VARIANCE PARAMETER ESTIMATION METHODS WITH RE-USE OF DATA

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VARIANCE PARAMETER ESTIMATION METHODS WITH RE-USE OF DATA

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To my family,

Gültekin Kuyzu

and

Seyfettin, Ruhiye, Mustafa Cemil, Mesut Meterelliyoz

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CHAPTER I

INTRODUCTION

A fundamental problem in simulation output analysis concerns the computation of point and confidence interval (CI) estimators for the mean, μ, of a stationary discrete-time stochastic process $\{X_j : j = 1, 2, \ldots\}$. The point estimation of μ is an "easy" problem when the underlying system starts in steady state; the sample mean $\bar{X}_n \equiv n^{-1} \sum_{j=1}^{n} X_j$ based on the finite sample $X_1, \ldots, X_n$ is an unbiased estimator of μ and usually is the estimator of choice. In order to provide a measure of the sample mean's precision, an estimate of $\mathrm{Var}(\bar{X}_n)$ also needs to be calculated. If the X_j's are independent and identically distributed (i.i.d.) random variables, then the sample variance $S_X^2(n) \equiv (n-1)^{-1} \sum_{j=1}^{n} (X_j - \bar{X}_n)^2$ is an unbiased estimator of the population variance $\sigma_X^2 \equiv \mathrm{E}[(X_1 - \mu)^2]$. In this case, $\mathrm{Var}(\bar{X}_n)$ can be estimated by $S_X^2(n)/n$. By the Central Limit Theorem (CLT), an asymptotically ($n \to \infty$) valid $100(1-\alpha)\%$ CI for μ is

$$\bar{X}_n \pm t_{1-\alpha/2,n-1} \frac{S_X(n)}{\sqrt{n}},$$

where $t_{\gamma,k}$ is the γ-quantile of Student's t distribution with k degrees of freedom.

Unfortunately, the X_i are typically correlated in simulation problems. While $\bar{X}_n$ remains unbiased for μ, $S_X^2(n)$ can be severely biased for σ_X^2. In fact, if the autocovariance function $R_j \equiv \mathrm{Cov}(X_1, X_{1+j})$, for $j = 0, 1, \ldots$, is positive, it can be shown that $\mathrm{E}[S_X^2(n)/n] \ll \mathrm{Var}(\bar{X}_n)$ (Law [24], pp. 230–231). In this case, valid CIs for μ can still be obtained based on estimators of the quantities $\{\sigma_n^2 \equiv n\mathrm{Var}(\bar{X}_n) : n = 1, 2, \ldots\}$ or their limit $\sigma^2 \equiv \lim_{n\to\infty} \sigma_n^2$, which is called the (asymptotic) *variance parameter* of the process.

In the literature, one can find many techniques for estimating the quantities $\{\sigma_n^2 :$

$n = 1, 2, \ldots\}$ and σ^2, such as nonoverlapping batch means (NBM) (Fishman [13]), overlapping batch means (OBM) (Meketon and Schmeiser [26]), and standardized time series (STS) (Schruben [30]).

Some of the techniques above group observations into nonoverlapping or overlapping batches. A method relying on nonoverlapping batches typically divides the data into adjacent disjoint batches of equal size, calculates a corresponding estimator from each batch separately, and then forms a grand estimator based on these batch estimators. The NBM estimator for the variance parameter, σ^2, as well as the STS area, Cramér–von Mises (CvM) (see, e.g. Alexopoulos et al. [4]), folded area, and folded CvM estimators (Antonini [6]), all use nonoverlapping batches. To obtain the NBM estimator, one computes the sample mean from each batch and multiplies the sample variance of the batch means by the batch size. For STS area and CvM estimators, one forms an STS based on each batch, computes the respective quantities based on weighted areas of functions of the STSs, and averages these quantities. On the other hand, the folded area and folded CvM estimation methods first "fold" STSs from each batch and then use these folded STSs analogously to non-folded area and CvM estimators to estimate the variance parameter. All methods above assume that the relevant estimators obtained from different nonoverlapping batches are approximately i.i.d. random variables.

The OBM, STS overlapping area, and STS overlapping CvM estimators are based on a similar approach as their nonoverlapping counterparts, but use overlapping batches to obtain the respective variance parameter estimators. Asymptotically (as the batch size, hence the sample size, approaches infinity), the estimators obtained by overlapping batches maintain the same bias but usually have smaller variances than the respective estimators based on nonoverlapping batches.

This thesis makes the following contributions to variance estimation in steady-state simulations. First, it obtains additional theoretical properties of (batched)

folded area and CvM estimators. Second, it introduces two new classes of variance parameter estimators. The first class, namely, folded overlapping area (FOA) estimators, is based on the concepts of overlapping batches and folded STSs. The second class, namely, reflected estimators, is based on reflections of the STS formed by the entire sample. Both folding and reflection operations are predicated on the concept of data "re-use". For instance, the folding operation on the original STS yields another STS; both STSs converge (as the sample size goes to infinity) to Brownian bridges with known cross-covariance structure, but the area estimators obtained from the two STSs are asymptotically independent scaled chi-squared random variables. As a result, one can obtain asymptotically independent estimators of σ^2 based on a single data set (sample path). The third contribution of this thesis is the development of linear combinations of folded and reflected estimators with substantially smaller mean squared error (MSE) than the constituent estimators.

The estimators of Calvin and Nakayama [11] are based on another type of data re-use. Specifically, they obtain permutations of path segments and develop an STS estimator by averaging over the permuted sample paths. In addition, Calvin [10] develops a method that is based on iterated integrations of the sample path.

The remainder of this thesis is organized as follows. Chapter 2 reviews background material. Chapter 3 discusses several folded standardized time series estimators along with their linear combinations and establishes their properties. Chapter 4 introduces the folded overlapping area estimators, obtains their first two moments. and describes approximate confidence intervals for μ and σ^2. Chapter 5 studies various reflected estimators. The Appendix contains several long proofs and a few detailed results.

CHAPTER II

BACKGROUND

In this chapter, we list some assumptions and review results that are needed in the rest of this thesis. We start with a list of necessary assumptions and a review of some basic properties regarding the convergence of stationary processes in §2.1. We define the standardized time series (STS) and some of the variance parameter estimators based on STS in §2.2. Finally, the estimators based on nonoverlapping batches and estimators based on overlapping batches are discussed in §2.3 and in §2.4, respectively.

2.1 Basics and Assumptions

We start with a Functional Central Limit Theorem (FCLT) which holds for several stationary processes, e.g., stationary strongly mixing processes, associated stationary processes, and regenerative processes (see, e.g. Glynn and Iglehart [16]).

Assumption FCLT Suppose that the series $\sigma^2 \equiv \sum_{j=-\infty}^{\infty} R_j$ converges absolutely and $\sigma^2 > 0$. For each positive integer n, let

$$Y_n(t) \equiv \frac{\lfloor nt \rfloor (\bar{X}_{\lfloor nt \rfloor} - \mu)}{\sigma\sqrt{n}}, \quad \text{for } t \in [0,1], \tag{1}$$

where $\lfloor \cdot \rfloor$ is the greatest integer function. Then

$$Y_n(\cdot) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{W}(\cdot),$$

where $\mathcal{W}(\cdot)$ is a standard Brownian motion process on [0,1] and $\xrightarrow[n\to\infty]{\mathcal{D}}$ denotes weak convergence (as $n \to \infty$) in the Skorohod space $D[0,1]$ of real-valued functions on $[0,1]$ that are right-continuous with left-hand limits (see Billingsley [9]).

Often, we will refer to the following assumption.

Uniform Integrability (Karr [22]) Let $(\Omega, \mathbb{F}, \mathrm{P})$ denote a probability space, where Ω is the sample space, $\mathbb{F}$ is a sigma-field, and P is a probability measure defined on

$\mathbb{F}$. The random sequence $\{X_j : j \geq 1\}$ defined on a probability space $(\Omega, \mathbb{F}, \mathrm{P})$ is said to be uniformly integrable if

$$\lim_{\alpha \to \infty} \sup_j \int_{\{|X_j| \geq \alpha\}} |X_j| \, \mathrm{dP} \equiv \mathrm{E}(|X_j| \mathbb{I}_{\{|X_j| \geq \alpha\}}) = 0,$$

where $\mathbb{I}_{\{|X_j| \geq \alpha\}}$ is the indicator function of the event $\{|X_j| \geq \alpha\}$. Also, notice that if the $\{X_j : j \geq 1\}$ are uniformly integrable, then $\sup_j \mathrm{E}(|X_j|) < \infty$.

Remark 1 A sufficient condition for uniform integrability of X_n is that $\mathrm{E}[X_n^{1+\epsilon}]$ is finite for all n (Karr [22]). $\triangleleft$

Some of our results make use of a generalized version of the continuous mapping theorem (CMT) given below.

Theorem 1 (Billingsley [9]) Let P_n, $n = 1, 2, \ldots$, and P be probability measures on $(\Omega, \mathbb{F})$, where Ω is a metric space equipped with a sigma-field $\mathbb{F}$. Let h_n and h be measurable mappings from Ω to another metric space Ω'. Let E be the set of $x \in \Omega$ such that $h_n(x_n) \to h(x)$ fails to hold for some sequence $x_n \to x$. Assume that E lies in $\mathbb{F}$. Under these conditions,

$$\text{if } \mathrm{P}_n \xrightarrow[n \to \infty]{\mathcal{D}} \mathrm{P} \text{ and } \mathrm{P}(E) = 0, \text{ then } \mathrm{P}_n(h_n^{-1}) \xrightarrow[n \to \infty]{\mathcal{D}} \mathrm{P}(h^{-1}) \, .$$

Remark 2 If $h_n = h$, the generalized CMT reduces to the CMT — that is, Theorem 5.1 of Billingsley [9]. $\triangleleft$

Throughout this thesis, we refer to the following assumptions.

Assumptions A

1. The process $\{X_j : j \geq 1\}$ is stationary.

2. The process $\{X_j : j \geq 1\}$ satisfies Assumption FCLT.

3. The autocovariance function decays exponentially, i.e., $|R_j| = O(\delta^j)$, $j = 1, 2, \ldots$, for some $\delta \in (0, 1)$. Further, the series $\sum_{j=-\infty}^{\infty} R_j = \sigma^2 \in (0, \infty)$.

We also need the following assumptions on the weight functions that will be used in this document.

Assumptions F

1. The function $f(\cdot)$ is normalized so that $\mathrm{Var}\left[\int_0^1 f(t)\mathcal{B}(t)\,\mathrm{d}t\right] = 1$, where $\mathcal{B}(\cdot)$ is a standard Brownian bridge process on $[0,1]$.

2. $f(t)$ is twice continuously differentiable in $[0,1]$.

3. $f(\cdot)$ is symmetric about $t = 1/2$; that is, $f(t) = f(1-t)$ for $t \in [0,1]$.

Assumptions G

1. The function $g(\cdot)$ is normalized so that $\mathrm{E}\left[\int_0^1 g(t)\mathcal{B}^2(t)\,\mathrm{d}t\right] = 1$.

2. $g(t)$ is twice continuously differentiable in $[0,1]$.

First, note that $\mathcal{B}(\cdot)$ is independent of $\mathcal{W}(1)$. Further, $\mathcal{B}(\cdot)$ can be expressed as $\mathcal{B}(t) = t\mathcal{W}(1) - \mathcal{W}(t)$, for $t \in [0,1]$, and all finite-dimensional distributions of $\mathcal{B}(\cdot)$ are normal with $\mathrm{E}[\mathcal{B}(t)] = 0$ and $\mathrm{Cov}(\mathcal{B}(s), \mathcal{B}(t)) = \min(s,t) - st$ for $0 \le s, t \le 1$. In the rest of this thesis we let $\gamma_i \equiv 2\sum_{\ell=1}^{\infty} \ell^i R_\ell$ and $\gamma_{i,j} \equiv \sum_{\ell=1}^{j} \ell^i R_\ell$, where $i = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots$. We use the "big-oh" notation $p(n) = O(q(n))$ to mean that there are positive constants c and k such that $0 \le p(n) \le cq(n)$ for all $n \ge k$, and we use the "little-oh" notation $p(n) = o(q(n))$ to mean that $\lim_{n\to\infty} p(n)/q(n) = 0$. We also define the functions $F(\cdot)$ and $\bar{F}(\cdot)$ by

$$F(t) \equiv \int_0^t f(s)\,\mathrm{d}s \quad \text{and} \quad \bar{F}(t) \equiv \int_0^t F(s)\,\mathrm{d}s,$$

for $0 \le t \le 1$, and we let $F \equiv F(1)$ and $\bar{F} \equiv \bar{F}(1)$. Finally, for $0 \le t \le 1$, we define $G(t) \equiv \int_0^t g(s)\,\mathrm{d}s$, $G \equiv G(1)$, $\bar{G}(t) \equiv \int_0^t G(s)\,\mathrm{d}s$, and $\bar{G} \equiv \bar{G}(1)$.

2.2 Standardized Time Series Estimators

In this section we define the standardized time series and review several variance parameter estimation methods that depend on the STS.

Standardized Time Series The STS based on the sample $X_1, \ldots, X_n$ is defined as

$$T_n(t) \equiv \frac{\lfloor nt \rfloor (\bar{X}_n - \bar{X}_{\lfloor nt \rfloor})}{\sigma\sqrt{n}}, \quad \text{for } t \in [0,1]. \tag{2}$$

Under Assumptions A, it can be shown that (Schruben [30])

$$\left[\sqrt{n}(\bar{X}_n - \mu), \sigma T_n(\cdot)\right] \xrightarrow[n\to\infty]{\mathcal{D}} \left[\sigma\mathcal{W}(1), \sigma\mathcal{B}(\cdot)\right]. \tag{3}$$

2.2.1 Weighted Area Estimator

Goldsman et al. [18] introduced the weighted area estimator, which is based on the square of the weighted area under the STS (2), and its limiting functional. These are defined by

$$\mathcal{A}(f;n) \equiv \left[\frac{1}{n}\sum_{j=1}^{n} f\left(\tfrac{j}{n}\right)\sigma T_n\left(\tfrac{j}{n}\right)\right]^2 \quad \text{and} \quad \mathcal{A}(f) \equiv \left[\int_0^1 f(t)\sigma\mathcal{B}(t)\,\mathrm{d}t\right]^2,$$

respectively. The weight function $f(\cdot)$ is assumed to satisfy Assumptions F.

Under Assumptions A, the CMT implies that $\int_0^1 f(t)\sigma B(t)\,\mathrm{d}t \stackrel{\mathcal{D}}{=} \sigma N(0,1)$ and $\mathcal{A}(f;n) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{A}(f) \stackrel{\mathcal{D}}{=} \sigma^2\chi_1^2$, where $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution, $N(0,1)$ denotes a standard normal random variable, and χ_v^2 denotes a chi-squared random variable with v degrees of freedom.

Under Assumptions A and F, Song and Schmeiser [32] and Alexopoulos et al. [4] show that the expected value of the weighted area estimator is

$$\mathrm{E}[\mathcal{A}(f,n)] = \frac{1}{n^3}\left(R_0\sum_{j=1}^{n} h^2(j;n) + 2\sum_{i=1}^{n-1} R_i \sum_{j=1}^{n-i} h(j;n)h(j+i;n)\right) \tag{4}$$

$$= \sigma^2 - \frac{[(F-\bar{F})^2 + \bar{F}^2]\gamma_1}{2n} + o(1/n), \tag{5}$$

where $h(j;n) \equiv \sum_{\ell=1}^{n} \frac{\ell}{n} f\left(\frac{\ell}{n}\right) - \sum_{\ell=j}^{n} f\left(\frac{\ell}{n}\right)$, for $j = 1, 2, \ldots, n$.

Further, if we assume that the sequence $\{\mathcal{A}^2(f;n) : n = 1, 2, \ldots\}$ is uniformly integrable, then

$$\lim_{n\to\infty} \mathrm{Var}[\mathcal{A}(f;n)] = \mathrm{Var}[\mathcal{A}(f)] = 2\sigma^4. \tag{6}$$

Remark 3 Note that the limiting variance (6) does not depend on the weight function $f(\cdot)$. $\triangleleft$

Example 1 The most basic area estimator of Schruben [30] uses the constant weight function $f_0(t) \equiv \sqrt{12}$, $t \in [0,1]$. In this case, Equation (4) gives

$$\mathrm{E}[\mathcal{A}(f_0;n)] = \sigma^2 - \frac{3\gamma_1}{n} - \frac{\sigma^2}{n^2} + \frac{\gamma_1 + 2\gamma_3}{n^3} + O(\delta^n) \tag{7}$$

$$= \sigma^2 - 3\gamma_1/n + o(1/n). \quad \triangleleft \tag{8}$$

Example 2 We say that an estimator of σ^2 is *first-order unbiased* if it has bias of the form $o(1/n)$. Goldsman et al. [18] showed that the quadratic weight function $f_2(t) \equiv \sqrt{840}(3t^2 - 3t + 1/2)$, $t \in [0,1]$, results in a first-order unbiased estimator $\mathcal{A}(f_2;n)$. In addition, Equation (4) yields

$$\mathrm{E}[\mathcal{A}(f_2;n)] = \sigma^2 + \frac{7(\sigma^2 - 6\gamma_2)}{2n^2} + \frac{35(\gamma_1 + 2\gamma_3)}{2n^3} + O(1/n^4). \quad \triangleleft$$

Example 3 The trigonometric weight functions of the form $f_{\cos,j}(t) \equiv \sqrt{8}\pi j \cos(2\pi j t)$ with $j = 1, 2, \ldots$, $t \in [0,1]$, yield asymptotically independent, first-order unbiased estimators $\mathcal{A}(f_{\cos,j};n)$, $j = 1, 2, \ldots$ (Foley and Goldsman [15]). From Equation (4), we get

$$\mathrm{E}[\mathcal{A}(f_{\cos,1};n)] = \sigma^2 + \frac{\pi^2(\sigma^2 - 6\gamma_2)}{3n^2} + \frac{2\pi^2(\gamma_1 + 2\gamma_3)}{3n^3} + O(1/n^4). \quad \triangleleft$$

2.2.2 Weighted Cramér–von Mises (CvM) Estimator

The CvM estimator for σ^2, introduced by Goldsman et al. [19], is the weighted area under the square of the STS. This estimator and its limiting functional are

$$\mathcal{C}(g;n) \equiv \frac{1}{n}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\sigma^2 T_n^2\Big(\tfrac{j}{n}\Big) \quad \text{and} \quad \mathcal{C}(g) \equiv \int_0^1 g(t)[\sigma\mathcal{B}(t)]^2 \,\mathrm{d}t,$$

respectively. Under Assumptions A and G, the CMT implies that $C(g;n) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{C}(g)$. In this case, Aktaran-Kalaycı et al. [1] showed that

$$\begin{aligned}\mathrm{E}[\mathcal{C}(g;n)] &= \frac{1}{n^3}\Big[R_0 - 2\gamma_{0,n-1} + \frac{2\gamma_{1,n-1}}{n}\Big]\sum_{k=1}^{n-1} g\Big(\tfrac{k}{n}\Big)k(n-k) \\ &\quad + 2\sum_{i=1}^{n-1} R_i \sum_{k=1}^{n-i}\Big[g\Big(\tfrac{k+i}{n}\Big) + g\Big(\tfrac{k}{n}\Big)\Big]k(n-k-i) \qquad (9)\\ &= \sigma^2 - \frac{\gamma_1}{n}(G-1) + o(1/n), \qquad (10)\end{aligned}$$

where the last equality follows from Goldsman et al. [19].

Further, if we assume that the sequence $\{\mathcal{C}^2(g;n) : n = 1, 2, \ldots\}$ is uniformly integrable, we have

$$\lim_{n\to\infty} \mathrm{Var}[\mathcal{C}(g;n)] = \mathrm{Var}[\mathcal{C}(g)] = 4\sigma^4 \int_0^1 g(t)(1-t)^2 \int_0^t g(s)s^2 \,\mathrm{d}s\,\mathrm{d}t. \qquad (11)$$

Remark 4 In this case, the limiting variance (11) depends on the weight function $g(\cdot)$. $\triangleleft$

Example 4 For the constant weight function $g_0(t) \equiv 6$, $t \in [0,1]$, Equation (9) yields

$$\begin{aligned}\mathrm{E}[\mathcal{C}(g_0;n)] &= \sigma^2 - \frac{5\gamma_1}{n} + \frac{6\gamma_2 - \sigma^2}{n^2} + \frac{\gamma_1 - 2\gamma_3}{n^3} + O(\delta^n) \\ &= \sigma^2 - 5\gamma_1/n + o(1/n).\end{aligned}$$

Also, $\mathrm{Var}[\mathcal{C}(g_0)] = 4\sigma^2/5$. $\triangleleft$

Example 5 The weight function $g_{2,c}(t) \equiv 51 - c/2 + ct - 150t^2$, for $t \in [0,1]$ and $c \in \mathbb{R}$ yields a first-order unbiased estimator with

$$\mathrm{Var}[\mathcal{C}(g_{2,c})] = \frac{(c^2 - 300c + 26856)\sigma^4}{2520}.$$

This limiting variance is minimized when $c = 150$, and $g_2^\star(t) \equiv g_{2,150}(t) = -24 + 150t - 150t^2$ has

$$\mathrm{Var}[\mathcal{C}(g_2^\star)] = \frac{121\sigma^4}{70} > \mathrm{Var}[\mathcal{C}(g_0)].$$

The expected value from Equation (9) is

$$\mathrm{E}[\mathcal{C}(g_2^\star; n)] = \sigma^2 + \frac{4(\sigma^2 - 6\gamma_2)}{n^2} + \frac{29(2\gamma_3 - \gamma_1)}{n^3} + O(1/n^4). \quad \triangleleft$$

Example 6 The first-order unbiased quartic weight function that minimizes the limiting variance is (Goldsman et al. [19])

$$g_4^\star(t) \equiv -\frac{1310}{21} + \frac{19270t}{21} - \frac{25230t^2}{7} + \frac{16120t^3}{3} - \frac{8060t^4}{3}, \quad \text{for } t \in [0, 1],$$

with limiting variance $\mathrm{Var}[\mathcal{C}(g_4^\star)] = 1.042\sigma^4$. Equation (9) gives the expected value as

$$\mathrm{E}[\mathcal{C}(g_4^*; n)] = \sigma^2 + \frac{655(\sigma^2 - 6\gamma_2)}{63n^2} + \frac{10290(2\gamma_3 - \gamma_1)}{63n^3} + O(1/n^4). \quad \triangleleft$$

2.3 Estimators Based on Nonoverlapping Batches

In this section, we describe batched versions of the estimators discussed in §2.2. We form b nonoverlapping batches each consisting of m observations (assuming $n = bm$). Specifically, batch i consists of observations $\{X_{(i-1)m+j} : j = 1, \ldots, m\}$. The STS from batch i is

$$T_{i,m}(t) \equiv \frac{\lfloor mt \rfloor (\bar{X}_{i,m} - \bar{X}_{i,\lfloor mt \rfloor})}{\sigma\sqrt{m}}, \quad \text{for } t \in [0, 1] \text{ and } i = 1, \ldots, b,$$

where

$$\bar{X}_{i,j} \equiv \frac{1}{j} \sum_{\ell=1}^{j} X_{(i-1)m+\ell}, \quad \text{for } i = 1, \ldots, b \text{ and } j = 1, \ldots, m. \tag{12}$$

2.3.1 Batched Area Estimator

The area estimator computed from batch i is

$$A_i(f; m) \equiv \left[\frac{1}{m} \sum_{j=1}^{m} f\left(\tfrac{j}{m}\right) \sigma T_{i,m}(\tfrac{j}{m}) \right]^2, \quad \text{for } i = 1, \ldots, b,$$

and the batched area estimator for σ^2 is the sample mean of the $A_i(f; m)$:

$$\mathcal{A}(f; b, m) \equiv \frac{1}{b} \sum_{i=1}^{b} A_i(f; m).$$

Since, under Assumptions A, the $\{T_{i,m}(\cdot) : i = 1, \ldots, b\}$ converge to independent Brownian bridge processes as m becomes large (with fixed b), we conclude that $\{A_i(f; m) : i = 1, \ldots, b\}$ are asymptotically independent as $m \to \infty$; and under Assumptions A and F, generalized CMT implies that

$$\mathcal{A}(f; b, m) \xrightarrow[m\to\infty]{\mathcal{D}} \frac{\sigma^2 \chi_b^2}{b}.$$

Remark 5 We can obtain the expected value for $\mathcal{A}(f; b, m)$ if we replace n by m in Equations (4) and (5), i.e.,

$$\mathrm{E}[\mathcal{A}(f, m)] = \frac{1}{m^3}\left(R_0 \sum_{j=1}^{m} h^2(j; m) + 2\sum_{i=1}^{m-1} R_i \sum_{j=1}^{m-i} h(j; m)h(j + i; m)\right) \tag{13}$$

$$= \sigma^2 - \frac{[(F - \bar{F})^2 + \bar{F}^2]\gamma_1}{2n} + o(1/m), \tag{14}$$

where $h(j; m) \equiv \sum_{\ell=1}^{m} \frac{\ell}{m} f\left(\frac{\ell}{m}\right) - \sum_{\ell=j}^{m} f\left(\frac{\ell}{m}\right)$, for $j = 1, 2, \ldots, m$. Also, for the weight functions introduced in Examples 1–3, we can obtain the expected value results for the batched area estimators if we replace n by m. $\triangleleft$

Further, if we assume that the sequence $\{\mathcal{A}^2(f; b, m) : m = 1, 2, \ldots\}$ is uniformly integrable, we have

$$\lim_{m\to\infty} b\mathrm{Var}[\mathcal{A}(f; b, m)] = \lim_{m\to\infty} b\mathrm{Var}[A_1(f; m)] = \mathrm{Var}[\mathcal{A}(f)] = 2\sigma^4.$$

2.3.2 Batched CvM Estimator

The CvM estimator obtained from batch i is

$$C_i(g; m) \equiv \frac{1}{m}\sum_{j=1}^{m} g\left(\tfrac{j}{m}\right)\sigma^2 T_{i,m}^2\left(\tfrac{j}{m}\right), \quad \text{for } i = 1, \ldots, b,$$

and the batched CvM estimator for σ^2 is the sample mean of the $C_i(g; b, m)$:

$$\mathcal{C}(g; b, m) \equiv \frac{1}{b}\sum_{i=1}^{b} C_i(g; m).$$

Remark 6 The expected values of the batched CvM estimators can be obtained if we replace n by m in Equations (9) and (10), i.e.,

$$\begin{aligned}\mathrm{E}[\mathcal{C}(g;m)] &= \frac{1}{m^3}\Big[R_0 - 2\gamma_{0,m-1} + \frac{2\gamma_{1,m-1}}{m}\Big]\sum_{k=1}^{m-1} g\Big(\tfrac{k}{m}\Big)k(m-k) \\ &\quad + 2\sum_{i=1}^{m-1} R_i \sum_{k=1}^{m-i}\Big[g\Big(\tfrac{k+i}{m}\Big) + g\Big(\tfrac{k}{m}\Big)\Big]k(m-k-i) \qquad (15)\\ &= \sigma^2 - \frac{\gamma_1}{m}(G-1) + o(1/m). \qquad (16)\end{aligned}$$

In addition, we can obtain the expected value of $\mathcal{C}(g;b,m)$ for different weight functions if we replace n with m in Examples 4–6. $\triangleleft$

Further, if we assume that the sequence $\{\mathcal{C}^2(g;b,m) : m = 1,2,\ldots\}$ is uniformly integrable, then

$$\lim_{m\to\infty} b\mathrm{Var}[\mathcal{C}(g;b,m)] = \mathrm{Var}[\mathcal{C}(g)] = 4\sigma^2\int_0^1 g(t)(1-t)^2\int_0^t g(s)s^2\,\mathrm{d}s\,\mathrm{d}t. \qquad (17)$$

2.3.3 Nonoverlapping Batch Means Estimator

The NBM estimator for σ^2 is probably the most popular one and serves as a benchmark for comparison with other estimators because of its simplicity and computational performance.

The quantities $\bar{X}_{i,m}$, $i = 1,\ldots,b$, defined in (12) are referred to as the *batch means* of the data $\{X_j : j = 1,\ldots,n\}$. One can show (Law and Carson [23]) that as $m \to \infty$, the batch means become approximately i.i.d. normal random variables. The NBM estimator is

$$\mathcal{N}(b,m) \equiv \frac{m}{b-1}\sum_{i=1}^{b}(\bar{X}_{i,m} - \bar{X}_n)^2.$$

It has been shown that (Steiger and Wilson [34]) for fixed b,

$$\mathcal{N}(b,m) \xrightarrow[m\to\infty]{\mathcal{D}} \frac{\sigma^2\chi^2_{b-1}}{b-1}.$$

Aktaran-Kalaycı et al. [1] and Goldsman and Meketon [17] obtained the following

expressions for the expected value of the NBM estimator;

$$\mathrm{E}[\mathcal{N}(b,m)] = R_0 + \frac{b\gamma_{0,m-1} - \gamma_{0,n-1}}{b-1} + \frac{\gamma_{1,n-1} - b^2\gamma_{1,m-1}}{n(b-1)} \tag{18}$$

$$= \sigma^2 - \frac{(b+1)\gamma_1}{bm} + o(1/m). \tag{19}$$

For fixed b, one can also show that

$$\lim_{m\to\infty}(b-1)\mathrm{Var}[\mathcal{N}(b,m)] = 2\sigma^4.$$

2.4 Estimators Based on Overlapping Batches

The overlapping variance estimators in Alexopoulos et al. [4] are based on $n-m+1$ overlapping batches of size m each. Specifically, the observations $\{X_{i+j} : j = 0,\ldots,m-1\}$ constitute overlapping batch i for $i = 1,\ldots,n-m+1$. We define $b \equiv n/m$ under the understanding that b is no longer the number of batches. The overlapping STS computed from batch i is

$$T^{\mathrm{o}}_{i,m}(t) \equiv \frac{\lfloor mt\rfloor(\bar{X}^{\mathrm{o}}_{i,m} - \bar{X}^{\mathrm{o}}_{i,\lfloor mt\rfloor})}{\sigma\sqrt{m}}, \quad \text{for } t\in[0,1] \text{ and } i = 1,\ldots,n-m+1,$$

where

$$\bar{X}^{\mathrm{o}}_{i,j} \equiv \frac{1}{j}\sum_{\ell=0}^{j-1} X_{i+\ell}, \quad \text{for } i = 1,\ldots,n-m+1 \text{ and } j = 1,\ldots,m$$

is the sample average of the first j observations from batch i. In addition we define $\mathcal{B}_{\mathcal{W},s}(\cdot)$, a standard Brownian bridge starting at time s, as

$$\mathcal{B}_{\mathcal{W},s}(t) \equiv t[\mathcal{W}(s+1) - \mathcal{W}(s)] - [\mathcal{W}(s+t) - \mathcal{W}(s)], \quad \text{for } t\in[0,1] \text{ and } s\in[0,b-1].$$

If Assumptions A hold, then for fixed s we have

$$\sigma T^{o}_{\lfloor sm\rfloor,m}(\cdot) \xrightarrow[m\to\infty]{\mathcal{D}} \sigma\mathcal{B}_{\mathcal{W},s}(\cdot), \quad \text{for each } s\in[0,b-1].$$

2.4.1 Overlapping Area Estimator

The overlapping area estimator computed from batch i is

$$A^{\mathrm{o}}_i(f;m) \equiv \left[\frac{1}{m}\sum_{j=1}^{m} f\left(\tfrac{j}{m}\right)\sigma T^{\mathrm{o}}_{i,m}\left(\tfrac{j}{m}\right)\right]^2, \quad \text{for } i = 1,\ldots,n-m+1,$$

and the overlapping area estimator for σ^2 is the sample mean of the $A_i^{\mathrm{o}}(f;m)$:

$$\mathcal{A}^{\mathrm{o}}(f;b,m) \equiv \frac{1}{n-m+1} \sum_{i=1}^{n-m+1} A_i^{\mathrm{o}}(f;m).$$

Alexopoulos et al. [4] show that under Assumptions A and F,

$$\mathcal{A}^{\mathrm{o}}(f;b,m) \underset{m\to\infty}{\xrightarrow{\mathcal{D}}} \mathcal{A}^{\mathrm{o}}(f;b) \equiv \frac{1}{b-1} \int_0^{b-1} \left[\sigma \int_0^1 f(u)\mathcal{B}_{\mathcal{W},s}(u)\,\mathrm{d}u\right]^2 \mathrm{d}s, \quad \text{for } b \geq 2.$$

Remark 7 The same argument as in Remark 5 holds for the expected value of the overlapping area estimator. ◁

Further, if the sequence $\{[\mathcal{A}^{\mathrm{o}}(f;b,m)]^2 : m = 1,2,\ldots\}$ is uniformly integrable, then for fixed b,

$$\mathrm{Var}[\mathcal{A}^{\mathrm{o}}(f;b,m)] \underset{m\to\infty}{\longrightarrow} \mathrm{Var}(\mathcal{A}^{\mathrm{o}}(f;b)) = \frac{4\sigma^4}{(b-1)^2} \int_0^1 (b-1-y)[p(0,y)]^2\,\mathrm{d}y, \qquad (20)$$

where

$$p(0,y) \equiv \bar{F}[F(1-y) - \bar{F}(1-y) - \bar{F}y] + \bar{F}(y)\bar{F} - \int_0^{1-y} f(u)\bar{F}(y+u)\,\mathrm{d}u,$$

for $y \in [0,1]$.

Example 7 For the constant weight function $f_0(\cdot)$, the asymptotic variance of the overlapping area estimator as $m \to \infty$ is given by

$$\mathrm{Var}[\mathcal{A}^{\mathrm{o}}(f_0;b)] = \frac{24b-31}{35(b-1)^2}\sigma^4 \approx \frac{24}{35b}\sigma^4,$$

with the approximate result holding for large b. ◁

Example 8 The asymptotic variance using the weight function $f_2(\cdot)$ is given by

$$\mathrm{Var}[\mathcal{A}^{\mathrm{o}}(f_2;b)] = \frac{3514b-4359}{4290(b-1)^2}\sigma^4 \approx \frac{3514}{4290b}\sigma^4. \quad \triangleleft$$

Example 9 The trigonometric weight functions $f_{\cos,j}(\cdot)$, $j = 1,2,\ldots$, yield

$$\mathrm{Var}[\mathcal{A}^{\mathrm{o}}(f_{\cos,j};b)] = \frac{(16\pi^2j^2+30)b - (20\pi^2j^2+33)}{22\pi^2j^2(b-1)^2}\sigma^4 \approx \frac{8\pi^2j^2+15}{12\pi^2j^2b}\sigma^4. \quad \triangleleft$$

Remark 8 Contrary to the results for weighted area estimators based on nonoverlapping batches, the asymptotic variance for the overlapping area estimator depends on the weight function. ◁

2.4.2 Overlapping CvM Estimator

The overlapping CvM estimator computed from overlapping batch i is

$$C_i^{\mathrm{o}}(g;m) \equiv \frac{1}{m}\sum_{j=1}^{m} g\Big(\tfrac{j}{m}\Big)\sigma^2\left[T_{i,m}^{\mathrm{o}}\big(\tfrac{j}{m}\big)\right]^2, \quad \text{for } i = 1,\ldots,n-m+1,$$

and the overlapping CvM estimator for σ^2 is the sample mean of the $C_i^{\mathrm{o}}(g;m)$:

$$\mathcal{C}^{\mathrm{o}}(g;b,m) \equiv \frac{1}{n-m+1}\sum_{i=1}^{n-m+1} C_i^{\mathrm{o}}(g;m).$$

Alexopoulos et al. [4] show that under Assumptions A and G,

$$\mathcal{C}^{\mathrm{o}}(g;b,m) \xrightarrow[m\to\infty]{\mathcal{D}} \mathcal{C}^{\mathrm{o}}(g;b) \equiv \frac{1}{b-1}\int_0^{b-1}\int_0^1 g(u)\sigma^2\mathcal{B}^2_{\mathcal{W},s}(u)\,\mathrm{d}u\,\mathrm{d}s.$$

Remark 9 The same argument as in Remark 6 holds for the expected value of the overlapping CvM estimator. ◁

Further, if the sequence $\{[\mathcal{C}^{\mathrm{o}}(g;b,m)]^2 : m = 1,2,\ldots\}$ is uniformly integrable, then for fixed b,

$$\mathrm{Var}[\mathcal{C}^{\mathrm{o}}(g;b,m)] \underset{m\to\infty}{\longrightarrow} \mathrm{Var}(\mathcal{C}^{\mathrm{o}}(g;b)) = \frac{4\sigma^4}{(b-1)^2}\int_0^1 (b-1-y)q(y)\,\mathrm{d}y,$$

where

$$q(y) \equiv \int_0^1\int_0^1 g(u)g(v)\mathrm{Cov}^2[\mathcal{B}_{\mathcal{W},0}(u),\mathcal{B}_{\mathcal{W},y}(v)]\,\mathrm{d}u\,\mathrm{d}v,$$

for $y \in [0,1]$, and this covariance is a little nasty to calculate.

Example 10 For the constant weight function $g_0(\cdot)$, the asymptotic variance is

$$\mathrm{Var}[\mathcal{C}^{\mathrm{o}}(g_0;b)] = \frac{88b-115}{210(b-1)^2}\sigma^4 \approx \frac{44}{105b}\sigma^4,$$

with the approximate result holding for large b. ◁

Example 11 The asymptotic variance of the overlapping CvM estimator based on the weight function $g_{2,c}(\cdot)$ in Example 5 is given by

$$\mathrm{Var}[\mathcal{C}^{\mathrm{o}}(g_{2,c};b)] = \frac{3876480b + 187c^2 - 56100c - 690300}{4989600(b-1)^2}\sigma^4.$$

This quantity is minimized when $c = 150$ (as in the case of nonoverlapping batches); then

$$\mathrm{Var}[\mathcal{C}^{\mathrm{o}}(g^{\star}_{2,150}; b)] = \frac{10768b - 13605}{13860(b-1)^2}\sigma^4 \approx \frac{10768}{13860b}\sigma^4. \quad \triangleleft$$

Example 12 The weight function $g^{\star}_4(\cdot)$ from Example 6 gives the following asymptotic variance:

$$\mathrm{Var}[\mathcal{C}^{\mathrm{o}}(g^{\star}_4; b)] \approx \frac{0.477\sigma^4}{b}. \quad \triangleleft$$

2.4.3 Overlapping Batch Means Estimator

The OBM estimator for σ^2 due to Meketon and Schmeiser [26] is given by

$$\mathcal{O}(b, m) \equiv \frac{nm}{(n-m+1)(n-m)} \sum_{i=1}^{n-m+1} (\bar{X}^{\mathrm{o}}_{i,m} - \bar{X}_n)^2.$$

Under Assumptions A, the expected value of $\mathcal{O}(b, m)$ is (Aktaran-Kalaycı et al. [1])

$$\begin{aligned} \mathrm{E}[\mathcal{O}(b, m)] &= \sigma^2 - \frac{\gamma_1(b^2+1)}{n(b-1)} + \frac{\gamma_1 + \gamma_2}{(n-m)(n-m+1)} + O(\delta^m) \\ &= \sigma^2 - \frac{(b^2+1)\gamma_1}{mb(b-1)} + o(1/m), \end{aligned}$$

for $b = n/m \geq 2$. The asymptotic variance of $\mathcal{O}(b, m)$ is (Damerdji [12])

$$\lim_{m\to\infty} \mathrm{Var}[\mathcal{O}(b, m)] = \frac{(4b^3 - 11b^2 + 4b + 6)\sigma^4}{3(b-1)^4} \approx \frac{4\sigma^4}{3b}.$$

2.5 Summary

This chapter presented the basic assumptions and reviewed various existing estimators for the asymptotic variance parameter σ^2 of a stationary discrete-time stochastic process. Chapter 3 proceeds with the folded area and CvM estimators.

CHAPTER III

FOLDED STANDARDIZED TIME SERIES ESTIMATORS

In this chapter, we start by reviewing the folded estimators discussed in Alexopoulos et al. [5]. First, we define the folding operation on Brownian bridges and STSs. Second, we discuss how to use the folded STSs to develop folded versions of area and CvM estimators. After these necessary information, we start discussing our findings on folded area and CvM estimators. We obtain detailed expressions for the expected value of these estimators. In addition, we present linear combinations of folded estimators, and analyze their expected value and variance behavior. We also study batched folded area and CvM estimators obtained from nonoverlapping batches. Similarly, we list expected value and variance results for these folded and batched estimators, as well as their linear combinations. Finally, we provide Monte Carlo simulation results for these estimators.

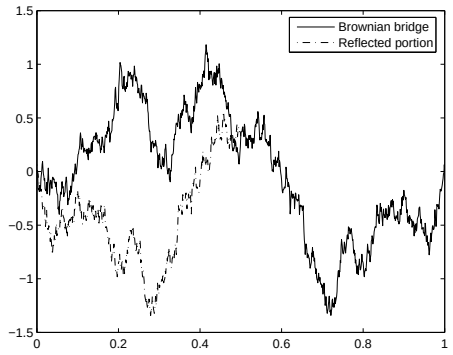
3.1 Folding Operation

This operation starts with a Brownian bridge, reflects its second half portion through the first half (Figure 1(a)), takes the difference between the two portions in $[0, 1/2]$, and stretches the difference over the interval $[0, 1]$ as shown in Figure 1(b) (see Antonini [6] and Shorack and Wellner [31]).

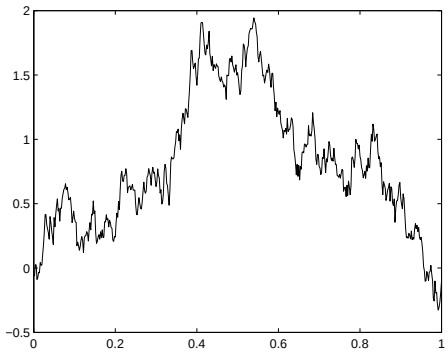
The level-k folded Brownian bridge, denoted by $\mathcal{B}_{(k)}(\cdot)$, is defined recursively from the level-$(k-1)$ Brownian bridge as follows:

$$\mathcal{B}_{(k)}(t) \equiv \mathcal{B}_{(k-1)}\Big(\tfrac{t}{2}\Big) - \mathcal{B}_{(k-1)}\Big(1 - \tfrac{t}{2}\Big), \quad \text{for } t \in [0, 1], \tag{21}$$

where $\mathcal{B}_{(0)}(t) \equiv \mathcal{B}(t)$ is also called the level-0 Brownian bridge. Indeed, it can be shown that $\mathcal{B}_{(k)}(\cdot)$, $k \geq 1$, are also Brownian bridge processes (Antonini [6]).



(a) Reflection around $t = \frac{1}{2}$



(b) Differencing and Stretching

Figure 1: Level-1 Folded Brownian Bridge

The level-k folded STS, $T_{(k),n}(\cdot)$, is computed in an analogous fashion:

$$T_{(k),n}(t) \equiv T_{(k-1),n}\Big(\tfrac{t}{2}\Big) - T_{(k-1),n}\Big(1 - \tfrac{t}{2}\Big), \quad \text{for } t \in [0,1] \text{ and } k = 1, 2, \ldots,$$

where we refer to $T_{(0),n}(\cdot) \equiv T_n(\cdot)$ as the level-0 STS. Lemma 1 expresses the folded STS in terms of the original observations. Henceforth, we define $S_j \equiv \sum_{\ell=1}^{j} X_\ell$, for $j = 1, 2, \ldots, n$, with $S_0 \equiv 0$.

Lemma 1 (Antonini [6]) For $k = 1, 2, \ldots$ and $j = 0, 1, \ldots, n$,

$$\begin{aligned}&\sigma\sqrt{n}\, T_{(k),n}(\tfrac{j}{n}) \\ &\quad = \sum_{\ell=1}^{2^{k-1}} \left\{ \left(\left\lfloor \frac{n(\ell-1)}{2^{k-1}} + \frac{j}{2^k} \right\rfloor - \left\lfloor \frac{n\ell}{2^{k-1}} - \frac{j}{2^k} \right\rfloor \right) \bar{X}_n + S_{\lfloor \frac{n\ell}{2^{k-1}} - \frac{j}{2^k} \rfloor} - S_{\lfloor \frac{n(\ell-1)}{2^{k-1}} + \frac{j}{2^k} \rfloor} \right\}.\end{aligned}$$

The following result from Antonini [6] shows that the joint distribution of folded STSs converges to the joint distribution of the analogous folded Brownian bridges.

Theorem 2 If Assumptions A hold, then for $k = 0, 1, \ldots$, we have

$$\boldsymbol{T}(\cdot) \equiv \Big[T_{(0),n}(\cdot), \ldots, T_{(k),n}(\cdot)\Big] \xrightarrow[n\to\infty]{\mathcal{D}} \boldsymbol{\mathcal{B}}(\cdot) \equiv \Big[\mathcal{B}_{(0)}(\cdot), \ldots, \mathcal{B}_{(k)}(\cdot)\Big],$$

where $\{\boldsymbol{\mathcal{B}}(t) : t \in [0,1]\}$ is a multivariate Brownian bridge process whose component univariate Brownian bridge processes have the following cross-covariances: for $s, t \in [0,1]$ and $j, \ell \in \{0, 1, 2, \ldots, k\}$,

$$\mathrm{Cov}[\mathcal{B}_{(j)}(s), \mathcal{B}_{(\ell)}(t)] = \mathrm{E}[\mathcal{B}_{(0)}(s), \mathcal{B}_{(|j-\ell|)}(t)].$$

Moreover, $\sqrt{n}\big(\bar{X}_n - \mu\big)$ and $\boldsymbol{T}(\cdot)$ are asymptotically independent as $n \to \infty$.

Remark 10 The univariate Brownian bridge processes $\mathcal{B}_{(0)}(\cdot)$, ..., $\mathcal{B}_{(k)}(\cdot)$ that constitute the components of $\boldsymbol{\mathcal{B}}(\cdot)$ are not independent. In addition, because $\boldsymbol{\mathcal{B}}(\cdot)$ is a multivariate Gaussian process with mean $\mathrm{E}[\boldsymbol{\mathcal{B}}(t)] = \mathbf{0}_{k+1}$, the $(k+1)$-dimensional vector of zeros, for all $t \in [0,1]$, Theorem 2 completely characterizes the asymptotic probability law governing the behavior of the multivariate STS process $\{\boldsymbol{T}(t) : t \in [0,1]\}$ as the simulation run length $n \to \infty$. $\triangleleft$

3.2 Folded Area Estimator

As in Alexopoulos et al. [5], the level-k folded area estimator $\mathcal{A}_{(k)}(f;n)$ and its limiting functional $\mathcal{A}_{(k)}(f)$ are

$$\mathcal{A}_{(k)}(f;n) \equiv \left[\frac{1}{n}\sum_{j=1}^{n} f\Big(\tfrac{j}{n}\Big)\sigma T_{(k),n}\Big(\tfrac{j}{n}\Big)\right]^2 \quad \text{and} \quad \mathcal{A}_{(k)}(f) \equiv \left[\int_0^1 f(t)\sigma\mathcal{B}_{(k)}(t)\,\mathrm{d}t\right]^2,$$

respectively.

Theorem 3 shows that the folded area estimators have the anticipated joint convergence.

Theorem 3 (Alexopoulos et al. [5]) If Assumptions A and F hold, then

$$\boldsymbol{\mathcal{A}}(f;n) \equiv [\mathcal{A}_{(0)}(f;n), \ldots, \mathcal{A}_{(k)}(f;n)] \xrightarrow[n\to\infty]{\mathcal{D}} \boldsymbol{\mathcal{A}}(f) \equiv [\mathcal{A}_{(0)}(f), \ldots, \mathcal{A}_{(k)}(f)].$$

Further, $\sqrt{n}\big(\bar{X}_n - \mu\big)$ is asymptotically independent of $\boldsymbol{\mathcal{A}}(f;n)$ as $n \to \infty$.

The following remarkable result is also from Alexopoulos et al. [5].

Corollary 1 Under the conditions of Theorem 3, the random variables $\{\mathcal{A}_{(k)}(f) : k = 0, 1, \ldots\}$ are i.i.d. $\sigma^2\chi^2_1$; and thus, $\mathcal{A}_{(0)}(f;n), \ldots, \mathcal{A}_{(k)}(f;n)$ are asymptotically i.i.d. $\sigma^2\chi^2_1$.

3.2.1 Expected Value

The following theorem gives a detailed expression for expected value of the level-1 folded area estimator. The more compact expression (22) is due to Antonini [6]. The proof is in Appendix A.1.

Theorem 4 Under Assumptions A and F, and for even n,

$$
\begin{aligned}
&\mathrm{E}[\mathcal{A}_{(1)}(f;n)] \\
&= \frac{1}{n^2}\left[\sum_{j=1}^{n}\left(\tfrac{j}{n}-1\right)f\left(\tfrac{j}{n}\right)\right]^2 (R_0 + 2\gamma_{0,n-1} - \tfrac{2}{n}\gamma_{1,n-1}) \\
&\quad - \frac{2}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{n} f\left(\tfrac{j}{n}\right)f\left(\tfrac{\ell}{n}\right)\Big[\Big(\tfrac{(j-n)(\ell-n)}{n} + \lfloor \tfrac{j}{2}\rfloor\Big)R_0 + \left(\tfrac{j}{n}-1\right)\Big[\left\lfloor\tfrac{\ell}{2}\right\rfloor \gamma_{0,\lfloor\frac{\ell}{2}\rfloor-1} - \gamma_{1,\lfloor\frac{\ell}{2}\rfloor-1} \\
&\qquad + \left(\left\lfloor\tfrac{\ell}{2}\right\rfloor - n\right)\gamma_{0,n-\lfloor\frac{\ell}{2}\rfloor} + \gamma_{1,n-\lfloor\frac{\ell}{2}\rfloor} + \left(n - \left\lfloor n-\tfrac{\ell}{2}\right\rfloor\right)\gamma_{0,n-\lfloor n-\frac{\ell}{2}\rfloor} - \gamma_{1,n-\lfloor n-\frac{\ell}{2}\rfloor} \\
&\qquad - \left\lfloor n-\tfrac{\ell}{2}\right\rfloor \gamma_{0,\lfloor n-\frac{\ell}{2}\rfloor-1} + \gamma_{1,\lfloor n-\frac{\ell}{2}\rfloor-1}\Big] + \left(\left\lfloor\tfrac{j}{2}\right\rfloor - \left\lfloor n-\tfrac{\ell}{2}\right\rfloor\right)\gamma_{0,\lfloor n-\frac{\ell}{2}\rfloor-\lfloor\frac{j}{2}\rfloor} \\
&\qquad + \gamma_{1,\lfloor n-\frac{\ell}{2}\rfloor-\lfloor\frac{j}{2}\rfloor}\Big] \\
&\quad + \frac{1}{n^3}\sum_{j=1}^{n}\sum_{\ell=j}^{n} f\left(\tfrac{j}{n}\right)f\left(\tfrac{\ell}{n}\right)\Big[\Big(\lfloor\tfrac{j}{2}\rfloor + \lfloor n-\tfrac{\ell}{2}\rfloor\Big)R_0 + \left(\left\lfloor\tfrac{j}{2}\right\rfloor - \left\lfloor\tfrac{\ell}{2}\right\rfloor\right)\gamma_{0,\lfloor\frac{\ell}{2}\rfloor-\lfloor\frac{j}{2}\rfloor} + \gamma_{1,\lfloor\frac{\ell}{2}\rfloor-\lfloor\frac{j}{2}\rfloor} \\
&\qquad + \left(\lfloor n-\tfrac{\ell}{2}\rfloor - \lfloor n-\tfrac{j}{2}\rfloor\right)\gamma_{0,\lfloor n-\frac{j}{2}\rfloor-\lfloor n-\frac{\ell}{2}\rfloor} + \gamma_{1,\lfloor n-\frac{j}{2}\rfloor-\lfloor n-\frac{\ell}{2}\rfloor}\Big] \\
&\quad + \frac{1}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{j-1} f\left(\tfrac{j}{n}\right)f\left(\tfrac{\ell}{n}\right)\Big[\Big(\left\lfloor\tfrac{\ell}{2}\right\rfloor + \left\lfloor n-\tfrac{j}{2}\right\rfloor\Big)R_0 + \left(\left\lfloor\tfrac{\ell}{2}\right\rfloor - \left\lfloor\tfrac{j}{2}\right\rfloor\right)\gamma_{0,\lfloor\frac{j}{2}\rfloor-\lfloor\frac{\ell}{2}\rfloor} + \gamma_{1,\lfloor\frac{j}{2}\rfloor-\lfloor\frac{\ell}{2}\rfloor} \\
&\qquad + \left(\left\lfloor n-\tfrac{j}{2}\right\rfloor - \left\lfloor n-\tfrac{\ell}{2}\right\rfloor\right)\gamma_{0,\lfloor n-\frac{\ell}{2}\rfloor-\lfloor n-\frac{j}{2}\rfloor} + \gamma_{1,\lfloor n-\frac{\ell}{2}\rfloor-\lfloor n-\frac{j}{2}\rfloor}\Big] \\
&= \sigma^2 - \frac{\bar{F}^2\gamma_1}{n} + o(1/n). \qquad (22)
\end{aligned}
$$

Example 13 For the constant weight function $f_0(\cdot)$, Theorem 4 yields

$$\begin{aligned}
\mathrm{E}[\mathcal{A}_{(1)}(f_0;n)] &= \Big(1-\frac{1}{n^2}\Big)R_0 + \sum_{j=1}^{n/2-1}\Big(2-\frac{6j}{n}-\frac{2}{n^2}-\frac{24j^2}{n^2}+\frac{6j}{n^3}+\frac{48j^3}{n^3}\Big)R_j \qquad (23)\\
&\quad + \sum_{j=n/2}^{n-1}\Big(-2-\frac{6j}{n}+\frac{2}{n^2}+\frac{24j^2}{n^2}-\frac{2j}{n^3}-\frac{16j^3}{n^3}\Big)R_j\\
&= \sigma^2 - \frac{3\gamma_1}{n} - \frac{\sigma^2+12\gamma_2}{n^2} + \frac{3(\gamma_1+8\gamma_3)}{n^3} + O(\delta^n) \qquad (24)\\
&= \sigma^2 - 3\gamma_1/n + o(1/n). \qquad (25)
\end{aligned}$$

Note that, although Equation (24) has different terms than Equation (7), Equation (25) is the same as Equation (8). $\triangleleft$

Example 14 The weight functions $f_2(\cdot)$ and $f_{\cos,j}(\cdot)$, $j = 1, 2, \ldots$, from Examples 2 and 3 yield first-order unbiased folded area estimators as well. From Theorem 4, we can see that

$$\begin{aligned}
&\mathrm{E}[\mathcal{A}_{(1)}(f_2;n)]\\
&= \Big(1+\frac{7}{2n^2}+\frac{63}{2n^4}-\frac{36}{n^6}\Big)R_0 + \sum_{j=1}^{\frac{n}{2}}\Bigg(2+\frac{7-168j^2}{n^2}+\frac{105j+840j^3}{n^3}-\frac{1680j^4-63}{n^4}\\
&\qquad + \frac{840j^2}{n^4} - \frac{441j-840j^3-2016j^5}{n^5} - \frac{72-756j^2-1680j^4+2688j^6}{n^6}\\
&\qquad + \frac{216j-504j^3-2016j^5+2304j^7}{n^7}\Bigg)R_j\\
&\quad + \sum_{j=\frac{n}{2}+1}^{n-1}(n-j)\left(\frac{40}{n}-\frac{128j}{n^2}+\frac{203+40j^2}{n^3}-\frac{672j+240j^3}{n^4}-\frac{315-168j^2}{n^5}\right.\\
&\qquad \left. - \frac{1440j^4}{n^5} + \frac{588j+1008j^3-1920j^5}{n^6} + \frac{72-168j^2-672j^4+768j^6}{n^7}\right)R_j\\
&= \sigma^2 + \frac{7(\sigma^2-168\gamma_2)}{2n^2} + \frac{105(\gamma_1+8\gamma_3)}{2n^3} + O(1/n^4). \quad \triangleleft
\end{aligned}$$

Remark 11 Notice that the multipliers of γ_i associated with the second- and third-order bias terms tend to be higher for the level-1 estimators than for the corresponding level-0 estimators. $\triangleleft$

Further, if we assume that the sequence $\{\mathcal{A}^2_{(1)}(f;n) : n = 1, 2, \ldots\}$ is uniformly integrable, then

$$\lim_{n\to\infty} \mathrm{Var}[\mathcal{A}_{(1)}(f;n)] = \mathrm{Var}[\mathcal{A}_{(1)}(f)] = 2\sigma^4. \tag{26}$$

Remark 12 As with the weighted area estimator, the limiting variance (26) does not depend on the weight function. $\triangleleft$

3.2.2 Linear Combinations of Folded Area Estimators

At this point, we have a number of individual level-0 and level-1 folded area variance estimators, some of which are first-order unbiased. All area estimators—whether or not they are folded—have asymptotic variance of $2\sigma^4$.

We can form new estimators with lower variance by taking linear combinations of the previously described estimators. The calculation of the expected value of such a linear combination is easy. In order to calculate the variance, we will need to obtain the covariances of the constituents of the linear combination. Theorem 5 computes such covariances for level-0 and level-k folded area estimators.

Theorem 5 (Antonini et al. [7]) Suppose that Assumptions A hold and that $f_x(\cdot)$ and $f_y(\cdot)$ are weight functions satisfying Assumptions F. In addition, suppose that the sequence $\{\mathcal{A}_{(0)}(f_x;n)\mathcal{A}_{(k)}(f_y;n) : n = 1, 2, \ldots\}$ is uniformly integrable. Then for $k \geq 1$, we have

$$\begin{aligned}\mathrm{Cov}[\mathcal{A}_{(0)}(f_x;n), \mathcal{A}_{(k)}(f_y;n)] &\underset{n\to\infty}{\longrightarrow} \mathrm{Cov}[\mathcal{A}_{(0)}(f_x), \mathcal{A}_{(k)}(f_y)] \\ &= 2\sigma^4 \left[\sum_{j=1}^{2^{k-1}} \int_0^1 f_y(s) \left\{\bar{F}_x\left(\frac{j}{2^{k-1}} - \frac{s}{2^k}\right) - \bar{F}_x\left(\frac{j-1}{2^{k-1}} + \frac{s}{2^k}\right)\right\} \mathrm{d}s - \bar{F}_x\bar{F}_y\right]^2, \end{aligned} \tag{27}$$

where $\bar{F}_x(u) = \int_0^u F_x(t)\,\mathrm{d}t$ and $F_x(u) = \int_0^u f_x(t)\,\mathrm{d}t$ for $0 < u < 1$.

Example 15 Consider the weight functions $f_0(\cdot)$, $f_2(\cdot)$, and $f_{\cos,j}(\cdot)$. By Theorem 5, we have:

- $\mathrm{Cov}[\mathcal{A}_{(0)}(f_0), \mathcal{A}_{(k)}(f_0)] = 0$, for any level $k \geq 1$, an obvious fact in light of Corollary 1.

- $\mathrm{Cov}[\mathcal{A}_{(0)}(f_2), \mathcal{A}_{(k)}(f)] = 0$, for any level $k \geq 1$ and weight function $f(\cdot)$ satisfying Assumptions F; this follows since $\bar{F} = 0$ and

$$\sum_{j=1}^{2^{k-1}} \left\{ \bar{F}\left(\frac{j}{2^{k-1}} - \frac{s}{2^k}\right) - \bar{F}\left(\frac{j-1}{2^{k-1}} + \frac{s}{2^k}\right) \right\} \;=\; 0.$$

- $\mathrm{Cov}[\mathcal{A}_{(0)}(f_{\cos,j}), \mathcal{A}_{(k)}(f)] = 0$ for any $j, k \geq 1$ and any weight function $f(\cdot)$ satisfying Assumptions F, by the same reasoning as above.

- By the joint normality of the limiting area functionals, zero covariance implies independence.

These pleasant results will allow us to construct simple linear combinations of area estimators that have about the same bias as their individual components, but at the same time achieve a 50% variance reduction. For example, if we define the linear combination estimator $\bar{\mathcal{A}}_{(0,1)}(f;n) \equiv [\mathcal{A}_{(0)}(f;n) + \mathcal{A}_{(1)}(f;n)]/2$ and the limiting functional $\bar{\mathcal{A}}_{(0,1)}(f) \equiv [\mathcal{A}_{(0)}(f) + \mathcal{A}_{(1)}(f)]/2$ for any weight function $f(\cdot)$ satisfying Assumptions F, we see that Equation (4) and Theorem 4 give

$$\mathrm{E}[\bar{\mathcal{A}}_{(0,1)}(f;n)] = \sigma^2 - \frac{[(F-\bar{F})^2 + 3\bar{F}^2]\gamma_1}{4n} + O(1/n^2),$$

which converges to σ^2 as $n \to \infty$, and is first-order unbiased for weight functions $f_2(\cdot)$ and $f_{\cos,j}(\cdot)$. Further, for $f = f_0$, f_2, or $f_{\cos,j}$, we have $\mathrm{Cov}[\mathcal{A}_{(0)}(f), \mathcal{A}_{(1)}(f)] = 0$, so that

$$\mathrm{Var}[\bar{\mathcal{A}}_{(0,1)}(f;n)] \underset{n\to\infty}{\longrightarrow} \mathrm{Var}[\bar{\mathcal{A}}_{(0,1)}(f)] = \frac{1}{4}\Big(\mathrm{Var}[\mathcal{A}_{(0)}(f)] + \mathrm{Var}[\mathcal{A}_{(1)}(f)]\Big) = \sigma^4. \quad \triangleleft$$

3.3 Batched Folded Area Estimator

We define the level-k folded STS obtained from nonoverlapping batch i as

$$T_{(k),i,m}(t) \equiv T_{(k-1),i,m}(\tfrac{t}{2}) - T_{(k-1),i,m}(1-\tfrac{t}{2}), \quad \text{for } t \in [0,1],\ k = 1, 2, \ldots \text{ and } i = 1, \ldots, b.$$

For each $k = 0, 1, \ldots$ and $i = 1, 2, \ldots, b$, the level-k folded area estimator for σ^2 from the ith batch and the respective square of the weighted area under the level-k Brownian bridge are

$$A_{(k),i}(f;m) \equiv \left[\frac{1}{m}\sum_{j=1}^{m} f\left(\tfrac{j}{m}\right)\sigma T_{(k),i,m}\left(\tfrac{j}{m}\right)\right]^2 \text{ and } A_{(k),i}(f) \equiv \left[\int_0^1 f(t)\sigma\mathcal{B}_{(k),i}(t)\,\mathrm{d}t\right]^2.$$

Remark 13 Corollary 1 and the fact that, for fixed k, $\{\mathcal{B}_{(k),i}(\cdot) : i = 1, 2, \ldots, b\}$ are independent Brownian bridges imply that $\{A_{(k),i}(f) : k = 0, 1, \ldots; i = 1, 2, \ldots, b\}$ are i.i.d. $\sigma^2\chi_1^2$. $\triangleleft$

The respective level-k folded batched area estimator for σ^2 is

$$\mathcal{A}_{(k)}(f;b,m) \equiv \frac{1}{b}\sum_{i=1}^{b} A_{(k),i}(f;m).$$

The next theorem (which follows from Remark 13) and Corollary 1 give limiting results for the level-k batched folded area estimator.

Theorem 6 Under Assumptions A and F, for fixed b and $k = 0, 1, \ldots$,

$$\mathcal{A}_{(k)}(f;b,m) \xrightarrow[m\to\infty]{\mathcal{D}} \frac{1}{b}\sum_{i=1}^{b} A_{(k),i}(f) \sim \frac{\sigma^2\chi_b^2}{b}.$$

Remark 14 We can find the expected value of the level-1 batched folded area estimator if we replace n with m in Theorem 4. Hence we get

$$\mathrm{E}[\mathcal{A}_{(1)}(f;b,m)] = \sigma^2 - \frac{\bar{F}^2\gamma_1}{m} + o(1/m). \tag{28}$$

Also, for different weight functions, the expected results can be found if we do the same replacement in Examples 13 and 14. $\triangleleft$

Corollary 2 Suppose that the Assumptions A and F hold, and further, for fixed $k = 0, 1, \ldots$, the sequence $\left\{\mathcal{A}_{(k)}^2(f;b,m) : m = 1, 2, \ldots\right\}$ is uniformly integrable. Then

$$\mathrm{E}[\mathcal{A}_{(k)}(f;b,m)] \xrightarrow[m\to\infty]{} \sigma^4 \quad \text{and} \quad \mathrm{Var}[\mathcal{A}_{(k)}(f;b,m)] \xrightarrow[m\to\infty]{} \frac{2\sigma^4}{b}.$$

The next theorem, which is analogous to Theorem 5, gives the asymptotic covariance between batched folded area estimators from levels $k \geq 0$.

Theorem 7 (Antonini et al. [7]) Suppose that Assumptions A hold and that $f_x(\cdot)$ and $f_y(\cdot)$ are weight functions satisfying Assumptions F. In addition, suppose that the sequences $\{A_{(0),i}(f_x; b, m)A_{(k),\ell}(f_y; b, m) : m = 1, 2, \ldots\}$, for $i, \ell \in \{1, 2, \ldots, b\}$, are uniformly integrable. Then for $k \geq 1$ and fixed b, we have

$$\mathrm{Cov}[\mathcal{A}_{(0)}(f_x; b, m), \mathcal{A}_{(k)}(f_y; b, m)] \underset{m\to\infty}{\longrightarrow} \frac{\mathrm{Cov}[\mathcal{A}_{(0)}(f_x), \mathcal{A}_{(k)}(f_y)]}{b},$$

where $\mathrm{Cov}[\mathcal{A}_{(0)}(f_x), \mathcal{A}_{(k)}(f_y)]$ is given in Equation (27).

3.3.1 Linear Combinations of Batched Folded Area Estimators

Theorem 7 suggests that we can combine batching and folding over multiple levels. To this end, we have a simple version of such an estimator, constructed only from the level-0 and level-1 folding.

The linearly combined batched folded area estimator for σ^2 is

$$\bar{\mathcal{A}}_{(0,1)}(f; b, m) \equiv \frac{\mathcal{A}_{(0)}(f; b, m) + \mathcal{A}_{(1)}(f; b, m)}{2}.$$

The following result follows from Corollary 2 and Theorem 7.

Corollary 3 Suppose that Assumptions A and F hold, and further, for fixed b, the sequence $\{\bar{\mathcal{A}}_{(0,1)}(f; b, m) : m = 1, 2, \ldots\}$ is uniformly integrable. Then

$$\mathrm{E}[\bar{\mathcal{A}}_{(0,1)}(f; b, m)] \underset{m\to\infty}{\longrightarrow} \sigma^4 \quad \text{and} \quad \mathrm{Var}[\bar{\mathcal{A}}_{(0,1)}(f; b, m)] \underset{m\to\infty}{\longrightarrow} \frac{\sigma^4}{b}.$$

3.4 Folded CvM Estimator

A similar development holds for the folded CvM estimators.

Definition 1 For $k = 0, 1, \ldots$, the level-k folded CvM estimator for σ^2 is

$$\mathcal{C}_{(k)}(g; n) \equiv \frac{1}{n}\sum_{j=1}^{n} g(\tfrac{j}{n})\sigma^2 T^2_{(k),n}(\tfrac{j}{n}),$$

where $g(\cdot)$ is a weight function satisfying Assumptions G. For $k = 0, 1, \ldots$, the weighted area under the square of the level-k Brownian bridge is

$$\mathcal{C}_{(k)}(g) \equiv \int_0^1 g(t)\sigma^2 \mathcal{B}^2_{(k)}(t)\,\mathrm{d}t. \quad \triangleleft$$

Theorem 8 (Antonini et al. [7]) If Assumptions A and G hold, then

$$\boldsymbol{\mathcal{C}}(g;n) \equiv [\mathcal{C}_{(0)}(g;n), \ldots, \mathcal{C}_{(k)}(g;n)] \xrightarrow[n\to\infty]{\mathcal{D}} \boldsymbol{\mathcal{C}}(g) \equiv [\mathcal{C}_{(0)}(g), \ldots, \mathcal{C}_{(k)}(g)].$$

Corollary 4 Suppose that Assumptions A and G hold and further, for fixed $k = 0, 1, \ldots$, the sequence $\left\{\mathcal{C}^2_{(k)}(g;n) : n = 1, 2, \ldots\right\}$ is uniformly integrable. Then

$$\lim_{n\to\infty} \mathrm{E}[\mathcal{C}_{(k)}(g;n)] = \mathrm{E}[\mathcal{C}_{(k)}(g)] = \sigma^2$$

and

$$\lim_{n\to\infty} \mathrm{Var}[\mathcal{C}_{(k)}(g;n)] = \mathrm{Var}[\mathcal{C}_{(k)}(g)] = 4\sigma^4 \int_0^1 g(t)(1-t)^2 \int_0^t g(s)s^2\,\mathrm{d}s\,\mathrm{d}t.$$

Remark 15 Unlike the area estimator, the asymptotic variance of the level-k folded CvM estimator does indeed depend on the weight function $g(\cdot)$. $\triangleleft$

Theorem 9 gives the first-order bias of the level-1 folded CvM estimator.

Theorem 9 (Antonini [6]) If Assumptions A and G hold, and n is even, then

$$\mathrm{E}[\mathcal{C}_{(1)}(g;n)] = \sigma^2 - \frac{(G + \bar{G} - 1)\gamma_1}{n} + O(1/n^2).$$

Example 16 The constant weight function $g_0(t) = 6$ yields $\mathrm{E}[\mathcal{C}_{(1)}(g_0, n)] = \sigma^2 - 8\gamma_1/n + O(1/n^2)$ and $\mathrm{Var}[\mathcal{C}_{(1)}(g_0, n)] = 0.8\sigma^4$. This estimator has the same asymptotic variance as, but significantly larger bias than, the original level-0 CvM estimator $\mathcal{C}_{(0)}(g_0, n)$, whose bias is about $5\gamma_1/n$ (see Example 4). $\triangleleft$

Remark 16 We will refer to the weight function $g_2^\star(\cdot)$ in Example 5 as $g_{0,2}^\star(\cdot)$ with the subscript 0 indicating that this weight function is used in level-0 CvM estimators. Similarly, we will use $g_{0,4}^\star(\cdot)$ to denote the weight function $g_4^\star(\cdot)$ in Example 6. $\triangleleft$

Example 17 By using Lagrange multipliers as in Goldsman et al. [19], we can find polynomial weights that minimize the limiting variance of the level-1 folded CvM estimator for σ^2 while satisfying the first-order unbiasedness and normalizing constraints. That is, we identify a function $g(\cdot)$ that minimizes $\text{Var}[\mathcal{C}_{(1)}(g)]$ subject to Assumptions G and $G + \bar{G} = 1$. For example, the minimum-variance, first-order unbiased quadratic weight is $g^\star_{1,2}(t) \equiv -180t^2 + 168t - 24$, resulting in a limiting variance of $\text{Var}[\mathcal{C}_{(1)}(g^\star_{1,2})] = 72\sigma^4/35 \doteq 2.057\sigma^4$—a bit greater than that of the analogous level-0 quadratically weighted CvM estimator, which equals $1.729\sigma^4$ (see Example 5). The first subscript of the weight again denotes the folding level. $\triangleleft$

Example 18 Similarly (see Antonini [6] for details), the asymptotically variance-optimal, first-order unbiased, level-1 quartic weight is

$$g^\star_{1,4}(t) \;\equiv\; -60 + \frac{2840t}{3} - 3860t^2 + 5920t^3 - \frac{9100t^4}{3},$$

for which the limiting variance is $\text{Var}[\mathcal{C}_{(1)}(g^\star_{1,4})] = 2360\sigma^4/2079 \approx 1.135\sigma^4$. This variance is a bit larger than that of its level-0 analog, which is $1.042\sigma^4$ (Example 6). $\triangleleft$

Remark 17 In order to achieve further variance reductions, we can continue to increase the degree of the polynomial weight function. However, the magnitudes of the resulting coefficients become quite large, and one must be careful to avoid round-off errors as well as deleterious second-order effects for small sample sizes. $\triangleleft$

3.4.1 Fine-tuned Expected Value

We proceed with the derivation of detailed expressions for the expectations of the various level-1 folded CvM estimators. The proof of Theorem 10 is in Appendix A.2.

Theorem 10 Suppose that Assumptions A and G hold, and n is even. Then, for the constant weight function $g_0(\cdot)$,

$$\text{E}[\mathcal{C}_{(1)}(g_0; n)] = R_0\Big(1 - \frac{1}{n^2}\Big) + 2\sum_{j=1}^{n-1} R_j\Big(1 - \frac{8j}{n} - \frac{1}{n^2} + \frac{15j^2}{n^2} + \frac{j}{n^3} - \frac{8j^3}{n^3}\Big).$$

In addition, the quadratic weight function $g^{\star}_{1,2}(\cdot)$ yields

$$\begin{aligned}
&\mathrm{E}[\mathcal{C}_{(1)}(g^{\star}_{1,2};n)] \\
&= R_0\Big(1+\frac{5}{n^2}-\frac{6}{n^4}\Big)+2\sum_{j=1}^{n/2-1}R_j\Big(1+\frac{5-66j^2}{n^2}-\frac{48j-288j^3}{n^3} \\
&\qquad -\frac{6-131j^2+479j^4}{n^4}+\frac{288j^5}{n^5}+\frac{6j}{n^5}-\frac{120j^3}{n^5}\Big) \\
&\quad +2\sum_{j=n/2}^{n-1}R_j\Big(-31+\frac{256j}{n}+\frac{37-834j^2}{n^2}-\frac{176j-1312j^3}{n^3} \\
&\qquad -\frac{6-259j^2+991j^4}{n^4}+\frac{6j-120j^3+288j^5}{n^5}\Big),
\end{aligned}$$

and the quartic weight function $g^{\star}_{1,4}(\cdot)$ gives

$$\begin{aligned}
&\mathrm{E}[\mathcal{C}_{(1)}(g^{\star}_{1,4};n)] \\
&= R_0\Big(1+\frac{110}{9n^2}-\frac{769}{9n^4}+\frac{650}{9n^6}\Big) \\
&\quad +2\sum_{j=1}^{n/2-1}R_j\Big(1+\frac{110/3-490j^2/}{3n^2}-\frac{790j-4600j^3}{3n^3}-\frac{769-14645j^2+60905j^4}{9n^4} \\
&\qquad +\frac{5642j/3-63340j^3/3+47876j^5}{3n^5}+\frac{1300/3-5827j^2+109715j^4/3}{3n^6} \\
&\qquad -\frac{172814j^6/3}{3n^6}-\frac{650j-36400j^3+145600j^5-166400j^7}{9n^7}\Big) \\
&\quad +\sum_{j=n/2}^{n-1}R_j\Big(-\frac{3626}{3}+\frac{116992j}{9n}+\frac{22300/3-177620j^2}{3n^2}-\frac{167620j-1330640j^3}{9n^3} \\
&\qquad -\frac{12722/3-165470j^2+655430j^4}{3n^4}+\frac{56020j/3-723160j^3/3+572936j^5}{3n^5} \\
&\qquad +\frac{1300/3-26566j^2+517670j^4/3-822812j^6/3}{3n^6} \\
&\qquad -\frac{1300j-36400j^3+145600j^5-166400j^7}{9n^7}\Big).
\end{aligned}$$

Remark 18 From Theorem 10, we can readily obtain the following simpler expressions:

$$\begin{aligned}
\mathrm{E}[\mathcal{C}_{(1)}(g_0;n)] &= \sigma^2-\frac{8\gamma_1}{n}+\frac{15\gamma_2-\sigma^2}{n^2}+\frac{\gamma_1-8\gamma_3}{n^3}+O(\delta^n) \\
\mathrm{E}[\mathcal{C}_{(1)}(g^{\star}_{1,2};n)] &= \sigma^2+\frac{5\sigma^2-66\gamma_2}{n^2}+\frac{48(6\gamma_3-\gamma_1)}{n^3}+O(1/n^4) \\
\mathrm{E}[\mathcal{C}_{(1)}(g^{\star}_{1,4};n)] &= \sigma^2+\frac{10(11\sigma^2-294\gamma_2)}{9n^2}+\frac{20(460\gamma_3-79\gamma_1)}{3n^3}+O(1/n^4). \quad \triangleleft
\end{aligned}$$

As in the case of the area estimators, we see that the various bias terms tend to be higher for the level-1 CvM estimators than for the corresponding level-0 CvM estimators.

3.4.2 Linear Combinations of Folded CvM Estimators

The next theorem derives the covariance between folded CvM estimators at different levels. Its proof is in Appendix A.3.

Theorem 11 Suppose that Assumptions A hold and that $g_x(\cdot)$ and $g_y(\cdot)$ are weight functions satisfying Assumptions G. In addition, suppose that the sequence $\{\mathcal{C}_{(0)}(g_x;n)\mathcal{C}_{(1)}(g_y;n) : n = 1, 2, \ldots\}$ is uniformly integrable. Then,

$$\begin{aligned}
&\mathrm{Cov}[\mathcal{C}_{(0)}(g_x;n), \mathcal{C}_{(1)}(g_y;n)] \underset{n\to\infty}{\longrightarrow} \mathrm{Cov}[\mathcal{C}_{(0)}(g_x), \mathcal{C}_{(1)}(g_y)] \\
&\quad = 2\sigma^4 \int_0^1 g_y(s) \left[\int_0^{\frac{s}{2}} g_x(t) t^2 (1-s)^2 \, \mathrm{d}t + \int_{\frac{s}{2}}^{1-\frac{s}{2}} g_x(t) s^2 \left(\tfrac{1}{2} - t\right)^2 \, \mathrm{d}t \right. \\
&\qquad \left. + \int_{1-\frac{s}{2}}^1 g_x(t)(1-s)^2(1-t)^2 \, \mathrm{d}t \right] \mathrm{d}s.
\end{aligned}$$

At this point, we can take advantage of the covariance information to obtain improved estimators for σ^2. As a simple example, let us define the linear combination $\bar{C}_{(0,1)}(g_x, g_y; \alpha; n) \equiv \alpha C_{(0)}(g_x;n) + (1-\alpha) C_{(1)}(g_y;n)$ and the limiting functional $\bar{C}_{(0,1)}(g_x, g_y; \alpha) \equiv \alpha C_{(0)}(g_x) + (1-\alpha) C_{(1)}(g_y)$ for any real α and appropriate weight functions $g_x(\cdot)$ and $g_y(\cdot)$ satisfying Assumption G. Then Equation (10) and Theorem 9 give

$$\mathrm{E}[\bar{\mathcal{C}}_{(0,1)}(g_x, g_y; \alpha; n)] = \sigma^2 - \frac{[\alpha G_x + (1-\alpha)(G_y + \bar{G}_y) - 1]\gamma_1}{n} + O(1/n^2), \qquad (29)$$

which converges to σ^2 as $n \to \infty$. Further, under the conditions of Theorem 11,

$$\begin{aligned}
&\mathrm{Var}[\bar{\mathcal{C}}_{(0,1)}(g_x, g_y; \alpha; n)] \underset{n\to\infty}{\longrightarrow} \mathrm{Var}[\bar{\mathcal{C}}_{(0,1)}(g_x, g_y; \alpha)] \\
&= \alpha^2 \mathrm{Var}[\mathcal{C}_{(0)}(g_x)] + (1-\alpha)^2 \mathrm{Var}[\mathcal{C}_{(1)}(g_y)] + 2\alpha(1-\alpha)\mathrm{Cov}[\mathcal{C}_{(0)}(g_x), \mathcal{C}_{(1)}(g_y)]. \qquad (30)
\end{aligned}$$

The limiting variance can be minimized for the α-value

$$\alpha^\star \;\equiv\; \frac{\mathrm{Var}[\mathcal{C}_{(1)}(g_y)] - \mathrm{Cov}[\mathcal{C}_{(0)}(g_x), \mathcal{C}_{(1)}(g_y)]}{\mathrm{Var}[\mathcal{C}_{(0)}(g_x)] + \mathrm{Var}[\mathcal{C}_{(1)}(g_y)] - 2\mathrm{Cov}[\mathcal{C}_{(0)}(g_x), \mathcal{C}_{(1)}(g_y)]}. \tag{31}$$

Example 19 Consider the linear combination $\bar{\mathcal{C}}_{(0,1)}(g_0, g_0; \alpha; n)$. By Equation (29),

$$\mathrm{E}[\bar{\mathcal{C}}_{(0,1)}(g_0, g_0; \alpha; n)] = \sigma^2 - \frac{13\gamma_1}{2n} + O(1/n^2),$$

indicating very high first-order bias. In addition, by Theorem 11, we have

$$\mathrm{Cov}[\mathcal{C}_{(0)}(g_0; n), \mathcal{C}_{(1)}(g_0; n)] \underset{n\to\infty}{\longrightarrow} \mathrm{Cov}[\mathcal{C}_{(0)}(g_0), \mathcal{C}_{(1)}(g_0)] \;=\; \frac{\sigma^4}{5}. \tag{32}$$

From Examples 4 and 16 and Equations (31) and (32) we obtain $\alpha^\star = 0.5$; and then Equation (30) implies

$$\mathrm{Var}[\bar{\mathcal{C}}_{(0,1)}(g_0, g_0; \alpha^\star, n)] \underset{n\to\infty}{\longrightarrow} \mathrm{Var}[\bar{\mathcal{C}}_{(0,1)}(g_0, g_0; \alpha^\star)] \;=\; \frac{\sigma^4}{2},$$

a variance that is significantly smaller than the limiting variances of the components of the linear combination, since $\mathrm{Var}[\mathcal{C}_{(0)}(g_0)] = \mathrm{Var}[\mathcal{C}_{(1)}(g_0)] = 0.8\sigma^4$. $\triangleleft$

Example 20 Consider the weight functions $g^\star_{0,2}(\cdot)$ and $g^\star_{1,2}(\cdot)$ from Examples 5 and 17, respectively. Performing the same machinations as in Example 19, we obtain

$$\mathrm{E}[\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,2}, g^\star_{1,2}; \alpha; n)] \;=\; \sigma^2 + O(1/n^2),$$

so that the combined estimator is first-order unbiased (like its components). By Theorem 11, we have

$$\mathrm{Cov}[\mathcal{C}_{(0)}(g^\star_{0,2}; n), \mathcal{C}_{(1)}(g^\star_{1,2}; n)] \underset{n\to\infty}{\longrightarrow} \mathrm{Cov}[\mathcal{C}_{(0)}(g^\star_{0,2}), \mathcal{C}_{(1)}(g^\star_{1,2})] \;=\; \frac{3\sigma^4}{28}.$$

As above, $\alpha^\star = 0.546$; and then Equation (30) implies

$$\mathrm{Var}[\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,2}, g^\star_{1,2}; \alpha^\star; n)] \to \mathrm{Var}[\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,2}, g^\star_{1,2}; \alpha^\star)] \;=\; 0.9924\sigma^4,$$

We note that even the naive choice of $\alpha = 0.5$ yields $\mathrm{Var}[\mathcal{C}(g^\star_{0,2}, g^\star_{1,2}; 0.5)] = \sigma^4$, which is nearly as good as that of the linear-combination estimator using the optimal $\alpha^\star$. $\triangleleft$

Example 21 Finally, consider the weight functions $g^\star_{0,4}(\cdot)$ and $g^\star_{1,4}(\cdot)$ from Examples 6 and 18, respectively. Proceeding in a similar fashion, we obtain

$$\mathrm{E}[\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,4}, g^\star_{1,4}; \alpha; n)] \;=\; \sigma^2 + O(1/n^2),$$

so that the combined estimator is first-order unbiased. By Theorem 11, we have

$$\mathrm{Cov}[\mathcal{C}_{(0)}(g^\star_{0,4}; n), \mathcal{C}_{(1)}(g^\star_{1,4}; n)] \;\underset{n\to\infty}{\longrightarrow}\; \mathrm{Cov}[\mathcal{C}_{(0)}(g^\star_{0,4}), \mathcal{C}_{(1)}(g^\star_{1,4})] \;=\; \frac{82091\sigma^4}{261954}. \qquad (33)$$

Examples 6 and 18 and Equations (31) and (33) imply $\alpha^\star = 0.5301$; and then Equation (30) yields

$$\mathrm{Var}[\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,4}, g^\star_{1,4}; \alpha; n)] \;\underset{n\to\infty}{\longrightarrow}\; \mathrm{Var}[\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,4}, g^\star_{1,4}; \alpha)] \;=\; 0.6995\sigma^4.$$

Yet again, we have a first-order unbiased linearly combined estimator with a limiting variance that is substantially smaller than those of its components, as $\mathrm{Var}[\mathcal{C}_{(0)}(g^\star_{0,4})] \doteq 1.042\sigma^4$ and $\mathrm{Var}[\mathcal{C}_{(1)}(g^\star_{1,4})] \doteq 1.135\sigma^4$. Note that the choice of $\alpha = 0.5$ yields $\mathrm{Var}[\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,4}, g^\star_{1,4}; 0.5)] \doteq 0.701\sigma^4$—very close to the optimal result. $\triangleleft$

Remark 19 Since the calculation of the level-k folded STS from the level-$(k-1)$ STS takes $O(n)$ time, and computation of the various area and CvM estimators given the underlying STS takes $O(n)$ time, our level-k folded estimators can be computed in $O(kn)$ time. $\triangleleft$

3.5 Batched Folded CvM Estimator

Definition 2 For each $k = 0, 1, \ldots$ and $i = 1, 2, \ldots, b$, the level-k folded CvM estimator for σ^2 from the ith batch and the respective weighted area under the square of the level-k Brownian bridge are

$$C_{(k),i}(g; m) \;\equiv\; \frac{1}{m}\sum_{j=1}^{m} g(\tfrac{j}{m})\sigma^2 T^2_{(k),i,m}(\tfrac{j}{m}) \text{ and } C_{(k),i}(g) \;\equiv\; \sigma^2 \int_0^1 g(t)\mathcal{B}^2_{(k),i}(t)\,\mathrm{d}t,$$

where $g(\cdot)$ is a weight function satisfying Assumptions G.

Definition 3 For each $k = 0, 1, \ldots$, the level-k batched folded CvM estimator for σ^2 is

$$\mathcal{C}_{(k)}(g; b, m) \equiv \frac{1}{b}\sum_{i=1}^{b} C_{(k),i}(g; m).$$

The next theorem (which follows from Slutsky's theorem; see Karr [22]) and corollary give limiting results for the level-k batched folded CvM estimator.

Theorem 12 If Assumptions A and G hold, then

$$\mathcal{C}_{(k)}(g; b, m) \xrightarrow[m\to\infty]{\mathcal{D}} \frac{1}{b}\sum_{i=1}^{b} C_{(k),i}(g).$$

We now give results that are analogous to those in Corollary 4 and Theorem 9 for the folded CvM estimator.

Corollary 5 Suppose Assumptions A and G hold, and that the sequence $\{C^2_{(1),i}(g; m) : m = 1, 2, \ldots\}$ is uniformly integrable for each $i = 1, 2, \ldots, b$. Then

$$\mathrm{E}[\mathcal{C}_{(1)}(g; b, m)] = \mathrm{E}[C_{(1),i}(g; m)] = \sigma^2 - \frac{(G + \bar{G} - 1)\gamma_1}{m} + O(1/m^2). \tag{34}$$

Further, for fixed b we have

$$\mathrm{Var}[\mathcal{C}_{(k)}(g; b, m)] \xrightarrow[m\to\infty]{} \frac{\mathrm{Var}[\mathcal{C}_{(k)}(g)]}{b} = \frac{4\sigma^4}{b}\int_0^1 g(t)(1-t)^2 \int_0^t g(s)s^2 \,\mathrm{d}s \,\mathrm{d}t. \tag{35}$$

Remark 20 We can obtain fine-tuned results on the expected values of the level-1 folded batched CvM estimators by replacing n by m in Theorem 10. ◁

3.5.1 Linear Combinations of Batched Folded CvM Estimators

The next result, which is analogous to Theorem 11, gives the asymptotic covariance between batched folded CvM estimators from levels $k \geq 0$. Its proof is similar to that of Theorem 7 (though Theorem 11 only gives explicit expressions for the case $k = 1$).

Theorem 13 Suppose Assumptions A and G hold and $\{C_{(0),i}(g_x; m)C_{(k),\ell}(g_y; m) : m = 1, 2, \ldots\}$, for $i, \ell \in \{1, 2, \ldots, b\}$, are uniformly integrable. Then for $k \geq 1$ and fixed b,

$$\mathrm{Cov}[\mathcal{C}_{(0)}(g_x; b, m), \mathcal{C}_{(k)}(g_y; b, m)] \xrightarrow[m\to\infty]{} \frac{\mathrm{Cov}[\mathcal{C}_{(0)}(g_x), \mathcal{C}_{(k)}(g_y)]}{b}.$$

Similar to our work in §3.4.2, we define the linearly combined batched folded CvM estimator as

$$\bar{\mathcal{C}}_{(0,1)}(g_x, g_y; \alpha; b, m) \equiv \alpha\mathcal{C}_{(0)}(g_x; b, m) + (1-\alpha)\mathcal{C}_{(1)}(g_y; b, m).$$

The next result follows from Corollary 5 and Theorem 13.

Corollary 6 Suppose Assumptions A and G hold, and further, for fixed b, the sequence $\left\{\bar{\mathcal{C}}_{(0,1)}(g_x, g_y; \alpha; b, m) : m = 1, 2, \ldots\right\}$ is uniformly integrable. Then

$$\mathrm{E}[\bar{\mathcal{C}}_{(0,1)}(g_x, g_y; \alpha; b, m)] \underset{m\to\infty}{\longrightarrow} \sigma^2$$

and

$$b\,\mathrm{Var}[\bar{\mathcal{C}}_{(0,1)}(g_x, g_y; \alpha; b, m)]$$
$$\underset{m\to\infty}{\longrightarrow} \alpha^2\mathrm{Var}[\mathcal{C}_{(0)}(g_x)] + (1-\alpha)^2\mathrm{Var}[\mathcal{C}_{(1)}(g_y)] + 2\alpha(1-\alpha)\mathrm{Cov}[\mathcal{C}_{(0)}(g_x), \mathcal{C}_{(1)}(g_y)].$$

Table 1 lists the asymptotic bias and variance of some of the variance estimators discussed so far. The linearly combined folded CvM estimators are based on the optimal $\alpha^\star$ values obtained in Examples 19–21 and Corollary 6.

3.6 Examples

In this section, we illustrate the performance of the level-1 folded area and CvM estimators as applied to a number of stochastic processes. First, in §3.6.1 we consider an i.i.d. Gaussian process. Then in §3.6.2 we conduct Monte Carlo studies based on a stationary first-order autoregressive Gaussian process and the delay-time process in a stationary M/M/1 system. §3.6.3 compares the expected values of level-1 folded estimators and some of the other estimators. In §3.6.4, we estimate the limiting distributions of folded estimators discussed in this chapter. Finally, in §3.6.5 we undertake a brief analysis for level-k folded area estimators with $k = 2, 3, 4$.

Table 1: Approximate Asymptotic Bias and Variance for Different Estimators

Area	(m/γ_1)Bias	(b/σ^4)Var	CvM	(m/γ_1)Bias	(b/σ^4)Var
$\mathcal{A}(f;b,m)$	Eq. (14)	2	$\mathcal{C}(g;b,m)$	Eq. (16)	Eq. (17)
$\mathcal{A}(f_0;b,m)$	3	2	$\mathcal{C}(g_0;b,m)$	5	0.8
$\mathcal{A}(f_2;b,m)$	$o(1)$	2	$\mathcal{C}(g^\star_{0,2};b,m)$	$o(1)$	1.723
$\mathcal{A}(f_{\cos,1};b,m)$	$o(1)$	2	$\mathcal{C}(g^*_{0,4};b,m)$	$o(1)$	1.042
$\mathcal{A}_{(1)}(f;b,m)$	Eq. (28)	2	$\mathcal{C}_{(1)}(g;b,m)$	Eq. (34)	Eq. (35)
$\mathcal{A}_{(1)}(f_0;b,m)$	3	2	$\mathcal{C}_{(1)}(g_0;b,m)$	8	0.8
$\mathcal{A}_{(1)}(f_2;b,m)$	$o(1)$	2	$\mathcal{C}_{(1)}(g^\star_{1,2};b,m)$	$o(1)$	2.057
$\mathcal{A}_{(1)}(f_{\cos,1};b,m)$	$o(1)$	2	$\mathcal{C}_{(1)}(g^\star_{1,4};b,m)$	$o(1)$	1.135
$\bar{\mathcal{A}}_{(0,1)}(f_0;b,m)$	3	1	$\bar{\mathcal{C}}_{(0,1)}(g_0,g_0;\alpha^\star;b,m)$	6.5	0.5
$\bar{\mathcal{A}}_{(0,1)}(f_2;b,m)$	$o(1)$	1	$\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,2},g^\star_{1,2};\alpha^\star;b,m)$	$o(1)$	0.99
$\bar{\mathcal{A}}_{(0,1)}(f_{cos,1};b,m)$	$o(1)$	1	$\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,4},g^\star_{1,4};\alpha^\star;b,m)$	$o(1)$	0.70
$\mathcal{N}(b,m)$	1	2	$\mathcal{O}(b,m)$	1	1.333

3.6.1 I.i.d. Gaussian Process

In the case of i.i.d standard normal random variables, we can calculate exactly the biases and variances for the level-1 folded estimators. Table 2 lists these moments. Note that in this special case, the usual sample variance is unbiased and has variance $2/(n-1)$, so it is the "best" estimator for $\sigma^2 = 1$. We see that the entries in Table 2 indeed match up nicely with those of Table 1 with $\gamma_1 = 0$.

Table 2: Bias and Variance for Folded Estimators in i.i.d. Normal Case

Estimator	Bias	Variance
$\mathcal{A}_{(1)}(f_0; b, m)$	$-\frac{1}{m^2}$	$\frac{2}{b}$
$\mathcal{A}_{(1)}(f_2; b, m)$	$\frac{7}{2m^2} + \frac{63}{2m^4} - \frac{36}{m^6}$	$\frac{2}{b}$
$\mathcal{C}_{(1)}(g_0; b, m)$	$-\frac{1}{m^2}$	$\frac{0.8}{b}$
$\mathcal{C}_{(1)}(g^\star_{1,2}; b, m)$	$\frac{5}{m^2} - \frac{6}{m^4}$	$\frac{2.057}{b}$
$\mathcal{C}_{(1)}(g^\star_{1,4}; b, m)$	$\frac{110}{9\,m^2} - \frac{769}{9\,m^4} + \frac{650}{9\,m^6}$	$\frac{1.135}{b}$

3.6.2 Monte Carlo Examples

In this section, we examine the performance of the folded estimators using the following processes.

1. First-order autoregressive (AR(1)) process: A stationary (Gaussian) AR(1) process is defined by $X_j = \phi X_{j-1} + \epsilon_j$ for $j = 1, 2, \ldots$, where $1 < \phi < 1$, X_0 is a standard normal random variable, and the ϵ_i are i.i.d. $N(0, 1-\phi^2)$ random variables that are independent of X_0. This process has covariance function $R_j = \phi^{|j|}$ for all $j = 0, \pm 1, \pm 2, \ldots$, so that $\sigma^2 = (1+\phi)/(1-\phi)$ and $\gamma_1 = 2\phi/(1-\phi)^2$. In our examples in this thesis, we chose $\phi = 0.9$, hence $\sigma^2 = 19$.

2. M/M/1 delay-time process: We consider the stationary delay-time process for an M/M/1 queue with interarrival rate $\rho < 1$ and service rate 1. The variance parameter for this process is $\sigma^2 = \rho(2 + 5\rho - 4\rho^2 + \rho^3)/(1-\rho)^4$ (Steiger and

Wilson [34]). In the following examples, we consider an arrival rate of 0.8 and service rate of 1.0 (hence $\rho = 0.8$). The variance parameter for this process is $\sigma^2 = 1976$.

Tables 3 and 4 contain the estimated expected values and standard deviations of the folded variance estimators and their counterparts for the AR(1) process, respectively. Similarly, in Tables 5 and 6 we present the corresponding estimated expected values and standard deviations for the M/M/1 delay-time process. All entries in Tables 3–6 are based on $b = 32$ batches and 10,000 independent replications. We employ common random numbers across all variance estimation methods, and we use the combined pseudo-random number generator described in Figure 1 of L'Ecuyer [25]. The standard errors of the point estimators in Table 4 (6) have an upper bound of about 0.05 (10.3, respectively).

Based on Tables 3–6, the following conclusions can be drawn:

- The estimated expected values of all variance estimators converge to σ^2 as the batch size m becomes large, in accordance with theory.

- For small values of m, the level-0 and 1 batched area and CvM estimators with constant weights $f_0(\cdot)$ and $g_0(\cdot)$, respectively, are much more biased than the other estimators. This is consistent with the bias results from Table 1.

- For small m, the level-1 folded estimators are more biased for σ^2 than the analogous level-0 estimators. This increase in bias is primarily due to the larger second-order terms in the expectations of the level-1 estimators (Theorem 4 and Theorem 10).

- For small m, the estimator $\mathcal{C}_{(1)}(g^\star_{1,2}; b, m)$ appears to have the smallest bias among the level-1 CvM estimators under study here. This makes sense in light of Example 18 and Remark 20, especially after we notice that $\gamma_2 \gg \sigma^2$ for the positively autocorrelated AR(1) and M/M/1 processes under consideration.

- We see from the M/M/1 results in Table 6 that the estimated standard deviations require very large batch size m before approaching their asymptotic values. For instance, we know from Table 1 that

$$\sqrt{\mathrm{Var}[\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,4}, g^\star_{1,4}; \alpha^\star; b, m)]} \underset{m\to\infty}{\longrightarrow} \sqrt{0.6995\sigma^4/b} \;=\; 292,$$

  yet even for $m = 32{,}768$, the estimated standard deviation is still 364.

- The combined area estimators $\bar{\mathcal{A}}_{(0,1)}(f; \alpha; b, m)$ and $\bar{\mathcal{C}}_{(0,1)}(g_x, g_y; \alpha; b, m)$ perform as expected. Their bias is between the biases of the constituent level-0 and level-1 estimators for small m, and dissipates for large m. The bonus from the combined estimators is the substantially reduced variance compared to their constituents. For large m, the standard deviation of a combined area estimator is about a factor of $\sqrt{2}$ smaller than those of its constituents, which is reasonable since the constituents are asymptotically independent (Example 15 and Table 1). The reduction in standard deviation achieved by the combined CvM estimators is a bit smaller due to the positive correlation between the level-0 and level-1 estimators, but is in line with the theory (Examples 19–21 and Table 1).

3.6.3 Expected Value Examples

Figure 2 plots the expected values of various level-0 and level-1 batched area and CvM estimators, as well as the NBM estimator, as functions of the batch size m for the AR(1) process introduced in §3.6.2. By Theorems 4 and 10, the expected values of these area and CvM estimators depend on the batch size, but not on the number of batches; however, since the expected value of $\mathcal{N}(b, m)$ depends on both m and b, we decided to use $b = 32$ batches, a realistic number in practice. The figure shows that, as m becomes large, the expected value of level-0 estimators tend to converge to $\sigma^2 = 19$ more quickly than their level-1 counterparts. The STS estimators incorporating first-order unbiased weight functions (i.e., $\mathcal{A}_{(k)}(f_2; b, m)$ and $\mathcal{C}_{(k)}(g^\star_{k,2}; b, m)$ for $k = 0, 1$)

Table 3: Estimated Expected Values of Variance Estimators for the AR(1) Process for $b = 32$

	Level-0 Estimators				
m	$\mathcal{A}_{(0)}(f_0; b, m)$	$\mathcal{A}_{(0)}(f_2; b, m)$	$\mathcal{C}_{(0)}(g_0; b, m)$	$\mathcal{C}_{(0)}(g^\star_{0,2}; b, m)$	$\mathcal{C}_{(0)}(g^\star_{0,4}; b, m)$
512	17.98	18.78	17.34	18.76	18.42
1024	18.47	18.94	18.15	18.93	18.84
2048	18.77	19.05	18.61	19.04	18.99
4096	18.87	19.00	18.79	18.99	19.01
	Level-1 Folded Estimators				
m	$\mathcal{A}_{(1)}(f_0; b, m)$	$\mathcal{A}_{(1)}(f_2; b, m)$	$\mathcal{C}_{(1)}(g_0; b, m)$	$\mathcal{C}_{(1)}(g^\star_{1,2}; b, m)$	$\mathcal{C}_{(1)}(g^\star_{1,4}; b, m)$
512	17.81	18.19	16.38	18.34	17.68
1024	18.48	18.82	17.67	18.87	18.62
2048	18.79	18.97	18.36	18.98	18.95
4096	18.91	19.08	18.69	19.07	18.99
	Linearly Combined Estimators				
m	$\bar{\mathcal{A}}_{(0,1)}(f_0; b, m)$	$\bar{\mathcal{A}}_{(0,1)}(f_2; b, m)$	$\bar{\mathcal{C}}_{(0,1)}(g_0, g_0; \alpha^\star; b, m)$	$\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,2}, g^\star_{1,2}; \alpha^\star; b, m)$	$\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,4}, g^\star_{1,4}; \alpha^\star; b, m)$
512	17.89	18.48	16.86	18.57	18.07
1024	18.47	18.88	17.91	18.91	18.74
2048	18.78	19.01	18.48	19.01	18.97
4096	18.89	19.04	18.74	19.03	19.00

Table 4: Estimated Standard Deviations of Variance Estimators for the AR(1) Process for $b = 32$

	Level-0 Estimators				
m	$\mathcal{A}_{(0)}(f_0; b, m)$	$\mathcal{A}_{(0)}(f_2; b, m)$	$\mathcal{C}_{(0)}(g_0; b, m)$	$\mathcal{C}_{(0)}(g^\star_{0,2}; b, m)$	$\mathcal{C}_{(0)}(g^\star_{0,4}; b, m)$
512	4.54	4.74	2.91	4.35	3.30
1024	4.63	4.75	2.95	4.37	3.35
2048	4.71	4.76	2.99	4.40	3.42
4096	4.74	4.74	3.00	4.42	3.42
	Level-1 Folded Estimators				
m	$\mathcal{A}_{(1)}(f_0; b, m)$	$\mathcal{A}_{(1)}(f_2; b, m)$	$\mathcal{C}_{(1)}(g_0; b, m)$	$\mathcal{C}_{(1)}(g^\star_{1,2}; b, m)$	$\mathcal{C}_{(1)}(g^\star_{1,4}; b, m)$
512	4.49	4.53	2.85	4.59	3.30
1024	4.58	4.64	2.91	4.68	3.45
2048	4.69	4.77	2.97	4.81	3.52
4096	4.67	4.67	2.96	4.76	3.55
	Linearly Combined Estimators				
m	$\bar{\mathcal{A}}_{(0,1)}(f_0; b, m)$	$\bar{\mathcal{A}}_{(0,1)}(f_2; b, m)$	$\bar{\mathcal{C}}_{(0,1)}(g_0, g_0; \alpha^\star; b, m)$	$\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,2}, g^\star_{1,2}; \alpha^\star; b, m)$	$\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,4}, g^\star_{1,4}; \alpha^\star; b, m)$
512	3.19	3.27	2.27	3.25	2.67
1024	3.25	3.31	2.31	3.28	2.72
2048	3.33	3.38	2.36	3.34	2.79
4096	3.32	3.33	2.35	3.32	2.79

Table 5: Estimated Expected Values of Variance Estimators for the M/M/1 Process for $b = 32$

	Level-0 Estimators				
m	$\mathcal{A}_{(0)}(f_0; b, m)$	$\mathcal{A}_{(0)}(f_2; b, m)$	$\mathcal{C}_{(0)}(g_0; b, m)$	$\mathcal{C}_{(0)}(g^\star_{0,2}; b, m)$	$\mathcal{C}_{(0)}(g^\star_{0,4}; b, m)$
2048	1849	1942	1760	1935	1883
4096	1900	1968	1858	1964	1940
8192	1939	1972	1917	1974	1968
16384	1960	1980	1949	1980	1978
32768	1962	1969	1957	1970	1968
	Level-1 Folded Estimators				
m	$\mathcal{A}_{(1)}(f_0; b, m)$	$\mathcal{A}_{(1)}(f_2; b, m)$	$\mathcal{C}_{(1)}(g_0; b, m)$	$\mathcal{C}_{(1)}(g^\star_{1,2}; b, m)$	$\mathcal{C}_{(1)}(g^\star_{1,4}; b, m)$
2048	1803	1833	1625	1861	1767
4096	1896	1931	1791	1939	1901
8192	1947	1972	1889	1973	1964
16384	1960	1977	1930	1979	1972
32768	1964	1969	1950	1969	1975
	Linearly Combined Estimators				
m	$\bar{\mathcal{A}}_{(0,1)}(f_0; b, m)$	$\bar{\mathcal{A}}_{(0,1)}(f_2; b, m)$	$\bar{\mathcal{C}}_{(0,1)}(g_0, g_0; \alpha^\star; b, m)$	$\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,2}, g^\star_{1,2}; \alpha^\star; b, m)$	$\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,4}, g^\star_{1,4}; \alpha^\star; b, m)$
2048	1826	1887	1692	1902	1829
4096	1898	1949	1825	1953	1922
8192	1943	1972	1903	1973	1966
16384	1960	1979	1939	1979	1975
32768	1963	1969	1953	1970	1971

Table 6: Estimated Standard Deviations of Variance Estimators for the M/M/1 Process for $b = 32$

	Level-0 Estimators				
m	$\mathcal{A}_{(0)}(f_0; b, m)$	$\mathcal{A}_{(0)}(f_2; b, m)$	$\mathcal{C}_{(0)}(g_0; b, m)$	$\mathcal{C}_{(0)}(g^\star_{0,2}; b, m)$	$\mathcal{C}_{(0)}(g^\star_{0,4}; b, m)$
2048	1010	1026	744	924	874
4096	809	831	606	738	705
8192	700	688	510	623	579
16384	614	611	433	557	488
32768	555	550	377	507	423
	Level-1 Folded Estimators				
m	$\mathcal{A}_{(1)}(f_0; b, m)$	$\mathcal{A}_{(1)}(f_2; b, m)$	$\mathcal{C}_{(1)}(g_0; b, m)$	$\mathcal{C}_{(1)}(g^\star_{1,2}; b, m)$	$\mathcal{C}_{(1)}(g^\star_{1,4}; b, m)$
2048	951	852	656	858	764
4096	790	787	568	756	662
8192	727	677	513	669	587
16384	604	607	424	598	497
32768	557	542	375	547	442
	Linearly Combined Estimators				
m	$\bar{\mathcal{A}}_{(0,1)}(f_0; b, m)$	$\bar{\mathcal{A}}_{(0,1)}(f_2; b, m)$	$\bar{\mathcal{C}}_{(0,1)}(g_0, g_0; \alpha^\star; b, m)$	$\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,2}, g^\star_{1,2}; \alpha^\star; b, m)$	$\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,4}, g^\star_{1,4}; \alpha^\star; b, m)$
2048	829	793	640	796	744
4096	668	677	538	655	616
8192	576	549	460	539	516
16384	478	475	376	466	425
32768	412	408	320	405	364

converge more quickly than do the other estimators ($\mathcal{A}_{(k)}(f_0; b, m)$ and $\mathcal{C}_{(k)}(g_0; b, m)$ for $k = 0, 1$, and $\mathcal{N}(b, m)$). $\triangleleft$

3.6.4 Density Estimation

Alexopoulos et al. [3] used the method of Satterthwaite [29] to develop distributional approximations for the distributions of (overlapping) area and CvM estimators as the batch size grows. In this section, we estimate the distributions of the level-0 and 1 folded area and CvM estimators for quadratic weight functions. Similar approximations exist for the remaining weight functions. First, we find approximations to the theoretical distributions of $\mathcal{A}_{(k)}(f; b, m)$ and $\mathcal{C}_{(k)}(g; b, m)$, for $k = 0, 1, 2, \ldots$, and sufficiently large values of m.

3.6.4.1 Batched Folded Area Estimators

Using an argument that is similar to that of §4.3 of Alexopoulos et al. [3], we obtain the approximation

$$\mathcal{A}_{(k)}(f; b, m) \dot{\sim} \mathrm{E}[\mathcal{A}_{(k)}(f; b, m)]\chi^2_{\nu_{\text{eff}}}/\nu_{\text{eff}}, \text{ where } \nu_{\text{eff}} = \left[\!\!\left[\frac{2\mathrm{E}^2[\mathcal{A}_{(k)}(f; b, m)]}{\mathrm{Var}[\mathcal{A}_{(k)}(f; b, m)]} \right]\!\!\right], \quad (36)$$

for $k = 0, 1, 2, \ldots$, where $[\![\cdot]\!]$ denotes rounding towards the nearest integer and the quantity ν_{eff} is called the "effective" degrees of freedom.

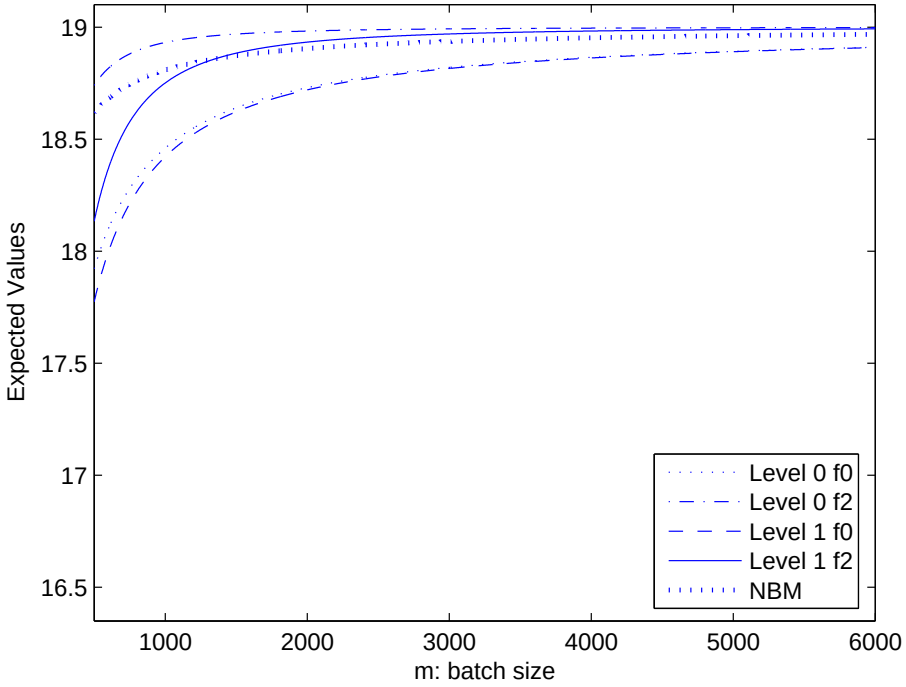
For example, for the first-order unbiased quadratic weight function $f_2(\cdot)$, we see that

$$\mathrm{E}[\mathcal{A}_{(0)}(f_2; b, m)] \approx \sigma^2 \quad \text{and} \quad \nu_{f_2,\text{eff}} = b; \quad (37)$$

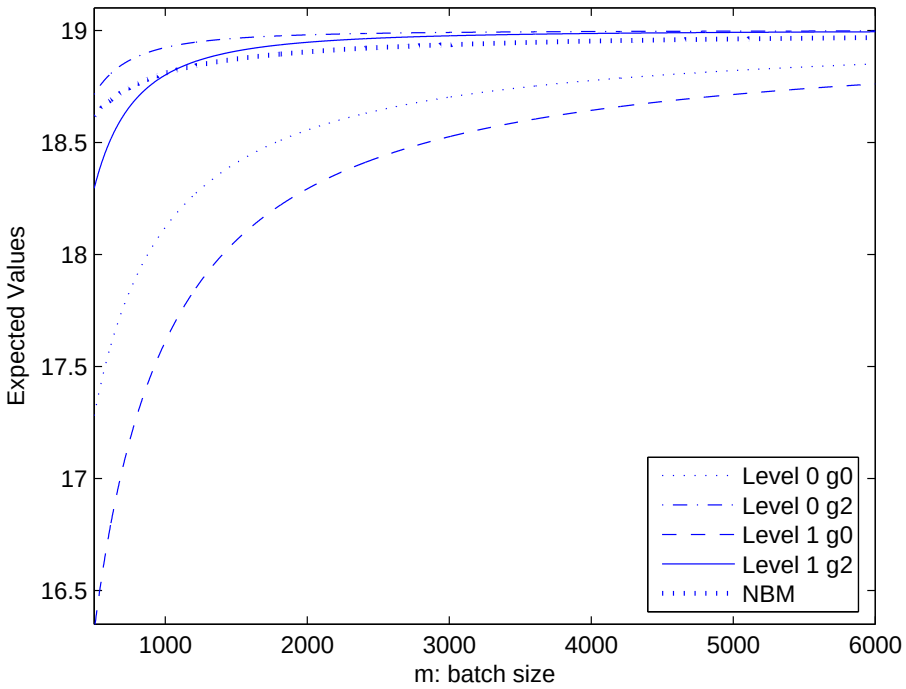
hence from Equation (36) we have

$$\mathcal{A}_{(0)}(f_2; b, m) \dot{\sim} \sigma^2\chi^2_b/b. \quad (38)$$

To compare our run results with these theoretical results, we generated 1,000,000 independent sample paths of the stationary AR(1) process. Each sample path contained $n = 32768$ observations and all variance estimators were computed using a



(a) Area Estimators



(b) CvM Estimators

Figure 2: Expected Values of Folded Estimators Based on the AR(1) Process

batch size of $m = 2048$. Figure 3(a) displays the empirical probability density function (p.d.f.) of $\mathcal{A}_{(0)}(f_2; 16, 2048)$ (drawn as a frequency polygon as discussed in the §3.3 of Hald [20]) and the fitted empirical p.d.f. based on Equations (37) and (38). We see that an appropriately scaled chi-squared random variable provides an excellent approximation to the distribution of $\mathcal{A}_{(0)}(f_2; 16, 2048)$.

For the level -1 folded area estimator with the first-order unbiased quadratic weight function $f_2(t)$, we see that

$$\mathrm{E}[\mathcal{A}_{(1)}(f_2; b, m)] \approx \sigma^2 \quad \text{and} \quad \nu_{f_2,\mathrm{eff}} = b; \tag{39}$$

hence from Equation (36) we have

$$\mathcal{A}_{(1)}(f_2; b, m) \dot{\sim} \sigma^2 \chi^2_b / b. \tag{40}$$

Figure 3(b) displays the empirical p.d.f. of $\mathcal{A}_{(1)}(f_2; 16, 2048)$ and the fitted p.d.f. based on Equations (39) and (40). Again we see that an appropriately scaled chi-squared random variable is a very good approximation to the distribution of $\mathcal{A}_{(1)}(f_2; 16, 2048)$.

3.6.4.2 Batched Folded CvM Estimators

Similar to Equation (36), we obtain the following equation for batched folded CvM estimators:

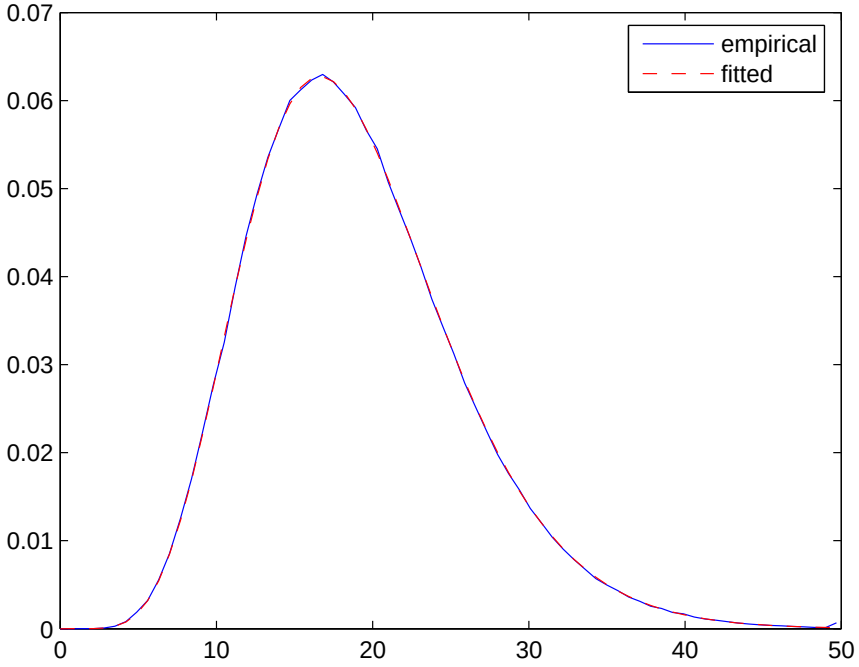
$$\mathcal{C}_{(k)}(g; b, m) \dot{\sim} \mathrm{E}[\mathcal{C}_{(k)}(g; b, m)] \chi^2_{\nu_{\mathrm{eff}}} / \nu_{\mathrm{eff}}, \text{ where } \nu_{\mathrm{eff}} = \left[\!\!\left[\frac{2\mathrm{E}^2[\mathcal{C}_{(k)}(g; b, m)]}{\mathrm{Var}[\mathcal{C}_{(k)}(g; b, m)]} \right]\!\!\right], \tag{41}$$

for $k = 0, 1, 2, \ldots$. If we follow a similar argument to that of §3.6.4.1 for the weight function $g^\star_{0,2}(\cdot)$, we can see that

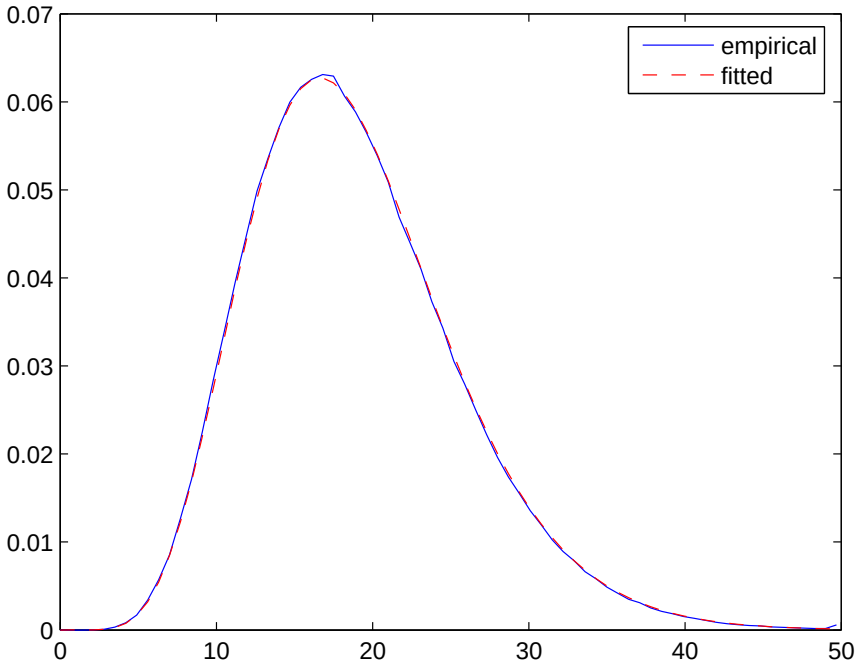
$$\mathrm{E}[\mathcal{C}_{(0)}(g^\star_{0,2}; b, m)] \approx \sigma^2 \quad \text{and} \quad \nu_{g^\star_{0,2},\mathrm{eff}} = \left[\!\!\left[\frac{140b}{121} \right]\!\!\right]; \tag{42}$$

hence from Equation (41) we have

$$\mathcal{C}_{(0)}(g^\star_{0,2}; b, m) \dot{\sim} \sigma^2 \chi^2_{\nu_{g^\star_{0,2},\mathrm{eff}}} / \nu_{g^\star_{0,2},\mathrm{eff}}. \tag{43}$$



(a) $\mathcal{A}_{(0)}(f_2; 16, 2048)$



(b) $\mathcal{A}_{(1)}(f_2; 16, 2048)$

Figure 3: Empirical and Fitted p.d.f.'s for Folded Area Estimators Based on the AR(1) Process

Figure 4(a) displays the empirical p.d.f. of $\mathcal{C}_{(0)}(g^\star_{0,2}; 16, 2048)$ and the fitted p.d.f. based on Equations (42) and (43). It can be seen that an appropriately scaled chi-squared random variable is a very good approximation to the distribution of $\mathcal{C}_{(0)}(g^\star_{0,2}; 16, 2048)$.

Also, for level-1 folded CvM estimators with weight function $g^\star_{1,2}(\cdot)$, we have

$$\mathrm{E}[\mathcal{C}_{(1)}(g^\star_{1,2}; b, m)] \approx \sigma^2 \quad \text{and} \quad \nu_{g^\star_{1,2},\mathrm{eff}} = \left[\!\!\left[\frac{70b}{72}\right]\!\!\right]; \tag{44}$$

hence from Equation (41) we have

$$\mathcal{C}_{(1)}(g^\star_{1,2}; b, m) \dot{\sim} \sigma^2 \chi^2_{\nu_{g^\star_{1,2},\mathrm{eff}}} / \nu_{g^\star_{1,2},\mathrm{eff}}. \tag{45}$$

Figure 4(b) displays the empirical p.d.f. of $\mathcal{C}_{(1)}(g^\star_{1,2}; 16, 2048)$ and the fitted p.d.f. based on Equations (44) and (45). Hence, an appropriately scaled chi-squared random variable is a very good approximation to the distribution of $\mathcal{C}_{(1)}(g^\star_{1,2}; 16, 2048)$.

We now show how one can construct approximate CIs for the parameters μ and σ^2. If $\mathcal{A}_{(k)}(f; b, m)$, $k = 1, 2, \ldots$, is a first-order unbiased estimator for σ^2, then for $\alpha \in (0, 1)$ and a sufficiently large batch size m, an approximate $100(1-\alpha)\%$ two-sided CI for σ^2 is given by

$$\frac{\nu_{\mathrm{eff}} \mathcal{A}_{(k)}(f; b, m)}{\chi^2_{1-\alpha/2,\nu_{\mathrm{eff}}}} \leq \sigma^2 \leq \frac{\nu_{\mathrm{eff}} \mathcal{A}_{(k)}(f; b, m)}{\chi^2_{\alpha/2,\nu_{\mathrm{eff}}}}, \tag{46}$$

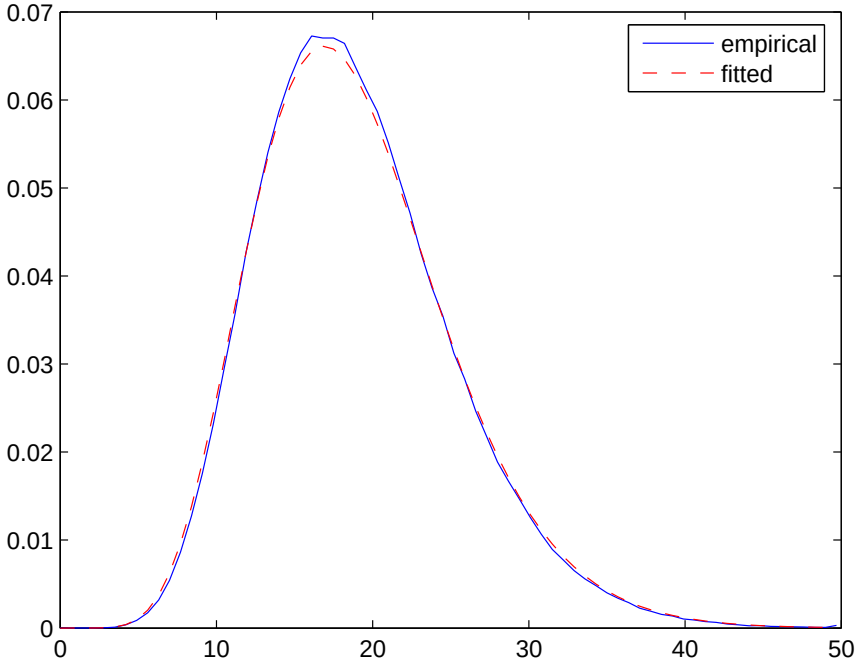
where the $\chi^2_{\beta,\nu}$ denotes the β-quantile of the χ^2 distribution with ν degrees of freedom.

Similarly, an approximate $100(1-\alpha)\%$ two-sided CI for σ^2 for CvM estimators can be obtained as follows:

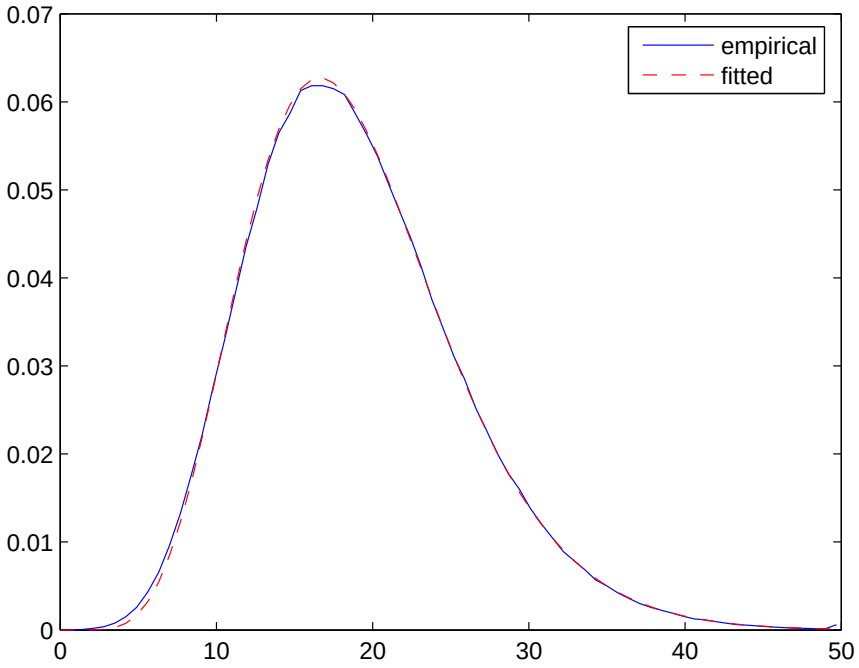
$$\frac{\nu_{\mathrm{eff}} \mathcal{C}_{(k)}(g; b, m)}{\chi^2_{1-\alpha/2,\nu_{\mathrm{eff}}}} \leq \sigma^2 \leq \frac{\nu_{\mathrm{eff}} \mathcal{C}_{(k)}(g; b, m)}{\chi^2_{\alpha/2,\nu_{\mathrm{eff}}}}. \tag{47}$$

We can also obtain an estimated CI for the mean μ, provided that the batch size m is sufficiently large. First, Equation (3) implies that for large n we have $\bar{X}_n \dot{\sim} N(\mu, \sigma^2/n)$ and $\bar{X}_n$ is approximately independent of the STS formed from $\{X_j, j = 1, \ldots, n\}$. It follows that for sufficiently large m,

$$\frac{\bar{X}_n - \mu}{\sqrt{\mathcal{A}_{(k)}(f; b, m)/n}} \dot{\sim} t_{\nu_{\mathrm{eff}}} \tag{48}$$



(a) $\mathcal{C}_{(0)}(g^{\star}_{0,2}; 16, 2048)$



(b) $\mathcal{C}_{(1)}(g^{\star}_{1,2}; 16, 2048)$

Figure 4: Empirical and Fitted p.d.f.'s for Folded CvM Estimators Based on the AR(1) Process

Table 7: Critical Values and Coverages of CIs for σ^2 and μ, for $b = 16$ and $m = 2048$

Estimator	σ^2			μ	
	$\frac{\nu_{\text{eff}}}{\chi^2_{1-\alpha/2,\nu_{\text{eff}}}}$	$\frac{\nu_{\text{eff}}}{\chi^2_{\alpha/2,\nu_{\text{eff}}}}$	Coverage	$t_{1-\alpha/2,\nu_{\text{eff}}}$	Coverage
$\mathcal{A}_{(0)}(f_2; b, m)$	0.6085	2.0095	0.8997	0.0096	0.9003
$\mathcal{A}_{(1)}(f_2; b, m)$	0.6085	2.0095	0.9002	0.0096	0.8994
$\bar{\mathcal{A}}_{(0,1)}(f_2; b, m)$	0.6927	1.5943	0.8997	0.0094	0.9002
$\mathcal{C}_{(0)}(g^\star_{0,2}; b, m)$	0.6033	1.8780	0.8981	0.0096	0.8999
$\mathcal{C}_{(1)}(g^\star_{1,2}; b, m)$	0.6085	2.0095	0.8952	0.0096	0.8988
$\bar{\mathcal{C}}_{(0,1)}(g^\star_{0,2}, g^\star_{1,2}; \alpha^\star; b, m)$	0.6927	1.5943	0.9026	0.0094	0.9005

for $k = 0, 1, 2, \ldots$, where t_ν denotes a random variable having Student's t-distribution with ν degrees of freedom. Hence an approximate $100(1-\alpha)\%$ two-sided CI for μ is

$$\bar{X}_n - t_{1-\alpha/2,\nu_{\text{eff}}}\sqrt{\mathcal{A}_{(k)}(f; b, m)/n} \le \mu \le \bar{X}_n + t_{1-\alpha/2,\nu_{\text{eff}}}\sqrt{\mathcal{A}_{(k)}(f; b, m)/n}. \tag{49}$$

Following a similar argument for CvM estimators, we get the following approximate $100(1-\alpha)\%$ two-sided CI for μ:

$$\bar{X}_n - t_{1-\alpha/2,\nu_{\text{eff}}}\sqrt{\mathcal{C}_{(k)}(g; b, m)/n} \le \mu \le \bar{X}_n + t_{1-\alpha/2,\nu_{\text{eff}}}\sqrt{\mathcal{C}_{(k)}(g; b, m)/n}. \tag{50}$$

Table 7 lists the critical values and coverage probabilities for approximate 90% two-sided CIs for σ^2 and μ obtained from (46)–(47) and (49)–(50), respectively, for the AR(1) process. It can be seen that all empirical coverages are very close to the nominal value, illustrating the validity of the CIs.

3.6.5 Multi-level Folding

In this section we study the effect of multi-level folding on the first moment of the folded area estimators. A Monte Carlo study revealed that the bias of level-k folded area estimators increases as the level k increases. Table 8 illustrates these findings for the AR(1) process using folded area estimators with weights $f_0(\cdot)$ and $f_2(\cdot)$. At level 0, the weight function $f_2(\cdot)$ yields a first-order unbiased estimator with less bias

Table 8: Estimated Expected Values of Multi-level Folded Area Estimators for the AR(1) Process for $b = 20$

m	$\mathcal{A}_{(0)}(f_0; b, m)$	$\mathcal{A}_{(1)}(f_0; b, m)$	$\mathcal{A}_{(2)}(f_0; b, m)$	$\mathcal{A}_{(3)}(f_0; b, m)$
100	13.80	11.83	7.30	3.05
300	17.22	16.88	15.37	11.03
500	17.94	17.80	17.18	14.91
m	$\mathcal{A}_{(0)}(f_2; b, m)$	$\mathcal{A}_{(1)}(f_2; b, m)$	$\mathcal{A}_{(2)}(f_2; b, m)$	$\mathcal{A}_{(3)}(f_2; b, m)$
100	14.92	11.06	6.12	2.29
300	18.34	17.05	14.25	9.51
500	18.77	18.16	16.58	13.33

than the constant weight $f_0(\cdot)$. But, Table 8 shows that, for small sample sizes, the bias of estimators based on the weight $f_2(\cdot)$ deteriorates faster (as the folding level increases) than the bias of estimators based on $f_0(\cdot)$. This puzzling behavior is the subject of ongoing research.

3.7 Summary

This chapter presented various folded estimators, their batched versions obtained from nonoverlapping batches, and their linear combinations. We extended the results in Antonini [6] and Antonini et al. [7] by obtaining detailed expressions for the expectation of folded estimators at level 1, conducting a detailed Monte Carlo study as the batch size increases while the number of batches remains constant, describing distributional approximations for the batched estimators, and constructing CIs for the mean and the variance parameter of the underlying process. Chapter 4 proceeds with folded overlapping area estimators, which combine the overlapping operation discussed in §2.4 with the folding operation that was reviewed in §3.1.

CHAPTER IV

FOLDED OVERLAPPING AREA ESTIMATORS

4.1 Introduction

In this chapter we introduce and study the folded overlapping area estimators for σ^2 and list the main results. These estimators are based on a combination of the folding and overlapping operations. As discussed in Chapter 3, linear combinations of folded estimators can have smaller variance than their "unfolded" analogues, whereas Alexopoulos et al. [3] show that estimators based on overlapping batches are usually less variable than their competitors based on nonoverlapping batches. Hence, the combination of these two techniques should be expected to reduce the variance of the variance parameter estimator even more than applying each technique separately. In §4.2 we introduce the level-k folded STS and overlapping area estimator. §4.3 lists the limiting properties, and obtains approximations for the first two moments for the proposed folded overlapping area (FOA) estimator. In §4.4 we introduce linearly combined estimators obtained from FOA estimators from levels 0 and 1. §4.5 lists algorithms to compute the level-1 FOA estimator in order-of-sample-size time for constant and quadratic weight functions. Finally, in §4.6 we conduct Monte Carlo studies.

4.2 Folded Overlapping Area Estimator

The level-k folded STS obtained from overlapping batch i is

$$T^{\mathrm{o}}_{(k),i,m}(t) \equiv T^{\mathrm{o}}_{(k-1),i,m}(\tfrac{t}{2}) - T^{\mathrm{o}}_{(k-1),i,m}(1-\tfrac{t}{2}),$$

where $T^{\mathrm{o}}_{(0),i,m}(t) \equiv T^{\mathrm{o}}_{i,m}(t)$, for $t \in [0,1]$, $k = 1, 2, \ldots$, and $i = 1, \ldots, n-m+1$. The level-k FOA estimator from batch i is

$$A^{\mathrm{o}}_{(k),i}(f;m) \equiv \left[\frac{1}{m}\sum_{j=1}^{m} f(\tfrac{j}{m})\sigma T^{\mathrm{o}}_{(k),i,m}(\tfrac{j}{m})\right]^2, \quad \text{for } k = 0, 1, 2, \ldots \text{ and } i = 1, \ldots, n-m+1.$$

Averaging these estimators from all $n-m+1$ batches gives the level-k FOA estimator for σ^2:

$$\mathcal{A}^{\mathrm{o}}_{(k)}(f;b,m) \equiv \frac{1}{n-m+1}\sum_{i=1}^{n-m+1} A^{\mathrm{o}}_{(k),i}(f;m).$$

4.3 Properties of the Folded Overlapping Area Estimator

Let $\mathcal{B}_{(k),s}(\cdot)$ denote the level-k folded Brownian bridge on $[0,1]$ starting at time s. In particular, we denote the level-0 Brownian bridge starting at time s by

$$\mathcal{B}_{(0),s}(t) \equiv t[\mathcal{W}(s+1) - \mathcal{W}(s)] - [\mathcal{W}(s+t) - \mathcal{W}(s)], \quad \text{for } t \in [0,1] \text{ and } s \in [0, b-1].$$

The analogous folded Brownian bridge $\mathcal{B}_{(k),s}(\cdot)$ can be defined recursively as follows:

$$\mathcal{B}_{(k),s}(t) \equiv \mathcal{B}_{(k-1),s}(\tfrac{t}{2}) - \mathcal{B}_{(k-1),s}(1-\tfrac{t}{2}), \quad \text{for } t \in [0,1].$$

The next theorem gives the limiting distribution of the level-k FOA estimator $\mathcal{A}^{\mathrm{o}}_{(k)}(f;b,m)$. Its proof, along with several auxiliary results, is given in Appendix A.4.

Theorem 14 If Assumptions A and F hold, then

$$\mathcal{A}^{\mathrm{o}}_{(k)}(f;b,m) \xrightarrow[m\to\infty]{\mathcal{D}} \mathcal{A}^{\mathrm{o}}_{(k)}(f;b) \equiv \frac{1}{b-1}\int_0^{b-1}\left[\sigma\int_0^1 f(u)\mathcal{B}_{(k),s}(u)\,\mathrm{d}u\right]^2 \mathrm{d}s,$$

for fixed $b \geq 2$ and $k = 1, 2, \ldots$.

The following theorem obtains the expected value of the level-1 FOA estimators, which is indeed the same result as in Remark 14.

Theorem 15 If Assumptions A and F hold and m is even, then

$$\mathrm{E}[\mathcal{A}^{\mathrm{o}}_{(1)}(f;b,m)] = \mathrm{E}[A^{\mathrm{o}}_{(1),1}(f;m)] = \sigma^2 - \frac{\bar{F}^2\gamma_1}{m} + O(1/m^2). \tag{51}$$

The next theorem gives the asymptotic variance of the level-1 FOA estimator. The proof follows from the generalized CMT.

Theorem 16 If Assumptions A and F hold and for each fixed b the sequence $\{[\mathcal{A}^{\mathrm{o}}_{(1)}(f;b,m)]^2 : m = 1,2,\ldots\}$ is uniformly integrable, then

$$\mathrm{Var}[\mathcal{A}^{\mathrm{o}}_{(1)}(f;b,m)] \underset{m\to\infty}{\longrightarrow} \mathrm{Var}[\mathcal{A}^{\mathrm{o}}_{(1)}(f;b)].$$

By Lemma 2 in Alexopoulos et al. [4], we can write the asymptotic variance of the FOA estimator as

$$\mathrm{Var}[\mathcal{A}^{\mathrm{o}}_{(1)}(f;b)] = \frac{4\sigma^4}{(b-1)^2}\int_0^1 (b-1-y)q^2(0,y)\,\mathrm{d}y, \tag{52}$$

where

$$q(0,y) \equiv \int_0^1\int_0^1 f(u)f(v)\mathrm{Cov}[B_{(1),0}(u), B_{(1),y}(v)]\,\mathrm{d}u\,\mathrm{d}v, \tag{53}$$

for $y \in [0,1]$. A detailed expansion of (52) is given in Appendix A.5.

Example 22 For the constant weight function $f_0(\cdot)$, Equations (52) and (53) yield

$$\mathrm{Var}[\mathcal{A}^{\mathrm{o}}_{(1)}(f_0;b,m)] \underset{m\to\infty}{\longrightarrow} \mathrm{Var}[\mathcal{A}^{\mathrm{o}}_{(1)}(f_0;b)] = \frac{23b-29}{35(b-1)^2}\sigma^4 \approx \frac{23}{35b}\sigma^4. \quad \triangleleft$$

Example 23 For the quadratic weight function $f_2(\cdot)$, we have

$$\mathrm{Var}[\mathcal{A}^{\mathrm{o}}_{(1)}(f_2;b,m)] \underset{m\to\infty}{\longrightarrow} \mathrm{Var}[\mathcal{A}^{\mathrm{o}}_{(1)}(f_2;b)] = \frac{4639b-5782}{8580(b-1)^2}\sigma^4 \approx \frac{4639}{8580b}\sigma^4. \quad \triangleleft$$

Example 24 For the trigonometric weight function $f_{\cos,1}(\cdot)$, we have

$$\begin{aligned}\mathrm{Var}[\mathcal{A}^{\mathrm{o}}_{(1)}(f_{\cos,1};b,m)] \underset{m\to\infty}{\longrightarrow} \mathrm{Var}[\mathcal{A}^{\mathrm{o}}_{(1)}(f_{\cos,1};b)] &= \frac{(600+128\pi^2)b-(160\pi^2+732)}{384\pi^2(b-1)^2}\sigma^4 \\ &\approx \frac{233}{474b}\sigma^4. \quad \triangleleft\end{aligned}$$

Table 9 lists approximations for the asymptotic bias and variance of several variance estimators. We see that the asymptotic variances of the level-1 FOA estimators are lower than those of the NBM estimator, the OBM estimator, the area estimator, the overlapping area estimator, and the level-1 folded (nonoverlapping) area estimator for all weight functions under consideration.

Table 9: Approximate Asymptotic Bias and Variance for Different Variance Estimators

Nonoverlapping	(m/γ_1)Bias	(b/σ^4)Var	Overlapping	(m/γ_1)Bias	(b/σ^4)Var
$\mathcal{A}(f;b,m)$	Eq.(14)	2	$\mathcal{A}^{\mathrm{o}}(f;b,m)$	Eq.(14)	Eq.(20)
$\mathcal{A}(f_0;b,m)$	3	2	$\mathcal{A}^{\mathrm{o}}(f_0;b,m)$	3	0.686
$\mathcal{A}(f_2;b,m)$	$o(1)$	2	$\mathcal{A}^{\mathrm{o}}(f_2;b,m)$	$o(1)$	0.819
$\mathcal{A}(f_{\cos,1};b,m)$	$o(1)$	2	$\mathcal{A}^{\mathrm{o}}(f_{\cos,1};b,m)$	$o(1)$	0.793
$\mathcal{A}_{(1)}(f;b,m)$	Eq.(28)	2	$\mathcal{A}^{\mathrm{o}}_{(1)}(f;b,m)$	Eq.(51)	Eq.(52)
$\mathcal{A}_{(1)}(f_0;b,m)$	3	2	$\mathcal{A}^{\mathrm{o}}_{(1)}(f_0;b,m)$	3	0.657
$\mathcal{A}_{(1)}(f_2;b,m)$	$o(1)$	2	$\mathcal{A}^{\mathrm{o}}_{(1)}(f_2;b,m)$	$o(1)$	0.541
$\mathcal{A}_{(1)}(f_{\cos,1};b,m)$	$o(1)$	2	$\mathcal{A}^{\mathrm{o}}_{(1)}(f_{\cos,1};b,m)$	$o(1)$	0.492
$\mathcal{N}(b,m)$	1	2	$\mathcal{O}(b,m)$	1	1.333

4.4 Linear Combinations

We have seen that level-1 FOA estimators appear to be more biased than the respective level-0 overlapping area estimators. Therefore, linear combinations of level-0 and level-1 estimators could lead to estimators with smaller bias than the level-1 FOA estimators. Unfortunately, FOA estimators from different levels are correlated. For example, we ran 1,000,000 independent replications to approximate the correlation between different levels of estimators with $b = 20$ and $m = 1000$. For the weight function $f_0(\cdot)$, the correlation between $\mathcal{A}^{\text{o}}(f_0; b, m)$ and $\mathcal{A}^{\text{o}}_{(1)}(f_0; b, m)$ is 0.599. Similarly, for the weight function $f_2(\cdot)$, the correlation is 0.765.

In any case, consider the linearly combined estimator with weight function $f_0(\cdot)$

$$\bar{\mathcal{A}}^{\text{o}}_{(0,1)}(f_0; b, m) = \alpha \mathcal{A}^{\text{o}}_{(0)}(f_0; b, m) + (1 - \alpha)\mathcal{A}^{\text{o}}_{(1)}(f_0; b, m).$$

The limiting variance of $\bar{\mathcal{A}}^{\text{o}}_{(0,1)}(f_0; b, m)$ is

$$\begin{aligned}\lim_{m\to\infty} \text{Var}[\bar{\mathcal{A}}^{\text{o}}_{(0,1)}(f_0; b, m)] &= \text{Var}[\bar{\mathcal{A}}^{\text{o}}_{(0,1)}(f_0; \alpha; b)] \\ &= \alpha^2 \text{Var}[\mathcal{A}^{\text{o}}_{(0)}(f_0; b)] + (1 - \alpha)^2 \text{Var}[\mathcal{A}^{\text{o}}_{(1)}(f_0; b)] \\ &\quad + 2\alpha(1 - \alpha)\text{Cov}[\mathcal{A}^{\text{o}}_{(0)}(f_0; b), \mathcal{A}^{\text{o}}_{(1)}(f_0; b)],\end{aligned}$$

where $\text{Var}[\mathcal{A}^{\text{o}}_{(0)}(f_0; b)] \approx 0.686\sigma^4/b$ and $\text{Var}[\mathcal{A}^{\text{o}}_{(1)}(f_0; b)] \approx 0.657\sigma^4/b$. The value of α that minimizes this variance is $\alpha^\star = 0.512$; hence $\text{Var}[\bar{\mathcal{A}}^{\text{o}}_{(0,1)}(f_0; \alpha^\star; b)] = 0.64\sigma^4/b$, which is greater than the variance of the individual estimators in the linear combination.

Similarly, for the weight function $f_2(\cdot)$, we obtain $\alpha^\star = 0.59$, and $\text{Var}[\mathcal{A}^{\text{o}}_{(0,1)}(f_2; \alpha^\star; b)] \approx 0.75\sigma^4/b$. This result is also greater than variance of individual estimators since $\text{Var}[\mathcal{A}^{\text{o}}_{(0)}(f_2; b)] \approx 0.819\sigma^4/b$ and $\text{Var}[\mathcal{A}^{\text{o}}_{(1)}(f_2; b)] \approx 0.541\sigma^4/b$. Hence, for FOA estimators, linear combinations do not seem to yield the desired variance reduction. Further analysis is part of our future research.

4.5 Computational Complexity

The ability to compute the proposed FOA variance estimators with only order-of-sample-size work is very important. In this section, we show how to calculate the level-1 FOA estimators for a polynomial weight function of degree ≤ 2 in $O(n)$ time and provide the formal algorithms.

Let $S_{\ell,j} \equiv \sum_{t=0}^{j-1} X_{\ell+t}$, for $\ell \geq 1$ and $j \geq 1$, and note that for $i = 1, \ldots, n-m+1$ and $j = 1, \ldots, m$, the level-1 folded STS from overlapping batch i can be written as

$$T^{\mathrm{o}}_{(1),i,m}(\tfrac{j}{m}) = \frac{1}{\sigma\sqrt{m}}\Big((j-m)\bar{X}^{\mathrm{o}}_{i,m} - S_{i,\lfloor\frac{j}{2}\rfloor} + S_{i,\lfloor m-\frac{j}{2}\rfloor}\Big).$$

Hence, we can rewrite the level-1 folded overlapping area estimator from batch i as

$$A^{\mathrm{o}}_{(1),i}(f;m) = \left[\frac{1}{m^{3/2}}\sum_{j=1}^{m} f\Big(\tfrac{j}{m}\Big)\Big((j-m)\bar{X}^{\mathrm{o}}_{i,m} - S_{i,\lfloor\frac{j}{2}\rfloor} + S_{i,\lfloor m-\frac{j}{2}\rfloor}\Big)\right]^2,$$

for $i = 1, \ldots, n-m+1$. First, we show how $A^{\mathrm{o}}_{(1),i}(f;m)$ can be computed recursively in $O(1)$ time for $i = 1, \ldots, n-m+1$ and $m = 1, 2, \ldots$. To this end, we have

$$\begin{aligned}
A^{\mathrm{o}}_{(1),i}(f;m) &= \Bigg[\frac{1}{m^{3/2}}\sum_{j=1}^{m} f\Big(\tfrac{j}{m}\Big)\Big((j-m)(\bar{X}^{\mathrm{o}}_{i-1,m} + \tfrac{X_{m+i}}{m} - \tfrac{X_i}{m}) \\
&\qquad - (S_{i-1,\lfloor\frac{j}{2}\rfloor} + X_{i+\lfloor\frac{j}{2}\rfloor} - X_i) + (S_{i-1,\lfloor m-\frac{j}{2}\rfloor} + X_{i+\lfloor m-\frac{j}{2}\rfloor} - X_i)\Big)\Bigg]^2 \\
&= \Bigg[\underbrace{\frac{1}{m^{3/2}}\sum_{j=1}^{m} f\Big(\tfrac{j}{m}\Big)\Big((j-m)\bar{X}^{\mathrm{o}}_{i-1,m} - S_{i-1,\lfloor\frac{j}{2}\rfloor} + S_{i-1,\lfloor m-\frac{j}{2}\rfloor}\Big)}_{=(A^{\mathrm{o}}_{(1),i-1}(f;m))^{1/2}} \\
&\qquad + \underbrace{\frac{1}{m^{3/2}}\sum_{j=1}^{m} f\Big(\tfrac{j}{m}\Big)\Big[\Big(\tfrac{j}{m}-1\Big)(X_{m+i} - X_i) - X_{i+\lfloor\frac{j}{2}\rfloor} + X_{i+\lfloor m-\frac{j}{2}\rfloor}\Big]}_{\equiv D_i(f;m)}\Bigg]^2 \\
&= [(A^{\mathrm{o}}_{(1),i-1}(f;m))^{1/2} + D_i(f;m)]^2.
\end{aligned}$$

Now write

$$m^{3/2}\, D_i(f;m) = (X_{m+i} - X_i)\sum_{j=1}^{m} f\Big(\tfrac{j}{m}\Big)(\tfrac{j}{m} - 1) + \sum_{j=1}^{m} f\Big(\tfrac{j}{m}\Big)(X_{i+\lfloor m-\frac{j}{2}\rfloor} - X_{i+\lfloor\frac{j}{2}\rfloor}),$$

where we note that the quantity $\sum_{j=1}^{m} f\Big(\frac{j}{m}\Big)(\frac{j}{m}-1)$ can be computed a priori in $O(m)$ time. If we show that each of the quantities $K_i(f;m) \equiv \sum_{j=1}^{m} f\Big(\frac{j}{m}\Big)(X_{i+\lfloor m-\frac{j}{2}\rfloor} - X_{i+\lfloor\frac{j}{2}\rfloor})$ can also be computed recursively in $O(1)$ time for $i = 1,\ldots,n-m+1$, then the entire estimator $\mathcal{A}^{o}_{(1)}(f;b,m)$ can be computed in $O(n)$ time. To analyze $K_i(f;m)$, we consider specific weight functions separately. For the remainder of this section, we assume that m is even. Similar results can be obtained for odd m.

4.5.1 Constant Weight Function

Let $f(t) = c$, where c is a constant. Then

$$\begin{aligned} K_i(f;m) &= c\sum_{j=1}^{m}\Big(X_{i+\lfloor m-\frac{j}{2}\rfloor} - X_{i+\lfloor\frac{j}{2}\rfloor}\Big) \\ &= K_{i-1}(f;m) + c(2X_{m-1+i} + X_i + X_{i-1} - 3X_{\frac{m}{2}-1+i} - X_{\frac{m}{2}+i}), \end{aligned}$$

for $i = 2,\ldots,n-m+1$. Notice that after the quantity $K_0(f;m)$ is computed in $O(m)$ time, each of the remaining quantities $K_i(f;m)$, $i \geq 1$, can be calculated recursively in $O(1)$ time.

4.5.2 Linear Weight Function

Let $f(t) = t$. Then

$$\begin{aligned} K_i(f;m) &= \sum_{j=1}^{m}\frac{j}{m}\Big(X_{i+\lfloor m-\frac{j}{2}\rfloor} - X_{i+\lfloor\frac{j}{2}\rfloor}\Big) \\ &= K_{i-1}(f;m) + \frac{3}{m}X_{m-1+i} + \frac{4S_{i,m-1}}{m} - 3X_{\frac{m}{2}+i-1} - X_{\frac{m}{2}+i} + \frac{X_{i-1}}{m}, \end{aligned}$$

for $i = 2,\ldots,n-m+1$. Since $S_{i,m-1} = S_{i-1,m-1} - X_{i-1} + X_{i+m-2}$ for $i = 1,\ldots,n-m+1$, $S_{i,m-1}$ and, therefore $K_i(f;m)$, can be updated in $O(1)$ time.

4.5.3 Quadratic Weight Function

Let $f(t) = t^2$. Then

$$\begin{aligned} K_i(f;m) &= \sum_{j=1}^{m}(\tfrac{j}{m})^2\Big(X_{i+\lfloor m-\frac{j}{2}\rfloor} - X_{i+\lfloor\frac{j}{2}\rfloor}\Big) \\ &= K_{i-1}(f;m) + \frac{5}{m^2}X_{m-1+i} + \frac{L_i}{m^2} - \frac{A_1}{m^2}X_{\frac{m}{2}+i} - \frac{A_2}{m^2}X_{\frac{m}{2}+i-1} + \frac{X_{i-1}}{m^2}, \end{aligned}$$

for $i = 2, \ldots, n-m+1$, where

$$L_i \equiv \sum_{j=i}^{\frac{m}{2}-2+i} [12 + 16(j-i)]X_j + \sum_{j=\frac{m}{2}+1+i}^{m-2+i} \Big\{A_3 - 16\Big[j - \Big(\tfrac{m}{2} + i + 1\Big)\Big]\Big\}X_j,$$

and

$$\begin{aligned} A_1 &\equiv (m-2)^2 + (m-3)^2 - (m-1)^2, \\ A_2 &\equiv 2(m-1)^2 + (m-2)^2, \\ A_3 &\equiv (m-2)^2 + (m-3)^2 - (m-4)^2 - (m-5)^2. \end{aligned}$$

First, we write

$$\begin{aligned} L_i &= \sum_{j=i}^{\frac{m}{2}-2+i} [12 + 16(j-i)]X_j + \sum_{j=\frac{m}{2}+1+i}^{m-2+i} \Big\{A_3 - 16\Big[j - \Big(\tfrac{m}{2} + i + 1\Big)\Big]\Big\}X_j \\ &= (12 - 16i) \underbrace{\sum_{j=i}^{\frac{m}{2}-2+i} X_j}_{\equiv M_{i,1}} + \Big[A_3 + 16\Big(\tfrac{m}{2} + 1 + i\Big)\Big] \underbrace{\sum_{j=\frac{m}{2}+1+i}^{m-2+i} X_j}_{\equiv M_{i,2}} + 16 \underbrace{\sum_{j=i}^{\frac{m}{2}-2+i} jX_j}_{\equiv M_{i,3}} \\ &\quad - 16 \underbrace{\sum_{j=\frac{m}{2}+1+i}^{m-2+i} jX_j}_{\equiv M_{i,4}}. \end{aligned}$$

Note that for $i = 2, \ldots, n-m+1$,

$$\begin{aligned} M_{i,1} &= M_{i-1,1} - X_{i-1} + X_{\frac{m}{2}-2+i}, \\ M_{i,2} &= M_{i-1,2} - X_{\frac{m}{2}+i} + X_{m-2+i}, \\ M_{i,3} &= M_{i-1,3} - (i-1)X_{i-1} + \Big(\frac{m}{2} - 2 + i\Big)X_{\frac{m}{2}-2+i}, \\ M_{i,4} &= M_{i-1,4} - \Big(\frac{m}{2} + i\Big)X_{\frac{m}{2}+i} + (m-2+i)X_{m-2+i}. \end{aligned}$$

Since all of the $M_{i,j}$'s can be updated in $O(1)$ time, so can L_i for $i = 2, \ldots, n-m+1$ and $j = 1, 2, 3, 4$. As a result, $K_i(f; m)$, for $i = 1, \ldots, n-m+1$, can also be updated from available information in $O(1)$ time.

4.5.4 Algorithms

Algorithms 1–3 below list pseudocodes for computing the level-1 FOA estimators in $O(n)$ time for each of the weight functions, $f(t) = c$, $f(t) = t$, and $f(t) = t^2$ respectively. Although, these are not the weight functions that satisfy Assumption F, they are the components of all such weight functions. The computations for arbitrary second-degree polynomial weights involve simple augmentations.

4.6 Examples

In this section, we illustrate the performance of the level-1 FOA estimators based on the stochastic processes in §3.6. First, in §4.6.1 we consider an i.i.d. Gaussian process. Then in §4.6.2 we conduct Monte Carlo studies based on a stationary first-order autoregressive Gaussian process and the delay-time process in the stationary M/M/1 system. In §4.6.3 we discuss distributional approximations for level-1 FOA estimators.

4.6.1 I.i.d. Gaussian Process

Table 10 contains results for the i.i.d. Gaussian process in §3.6.1 for $b \geq 2$. The entries indeed match up nicely with those of Table 9 with $\gamma_1 = 0$.

Table 10: Bias and Variance for FOA Estimators in the I.i.d. Normal Case

Estimator	Bias	Variance
$\mathcal{A}^{\circ}_{(1)}(f_0; b, m)$	$-\frac{1}{m^2}$	$\frac{0.66b-0.82}{(b-1)^2}$
$\mathcal{A}^{\circ}_{(1)}(f_2; b, m)$	$\frac{7}{2m^2} + \frac{63}{2m^4} - \frac{36}{m^6}$	$\frac{0.54b-0.68}{(b-1)^2}$

4.6.2 Monte Carlo Examples

In this section, we examine the empirical performance of the FOA estimators using the AR(1) and M/M/1 processes from §3.6.2.

Algorithm 1 Computation FOA Estimators with Constant Weight Functions ($f(t) = c$)

Step 1: Initialization

$s_\ell \leftarrow 0$ for $\ell = 1, \ldots, 5,\ A^{\circ}_{(1),1} \leftarrow 0,\ i \leftarrow 1$

Step 2: Calculate $A^{\circ}_{(1),1}(f; m)$ and K_1

repeat
  $j \leftarrow 1$
  repeat
    $s_1 \leftarrow s_1 + X_j$
  until $j = \left\lfloor \frac{i}{2} \right\rfloor$
  $j \leftarrow \left\lfloor m - \frac{i}{2} \right\rfloor + 1$
  repeat
    $s_2 \leftarrow s_2 + X_j$
  until $j = m$
  $j \leftarrow \left\lfloor \frac{i}{2} \right\rfloor + 1$
  repeat
    $s_3 \leftarrow s_3 + X_j$
  until $j = \left\lfloor m - \frac{i}{2} \right\rfloor$
  $A^{\circ}_{(1),1} \leftarrow A^{\circ}_{(1),1} + c\Big[\Big(\frac{i}{m} - 1\Big)(s_1 + s_2) + s_3\Big(\frac{i}{m}\Big)\Big]$
  $s_\ell \leftarrow 0$ for $\ell = 1, 2, 3$
  $s_4 \leftarrow s_4 + X_{1+\lfloor m - i/2 \rfloor} - X_{1+\lfloor i/2 \rfloor}$
  $s_5 \leftarrow s_5 + c\Big(\frac{i}{m} - 1\Big)$
until $i = m$
$A^{\circ}_{(1),1} \leftarrow A^{\circ}_{(1),1}/m^{3/2}$
$K_1 \leftarrow c * s_4$

Step 3:

$i \leftarrow 2,\ \text{sum} \leftarrow [A^{\circ}_{(1),1}]^2$
repeat
  $K_i \leftarrow K_{i-1} + c(2X_{m-1+i} + X_i + X_{i-1} - 3X_{m/2-1+i} - X_{m/2+i})$
  $A^{\circ}_{(1),i} \leftarrow A^{\circ}_{(1),i-1} + [s_5(X_{m+i} - X_i) + K_i]/m^{3/2}$
  $\text{sum} \leftarrow \text{sum} + [A^{\circ}_{(1),i}]^2$
until $i = n - m + 1$

Return $A^{\circ}_{(1)} \leftarrow \text{sum}/(n - m + 1)$

Algorithm 2 Computation of FOA Estimators with Linear Weight Functions ($f(t) = t$)

Step 1: Initialization

$s_\ell \leftarrow 0$ for $\ell = 1, \ldots, 5,\ A^{\circ}_{(1),1} \leftarrow 0,\ i \leftarrow 1,\ K_1 \leftarrow 0$

Step 2: Calculate $A^{\circ}_{(1),1}(f; m)$ and K_1

repeat
  $j \leftarrow 1$
  repeat
    $s_1 \leftarrow s_1 + X_j$
  until $j = \left\lfloor \frac{i}{2} \right\rfloor$
  $j \leftarrow \left\lfloor m - \frac{i}{2} \right\rfloor + 1$
  repeat
    $s_2 \leftarrow s_2 + X_j$
  until $j = m$
  $j \leftarrow \left\lfloor \frac{i}{2} \right\rfloor + 1$
  repeat
    $s_3 \leftarrow s_3 + X_j$
  until $j = \left\lfloor m - \frac{i}{2} \right\rfloor$
  $A^{\circ}_{(1),1} \leftarrow A^{\circ}_{(1),1} + f_i\Big[\Big(\frac{i}{m} - 1\Big)(s_1 + s_2) + s_3\Big(\frac{i}{m}\Big)\Big]$
  $s_\ell \leftarrow 0$ for $\ell = 1, 2, 3$
  $s_4 \leftarrow s_4 + f_i\Big(\frac{i}{m} - 1\Big)$
  $K_1 \leftarrow K_1 + f_i(X_{1+\lfloor m-i/2 \rfloor} - X_{1+\lfloor i/2 \rfloor})$
until $i = m$
$j \leftarrow 2$
repeat
  $s_5 \leftarrow s_5 + 4X_j$
until $j = m$
$A^{\circ}_{(1),1} \leftarrow A^{\circ}_{(1),1}/m^{3/2}$

Step 3:

$i \leftarrow 2,\ \text{sum} \leftarrow [A^{\circ}_{(1)}]^2$
repeat
  $K_i \leftarrow K_{i-1} + (3X_{m-1+i} + s_5 + X_{i-1})/m - 3X_{m/2-1+i} - X_{m/2+i}$
  $s_5 \leftarrow s_5 - 4(X_i - X_{m+i-1})$
  $A^{\circ}_{(1),i} \leftarrow A^{\circ}_{(1),i-1} + [(X_{m+i} - X_i)s_4 + K_i]/m^{3/2}$
  $\text{sum} \leftarrow \text{sum} + [A^{\circ}_{(1),i}]^2$
until $i = n - m + 1$

Return $A^{\circ}_{(1)} \leftarrow \text{sum}/(n - m + 1)$

Algorithm 3 Computation of FOA Estimators with Quadratic Weight Functions ($f(t) = t^2$)

Step 1: Initialization

$s_\ell \leftarrow 0$ for $\ell = 1, \ldots, 4,\ A^{\circ}_{(1),1} \leftarrow 0,\ i \leftarrow 1,\ K_1 \leftarrow 0,\ M_{1,\ell} \leftarrow 0$ for $\ell = 1, \ldots, 4,$
$A_1 \leftarrow (m-2)^2 + (m-3)^2 - (m-1)^2,\ A_2 \leftarrow 2(m-1)^2 + (m-2)^2,$
$A_3 \leftarrow (m-2)^2 + (m-3)^2 - (m-4)^2 - (m-5)^2,\ A_4 \leftarrow (m-1)^2 + (m-2)^2 - (m-3)^2 - (m-4)^2$

Step 2: Calculate $M_{1,1}, M_{1,2}, M_{1,3}, M_{1,4}, A^{\circ}_{(1),1}(f; m)$ **and** K_1

repeat
  $j \leftarrow 1$
  repeat
    $s_1 \leftarrow s_1 + X_j$
  until $j = \left\lfloor \frac{i}{2} \right\rfloor$
  $j \leftarrow \left\lfloor m - \frac{i}{2} \right\rfloor + 1$
  repeat
    $s_2 \leftarrow s_2 + X_j$
  until $j = m$
  $j \leftarrow \left\lfloor \frac{i}{2} \right\rfloor + 1$
  repeat
    $s_3 \leftarrow s_3 + X_j$
  until $j = \left\lfloor m - \frac{i}{2} \right\rfloor$
  $A^{\circ}_{(1),1} \leftarrow A^{\circ}_{(1),1} + f_i\Big[\Big(\frac{i}{m} - 1\Big)(s_1 + s_2) + s_3\Big(\frac{i}{m}\Big)\Big]$
  $s_\ell \leftarrow 0$ for $\ell = 1, 2, 3$
  $s_4 \leftarrow s_4 + f_i\Big(\frac{i}{m} - 1\Big),\ K_1 \leftarrow K_1 + f_i(X_{1+\lfloor m-i/2 \rfloor} - X_{1+\lfloor i/2 \rfloor}),$
  if $i \le (m/2 - 1)$ **then**
    $M_{1,1} \leftarrow M_{1,1} + X_i,\ M_{1,3} \leftarrow M_{1,3} + iX_i$
  else if $i \ge m/2 + 1 + i$ and $i \le (m - 2 + i)$ **then**
    $M_{1,2} \leftarrow M_{1,2} + X_i,\ M_{1,4} \leftarrow M_{1,4} + iX_i$
  end if
until $i = m$
$A^{\circ}_{(1),1} \leftarrow A^{\circ}_{(1),1}/m^{3/2}$

Step 3:

$i \leftarrow 2,\ \text{sum} \leftarrow [A^{\circ}_{(1)}]^2$
repeat
  $M_{1,1} \leftarrow M_{1,1} - X_{i-1} + X_{m/2-2+i},\ M_{1,2} \leftarrow M_{1,2} - X_{m/2+i} + X_{m-2+i}$
  $M_{1,3} \leftarrow M_{1,3} - (i-1)X_{i-1} + (m/2 - 2 + i)X_{m/2-2+i}$
  $M_{1,4} \leftarrow M_{1,4} - (m/2 + i)X_{m/2+i} + (m - 2 + i)X_{m-2+i}$
  $L_i \leftarrow (12 - 16i)M_{1,1} + [A_3 + 16(m/2 + 1 + i)]M_{1,2} + 16(M_{1,3} - M_{1,4})$
  $K_i \leftarrow K_{i-1} + (5X_{m-1+i} + L_i - A_2X_{m/2-1+i} - A_1X_{m/2+i} + X_{i-1})/m^2$
  $A^{\circ}_{(1),i} \leftarrow A^{\circ}_{(1),i-1} + [s_4(X_{m+i} - X_i) + K_i]/m^{3/2},\ \text{sum} \leftarrow \text{sum} + [A^{\circ}_{(1),i}]^2$
until $i = n - m + 1$

Return $A^{\circ}_{(1)} \leftarrow \text{sum}/(n - m + 1)$

Tables 11 and 12 contain the estimated expected values and standard deviations of the FOA variance estimators and their counterparts for the AR(1) process. Similarly, in Tables 13 and 14 we present the corresponding estimated expected values and standard devations for the M/M/1 delay-time process. All entries in Tables 11–14 are based on 10,000 independent replications. In Tables 12 and 14, the rows labeled "$m \to \infty$" provide the asymptotic standard deviations of the respective variance estimators, which are summarized in Table 9.

Table 11: Estimated Expected Values of Variance Estimators for the AR(1) Process for $\phi = 0.9$, $\sigma^2 = 19$ and $b = 32$

	Level-0 Folded Area Estimators			
m	$\mathcal{A}_{(0)}(f_0; b, m)$	$\mathcal{A}^{\circ}_{(0)}(f_0; b, m)$	$\mathcal{A}_{(0)}(f_2; b, m)$	$\mathcal{A}^{\circ}_{(0)}(f_2; b, m)$
512	17.98	17.94	18.78	18.75
1024	18.47	18.43	18.94	18.88
2048	18.77	18.72	19.05	18.96
4096	18.87	18.86	19.00	18.98
	Level-1 Folded Area Estimators			
m	$\mathcal{A}_{(1)}(f_0; b, m)$	$\mathcal{A}^{\circ}_{(1)}(f_0; b, m)$	$\mathcal{A}_{(1)}(f_2; b, m)$	$\mathcal{A}^{\circ}_{(1)}(f_2; b, m)$
512	17.81	17.82	18.19	18.18
1024	18.48	18.46	18.82	18.78
2048	18.79	18.75	18.97	18.97
4096	18.91	18.87	19.08	18.98

From Tables 11 and 13, we can make the following conclusions concerning the expected values of the FOA estimators:

- The expected values of all estimators converge to σ^2 as m increases, as dictated by our theoretical results. For small values of m, the level-1 FOA estimators have larger bias than corresponding level-0 variance estimators mentioned above.

- For the M/M/1 process the estimated expected values of all variance estimates

Table 12: Estimated Standard Deviations of Variance Estimators for the AR(1) Process for $\phi = 0.9$, $\sigma^2 = 19$ and $b = 32$

	Level-0 Folded Area Estimators			
m	$\mathcal{A}_{(0)}(f_0; b, m)$	$\mathcal{A}^{\circ}_{(0)}(f_0; b, m)$	$\mathcal{A}_{(0)}(f_2; b, m)$	$\mathcal{A}^{\circ}_{(0)}(f_2; b, m)$
512	4.54	2.79	4.74	3.03
1024	4.63	2.84	4.75	3.09
2048	4.71	2.81	4.76	3.07
4096	4.74	2.82	4.74	3.09
$\to \infty$	4.75	2.78	4.75	3.04
	Level-1 Folded Area Estimators			
m	$\mathcal{A}_{(1)}(f_0; b, m)$	$\mathcal{A}^{\circ}_{(1)}(f_0; b, m)$	$\mathcal{A}_{(1)}(f_2; b, m)$	$\mathcal{A}^{\circ}_{(1)}(f_2; b, m)$
512	4.49	2.72	4.53	2.45
1024	4.58	2.76	4.64	2.49
2048	4.69	2.77	4.77	2.51
4096	4.67	2.74	4.67	2.48
$\to \infty$	4.75	2.72	4.75	2.47

first increase and then decrease with m. This phenomenon was first observed by Sargent et al. [28].

Based on Tables 12 and 14, we can make the following conclusions concerning the variances of the FOA estimators:

- For the AR(1) process, all variance estimators converge to the theoretical asymptotic values quite quickly as m increases. We can see that the estimated standard deviations for the M/M/1 process converge to the theoretical values as m increases, but at a slower rate than for the AR(1) process.

- The FOA estimators have smaller variances for both weight functions than their competitors listed in Tables 12 and 14.

Table 13: Estimated Expected Values of Variance Estimators for the M/M/1 Delay-Time Process for $\rho = 0.8$, $\sigma^2 = 1976$ and $b = 32$

	Level-0 Folded Area Estimators			
m	$\mathcal{A}_{(0)}(f_0; b, m)$	$\mathcal{A}^{\circ}_{(0)}(f_0; b, m)$	$\mathcal{A}_{(0)}(f_2; b, m)$	$\mathcal{A}^{\circ}_{(0)}(f_2; b, m)$
2048	1849	1819	1942	1950
4096	1900	1899	1968	1968
8192	1939	1959	1972	1970
16384	1960	1965	1980	1976
32768	1962	1964	1969	1973
	Level-1 Folded Area Estimators			
m	$\mathcal{A}_{(1)}(f_0; b, m)$	$\mathcal{A}^{\circ}_{(1)}(f_0; b, m)$	$\mathcal{A}_{(1)}(f_2; b, m)$	$\mathcal{A}^{\circ}_{(1)}(f_2; b, m)$
2048	1803	1806	1833	1832
4096	1896	1894	1931	1929
8192	1947	1942	1972	1969
16384	1960	1960	1977	1976
32768	1964	1959	1969	1969

4.6.3 Density Estimation

In this section, we use the approach from §3.6.4 to estimate the limiting distributions of the level-1 FOA estimators for sufficiently large values of m. Using an argument that is similar to that of §3.6.4, we obtain the approximation

$$\mathcal{A}^{\circ}_{(k)}(f; b, m) \dot{\sim} \mathrm{E}[\mathcal{A}^{\circ}_{(k)}(f; b, m)]\chi^2_{\nu_{\text{eff}}}/\nu_{\text{eff}}, \text{ where } \nu_{\text{eff}} = \left[\!\!\left[\frac{2\mathrm{E}^2[\mathcal{A}^{\circ}_{(k)}(f; b, m)]}{\mathrm{Var}[\mathcal{A}^{\circ}_{(k)}(f; b, m)]}\right]\!\!\right], \quad (54)$$

for $k = 1, 2, \ldots$.

For the FOA estimator with the constant weight function $f_0(t)$, we see that

$$\mathrm{E}[\mathcal{A}^{\circ}_{(1)}(f_0; b, m)] \approx \sigma^2 + 3\gamma/m \quad \text{and} \quad \nu_{0,\text{eff}} = \left[\!\!\left[\frac{(\sigma^2 + 3\gamma/m)^2 70(b-1)^2}{(23b - 29)\sigma^4}\right]\!\!\right]; \quad (55)$$

hence from Equation (54), we have

$$\mathcal{A}^{\circ}_{(1)}(f_0; b, m) \dot{\sim} (\sigma^2 + 3\gamma/m)\chi^2_{\nu_{0,\text{eff}}}/\nu_{0,\text{eff}}. \quad (56)$$

To evaluate the approximation in Equation (54), we generated 1,000,000 independent sample paths of the stationary AR(1) process. Each sample path contained

Table 14: Estimated Standard Deviations of Variance Estimators for the M/M/1 Delay-Time Process for $\rho = 0.8$, $\sigma^2 = 1976$ and $b = 32$

	Level-0 Folded Area Estimators			
m	$\mathcal{A}_{(0)}(f_0; b, m)$	$\mathcal{A}^{\circ}_{(0)}(f_0; b, m)$	$\mathcal{A}_{(0)}(f_2; b, m)$	$\mathcal{A}^{\circ}_{(0)}(f_2; b, m)$
2048	1010	772	1026	831
4096	809	633	831	704
8192	700	487	688	517
16384	614	407	611	432
32768	555	352	550	367
$\to \infty$	494	289	494	316
	Level-1 Folded Area Estimators			
m	$\mathcal{A}_{(1)}(f_0; b, m)$	$\mathcal{A}^{\circ}_{(1)}(f_0; b, m)$	$\mathcal{A}_{(1)}(f_2; b, m)$	$\mathcal{A}^{\circ}_{(1)}(f_2; b, m)$
2048	951	701	852	675
4096	790	579	787	575
8192	727	485	677	477
16384	604	404	607	389
32768	557	345	542	314
$\to \infty$	494	283	494	257

n = 20,000 observations and all variance estimators were computed using a batch size of $m = 1000$. Figure 5(a) displays the empirical probability density function of $\mathcal{A}^{\circ}_{(1)}(f_0; 20, 1000)$ and the fitted p.d.f. based on Equations (55) and (56). We see that an appropriately scaled chi-squared random variable provides a good approximation to the distribution of $\mathcal{A}^{\circ}_{(1)}(f_0; 20, 1000)$.

For the level-1 FOA estimator with the first-order unbiased quadratic weight function $f_2(\cdot)$, we have

$$\mathrm{E}[\mathcal{A}^{\circ}_{(1)}(f_2; b, m)] \approx \sigma^2 \quad \text{and} \quad \nu_{2,\mathrm{eff}} = \left[\!\!\left[\frac{(17160(b-1)^2}{(4639b - 5782)} \right]\!\!\right]; \tag{57}$$

hence from Equation (54) we have

$$\mathcal{A}^{\circ}_{(1)}(f_2; b, m) \dot{\sim} \sigma^2 \chi^2_{\nu_{2,\mathrm{eff}}} / \nu_{2,\mathrm{eff}}. \tag{58}$$

Figure 5(b) displays the empirical p.d.f. of $\mathcal{A}^{\circ}_{(1)}(f_2; 20, 1000)$ and the fitted p.d.f. based

on Equations (57) and (58). Again we see that an appropriately scaled chi-squared random variable is a good approximation to the distribution of $\mathcal{A}^{\circ}_{(1)}(f_2; 20, 1000)$.

As in §3.6.4, we can obtain approximate CIs for the parameters μ and σ^2. For example, if $\mathcal{A}^{\circ}_{(k)}(f; b, m)$, $k = 1, 2, \ldots$, is a first-order unbiased estimator for σ^2, then for $\alpha \in (0, 1)$ and a sufficiently large batch size m, an approximate $100(1-\alpha)\%$ two-sided CI for σ^2 is

$$\frac{\nu_{\text{eff}}\mathcal{A}^{\circ}_{(k)}(f; b, m)}{\chi^2_{1-\alpha/2,\nu_{\text{eff}}}} \leq \sigma^2 \leq \frac{\nu_{\text{eff}}\mathcal{A}^{\circ}_{(k)}(f; b, m)}{\chi^2_{\alpha/2,\nu_{\text{eff}}}}. \tag{59}$$

Example 25 Consider the FOA variance estimator based on the quadratic weight function $f_2(\cdot)$, for which Equations (57) and (58) yield $\mathcal{A}^{\circ}_1(f_2; 20, 1000) \dot{\sim} \sigma^2\chi^2_{71}/71$. The two-sided 90% CI for σ^2 is

$$0.7745\mathcal{A}^{\circ}_{(1)}(f_2; 20, 1000) \leq \sigma^2 \leq 1.3498\mathcal{A}^{\circ}_{(1)}(f_2; 20, 1000). \tag{60}$$

We used the 1,000,000 independent realizations of $\mathcal{A}^{\circ}_{(1)}(f_2; 20, 1000)$ computed from the AR(1) process to estimate the coverage probability of the CI defined by (60). The estimated coverage was 0.9. We also obtained an empirical coverage probability of 0.9023 for the two-sided 90% CI (59) for σ^2 based on $\mathcal{A}^{\circ}_{(1)}(f_0; 20, 1000)$, even though $\mathcal{A}^{\circ}_{(1)}(f_0; 20, 1000)$ is a biased estimator for σ^2. ◁

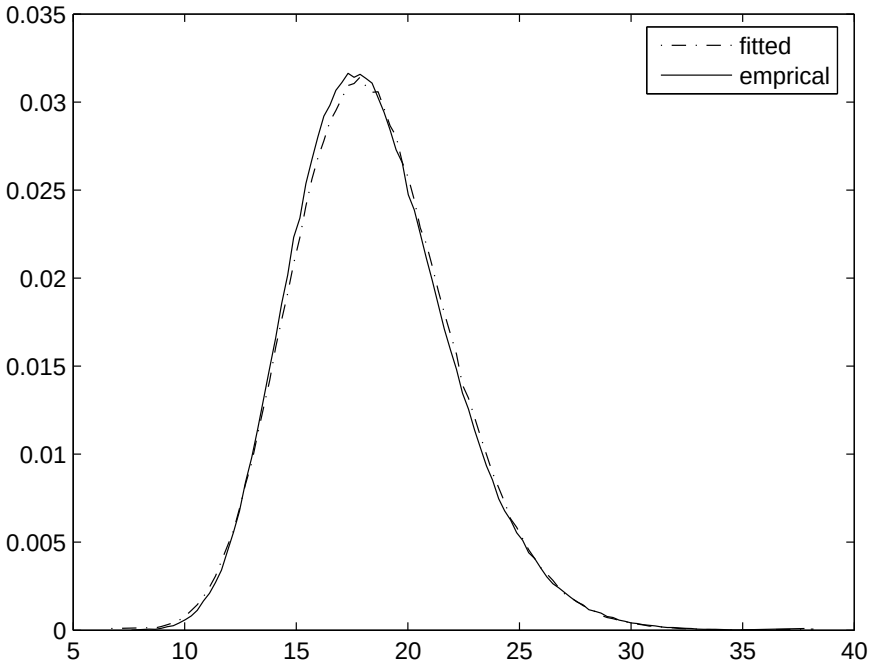
An approximate $100(1-\alpha)\%$ two-sided CI for μ is given by

$$\bar{X}_n - t_{1-\alpha/2,\nu_{\text{eff}}}\sqrt{\mathcal{A}^{\circ}_{(k)}(f; b, m)/n} \leq \mu \leq \bar{X}_n + t_{1-\alpha/2,\nu_{\text{eff}}}\sqrt{\mathcal{A}^{\circ}_{(k)}(f; b, m)/n}. \tag{61}$$

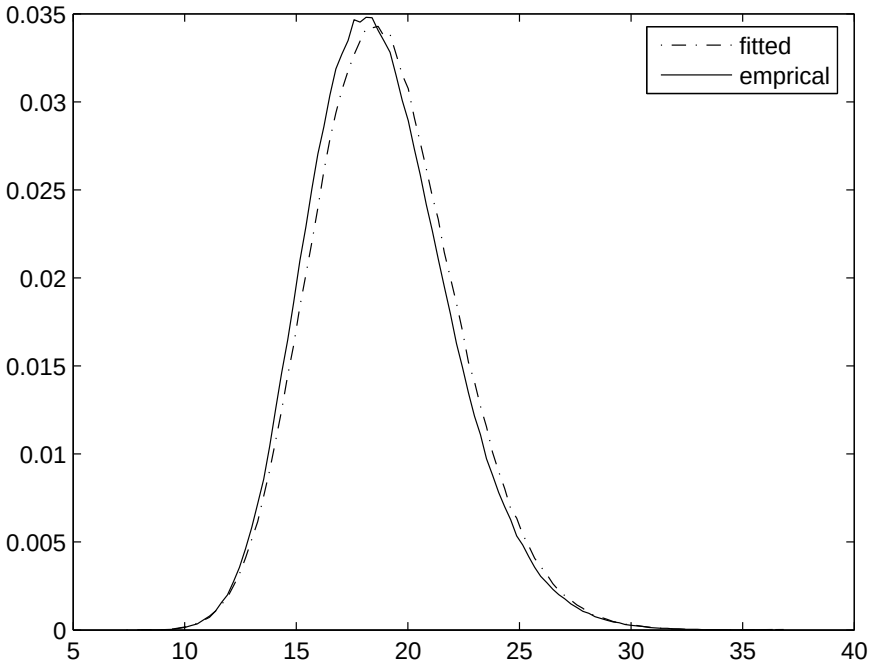
Example 26 Consider the FOA estimator using the quadratic weight function $f_2(\cdot)$. The two-sided 90% CI for μ is

$$\bar{X}_n - 0.01178\sqrt{\mathcal{A}^{\circ}_{(1)}(f_2; 20, 1000)} \leq \mu \leq \bar{X}_n + 0.01178\sqrt{\mathcal{A}^{\circ}_{(1)}(f_2; 20, 1000)}. \tag{62}$$

We used the 1,000,000 independent realizations of $\mathcal{A}^{\circ}_{(1)}(f_2; 20, 1000)$ computed from the AR(1) process to estimate the coverage probability of the CI defined by Equation (62). The estimated coverage was 0.8973. Similarly, we obtained an estimated coverage probability of 0.8944 for the two-sided 90% CI using $\mathcal{A}^{\circ}_{(1)}(f_0; 20, 1000)$. ◁



(a) $\mathcal{A}^{\mathrm{o}}_{(1)}(f_0; 20, 1000)$



(b) $\mathcal{A}^{\mathrm{o}}_{(1)}(f_2; 20, 1000)$

Figure 5: Empirical and Fitted p.d.f.'s for FOA Estimators Based on the AR(1) Process

4.7 Summary

This chapter studied folded estimators based on overlapping batches. Specifically, we obtained the first two moments of level-1 FOA estimators, their limiting distribution as the batch size goes to infinity while the ratio of the sample size to the batch size remains constant, and developed algorithms for computing them in linear time for polynomial weights of degree ≤ 2. We also analyzed their performance via a Monte Carlo study involving an AR(1) process and the delay-time process in a stationary M/M/1 system. At level 1 and for a given weight function, the FOA estimators are significantly less variable than their level-1 counterparts based on nonoverlapping batches, and level-0 counterparts based on both nonoverlapping and overlapping batches. However, level-1 FOA estimators exhibit more small-sample bias compared to level-0 counterparts. Also, the correlation between folded overlapping estimators at levels 0 and 1 reduces the potential benefits of linear combinations. Using an approach analogous to Alexopoulos et al. [4], we also showed that the FOA estimators can be approximated (for sufficiently large batch sizes) by properly rescaled χ^2 distributions (with appropriate degrees of freedom). Finally, we constructed CIs for the variance parameter as well as the mean of the underlying process. The confidence intervals exhibited nearly nominal coverage.

Chapter 5 proposes estimators based on reflections applied to the STS corresponding to the entire sample. Reflections are another method of data re-use, which we shall study in the next chapter.

CHAPTER V

REFLECTED ESTIMATORS

5.1 Introduction

Assumptions A show that certain functionals of a stationary process converge in distribution to a Brownian motion process. For the Brownian motion processes, the following reflection principle holds.

Reflection Principle: If $\mathcal{W}(t)$ is a Brownian motion on $[0, 1]$, then

$$\mathcal{W}_c^\star(t) = \begin{cases} \mathcal{W}(t) & \text{if } t < c \\ 2\mathcal{W}(c) - \mathcal{W}(t) & \text{if } t \geq c \end{cases}$$

is also a Brownian motion process, where $c \in [0, 1]$ is any reflection point. Note that the processes $\mathcal{W}(\cdot)$ and $\mathcal{W}_c^\star(\cdot)$ are correlated.

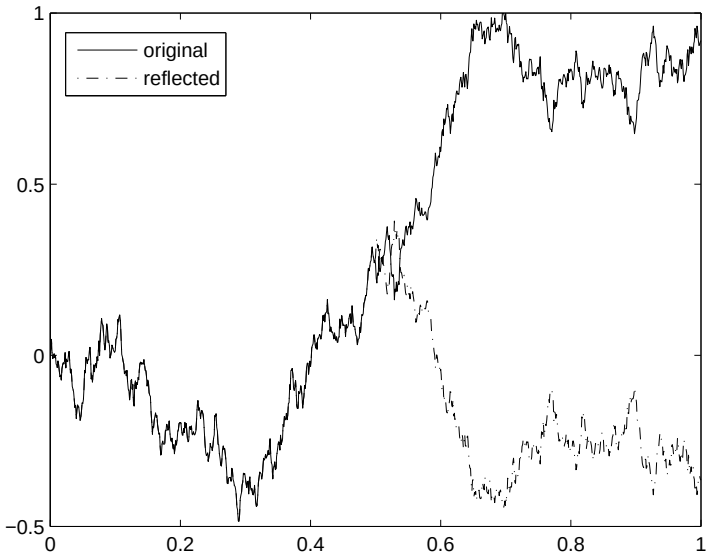


Figure 6: Original and Reflected Brownian Motion Processes

Figure 6 shows an example of the original and reflected Brownian motion processes, corresponding to the reflection point $c = 0.5$. The reflection principle tells us that Brownian motion processes reflected after they hit a point preserve the same distributional properties as the original Brownian motion. Hence, we can use a sample path from a Brownian motion process to generate several other different sample paths from the same process. Therefore, a set of data from a simulation output process can be re-used to obtain different sample data sets, where their respective functionals still converge in distribution to Brownian motion processes. The new paths generated through reflection are generally correlated.

First of all, assume that Assumptions A hold. Without loss of generality, we will assume the mean μ for this process to be zero. For processes with mean different than zero, we can take the difference of two independent replications, and the variance of the sample mean of the difference is approximately σ^2/n. Hence the assumption of zero mean is legitimate. Define

$$X^{\star}_{c,j} \equiv \begin{cases} X_j & \text{if } 0 \le j \le \lfloor cn \rfloor \\ -X_j & \text{if } \lfloor cn \rfloor + 1 \le j \le n \end{cases}$$

for $j = 1, \ldots, n$. Let $S^{\star}_{c,k} \equiv \sum_{j=1}^{k} X^{\star}_{c,j}$ for $k = 1, 2, \ldots$. Therefore, for $t, c \in [0, 1]$,

$$S^{\star}_{c,\lfloor nt \rfloor} \equiv \begin{cases} S_{\lfloor nt \rfloor} & \text{if } t \le c \\ 2S_{\lfloor nc \rfloor} - S_{\lfloor nt \rfloor} & \text{if } t > c. \end{cases}$$

Further define $X^{\star}_{c,n}(t) \equiv S^{\star}_{c,\lfloor nt \rfloor}/(\sigma\sqrt{n})$, and note that

$$\begin{aligned} X^{\star}_{c,n}(t) &= \begin{cases} \frac{S_{\lfloor nt \rfloor}}{\sigma\sqrt{n}} & \text{if } t \le c \\ \frac{2S_{\lfloor nc \rfloor}}{\sigma\sqrt{n}} - \frac{S_{\lfloor nt \rfloor}}{\sigma\sqrt{n}} & \text{if } t > c \end{cases} \\ &\xrightarrow[n\to\infty]{\mathcal{D}} \begin{cases} \mathcal{W}(t) & \text{if } t \le c \\ 2\mathcal{W}(c) - \mathcal{W}(t) & \text{if } t > c \end{cases} \\ &\equiv \mathcal{W}^{\star}_{c}(t). \end{aligned}$$

We have shown that the process $X^\star_{c,n}(\cdot)$ obtained from the reflected data $\{X^\star_{c,j} : j = 1, \ldots, n\}$ converges to the Brownian motion that results from a reflection at point c of the limit of the original process $X_n(\cdot)$.

The rest of this chapter proceeds as follows. In §5.2, we introduce the reflected NBM estimator for σ^2. In §5.3, we propose reflected estimators based on STS. Specifically, §5.4 and §5.5 discuss reflected area and reflected CvM estimators, while §5.4.1 and §5.5.1 study linear combinations of reflected area and reflected CvM estimators, respectively.

5.2 Reflected Nonoverlapping Batch Means Estimators

Recall that the nonoverlapping batch means estimator is

$$\mathcal{N}(b,m) = \frac{m}{b-1}\sum_{i=1}^{b}(\bar{X}_{i,m} - \bar{X}_n)^2.$$

We define the reflected NBM estimator as

$$\begin{aligned}
\mathcal{N}^\star_c(b,m) &\equiv \frac{m}{b-1}\sum_{i=1}^{b}(\bar{X}^\star_{c,i,m} - \bar{X}^\star_{c,n})^2 \\
&= \frac{m}{b-1}\sum_{i=1}^{b}\Big[(\bar{X}^\star_{c,i,m})^2 - 2\bar{X}^\star_{c,i,m}\bar{X}^\star_{c,n} + (\bar{X}^\star_{c,n})^2\Big] \\
&= \frac{m}{b-1}\left[\sum_{i=1}^{b}(\bar{X}^\star_{c,i,m})^2 - 2\bar{X}^\star_{c,n}\sum_{i=1}^{b}\bar{X}^\star_{c,i,m} + b(\bar{X}^\star_{c,n})^2\right], \qquad (63)
\end{aligned}$$

where $\bar{X}^\star_{c,i,m} \equiv \frac{1}{m}\sum_{j=1}^{m} X^\star_{c,(i-1)m+j}$ for $i = 1, \ldots, b$, and $\bar{X}^\star_{c,n} \equiv \frac{1}{n}\sum_{j=1}^{n} X^\star_{c,j}$.

5.2.1 Reflecting an Entire Batch

First, we consider reflecting an entire batch. We have b batches, and for each reflected batch we obtain two different sample paths; one with original and the other with negative of the original observations. Therefore, reflecting b batches yields 2^b possible combinations of reflected paths. We can see that in 2^{b-1} reflected paths the values in an arbitrary batch i are $X_{(i-1)m+1}, \ldots, X_{im}$, while in 2^{b-1} reflected paths the data points are $-X_{(i-1)m+1}, \ldots, -X_{im}$. For instance, consider the trivial case for which

$m = 1$ and $b = 2$; hence X_1 and X_2 are the only observations in the first and second batches, respectively. If we consider reflecting at the beginning of a batch, the possible sample paths are $\{X_1, X_2\}$, $\{X_1, -X_2\}$, $\{-X_1, X_2\}$ and $\{-X_1, -X_2\}$. Hence, there are 2^2 possible combinations of reflected paths.

Under this setting, the reflected NBM estimator, say $\mathcal{N}^{\star}_{1/b}(b, m)$, is the average of all of the potential 2^b NBM estimators obtained from the reflected paths. We proceed with the computation of this estimator. The first summation in Equation (63) becomes

$$\begin{aligned}\frac{m}{b-1}\sum_{i=1}^{b}(\bar{X}^{\star}_{c,i,m})^2 &= \frac{1}{2^b m(b-1)}\left\{\frac{2^b}{2}\sum_{i=1}^{b}\left[\left(\sum_{j=1}^{m}X_{(i-1)m+j}\right)^2+\left(\sum_{j=1}^{m}(-X_{(i-1)m+j})\right)^2\right]\right\}\\ &= \frac{1}{m(b-1)}\sum_{i=1}^{b}\left(\sum_{j=1}^{m}X_{(i-1)m+j}\right)^2 = \frac{m}{b-1}\sum_{i=1}^{b}(\bar{X}_{i,m})^2. \qquad (64)\end{aligned}$$

Similarly the second and third summation terms in Equation (63) become:

$$\begin{aligned}&-\frac{2m}{b-1}\bar{X}^{\star}_{c,n}\sum_{i=1}^{b}\bar{X}^{\star}_{c,i,m}+\frac{mb}{b-1}(\bar{X}^{\star}_{c,n})^2 = -\frac{2m}{b(b-1)}\left(\sum_{i=1}^{b}\bar{X}^{\star}_{c,i,m}\right)^2+\frac{m}{b(b-1)}\left(\sum_{i=1}^{b}\bar{X}^{\star}_{c,i,m}\right)^2\\ &= -\frac{m}{b(b-1)}\left(\sum_{i=1}^{b}\bar{X}^{\star}_{c,i,m}\right)^2 = -\frac{m}{b(b-1)}\sum_{i=1}^{b}(\bar{X}_{i,m})^2. \qquad (65)\end{aligned}$$

Substitution of Equations (64) and (65) into Equation (63) yields

$$\mathcal{N}^{\star}_{1/b}(b, m) = \frac{m}{b}\sum_{i=1}^{b}\bar{X}^2_{i,m}.$$

The following theorem gives the expected value of this estimator.

Theorem 17 If Assumptions A hold, then

$$\mathrm{E}[\mathcal{N}^{\star}_{1/b}(b, m)] = \sigma^2 - \frac{\gamma_1}{m} + o(m^{-1}).$$

Proof:

$$\begin{aligned}
\mathrm{E}[\mathcal{N}^{\star}_{1/b}(b,m)] &= \frac{m}{b}\sum_{i=1}^{b}\mathrm{E}[(\bar{X}_{i,m})^2] = \frac{1}{mb}\sum_{i=1}^{b}\mathrm{E}\Big[\Big(\sum_{j=1}^{m}X_{(i-1)m+j}\Big)^2\Big] \\
&= \frac{1}{mb}\sum_{i=1}^{b}\mathrm{E}\Big[\sum_{j=1}^{m}(X_{(i-1)m+j})^2 + 2\sum_{\substack{1\le j,\ell\le m\\ j<\ell}}X_{(i-1)m+j}X_{(i-1)m+\ell}\Big] \\
&= \frac{1}{mb}\sum_{i=1}^{b}\sum_{j=1}^{m}\mathrm{Var}(X_{(i-1)m+j}) + \frac{2}{mb}\sum_{j=1}^{b}\sum_{\substack{1\le j,\ell\le m\\ j<\ell}}R_{\ell-j} \\
&= \frac{1}{mb}\sum_{i=1}^{b}\sum_{j=1}^{m}R_0 + 2\sum_{j=1}^{m-1}R_j - \frac{2}{m}\sum_{j=1}^{m-1}jR_j \\
&= \sigma^2 - \frac{\gamma_1}{m} + o(m^{-1}),
\end{aligned}$$

with the last equality following from Lemmas 2 and 12 in Appendix A.1. □

5.2.2 Reflection in the Middle of a Batch

We now analyze reflected batch means estimators with reflection in the middle of an arbitrary batch. If the batch size m is even and the reflection is in batch i, the observations in that batch will be $X_{(i-1)m+1},\ldots,X_{(i-1)m+m/2},-X_{(i-1)m+m/2+1},\ldots,-X_{im}$. Then the reflected batch means estimator is given by

$$\begin{aligned}
\mathcal{N}^{\star}_{1/2,i}(b,m) \equiv{}& \frac{m}{b-1}\Bigg[\sum_{j=1}^{i-1}(\bar{X}_{j,m}-\bar{X}^{\star}_{1/2,n})^2 + \sum_{j=i+1}^{b}(-\bar{X}_{j,m}-\bar{X}^{\star}_{1/2,n})^2 \\
&+ \frac{1}{m}\Bigg(\sum_{j=1}^{m/2}X_{(i-1)m+j} - \sum_{j=m/2+1}^{m}X_{(i-1)m+j} - \bar{X}^{\star}_{1/2,n}\Bigg)^2\Bigg] \\
={}& \frac{m}{b-1}\Bigg\{\sum_{j=1}^{i-1}\bar{X}^2_{j,m} + \frac{1}{m}\Bigg[\sum_{j=1}^{m/2}X_{(i-1)m+j} - \sum_{j=1}^{m/2}X_{(i-1)m+m/2+j}\Bigg]^2 + \sum_{j=i+1}^{b}\bar{X}^2_{j,m}\Bigg\} \\
&- \frac{m}{b(b-1)}\Bigg\{\sum_{j=1}^{i-1}\bar{X}^2_{j,m} + \frac{1}{m}\Bigg[\sum_{j=1}^{m/2}X_{(i-1)m+j} - \sum_{j=1}^{m/2}X_{(i-1)m+m/2+j}\Bigg] \\
&- \sum_{j=i+1}^{b}\bar{X}_{j,m}\Bigg\}^2.
\end{aligned}$$

Theorem 18 gives the expected value of the estimator $\mathcal{N}^{\star}_{1/2,i}(b,m)$. Its proof is omitted here because it is very similar to the proof of Theorem 17.

Theorem 18 If Assumptions A hold, then

$$\mathrm{E}[\mathcal{N}^{\star}_{1/2,i}(b,m)] = \sigma^2 - \frac{(b+3)}{mb}\gamma_1 + o(m^{-1}).$$

When we compare this expected value with that of Equation (18), we can see that reflecting in the middle of one arbitrary batch increases the bias by $2\gamma_1/(mb)$. On the other hand, the limiting variance of the reflected estimator is the same as the limiting variance of the regular NBM estimator.

We now look at the average of $\mathcal{N}(b,m)$ and $\mathcal{N}^{\star}_{1/2,i}(b,m)$, say $\bar{\mathcal{N}}_i(b,m) \equiv [\mathcal{N}(b,m) + \mathcal{N}^{\star}_{1/2,i}(b,m)]/2$. The expected value of $\bar{\mathcal{N}}_i(b,m)$ is $\sigma^2 - \frac{b+2}{mb}\gamma_1 + o(m^{-1})$; hence the bias does not change significantly from the bias of $\mathcal{N}(b,m)$. In order to approximate the asymptotic variance of $\bar{\mathcal{N}}_i(b,m)$, we conducted a Monte Carlo experiment with 10,000 replications using the AR(1) process discussed in §3.6.2 with $m = 5000$ and $b = 20$. When we reflect in the middle of batch $i = 10$, the estimated variance of $\bar{\mathcal{N}}_{10}(20, 5000)$ is 36. Further, reflection in the middle of the second batch yields an estimated variance of 36.48 and reflection in the middle of batch 17 yields an estimated variance of 36.49. Recall that the asymptotic variance of $\mathcal{N}(b,m)$ (or $\mathcal{N}^{\star}_{1/2,i}(b,m)$) is 38. The experiments showed that there is large positive correlation between $\mathcal{N}(b,m)$ and $\mathcal{N}^{\star}_{1/2,i}(b,m)$; hence we do not get a significant reduction in variance with the linear combination.

5.3 Reflected Standardized Time Series Estimators

In Chapter 2, we reviewed estimators based on the standardized time series $T_n(\cdot)$, and showed that their limiting functionals are Brownian bridge processes. For $t \in [0,1]$, the reflected STS with reflection point c is

$$\begin{aligned} T^{\star}_{c,n}(t) &\equiv \frac{\lfloor nt \rfloor (\bar{X}^{\star}_{c,n} - \bar{X}^{\star}_{c,\lfloor nt \rfloor})}{\sigma\sqrt{n}} \\ &= \frac{tS^{\star}_{c,n}}{\sigma\sqrt{n}} - \frac{S^{\star}_{c,\lfloor nt \rfloor}}{\sigma\sqrt{n}} - \frac{(nt - \lfloor nt \rfloor)\bar{X}^{\star}_{c,n}}{\sigma\sqrt{n}}. \end{aligned}$$

The reflected Brownian bridge process with reflection point c is defined as

$$\mathcal{B}_c^\star(t) \equiv t\mathcal{W}_c^\star(1) - \mathcal{W}_c^\star(t) = \begin{cases} t[2\mathcal{W}(c) - \mathcal{W}(1)] - \mathcal{W}(t) & \text{if } 0 \le t \le c \\ \mathcal{W}(t) + 2(t-1)\mathcal{W}(c) - t\mathcal{W}(1) & \text{if } c < t \le 1. \end{cases} \tag{66}$$

It is easy to verify that $\mathcal{B}_c^\star(t)$ is indeed a Brownian bridge process. To do so, we can easily show that $\mathrm{Cov}(\mathcal{B}_c^\star(s), \mathcal{B}_c^\star(t)) = \min(s,t) - st$ for $0 \le s,t \le 1$; hence the covariance structure is preserved.

Assumptions A imply

$$T_{c,n}^\star(t) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{B}_c^\star(t)$$

since we have

$$\frac{(nt - \lfloor nt \rfloor)\bar{X}_{c,n}^\star}{\sigma\sqrt{n}} \xrightarrow{P} 0,$$

where $\xrightarrow{P}$ denotes convergence in probability as $n \to \infty$.

5.4 Reflected Weighted Area Estimators

We define the reflected weighted area estimator $\mathcal{A}_c^\star(f;n)$ with reflection point $c \in [0,1]$, and its limiting functional $\mathcal{A}_c^\star(f)$ as

$$\mathcal{A}_c^\star(f;n) \equiv \left[\frac{1}{n}\sum_{j=1}^{n} f\left(\tfrac{j}{n}\right)\sigma T_{c,n}^\star(\tfrac{j}{n})\right]^2 \quad \text{and} \quad \mathcal{A}_c^\star(f) \equiv \left[\int_0^1 f(t)\sigma\mathcal{B}_c^\star(t)\,\mathrm{d}t\right]^2,$$

respectively, where $f(\cdot)$ is a weight function satisfying Assumptions F. Under Assumptions A, it can be shown that $\mathcal{A}_c^\star(f;n) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{A}_c^\star(f)$. The following theorem gives the expected value results of $\mathcal{A}_c^\star(f;n)$ for the weight functions $f_0(\cdot)$ and $f_2(\cdot)$. We assume cn to be integer. The proof is in Appendix A.6.

Theorem 19 If $c \in [0,1]$ and Assumptions A and F hold, then

$$\begin{aligned}
\mathrm{E}[\mathcal{A}_c^\star(f_0,n)] &= \Big(1-\frac{1}{n^3}\Big)R_0 + 12\Bigg[\sum_{j=1}^{n-cn-1} R_j\Big(-\frac{2c^2j-2cj+j/2-2c}{n}+\frac{4c/3-1/6}{n^2} \\
&\qquad + \frac{j^3/3+j/6}{n^3}-\frac{4c^3}{3}+2c^2+\frac{1}{6}\Big) \\
&\quad + \sum_{j=n-cn}^{n-1} R_j\Big(-\frac{j/2-2}{n}-\frac{2j-7/6}{n^2}-\frac{j^3/3+7j/6}{n^3}+\frac{5}{6}\Big) \\
&\quad + \sum_{j=1}^{cn-1} R_j\Big(-\frac{2c^2j-2cj+2c}{n}+\frac{2j-4c/3}{n^2}+\frac{2j^3+4j}{3n^3}+\frac{4c^3}{3}-2c^2\Big)\Bigg] \\
&= \sigma^2 - (24c^2-24c+3)\tfrac{\gamma_1}{n} + O(1/n^2) \qquad (67)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}[\mathcal{A}_c^\star(f_2;n)] &= \Big(\frac{7}{2n^2}+\frac{63}{2n^4}-\frac{36}{n^6}+1\Big)R_0 \\
&\quad + 840\Bigg[\sum_{j=1}^{cn} R_j\Big(-\frac{2c^6j-6c^5j+13c^4j/2-3c^3j+c^2j/2}{n}+\frac{12c^5j^2/5}{n^2} \\
&\qquad - \frac{3c^2j^2/2+6c^4j^2-5c^3j^2+7c^5/5-7c^4/2+3c^3-c^2+c/3}{n^2} \\
&\qquad - \frac{c^4j^3-2c^3j^3+c^2j^3-j^3/6-7c^4j/2+7c^3j-9c^2j/2+cj}{n^3} \\
&\qquad - \frac{j/3-c/2}{n^3} - \frac{3c^3j^2-9c^2j^2/2+2cj^2-j/2-c^3+3c^2/2}{n^4} \\
&\qquad + \frac{2c/5}{n^4} - \frac{j^5/10-c^2j^3+cj^3-j^3/2+3c^2j/2-3cj/2+2j/5}{n^5} \\
&\qquad + \frac{3cj^2/5-6c/35}{n^6} + \frac{j^7/35-j^5/10-j^3/10+6j/35}{n^7} + \frac{4c^7}{7} \\
&\qquad - 2c^6+\frac{13c^5}{5}-\frac{3c^4}{2}+\frac{c^3}{3}\Big) \\
&\quad + \sum_{j=1}^{n-1} R_j\Big(\frac{j^2/20+29/120}{n^2}-\frac{j^3/12+7j/24-1/2}{12n^3}+\frac{j^2/4-j/2}{n^4} \\
&\qquad + \frac{7}{40n^4}+\frac{j^5/20-j^3/4+3j/40}{n^5}-\frac{3j^2/10-3/35}{n^6} \\
&\qquad - \frac{j^7/70-j^5/20-j^3/20+3j/35}{n^7}-\frac{1}{420}\Big) \\
&\quad + \sum_{j=1}^{n-cn-1} R_j\Big(-\frac{2c^6j-6c^5j+13c^4j/2-3c^3j+c^2j/2}{n}-\frac{12c^5j^2/5}{n^2} \\
&\qquad + \frac{6c^4j^2+3c^2j^2/2-5c^3j^2-j^2/20+7c^5/5-7c^4/2+3c^3-c^2}{n^2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{c/3-7/30}{n^2}-\frac{c^4j^3-2c^3j^3+c^2j^3-j^3/6-7c^4j/2+7c^3j}{n^3}\\
&+\frac{9c^2j/2-cj+j/3+c/2-1/2}{n^3}+\frac{3c^3j^2-9c^2j^2/2+2cj^2}{n^4}\\
&+\frac{j/2-j^2/2-c^3+3c^2/2-2c/5-1/10}{n^4}-\frac{j^5/10-c^2j^3+cj^3}{n^5}\\
&-\frac{3c^2j/2-j^3/2-3cj/2+2j/5}{n^5}+\frac{3j^2/5+6c/35-6/35}{n^6}\\
&+\frac{j^7/35-j^5/10-j^3/10+6j/35}{n^7}-\frac{4c^7}{7}+2c^6-\frac{13c^5}{5}+\frac{3c^4}{2}\\
&-\frac{c^3}{3}+\frac{1}{210}\Big)\Big]\\
&=\sigma^2-420(4c^6-12c^5+13c^4-6c^3+c^2)\tfrac{\gamma_1}{n}+O(1/n^2). \qquad (68)
\end{aligned}
$$

Note that the first-order bias term in the expected value (68) is minimized in $[0,1]$ when $c=1/2$; this choice makes this bias equal to zero. Hence, reflection in the middle of the sample yields a first-order unbiased estimator. The same argument holds for $c=0$ and $c=1$ as expected.

5.4.1 Linear Combinations of Reflected Area Estimators

In §5.4, we saw that the biases of reflected area estimators depend on the reflection points, while the limiting variances of these estimators remain unchanged. However, we can obtain better estimators in terms of variance if we find appropriate linear combinations of the original and reflected estimators. This is consistent with the concept of data re-use. To this end, we consider linear combinations of the form

$$\sum_{j=1}^{k}\alpha_j\mathcal{A}^{\star}_{c_j}(f),$$

where $0\le c_1\le c_2\le\cdots\le c_k\le 1$ are the reflection points. Let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_k)$, and $\boldsymbol{c}=(c_1,c_2,\ldots,c_k)$. We can obtain the estimator with minimum asymptotic variance if we solve the following minimization problem:

$$
\begin{aligned}
\min_{\boldsymbol{\alpha},\boldsymbol{c}}\quad & \mathrm{Var}\left[\sum_{j=1}^{k}\alpha_j\mathcal{A}^{\star}_{c_j}(f)\right] \qquad (69)\\
\text{subject to}\quad & \sum_{j=1}^{k}\alpha_j=1
\end{aligned}
$$

$$0 \leq c_1 \leq c_2 \leq \cdots \leq c_k \leq 1$$

$$\alpha_j \in \mathbb{R}, 0 \leq c_j \leq 1;\ j = 1, \ldots, k.$$

5.4.1.1 Covariance Between Limiting Functionals of Reflected Area Estimators

To solve Problem (69), we first need to obtain the covariance between the limiting functionals of two reflected weighted area estimators with reflection points $c_1 \leq c_2$. We have

$$\begin{aligned}
\mathrm{Cov}[\mathcal{A}^{\star}_{c_1}(f), \mathcal{A}^{\star}_{c_2}(f)] &= \mathrm{Cov}\left[\left(\int_0^1 f(t)\sigma\mathcal{B}^{\star}_{c_1}(t)\mathrm{d}t\right)^2, \left(\int_0^1 f(y)\sigma\mathcal{B}^{\star}_{c_2}(y)\,\mathrm{d}y\right)^2\right] \\
&= 2\mathrm{Cov}^2\left[\int_0^1 f(t)\sigma\mathcal{B}^{\star}_{c_1}(t)\mathrm{d}t, \int_0^1 f(y)\sigma\mathcal{B}^{\star}_{c_2}(y)\,\mathrm{d}y\right] \\
&\qquad \text{(see Patel and Read [27])} \\
&= 2\sigma^4\left(\int_0^1\int_0^1 f(t)f(y)\mathrm{Cov}(B^{\star}_{c_1}(t), B^{\star}_{c_2}(y))\,\mathrm{d}y\,\mathrm{d}t\right)^2. \qquad (70)
\end{aligned}$$

First, we will calculate the covariance term in Equation (70) using the definition of the reflected Brownian bridge process in (66). We write

$$\mathrm{Cov}[\mathcal{B}^{\star}_{c_1}(t), \mathcal{B}^{\star}_{c_2}(y)]$$

$$= \begin{cases}
\mathrm{Cov}[\mathcal{W}(t) - t(2\mathcal{W}(c_1) - \mathcal{W}(1)), \mathcal{W}(y) - y(2\mathcal{W}(c_2) - \mathcal{W}(1))] \\
\qquad\qquad \text{if } 0 \leq t \leq c_1 \text{ and } 0 \leq y \leq c_2 \\
\mathrm{Cov}[\mathcal{W}(t) - t(2\mathcal{W}(c_1) - \mathcal{W}(1)), -\mathcal{W}(y) + 2(1-y)\mathcal{W}(c_2) + y\mathcal{W}(1))] \\
\qquad\qquad \text{if } 0 \leq t \leq c_1 \text{ and } c_2 < y \leq 1 \\
\mathrm{Cov}[-\mathcal{W}(t) + 2(1-t)\mathcal{W}(c_1) + t\mathcal{W}(1)), \mathcal{W}(y) - y(2\mathcal{W}(c_2) - \mathcal{W}(1))] \\
\qquad\qquad \text{if } c_1 < t \leq 1 \text{ and } 0 \leq y \leq c_2 \\
\mathrm{Cov}[-\mathcal{W}(t) + 2(1-t)\mathcal{W}(c_1) + t\mathcal{W}(1)), -\mathcal{W}(y) + 2(1-y)\mathcal{W}(c_2) + y\mathcal{W}(1))] \\
\qquad\qquad \text{if } c_1 < t \leq 1 \text{ and } c_2 < y \leq 1
\end{cases}$$

$$
= \begin{cases}
\min(t,y) - 2y\min(t,c_2) + 3ty - 2t\min(c_1,y) + 4ty\min(c_1,c_2) & \\
\quad - 2tyc_1 - 2ytc_2 & \text{if } 0 \le t \le c_1 \text{ and } 0 \le y \le c_2 \\
-\min(t,y) + 2(1-y)\min(t,c_2) + ty + 2t\min(y,c_1) - 4t(1-y)\min(c_1,c_2) & \\
\quad - 2tyc_1 + 2t(1-y)c_2 & \text{if } 0 \le t \le c_1 \text{ and } c_2 < y \le 1 \\
-\min(t,y) + 2y\min(t,c_2) + ty + 2(1-t)\min(y,c_1) - 4y(1-t)\min(c_1,c_2) & \\
\quad + 2(1-t)yc_1 - 2tyc_2 & \text{if } c_1 < t \le 1 \text{ and } 0 \le y \le c_2 \\
\min(t,y) - 2(1-y)\min(t,c_2) - 2(1-t)\min(c_1,y) + 2y(1-t)c_1 - ty & \\
\quad + 4(1-t)(1-y)\min(c_1,c_2) + 2t(1-y)c_2 & \text{if } c_1 < t \le 1 \text{ and } c_2 < y \le 1.
\end{cases} \tag{71}
$$

Substitution of Equation (71) into Equation (70) yields

$$
\begin{aligned}
&\mathrm{Cov}[\mathcal{A}^\star_{c_1}(f), \mathcal{A}^\star_{c_2}(f)] \\
&= 2\sigma^4 \Bigg[\int_0^{c_1} \int_0^{c_2} f(t)f(y)(\min(t,y) - 2yt - 2t\min(c_1,y) + 4ytc_1 - 2ytc_1 \\
&\qquad - 2ytc_2 + 3yt)\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int_0^{c_1} \int_{c_2}^{1} f(t)f(y)(-t + 2(1-y)t + ty + 2tc_1 - 4t(1-y)c_1 - 2tyc_1 \\
&\qquad - ty + 2t(1-y)c_2 + ty)\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int_{c_1}^{1} \int_0^{c_2} f(t)f(y)(-\min(t,y) + 2y\min(t,c_2) + 2(1-t)\min(y,c_1) \\
&\qquad - 4y(1-t)c_1 + 2y(1-t)c_1 + ty - 2tyc_2)\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int_{c_1}^{1} \int_{c_2}^{1} f(t)f(y)(\min(t,y) - 2(1-y)\min(t,c_2) - 2(1-t)c_1 \\
&\qquad + 4(1-y)(1-t)c_1 + 2y(1-t)c_1 - ty + 2t(1-y)c_2)\,\mathrm{d}y\,\mathrm{d}t \Bigg]^2 \\
&= 2\sigma^4 \Bigg[\int_0^{c_1} \int_0^{t} f(t)f(y)(y - yt + 2ytc_1 - 2ytc_2)\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int_0^{c_1} \int_t^{c_1} f(t)f(y)(t + yt - 2yt + 2ytc_1 - 2ytc_2)\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int_0^{c_1} \int_{c_1}^{c_2} f(t)f(y)(t + yt - 2tc_1 + 2ytc_1 - 2ytc_2)\,\mathrm{d}y\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^{c_1}\int_{c_2}^1 f(t)f(y)(-2tc_1 + 2tyc_1 - yt + t + 2tc_2 - 2ytc_2)\,\mathrm{d}y\,\mathrm{d}t \\
&+ \int_{c_1}^{c_2}\int_0^{c_1} f(t)f(y)(-y + 3yt + 2y(1-t) - 2y(1-t)c_1 - 2ytc_2)\,\mathrm{d}y\,\mathrm{d}t \\
&+ \int_{c_2}^1\int_0^{c_1} f(t)f(y)(-y + yt + 2y(1-t) + 2yc_2 - 2y(1-t)c_1 - 2ytc_2)\,\mathrm{d}y\,\mathrm{d}t \\
&+ \int_{c_2}^1\int_{c_1}^{c_2} f(t)f(y)(-y + 2yc_2 + 2c_1(1-t) - 2yc_1(1-t) + yt - 2ytc_2)\,\mathrm{d}y\,\mathrm{d}t \\
&+ \int_{c_1}^{c_2}\int_{c_1}^{t} f(t)f(y)(-y + 3yt + 2(1-t)c_1 - 2yc1(1-t) - 2ytc_2)\,\mathrm{d}y\,\mathrm{d}t \\
&+ \int_{c_1}^{c_2}\int_{t}^{c_2} f(t)f(y)(-t + 3yt + 2(1-t)c_1 - 2y(1-t)c_1 - 2ytc_2)\,\mathrm{d}y\,\mathrm{d}t \\
&+ \int_{c_1}^{c_2}\int_{c_2}^1 f(t)f(y)(t - yt - 2t(1-y) + 2(1-t)c_1 - 2y(1-t)c_1 + 2tc_2 \\
&\qquad - 2ytc_2)\,\mathrm{d}y\,\mathrm{d}t \\
&+ \int_{c_2}^1\int_{c_2}^{t} f(t)f(y)(y - yt - 2(1-y)c_2 + 2c_1(1-t) - 2y(1-t)c_1 + 2tc_2 \\
&\qquad - 2ytc_2)\,\mathrm{d}y\,\mathrm{d}t \\
&+ \int_{c_2}^1\int_{t}^{1} f(t)f(y)(t - yt - 2(1-y)c_2 + 2c_1(1-t) - 2y(1-t)c_1 + 2tc_2 \\
&\qquad - 2ytc_2)\,\mathrm{d}y\,\mathrm{d}t\Bigg]^2.
\end{aligned}
$$

Example 27 After some algebra, we can show that

$$
\mathrm{Cov}[\mathcal{A}^\star_{c_1}(f_0), \mathcal{A}^\star_{c_2}(f_0)] = 2\sigma^4[1 + 8(c_1^3 - c_2^3) - 12(c_1^2 - c_2^2) + 6(c_1 - c_2)]^2. \quad \triangleleft
$$

Example 28 For the weight function $f_2(\cdot)$, we have

$$
\begin{aligned}
\mathrm{Cov}[\mathcal{A}^\star_{c_1}(f_2), \mathcal{A}^\star_{c_2}(f_2)] = 2\sigma^4[1 &+ 240(c_1^7 - c_2^7) - 840(c_1^6 - c_2^6) + 1092(c_1^5 - c_2^5) \\
&- 630(c_1^4 - c_2^4) + 140(c_1^3 - c_2^3)]^2. \quad \triangleleft
\end{aligned}
$$

5.4.1.2 Convexity Analysis

First we investigate the convexity of the objective function (69). For $k = 2$ and weight function $f_0(\cdot)$, we express the objective function as $h(\boldsymbol{\alpha}, \boldsymbol{c}) \equiv 2\alpha_1^2\sigma^4 + 2\alpha_2^2\sigma^4 + 4\alpha_1\alpha_2\theta^2\sigma^4$, where $\theta = [1 + 6(c_1 - c_2) - 12(c_1^2 - c_2^2) + 8(c_1^3 - c_2^3)]$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, and

$\boldsymbol{c} = (c_1, c_2)$. For $h(\boldsymbol{\alpha}, \boldsymbol{c})$ to be a convex function, the Hessian matrix of $h(\boldsymbol{\alpha}, \boldsymbol{c})$ should be a positive semi-definite (p.s.d.) matrix. We first obtain the gradient of $h(\boldsymbol{\alpha}, \boldsymbol{c})/\sigma^4$:

$$\sigma^{-4} \bigtriangledown h(\boldsymbol{\alpha}, \boldsymbol{c}) = \begin{bmatrix} 4\alpha_1 + 4\alpha_2\theta^2 \\ 4\alpha_2 + 4\alpha_1\theta^2 \\ 8\alpha_1\alpha_2\theta(6 - 24c_1 + 24c_1^2) \\ 8\alpha_1\alpha_2\theta(-6 + 24c_2 - 24c_2^2) \end{bmatrix}.$$

The Hessian matrix $\sigma^{-4} \bigtriangledown^2 h(\boldsymbol{\alpha}, \boldsymbol{c})$ is

$$\begin{bmatrix} 4 & 4\theta^2 & 8\alpha_2\theta(6 - 24c_1 + 24c_1^2) & 8\alpha_2\theta(-6 + 24c_2 - 24c_2^2) \\ 4\theta^2 & 4 & 8\alpha_1\theta(6 - 24c_1 + 24c_1^2) & 8\alpha_1\theta(-6 + 24c_2 - 24c_2^2) \\ 8\alpha_2\theta(6 - 24c_1 + 24c_1^2) & 8\alpha_1\theta(6 - 24c_1 + 24c_1^2) & J_1 & J_2 \\ 8\alpha_2\theta(-6 + 24c_2 - 24c_2^2) & 8\alpha_1\theta(-6 + 24c_2 - 24c_2^2) & J_3 & J_4 \end{bmatrix},$$

where

$$\begin{aligned} J_1 &\equiv 8\alpha_1\alpha_2[(6 - 24c_1 + 24c_1^2)^2 + \theta(-24 + 48c_1)] \\ J_2 &\equiv 8\alpha_1\alpha_2(-6 + 24c_2 - 24c_2^2)(6 - 24c_1 + 24c_1^2) \\ J_3 &\equiv 8\alpha_1\alpha_2(6 - 24c_1 + 24c_1^2)(-6 + 24c_2 - 24c_2^2) \\ J_4 &\equiv 8\alpha_1\alpha_2[(-6 + 24c_2 - 24c_2^2)^2 + \theta(24 - 48c_2)]. \end{aligned}$$

For this Hessian matrix to be p.s.d., all principle minor determinants must be positive. We can see that

$$\begin{aligned} \det \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} &= J_1 \times J_4 - J_2 \times J_3 \\ &= -36864\alpha_1^2\alpha_2^2(2c_1 - 1)(2c_2 - 1)(2.5\theta^2 - 1.5\theta) \end{aligned} \tag{72}$$

Let us consider the last expression of Equation (72). For $\boldsymbol{\alpha} = (0.1, 0.9)$ and $\boldsymbol{c} = (0.2, 0.3)$, we obtain $\theta = 0.848$, and Equation (72) takes the value of -37.68. Hence, the objective function of Problem (69) is not convex. This means that there is no unique optimal solution for the minimization Problem (69). We solved this problem approximately using the pattern search algorithm in MATLAB (Hanselman and Littlefield [21]). The same analysis holds for $k > 2$ estimators, as well as for other weight functions.

Table 15 below lists the optimal reflection points and optimal weights of each estimator in various linear combinations. For $k \geq 8$ estimators, the variance reduction becomes insignificant. Hence, 7 estimators suffice to realize the advantage of linearly combined estimator in terms of variance reduction.

Note that the reflection points do not seem to have a pattern. Using the above optimal reflection point and weight combinations, the respective expected values and variances for the AR(1) process with n =10,000 can be found in the Table 16, along with the simulated results. It can be seen that the point estimators are very close to the theoretical values.

Recall that the asymptotic variance of the area estimator for one long batch of observations for the AR(1) process under study is $2\sigma^4 = 722$. Based on Table 16 and Figure 7, the combination of merely two reflected estimators induces a reduction in variance of about 50%, while the combination of 7 estimators yields an additional reduction of about 10%.

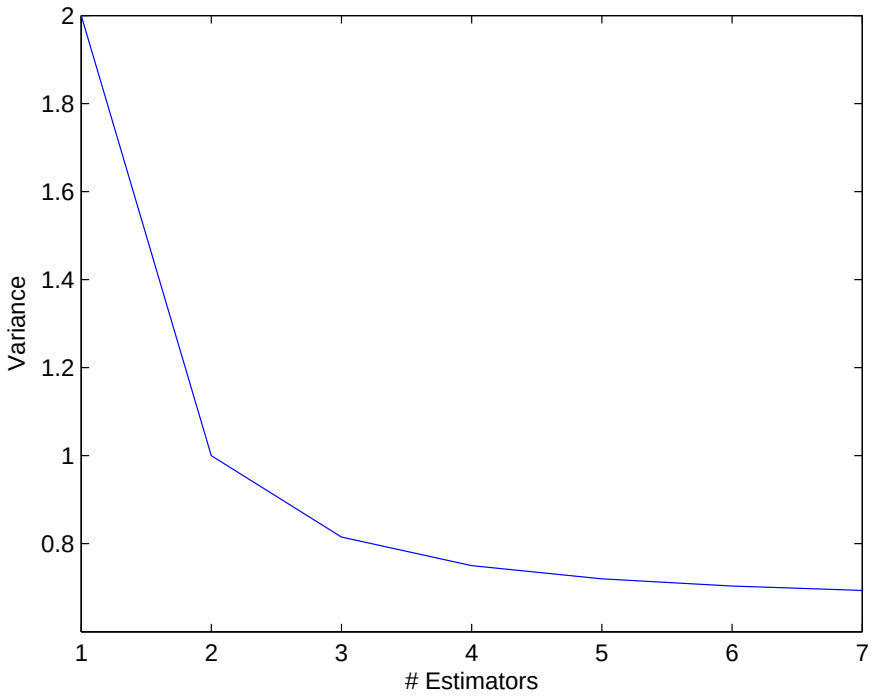
On the other hand, as seen in Equations (67) and (68), the expected value of reflected area estimators depends on the reflection points. Hence, the bias may increase based on the reflection points, as seen in Table 16. Fortunately, this potential increase in bias is insignificant with respect to its influence on the substantial reduction in MSE.

Table 15: Optimal Reflection Points and Weights for Linearly Combined Reflected Area Estimators

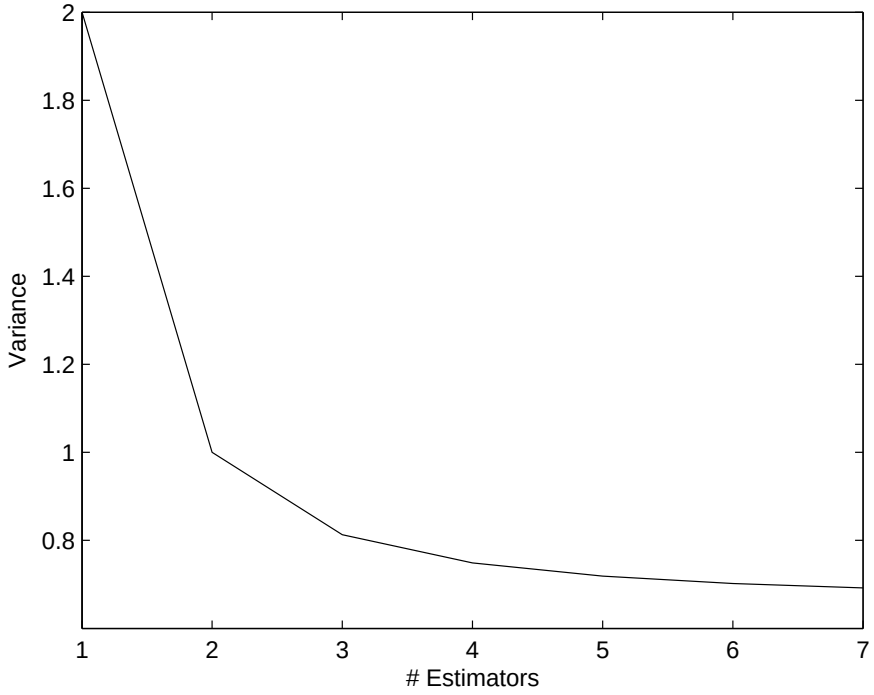
# Estimators	f_0	
	Reflection Points	Weights
2	$0, 0.5$	$0.5, 0.5$
3	$0, 0.153, 0.847$	$0.33, 0.33, 0.33$
4	$0, 0.103, 0.491, 0.897$	$0.25, 0.25, 0.25, 0.25$
5	$0.000034, 0.078, 0.208, 0.793, 0.922$	$0.2, 0.2, 0.2, 0.2, 0.2$
6	$0.057, 0.143, 0.346, 0.836, 0.93, 0.995$	$0.167, 0.167, 0.167, 0.167, 0.167, 0.167$
7	$0.053, 0.123, 0.239, 0.761, 0.877, 0.947, 1$	$0.143, 0.143, 0.143, 0.143, 0.143, 0.143, 0.143$
# Estimators	f_2	
	Reflection Points	Weights
2	$0, 0.5$	$0.5, 0.5$
3	$0.208, 0.65, 0.86$	$0.33, 0.33, 0.33$
4	$0.06, 0.23, 0.595, 0.784$	$0.25, 0.25, 0.25, 0.25$
5	$0.064, 0.204, 0.32, 0.699, 0.809$	$0.2, 0.2, 0.2, 0.2, 0.2$
6	$0.000017, 0.1797, 0.268, 0.492, 0.732, 0.82$	$0.167, 0.167, 0.167, 0.167, 0.167, 0.167$
7	$0.035, 0.168, 0.243, 0.335, 0.7, 0.7597, 0.835$	$0.143, 0.143, 0.143, 0.143, 0.143, 0.143, 0.143$

Table 16: Theoretical and Estimated Means and Variances of Linearly Combined Reflected Area Estimators

# Estimators	f_0			
	Theoretical Mean	Estimated Mean	Theoretical Variance	Estimated Variance
1	18.95	18.90	722.00	710.35
2	19.00	18.88	361.00	354.45
3	18.98	18.84	294.15	289.38
4	18.99	18.85	270.75	264.99
5	18.99	18.82	259.92	253.66
6	19.03	18.79	254.04	247.68
7	18.99	18.83	250.49	245.67
# Estimators	f_2			
	Theoretical Mean	Estimated Mean	Theoretical Variance	Estimated Variance
1	19.00	18.97	722.00	721.09
2	19.00	18.95	361.00	358.77
3	18.94	18.90	293.56	291.93
4	18.96	18.94	270.21	270.55
5	18.95	18.92	259.42	259.12
6	18.99	18.92	253.53	253.94
7	18.93	18.92	249.99	250.17



(a) Weight f_0



(b) Weight f_2

Figure 7: Theoretical Variances of Linearly Combined Reflected Area Estimators ($\times\sigma^4$)

5.5 Reflected Cramér–von Mises Estimators

We define the reflected CvM estimator $\mathcal{C}_c^\star(g;n)$ and its limiting functional $\mathcal{C}_c^\star(g)$, respectively, as

$$\mathcal{C}_c^\star(g;n) \equiv \frac{1}{n}\sum_{j=1}^{n} \sigma^2 g\Big(\tfrac{j}{n}\Big)\Big[T_{c,n}^\star(\tfrac{j}{n})\Big]^2 \quad \text{and} \quad \mathcal{C}_c^\star(g) \equiv \int_0^1 g(t)\sigma^2\Big[\mathcal{B}_c^\star(t)\Big]^2 \,\mathrm{d}t,$$

where $c \in [0,1]$ is any reflection point and $g(\cdot)$ is a weight function satisfying Assumptions G. Under Assumptions A, it can be shown that $\mathcal{C}_c^\star(g;n) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{C}_c^\star(g)$.

The following theorem gives the expected value of $\mathcal{C}_c^\star(g;n)$ for different weight functions. We assume cn to be integer. It's proof is in Appendix A.7.

Theorem 20 Under Assumptions A and G, for $c \in (0,1)$,

$$\begin{aligned}
\mathrm{E}[\mathcal{C}_c^\star(g_0,n)] &= \Big(1-\frac{1}{n^2}\Big)R_0 + \sum_{j=1}^{n-1} R_j\Big(\frac{10j}{n} - \frac{12j^2-2n^2}{n^2} + \frac{4j^3-2j}{n^3} - 2\Big) \\
&\quad + \sum_{j=1}^{cn-1} R_j\Big(-\frac{12c^2j+8j}{n} + \frac{12cj^2+12j^2-4c}{n^2} - \frac{8j^3-4j}{n^3} \\
&\qquad + 8c^3 - 12c^2 + 8c\Big) \\
&\quad + \sum_{j=1}^{n-cn-1} R_j\Big(-\frac{12c^2j-24cj+20j}{n} - \frac{12cj^2-24j^2-4c+4}{n^2} \\
&\qquad - \frac{8j^3-4j}{n^3} - 8c^3 + 12c^2 - 8c + 4\Big) \\
&= \sigma^2 - (12c^2-12c+9)\,\frac{\gamma_1}{n} + o(1/n)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}[\mathcal{C}_c^\star(g_2^\star;n)] &= \Big(1+\frac{4}{n^2}-\frac{5}{n^4}\Big)R_0 \\
&\quad + \sum_{j=1}^{cn-1} R_j\Bigg(\frac{150c^4j-200c^3j+48c^2j+2j}{n} - \frac{300c^3j^2-300c^2j^2+48cj^2}{n^2} \\
&\qquad + \frac{100c^3-48j^2-150c^2+66c}{n^2} + \frac{300c^2j^3-200cj^3+132j^3}{n^3} \\
&\qquad + \frac{100cj-66j-150c^2j}{n^3} - \frac{150cj^4+50j^4-150cj^2-50j^2+20c}{n^4} \\
&\qquad + \frac{60j^5-100j^3+20j}{n^5} - 60c^5 + 150c^4 - 132c^3 + 48c^2 - 2c\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=1}^{n-cn-1} R_j \Bigg(\frac{150c^4 j - 400c^3 j + 348c^2 j - 96cj}{n} + \frac{300c^3 j^2 - 600c^2 j^2}{n^2} \\
&\quad - \frac{96j^2 - 348cj^2 + 100c^3 - 150c^2 + 66c - 16}{n^2} + \frac{300c^2 j^3 - 400cj^3}{n^3} \\
&\quad + \frac{232j^3 - 150c^2 j + 200cj - 116j}{n^3} + \frac{150cj^4 - 200j^4}{n^4} \\
&\quad - \frac{150cj^2 + 200j^2 - 20c + 20}{n^4} + \frac{60j^5 - 100j^3 + 20j}{n^5} \\
&\quad + 60c^5 - 150c^4 + 132c^3 - 48c^2 + 2c + 4 \Bigg) \\
&+ \sum_{j=1}^{n-1} R_j \Bigg(\frac{48j^2 - 8}{n^2} - \frac{116j^3 - 58j}{n^3} + \frac{100j^4 - 100j^2 + 10}{n^4} \\
&\quad - \frac{30j^5 - 50j^3 + 10j}{n^5} - 2 \Bigg) \\
&= \sigma^2 - (-150c^4 + 300c^3 - 198c^2 + 48c + 1)\,\frac{\gamma_1}{n} + O(1/n^2).
\end{aligned}
$$

For $c = 0$ and $c = 1$, the expected value results are the same as in Examples 4 and 5.

5.5.1 Linear Combination of Reflected CvM Estimators

Similar to the analysis in §5.4.1, we study linear combinations of the form $\sum_{j=1}^{k} \alpha_j \mathcal{C}^{\star}_{c_j}(g)$. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, and $\boldsymbol{c} = (c_1, c_2, \ldots, c_k)$. We can obtain the estimator with minimum variance if we solve the following minimization problem:

$$
\begin{aligned}
\min_{\boldsymbol{\alpha},\boldsymbol{c}} \quad & \mathrm{Var}\left[\sum_{j=1}^{k} \alpha_j \mathcal{C}^{\star}_{c_j}(g) \right] \\
\text{subject to} \quad & \sum_{j=1}^{k} \alpha_j = 1 \\
& 0 \le c_1 \le c_2 \le \cdots \le c_k \le 1 \\
& \alpha_i \in \mathbb{R}, 0 \le c_j \le 1,\ j = 1, \ldots, k.
\end{aligned} \tag{73}
$$

5.5.1.1 Covariance Between Limiting Functionals of Reflected CvM Estimators

The covariance between the limiting functionals of reflected CvM estimators with reflection points $c_1 \le c_2$ can be written as:

$$
\mathrm{Cov}[\mathcal{C}^{\star}_{c_1}(g), \mathcal{C}^{\star}_{c_2}(g)] = \mathrm{Cov}\left[\int_0^1 g(t)\sigma^2 (\mathcal{B}^{\star}_{c_1}(t))^2 \,\mathrm{d}t, \int_0^1 g(y)\sigma^2 (\mathcal{B}^{\star}_{c_2}(y))^2 \,\mathrm{d}y \right]
$$

$$= \sigma^4 \int_0^1 \int_0^1 g(t)g(y)\mathrm{Cov}[(\mathcal{B}^\star_{c_1}(t))^2, (\mathcal{B}^\star_{c_2}(y))^2 \,\mathrm{d}t \,\mathrm{d}y$$
$$= 2\sigma^4 \int_0^1 \int_0^1 g(t)g(y)[\mathrm{Cov}(\mathcal{B}^\star_{c_1}(t), \mathcal{B}^\star_{c_2}(y))]^2 \,\mathrm{d}y \,\mathrm{d}t.$$

The covariance inside the integrand can be computed using Equation (71).

Example 29 After some algebra, we can show that

$$\begin{aligned}
&\mathrm{Cov}[\mathcal{C}^\star_{c_1}(g_0), \mathcal{C}^\star_{c_2}(g_0)] \\
&= \sigma^4\Big[154c_1^4 - 96c_1^3 + 32c_1^2 + \tfrac{32}{5}c_1 - \tfrac{32}{5}c_2 + \tfrac{224}{5}c_2^6 - 64c_2c_1^3 + 144c_1c_2^4 + 96c_2^2c_1^3 \\
&\quad - 32c_1^3c_2^3 - 448c_1^6c_2 + \tfrac{2816}{7}c_1^7c_2 - 16c_1^6c_2^2 - \tfrac{48}{5}c_1^5c_2 - \tfrac{144}{5}c_1c_2^5 + 192c_1c_2^2 \\
&\quad - \tfrac{672}{5}c_2^5 + 160c_2^4 + \tfrac{168}{5}c_1^5 - \tfrac{5184}{7}c_1^8c_2 + \tfrac{2592}{7}c_1^7c_2^2 - \tfrac{2704}{7}c_1^8 - 64c_1c_2 \\
&\quad - 96c_2^3 + 32c_2^2 + \tfrac{4}{5} + \tfrac{2592}{7}c_1^9 + \tfrac{3784}{7}c_1^7 - \tfrac{972}{5}c_1^6 - 256c_1c_2^3\Big]. \quad \triangleleft
\end{aligned}$$

Example 30 For the weight function $g_2^\star(\cdot)$, we have

$$\begin{aligned}
&\mathrm{Cov}[\mathcal{C}^\star_{c_1}(g_2^\star), \mathcal{C}^\star_{c_2}(g_2^\star)] \\
&= \sigma^4\Big[1536c_2^2c_1^3 + \tfrac{14513896}{7}c_2c_1^7 - \tfrac{198504}{5}c_2^5c_1 - \tfrac{117654294}{7}c_1^8c_2 - 1735545c_1^8c_2^2 + \tfrac{124908}{5}c_1^5 \\
&\quad + 13231400c_1^9c_2^2 + 77823990c_1^9c_2 - 55011600c_1^{10}c_2^2 - 3712c_1^3c_2^3 - 64c_1^3c_2 \\
&\quad - \tfrac{48252}{5}c_2^5 - \tfrac{7292261250}{11}c_1^{12}c_2 + \tfrac{8703375000}{13}c_1^{13}c_2 - 201000000c_1^{12}c_2^2 - 96c_2^3 \\
&\quad - \tfrac{3189937500}{7}c_1^{14}c_2 - 110448c_1^6c_2 + 1800c_2^4 + 2c_2^2 + \tfrac{4539828250}{11}c_1^{11} + 192c_1c_2^2 \\
&\quad - 84375000c_1^{14}c_2^2 + 200c_2c_1^4 + 15000c_1c_2^8 + 6000c_1^5c_2^4 - \tfrac{6381211350}{11}c_1^{12} \\
&\quad + 185625000c_1^{15}c_2 - 4800c_2^2c_1^4 + \tfrac{680352}{7}c_1^7c_2^2 - \tfrac{264400}{7}c_1c_2^7 + 17104c_1c_2^4 \\
&\quad + 3200c_1^3c_2^4 - 2500c_2^9c_1 + \tfrac{1488240000}{11}c_1^{11}c_2^2 + \tfrac{1983093750}{7}c_1^{15} - 33750000c_1^{16}c_2 \\
&\quad + 16875000c_1^{15}c_2^2 + 176625000c_1^{13}c_2^2 + 3400c_2^{10} - 224552445c_1^{10} - 17000c_2^9 \\
&\quad + \tfrac{643393892}{7}c_1^9 + \tfrac{248370}{7}c_2^8 - \tfrac{759377401}{28}c_1^8 - \tfrac{279480}{7}c_2^7 + 5350482c_1^7 + \tfrac{129984}{5}c_2^6 \\
&\quad - \tfrac{3063927}{5}c_1^6 + 11600c_2^3c_1^4 - 960c_1^3c_2^5 - \tfrac{1368}{5}c_1^5c_2 + \tfrac{284}{35}c_1 - 10000c_2^4c_1^4 - 96c_1^3 \\
&\quad + 2880c_1^5c_2^2 + 1704c_1^4 + 3000c_2^5c_1^4 - 230298400c_1^{10}c_2 + 2c_1^2 + \tfrac{5110495200}{11}c_1^{11}c_2
\end{aligned}$$

$$- 3536c_1c_2^3 + \tfrac{121}{70} + \tfrac{88959588750}{143}c_1^{13} + 51200c_1c_2^6 - 6960c_1^5c_2^3 - 101250000c_1^{16}$$
$$- \tfrac{284}{35}c_2 - 4c_1c_2 - 256c_1^6c_2^2 - \tfrac{45374812500}{91}c_1^{14} - 1800c_2^5c_1^5 + 16875000c_1^{17}\Big]. \quad \triangleleft$$

A similar analysis to that in §5.4.1.2 shows the objective function of Problem (73) is not convex in α and c for any weight function selected and for any $k \geq 2$. Hence, we again use the pattern search algorithm in MATLAB to solve this problem as well. Optimal reflection points and weights are given in Table 17 for various combinations of reflected estimators based on the same weight function. For the weight function $g_2^\star(\cdot)$, with more than three estimators in the linear combination, the degree of the polynomial objective function becomes very large. As a result, the pattern search algorithm did not terminate within the 3-day allocated time window. Hence for this weight function, we only list the results for 2 and 3 estimators in the linear combination.

Note that the weights of the individual estimators are not identical because of the covariance structure of the estimators, which depends on the reflection points as seen in Examples 29 and 30. Again, for the constant weight function $g_0(\cdot)$, we stop at 7 estimators since the variance reduction is not significant for linear combinations with a larger number of estimators. Using the optimal reflection point and weight combinations in Table 17, the theoretical expected value and variances for the AR(1) process with n =10,000 are listed in Table 18, along with the simulated results. The estimated results confirm the calculated variance reduction. For example, for the weight function $g_2^\star(\cdot)$, a comparison of an individual estimator with the linear combination estimator with two components suggests a variance reduction of about 50%. Parallel to the findings in Theorem 20, Table 18 suggests that reflection may increase the bias of CvM estimators depending on the reflection point. Fortunately, the increase in bias is very insignificant when compared to the amount of variance reduction.

Table 17: Optimal Reflection Points and Weights for Linearly Combined Reflected CvM Estimators

# Estimators	g_0	
	Reflection Points	Weights
2	$0.027, 0.508$	$0.5, 0.5$
3	$0.07, 0.321, 0.911$	$0.342, 0.334, 0.323$
4	$0.0, 0.051, 0.127, 0.517$	$0.251, 0.188, 0.163, 0.397$
5	$0.0, 0.036, 0.083, 0.148, 0.519$	$0.22, 0.145, 0.13, 0.118, 0.386$
6	$0.0, 0.028, 0.061, 0.104, 0.161, 0.52$	$0.201, 0.117, 0.108, 0.099, 0.093, 0.381$
7	$0.0, 0.023, 0.049, 0.08, 0.118, 0.169, 0.52$	$0.19, 0.098, 0.091, 0.085, 0.08, 0.077, 0.378$
# Estimators	$g_2^\star$	
	Reflection Points	Weights
2	$0.172, 0.824$	$0.5, 0.5$
3	$0.121, 0.391, 0.853$	$0.371, 0.272, 0.367$

Table 18: Theoretical and Estimated Means and Variances of Linearly Combined Reflected CvM Estimators

# Estimators	g_0			
	Theoretical Mean	Estimated Mean	Theoretical Variance	Estimated Variance
1	18.91	18.92	288.80	289.28
2	18.87	18.88	158.66	159.12
3	18.85	18.84	144.18	139.31
4	18.87	18.85	144.08	144.25
5	18.86	18.85	143.03	143.30
6	18.86	18.85	142.60	142.95
7	18.86	18.85	142.34	142.71
# Estimators	$g_2^\star$			
	Theoretical Mean	Estimated Mean	Theoretical Variance	Estimated Variance
1	19.00	19.01	624.01	622.05
2	18.93	18.88	314.70	301.19
3	18.94	18.91	259.73	246.26

5.6 Summary

This chapter introduced a new class of estimators based on reflections of a Brownian motion. We started with reflected versions of NBM estimators, but discovered that linear combination of such estimators did not yield a significant variance reduction. On the other hand, we developed reflected area and reflected CvM estimators, where the linear combinations of these estimators performed nicely in terms of variance reduction. We provided analysis on the optimal selection of weights and reflection points. A Monte Carlo study supported our theoretical findings. Chapter 6 proceeds with a comprehensive summary of this dissertation and a list of problems we intend to study in the future.

CHAPTER VI

CONCLUSIONS AND FUTURE WORK

6.1 Summary

This dissertation studied three classes of estimators for the asymptotic variance parameter of a stationary stochastic process. Chapter 3 extended the research at Antonini [6] on several fronts. We obtained detailed expressions for the expectation of the folded area and CvM estimators at levels 0 and 1; those expressions explained the puzzling increase in small-sample bias as the folding level increases. We also used batching and linear combinations of estimators from different levels to produce estimators with significantly smaller variance. Finally, we applied the technique of Satterthwaite [29] to obtain very accurate approximations of the limiting distributions of batched folded estimators (as the number of batches remains fixed while the batch size increases) as appropriately scaled chi-square distributions. These approximations were used to compute CIs for μ and σ^2.

To illustrate that this first class of variance estimators performs as advertised, we conducted exact and Monte Carlo studies involving AR(1) and M/M/1 delay-time processes. For large batch sizes, the level-1 estimators performed about the same as their level-0 counterparts; and linear combinations of the corresponding level-0 and level-1 estimators outperformed the individual estimators—as anticipated by the theory. Finally we studied folded batched area estimators at levels ≥ 2. Our brief study indicated that the bias of the estimators increases with the folding level.

The second class of variance estimators combines the concepts of folding and overlapping. Recall that the folding operation on a Brownian bridge yields a new Brownian bridge. Since the STS corresponding to the data $\{X_1, \ldots, X_n\}$ converges

to a Brownian bridge, the recursive application of folding applied to an STS results in STS area estimators that are asymptotically independent. In a sense, both overlapping and folding operations are based on the concept of data re-use.

In Chapter 4, we showed that folded overlapping area (FOA) estimators have almost the same bias but smaller variance than other estimators in the literature, such as the NBM, OBM, nonoverlapping area, overlapping area, and batched folded area estimators. We also obtained the asymptotic distribution of the proposed estimators along with detailed expressions for their first two moments. Further, we presented efficient algorithms to obtain these estimators in order-of-sample-size time. We analyzed linear combinations of level-0 and level-1 folded overlapping area estimators, but the linearly combined estimator did not have a significantly smaller variance than the individual estimators in the linear combination due to the excessive positive correlation between the constituent estimators. Monte Carlo examples based on a stationary Gaussian AR(1) process and the waiting time process in a stationary M/M/1 system illustrated the performance of the FOA estimators. We also showed that the FOA estimators can be approximated with a scaled chi-square distributions. In addition, we derived confidence intervals for the variance parameter and the mean of the underlying process. These CIs exhibited nearly nominal coverage.

In Chapter 5, we introduced reflected estimators. These estimators also take the advantage of data re-use. The idea is to combine estimators obtained from various reflections of the original (entire) sample path. In particular, we started with two different reflected NBM estimators, based on reflections at the beginning of each batch or in the middle of an arbitrary batch. A Monte Carlo example showed that linear combinations of reflected estimators and the original estimators do not yield significant reductions in variance because the constituent estimators are highly positively correlated.

We proceeded with the reflected version of an area estimator based on an arbitrary

point. The optimal weights and reflection points of the linearly combined estimators were obtained by solving non-convex optimization problems. These linear combinations yielded estimators with significantly smaller variances compared to the variances of the individual estimators. We found that the bias of estimators depended on the reflection points. Further, we performed similar analysis for reflected CvM estimators and obtained optimal linear combinations with minimum variance. To complement the theoretical work, we conducted Monte Carlo experiments which confirmed our theoretical findings.

6.2 Other Topics of Interest

A number of interesting problems on folded estimators are the subjects of ongoing research.

1. *Higher Levels.* Our theoretical and empirical analysis primarily concerned level-0 and level-1 folded estimators. What happens when we go to higher levels? Although we derived certain asymptotic properties related to the estimators' expected value and variance at higher levels, we did not perform a fine-tuned analysis of estimators from those levels. Preliminary Monte Carlo analysis, as described in this thesis, indicates that bias becomes more problematic at higher levels. Another question worth asking is: How many levels can we take an estimator before the necessary asymptotics fail? (A similar question is addressed in Foley and Goldsman [15] with respect to the number of orthogonal weights that an area estimator could accommodate in practical situations.)

2. *Linear Combinations of Estimators.* Related to the above, we also intend to study the properties of different linear combinations of area and CvM estimators between and within higher levels. Estimators constructed with these ideas in mind will likely have comparatively lower variance than their constituents.

3. *Overlapping Folded CvM Estimators.* Another extension of our work is to formulate overlapping folded CvM estimators. We intend to develop similar analysis to that of Chapter 4.

4. *Sequential Procedures.* Procedures that deliver required batch and sample sizes along with final point and CI estimates have typically been based on nonoverlapping batch means, e.g., LABATCH.2 (Fishman and Yarberry [14]) and ASAP3 (Steiger et al. [35]). Rigorous sequential procedures based on folded STS estimators will give simulation practitioners and software developers more-powerful tools with which to conduct simulation output analysis.

In addition, future research topics on reflected estimators include the following:

1. *Algorithms for Obtaining Minimum Variance Reflected Estimators.* The pattern search algorithm proved to be very slow when applied to the variance minimization problem relative to reflected CvM estimators. We are looking for alternative algorithms to solve this problem.

2. *Batching.* What happens when batching (nonoverlapping or overlapping) is combined with reflection? In this thesis, we only analyzed non-batched versions of area and CvM estimators. We are interested in seeing if we can reduce variances further when reflection is applied to batched STS estimators.

Finally, we want to conduct a comprehensive comparison of various estimators in the literature and this thesis. The comparison will involve their properties such as bias, variance, MSE, coverage probabilities of CIs, as well as the associated computational times. This work will provide a very useful summary for simulation practitioners.

APPENDIX A

PROOFS AND FURTHER RESULTS

A.1 Expected Value of Folded Area Estimators: Proof of Theorem 4

We start with the derivation of the first equality in Theorem 4. We need the following results in the proof.

Lemma 2 (Goldsman and Meketon [17], Equation (4)) Under Assumptions A,

$$\mathrm{Var}(S_j) = j\sigma^2 - \gamma_1 - 2\sum_{\ell=j}^{\infty}(j-\ell)R_\ell = j\sigma^2 - \gamma_1 + o\,(1).$$

Now note that the following equality holds from Lemma 1:

$$T_{(1),n}\Big(\tfrac{j}{n}\Big) = \frac{1}{\sigma\sqrt{n}}\Big[(\tfrac{j}{n}-1)S_n - S_{\lfloor\frac{j}{2}\rfloor} + S_{\lfloor n-\frac{j}{2}\rfloor}\Big], \quad \text{for } j = 1,\ldots,n.$$

The expected value of $\mathcal{A}_{(1)}(f;n)$ can be obtained as follows:

$$\begin{aligned}
\mathrm{E}[\mathcal{A}_{(1)}(f;n)] &= \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n} f\Big(\tfrac{j}{n}\Big)\sigma T_{(1),n}\Big(\tfrac{j}{n}\Big)\right]^2 \\
&= \mathrm{Var}\left[\frac{1}{n}\sum_{j=1}^{n} f\Big(\tfrac{j}{n}\Big)\sigma T_{(1),n}\Big(\tfrac{j}{n}\Big)\right] \\
&= \mathrm{Cov}\left[\frac{1}{n}\sum_{j=1}^{n} f\Big(\tfrac{j}{n}\Big)\sigma T_{(1),n}\Big(\tfrac{j}{n}\Big), \frac{1}{n}\sum_{\ell=1}^{n} f\Big(\tfrac{\ell}{n}\Big)\sigma T_{(1),n}\Big(\tfrac{\ell}{n}\Big)\right] \\
&= \frac{\sigma^2}{n^2}\sum_{j=1}^{n}\sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big)\mathrm{Cov}\left(T_{(1),n}\Big(\tfrac{j}{n}\Big), T_{(1),n}\Big(\tfrac{\ell}{n}\Big)\right), \qquad (74)
\end{aligned}$$

where

$$
\begin{aligned}
&n\sigma^2 \mathrm{Cov}\Big[T_{(1),n}\Big(\tfrac{j}{n}\Big), T_{(1),n}\Big(\tfrac{\ell}{n}\Big)\Big] \\
&= \mathrm{Cov}\Big[\Big(\tfrac{j}{n} - n\Big)S_n - S_{\lfloor \frac{j}{2} \rfloor} + S_{\lfloor n - \frac{j}{2} \rfloor}, \Big(\tfrac{\ell}{n} - n\Big)S_n - S_{\lfloor \frac{\ell}{2} \rfloor} + S_{\lfloor n - \frac{\ell}{2} \rfloor}\Big] \\
&= \Big[\Big(\tfrac{j}{n} - n\Big)\Big(\tfrac{\ell}{n} - n\Big)\Big]\mathrm{Var}(S_n) - \Big(\tfrac{j}{n} - n\Big)\mathrm{Cov}\Big(S_n, S_{\lfloor \frac{\ell}{2} \rfloor}\Big) + \Big(\tfrac{j}{n} - n\Big)\mathrm{Cov}\Big(S_n, S_{\lfloor n - \frac{\ell}{2} \rfloor}\Big) \\
&\qquad + \Big(\tfrac{\ell}{n} - n\Big)\mathrm{Cov}\Big(S_n, S_{\lfloor \frac{j}{2} \rfloor}\Big) + \mathrm{Cov}\Big(S_{\lfloor \frac{j}{2} \rfloor}, S_{\lfloor \frac{\ell}{2} \rfloor}\Big) - \mathrm{Cov}\Big(S_{\lfloor \frac{j}{2} \rfloor}, S_{\lfloor n - \frac{\ell}{2} \rfloor}\Big) \\
&\qquad + \Big(\tfrac{\ell}{n} - n\Big)\mathrm{Cov}\Big(S_n, S_{\lfloor n - \frac{j}{2} \rfloor}\Big) - \mathrm{Cov}\Big(S_{\lfloor n - \frac{j}{2} \rfloor}, S_{\lfloor \frac{\ell}{2} \rfloor}\Big) + \mathrm{Cov}\Big(S_{\lfloor n - \frac{j}{2} \rfloor}, S_{\lfloor n - \frac{\ell}{2} \rfloor}\Big).
\end{aligned} \tag{75}
$$

If we plug Equation (75) into Equation (74), and collect similar terms together, we get

$$
\begin{aligned}
\mathrm{E}[\mathcal{A}_{(1)}(f; n)] = \frac{1}{n^3}\Bigg[\sum_{j=1}^{n}\Big(\tfrac{j}{n} - 1\Big)f\Big(\tfrac{j}{n}\Big)\Bigg]^2 \mathrm{Var}(S_n) \\
- \frac{2}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{n}\Big(\tfrac{j}{n} - 1\Big)f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\mathrm{Cov}[S_n, S_{\lfloor \frac{\ell}{2} \rfloor}] \\
+ \frac{2}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{n}\Big(\tfrac{j}{n} - 1\Big)f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\mathrm{Cov}[S_n, S_{\lfloor n - \frac{\ell}{2} \rfloor}] \\
- \frac{2}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\mathrm{Cov}[S_{\lfloor \frac{j}{2} \rfloor}, S_{\lfloor n - \frac{\ell}{2} \rfloor}] \\
+ \frac{1}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\mathrm{Cov}[S_{\lfloor \frac{j}{2} \rfloor}, S_{\lfloor \frac{\ell}{2} \rfloor}] \\
+ \frac{1}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\mathrm{Cov}[S_{\lfloor n - \frac{j}{2} \rfloor}, S_{\lfloor n - \frac{\ell}{2} \rfloor}].
\end{aligned} \tag{76}
$$

Lemmas 3–7 below obtain the covariance terms in the sums of Equation (76).

Lemma 3

$$
\begin{aligned}
\mathrm{Cov}[S_n, S_{\lfloor \frac{\ell}{2} \rfloor}] = \Big\lfloor \tfrac{\ell}{2} \Big\rfloor \gamma_{0,\lfloor \frac{\ell}{2} \rfloor - 1} - \gamma_{1,\lfloor \frac{\ell}{2} \rfloor - 1} + n\gamma_{0,n-1} - \gamma_{1,n-1} + \Big(\Big\lfloor \tfrac{\ell}{2} \Big\rfloor - n\Big)\gamma_{0,n-\lfloor \frac{\ell}{2} \rfloor} \\
+ \gamma_{1,n-\lfloor \frac{\ell}{2} \rfloor} + \Big\lfloor \tfrac{\ell}{2} \Big\rfloor R_0.
\end{aligned}
$$

Proof:

$$\begin{aligned}
\mathrm{Cov}[S_n, S_{\lfloor \frac{\ell}{2} \rfloor}] &= \sum_{t=1}^{\lfloor \frac{\ell}{2} \rfloor} \left(\sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} R_{|t-a|} + \sum_{a=\lfloor \frac{\ell}{2} \rfloor + 1}^{n} R_{a-t} \right) \\
&= \sum_{t=1}^{\lfloor \frac{\ell}{2} \rfloor} \left(\sum_{a=1}^{t-1} R_a + R_0 + \sum_{a=1}^{n-t} R_a \right) \\
&= \sum_{t=1}^{\lfloor \frac{\ell}{2} \rfloor} \sum_{a=1}^{t-1} R_a + \left\lfloor \frac{\ell}{2} \right\rfloor R_0 + \sum_{t=1}^{\lfloor \frac{\ell}{2} \rfloor} \sum_{a=1}^{n-t} R_a \\
&= \sum_{t=1}^{\lfloor \frac{\ell}{2} \rfloor - 1} (\left\lfloor \frac{\ell}{2} \right\rfloor - t) R_t + \left\lfloor \frac{\ell}{2} \right\rfloor R_0 + \sum_{t=1}^{\lfloor \frac{\ell}{2} \rfloor - 1} t R_{n-t} + \left\lfloor \frac{\ell}{2} \right\rfloor \sum_{t=1}^{n - \lfloor \frac{\ell}{2} \rfloor} R_t \\
&= \sum_{t=1}^{\lfloor \frac{\ell}{2} \rfloor - 1} \Big(\left\lfloor \frac{\ell}{2} \right\rfloor - t \Big) R_t + \left\lfloor \frac{\ell}{2} \right\rfloor R_0 + n \sum_{t=1}^{n-1} R_t - \sum_{l=1}^{n-1} t R_t \\
&\quad - n \sum_{t=1}^{n - \lfloor \frac{\ell}{2} \rfloor} R_t + \sum_{t=1}^{n - \lfloor \frac{\ell}{2} \rfloor} t R_t + \left\lfloor \frac{\ell}{2} \right\rfloor \sum_{t=1}^{n - \lfloor \frac{\ell}{2} \rfloor} R_t \\
&= \left\lfloor \frac{\ell}{2} \right\rfloor \gamma_{0, \lfloor \frac{\ell}{2} \rfloor - 1} - \gamma_{1, \lfloor \frac{\ell}{2} \rfloor - 1} + n \gamma_{0, n-1} - \gamma_{1, n-1} + \Big(\left\lfloor \frac{\ell}{2} \right\rfloor - n \Big) \gamma_{0, n - \lfloor \frac{\ell}{2} \rfloor} \\
&\quad + \gamma_{1, n - \lfloor \frac{\ell}{2} \rfloor} + \left\lfloor \frac{\ell}{2} \right\rfloor R_0.
\end{aligned}$$

□

Lemma 4

$$\begin{aligned}
\mathrm{Cov}[S_n, S_{\lfloor n - \frac{\ell}{2} \rfloor}] &= \left\lfloor n - \frac{\ell}{2} \right\rfloor \gamma_{0, \lfloor n - \frac{\ell}{2} \rfloor - 1} - \gamma_{1, \lfloor n - \frac{\ell}{2} \rfloor - 1} + \Big(\left\lfloor n - \frac{\ell}{2} \right\rfloor - n \Big) \gamma_{0, n - \lfloor n - \frac{\ell}{2} \rfloor} \\
&\quad + \gamma_{1, n - \lfloor n - \frac{\ell}{2} \rfloor} + n \gamma_{0, n-1} - \gamma_{1, n-1} + \left\lfloor n - \frac{\ell}{2} \right\rfloor R_0.
\end{aligned}$$

Proof:

$$\begin{aligned}
\mathrm{Cov}[S_n, S_{\lfloor n - \frac{\ell}{2} \rfloor}] &= \sum_{t=1}^{\lfloor n - \frac{\ell}{2} \rfloor} \left(\sum_{a=1}^{\lfloor n - \frac{\ell}{2} \rfloor} R_{|t-a|} + \sum_{a=\lfloor n - \frac{\ell}{2} \rfloor + 1}^{n} R_{a-t} \right) \\
&= \sum_{t=1}^{\lfloor n - \frac{\ell}{2} \rfloor} \left(\sum_{a=1}^{t-1} R_a + R_0 + \sum_{a=1}^{n-t} R_a \right) \\
&= \sum_{t=1}^{\lfloor n - \frac{\ell}{2} \rfloor} \sum_{a=1}^{t-1} R_a + \left\lfloor n - \frac{\ell}{2} \right\rfloor R_0 + \sum_{t=1}^{\lfloor n - \frac{\ell}{2} \rfloor} \sum_{a=1}^{n-t} R_a
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=1}^{\lfloor n-\frac{\ell}{2}\rfloor-1} \big(\Big\lfloor n-\tfrac{\ell}{2}\Big\rfloor - t\big)R_t + \Big\lfloor n-\tfrac{\ell}{2}\Big\rfloor R_0 \\
&\quad + \sum_{t=1}^{\lfloor n-\frac{\ell}{2}\rfloor-1} tR_{n-t} + \Big\lfloor n-\tfrac{\ell}{2}\Big\rfloor \sum_{t=1}^{n-\lfloor n-\frac{\ell}{2}\rfloor} R_t \\
&= \sum_{t=1}^{\lfloor n-\frac{\ell}{2}\rfloor-1} \big(\Big\lfloor n-\tfrac{\ell}{2}\Big\rfloor - t\big)R_t + \Big\lfloor n-\tfrac{\ell}{2}\Big\rfloor R_0 + n\sum_{t=1}^{n-1} R_t - \sum_{t=1}^{n-1} tR_t \\
&\quad - n\sum_{t=1}^{n-\lfloor n-\frac{\ell}{2}\rfloor} R_t + \sum_{t=1}^{n-\lfloor n-\frac{\ell}{2}\rfloor} tR_t + \Big\lfloor n-\tfrac{\ell}{2}\Big\rfloor \sum_{t=1}^{n-\lfloor n-\frac{\ell}{2}\rfloor} R_t \\
&= \Big\lfloor n-\tfrac{\ell}{2}\Big\rfloor \gamma_{0,\lfloor n-\frac{\ell}{2}\rfloor-1} - \gamma_{1,\lfloor n-\frac{\ell}{2}\rfloor-1} + \Big(\Big\lfloor n-\tfrac{\ell}{2}\Big\rfloor - n\Big)\gamma_{0,n-\lfloor n-\frac{\ell}{2}\rfloor} \\
&\quad + \gamma_{1,n-\lfloor n-\frac{\ell}{2}\rfloor} + n\gamma_{0,n-1} - \gamma_{1,n-1} + \Big\lfloor n-\tfrac{\ell}{2}\Big\rfloor R_0.
\end{aligned}
$$

□

Lemma 5

$$
\begin{aligned}
\mathrm{Cov}[S_{\lfloor\frac{j}{2}\rfloor}, S_{\lfloor n-\frac{\ell}{2}\rfloor}] &= \Big\lfloor\tfrac{j}{2}\Big\rfloor \gamma_{0,\lfloor\frac{j}{2}\rfloor-1} - \gamma_{1,\lfloor\frac{j}{2}\rfloor-1} + \Big(\Big\lfloor\tfrac{j}{2}\Big\rfloor - \Big\lfloor n-\tfrac{\ell}{2}\Big\rfloor\Big)\gamma_{0,\lfloor n-\frac{\ell}{2}\rfloor-\lfloor\frac{j}{2}\rfloor} + \gamma_{1,\lfloor n-\frac{\ell}{2}\rfloor-\lfloor\frac{j}{2}\rfloor} \\
&\quad + \Big\lfloor n-\tfrac{\ell}{2}\Big\rfloor \gamma_{0,\lfloor n-\frac{\ell}{2}\rfloor-1} - \gamma_{1,\lfloor n-\frac{\ell}{2}\rfloor-1} + \Big\lfloor\tfrac{j}{2}\Big\rfloor R_0.
\end{aligned}
$$

Proof:

$$
\begin{aligned}
\mathrm{Cov}[S_{\lfloor\frac{j}{2}\rfloor}, S_{\lfloor n-\frac{\ell}{2}\rfloor}] &= \sum_{t=1}^{\lfloor\frac{j}{2}\rfloor}\Bigg(\sum_{a=1}^{\lfloor\frac{j}{2}\rfloor} R_{|t-a|} + \sum_{a=\lfloor\frac{j}{2}\rfloor+1}^{\lfloor n-\frac{\ell}{2}\rfloor} R_{a-t}\Bigg) \\
&= \sum_{t=1}^{\lfloor\frac{j}{2}\rfloor}\Bigg(\sum_{a=1}^{t-1} R_a + R_0 + \sum_{a=1}^{\lfloor n-\frac{\ell}{2}\rfloor-t} R_a\Bigg) \\
&= \sum_{t=1}^{\lfloor\frac{j}{2}\rfloor}\sum_{a=1}^{t-1} R_a + \Big\lfloor\tfrac{j}{2}\Big\rfloor R_0 + \sum_{t=1}^{\lfloor\frac{j}{2}\rfloor}\sum_{a=1}^{\lfloor n-\frac{\ell}{2}\rfloor-t} R_a \\
&= \sum_{t=1}^{\lfloor\frac{j}{2}\rfloor-1} \big(\Big\lfloor\tfrac{j}{2}\Big\rfloor - t\big)R_t + \Big\lfloor\tfrac{j}{2}\Big\rfloor R_0 + \sum_{t=1}^{\lfloor\frac{j}{2}\rfloor-1} tR_{\lfloor n-\frac{\ell}{2}\rfloor-t} + \Big\lfloor\tfrac{j}{2}\Big\rfloor \sum_{t=1}^{\lfloor n-\frac{\ell}{2}\rfloor-\lfloor\frac{j}{2}\rfloor} R_t \\
&= \sum_{t=1}^{\lfloor\frac{j}{2}\rfloor-1} \big(\Big\lfloor\tfrac{j}{2}\Big\rfloor - t\big)R_t + \Big\lfloor\tfrac{j}{2}\Big\rfloor R_0 + \Big\lfloor n-\tfrac{\ell}{2}\Big\rfloor \sum_{t=1}^{\lfloor n-\frac{\ell}{2}\rfloor-1} R_t - \sum_{t=1}^{\lfloor n-\frac{\ell}{2}\rfloor-1} tR_t \\
&\quad - \Big\lfloor n-\tfrac{\ell}{2}\Big\rfloor \sum_{t=1}^{\lfloor n-\frac{\ell}{2}\rfloor-\lfloor\frac{j}{2}\rfloor} R_t + \sum_{t=1}^{\lfloor n-\frac{\ell}{2}\rfloor-\lfloor\frac{j}{2}\rfloor} tR_t + \Big\lfloor\tfrac{j}{2}\Big\rfloor \sum_{t=1}^{\lfloor n-\frac{\ell}{2}\rfloor-\lfloor\frac{j}{2}\rfloor} R_t
\end{aligned}
$$

$$= \left\lfloor \tfrac{j}{2} \right\rfloor \gamma_{0,\lfloor \frac{j}{2} \rfloor - 1} - \gamma_{1,\lfloor \frac{j}{2} \rfloor - 1} + \Big(\left\lfloor \tfrac{j}{2} \right\rfloor - \left\lfloor n - \tfrac{\ell}{2} \right\rfloor \Big) \gamma_{0,\lfloor n - \frac{\ell}{2} \rfloor - \lfloor \frac{j}{2} \rfloor} + \gamma_{1,\lfloor n - \frac{\ell}{2} \rfloor - \lfloor \frac{j}{2} \rfloor}$$
$$+ \left\lfloor n - \tfrac{\ell}{2} \right\rfloor \gamma_{0,\lfloor n - \frac{\ell}{2} \rfloor - 1} - \gamma_{1,\lfloor n - \frac{\ell}{2} \rfloor - 1} + \left\lfloor \tfrac{j}{2} \right\rfloor R_0.$$

□

Lemma 6 For $j \leq \ell$,

$$\mathrm{Cov}[S_{\lfloor \frac{j}{2} \rfloor}, S_{\lfloor \frac{\ell}{2} \rfloor}] = \left\lfloor \tfrac{j}{2} \right\rfloor \gamma_{0,\lfloor \frac{j}{2} \rfloor - 1} - \gamma_{1,\lfloor \frac{j}{2} \rfloor - 1} + \Big(\left\lfloor \tfrac{j}{2} \right\rfloor - \left\lfloor \tfrac{\ell}{2} \right\rfloor \Big) \gamma_{0,\lfloor \frac{\ell}{2} \rfloor - \lfloor \frac{j}{2} \rfloor} + \gamma_{1,\lfloor \frac{\ell}{2} \rfloor - \lfloor \frac{j}{2} \rfloor}$$
$$+ \left\lfloor \tfrac{\ell}{2} \right\rfloor \gamma_{0,\lfloor \frac{\ell}{2} \rfloor - 1} - \gamma_{1,\lfloor \frac{\ell}{2} \rfloor - 1} + \left\lfloor \tfrac{j}{2} \right\rfloor R_0.$$

while for $j > \ell$,

$$\mathrm{Cov}[S_{\lfloor \frac{j}{2} \rfloor}, S_{\lfloor \frac{\ell}{2} \rfloor}] = \left\lfloor \tfrac{\ell}{2} \right\rfloor \gamma_{0,\lfloor \frac{\ell}{2} \rfloor - 1} - \gamma_{1,\lfloor \frac{\ell}{2} \rfloor - 1} + \Big(\left\lfloor \tfrac{\ell}{2} \right\rfloor - \left\lfloor \tfrac{j}{2} \right\rfloor \Big) \gamma_{0,\lfloor \frac{j}{2} \rfloor - \lfloor \frac{\ell}{2} \rfloor} + \gamma_{1,\lfloor \frac{j}{2} \rfloor - \lfloor \frac{\ell}{2} \rfloor}$$
$$+ \left\lfloor \tfrac{j}{2} \right\rfloor \gamma_{0,\lfloor \frac{j}{2} \rfloor - 1} - \gamma_{1,\lfloor \frac{j}{2} \rfloor - 1} + \left\lfloor \tfrac{\ell}{2} \right\rfloor R_0.$$

Proof: We will consider the case $j \leq \ell$. The proof for $j > \ell$ is similar.

$$\begin{aligned}
\mathrm{Cov}[S_{\lfloor \frac{j}{2} \rfloor}, S_{\lfloor \frac{\ell}{2} \rfloor}] &= \sum_{t=1}^{\lfloor \frac{j}{2} \rfloor} \Bigg(\sum_{a=1}^{\lfloor \frac{j}{2} \rfloor} R_{|t-a|} + \sum_{a=\lfloor \frac{j}{2} \rfloor + 1}^{\lfloor \frac{\ell}{2} \rfloor} R_{a-t} \Bigg) \\
&= \sum_{t=1}^{\lfloor \frac{j}{2} \rfloor} \Bigg(\sum_{a=1}^{t-1} R_a + R_0 + \sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor - t} R_a \Bigg) \\
&= \sum_{t=1}^{\lfloor \frac{j}{2} \rfloor} \sum_{a=1}^{t-1} R_a + \left\lfloor \tfrac{j}{2} \right\rfloor R_0 + \sum_{t=1}^{\lfloor \frac{j}{2} \rfloor} \sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor - t} R_a \\
&= \sum_{t=1}^{\lfloor \frac{j}{2} \rfloor - 1} \Big(\left\lfloor \tfrac{j}{2} \right\rfloor - t \Big) R_t + \left\lfloor \tfrac{j}{2} \right\rfloor R_0 + \sum_{t=1}^{\lfloor \frac{j}{2} \rfloor - 1} t R_{\lfloor \frac{\ell}{2} \rfloor - t} + \left\lfloor \tfrac{j}{2} \right\rfloor \sum_{t=1}^{\lfloor \frac{\ell}{2} \rfloor - \lfloor \frac{j}{2} \rfloor} R_t \\
&= \sum_{t=1}^{\lfloor \frac{j}{2} \rfloor - 1} \Big(\left\lfloor \tfrac{j}{2} \right\rfloor - t \Big) R_t + \left\lfloor \tfrac{j}{2} \right\rfloor R_0 + \left\lfloor \tfrac{\ell}{2} \right\rfloor \sum_{t=1}^{\lfloor \frac{\ell}{2} \rfloor - 1} R_t - \sum_{t=1}^{\lfloor \frac{\ell}{2} \rfloor - 1} t R_t \\
&\quad - \left\lfloor \tfrac{\ell}{2} \right\rfloor \sum_{t=1}^{\lfloor \frac{\ell}{2} \rfloor - \lfloor \frac{j}{2} \rfloor} R_t + \sum_{t=1}^{\lfloor \frac{\ell}{2} \rfloor - \lfloor \frac{j}{2} \rfloor} t R_t + \left\lfloor \tfrac{j}{2} \right\rfloor \sum_{t=1}^{\lfloor \frac{\ell}{2} \rfloor - \lfloor \frac{j}{2} \rfloor} R_t \\
&= \left\lfloor \tfrac{j}{2} \right\rfloor \gamma_{0,\lfloor \frac{j}{2} \rfloor - 1} - \gamma_{1,\lfloor \frac{j}{2} \rfloor - 1} + \Big(\left\lfloor \tfrac{j}{2} \right\rfloor - \left\lfloor \tfrac{\ell}{2} \right\rfloor \Big) \gamma_{0,\lfloor \frac{\ell}{2} \rfloor - \lfloor \frac{j}{2} \rfloor} + \gamma_{1,\lfloor \frac{\ell}{2} \rfloor - \lfloor \frac{j}{2} \rfloor} \\
&\quad + \left\lfloor \tfrac{\ell}{2} \right\rfloor \gamma_{0,\lfloor \frac{\ell}{2} \rfloor - 1} - \gamma_{1,\lfloor \frac{\ell}{2} \rfloor - 1} + \left\lfloor \tfrac{j}{2} \right\rfloor R_0.
\end{aligned}$$

□

The proof of Lemma 7 is very similar to the proof of Lemma 6.

Lemma 7 For $j \le \ell$,

$$\begin{aligned}
&\mathrm{Cov}[S_{\lfloor n-\frac{j}{2}\rfloor}, S_{\lfloor n-\frac{\ell}{2}\rfloor}] \\
&= \left\lfloor n-\tfrac{j}{2}\right\rfloor \gamma_{0,\lfloor n-\frac{j}{2}\rfloor-1} - \gamma_{1,\lfloor n-\frac{j}{2}\rfloor-1} + \Big(\left\lfloor n-\tfrac{\ell}{2}\right\rfloor - \left\lfloor n-\tfrac{j}{2}\right\rfloor\Big)\gamma_{0,\lfloor n-\frac{j}{2}\rfloor-\lfloor n-\frac{\ell}{2}\rfloor} \\
&\quad + \gamma_{1,\lfloor n-\frac{j}{2}\rfloor-\lfloor n-\frac{\ell}{2}\rfloor} + \left\lfloor n-\tfrac{\ell}{2}\right\rfloor \gamma_{0,\lfloor n-\frac{\ell}{2}\rfloor-1} - \gamma_{1,\lfloor n-\frac{\ell}{2}\rfloor-1} + \left\lfloor n-\tfrac{j}{2}\right\rfloor R_0
\end{aligned}$$

while for $j > \ell$,

$$\begin{aligned}
&\mathrm{Cov}[S_{\lfloor n-\frac{j}{2}\rfloor}, S_{\lfloor n-\frac{\ell}{2}\rfloor}] \\
&= \left\lfloor n-\tfrac{\ell}{2}\right\rfloor \gamma_{0,\lfloor n-\frac{\ell}{2}\rfloor-1} - \gamma_{1,\lfloor n-\frac{\ell}{2}\rfloor-1} + \Big(\left\lfloor n-\tfrac{j}{2}\right\rfloor - \left\lfloor n-\tfrac{\ell}{2}\right\rfloor\Big)\gamma_{0,\lfloor n-\frac{\ell}{2}\rfloor-\lfloor n-\frac{j}{2}\rfloor} \\
&\quad + \gamma_{1,\lfloor n-\frac{\ell}{2}\rfloor-\lfloor n-\frac{j}{2}\rfloor} + \left\lfloor n-\tfrac{j}{2}\right\rfloor \gamma_{0,\lfloor n-\frac{j}{2}\rfloor-1} - \gamma_{1,\lfloor n-\frac{j}{2}\rfloor-1} + \left\lfloor n-\tfrac{\ell}{2}\right\rfloor R_0.
\end{aligned}$$

The first equality in the expected value result of Theorem 4 follows by combining Lemmas 2–7 with Equation (76), and applying some additional algebra. In order to establish the second equality we need several additional auxiliary definitions and results. Let

$$\begin{aligned}
F_{D,n}(t) &\equiv \frac{1}{n}\sum_{j=1}^{\lfloor mt\rfloor} f\Big(\tfrac{j}{n}\Big), \quad \text{for } 0 \le t \le 1, \\
F_{D,n} &\equiv F_{D,n}(1) = \frac{1}{n}\sum_{j=1}^{n} f\Big(\tfrac{j}{n}\Big), \\
\bar{F}_{D,n} &\equiv \frac{1}{n}\sum_{j=1}^{n-1} F_{D,n}\Big(\tfrac{j}{n}\Big).
\end{aligned}$$

Lemma 8

$$\bar{F}_{D,n} = \frac{1}{n}\sum_{j=1}^{n}\Big(1-\tfrac{j}{n}\Big)f\Big(\tfrac{j}{n}\Big) = \frac{1}{n^2}\sum_{j=1}^{n}\Big(\left\lfloor n-\tfrac{j}{2}\right\rfloor - \left\lfloor \tfrac{j}{2}\right\rfloor\Big)f\Big(\tfrac{j}{n}\Big).$$

Proof: Similar to the work in Foley and Goldsman [15],

$$n\bar{F}_{D,n} = \sum_{j=1}^{n-1} F_{D,n}\Big(\tfrac{j}{n}\Big) = \sum_{j=1}^{n-1}\tfrac{1}{n}\sum_{\ell=1}^{j} f\Big(\tfrac{\ell}{n}\Big) = \sum_{j=1}^{n}\Big(1-\tfrac{j}{n}\Big)f\Big(\tfrac{j}{n}\Big).$$

The second result follows from the fact that $\lfloor n-\frac{j}{2}\rfloor - \lfloor\frac{j}{2}\rfloor = n-j$, $j = 1, 2, \ldots, n$. □

Lemma 9

$$F_{D,n} - \bar{F}_{D,n} \;=\; \frac{1}{n^2}\sum_{j=1}^{n} j f\Big(\tfrac{j}{n}\Big).$$

Proof: The result follows from Lemma 8 and the definition of $F_{D,n}$. □

During the proof of the second equality of Theorem 4, we replace the discrete approximations to certain integrals with their respective integrals plus appropriate error terms. We set up the necessary results with the Trapezoid Rule and the following lemmas.

Trapezoid Rule (Atkinson [8]) Suppose $\varphi(t)$ is a function with two continuous derivatives on $[a, b]$. Define $h \equiv (b-a)/n$ and $x_j \equiv a + jh$, for $j = 1, 2, \ldots, n$. Then

$$\int_a^b \varphi(t)\,dt \;=\; h\sum_{j=1}^{n-1}\varphi(x_j) + \frac{h}{2}[\varphi(a)+\varphi(b)] + O(n^{-2}).$$

Lemma 10 For $j = 0, 1, 2, \ldots$,

$$F(\tfrac{j}{n}) \;=\; F_{D,n}\Big(\tfrac{j}{n}\Big) + \frac{f(0) - f\Big(\tfrac{j}{n}\Big)}{2n} + O(n^{-2}).$$

Proof: By the Trapezoid Rule with $a = 0$, $b = j/n$, and $h = (b-a)/j = 1/n$, we have

$$F(\tfrac{j}{n}) = \int_0^{j/n} f(t)\,dt \;=\; \frac{1}{n}\sum_{j=1}^{j-1} f\Big(\tfrac{j}{n}\Big) + \frac{f(0) + f\Big(\tfrac{j}{n}\Big)}{2n} + O(n^{-2}),$$

and the result follows from the definition of $F_{D,n}(\frac{j}{n})$. □

Lemma 11

$$\bar{F} \;=\; \bar{F}_{D,n} + \frac{f(0)}{2n} + O(n^{-2}).$$

Proof: Applying the Trapezoid Rule, we have

$$\begin{aligned}
\bar{F} &= \int_0^1 F(t)\,dt \;=\; \frac{1}{n}\sum_{j=1}^{n-1} f\Big(\tfrac{j}{n}\Big) + \frac{F(0)+F(1)}{2n} + O(n^{-2}) \\
&= \frac{1}{n}\sum_{j=1}^{n} f\Big(\tfrac{j}{n}\Big) - \frac{F}{2n} + O(n^{-2}) \\
&= \frac{1}{n}\sum_{j=1}^{n}\left[F_{D,n}\Big(\tfrac{j}{n}\Big) + \frac{f(0)-f\Big(\frac{j}{n}\Big)}{2n} + O(n^{-2})\right] - \frac{F_{D,n}+O(n^{-1})}{2n} \\
&= \frac{1}{n}\sum_{j=1}^{n-1} F_{D,n}\Big(\tfrac{j}{n}\Big) + \frac{f(0)}{2n} + O(n^{-2}).
\end{aligned}$$

where the third equality follows from Lemma 10. □

Lemma 12 (Goldsman and Meketon [17], Lemma 1) Under Assumptions A, for $\ell > j$,

$$\mathrm{Cov}(S_\ell, S_j) = \frac{1}{2}[\mathrm{Var}(S_\ell) + \mathrm{Var}(S_j) - \mathrm{Var}(S_{l-j})] = j\sigma^2 - \frac{\gamma_1}{2} + o\,(1).$$

Combining Lemmas 2 and 12, and Equation (76) gives

$$\begin{aligned}
&\mathrm{E}[\mathcal{A}_{(1)}(f;n)] \\
&= \frac{1}{n^3}\left[\sum_{j=1}^{n}\Big(\tfrac{j}{n}-1\Big)f\Big(\tfrac{j}{n}\Big)\right]^2[\sigma^2 n - \gamma_1 + o\,(1)] \\
&\quad - \frac{2}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{n}\Big(\tfrac{j}{n}-1\Big)f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)[(\ell-n)\sigma^2 + o\,(1)] \\
&\quad - \frac{2}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\{\lfloor\tfrac{j}{2}\rfloor\sigma^2 - \tfrac{\gamma_1}{2} + o\,(1)\} \\
&\quad + \frac{1}{n^3}\sum_{j=1}^{n}\sum_{\ell=j}^{n} f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\{\lfloor\tfrac{j}{2}\rfloor\sigma^2 - \tfrac{\gamma_1}{2} + o\,(1)\} \\
&\quad + \frac{1}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{j-1} f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\{\lfloor\tfrac{\ell}{2}\rfloor\sigma^2 - \tfrac{\gamma_1}{2} + o\,(1)\} \\
&\quad + \frac{1}{n^3}\sum_{j=1}^{n}\sum_{\ell=j}^{n} f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\{\lfloor n - \tfrac{\ell}{2}\rfloor\sigma^2 - \tfrac{\gamma_1}{2} + o\,(1)\} \\
&\quad + \frac{1}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{j-1} f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\{\lfloor n - \tfrac{j}{2}\rfloor\sigma^2 - \tfrac{\gamma_1}{2} + o\,(1)\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n^3}\Big(\sum_{j=1}^{n}(j-n)f\Big(\tfrac{j}{n}\Big)\Big)^2\{\tfrac{\sigma^2}{n}-\tfrac{\gamma_1}{n^2}+o\,(1)\}\\
&\quad-\frac{2}{n^4}\Big(\sum_{j=1}^{n}(j-n)f\Big(\tfrac{j}{n}\Big)\sum_{\ell=1}^{n}(\ell-n)f\Big(\tfrac{\ell}{n}\Big)\sigma^2+o\,(1)\Big)\\
&\quad-\frac{2}{n^3}\Big(\sum_{j=1}^{n}\lfloor\tfrac{j}{2}\rfloor f\Big(\tfrac{j}{n}\Big)\sigma^2\sum_{\ell=1}^{n}f\Big(\tfrac{\ell}{n}\Big)-\sum_{j=1}^{n}\sum_{\ell=1}^{n}f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\tfrac{\gamma_1}{2}+o\,(1)\Big)\\
&\qquad+\frac{1}{n^3}\Big(\sum_{j=1}^{n}\lfloor\tfrac{j}{2}\rfloor f\Big(\tfrac{j}{n}\Big)\sigma^2\sum_{\ell=j}^{n}f\Big(\tfrac{\ell}{n}\Big)-\sum_{j=1}^{n}\sum_{\ell=j}^{n}f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\tfrac{\gamma_1}{2}+o\,(1)\Big)\\
&\qquad+\frac{1}{n^3}\Big(\sum_{j=1}^{n}f\Big(\tfrac{j}{n}\Big)\sum_{\ell=1}^{j-1}\lfloor\tfrac{\ell}{2}\rfloor f\Big(\tfrac{\ell}{n}\Big)\sigma^2-\sum_{j=1}^{n}\sum_{\ell=1}^{j-1}f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\tfrac{\gamma_1}{2}+o\,(1)\Big)\\
&\qquad+\frac{1}{n^3}\Big(\sum_{j=1}^{n}f\Big(\tfrac{j}{n}\Big)\sum_{\ell=j}^{n}\lfloor n-\tfrac{\ell}{2}\rfloor f\Big(\tfrac{\ell}{n}\Big)\sigma^2-\sum_{j=1}^{n}\sum_{\ell=j}^{n}f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\tfrac{\gamma_1}{2}+o\,(1)\Big)\\
&\qquad+\frac{1}{n^3}\Big(\sum_{j=1}^{n}\sum_{\ell=1}^{j-1}f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\tfrac{-\gamma_1}{2}+o\,(1)+\sum_{j=1}^{n}\lfloor n-\tfrac{j}{2}\rfloor f\Big(\tfrac{j}{n}\Big)\sigma^2\sum_{\ell=1}^{j-1}f\Big(\tfrac{\ell}{n}\Big)\Big)\\
&= \frac{1}{n^4}\Big(\sum_{j=1}^{n}(j-n)f\Big(\tfrac{j}{n}\Big)\Big)^2\sigma^2-\frac{1}{n^5}\Big(\sum_{j=1}^{n}(j-n)f\Big(\tfrac{j}{n}\Big)\Big)^2\gamma_1\\
&\qquad-\frac{2}{n^4}\Big(\sum_{j=1}^{n}(j-n)f\Big(\tfrac{j}{n}\Big)\Big)^2\sigma^2-\frac{2}{n^3}\Big(\sum_{j=1}^{n}\lfloor\tfrac{j}{2}\rfloor f\Big(\tfrac{j}{n}\Big)\sigma^2\sum_{\ell=1}^{n}f\Big(\tfrac{\ell}{n}\Big)\Big)\\
&\qquad+\frac{1}{n^3}\Big(\sum_{j=1}^{n}\lfloor\tfrac{j}{2}\rfloor f\Big(\tfrac{j}{n}\Big)\sigma^2\sum_{\ell=j}^{n}f\Big(\tfrac{\ell}{n}\Big)\Big)+\frac{1}{n^3}\Big(\sum_{j=1}^{n}f\Big(\tfrac{j}{n}\Big)\sum_{\ell=1}^{j-1}\lfloor\tfrac{\ell}{2}\rfloor f\Big(\tfrac{\ell}{n}\Big)\sigma^2\Big)\\
&\qquad+\frac{1}{n^3}\Big(\sum_{j=1}^{n}f\Big(\tfrac{j}{n}\Big)\sum_{\ell=j}^{n}\lfloor n-\tfrac{\ell}{2}\rfloor f\Big(\tfrac{\ell}{n}\Big)\sigma^2\Big)+\frac{1}{n^3}\Big(\sum_{j=1}^{n}\lfloor n-\tfrac{j}{2}\rfloor\sigma^2 f\Big(\tfrac{j}{n}\Big)\sum_{\ell=1}^{j-1}f\Big(\tfrac{\ell}{n}\Big)\Big)+o\,(n^{-1})\\
&= -\frac{1}{n^4}\Big(\sum_{j=1}^{n}(j-n)f\Big(\tfrac{j}{n}\Big)\Big)^2\sigma^2-\frac{1}{n^5}\Big(\sum_{j=1}^{n}(j-n)f\Big(\tfrac{j}{n}\Big)\Big)^2\gamma_1\\
&\qquad+\frac{1}{n^3}\Big(\sum_{j=1}^{n}\sum_{\ell=j}^{n}\Big[\lfloor\tfrac{j}{2}\rfloor f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\sigma^2+\lfloor n-\tfrac{\ell}{2}\rfloor f\Big(\tfrac{\ell}{n}\Big)f\Big(\tfrac{j}{n}\Big)\sigma^2\Big]\Big)\\
&\qquad+\frac{1}{n^3}\Big(\sum_{j=1}^{n}\sum_{\ell=1}^{j-1}\Big[\lfloor\tfrac{\ell}{2}\rfloor f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\sigma^2+\lfloor n-\tfrac{j}{2}\rfloor f\Big(\tfrac{\ell}{n}\Big)f\Big(\tfrac{j}{n}\Big)\sigma^2\Big]\Big)\\
&\qquad-\frac{2}{n^3}\Big(\sum_{j=1}^{n}\sum_{\ell=1}^{n}\lfloor\tfrac{j}{2}\rfloor f\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big)\sigma^2\Big)+o\,(n^{-1})
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{n^4}\left(\sum_{j=1}^{n}(j-n)f\left(\tfrac{j}{n}\right)\right)^2(\sigma^2+\tfrac{\gamma_1}{n}) - \frac{2}{n^3}\left(\sum_{j=1}^{n}\sum_{\ell=1}^{n}\lfloor\tfrac{j}{2}\rfloor f\left(\tfrac{j}{n}\right)f\left(\tfrac{\ell}{n}\right)\sigma^2\right)\\
&\quad + \frac{\sigma^2}{n^3}\left(\sum_{j=1}^{n}\lfloor\tfrac{j}{2}\rfloor f\left(\tfrac{j}{n}\right)\sum_{\ell=j}^{n}f\left(\tfrac{\ell}{n}\right) + \sum_{j=1}^{n}\lfloor n-\tfrac{j}{2}\rfloor f\left(\tfrac{j}{n}\right)\sum_{\ell=1}^{j-1}f\left(\tfrac{\ell}{n}\right)\right.\\
&\qquad \left. + \sum_{j=1}^{n}f\left(\tfrac{j}{n}\right)\sum_{\ell=j}^{n}\lfloor n-\tfrac{\ell}{2}\rfloor f\left(\tfrac{\ell}{n}\right) + \sum_{j=1}^{n}f\left(\tfrac{j}{n}\right)\sum_{\ell=1}^{j-1}\lfloor\tfrac{\ell}{2}\rfloor f\left(\tfrac{\ell}{n}\right)\right) + o\left(n^{-1}\right)\\
&= -\frac{1}{n^4}\left(\sum_{j=1}^{n}(j-n)f\left(\tfrac{j}{n}\right)\right)^2(\sigma^2+\tfrac{\gamma_1}{n}) - \frac{2}{n^3}\left(\sum_{j=1}^{n}\sum_{\ell=1}^{n}\lfloor\tfrac{j}{2}\rfloor f\left(\tfrac{j}{n}\right)f\left(\tfrac{\ell}{n}\right)\sigma^2\right)\\
&\quad + \frac{\sigma^2}{n^3}\left(\sum_{j=1}^{n}\lfloor\tfrac{j}{2}\rfloor f\left(\tfrac{j}{n}\right)\sum_{\ell=j}^{n}f\left(\tfrac{\ell}{n}\right) + \sum_{j=1}^{n}\lfloor\tfrac{j}{2}\rfloor f\left(\tfrac{j}{n}\right)\sum_{\ell=1}^{j-1}f\left(\tfrac{\ell}{n}\right)\right.\\
&\qquad - \sum_{j=1}^{n}jf\left(\tfrac{j}{n}\right)\sum_{\ell=1}^{j-1}f\left(\tfrac{\ell}{n}\right) + \sum_{j=1}^{n}nf\left(\tfrac{j}{n}\right)\sum_{\ell=1}^{j-1}f\left(\tfrac{\ell}{n}\right) + \sum_{j=1}^{n}f\left(\tfrac{j}{n}\right)\sum_{\ell=j}^{n}\lfloor\tfrac{\ell}{2}\rfloor f\left(\tfrac{\ell}{n}\right)\\
&\qquad \left. - \sum_{j=1}^{n}f\left(\tfrac{j}{n}\right)\sum_{\ell=j}^{n}\ell f\left(\tfrac{\ell}{n}\right) + \sum_{j=1}^{n}f\left(\tfrac{j}{n}\right)\sum_{\ell=j}^{n}nf\left(\tfrac{\ell}{n}\right) + \sum_{j=1}^{n}f\left(\tfrac{j}{n}\right)\sum_{\ell=1}^{j-1}\lfloor\tfrac{\ell}{2}\rfloor f\left(\tfrac{\ell}{n}\right)\right) + o\left(n^{-1}\right)\\
&= -\frac{1}{n^4}\left(\sum_{j=1}^{n}(j-n)f\left(\tfrac{j}{n}\right)\right)^2(\sigma^2+\tfrac{\gamma_1}{n}) - \frac{2}{n^3}\left(\sum_{j=1}^{n}\sum_{\ell=1}^{n}\lfloor\tfrac{j}{2}\rfloor f\left(\tfrac{j}{n}\right)f\left(\tfrac{\ell}{n}\right)\sigma^2\right)\\
&\quad + \frac{\sigma^2}{n^3}\left(\sum_{j=1}^{n}\sum_{\ell=1}^{n}\lfloor\tfrac{j}{2}\rfloor f\left(\tfrac{j}{n}\right)f\left(\tfrac{\ell}{n}\right) + \sum_{j=1}^{n}\sum_{\ell=1}^{n}\lfloor\tfrac{j}{2}\rfloor f\left(\tfrac{j}{n}\right)f\left(\tfrac{\ell}{n}\right) + n\sum_{j=1}^{n}\sum_{\ell=1}^{n}f\left(\tfrac{j}{n}\right)f\left(\tfrac{\ell}{n}\right)\right.\\
&\qquad \left. - \left[\sum_{j=1}^{n}jf\left(\tfrac{j}{n}\right)\sum_{\ell=1}^{j-1}f\left(\tfrac{\ell}{n}\right) + \sum_{j=1}^{n}f\left(\tfrac{j}{n}\right)\sum_{\ell=j}^{n}\ell f\left(\tfrac{\ell}{n}\right)\right]\right) + o\left(n^{-1}\right)\\
&= -\frac{1}{n^4}\left(\sum_{j=1}^{n}(j-n)f\left(\tfrac{j}{n}\right)\right)^2(\sigma^2+\tfrac{\gamma_1}{n}) + \frac{\sigma^2}{n^2}\sum_{j=1}^{n}\sum_{\ell=1}^{n}f\left(\tfrac{j}{n}\right)f\left(\tfrac{\ell}{n}\right)\\
&\quad - \frac{\sigma^2}{n^3}\sum_{j=1}^{n}jf\left(\tfrac{j}{n}\right)\sum_{\ell=1}^{j-1}f\left(\tfrac{\ell}{n}\right) - \frac{\sigma^2}{n^3}\sum_{j=1}^{n}f\left(\tfrac{j}{n}\right)\sum_{\ell=j}^{n}\ell f\left(\tfrac{\ell}{n}\right) + o\left(n^{-1}\right)\\
&= -\frac{1}{n^4}\left(\sum_{j=1}^{n}(j-n)f\left(\tfrac{j}{n}\right)\right)^2(\sigma^2+\tfrac{\gamma_1}{n}) + \frac{\sigma^2}{n^2}\left(\sum_{j=1}^{n}f\left(\tfrac{j}{n}\right)\right)^2\\
&\quad - \frac{\sigma^2}{n^3}\left(\sum_{j=1}^{n}f\left(\tfrac{j}{n}\right)\left[\sum_{\ell=1}^{j-1}jf\left(\tfrac{\ell}{n}\right) + \sum_{\ell=j}^{n}\ell f\left(\tfrac{\ell}{n}\right)\right]\right) + o\left(n^{-1}\right)\\
&= -\frac{1}{n^4}\left(\sum_{j=1}^{n}(j-n)f\left(\tfrac{j}{n}\right)\right)^2(\sigma^2+\tfrac{\gamma_1}{n}) + \frac{\sigma^2}{n^2}\left(\sum_{j=1}^{n}f\left(\tfrac{j}{n}\right)\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{\sigma^2}{n^3}\Bigg(2\sum_{j=1}^{n}\sum_{\ell=1}^{j} jf\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big) - \sum_{j=1}^{n} jf^2\Big(\tfrac{j}{n}\Big)\Bigg) + o\left(n^{-1}\right) \\
&= -\Bigg(\frac{1}{n^4}\Bigg[\sum_{j=1}^{n} jf\Big(\tfrac{j}{n}\Big)\Bigg]^2 + \frac{1}{n^4}\Bigg[\sum_{j=1}^{n} nf\Big(\tfrac{j}{n}\Big)\Bigg]^2 - \frac{2}{n^4}\sum_{j=1}^{n} jf\Big(\tfrac{j}{n}\Big)\sum_{\ell=1}^{n} nf\Big(\tfrac{\ell}{n}\Big)\Bigg)(\sigma^2 + \tfrac{\gamma_1}{n}) \\
&\quad + \frac{\sigma^2}{n^2}\Bigg(\sum_{j=1}^{n} f\Big(\tfrac{j}{n}\Big)\Bigg)^2 - \frac{2\sigma^2}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{j} jf\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big) + \frac{\sigma^2}{n^3}\sum_{j=1}^{n} jf^2\Big(\tfrac{j}{n}\Big) + o\left(n^{-1}\right) \\
&= -\frac{\sigma^2}{n^4}\Bigg[\sum_{j=1}^{n} jf\Big(\tfrac{j}{n}\Big)\Bigg]^2 - \frac{\sigma^2}{n^2}\Bigg[\sum_{j=1}^{n} f\Big(\tfrac{j}{n}\Big)\Bigg]^2 + \frac{2\sigma^2}{n^3}\sum_{j=1}^{n} jf\Big(\tfrac{j}{n}\Big)\sum_{\ell=1}^{n} f\Big(\tfrac{\ell}{n}\Big) \\
&\quad - \frac{\gamma_1}{n^5}\Bigg(\sum_{j=1}^{n}(j-n)f\Big(\tfrac{j}{n}\Big)\Bigg)^2 + \frac{\sigma^2}{n^2}\Bigg(\sum_{j=1}^{n} f\Big(\tfrac{j}{n}\Big)\Bigg)^2 \\
&\quad - \frac{2\sigma^2}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{j} jf\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big) + \frac{\sigma^2}{n^3}\sum_{j=1}^{n} jf^2\Big(\tfrac{j}{n}\Big) + o\left(n^{-1}\right) \\
&= -\frac{\sigma^2}{n^4}\Bigg[\sum_{j=1}^{n} jf\Big(\tfrac{j}{n}\Big)\Bigg]^2 + \frac{2\sigma^2}{n^3}\sum_{j=1}^{n} jf\Big(\tfrac{j}{n}\Big)\sum_{\ell=1}^{n} f\Big(\tfrac{\ell}{n}\Big) \\
&\quad - \frac{2\sigma^2}{n^3}\sum_{j=1}^{n}\sum_{\ell=1}^{j} jf\Big(\tfrac{j}{n}\Big)f\Big(\tfrac{\ell}{n}\Big) + \frac{\sigma^2}{n^3}\sum_{j=1}^{n} jf^2\Big(\tfrac{j}{n}\Big) \\
&\quad - \frac{\gamma_1}{n^5}\Bigg(\sum_{j=1}^{n}(j-n)f\Big(\tfrac{j}{n}\Big)\Bigg)^2 + o\left(n^{-1}\right) \\
&= \sigma^2\Bigg[-(F_{D,n} - \bar{F}_{D,n})^2 + 2(F_{D,n} - \bar{F}_{D,n})F_{D,n} - \frac{2}{n^2}\sum_{j=1}^{m} jf\Big(\tfrac{j}{n}\Big)F_{D,n}\Big(\tfrac{j}{n}\Big) \\
&\quad + \frac{1}{n^3}\sum_{j=1}^{n} jf^2\Big(\tfrac{j}{n}\Big)\Bigg] - \frac{\bar{F}_{D,n}^2\gamma_1}{n} + o\left(n^{-1}\right),
\end{aligned}
$$

where the last equality follows from definitions of $F_{D,n}$ and $\bar{F}_{D,n}$, and Lemmas 8 and 9. After some algebra we get

$$
\begin{aligned}
\mathrm{E}[\mathcal{A}_{(1)}(f;n)] = \sigma^2\Big[F_{D,n}^2 - \bar{F}_{D,n}^2 - \frac{2}{n^3}\sum_{j=1}^{m} jf\Big(\tfrac{j}{n}\Big)F_{D,n}\Big(\tfrac{j}{n}\Big) + \frac{1}{n^3}\sum_{j=1}^{m} jf^2\Big(\tfrac{j}{n}\Big)\Big] \\
- \frac{\bar{F}^2\gamma_1}{n} + o\left(n^{-1}\right). \qquad (77)
\end{aligned}
$$

Finally, we write an integral expression for the coefficient of σ^2 in Equation (77). By Lemmas 10 and 11, we have

$$\begin{aligned}
&\left[F^2 - \frac{(f(0)-f(1))F}{n}\right] - \left[\bar{F}^2 - \frac{f(0)\bar{F}}{n}\right] \\
&\qquad - \frac{2}{n^2}\sum_{j=1}^{n} jf\left(\tfrac{j}{n}\right)\left[F\left(\tfrac{j}{n}\right) - \frac{f(0)-f\left(\frac{j}{n}\right)}{2n}\right] + \frac{1}{n^3}\sum_{j=1}^{n} jf^2\left(\tfrac{j}{n}\right) + o(n^{-1}) \\
&= F^2 - \bar{F}^2 + \frac{(f(1)-f(0))F + f(0)\bar{F}}{n} - \frac{2}{n^2}\sum_{j=1}^{n} jf\left(\tfrac{j}{n}\right)F\left(\tfrac{j}{n}\right) \\
&\qquad + \frac{f(0)}{n^3}\sum_{j=1}^{n} jf\left(\tfrac{j}{n}\right) + o(n^{-1}) \\
&= F^2 - \bar{F}^2 - \frac{(f(1)+f(0))F - f(0)\bar{F}}{n} - \frac{2}{n^2}\sum_{j=1}^{n-1} jf\left(\tfrac{j}{n}\right)F\left(\tfrac{j}{n}\right) \\
&\qquad + \frac{f(0)}{n^3}\sum_{j=1}^{n-1} jf\left(\tfrac{j}{n}\right) + o(n^{-1}) \\
&= F^2 - \bar{F}^2 - \frac{(f(1)+f(0))F - f(0)\bar{F}}{n} \\
&\qquad - 2\left[\int_0^1 tf(t)F(t)\,dt - \frac{f(1)F}{2n}\right] + \frac{f(0)}{n}\left[\int_0^1 tf(t)\,\mathrm{d}t - \frac{f(1)}{2n}\right] + o(n^{-1}) \\
&\qquad \text{(by the Trapezoid Rule)} \\
&= F^2 - \bar{F}^2 - 2\int_0^1 tf(t)F(t)\,\mathrm{d}t + o\left(n^{-1}\right) \\
&\qquad \text{(after integration by parts on } \textstyle\int_0^1 tf(t)\,\mathrm{d}t \text{ and algebra)} \\
&= \int_0^1 F^2(t)\,\mathrm{d}t - \bar{F}^2 + o(n^{-1}) \quad \text{(by Equation (A-2) from Goldsman et al. [18])}, \\
&= 1 + o(n^{-1}) \quad \text{(by Equation (2) from Goldsman et al. [18] and Assumptions A)}.
\end{aligned}$$

□

A.2 Expected Value of Folded CvM Estimators: Proof of Theorem 10

We have

$$\mathrm{E}[\mathcal{C}_{(1)}(g;n)] = \frac{1}{n}\sum_{j=1}^{n} g\left(\tfrac{j}{n}\right)\sigma^2\mathrm{E}[\{T_{(1)}\left(\tfrac{j}{n}\right)\}^2]$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\sigma^2 \mathrm{E}\Big[\Big(\frac{(j-n)\bar{X}_n - S_{\lfloor \frac{j}{2}\rfloor} + S_{\lfloor n-\frac{j}{2}\rfloor}}{\sigma\sqrt{n}}\Big)^2\Big] \\
&= \frac{1}{n^2}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\mathrm{E}\Big[\Big((j-n)\bar{X}_n - S_{\lfloor \frac{j}{2}\rfloor} + S_{\lfloor n-\frac{j}{2}\rfloor}\Big)^2\Big] \\
&= \frac{1}{n^2}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\mathrm{Var}[(j-n)\bar{X}_n - S_{\lfloor \frac{j}{2}\rfloor} + S_{\lfloor n-\frac{j}{2}\rfloor}] \\
&= \frac{1}{n^2}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\Big[\Big(\tfrac{j}{n}-1\Big)^2\mathrm{Var}(S_n) + \mathrm{Var}(S_{\lfloor \frac{j}{2}\rfloor}) + \mathrm{Var}(S_{\lfloor n-\frac{j}{2}\rfloor}) \\
&\qquad - 2\Big(\tfrac{j}{n}-1\Big)\mathrm{Cov}(S_n, S_{\lfloor \frac{j}{2}\rfloor}) + 2\Big(\tfrac{j}{n}-1\Big)\mathrm{Cov}(S_n, S_{\lfloor n-\frac{j}{2}\rfloor}) \\
&\qquad - 2\mathrm{Cov}(S_{\lfloor \frac{j}{2}\rfloor}, S_{\lfloor n-\frac{j}{2}\rfloor})\Big].
\end{aligned}
$$

Using Lemmas 2–5, after some algebra we have

$$
\begin{aligned}
&\mathrm{E}[\mathcal{C}_{(1)}(g;n)] = \\
&\quad \frac{1}{n^2}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\Big[\Big(j-\tfrac{j^2}{n}\Big)R_0 + \tfrac{2(n-j)^2}{n}\gamma_{0,n-1} - \tfrac{2(n-j)^2}{n^2}\gamma_{1,n-1} - 2(j-n)\gamma_{0,n-j} - 2\gamma_{1,n-j} \\
&\qquad + 2\Big(\tfrac{j}{n}-1\Big)\Big\{-\lfloor\tfrac{j}{2}\rfloor\gamma_{0,\lfloor\frac{j}{2}\rfloor-1} + \gamma_{1,\lfloor\frac{j}{2}\rfloor-1} + \lfloor n-\tfrac{j}{2}\rfloor\gamma_{0,\lfloor n-\frac{j}{2}\rfloor-1} - \gamma_{1,\lfloor n-\frac{j}{2}\rfloor-1} \\
&\qquad - \Big(\lfloor\tfrac{j}{2}\rfloor - n\Big)\gamma_{0,n-\lfloor\frac{j}{2}\rfloor} - \gamma_{1,n-\lfloor\frac{j}{2}\rfloor} + \Big(\lfloor n-\tfrac{j}{2}\rfloor - n\Big)\gamma_{0,n-\lfloor n-\frac{j}{2}\rfloor} + \gamma_{1,n-\lfloor n-\frac{j}{2}\rfloor}\Big\}\Big].
\end{aligned} \tag{78}
$$

In order to calculate the results in the Theorem, we will use the following relations. For any function $p(j)$,

$$
\sum_{j=1}^{z} p(j)\gamma_{i,j} = \sum_{\ell=1}^{z} \ell^i R_\ell \sum_{j=\ell}^{z} p(j), \tag{79}
$$

for $i = 0, 1, 2, \ldots$ and $z = 1, 2, \ldots$. Using Equation (79) and some algebra, we get

$$
\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)(j-n)\sum_{\ell=1}^{n-j} R_\ell = \sum_{\ell=1}^{n-1} R_\ell \sum_{j=\ell}^{n-1} g\Big(\tfrac{n-j}{n}\Big)(-j), \tag{80}
$$

$$
\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)(j-n)\sum_{\ell=1}^{n-j} \ell R_\ell = \sum_{\ell=1}^{n-1} \ell R_\ell \sum_{j=\ell}^{n-1} g\Big(\tfrac{n-j}{n}\Big), \tag{81}
$$

$$\sum_{\ell=1}^{n} g\Big(\tfrac{\ell}{n}\Big)\Big(\tfrac{\ell}{n}-1\Big)\lfloor\tfrac{\ell}{2}\rfloor \sum_{\ell=1}^{\lfloor\frac{\ell}{2}\rfloor-1} R_\ell = \sum_{\ell=1}^{n/2-2} R_\ell \sum_{\ell=\ell}^{n/2-2} \Big[g\Big(\tfrac{2(\ell+1)}{n}\Big)\Big(\tfrac{2(\ell+1)}{n}-1\Big) + g\Big(\tfrac{2(\ell+1)+1}{n}\Big)\Big(\tfrac{2(\ell+1)+1}{n}-1\Big)\Big](\ell+1), \tag{82}$$

$$\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\Big(\tfrac{j}{n}-1\Big) \sum_{\ell=1}^{\lfloor\frac{j}{2}\rfloor-1} \ell R_\ell = \sum_{\ell=1}^{n/2-2} \ell R_\ell \sum_{j=\ell}^{n/2-2} \Big[g\Big(\tfrac{2(j+1)}{n}\Big)\Big(\tfrac{2(j+1)}{n}-1\Big) + g\Big(\tfrac{2(j+1)+1}{n}\Big)\Big(\tfrac{2(j+1)+1}{n}-1\Big)\Big], \tag{83}$$

$$\begin{aligned}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\Big(\tfrac{j}{n}-1\Big)\lfloor n-\tfrac{j}{2}\rfloor \sum_{\ell=1}^{\lfloor n-\frac{j}{2}\rfloor-1} R_\ell &= \sum_{\ell=1}^{n-2} R_\ell \sum_{j=\ell}^{n-2}\Big[g\Big(\tfrac{2(n-j)-2}{n}\Big)\Big(\tfrac{2(n-j)-2}{n}-1\Big) \\ &\quad + g\Big(\tfrac{2(n-j)-3}{n}\Big)\Big(\tfrac{2(n-j)-3}{n}-1\Big)\Big](j+1) \\ &\quad - \sum_{\ell=1}^{n/2-2} R_\ell \sum_{j=\ell}^{n/2-2} \Big[g\Big(\tfrac{2(n-j)-2}{n}\Big)\Big(\tfrac{2(n-j)-2}{n}-1\Big) \\ &\quad + g\Big(\tfrac{2(n-j)-3}{n}\Big)\Big(\tfrac{2(n-j)-3}{n}-1\Big)\Big](j+1),\end{aligned} \tag{84}$$

$$\begin{aligned}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\Big(\tfrac{j}{n}-1\Big) \sum_{\ell=1}^{\lfloor n-\frac{j}{2}\rfloor-1} \ell R_\ell &= \sum_{\ell=1}^{n-2} \ell R_\ell \sum_{j=\ell}^{n-2}\Big[g\Big(\tfrac{2(n-j)-2}{n}\Big)\Big(\tfrac{2(n-j)-2}{n}-1\Big) \\ &\quad + g\Big(\tfrac{2(n-j)-3}{n}\Big)\Big(\tfrac{2(n-j)-3}{n}-1\Big)\Big] \\ &\quad - \sum_{\ell=1}^{n/2-2} \ell R_\ell \sum_{j=\ell}^{n/2-2} \Big[g\Big(\tfrac{2(n-j)-2}{n}\Big)\Big(\tfrac{2(n-j)-2}{n}-1\Big) \\ &\quad + g\Big(\tfrac{2(n-j)-3}{n}\Big)\Big(\tfrac{2(n-j)-3}{n}-1\Big)\Big],\end{aligned} \tag{85}$$

$$\begin{aligned}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\Big(\tfrac{j}{n}-1\Big)\Big(\lfloor\tfrac{j}{2}\rfloor-n\Big) \sum_{\ell=1}^{n-\lfloor\frac{j}{2}\rfloor} R_\ell &= \sum_{\ell=1}^{n-1} R_\ell \sum_{j=\ell}^{n-1}\Big[g\Big(\tfrac{2(n-j)+1}{n}\Big)\Big(\tfrac{2(n-j)+1}{n}-1\Big) \\ &\quad + g\Big(\tfrac{2(n-j)}{n}\Big)\Big(\tfrac{2(n-j)}{n}-1\Big)\Big](-j) \\ &\quad - \sum_{\ell=1}^{n/2} R_\ell \sum_{j=\ell}^{n/2}\Big[g\Big(\tfrac{2(n-j)+1}{n}\Big)\Big(\tfrac{2(n-j)+1}{n}-1\Big) \\ &\quad + g\Big(\tfrac{2(n-j)}{n}\Big)\Big(\tfrac{2(n-j)}{n}-1\Big)\Big](-j) \\ &\quad + g\Big(\tfrac{1}{n}\Big)\Big(\tfrac{1}{n}-1\Big)(-n)\sum_{\ell=1}^{n} R_\ell,\end{aligned} \tag{86}$$

$$\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\Big(\tfrac{j}{n}-1\Big)\sum_{\ell=1}^{n-\lfloor\frac{j}{2}\rfloor}\ell R_\ell = \sum_{\ell=1}^{n-1}\ell R_\ell \sum_{j=\ell}^{n-1}\Big[g\Big(\tfrac{2(n-j)+1}{n}\Big)\Big(\tfrac{2(n-j)+1}{n}-1\Big) + g\Big(\tfrac{2(n-j)}{n}\Big)\Big(\tfrac{2(n-j)}{n}-1\Big)\Big] - \sum_{\ell=1}^{n/2}\ell R_\ell \sum_{j=\ell}^{n/2}\Big[g\Big(\tfrac{2(n-j)+1}{n}\Big)\Big(\tfrac{2(n-j)+1}{n}-1\Big) + g\Big(\tfrac{2(n-j)}{n}\Big)\Big(\tfrac{2(n-j)}{n}-1\Big)\Big] + g\Big(\tfrac{1}{n}\Big)\Big(\tfrac{1}{n}-1\Big)\sum_{\ell=1}^{n}\ell R_\ell, \tag{87}$$

$$\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\Big(\tfrac{j}{n}-1\Big)\Big(\lfloor n-\tfrac{j}{2}\rfloor - n\Big)\sum_{\ell=1}^{n-\lfloor n-\frac{j}{2}\rfloor} R_\ell = \sum_{\ell=1}^{n/2} R_\ell \sum_{j=\ell}^{n/2}\Big[g\Big(\tfrac{2j-1}{n}\Big)\Big(\tfrac{2j-1}{n}-1\Big) + g\Big(\tfrac{2j}{n}\Big)\Big(\tfrac{2j}{n}-1\Big)\Big](-j), \tag{88}$$

$$\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\Big(\tfrac{j}{n}-1\Big)\Big(\lfloor n-\tfrac{j}{2}\rfloor - n\Big)\sum_{\ell=1}^{n-\lfloor n-\frac{j}{2}\rfloor} \ell R_\ell = \sum_{\ell=1}^{n/2}\ell R_\ell \sum_{j=\ell}^{n/2}\Big[g\Big(\tfrac{2j-1}{n}\Big)\Big(\tfrac{2j-1}{n}-1\Big) + g\Big(\tfrac{2j}{n}\Big)\Big(\tfrac{2j}{n}-1\Big)\Big]. \tag{89}$$

Given the appropriate weight function, Theorem 10 follows from Equations (78) and (80)–(89), and after some algebra. □

A.3 Covariances of Folded CvM Estimators: Proof of Theorem 11

First, it is easy to show that

$$\mathrm{Cov}[\mathcal{B}_{(0)}(t),\mathcal{B}_{(1)}(s)] = \begin{cases} t(1-s) & \text{if } t < \frac{s}{2} \\ s(\frac{1}{2}-t) & \text{if } \frac{s}{2} < t < 1-\frac{s}{2}. \\ (s-1)(1-t) & \text{if } 1-\frac{s}{2} < t. \end{cases} \tag{90}$$

Then we have

$$\begin{aligned}\mathrm{Cov}[\mathcal{C}_{(0)}(g_x),\mathcal{C}_{(1)}(g_y)] &= \mathrm{Cov}\left[\int_0^1 g_x(t)(\sigma\mathcal{B}_{(0)}(t))^2\,\mathrm{d}t, \int_0^1 g_y(s)(\sigma\mathcal{B}_{(1)}(s))^2\,\mathrm{d}s\right] \\ &= \sigma^4\int_0^1\int_0^1 g_x(t)g_y(s)\mathrm{Cov}[\mathcal{B}^2_{(0)}(t),\mathcal{B}^2_{(1)}(s)]\,\mathrm{d}t\,\mathrm{d}s \\ &= 2\sigma^4\int_0^1\int_0^1 g_x(t)g_y(s)\mathrm{Cov}^2[\mathcal{B}_{(0)}(t),\mathcal{B}_{(1)}(s)]\,\mathrm{d}t\,\mathrm{d}s,\end{aligned}$$

where the last equality follows from Patel and Read [27], and the result follows from Equation (90). $\square$

A.4 Asymptotic Distribution of Folded Overlapping Area Estimator: Proof of Theorem 14

The proof requires several auxiliary definitions and results.

Definition 4 Let $D[0,b]$ denote the space of functions on $[0,b]$ that are right-continuous with left-hand limits. The bridging map $\Theta : (Z,s) \in D[0,b] \times [0,b-1] \to \Theta_{Z,s} \in D[0,1]$ is defined by

$$\Theta_{Z,s}(t) \equiv t[Z(s+1) - Z(s)] - [Z(s+t) - Z(s)], \quad \text{for } t \in [0,1].$$

For each $s \in [0,b-1]$, we have $\Theta_{\mathcal{W},s}(t) = \mathcal{B}_{(0),s}(t)$, for $t \in [0,1]$; thus $\Theta_{\mathcal{W},s}(\cdot)$ is a Brownian bridge starting at time s.

Definition 5 The folding operation $\Psi : Z \in D[0,1] \to \Psi_Z \in D[0,1]$ is defined by

$$\Psi_Z(t) \equiv Z(\tfrac{t}{2}) - Z(1 - \tfrac{t}{2}), \quad \text{for } t \in [0,1]. \tag{91}$$

Definition 6 Let

$$\boldsymbol{D_\Psi} \equiv \{x \in D[0,1] : \text{for some sequence } \{x_m\} \subset D[0,1] \text{ converging to } x \text{ as } m \to \infty, \text{ the sequence } \{\Psi_{x_m}\} \text{ does not converge to } \Psi_x\}.$$

Proposition 1 (Alexopoulos et al. [5]) If $\Psi_Z(\cdot)$ is defined by (91) with the event $\boldsymbol{D_\Psi}$, then

$$\Pr\{\mathcal{W} \in \boldsymbol{D_\Psi}\} = 0, \tag{92}$$

where $\Pr\{\cdot\}$ denotes the Wiener measure.

Proposition 1 shows that $\Psi(\cdot)$ is continuous almost surely with respect to the Wiener measure on $D[0,1]$. We have $T^{\circ}_{(k),i,m}(t) = \Psi_{T^{\circ}_{(k-1),i,m}}(t)$ for $t \in [0,1]$, $i = 1,\ldots,(b-1)m+1$ and $k = 1,2,\ldots$ from Equation (91). Hence, by Equation (92) and Assumptions A, the generalized CMT gives $\Psi_{T^{\circ}_{(k-1),i,m}}(\cdot) \xrightarrow[n\to\infty]{\mathcal{D}} \Psi_{\mathcal{B}_{(k-1),s}}(\cdot) = \mathcal{B}_{(k),s}(\cdot)$.

Definition 7 Suppose that the weight function $f(\cdot)$ satisfies Assumptions F. The folded overlapping area map $\Xi : Z \in D[0,b] \to \Xi(Z) \in \mathbb{R}$ is defined by

$$\Xi(Z) \equiv \frac{1}{b-1}\int_0^{b-1}\left[\sigma\int_0^1 f(u)\Psi_{\Theta_{Z,s}}(u)\right]^2 \mathrm{d}s. \tag{93}$$

For $m = 1,2,\ldots$, we define the approximate folded overlapping area map $\Xi_m : Z \in D[0,b] \to \Xi_m(Z) \in \mathbb{R}$ by

$$\Xi_m(Z) \equiv \frac{1}{(b-1)m+1}\sum_{i=1}^{(b-1)m+1}\left[\frac{1}{m}\sum_{j=1}^{m} f[u(j,m)]\sigma\Psi_{\Theta_{Z,s(i,m)}}[u(j,m)]\right]^2, \tag{94}$$

where for $m = 1,2,\ldots$, we take $s(i,m) \equiv (i-1)/m$, for $i = 1,\ldots,(b-1)m+1$ and $u(j,m) \equiv j/m$, for $j = 1,\ldots,m$.

For level-1 folding, from Equations (1), (91) and (94) and the definitions of $s(i,m)$ and $u(j,m)$, we have $\Psi_{\Theta_{Y_m,s(i,m)}}(u(j,m)) = \Psi_{T^{\circ}_{(0),i,m}}\left(\frac{j}{m}\right) = T^{\circ}_{(1),i,m}\left(\frac{j}{m}\right)$, for $m = 1,2,\ldots$, $i = 1,\ldots,(b-1)m+1$, and $j = 1,\ldots,m$. Hence,

$$\Xi_m(Y_m) = \frac{1}{(b-1)m+1}\sum_{i=1}^{(b-1)m+1}\left[\frac{1}{m}\sum_{j=1}^{m} f\left(\tfrac{j}{m}\right)\sigma T^{\circ}_{(1),i,m}\left(\tfrac{j}{m}\right)\right]^2 = A^{\circ}_{(1)}(f;b,m). \tag{95}$$

To prove the theorem, it remains to show that $\Xi(\cdot)$ is continuous almost surely with respect to the Wiener measure on $D[0,b]$. This follows from Proposition 2, which is preceded by Definitions 8 and 9.

Definition 8 Let Λ denote the class of strictly increasing, continuous mappings of $[0,b]$ onto itself such that for every $\lambda \in \Lambda_b$, we have $\lambda(0) = 0$ and $\lambda(b) = b$. If $X, Z \in D[0,b]$, then the Skorohod metric $\rho_b(X,Z)$ defining the "distance" between X and Z in $D[0,b]$ is the infimum of those positive ξ for which there exists a $\lambda \in \Lambda_b$ such that

$$\sup_{t\in[0,b]}|\lambda(t)-t| \le \xi \text{ and } \sup_{t\in[0,b]}|X(t)-Z[\lambda(t)]| \le \xi.$$

Definition 9 Let

$$\boldsymbol{D}_{\boldsymbol{\Xi}} \equiv \Big\{ x \in D[0,b] : \text{for some sequence } \{x_m\} \subset D[0,b] \text{ with } \lim_{m\to\infty} \rho_b(x_m, x) = 0, \text{ the sequence } \{\Xi_m(x_m)\} \text{ does not converge to } \Xi(x) \Big\}. \tag{96}$$

Proposition 2 (Alexopoulos et al. [4]) If the weight function $f(\cdot)$ satisfies Assumptions F and if $\Xi(\cdot)$ and $\Xi_m(\cdot)$ are defined by Equations (93) and (94), respectively, with the event $\boldsymbol{D}_{\boldsymbol{\Xi}}$ defined in (9), then

$$\Pr\{\mathcal{W} \in \boldsymbol{D}_{\boldsymbol{\Xi}}\} = 0. \tag{97}$$

Proof of Theorem 14: Combining Assumptions A, and Equations (92), (94), (95) and (97), we see that the result

$$\mathcal{A}^{\circ}_{(k)}(f;b,m) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{A}^{\circ}_{(k)}(f;b) = \frac{1}{b-1}\int_0^{b-1} \left[\sigma \int_0^1 f(u)\mathcal{B}_{(k),s}(u)\,\mathrm{d}u\right]^2$$

follows from the generalized CMT for $k = 1$. Using mathematical induction, one can show that the result holds for $k \geq 2$ as well. □

A.5 Expansion of Equation (52)*:*

We start with the covariance in the integrand of (53).

$$\begin{aligned}
&\mathrm{Cov}[\mathcal{B}_{(1),0}(u), \mathcal{B}_{(1),y}(v)] \\
&= \mathrm{Cov}[(u-1)(W(1)-W(0)) - (W(u/2)-W(1-u/2)), \\
&\quad (v-1)(W(y+1)-W(y)) - (W(y+v/2)-W(y+1-v/2))] \\
&= (u-1)(v-1)\mathrm{Cov}(W(1),W(y+1)) - (u-1)(v-1)\mathrm{Cov}(W(1),W(y)) \\
&\quad - (u-1)\mathrm{Cov}(W(1),W(y+v/2)) + (u-1)\mathrm{Cov}(W(1),W(y+1-v/2)) \\
&\quad - (v-1)\mathrm{Cov}(W(u/2),W(y+1)) + (v-1)\mathrm{Cov}(W(u/2),W(y)) \\
&\quad + \mathrm{Cov}(W(u/2),W(y+v/2)) - \mathrm{Cov}(W(u/2),W(y+1-v/2))
\end{aligned}$$

$$
\begin{aligned}
&+ (v-1)\mathrm{Cov}(W(y+1), W(1-u/2)) - (v-1)\mathrm{Cov}(W(y), W(1-u/2)) \\
&- \mathrm{Cov}(W(1-u/2), W(y+v/2)) \\
&+ \mathrm{Cov}(W(1-u/2), W(y+1-v/2)) \\
&= (u-1)(v-1) - (u-1)(v-1)y - (u-1)\min(1, y+v/2) \\
&+ (u-1)\min(1, y+1-v/2) - (v-1)u/2 + (v-1)\min(u/2, y) \\
&+ \min(u/2, y+v/2) - \min(u/2, y+1-v/2) + (v-1)(1-u/2) \\
&- (v-1)\min(y, 1-u/2) - \min(1-u/2, y+v/2) \\
&+ \min(1-u/2, y+1-v/2) \\
&= -uvy + uy + vy - y - (u-1)\min(1, y+v/2) \\
&+ (u-1)\min(1, y+1-v/2) + (v-1)\min(u/2, y) + \min(u/2, y+v/2) \\
&- \min(u/2, y+1-v/2) - (v-1)\min(y, 1-u/2) \\
&- \min(1-u/2, y+v/2) + \min(1-u/2, y+1-v/2), \qquad (98)
\end{aligned}
$$

for $u, v, y \in [0, 1]$.

We proceed with the derivation of each term in Equation (98). Since $u, v, y \in [0, 1]$, we have

$$
\begin{aligned}
\min(1, y+v/2) &= \begin{cases} y+v/2 & \text{if } v < 2-2y \\ 1 & \text{if } v \geq 2-2y \end{cases} \\
&= \begin{cases} y+v/2 & \text{if } 0 \leq 2y \leq 1 \leq 2-v \leq 2 \\ y+v/2 & \text{if } 1 \leq 2y \leq 2-v \leq 2 \\ 1 & \text{if } 1 \leq 2-v \leq 2y \leq 2, \end{cases} \\
\min(1, y+1-v/2) &= \begin{cases} 1 & \text{if } v \leq 2y \\ y+1-v/2 & \text{if } v > 2y \end{cases}
\end{aligned}
$$

$$= \begin{cases} 1 & \text{if } 0 \le v \le 1 \le 2y \le 2 \\ 1 & \text{if } 0 \le v \le 2y \le 1 \\ y + 1 - v/2 & \text{if } 0 \le 2y \le v \le 1, \end{cases}$$

$$\min(u/2, y) = \begin{cases} u/2 & \text{if } u < 2y \\ y & \text{if } u \ge 2y \end{cases}$$

$$= \begin{cases} u/2 & \text{if } 0 \le u \le 2y \le 1 \\ u/2 & \text{if } 0 \le u \le 1 \le 2y \le 2 \\ y & \text{if } 0 \le 2y \le u \le 1, \end{cases}$$

$$\min(u/2, y + v/2) = \begin{cases} u/2 & \text{if } u \le 2y + v \\ y + v/2 & \text{if } u > 2y + v \end{cases}$$

$$= \begin{cases} u/2 & \text{if } 0 \le u \le 2y + v \le 1 \\ u/2 & \text{if } 0 \le u \le 1 \le 2y + v \le 2 \\ u/2 & \text{if } 0 \le u \le 1 \le 2 \le 2y + v \le 3 \\ y + v/2 & \text{if } 0 \le 2y + v \le u \le 1, \end{cases}$$

$$\min(u/2, y + 1 - v/2) = \begin{cases} u/2 & \text{if } u \le 2y + 2 - v \\ y + 1 - v/2 & \text{if } u > 2y + 2 - v \end{cases}$$

$$= u/2 \quad \text{for all } u, v, y,$$

$$\min(y, 1 - u/2) = \begin{cases} y & \text{if } u \le 2 - 2y \\ 1 - u/2 & \text{if } u > 2 - 2y \end{cases}$$

$$= \begin{cases} y & \text{if } 0 \le 2y \le 1 \le 2-u \le 2 \\ y & \text{if } 1 \le 2y \le 2-u \le 2 \\ 1-u/2 & \text{if } 1 \le 2-u \le 2y \le 2, \end{cases}$$

$$\min(1-u/2, y+v/2) = \begin{cases} y+v/2 & \text{if } u \le 2-2y-v \\ 1-u/2 & \text{if } u > 2-2y-v \end{cases}$$

$$= \begin{cases} y+v/2 & \text{if } 0 \le u \le 2-2y-v \le 1 \\ y+v/2 & \text{if } 0 \le u \le 1 \le 2-2y-v \le 2 \\ 1-u/2 & \text{if } 0 \le 2-2y-v \le u \le 1 \\ 1-u/2 & \text{if } 2-2y-v \le 0 \le u \le 1, \end{cases}$$

$$\min(1-u/2, y+1-v/2) = \begin{cases} y+1-v/2 & \text{if } u \le v-2y \\ 1-u/2 & \text{if } u > v-2y \end{cases}$$

$$= \begin{cases} y+1-v/2 & \text{if } 0 \le u \le v-2y \le 1 \\ 1-u/2 & \text{if } 0 \le v-2y \le u \le 1 \\ 1-u/2 & \text{if } v-2y \le 0 \le u \le 1. \end{cases}$$

We can now write $q^2(0,y)$ as follows:

$$\begin{aligned} q^2(0,y) = \Bigg(\int_0^1 \int_0^1 f(u)f(v) \Big\{ & (-uvy + uy + vy - y) - (u-1)\min(1, y+v/2) \\ & + (u-1)\min(1, y+1-v/2) + (v-1)\min(u/2, y) + \min(u/2, y+v/2) \\ & - \min(u/2, y+1-v/2) - (v-1)\min(y, 1-u/2) - \min(1-u/2, y+v/2) \\ & + \min(1-u/2, y+1-v/2) \Big\} \, \mathrm{d}u \, \mathrm{d}v \Bigg)^2 \\ = \Bigg(& \int_0^1 \int_0^1 f(u)f(v)(-uvy + uy + vy - y) \, \mathrm{d}u \, \mathrm{d}v \\ & - \int_0^1 \int_0^1 f(u)f(v)(u-1)\min(1, y+v/2) \, \mathrm{d}u \, \mathrm{d}v \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^1 \int_0^1 f(u)f(v)(u-1)\min(1, y+1-v/2)\,\mathrm{d}u\,\mathrm{d}v \\
&+ \int_0^1 \int_0^1 f(u)f(v)(v-1)\min(u/2, y)\,\mathrm{d}u\,\mathrm{d}v \\
&+ \int_0^1 \int_0^1 f(u)f(v)\min(u/2, y+v/2)\,\mathrm{d}u\,\mathrm{d}v \\
&- \int_0^1 \int_0^1 f(u)f(v)\min(u/2, y+1-v/2)\,\mathrm{d}u\,\mathrm{d}v \\
&- \int_0^1 \int_0^1 f(u)f(v)(v-1)\min(y, 1-u/2)\,\mathrm{d}u\,\mathrm{d}v \\
&- \int_0^1 \int_0^1 f(u)f(v)\min(1-u/2, y+v/2)\,\mathrm{d}u\,\mathrm{d}v \\
&+ \int_0^1 \int_0^1 f(u)f(v)\min(1-u/2, y+1-v/2)\,\mathrm{d}u\,\mathrm{d}v\Bigg)^2. \qquad (99)
\end{aligned}
$$

We denote the double integrals in Equation (99) as $I_1, I_2, \ldots, I_9$, respectively. Then

$$
\begin{aligned}
q^2(0,y) &= (I_1 - I_2 + I_3 + I_4 + I_5 - I_6 - I_7 - I_8 + I_9)^2 \\
&= I_1^2 - 2I_1I_2 + 2I_1I_3 + 2I_1I_4 + 2I_1I_5 - 2I_1I_6 - 2I_1I_7 - 2I_1I_8 + 2I_1I_9 \\
&\quad + I_2^2 - 2I_2I_3 - 2I_2I_4 - 2I_2I_5 + 2I_2I_6 + 2I_2I_7 + 2I_2I_8 - 2I_2I_9 + I_3^2 \\
&\quad + 2I_3I_4 + 2I_3I_5 - 2I_3I_6 - 2I_3I_7 - 2I_3I_8 + 2I_3I_9 + I_4^2 + 2I_4I_5 \\
&\quad - 2I_4I_6 - 2I_4I_7 - 2I_4I_8 + 2I_4I_9 + I_5^2 - 2I_5I_6 - 2I_5I_7 - 2I_5I_8 + 2I_5I_9 \\
&\quad + I_6^2 + 2I_6I_7 + 2I_6I_8 - 2I_6I_9 + I_7^2 + 2I_7I_8 - 2I_7I_9 + I_8^2 - 2I_8I_9 + I_9^2.
\end{aligned}
$$

Substitution of this last expression into the integral of Equation (52) yields

$$
\begin{aligned}
&\int_0^1 (b-1-y)q^2(0,y)\,\mathrm{d}y \\
&= \int_0^1 (b-1-y)(I_1^2 - 2I_1I_2 + 2I_1I_3 + 2I_1I_4 + 2I_1I_5 - 2I_1I_6 - 2I_1I_7 - 2I_1I_8 \\
&\quad + 2I_1I_9 + I_2^2 - 2I_2I_3 - 2I_2I_4 - 2I_2I_5 + 2I_2I_6 + 2I_2I_7 + 2I_2I_8 - 2I_2I_9 \\
&\quad + I_3^2 + 2I_3I_4 + 2I_3I_5 - 2I_3I_6 - 2I_3I_7 - 2I_3I_8 + 2I_3I_9 + I_4^2 + 2I_4I_5 \\
&\quad - 2I_4I_6 - 2I_4I_7 - 2I_4I_8 + 2I_4I_9 + I_5^2 - 2I_5I_6 - 2I_5I_7 - 2I_5I_8 + 2I_5I_9 \\
&\quad + I_6^2 + 2I_6I_7 + 2I_6I_8 - 2I_6I_9 + I_7^2 + 2I_7I_8 - 2I_7I_9 + I_8^2 - 2I_8I_9 + I_9^2)\,\mathrm{d}y \\
&= \int_0^1 (b-1-y)\Big(\int_0^1 \int_0^1 f(u)f(v)(-uvy + uy + vy - y)\,\mathrm{d}u\,\mathrm{d}v\Big)^2\,\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&- 2\int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\int_0^1\int_0^1(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}y\\
&- 2\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\Big(\int_0^1\int_0^{2-2y}(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u+\int_0^1\int_{2-2y}^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\,\mathrm{d}y\\
&+ 2\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\int_0^1\int_0^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}y\\
&+ 2\int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\Big(\int_0^1\int_0^{2y}(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u+\int_0^1\int_{2y}^1(u-1)(y+1-v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\,\mathrm{d}y\\
&+ 2\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\int_0^1\int_0^1(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&+ 2\int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\Big(\int_0^1\int_0^{2y}(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_0^1\int_{2y}^1(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+ 2\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\Big(\int_{2-2y}^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_0^{2-2y}\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+ 2\int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\Big(\int_0^{1-2y}\int_0^{2y+v}(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_{1-2y}^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^{1-2y}\int_{2y+v}^1(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&- 2\int_0^1(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\int_0^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&- 2\int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\int_0^1\int_0^1(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&- 2\int_{1/2}^{1}(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\Big(\int_0^1\int_0^{2-2y}(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_0^1\int_{2-2y}^1(v-1)(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&- 2\int_{0}^{1/2}(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\Big(\int_0^{1-2y}\int_0^1(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_{1-2y}^1\int_0^{2-2y-v}(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\qquad+\int_{1-2y}^1\int_{2-2y-v}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&- 2\int_{1/2}^{1}(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\Big(\int_0^{2-2y}\int_0^{2-2y-v}(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_0^{2-2y}\int_{2-2y-v}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_{2-2y}^1\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+ 2\int_{0}^{1/2}(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\Big(\int_{2y}^1\int_0^{v-2y}(y+1-v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_{2y}^1\int_{v-2y}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^{2y}\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+ 2\int_{1/2}^{1}(b-1-y)\Big(\int_0^1\int_0^1 f(u)f(v)(-uvy+uy+vy-y)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad\times\int_0^1\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&+\int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^1(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)^2\\
&+\int_{1/2}^{1}(b-1-y)\Big(\int_0^1\int_0^{2-2y}(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad+\int_0^1\int_{2-2y}^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)^2\,\mathrm{d}y\\
&- 2\int_{0}^{1/2}(b-1-y)\Big(\int_0^1\int_0^1(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\\
&\quad\times\Big(\int_0^1\int_0^{2y}(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u+\int_0^1\int_{2y}^1(u-1)(y+1-v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&- 2\int_{1/2}^{1}(b-1-y)\Big(\int_0^1\int_0^{2-2y}(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad+\int_0^1\int_{2-2y}^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\times\int_0^1\int_0^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}y\\
&- 2\int_{0}^{1/2}(b-1-y)\Big(\int_0^1\int_0^1(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad\times\Big(\int_0^1\int_0^{2y}(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_0^1\int_{2y}^1(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&-2\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^{2-2y}(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad+\int_0^1\int_{2-2y}^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\int_0^1\int_0^1(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&-2\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^{2-2y}(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad+\int_0^1\int_{2-2y}^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\Big(\int_{2-2y}^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^{2-2y}\int_0^1u/2f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&-2\int_0^{1/2}(b-1-y)\int_0^1\int_0^1(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad\times\Big(\int_0^{1-2y}\int_0^{2y+v}(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_{1-2y}^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^{1-2y}\int_{2y+v}^1(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+2\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^{2-2y}(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad+\int_0^1\int_{2-2y}^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\times\int_0^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&+2\int_0^{1/2}(b-1-y)\int_0^1\int_0^1(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\int_0^1\int_0^1u/2f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&+2\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^{2-2y}(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad+\int_0^1\int_{2-2y}^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\Big(\int_0^1\int_0^{2-2y}(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^1\int_{2-2y}^1(v-1)(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+2\int_0^{1/2}(b-1-y)\int_0^1\int_0^1(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad\times\int_0^1\int_0^1(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&+2\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^{2-2y}(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad+\int_0^1\int_{2-2y}^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\Big(\int_0^{2-2y}\int_0^{2-2y-v}(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^{2-2y}\int_{2-2y-v}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_{2-2y}^1\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+2\int_0^{1/2}(b-1-y)\int_0^1\int_0^1(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
&\quad\times\Big(\int_0^{1-2y}\int_0^1 (y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_{1-2y}^1\int_0^{2-2y-v}(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_{1-2y}^1\int_{2-2y-v}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&-2\int_0^{1/2}(b-1-y)\int_0^1\int_0^1(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad\times\Big(\int_{2y}^1\int_0^{v-2y}(y+1-v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_{2y}^1\int_{v-2y}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^{2y}\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&-2\int_{1/2}^1(b-1-y)\int_0^1\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad\times\Big(\int_0^1\int_0^{2-2y}(u-1)(y+v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u+\int_0^1\int_{2-2y}^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\,\mathrm{d}y\\
&+\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)^2+\int_0^{1/2}(b-1-y)\\
&\quad\times\Big(\int_0^1\int_0^{2y}(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u+\int_0^1\int_{2y}^1(u-1)(y+1-v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)^2\,\mathrm{d}y\\
&+2\int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^{2y}(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad+\int_0^1\int_{2y}^1(u-1)(y+1-v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\\
&\quad\times\Big(\int_0^1\int_0^{2y}(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_0^1\int_{2y}^1(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+2\int_{1/2}^1(b-1-y)\int_0^1\int_0^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\times\int_0^1\int_0^1(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&+2\int_{1/2}^1(b-1-y)\int_0^1\int_0^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad\times\Big(\int_{2-2y}^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_0^{2-2y}\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+2\int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^{2y}(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad+\int_0^1\int_{2y}^1(u-1)(y+1-v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\Big(\int_0^{1-2y}\int_0^{2y+v}(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}\\
&\quad+\int_{1-2y}^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_0^{1-2y}\int_{2y+v}^1(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&-2\int_{1/2}^1(b-1-y)\int_0^1\int_0^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\int_0^1\int_0^1u/2f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}y\\
&-2\int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^{2y}(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad+\int_0^1\int_{2y}^1(u-1)(y+1-v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\Big(\int_0^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\Big)
\end{aligned}
$$

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\begin{aligned}
&- 2\int_0^{1/2}(b-1-y)\int_0^1\int_0^1(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big(\int_0^1\int_0^{2y}(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad+\int_0^1\int_{2y}^1(u-1)(y+1-v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\,\mathrm{d}y\\
&- 2\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^{2-2y}(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^1\int_{2-2y}^1(v-1)(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\int_0^1\int_0^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}y\\
&- 2\int_{1/2}^1(b-1-y)\int_0^1\int_0^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad\times\Big(\int_0^{2-2y}\int_0^{(}2-2y-v)y+v/2f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^{2-2y}\int_{2-2y-v}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_{2-2y}^1\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&- 2\int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^{2y}(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad+\int_0^1\int_{2y}^1(u-1)(y+1-v/2)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\Big(\int_0^{1-2y}\int_0^1(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_{1-2y}^1\int_0^{2-2y-v}(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_{1-2y}^1\int_{2-2y-v}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+ 2\int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^{2y}(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\\
&\quad+\int_0^1\int_{2y}^1(u-1)y+1-v/2f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\Big)\Big(\int_{2y}^1\int_0^{v-2y}(y+1-v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_{2y}^1\int_{v-2y}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_0^{2y}\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+ 2\int_{1/2}^1(b-1-y)\int_0^1\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\int_0^1\int_0^1(u-1)f(u)f(v)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}y\\
&+ \int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^1(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)^2\,\mathrm{d}y\\
&+ \int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^{2y}(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^1\int_{2y}^1(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)^2\,\mathrm{d}y\\
&+ 2\int_{1/2}^1(b-1-y)\int_0^1\int_0^1(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad\times\Big(\int_{2-2y}^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_0^{2-2y}\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+ 2\int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^{2y}(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^1\int_{2y}^1(v-1)y)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\Big(\int_0^{1-2y}\int_0^{2y+v}(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}
$$

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\begin{aligned}
&\quad + \int_{1-2y}^{1}\int_{0}^{1}(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{0}^{1-2y}\int_{2y+v}^{1}(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&- 2\int_{1/2}^{1}(b-1-y)\int_{0}^{1}\int_{0}^{1}(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad \times \int_{0}^{1}\int_{0}^{1}(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&- 2\int_{0}^{1/2}(b-1-y)\Big(\int_{0}^{1}\int_{0}^{2y}(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad + \int_{0}^{1}\int_{2y}^{1}(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\int_{0}^{1}\int_{0}^{1}(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&- 2\int_{0}^{1/2}(b-1-y)\int_{0}^{1}\int_{0}^{1}(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad \times \Big(\int_{0}^{1}\int_{0}^{2y}(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{0}^{1}\int_{2y}^{1}(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&- 2\int_{1/2}^{1}(b-1-y)\Big(\int_{0}^{1}\int_{0}^{2-2y}(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad + \int_{0}^{1}\int_{2-2y}^{1}(v-1)(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\int_{0}^{1}\int_{0}^{1}(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&- 2\int_{1/2}^{1}(b-1-y)\int_{0}^{1}\int_{0}^{1}(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad \times \Big(\int_{0}^{2-2y}\int_{0}^{2-2y-v}(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{0}^{2-2y}\int_{2-2y-v}^{1}(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad + \int_{2-2y}^{1}\int_{0}^{1}(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&- 2\int_{0}^{1/2}(b-1-y)\Big(\int_{0}^{1}\int_{0}^{2y}(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{0}^{1}\int_{2y}^{1}(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad \times \Big(\int_{0}^{1-2y}\int_{0}^{1}(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{1-2y}^{1}\int_{0}^{2-2y-v}(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad + \int_{1-2y}^{1}\int_{2-2y-v}^{1}(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+ 2\int_{1/2}^{1}(b-1-y)\int_{0}^{1}\int_{0}^{1}(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\int_{0}^{1}\int_{0}^{1}(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&+ 2\int_{0}^{1/2}(b-1-y)\Big(\int_{0}^{1}\int_{0}^{2y}(v-1)(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{0}^{1}\int_{2y}^{1}(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\\
&\quad \times \Big(\int_{2y}^{1}\int_{0}^{v-2y}(y-v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{2y}^{1}\int_{v-2y}^{1}(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad + \int_{0}^{2y}\int_{0}^{1}(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+ \int_{1/2}^{1}(b-1-y)\Big(\int_{2-2y}^{1}\int_{0}^{1}(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{0}^{2-2y}\int_{0}^{1}(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)^2\,\mathrm{d}y\\
&+ \int_{0}^{1/2}(b-1-y)\Big(\int_{0}^{1-2y}\int_{0}^{2y+v}(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{1-2y}^{1}\int_{0}^{1}(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}
$$

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\begin{aligned}
&\quad + \int_0^{1-2y} \int_{2y+v}^1 (y+v/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big)^2 \,\mathrm{d}y \\
&- 2 \int_{1/2}^1 (b-1-y) \Big(\int_{2-2y}^1 \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v + \int_0^{2-2y} \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big) \\
&\quad \times \int_0^1 \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}y \\
&- 2 \int_0^{1/2} (b-1-y) \Big(\int_0^{1-2y} \int_0^{2y+v} (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v + \int_{1-2y}^1 \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \\
&\quad + \int_0^{1-2y} \int_{2y+v}^1 (y+v/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big) \int_0^1 \int_0^1 u/2 f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}y \\
&- 2 \int_{1/2}^1 (b-1-y) \Big(\int_{2-2y}^1 \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v + \int_0^{2-2y} \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big) \\
&\quad \times \Big(\int_0^1 \int_0^{2-2y} (v-1) y f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v + \int_0^1 \int_{2-2y}^1 (v-1)(1-u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big) \,\mathrm{d}y \\
&- 2 \int_0^{1/2} (b-1-y) \Big(\int_0^{1-2y} \int_0^{2y+v} (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v + \int_{1-2y}^1 \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \\
&\quad + \int_0^{1-2y} \int_{2y+v}^1 (y+v/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big) \int_0^1 \int_0^1 (v-1) y f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}y \\
&- 2 \int_{1/2}^1 (b-1-y) \Big(\int_{2-2y}^1 \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v + \int_0^{2-2y} \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big) \\
&\quad \times \Big(\int_0^{2-2y} \int_0^{2-2y-v} (y+v/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v + \int_0^{2-2y} \int_{2-2y-v}^1 (1-u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \\
&\quad + \int_{2-2y}^1 \int_0^1 (1-u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big) \,\mathrm{d}y \\
&- 2 \int_0^{1/2} (b-1-y) \Big(\int_0^{1-2y} \int_0^{2y+v} (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v + \int_{1-2y}^1 \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \\
&\quad + \int_0^{1-2y} \int_{2y+v}^1 (y+v/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big) \Big(\int_0^{1-2y} \int_0^1 (y+v/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \\
&\quad + \int_{1-2y}^1 \int_0^{2-2y-v} (y+v/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v + \int_{1-2y}^1 \int_{2-2y-v}^1 (1-u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big) \,\mathrm{d}y \\
&+ 2 \int_{1/2}^1 (b-1-y) \Big(\int_{2-2y}^1 \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v + \int_0^{2-2y} \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big) \\
&\quad \times \int_0^1 \int_0^1 (1-u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}y \\
&+ 2 \int_0^{1/2} (b-1-y) \Big(\int_0^{1-2y} \int_0^{2y+v} (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v + \int_{1-2y}^1 \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \\
&\quad + \int_0^{1-2y} \int_{2y+v}^1 (y+v/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big) \Big(\int_{2y}^1 \int_0^{v-2y} (y+1-v/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \\
&\quad + \int_{2y}^1 \int_{v-2y}^1 (1-u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v + \int_0^{2y} \int_0^1 (1-u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big) \,\mathrm{d}y \\
&+ \int_0^1 (b-1-y) \Big(\int_0^1 \int_0^1 (u/2) f(u) f(v) \,\mathrm{d}u \,\mathrm{d}v \Big)^2 \,\mathrm{d}y
\end{aligned}
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\begin{aligned}
&+ 2\int_0^{1/2}(b-1-y)\int_0^1\int_0^1(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\int_0^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&+ 2\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^{2-2y}(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^1\int_{2-2y}^1(v-1)(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\int_0^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&+ 2\int_0^{1/2}(b-1-y)\int_0^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big(\int_0^{1-2y}\int_0^1(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_{1-2y}^1\int_0^{2-2y-v}(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_{1-2y}^1\int_{2-2y-v}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+ 2\int_{1/2}^1(b-1-y)\int_0^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big(\int_0^{2-2y}\int_0^{2-2y-v}(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^{2-2y}\int_{2-2y-v}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_{2-2y}^1\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&- 2\int_0^{1/2}(b-1-y)\int_0^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big(\int_{2y}^1\int_0^{v-2y}(y+1-v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_{2y}^1\int_{v-2y}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_0^{2y}\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&- 2\int_{1/2}^1(b-1-y)\int_0^1\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\int_0^1\int_0^1(u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\\
&+\int_0^{1/2}(b-1-y)\Big(\int_0^1\int_0^1(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)^2\,\mathrm{d}y\\
&+\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^{2-2y}(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^1\int_{2-2y}^1(v-1)(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)^2\,\mathrm{d}y\\
&+ 2\int_0^{1/2}(b-1-y)\int_0^1\int_0^1(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big(\int_0^{1-2y}\int_0^1(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_{1-2y}^1\int_0^{2-2y-v}(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_{1-2y}^1\int_{2-2y-v}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&+ 2\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^{2-2y}(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^1\int_{2-2y}^1(v-1)(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\Big(\int_0^{2-2y}\int_0^{2-2y-v}(y+v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_0^{2-2y}\int_{2-2y-v}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_{2-2y}^1\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&- 2\int_0^{1/2}(b-1-y)\int_0^1\int_0^1(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big(\int_{2y}^1\int_0^{v-2y}(y+1-v/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad+\int_{2y}^1\int_{v-2y}^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v+\int_0^{2y}\int_0^1(1-u/2)f(u)f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&- 2\int_{1/2}^1(b-1-y)\Big(\int_0^1\int_0^{2-2y}(v-1)yf(u)f(v)\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \int_0^1 \int_{2-2y}^1 (v-1)(1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v\Big) \int_0^1 \int_0^1 (1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y \\
&+ \int_0^{1/2} (b-1-y)\Big(\int_0^{1-2y} \int_0^1 (y+v/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v \\
&\quad + \int_{1-2y}^1 \int_0^{2-2y-v} (y+v/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{1-2y}^1 \int_{2-2y-v}^1 (1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)^2\,\mathrm{d}y \\
&+ \int_{1/2}^1 (b-1-y)\Big(\int_0^{2-2y} \int_0^{2-2y-v} (y+v/2) y f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v \\
&\quad + \int_0^{2-2y} \int_{2-2y-v}^1 (1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v \int_{2-2y}^1 \int_0^1 (1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)^2\,\mathrm{d}y \\
&- 2\int_0^{1/2} (b-1-y)\Big(\int_0^{1-2y} \int_0^1 (y+v/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v \\
&\quad + \int_{1-2y}^1 \int_0^{2-2y-v} (y+v/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{1-2y}^1 \int_{2-2y-v}^1 (1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v\Big) \\
&\quad \times \Big(\int_{2y}^1 \int_0^{v-2y} (y+1-v/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{2y}^1 \int_{v-2y}^1 (1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v \\
&\quad + \int_0^{2y} \int_0^1 (1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)\,\mathrm{d}y \\
&- 2\int_{1/2}^1 (b-1-y)\Big(\int_0^{2-2y} \int_0^{2-2y-v} (y+v/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v \\
&\quad + \int_0^{2-2y} \int_{2-2y-v}^1 (1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{2-2y}^1 \int_0^1 (1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v\Big) \\
&\quad \times \int_0^1 \int_0^1 (1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y \\
&+ \int_0^{1/2} (b-1-y)\Big(\int_{2y}^1 \int_0^{v-2y} (y+1-v/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v \\
&\quad + \int_{2y}^1 \int_{v-2y}^1 (1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v + \int_0^{2y} \int_0^1 (1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)^2\,\mathrm{d}y \\
&+ \int_{1/2}^1 (b-1-y)\Big(\int_0^1 \int_0^1 (1-u/2) f(u) f(v)\,\mathrm{d}u\,\mathrm{d}v\Big)^2\,\mathrm{d}y.
\end{aligned}
$$

If we substitute this expression in Equation (52), after some algebra we can obtain the detailed expression for the variance result of level-1 FOA estimators.

A.6 Expected Value for Reflected Area Estimators: Proof of Theorem 19

We have

$$
\mathrm{E}[\mathcal{A}_c^\star(n)] = \mathrm{Var}\left[\frac{1}{n}\sum_{j=1}^n f\Big(\tfrac{j}{n}\Big)\sigma T_{c,n}^\star(\tfrac{j}{n})\right]
$$

$$= \frac{\sigma^2}{n^2} \sum_{j=1}^{n} \sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \mathrm{Cov}[T^{\star}_{c,n}(\tfrac{j}{n}), T^{\star}_{c,n}(\tfrac{\ell}{n})]$$

$$= \frac{1}{n^3} \sum_{j=1}^{n} \sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \Bigg[\sum_{t=1}^{j} \sum_{a=1}^{\ell} \mathrm{Cov}(X^{\star}_{c,t}, X^{\star}_{c,a}) - \frac{\ell}{n} \sum_{t=1}^{j} \sum_{a=1}^{n} \mathrm{Cov}(X^{\star}_{c,t}, X^{\star}_{c,a})$$

$$- \frac{j}{n} \sum_{t=1}^{\ell} \sum_{a=1}^{n} \mathrm{Cov}(X^{\star}_{c,t}, X^{\star}_{c,a}) + \frac{j\ell}{n^2} \sum_{t=1}^{n} \sum_{a=1}^{n} \mathrm{Cov}(X^{\star}_{c,t}, X^{\star}_{c,a}) \Bigg]. \tag{100}$$

We proceed with the analysis of each of the four summation terms. The first term is

$$\sum_{j=1}^{n} \sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \sum_{t=1}^{j} \sum_{a=1}^{\ell} \mathrm{Cov}(X^{\star}_{c,t}, X^{\star}_{c,a})$$

$$= \sum_{j=1}^{cn} \sum_{\ell=1}^{cn} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \sum_{t=1}^{j} \sum_{a=1}^{\ell} \mathrm{Cov}(X_t, X_a)$$

$$+ \sum_{j=1}^{cn} \sum_{\ell=cn+1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \sum_{t=1}^{j} \sum_{a=1}^{cn} \mathrm{Cov}(X_t, X_a)$$

$$- \sum_{j=1}^{cn} \sum_{\ell=cn+1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \sum_{t=1}^{j} \sum_{a=cn+1}^{\ell} \mathrm{Cov}(X_t, X_a)$$

$$+ \sum_{j=cn+1}^{n} \sum_{\ell=1}^{cn} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \sum_{t=1}^{cn} \sum_{a=1}^{\ell} \mathrm{Cov}(X_t, X_a)$$

$$- \sum_{j=cn+1}^{n} \sum_{\ell=1}^{cn} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \sum_{t=cn+1}^{j} \sum_{a=1}^{\ell} \mathrm{Cov}(X_t, X_a)$$

$$+ \sum_{j=cn+1}^{n} \sum_{\ell=cn+1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \sum_{t=1}^{cn} \sum_{a=1}^{cn} \mathrm{Cov}(X_t, X_a)$$

$$- \sum_{j=cn+1}^{n} \sum_{\ell=cn+1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \sum_{t=1}^{cn} \sum_{a=cn+1}^{\ell} \mathrm{Cov}(X_t, X_a)$$

$$- \sum_{j=cn+1}^{n} \sum_{\ell=cn+1}^{cn} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \sum_{t=cn+1}^{j} \sum_{a=1}^{cn} \mathrm{Cov}(X_t, X_a)$$

$$+ \sum_{j=cn+1}^{n} \sum_{\ell=cn+1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \sum_{t=cn+1}^{j} \sum_{a=cn+1}^{\ell} \mathrm{Cov}(X_t, X_a). \tag{101}$$

Similarly, the second covariance term in Equation (100) can be written as

$$\sum_{j=1}^{n} \sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \tfrac{\ell}{n} \sum_{t=1}^{j} \sum_{a=1}^{n} \mathrm{Cov}(X^{\star}_{c,t}, X^{\star}_{c,a})$$

$$= \sum_{j=1}^{cn} \sum_{\ell=1}^{cn} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \tfrac{\ell}{n} \sum_{t=1}^{j} \sum_{a=1}^{n} \mathrm{Cov}(X^{\star}_{c,t}, X^{\star}_{c,a})$$

$$+ \sum_{j=cn+1}^{n} \sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \tfrac{\ell}{n} \sum_{t=1}^{j} \sum_{a=1}^{n} \mathrm{Cov}(X^{\star}_{c,t}, X^{\star}_{c,a})$$

$$
\begin{aligned}
&= \sum_{j=1}^{cn}\sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \tfrac{\ell}{n} \sum_{t=1}^{j}\sum_{a=1}^{cn} \mathrm{Cov}(X_t, X_a) \\
&\quad - \sum_{j=1}^{cn}\sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \tfrac{\ell}{n} \sum_{t=1}^{j}\sum_{a=cn+1}^{n} \mathrm{Cov}(X_t, X_a) \\
&\quad + \sum_{j=cn+1}^{n}\sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \tfrac{\ell}{n} \sum_{t=1}^{cn}\sum_{a=1}^{cn} \mathrm{Cov}(X_t, X_a) \\
&\quad - \sum_{j=cn+1}^{n}\sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \tfrac{\ell}{n} \sum_{t=1}^{cn}\sum_{a=cn+1}^{n} \mathrm{Cov}(X_t, X_a) \\
&\quad - \sum_{j=cn+1}^{n}\sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \tfrac{\ell}{n} \sum_{t=cn+1}^{j}\sum_{a=1}^{cn} \mathrm{Cov}(X_t, X_a) \\
&\quad + \sum_{j=cn+1}^{n}\sum_{\ell=1}^{n} f\Big(\tfrac{j}{n}\Big) f\Big(\tfrac{\ell}{n}\Big) \tfrac{\ell}{n} \sum_{t=cn+1}^{j}\sum_{a=cn+1}^{n} \mathrm{Cov}(X_t, X_a). \qquad (102)
\end{aligned}
$$

Similar calculations can be carried out for the third and forth summation terms in Equation (100). It can be seen that Equations (101) and (102) contain similar covariance terms. We first analyze these covariance terms for $c \in (0,1)$. We have

$$
\begin{aligned}
&\sum_{t=1}^{j}\sum_{a=1}^{\ell} \mathrm{Cov}(X_t, X_a) = \\
&= \begin{cases}
\sum_{t=1}^{j}\Big(\sum_{a=1}^{j} R_{|t-a|} + \sum_{a=j+1}^{\ell} R_{|a-t|}\Big) & \\
\quad = \sum_{t=1}^{j}\Big(\sum_{a=1}^{t-1} R_a + R_0 + \sum_{a=1}^{\ell-t} R_a\Big) & \\
\quad = \sum_{t=1}^{j-1}(j-t)R_t + jR_0 + j\sum_{t=1}^{\ell-j} R_t + \sum_{t=\ell-j+1}^{\ell-1}(\ell-t)R_t & \text{for } j \le \ell \\
\sum_{t=1}^{\ell-1}(\ell-t)R_t + \ell R_0 + \ell\sum_{t=1}^{j-\ell} R_t + \sum_{t=j-\ell+1}^{j-1}(j-t)R_t & \text{for } j > \ell.
\end{cases} \qquad (103)
\end{aligned}
$$

For $j \le cn$,

$$
\sum_{t=1}^{j}\sum_{a=1}^{cn} \mathrm{Cov}(X_t, X_a) = \sum_{t=1}^{j-1}(j-t)R_t + jR_0 + j\sum_{t=1}^{cn-j} R_t + \sum_{t=cn-j+1}^{cn-1}(cn-t)R_t. \qquad (104)
$$

For $j \le cn < \ell$,

$$
\begin{aligned}
\sum_{t=1}^{j}\sum_{a=cn+1}^{\ell} \mathrm{Cov}(X_t, X_a) &= \sum_{t=1}^{j}\sum_{a=1}^{\ell} \mathrm{Cov}(X_t, X_a) - \sum_{t=1}^{j}\sum_{a=1}^{cn} \mathrm{Cov}(X_a, X_t) \\
&= j\Big(\sum_{t=1}^{\ell-j} R_t - \sum_{t=1}^{cn-j} R_t\Big) + \sum_{t=\ell-j+1}^{\ell-1}(\ell-t)R_t - \sum_{t=cn-j+1}^{cn-1}(cn-t)R_t. \qquad (105)
\end{aligned}
$$

For $\ell \le cn$,

$$\sum_{t=1}^{cn}\sum_{a=1}^{\ell}\mathrm{Cov}(X_t,X_a)=\sum_{t=1}^{\ell-1}(\ell-t)R_t+\ell R_0+\ell\sum_{t=1}^{cn-\ell}R_t+\sum_{t=cn-\ell+1}^{cn-1}(cn-t)R_t \tag{106}$$

For $\ell \le cn < j$,

$$\sum_{t=cn+1}^{j}\sum_{a=1}^{\ell}\mathrm{Cov}(X_t,X_a)=\ell\Big(\sum_{t=1}^{j-\ell}R_t-\sum_{t=1}^{cn-\ell}R_t\Big)+\sum_{t=j-\ell+1}^{j-1}(j-t)R_t-\sum_{t=cn-\ell+1}^{cn-1}(cn-t)R_t. \tag{107}$$

$$\sum_{t=1}^{cn}\sum_{a=1}^{cn}\mathrm{Cov}(X_t,X_a)=2\sum_{t=1}^{cn-1}(cn-t)R_t+cnR_0. \tag{108}$$

For $\ell > cn$,

$$\sum_{t=1}^{cn}\sum_{a=cn+1}^{\ell}\mathrm{Cov}(X_t,X_a)=cn\sum_{t=1}^{\ell-cn}R_t+\sum_{t=\ell-cn+1}^{\ell-1}(\ell-t)R_t-\sum_{t=1}^{cn-1}(cn-t)R_t. \tag{109}$$

For $j > cn$,

$$\sum_{t=cn+1}^{j}\sum_{a=1}^{cn}\mathrm{Cov}(X_t,X_a)=cn\sum_{t=1}^{j-cn}R_t+\sum_{t=j-cn+1}^{j-1}(j-t)R_t-\sum_{t=1}^{cn-1}(cn-t)R_t. \tag{110}$$

For $j, \ell > cn$,

$$\begin{aligned}
&\sum_{t=cn+1}^{j}\sum_{a=cn+1}^{\ell}\mathrm{Cov}(X_t,X_a)\\
&=\Big(\sum_{t=1}^{j}\sum_{a=cn+1}^{\ell}-\sum_{t=1}^{cn}\sum_{a=cn+1}^{\ell}\Big)\mathrm{Cov}(X_t,X_a)\\
&=\Big(\sum_{t=1}^{j}\sum_{a=1}^{\ell}-\sum_{t=1}^{j}\sum_{a=1}^{cn}-\sum_{t=1}^{cn}\sum_{a=1}^{\ell}+\sum_{t=1}^{cn}\sum_{a=1}^{cn}\Big)\mathrm{Cov}(X_t,X_a)\\
&=\begin{cases}
\sum_{t=1}^{j-1}(j-t)R_t+(j-cn)R_0+j\sum_{t=1}^{\ell-j}R_t-cn\sum_{t=1}^{j-cn}R_t-cn\sum_{t=1}^{\ell-cn}R_t\\
\quad+\sum_{t=\ell-j+1}^{\ell-1}(\ell-t)R_t-\sum_{t=j-cn+1}^{j-1}(j-t)R_t-\sum_{t=\ell-cn+1}^{\ell-1}(\ell-t)R_t,\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for } j\le\ell\\
\sum_{t=1}^{\ell-1}(\ell-t)R_t+(\ell-cn)R_0+\ell\sum_{t=1}^{j-\ell}R_t-cn\sum_{t=1}^{j-cn}R_t-cn\sum_{t=1}^{\ell-cn}R_t\\
\quad+\sum_{t=j-\ell+1}^{j-1}(j-t)R_t-\sum_{t=j-cn+1}^{j-1}(j-t)R_t-\sum_{t=\ell-cn+1}^{\ell-1}(\ell-t)R_t,\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for } j>\ell.
\end{cases}
\end{aligned} \tag{111}$$

Substitution of Equations (103)–(111) in Equations (101), (102) and the remaining terms of Equation (100) and some algebra yield the desired expected value. The cases $c = 0$ and $c = 1$ are covered by the results in Examples 1 and 2. $\square$

A.7 Expected Value for Reflected CvM Estimators: Proof of Theorem 20

We have

$$
\begin{aligned}
\mathrm{E}[\mathcal{C}_c^\star(g;n)] &= \frac{1}{n}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\sigma^2\mathrm{E}[\{T_{c,n}^\star(\tfrac{j}{n})\}^2] \\
&= \frac{1}{n}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\sigma^2\mathrm{E}\left[\left(\frac{j(\bar{X}_{c,j}^\star - \bar{X}_{c,n}^\star)}{\sigma\sqrt{n}}\right)^2\right] \\
&= \frac{1}{n^2}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\mathrm{E}\left[\left(\sum_{\ell=1}^{j} X_{c,\ell}^\star\right)^2 + \frac{j^2}{n^2}\left(\sum_{\ell=1}^{n} X_{c,\ell}^\star\right)^2 - \frac{2j}{n}\sum_{\ell=1}^{j} X_{c,\ell}^\star \sum_{t=1}^{n} X_{c,t}^\star\right] \\
&= \frac{1}{n^2}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)\mathrm{E}\left[\left(\sum_{\ell=1}^{j} X_{c,\ell}^\star\right)^2\right] + \frac{1}{n^4}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)j^2\mathrm{E}\left[\left(\sum_{\ell=1}^{n} X_{c,\ell}^\star\right)^2\right] \\
&\quad - \frac{2}{n^3}\sum_{j=1}^{n} j g\Big(\tfrac{j}{n}\Big)\mathrm{E}\left[\sum_{\ell=1}^{j} X_{c,\ell}^\star \sum_{t=1}^{n} X_{c,t}^\star\right] \\
&= \frac{1}{n^2}\left\{\sum_{j=1}^{cn} g\Big(\tfrac{j}{n}\Big)\mathrm{E}\left[\left(\sum_{\ell=1}^{j} X_\ell\right)^2\right] + \sum_{j=cn+1}^{n} g\Big(\tfrac{j}{n}\Big)\mathrm{E}\left[\left(\sum_{\ell=1}^{cn} X_\ell - \sum_{\ell=cn+1}^{j} X_\ell\right)^2\right]\right\} \\
&\quad + \frac{1}{n^4}\sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big)j^2\mathrm{E}\left[\left(\sum_{\ell=1}^{cn} X_\ell - \sum_{\ell=cn+1}^{n} X_\ell\right)^2\right] \\
&\quad - \frac{2}{n^3}\sum_{j=1}^{cn} j g\Big(\tfrac{j}{n}\Big)\mathrm{E}\left[\sum_{\ell=1}^{j}\sum_{t=1}^{cn} X_t - \sum_{\ell=1}^{j} X_\ell \sum_{t=cn+1}^{n} X_t\right] \\
&\quad - \frac{2}{n^3}\sum_{j=cn+1}^{n} j g\Big(\tfrac{j}{n}\Big)\mathrm{E}\left[\sum_{\ell=1}^{cn} X_t \sum_{t=1}^{cn} X_t - \sum_{\ell=1}^{cn} X_\ell \sum_{t=cn+1}^{n} X_t\right. \\
&\quad \left. - \sum_{\ell=cn+1}^{j} X_\ell \sum_{t=1}^{cn} X_t + \sum_{\ell=cn+1}^{j} X_\ell \sum_{t=cn+1}^{n} X_t\right] \\
&= \frac{1}{n^2}\sum_{j=1}^{cn} g\Big(\tfrac{j}{n}\Big)\sum_{\ell=1}^{t}\sum_{t=1}^{j}\mathrm{Cov}(X_\ell, X_t)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{n^2} \sum_{j=cn+1}^{n} g\Big(\tfrac{j}{n}\Big) \Big[\sum_{\ell=1}^{cn} \sum_{t=1}^{cn} \mathrm{Cov}(X_\ell, X_t) - 2 \sum_{\ell=1}^{cn} \sum_{t=cn+1}^{j} \mathrm{Cov}(X_\ell, X_t) \\
&\quad + \sum_{\ell=cn+1}^{j} \sum_{t=cn+1}^{j} \mathrm{Cov}(X_\ell, X_t) \Big] \\
&+ \frac{1}{n^4} \sum_{j=1}^{n} g\Big(\tfrac{j}{n}\Big) j^2 \Big[\sum_{\ell=1}^{cn} \sum_{t=1}^{cn} \mathrm{Cov}(X_\ell, X_t) - 2 \sum_{\ell=1}^{cn} \sum_{t=cn+1}^{n} \mathrm{Cov}(X_\ell, X_t) \\
&\quad + \sum_{\ell=cn+1}^{n} \sum_{t=cn+1}^{n} \mathrm{Cov}(X_\ell, X_t) \Big] \\
&- \frac{2}{n^3} \sum_{j=cn+1}^{n} g\Big(\tfrac{j}{n}\Big) j \Big[\sum_{\ell=1}^{j} \sum_{t=1}^{cn} \mathrm{Cov}(X_\ell, X_t) - \sum_{\ell=1}^{j} \sum_{t=cn+1}^{n} \mathrm{Cov}(X_\ell, X_t) \Big] \\
&- \frac{2}{n^3} \sum_{j=cn+1}^{n} g\Big(\tfrac{j}{n}\Big) j \Big[\sum_{\ell=1}^{cn} \sum_{t=1}^{cn} \mathrm{Cov}(X_\ell, X_t) - \sum_{\ell=1}^{cn} \sum_{t=cn+1}^{n} \mathrm{Cov}(X_\ell, X_t) \\
&\quad - \sum_{\ell=cn+1}^{j} \sum_{t=1}^{cn} \mathrm{Cov}(X_\ell, X_t) + \sum_{\ell=cn+1}^{j} \sum_{t=cn+1}^{n} \mathrm{Cov}(X_\ell, X_t) \Big] \qquad (112)
\end{aligned}
$$

Substitution of Equations (103)–(111) into Equation (112) and some additional algebra complete the proof for $c \in (0, 1)$. The cases $c = 0$ and $c = 1$ are covered by Examples 4 and 5. $\square$

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