

SYMMETRY, ISOTOPY, AND IRREGULAR COVERS

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Rebecca R. Winarski

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Approved by:

Professor Dan Margalit, Advisor
School of Mathematics
Georgia Institute of Technology

Professor Igor Belegradek
School of Mathematics
Georgia Institute of Technology

Professor Tara Brendle
Department of Mathematics
University of Glasgow

Professor John Etnyre
School of Mathematics
Georgia Institute of Technology

Professor Thang Le
School of Mathematics
Georgia Institute of Technology

Date Approved: March 14, 2014

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SUMMARY

We say that a covering space of the surface S over X has the Birman–Hilden property if the subgroup of the mapping class group of X consisting of mapping classes that have representatives that lift to S embeds in the mapping class group of S modulo the group of deck transformations. We identify one necessary condition and one sufficient condition for when a covering space has this property. We give new explicit examples of irregular branched covers that do not satisfy the necessary condition as well as explicit covers that satisfy the sufficient condition. Our criteria are conditions on simple closed curves, and our proofs use the combinatorial topology of curves on surfaces.

CHAPTER I

INTRODUCTION

A general problem is to understand all (injective) homomorphisms between (finite index subgroups of) mapping class groups of surfaces, see e.g. [1, 2, 13, 14, 19]. Birman and Hilden proved that if the surface \tilde{X} and map $\tilde{X} \rightarrow X$ is a regular branched covering space, there is an embedding of a finite-index subgroup of the mapping class group of X to a quotient of the mapping class group of \tilde{X} by a finite subgroup. The goal of this thesis is to extend their results to certain irregular covers and describe a class of covers where their results do not extend.

Let S be a surface, by which we mean the space obtained from a closed, oriented 2-manifold by removing finitely many points and choosing finitely many points to be distinguished, or *marked*. The mapping class group of S , denoted $\text{Mod}(S)$, is the group of diffeomorphisms of S that leave the set of marked points invariant, up to isotopies that leave the set of marked points invariant.

Throughout the thesis we assume all covering spaces of surfaces are connected, oriented and possibly branched. Fix such a covering map $p : \tilde{X} \rightarrow X$. If the covering space is branched, we regard the branch points as marked points in X . In fact, we will assume that the only marked points in X are the branch points. In particular, the mapping class group of X must fix the set of branch points. Let $\text{LMod}(X)$ be the finite-index subgroup of $\text{Mod}(X)$ consisting of the isotopy classes of the diffeomorphisms of X that lift to diffeomorphisms of \tilde{X} . Let $\text{SMod}(\tilde{X})$ be the subgroup of $\text{Mod}(\tilde{X})$ consisting of the isotopy classes of the fiber preserving, or *symmetric*, diffeomorphisms of \tilde{X} . Note that by isotopy, we mean arbitrary isotopy, not necessarily isotopy through symmetric diffeomorphisms.

When $\chi(\tilde{X}) < 0$, the group of deck transformations $\text{Deck}(p)$ maps injectively to a subgroup of $\text{SMod}(\tilde{X})$ by the Nielsen Realization Theorem. Since isotopies of X lift to isotopies of \tilde{X} , there is a natural surjective homomorphism $\Phi : \text{LMod}(X) \rightarrow \text{SMod}(\tilde{X})/\text{Deck}(p)$ that maps $f \in \text{LMod}(X)$ to the class of a lift of a representative of f in $\text{SMod}(\tilde{X})$. We say that $p : \tilde{X} \rightarrow X$ has the *Birman–Hilden property* if the map Φ is an isomorphism.

In this paper, we investigate the question of which covering spaces have the Birman–Hilden property. The Birman–Hilden property has been studied in various cases:

- *Regular covers:* Birman and Hilden proved that regular covering spaces have the Birman–Hilden property [7, 8]. MacLachlan and Harvey gave a new proof using Teichmüller theory [18].
- *Unbranched covers:* Aramayona, Leininger, and Souto proved that unbranched cover spaces have the Birman–Hilden property [2, Proposition 5].
- *Irregular branched covers:* It is easy to see that the 3-fold irregular branched covering space of a surface of genus g over a sphere with $2g+4$ branch points does not have the Birman–Hilden property [9, 12], and we explain this in Chapter 3. In contrast, Aramayona, Leininger, and Souto found an example of an irregular branched covering space that does have the Birman–Hilden property [2].

In summary, the Birman–Hilden property is understood for all covering spaces except irregular branched covering spaces where there are both examples that have the Birman–Hilden property and examples that do not have the Birman–Hilden property. However, the irregular covers comprise an interesting class of examples. To construct an embedding for $\text{LMod}(X)$ into $\text{SMod}(\tilde{X})$, the group $\text{Deck}(p)$ must be trivial. If $\text{Deck}(p)$ is trivial, then either p is trivial or p is irregular.

Let $\tilde{S} \rightarrow S_0$ be the two-fold regular branched cover over the sphere. Let $M^4 \rightarrow S_0$ be a Lefschetz fibration. If the monodromy of the fibration represents an element of $\text{SMod}(\tilde{S})$, then M is a 2-fold branched cover of $S_0 \times S_0$. Fuller shows that the fact that $\tilde{S} \rightarrow S_0$ has the Birman–Hilden property is necessary to prove this result [12]. Irregular branched covers that have the Birman–Hilden property may correspond to other Lefschetz fibrations $N \rightarrow S_0$, where N is a cover of $S_0 \times S_0$. Such covers would give rise to a richer class of Lefschetz fibrations.

In this thesis we give one new sufficient condition and one new necessary condition for the Birman–Hilden property. We also introduce new explicit examples that satisfy our sufficient condition and new explicit examples that do not satisfy our necessary condition.

Full ramification. We say a (possibly branched) covering space consisting of the surface \tilde{X} and the map $\tilde{X} \rightarrow X$ is *fully ramified* (FR) if there is no unramified preimage in \tilde{X} of a branch point.

Theorem 1.0.1. *Let the surface \tilde{X} and the map $p : \tilde{X} \rightarrow X$ be a finite covering space such that $\chi(\tilde{X}) < 0$. If p is fully ramified, then p has the Birman–Hilden property.*

Theorem 1.0.1 unifies the results of Birman–Hilden and Aramayona–Leininger–Souto into a single proof. We also introduce new ideas which lead to a new class of examples that have the Birman–Hilden property. An example is shown in Figure 2 and a generalized procedure for constructing covering spaces with the Birman–Hilden property is shown in Figure 3.

Farb and Margalit gave a new proof of the Birman–Hilden property for the regular branched covering space induced by the hyperelliptic involution of a surface [11, Theorem 9.2]. We generalize their approach, which translates the Birman–Hilden property, a property of diffeomorphisms, into a property of simple closed curves.

Theorem 1.0.1 can also be proved using Teichmüller theory. We sketch a proof in Chapter 4.2.

Curve lifting property. An essential curve is a curve that is not isotopic to a single point. We say that a covering space of surfaces has the *weak curve lifting property* if the preimage in \tilde{X} of every essential simple closed curve in X has at least one essential connected component.

Proposition 1.0.2. *Let the surface \tilde{X} and the map $p : \tilde{X} \rightarrow X$ be a finite covering space such that $\chi(\tilde{X}) < 0$. If $\tilde{X} \rightarrow X$ has the Birman–Hilden property, then it has the weak curve lifting property.*

We prove Proposition 1.0.2 in Chapter 3.

The weak curve lifting property can be checked algorithmically. Theorem 3.3.2 gives an algorithm for choosing a finite set of simple closed curves sufficient for checking the weak curve lifting property.

As a consequence of Proposition 1.0.2, we have Theorem 1.0.3. We say that a covering space consisting of the surface \tilde{X} and the map $p : \tilde{X} \rightarrow X$ is *exceptional* if one of the following holds:

- X has fewer than 2 branch points,
- the degree p is less than 3, or
- the complement of the branch points in X is a sphere with 3 punctures.

We define simple covers in Chapter 5. Simple covers of degree less than 3 are fully ramified and therefore have the Birman–Hilden property by Theorem 1.0.1.

Theorem 1.0.3. *Let the surface \tilde{X} and the map $p : \tilde{X} \rightarrow X$ be a non-exceptional covering space such that $\chi(\tilde{X}) < 0$. If p is simple, then $\tilde{X} \rightarrow X$ does not have the Birman–Hilden property.*

We prove Theorem 1.0.3 in Chapter 5. The 3-fold irregular branched covering space of a surface of genus g over a sphere with $2g + 4$ branch points described above

is a simple cover, so Theorem 1.0.3 generalizes this example. Work of Birman and Wajnryb uses the 3-fold simple cover of the genus g surface over a sphere with $2g + 4$ branch points to obtain a presentation for $\text{Mod}(S_g)$. Theorem 1.0.3 suggests their work can be extended by using other simple covers.

Curve complex. It is well known that the mapping class group of a surface S is the group of automorphisms of the curve complex $\mathcal{C}(S)$ of S . Let $\tilde{X} \rightarrow X$ be a covering map. It is natural to ask if there is any relationship between $\mathcal{C}(X)$ and $\mathcal{C}(\tilde{X})$. Luo showed that there is an isomorphism between $\mathcal{C}(X)$ and $\mathcal{C}(\tilde{X})$ in only three cases [17]. Rafi and Schleimer showed that there is a quasi-isometric embedding from $\mathcal{C}(X)$ to $\mathcal{C}(\tilde{X})$ when $\tilde{X} \rightarrow X$ is an unbranched cover, but there is not a quasi-isometric embedding from $\mathcal{C}(X)$ to $\mathcal{C}(\tilde{X})$ for general irregular branched covers [20]. Further research could pursue necessary and sufficient conditions for the existence of an quasi-isometric embedding from $\mathcal{C}(X)$ to $\mathcal{C}(\tilde{X})$. In particular: if $\tilde{X} \rightarrow X$ has the Birman–Hilden property, is there a quasi-isometric embedding from $\mathcal{C}(X)$ to $\mathcal{C}(\tilde{X})$?

Ramification number. We say that a covering space of surfaces has the *equal ramification number property* if for every point $x \in X$, all points of $p^{-1}(x)$ have the same ramification number (see Chapter 2 for the definition of ramification number). Regular covers and unbranched covering spaces have the equal ramification number property. A covering space that has the equal ramification number property is fully ramified. Therefore by Theorem 1.0.1, covering spaces that have the equal ramification number property have the Birman–Hilden property. There are some irregular, branched covers that have equal ramification number property.

One might guess that the Birman–Hilden property is equivalent to the equal ramification number property. However, Aramayona, Leininger, and Souto give an example of a covering space that has unramified preimage and the Birman–Hilden property [2, Section 4].

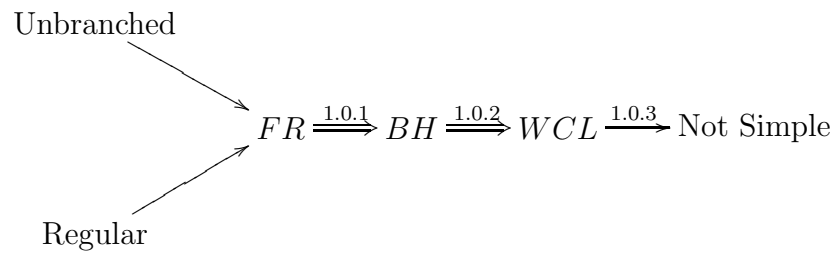


Figure 1: Implications of relevant properties of finite covers. The double arrows represent the main results of the paper. The single arrows are implications proved in corollaries.

CHAPTER II

PRELIMINARIES

In this chapter, we give definitions related to branched covering spaces. We also show a construction of irregular branched covers that are fully ramified, hence have the Birman–Hilden property.

2.1 *Branched covers*

Let X be a surface. A *branched cover* over X is a surface \tilde{X} and a map $p : \tilde{X} \rightarrow X$ such that for some finite set B in X we have:

1. $p|_{\tilde{X} \setminus p^{-1}(B)}$ is a covering map;
2. for each $x \in B$ there exists an open neighborhood $U \subset X$, called a *standard neighborhood*, such that
 - (a) $U \cap B = \{x\}$
 - (b) Each component V_i of $p^{-1}(U)$ is an open disk and contains exactly one point $\tilde{x}_i \in p^{-1}(x)$
 - (c) Each $p|_{V_i \setminus \tilde{x}_i}$ is covering map $V_i \setminus \tilde{x}_i \rightarrow U \setminus x$; we refer to the degree of the cover, m_i , as the *ramification number* of \tilde{x}_i
 - (d) There is at least one \tilde{x}_i such that $m_i > 1$

We say that a preimage \tilde{x}_i of x is *ramified* if its ramification number m_i is greater than 1; otherwise it is *unramified*. Each point in B is called a *branch point*. A branch point may have preimages with different ramification numbers.

Recall that in a branched cover of surfaces $p : \tilde{X} \rightarrow X$, each branch point of X is considered as a marked point. In particular, $\text{Mod}(X)$ is defined relative to the

branch points. Throughout the paper, let X° be the (noncompact) surface that is the complement of the branch points in X . Let the surface \tilde{X} and $p : \tilde{X} \rightarrow X$ be a branched covering space of surfaces. Then we will define $\tilde{X}^\circ = p^{-1}(X^\circ)$.

A *deck transformation* of a branched cover of surfaces $p : \tilde{X} \rightarrow X$ is a homeomorphism of \tilde{X} that restricts to a deck transformation of the covering space consisting of the surface \tilde{X} and the map $\tilde{X}^\circ \rightarrow X^\circ$. A covering space consisting of the surface \tilde{X} and the map $\tilde{X} \rightarrow X$ is *regular* if the group of deck transformations acts transitively on the fibers of each point in X , and is *irregular* otherwise.

2.1.1 Simple closed curves and branched covers

Let the surface \tilde{X} and the map $p : \tilde{X} \rightarrow X$ be a covering space with branch points $B \subset X$. A *simple closed curve* in X is the image of a continuous injective map $S^1 \rightarrow X^\circ \subseteq X$. In particular, there is a bijective correspondence between simple closed curves in X and simple closed curves in X° . There is not a bijective correspondence between simple closed curves in \tilde{X} and simple closed curves in \tilde{X}° because simple closed curves in \tilde{X} are allowed to contain points of $p^{-1}(B)$. As a result, curves that are isotopic in X are isotopic in X° , but curves that are isotopic in \tilde{X} might not be isotopic in \tilde{X}° .

A *multicurve* in a surface S is a set of pairwise disjoint simple closed curves $\{\alpha_1, \dots, \alpha_k\}$ in S . We say that two multicurves $\alpha = \{\alpha_1, \dots, \alpha_k\}$ and $\beta = \{\beta_1, \dots, \beta_l\}$ are isotopic if $k = l$, and they are homotopic through k -component multicurves.

2.2 Constructing irregular branched covers

One method of constructing irregular covering spaces is by cutting surfaces and sewing their resulting boundary components as follows. Let S_g^b denote a surface of genus g with b boundary components. Figure 2 shows a 6-fold covering space of a torus with two branch points by S_4^0 . We construct the covering map in 3 steps:

1. Cut S_4^0 .

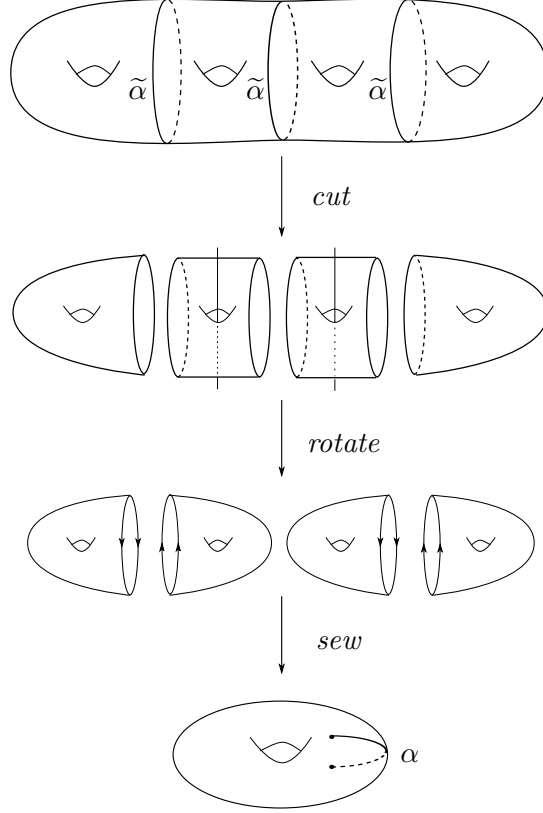


Figure 2: The 6-fold branched, irregular covering of the twice marked torus comes from cutting along the curves marked and identifying along boundaries

2. Map each S_1^d to S_1^1 by any covering map.
3. Sew the boundary of each S_1^1 .

The covering map is constructed by cutting S_4^0 along the curves $\tilde{\alpha}$ into four tori with one or two boundary components as shown in Figure 2. Then take a quotient by rotation of each S_1^d by $\frac{2\pi}{d}$. We then have four disjoint copies of S_1^1 . There is a map from the four disjoint copies of S_1^1 to the torus with two branch points induced by identifying the top and bottom of the boundary component of each S_1^1 as indicated by the arrows. This procedure constructs a well-defined 6-fold cover of S_4^0 over the torus with two branch points.

All three preimages of each branch point are ramified, so the covering space is fully ramified. Therefore by Theorem 1.0.1 the covering space described above has

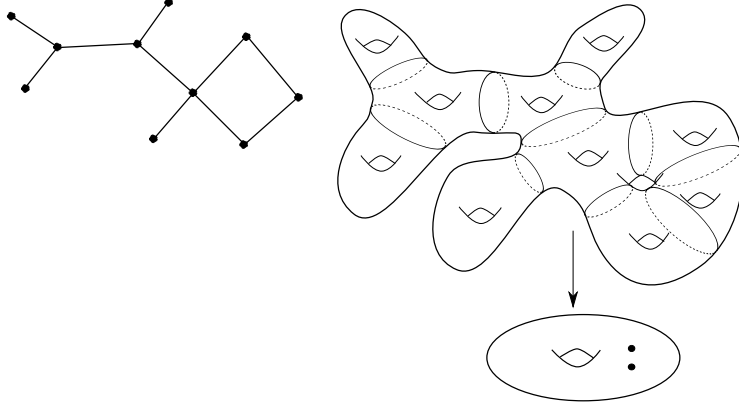


Figure 3: Constructing an irregular branched cover from a graph.

the Birman–Hilden property.

One can generalize the cut-and-identify construction in Figure 2 to make a family of covering spaces of the torus with two branch points as follows. In fact, we can make a covering space of any even degree.

Theorem 2.2.1. *Let G be a finite graph. Let X be a torus with two branch points. One may construct a finite covering space of surfaces $\tilde{X} \rightarrow X$ corresponding to G that has the Birman–Hilden property.*

Proof. We first construct \tilde{X} . For each vertex of G of degree d , start with one copy of S_1^d . Then identify the boundaries of two components if and only if there is an edge between the corresponding vertices of G . Figure 3 shows the surface constructed from a graph.

To create the covering map, we repeat the procedure as above. First cut \tilde{X} at each corresponding edge of G . For every integer $d \geq 1$, take the quotient by rotation of S_1^d by $2\pi/d$ of each S_1^d to obtain a copy of S_1^1 , analogously to the example in Figure 2 above. Then we have a disjoint union of components each of which is S_1^1 . For each torus, identify the top and bottom of the boundary component, creating two fixed points. We then have a cover of the torus with two branch points by a disjoint union of tori with two marked points.

As above, one can show that $\tilde{X} \rightarrow X$ is a branched cover. All preimages of both branch points have ramification number 2. Therefore the covering space is fully ramified.

By Theorem 1.0.1, $\tilde{X} \rightarrow X$ has the Birman–Hilden property. \square

By choosing G to be a planar graph with no automorphisms, we construct $\tilde{X} \rightarrow X$ to be an irregular branched cover of surfaces.

2.3 Applications

The work of Birman and Hilden has been used to better understand the mapping class group and low-dimensional manifolds.

2.3.1 Representations of the braid group

The braid group, which we denote B_k , is the mapping class group of a disk with n punctures. The braid group is a classical group and is better understood than the mapping class group of a surface of higher genus.

A *representation* of a group G is a homomorphism from G to a group of matrices over a commutative ring. A representation is said to be *faithful* if it is injective. If there is a faithful representation of G , then G is said to be *linear*.

Let $S_{g,k}$ be a surface of genus g with k punctures. One of the biggest open questions regarding mapping class groups is whether or not $\text{Mod}(S_{g,k})$ is linear over \mathbb{R} . The answer is known for certain cases. In particular, it is a basic fact of mapping class groups that $\text{Mod}(S_{1,0}) = \text{Mod}(S_{1,1}) = \text{SL}_2(\mathbb{Z})$. That is, the mapping class groups of the torus and the torus with one branch point are linear over \mathbb{R} .

Theorem 2.3.1 (Bachmuth [3], Bigelow [5], Krammer [16]). *The braid group, B_k , is linear over \mathbb{R} for all k .*

Theorem 2.3.1 identifies additional structure of the mapping class group. Using

Theorem 2.3.1 together with the work of Birman and Hilden [7][Theorem 1], Bigelow–Budney and Korkmaz were able to show the Theorem 2.3.2.

Bigelow and Budney used the hyperelliptic involution to construct a representation of $\text{Mod}(S_2)$. They start with \mathcal{K} , a faithful representation of $\text{Mod}(S_{0,6})$. Let $S_2 \rightarrow S_{0,6}$ be the covering space induced by the hyperelliptic involution σ of S_2 . The representation κ extends to a representation of S_2 where the kernel is σ . We will also refer to this representation as κ . Additionally, the action of $\text{Mod}(S_2)$ on $H_1(S_2, \mathbb{Z})$ induces a representation ρ of $\text{Mod}(S_2)$. The representation ρ maps σ to $-I$. Thus the direct sum $\kappa \oplus \rho$ is a faithful representation of $\text{Mod}(S_2)$.

Theorem 2.3.2 (Bigelow–Budney [6], Korkmaz [15]). *Let S_2 be a surface of genus 2. Then $\text{Mod}(S_2)$ is linear.*

Let $\sigma : S_2 \rightarrow S_{0,6}$ be the double branched cover given by a hyperelliptic involution. For this covering space $\text{SMod}(S_2) = \text{Mod}(S_2)$. By the work of Birman and Hilden, it is known that there is an isomorphism between B_6 and $\text{Mod}(S_2)/\text{Deck}(\sigma)$. Therefore $\text{Mod}(S_2)$ is linear. For a surface of genus $g \geq 3$, it is only known that $\text{SMod}(S_g)$ is linear.

Due to Theorem 1.0.1 one may find additional covering spaces consisting of a surface \tilde{X} and a map $p : \tilde{X} \rightarrow S_{0,k}$ that have $\text{SMod}(\tilde{X})/\text{Deck}(p) \cong B_k$. However, we are particularly interested in covering spaces where $\text{SMod}(\tilde{X}) = \text{Mod}(\tilde{X})$, but not much is known about this problem. In general, $\text{SMod}(\tilde{X})$ is infinite index in $\text{Mod}(\tilde{X})$.

2.3.2 Injections of mapping class groups

In order to understand the structure of the mapping class group, it is useful to study homomorphisms between (subgroups of) mapping class groups. Covering spaces that have the Birman–Hilden property naturally correspond to injective homomorphisms between subgroups of mapping class groups. Aramayona, Leininger, and Souto use

covering spaces that have the Birman–Hilden property to find injective homomorphisms [2, Theorem 1].

Theorem 2.3.3 (Aramayona, Leininger, Souto). *For every $g \geq 0$, there exists $g' > g$ and an injective homomorphism: $\phi : \text{Mod}(S_{g,0}) \rightarrow \text{Mod}(S_{g',0})$.*

In fact, Aramayona and Souto prove that with sufficient restrictions on genus, all homomorphisms between mapping class groups of surfaces X, S , $\text{Mod}(X) \rightarrow \text{Mod}(S)$, are induced by embeddings of $X \rightarrow S$ [1, Theorem 1].

Irregular branched covers with the Birman–Hilden property induce embeddings of braid groups, and mapping class groups of surfaces with branch points in to closed surfaces.

CHAPTER III

THE WEAK CURVE LIFTING PROPERTY

In this chapter, we prove Proposition 1.0.2, which states that the Birman–Hilden property implies the weak curve lifting property. Additionally we prove that there is an algorithm to check the weak curve lifting property.

3.1 *Proof of Proposition 1.0.2*

First we prove that covering spaces with the Birman–Hilden property have the weak curve lifting property.

Proof of Proposition 1.0.2. Suppose that the covering space consisting of the surface \tilde{X} and the map $p : \tilde{X} \rightarrow X$ does not have the weak curve lifting property, that is, there exists an essential simple closed curve γ in X such that $p^{-1}(\gamma)$ has no essential connected component. We want to show that $\Phi : \text{LMod}(X) \rightarrow \text{SMod}(\tilde{X})/\text{Deck}(p)$ has nontrivial kernel.

Let φ be a diffeomorphism of X in the class of the Dehn twist about γ . Since $\text{LMod}(X)$ has finite index in $\text{Mod}(X)$, there is some integer k such that the class of φ^k is in $\text{LMod}(X)$. The isotopy class of φ^k is a nontrivial mapping class because it is a power of a Dehn twist about an essential simple closed curve [11, Proposition 3.1].

Since φ^k is supported on a neighborhood of γ , there is a lift $\tilde{\varphi}^k : \tilde{X} \rightarrow \tilde{X}$ of φ^k that is supported on a neighborhood of the components of $p^{-1}(\gamma)$, each of which is an annulus. Therefore $\tilde{\varphi}^k$ is a composition of Dehn twists in these annuli. Since all components of $p^{-1}(\gamma)$ are inessential, the Dehn twist around each component of $p^{-1}(\gamma)$ is isotopic to $id_{\tilde{X}}$. Therefore $\tilde{\varphi}^k$ is isotopic to $id_{\tilde{X}}$, and hence its class is the trivial mapping class. Since φ^k does not represent the trivial mapping class but is in

the kernel of Φ , p does not have the Birman–Hilden property. \square

3.2 A cover without the Birman–Hilden property

Consider the 3-fold simple branched covering space of surfaces $p : \tilde{X} \rightarrow X$ where \tilde{X} is a closed surface of genus g and X is a sphere with $2g+4$ branch points; (this covering space is unique up to isomorphism). The example where $g = 3$ is shown in Figure 4. The preimages of the branch points are shown but, as usual, are not thought of as marked points in \tilde{X} .

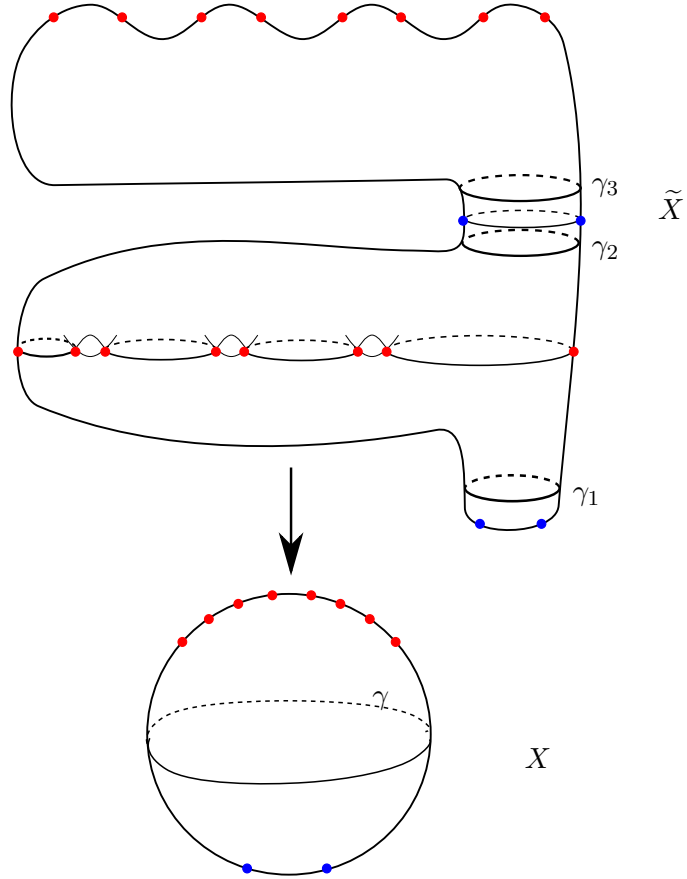


Figure 4: An irregular branched cover that does not have the Birman–Hilden property.

We can choose p so that the preimage of the simple closed curve γ in Figure 4 is the union of $\tilde{\gamma}_1, \tilde{\gamma}_2$, and $\tilde{\gamma}_3$ in \tilde{X} . Since each $\tilde{\gamma}_i$ is inessential, the covering map

p does not have the weak curve lifting property, and therefore does not have the Birman–Hilden property by Proposition 1.0.2.

The covering space described is one example of a simple cover that does not have the Birman–Hilden property. See Fuller’s paper [12] for more discussion of this example. Berstein and Edmonds [4] generalize this example to show that branched simple covers of the sphere do not have the Birman–Hilden property. Theorem 1.0.3, proven in Chapter 5 below, says that no simple covers with degree at least 3 and at least two branch points have the weak curve lifting property. Therefore no such covering spaces have the Birman–Hilden property, further generalizing Berstein and Edmond’s result.

3.3 *Checking the weak curve lifting property*

The usefulness of the weak curve lifting property lies in the fact that given a covering space consisting of the surface \tilde{X} and the map $\tilde{X} \rightarrow X$, one only needs to check a finite number of curves in X in order to determine whether or not it holds. Our next goal is to show that there is an algorithm to select a sufficient set of simple closed curves, as we will prove in Theorem 3.3.2 below. We begin by finding coset representatives of $\text{LMod}(X)$ for unbranched covers $\tilde{X} \rightarrow X$ in Proposition 3.3.1.

Proposition 3.3.1. *Let $p : \tilde{X} \rightarrow X$ be an n -fold unbranched covering space of surfaces. There is an algorithm to find coset representatives of $\text{LMod}(X)$ in $\text{Mod}(X)$.*

Proof. A diffeomorphism of a surface S lifts if and only if it preserves the subgroup of $\pi_1(S, s_0)$ that lift for some s_0 . Therefore we fix a basepoint x_0 in X and consider it as a marked point, and choose $\tilde{x}_0 \in p^{-1}(x_0)$. The mapping class group relative to the base point, denoted $\text{Mod}(X, x_0)$, is the group of isotopy classes of diffeomorphisms of X that fix x_0 . We define $\text{LMod}(X, x_0)$ to be the subgroup of $\text{Mod}(X, x_0)$ consisting of mapping classes that have representatives that lift to $\text{Mod}(\tilde{X}, \tilde{x}_0)$. We

first find coset representatives of $\text{LMod}(X, x_0)$ in $\text{Mod}(X, x_0)$ by considering the action of $\text{Mod}(X, x_0)$ on $\pi_1(X, x_0)$, and to do this we must fix the basepoint. Then we forget the basepoint and the cosets of $\text{LMod}(X, x_0)$ in $\text{Mod}(X, x_0)$ map to cosets of $\text{LMod}(X)$ in $\text{Mod}(X)$.

Let $H_0 = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Then H_0 is an index n subgroup of $\pi_1(X, x_0)$. There is an action of $\text{Mod}(X, x_0)$ on the set of index n subgroups of $\pi_1(X, x_0)$; denote the orbit of H_0 by \mathcal{H} . The action of $\text{Mod}(X, x_0)$ on the subgroups of $\pi_1(X, x_0)$ induces a map $P : \text{Mod}(X, x_0) \rightarrow \Sigma_{\mathcal{H}}$.

We have two facts:

1. $\text{LMod}(X, x_0)$ is the stabilizer of H_0 in $\text{Mod}(X, x_0)$.
2. \mathcal{H} is finite.

The first follows from the lifting criterion for covering spaces and the fact that all isotopies of X lift to isotopies of \tilde{X} . The second follows from the fact that there are finitely many index n subgroups of any finitely generated group.

To find coset representatives of $\text{LMod}(X, x_0)$, it suffices to find a finite set $F \subset \text{Mod}(X, x_0)$ such that for all $H \in \mathcal{H}$ there exists $f \in F$ such that $f \cdot H_0 = H$.

Let $\{g_i\}$ be any finite generating set of $\text{Mod}(X, x_0)$. The set $\{P(g_i)\}$ generates a subgroup of $\Sigma_{\mathcal{H}}$. Because $\Sigma_{\mathcal{H}}$ is finite, there is an element in $\text{Im}(P)$ whose word length L with respect to $\{P(g_i)\}$ is maximal. Let F be the finite set of elements in $\text{Mod}(X, x_0)$ of word length at most L with respect to $\{g_i\}$.

By construction $P(F) = \text{Im}(P)$. In other words, any permutation of elements of \mathcal{H} obtained by the action of $\text{Mod}(X, x_0)$ has a preimage under P in F . Therefore, for all $H \in \mathcal{H}$, there is $f \in F$ such that $f \cdot H_0 = H$.

The forgetful homomorphism $\mathcal{F} : \text{Mod}(X, x_0) \rightarrow \text{Mod}(X)$ is surjective. Additionally, any diffeomorphism of X that lifts to \tilde{X} can be modified by isotopy to fix x_0 . Since every isotopy of X lifts to an isotopy of \tilde{X} , the restriction of \mathcal{F} to

$\text{LMod}(X, x_0)$ surjects onto $\text{LMod}(X)$. The image $\{\mathcal{F}(f_j)\}$ of the coset representatives of $\text{LMod}(X, x_0)$ in $\text{Mod}(X, x_0)$ is a set of coset representatives of the subgroup $\text{LMod}(X)$ in $\text{Mod}(X)$. \square

We recall the notation that X° is the surface X with the branch points removed and $\tilde{X}^\circ = p^{-1}(X^\circ)$. Denote the set of isotopy classes of simple closed curves in a surface S by $\mathcal{G}(S)$. Denote the set of isotopy classes of multicurves in S by $\mathcal{M}(S)$.

Theorem 3.3.2. *Let the surface \tilde{X} and the map $p : \tilde{X} \rightarrow X$ be a finite covering space of surfaces such that $\chi(\tilde{X}) < 0$. There is an algorithm to check whether or not p has the weak curve lifting property.*

Proof. The embedding $X^\circ \rightarrow X$ takes simple closed curves in X° to simple closed curves in X . Then there is a sequence of maps as follows; we denote their composition by Ψ :

$$\Psi : \mathcal{G}(X) \xrightarrow{\cong} \mathcal{G}(X^\circ) \xrightarrow{\text{preimage}} \mathcal{M}(\tilde{X}^\circ) \xrightarrow{\text{fill in}} \mathcal{M}(\tilde{X})$$

The second map takes a simple closed curve in X° to its preimage in \tilde{X}° , and the third map takes multicurves in the one induced by the inclusion $\tilde{X}^\circ \rightarrow \tilde{X}$.

If $c, c' \in \mathcal{G}(X)$ differ by an element of $\text{LMod}(X)$, then $\Psi(c)$ and $\Psi(c')$ have the same number of essential and inessential components. Therefore to check the weak curve lifting property, it is sufficient to compute $\Psi(c)$ for a set of representatives of $\mathcal{G}(X)/\text{LMod}(X)$.

It follows from the classification of surfaces that the set $\mathcal{G}(X)/\text{Mod}(X)$ is finite [11, Section 1.3]. Let $\{\gamma_i\}$ be a set of representatives of the quotient.

By Proposition 3.3.1, we can choose a finite set of coset representatives $\{h_j\}$ of $\text{LMod}(X^\circ)$ in $\text{Mod}(X^\circ)$. Therefore it is sufficient to check the weak curve lifting property by calculating the image under Ψ of the isotopy classes of simple closed curves in the set $\{h_j(\gamma_i)\}$. \square

CHAPTER IV

PROOF OF THEOREM 1.0.1

In this chapter we will prove Theorem 1.0.1. As an intermediate step we define the isotopy projection property and prove it implies the Birman–Hilden property.

4.1 Isotopy Projection Property

We say a covering space consisting of the surface \tilde{X} and the map $p : \tilde{X} \rightarrow X$ has the *isotopy projection property* if the following holds:

For any pair of simple closed curves α and β in X such that $p^{-1}(\alpha)$ and $p^{-1}(\beta)$ are isotopic multicurves, we have that α and β are isotopic.

When convenient, we will also say that the covering map p has the isotopy projection property if the covering space consisting of \tilde{X} and $p : \tilde{X} \rightarrow X$ has the isotopy projection property.

Recall that branch points in X are marked points, but their preimages in \tilde{X} are not marked. Therefore homotopies in \tilde{X} may not project to homotopies in X that fix the set of marked points.

We will use the fact that the isotopy projection property implies the Birman–Hilden property to prove Theorem 1.0.1.

Proposition 4.1.1. *Let the surface \tilde{X} and the map $p : \tilde{X} \rightarrow X$ be a finite covering space such that $\chi(\tilde{X}) < 0$. The covering map p has the isotopy projection property if and only if it has the Birman–Hilden property.*

Below we prove the forward direction of Proposition 4.1.1, namely, that the isotopy projection property implies the Birman–Hilden property. The other direction is left as an exercise, and is not used in this paper.

4.1.1 Bigons

A *bigon* is a cell structure on a disk with one 2-cell, two 1-cells, and two 0-cells. A bigon in a surface X is an embedding of the cell complex into X . We call the 1-cells *edges* of the bigon and the 0-cells *vertices*. We say a bigon is bounded by two curves α and β if the edges of the bigon are contained in α and β respectively. Let B be a bigon in X bounded by α and β . We let B_α and B_β be the edges of B contained in α and β . An *innermost bigon* is a bigon bounded by α and β that does not contain any other bigons bounded by α and β .

4.1.2 Finite order elements

In order to prove Proposition 4.1.1, we need the following lemma about finite order elements.

Proposition 4.1.2. *Let the surface \tilde{X} and the map $p : \tilde{X} \rightarrow X$ be a finite covering space such that $\chi(\tilde{X}) < 0$. The kernel of $\Phi : \text{LMod}(X) \rightarrow \text{SMod}(\tilde{X})/\text{Deck}(p)$ is torsion free.*

Proof. Let f be a mapping class of order k in the kernel of Φ . By a classical theorem of Fenchel and Nielsen [11, Theorem 7.1], there is a diffeomorphism φ of order k that represents f . There is a lift $\tilde{\varphi} : \tilde{X} \rightarrow \tilde{X}$ of the diffeomorphism φ that is isotopic to the identity in \tilde{X} . Then φ^k is the identity on X and lifts to $\tilde{\varphi}^k$, which is a deck transformation. Thus $\tilde{\varphi}$ is a finite-order diffeomorphism of \tilde{X} . Since $\chi(\tilde{X}) < 0$, the only finite-order diffeomorphism of \tilde{X} that is isotopic to $id_{\tilde{X}}$ is $id_{\tilde{X}}$ itself. Therefore, $\tilde{\varphi}$ is exactly $id_{\tilde{X}}$, so φ is exactly id_X . \square

4.1.3 Finishing the proof

We are now ready to prove that the isotopy projection property implies the Birman–Hilden property.

Theorem 4.1.3 (Casson–Bleiler). *Let S be a surface, with $\chi(S) < 0$. Let φ be a diffeomorphism of S . If for every essential simple closed curve γ , there is some $k > 0$ such that $\phi^k(\gamma)$ is isotopic to γ , then φ is finite order.*

A proof is given in [10].

Proof of Proposition 4.1.1. Suppose $p : \tilde{X} \rightarrow X$ has the isotopy projection property. We want to show that the map $\Phi : \text{LMod}(X) \rightarrow \text{SMod}(\tilde{X})/\text{Deck}(p)$ is injective.

Fix an essential simple closed curve γ in X . Let $\tilde{\gamma} = p^{-1}(\gamma)$ in \tilde{X} .

Let φ be a diffeomorphism of X whose class is in $\ker \Phi$. We need to show that φ is isotopic to id_X . There is a lift $\tilde{\varphi} : \tilde{X} \rightarrow \tilde{X}$ of φ that is isotopic to the identity.

Since $\tilde{\varphi}$ is isotopic to $\text{id}_{\tilde{X}}$, the multicurve $\tilde{\varphi}(\tilde{\gamma})$ is isotopic to $\tilde{\gamma}$. Because p has the isotopy projection property, γ is isotopic to $\varphi(\gamma)$. Since this is true for any essential simple closed curve γ , Theorem 4.1.3 implies φ is finite order.

By Proposition 4.1.2, φ is isotopic to the identity. □

4.2 Full Ramification

Now we prove Theorem 1.0.1. We recall that a covering space is fully ramified if it has no unramified preimages of branch points. Theorem 1.0.1 states that covering spaces that are fully ramified have the Birman–Hilden property.

To prove Theorem 1.0.1, we prove that coverings spaces of surfaces that are fully ramified have the isotopy projection property.

We fix the following notation: the surface \tilde{X} and the map $p : \tilde{X} \rightarrow X$ is a finite covering space such that $\chi(\tilde{X}) < 0$, α and β are transverse simple closed curves in X , and $\tilde{\alpha}$ and $\tilde{\beta}$ are the multicurves $p^{-1}(\alpha)$ and $p^{-1}(\beta)$ in \tilde{X} . Also, \bar{X} is the surface obtained from X by forgetting that the branch points are marked.

Before we prove Theorem 1.0.1, we prove Lemmas 4.2.1 and 4.2.2, which roughly state that bigons in \tilde{X} project to bigons in X and disjoint isotopic curves in \tilde{X} project to disjoint isotopic curves in X .

We recall that simple closed curves are isotopic in X if and only if they are isotopic in X° but simple closed curves in \tilde{X} are not necessarily isotopic in \tilde{X}° .

We will repeatedly use the following basic fact about covering spaces: if C is any subset of X , then $p|_{p^{-1}(C)}$ is a covering map.

Lemma 4.2.1. *Let \tilde{B} be an innermost bigon bounded by $\tilde{\alpha}$ and $\tilde{\beta}$ in \tilde{X} . Then $p(\tilde{B})$ is an innermost bigon bounded by α and β in \bar{X} . Additionally, \tilde{B} does not contain any ramified points.*

Lemma 4.2.1 implies in particular that if a covering space is fully ramified then innermost bigons in \tilde{X} do not contain preimages of branch points.

Proof. We first show that $p|_{\partial\tilde{B}}$ is injective. We deal with vertices, then edges. Let \tilde{y} and \tilde{z} be the two vertices of \tilde{B} and let $y = p(\tilde{y})$ and $z = p(\tilde{z})$. Because $\tilde{\alpha}$ and $\tilde{\beta}$ have opposite signs of intersection at \tilde{y} and \tilde{z} (after choosing arbitrary orientations of α and β), y and z are distinct points.

Now we consider edges of $p|_{\partial\tilde{B}}$. Since p is a covering map, $p|_{\tilde{B}_\alpha}$ is an immersion. Since \tilde{B} is innermost, the interior of \tilde{B}_α maps to $\alpha \setminus \beta$. And since $y \neq z$, $p|_{\tilde{B}_\alpha}$ is injective. By symmetry, $p|_{\tilde{B}_\beta}$ is injective.

We now show that $p(\tilde{B})$ is an innermost bigon in X . Since $p|_{\partial\tilde{B}}$ is injective, $p(\partial\tilde{B})$ is a simple closed curve formed from one arc of α and one arc of β . Moreover the arc of α only intersects β at its endpoints and vice versa. The image of the disk \tilde{B} under p defines a nullhomotopy of $p(\partial\tilde{B})$. Also, the interior $p(\text{int}\tilde{B})$ is disjoint from $\alpha \cup \beta$. It follows that $p(\partial\tilde{B})$ is the boundary of an innermost bigon B bounded by $\alpha \cup \beta$ and that $B = p(\tilde{B})$.

It remains to show that that \tilde{B} does not contain any ramified points. First we show that $p' = p|_{\tilde{B}}$ is a covering map of $B = p(\tilde{B})$. It suffices to show \tilde{B} is a connected component of $p^{-1}(B)$. The interior of B lies only to one side of ∂B and is disjoint from $\alpha \cup \beta$, therefore $p^{-1}(p(\tilde{B})) \setminus (\tilde{\alpha} \cup \tilde{\beta})$ lies only to one side of $\partial\tilde{B}$ and is disjoint from

$\tilde{\alpha} \cup \tilde{\beta}$. It follows that \tilde{B} is the entire connected component of $p^{-1}(B) = p^{-1}(p(\tilde{B}))$ containing \tilde{B} .

The restriction of p' to $\partial\tilde{B}$ is a covering map of ∂B . Because $p|_{\partial\tilde{B}} = p'|_{\partial\tilde{B}}$ is injective, the degree of p' is 1. Since every nontrivially branched cover has degree greater than 1, $p|_{\tilde{B}}$ is unbranched, hence \tilde{B} does not contain any ramified points. \square

Lemma 4.2.2. *If $\tilde{\alpha}$ and $\tilde{\beta}$ are isotopic and disjoint in \tilde{X} , then α and β are isotopic in X .*

Proof. Any isotopy between $\tilde{\alpha}$ and $\tilde{\beta}$ in \tilde{X} projects to a homotopy between α and β in \bar{X} . Since α and β are homotopic in \bar{X} and disjoint, they bound an annulus in \bar{X} . Therefore they bound an annulus A in X , though A might contain branch points. Our goal is to show that A does not contain any branch points.

Because A is an annulus in X , α and β divide X into two closed subsurfaces, A and Z with $\partial A = \partial Z = \alpha \cup \beta$ sharing both boundary components α and β . Let $\tilde{A} = p^{-1}(A)$ and $\tilde{Z} = p^{-1}(Z)$.

Claim: Either \tilde{A} or \tilde{Z} is a disjoint union of annuli in \tilde{X} .

Assuming the claim, suppose \tilde{A} is a disjoint union of annuli. The restriction of p to \tilde{A} is a (possibly branched) covering space. The Euler characteristic $\chi(\tilde{A})$ is 0. By the multiplicativity of Euler characteristic, $\chi(p(\tilde{A})) = \chi(A)$ is also 0. By the Riemann–Hurwitz formula [11, Section 7.2.2], an annulus with branch points has negative Euler characteristic, therefore A has no branch points.

Proof of claim. We will call the components of \tilde{A} and \tilde{Z} regions of \tilde{X} . Each region in \tilde{X} has at least one boundary component that is a component of $\tilde{\alpha}$ and at least one boundary component that is a component of $\tilde{\beta}$. Similarly, each component of $\tilde{\alpha} \cup \tilde{\beta}$ is a boundary component of exactly one \tilde{A} -region and exactly one \tilde{Z} -region. Because $\tilde{\alpha}$ and $\tilde{\beta}$ are isotopic, either the \tilde{A} -region or \tilde{Z} -region bounded by any component of $\tilde{\alpha} \cup \tilde{\beta}$ is an annulus.

Suppose for every component of $\tilde{\alpha} \cup \tilde{\beta}$, exactly one of the adjacent regions is an annulus. Then there is one annular region and one non-annular region adjacent to each component of $\tilde{\alpha} \cup \tilde{\beta}$. The annular region always occurs on the same side of a component of $\tilde{\alpha}$ and the same side of a component of $\tilde{\beta}$, according to an arbitrary choice on the orientation of α and β respectively. Therefore either all of the components of \tilde{A} or all of the components of \tilde{Z} are annuli and the claim is proved.

Suppose now there is some component of $\tilde{\alpha} \cup \tilde{\beta}$ such that both the adjacent regions are annuli. We say two such annular regions are *adjacent*. The adjacency of annular regions generates an equivalence relation of annular regions of \tilde{X} . Let $\{\mathcal{F}_i\}$ be the set of equivalence classes of annular regions of \tilde{X} . Where $|\mathcal{F}_i| \geq 2$, there exist two annuli in each \mathcal{F}_i that are adjacent to only one annulus in \mathcal{F}_i ; call these *end annuli*. Where $|\mathcal{F}_i| = 1$, \mathcal{F}_i consists of a single annulus, which is also an end annulus.

Because $\tilde{\alpha}$ and $\tilde{\beta}$ are isotopic in \tilde{X} , there are the same number of components of $\tilde{\alpha}$ and $\tilde{\beta}$ in the union of the elements of \mathcal{F}_i for any i . It follows that $|\mathcal{F}_i|$ is odd. If the end annuli of some \mathcal{F}_i are in \tilde{A} , then the adjacent non-annular regions are in \tilde{Z} . All annular regions in \tilde{X} are in some class \mathcal{F}_i . By replacing annular regions in the case above with the \mathcal{F}_i , we see that the \mathcal{F}_i alternate with the non-annular regions in \tilde{Z} . Then end annuli of all \mathcal{F}_i are in \tilde{A} , and the claim is proved. \square

We may now prove Theorem 1.0.1.

Proof of Theorem 1.0.1. By Proposition 4.1.1, it is enough to show that p has the isotopy projection property. Let α and β be transverse simple closed curves in X . We need to show that if $\tilde{\alpha} = p^{-1}(\alpha)$ and $\tilde{\beta} = p^{-1}(\beta)$ are isotopic, then α and β are isotopic. If $\tilde{\alpha}$ and $\tilde{\beta}$ are not disjoint, $\tilde{\alpha}$ and $\tilde{\beta}$ bound an innermost bigon \tilde{B} . By assumption, there are no preimages of branch points with unramified preimage. Therefore Lemma 4.2.1 says that $p(\tilde{B})$ is an innermost bigon in $\alpha \cup \beta$ in X and it does not contain any branch points. There is then an isotopy of α in X that reduces

the number of intersections of α and β (by passing α through $p(\tilde{B})$). We continue reducing intersections of α and β by isotopy of α in X until α and β are disjoint, hence $\tilde{\alpha}$ and $\tilde{\beta}$ are also disjoint. Once $\tilde{\alpha}$ and $\tilde{\beta}$ are disjoint, isotopic multicurves, Lemma 4.2.2 implies that α and β are isotopic. \square

Theorem 1.0.1 can also be proved using quadratic differentials. We sketch the proof. Because the branch points are treated as marked points, a quadratic differential on X may have a pole at a branch point. However, quadratic differentials on \tilde{X} are not allowed to have any poles. When the preimages of all branch points are ramified, a quadratic differential on X lifts to a quadratic differential on \tilde{X} [11, Section 11.3.3]. A given surface S (and hyperbolic structure) and a quadratic differential on S determine a Teichmüller geodesic in the Teichmüller space of S . Since all quadratic differentials on X lift to quadratic differentials on \tilde{X} , Teichmüller geodesics in Teichmüller space of X lift to Teichmüller geodesics in the Teichmüller space of \tilde{X} . Therefore the Teichmüller space of X isometrically embeds in the Teichmüller space of \tilde{X} , which is equivalent to the Birman–Hilden property by the work of MacLachlan and Harvey [18].

CHAPTER V

SIMPLE COVERS

A *simple cover* of surfaces is a cover such that each branch point has one preimage with ramification number 2 and all other preimages are unramified. If the degree of a simple cover is n , then each branch point has $n - 1$ preimages.

5.1 Colorings

We first develop a consistent way of labeling preimages of branch points. The ideas in this chapter are inspired by the work of Bernstein and Edmonds [4].

Let the surface \tilde{X} and the map $\tilde{X} \rightarrow X$ be an n -fold branched covering space and fix some orientation of X . Fix a base point in X that is not a branch point; call it x_0 . Label the points in $p^{-1}(x_0)$ with the numbers $1, \dots, n$ and fix throughout. Let x_1, \dots, x_k be the branch points of the cover. Fix a set of simple arcs $\gamma_1, \dots, \gamma_k$ in X that are disjoint on their interiors and such that γ_i connects x_0 and x_i . Such a set of arcs is called a *Hurwitz system*. Let $\rho : \pi_1(X^\circ, x_0) \rightarrow \Sigma_n$ be the monodromy homomorphism of the unbranched cover $\tilde{X}^\circ \rightarrow X^\circ$ with respect to our labeling of the points in $p^{-1}(x_0)$. Let $c_i \in \pi_1(X^\circ, x_0)$ be the homotopy class of the counterclockwise-oriented boundary of a neighborhood of $\gamma_i \subset X$. We define the *coloring* of x_i , denoted $\text{color}(x_i)$, as $\text{color}(x_i) = \rho(c_i)$.

Alternatively, we can think of coloring the preimages of the branch point x_i instead of coloring x_i itself. Let $x_i^1, \dots, x_i^{\ell_i}$ be the preimages of x_i under p . Each arc of $p^{-1}(\gamma_i)$ connects some point of $p^{-1}(x_0)$ to some x_i^r . The arcs of $p^{-1}(\gamma_i)$ containing a given x_i^r can be ordered cyclically counterclockwise near x_i^r . This defines a cyclic ordering of the points of $p^{-1}(x_0)$ to which $p^{-1}(\gamma_i)$ connects x_i^r . Define $\text{color}(x_i^r)$ as the cycle in Σ_n that cyclically permutes the labels of the points; see Figure 5. Then it is easy to

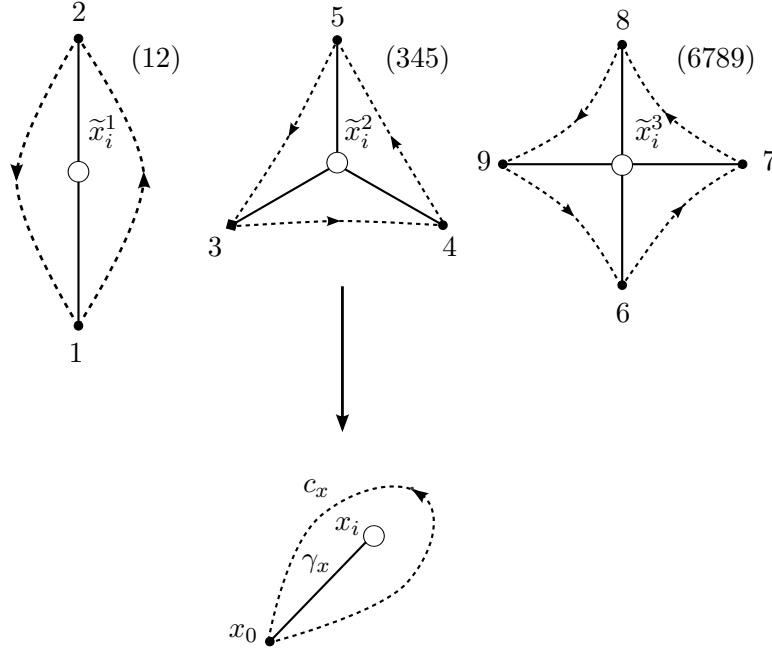


Figure 5: The coloring $(12)(345)(6789)$.

see that

$$\text{color}(x_i) = \prod_{r=1}^{\ell_i} \text{color}(x_i^r).$$

Coloring graph. Let the surface \tilde{X} and the map $\tilde{X} \rightarrow X$ be a branched covering space and x_i and x_j be branch points in X .

Let $x_i^1, \dots, x_i^{\ell_i}$ in \tilde{X} be the elements of $p^{-1}(x_i)$ and $x_j^1, \dots, x_j^{\ell_j}$ the elements of $p^{-1}(x_j)$. We form a loopless bipartite multigraph Γ_{ij} as follows: there is a vertex for every x_i^r and x_j^q . There is one edge of Γ_{ij} between the vertices corresponding to x_i^r and x_j^q for every element of the set $\{1, \dots, n\}$ in the support of both $\text{color}(x_i^r)$ and $\text{color}(x_j^q)$ (the support of a permutation $\sigma \in \Sigma_n$ is any number $1, \dots, n$ such that $\sigma \cdot n \neq n$). We note that the degree of the vertex of Γ_{ij} corresponding to x_i^ℓ is the ramification number of x_i^ℓ .

We now give an alternative description of the edges of the vertices of Γ_{ij} . Let $\gamma = \gamma_i \cup \gamma_j$. As above, the vertices are the elements of x_i^r and x_j^q and there is one edge between the vertices x_i^r and x_j^q for each arc of $p^{-1}(\gamma)$ connecting x_i^r and x_j^q .

5.2 Simple covers do not have the Birman–Hilden property

Before we prove Theorem 1.0.3, we need a lemma.

Lemma 5.2.1. *Let the surface \tilde{X} and the map $p : \tilde{X} \rightarrow X$ be a finite covering space where $n \geq 3$ and $\chi(\tilde{X}) < 0$, and there are at least two branch points x_1, \dots, x_k in X . Choose a base point x_0 that is not a branch point. There exists a Hurwicz system $\gamma_1, \dots, \gamma_k$ in X where with respect to $\gamma_1, \dots, \gamma_k$, $\text{color}(x_i) \neq \text{color}(x_j)$ for some $1 \leq i, j \leq k$.*

Proof. First choose any Hurwicz system $\{\gamma_1, \dots, \gamma_k\}$. As above, let $c_i \in \pi_1(X^\circ, x_0)$ be the homotopy class of the boundary of a regular neighborhood of $\gamma_i \subset X$ that is oriented counterclockwise. To prove the lemma we consider two cases: when X° is a punctured sphere and when X° is a punctured surface of genus g with $g \geq 1$. In both cases since the cover is simple $\text{color}(x_i)$ is a transposition for all i .

Suppose X° is a punctured sphere. In this case $\{c_1, \dots, c_k\}$ generate $\pi_1(X^\circ, x_0)$. As $\rho(c_i) = \text{color}(x_i)$, the cycles $\{\text{color}(x_1), \dots, \text{color}(x_k)\}$ generate $\rho(\pi_1(X^\circ, x_0))$. Since $n \geq 3$, and \tilde{X} is connected, $\rho(\pi_1(X^\circ, x_0))$ is generated by more than one transposition. Hence there exist x_i and x_j such that $\text{color}(x_i) \neq \text{color}(x_j)$.

Now suppose X has positive genus. We will assume that with respect to the Hurwicz system $\gamma_1, \dots, \gamma_k$, $\text{color}(x_i) = \text{color}(x_j)$ for all i and j . (Otherwise we have nothing to show). Without loss of generality $\text{color}(x_i) = (12)$. There exists an embedded disk $D \subset X$ with $x_0 \in \partial D$ and $\gamma_i \subset D$ for all i as in Figure 6.

Choose a nonseparating simple loop δ in X° that is disjoint from the interior of D and where $\rho(\delta) \cdot 1 \notin \{1, 2\}$. Such δ exists because $\pi_1(X^\circ, x_0)$ is generated by the c_i and nonseparating simple loops in X° disjoint from the interior of D and because $\rho(\pi_1(X^\circ, x_0))$ is transitive. Up to homeomorphism of X and relabeling the c_i , we have the configuration shown in Figure 6. The conjugate $\delta c_1 \delta^{-1}$ is an element of $\pi_1(X^\circ, x_0)$ that collapses in X to the arc γ'_1 shown in Figure 6. The arc γ'_1 is disjoint from the

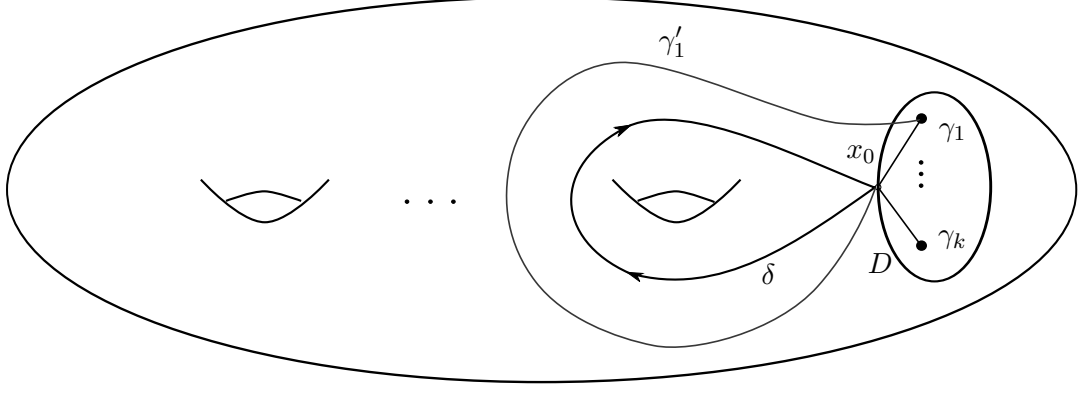


Figure 6: The picture used in the proof of Lemma 5.2.1.

interior of each $\gamma_i, i \neq 1$. We also have that $\rho(\delta c_1 \delta^{-1}) = (\rho(\delta) \cdot 1 \rho(\delta) \cdot 2) \neq (12)$. Therefore with respect to $\{\gamma'_1, \gamma_2, \dots, \gamma_k\}$, $\text{color}(x_1)$ is not equal to $\text{color}(x_i), i \neq 1$. \square

We now prove Theorem 1.0.3, that simple covers do not have the Birman–Hilden property when the degree of the covering space is at least 3.

Proof of Theorem 1.0.3. Recall that by hypothesis $\tilde{X} \rightarrow X$ has at least two branch points and that X° is not a pair of pants.

Label the branch points x_1, \dots, x_k . Fix a Hurwitz system $\gamma_1, \dots, \gamma_k$ so that there exist branch points x_i and x_j with $\text{color}(x_i) \neq \text{color}(x_j)$. This is possible by Lemma 5.2.1.

Because there are only two vertices of Γ_{ij} with degree larger than one, circuits in Γ_{ij} can only occur when there are two edges connecting the vertices of degree 2. Since $\text{color}(x_i) \neq \text{color}(x_j)$, this is not the case. It follows that Γ_{ij} is a forest.

Let α be a simple closed curve in X homotopic to the boundary of the neighborhood of $\gamma_i \cup \gamma_j$. Then $p^{-1}(\alpha)$ is homotopic to the boundary of a neighborhood of $p^{-1}(\gamma_i \cup \gamma_j)$. The simple closed curve α is essential because it bounds a disk containing two branch points and X° is not a sphere with three punctures. Since Γ_{ij} is a forest,

all components of $p^{-1}(\alpha)$ are trivial. Therefore p does not have the weak curve lifting property, and by Proposition 1.0.2 it does not have the Birman–Hilden property. \square

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