{(\Delta_\Sigma^0 f, f)_{L^2}}{\|f\|_{L^2}^2}\right\}. \tag{164}$$

In other words $\mu_{1,N}^2$ is equal to the first eigenvalue of the scalar Laplacian Δ_Σ^0 on Σ.

In order to calculate $\mu_{1,T}^2$ we first calculate the orthogonal basis of eigenforms on $\mathcal{H}\cap\Omega_T^1(P)$. Let $\{\beta_m\}$ be an orthonormal basis of Δ_Σ^1-eigenforms on $\mathrm{Ker}(\Delta_\Sigma^1)\oplus\mathrm{Im}(\delta) \subset L^2(\Lambda^1T^*\Sigma)$, define for all $m, n \in \mathbb{Z}^+$:

$$\begin{gathered}
h_0 = g_0 = 1, \quad h_n = \cos\big(\frac{2\pi nt}{l}\big),\ g_n = \sin\big(\frac{2\pi nt}{l}\big), \\
\alpha_{n\,m}^g = g_n\,\beta_m, \qquad \text{and} \qquad \alpha_{n\,m}^h = h_n\,\beta_m\,.
\end{gathered} \tag{165}$$

Clearly, $\{\alpha_{n\,m}^g, \alpha_{n\,m}^h\}$ is a set of 1-eigenforms of Δ_P^1 on $\mathcal{H}\cap\Omega_T^1(P)$. Indeed, from Equation (155)

$$\Delta_P^1\alpha_{mn}^r = \gamma_{mn}^r\alpha_{mn}^r, \quad \gamma_{mn}^r = \Big(\frac{2\pi n}{l}\Big)^2 + \tilde\nu_m\,,$$

where $\tilde\nu_m$ is the m-th eigenvalue of Δ_Σ^1. One easily shows that $\{\alpha_{n\,m}^g, \alpha_{n\,m}^h\}$ is an orthonormal basis of $\mathcal{H}\cap\Omega_T^1(P)$. Consequently, all eigenforms of Δ_P^1 on $\mathcal{H}\cap\Omega_T^1(P)$

are listed in Equation (165), and we have

$$\mu_{1,T}^2 = \min\left\{\left(\frac{2\pi}{l}\right)^2, \tilde{\nu}_1\right\}. \tag{166}$$

It remains to show that $\tilde{\nu}_1 = \nu_1$. By the Hodge decomposition theorem:

$$\Omega^1(\Sigma) = \mathrm{Ker}(\Delta^1_\Sigma) \oplus \mathrm{Im}(d_\Sigma) \oplus \mathrm{Im}(\delta_\Sigma).$$

Moreover, $\Omega^0(\Sigma) \simeq \Omega^2(\Sigma)$ through the Hodge-star isometry, therefore $\mathrm{Im}(\delta_\Sigma) = \{*_\Sigma\, d\, f; f \in C^\infty(\Sigma)\}$. Since Δ^1_Σ commutes with $*_\Sigma\, d$, any ν_m-eigenfunction f_m results in a ν_m-eigenform $*_\Sigma\, d\, f_m$. Therefore $\tilde{\nu}_1 = \nu_1$ and the lemma follows from (163), (164), and (166). □

Combining Theorem 2.4.3 of Chapter 2 with Lemma 4.2.2 results in the following;

Theorem 4.2.3. *Let $\Sigma \neq S^2$ be an orientable surface of an arbitrary nonzero genus. One can prescribe a metric g_Σ on Σ such that there exists an overtwisted curl eigenfield v on the product manifold $(S^1 \times \Sigma, 1 \oplus g_\Sigma)$ which minimizes the energy* (147) *on the coadjoint orbit Ψ_α.*

Proof. In the first step we choose a metric g_Σ on $\Sigma \neq S^2$ constructed in Theorem 2.4.3 and assume that the length of fibres in $(S^1 \times \Sigma, 1 \oplus g_\Sigma)$ is equal to l. By Lemma 4.2.2 we may choose l small, so that the first eigenvalue satisfies $\mu_1 = \nu_1$. The proof of 4.2.2 implies that the corresponding eigenspace $E^\Delta(\mu_1^2)$ is spanned by two independent co-closed μ_1^2-eigenforms of Δ^1_P:

$$\alpha_1 = f_1\, \eta, \quad \text{and} \quad \alpha_2 = *\, d_\Sigma f_1.$$

($E^\Delta(\mu_1^2)$ is 2-dimensional since g_Σ is a generic metric.)

By previous considerations $E^\Delta(\mu_1^2) = E^{*\,d}(\mu_1) \oplus E^{*\,d}(-\mu_1)$ and $E^\Delta(\mu_1^2)$ is spanned by two independent $\pm\mu_1$-curl eigenforms. Choosing any linear combination of α_1 and α_2 Lemma 4.1.2 leads to $\pm\mu_1$-curl eigenforms given by,

$$\beta_\pm = f_1 \eta \pm *_\Sigma\, d\, f_1.$$

These forms are nonvanishing since the set of zeros is clearly equal to the singular part of the nodal set of f_1 and by Theorem 2.2.7 the singular part is empty for a generic choice of metric. Both forms $\beta_\pm$ are S^1-invariant and overtwisted by Theorem 3.2.12. Indeed, the projection of the characteristic surface Γ_{S^1} of $\alpha_\pm$ onto Σ (i.e. the dividing set $\pi(\Gamma_{S^1})$) is equal to the nodal set of f_1. By the choice of the metric g_Σ, $\pi(\Gamma_{S^1})$ bounds a disc. Now, the dual curl eigenfields $\beta^\#_\pm$ minimize energy (147) on $\Psi_{\beta_\pm}$ due to Proposition 4.1.1. $\square$

Remark 4.2.4. By perturbing a product metric on P the eigenvalues $\pm\mu_1$ "split apart" giving only simple eigenvalues and only one minimal eigenvalue. If the perturbation is small then the resulting eigenform will be C^0-close to β_+(or β_-), and define an isotopic contact structure (by Grey's Theorem, see e.g. [36]), which in turn must be an overtwisted minimizer.

4.3 Tight energy minimizers

So far the only known examples of curl eigenfields minimizing the energy were ABC-fields on the flat T^3 (see Equation (4) in Chapter 1) and Hopf fields on round S^3, [5]. Both of them are tight principal curl-eigenfields. In the following paragraphs we indicate examples of tight principal curl eigenfields on S^1-bundles. This may support the claim that Conjecture 4.0.10 may be valid for certain classes of contact manifolds.

First, we review relevant facts from Section 3.4 in the current setting. Let (M, g_M) be a Riemannian 3-manifold which admits a unit Killing vector field X orthogonal to a contact structure $\xi = X^\perp$. Choosing a local frame of vector fields $\{e_1, e_2, e_3\}$, $e_1 = X$, the Cartan's structure equations imply that the dual 1-form $\eta = g_M(X, \cdot)$ satisfies

$$*\, d\eta = 2\,\lambda\,\eta, \qquad \lambda \in C^\infty(P). \tag{167}$$

In terms of Christoffel symbols ($\nabla_i e_j = \omega^k_{i\,j}\, e_k$) we obtain

$$-\omega^2_{3\,1} = \omega^3_{2\,1} = \lambda. \tag{168}$$

(Notice that the parameter $\mathcal{E} = \iota(X) * d\eta_1 = 2\lambda$, appears in Chapter 3.)

In the case $\lambda(x) = \lambda = \text{const}$, η defines a curl eigenform on M. Now, we focus on the special case of (M, η, g_M), namely the case of a principal S^1-bundle P, $\pi : P \mapsto \Sigma$ over a closed orientable surface Σ equipped with the unit Killing vector field tangent to the fiberes of P. By Theorem 3.2.12 η is necessarily a tight curl eigenform since the contact plane distribution is S^1-invariant and orthogonal to the fibers. In the following proposition we denote by $\mathcal{H}_{S^1}$ the subspace of S^1-invariant 1-forms in $\mathcal{H} \subset \Omega^1(P)$.

Proposition 4.3.1. *Any curl eigenform η defined on (P, g_P) by (167) is always energy-minimizing on $\mathcal{H}_{S^1} \cap \Psi_\eta$. Let ν be the first nonzero eigenvalue of the scalar Laplacian Δ^0_Σ on Σ. If $\nu > 3\lambda^2$ then η is a principal curl eigenform on $\mathcal{H}_{S^1}$.*

Proof. We provide the proof of the first claim for $\lambda > 0$ (in the case of $\lambda < 0$ the reasoning is analogous). The space $\mathcal{H}_{S^1}$ decomposes as

$$\mathcal{H}_{S^1} = \mathcal{H}^+_{S^1} \oplus \mathcal{H}^-_{S^1},$$

where $\mathcal{H}^\pm_{S^1}$ is a subspace spanned by positive/negative curl eigenforms. We need to show that η is an energy minimizer on $\mathcal{H}_{S^1} \cap \Psi_\eta$. Given a volume preserving diffeomorphism $\varphi : P \to P$ we denote $\eta_\varphi = \varphi_*(\eta) \in \Psi_\eta$. Under the assumptions on the φ action, $\eta_\varphi \in \mathcal{H}_{S^1} \cap \Psi_\eta$. We expand η_φ in the eigenbasis of curl eigenforms (149), $\eta_\varphi = \sum_{i\geq 0} c^+_i \alpha^+_i + \sum_{i<0} c^-_i \alpha^-_i$, where $\{\alpha^\pm_i\}$ span $\mathcal{H}_{S^1}$. Since the helicity $(*d^{-1}\eta_\varphi, \eta_\varphi)$ is invariant under φ, as in (150), we obtain

$$0 < \frac{E(\eta)}{2\lambda} = (*d^{-1}\eta, \eta) = (*d^{-1}\eta_\varphi, \eta_\varphi) = \sum_{i\geq 0} \frac{(c^+)^2}{\mu^+_i} + \sum_{i<0} \frac{(c^-)^2}{\mu^-_i}$$

where $\mu^\pm_i$, positive/negative eigenvalues of $*d$ on $\mathcal{H}_{S^1}$. Since the second sum is negative we can estimate $\mu^+_1(*d^{-1}\eta, \eta) \leq E(\eta_\varphi)$. To finish the proof it suffices to show that $2\lambda = \mu^+_1$. Then we obtain $E(\eta) \leq E(\eta_\varphi)$ which proves the claim.

Now we show the equality $2\lambda = \mu_1^+$, let $\alpha_1 = a_i\eta^i = f\eta + \beta$ be the curl eigenform satisfying

$$*d\,\alpha_1 = \mu_1^+\alpha_1. \tag{169}$$

Since, $\mathcal{L}_X\alpha_1 = 0$ and the fibres of $\pi : P \mapsto \Sigma$ are totally geodesic, Theorem 3.3.7 tells us the following equation for f:

$$\Delta_P^0 f = \mu_1^+(\mu_1^+ - 2\lambda)f. \tag{170}$$

Equations (108), in Chapter 3, imply that, for α_1 to be nontrivial, f cannot be a constant zero function. Since Δ_P^0 is a positive operator we conclude that $\mu_1^+ \geq 2\lambda$; consequently, $\mu_1^+ = 2\lambda$ because μ_1^+ is the first positive eigenvalue.

Theorem 3.3.5 guarantees that Equation (170) is valid for any S^1-invariant μ-eigenform. By Lemma 3.4.3, $\pi : P \mapsto \Sigma$ defines a Riemannian submersion: $\Delta_P^0(h \circ \pi) = \pi \circ \Delta_\Sigma^0 h$, $h \in C^\infty(\Sigma)$, and the proof of the second statement follows from the equation: $\nu = \mu(\mu - 2\lambda)$. Indeed, for $\gamma^2 = \nu$, $\gamma > 0$, we obtain

$$\mu^2 - 2\,\lambda\,\mu + \gamma^2 = 0, \qquad \overline{\mu} = \lambda + \sqrt{\lambda^2 + \gamma^2},\ \underline{\mu} = \lambda - \sqrt{\lambda^2 + \gamma^2}, \tag{171}$$

where $\overline{\mu}$, $\underline{\mu}$ are the roots of the equation. Consequently,

$$\mu_1^- \in (-\infty, -\gamma),\ \text{if} \quad \mu_1^+ = \lambda > 0$$
$$\mu_1^+ \in (\gamma, +\infty),\ \text{if} \quad \mu_1^- = \lambda < 0,$$

and it suffices to assume $\nu = \gamma^2 > 3\lambda^2$ for λ to be the principal eigenvalue of $*\,d$ on $\mathcal{H}_{S^1}$. □

We may think about $\lambda \neq 0$ as a "topological deviation" from the $\lambda = 0$ case. Recall that Hopf fields are principal curl eigenfields of $*\,d$ on $\mathcal{H}$ and therefore, by Proposition 4.1.1, energy minimizers. Lemma 4.2.2 and 4.1.2, characterize the principal curl eigenfields on products: $S^1 \times \Sigma$. The next theorem provides conditions for η to become the principal curl eigenfield on (P, g_P).

By Lemma 3.4.3 we may prescribe a metric g_Σ on the base Σ such that a local frame $\{e_1 = X, e_2, e_3\}$ on P satisfies

$$[e_1, e_i] = 0, \qquad i = 1, 2, 3. \tag{172}$$

Consequently,

$$\mathcal{L}_X \eta_i = 0, \qquad \text{for} \quad \eta_i = g_P(e_i, \cdot\,). \tag{173}$$

Theorem 4.3.2. *The curl eigenform η defined by* (167) *on* (P, g_P) *such that* (172), (173) *hold is a principal curl eigenform on $\mathcal{H}$ if*

$$\lambda^2 < \min\big(\frac{\nu}{3}, \frac{4\pi^2}{l^2}\big), \tag{174}$$

where ν is the first nonzero eigenvalue of the scalar Laplacian Δ^0_Σ on Σ, and l is the length of the fibre.

Proof. Observe that on (P, g_P) the operator $-\mathcal{L}_X^2$ commutes with $\Delta^1_P = \delta\, d + d\, \delta$. We check that $\mathcal{L}_X : \Omega^*(P) \to \Omega^*(P)$ commutes with the Hodge star operator $* : \Omega^*(P) \to \Omega^*(P)$. Indeed since $\mathcal{L}_X$ respects the wedge product:

$$\mathcal{L}_X(\omega_1 \wedge \omega_2) = \mathcal{L}_X\omega_1 \wedge \omega_2 + \omega_1 \wedge \mathcal{L}_X\omega_2, \quad \omega_1 \in \Omega^j(P), \omega_2 \in \Omega^k(P),$$

Equation (173) implies that for any k-form $\alpha = \sum_I a_I \omega_I$, where $\omega_I = \eta_{i_1} \wedge \eta_{i_2} \wedge \ldots \wedge \eta_{i_k}$, $I = (i_1, \ldots, i_k)$, we have

$$\mathcal{L}_X \alpha = \sum_I (\mathcal{L}_X a_I)\omega_I, \qquad \text{since } \mathcal{L}_X\omega_I = 0.$$

Consequently, $*\,\mathcal{L}_X = \mathcal{L}_X\, *$ follows from the definition of the Hodge star operator, namely

$$*\mathcal{L}_X\alpha = \sum_I (\mathcal{L}_X a_I) * \omega_I \tag{175}$$

$$= \mathcal{L}_X \sum_I a_I * \omega_I = \mathcal{L}_X * \alpha, \quad \text{since} \quad \mathcal{L}_X\omega_I = \mathcal{L}_X * \omega_I = 0.$$

Hence, we obtain $\mathcal{L}_X \Delta_P^1 = \Delta_P^1 \mathcal{L}_X$ because $\mathcal{L}_X$ commutes with an exterior derivative d and $\delta = \pm * d *$. Consequently,

$$(-\mathcal{L}_X^2)\Delta_P^1 = \Delta_P^1(-\mathcal{L}_X^2). \tag{176}$$

Therefore, we may define the following decomposition;

$$\Delta_P^1 = -\mathcal{L}_X^2 + \Delta_P^H\,, \tag{177}$$

where $\Delta_P^H = \Delta_P^1 + \mathcal{L}_X^2$ and we call Δ_P^H the horizontal Laplacian. Both $-\mathcal{L}_X^2$ and Δ_P^H are not elliptic, since take into account only derivatives in certain directions. But $-\mathcal{L}_X^2$ and Δ_P^H have discrete spectra and commute by (176). Consequently, these operators are simultaneously diagonalizable in a suitable L^2-orthonormal basis of Δ_P^1-eigenforms (see also [12]). The decomposition in (177) and (176) implies that any eigenvalue τ of Δ_P^1 is a sum $\tau = \psi + \mu$ of an eigenvalue ψ of $-\mathcal{L}_X^2$, and μ of Δ_P^H.

If $-\mathcal{L}_X^2 \alpha = \psi\,\alpha$, $\alpha = a_i\,\eta_i$, then

$$-\mathcal{L}_X^2 a_i = \psi\, a_i.$$

Solving this equation in a local trivialization: $(t, q) \in U \simeq S^1 \times V \subset P$, $V \subset \Sigma$, gives us $-\mathcal{L}_X^2 a_i = -\partial_t^2 a_i(t,q) = \psi\, a_i(t,q)$. For a fixed q: $a_i(t,q) = A_q\,\cos(\psi\, t) + B_q\,\sin(\psi\, t)$, therefore $\psi = \big(2\,\pi\, n/l\big)^2$, $n \in \mathbb{N}$. Now, the theorem is a consequence of $\mathcal{H}_{S^1} = \mathrm{Ker}(-\mathcal{L}_X^2) \cap \mathcal{H}$, and Proposition 4.3.1. $\square$

REFERENCES

[1] Alessandrini, G., "Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains," *Comment. Math. Helv.*, vol. 69, no. 1, pp. 142–154, 1994.

[2] Anné, C., "Perturbation du spectre $X - TUB^{\epsilon}Y$ (conditions de Neumann)," in *Séminaire de Théorie Spectrale et Géométrie, No. 4, Année 1985–1986*, pp. 17–23, Saint: Univ. Grenoble I, 1986.

[3] Anné, C., "Spectre du laplacien et écrasement d'anses," *Ann. Sci. École Norm. Sup. (4)*, vol. 20, no. 2, pp. 271–280, 1987.

[4] Anné, C. and Colbois, B., "Opérateur de Hodge-Laplace sur des variétés compactes privées d'un nombre fini de boules," *J. Funct. Anal.*, vol. 115, no. 1, pp. 190–211, 1993.

[5] Arnold, V. and Khesin, B., *Topological methods in hydrodynamics*, vol. 125 of *Applied Mathematical Sciences*. New York: Springer-Verlag, 1998.

[6] Aronszajn, N., "A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order," *J. Math. Pures Appl. (9)*, vol. 36, pp. 235–249, 1957.

[7] Aubin, T., *Some nonlinear problems in Riemannian geometry.* Springer Monographs in Mathematics, Berlin: Springer-Verlag, 1998.

[8] Bando, S. and Urakawa, H., "Generic properties of the eigenvalue of the Laplacian for compact Riemannian manifolds," *Tôhoku Math. J. (2)*, vol. 35, no. 2, pp. 155–172, 1983.

[9] Bär, C., "On nodal sets for Dirac and Laplace operators," *Comm. Math. Phys.*, vol. 188, no. 3, pp. 709–721, 1997.

[10] Belgun, F., "Normal CR structures on S^3," *Math. Z.*, vol. 244, no. 1, pp. 125–151, 2003.

[11] Bennequin, D., "Links and Pfaffian equations," *Uspekhi Mat. Nauk*, vol. 44, no. 3(267), pp. 3–53, 208, 1989.

[12] Bérard-Bergery, L. and Bourguignon, J., "Laplacians and Riemannian submersions with totally geodesic fibres," *Illinois J. Math.*, vol. 26, no. 2, pp. 181–200, 1982.

[13] Blair, D., *Riemannian geometry of contact and symplectic manifolds*, vol. 203 of *Progress in Mathematics.* Boston, MA: Birkhäuser Boston Inc., 2002.

[14] Bott, R. and Tu, L., *Differential forms in algebraic topology*, vol. 82 of *Graduate Texts in Mathematics*. New York: Springer-Verlag, 1982.

[15] Brin, M., *Seifert Fibered Spaces: Notes for a course given in the Spring of 1993*. Online notes.

[16] Bryant, R. L., Chern, S. S., Gardner, R. B., Goldschmidt, H. L., and Griffiths, P. A., *Exterior differential systems*, vol. 18 of *Mathematical Sciences Research Institute Publications*. New York: Springer-Verlag, 1991.

[17] Buser, P., "Riemannsche Flächen mit Eigenwerten in (0, 1/4)," *Comment. Math. Helv.*, vol. 52, no. 1, pp. 25–34, 1977.

[18] Chandrasekhar, S. and Kendall, P., "On force-free magnetic fields," *Astrophys. J.*, vol. 126, pp. 457–460, 1957.

[19] Chavel, I., *Eigenvalues in Riemannian geometry*, vol. 115 of *Pure and Applied Mathematics*. Orlando, FL: Academic Press Inc., 1984. Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk.

[20] Cheng, S. Y., "Eigenfunctions and nodal sets," *Comment. Math. Helv.*, vol. 51, no. 1, pp. 43–55, 1976.

[21] Chern, S. S. and Hamilton, R. S., "On Riemannian metrics adapted to three-dimensional contact manifolds," in *Workshop Bonn 1984 (Bonn, 1984)*, vol. 1111 of *Lecture Notes in Math.*, pp. 279–308, Berlin: Springer, 1985.

[22] do Carmo, M. P., *Differential forms and applications*. Universitext, Berlin: Springer-Verlag, 1994. Translated from the 1971 Portuguese original.

[23] Dong, R.-T., "Nodal sets of eigenfunctions on Riemann surfaces," *J. Differential Geom.*, vol. 36, no. 2, pp. 493–506, 1992.

[24] Eliashberg, Y. Givental, A. and Hofer, H., "An introduction to symplectic field theory," in *GAFA 2000*, Special Volume, Part II, pp. 560–673, Basel: Birkhuser Verlag, 2000.

[25] Eliashberg, Y., "Classification of overtwisted contact structures on 3-manifolds," *Invent. Math.*, vol. 98, no. 3, pp. 623–637, 1989.

[26] Eliashberg, Y., "Contact 3-manifolds twenty years since J. Martinet's work," *Ann. Inst. Fourier (Grenoble)*, vol. 42, no. 1-2, pp. 165–192, 1992.

[27] Etnyre, J., "Tight contact structures on lens spaces," *Commun. Contemp. Math.*, vol. 2, no. 4, pp. 559–577, 2000.

[28] Etnyre, J. and Ghrist, R., "Contact topology and hydrodynamics. I. Beltrami fields and the Seifert conjecture," *Nonlinearity*, vol. 13, no. 2, pp. 441–458, 2000.

[29] Etnyre, J. and Ghrist, R., "Contact topology and hydrodynamics. III. Knotted orbits," *Trans. Amer. Math. Soc.*, vol. 352, no. 12, pp. 5781–5794 (electronic), 2000.

[30] Etnyre, J. and Ghrist, R., "Contact topology and hydrodynamics. II. Solid tori," *Ergodic Theory Dynam. Systems*, vol. 22, no. 3, pp. 819–833, 2002.

[31] Etnyre, J. and Ghrist, R., "Generic hydrodynamic instability of curl eigenfields," *to appear, SIAM J. Appl. Dynamical Systems*, 2003.

[32] Etnyre, J., Ng, L., and Sabloff, J., "Invariants of Legendrian knots and coherent orientations," *J. Symplectic Geom.*, vol. 1, no. 2, pp. 321–367, 2002.

[33] Evans, L. C., *Partial differential equations*, vol. 19 of *Graduate Studies in Mathematics*. Providence, RI: American Mathematical Society, 1998.

[34] Freitas, P., "Closed nodal lines and interior hot spots of the second eigenfunction of the Laplacian on surfaces," *Indiana Univ. Math. J.*, vol. 51, no. 2, pp. 305–316, 2002.

[35] Gauld, D., *Differential topology*, vol. 72 of *Monographs and Textbooks in Pure and Applied Mathematics*. New York: Marcel Dekker Inc., 1982. An introduction.

[36] Geiges, H., "Contact geometry," arXiv:math.SG/0307242.

[37] Ghrist, R., "Steady nonintegrable high-dimensional fluids," *Lett. Math. Phys.*, vol. 55, no. 3, pp. 193–204, 2001. Topological and geometrical methods (Dijon, 2000).

[38] Ghrist, R. and Komendarczyk, R., "Overtwisted energy-minimizing curl eigenfields," *to appear in Nonlinearity.*, 2005.

[39] Gilbarg, D. and Trudinger, N., *Elliptic partial differential equations of second order*, vol. 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Berlin: Springer-Verlag, second ed., 1983.

[40] Gilkey, P., Leahy, J., and Park, J., *Spinors, spectral geometry, and Riemannian submersions*, vol. 40 of *Lecture Notes Series*. Seoul: Seoul National University Research Institute of Mathematics Global Analysis Research Center, 1998.

[41] Giroux, E., "Convexité en topologie de contact," *Comment. Math. Helv.*, vol. 66, no. 4, pp. 637–677, 1991.

[42] Giroux, E., "Structures de contact sur les variétés fibrées en cercles audessus d'une surface," *Comment. Math. Helv.*, vol. 76, no. 2, pp. 218–262, 2001.

[43] Guilfoyle, B., "Einstein metrics adapted to contact structures on 3-manifolds," *preprint arXiv:math.DG/0012027v1*, 2002.

[44] Hatcher, A., *Algebraic topology.* Cambridge: Cambridge University Press, 2002.

[45] Hempel, J., *3-manifolds.* AMS Chelsea Publishing, Providence, RI, 2004. Reprint of the 1976 original.

[46] Hoffmann-Ostenhof, M., Hoffmann-Ostenhof, T., and Nadirashvili, N., "On the nodal line conjecture," in *Advances in differential equations and mathematical physics (Atlanta, GA, 1997)*, vol. 217 of *Contemp. Math.*, pp. 33–48, Providence, RI: Amer. Math. Soc., 1998.

[47] Honda, K., "On the classification of tight contact structures. I," *Geom. Topol.*, vol. 4, pp. 309–368 (electronic), 2000.

[48] Honda, K., Kazez, W., and Matić, G., "Convex decomposition theory," *Int. Math. Res. Not.*, no. 2, pp. 55–88, 2002.

[49] Jost, J., *Riemannian geometry and geometric analysis.* Universitext, Berlin: Springer-Verlag, third ed., 2002.

[50] Kobayashi, S., *Transformation groups in differential geometry.* Classics in Mathematics, Berlin: Springer-Verlag, 1995. Reprint of the 1972 edition.

[51] Kobayashi, S. and Nomizu, K., *Foundations of differential geometry. Vol. I.* Wiley Classics Library, New York: John Wiley & Sons Inc., 1996. Reprint of the 1963 original, A Wiley-Interscience Publication.

[52] Komendarczyk, R., "On the contact geometry of nodal sets, math.dg/0402070," *to appear in Trans. of Amer. Math. Soc.*, 2004.

[53] Lutz, R., "Structures de contact sur les fibrés principaux en cercles de dimension trois," *Ann. Inst. Fourier (Grenoble)*, vol. 27, 1977.

[54] Martinet, J., "Formes de contact sur les variétés de dimension 3," in *Proceedings of Liverpool Singularities Symposium, II (1969/1970)*, (Berlin), pp. 142–163. Lecture Notes in Math., Vol. 209, Springer, 1971.

[55] Melas, A., "On the nodal line of the second eigenfunction of the Laplacian in $\mathbf{R}^2$," *J. Differential Geom.*, vol. 35, no. 1, pp. 255–263, 1992.

[56] Morgan, J. W., "Nonsingular Morse-Smale flows on 3-dimensional manifolds," *Topology*, vol. 18, no. 1, pp. 41–53, 1979.

[57] Neumann, W. and Norbury, P., *Notes on Geometry and 3-Manifolds.* Online notes, 2003.

[58] Nicolaescu, L., "Adiabatic limits of the Seiberg-Witten equations on Seifert manifolds," *Comm. Anal. Geom.*, vol. 6, no. 2, pp. 331–392, 1998.

[59] Payne, L., "Isoperimetric inequalities and their applications," *SIAM Rev.*, vol. 9, pp. 453–488, 1967.

[60] Rauch, J. and Taylor, M., "Potential and scattering theory on wildly perturbed domains," *J. Funct. Anal.*, vol. 18, pp. 27–59, 1975.

[61] Rosenberg, S., *The Laplacian on a Riemannian manifold*, vol. 31 of *London Mathematical Society Student Texts*. Cambridge: Cambridge University Press, 1997. An introduction to analysis on manifolds.

[62] Savo, A., "Eigenvalue estimates and nodal length of eigenfunctions," in *Steps in differential geometry (Debrecen, 2000)*, pp. 295–301, Inst. Math. Inform., Debrecen, 2001.

[63] Schoen, R. and Yau, S.-T., *Lectures on differential geometry*. Conference Proceedings and Lecture Notes in Geometry and Topology, I, Cambridge, MA: International Press, 1994.

[64] Scott, P., "The geometries of 3-manifolds," *Bull. London Math. Soc.*, vol. 15, no. 5, pp. 401–487, 1983.

[65] Takahashi, J., "Collapsing of connected sums and the eigenvalues of the Laplacian," *J. Geom. Phys.*, vol. 40, no. 3-4, pp. 201–208, 2002.

[66] Uhlenbeck, K., "Generic properties of eigenfunctions," *Amer. J. Math.*, vol. 98, no. 4, pp. 1059–1078, 1976.

[67] Yoshida, Z. and Giga, Y., "Remarks on spectra of operator rot," *Math. Z.*, vol. 204, no. 2, pp. 235–245, 1990.