TWO PROBLEMS IN MATHEMATICAL PHYSICS: VILLANI'S CONJECTURE AND TRACE INEQUALITY FOR THE FRACTIONAL LAPLACIAN

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TWO PROBLEMS IN MATHEMATICAL PHYSICS: VILLANI'S CONJECTURE AND TRACE INEQUALITY FOR THE FRACTIONAL LAPLACIAN

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To my wife,

The love of my life

To my family,

near and far

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'No man is an island, entire of itself' wrote John Donne in 1624, referring to the interconnectedness of humanity. This known phrase strikes a cord with me as I think about the path leading to this dissertation, a path I could have never walked alone and unaided. I'd like to take this opportunity and thank those who made this possible.

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Contents

SUMMARY

This dissertation consists of two problems in the field of Mathematical Physics.

The first part of our dissertation deals with a celebrated conjecture by Villani (See [22]). Taking ideas that were presented in [4] one step forward we manage to give an upper bound to the entropy production, showing that Villani's conjecture is true for all practical purposes.

The second part of our dissertation deals with developing a new trace inequality for the fractional Laplacian. We show that the new inequality is sharp and continue to give a complete characterization for the functions who minimize it, along with the space where it is most natural.

Chapter I

INTRODUCTION

The Journal of Mathematical Physics defines 'Mathematical Physics' as 'The application of mathematics to problems in physics and the development of mathematical methods suitable for such applications and for the formulation of physical theories'. I find that this definition is a good phrasing of my own views. Mathematics has gone a long way since the 17th century and the scientific revolution, and while at our stage of knowledge and specialty Mathematics is a world of its own I still find that my mathematical intuition and understanding rely heavily on my ability to relate the problem to some physical situation.

This dissertation deals with two different problems in the field of Mathematical Physics. As such, it consists of two main chapters, each dedicated to one problem, and an Appendix for additional proofs.

The second chapter is dedicated to an almost solution of Villani's conjecture, a known conjecture related to a Statistical Mechanics model invented by Kac ([16]) in 1956, dealing with equilibrium of a system with large amount of particles. In 2003 Villani conjectured that the time it will take the system to equilibrate is proportional to the number of particles in the system. Our main result of the chapter is a proof of that conjecture, up to an ϵ, showing that for all practical purposes we can consider it to be true. This result have been published in the Kinetic and Related Models Journal (See [8]).

The third chapter is dedicated to a newly developed trace inequality connecting between the fractional Laplacian of a function and its restriction to the intersection of the hyperplanes $x_n = 0, \dots, x_{n-j+1} = 0$, where $1 \leq j < n$. In this chapter not only will we manage to prove the inequality, but also show that it is sharp, and classify completely all the functions that attain equality. The results in this chapter are the product of a joint work with Prof. Michael Loss and will be published in the Proceedings of the American Mathematical Society Journal (See [9]).

The structure of the dissertation will be as followed:

Chapter 2 is divided into seven sections. Sections 2.1 to 2.3 are devoted to background material, motivation, and a small summary of known results including Villani's conjecture. Section 2.4 describes the properties of an important function that will be used thoroughly throughout the chapter. Section 2.5 is the main theoretical section of this chapter, consisting of a central limit theorem that will allow us to get an asymptotic approximation which will play a key role in our proof. Section 2.6 is the main computational section of the chapter. Following ideas presented in Section 2.3 and results from Sections 2.4 and 2.5 we will present an proof to Villani's conjecture, up to an ϵ. The last section of the chapter, Section 2.7, is dedicated to a few last remarks about the material presented in the chapter.

Chapter 3 is divided into eight sections. Sections 3.1 and 3.2 set the background tone and motivation for our investigation of the new trace inequality. Section 3.3 consists of our main inequality, and an initial investigation of it. In Section 3.4 we will extend the class of functions we're allowed to use in the inequality, and classify the functions that will attain equality. Section 3.5 will introduce another trace inequality, that while similar in nature to our main inequality, still posses some interesting features. Section 3.6 will discuss an important boundary case and Section 3.7 will contain a few last remarks on the material presented in the chapter.

Without further ado, let us begin!

Chapter II

VILLANI'S CONJECTURE AND KAC'S MODEL

2.1 The Boltzmann Equation and Kac's Model

One of the most important equations in non-equilibrium statistical mechanics is the Boltzmann equation, describing the time evolution of the density function $f(\overrightarrow{x}, \overrightarrow{v}, t)$, where $f(\overrightarrow{x}, \overrightarrow{v}, t)$ is defined as the number of particles in an infinitesimal rectangle of volume $d\overrightarrow{x}d\overrightarrow{v}$ about $(\overrightarrow{x}, \overrightarrow{v})$ at time t, where $\overrightarrow{x}$ and $\overrightarrow{v}$ represent position and velocity respectfully. The time evolution of the density function is given by

$$\frac{\partial f}{\partial t}(\overrightarrow{x}, \overrightarrow{v}, t) + \overrightarrow{v} \circ \nabla_{\overrightarrow{x}} f(\overrightarrow{x}, \overrightarrow{v}, t) + \frac{\overrightarrow{F}(\overrightarrow{x}, \overrightarrow{v}, t)}{m} \circ \nabla_{\overrightarrow{v}} f(\overrightarrow{x}, \overrightarrow{v}, t) = \frac{df}{dt}|_{collision}(\overrightarrow{x}, \overrightarrow{v}, t)$$

where $\overrightarrow{F}(\overrightarrow{x}, \overrightarrow{v}, t)$ is the external force acting on the system of particles and m is the mass of the particles. This follows from the fact that at time $t + dt$ the position and velocity of the particles is given by $\overrightarrow{x} + \overrightarrow{v}dt$ and $\overrightarrow{v} + \frac{\overrightarrow{F}}{m}dt$ respectfully. The real problem is specifying what $\frac{df}{dt}|_{collision}(\overrightarrow{x}, \overrightarrow{v}, t)$ is. Boltzmann determined the collision term resulting solely from collisions of two particles that are assumed to be uncorrelated prior to the collision ('Stosszahlanastz' as coined by Boltzmann, also known as the 'molecular chaos assumption'). The effect of the collisions is expressed in terms of a function $\sigma(\Omega, |\overrightarrow{v_1} - \overrightarrow{v_2}|)$ representing the differential scattering cross section, describing the probability for the change of velocities $(\overrightarrow{v_1}, \overrightarrow{v_2}) \to (\overrightarrow{v_1'}, \overrightarrow{v_2'})$, where Ω denoted the relative orientation of the vectors $(\overrightarrow{v_2'} - \overrightarrow{v_1'})$ and $(\overrightarrow{v_2} - \overrightarrow{v_1})$. The collision term is given by

$$\int d\Omega \int d\overrightarrow{v_1}\sigma(\Omega, |\overrightarrow{v_1} - \overrightarrow{v_2}|)|\overrightarrow{v_1} - \overrightarrow{v_2}| \left(f(\overrightarrow{x}, \overrightarrow{v_1'}, t) f(\overrightarrow{x}, \overrightarrow{v_2'}, t) - f(\overrightarrow{x}, \overrightarrow{v_1}, t) f(\overrightarrow{x}, \overrightarrow{v_2}, t) \right)$$

In 1956 Marc Kac developed a linear model from which a simple version of the spatially homogenous Boltzmann equation appeared under certain conditions. In [16]

Kac considered a system of N particles in one dimension that interact through random binary collisions: if $v_1, \ldots, v_N$ are the velocities of the N particles, a collision can occur between any two particles, leaving the rest unperturbed. If the ith particle and the jth particle collided, their velocities change from (v_i, v_j) to $(v_i \cos\vartheta + v_j \sin\vartheta, -v_i \sin\vartheta + v_j \cos\vartheta)$, where ϑ is a random angle. While this model doesn't conserve momentum, it does conserve the total kinetic energy.

Given a probability density for 'scaterring' in an angle ϑ, this Possion-like process yields a time evolution equation for the density function F. In the case of a constant density, and a spatially independent density function the equation is given by

$$\frac{\partial F}{\partial t}\left(v_1, \ldots, v_N, t\right) = -N(I-Q)F\left(v_1, \ldots, v_N, t\right) \tag{2.1.1}$$

where

$$QF\left(v_1, \ldots, v_N\right)$$

$$= \frac{1}{2\pi \begin{pmatrix} N \\ 2 \end{pmatrix}} \sum_{i<j} \int_0^{2\pi} F\left(v_1, \ldots, v_i \cos\vartheta + v_j \sin\vartheta, \ldots, -v_i \sin\vartheta + v_j \cos\vartheta, \ldots, v_N\right) d\vartheta$$

We note that a beautiful probabilistic explanation to (2.1.1) and the entire process can be found in [3].

Next in his paper, Kac noticed that if he defined the marginals

$$f_n(v_1, \ldots, v_n) = \int_{\sum_{i=n+1}^N v_i^2 = E - \sum_{i=1}^n v_i^2} F\left(v_1, \ldots, v_N\right) ds^{N-n}$$

where E is the fixed total energy and ds^{N-n} is the uniform measure on $\mathbb{S}^{N-n-1}\left(\sqrt{E - \sum_{i=1}^n v_i^2}\right)$, then equation (2.1.1), which was coined as 'Kac's Master Equation', implies a similar equation to the Boltzmann equation for the the first marginal f_1! To get the exact

Boltzmann equation we must have

$$f_n(v_1, v_2, \ldots, v_n, t) \approx f_1(v_1, t) \cdot \cdots \cdot f_1(v_n, t)$$

in some sense. The above observation prompted Kac to define what he called 'The Boltzmann Property': thinking of each particle as of unit energy particle, a sequence of density functions $F_N(v_1, \ldots, v_N)$ on $\mathbb{S}^{N-1}(\sqrt{N})$ is said to have the Boltzmann property if

$$\lim_{N\to\infty} f_k^{(N)}(v_1, \ldots, v_k) = \lim_{N\to\infty} \Pi_{i=1}^k f_1^{(N)}(v_i)$$

in some weak sense, where $f_k^{'(N)}$ is the kth marginal of F_N. In his original paper, Kac didn't define the convergence rigorously. A complete explanation with the right type of convergence can be found in [4].

Intuitively 'The Boltzmann property' means that as the number of particles get larger, each given k particles become more and more independent. Kac proceeded to prove that if $F_N(v_1, \ldots, v_N, t)$ is the solution to the master equation (2.1.1) with initial condition $F_N(v_1, \ldots, v_N, 0) = F_N(v_1, \ldots, v_N)$ where $F_N(v_1, \ldots, v_N)$ has the Boltzmann property, then $F_N(v_1, \ldots, v_N, t)$ will have the Boltzmann property for any t. This is now known as 'Propogation of Chaos'. Moreover, in this case the time evolution equation that $f_1(v,t) = \lim_{N\to\infty} f_1^{(N)}(v,t)$ satisfies is

$$\frac{\partial f}{\partial t}(v,t) \tag{2.1.2}$$

$$= \tfrac{1}{2\pi}\int_{\mathbb{R}} d\omega \int_0^{2\pi} d\vartheta \left(f_1\left(v\cos\vartheta + \omega\sin\vartheta, t\right) f_1\left(-v\sin\vartheta + \omega\cos\vartheta, t\right) - f_1(v,t)f_1(\omega,t)\right)$$

which is the Boltzmann equation in the spatially homogeneous, no external force case.

2.2 Kac's Conjecture and the gap problem

Another observation made by Kac was that any solution to the master equation will converge to the equilibrium state, represented by the constant function, as the time goes to infinity. This is not hard to see since the operator Q, given in the master equation (2.1.1), is self adjoint, bounded, satisfies $Q \leq I$ and $\dim\ker(Q-I) = 1$. Indeed, the solution to the master equation with initial condition $F(v_1,\dots,v_N,0) = F(v_1,\dots,v_N)$ is given by

$$F(v_1,\dots,v_N,t) = e^{-N(I-Q)t}F(v_1,\dots,v_N)$$

That along with the fact that Q has a one dimensional eigenspace for the eigenvalue 1 and $Q \leq I$ shows that in a weak sense $F_N(v_1,\dots,v_N,t)$ will converge to a function in the above eigenspace. The fact that the eigenspace for the eigenvalue 1 is spanned by the constant function shows the equilibrium convergence. Normalizing the measure implies $F_N(v_1,\dots,v_N,t)$ will converge weakly to the function 1.

Since every solution converges in a weak sense ($L^2\left(\mathbb{S}^{N-1}(\sqrt{N})\right)$ sense) a natural question to ask is how quickly will the convergence occur? This prompted Kac to define the *spectral gap*

$$\Delta_N = \inf\left\{\langle\varphi, N(I-Q)\varphi\rangle \;\middle|\; \varphi \in L^2\left(\mathbb{S}^{N-1}(\sqrt{N})\right),\ \langle\varphi,1\rangle = 0,\ \langle\varphi,\varphi\rangle = 1\right\}$$

Any solution of the master equation satisfies

$$\|F(v_1,\dots,v_N,t)-1\|_{L^2\left(\mathbb{S}^{N-1}\sqrt{N}\right)} \leq e^{-\Delta_N t}\,\|F(v_1,\dots,v_N,0)-1\|_{L^2\left(\mathbb{S}^{N-1}\sqrt{N}\right)} \tag{2.2.1}$$

(See Lemma A.1.1 in the Appendix).

In hope for an exponential convergence rate Kac conjectured that

$$\liminf_{N\to\infty} \Delta_N > 0$$

which will give a uniform bound in the exponent. The conjecture turned out to be true as was first proved by Janvresse in [14]. Her proof, however, didn't reveal what the spectral gap is. Later on the same year Carlen, Carvalho and Loss managed to find the exact value of Δ_N and showed it to be

$$\Delta_N = \frac{N+2}{2(N-1)}$$

as well as finding a function attaining the above value (See [3]).

After 44 years Kac's conjecture was proved. Is it enough? Unfortunately the answer is no.

While the exponent appearing in the relaxation estimation is not affected by N, the initial condition can, and in most natural cases, is. A density function which satisfies $F(v_1, \ldots, v_N, 0) \approx \Pi_{i=1}^N f(v_i)$ would generate a very large $L^2\left(\mathbb{S}^{N-1}(\sqrt{N})\right)$ norm. Indeed, one can find many sequences of density functions that satisfy

$$\|F(v_1, \ldots, v_n)\|_{L^2\left(\mathbb{S}^{N-1}(\sqrt{N})\right)} \geq C^N$$

where $C > 1$. This implies that the estimation (2.2.1) would yield time proportional to N and not the desired exponential decay Kac wanted.

2.3 Entropy and Vaillani's Conjecture

Seeing how Kac's conjecture didn't help in showing a fast relaxation time, a different approach was taken. In many subjects related to Statistical Mechanics a good quantity to investigate is the entropy:

Given a density function $F_N(v_1, \ldots, v_N)$ on $\mathbb{S}^{N-1}(\sqrt{N})$ we define

$$H_N(f) = \int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N(v_1, \ldots, v_N) \log\left(F_N(v_1, \ldots, v_N)\right) d\sigma^N$$

where $d\sigma^N$ is the uniform probability measure of $\mathbb{S}^{N-1}(\sqrt{N})$.

A well known inequality by Csiszar, Kullback, Leibler and Pinsker asserts that

$$\left\|F_N d\sigma^N - d\sigma^N\right\|^2_{\text{Total Variation}} \leq 2H_N(F_N)$$

Given $F_N(v_1, \ldots, v_N, t)$ that solves the master equation we find that

$$\frac{\partial H_N(F_N)}{\partial t} = \left\langle \frac{\frac{\partial F_N}{\partial t}}{F_N}, F_N \right\rangle + \left\langle \log\left(F_N\right), N(Q-1)F_N \right\rangle$$

$$= -\left\langle N(I-Q)F_N, 1 \right\rangle + \left\langle \log\left(F_N\right), N(Q-1)F_N \right\rangle$$

$$= N\left\langle F_N, (Q-I)1 \right\rangle + \left\langle \log\left(F_N\right), N(Q-1)F_N \right\rangle = \left\langle \log\left(F_N\right), N(Q-1)F_N \right\rangle$$

In a similar way to the spectral gap we define the *entropy production*

$$\Gamma_N = \inf \frac{\langle \log\left(\psi(v_1,\dots,v_N)\right), N(I-Q)\psi(v_1,\dots,v_N)\rangle}{H_N(\psi(v_1,\dots,v_N))}$$

where the infimum is taken over all density functions $\psi \in L^2\left(\mathbb{S}^{N-1}(\sqrt{N})\right)$ which are symmetric in all their variables.

Much like (2.2.1), the entropy production gives us a relaxation estimation:

$$\left\|F_N(v_1,\dots,v_N,t)d\sigma^N - d\sigma^N\right\|^2_{\text{Total Variation}} \leq 2e^{-\Gamma_N t} H_N(F_N(v_1,\dots,v_N,0)) \tag{2.3.1}$$

but with one crucial difference: The extensivity of the entropy. Intuitively speaking, if $F_N(v_1,\dots,v_N,t) \approx \Pi_{i=1}^N f(v_i,t)$ then

$$H_N\left(F_N(v_1,\dots,v_N,t)\right) \approx \int_{\mathbb{S}^{N-1}(\sqrt{N})} \Pi_{i=1}^N f(v_i,t) \left(\sum_{k=1}^N \log f(v_k,t)\right) d\sigma^N$$

$$= N \int_{\mathbb{S}^{N-1}(\sqrt{N})} \Pi_{i=1}^N f(v_i,t) \log\varphi(v_1,t) d\sigma^N \approx N \int_{\mathbb{R}} f(v_1,t)\log\frac{\varphi(v_1,t)}{\gamma(v_1)} dv_1$$

$$= N \cdot H(f(v,t)|\gamma(v))$$

where $\gamma(v)$ is the standard Gaussian. While being informal, the above property is indeed satisfied in the constructions related to the desired proofs. The extensivity of the entropy implies that

$$\left\|F_N(v_1,\dots,v_N,t)d\sigma^N - d\sigma^N\right\|^2_{\text{Total Variation}} \leq 2Ne^{-\Gamma_N t} H(f(v,0)|\gamma(v))$$

and so if we can prove that $\Gamma_N \geq C > 0$ independently of N, we will manage to achieve a far superior relaxation rate than that of the spectral gap!

Unfortunately, the evaluation of the entropy production is far more difficult and delicate than that of the the spectral gap. In [22] Villani managed to show that

$$\Gamma_N \geq \frac{2}{N-1}$$

and proceeded to conjecture that this is of optimal order, i.e.

$$\Gamma_N = O\left(\frac{1}{N}\right)$$

This will, of course, be disastrous for the relaxation time (as it will still imply a relaxation time of order N) but poses an interesting mathematical question.

A step towards the proof of the conjecture was done in 2010 by Carlen, Carvalho, Le Roux, Loss, and Villani. They managed to show that

Theorem 2.3.1. *(Carlen, Carvalho, Le Roux, Loss and Villani) For any $c > 0$ there is a probability density $f(v)$ on $\mathbb{R}$ with $\int_{\mathbb{R}} v f(v) dv = 0$ and $\int_{\mathbb{R}} v^2 f(v) dv = 1$, and a family of functions $\{F_N\}_{N \in \mathbb{N}}$ that have the Boltzmann property with $f_1(v,0) = f(v)$ such that*

$$\limsup_{N\to\infty} \frac{\langle \log(F_N), N(I-Q)F_N \rangle}{H_N(F_N)} \leq c$$

In particular, for each $c > 0$ the density function f is smooth, bounded and has moments of all orders.

(See [4]). While the theorem doesn't give us an expression for Γ_N, it does prove that

$$\lim_{N\to\infty} \Gamma_N = 0$$

as expected. The main result of this chapter is an upper bound for Γ_N that, while it, doesn't prove the exact conjecture, gets as close as possible to it:

Theorem. *Let $0 < \eta < 1$. There exists a constant C_η depending only on η such that*

$$\Gamma_N \leq \frac{C_\eta}{N^\eta}$$

(See Theorem 2.6.9 in Section 2.6).

Before we venture into the calculation and proof, we take a moment to shortly explain how Carlen, Carvalho, Le Roux, Loss and Villani proved Theorem 2.3.1. While our proof uses different computations, the idea behind the two proofs is the same.

The Boltzmann Equation arising from Kac's model (equation (2.1.2)) has a very natural stationary state, which is very common in Statistical Mechanics: the maxwellian function $M_a(v) = \frac{e^{-\frac{v^2}{2a}}}{\sqrt{2\pi a}}$. In [2] Bobylev and Cercignani exploited the maxwellians to create a one variable density function which is a superposition of two stationary states

$$f_\delta(v) = \delta M_{\frac{1}{2\delta}}(v) + (1-\delta) M_{\frac{1}{2(1-\delta)}}(v)$$

for a given fixed δ. The idea behind this is that each part in f_δ has the same energy

$$\int v^2 \delta M_{\frac{1}{2\delta}}(v) dv = \int v^2 (1-\delta) M_{\frac{1}{2(1-\delta)}}(v) dv = \frac{1}{2}$$

while obviously $\delta M_{\frac{1}{2\delta}}(v)$ represents far less 'mass' than $(1-\delta) M_{\frac{1}{2(1-\delta)}}(v)$ when δ is small. The attempt to equilibrate a large 'mass' and a small 'mass' with the same amount of energy is exactly the situation which will create the low entropy production.

Carlen, Carvalho, Le Roux, Loss and Villani defined the N particle function

$$F_N(v_1, \dots, v_N) = \frac{\Pi_{i=1}^N f_\delta(v_i)}{Z_N(f, \sqrt{N})}$$

where $Z_N(f, \sqrt{N})$ is the normalization function

$$Z_N(f, r) = \int_{\mathbb{S}^{N-1}(r)} \Pi_{i=1}^N f_\delta(v_i) d\sigma_r^N$$

and $d\sigma_r^N$ is the uniform probability measure on $\mathbb{S}^{N-1}(r)$. Using an asymptotic expression to $Z_N(f, \sqrt{N})$ (a central limit theorem) the authors showed that

$$\limsup_{N\to\infty} \frac{\langle \log(F_N), N(I-Q)F_N \rangle}{N} \leq -2\delta \log \delta + \log \pi + 4\delta^2 \tag{2.3.2}$$

$$\lim_{N\to\infty} \frac{H_N(F_N)}{N} = H(f_\delta|\gamma) = \int_{\mathbb{R}} f_\delta(v) \log \frac{f_\delta(v)}{\gamma(v)} dv \tag{2.3.3}$$

where $\gamma(v)$ is the standard Gaussian, and

$$\lim_{\delta\to 0} H(f_\delta|\gamma) = \frac{\log 2}{2} \tag{2.3.4}$$

Combining (2.3.2), (2.3.3) and (2.3.4) along with a suitable choice for δ gives Theorem 2.3.1.

Our proof will follow the same route, but will allow the parameter δ to be dependent in N.

We start by discussing several properties of the normalization function $Z_N(f, r)$.

2.4 The normalization function $Z_N(f,r)$

The key to the computation of the entropy production lies with the normalization function $Z_N(f,r)$. The probabilistic nature of the subject prompts us to use probabilistic techniques in order to understand $Z_N(f,r)$ better. The main goal of this section is to find a formula for $Z_N(f,r)$ that will serve us in the following sections. The results presented in this section can also be found in [4].

Lemma 2.4.1. *Let f be a density function for the real valued random variable V. Then the density function for the random variable V^2 is given by*

$$h(u) = \frac{f(\sqrt{u}) + f(-\sqrt{u})}{2\sqrt{u}}$$

for $u > 0$.

Proof. For any continuous function $\varphi = \varphi(|x|) = \varphi(r)$ we find that

$$\mathbb{E}\varphi = \int_0^\infty \varphi(r) \cdot (f(r) + f(-r))\, dr$$

On the other hand

$$\mathbb{E}\varphi = \int_0^\infty \varphi\left(\sqrt{t}\right) h(t)dt = \int_0^\infty \varphi(r) \cdot 2r \cdot h\left(r^2\right) dr$$

Since φ was arbitrary we find that

$$2r \cdot h\left(r^2\right) = f(r) + f(-r)$$

or

$$h(u) = \frac{f(\sqrt{u}) + f(-\sqrt{u})}{2\sqrt{u}}$$

□

Next we extend Lemma 2.4.1 to find the interpretation of $Z_N(f,r)$.

Lemma 2.4.2. *Let $V_1, \ldots, V_N$ be independent real valued random variables with identical density function $f(v)$. Then the density function for $S_N = \sum_{i=1}^N V_i^2$ is given by $s_N(u) = \frac{|\mathbb{S}^{N-1}|}{2} u^{\frac{N}{2}-1} Z_N(f, \sqrt{u})$.*

Proof. The proof follows the same route as Lemma 2.4.1. For any continuous $\varphi = \varphi\left(|(v_1, \dots, v_N)|\right) = \varphi(r)$ we find that

$$\mathbb{E}\varphi = \int_{\mathbb{R}^N} \varphi(v_1, \dots, v_N) \cdot \Pi_{i=1}^N f(v_i) dv_1 \dots dv_N = \int_0^\infty \varphi(r) \left(\int_{\mathbb{S}^{N-1}(r)} \Pi_{i=1}^N f(v_i) ds_r^N \right) dr$$

$$= \int_0^\infty \varphi(r) |\mathbb{S}^{N-1}| r^{N-1} \left(\int_{\mathbb{S}^{N-1}(r)} \Pi_{i=1}^N f(v_i) d\sigma_r^N \right) dr = \int_0^\infty \varphi(r) |\mathbb{S}^{N-1}| r^{N-1} Z_N(f, r) dr$$

On the other hand

$$\mathbb{E}\varphi = \int_0^\infty \varphi(\sqrt{t}) s_N(t) dt = \int_0^\infty \varphi(r) \cdot 2r \cdot s_N\left(r^2\right) dr$$

Since φ is arbitrary

$$2r \cdot s_N\left(r^2\right) = |\mathbb{S}^{N-1}| r^{N-1} Z_N(f, r)$$

or

$$s_N(u) = \frac{|\mathbb{S}^{N-1}|}{2} u^{\frac{N}{2}-1} Z_N(f, \sqrt{u})$$

$\square$

Lastly, we combine the above lemmas to get a formula for the normalization function.

Lemma 2.4.3. *Let f be a density function on $\mathbb{R}$, then*

$$Z_N(f, \sqrt{r}) = \frac{2h^{*N}(r)}{|\mathbb{S}^{N-1}| r^{\frac{N}{2}-1}}$$

where h^{*N} is the N-fold convolution of h, defined in Lemma 2.4.1.

Proof. Thinking of f as the density function for N independent random variables $V_1, \dots, V_N$ we find from Lemma 2.4.2 that

$$Z_N(f, \sqrt{r}) = \frac{2s_N(r)}{|\mathbb{S}^{N-1}| r^{\frac{N}{2}-1}}$$

where s_N is the density function for $S_N = \sum_{i=1}^N V_i^2$. On the other hand we know that $V_1^2, \dots, V_N^2$ have the same density function, given by h from Lemma 2.4.1, and as $V_1, \dots, V_N$ are independent a known result in probability theory (See for instance [7]) tells us that

$$s_N(u) = h^{*N}(u)$$

where h^{*N} is the N-fold convolution of h. We thus conclude that

$$Z_N(f, \sqrt{r}) = \frac{2h^{*N}(r)}{|\mathbb{S}^{N-1}| r^{\frac{N}{2}-1}}$$

$\square$

Armed with the formula for the normalization function we're now ready to find its asymptotic behavior.

2.5 A Central Limit Theorem

In order for us to be able prove our main result the asymptotic behavior for $Z_N(f, r)$ is needed. The formula given in Lemma 2.4.3 ties the function $Z_N(f, r)$ to the N-fold convolution of the density function h (given in Lemma 2.4.1). As such we'll employ techniques involving the Fourier transform in order to evaluate the normalization function.

Unlike many other central limit theorems, the theorem we'll present here gives us a uniform estimation on the convergence of the N-fold convolution of the *density function* to the Gaussian function, along with an explicit error estimation. The explicit error estimation is crucial to our main theorem as it will allow to change the 'one particle generating function' f as N changes and still get the same result. The only other similar convergence theorems we're aware of appear in [4] and in [12]. Our own starting point is much the same, though as the proof progresses the difference become very substantial.

The specific N particle function we'll construct as a test function for the entropy production has the property that the Fourier transform of the function h associated to its 'one particle generating function' f splits the line into two natural domains:

One where we can use analytic expansion, and one where the decay is dominated by an exponential function. The radius of the separating circle would depend on a parameter $\delta = \delta_N$ that we'll exploit later on to get the final conclusion. While this is the case arising in our specific construction, we believe that it's a natural way to view the problem. Even though we have yet to attempt any different test functions we think that similar situation would happen in a larger class of functions created from one particle function. As a result, we tried to make the Theorems of this section as general as we can make them.

Before we begin with the 'heavy' computations we'll state a few technical lemmas whose proofs can be found at the Appendix and that would serve us throughout this section.

Lemma 2.5.1. *For any $a, \eta > 0$ we have that*

$$\frac{\sqrt{2\pi}}{a} \cdot \sqrt{1 - e^{-\frac{a\eta^2}{2}}} \leq \int_{|x|<\eta} e^{-\frac{a^2x^2}{2}} dx \leq \frac{\sqrt{2\pi}}{a} \cdot \sqrt{1 - e^{-a^2\eta^2}}$$

and

$$\int_{|x|>\eta} e^{-\frac{a^2x^2}{2}} dx \leq \frac{\sqrt{2\pi} \cdot e^{-\frac{a^2\eta^2}{2}}}{a}$$

Lemma 2.5.2. *For any $a > 0$ and $k_0, m \in \mathbb{N}$ we have that*

$$\sum_{k=k_0+1}^{m} \frac{e^{-\frac{a^2k}{2}}}{\sqrt{k}} \leq \frac{\sqrt{2\pi} \cdot e^{-\frac{a^2k_0}{2}}}{a}$$

$$\sum_{k=k_0+1}^{m} \frac{1}{\sqrt{k}} \leq 2\sqrt{m}$$

(See Lemmas A.1.2 and A.1.3 in the Appendix)

While continuing to read this section, please keep in mind the following: the function g will represent the Fourier transform of the function h, connected to the one particle generating function f via Lemma 2.4.1. We'll start by exploring the domain outside the radius of analiticity, and then point our attention to the domain where analytic expansion is possible. The parameter δ itself should be thought of as a function of N that goes to zero as N goes to infinity.

We'll denote by $\gamma_1(\xi) = e^{-2\pi i\zeta} \cdot e^{-2\pi^2\xi^2\Sigma_\delta^2}$, where Σ_δ is a function of δ which we'll introduce later on.

Lemma 2.5.3. *Let $g_\delta(\xi)$ be such that*

(i) for $|\xi| > c\delta$ $|g_\delta(\xi)| \leq 1 - \alpha(\delta)$, where $0 < \alpha(\delta) < 1$.

(ii) $|g_\delta(\xi)| \leq 1$ for all ξ.

Then

$$\int_{|\xi|>c\delta} \left|g_\delta^N(\xi) - \gamma_1^N(\xi)\right| d\xi$$

$$\leq 2\int_{|\xi|>c\delta} |g_\delta(\xi)|^{N-1} d\xi + \frac{(1-\alpha(\delta))^{\frac{N}{2}-1}}{\pi c\delta\Sigma_\delta^2} + \frac{1}{\pi c\delta\Sigma_\delta^2} \cdot e^{-(1+N)\pi^2c^2\delta^2\Sigma_\delta^2}$$

Proof. We have that

$$\int_{|\xi|>c\delta} \left|g_\delta^N(\xi) - \gamma_1^N(\xi)\right| d\xi = \int_{|\xi|>c\delta} |g_\delta(\xi) - \gamma_1(\xi)| \cdot \left|\sum_{k=0}^{N-1} g_\delta^{N-k-1}(\xi)\gamma_1^k(\xi)\right| d\xi$$

Since $|\gamma_1(\xi)| = e^{-2\pi^2\xi^2\Sigma_\delta^2} \leq 1$ we find that

$$\int_{|\xi|>c\delta} \left|g_\delta^N(\xi) - \gamma_1^N(\xi)\right| d\xi \leq 2\int_{|\xi|>c\delta} \sum_{k=0}^{N-1} \left|g_\delta^{N-k-1}(\xi)\right| \left|\gamma_1^k(\xi)\right| d\xi$$

$$\leq 2\int_{|\xi|>c\delta} |g_\delta(\xi)|^{N-1} d\xi + 2\sum_{k=1}^{N-1} (1-\alpha(\delta))^{N-k-1} \int_{|\xi|>c\delta} e^{-2k\pi^2\xi^2\Sigma_\delta^2} d\xi$$

The last inequality is valid due to (i).

We notice that as k gets larger the expression $(1-\alpha(\delta))^{N-k-1}$ gets bigger while $\int_{|\xi|>c\delta} e^{-2k\pi^2\xi^2\Sigma_\delta^2} d\xi$ gets smaller, and vice versa as k gets smaller. As such we proceed to divide the sum from $k = 1$ to $k = N-1$ to two sums, each with a definite dominating small element, and a sum we can estimate easily.

$$\int_{|\xi|>c\delta} \left|g_\delta^N(\xi) - \gamma_1^N(\xi)\right| d\xi \leq 2\int_{|\xi|>c\delta} |g_\delta(\xi)|^{N-1} d\xi$$

$$+2\sum_{k=1}^{\left[\frac{N}{2}\right]}(1-\alpha(\delta))^{N-k-1}\int_{|\xi|>c\delta}e^{-2k\pi^2\xi^2\Sigma_\delta^2}d\xi+2\sum_{k=\left[\frac{N}{2}\right]+1}^{N-1}(1-\alpha(\delta))^{N-k-1}\int_{|\xi|>c\delta}e^{-2k\pi^2\xi^2\Sigma_\delta^2}d\xi$$

$$\leq 2\int_{|\xi|>c\delta}|g_\delta(\xi)|^{N-1}\,d\xi$$

$$+2\,(1-\alpha(\delta))^{N-\left[\frac{N}{2}\right]-1}\sum_{k=1}^{\left[\frac{N}{2}\right]}\int_{|\xi|>c\delta}e^{-2k\pi^2\xi^2\Sigma_\delta^2}d\xi+2\sum_{k=\left[\frac{N}{2}\right]+1}^{N-1}1\cdot\int_{|\xi|>c\delta}e^{-2k\pi^2\xi^2\Sigma_\delta^2}d\xi$$

Using Lemma 2.5.1 and 2.5.2 we conclude that

$$\sum_{k=k_0}^{N-1}\int_{|\xi|>c\delta}e^{-2k\pi^2\xi^2\Sigma_\delta^2}d\xi\leq\sum_{k=k_0}^{N-1}\frac{\sqrt{2\pi}\cdot e^{-\frac{4k\pi^2c^2\delta^2\Sigma_\delta^2}{2}}}{\sqrt{4k\pi^2\Sigma_\delta^2}}$$

$$=\frac{1}{\sqrt{2\pi\Sigma_\delta^2}}\cdot\sum_{k=k_0}^{N-1}\frac{e^{-\frac{4k\pi^2c^2\delta^2\Sigma_\delta^2}{2}}}{\sqrt{k}}\leq\frac{1}{2\pi c\delta\Sigma_\delta^2}\cdot e^{-2k_0\pi^2c^2\delta^2\Sigma_\delta^2}$$

Hence

$$\int_{|\xi|>c\delta}\left|g_\delta^N(\xi)-\gamma_1^N(\xi)\right|d\xi\leq 2\int_{|\xi|>c\delta}|g_\delta(\xi)|^{N-1}\,d\xi$$

$$+\frac{2\,(1-\alpha(\delta))^{N-\left[\frac{N}{2}\right]-1}}{2\pi c\delta\Sigma_\delta^2}\cdot e^{-2\pi^2c^2\delta^2\Sigma_\delta^2}+\frac{1}{2\pi c\delta\Sigma_\delta^2}\cdot e^{-2\left(\left[\frac{N}{2}\right]+1\right)\pi^2c^2\delta^2\Sigma_\delta^2}$$

$$\leq 2\int_{|\xi|>c\delta}|g_\delta(\xi)|^{N-1}\,d\xi+\frac{(1-\alpha(\delta))^{\frac{N}{2}-1}}{\pi c\delta\Sigma_\delta^2}+\frac{1}{\pi c\delta\Sigma_\delta^2}\cdot e^{-(1+N)\pi^2c^2\delta^2\Sigma_\delta^2}$$

which is the desired result. $\square$

Lemma 2.5.4. *Let $g_\delta(\xi)$ be such that*

(i) there exist $M_0, M_1, M_2>0$, independent of δ, such that $\sup_{|\xi|<c\delta}|g_\delta(\xi)-\gamma_1(\xi)|\leq\left(\frac{M_0}{\delta^2}+\frac{M_1}{\delta}+M_2\right)|\xi|^3$.

(ii) for $c\delta^{1+\beta}<|\xi|<c\delta$ $|g_\delta(\xi)|\leq 1-\alpha_\beta(\delta)$ where $0<\alpha_\beta(\delta)<1$, $\beta>0$ and $0<\delta<1$.

(iii) $|g_\delta(\xi)|\leq 1$ for all ξ.

Then

$$\int_{|\xi|<c\delta}\left|g_\delta^N(\xi)-\gamma_1^N(\xi)\right|d\xi\leq\frac{c^4\delta^2\left(M_0+M_1\delta+M_2\delta^2\right)}{2}$$

$$+\frac{c^3\delta\sqrt{N}\left(M_0+M_1\delta+M_2\delta^2\right)\left(1-\alpha_\beta(\delta)\right)^{\frac{N}{2}-1}}{\sqrt{\pi}\Sigma_\delta}+\frac{c^3\delta^{1-\beta}\left(M_0+M_1\delta+M_2\delta^2\right)e^{-\pi^2(N-1)c^2\delta^{2+2\beta}\Sigma_\delta^2}}{2\pi c\delta\Sigma_\delta^2\cdot\sqrt{1-e^{-2\pi^2Nc^2\delta^2\Sigma_\delta^2}}}$$

$$+\frac{2c^3\left(M_0+M_1\delta+M_2\delta^2\right)\sqrt{N}\delta^{1+3\beta}}{\sqrt{2\pi\Sigma_\delta^2}}$$

Proof. Just like Lemma 2.5.3 we have that

$$\int_{|\xi|<c\delta}\left|g_\delta^N(\xi)-\gamma_1^N(\xi)\right|d\xi\leq\sum_{k=0}^{N-1}\int_{|\xi|<c\delta}\left|g_\delta(\xi)-\gamma_1(\xi)\right|\left|g_\delta(\xi)\right|^{N-k-1}\left|\gamma_1(\xi)\right|^k d\xi$$

$$\leq\int_{|\xi|<c\delta}\left(\frac{M_0}{\delta^2}+\frac{M_1}{\delta}+M_2\right)|\xi|^3\left|g_\delta(\xi)\right|^{N-1}d\xi$$

$$+\sum_{k=1}^{N-1}\int_{|\xi|<c\delta}\left(\frac{M_0}{\delta^2}+\frac{M_1}{\delta}+M_2\right)|\xi|^3\left|g_\delta(\xi)\right|^{N-k-1}\left|\gamma_1(\xi)\right|^k d\xi$$

Since $|g_\delta(\xi)|\leq 1$

$$\int_{|\xi|<c\delta}\left|g_\delta^N(\xi)-\gamma_1^N(\xi)\right|d\xi \tag{2.5.1}$$

$$\leq\int_{|\xi|<c\delta}\left(\frac{M_0}{\delta^2}+\frac{M_1}{\delta}+M_2\right)|\xi|^3d\xi+\sum_{k=1}^{N-1}\int_{|\xi|<c\delta}\left(\frac{M_0}{\delta^2}+\frac{M_1}{\delta}+M_2\right)|\xi|^3\left|g_\delta(\xi)\right|^{N-k-1}\left|\gamma_1(\xi)\right|^k d\xi$$

$$=\frac{c^4\delta^2\left(M_0+M_1\delta+M_2\delta^2\right)}{2}+\sum_{k=1}^{N-1}\int_{c\delta^{1+\beta}<|\xi|<c\delta}\left(\frac{M_0}{\delta^2}+\frac{M_1}{\delta}+M_2\right)|\xi|^3\left|g_\delta(\xi)\right|^{N-k-1}\left|\gamma_1(\xi)\right|^k d\xi$$

$$+\sum_{k=1}^{N-1}\int_{|\xi|<c\delta^{1+\beta}}\left(\frac{M_0}{\delta^2}+\frac{M_1}{\delta}+M_2\right)|\xi|^3\left|g_\delta(\xi)\right|^{N-k-1}\left|\gamma_1(\xi)\right|^k d\xi$$

Using (ii) and a similar idea of sum separation as in Lemma 2.5.3 yields

$$\sum_{k=1}^{N-1}\int_{c\delta^{1+\beta}<|\xi|<c\delta}\left(\frac{M_0}{\delta^2}+\frac{M_1}{\delta}+M_2\right)|\xi|^3\left|g_\delta(\xi)\right|^{N-k-1}\left|\gamma_1(\xi)\right|^k d\xi$$

$$\leq c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)\sum_{k=1}^{N-1}\left(1-\alpha_\beta(\delta)\right)^{N-k-1}\int_{c\delta^{1+\beta}<|\xi|<c\delta}e^{-2k\pi^2\xi^2\Sigma_\delta^2}d\xi$$

$$=c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)\sum_{k=1}^{\left[\frac{N}{2}\right]}\left(1-\alpha_\beta(\delta)\right)^{N-k-1}\int_{c\delta^{1+\beta}<|\xi|<c\delta}e^{-2k\pi^2\xi^2\Sigma_\delta^2}d\xi$$

$$+c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)\sum_{k=\left[\frac{N}{2}\right]+1}^{N-1}\left(1-\alpha_\beta(\delta)\right)^{N-k-1}\int_{c\delta^{1+\beta}<|\xi|<c\delta}e^{-2k\pi^2\xi^2\Sigma_\delta^2}d\xi$$

$$\leq c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)\left(1-\alpha_\beta(\delta)\right)^{N-\left[\frac{N}{2}\right]-1}\sum_{k=1}^{\left[\frac{N}{2}\right]}\int_{|\xi|<c\delta}e^{-2k\pi^2\xi^2\Sigma_\delta^2}d\xi$$

$$+c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)\sum_{k=\left[\frac{N}{2}\right]+1}^{N-1}\int_{c\delta^{1+\beta}<|\xi|<c\delta}e^{-2k\pi^2\xi^2\Sigma_\delta^2}d\xi$$

Using Lemma 2.5.1 and 2.5.2 gives

$$\sum_{k=1}^{N-1}\int_{c\delta^{1+\beta}<|\xi|<c\delta}\left(\frac{M_0}{\delta^2}+\frac{M_1}{\delta}+M_2\right)|\xi|^3\left|g_\delta(\xi)\right|^{N-k-1}\left|\gamma_1(\xi)\right|^k d\xi$$

$$\leq c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)\left(1-\alpha_\beta(\delta)\right)^{\frac{N}{2}-1}\sum_{k=1}^{\left[\frac{N}{2}\right]}\frac{\sqrt{1-e^{-4\pi^2kc^2\delta^2\Sigma_\delta^2}}}{\sqrt{2\pi\Sigma_\delta^2k}}$$

$$+c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)\sum_{k=\left[\frac{N}{2}\right]+1}^{N-1}\left(\int_{|\xi|<c\delta}e^{-2k\pi^2\xi^2\Sigma_\delta^2}d\xi-\int_{|\xi|<c\delta^{1+\beta}}e^{-2k\pi^2\xi^2\Sigma_\delta^2}d\xi\right)$$

$$\leq c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)\left(1-\alpha_\beta(\delta)\right)^{\frac{N}{2}-1}\sum_{k=1}^{\left[\frac{N}{2}\right]}\frac{1}{\sqrt{2\pi\Sigma_\delta^2k}}$$

$$+c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)\sum_{k=\left[\frac{N}{2}\right]+1}^{N-1}\frac{\left(\sqrt{1-e^{-4\pi^2kc^2\delta^2\Sigma_\delta^2}}-\sqrt{1-e^{-2\pi^2kc^2\delta^{2+2\beta}\Sigma_\delta^2}}\right)}{\sqrt{2\pi k\Sigma_\delta^2}}$$

$$\leq\frac{c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)\left(1-\alpha_\beta(\delta)\right)^{\frac{N}{2}-1}}{\sqrt{2\pi\Sigma_\delta^2}}\cdot\sqrt{4\left[\frac{N}{2}\right]}$$

$$+\frac{c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)}{\sqrt{2\pi\Sigma_\delta^2}}\sum_{k=\left[\frac{N}{2}\right]+1}^{N-1}\frac{1}{\sqrt{k}}\cdot\frac{e^{-2\pi^2kc^2\delta^{2+2\beta}\Sigma_\delta^2}-e^{-4\pi^2kc^2\delta^2\Sigma_\delta^2}}{\left(\sqrt{1-e^{-4\pi^2kc^2\delta^2\Sigma_\delta^2}}+\sqrt{1-e^{-2\pi^2kc^2\delta^{2+2\beta}\Sigma_\delta^2}}\right)}$$

$$\leq\frac{c^3\delta\sqrt{N}\left(M_0+M_1\delta+M_2\delta^2\right)\left(1-\alpha_\beta(\delta)\right)^{\frac{N}{2}-1}}{\sqrt{\pi\Sigma_\delta^2}}$$

$$+\frac{c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)}{\sqrt{2\pi\Sigma_\delta^2}}\sum_{k=\left[\frac{N}{2}\right]+1}^{N-1}\frac{1}{\sqrt{k}}\cdot\frac{e^{-2\pi^2kc^2\delta^{2+2\beta}\Sigma_\delta^2}}{\sqrt{1-e^{-4\pi^2kc^2\delta^2\Sigma_\delta^2}}}$$

$$\leq\frac{c^3\delta\sqrt{N}\left(M_0+M_1\delta+M_2\delta^2\right)\left(1-\alpha_\beta(\delta)\right)^{\frac{N}{2}-1}}{\sqrt{\pi\Sigma_\delta^2}}$$

$$+\frac{c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)}{\sqrt{2\pi\Sigma_\delta^2}\cdot\sqrt{1-e^{-4\pi^2\left(\left[\frac{N}{2}\right]+1\right)c^2\delta^2\Sigma_\delta^2}}}\sum_{k=\left[\frac{N}{2}\right]+1}^{N-1}\frac{e^{-2\pi^2kc^2\delta^{2+2\beta}\Sigma_\delta^2}}{\sqrt{k}}$$

$$\le \frac{c^3\delta\sqrt{N}\left(M_0+M_1\delta+M_2\delta^2\right)\left(1-\alpha_\beta(\delta)\right)^{\frac{N}{2}-1}}{\sqrt{\pi\Sigma_\delta^2}}$$

$$+\frac{c^3\delta\left(M_0+M_1\delta+M_2\delta^2\right)}{\sqrt{\Sigma_\delta^2}\cdot\sqrt{1-e^{-2\pi^2Nc^2\delta^2\Sigma_\delta^2}}}\cdot\frac{e^{-2\pi^2\left[\frac{N}{2}\right]c^2\delta^{2+2\beta}\Sigma_\delta^2}}{\sqrt{4\pi^2c^2\delta^{2+2\beta}\Sigma_\delta^2}}$$

$$\le \frac{c^3\delta\sqrt{N}\left(M_0+M_1\delta+M_2\delta^2\right)\left(1-\alpha_\beta(\delta)\right)^{\frac{N}{2}-1}}{\sqrt{\pi}\Sigma_\delta}+\frac{c^3\delta^{1-\beta}\left(M_0+M_1\delta+M_2\delta^2\right)e^{-\pi^2(N-1)c^2\delta^{2+2\beta}\Sigma_\delta^2}}{2\pi c\delta\Sigma_\delta^2\cdot\sqrt{1-e^{-2\pi^2Nc^2\delta^2\Sigma_\delta^2}}}$$

so

$$\sum_{k=1}^{N-1}\int_{c\delta^{1+\beta}<|\xi|<c\delta}\left(\frac{M_0}{\delta^2}+\frac{M_1}{\delta}+M_2\right)|\xi|^3\left|g_\delta(\xi)\right|^{N-k-1}\left|\gamma_1(\xi)\right|^k d\xi \tag{2.5.2}$$

$$\le \frac{c^3\delta\sqrt{N}\left(M_0+M_1\delta+M_2\delta^2\right)\left(1-\alpha_\beta(\delta)\right)^{\frac{N}{2}-1}}{\sqrt{\pi}\Sigma_\delta}+\frac{c^3\delta^{1-\beta}\left(M_0+M_1\delta+M_2\delta^2\right)e^{-\pi^2(N-1)c^2\delta^{2+2\beta}\Sigma_\delta^2}}{2\pi c\delta\Sigma_\delta^2\cdot\sqrt{1-e^{-2\pi^2Nc^2\delta^2\Sigma_\delta^2}}}$$

Also, since $|g_\delta(\xi)|\le 1$ we have that

$$\sum_{k=1}^{N-1}\int_{|\xi|<c\delta^{1+\beta}}\left(\frac{M_0}{\delta^2}+\frac{M_1}{\delta}+M_2\right)|\xi|^3\left|g_\delta(\xi)\right|^{N-k-1}\left|\gamma_1(\xi)\right|^k d\xi$$

$$\le c^3\left(M_0+M_1\delta+M_2\delta^2\right)\delta^{1+3\beta}\cdot\sum_{k=1}^{N-1}\int_{|\xi|<c\delta^{1+\beta}}e^{-2k\pi^2\xi^2\Sigma_\delta^2}d\xi$$

$$\le c^3\left(M_0+M_1\delta+M_2\delta^2\right)\delta^{1+3\beta}\cdot\sum_{k=1}^{N-1}\frac{\sqrt{1-e^{-4k\pi^2c^2\delta^{2+2\beta}\Sigma_\delta^2}}}{\sqrt{2\pi k\Sigma_\delta^2}}$$

$$\le \frac{c^3\left(M_0+M_1\delta+M_2\delta^2\right)\delta^{1+3\beta}}{\sqrt{2\pi\Sigma_\delta^2}}\cdot\sum_{k=1}^{N-1}\frac{1}{\sqrt{k}}$$

$$\le \frac{2c^3\left(M_0+M_1\delta+M_2\delta^2\right)\sqrt{N}\delta^{1+3\beta}}{\sqrt{2\pi\Sigma_\delta^2}}$$

so

$$\sum_{k=1}^{N-1}\int_{|\xi|<c\delta^{1+\beta}}\left(\frac{M_0}{\delta^2}+\frac{M_1}{\delta}+M_2\right)|\xi|^3\left|g_\delta(\xi)\right|^{N-k-1}\left|\gamma_1(\xi)\right|^k d\xi \tag{2.5.3}$$

$$\le \frac{2c^3\left(M_0+M_1\delta+M_2\delta^2\right)\sqrt{N}\delta^{1+3\beta}}{\sqrt{2\pi\Sigma_\delta^2}}$$

Combining (2.5.1), (2.5.2) and (2.5.3) gives the desired result. $\square$

Now that we have proved Lemma 2.5.3 and 2.5.4 we can turn our attention to the main theorem of this section, giving us the tool to approximate $Z_N(f,r)$.

Theorem 2.5.5. *Let $h_\delta(x) = h_{\delta_N}(x)$ be a continuous $L^1(\mathbb{R})$ function such that $g_\delta(\xi) = \widehat{h_\delta}(\xi)$ satisfies*

(i) for $|\xi| > c\delta_N$ $|g_{\delta_N}(\xi)| \leq 1 - \alpha(\delta_N)$, where $0 < \alpha(\delta_N) < 1$.

(ii) there exist $M_0, M_1, M_2 > 0$, independent of δ_N, such that $\sup_{|\xi|<c\delta_N} |g_{\delta_N}(\xi) - \gamma_1(\xi)| \leq \left(\frac{M_0}{\delta_N^2} + \frac{M_1}{\delta_N} + M_2\right)|\xi|^3$.

(iii) for $c\delta_N^{1+\beta} < |\xi| < c\delta_N$ $|g_{\delta_N}(\xi)| \leq 1 - \alpha_\beta(\delta_N)$ where $0 < \alpha_\beta(\delta_N) < 1$ and $0 < \beta < 1$.

(vi) $|g_{\delta_N}(\xi)| \leq 1$ for all ξ.

and if

(a) δ_N, $\alpha(\delta_N)$ and $\alpha_\beta(\delta_N)$ are of order of a negative power of N.

(b) $\alpha(\delta_N)N \underset{N\to\infty}{\longrightarrow} \infty$.

(c) $\alpha_\beta(\delta_N)N \underset{N\to\infty}{\longrightarrow} \infty$.

(d) $\Sigma_{\delta_N}^2 \delta_N^{2+2\beta} N \underset{N\to\infty}{\longrightarrow} \infty$.

(e) $\delta_N^{1+3\beta} N \underset{N\to\infty}{\longrightarrow} 0$.

(f^0) $\sqrt{N}\Sigma_{\delta_N} \int_{|\xi|>c\delta_N} |g_{\delta_N}(\xi)|^{N-1} d\xi \underset{N\to\infty}{\longrightarrow} 0$.

(g) $\delta_N^{\frac{3}{2}(1-\beta)}\Sigma_{\delta_N}$ is bounded.

Then

$$\sup_u \left| h_{\delta_N}^{*N}(u) - \frac{1}{\sqrt{N}\Sigma_{\delta_N}} \cdot \frac{e^{-\frac{(u-N)^2}{2N\Sigma_{\delta_N}^2}}}{\sqrt{2\pi}} \right| \leq \frac{\epsilon(N)}{\sqrt{N}\Sigma_{\delta_N}} \tag{2.5.4}$$

where $h_{\delta_N}^{*N}(x)$ is the N-fold convolution of h_{δ_N} and $\epsilon(N) \underset{N\to\infty}{\longrightarrow} 0$. Moreover if for a fixed $j \in \left\{0, 1, \ldots, \left[\frac{N}{2}\right]\right\}$ we have that

(f^j) $\sqrt{N-j}\Sigma_{\delta_N} \int_{|\xi|>c\delta_N} |g_{\delta_N}(\xi)|^{N-j-1} \underset{N\to\infty}{\longrightarrow} 0$.

instead of condition (f^0) then

$$\sup_u \left| h_{\delta_N}^{*N-j}(u) - \frac{1}{\sqrt{N-j}\Sigma_{\delta_N}} \cdot \frac{e^{-\frac{(u-N+j)^2}{2(N-j)\Sigma_{\delta_N}^2}}}{\sqrt{2\pi}} \right| \leq \frac{\epsilon_j(N)}{\sqrt{N-j}\Sigma_{\delta_N}} \tag{2.5.5}$$

where $\epsilon_j(N) \underset{N\to\infty}{\longrightarrow} 0$.

Proof. We start by noticing that

$$\frac{1}{\sqrt{N}\Sigma_\delta}\cdot\widehat{\frac{e^{-\frac{(x-N)^2}{2N\Sigma_\delta^2}}}{\sqrt{2\pi}}}(\xi)=\frac{1}{\sqrt{2\pi N}\Sigma_\delta}\int_{\mathbb{R}}e^{-\frac{(x-N)^2}{2N\Sigma_\delta^2}}\cdot e^{-2\pi i\xi x}dx$$

$$\underset{y=\frac{x-N}{\sqrt{N}\Sigma_\delta}}{=}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-\frac{y^2}{2}}\cdot e^{-2\pi i\xi\left(\sqrt{N}\Sigma_\delta y+N\right)}dy=\frac{e^{-2\pi iN\xi}}{\sqrt{2\pi}}\cdot e^{\frac{(2\pi i\xi\sqrt{N}\Sigma_\delta)^2}{2}}\int_{\mathbb{R}}e^{-\frac{(y+2\pi i\zeta\sqrt{N}\Sigma_\delta)^2}{2}}dy$$

$$=\left(e^{-2\pi i\xi}\cdot e^{-2\pi^2\xi^2\Sigma_\delta^2}\right)^N=\gamma_1^N(\xi)$$

Since (2.5.4) follows from (2.5.5) for $j=0$ we'll only prove the second part of the theorem. Using Lemma 2.5.3 and 2.5.4 we find that

$$\sup_u\left|h_{\delta_N}^{*N-j}(u)-\frac{1}{\sqrt{N-j}\Sigma_{\delta_N}}\cdot\frac{e^{-\frac{(u-N+j)^2}{2(N-j)\Sigma_{\delta_N}^2}}}{\sqrt{2\pi}}\right|$$

$$\leq\int_{\mathbb{R}}\left|g_\delta^{N-j}(\xi)-\gamma_1^{N-j}(\xi)\right|d\xi$$

$$=\int_{|\xi|<c\delta}\left|g_\delta^{N-j}(\xi)-\gamma_1^{N-j}(\xi)\right|d\xi+\int_{|\xi|>c\delta}\left|g_\delta^{N-j}(\xi)-\gamma_1^{N-j}(\xi)\right|d\xi$$

$$\leq\frac{1}{\sqrt{N-j}\Sigma_{\delta_N}}\left(\frac{c^4\sqrt{(N-j)\delta_N^{1+3\beta}}\delta_N^{\frac{3}{2}(1-\beta)}\Sigma_{\delta_N}\left(M_0+M_1\delta_N+M_2\delta_N^2\right)}{2}\right.$$

$$+\frac{c^3\delta_N(N-j)\left(M_0+M_1\delta_N+M_2\delta_N^2\right)\left(1-\alpha_\beta(\delta_N)\right)^{\frac{N-j}{2}-1}}{\sqrt{\pi}}$$

$$+\frac{c^3\sqrt{N-j}\delta_N^{1-\beta}\left(M_0+M_1\delta_N+M_2\delta_N^2\right)e^{-\pi^2(N-j-1)c^2\delta_N^{2+2\beta}\Sigma_{\delta_N}^2}}{2\pi c\delta_N\Sigma_{\delta_N}\cdot\sqrt{1-e^{-2\pi^2(N-j)c^2\delta_N^2\Sigma_{\delta_N}^2}}}$$

$$+\frac{2c^3\left(M_0+M_1\delta_N+M_2\delta_N^2\right)(N-j)\delta_N^{1+3\beta}}{\sqrt{2\pi}}+2\sqrt{N-j}\Sigma_{\delta_N}\int_{|\xi|>c\delta_N}\left|g_{\delta_N}(\xi)\right|^{N-j-1}d\xi$$

$$\left.+2\left(1-\alpha(\delta_N)\right)^{\frac{N-j}{2}-1}\cdot\frac{\sqrt{N-j}}{2\pi c\delta_N\Sigma_{\delta_N}}+\frac{\sqrt{N-j}}{\pi c\delta_N\Sigma_{\delta_N}}\cdot e^{-(1+N-j)\pi^2c^2\delta_N^2\Sigma_{\delta_N}^2}\right)$$

$$=\frac{\epsilon_j(N)}{\sqrt{N-j}\Sigma_{\delta_N}}$$

Conditions (e) and (g) imply that

$$\sqrt{(N-j)\delta_N^{1+3\beta}}\delta_N^{\frac{3}{2}(1-\beta)}\Sigma_{\delta_N}\leq\sqrt{N\delta_N^{1+3\beta}}\delta_N^{\frac{3}{2}(1-\beta)}\Sigma_{\delta_N}\underset{N\to\infty}{\longrightarrow}0$$

Conditions (a), (c) and the fact that $j \leq \frac{N}{2}$ imply that

$$\delta_N(N-j)\left(1-\alpha_\beta(\delta_N)\right)^{\frac{N-j-2}{2}} = \delta_N(N-j)\left(\left(1-\alpha_\beta(\delta_N)\right)^{\frac{1}{\alpha_\beta(\delta_N)}}\right)^{\frac{\alpha_\beta(\delta_N)(N-j-2)}{2}}$$

$$\leq \delta_N N\left(\left(1-\alpha_\beta(\delta_N)\right)^{\frac{1}{\alpha_\beta(\delta_N)}}\right)^{\frac{\alpha_\beta(\delta_N)(N-4)}{4}} \underset{N\to\infty}{\longrightarrow} 0$$

Also by condition (d):

$$(N-j)\delta_N^2\Sigma_{\delta_N}^2 \geq (N-j-1)\delta_N^{2+2\beta}\Sigma_{\delta_N}^2 \geq \frac{(N-2)\delta_N^{2+2\beta}\Sigma_{\delta_N}^2}{2} \underset{N\to\infty}{\longrightarrow} \infty$$

and so along with condition (a) we have that

$$(N-j)\delta_N e^{-\pi^2(N-j-1)c^2\delta_N^{2+2\beta}\Sigma_{\delta_N}^2} \leq N\delta_N e^{-\pi^2(N-j-1)c^2\delta_N^{2+2\beta}\Sigma_{\delta_N}^2} \underset{N\to\infty}{\longrightarrow} 0$$

$$\delta_N^{1+\beta}\Sigma_{\delta_N}\sqrt{N-j}\cdot\sqrt{1-e^{-2\pi^2(N-j)c^2\delta_N^2\Sigma_{\delta_N}^2}} \underset{N\to\infty}{\longrightarrow} \infty$$

which implies

$$\frac{c^3\sqrt{N-j}\,\delta_N^{1-\beta}\left(M_0+M_1\delta_N+M_2\delta_N^2\right)e^{-\pi^2(N-j-1)c^2\delta_N^{2+2\beta}\Sigma_{\delta_N}^2}}{2\pi c\delta_N\Sigma_{\delta_N}\cdot\sqrt{1-e^{-2\pi^2(N-j)c^2\delta_N^2\Sigma_{\delta_N}^2}}}$$

$$= \frac{c^3(N-j)\delta_N\left(M_0+M_1\delta_N+M_2\delta_N^2\right)e^{-\pi^2(N-j-1)c^2\delta_N^{2+2\beta}\Sigma_{\delta_N}^2}}{2\pi c\delta_N^{1+\beta}\Sigma_{\delta_N}\sqrt{N-j}\cdot\sqrt{1-e^{-2\pi^2(N-j)c^2\delta_N^2\Sigma_{\delta_N}^2}}} \underset{N\to\infty}{\longrightarrow} 0$$

We also notice that conditions (e) and (f^j), along with the fact that $j \leq \frac{N}{2}$ imply that

$$(N-j)\delta_N^{1+3\beta} \leq N\delta_N^{1+3\beta} \underset{N\to\infty}{\longrightarrow} 0$$

$$\sqrt{N-j}\,\Sigma_{\delta_N}\int_{|\xi|>c\delta_N}\left|g_{\delta_N}(\xi)\right|^{N-j-1}d\xi \underset{N\to\infty}{\longrightarrow} 0$$

Conditions (a), (b), (d) and the fact that $j \leq \frac{N}{2}$ show that

$$\left(1-\alpha(\delta_N)\right)^{\frac{N-j}{2}-1}\cdot\frac{\sqrt{N-j}}{2\pi c\delta_N\Sigma_{\delta_N}} = \frac{(N-j)\delta_N^\beta\left(\left(1-\alpha(\delta_N)\right)^{\alpha(\delta_N)}\right)^{\frac{\alpha(\delta_N)(N-j-2)}{2}}}{2\pi c\sqrt{N-j}\,\delta_N^{1+\beta}\Sigma_{\delta_N}}$$

$$\leq \frac{N\delta_N^\beta\left(\left(1-\alpha(\delta_N)\right)^{\alpha(\delta_N)}\right)^{\frac{\alpha(\delta_N)(N-4)}{4}}}{\sqrt{2}\pi c\sqrt{N}\delta_N^{1+\beta}\Sigma_{\delta_N}} \underset{N\to\infty}{\longrightarrow} 0$$

Lastly,

$$\frac{\sqrt{N-j}}{\pi c\delta_N\Sigma_{\delta_N}}\cdot e^{-(1+N-j)\pi^2c^2\delta_N^2\Sigma_{\delta_N}^2}=\frac{(N-j)\delta_N^{\beta}}{\pi c\sqrt{N-j}\delta_N^{1+\beta}\Sigma_{\delta_N}}\cdot e^{-(1+N-j)\pi^2c^2\delta_N^2\Sigma_{\delta_N}^2}$$

which we saw converges to zero.

Combining all the information presented we find that $\epsilon_j(N) \underset{N\to\infty}{\longrightarrow} 0$. □

Remark 2.5.6. Conditions (a) to (g) were designed so that $\epsilon_j(N)$ will converge to zero. Looking over the proof of Theorem 2.5.5 we see that the constants M_0, M_1 and M_2 play a role in the convergence. For instance: if $M_0 = 0$ then many terms in the expression for $\epsilon_j(N)$ would have an extra factor of δ_N - making the convergence faster and allowing us to weaken conditions (a) to (h). Unfortunately, this is not the case in our constructed sequence (to appear in the next section) but it may be the case for a different type of construction.

We are now ready to construct our sequence of density functions that will yield an upper bound to the entropy production, proving Villani's conjecture, up to an ϵ.

2.6 The main result: A Proof of Villani's Conjecture, up to an ϵ

The route we'll take in this section was outlined in Section 2.3.

We define our one particle generating function to be

$$f_\delta(v)=\delta M_{\frac{1}{2\delta}}(v)+(1-\delta)M_{\frac{1}{2(1-\delta)}}(v)$$

where $M_a(v)=\frac{e^{-\frac{v^2}{2a}}}{\sqrt{2\pi a}}$ and $0<\delta<1$. Since M_a is a density function and f_δ is a convex combination of two M_a-s, we conclude that f_δ itself is a density function.

Let

$$h_\delta(u)=\frac{f_\delta(\sqrt{u})+f_\delta(-\sqrt{u})}{2\sqrt{u}}$$

for $u>0$.

We'll start this section with finding properties of h_δ.

Theorem 2.6.1. *Let h_δ be defined as above. Then*

(i) h_δ is a continuous density function on $(0,\infty)$.

(ii) $\int_0^\infty u h_\delta(u)du = 1$.

(iii) $\int_0^\infty u^2 h_\delta(u)du = \frac{3}{4\delta(1-\delta)}$.

(iv) $\widehat{h_\delta}(\xi) = \frac{\delta}{\sqrt{1+\frac{2\pi i\xi}{\delta}}} + \frac{1-\delta}{\sqrt{1+\frac{2\pi i\xi}{1-\delta}}}$.

Proof. Clearly h_δ is continuous on $(0,\infty)$ as f_δ is smooth on $\mathbb{R}$. Next we see that

$$\int_0^\infty u^m h_\delta(u)du = \frac{1}{2}\int_0^\infty \frac{u^m f_\delta(\sqrt{u})}{\sqrt{u}}du + \frac{1}{2}\int_0^\infty \frac{u^m f_\delta(-\sqrt{u})}{\sqrt{u}}du$$

using the substitution $v = \sqrt{u}$ in the first integration and $v = -\sqrt{u}$ in the second integration yields

$$\int_0^\infty u^m h_\delta(u)du = \int_0^\infty v^{2m} f_\delta(v)dv + \int_{-\infty}^0 v^{2m} f_\delta(v)dv = \int_{\mathbb{R}} v^{2m} f_\delta(v)dv \tag{2.6.1}$$

For $m \geq 1$ we have that

$$\int_{\mathbb{R}} v^{2m} M_a(v)dv = \frac{1}{\sqrt{2\pi a}}\int_{\mathbb{R}} v^{2m} e^{-\frac{v^2}{2a}}dv \underset{x=\frac{v}{\sqrt{a}}}{=} \frac{a^m}{\sqrt{2\pi}} \cdot \int_{\mathbb{R}} x^{2m} e^{-\frac{x^2}{2}}dx$$

$$= \frac{a^m}{\sqrt{2\pi}} \cdot x^{2m-1} \cdot \left(-e^{-\frac{x^2}{2}}\right)\Big|_{-\infty}^{\infty} + \frac{a^m}{\sqrt{2\pi}} \cdot (2m-1)\int_{\mathbb{R}} x^{2m-2}e^{-\frac{x^2}{2}}dx$$

$$= \frac{a^m}{\sqrt{2\pi}} \cdot (2m-1)\int_{\mathbb{R}} x^{2m-2}e^{-\frac{x^2}{2}}dx = \cdots = \frac{a^m}{\sqrt{2\pi}} \cdot (2m-1)\cdot(2m-3)\cdot\cdots\cdot 1 \cdot \int e^{-\frac{x^2}{2}}dx$$

$$= (2m-1)\cdot(2m-3)\cdot\cdots\cdot 1\cdot a^m$$

We find that

$$\int_0^\infty u^m h_\delta(u)du = \delta\int_{\mathbb{R}} v^{2m} M_{\frac{1}{2\delta}}(v)dv + (1-\delta)\int_{\mathbb{R}} v^{2m} M_{\frac{1}{2(1-\delta)}}(v)dv$$

$$= (2m-1)\cdot(2m-3)\cdot\cdots\cdot 1\left(\delta\cdot\left(\frac{1}{2\delta}\right)^m + (1-\delta)\cdot\left(\frac{1}{2(1-\delta)}\right)^m\right)$$

Hence

$$\int_0^\infty h_\delta(u)du = \delta\int_{\mathbb{R}} M_{\frac{1}{2\delta}}(v)dv + (1-\delta)\int_{\mathbb{R}} M_{\frac{1}{2(1-\delta)}}(v)dv = \delta + (1-\delta) = 1$$

$$\int_0^\infty u h_\delta(u)du = \left(\delta \cdot \left(\frac{1}{2\delta}\right) + (1-\delta) \cdot \left(\frac{1}{2(1-\delta)}\right)\right) = 1$$

$$\int_0^\infty u^2 h_\delta(u)du = 3\left(\delta \cdot \left(\frac{1}{2\delta}\right)^2 + (1-\delta) \cdot \left(\frac{1}{2(1-\delta)}\right)^2\right)$$

$$= \frac{3}{4}\left(\frac{1}{\delta} + \frac{1}{1-\delta}\right) = \frac{3}{4\delta(1-\delta)}$$

which proves (i), (ii) and (iii).

In order to prove (iv) we notice that due to fact that M_a is a Schwartz class function we have that

$$\frac{d}{d\xi}\int_{\mathbb{R}} M_a(u) \cdot e^{-2\pi i \xi u^2} du = \int_{\mathbb{R}} \left(-2\pi i u^2\right) M_a(u) \cdot e^{-2\pi i \xi u^2} du$$

(differentiation under the sign of integration is allowed)

Also since $\frac{d}{du}M_a(u) = -\frac{u}{a}M_a(u)$ we find that

$$\frac{d}{d\xi}\int_{\mathbb{R}} M_a(u) \cdot e^{-2\pi i \xi u^2} du = \int_{\mathbb{R}} \left(2\pi i a u \cdot e^{-2\pi i \xi u^2}\right) \cdot \frac{d}{du}M_a(u)du$$

$$= 2\pi i a u \cdot e^{-2\pi i \xi u^2} \cdot M_a(u)|_{-\infty}^{\infty} - 2\pi i a \int_{\mathbb{R}} \left(1 - 4\pi i \xi u^2\right) M_a(u) \cdot e^{-2\pi i \xi u^2} du$$

$$= -2\pi i a \int_{\mathbb{R}} M_a(u) \cdot e^{-2\pi i \xi u^2} du - 4\pi i \xi a \cdot \frac{d}{d\xi}\int_{\mathbb{R}} M_a(u) \cdot e^{-2\pi i \xi u^2} du$$

Thus

$$\frac{d}{d\xi}\int_{\mathbb{R}} M_a(u) \cdot e^{-2\pi i \xi u^2} du = \frac{-2\pi i a}{1 + 4\pi i a \xi}\int_{\mathbb{R}} M_a(u) \cdot e^{-2\pi i \xi u^2} du$$

For $a > 0$ the initial value problem

$$\frac{d}{d\xi}\varphi(\xi) = \frac{-2\pi i a}{1 + 4\pi i a \xi}\varphi(\xi), \ \xi \in \mathbb{R}$$

$$\varphi(0) = 1$$

has a unique solution, which must be $\int_{\mathbb{R}} M_a(u) \cdot e^{-2\pi i \xi u^2} du$ by the above computation.

Since

$$\frac{d}{d\xi}\left(\frac{1}{\sqrt{1 + 4\pi i a \xi}}\right) = \frac{-4\pi i a}{2\left(1 + 4\pi i a \xi\right)^{\frac{3}{2}}} = \frac{-2\pi i a}{1 + 4\pi i a \xi} \cdot \left(\frac{1}{\sqrt{1 + 4\pi i a \xi}}\right)$$

(notice that the root is well defined as $\mathrm{Re}\,(1+4\pi ia\xi)=1$), and

$$\left(\frac{1}{\sqrt{1+4\pi ia\xi}}\right)|_{\xi=0}=1$$

we conclude that

$$\int_{\mathbb{R}} M_a(u)\cdot e^{-2\pi i\xi u^2}du=\frac{1}{\sqrt{1+4\pi ia\xi}}$$

Finally we have that, similarly to (2.6.1)

$$\widehat{h_\delta}(\xi)=\int_0^\infty h_\delta(u)e^{-2\pi i\xi u}du=\int_{\mathbb{R}} f_\delta(u)e^{-2\pi i\xi u^2}du$$

$$=\delta\int_{\mathbb{R}} M_{\frac{1}{2\delta}}(u)\cdot e^{-2\pi i\xi u^2}du+(1-\delta)\int_{\mathbb{R}} M_{\frac{1}{2(1-\delta)}}(u)\cdot e^{-2\pi i\xi u^2}du$$

$$=\frac{\delta}{\sqrt{1+\frac{2\pi i\xi}{\delta}}}+\frac{1-\delta}{\sqrt{1+\frac{2\pi i\xi}{1-\delta}}}$$

concluding the proof of (iv). □

Next on the list is finding an asymptotic expression to $Z_N(f_{\delta_N}.r)$. In order to do that we need to check that the conditions of Theorem 2.5.5 pertaining to h_{δ_N} are true. Before we begin, we need to specify what Σ_δ^2 is, as $\gamma_1(\xi)=e^{-2\pi i\zeta}\cdot e^{-2\pi^2\xi^2\Sigma_\delta^2}$ depends on it. Since it is a central limit theorem we're after, the natural selection would be the variance of the random variable with density function h_δ, which is exactly what we'll choose.

We define

$$\Sigma_\delta^2=\int_0^\infty u^2h_\delta(u)du-\left(\int_0^\infty uh_\delta(u)du\right)^1=\frac{3}{4\delta(1-\delta)}-1$$

Theorem 2.6.2. *Let $g_\delta(\xi)=\widehat{h_\delta}(\xi)$ where $\delta<\frac{1}{2}$. Then*

(i) for $|\xi|>\frac{\delta}{4\pi}$ $|g_\delta(\xi)|\le 1-\delta\left(1-\sqrt[4]{\frac{4}{5}}\right)+\rho_1(\delta)$ where $\frac{\rho_1(\delta)}{\delta}\underset{\delta\to 0}{\longrightarrow}0$.

(ii) there exist $M_0,M_1,M_2>0$, independent of δ, such that $\sup_{|\xi|<\frac{\delta}{4\pi}}|g_\delta(\xi)-\gamma_1(\xi)|\le\left(\frac{M_0}{\delta^2}+\frac{M_1}{\delta}+M_2\right)|\xi|^3$.

(iii) for $0<\beta<1$ and $\frac{\delta^{1+\beta}}{4\pi}<|\xi|<\frac{\delta}{4\pi}$ we have that $|g_\delta(\xi)|\le 1-\frac{\delta^{1+2\beta}}{16}+\rho_2(\delta)$ where $\frac{\rho_2(\delta)}{\delta^{1+2\beta}}\underset{\delta\to 0}{\longrightarrow}0$.

(iv) $|g_\delta(\xi)| \leq 1$ *for all* ξ.

(v) for a fixed $j < N-3$ *we have that*

$$\int_{|\xi|>\frac{\delta}{4\pi}} |g_{\delta_N}(\xi)|^{N-j-1} d\xi \leq \frac{\left(1-\delta\left(1-\sqrt[4]{\frac{4}{5}}\right)+\rho_1(\delta)\right)^{N-j-1}}{\pi} + \frac{2}{\pi(N-j-3)}$$

.

Proof. *(i)* Since $\left|\sqrt{1+ix}\right| = \sqrt{|1+ix|} = \sqrt[4]{1+x^2}$ for any $x \in \mathbb{R}$ we find that for $|\xi| > \frac{\delta}{4\pi}$

$$|g_\delta(\xi)| \leq \frac{\delta}{\left|\sqrt{1+\frac{2\pi i\xi}{\delta}}\right|} + \frac{1-\delta}{\left|\sqrt{1+\frac{2\pi i\xi}{1-\delta}}\right|}$$

$$= \frac{\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{\delta^2}}} + \frac{1-\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{(1-\delta)^2}}} \leq \frac{\delta}{\sqrt[4]{\frac{5}{4}}} + \frac{1-\delta}{\sqrt[4]{1+\frac{\delta^2}{4(1-\delta)^2}}}$$

Using the expansion

$$\frac{1}{\sqrt[4]{1+x}} = 1 - \frac{x}{4} + x^2 \cdot \eta(x)$$

where η is analytic in $|x| < \frac{1}{2}$, and the fact that for $0 < \delta < \frac{1}{2}$ we have that

$$\frac{\delta^2}{4(1-\delta)^2} < \frac{\delta^2}{4\left(1-\frac{1}{2}\right)^2} = \delta^2 < \frac{1}{4}$$

We find that

$$|g_\delta(\xi)| \leq \sqrt[4]{\frac{4}{5}} \cdot \delta + (1-\delta)\left(1 - \frac{\delta^2}{16(1-\delta)^2} + \frac{\delta^4}{16(1-\delta)^4} \cdot \eta\left(\frac{\delta^2}{4(1-\delta)^2}\right)\right)$$

$$= 1 - \delta\left(1 - \sqrt[4]{\frac{4}{5}}\right) + \rho_1(\delta)$$

where $\frac{\rho_1(\delta)}{\delta} \underset{\delta\to 0}{\longrightarrow} 0$.

(ii) Using the expansions

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3}{8}x^2 + x^3 \cdot \phi(x)$$

$$e^x = 1 + x + \frac{x^2}{2} + x^3 \cdot \psi(x)$$

where $\phi(x)$ is analytic in $|x| < \frac{1}{2}$ and $\psi(x)$ is entire. we find that

$$g_\delta(\xi) = \delta\left(1 - \frac{\pi i\xi}{\delta} - \frac{3\pi^2\xi^2}{2\delta^2} - \frac{8\pi^3 i\xi^3}{\delta^3}\phi\left(\frac{2\pi i\xi}{\delta}\right)\right)$$

$$+(1-\delta)\left(1 - \frac{\pi i\xi}{1-\delta} - \frac{3\pi^2\xi^2}{2(1-\delta)^2} - \frac{8\pi^3 i\xi^3}{(1-\delta)^3}\phi\left(\frac{2\pi i\xi}{1-\delta}\right)\right)$$

$$= 1 - 2\pi i\xi - \xi^2\left(\frac{3\pi^2}{2\delta} + \frac{3\pi^2}{2(1-\delta)}\right) - 8\pi^3 i\xi^3\left(\frac{1}{\delta^2}\cdot\phi\left(\frac{2\pi i\xi}{\delta}\right) + \frac{1}{(1-\delta)^2}\cdot\phi\left(\frac{2\pi i\xi}{1-\delta}\right)\right)$$

$$= 1 - 2\pi i\xi - 2\pi^2\xi^2\left(\Sigma_\delta^2 + 1\right) - 8\pi^3 i\xi^3\left(\frac{1}{\delta^2}\cdot\phi\left(\frac{2\pi i\xi}{\delta}\right) + \frac{1}{(1-\delta)^2}\cdot\phi\left(\frac{2\pi i\xi}{1-\delta}\right)\right)$$

and

$$\gamma_1(\xi) = e^{-2\pi i\xi}\cdot e^{-2\pi^2\xi^2\Sigma_\delta^2}$$

$$= \left(1 - 2\pi i\xi - 2\pi^2\xi^2 - 8\pi^3 i\xi^3\psi\left(-2\pi i\xi\right)\right)\cdot\left(1 - 2\pi^2\Sigma_\delta^2\xi^2 + 2\pi^4\Sigma_\delta^4\xi^4 - 8\pi^6\Sigma_\delta^6\xi^6\psi\left(-2\pi^2\Sigma_\delta^2\xi^2\right)\right)$$

$$= 1 - 2\pi i\xi - 2\pi^2\xi^2\left(\Sigma_\delta^2 + 1\right) + 4\pi^3 i\xi^3\Sigma_\delta^2 + 2\pi^4\xi^4\left(\Sigma_\delta^4 + 2\Sigma_\delta^2\right)$$

$$-4\pi^5 i\Sigma_\delta^4\xi^5 - 4\pi^6\Sigma_\delta^4\xi^6 - 8\pi^3 i\xi^3\psi\left(-2\pi i\xi\right)\cdot e^{-2\pi^2\xi^2\Sigma_\delta^2} - 8\pi^6\Sigma_\delta^6\xi^6\psi\left(-2\pi^2\Sigma_\delta^2\xi^2\right)\cdot e^{-2\pi i\xi}$$

From the above we conclude that

$$|g_\delta(\xi) - \gamma_1(\xi)| \le |\xi|^3\left(\frac{8\pi^3}{\delta^2}\cdot\left|\phi\left(\frac{2\pi i\xi}{\delta}\right)\right| + \frac{8\pi^3}{(1-\delta)^2}\cdot\left|\phi\left(\frac{2\pi i\xi}{1-\delta}\right)\right|\right.$$

$$+4\pi^3\Sigma_\delta^2 + 2\pi^4|\xi|\left(\Sigma_\delta^4 + 2\Sigma_\delta^2\right)$$

$$+4\pi^5\Sigma_\delta^4|\xi|^2 + 4\pi^6\Sigma_\delta^4|\xi|^3$$

$$\left.+8\pi^3\left|\psi\left(-2\pi i\xi\right)\right| + 8\pi^6\Sigma_\delta^6|\xi|^3\left|\psi\left(-2\pi^2\Sigma_\delta^2\xi^2\right)\right|\right)$$

Denoting $M_\phi = \sup_{|x|\le\frac{1}{2}}|\phi(x)|$ and $M_\psi = \sup_{|x|\le\frac{1}{2}}|\psi(x)|$ and noticing that for $|\xi| < \frac{\delta}{4\pi}$ and $\delta < \frac{1}{2}$ we have

$$\Sigma_\delta^2 < \frac{3}{4\delta(1-\delta)} < \frac{3}{2\delta}$$

$$\left|\frac{2\pi i\xi}{\delta}\right| < \frac{1}{2}$$

$$\left|\frac{2\pi i\xi}{1-\delta}\right| < 4\pi|\xi| < \delta < \frac{1}{2}$$

we find that

$$\frac{8\pi^3}{\delta^2} \cdot \left|\phi\left(\frac{2\pi i\xi}{\delta}\right)\right| \leq \frac{8\pi^3 M_\phi}{\delta^2}$$

$$\frac{8\pi^3}{(1-\delta)^2} \cdot \left|\phi\left(\frac{2\pi i\xi}{1-\delta}\right)\right| \leq 32\pi^3 M_\phi$$

$$4\pi^3\Sigma_\delta^2 < \frac{6\pi^3}{\delta}$$

$$2\pi^4|\xi|\left(\Sigma_\delta^4 + 2\Sigma_\delta^2\right) \leq \frac{\pi^3\delta}{2}\left(\frac{9}{4\delta^2} + \frac{3}{\delta}\right) = \frac{9\pi^3}{8\delta} + \frac{3\pi^3}{2}$$

$$4\pi^5\Sigma_\delta^4|\xi|^2 \leq \frac{\pi^3\delta^2}{4} \cdot \frac{9}{4\delta^2} = \frac{9\pi^3}{16}$$

$$4\pi^6\Sigma_\delta^4|\xi|^3 \leq \frac{\pi^3\delta^3}{16} \cdot \frac{9}{4\delta^2} = \frac{9\pi^3\delta}{64} < \frac{9\pi^3}{128}$$

We also have that

$$|-2\pi i\xi| < \frac{\delta}{2} < \frac{1}{4}$$

$$\left|-2\pi^2\Sigma_\delta^2\xi^2\right| \leq \frac{\delta^2}{2} \cdot \frac{3}{2\delta} = \frac{3\delta}{4} < \frac{3}{8} < \frac{1}{2}$$

And as such

$$8\pi^3\left|\psi\left(-2\pi i\xi\right)\right| \leq 8\pi^3 M_\psi$$

$$8\pi^6\Sigma_\delta^6|\xi|^3\left|\psi\left(-2\pi^2\Sigma_\delta^2\xi^2\right)\right| \leq \frac{\pi^3\delta^3}{8} \cdot \frac{27}{8\delta^3} \cdot M_\psi = \frac{27\pi^3}{64}M_\psi$$

Defining $M_0 = 8\pi^3 M_\phi$, $M_1 = 6\pi^3 + \frac{9\pi^3}{8}$ and $M_2 = 32\pi^3 M_\phi + \frac{3\pi^3}{2} + \frac{9\pi^3}{16} + \frac{9\pi^3}{128} + 8\pi^3 M_\psi + \frac{27\pi^3}{64}M_\psi$ gives us

$$\sup_{|\xi|<\frac{\delta}{4\pi}} |g_\delta(\xi) - \gamma_1(\xi)| \leq \left(\frac{M_0}{\delta^2} + \frac{M_1}{\delta} + M_2\right)|\xi|^3$$

(iii) Much like the proof of (i), for $|\xi| > \frac{\delta^{1+\beta}}{4\pi}$

$$|g_\delta(\xi)| \leq \frac{\delta}{\sqrt[4]{1 + \frac{4\pi^2\xi^2}{\delta^2}}} + \frac{1-\delta}{\sqrt[4]{1 + \frac{4\pi^2\xi^2}{(1-\delta)^2}}} \leq \frac{\delta}{\sqrt[4]{1 + \frac{\delta^{2\beta}}{4}}} + \frac{1-\delta}{\sqrt[4]{1 + \frac{\delta^{2+2\beta}}{4(1-\delta)^2}}}$$

Since $\frac{\delta^{2\beta}}{4} < \frac{1}{4}$ and $\frac{\delta^{2+2\beta}}{4(1-\delta)^2} < \delta^{2+2\beta} < \frac{1}{4}$ we find that

$$|g_\delta(\xi)| \leq \delta\left(1 - \frac{\delta^{2\beta}}{16} + \frac{\delta^{4\beta}}{16} \cdot \eta\left(\frac{\delta^{2\beta}}{4}\right)\right) + (1-\delta)\left(1 - \frac{\delta^{2+2\beta}}{16(1-\delta)^2} + \frac{\delta^{4+4\beta}}{16(1-\delta)^4}\eta \cdot \left(\frac{\delta^{2+2\beta}}{4(1-\delta)^2}\right)\right)$$

$$= 1 - \frac{\delta^{1+2\beta}}{16} + \rho_2(\delta)$$

where $\frac{\rho_2(\delta)}{\delta^{1+2\beta}} \underset{\delta \to 0}{\longrightarrow} 0$.

(iv) Since h_δ is a density function we have that for all ξ

$$|g_\delta(\xi)| \leq \|h_\delta\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} h_\delta(u)du = 1$$

(v) For a fixed j

$$\int_{|\xi|>\frac{\delta}{4\pi}} |g_{\delta_N}(\xi)|^{N-j-1} d\xi \leq \int_{|\xi|>\frac{\delta}{4\pi}} \left(\frac{\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{\delta^2}}} + \frac{1-\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{(1-\delta)^2}}} \right)^{N-j-1} d\xi$$

$$= \int_{\frac{\delta}{4\pi}<|\xi|<\frac{1}{2\pi}} \left(\frac{\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{\delta^2}}} + \frac{1-\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{(1-\delta)^2}}} \right)^{N-j-1} d\xi$$

$$+ \int_{|\xi|>\frac{1}{2\pi}} \left(\frac{\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{\delta^2}}} + \frac{1-\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{(1-\delta)^2}}} \right)^{N-j-1} d\xi$$

Using (i) we have that

$$\int_{\frac{\delta}{4\pi}<|\xi|<\frac{1}{2\pi}} \left(\frac{\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{\delta^2}}} + \frac{1-\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{(1-\delta)^2}}} \right)^{N-j-1} d\xi \tag{2.6.2}$$

$$\leq \int_{\frac{\delta}{4\pi}<|\xi|<\frac{1}{2\pi}} \left(1 - \delta\left(1 - \sqrt[4]{\frac{4}{5}}\right) + \rho_1(\delta) \right)^{N-j-1} d\xi$$

$$\leq \frac{\left(1 - \delta\left(1 - \sqrt[4]{\frac{4}{5}}\right) + \rho_1(\delta)\right)^{N-j-1}}{\pi}$$

Also

$$\int_{|\xi|>\frac{1}{2\pi}} \left(\frac{\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{\delta^2}}} + \frac{1-\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{(1-\delta)^2}}} \right)^{N-j-1} d\xi$$

$$\leq \int_{|\xi|>\frac{1}{2\pi}} \left(\frac{\delta^{\frac{2}{3}}}{\sqrt{2\pi\xi}} + \frac{(1-\delta)^{\frac{3}{2}}}{\sqrt{2\pi\xi}} \right)^{N-j-1} d\xi$$

$$= \frac{2\left(\delta^{\frac{2}{3}} + (1-\delta)^{\frac{3}{2}}\right)^{N-j-1}}{(2\pi)^{\frac{N-j-1}{2}}} \int_{\frac{1}{2\pi}}^{\infty} \xi^{-\left(\frac{N-j-1}{2}\right)} d\xi$$

$$= \frac{2\left(\delta^{\frac{2}{3}} + (1-\delta)^{\frac{3}{2}}\right)^{N-j-1}}{(2\pi)^{\frac{N-j-1}{2}}} \cdot \left(-\frac{2\xi^{-\left(\frac{N-j-3}{2}\right)}}{N-j-3}\right)\Big|_{\frac{1}{2\pi}}^{\infty}$$

$$= \left[\frac{4\left(\delta^{\frac{2}{3}} + (1-\delta)^{\frac{3}{2}}\right)^{N-j-1}}{(2\pi)^{\frac{N-j-1}{2}}}\right] \cdot \frac{(2\pi)^{\frac{N-j-3}{2}}}{N-j-3} = \frac{2\left(\delta^{\frac{2}{3}} + (1-\delta)^{\frac{3}{2}}\right)^{N-j-1}}{\pi(N-j-3)}$$

For $0 < \delta < 1$ we have that $\delta^{\frac{2}{3}} + (1-\delta)^{\frac{3}{2}} \leq \delta + (1-\delta) = 1$ and as such

$$\int_{|\xi|>\frac{1}{2\pi}} \left(\frac{\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{\delta^2}}} + \frac{1-\delta}{\sqrt[4]{1+\frac{4\pi^2\xi^2}{(1-\delta)^2}}}\right)^{N-j-1} d\xi \leq \frac{2}{\pi(N-j-3)}i \tag{2.6.3}$$

Combining (2.6.2) and (2.6.3) we get that

$$\int_{|\xi|>\frac{\delta}{4\pi}} |g_{\delta_N}(\xi)|^{N-j-1}\, d\xi \leq \frac{\left(1-\delta\left(1-\sqrt[4]{\frac{4}{5}}\right)+\rho_1(\delta)\right)^{N-j-1}}{\pi} + \frac{2}{\pi(N-j-3)}$$

as required. $\square$

Now that we've checked that h_δ is a good candidate to use Theorem 2.5.4 we can present the asymptotic behavior of the normalization function under some conditions on δ_N.

Theorem 2.6.3. *Let $f_{\delta_N}(v) = \delta_N M_{\frac{1}{2\delta_N}}(v) + (1-\delta_N) M_{\frac{1}{2(1-\delta_N)}}(v)$ where $0 < \delta_N < \frac{1}{2}$ and*

(a') δ_N is of order of a negative power of N.

(b') $\delta_N^{1+2\beta} \cdot N \underset{N\to\infty}{\longrightarrow} \infty$.

(c') $\delta_N^{1+3\beta} N \underset{N\to\infty}{\longrightarrow} 0$.

Then for a fixed $j \in \left\{0, 1, \ldots, \left[\frac{N}{2}\right]\right\}$, $j < N-3$ and any $0 < \beta \leq \frac{2}{3}$ we have that

$$Z_{N-j}\left(f_{\delta_N}, \sqrt{u}\right) = \frac{2}{\sqrt{N-j} \cdot \Sigma_{\delta_N} \cdot |\mathbb{S}^{N-j-1}| u^{\frac{N-j}{2}-1}} \left(\frac{e^{-\frac{(u-N+j)^2}{2(N-j)\Sigma_{\delta_N}^2}}}{\sqrt{2\pi}} + \lambda_j(N-j,u)\right)$$

where $\sup_{u\in\mathbb{R}} |\lambda_j(N-j,u)| \leq \epsilon_j(N)$ *and* $\lim_{N\to\infty} \epsilon_j(N) = 0$.

Proof. We'll check the conditions of Theorem 2.5.5: Property (i) in Theorem 2.6.1 shows that h_{δ_N} is continuous and in $L^1(\mathbb{R})$. Properties (i) to (iv) of Theorem 2.6.2 corresponds to conditions (i) to (vi) of Theorem 2.5.5 with

$$\alpha(\delta_N) = \delta_N \left(\left(1 - \sqrt[4]{\frac{4}{5}} \right) - \frac{\rho_1(\delta_N)}{\delta_N} \right)$$

$$\alpha_\beta(\delta_N) = \delta_N^{1+2\beta} \left(\frac{1}{16} - \frac{\rho_2(\delta_N)}{\delta_N^{1+2\beta}} \right)$$

and $c = \frac{1}{4\pi}$. Next we check conditions (a) to (g):

Condition (a) is satisfied due to condition (a') and the definition of $\alpha(\delta_N)$ and $\alpha_\beta(\delta_N)$.

Condition (b) is satisfied since

$$\alpha(\delta_N)N = N\delta_N \left(\left(1 - \sqrt[4]{\frac{4}{5}} \right) - \frac{\rho_1(\delta_N)}{\delta_N} \right)$$

$$\geq N\delta_N^{1+2\beta} \left(\left(1 - \sqrt[4]{\frac{4}{5}} \right) - \frac{\rho_1(\delta_N)}{\delta_N} \right) \underset{N\to\infty}{\longrightarrow} \infty$$

by condition (b').

Condition (c) is satisfied since

$$\alpha_\beta(\delta_N)N = N\delta_N^{1+2\beta} \left(\frac{1}{16} - \frac{\rho_2(\delta_N)}{\delta_N^{1+2\beta}} \right) \underset{N\to\infty}{\longrightarrow} \infty$$

by condition (b').

Condition (d) is satisfied since $\Sigma_\delta = \sqrt{\frac{3}{4\delta(1-\delta)} - 1}$ and

$$\Sigma_{\delta_N}^2 \delta_N^{2+2\beta} N = \left(\frac{3}{4(1-\delta_N)} - \delta_N \right) \delta_N^{1+2\beta} N \underset{N\to\infty}{\longrightarrow} \infty$$

by condition (b').

Condition (e) is satisfied due to condition (c').

Condition (f^j) follows immediately from property (v) of Theorem 2.6.2 and a similar computation to that presented in the proof of Theorem 2.5.5.

Condition (g) is satisfied since

$$\delta_N^{\frac{3}{2}(1-\beta)}\Sigma_{\delta_N} = \sqrt{\frac{3\delta_N^{2-3\beta}}{4(1-\delta_N)} - \delta^{3(1-\beta)}}$$

and δ_N goes to zero while $0 < \beta \leq \frac{2}{3}$.

Since all the conditions are met, Theorem 2.5.5 assures us that

$$\sup_u \left| h_{\delta_N}^{*N-j}(u) - \frac{1}{\sqrt{N-j}\Sigma_{\delta_N}} \cdot \frac{e^{-\frac{(u-N+j)^2}{2(N-j)\Sigma_{\delta_N}^2}}}{\sqrt{2\pi}} \right| \leq \frac{\epsilon_j(N)}{\sqrt{N-j}\Sigma_{\delta_N}}$$

where $\epsilon_j(N) \underset{N\to\infty}{\longrightarrow} 0$. Defining $\lambda_j(N-j,u) = \sqrt{N-j}\Sigma_{\delta_N} h_{\delta_N}^{*N-j}(u) - \frac{e^{-\frac{(u-N+j)^2}{2(N-j)\Sigma_{\delta_N}^2}}}{\sqrt{2\pi}}$ and using the expression for $Z_N(f,\sqrt{u})$ from Lemma 2.4.3 we find that

$$Z_{N-j}(f,\sqrt{u}) = \frac{2h^{*N-j}(u)}{|\mathbb{S}^{N-j-1}|u^{\frac{N-j}{2}-1}}$$

$$= \frac{2}{\sqrt{N-j}\cdot\Sigma_{\delta_N}\cdot|\mathbb{S}^{N-j-1}|u^{\frac{N-j}{2}-1}} \left(\frac{e^{-\frac{(u-N+j)^2}{2(N-j)\Sigma_{\delta_N}^2}}}{\sqrt{2\pi}} + \lambda_j(N-j,u) \right)$$

Clearly $\sup_u |\lambda_j(N-j,u)| \leq \epsilon_j(N)$ and so the claim is proved. □

With the asymptotic expression in hand we're finally ready to estimate the entropy production.

Defining

$$F_N(v_1,\dots,v_N) = \frac{\Pi_{i=1}^N f_{\delta_N}(v_i)}{Z_N(f_{\delta_N},\sqrt{N})}$$

we will show that

$$\frac{\langle \log F_N, N(I-Q)F_N\rangle}{N} \leq C_{type-\delta}\left(-\delta_N \log\delta_N\right)$$

and

$$\lim_{N\to\infty} \frac{\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N\left(v_1,\dots,v_N\right)\log F_N\left(v_1,\dots,v_N\right)d\sigma^N}{N} = \lim_{N\to\infty}\frac{H_N\left(F_N\right)}{N} = \frac{\log 2}{2}$$

where $C_{type-\delta}$ is a constant depending only on the behavior of δ_N. In order to do that we will need the next technical lemma whose proof can be found in the Appendix (See Lemma A.1.5 in the Appendix)

Lemma 2.6.4. *Let $f(v_1,\dots,v_j)$ and $g(v_{j+1},\dots,v_N)$ be continuous functions on $\mathbb{R}^j$ and $\mathbb{R}^{N-j}$ respectively. Then*

$$\int_{\mathbb{S}^{N-1}(r)} f(v_1,\dots,v_j)\cdot g(v_{j+1},\dots,v_N)\,d\sigma_r^N$$

$$= \frac{\left|\mathbb{S}^{N-j-1}\right|}{\left|\mathbb{S}^{N-1}\right| r^{N-2}} \int_{\sum_{i=1}^j v_i^2\le r^2} f(v_1,\dots,v_j)\left(r^2-\sum_{i=1}^j v_i^2\right)^{\frac{N-j-2}{2}}$$

$$\left(\int_{\mathbb{S}^{N-j-1}\left(\sqrt{r^2-\sum_{i=1}^j v_i^2}\right)} g\,d\sigma^{N-j}_{\sqrt{r^2-\sum_{i=1}^j v_i^2}}\right) dv_1\dots dv_j$$

Theorem 2.6.5. *Let $F_N = \frac{\Pi_{i=1}^N f_{\delta_N}(v_i)}{Z_N(f_{\delta_N},\sqrt{N})}$ where $0<\delta_N<\frac{1}{2}$, $0<\beta\le\frac{2}{3}$ and δ_N satisfies conditions (a') to (c') in Theorem 2.6.3. Then there exists a constant $c_{type-\delta}$ depending only on the behavior of δ_N such that*

$$\frac{\langle\log F_N, N(I-Q)F_N\rangle}{N} \le c_{type-\delta}\left(-\delta_N\log\delta_N\right)$$

Proof. Denoting

$$R_{i.j}(\vartheta)(v_1,\dots,v_N) = (v_1,\dots,v_{i-1},v_i(\vartheta),v_{i+1},\dots,v_{j-1},v_j(\vartheta),v_{j+1},\dots,v_N)$$

where

$$v_i(\vartheta) = v_i\cos\vartheta + v_j\sin\vartheta$$

$$v_j(\vartheta) = -v_i\sin\vartheta + v_j\cos\vartheta$$

we have that

$$N(I-Q)F_N(v_1,\dots,v_N)$$

$$= N\cdot\frac{1}{2\pi}\cdot\frac{1}{\begin{pmatrix}N\\2\end{pmatrix}}\sum_{i<j}\int_0^{2\pi}\left(F_N(v_1,\dots,v_N) - F_N\left(R_{i,j}(\vartheta)(v_1,\dots,v_N)\right)\right)d\vartheta$$

$$= \frac{1}{\pi(N-1)}\sum_{i<j}\int_0^{2\pi}\left(F_N(v_1,\dots,v_N) - F_N\left(R_{i,j}(\vartheta)(v_1,\dots,v_N)\right)\right)d\vartheta$$

$$= \frac{1}{\pi(N-1)Z_N(f_{\delta_N},\sqrt{N})}\sum_{i<j}\int_0^{2\pi}\left(\Pi_{k=1,k\neq i,j}^N f_{\delta_N}(v_k)\right)\cdot\left(f_{\delta_N}(v_i)f_{\delta_N}(v_j)-f_{\delta_N}(v_i(\vartheta))f_{\delta_N}(v_j(\vartheta))\right)d\vartheta$$

(the operator Q was defined in Section 2.1). Also

$$\log F_N = \sum_{l=1}^N \log\left(f_{\delta_N}(v_l)\right) - \log Z_N(f_{\delta_N},\sqrt{N})$$

Remembering that for any constant function c we have

$$\langle c, N(I-Q)F_N\rangle = \langle N(I-Q)c, F_N\rangle = \langle 0, F_N\rangle = 0$$

(See Section 2.2), we find that

$$\langle \log F_N, N(I-Q)F_N\rangle = \sum_{l=1}^N \langle \log\left(f_{\delta_N}(v_l)\right), N(I-Q)F_N\rangle$$

$$= \frac{1}{Z_N(f_\delta,\sqrt{N})(N-1)\pi}\sum_{l=1}^N\sum_{i<j}\int_{\mathbb{S}^{N-1}(\sqrt{N})}\log f_{\delta_N}(v_l)$$

$$\cdot\left(\int_0^{2\pi}\left(\Pi_{k=1,k\neq i,j}^N f_{\delta_N}(v_k)\right)\cdot\left(f_{\delta_N}(v_i)f_{\delta_N}(v_j)-f_{\delta_N}(v_i(\vartheta))f_{\delta_N}(v_j(\vartheta))\right)d\vartheta\right)d\sigma^N$$

For a fixed i,j we find that if $l\neq i.j$ then by Lemma 2.6.4

$$\int_{\mathbb{S}^{N-1}(\sqrt{N})}\int_0^{2\pi}\log f_{\delta_N}(v_l)\left(\Pi_{k=1,k\neq i,j}^N f_{\delta_N}(v_k)\right)\cdot\left(f_{\delta_N}(v_i)f_{\delta_N}(v_j)-f_{\delta_N}(v_i(\vartheta))f_{\delta_N}(v_j(\vartheta))\right)d\vartheta d\sigma^N$$

$$= \frac{|\mathbb{S}^1|}{|\mathbb{S}^{N-1}|N^{\frac{N-2}{2}}}\int_0^{2\pi}\int_{\sum_{m=1,m\neq i,j}^N v_m^2\leq N}\log f_{\delta_N}(v_l)\left(\Pi_{k=1,k\neq i,j}^N f_{\delta_N}(v_k)\right)$$

$$\left(\int_{\mathbb{S}^1\left(\sqrt{N-\sum_{m=1,m\neq i,j}^N v_m^2}\right)}\left(f_{\delta_N}(v_i)f_{\delta_N}(v_j)-f_{\delta_N}(v_i(\vartheta))f_{\delta_N}(v_j(\vartheta))\right)d\sigma^2_{\sqrt{r^2-\sum_{m=1,m\neq i,j}^N v_m^2}}\right)dv_1\dots dv_{N-2}$$

Since $\mathbb{S}^1$ with the uniform measure is invariant under rotation, we have that for a given ϑ

$$\int_{\mathbb{S}^1\left(\sqrt{N-\sum_{m=1,m\neq i,j}^N v_m^2}\right)} f_{\delta_N}(v_i)f_{\delta_N}(v_j)d\sigma^2_{\sqrt{r^2-\sum_{m=1,m\neq i,j}^N v_m^2}}$$

$$= \int_{\mathbb{S}^1\left(\sqrt{N-\sum_{m=1,m\neq i,j}^N v_m^2}\right)} f_{\delta_N}(v_i(\vartheta))f_{\delta_N}(v_j(\vartheta))d\sigma^2_{\sqrt{r^2-\sum_{m=1,m\neq i,j}^N v_m^2}}$$

We conclude that only $l=i$ or $l=j$ contribute to the sum. Hence

$$\langle \log F_N, N(I-Q)F_N\rangle$$

$$= \frac{1}{Z_N(f_\delta,\sqrt{N})(N-1)\pi} \sum_{i<j} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \left(\log f_{\delta_N}(v_i) + \log f_{\delta_N}(v_j)\right)$$

$$\cdot \left(\int_0^{2\pi} \left(\Pi_{k=1,k\neq i,j}^N f_{\delta_N}(v_k)\right) \cdot \left(f_{\delta_N}(v_i) f_{\delta_N}(v_j) - f_{\delta_N}(v_i(\vartheta)) f_{\delta_N}(v_j(\vartheta))\right) d\vartheta \right) d\sigma^N$$

Next we notice that by renaming i as j and vice versa

$$\sum_{i<j} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log f_{\delta_N}(v_i)$$

$$\cdot \left(\int_0^{2\pi} \left(\Pi_{k=1,k\neq i,j}^N f_{\delta_N}(v_k)\right) \cdot \left(f_{\delta_N}(v_i) f_{\delta_N}(v_j) - f_{\delta_N}(v_i(\vartheta)) f_{\delta_N}(v_j(\vartheta))\right) d\vartheta \right) d\sigma^N$$

$$= \sum_{j<i} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log f_{\delta_N}(v_j)$$

$$\cdot \left(\int_0^{2\pi} \left(\Pi_{k=1,k\neq i,j}^N f_{\delta_N}(v_k)\right) \cdot \left(f_{\delta_N}(v_j) f_{\delta_N}(v_i) - f_{\delta_N}(v_j(-\vartheta)) f_{\delta_N}(v_i(-\vartheta))\right) d\vartheta \right) d\sigma^N$$

$$\underset{\vartheta=-\vartheta}{=} \sum_{j<i} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log f_{\delta_N}(v_j)$$

$$\cdot \left(\int_0^{2\pi} \left(\Pi_{k=1,k\neq i,j}^N f_{\delta_N}(v_k)\right) \cdot \left(f_{\delta_N}(v_i) f_{\delta_N}(v_j) - f_{\delta_N}(v_i(\vartheta)) f_{\delta_N}(v_j(\vartheta))\right) d\vartheta \right) d\sigma^N$$

As such

$$\langle \log F_N, N(I-Q)F_N \rangle == \frac{1}{Z_N(f_\delta,\sqrt{N})(N-1)\pi} \sum_{i=1}^N \sum_{j\neq i} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log f_{\delta_N}(v_i)$$

$$\cdot \left(\int_0^{2\pi} \left(\Pi_{k=1,k\neq i,j}^N f_{\delta_N}(v_k)\right) \cdot \left(f_{\delta_N}(v_i) f_{\delta_N}(v_j) - f_{\delta_N}(v_i(\vartheta)) f_{\delta_N}(v_j(\vartheta))\right) d\vartheta \right) d\sigma^N$$

For a given i, the transformation that replaces v_1 with v_i and vice versa is invariant under the uniform measure, and so

$$\langle \log F_N, N(I-Q)F_N \rangle = \frac{1}{Z_N(f_\delta,\sqrt{N})(N-1)\pi} \sum_{i=1}^N \sum_{j\neq 1} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log f_{\delta_N}(v_1)$$

$$\cdot \left(\int_0^{2\pi} \left(\Pi_{k=1,k\neq 1,j}^N f_{\delta_N}(v_k)\right) \cdot \left(f_{\delta_N}(v_1) f_{\delta_N}(v_j) - f_{\delta_N}(v_1(\vartheta)) f_{\delta_N}(v_j(\vartheta))\right) d\vartheta \right) d\sigma^N$$

$$= \frac{N}{Z_N(f_\delta,\sqrt{N})(N-1)\pi} \sum_{j\neq 1} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log f_{\delta_N}(v_1)$$

$$\cdot \left(\int_0^{2\pi} \left(\Pi_{k=1,k\neq 1,j}^N f_{\delta_N}(v_k)\right) \cdot \left(f_{\delta_N}(v_1) f_{\delta_N}(v_j) - f_{\delta_N}(v_1(\vartheta)) f_{\delta_N}(v_j(\vartheta))\right) d\vartheta \right) d\sigma^N$$

Using the same argument with v_j and v_2 we find that

$$\langle \log F_N, N(I-Q)F_N \rangle = \frac{N}{Z_N(f_\delta, \sqrt{N})\pi} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log f_{\delta_N}(v_1)$$

$$\cdot \left(\int_0^{2\pi} \left(\Pi_{k=3}^N f_{\delta_N}(v_k) \right) \cdot \left(f_{\delta_N}(v_1) f_{\delta_N}(v_2) - f_{\delta_N}(v_1(\vartheta)) f_{\delta_N}(v_2(\vartheta)) \right) d\vartheta \right) d\sigma^N$$

Using Lemma 2.6.4 we conclude that

$$\langle \log F_N, N(I-Q)F_N \rangle = \frac{N}{Z_N(f_\delta, \sqrt{N})\pi} \cdot \frac{|\mathbb{S}^{N-3}|}{|\mathbb{S}^{N-1}| N^{\frac{N-2}{2}}}$$

$$\int_0^{2\pi} \int_{v_1^2+v_2^2 \leq N} \log f_{\delta_N}(v_1) \left(f_{\delta_N}(v_1) f_{\delta_N}(v_2) - f_{\delta_N}(v_1(\vartheta)) f_{\delta_N}(v_2(\vartheta)) \right) \left(N - v_1^2 - v_2^2 \right)^{\frac{N-4}{2}}$$

$$\left(\int_{\mathbb{S}^{N-3}\left(\sqrt{N-v_1^2-v_2^2}\right)} \left(\Pi_{k=3}^N f_{\delta_N}(v_k) \right) d\sigma^{N-2}_{\sqrt{N-v_1^2-v_2^2}} \right) dv_1 dv_2 d\vartheta$$

$$= \frac{|\mathbb{S}^{N-3}| N}{|\mathbb{S}^{N-1}| N^{\frac{N-2}{2}} \pi} \int_0^{2\pi} \int_{v_1^2+v_2^2 \leq N} \log f_{\delta_N}(v_1) \left(f_{\delta_N}(v_1) f_{\delta_N}(v_2) - f_{\delta_N}(v_1(\vartheta)) f_{\delta_N}(v_2(\vartheta)) \right)$$

$$\cdot \left(N - v_1^2 - v_2^2 \right)^{\frac{N-4}{2}} \frac{Z_{N-2}\left(f_{\delta_N} . \sqrt{N - v_1^2 - v_2^2} \right)}{Z_N(f_{\delta_N}, \sqrt{N})} dv_1 dv_2 d\vartheta$$

and here we finally use Theorem 2.6.3. For $N \geq 4$ (which means we're allowed to use $j = 2$)

$$Z_{N-2}\left(f_{\delta_N} . \sqrt{N - v_1^2 - v_2^2} \right)$$

$$= \frac{2}{\sqrt{N-2} \cdot \Sigma_{\delta_N} \cdot |\mathbb{S}^{N-3}| \left(N - v_1^2 - v_2^2 \right)^{\frac{N-4}{2}}} \left(\frac{e^{-\frac{\left(-v_1^2 - v_2^2 + 2 \right)^2}{2(N-2)\Sigma_{\delta_N}^2}}}{\sqrt{2\pi}} + \lambda_2 \left(N-2, N - v_1^2 - v_2^2 \right) \right)$$

$$Z_N(f_{\delta_N}, \sqrt{N}) = \frac{2}{\sqrt{N} \cdot \Sigma_{\delta_N} \cdot |\mathbb{S}^{N-1}| N^{\frac{N-2}{2}}} \left(\frac{1}{\sqrt{2\pi}} + \lambda_0(N, N) \right)$$

And so

$$\frac{|\mathbb{S}^{N-3}| N}{|\mathbb{S}^{N-1}| N^{\frac{N-2}{2}}} \cdot \left(N - v_1^2 - v_2^2 \right)^{\frac{N-4}{2}} \frac{Z_{N-2}\left(f_{\delta_N} . \sqrt{N - v_1^2 - v_2^2} \right)}{Z_N(f_{\delta_N}, \sqrt{N})}$$

$$= \sqrt{\frac{N}{N-2}} \cdot N \cdot \frac{e^{-\frac{\left(-v_1^2 - v_2^2 + 2 \right)^2}{2(N-2)\Sigma_{\delta_N}^2}} + \sqrt{2\pi} \lambda_2 \left(N-2, N - v_1^2 - v_2^2 \right)}{1 + \sqrt{2\pi} \lambda_0 (N, N)}$$

which allows us to rewrite

$$
\begin{aligned}
&\langle \log F_N, N(I-Q)F_N\rangle \qquad (2.6.4)\\
&= \frac{N}{\pi\sqrt{1-\frac{2}{N}}}\int_0^{2\pi}\int_{v_1^2+v_2^2\le N} \log f_{\delta_N}(v_1)\left(f_{\delta_N}(v_1)f_{\delta_N}(v_2)-f_{\delta_N}(v_1(\vartheta))f_{\delta_N}(v_2(\vartheta))\right)\\
&\qquad\cdot\frac{e^{-\frac{\left(2-v_1^2-v_2^2\right)}{(N-2)\Sigma_\delta^2}}+\sqrt{2\pi}\lambda_2\left(N-2.N-v_1^2-v_2^2\right)}{1+\sqrt{2\pi}\lambda_0(N,N)}dv_1dv_2d\vartheta
\end{aligned}
$$

Using the invariance of $\{v_1^2+v_2^2\le N\}$ under rotation, and the notation $(f\otimes g)(x,y)=f(x)g(y)$, $f^{\otimes 2}=f\otimes f$ we find that

$$
\begin{aligned}
&\int_{v_1^2+v_2^2\le N} \log f_{\delta_N}(v_1)\left(f_{\delta_N}(v_1)f_{\delta_N}(v_2)-f_{\delta_N}(v_1(\vartheta))f_{\delta_N}(v_2(\vartheta))\right)\\
&\qquad\cdot\frac{e^{-\frac{\left(2-v_1^2-v_2^2\right)}{(N-2)\Sigma_\delta^2}}+\sqrt{2\pi}\lambda_2\left(N-2.N-v_1^2-v_2^2\right)}{1+\sqrt{2\pi}\lambda_0(N,N)}dv_1dv_2\\
&=\int_{v_1^2+v_2^2\le N} \log\left(f_{\delta_N}\otimes 1\right)(v_1,v_2)\left(f_{\delta_N}^{\otimes 2}(v_1,v_2)-f_{\delta_N}^{\otimes 2}(v_1(\vartheta),v_2(\vartheta))\right)\\
&\qquad\cdot\frac{e^{-\frac{\left(2-v_1^2-v_2^2\right)}{(N-2)\Sigma_\delta^2}}+\sqrt{2\pi}\lambda_2\left(N-2.N-v_1^2-v_2^2\right)}{1+\sqrt{2\pi}\lambda_0(N,N)}dv_1dv_2\\
&=\int_{v_1^2+v_2^2\le N} \log\left(f_{\delta_N}\otimes 1\right)\left(R_{1,2,-\vartheta}(v_1,v_2)\right)\left(f_{\delta_N}^{\otimes 2}\left(R_{1,2,-\vartheta}(v_1,v_2)\right)-f_{\delta_N}^{\otimes 2}\left(R_{1,2,-\vartheta}(v_1(\vartheta),v_2(\vartheta))\right)\right)\\
&\qquad\cdot\frac{e^{-\frac{\left(2-v_1^2(-\vartheta)-v_2^2(-\vartheta)\right)}{(N-2)\Sigma_\delta^2}}+\sqrt{2\pi}\lambda_2\left(N-2.N-v_1^2(-\vartheta)-v_2^2(-\vartheta)\right)}{1+\sqrt{2\pi}\lambda_0(N,N)}dv_1dv_2\\
&=\int_{v_1^2+v_2^2\le N} \log f_{\delta_N}(v_1(-\vartheta))\left(f_{\delta_N}(v_1(-\vartheta))f_{\delta_N}(v_2(-\vartheta))-f_{\delta_N}(v_1)f_{\delta_N}(v_2)\right)\\
&\qquad\cdot\frac{e^{-\frac{\left(2-v_1^2-v_2^2\right)}{(N-2)\Sigma_\delta^2}}+\sqrt{2\pi}\lambda_2\left(N-2.N-v_1^2-v_2^2\right)}{1+\sqrt{2\pi}\lambda_0(N,N)}dv_1dv_2
\end{aligned}
$$

Thus, by using the substitution $-\vartheta=\vartheta$ and combining the above with (2.6.4) we see that

$$
\begin{aligned}
&\langle \log F_N, N(I-Q)F_N\rangle \qquad (2.6.5)\\
&= \frac{N}{\pi\sqrt{1-\frac{2}{N}}}\int_0^{2\pi}\int_{v_1^2+v_2^2\le N} \log f_{\delta_N}(v_1(\vartheta))\left(f_{\delta_N}(v_1(\vartheta))f_{\delta_N}(v_2(\vartheta))-f_{\delta_N}(v_1)f_{\delta_N}(v_2)\right)
\end{aligned}
$$

$$\cdot \frac{e^{-\frac{\left(2-v_1^2-v_2^2\right)}{(N-2)\Sigma_\delta^2}} + \sqrt{2\pi}\lambda_2\left(N-2.N-v_1^2-v_2^2\right)}{1+\sqrt{2\pi}\lambda_0(N,N)} dv_1 dv_2 d\vartheta$$

We also notice that if we replace v_1 with v_2 in (2.6.4) we find that

$$\langle \log F_N, N(I-Q)F_N \rangle \tag{2.6.6}$$

$$= \frac{N}{\pi\sqrt{1-\frac{2}{N}}} \int_0^{2\pi} \int_{v_1^2+v_2^2 \leq N} \log f_{\delta_N}(v_2)\left(f_{\delta_N}(v_1) f_{\delta_N}(v_2) - f_{\delta_N}(v_1(\vartheta)) f_{\delta_N}(v_2(\vartheta))\right)$$

$$\cdot \frac{e^{-\frac{\left(2-v_1^2-v_2^2\right)}{(N-2)\Sigma_\delta^2}} + \sqrt{2\pi}\lambda_2\left(N-2.N-v_1^2-v_2^2\right)}{1+\sqrt{2\pi}\lambda_0(N,N)} dv_1 dv_2 d\vartheta$$

and similarly to (2.6.5)

$$\langle \log F_N, N(I-Q)F_N \rangle \tag{2.6.7}$$

$$= \frac{N}{\pi\sqrt{1-\frac{2}{N}}} \int_0^{2\pi} \int_{v_1^2+v_2^2 \leq N} \log f_{\delta_N}(v_2(\vartheta))\left(f_{\delta_N}(v_1(\vartheta)) f_{\delta_N}(v_2(\vartheta)) - f_{\delta_N}(v_1) f_{\delta_N}(v_2)\right)$$

$$\cdot \frac{e^{-\frac{\left(2-v_1^2-v_2^2\right)}{(N-2)\Sigma_\delta^2}} + \sqrt{2\pi}\lambda_2\left(N-2.N-v_1^2-v_2^2\right)}{1+\sqrt{2\pi}\lambda_0(N,N)} dv_1 dv_2 d\vartheta$$

Combining (2.6.4), (2.6.5), (2.6.6) and (2.6.7) gives us

$$\langle \log F_N, N(I-Q)F_N \rangle = \frac{N}{4\pi\sqrt{1-\frac{2}{N}}} \tag{2.6.8}$$

$$\int_0^{2\pi} \int_{v_1^2+v_2^2 \leq N} \left(\log f_{\delta_N}(v_1) + \log f_{\delta_N}(v_2) - \log f_{\delta_N}(v_1(\vartheta)) - \log f_{\delta_N}(v_2(\vartheta))\right)$$

$$\cdot \left(f_{\delta_N}(v_1) f_{\delta_N}(v_2) - f_{\delta_N}(v_1(\vartheta)) f_{\delta_N}(v_2(\vartheta))\right)$$

$$\cdot \frac{e^{-\frac{\left(2-v_1^2-v_2^2\right)}{(N-2)\Sigma_\delta^2}} + \sqrt{2\pi}\lambda_2\left(N-2.N-v_1^2-v_2^2\right)}{1+\sqrt{2\pi}\lambda_0(N,N)} dv_1 dv_2 d\vartheta$$

$$= \frac{N}{4\pi\sqrt{1-\frac{2}{N}}} \int_0^{2\pi} \int_{v_1^2+v_2^2 \leq N} \left(\log\left(f_{\delta_N}(v_1) f_{\delta_N}(v_2)\right) - \log\left(f_{\delta_N}(v_1(\vartheta)) f_{\delta_N}(v_2(\vartheta))\right)\right)$$

$$\cdot \left(f_{\delta_N}(v_1) f_{\delta_N}(v_2) - f_{\delta_N}(v_1(\vartheta)) f_{\delta_N}(v_2(\vartheta))\right)$$

$$\cdot \frac{e^{-\frac{\left(2-v_1^2-v_2^2\right)}{(N-2)\Sigma_\delta^2}} + \sqrt{2\pi}\lambda_2\left(N-2.N-v_1^2-v_2^2\right)}{1+\sqrt{2\pi}\lambda_0(N,N)} dv_1 dv_2 d\vartheta$$

Due to the monotonicity of the logarithm we know that

$$(\log x - \log y)(x - y) \geq 0$$

for any $x, y > 0$. That along with the fact that $\sup_{u\in\mathbb{R}} |\lambda_j(N-j,u)| \leq \epsilon_j(N)$ and $\langle \log F_N, N(I-Q)F_N\rangle \geq 0$ (See Lemma A.1.6 in the Appendix) shows us that

$$\langle \log F_N, N(I-Q)F_N\rangle = |\langle \log F_N, N(I-Q)F_N\rangle|$$

$$\leq \frac{N}{4\pi\sqrt{1-\frac{2}{N}}}\int_0^{2\pi}\int_{v_1^2+v_2^2\leq N} |\log\left(f_{\delta_N}(v_1)f_{\delta_N}(v_2)\right) - \log\left(f_{\delta_N}(v_1(\vartheta))f_{\delta_N}(v_2(\vartheta))\right)|$$

$$\cdot |f_{\delta_N}(v_1)f_{\delta_N}(v_2) - f_{\delta_N}(v_1(\vartheta))f_{\delta_N}(v_2(\vartheta))|$$

$$\cdot\frac{e^{-\frac{\left(2-v_1^2-v_2^2\right)}{(N-2)\Sigma_\delta^2}} + \sqrt{2\pi}\left|\lambda_2\left(N-2.N-v_1^2-v_2^2\right)\right|}{\left|1+\sqrt{2\pi}\lambda_0(N,N)\right|}dv_1dv_2d\vartheta$$

$$\leq \frac{N}{4\pi\sqrt{1-\frac{2}{N}}}\cdot\frac{1+\sqrt{2\pi}\epsilon_2(N)}{\left|1+\sqrt{2\pi}\lambda_0(N,N)\right|}$$

$$\int_0^{2\pi}\int_{v_1^2+v_2^2\leq N}\left(\log\left(f_{\delta_N}(v_1)f_{\delta_N}(v_2)\right) - \log\left(f_{\delta_N}(v_1(\vartheta))f_{\delta_N}(v_2(\vartheta))\right)\right)$$

$$\cdot\left(f_{\delta_N}(v_1)f_{\delta_N}(v_2) - f_{\delta_N}(v_1(\vartheta))f_{\delta_N}(v_2(\vartheta))\right)dv_1dv_2d\vartheta$$

Much like (2.6.8) we can 'untangle' the above expression and get that

$$\langle \log F_N, N(I-Q)F_N\rangle \leq \frac{N}{\pi\sqrt{1-\frac{2}{N}}}\cdot\frac{1+\sqrt{2\pi}\epsilon_2(N)}{\left|1+\sqrt{2\pi}\lambda_0(N,N)\right|} \tag{2.6.9}$$

$$\int_0^{2\pi}\int_{v_1^2+v_2^2\leq N}\log f_{\delta_N}(v_1)\left(f_{\delta_N}(v_1)f_{\delta_N}(v_2) - f_{\delta_N}(v_1(\vartheta))f_{\delta_N}(v_2(\vartheta))\right)dv_1dv_2d\vartheta$$

$$= \frac{N}{\pi\sqrt{1-\frac{2}{N}}}\cdot\frac{1+\sqrt{2\pi}\epsilon_2(N)}{\left|1+\sqrt{2\pi}\lambda_0(N,N)\right|}$$

$$\int_0^{2\pi}\int_{v_1^2+v_2^2\leq N}\left(-\log f_{\delta_N}(v_1)\right)\left(f_{\delta_N}(v_1(\vartheta))f_{\delta_N}(v_2(\vartheta)) - f_{\delta_N}(v_1)f_{\delta_N}(v_2)\right)dv_1dv_2d\vartheta$$

Remembering that $f_\delta = \delta M_{\frac{1}{2\delta}} + (1-\delta)M_{\frac{1}{2(1-\delta)}}$ and noticing that

$$M_a(v_1(\vartheta))\cdot M_a(v_2(\vartheta)) = \frac{1}{2\pi}e^{-\frac{v_1^2(\vartheta)}{2a}}\cdot e^{-\frac{v_2^2(\vartheta)}{2a}} = \frac{1}{2\pi}e^{-\frac{v_1^2(\vartheta)+v_2^2(\vartheta)}{2a}}$$

$$= \frac{1}{2\pi} e^{-\frac{v_1^2+v_2^2}{2a}} = M_a(v_1) \cdot M_a(v_2)$$

and

$$f_\delta(x) f_\delta(y) = \delta^2 M_{\frac{1}{2\delta}}(x) M_{\frac{1}{2\delta}}(y) + \delta(1-\delta) M_{\frac{1}{2\delta}}(x) M_{\frac{1}{2(1-\delta)}}(y)$$

$$\delta(1-\delta) M_{\frac{1}{2\delta}}(y) M_{\frac{1}{2(1-\delta)}}(x) + (1-\delta)^2 M_{\frac{1}{2(1-\delta)}}(x) M_{\frac{1}{2(1-\delta)}}(y)$$

we find that

$$f_\delta(v_1(\vartheta)) f_\delta(v_2(\vartheta)) - f_\delta(v_1) f_\delta(v_2) \tag{2.6.10}$$

$$= \delta(1-\delta) \left(M_{\frac{1}{2\delta}}(v_1(\vartheta)) M_{\frac{1}{2(1-\delta)}}(v_2(\vartheta)) - M_{\frac{1}{2\delta}}(v_1) M_{\frac{1}{2(1-\delta)}}(v_2) \right)$$

$$+ \delta(1-\delta) \left(M_{\frac{1}{2\delta}}(v_2(\vartheta)) M_{\frac{1}{2(1-\delta)}}(v_1(\vartheta)) - M_{\frac{1}{2\delta}}(v_2) M_{\frac{1}{2(1-\delta)}}(v_1) \right)$$

$$\leq \delta(1-\delta) \left(M_{\frac{1}{2\delta}}(v_1(\vartheta)) M_{\frac{1}{2(1-\delta)}}(v_2(\vartheta)) + M_{\frac{1}{2\delta}}(v_2(\vartheta)) M_{\frac{1}{2(1-\delta)}}(v_1(\vartheta)) \right)$$

Also, since the logarithm is an increasing function and M_a is a positive function we find that

$$-\log f_\delta(v_1) = -\log \left(\delta M_{\frac{1}{2\delta}}(v_1) + (1-\delta) M_{\frac{1}{2(1-\delta)}}(v_1) \right) \tag{2.6.11}$$

$$\leq -\log \left(\delta M_{\frac{1}{2\delta}}(v_1) \right) = -\log \left(\frac{\delta^{\frac{3}{2}}}{\sqrt{\pi}} e^{-\delta v_1^2} \right) = -\frac{3 \log \delta}{2} + \frac{\log \pi}{2} + \delta v_1^2$$

$$\leq -\frac{3 \log \delta}{2} + \frac{\log \pi}{2} + \delta \left(v_1^2 + v_2^2 \right)$$

$$= -\frac{3 \log \delta}{2} + \frac{\log \pi}{2} + \delta \left(v_1^2(\vartheta) + v_2^2(\vartheta) \right)$$

and since

$$f_\delta(v) = \frac{\delta^{\frac{3}{2}}}{\sqrt{\pi}} e^{-\delta v^2} + \frac{(1-\delta)^{\frac{3}{2}}}{\sqrt{\pi}} e^{-(1-\delta)v^2} \leq \frac{\delta^{\frac{3}{2}}}{\sqrt{\pi}} + \frac{(1-\delta)^{\frac{3}{2}}}{\sqrt{\pi}} \tag{2.6.12}$$

$$\leq \frac{\delta}{\sqrt{\pi}} + \frac{1-\delta}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} < 1$$

we have that $-\log f_\delta(v_1) > 0$. Combining this with (2.6.9), (2.6.10) and (2.6.11) yields

$$\langle \log F_N, N(I-Q)F_N \rangle \leq \frac{N}{\pi\sqrt{1-\frac{2}{N}}} \cdot \frac{1+\sqrt{2\pi}\epsilon_2(N)}{\left|1+\sqrt{2\pi}\lambda_0(N,N)\right|}$$

$$\int_0^{2\pi}\int_{v_1^2+v_2^2\le N}\left(-\frac{3\log\delta_N}{2}+\frac{\log\pi}{2}+\delta_N\left(v_1^2(\vartheta)+v_2^2(\vartheta)\right)\right)$$
$$\cdot\delta_N(1-\delta_N)\left(M_{\frac{1}{2\delta_N}}(v_1(\vartheta))M_{\frac{1}{2(1-\delta_N)}}(v_2(\vartheta))+M_{\frac{1}{2\delta_N}}(v_2(\vartheta))M_{\frac{1}{2(1-\delta_N)}}(v_1(\vartheta))\right)dv_1dv_2d\vartheta$$

Using the rotation invariance of the the disc along with its invariance under the transformation switching v_1 and v_2, we find that

$$\langle\log F_N,N(I-Q)F_N\rangle\le\frac{2N}{\pi\sqrt{1-\frac{2}{N}}}\cdot\frac{\left(1+\sqrt{2\pi}\epsilon_2(N)\right)\delta_N(1-\delta_N)}{\left|1+\sqrt{2\pi}\lambda_0(N,N)\right|}$$
$$\int_0^{2\pi}\int_{v_1^2+v_2^2\le N}\left(-\frac{3\log\delta_N}{2}+\frac{\log\pi}{2}+\delta_N\left(v_1^2+v_2^2\right)\right)\left(M_{\frac{1}{2\delta_N}}(v_1)M_{\frac{1}{2(1-\delta_N)}}(v_2)\right)dv_1dv_2d\vartheta$$
$$=\frac{4N}{\sqrt{1-\frac{2}{N}}}\cdot\frac{\left(1+\sqrt{2\pi}\epsilon_2(N)\right)\delta_N(1-\delta_N)}{\left|1+\sqrt{2\pi}\lambda_0(N,N)\right|}$$
$$\int_{v_1^2+v_2^2\le N}\left(-\frac{3\log\delta_N}{2}+\frac{\log\pi}{2}+\delta_N\left(v_1^2+v_2^2\right)\right)\left(M_{\frac{1}{2\delta_N}}(v_1)M_{\frac{1}{2(1-\delta_N)}}(v_2)\right)dv_1dv_2$$

Increasing the domain of integration from $\{v_1^2+v_2^2\le N\}$ to $\mathbb{R}^2$ only increases the above expression, and so

$$\langle\log F_N,N(I-Q)F_N\rangle\le\frac{4N}{\sqrt{1-\frac{2}{N}}}\cdot\frac{\left(1+\sqrt{2\pi}\epsilon_2(N)\right)\delta_N(1-\delta_N)}{\left|1+\sqrt{2\pi}\lambda_0(N,N)\right|}$$
$$\int_{\mathbb{R}^2}\left(-\frac{3\log\delta_N}{2}+\frac{\log\pi}{2}+\delta_N\left(v_1^2+v_2^2\right)\right)\left(M_{\frac{1}{2\delta_N}}(v_1)M_{\frac{1}{2(1-\delta_N)}}(v_2)\right)dv_1dv_2$$
$$=\frac{4N}{\sqrt{1-\frac{2}{N}}}\cdot\frac{\left(1+\sqrt{2\pi}\epsilon_2(N)\right)\delta_N(1-\delta_N)}{\left|1+\sqrt{2\pi}\lambda_0(N,N)\right|}$$
$$\Bigg[\left(-\frac{3\log\delta_N}{2}+\frac{\log\pi}{2}\right)\int_{\mathbb{R}}\int_{\mathbb{R}}M_{\frac{1}{2\delta_N}}(v_1)M_{\frac{1}{2(1-\delta_N)}}(v_2)dv_1dv_2$$
$$+\delta_N\int_{\mathbb{R}}\int_{\mathbb{R}}v_1^2M_{\frac{1}{2\delta_N}}(v_1)M_{\frac{1}{2(1-\delta_N)}}(v_2)dv_1dv_2$$
$$\delta_N\int_{\mathbb{R}}\int_{\mathbb{R}}v_2^2M_{\frac{1}{2\delta_N}}(v_1)M_{\frac{1}{2(1-\delta_N)}}(v_2)dv_1dv_2\Bigg]$$

Since $\int_{\mathbb{R}}M_a(v)dv=1$ and $\int_{\mathbb{R}}v^2M_a(v)dv=a$ (See the proof of Lemma 2.6.1) we conclude that

$$\langle\log F_N,N(I-Q)F_N\rangle\le\frac{4N}{\sqrt{1-\frac{2}{N}}}\cdot\frac{\left(1+\sqrt{2\pi}\epsilon_2(N)\right)\delta_N(1-\delta_N)}{\left|1+\sqrt{2\pi}\lambda_0(N,N)\right|}$$

$$\left(-\frac{3\log\delta_N}{2}+\frac{\log\pi}{2}+\delta_N\cdot\frac{1}{2\delta_N}+\delta_N\cdot\frac{1}{2(1-\delta_N)}\right)$$

Put differently

$$\frac{\langle\log F_N, N(I-Q)F_N\rangle}{N\left(-\delta_N\log\delta_N\right)}\leq\frac{4}{\sqrt{1-\frac{2}{N}}}\cdot\frac{\left(1+\sqrt{2\pi}\epsilon_2(N)\right)(1-\delta_N)}{\left|1+\sqrt{2\pi}\lambda_0(N,N)\right|}$$

$$\cdot\left(\frac{3}{2}-\frac{\log\pi}{2\log\delta_N}-\frac{1}{2\log\delta_N}-\frac{\delta_N}{2(1-\delta_N)\log\delta_N}\right)$$

Since δ_N satisfies conditions (a') of Theorem 2.6.3 and $|\lambda_0(N,N)|\leq\epsilon_0(N)\underset{N\to\infty}{\longrightarrow}0$ we conclude that

$$\frac{4}{\sqrt{1-\frac{2}{N}}}\cdot\frac{\left(1+\sqrt{2\pi}\epsilon_2(N)\right)(1-\delta_N)}{\left|1+\sqrt{2\pi}\lambda_0(N,N)\right|}\underset{N\to\infty}{\longrightarrow}4$$

$$\frac{3}{2}-\frac{\log\pi}{2\log\delta_N}-\frac{1}{2\log\delta_N}-\frac{\delta_N}{2(1-\delta_N)\log\delta_N}\underset{N\to\infty}{\longrightarrow}\frac{3}{2}$$

and thus there exists a constant $\tilde{c}_{type-\delta}$, depending only on the behavior of δ_N such that

$$\frac{4}{\sqrt{1-\frac{2}{N}}}\cdot\frac{\left(1+\sqrt{2\pi}\epsilon_2(N)\right)(1-\delta_N)}{\left|1+\sqrt{2\pi}\lambda_0(N,N)\right|}\cdot\left(\frac{3}{2}-\frac{\log\pi}{2\log\delta_N}-\frac{1}{2\log\delta_N}-\frac{\delta_N}{2(1-\delta_N)\log\delta_N}\right)$$

$$\leq\tilde{c}_{type-\delta}$$

This proves that for all $N\geq 4$ (which was needed for the approximation of Z_{N-2})

$$\frac{\langle\log F_N, N(I-Q)F_N\rangle}{N}\leq\tilde{c}_{type-\delta}\left(-\delta_N\log\delta_N\right)$$

Adding the cases $N=2,3$ leads us to find a constant $c_{type-\delta}$ such that

$$\frac{\langle\log F_N, N(I-Q)F_N\rangle}{N}\leq c_{type-\delta}\left(-\delta_N\log\delta_N\right)$$

for all $N\geq 2$, as was claimed. □

Theorem 2.6.6. *Let* $F_N=\frac{\Pi_{i=1}^N f_{\delta_N}(v_i)}{Z_N(f_{\delta_N},\sqrt{N})}$ *where* $0<\delta_N<\frac{1}{2}$, $0<\beta\leq\frac{2}{3}$ *and* δ_N *satisfies conditions* (a') *to* (c') *in Theorem 2.6.3. Then*

$$\lim_{N\to\infty}\frac{\int_{\mathbb{S}^{N-1}(\sqrt{N})}F_N(v_1,\dots,v_N)\log F_N(v_1,\dots,v_N)\,d\sigma^N}{N}=\frac{\log 2}{2}$$

Proof. We have that

$$\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N(v_1,\dots,v_N)\log F_N(v_1,\dots,v_N)\,d\sigma^N$$

$$= \frac{1}{Z_N(f_{\delta_N},\sqrt{N})}\int_{\mathbb{S}^{N-1}(\sqrt{N})} \Pi_{i=1}^N f_{\delta_N}(v_i)\left(\sum_{k=1}^N \log f_{\delta_N}(v_k) - \log Z_N(f_{\delta_N},\sqrt{N})\right) d\sigma^N$$

$$= \sum_{k=1}^N \frac{1}{Z_N(f_{\delta_N},\sqrt{N})}\int_{\mathbb{S}^{N-1}(\sqrt{N})} \left(\Pi_{i=1}^N f_{\delta_N}(v_i)\right)\log f_{\delta_N}(v_k)d\sigma^N$$

$$-\log Z_N(f_{\delta_N},\sqrt{N})\cdot\frac{1}{Z_N(f_{\delta_N},\sqrt{N})}\int_{\mathbb{S}^{N-1}(\sqrt{N})}\Pi_{i=1}^N f_{\delta_N}(v_i)d\sigma^N$$

$$= \left(\sum_{k=1}^N \frac{1}{Z_N(f_{\delta_N},\sqrt{N})}\int_{\mathbb{S}^{N-1}(\sqrt{N})} \left(\Pi_{i=1}^N f_{\delta_N}(v_i)\right)\log f_{\delta_N}(v_k)d\sigma^N\right) - \log Z_N(f_{\delta_N},\sqrt{N})$$

For a fixed k, switching between v_k and v_1 is invariant under the uniform measure and as such

$$\sum_{k=1}^N \frac{1}{Z_N(f_{\delta_N},\sqrt{N})}\int_{\mathbb{S}^{N-1}(\sqrt{N})} \left(\Pi_{i=1}^N f_{\delta_N}(v_i)\right)\log f_{\delta_N}(v_k)d\sigma^N$$

$$= \frac{N}{Z_N(f_{\delta_N},\sqrt{N})}\int_{\mathbb{S}^{N-1}(\sqrt{N})} \left(\Pi_{i=1}^N f_{\delta_N}(v_i)\right)\log f_{\delta_N}(v_1)d\sigma^N$$

Using Lemma 2.6.4 we find that

$$\int_{\mathbb{S}^{N-1}(\sqrt{N})} \left(\Pi_{i=1}^N f_{\delta_N}(v_i)\right)\log f_{\delta_N}(v_1)d\sigma^N$$

$$= \frac{|\mathbb{S}^{N-2}|}{|\mathbb{S}^{N-1}|N^{\frac{N-2}{2}}}\int_{-\sqrt{N}}^{\sqrt{N}} f_{\delta_N}(v_1)\log f_{\delta_N}(v_1)\left(N-v_1^2\right)^{\frac{N-3}{2}}$$

$$\left(\int_{\mathbb{S}^{N-2}\left(\sqrt{N-v_1^2}\right)} \left(\Pi_{i=2}^N f_{\delta_N}(v_i)\right) d\sigma^{N-1}_{\sqrt{N-v_1^2}}\right) dv_1$$

$$= \frac{|\mathbb{S}^{N-2}|}{|\mathbb{S}^{N-1}|N^{\frac{N-2}{2}}}\int_{-\sqrt{N}}^{\sqrt{N}} f_{\delta_N}(v_1)\log f_{\delta_N}(v_1)\left(N-v_1^2\right)^{\frac{N-3}{2}}\cdot Z_{N-1}\left(f_{\delta_N},\sqrt{N-v_1^2}\right)dv_1$$

Using Theorem 2.6.3 for $N \geq 4$ and $j = 0,1$ we have

$$Z_{N-1}\left(f_{\delta_N}.\sqrt{N-v_1^2}\right)$$

$$= \frac{2}{\sqrt{N-1}\cdot\Sigma_{\delta_N}\cdot|\mathbb{S}^{N-2}|\left(N-v_1^2\right)^{\frac{N-3}{2}}}\left(\frac{e^{-\frac{\left(-v_1^2+1\right)^2}{2(N-1)\Sigma_{\delta_N}^2}}}{\sqrt{2\pi}}+\lambda_1\left(N-1,N-v_1^2\right)\right)$$

$$Z_N(f_{\delta_N},\sqrt{N}) = \frac{2}{\sqrt{N}\cdot\Sigma_{\delta_N}\cdot|\mathbb{S}^{N-1}|N^{\frac{N-2}{2}}}\left(\frac{1}{\sqrt{2\pi}}+\lambda_0(N,N)\right)$$

Thus

$$\frac{|\mathbb{S}^{N-2}|}{|\mathbb{S}^{N-1}|N^{\frac{N-2}{2}}}\cdot\left(N-v_1^2\right)^{\frac{N-3}{2}}\cdot\frac{Z_{N-1}\left(f_{\delta_N},\sqrt{N-v_1^2}\right)}{Z_N(f_{\delta_N},\sqrt{N})}$$

$$= \sqrt{\frac{N}{N-1}}\cdot\frac{e^{-\frac{\left(-v_1^2+1\right)^2}{2(N-1)\Sigma_{\delta_N}^2}}+\sqrt{2\pi}\lambda_1\left(N-1,N-v_1^2\right)}{1+\sqrt{2\pi}\lambda_0(N,N)}$$

and as such

$$\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N\left(v_1,\dots,v_N\right)\log F_N\left(v_1,\dots,v_N\right)d\sigma^N \tag{2.6.13}$$

$$= \frac{N}{\sqrt{1-\frac{1}{N}}}\int_{-\sqrt{N}}^{\sqrt{N}} f_{\delta_N}(v_1)\log f_{\delta_N}(v_1)\cdot\frac{e^{-\frac{\left(-v_1^2+1\right)^2}{2(N-1)\Sigma_{\delta_N}^2}}+\sqrt{2\pi}\lambda_1\left(N-1,N-v_1^2\right)}{1+\sqrt{2\pi}\lambda_0(N,N)}dv_1$$

$$-\log Z_N(f_{\delta_N},\sqrt{N})$$

Next we notice that

$$|\mathbb{S}^{N-1}| = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} = \frac{(2\pi e)^{\frac{N}{2}}}{\sqrt{\pi N}\cdot N^{\frac{N-2}{2}}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right)}$$

and as such

$$Z_N(f_{\delta_N},\sqrt{N}) = \frac{2\cdot\sqrt{\pi N}\cdot N^{\frac{N-2}{2}}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right)}{\sqrt{N}\Sigma_{\delta_N}N^{\frac{N-2}{2}}\left(2\pi e\right)^{\frac{N}{2}}}\left(\frac{1}{\sqrt{2\pi}}+\lambda_0(N,N)\right)$$

$$= \frac{\sqrt{2}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right)}{\Sigma_{\delta_N}\left(2\pi e\right)^{\frac{N}{2}}}\left(1+\sqrt{2\pi}\lambda_0(N,N)\right)$$

which implies

$$\log Z_N(f_{\delta_N},\sqrt{N}) = \log\left(\sqrt{2}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right)\left(1+\sqrt{2\pi}\lambda_0(N,N)\right)\right) \tag{2.6.14}$$

$$-\frac{N}{2}\log\left(2\pi e\right)-\frac{1}{2}\cdot\log\left(\frac{3}{4\delta_N(1-\delta_N)}-1\right)$$

Combining (2.6.13) and (2.6.14) yields

$$\frac{\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N\left(v_1,\dots,v_N\right)\log F_N\left(v_1,\dots,v_N\right)d\sigma^N}{N} = \frac{1}{\sqrt{1-\frac{1}{N}}\left(1+\sqrt{2\pi}\lambda_0(N,N)\right)} \tag{2.6.15}$$

$$\int_{-\sqrt{N}}^{\sqrt{N}} f_{\delta_N}(v_1)\log f_{\delta_N}(v_1)\left(e^{-\frac{\left(-v_1^2+1\right)^2}{2(N-1)\Sigma_{\delta_N}^2}}+\sqrt{2\pi}\lambda_1\left(N-1,N-v_1^2\right)\right)dv_1$$

$$-\frac{\log\left(\sqrt{2}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right)\left(1+\sqrt{2\pi}\lambda_0(N,N)\right)\right)}{N}+\frac{1}{2}\log\left(2\pi e\right)+\frac{1}{2N}\cdot\log\left(\frac{3}{4\delta_N(1-\delta_N)}-1\right)$$

We'll show that each term in (2.6.15) converges as N goes to infinity.

$$\int_{-\sqrt{N}}^{\sqrt{N}} f_{\delta_N}(v_1)\log f_{\delta_N}(v_1)\left(e^{-\frac{\left(-v_1^2+1\right)^2}{2(N-1)\Sigma_{\delta_N}^2}}+\sqrt{2\pi}\lambda_1\left(N-1,N-v_1^2\right)\right)dv_1$$

$$\int_{\mathbb{R}} f_{\delta_N}(v_1)\log f_{\delta_N}(v_1)\chi_{[-\sqrt{N},\sqrt{N}]}(v_1)\left(e^{-\frac{\left(-v_1^2+1\right)^2}{2(N-1)\Sigma_{\delta_N}^2}}+\sqrt{2\pi}\lambda_1\left(N-1,N-v_1^2\right)\right)dv_1$$

Since $0<f_{\delta_N}<1$ (See (2.6.12) in the proof of Theorem 2.6.5) and $\sup_{u\in\mathbb{R}}|\lambda_1(N-1,u)|\leq \epsilon_1(N)$ we have that

$$\left|f_{\delta_N}(v_1)\log f_{\delta_N}(v_1)\chi_{[-\sqrt{N},\sqrt{N}]}(v_1)\left(e^{-\frac{\left(-v_1^2+1\right)^2}{2(N-1)\Sigma_{\delta_N}^2}}+\sqrt{2\pi}\lambda_1\left(N-1,N-v_1^2\right)\right)\right| \tag{2.6.16}$$

$$\leq -f_{\delta_N}(v_1)\log f_{\delta_N}(v_1)\left(1+\sqrt{2\pi}\epsilon_1(N)\right)$$

The logarithm is an increasing function and M_a is a positive function, and so

$$-\log f_{\delta_N}(v_1) = -\log\left(\delta_N M_{\frac{1}{2\delta_N}}(v_1)+(1-\delta_N)M_{\frac{1}{2(1-\delta_N)}}(v_1)\right)$$

$$\leq \min\left(-\log\left(\delta_N M_{\frac{1}{2\delta_N}}(v_1)\right),-\log\left((1-\delta_N)M_{\frac{1}{2(1-\delta_N)}}(v_1)\right)\right)$$

which implies

$$-f_{\delta_N}(v_1)\log f_{\delta_N}(v_1) = \delta_N M_{\frac{1}{2\delta_N}}(v_1)\cdot(-\log f_{\delta_N}(v_1))+(1-\delta_N)M_{\frac{1}{2(1-\delta_N)}}(v_1)\cdot(-\log f_{\delta_N}(v_1))$$

$$\leq -\delta_N M_{\frac{1}{2\delta_N}}(v_1)\log\left(\delta_N M_{\frac{1}{2\delta_N}}(v_1)\right) - (1-\delta_N)M_{\frac{1}{2(1-\delta_N)}}(v_1)\log\left((1-\delta_N)M_{\frac{1}{2(1-\delta_N)}}(v_1)\right)$$

Define

$$g_N(v_1) = -\delta_N M_{\frac{1}{2\delta_N}}(v_1)\log\left(\delta_N M_{\frac{1}{2\delta_N}}(v_1)\right) - (1-\delta_N)M_{\frac{1}{2(1-\delta_N)}}(v_1)\log\left((1-\delta_N)M_{\frac{1}{2(1-\delta_N)}}(v_1)\right)$$

Since $\delta_N \underset{N\to\infty}{\longrightarrow} 0$ from condition (a') of Theorem 2.6.3 we conclude that

$$\delta_N M_{\frac{1}{2\delta_N}}(v_1) = \frac{\delta_N^{\frac{3}{2}}}{\sqrt{\pi}} \cdot e^{-\delta_N v_1^2} \underset{N\to\infty}{\longrightarrow} 0 \tag{2.6.17}$$

$$(1-\delta_N)M_{\frac{1}{2(1-\delta_N)}}(v_1) = \frac{(1-\delta_N)^{\frac{3}{2}}}{\sqrt{\pi}} \cdot e^{-(1-\delta_N)v_1^2} \underset{N\to\infty}{\longrightarrow} \frac{e^{-v_1^2}}{\sqrt{\pi}} = M_{\frac{1}{2}}(v)$$

and as such

$$g_N(v_1) \underset{N\to\infty}{\longrightarrow} -M_{\frac{1}{2}}(v_1)\log\left(M_{\frac{1}{2}}(v_1)\right) \tag{2.6.18}$$

pointwise. On the other hand, since

$$\frac{1}{2a}\int_{\mathbb{R}} M_a(v)\log\left(\frac{M_a(v)}{2a}\right)dv = \frac{1}{2a}\int_{\mathbb{R}} M_a(v)\log\left(\frac{1}{\sqrt{\pi}\cdot(2a)^{\frac{3}{2}}}\cdot e^{-\frac{v^2}{2a}}\right)dv$$

$$= \frac{-3\log(2a)-\log\pi}{4a}\int_{\mathbb{R}} M_a(v)dv - \frac{1}{4a^2}\int_{\mathbb{R}} M_a(v)v^2dv$$

$$= \frac{-3\log(2a)-\log\pi}{4a} - \frac{1}{4a} = \frac{-3\log(2a)-\log\pi-1}{4a}$$

(the last equality is due the computation in the proof of Lemma 2.6.1), we find that

$$\begin{aligned}\int_{\mathbb{R}} g_N(v_1)dv_1 &= -\frac{\delta_N\left(-3\log\left(\frac{1}{\delta_N}\right)-\log\pi-1\right)}{2} - \frac{(1-\delta_N)\left(-3\log\left(\frac{1}{(1-\delta_N)}\right)-\log\pi-1\right)}{2} \\ &= \frac{\delta_N\left(-3\log\delta_N+\log\pi+1\right)+(1-\delta_N)\left(-3\log(1-\delta_N)+\log\pi+1\right)}{2} \\ &\underset{N\to\infty}{\longrightarrow} \frac{\log\pi+1}{2} = -\int_{\mathbb{R}} M_{\frac{1}{2}}(v_1)\log\left(M_{\frac{1}{2}}(v_1)\right)dv_1\end{aligned} \tag{2.6.19}$$

Lastly (2.6.17) tells us that

$$f_{\delta_N}(v_1)\log f_{\delta_N}(v_1)\chi_{[-\sqrt{N},\sqrt{N}]}(v_1) \underset{N\to\infty}{\longrightarrow} M_{\frac{1}{2}}(v_1)\log\left(M_{\frac{1}{2}}(v_1)\right)$$

pointwise, and since $|\sup_u \lambda_1(N-1,u)| \leq \epsilon_1(N) \underset{N\to\infty}{\longrightarrow} 0$ and

$$N\Sigma_{\delta_N}^2 = N\left(\frac{3}{4\delta_N(1-\delta_N)} - 1\right) \underset{N\to\infty}{\longrightarrow} \infty$$

we have that

$$f_{\delta_N}(v_1)\log f_{\delta_N}(v_1)\chi_{[-\sqrt{N},\sqrt{N}]}(v_1)\left(e^{-\frac{\left(-v_1^2+1\right)^2}{2(N-1)\Sigma_{\delta_N}^2}} + \sqrt{2\pi}\lambda_1\left(N-1,N-v_1^2\right)\right) \tag{2.6.20}$$

$$\underset{N\to\infty}{\longrightarrow} M_{\frac{1}{2}}(v_1)\log\left(M_{\frac{1}{2}}(v_1)\right)$$

pointwise. Combining (2.6.16), (2.6.18), (2.6.19), (2.6.20) and the generalized Dominated Convergence Theorem gives

$$\int_{-\sqrt{N}}^{\sqrt{N}} f_{\delta_N}(v_1)\log f_{\delta_N}(v_1)\left(e^{-\frac{\left(-v_1^2+1\right)^2}{2(N-1)\Sigma_{\delta_N}^2}} + \sqrt{2\pi}\lambda_1\left(N-1,N-v_1^2\right)\right)dv_1 \tag{2.6.21}$$

$$\underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} M_{\frac{1}{2}}(v_1)\log\left(M_{\frac{1}{2}}(v_1)\right)dv_1 = -\frac{\log\pi+1}{2}$$

Next we notice that since $|\lambda_0(N,N)| \leq \epsilon_0(N) \underset{N\to\infty}{\longrightarrow} 0$

$$\frac{\log\left(\sqrt{2}\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right)\left(1+\sqrt{2\pi}\lambda_0(N,N)\right)\right)}{N} \underset{N\to\infty}{\longrightarrow} 0 \tag{2.6.22}$$

Also,

$$\frac{1}{2N}\cdot\log\left(\frac{3}{4\delta_N(1-\delta_N)}-1\right) = \frac{1}{2N}\log\left(\frac{1}{\delta_N}\right) + \frac{1}{2N}\log\left(\frac{3}{4(1-\delta_N)}-\delta_N\right)$$

$$= -\frac{1}{2N\delta_N}\cdot\delta_N\log\delta_N + \frac{1}{2N}\log\left(\frac{3}{4(1-\delta_N)}-\delta_N\right)$$

$$= -\frac{\delta_N^{2\beta}}{2N\delta_N^{1+2\beta}}\cdot\delta_N\log\delta_N + \frac{1}{2N}\log\left(\frac{3}{4(1-\delta_N)}-\delta_N\right)$$

Using condition (b') of Theorem 2.6.3 we find that

$$\frac{1}{2N}\cdot\log\left(\frac{3}{4\delta_N(1-\delta_N)}-1\right) \underset{N\to\infty}{\longrightarrow} 0 \tag{2.6.23}$$

Finally, (2.6.15), (2.6.21), (2.6.22), (2.6.23) and the fact that $\lambda_0(N,N) \underset{N\to\infty}{\longrightarrow} 0$ show that

$$\frac{\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N\left(v_1,\dots,v_N\right)\log F_N\left(v_1,\dots,v_N\right) d\sigma^N}{N}$$
$$\underset{\to\infty}{\longrightarrow} -\frac{\log\pi+1}{2}+\frac{1}{2}\log\left(2\pi e\right)=\frac{\log 2}{2}$$

which is the desired result. $\square$

The last two theorems allow us to conclude the following:

Theorem 2.6.7. *Let* $F_N = \frac{\Pi_{i=1}^N f_{\delta_N}(v_i)}{Z_N(f_{\delta_N},\sqrt{N})}$ *where* $0<\delta_N<\frac{1}{2}$, $0<\beta\leq\frac{2}{3}$ *and* δ_N *satisfies conditions* (a') *to* (c') *in Theorem 2.6.3. Then there exists a constant* $C_{type-\delta}$ *depending only on the behavior of* δ_N *such that*

$$\frac{\langle\log F_N, N(I-Q)F_N\rangle}{\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N\left(v_1,\dots,v_N\right)\log F_N\left(v_1,\dots,v_N\right) d\sigma^N}\leq C_{type-\delta}\left(-\delta_N\log\delta_N\right)$$

In particular

$$\Gamma_N\leq C_{type-\delta}\left(-\delta_N\log\delta_N\right)$$

Proof. This follows immediately from Theorems 2.6.5 and 2.6.6. Since

$$\frac{\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N\left(v_1,\dots,v_N\right)\log F_N\left(v_1,\dots,v_N\right) d\sigma^N}{N}\neq 0$$

(See Lemma A.1.6 in the Appendix) and it converges to $\frac{\log 2}{2}$ we know that it is bounded from below by a positive constant α. As such

$$\frac{\langle\log F_N, N(I-Q)F_N\rangle}{\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N\left(v_1,\dots,v_N\right)\log F_N\left(v_1,\dots,v_N\right) d\sigma^N}\leq \frac{c_{type-\delta}}{\alpha}\left(-\delta_N\log\delta_N\right)$$

where $c_{type-\delta}$ is the constant found in Theorem 2.6.5. This concludes the first part of the theorem.

Since

$$\Gamma_N=\inf\frac{\langle\log\psi_N, N(I-Q)\psi_N\rangle}{\int_{\mathbb{S}^{N-1}(\sqrt{N})}\psi_N\left(v_1,\dots,v_N\right)\log\psi_N\left(v_1,\dots,v_N\right) d\sigma^N}$$

where ψ_N is a density function on $\mathbb{S}^{N-1}(\sqrt{N})$ we find that

$$\Gamma_N\leq\frac{c_{type-\delta}}{\alpha}\left(-\delta_N\log\delta_N\right)$$

which is the second part of the theorem. $\square$

Theorem 2.6.8. *Let* $F_N = \frac{\Pi_{i=1}^N f_{\delta_N}(v_i)}{Z_N(f_{\delta_N},\sqrt{N})}$ *where* $\delta_N = \frac{1}{N^{1-2\beta}}$*, and* $\beta > 0$*. Then there exists a constant* C_β *depending only on* β *such that*

$$\frac{\langle \log F_N, N(I-Q)F_N \rangle}{\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N(v_1,\dots,v_N) \log F_N(v_1,\dots,v_N)\, d\sigma^N} \leq \frac{C_\beta \log N}{N^{1-2\beta}}$$

In particular

$$\Gamma_N \leq \frac{C_\beta \log N}{N^{1-2\beta}}$$

Proof. Without loss of generality we can assume that $\beta < \frac{1}{6}$.

Since

$$-\log\left(\frac{1}{N^{1-2\beta}}\right) \cdot \frac{1}{N^{1-2\beta}} = \frac{(1-2\beta)\log N}{N^{1-2\beta}}$$

this will follow immediately from Theorem 2.6.7 if we can show that conditions (a') to (c') of Theorem 2.6.3 are satisfied.

(a') is obviously true since δ_N is a negative power of N.

For (b') we notice that

$$\delta_N^{1+2\beta} N = \frac{N}{N^{1-4\beta^2}} = N^{4\beta^2} \underset{N\to\infty}{\longrightarrow} \infty$$

For (c') we have that since $0 < \beta < \frac{1}{6}$

$$\delta_N^{1+3\beta} N = N^{-(1+3\beta)(1-2\beta)} \cdot N = N^{6\beta^2-\beta} = N^{\beta(6\beta-1)} \underset{N\to\infty}{\longrightarrow} 0$$

Obviously $\delta_N < \frac{1}{2}$ for $N \geq 3$ and the addition of the case $N = 2$ may only change the constant C_β slightly. □

Theorem 2.6.9. *Let* $0 < \eta < 1$*. There exists a constant* C_η *depending only on* η *such that*

$$\Gamma_N \leq \frac{C_\eta}{N^\eta}$$

Proof. Given any $0 < \eta < 1$ we can find $\epsilon > 0$ such that $\eta < \frac{1}{1+\epsilon}$ (for instance $\epsilon = \frac{1}{2}\left(\frac{1}{\eta} - 1\right)$). Choose

$$\beta = \frac{1-\eta(1+\epsilon)}{2}$$

By Theorem 2.6.8, we can find a constant $C_{\beta(\eta)}$ such that

$$\Gamma_N \leq \frac{C_{\beta(\eta)} \log N}{N^{1-2\beta}}$$

Since $1 - 2\beta = \eta(1+\epsilon)$ we have that

$$\Gamma_N \leq \frac{C_\beta \log N}{N^{\eta\epsilon}} \cdot \frac{1}{N^\eta}$$

and since the $\eta\epsilon > 0$ we can find another constant D_η such that $\frac{\log N}{N^{\eta\epsilon}} \leq D_\eta$ for all $N \geq 2$. Thus

$$\Gamma_N \leq \frac{C_{\beta(\eta)} D_\eta}{N^\eta}$$

which is the desired result. $\square$

The last section of this chapter will be devoted to a few last remarks.

2.7 Last Remarks

For all practical purposes Theorem 2.6.9 tells us that the entropy production approach is not better than that of the spectral gap. We'll still have to wait time almost proportional to N in order to see every system of N particles equilibrate. Is there no hope? A careful look at our results raises the following question:

Problem. In our result, as in [4] , the fourth moment of the one particle generating function played a major role via the central limit theorem. In both, the sequence of test functions had the property that its fourth moment, Σ_{δ_N}, was unbounded as N went to infinity. Will we get a better estimate on Γ_N if we restrict ourselves to the case where the fourth moment of the test functions is bounded uniformly in N?

We still don't have any ideas if the above is true or false. Another, more academic, question is also natural:

Problem. Can the methods we employed in this chapter be used to prove or disprove Villani's conjecture?

To this question we believe the answer is no. The purpose of the the technique we developed was to estimate the entropy production via a known sequence δ_N. Hoping to be able to use some negative power of N as δ_N proved to be possible but with restriction: conditions (b') and (c') from Theorem 2.6.3

$$\delta_N^{1+2\beta} \cdot N \underset{N\to\infty}{\longrightarrow} \infty$$

$$\delta_N^{1+3\beta} N \underset{N\to\infty}{\longrightarrow} 0$$

This gives a very tight choice on possible δ_N's and we feel we exploited it to the fullest. There is a chance that one can pick a better one particle generating function, and by that get different function $\alpha(\delta)$, $\alpha_\beta(\delta)$ in an equivalent Theorem to Theorem 2.6.2, leading to a possible better upper bound, but we believe that our functions are very natural and optimize the problem. We feel that in order to prove or disprove Villani's conjecture new techniques are needed, and we hope to be able to see the conjecture settled in the near future.

Chapter III

TRACE INEQUALITY FOR THE FRACTIONAL LAPLACIAN

3.1 Relativistic Energy and the Fractional Laplacian

The beginning of the 20th century was filled with great discoveries in the world of Physics. One of the biggest and most influential, emerging in 1925, was Quantum Mechanics. Quantum Mechanics provided a description to the dual wave-particle properties of matter and investigated the subatomic level with incredible accuracy. The combination of ideas from Statistical Mechanics, Classical Mechanics, Probability Theory and the Physics of Waves resulted in a robust theory capable of explaining and predicting many unexplained and unknown phenomena.

One of the crucial ideas in Quantum Mechanics is the introduction of the state function $\psi(x)$, whose square is the density function for probability to find the particle at position x. Due to wave-particle duality, the square of its Fourier transform, $\hat{\psi}(p)$, represents the density function for probability to find the particle the at momentum p.

The main tool to understand phenomena in Quantum Mechanics is the Schrodinger equation, which is the 'wave equation' for the state function $\psi(x)$. The roots of the equation lie in the classical energy equation

$$E = \frac{p^2}{2m} + U$$

where $\frac{p^2}{2m}$ is the kinetic energy term and U is the potential energy term.

Incorporating this into Quantum Mechanics we find that the correct expression for the kinetic energy in Quantum Mechanics is:

$$\frac{1}{2m} \int_{\mathbb{R}^n} |p|^2 \left| \widehat{\psi}(p) \right|^2 dp$$

The fact that for nice enough function f, for instance Schwartz class, we have that

$$\langle f, (-\Delta) f \rangle = (2\pi)^2 \int_{\mathbb{R}^n} |p|^2 \left| \widehat{f}(p) \right|^2 dp \tag{3.1.1}$$

insinuates that we should connect the kinetic term $\frac{p^2}{2m}$ to the operator $\frac{1}{2m}(-\Delta)$, which is indeed what Schrodinger did in his equation.

In 1928 Quantum Mechanics took another leap forward by integrating Einstein's special relativity into itself, resulting in the celebrated Dirac Equation. The main point behind the equation was that in relativity the kinetic energy is not given by $E = \frac{p^2}{2m}$ but by $E = |p|c$, where c is the speed of light. Dirac equation is far more complicated than Schrodinger's, but it managed to include a new property of matter and energy called 'Spin'. It also managed to correctly explain some matter-energy phenomena that were a mystery until then. For our discussion though, the interesting part is that the new kinetic energy expression is

$$c \int_{\mathbb{R}^n} |p| \left| \widehat{\psi}(p) \right|^2 dp$$

The resemblance with the classical kinetic energy, and its interpretation as a partial differential equation related to the Laplacian, prompts us to define the operator $\sqrt{-\Delta}$ as

$$\left\langle f, \sqrt{-\Delta} f \right\rangle = 2\pi \int_{\mathbb{R}^n} |p| \left| \widehat{f}(p) \right|^2 dp \tag{3.1.2}$$

or $\left(\widehat{\sqrt{-\Delta} f}\right)(p) = |p| \widehat{f}(p)$.

Mathematically speaking, the language of Schrodinger's equation is the language of the Sobolev space $H^1(\mathbb{R}^n)$: The space of all $L^2(\mathbb{R}^n)$ function that have weak derivative in $L^2(\mathbb{R}^n)$. The language of Dirac's equation is a that of the fractional Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^n)$: The space of all $L^2(\mathbb{R}^n)$ function f such that their Fourier transform $\widehat{f}$ satisfies the condition $\int_{\mathbb{R}^n} |p| \left|\widehat{f}(p)\right|^2 dp < \infty$. This is the first and simplest example of the fractional Laplacian.

In general we can define the fractional Laplacian of power α as the operator

$$\widehat{(-\Delta)^\alpha} f(p) = |p|^{2\alpha} \widehat{f}(p) \tag{3.1.3}$$

when the right hand side makes sense. This operator, besides being a natural generalization of the classical and relativistic operators, has its own merits: it is connected to fractal stochastic process and stable Levy process (and as such to finances), it is connected to nonlinear diffusion processes and in pure mathematics it is an example for a pseudo- differential operators, arising naturally in the subject of Harmonic Analysis.

In this chapter we will keep the definition of the fractional Laplacian as in (3.1.3) when we can. Also, motivated by (3.1.1) and (3.1.2) we define

$$\langle f, (-\Delta)^\alpha f \rangle = (2\pi)^{2\alpha} \int_{\mathbb{R}^n} |p|^{2\alpha} \left|\widehat{f}(p)\right|^2 dp \tag{3.1.4}$$

This chapter is devoted to a new trace inequality connected to the fractional Laplacian. Before we begin with our new results, we will mention what have been done so far.

3.2 Known Sharp Trace Inequalities connected to Fractional Laplacian

Trace inequalities are very common in Mathematics and provide a way to connect between 'boundary values' of a function and 'interior values' of its derivatives - usually in an integral form. Sharp trace inequalities pose a far greater tool, as they distill the inequality to its truest form, usually with the classification of possibilities to attain equality in the inequality. Physically speaking, sharp trace inequality can reflect a connection between some sort of density of charge on the boundary and the total energy inside the domain, it is connected to capacitance problems, and many more examples.

A prime example for such sharp inequality is the inequality found by Jose' F. Escobar in [10] showing that

$$\left(\int_{\mathbb{R}^{n-1}} |\tau f(x)|^{\frac{2(n-1)}{n-2}}\, dx\right)^{\frac{n-2}{(n-1)}} \leq \frac{1}{\sqrt{\pi}\cdot(n-2)} \cdot \left\{\frac{\Gamma(n-1)}{\Gamma\left(\frac{n-1}{2}\right)}\right\}^{\frac{1}{n}} \cdot \int_{\mathbb{H}^n} |\nabla f(x,t)|^2\, dxdt \tag{3.2.1}$$

where $\mathbb{H}^n = \{(x,t)\,|\, x\in\mathbb{R}^n,\ t>0\}$ and τf is the trace of the function on the boundary of $\mathbb{H}^n$. Escobar managed to show that the inequality is sharp and completely classify the functions which give equality. Different proofs for (3.2.1) were found by Beckner in [1], Carlen and Loss in [5] and Maggi and Villani in [20] whose approach to the problem has been generalized by Nazaret in [21].

In view of such inequality a desire to try and find a similar one for the fractional Laplacian is natural. In [24] Xiao managed to show that for $\alpha\in(0,1)$

$$\left(\int_{\mathbb{R}^{n-1}} |g(x)|^{\frac{2n}{n-2\alpha}}\, dx\right)^{\frac{n-2\alpha}{n}} \leq \frac{2^{1-4\alpha}}{\pi^{\alpha}\cdot\Gamma(2-2\alpha)} \cdot \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma\left(\frac{n+2\alpha}{2}\right)} \cdot \left\{\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right\}^{\frac{2\alpha}{n}} \cdot \int_{\mathbb{H}^n} |\nabla f(x,t)|^2\, t^{1-2\alpha} dxdt \tag{3.2.2}$$

where $f(x,t) = e^{\sqrt{-\Delta}t} g(x)$. The right hand side can be rewritten as $\langle g, (-\Delta)^{\alpha} g\rangle$ (up to a constant), which gives the connection with the fractional Laplacian. However, this implies that (3.2.2) is nothing more than a Sobolev inequality for the fractional Laplacian on $\mathbb{R}^{n-1}$ (one that can be found in [6]) and not a true trace inequality.

The inequality we develop here is closer in spirit to Escobar's inequality.

3.3 The Main Trace Inequality

We start this section with two known results that will play a major role in our discussion. The first is the case of equality in Hardy-Littlewood-Sobolev inequality, originally proven by Lieb in [18], and the second is the Fourier transform of $|x|^{\alpha-n}$. Proves for both theorems can be found in [19].

Theorem 3.3.1. *(Hardy-Littlewood-Sobolev inequality) Let* $0<\lambda<n$, $q=\frac{2n}{2n-\lambda}$ *and* $f,h\in L^q(\mathbb{R}^n)$ *. Then*

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)h(y)}{|x-y|^{\lambda}}dxdy\right|\leq \pi^{\frac{\lambda}{2}}\cdot\frac{\Gamma\left(\frac{n-\lambda}{2}\right)}{\Gamma\left(\frac{2n-\lambda}{2}\right)}\cdot\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}\right)^{\frac{\lambda-n}{n}}\|f\|_{L^q}\|h\|_{L^q} \tag{3.3.1}$$

The inequality is sharp and there is equality in (3.3.1) if and only if $h=const\cdot f$ *and*

$$f(x)=\frac{A}{\left(\gamma^2+|x-a|^2\right)^{\frac{2n-\lambda}{2}}}$$

for some $A\in\mathbb{C}$, $0\neq\gamma\in\mathbb{R}$ *and* $a\in\mathbb{R}^n$.

Theorem 3.3.2. *If* $0<\alpha<\frac{n}{2}$ *and if* $f\in L^q(\mathbb{R}^n)$ *with* $q=\frac{2n}{n+2\alpha}$, *then* $\widehat{f}$ *exists. Moreover, with* $c_\alpha=\frac{\Gamma\left(\frac{\alpha}{2}\right)}{\pi^{\frac{\alpha}{2}}}$, *the function* $g=c_{n-\alpha}|x|^{\alpha-n}*f$ *is in* $L^2(\mathbb{R}^n)$ *and*

$$c_\alpha\frac{\widehat{f}(p)}{|p|^{\alpha}}=\widehat{g}(p)$$

in that case we have

$$c_{2\alpha}\int_{\mathbb{R}^n}\frac{\left|\widehat{f}(p)\right|^2}{|p|^{2\alpha}}dp=c_{n-2\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\overline{f(x)}f(y)}{|x-y|^{n-2\alpha}}dxdy$$

We are now ready to state our main trace inequality, at least for nice functions.

Theorem 3.3.3. *Let* $\frac{1}{2}<\alpha<\frac{n}{2}$. *For any* $f\in S(\mathbb{R}^n)$ *define* $\tau f(x')=f(x',0)$ *where* $x'\in\mathbb{R}^{n-1}$. *Then*

$$\|\tau f\|^2_{L^{\frac{2(n-1)}{n-2\alpha}}}\leq C_{\alpha,n}\left\langle f,(-\Delta)^{\alpha}f\right\rangle \tag{3.3.2}$$

where

$$C_{\alpha,n}=\frac{1}{2^{2\alpha}\pi^{\alpha}}\cdot\frac{\Gamma\left(\frac{n-2\alpha}{2}\right)\Gamma\left(\frac{2\alpha-1}{2}\right)}{\Gamma(\alpha)\Gamma\left(\frac{n+2\alpha-2}{2}\right)}\cdot\left(\frac{\Gamma(n-1)}{\Gamma\left(\frac{n-1}{2}\right)}\right)^{\frac{2\alpha-1}{n-1}}$$

Proof. We start by noticing that the inversion formula for Fourier transform states that

$$\tau f(x')=f(x',0)=\int_{\mathbb{R}^n}\widehat{f}(p',p'')e^{2\pi i(x',0)\circ(p',p'')}dp'dp''$$

Since $\widehat{f}\in S(\mathbb{R}^n)$ we have

$$\tau f(x')=\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}\widehat{f}(p',p'')dp''\right)e^{2\pi i x'\circ p'}dp'$$

Clearly $\int_{\mathbb{R}} \widehat{f}(p', p'')\, dp'' \in L^2(\mathbb{R}^{n-1}) \cap L^1(\mathbb{R}^{n-1})$ since for every $k \in \mathbb{N}$ there exists C_k such that

$$\left|\widehat{f}(p)\right| \leq \frac{C_k}{\left(1 + |p|^2\right)^k}$$

An easy result from Fourier Analysis shows that

$$\widehat{\tau f}(p') = \int_{\mathbb{R}} \widehat{f}(p', p'')\, dp''$$

(See Theorem A.2.1 in the Appendix).

Let $g \in S(\mathbb{R}^{n-1})$. We have

$$|\langle \tau f, g\rangle|^2 = \left|\int_{\mathbb{R}^{n-1}} (\tau f)(x')\, \overline{g(x')} dx'\right|^2 = \left|\int_{\mathbb{R}^{n-1}} \widehat{\tau f}(p')\, \overline{\widehat{g}(p')} dp'\right|^2$$

$$= \left|\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \widehat{f}(p', p'')\, dp''\right) \overline{\widehat{g}(p')} dp'\right|^2 \leq \left(\int_{\mathbb{R}^n} \left|\widehat{f}(p', p'')\right| |\widehat{g}(p')|\, dp' dp''\right)^2$$

$$= \left(\int_{\mathbb{R}^n} \left|\widehat{f}(p', p'')\right| |p|^\alpha \frac{|\widehat{g}(p')|}{|p|^\alpha} dp' dp''\right)^2$$

Using the Cauchy-Schwarz inequality we get

$$|\langle \tau f, g\rangle|^2 \leq \left(\int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 |p|^{2\alpha}\, dp\right) \left(\int_{\mathbb{R}^n} \frac{|\widehat{g}(p')|^2}{\left(|p'|^2 + (p'')^2\right)^\alpha} dp' dp''\right) \tag{3.3.3}$$

$$= \frac{\langle f, (-\Delta)^\alpha f\rangle}{(2\pi)^{2\alpha}} \left(\int_{\mathbb{R}^n} \frac{|\widehat{g}(p')|^2}{\left(|p'|^2 + (p'')^2\right)^\alpha} dp' dp''\right)$$

Since $\alpha > \frac{1}{2}$ we have that

$$\int_{\mathbb{R}} \frac{1}{\left(|p'|^2 + (p'')^2\right)^\alpha} dp'' \underset{p''=|p'|u}{=} \frac{|p'|}{|p'|^{2\alpha}} \int_{\mathbb{R}} \frac{du}{(1+u^2)^\alpha} = \frac{2}{|p'|^{2\alpha-1}} \cdot \int_0^\infty \frac{du}{(1+u^2)^\alpha} \tag{3.3.4}$$

$$\underset{t=u^2}{=} \frac{1}{|p'|^{2\alpha-1}} \cdot \int_0^\infty t^{-\frac{1}{2}} (1+t)^{-\alpha}\, dt = \frac{1}{|p'|^{2\alpha-1}} \cdot \int_0^\infty \frac{t^{\frac{1}{2}-1}}{(1+t)^{\frac{1}{2}+\frac{2\alpha-1}{2}}} dt$$

$$= \frac{B\left(\frac{1}{2}, \frac{2\alpha-1}{2}\right)}{|p'|^{2\alpha-1}} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2\alpha-1}{2}\right)}{|p'|^{2\alpha-1}\,\Gamma(\alpha)} = \frac{\sqrt{\pi}\cdot\Gamma\left(\frac{2\alpha-1}{2}\right)}{|p'|^{2\alpha-1}\,\Gamma(\alpha)}$$

As such

$$\int_{\mathbb{R}^n} \frac{|\widehat{g}(p')|^2}{\left(|p'|^2 + (p'')^2\right)^\alpha} dp' dp'' = \sqrt{\pi} \cdot \frac{\Gamma\left(\frac{2\alpha-1}{2}\right)}{\Gamma(\alpha)} \cdot \int_{\mathbb{R}^{n-1}} \frac{|\widehat{g}(p')|^2}{|p'|^{2\alpha-1}} dp' \tag{3.3.5}$$

Since $0 < 2\alpha - 1 < n - 1$ and $g \in S(\mathbb{R}^{n-1})$, we can conclude from Theorem 3.3.2 that

$$\int_{\mathbb{R}^{n-1}} \frac{|\widehat{g}(p')|^2}{|p'|^{2\alpha-1}} dp' = \frac{c_{(n-1)-(2\alpha-1)}}{c_{2\alpha-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{\overline{g(x)}g(y)}{|x-y|^{(n-1)-(2\alpha-1)}} dxdy \tag{3.3.6}$$
$$= \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)}{\pi^{\frac{n-2\alpha}{2}}} \cdot \frac{\pi^{\frac{2\alpha-1}{2}}}{\Gamma\left(\frac{2\alpha-1}{2}\right)} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{\overline{g(x)}g(y)}{|x-y|^{(n-1)-(2\alpha-1)}} dxdy$$
$$= \frac{\pi^{2\alpha}}{\pi^{\frac{n+1}{2}}} \cdot \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma\left(\frac{2\alpha-1}{2}\right)} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{\overline{g(x)}g(y)}{|x-y|^{(n-1)-(2\alpha-1)}} dxdy$$

Combining (3.3.3), (3.3.5) and (3.3.6) gives us

$$|\langle \tau f, g\rangle|^2 \leq \frac{\langle f, (-\Delta)^\alpha f\rangle}{(2\pi)^{2\alpha}} \cdot \sqrt{\pi} \cdot \frac{\Gamma\left(\frac{2\alpha-1}{2}\right)}{\Gamma(\alpha)} \tag{3.3.7}$$
$$\cdot \frac{\pi^{2\alpha}}{\pi^{\frac{n+1}{2}}} \cdot \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma\left(\frac{2\alpha-1}{2}\right)} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{\overline{g(x)}g(y)}{|x-y|^{(n-1)-(2\alpha-1)}} dxdy$$

Using Theorem 3.3.1 with $n-1$ as the dimension and $\lambda = (n-1) - (2\alpha - 1)$ we conclude from (3.3.7) that

$$|\langle \tau f, g\rangle|^2 \leq \frac{\langle f, (-\Delta)^\alpha f\rangle}{(2\pi)^{2\alpha}} \cdot \sqrt{\pi} \cdot \frac{\Gamma\left(\frac{2\alpha-1}{2}\right)}{\Gamma(\alpha)} \cdot \frac{\pi^{2\alpha}}{\pi^{\frac{n+1}{2}}} \cdot \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma\left(\frac{2\alpha-1}{2}\right)} \cdot \tag{3.3.8}$$
$$\cdot \pi^{\frac{n-2\alpha}{2}} \cdot \frac{\Gamma\left(\frac{2\alpha-1}{2}\right)}{\Gamma\left(\frac{(n-1)+(2\alpha-1)}{2}\right)} \cdot \left(\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n-1)}\right)^{-\frac{2\alpha-1}{n-1}} \cdot \|g\|^2_{L^{\frac{2(n-1)}{(n-1)+(2\alpha-1)}}}$$
$$= \frac{1}{2^{2\alpha}\pi^\alpha} \cdot \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)\Gamma\left(\frac{2\alpha-1}{2}\right)}{\Gamma(\alpha)\Gamma\left(\frac{n+2\alpha-2}{2}\right)} \cdot \left(\frac{\Gamma(n-1)}{\Gamma\left(\frac{n-1}{2}\right)}\right)^{\frac{2\alpha-1}{n-1}} \cdot \langle f, (-\Delta)^\alpha f\rangle \cdot \|g\|^2_{L^{\frac{2(n-1)}{n+2\alpha-2}}}$$

Thus, for every $g \in S(\mathbb{R}^{n-1})$

$$\left|\left\langle \tau f, \frac{g}{\|g\|_{L^{\frac{2(n-1)}{n+2\alpha-2}}}}\right\rangle\right|^2 \leq \frac{1}{2^{2\alpha}\pi^\alpha} \cdot \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)\Gamma\left(\frac{2\alpha-1}{2}\right)}{\Gamma(\alpha)\Gamma\left(\frac{n+2\alpha-2}{2}\right)} \cdot \left(\frac{\Gamma(n-1)}{\Gamma\left(\frac{n-1}{2}\right)}\right)^{\frac{2\alpha-1}{n-1}} \cdot \langle f, (-\Delta)^\alpha f\rangle$$

and since $\frac{n+2\alpha-2}{2(n-1)} + \frac{n-2\alpha}{2(n-1)} = 1$ and $S(\mathbb{R}^{n-1})$ is dense in $L^q(\mathbb{R}^{n-1})$ for all $q \geq 1$ we have that

$$\|\tau f\|^2_{L^{\frac{2(n-1)}{n-2\alpha}}} = \sup_{g \in S(\mathbb{R}^{n-1})} \left\langle \tau f, \frac{g}{\|g\|_{L^{\frac{2(n-1)}{n+2\alpha-2}}}}\right\rangle^2$$
$$\leq \frac{1}{2^{2\alpha}\pi^\alpha} \cdot \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)\Gamma\left(\frac{2\alpha-1}{2}\right)}{\Gamma(\alpha)\Gamma\left(\frac{n+2\alpha-2}{2}\right)} \cdot \left(\frac{\Gamma(n-1)}{\Gamma\left(\frac{n-1}{2}\right)}\right)^{\frac{2\alpha-1}{n-1}} \cdot \langle f, (-\Delta)^\alpha f\rangle$$

which is the desired result. $\square$

A careful look at the proof reveals a few things. For starters, we didn't really need the requirement that f is a Schwartz class function, far weaker conditions would have worked. Secondly, we see that the inequalities we used to show (3.3.2) are all sharp inequalities that can be attained with a specific choice of functions (which we call minimizers for obvious reasons). This leads us to hope that our trace inequality is actually a sharp one and that we can classify its minimizers. Indeed,

- In order to get equality in the Cauchy-Schwarz inequality (3.3.3) we must have, up to a constant,

$$\widehat{f}(p) = \frac{\widehat{g}(p')}{|p|^{2\alpha}}$$

- In order to get equality in the Hardy-Littlewood-Sobolev inequality (3.3.8) g must be of the form

$$g(x') = \frac{A}{\left(\gamma^2 + |x' - a'|^2\right)^{\frac{n+2\alpha-2}{2}}}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a' \in \mathbb{R}^{n-1}$.

It is not so hard to notice that the function we've constructed is not a Schwartz function, and actually in many cases, not even an $L^2(\mathbb{R}^n)$ function. As such, our first goal will be to extent our trace inequality for a larger class of function, hoping to find the right space where the inequality is both natural and attainable.

Before we continue to do just that we notice that Theorem 3.3.3 can easily be extended to traces of an intersection on several hyperplanes in the following way:

Theorem 3.3.4. *Let $1 \leq j < n$ and $\frac{j}{2} < \alpha < \frac{n}{2}$. For any $f \in S(\mathbb{R}^n)$ we define $\tau_j f(x') = f(x', 0)$ where $x' \in \mathbb{R}^{n-j}$. Then*

$$\|\tau_j f\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}} \leq C_{j,\alpha,n} \langle f, (-\Delta)^\alpha f \rangle \tag{3.3.9}$$

where

$$C_{j,a,n} = \frac{1}{2^{2\alpha}\pi^\alpha} \cdot \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma(\alpha)\Gamma\left(\frac{n+2\alpha-2j}{2}\right)} \left\{\frac{\Gamma(n-j)}{\Gamma\left(\frac{n-j}{2}\right)}\right\}^{\frac{2\alpha-j}{n-j}}$$

Proof. The idea and proof are exactly like those of Theorem 3.3.3. We will repeat the steps for completion.

If $f \in S(\mathbb{R}^n)$ then $\tau_j f \in S(\mathbb{R}^{n-j})$ and $\widehat{f} \in S(\mathbb{R}^n)$. As such

$$\widehat{\tau_j f}(p') = \int_{\mathbb{R}^j} \widehat{f}(p', p'')\, dp''$$

Let $g \in S(\mathbb{R}^{n-j})$. We find that

$$|\langle \tau_j f, g\rangle|^2 = \left| \int_{\mathbb{R}^{n-j}} \left(\int_{\mathbb{R}^j} \widehat{f}(p', p'')\, dp'' \right) \overline{\widehat{g}(p')} dp' \right|^2 \leq \left(\int_{\mathbb{R}^n} \left| \widehat{f}(p', p'') \right| |\widehat{g}(p')|\, dp' dp'' \right)^2$$

Using Cauchy-Schwartz inequality we find that

$$|\langle \tau_j f, g\rangle|^2 \leq \left(\int_{\mathbb{R}^n} \left| \widehat{f}(p) \right|^2 |p|^{2\alpha}\, dp \right) \left(\int_{\mathbb{R}^n} \frac{|\widehat{g}(p')|^2}{\left(|p'|^2 + (p'')^2\right)^\alpha} dp' dp'' \right) \tag{3.3.10}$$

Denoting $D_{j,\alpha} = \int_{\mathbb{R}^j} \frac{1}{(1+|y|^2)^\alpha} dy$ we have that

$$\int_{\mathbb{R}^n} \frac{|\widehat{g}(p')|^2}{\left(|p'|^2 + |p''|^2\right)^\alpha} dp' dp'' = \int_{\mathbb{R}^{n-j}} |\widehat{g}(p')|^2 \left(\int_{\mathbb{R}^j} \frac{1}{\left(|p'|^2 + |p''|^2\right)^\alpha} dp'' \right) dp' \tag{3.3.11}$$

$$\underset{p''=|p'|y}{=} \int_{\mathbb{R}^{n-j}} \frac{|\widehat{g}(p')|^2}{|p'|^{2\alpha-j}} \left(\int_{\mathbb{R}^j} \frac{1}{\left(1 + |y|^2\right)^\alpha} dy \right) dp' = D_{j,\alpha} \int_{\mathbb{R}^{n-j}} \frac{|\widehat{g}(p')|^2}{|p'|^{2\alpha-j}} dp'$$

Since $g \in S(\mathbb{R}^j)$ and $0 < 2\alpha - j < n - j$ Theorem 3.3.2 assures us that

$$\int_{\mathbb{R}^{n-j}} \frac{|\widehat{g}(p')|^2}{|p'|^{2\alpha-j}} dp' = \frac{c_{(n-j)-(2\alpha-j)}}{c_{2\alpha-j}} \int_{\mathbb{R}^{n-j}} \int_{\mathbb{R}^{n-j}} \frac{\overline{g(x)} g(y)}{|x-y|^{(n-j)-(2\alpha-j)}} dx dy \tag{3.3.12}$$

$$= \frac{\pi^{\frac{2\alpha-j}{2}} \Gamma\left(\frac{n-2\alpha}{2}\right)}{\pi^{\frac{n-2\alpha}{2}} \Gamma\left(\frac{2\alpha-j}{2}\right)} \int_{\mathbb{R}^{n-j}} \int_{\mathbb{R}^{n-j}} \frac{g(x')\overline{g}(y')}{|x'-y'|^{n-2\alpha}} dx' dy'$$

Using Theorem 3.3.1 with $n-j$ as the dimension and $\lambda = (n-j) - (2\alpha - j)$ we find that

$$\int_{\mathbb{R}^{n-j}} \int_{\mathbb{R}^{n-j}} \frac{g(x')\overline{g}(y')}{|x'-y'|^{n-2\alpha}} dx' dy' \tag{3.3.13}$$

$$\leq \pi^{\frac{n-2\alpha}{2}} \cdot \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)}{\Gamma\left(\frac{n+2\alpha-2j}{2}\right)} \left\{ \frac{\Gamma\left(\frac{n-j}{2}\right)}{\Gamma(n-j)} \right\}^{-\frac{2\alpha-j}{n-j}} \|g\|^2_{L^{\frac{2(n-j)}{n+2\alpha-2j}}}$$

Combining (3.3.10), (3.3.11), (3.3.12) and (3.3.13) we conclude that

$$|\langle \tau_j f, g\rangle|^2 \leq \frac{D_{j,\alpha}}{2^{2\alpha}\pi^{\frac{2\alpha+j}{2}}} \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma\left(\frac{n+2\alpha-2j}{2}\right)} \left\{ \frac{\Gamma(n-j)}{\Gamma\left(\frac{n-j}{2}\right)} \right\}^{\frac{2\alpha-j}{n-j}} \cdot \langle f, (-\Delta)^\alpha f\rangle \cdot \|g\|^2_{L^{\frac{2(n-j)}{n+2\alpha-2j}}}$$

Using the density of $S(\mathbb{R}^{n-j})$ in $L^{\frac{2(n-j)}{n+2\alpha-2j}}(\mathbb{R}^{n-j})$ and the fact that $\frac{n-2\alpha}{2(n-j)}+\frac{n+2\alpha-2j}{2(n-j)}=1$, we conclude that

$$\|\tau_j f\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}} \leq \frac{D_{j,\alpha}}{2^{2\alpha}\pi^{\frac{2\alpha+j}{2}}} \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma\left(\frac{n+2\alpha-2j}{2}\right)} \left\{\frac{\Gamma(n-j)}{\Gamma\left(\frac{n-j}{2}\right)}\right\}^{\frac{2\alpha-j}{n-j}} \langle f, (-\Delta)^\alpha f\rangle \tag{3.3.14}$$

We only need to compute $D_{j,\alpha}$ in order to finish the proof. We notice that in the proof of Theorem 3.3.3 we showed that $D_{1,\alpha}=\sqrt{\pi}\cdot\frac{\Gamma\left(\frac{2\alpha-1}{2}\right)}{\Gamma(\alpha)}$. For $j>1$ we have that

$$D_{j,\alpha}=\int_{\mathbb{R}^j}\frac{1}{(1+|y|^2)^\alpha}dy=\int_{\mathbb{R}^{j-1}}\left(\int_{\mathbb{R}}\frac{1}{\left(1+|y'|^2+(y'')^2\right)^\alpha}dy''\right)dy'$$

$$\underset{y''=\sqrt{1+|y'|^2}t}{=}\int_{\mathbb{R}^{j-1}}\frac{1}{(1+|y'|^2)^{\alpha-\frac{1}{2}}}\left(\int_{\mathbb{R}}\frac{1}{(1+t^2)^\alpha}dt\right)dy'$$

$$=D_{1,\alpha}\int_{\mathbb{R}^{j-1}}\frac{1}{(1+|y'|^2)^{\alpha-\frac{1}{2}}}dy'=D_{1,\alpha}\cdot D_{j-1,\alpha-\frac{1}{2}}$$

Thus

$$D_{j,\alpha}=\left(\sqrt{\pi}\cdot\frac{\Gamma\left(\frac{2\alpha-1}{2}\right)}{\Gamma(\alpha)}\right)\cdot\left(\sqrt{\pi}\cdot\frac{\Gamma\left(\frac{2\left(\alpha-\frac{1}{2}\right)-1}{2}\right)}{\Gamma\left(\alpha-\frac{1}{2}\right)}\right)\cdot\dots\cdot\left(\sqrt{\pi}\cdot\frac{\Gamma\left(\frac{2\left(\alpha-\frac{j-1}{2}\right)-1}{2}\right)}{\Gamma\left(\alpha-\frac{j-1}{2}\right)}\right)=\pi^{\frac{j}{2}}\cdot\frac{\Gamma\left(\frac{2\alpha-j}{2}\right)}{\Gamma(\alpha)}$$

Plugging this in (3.3.14) we conclude that

$$\|\tau_j f\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}} \leq \frac{1}{2^{2\alpha}\pi^\alpha}\cdot\frac{\Gamma\left(\frac{2\alpha-j}{2}\right)\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma(\alpha)\Gamma\left(\frac{n+2\alpha-2j}{2}\right)}\left\{\frac{\Gamma(n-j)}{\Gamma\left(\frac{n-j}{2}\right)}\right\}^{\frac{2\alpha-j}{n-j}}$$

which is the desired result. □

From this point onward we'll deal with the more general inequality (3.3.9).

3.4 *The space* $D^\alpha(\mathbb{R}^n)$

As discussed in the previous section, our goal is to find the most natural space where (3.3.9) is not only true, but attainable. While the fractional Sobolev space $H^\alpha(\mathbb{R}^n)$, defined as the space of all $L^2(\mathbb{R}^n)$ functions f such that $\int_{\mathbb{R}^n}\left|\widehat{f}(p)\right||p|^{2\alpha}dp<\infty$, might seem right we must go a different route.

Definition 3.4.1. The space $D^\alpha(\mathbb{R}^n)$, where $0<\alpha<\frac{n}{2}$, is the space of all tempered distributions $f\in S'(\mathbb{R}^n)$ whose Fourier transform (in the distributional sense) $\widehat{f}$ is a function in $L^2(\mathbb{R}^n,|p|^{2\alpha}dp)$.

Theorem 3.4.2. *The space $D^\alpha(\mathbb{R}^n)$ is a Banach space under the norm*

$$\|f\|_{D^\alpha}=\left\|\widehat{f}\right\|_{L^2(|p|^{2\alpha}dp)}$$

Proof. We start by noticing that $\|\cdot\|_{D^\alpha}$ is indeed a norm since $\|\cdot\|_{L^2(|p|^{2\alpha}dp)}$ is, and $\widehat{f}=0$ if and only if $f=0$. The completeness is the only thing we really need to show. Let $f_k\in D^\alpha(\mathbb{R}^n)$ be a Cauchy sequence in $\|\cdot\|_{D^\alpha(\mathbb{R}^n)}$. This means that $\widehat{f_k}(p)|p|^\alpha$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. Since $L^2(\mathbb{R}^n)$ is complete we can find $F(p)|p|^\alpha\in L^2(\mathbb{R}^n)$ such that

$$\left\|\widehat{f_k}(p)-F(p)\right\|_{L^2(|p|^{2\alpha}dp)}\underset{k\to\infty}{\longrightarrow}0$$

In order to finish the proof we need to construct a distribution $f\in S'(\mathbb{R}^n)$ with $\widehat{f}=F$. Given any $g\in S(\mathbb{R}^n)$ Theorem 3.3.2 assures us that $\frac{\widehat{g}(p)}{|p|^\alpha}\in L^2(\mathbb{R}^n)$. We define

$$\langle f,g\rangle=\int_{\mathbb{R}^n}F(p)\overline{\widehat{g}(p)}dp=\int_{\mathbb{R}^n}F(p)|p|^\alpha\cdot\frac{\overline{\widehat{g}(p)}}{|p|^\alpha}dp$$

By Theorems 3.3.2 and 3.3.1 we find that

$$\int_{\mathbb{R}^n}\frac{|\widehat{g}(p)|^2}{|p|^{2\alpha}}dp=\pi^{2\alpha-\frac{n}{2}}\cdot\frac{\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma(\alpha)}\cdot\int_{\mathbb{R}^n}\frac{\overline{g(x)}g(y)}{|x-y|^{n-2\alpha}}dxdy \tag{3.4.1}$$

$$\le\pi^{2\alpha-\frac{n}{2}}\cdot\frac{\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma(\alpha)}\cdot\pi^{\frac{n}{2}-\alpha}\cdot\frac{\Gamma(\alpha)}{\Gamma\left(\frac{n+2\alpha}{2}\right)}\cdot\left\{\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right\}^{\frac{2\alpha}{n}}\cdot\|g\|^2_{L^{\frac{2n}{n+2\alpha}}}$$

$$=\pi^\alpha\cdot\frac{\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma\left(\frac{n+2\alpha}{2}\right)}\cdot\left\{\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right\}^{\frac{2\alpha}{n}}\cdot\|g\|^2_{L^{\frac{2n}{n+2\alpha}}}$$

Also, for any $r>1$ we have that

$$\left(\int_{\mathbb{R}^n}|g(x)|^rdx\right)^{\frac{1}{r}}=\left(\int_{|x|\le1}|g(x)|^rdx\right)^{\frac{1}{r}}+\left(\int_{|x|>1}|g(x)|^rdx\right)^{\frac{1}{r}}$$

$$\le\|g(x)\|_\infty|B^n|^{\frac{1}{r}}+\||x|^n\cdot g(x)\|_\infty\left(\int_{|x|>1}\frac{dx}{|x|^{nr}}\right)^{\frac{1}{r}}$$

$$= \|g(x)\|_\infty |B^n|^{\frac{1}{r}} + \||x|^n \cdot g(x)\|_\infty \left(|\mathbb{S}^{n-1}| \int_1^\infty \frac{|x|^{n-1} d|x|}{|x|^{nr}} \right)^{\frac{1}{r}}$$

$$= \|g(x)\|_\infty |B^n|^{\frac{1}{r}} + \||x|^n \cdot g(x)\|_\infty \left(|\mathbb{S}^{n-1}| \int_1^\infty \frac{d|x|}{|x|^{n(r-1)+1}} \right)^{\frac{1}{r}}$$

Since $n(r-1) > 0$ we have that the second term converges and we can conclude that for any $r > 1$ there exists $C_{r,n}$ such that

$$\|g\|_{L^r} \leq C_{r,n} \left(\|g(x)\|_\infty + \||x|^n \cdot g(x)\|_\infty \right)$$

From this, and (3.4.1) we find that

$$\left| \int_{\mathbb{R}^n} F(p) \overline{\widehat{g}(p)} dp \right| \leq \|F\|_{L^2(|p|^{2\alpha} dp)} \cdot \sqrt{\int_{\mathbb{R}^n} \frac{|\widehat{g}(p)|^2}{|p|^{2\alpha}} dp}$$

$$\leq \sqrt{\pi^\alpha \cdot \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma\left(\frac{n+2\alpha}{2}\right)} \cdot \left\{ \frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)} \right\}^{\frac{2\alpha}{n}}} \cdot \|F\|_{L^2(|p|^{2\alpha} dp)} \cdot C_{\frac{2n}{n+2\alpha}, n} \left(\|g(x)\|_\infty + \||x|^n \cdot g(x)\|_\infty \right)$$

i.e. $\langle f, g \rangle = \int_{\mathbb{R}^n} F(p) \overline{\widehat{g}(p)} dp$ indeed defines a distribution $f \in S'(\mathbb{R}^n)$. For any $g \in S(\mathbb{R}^n)$

$$\left\langle \widehat{f}, g \right\rangle = \langle f, \check{g} \rangle = \int_{\mathbb{R}^n} F(p) \overline{g(p)} dp = \langle F, g \rangle$$

Thus, $\widehat{f} = F$ and the proof is complete. $\square$

Remark 3.4.3. The proof of Theorem 3.4.2 shows us more than the fact that $D^\alpha(\mathbb{R}^n)$ is a Banach space. It gives us an identification between it and $L^2(\mathbb{R}^n, |p|^{2\alpha} dp)$. Indeed, the map

$$f \longmapsto \widehat{f}$$

is an isometry by the definition of $\|\cdot\|_{D^\alpha}$. On the other hand, the proof of Theorem 3.4.2 showed that for any $F \in L^2(\mathbb{R}^n, |p|^{2\alpha} dp)$ we can find $f \in D^\alpha(\mathbb{R}^n)$ such that $\widehat{f} = F$, i.e. the above map is an isometric isomorphism.

Before we can establish the trace inequality for $D^\alpha(\mathbb{R}^n)$ we'll need to know a few more things about the space.

Theorem 3.4.4. *Let $f \in L^q(\mathbb{R}^n)$ where $1 \leq q \leq 2$. If $\widehat{f} \in L^2(\mathbb{R}^n, |p|^{2\alpha}dp)$ then $f \in D^\alpha(\mathbb{R}^n)$.*

Proof. Clearly we can consider f as a tempered distribution since it's an $L^q(\mathbb{R}^n)$ function. The only thing we need to show is that the distributional Fourier transform is the same as the regular Fourier transform. In order to show that we prove that for any $g \in S(\mathbb{R}^n)$

$$\langle f, g\rangle = \int_{\mathbb{R}^n} \widehat{f}(p)\overline{\widehat{g}(p)}dp$$

Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of Schwartz functions that converges to f in $L^q(\mathbb{R}^n)$. From the theory of Fourier transforms on $L^q(\mathbb{R}^n)$, when $1 \leq q \leq 2$, we know that there exists $C_q > 0$ such that

$$\left\|\widehat{h}\right\|_{L^p(\mathbb{R}^n)} \leq C_q \|h\|_{L^q(\mathbb{R}^n)}$$

where $\frac{1}{p}+\frac{1}{q}=1$ (See [17]). As such, $\left\{\widehat{f_k}\right\}_{k\in\mathbb{N}}$ converges to $\widehat{f}$ in $L^p(\mathbb{R}^n)$ and

$$\langle f, g\rangle = \lim_{k\to\infty}\langle f_k, g\rangle = \lim_{k\to\infty}\int_{\mathbb{R}^n} \widehat{f_k}(p)\overline{\widehat{g}(p)}dp = \int_{\mathbb{R}^n} \widehat{f}(p)\overline{\widehat{g}(p)}dp$$

Now, if we denote by $\widehat{f}^d$ the distributional Fourier transform we find that for any $g \in S(\mathbb{R}^n)$

$$\left\langle \widehat{f}^d, g\right\rangle = \langle f, \check{g}\rangle = \int_{\mathbb{R}^n} \widehat{f}(p)\overline{g(p)}dp = \left\langle \widehat{f}, g\right\rangle$$

so $\widehat{f}^d = \widehat{f}$, and the proof is complete. □

Theorem 3.4.5. *The space $S(\mathbb{R}^n)$ is dense in $D^\alpha(\mathbb{R}^n)$.*

Proof. We'll start by showing that $H^\alpha(\mathbb{R}^n)$ is dense in $D^\alpha(\mathbb{R}^n)$. Theorem 3.4.4 assures us that $H^\alpha(\mathbb{R}^n) \subset D^\alpha(\mathbb{R}^n)$. Given $f \in D^\alpha(\mathbb{R}^n)$ we define $\widehat{f_k}(p) = \chi_{\left[\frac{1}{k},k\right]}(|p|)\,\widehat{f}(p)$. We have that

$$\int_{\mathbb{R}^n} \left|\widehat{f_k}(p)\right|^2 |p|^{2\alpha}dp \leq \int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 |p|^{2\alpha}dp < \infty$$

and

$$\int_{\mathbb{R}^n} \left|\widehat{f_k}(p)\right|^2 dp = \int_{\frac{1}{k}\leq|p|\leq k} \left|\widehat{f_k}(p)\right|^2 dp \leq k^{2\alpha}\int_{\frac{1}{k}\leq|p|\leq k} \left|\widehat{f_k}(p)\right|^2 |p|^{2\alpha}dp$$

$$= k^{2\alpha} \int_{\mathbb{R}^n} \left| \widehat{f}_k(p) \right|^2 |p|^{2\alpha} dp \leq k^{2\alpha} \int_{\mathbb{R}^n} \left| \widehat{f}(p) \right|^2 |p|^{2\alpha} dp < \infty$$

Thus $\widehat{f}_k \in L^2(\mathbb{R}^n)$ and has an inverse Fourier transform which we denote by f_k. Clearly from the above $f_k \in H^\alpha(\mathbb{R}^n)$.

Since $\left| \widehat{f}_k(p) \right|^2 |p|^{2\alpha} \leq \left| \widehat{f}(p) \right|^2 |p|^{2\alpha} \in L^1(\mathbb{R}^n)$ and $\left| \widehat{f}_k(p) \right|^2 |p|^{2\alpha} \underset{k\to\infty}{\longrightarrow} \left| \widehat{f}(p) \right|^2 |p|^{2\alpha}$ pointwise we find by the Dominated Convergence Theorem that

$$\|f_k - f\|_{D^\alpha} = \int_{\mathbb{R}^n} \left| \widehat{f}_k(p) - \widehat{f}(p) \right|^2 |p|^{2\alpha} dp \underset{k\to\infty}{\longrightarrow} 0$$

concluding that $H^\alpha(\mathbb{R}^n)$ is dense in $D^\alpha(\mathbb{R}^n)$.

Given any $\epsilon > 0$ and $f \in D^\alpha(\mathbb{R}^n)$ we can find $f_\epsilon \in H^\alpha(\mathbb{R}^n)$ such that $\|f_\epsilon - f\|_{D^\alpha} < \frac{\epsilon}{2}$. Using the fact that $S(\mathbb{R}^n)$ is dense in $H^\alpha(\mathbb{R}^n)$ (See Lemma A.2.2) we can find $g_\epsilon \in S(\mathbb{R}^n)$ such that

$$\|g_\epsilon - f_\epsilon\|_{H^\alpha} = \sqrt{\|g_\epsilon - f\|_{L^2}^2 + \|g_\epsilon - f\|_{D^\alpha}^2} < \frac{\epsilon}{2}$$

We have that

$$\begin{aligned} \|g_\epsilon - f\|_{D^\alpha} &\leq \|f_\epsilon - f\|_{D^\alpha} + \|g_\epsilon - f_\epsilon\|_{D^\alpha} \\ &< \frac{\epsilon}{2} + \|g_\epsilon - f_\epsilon\|_{H^\alpha} < \epsilon \end{aligned}$$

which concludes the proof. □

Theorem 3.4.5 immediately implies our trace inequality

Theorem 3.4.6. *Let $1 \leq j < n$ and $\frac{j}{2} < \alpha < \frac{n}{2}$. There exists a continuous linear operator $\tau_j : D^\alpha(\mathbb{R}^n) \to L^{\frac{2(n-j)}{n-2\alpha}}(\mathbb{R}^n)$ such that*

$$\|\tau_j f\|_{L^{\frac{2(n-j)}{n-2\alpha}}}^2 \leq C_{j,\alpha,n} \left\langle f, (-\Delta)^\alpha f \right\rangle$$

where

$$C_{j,a,n} = \frac{1}{2^{2\alpha}\pi^\alpha} \cdot \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma(\alpha)\Gamma\left(\frac{n+2\alpha-2j}{2}\right)} \left\{ \frac{\Gamma(n-j)}{\Gamma\left(\frac{n-j}{2}\right)} \right\}^{\frac{2\alpha-j}{n-j}}$$

Moreover, for any $f \in S(\mathbb{R}^n)$, $\tau_j f(x') = f(x', 0)$ where $x' \in \mathbb{R}^{n-j}$.

Proof. This follows immediately from Theorem 3.3.4 and 3.4.5. □

Surprisingly enough, Theorem 3.4.5 tells us more than only the trace inequality - it implies that $D^{\alpha}(\mathbb{R}^n)$ is actually a function space and not an abstract distribution space. While we don't need this in order to show that (3.3.9) is sharp and attainable in $D^{\alpha}(\mathbb{R}^n)$, we decided to include this in our discussion as it will give us another attribute of $D^{\alpha}(\mathbb{R}^n)$ and, as will be mentioned later, gives an alternative proof to our main inequality.

We begin with a Sobolev type theorem. This was originally proved in [6] but we repeat it here due to its simplicity and relevance.

Theorem 3.4.7. *Let $0 < \alpha < \frac{n}{2}$ and $f \in S(\mathbb{R}^n)$ then*

$$\|f\|_{L^{\frac{2n}{n-2\alpha}}} \leq c_{\alpha,n} \|f\|_{D^{\alpha}} \tag{3.4.2}$$

where

$$c_{\alpha,n} = \sqrt{\pi^{\alpha} \cdot \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma\left(\frac{n+2\alpha}{2}\right)} \cdot \left\{\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right\}^{\frac{2\alpha}{n}}}$$

Proof. The proof is similar to proofs presented in chapter 8 of [19] and our proof of Theorems 3.3.3 and 3.3.4. Given $g \in S(\mathbb{R}^n)$ we find that

$$|\langle f, g\rangle|^2 = \left|\int_{\mathbb{R}^n} \widehat{f}(p)\overline{\widehat{g}(p)}dp\right|^2 \leq \|f\|_{D^{\alpha}}^2 \cdot \int_{\mathbb{R}^n} \frac{|\widehat{g}(p)|^2}{|p|^{2\alpha}}dp$$

Using the (3.4.1) from the proof of Theorem 3.4.2 we find that

$$|\langle f, g\rangle|^2 \leq \|f\|_{D^{\alpha}}^2 \cdot \pi^{\alpha} \cdot \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma\left(\frac{n+2\alpha}{2}\right)} \cdot \left\{\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right\}^{\frac{2\alpha}{n}} \cdot \|g\|_{L^{\frac{2n}{n+2\alpha}}}^2$$

Since $\frac{n+2\alpha}{2n} + \frac{n-2\alpha}{2n} = 1$ and $S(\mathbb{R}^n)$ is dense in all $L^q(\mathbb{R}^n)$ spaces the result follows. □

The following is an improvement to the above theorem:

Theorem 3.4.8. *If $f \in D^{\alpha}(\mathbb{R}^n)$ then $f \in L^{\frac{2n}{n-2\alpha}}(\mathbb{R}^n)$ and*

$$\|f\|_{L^{\frac{2n}{n-2\alpha}}} \leq c_{\alpha,n} \|f\|_{D^{\alpha}}$$

where $c_{\alpha,n}$ was defined in Theorem 3.4.7.

Proof. Given $f \in D^{\alpha}(\mathbb{R}^n)$ we can find a sequence of functions $\{f_k\}_{k\in\mathbb{N}} \in S(\mathbb{R}^n)$ such that

$$\|f - f_k\|_{D^{\alpha}} \underset{k\to\infty}{\longrightarrow} 0$$

(this is due to Theorem 3.4.5). As such, $\{f_k\}_{k\in\mathbb{N}}$ is Cauchy in $D^{\alpha}(\mathbb{R}^n)$ and (3.4.2) implies that $\{f_k\}_{k\in\mathbb{N}}$ is Cauchy in $L^{\frac{2n}{n-2\alpha}}(\mathbb{R}^n)$. Since $L^{\frac{2n}{n-2\alpha}}(\mathbb{R}^n)$ is complete we can find $h_f \in L^{\frac{2n}{n-2\alpha}}(\mathbb{R}^n)$ such that

$$\|h_f - f_k\|_{L^{\frac{2n}{n-2\alpha}}} \underset{k\to\infty}{\longrightarrow} 0$$

We'll now show that $f = h_f$. The proof of Theorem 3.4.2 and Remark 3.4.3 showed that

$$\langle f, g\rangle = \int_{\mathbb{R}^n} \widehat{f}(p)\overline{\widehat{g}(p)}dp = \int_{\mathbb{R}^n} \widehat{f}(p)|p|^{\alpha} \cdot \frac{\overline{\widehat{g}(p)}}{|p|^{\alpha}}dp$$

for any $g \in S(\mathbb{R}^n)$. Since $\frac{\widehat{g}(p)}{|p|^{\alpha}} \in L^2(\mathbb{R}^n)$ (as seen in (3.4.1)) and $\|f - f_k\|_{D^{\alpha}} \underset{k\to\infty}{\longrightarrow} 0$ we can conclude that

$$\langle f, g\rangle = \lim_{k\to\infty} \int_{\mathbb{R}^n} \widehat{f}_k(p)|p|^{\alpha} \cdot \frac{\overline{\widehat{g}(p)}}{|p|^{\alpha}}dp = \lim_{k\to\infty} \int_{\mathbb{R}^n} \widehat{f}_k(p)\overline{\widehat{g}(p)}dp = \lim_{k\to\infty} \langle f_k, g\rangle = \langle h_f, g\rangle$$

which shows that $f = h_f$ and can be considered as a function. We also have that

$$\|f\|_{L^{\frac{2n}{n-2\alpha}}} = \lim_{k\to\infty} \|f_k\|_{L^{\frac{2n}{n-2\alpha}}} \leq \lim_{k\to\infty} c_{\alpha,n} \|f_k\|_{D^{\alpha}} = c_{\alpha,n} \|f\|_{D^{\alpha}}$$

and the proof is complete. $\square$

We turn our attention to the study of the minimizers, if there are any. The next technical Lemma is crucial in our discussion and is motivated by the proof of Theorem 3.3.4.

Lemma 3.4.9. *Let $1 \leq j < n$ and $\frac{j}{2} < \alpha < \frac{n}{2}$. For any $g \in L^{\frac{2(n-j)}{n+2\alpha-2j}}(\mathbb{R}^{n-j})$ and $f \in D^{\alpha}(\mathbb{R}^n)$ we have that*

$$\langle \tau_j f, g\rangle = \int_{\mathbb{R}^n} \widehat{f}(p', p'')\overline{\widehat{g}(p')}dp'dp''$$

where $p' \in \mathbb{R}^{n-j}$.

Proof. We start by noticing that since $\frac{j}{2} < \alpha < \frac{n}{2}$ we have that

$$1 < \frac{2(n-j)}{n+2\alpha-2j} < 2$$

so g has a Fourier transform, and the righthand side makes sense. The main idea behind the proof of this Lemma is using approximation by Schwartz functions, similar to the steps taken in the proof of Theorem 3.4.4. Let $f \in S(\mathbb{R}^n)$ and $g \in L^{\frac{2(n-j)}{n+2\alpha-2j}}(\mathbb{R}^{n-j})$. Since $S(\mathbb{R}^{n-j})$ is dense in $L^{\frac{2(n-j)}{n+2\alpha-2j}}(\mathbb{R}^{n-j})$ we can find a sequence of Schwartz functions $\{g_k\}_{k\in\mathbb{N}}$ such that $\|g_k - g\|_{L^{\frac{2(n-j)}{n+2\alpha-2j}}} \underset{k\to\infty}{\longrightarrow} 0$. We know that

$$\Phi_f(g) = \langle \tau_j f, g\rangle$$

is a bounded linear functional on $L^{\frac{2(n-j)}{n+2\alpha-2j}}(\mathbb{R}^{n-j})$ and so

$$\langle \tau_j f, g\rangle = \lim_{k\to\infty} \langle \tau_j f, g_k\rangle = \lim_{k\to\infty} \int_{\mathbb{R}^n} \widehat{f}(p', p'') \overline{\widehat{g_k}(p')} dp' dp''$$

Using Fubini's Theorem we find that

$$\int_{\mathbb{R}^n} \widehat{f}(p', p'') \widehat{g_k}(p') \, dp' dp'' = \int_{\mathbb{R}^{n-j}} \left(\int_{\mathbb{R}^j} \widehat{f}(p', p'') \, dp'' \right) \overline{\widehat{g_k}(p')} dp'$$

Since $\widehat{g_k} \underset{L^{\frac{2(n-j)}{n-2\alpha}}}{\longrightarrow} \widehat{g}$ and $\int_{\mathbb{R}^j} \widehat{f}(p', p'') \, dp'' \in L^{\frac{2(n-j)}{n+2\alpha-2j}}(\mathbb{R}^{n-j})$ (same explanation as given in the proof of Theorem 3.3.3) we conclude that

$$\lim_{k\to\infty} \int_{\mathbb{R}^{n-j}} \left(\int_{\mathbb{R}^j} \widehat{f}(p', p'') \, dp'' \right) \overline{\widehat{g_k}(p')} dp' = \int_{\mathbb{R}^{n-j}} \left(\int_{\mathbb{R}^j} \widehat{f}(p', p'') \, dp'' \right) \overline{\widehat{g}(p')} dp'$$

Using Fubini's Theorem again we see that

$$\langle \tau_j f, g\rangle = \int_{\mathbb{R}^n} \widehat{f}(p', p'') \overline{\widehat{g}(p')} dp' dp''$$

for all $f \in S(\mathbb{R}^n)$ and $g \in L^{\frac{2(n-j)}{n+2\alpha-2j}}(\mathbb{R}^{n-j})$.

Next, given $f \in D^\alpha(\mathbb{R}^n)$ and $g \in L^{\frac{2(n-j)}{n+2\alpha-2j}}(\mathbb{R}^{n-j})$ we can find a sequence of Schwartz functions $\{f_k\}_{k\in\mathbb{N}}$ such that

$$\|f_k - f\|_{D^\alpha} \underset{k\to\infty}{\longrightarrow} 0$$

(this is true due to Lemma 3.4.5). By the definition of τ_j we have that

$$\tau_j f = \lim_{k\to\infty} \tau_j f_k$$

in the $L^{\frac{2(n-j)}{n-2\alpha}}\left(\mathbb{R}^{n-j}\right)$ sense, and so

$$\begin{aligned}\langle \tau_j f, g\rangle &= \lim_{k\to\infty} \langle \tau_j f_k, g\rangle = \lim_{k\to\infty} \int_{\mathbb{R}^n} \widehat{f_k}\left(p', p''\right) \overline{\widehat{g}\left(p'\right)} dp' dp'' \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^n} \widehat{f_k}\left(p', p''\right) |p|^{\alpha} \frac{\overline{\widehat{g}\left(p'\right)}}{|p|^{\alpha}} dp' dp''\end{aligned}$$

We notice that

$$\int_{\mathbb{R}^n} \frac{\left|\widehat{g}\left(p'\right)\right|^2}{|p|^{2\alpha}} dp = \int_{\mathbb{R}^{n-j}} \left|\widehat{g}\left(p'\right)\right|^2 \left(\int_{\mathbb{R}^j} \frac{dp''}{|p|^{2\alpha}}\right) dp' = \pi^{\frac{j}{2}} \cdot \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)}{\Gamma(\alpha)} \int_{\mathbb{R}^{n-j}} \frac{\left|\widehat{g}\left(p'\right)\right|^2}{\left|p'\right|^{2\alpha-j}} dp'$$

as was shown in the proof of Theorem 3.3.4. Using Theorem 3.3.2 with $n-j$ as the dimension and $2\alpha - j$ replacing 2α, the fact that $g \in L^{\frac{2(n-j)}{n+2\alpha-2j}}\left(\mathbb{R}^{n-j}\right)$ and $\frac{j}{2} < \alpha < \frac{n}{2}$ implies that $\frac{\widehat{g}(p')}{|p'|^{\alpha-\frac{j}{2}}} \in L^2\left(\mathbb{R}^{n-j}\right)$. Thus

$$\int_{\mathbb{R}^n} \frac{\left|\widehat{g}\left(p'\right)\right|^2}{|p|^{2\alpha}} dp < \infty$$

i.e. $\frac{\widehat{g}(p')}{|p|^{\alpha}} \in L^2\left(\mathbb{R}^n\right)$. Since $\widehat{f}(p)|p|^{\alpha} = \lim_{k\to\infty} \widehat{f_k}(p)|p|^{\alpha}$ in the $L^2\left(\mathbb{R}^n\right)$ sense (by the definition) we conclude that

$$\begin{aligned}\langle \tau_j f, g\rangle &= \lim_{k\to\infty} \int_{\mathbb{R}^n} \widehat{f_k}\left(p', p''\right) |p|^{\alpha} \frac{\overline{\widehat{g}\left(p'\right)}}{|p|^{\alpha}} dp' dp'' = \int_{\mathbb{R}^n} \widehat{f}\left(p', p''\right) |p|^{\alpha} \frac{\overline{\widehat{g}\left(p'\right)}}{|p|^{\alpha}} dp' dp'' \\ &= \int_{\mathbb{R}^n} \widehat{f}\left(p', p''\right) \overline{\widehat{g}\left(p'\right)} dp' dp''\end{aligned}$$

which is the desired result. $\square$

The above theorem is the key to showing that (3.3.9) is sharp and attainable in $D^{\alpha}\left(\mathbb{R}^n\right)$.

Theorem 3.4.10. *If $f \in D^{\alpha}\left(\mathbb{R}^n\right)$ is a minimizer for (3.3.9) then $\widehat{f}$ must be of the form*

$$\widehat{f}(p) = \frac{\widehat{g_{j,HLS}}\left(p'\right)}{|p|^{2\alpha}} \tag{3.4.3}$$

where $p' \in \mathbb{R}^{n-j}$ and $g_{j,HLS}\left(x'\right) = \frac{A}{\left(\gamma^2 + |x'-a'|^2\right)^{\frac{n+2\alpha-2j}{2}}}$ for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a' \in \mathbb{R}^{n-j}$.

Proof. Assume that $f \in D^{\alpha}(\mathbb{R}^n)$ is a minimizer for (3.3.9). Let $g \in L^{\frac{2(n-j)}{n+2\alpha-2j}}(\mathbb{R}^{n-j}) = \left(L^{\frac{2(n-j)}{n-2\alpha}}(\mathbb{R}^{n-j})\right)^*$ be such that $\|g\|_{L^{\frac{2(n-j)}{n+2\alpha-2j}}} = 1$ and

$$\langle \tau_j f, g\rangle = \|\tau_j f\|_{L^{\frac{2(n-j)}{n-2\alpha}}}$$

We find that by Lemma 3.4.9

$$\|\tau_j f\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}} = |\langle \tau_j f, g\rangle|^2 = \left|\int_{\mathbb{R}^n} \widehat{f}(p', p'')\, \overline{\widehat{g}(p')} dp' dp''\right|^2$$

$$= \left|\int_{\mathbb{R}^n} \widehat{f}(p', p'')\, |p|^{\alpha} \frac{\overline{\widehat{g}(p')}}{|p|^{\alpha}} dp' dp''\right|^2$$

From this point we continue word to word as the proof of Theorem 3.3.4. Using Cauchy-Schwartz inequality and Theorem 3.3.2 we find that

$$\|\tau_j f\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}} \leq \left(\int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 |p|^{2\alpha}\, dp\right)\left(\int_{\mathbb{R}^n} \frac{|\widehat{g}(p')|^2}{|p|^{2\alpha}} dp\right)$$

$$= \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)\Gamma\left(\frac{n-2\alpha}{2}\right)}{2^{2\alpha}\cdot \pi^{\frac{n}{2}}\,\Gamma(\alpha)\,\Gamma\left(\frac{2\alpha-j}{2}\right)} \cdot \langle f, (-\Delta)^{\alpha} f\rangle \cdot \int_{\mathbb{R}^{n-j}}\int_{\mathbb{R}^{n-j}} \frac{g(x')\overline{g}(y')}{|x'-y'|^{n-2\alpha}} dx' dy'$$

Using Theorem 3.3.1 and the fact that $\|g\|_{L^{\frac{2(n-j)}{n+2\alpha-2j}}} = 1$ we find that

$$\|\tau_j f\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}} \leq \frac{1}{2^{2\alpha}\pi^{\alpha}} \cdot \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)\Gamma\left(\frac{n-2\alpha}{2}\right)}{\Gamma(\alpha)\,\Gamma\left(\frac{n+2\alpha-2j}{2}\right)} \left\{\frac{\Gamma\left(\frac{n-j}{2}\right)}{\Gamma(n-j)}\right\}^{-\frac{2\alpha-j}{n-j}} \cdot \langle f, (-\Delta)^{\alpha} f\rangle$$

$$= C_{j,\alpha,n}\, \langle f, (-\Delta)^{\alpha} f\rangle = \|\tau_j f\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}}$$

This implies that we must have had equality in every step of the way. Thus, equality in the Cauchy-Schwartz inequality implies that

$$\widehat{f}(p)\,|p|^{\alpha} = C \cdot \frac{\widehat{g}(p')}{|p|^{\alpha}}$$

for some $C \in \mathbb{C}$, and equality in the Hardy-Littlewood-Sobolev inequality for g implies that g must be of the form $g_{j,HLS}$. Since the constant C can be 'swallowed' in the general form of $g_{j,HLS}$ we obtain the desired result. $\square$

Finding what might be a minimizer is only half the story. Are functions of the form (3.4.3) in $D^{\alpha}(\mathbb{R}^n)$?

In order to show that it is indeed the case we need the next technical lemma:

Lemma 3.4.11. *Let* $g_j(x') = \left(\frac{1}{1+|x'|^2}\right)^{\frac{n-2j+2\alpha}{2}}$ *where* $x' \in \mathbb{R}^{n-j}$. *Then*

(i) $g_j \in L^q(\mathbb{R}^{n-j})$ *for all* $q \geq 1$ *when* $\frac{j}{2} < \alpha < \frac{n}{2}$. *In particular* $g_j \in L^{\frac{2(n-j)}{n+2\alpha-j}}(\mathbb{R}^{n-j})$ *and by Theorem 3.3.2*

$$\int_{\mathbb{R}^{n-j}} \frac{|\widehat{g_j}(p')|^2}{|p'|^{2\alpha-j}} dp' = \frac{\pi^{2\alpha-j}\Gamma\left(\frac{n-2\alpha}{2}\right)}{\pi^{\frac{n-j}{2}}\Gamma\left(\frac{2\alpha-j}{2}\right)} \int_{\mathbb{R}^{n-j}} \int_{\mathbb{R}^{n-j}} \frac{g_j(x')\overline{g_j}(y')}{|x'-y'|^{n-2\alpha}} dx'dy'$$

(ii) for $\frac{j}{2} < \alpha < \frac{n}{2}$, $\widehat{g_j}$ *decays faster than any polynomial at infinity. As such, along with (ii) we conclude that* $\widehat{g_j} \in L^1(\mathbb{R}^{n-j}) \cap C(\mathbb{R}^{n-j})$.

Proof. To prove (i) we notice that $g_j \in C^{\infty}(\mathbb{R}^{n-j})$, and so for any $q \geq 1$ we have that

$$\int_{\mathbb{R}^{n-j}} |g_j(x)|^q dx = \int_{|x|\leq 1} |g_j(x)|^q dx + \int_{|x|>1} |g_j(x)|^q dx \leq \|g\|_{\infty}^q \cdot \left|B^{n-j}\right| + \int_{|x|>1} |g_j(x)|^q dx$$

where $|B^{n-j}|$ is the volume of the $n-j$ dimensional unit ball B^{n-j}. We conclude that the convergence or divergence of $\int_{\mathbb{R}^{n-j}} |g_j(x)|^q dx$ depends solely on the behavior 'at infinity'. We also know that on $\mathbb{R}^k$

$$\int_{|x|\geq 1} \frac{dx}{|x|^{\beta}} = |\mathbb{S}^{k-1}| \int_1^{\infty} \frac{|x|^{k-1}}{|x|^{\beta}} d|x| = |\mathbb{S}^{k-1}| \int_1^{\infty} \frac{d|x|}{|x|^{\beta-k+1}}$$

so convergence will occur if and only if $\beta - k > 0$, i.e. $\beta > k$.

Since

$$|x'|^{n+2\alpha-2j} g_j(x') \underset{|x'|\to\infty}{\longrightarrow} 1$$

we know that g_j will be in $L^q(\mathbb{R}^{n-j})$ if and only if $\frac{1}{|x'|^{n+2\alpha-2j}} \in L^q(\mathbb{R}^{n-j} \setminus B^{n-j})$. This happens if and only if

$$q(n+2\alpha-2j) > n-j$$

Indeed, since $\frac{j}{2} < \alpha < \frac{n}{2}$ we have that for any $q \geq 1$

$$q(n+2\alpha-2j) \geq n+2\alpha-2j > n-j > 0$$

Also, $\frac{j}{2} < \alpha < \frac{n}{2}$ implies that $1 < \frac{2(n-j)}{n+2\alpha-j} < 2$ and so $g_j \in L^{\frac{2(n-j)}{n+2\alpha-j}}\left(\mathbb{R}^{n-j}\right)$ which proves the second part of (i).

The first part of (ii) follows from the observation that $g_j \in C^{\infty}\left(\mathbb{R}^{n-j}\right)$ and all of its derivatives are $L^1\left(\mathbb{R}^{n-j}\right)$ functions, along with known facts about the decay of the Fourier transform (see [13]). Indeed, if we have

$$f(x) = \frac{P(x)}{\left(1+|x|^2\right)^{\beta}}$$

where P is a polynomial then for any $1 \leq i \leq n$

$$\frac{\partial f}{\partial x_i}(x) = \frac{\frac{\partial P}{\partial x_i}(x)}{\left(1+|x|^2\right)^{\beta}} - \frac{2\beta P(x)}{\left(1+|x|^2\right)^{\beta}} \cdot \frac{x_i}{1+|x|^2}$$

Since $\deg\left(\frac{\partial P}{\partial x_i}(x)\right) \leq \deg\left(P(x)\right) - 1$ and $\frac{|x_1|}{\sqrt{1+|x|^2}} \leq 1$ we conclude that the behavior at infinity of $\frac{\partial f}{\partial x_i}$ is 'better' than that of f (in the sense of integral convergence). Thus, if $f \in L^1\left(\mathbb{R}^k\right)$ so would $\frac{\partial f}{\partial x_i}$ and by induction all the derivatives. This is our case with $P(x) = 1$ and $\beta = \frac{n+2\alpha-2j}{2}$.

The second part of (ii) follows immediately from the fact that $g_j \in L^1\left(\mathbb{R}^{n-j}\right)$, which implies that $\widehat{g_j} \in C\left(\mathbb{R}^{n-j}\right)$. $\square$

We're finally ready to show that $D^{\alpha}\left(\mathbb{R}^n\right)$ is indeed the right space.

Theorem 3.4.12. *Let $\widehat{f}(p) = \frac{\widehat{g_{j,HLS}}(p')}{|p|^{2\alpha}}$ where $p' \in \mathbb{R}^{n-j}$ and $g_{j,HLS}\left(x'\right) = \frac{A}{\left(\gamma^2+|x'-a'|^2\right)^{\frac{n+2\alpha-2j}{2}}}$ for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a' \in \mathbb{R}^{n-j}$. Then $\widehat{f}$ is the distributional Fourier transform of some $f \in D^{\alpha}\left(\mathbb{R}^n\right)$ and f is a minimizer for (3.3.9).*

Proof. We start by noting that with the notations of Theorem 3.4.10 and Lemma 3.4.11 we have

$$g_{j,HLS}\left(x'\right) = \frac{A}{\left(\gamma^2+|x'-a'|^2\right)^{\frac{n+2\alpha-2j}{2}}} = \frac{A}{|\gamma|^{n+2\alpha-2j}} \cdot \frac{1}{\left(1+\left|\frac{x'-a'}{\gamma}\right|^2\right)^{\frac{n+2\alpha-2j}{2}}}$$

$$= \frac{A}{|\gamma|^{n+2\alpha-2j}} \cdot g_j\left(\frac{x'-a'}{\gamma}\right)$$

so $g_{j,HLS}$ satisfies all the conclusions of Lemma 3.4.11.

We have that

$$\int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 |p|^{2\alpha} dp = \int_{\mathbb{R}^n} \frac{\left|\widehat{g_{j,HLS}}(p')\right|^2}{|p|^{2\alpha}} dp = \pi^{\frac{j}{2}} \cdot \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)}{\Gamma(\alpha)} \cdot \int_{\mathbb{R}^{n-j}} \frac{\left|\widehat{g_{j,HLS}}(p')\right|^2}{|p'|^{2\alpha-j}} dp'$$

Using Theorem 3.3.1, Theorem 3.3.2 and the fact that $g_{j,HLS}$ is the minimizer for the Hardy-Littlewood-Sobolev inequality we see that

$$\int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 |p|^{2\alpha} dp = \frac{\pi^{2\alpha}\Gamma\left(\frac{n-2\alpha}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(\alpha)} \int_{\mathbb{R}^{n-j}} \int_{\mathbb{R}^{n-j}} \frac{g_{j,HLS}(x')\overline{g_{j,HLS}}(y')}{|x'-y'|^{n-2\alpha}} dx'dy'$$

$$= \pi^{\alpha} \cdot \frac{\Gamma\left(\frac{n-2\alpha}{2}\right)\Gamma\left(\frac{2\alpha-j}{2}\right)}{\Gamma(\alpha)\Gamma\left(\frac{n+2\alpha-2j}{2}\right)} \left\{\frac{\Gamma\left(\frac{n-j}{2}\right)}{\Gamma(n-j)}\right\}^{-\frac{2\alpha-j}{n-j}} \|g_{j,HLS}\|^2_{L^{\frac{2(n-j)}{n+2\alpha-2j}}}$$

$$= (2\pi)^{2\alpha} C_{j,\alpha,n} \|g_{j,HLS}\|^2_{L^{\frac{2(n-j)}{n+2\alpha-2j}}} < \infty$$

i.e. $\widehat{f} \in L^2(\mathbb{R}^n, |p|^{2\alpha}dp)$. From Theorem 3.4.2 and Remark 3.4.3 we conclude that there exists $f \in D^{\alpha}(\mathbb{R}^n)$ such that $\widehat{f}$ is its distibutional Fourier transform.

In order to show that f is indeed a minimzer we note that by Lemma 3.4.9 and the above computation we have that

$$\|\tau_j f\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}} \geq \left|\left\langle \tau_j f, \frac{g_{j,HLS}}{\|g_{j,HLS}\|_{L^{\frac{2(n-j)}{n+2\alpha-2j}}}} \right\rangle\right|^2 = \frac{1}{\|g_{j,HLS}\|^2_{L^{\frac{2(n-j)}{n+2\alpha-2j}}}} \left|\int_{\mathbb{R}^n} \widehat{f}(p)\overline{\widehat{g_{j,HLS}}(p')} dp\right|^2$$

$$= \frac{1}{\|g_{j,HLS}\|^2_{L^{\frac{2(n-j)}{n+2\alpha-2j}}}} \cdot \left(\int_{\mathbb{R}^n} \frac{\left|\widehat{g_{j,HLS}}(p')\right|^2}{|p|^{2\alpha}} dp\right)^2 = \frac{\left(\int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 |p|^{2\alpha} dp\right)^2}{\|g_{j,HLS}\|^2_{L^{\frac{2(n-j)}{n+2\alpha-2j}}}}$$

$$(2\pi)^{2\alpha} C_{j,\alpha,n} \int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 |p|^{2\alpha} dp = C_{j,\alpha,n} \cdot \langle f, (-\Delta)^{\alpha} f\rangle$$

which concludes our proof. □

Before we finish this chapter we'd like to show two more things:

- Why $H^{\alpha}(\mathbb{R}^n)$ isn't the right space.
- Inequality (3.3.9) is actually sharp in $S(\mathbb{R}^n)$, though equality is unattainable.

Theorem 3.4.13. *Let $\frac{j}{2} < \alpha < \frac{n}{2}$ and $\widehat{f}(p) = \frac{\widehat{g_{j,HLS}}(p')}{|p|^{2\alpha}}$ where $p' \in \mathbb{R}^{n-j}$ and $g_{j,HLS}(x')$ was defined in Theorem 3.4.10. Then $f \in H^{\alpha}(\mathbb{R}^n)$ if and only if $\alpha < \frac{n}{4}$.*

Note that as $H^{\alpha}(\mathbb{R}^n)$ is contained in $D^{\alpha}(\mathbb{R}^n)$ Theorem 3.4.10 tells us that a function in $H^{\alpha}(\mathbb{R}^n)$ can attain equality in (3.3.9) if and only if it's of the form (3.4.3). As such, the above theorem tells us that for many choices of α we won't have a minimizer in $H^{\alpha}(\mathbb{R}^n)$.

Proof. Since $\alpha > \frac{j}{2}$ we have that $2\alpha > \frac{j}{2}$ and as shown in Theorem 3.3.4

$$\int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 dp = \int_{\mathbb{R}^n} \frac{\left|\widehat{g_{j,HLS}}(p')\right|^2}{|p|^{4\alpha}} dp = \pi^{\frac{j}{2}} \cdot \frac{\Gamma\left(\frac{4\alpha-j}{2}\right)}{\Gamma(2\alpha)} \cdot \int_{\mathbb{R}^{n-j}} \frac{\left|\widehat{g_{j,HLS}}(p')\right|^2}{|p'|^{4\alpha-j}} dp'$$

$$= \pi^{\frac{j}{2}} \cdot \frac{\Gamma\left(\frac{4\alpha-j}{2}\right)}{\Gamma(2\alpha)} \cdot \int_{|p'|\le 1} \frac{\left|\widehat{g_{j,HLS}}(p')\right|^2}{|p'|^{4\alpha-j}} dp' + \pi^{\frac{j}{2}} \cdot \frac{\Gamma\left(\frac{4\alpha-j}{2}\right)}{\Gamma(2\alpha)} \cdot \int_{|p'|>1} \frac{\left|\widehat{g_{j,HLS}}(p')\right|^2}{|p'|^{4\alpha-j}} dp'$$

Due to property (ii) in Lemma 3.4.11 we find that $\int_{|p'|>1} \frac{\left|\widehat{g_{j,HLS}}(p')\right|^2}{|p'|^{4\alpha-j}} dp' < \infty$. Since $\widehat{g_{j,HLS}}$ is continuous (property (ii) again) and $\widehat{g_{j,HLS}}(p') \underset{p'\to 0}{\longrightarrow} \widehat{g_{j,HLS}}(0) = \frac{A}{|\gamma|^{n+2\alpha-2j}} \cdot \left\|g_j\left(\frac{\cdot - a'}{\gamma}\right)\right\|_{L^1} \neq 0$ when $g_{j,HLS} \neq 0$, we find that $\int_{|p'|\le 1} \frac{\left|\widehat{g_{j,HLS}}(p')\right|^2}{|p'|^{4\alpha-j}} dp'$ will converge if and only if $\int_{|p'|\le 1} \frac{dp'}{|p'|^{4\alpha-j}}$ converges.

$$\int_{|p'|\le 1} \frac{dp'}{|p'|^{4\alpha-j}} = \left|\mathbb{S}^{n-j-1}\right| \cdot \int_0^1 \frac{|p'|^{n-j-1}}{|p'|^{4\alpha-j}} d|p'| = \left|\mathbb{S}^{n-j-1}\right| \cdot \int_0^1 \frac{d|p'|}{|p'|^{4\alpha-n+1}}$$

which will converge if and only if $4\alpha - n < 0$ or $\alpha < \frac{n}{4}$.

Thus, if $\alpha < \frac{n}{4}$ we have that $\widehat{f}$ is in $L^2(\mathbb{R}^n)$ and as such has an inverse Fourier transform f. We know that $\widehat{f} \in L^2(\mathbb{R}^n, |p|^{2\alpha}dp)$ (from Theorem 3.4.12) and as such $f \in H^{\alpha}(\mathbb{R}^n)$.

Conversely, if $f \in H^{\alpha}(\mathbb{R}^n)$ then $f \in L^2(\mathbb{R}^n)$ and so α must satisfy $\alpha < \frac{n}{4}$. □

Theorem 3.4.14. *Let $1 \le j < n$ and $\frac{j}{2} < \alpha < \frac{n}{2}$. For any $\epsilon > 0$ there exists $f_\epsilon \in S(\mathbb{R}^n)$ such that*

$$\left\|\tau_j f_\epsilon\right\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}} \ge (1-\epsilon) C_{j,\alpha,n} \cdot \langle f_\epsilon, (-\Delta)^{\alpha} f_\epsilon \rangle$$

Proof. This is a direct result of the density of $S(\mathbb{R}^n)$ in $D^\alpha(\mathbb{R}^n)$, but we'll show it for completion. Let $f \in D^\alpha(\mathbb{R}^n)$ be a minimzer for (3.3.9). Since $S(\mathbb{R}^n)$ is dense in $D^\alpha(\mathbb{R}^n)$ we can find a sequence of functions in $f_k \in S(\mathbb{R}^n)$ such that

$$\|f_k - f\|_{D^\alpha} \underset{k\to\infty}{\longrightarrow} 0$$

As such

$$\langle f_k, (-\Delta)^\alpha f_k\rangle = \int_{\mathbb{R}^n} \left|\widehat{f}_k(p)\right|^2 |p|^{2\alpha} dp \underset{k\to\infty}{\longrightarrow} \int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 |p|^{2\alpha} dp = \langle f, (-\Delta)^\alpha f\rangle$$

By the definition of τ_j we have that $\tau_j f = \lim_{k\to\infty} \tau_j f_k$ in the $L^{\frac{2(n-j)}{n-2\alpha}}(\mathbb{R}^{n-j})$ sense and so

$$\|\tau_j f_k\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}} \underset{k\to\infty}{\longrightarrow} \|\tau_j f\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}}$$

Thus, we can find k_η such that

$$\left\langle f_{k_\eta}, (-\Delta)^\alpha f_{k_\eta}\right\rangle \leq (1+\eta)\langle f, (-\Delta)^\alpha f\rangle$$

and

$$\left\|\tau_j f_{k_\eta}\right\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}} \geq (1-\eta)\|\tau_j f\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}}$$

which implies

$$\begin{aligned}\left\|\tau_j f_{k_\eta}\right\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}} &\geq (1-\eta)\|\tau_j f\|^2_{L^{\frac{2(n-j)}{n-2\alpha}}} \geq (1-\eta)C_{j,\alpha,n}\cdot\langle f, (-\Delta)^\alpha f\rangle \\ &\geq \frac{1-\eta}{1+\eta}\cdot C_{j,\alpha,n}\cdot\left\langle f_{k_\eta}, (-\Delta)^\alpha f_{k_\eta}\right\rangle\end{aligned}$$

For a given $\epsilon > 0$ picking η such that $\frac{1-\eta}{1+\eta} > 1-\epsilon$ concludes the proof. □

In the next section we will develop another trace type inequality, using similar methods to those we used to prove (3.3.9).

3.5 Another trace inequality

Our main trace inequality (3.3.9) connects the fractional Laplacian of a function to some $L^q(\mathbb{R}^n)$ norm of its restriction to the intersections of the hyperplanes $x_n = 0, \ldots, x_{n-j+1} = 0$. A different possibility we can investigate is an inequality connecting the fractional Laplacian of a function to the fractional Laplacian of appropriate order of its restriction to the intersection of the hyperplanes $x_n = 0, \ldots, x_{n-j+1} = 0$.

As usual, we start with $S(\mathbb{R}^n)$.

Theorem 3.5.1. *Let $1 \leq j < n$ and $\frac{j}{2} < \alpha < \frac{n}{2}$. For any $f \in S(\mathbb{R}^n)$ we have*

$$\left\langle \tau_j f, (-\Delta)^{\alpha-\frac{j}{2}} \tau_j f \right\rangle \leq \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)}{2^j \cdot \pi^{\frac{j}{2}} \cdot \Gamma(\alpha)} \cdot \langle f, (-\Delta)^{\alpha} f\rangle \tag{3.5.1}$$

where $\tau_j f$ was defined in Theorem 3.3.4.

Proof. As in the proof of Theorem 3.3.4 given $f \in S(\mathbb{R}^n)$, $g \in S(\mathbb{R}^{n-j})$ we have that

$$|\langle \tau f, g\rangle|^2 \leq \frac{\langle f, (-\Delta)^{\alpha} f\rangle}{(2\pi)^{2\alpha}} \cdot \pi^{\frac{j}{2}} \cdot \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)}{\Gamma(\alpha)} \cdot \int_{\mathbb{R}^{n-j}} \frac{|\widehat{g}(p')|^2}{|p'|^{2\alpha-j}} dp' \tag{3.5.2}$$

On the other hand

$$|\langle \tau f, g\rangle|^2 = \left|\int_{\mathbb{R}^{n-j}} \widehat{\tau f}(p') \overline{\widehat{g}(p')} dp'\right|^2 = \left|\int_{\mathbb{R}^{n-j}} \widehat{\tau f}(p') |p'|^{\alpha-\frac{j}{2}} \cdot \frac{\overline{\widehat{g}(p')}}{|p'|^{\alpha-\frac{j}{2}}} dp'\right|^2 \tag{3.5.3}$$

Denoting $\widehat{h}(p') = \frac{\widehat{g}(p')}{|p'|^{\alpha-\frac{j}{2}}}$ we find that $\widehat{h} \in L^2(\mathbb{R}^{n-j})$. Indeed, since $\widehat{g} \in S(\mathbb{R}^{n-j})$ we have that $\int_{|p'|\geq 1} \frac{|\widehat{g}(p')|^2}{|p'|^{2\alpha-j}} dp' < \infty$. Also, $\int_{|p'|<1} \frac{|\widehat{g}(p')|^2}{|p'|^{2\alpha-j}} dp' \leq \|\widehat{g}\|_{\infty}^2 \cdot \int_{|p'|<1} \frac{dp'}{|p'|^{2\alpha-j}} = \|\widehat{g}\|_{\infty}^2 \cdot |\mathbb{S}^{n-j-1}| \int_0^1 \frac{dp'}{|p'|^{2\alpha-n+1}}$, which will be finite since $\alpha < \frac{n}{2}$.

(3.5.2) and (3.5.3) can be rewritten as

$$\left|\int_{\mathbb{R}^{n-1}} \widehat{\tau f}(p') |p'|^{\alpha-\frac{j}{2}} \cdot \overline{\widehat{h}(p')} dp'\right|^2 \leq \frac{\pi^{\frac{j}{2}} \cdot \Gamma\left(\frac{2\alpha-j}{2}\right)}{(2\pi)^{2\alpha}\Gamma(\alpha)} \langle f, (-\Delta)^{\alpha} f\rangle \cdot \left\|\widehat{h}\right\|_{L^2} \tag{3.5.4}$$

It is easy to show that functions of the form $\frac{\widehat{g}(p')}{|p'|^{\alpha-\frac{j}{2}}}$ where $g \in S(\mathbb{R}^{n-j})$ are dense in $L^2(\mathbb{R}^{n-j})$ (See Lemma A.2.3 in the Appendix). As such (3.5.4) is valid for any $\widehat{h} \in L^2(\mathbb{R}^{n-j})$. This implies that $\widehat{\tau f}(p') |p'|^{\alpha-\frac{j}{2}} \in L^2(\mathbb{R}^{n-j})$ and

$$\int_{\mathbb{R}^{n-j}} \left|\widehat{\tau_j f}(p')\right|^2 |p'|^{2\alpha-j} dp' \leq \frac{\pi^{\frac{j}{2}} \cdot \Gamma\left(\frac{2\alpha-j}{2}\right)}{(2\pi)^{2\alpha}\Gamma(\alpha)} \langle f, (-\Delta)^{\alpha} f\rangle$$

or

$$\left\langle \tau_j f, (-\Delta)^{\alpha-\frac{j}{2}}\, \tau_j f \right\rangle \leq \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)}{2^j \cdot \pi^{\frac{j}{2}} \cdot \Gamma(\alpha)} \left\langle f, (-\Delta)^{\alpha} f \right\rangle$$

which is the desired result. □

The advantage of inequality (3.5.1) over (3.3.9) lies in its proof: we only used the Cauchy-Schwarz inequality, removing a restriction on possible minimizers imposed by the Hardy-Littlewood-Sobolev inequality! Indeed, we note the following theorem whose proof we'll leave to the Appendix:

Theorem 3.5.2. *Let $1 \leq j < n$ and $\frac{j}{2} < \alpha < \frac{n}{2}$. Given $g \in C_c^{\infty}\left(\mathbb{R}^{n-j} \setminus \{0\}\right)$, define $\widehat{f}(p) = \frac{g(p')}{|p|^{2\alpha}}$. Then $\widehat{f} \in L^q\left(\mathbb{R}^n\right)$ for any $q \geq 1$ and as such $f = \check{f}$ is well defined. Moreover, $f \in L^2\left(\mathbb{R}^n\right) \cap C\left(\mathbb{R}^n\right)$ and*

$$\left\langle \tau_j f, (-\Delta)^{\alpha-\frac{j}{2}}\, \tau_j f \right\rangle = \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)}{2^j \cdot \pi^{\frac{j}{2}} \cdot \Gamma(\alpha)} \cdot \left\langle f, (-\Delta)^{\alpha} f \right\rangle$$

Before continuing to the next section we'd like to observe that the trace inequality we developed here along with the Sobolev type inequality found in Theorem 3.4.7 can be combined together to give an alternative proof of our main inequality (3.3.9). We've decided not to take that path since we wanted a simple way to see what the minimizers were, and felt that proving (3.3.9) from scratch was more enlightening.

Our last theoretical section will investigate the case $\alpha = \frac{j}{2}$.

3.6 The case $\alpha = \frac{j}{2}$

Throughout this chapter we always demanded that α be bigger than $\frac{j}{2}$. Our computations showed why it was necessary - we had many integrals whose convergence depended on it. In this short section we'll see that it wasn't just a technicality for a tricky proof. We will show that no inequality of the form (3.3.9) is possible even for Schwartz functions when $\alpha = \frac{j}{2}$. Before we start we notice that when $\alpha = \frac{j}{2}$

$$\frac{2(n-j)}{n+2\alpha-2j} = 2$$

Theorem 3.6.1. *For any $M > 0$ there exists $f \in S(\mathbb{R}^n)$ such that*

$$\|\tau_j f\|_{L^2}^2 > M\left(f, (-\Delta)^{\frac{j}{2}} f\right)$$

Proof. Let $\beta > \frac{j}{2}$ and $\widehat{g} \in C_c^{\infty}(\mathbb{R}^{n-j} \setminus \{0\})$. Define $\widehat{f_{\beta,m}}(p) = \frac{\widehat{g}(p')}{|p|^{2\beta}} \cdot \omega_m(|p''|)$, where $p = (p', p'')$ and $\omega_m \in C_c^{\infty}(0, \infty)$ be such that $\omega_m|_{\left[\frac{1}{m}, m\right]} = 1$, $\text{supp}\omega_m \subset \left[\frac{1}{2m}, 2m\right]$ and $0 \le \omega_m(x) \le 1$ for all $x \in (0, \infty)$. $\widehat{f_{\beta,m}} \in C_c^{\infty}(\mathbb{R}^n)$ and as such it has an inverse Fourier transform in $S(\mathbb{R}^n)$. As shown in the proof of Theorem 3.3.4

$$\widehat{\tau_j f_{\beta,m}}(p') = \int_{\mathbb{R}^j} \widehat{f_{\beta,m}}(p', p'')\, dp''$$

and by Plancherel's equality

$$\|\tau f_{\beta,m}\|_{L^2}^2 = \int_{\mathbb{R}^{n-j}} \left(\int_{\mathbb{R}^j} \frac{\widehat{g}(p')}{|p|^{2\beta}} \omega_m(|p''|)\, dp''\right)^2 dp' = \int_{\mathbb{R}^{n-j}} |\widehat{g}(p')|^2 \left(\int_{\mathbb{R}^j} \frac{\omega_m(|p''|)}{|p|^{2\beta}} dp''\right)^2 dp'$$

Since $0 \le \frac{\omega_m(|p''|)}{|p|^{2\beta}} \le \frac{1}{|p|^{2\beta}}$, $\int_{\mathbb{R}^j} \frac{dp''}{|p|^{2\beta}} = \frac{\pi^{\frac{j}{2}} \cdot \Gamma\left(\frac{2\beta - j}{2}\right)}{\Gamma(\beta)} \cdot \frac{1}{|p'|^{2\beta - j}}$ and $\frac{\omega_m(|p''|)}{|p|^{2\beta}} \underset{m\to\infty}{\longrightarrow} \frac{1}{|p|^{2\beta}}$ pointwise we can use the Dominated Convergence Theorem to conclude that

$$\lim_{m\to\infty} \|\tau f_{\beta,m}\|_{L^2}^2 = \left(\frac{\pi^{\frac{j}{2}} \cdot \Gamma\left(\frac{2\beta - j}{2}\right)}{\Gamma(\beta)}\right)^2 \int_{\mathbb{R}^{n-j}} \frac{|\widehat{g}(p')|^2}{|p'|^{2(2\beta - j)}} dp'$$

On the other hand,

$$\int_{\mathbb{R}^n} |p|^j \left|\widehat{f_{\beta,m}}(p)\right|^2 dp = \int_{\mathbb{R}^n} \frac{|\widehat{g}(p')|^2}{|p|^{4\beta - j}} \omega_m^2(|p''|)\, dp$$

a similar discussion shows that

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} |p|^j \left|\widehat{f_{\beta,m}}(p)\right|^2 dp = \int_{\mathbb{R}^n} \frac{|\widehat{g}(p')|^2}{|p|^{4\beta - j}} dp = \pi^{\frac{j}{2}} \cdot \frac{\Gamma\left(\frac{4\beta - 2j}{2}\right)}{\Gamma(4\beta - j)} \int_{\mathbb{R}^{n-j}} \frac{|\widehat{g}(p')||^2}{|p'|^{2(2\beta - j)}} dp'$$

Since $\zeta\Gamma(\zeta) \underset{\zeta\to 0}{\longrightarrow} 1$ we have that

$$\frac{2\beta - j}{4} \cdot \frac{\Gamma^2\left(\frac{2\beta - j}{2}\right)}{\Gamma\left(\frac{4\beta - 2j}{2}\right)} \underset{\beta\to\frac{j}{2}}{\longrightarrow} 1$$

and so, for a given $M > 0$, we can find $\beta_M > \frac{j}{2}$ such that

$$\frac{\pi^{\frac{j}{2}} \cdot \Gamma^2\left(\frac{2\beta_M - j}{2}\right) \Gamma(4\beta_M - j)}{\Gamma^2(\beta_M)\, \Gamma\left(\frac{4\beta_M - 2j}{2}\right)} > 2(2\pi)^{\frac{j}{2}} \cdot M$$

For the fixed β_M we can find k_M such that

$$\frac{\|\tau f_{\beta_M,k_M}\|_{L^2}^2}{(2\pi)^{\frac{j}{2}} \int_{\mathbb{R}^n} |p|^j \left|\widehat{f_{\beta_M,k_M}}(p)\right|^2} \geq \frac{1}{2} \cdot \lim_{k\to\infty} \frac{\|\tau f_{\beta_M,k}\|_{L^2}^2}{(2\pi)^{\frac{j}{2}} \int_{\mathbb{R}^n} |p|^j \left|\widehat{f_{\beta_M,k}}(p)\right|^2}$$

$$= \frac{1}{2\,(2\pi)^{\frac{j}{2}}} \cdot \frac{\pi^{\frac{j}{2}} \cdot \Gamma^2\left(\frac{2\beta_M - j}{2}\right) \Gamma\left(4\beta_M - j\right)}{\Gamma^2\left(\beta_M\right) \Gamma\left(\frac{4\beta_M - 2j}{2}\right)} > M$$

Which concludes the proof. □

The last section of this chapter will be devoted to a few last remarks.

3.7 Last Remarks

A thing we may notice, looking at all the theorems presented in this chapter, is that we choose to restrict the original function f to the intersection of the hyperplanes $x_n = 0, \ldots, x_{n-j+1} = 0$. However, this seems more of a convenience than an actual necessity. Indeed, looking at all our formulas and remembering that

$$\widehat{f(\cdot - a)}(p) = e^{-2\pi i a \circ p} \widehat{f}(p)$$

we conclude that we can easily replace τ_j by $\tau_{j,a''}$ where

$$\tau_{j,a''} f\left(x'\right) = f\left(x', a''\right)$$

for $x' \in \mathbb{R}^{n-j}$ and $a'' \in \mathbb{R}^j$, and still obtain the same results. The fact that the set of minimizers we obtained is translation invariant (in the spatial sense) is not a big surprise!

Lastly, while we feel that we've exploited everything we can from (3.3.9) we still think that there is much more to be done concerning the fractional Laplacian, and are eager to learn more and think more on the subject.

Appendix A

HELPFUL ADDITIONS

In this Appendix we present proofs to several results we used in our main chapters, but felt they would hinder the flow of reading.

A.1 Additional Proofs to Chapter 2

Lemma A.1.1. *Any solution of the master equation (2.1.1) satisfies*

$$\|F(v_1,\ldots,v_N,t)-1\|_{L^2\left(\mathbb{S}^{N-1}\sqrt{N}\right)} \leq e^{-\Delta_N t}\,\|F(v_1,\ldots,v_N,0)-1\|_{L^2\left(\mathbb{S}^{N-1}\sqrt{N}\right)}$$

Proof. We know that $F(v_1,\ldots,v_N,0)$ is a density function, and as such $(F(v_1,\ldots,v_N),1) = 1$. Since $F(v_1,\ldots,v_N,t)$ solves the master equation we have that

$$\frac{d}{dt}\left(F(v_1,\ldots,v_N,t),1\right) = \left(\frac{\partial F}{\partial t}(v_1,\ldots,v_N,t),1\right) = -N\left((I-Q)F(v_1,\ldots,v_N,t),1\right)$$

$$= -N\left(F(v_1,\ldots,v_N,t),(I-Q)1\right) = 0$$

and hence $(F(v_1,\ldots,v_N,t),1) = 1$ for all t. Next we notice that

$$\frac{d}{dt}\|F(v_1,\ldots,v_N,t)-1\|^2_{L^2\left(\mathbb{S}^{N-1}\sqrt{N}\right)} = 2\left\langle\frac{\partial\left(F-1\right)}{\partial t},F-1\right\rangle = -2\left\langle N(I-Q)\left(F-1\right),\left(F-1\right)\right\rangle$$

and since $(F-1,1) = 1-1 = 0$ we find that

$$\frac{d}{dt}\|F(v_1,\ldots,v_N,t)-1\|^2_{L^2\left(\mathbb{S}^{N-1}\sqrt{N}\right)} \leq -2\Delta_N\,\|F(v_1,\ldots,v_N,0)-1\|^2_{L^2\left(\mathbb{S}^{N-1}\sqrt{N}\right)}$$

and so

$$e^{2\Delta_N t}\,\|f(v_1,\ldots,v_N,t)-1\|^2_{L^2\left(\mathbb{S}^{N-1}\sqrt{N}\right)} \leq \|f(v_1,\ldots,v_N)-1\|^2_{L^2\left(\mathbb{S}^{N-1}\sqrt{N}\right)}$$

which is the desired proof. $\square$

Lemma A.1.2. *For any $a, \eta > 0$ we have that*

$$\frac{\sqrt{2\pi}}{a} \cdot \sqrt{1 - e^{-\frac{a\eta^2}{2}}} \leq \int_{|x|<\eta} e^{-\frac{a^2x^2}{2}} dx \leq \frac{\sqrt{2\pi}}{a} \cdot \sqrt{1 - e^{-a^2\eta^2}}$$

and

$$\int_{|x|>\eta} e^{-\frac{a^2x^2}{2}} dx \leq \frac{\sqrt{2\pi} \cdot e^{-\frac{a^2\eta^2}{2}}}{a}$$

Proof. We have that

$$\int_{|x|<\eta} e^{-\frac{a^2x^2}{2}} dx = \sqrt{\int\int_{|x|,|y|<\eta} e^{-\frac{a^2(x^2+y^2)}{2}} dxdy} \leq \sqrt{\int\int_{x^2+y^2<2\eta^2} e^{-\frac{a^2(x^2+y^2)}{2}} dxdy}$$

$$= \sqrt{\int_0^{2\pi} \int_0^{\sqrt{2}\eta} re^{-\frac{a^2r^2}{2}} drd\vartheta} = \sqrt{2\pi} \cdot \sqrt{\frac{1 - e^{-a^2\eta^2}}{a^2}}$$

And

$$\int_{|x|<\eta} e^{-\frac{a^2x^2}{2}} dx \geq \sqrt{\int\int_{x^2+y^2<\eta^2} e^{-\frac{a^2(x^2+y^2)}{2}} dxdy} = \sqrt{2\pi} \cdot \sqrt{\frac{1 - e^{-\frac{a\eta^2}{2}}}{a^2}}$$

Similarly

$$\int_{|x|>\eta} e^{-\frac{a^2x^2}{2}} dx = \int_{\mathbb{R}} e^{-\frac{a^2x^2}{2}} dx - \int_{|x|<\eta} e^{-\frac{a^2x^2}{2}} dx = \frac{\sqrt{2\pi}}{a} - \int_{|x|<\eta} e^{-\frac{a^2x^2}{2}} dx$$

$$\leq \frac{\sqrt{2\pi}}{a} \left(1 - \sqrt{1 - e^{-\frac{a^2\eta^2}{2}}}\right) = \frac{\sqrt{2\pi} \cdot e^{-\frac{a^2\eta^2}{2}}}{a\left(1 + \sqrt{1 - e^{-\frac{a^2\eta^2}{2}}}\right)} \leq \frac{\sqrt{2\pi} \cdot e^{-\frac{a^2\eta^2}{2}}}{a}$$

□

Lemma A.1.3. *For any $a > 0$ and $k_0, m \in \mathbb{N}$ we have that*

$$\sum_{k=k_0+1}^{m} \frac{e^{-\frac{a^2k}{2}}}{\sqrt{k}} \leq \frac{\sqrt{2\pi} \cdot e^{-\frac{a^2k_0}{2}}}{a}$$

$$\sum_{k=k_0+1}^{m} \frac{1}{\sqrt{k}} \leq 2\sqrt{m}$$

Proof. Since $f(x) = \frac{e^{-\frac{a^2 x}{2}}}{\sqrt{x}}$ is a positive decreasing function on $(1, \infty)$ we have that

$$\sum_{k=k_0+1}^{m} \frac{e^{-\frac{a^2 k}{2}}}{\sqrt{k}} \leq \int_{k_0}^{m} \frac{e^{-\frac{a^2 x}{2}}}{\sqrt{x}} dx \underset{y=a\sqrt{x}}{=} \frac{2}{a} \int_{a\sqrt{k_0}}^{a\sqrt{m}} e^{-\frac{y^2}{2}} dy \leq \frac{2}{a} \int_{a\sqrt{k_0}}^{\infty} e^{-\frac{y^2}{2}} dy$$

$$= \frac{1}{a} \int_{|y|>a\sqrt{k_0}} e^{-\frac{y^2}{2}} dy \leq \frac{\sqrt{2\pi} \cdot e^{-\frac{a^2 k_0}{2}}}{a}$$

where we used Theorem A.1.2 in the last inequality. Similarly

$$\sum_{k=k_0+1}^{m} \frac{1}{\sqrt{k}} \leq \int_{k_0}^{m} \frac{dx}{\sqrt{x}} = 2\left(\sqrt{m} - \sqrt{k_0}\right) \leq 2\sqrt{m}$$

□

Lemma A.1.4. *Let $f(v_1, \dots, v_N)$ be a continuous function on $\mathbb{R}^N$ then*

$$\int_{\mathbb{S}^{N-1}(r)} f d\sigma_r^N = \frac{1}{|\mathbb{S}^{N-1}| r^{N-2}} \cdot \sum_{\epsilon=\{+,-\}} \int_{\sum_{i=1}^{N-1} v_i^2 \leq r^2} \frac{f\left(v_1, \dots, v_{N-1}, \epsilon\sqrt{r^2 - \sum_{i=1}^{N-1} v_i^2}\right)}{\sqrt{r^2 - \sum_{i=1}^{N-1} v_i^2}} dv_1 \dots dv_{N-1}$$

Proof. We start by noticing that

$$\int_{\mathbb{S}^{N-1}(r)} f d\sigma_r^N = \frac{1}{|\mathbb{S}^{N-1}| r^{N-1}} \int_{\mathbb{S}^{N-1}(r)} f ds_r^N$$

where ds_r^N is the uniform measure on $\mathbb{S}^{N-1}(r)$ induced from the regular measure on $\mathbb{R}^N$. Next we see that since we can think of the upper hemisphere, $\mathbb{S}_+^{N-1}(r)$, as the graph of the function $\gamma(v_1, \dots, v_{N-1}) = \sqrt{r^2 - \sum_{i=1}^{N-1} v_i^2}$. Thus, we can compute the surface element using the parametrization:

$$\Gamma(v_1, \dots, v_{N-1}) = (v_1, \dots, v_{N-1}, \gamma(v_1, \dots, v_{N-1}))$$

with the domain $D = \left\{\sum_{i=1}^{N-1} v_i^2 \leq r^2\right\}$.

As such

$$\frac{\partial \Gamma}{\partial v_i} = \left(0, \dots, 0, \underbrace{1}_{ith\,position}, 0, \dots, 0, \frac{\partial \gamma}{\partial v_i}\right)$$

The last vector we'll need for the surface element is a unit normal to $\mathbb{S}_+^{N-1}(r)$. This is easily seen to be

$$\widehat{n} = \frac{1}{\sqrt{|\nabla\gamma|^2+1}}\left(-\frac{\partial\gamma}{\partial v_1}, -\frac{\partial\gamma}{\partial v_2}, \dots, -\frac{\partial\gamma}{\partial v_{N-1}}, 1\right)$$

Thus, the surface element is given by

$$ds = \frac{1}{\sqrt{|\nabla\gamma|^2+1}} \cdot \det\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \frac{\partial\gamma}{\partial v_1} \\ 0 & 1 & 0 & \dots & 0 & \frac{\partial\gamma}{\partial v_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \frac{\partial\gamma}{\partial v_{N-1}} \\ -\frac{\partial\gamma}{\partial v_1} & -\frac{\partial\gamma}{\partial v_2} & -\frac{\partial\gamma}{\partial v_3} & \dots & -\frac{\partial\gamma}{\partial v_{N-1}} & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{|\nabla\gamma|^2+1}} \cdot \det\begin{pmatrix} 1 & 0 & \dots & 0 & \frac{\partial\gamma}{\partial v_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \frac{\partial\gamma}{\partial v_{N-1}} \\ -\frac{\partial\gamma}{\partial v_2} & -\frac{\partial\gamma}{\partial v_3} & \dots & -\frac{\partial\gamma}{\partial v_{N-1}} & 1 \end{pmatrix}$$

$$+\frac{(-1)^{N-1}\cdot\frac{\partial\gamma}{\partial v_1}}{\sqrt{|\nabla\gamma|^2+1}} \cdot \det\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{\partial\gamma}{\partial v_1} & -\frac{\partial\gamma}{\partial v_2} & -\frac{\partial\gamma}{\partial v_3} & \dots & -\frac{\partial\gamma}{\partial v_{N-1}} \end{pmatrix}$$

Since

$$\det\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{\partial\gamma}{\partial v_1} & -\frac{\partial\gamma}{\partial v_2} & -\frac{\partial\gamma}{\partial v_3} & \dots & -\frac{\partial\gamma}{\partial v_{N-1}} \end{pmatrix} = (-1)^{N-2}\det\begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ -\frac{\partial\gamma}{\partial v_2} & \dots & -\frac{\partial\gamma}{\partial v_{N-1}} & -\frac{\partial\gamma}{\partial v_1} \end{pmatrix}$$

$$= (-1)^{N-1}\cdot\frac{\partial\gamma}{\partial v_1}\cdot\det\begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ -\frac{\partial\gamma}{\partial v_2} & \dots & -\frac{\partial\gamma}{\partial v_{N-1}} & 1 \end{pmatrix} = (-1)^{N-1}\cdot\frac{\partial\gamma}{\partial v_1}$$

we conclude that

$$ds = \frac{1}{\sqrt{|\nabla\gamma|^2+1}} \cdot \det\begin{pmatrix} 1 & 0 & \dots & 0 & \frac{\partial\gamma}{\partial v_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \frac{\partial\gamma}{\partial v_{N-1}} \\ -\frac{\partial\gamma}{\partial v_2} & -\frac{\partial\gamma}{\partial v_3} & \dots & -\frac{\partial\gamma}{\partial v_{N-1}} & 1 \end{pmatrix} + \frac{\left(\frac{\partial\gamma}{\partial v_1}\right)^2}{\sqrt{|\nabla\gamma|^2+1}}$$

Continuing in the same way we find that

$$ds = \frac{1}{\sqrt{|\nabla\gamma|^2+1}}\left(\det\begin{pmatrix} 1 & \frac{\partial\gamma}{\partial v_{N-1}} \\ -\frac{\partial\gamma}{\partial v_{N-1}} & 1 \end{pmatrix} + \sum_{i=1}^{N-2}\left(\frac{\partial\gamma}{\partial v_i}\right)^2\right) = \sqrt{|\nabla\gamma|^2+1}$$

In our particular case, $\frac{\partial\gamma}{\partial v_i} = -\frac{v_i}{\sqrt{r^2-\sum_{i=1}^{N-1}v_i^2}}$ and so

$$ds = \sqrt{\frac{\sum_{i=1}^{N-1}v_i^2}{r^2-\sum_{i=1}^{N-1}v_i^2}+1} = \frac{r}{\sqrt{r^2-\sum_{i=1}^{N-1}v_i^2}}$$

Thus

$$\int_{\mathbb{S}_+^{N-1}(r)} f ds_r^N = \int_{\sum_{i=1}^{N-1}v_i^2\le r^2} \frac{rf\left(v_1,\dots,v_{N-1},\sqrt{r^2-\sum_{i=1}^{N-1}v_i^2}\right)}{\sqrt{r^2-\sum_{i=1}^{N-1}v_i^2}} dv_1\dots dv_{N-1}$$

In the same way

$$\int_{\mathbb{S}_-^{N-1}(r)} f ds_r^N = \int_{\sum_{i=1}^{N-1}v_i^2\le r^2} \frac{rf\left(v_1,\dots,v_{N-1},-\sqrt{r^2-\sum_{i=1}^{N-1}v_i^2}\right)}{\sqrt{r^2-\sum_{i=1}^{N-1}v_i^2}} dv_1\dots dv_{N-1}$$

Combining the two gives

$$\int_{\mathbb{S}^{N-1}(r)} f d\sigma_r^N = \frac{1}{|\mathbb{S}^{N-1}|r^{N-2}} \cdot \sum_{\epsilon=\{+,-\}} \int_{\sum_{i=1}^{N-1}v_i^2\le r^2} \frac{f\left(v_1,\dots,v_{N-1},\epsilon\sqrt{r^2-\sum_{i=1}^{N-1}v_i^2}\right)}{\sqrt{r^2-\sum_{i=1}^{N-1}v_i^2}} dv_1\dots dv_{N-1}$$

□

Lemma A.1.5. *Let $f(v_1,\dots,v_j)$ and $g(v_{j+1},\dots,v_N)$ be continuous functions on $\mathbb{R}^j$ and $\mathbb{R}^{N-j}$ respectively. Then*

$$\int_{\mathbb{S}^{N-1}(r)} f(v_1,\dots,v_j)\cdot g(v_{j+1},\dots,v_N)\, d\sigma_r^N$$

$$= \frac{\left|\mathbb{S}^{N-j-1}\right|}{\left|\mathbb{S}^{N-1}\right| r^{N-2}} \int_{\sum_{i=1}^{j} v_i^2 \le r^2} f\left(v_1, \ldots, v_j\right) \left(r^2 - \sum_{i=1}^{j} v_i^2\right)^{\frac{N-j-2}{2}}$$

$$\left(\int_{\mathbb{S}^{N-j-1}\left(\sqrt{r^2 - \sum_{i=1}^{j} v_i^2}\right)} g d\sigma^{N-j}_{\sqrt{r^2 - \sum_{i=1}^{j} v_i^2}}\right) dv_1 \ldots dv_j$$

Proof. By Lemma A.1.4

$$\int_{\mathbb{S}^{N-1}(r)} f\left(v_1, \ldots, v_j\right) \cdot g\left(v_{j+1}, \ldots, v_N\right) d\sigma_r^N$$

$$= \frac{1}{\left|\mathbb{S}^{N-1}\right| r^{N-2}} \cdot \sum_{\epsilon = \{+,-\}} \int_{\sum_{i=1}^{N-1} v_i^2 \le r^2} \frac{f\left(v_1, \ldots, v_j\right) \cdot g\left(v_{j+1}, \ldots, v_{N-1}, \epsilon\sqrt{r^2 - \sum_{i=1}^{N-1} v_i^2}\right)}{\sqrt{r^2 - \sum_{i=1}^{N-1} v_i^2}} dv_1 \ldots dv_{N-1}$$

$$= \frac{1}{\left|\mathbb{S}^{N-1}\right| r^{N-2}} \int_{\sum_{i=1}^{j} v_i^2 \le r^2} \frac{f\left(v_1, \ldots, v_j\right)}{\sqrt{r^2 - \sum_{i=1}^{j} v_i^2}} \cdot$$

$$\left(\sum_{\epsilon = \{+,-\}} \int_{\sum_{i=j+1}^{N-1} v_i^2 \le r^2 - \sum_{i=1}^{j} v_i^2} \frac{\sqrt{r^2 - \sum_{i=1}^{j} v_i^2} \cdot g\left(v_{j+1}, \ldots, v_{N-1}, \epsilon\sqrt{\left(r^2 - \sum_{i=1}^{j} v_i^2\right) - \sum_{i=j+1}^{N-1} v_i^2}\right)}{\sqrt{\left(r^2 - \sum_{i=1}^{j} v_i^2\right) - \sum_{i=j+1}^{N-1} v_i^2}}\right.$$

$$\left. \cdot\, dv_{j+1} \ldots dv_{N-1}\right) dv_1 \ldots dv_j$$

$$= \frac{1}{\left|\mathbb{S}^{N-1}\right| r^{N-2}} \int_{\sum_{i=1}^{j} v_i^2 \le r^2} \frac{f\left(v_1, \ldots, v_j\right)}{\sqrt{r^2 - \sum_{i=1}^{j} v_i^2}} \left(\int_{\mathbb{S}^{N-j-1}\left(\sqrt{r^2 - \sum_{i=1}^{j} v_i^2}\right)} g ds^{N-j}_{\sqrt{r^2 - \sum_{i=1}^{j} v_i^2}}\right) dv_1 \ldots dv_j$$

$$= \frac{\left|\mathbb{S}^{N-j-1}\right|}{\left|\mathbb{S}^{N-1}\right| r^{N-2}} \int_{\sum_{i=1}^{j} v_i^2 \le r} f\left(v_1, \ldots, v_j\right) \left(r^2 - \sum_{i=1}^{j} v_i^2\right)^{\frac{N-j-2}{2}}$$

$$\left(\int_{\mathbb{S}^{N-j-1}\left(\sqrt{r^2 - \sum_{i=1}^{j} v_i^2}\right)} g d\sigma^{N-j}_{\sqrt{r^2 - \sum_{i=1}^{j} v_i^2}}\right) dv_1 \ldots dv_j$$

□

Lemma A.1.6. *For any continuous density function on $\mathbb{S}^{N-1}(\sqrt{N})$, F_N, we have that*

$$\langle F_N, (I-Q)F_N \rangle \geq 0$$

Moreover, $\langle F_N, (I-Q)F_N \rangle = 0$ if and only if F_N is constant.

Proof. Using the definition of Q (given in Section 2.1) and the notation presented in Theorem 2.6.5 we find that

$$\langle F_N, N(I-Q)F_N \rangle$$

$$= N \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log F_N \left(v_1, \dots, v_N\right)$$

$$\cdot \left(F_N \left(v_1, \dots, v_N\right) - \frac{1}{2\pi \begin{pmatrix} N \\ 2 \end{pmatrix}} \sum_{i<j} \int_0^{2\pi} F_N \left(R_{i,j,\vartheta}\left(v_1, \dots, v_N\right)\right) d\vartheta \right) d\sigma^N$$

$$= \frac{N}{2\pi \begin{pmatrix} N \\ 2 \end{pmatrix}} \sum_{i<j} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log F_N \left(v_1, \dots, v_N\right)$$

$$\cdot \int_0^{2\pi} \left(F_N \left(v_1, \dots, v_N\right) - F_N \left(R_{i,j,\vartheta}\left(v_1, \dots, v_N\right)\right)\right) d\vartheta d\sigma^N$$

By the same argument that led us to equation (2.6.5) in Theorem 2.6.5 in Section 2.6 we find that

$$\frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log F_N \left(v_1, \dots, v_N\right) \left(F_N \left(v_1, \dots, v_N\right) - F_N \left(R_{i,j,\vartheta}\left(v_1, \dots, v_N\right)\right)\right) d\vartheta d\sigma^N$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log F_N \left(R_{i,j,\vartheta}\left(v_1, \dots, v_N\right)\right) \left(F_N \left(v_1, \dots, v_N\right) - F_N \left(R_{i,j,\vartheta}\left(v_1, \dots, v_N\right)\right)\right) d\vartheta d\sigma^N$$

And so

$$\langle F_N, N(I-Q)F_N \rangle$$

$$= \frac{N}{4\pi \begin{pmatrix} N \\ 2 \end{pmatrix}} \sum_{i<j} \int_0^{2\pi} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \left(\log F_N\left(v_1,\ldots,v_N\right) - \log F_N\left(R_{i,j,\vartheta}\left(v_1,\ldots,v_N\right)\right)\right)$$

$$\cdot \left(F_N\left(v_1,\ldots,v_N\right) - F_N\left(R_{i,j,\vartheta}\left(v_1,\ldots,v_N\right)\right)\right) d\vartheta d\sigma^N$$

Since $(\log x - \log y)(x-y) \geq 0$ (as mentioned in Theorem 2.6.5) we attain the desired result. Moreover, $\langle F_N, N(I-Q)F_N\rangle = 0$ if and only if

$$F_N\left(v_1,\ldots,v_N\right) = F_N\left(R_{i,j,\vartheta}\left(v_1,\ldots,v_N\right)\right)$$

for each i, j and ϑ which implies that F_N is constant. $\square$

A.2 Additional Proofs to Chapter 3

Theorem A.2.1. *Let $g \in L^2\left(\mathbb{R}^n\right) \cap L^1\left(\mathbb{R}^n\right)$ and define*

$$f(x) = \int_{\mathbb{R}^n} g(p) e^{2\pi i x \circ p} dp$$

Then $f \in L^2\left(\mathbb{R}^n\right) \cap C\left(\mathbb{R}^n\right)$ and $\widehat{f} = g$.

Proof. We notice that by the definition

$$f(x) = \widehat{g}(-x)$$

Using known properties of the Fourier transform (See for example [17]) we have that $f \in L^2\left(\mathbb{R}^n\right) \cap C\left(\mathbb{R}^n\right)$.

Let $g_n \in S\left(\mathbb{R}^n\right)$ be such that $\|g_n - g\|_{L^2} \underset{n\to\infty}{\longrightarrow} 0$. Define $f_n = \widehat{g_n}(-x)$. $f_n \in S\left(\mathbb{R}^n\right)$ and since the Fourier transform is an isometry on $L^2\left(\mathbb{R}^n\right)$ we have that

$$\|f_n - f\|_{L^2} = \|\widehat{g_n} - \widehat{g}\|_{L^2} = \|g_n - g\|_{L^2} \underset{n\to\infty}{\longrightarrow} 0$$

Again, using the fact that the Fourier transform is an isometry and that for $g \in S\left(\mathbb{R}^n\right)$, $\widehat{\widehat{g}(-x)}(p) = g(p)$ we find that

$$\widehat{f}(p) = \lim_{n\to\infty} \widehat{f_n}(p) = \lim_{n\to\infty} \widehat{\widehat{g_n}(-x)}(p) = \lim_{n\to\infty} g_n(p)$$

in $L^2(\mathbb{R}^n)$. But

$$\lim_{n\to\infty} g_n(p) = g(p)$$

in $L^2(\mathbb{R}^n)$ which implies that $\widehat{f} = g$. □

Lemma A.2.2. *$H^\alpha(\mathbb{R}^n)$ is a Hilbert space with the inner product*

$$\langle f, g\rangle_{H^\alpha} = \int_{\mathbb{R}^n} \widehat{f}(p)\overline{\widehat{g}(p)}dp + \int_{\mathbb{R}^n} \widehat{f}(p)\overline{\widehat{g}(p)}|p|^{2\alpha}dp$$

Moreover, $S(\mathbb{R}^n)$is dense in $H^\alpha(\mathbb{R}^n)$ as well as $H^l(\mathbb{R}^n)$ for any $l \geq \alpha$, $l \in \mathbb{N}$.

Proof. We have that

$$\langle f, f\rangle_{H^\alpha} = \int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 dp + \int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 |p|^{2\alpha}dp \geq \|f\|_{L^2}^2 > 0$$

which implies that $\langle f, f\rangle_{H^\alpha}$ only if $f = 0$. Given f, g and h in $H^\alpha(\mathbb{R}^n)$, $\alpha, \beta \in \mathbb{C}$ it is clear that

$$\langle f, g\rangle_{H^\alpha} = \overline{\langle g, f\rangle_{H^\alpha}}$$

and

$$\begin{aligned}\langle f, \alpha g + \beta h\rangle_{H^\alpha} &= \overline{\alpha}\left(\int_{\mathbb{R}^n} \widehat{f}(p)\overline{\widehat{g}(p)}dp + \int_{\mathbb{R}^n} \widehat{f}(p)\overline{\widehat{g}(p)}|p|^{2\alpha}dp\right)\\ &+\overline{\beta}\left(\int_{\mathbb{R}^n} \widehat{f}(p)\overline{\widehat{h}(p)}dp + \int_{\mathbb{R}^n} \widehat{f}(p)\overline{\widehat{h}(p)}|p|^{2\alpha}dp\right)\\ &= \overline{\alpha}\langle f, g\rangle_{H^\alpha} + \overline{\beta}\langle f, h\rangle_{H^\alpha}\end{aligned}$$

□

Thus $\langle \cdot, \cdot\rangle_{H^\alpha}$ is an inner product. Next we'll show completeness. Given a Cauchy sequence $\{f_k\}_{k\in\mathbb{N}}$ in the induced norm $\|\cdot\|_{H^\alpha}$ we find that

$$\|f_k - f_m\|_{H^\alpha} \geq \max\left(\int_{\mathbb{R}^n} \left|\widehat{f_k}(p) - \widehat{f_m}(p)\right|^2 dp, \int_{\mathbb{R}^n} \left|\widehat{f_k}(p)|p|^\alpha - \widehat{f_m}(p)|p|^\alpha\right|^2 dp\right)$$

implying that $\left\{\widehat{f_k}(p)\right\}_{k\in\mathbb{R}^n}$ and $\left\{\widehat{f_k}(p)|p|^\alpha\right\}_{k\in\mathbb{R}^n}$ are Cauchy sequences in $L^2(\mathbb{R}^n)$. Since $L^2(\mathbb{R}^n)$ is a Hilbert space there are $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}^n)$ such that $\left\|\widehat{f_k}(p) - \widehat{f}(p)\right\|_{L^2}^2 \xrightarrow[k\to\infty]{}$

0 and $\left\|\widehat{f}_k(p)|p|^\alpha - \widehat{g}(p)\right\|_{L^2}^2 \underset{k\to\infty}{\longrightarrow} 0$. By passing to subsequences we can assume that the convergence is also pointwise almost-everywhere. This implies that

$$\widehat{g}(p) = \lim_{k\to\infty} \widehat{f}_k(p)|p|^\alpha = \widehat{f}(p)|p|^\alpha \in L^2\left(\mathbb{R}^n\right)$$

Proof. We can conclude that $f \in H^\alpha\left(\mathbb{R}^n\right)$ and

$$\|f_k - f\|_{H^\alpha} = \int_{\mathbb{R}^n} \left|\widehat{f}(p) - \widehat{f}_k(p)\right|^2 dp + \int_{\mathbb{R}^n} \left|\widehat{f}_k(p)|p|^\alpha - \widehat{f}(p)|p|^\alpha\right|^2 dp \underset{k\to\infty}{\longrightarrow} 0$$

i.e. $H^\alpha\left(\mathbb{R}^n\right)$ is a Hilbert space.

Next, given any $l \geq \alpha$, $l \in \mathbb{N}$ we have that for any $f \in H^l\left(\mathbb{R}^n\right)$

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 |p|^{2\alpha} dp &= \int_{|p|<1} \left|\widehat{f}(p)\right|^2 |p|^{2\alpha} dp + \int_{|p|\geq 1} \left|\widehat{f}(p)\right|^2 |p|^{2\alpha} dp \\ &\leq \int_{|p|<1} \left|\widehat{f}(p)\right|^2 dp + \int_{|p|\geq 1} \left|\widehat{f}(p)\right|^2 |p|^{2l} dp \\ &\leq \int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 dp + \int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 |p|^{2l} dp < \infty\end{aligned}$$

This implies that $H^l\left(\mathbb{R}^n\right) \subset H^\alpha\left(\mathbb{R}^n\right)$ and

$$\|f\|_{H^\alpha}^2 = \int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 dp + \int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 |p|^{2\alpha} dp \tag{A.2.1}$$

$$\leq 2\int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 dp + \int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 |p|^{2l} dp \leq 2\,\|f\|_{H^l}^2$$

To prove density we define $\widehat{f}_k(p) = \widehat{f}(p)\chi_{[0,k]}\left(|p|\right)$ for a given $f \in H^\alpha\left(\mathbb{R}^n\right)$. We notice that $\left|\widehat{f}_k(p)\right| \leq \left|\widehat{f}(p)\right|$ and so $f_k \in L^2\left(\mathbb{R}^n\right)$. Let $f_k = \left(\widehat{f}_k\right)^{\vee}$ where $\check{g}$ is the inverse Fourier transform of g. We have that

$$\int_{\mathbb{R}^n} \left|\widehat{f}_k(p)\right|^2 |p|^{2s} dp \leq |k|^{2s} \int_{\mathbb{R}^n} \left|\widehat{f}_k(p)\right|^2 dp \leq |k|^{2s} \int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^2 dp < \infty$$

and so $f_k \in H^s\left(\mathbb{R}^n\right)$ for any $s \in \mathbb{R}_+$. Moreover, since $\left|\widehat{f}_k(p) - \widehat{f}(p)\right| \leq 2\left|\widehat{f}(p)\right|$ and $\widehat{f}_k(p) \underset{k\to\infty}{\longrightarrow} \widehat{f}(p)$ pointwise, the Dominated Convergence Theorem implies that

$$\|f_k - f\|_{H^\alpha} = \int_{\mathbb{R}^n} \left|\widehat{f}(p) - \widehat{f}_k(p)\right|^2 dp + \int_{\mathbb{R}^n} \left|\widehat{f}_k(p)|p|^\alpha - \widehat{f}(p)|p|^\alpha\right|^2 dp \underset{k\to\infty}{\longrightarrow} 0$$

which shows the density for $H^l(\mathbb{R}^n)$ when $l \geq \alpha$, $l \in \mathbb{N}$.

To show the density of $S(\mathbb{R}^n)$ in $H^\alpha(\mathbb{R}^n)$ we use the known result that $S(\mathbb{R}^n)$ is dense in $H^{[\alpha]+1}(\mathbb{R}^n)$ (See [11]). Given $f \in H^\alpha(\mathbb{R}^n)$ and $\epsilon > 0$ we can find $f_\epsilon \in H^{[\alpha]+1}(\mathbb{R}^n)$ such that $\|f_\epsilon - f\|_{H^\alpha} < \frac{\epsilon}{2}$. Next we find $g_\epsilon \in S(\mathbb{R}^n)$ such that $\|f_\epsilon - g_\epsilon\|_{H^{[\alpha]+1}} < \frac{\epsilon}{2\sqrt{2}}$. Using (A.2.1) we conclude that

$$\|g_\epsilon - f\|_{H^\alpha} \leq \|f_\epsilon - f\|_{H^\alpha} + \|f_\epsilon - g_\epsilon\|_{H^\alpha} \leq \frac{\epsilon}{2} + \sqrt{2}\,\|f_\epsilon - g_\epsilon\|_{H^{[\alpha]+1}} < \epsilon$$

completing the proof. □

Lemma A.2.3. *The set* $\left\{\frac{g(p)}{|p|^\beta} \,|\, g \in S(\mathbb{R}^n)\right\}$ *is dense in* $L^2(\mathbb{R}^n)$ *for any* $\beta < \frac{n}{2}$.

Proof. Since $g \in S(\mathbb{R}^n)$ we know that $\int_{|p|\geq 1} \frac{|g(p)|^2}{|p|^{2\beta}} dp < \infty$. Also,

$$\int_{|p|<1} \frac{|g(p)|^2}{|p|^{2\beta}} dp \leq \|g\|_\infty^2 \int_{|p|<1} \frac{dp}{|p|^{2\beta}} < \infty$$

since $\beta < \frac{n}{2}$. This implies that $\left\{\frac{g(p)}{|p|^\beta} \,|\, g \in S(\mathbb{R}^n)\right\} \subset L^2(\mathbb{R}^n)$. Given $f \in L^2(\mathbb{R}^n)$ we can find a function $f_\epsilon \in S(\mathbb{R}^n)$ such that $\|f_\epsilon - f\|_{L^2} < \frac{\epsilon}{2}$. Let ω_m be as defined in Theorem **??**. We have that $f_{\epsilon,m}(p) = f_\epsilon(p)\omega_m(|p|) \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, $|f_{\epsilon,m}(p)| \leq |f_\epsilon(p)|$ and $f_{\epsilon,m}(p) \underset{m\to\infty}{\longrightarrow} f_\epsilon(p)$ pointwise. Using the Dominated Convergence Theorem we conclude that $\|f_{\epsilon,m} - f_\epsilon\|_{L^2} \underset{m\to\infty}{\longrightarrow} 0$. We can find m_ϵ such that $\|f_{\epsilon,m_\epsilon} - f_\epsilon\| < \frac{\epsilon}{2}$ and conclude that $\|f_{\epsilon,m_\epsilon} - f\|_{L^2} < \epsilon$. Defining $g_\epsilon(p) = |p|^\beta f_{\epsilon,m_\epsilon}(p)$ we find that $g_\epsilon \in C_c^\infty(\mathbb{R}^n \setminus \{0\}) \subset S(\mathbb{R}^n)$ and

$$\left\|\frac{g_\epsilon(p)}{|p|^\beta} - f(p)\right\|_{L^2} = \|f_{\epsilon,m_\epsilon}(p) - f(p)\|_{L^2} < \epsilon$$

which is the desired result. □

Theorem A.2.4. *Let* $1 \leq j < n$ *and* $\frac{j}{2} < \alpha < \frac{n}{2}$. *Given* $g \in C_c^\infty(\mathbb{R}^{n-j} \setminus \{0\})$, *define* $\widehat{f}(p) = \frac{g(p')}{|p|^{2\alpha}}$. *Then* $\widehat{f} \in L^q(\mathbb{R}^n)$ *for any* $q \geq 1$ *and as such* $f = \check{f}$ *is well defined. Moreover,* $f \in L^2(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ *and*

$$\left\langle \tau_j f, (-\Delta)^{\alpha - \frac{j}{2}} \tau_j f \right\rangle = \frac{\Gamma\left(\frac{2\alpha - j}{2}\right)}{2^j \cdot \pi^{\frac{j}{2}} \cdot \Gamma(\alpha)} \cdot \langle f, (-\Delta)^\alpha f\rangle$$

Proof. Since $q\alpha \geq \alpha > \frac{j}{2}$ we find that

$$\int_{\mathbb{R}^n} \left|\widehat{f}(p)\right|^q dp = \int_{\mathbb{R}^n} \frac{|g(p')|^q}{|p|^{2q\alpha}} dp = \pi^{\frac{j}{2}} \cdot \frac{\Gamma\left(\frac{2q\alpha-j}{2}\right)}{\Gamma(q\alpha)} \int_{\mathbb{R}^{n-j}} \frac{|g(p')|^q}{|p|'^{2q\alpha-j}} dp'$$

as was shown in the proof of Theorem 3.3.4. Since $g \in C_c^\infty(\mathbb{R}^{n-j} \setminus \{0\})$ we have that $\int_{\mathbb{R}^{n-j}} \frac{|g(p')|^q}{|p|'^{2q\alpha-j}} dp'$ converges, and so $\widehat{f} \in L^q(\mathbb{R}^n)$ for any $q \geq 1$. This implies that $f = \check{f}$ is well defined and is indeed in $L^2(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

Using the inversion formula, we have that

$$f(x) = \int_{\mathbb{R}^n} \frac{g(p')}{|p|^{2\alpha}} e^{2\pi i x \circ p} dp$$

By continuity we find that

$$\tau_j f(x') = \int_{\mathbb{R}^n} \frac{g(p')}{|p|^{2\alpha}} e^{2\pi i (x',0) \circ p} dp = \int_{\mathbb{R}^n} \frac{g(p')}{|p|^{2\alpha}} e^{2\pi i x' \circ p'} dp$$

Since $\frac{g(p')}{|p|^{2\alpha}} \in L^1(\mathbb{R}^n)$ we have by Fubini's Formula that

$$\tau_j f(x') = \int_{\mathbb{R}^{n-j}} \left(\int_{\mathbb{R}^j} \frac{g(p')}{|p|^{2\alpha}} dp''\right) e^{2\pi i x' \circ p'} dp = \pi^{\frac{j}{2}} \cdot \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)}{\Gamma(\alpha)} \int_{\mathbb{R}^{n-j}} \frac{g(p')}{|p'|^{2\alpha-j}} e^{2\pi i x' \circ p'} dp$$

(again we used the fact that $\alpha > \frac{j}{2}$). $g \in C_c^\infty(\mathbb{R}^{n-j} \setminus \{0\})$ and as such $\frac{g(p')}{|p'|^{2\alpha-j}} \in L^q(\mathbb{R}^{n-j})$ for all $q \geq 1$. An easy result from Fourier Analysis shows that $\tau_j f \in L^2(\mathbb{R}^{n-j}) \cap C(\mathbb{R}^{n-j})$ and

$$\widehat{\tau_j f}(p') = \pi^{\frac{j}{2}} \cdot \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)}{\Gamma(\alpha)} \frac{g(p')}{|p'|^{2\alpha-j}} = \left(\int_{\mathbb{R}^j} \frac{g(p')}{|p|^{2\alpha}} dp''\right) = \left(\int_{\mathbb{R}^j} \widehat{f}(p', p'') dp''\right)$$

(See Lemma A.2.1 in the Appendix).

From all the above we can the steps in Theorem 3.5.1 are valid and

$$\left\langle \tau_j f, (-\Delta)^{\alpha-\frac{j}{2}} \tau_j f \right\rangle \leq \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)}{2^j \cdot \pi^{\frac{j}{2}} \cdot \Gamma(\alpha)} \cdot \langle f, (-\Delta)^\alpha f \rangle$$

On the other hand

$$\left\langle \tau_j f, (-\Delta)^{\alpha-\frac{j}{2}} \tau_j f \right\rangle = (2\pi)^{2\alpha-j} \int_{\mathbb{R}^{n-j}} \left|\widehat{\tau_j f}(p')\right|^2 |p'|^{2\alpha-j} dp'$$

$$= (2\pi)^{2\alpha-j} \cdot \left(\pi^{\frac{j}{2}} \cdot \frac{\Gamma\left(\frac{2\alpha-j}{2}\right)}{\Gamma(\alpha)}\right)^2 \cdot \int_{\mathbb{R}^{n-j}} \frac{|g(p')|^2}{|p'|^{2\alpha-j}} dp'$$

and

$$\langle f, (-\Delta)^{\alpha} f\rangle = (2\pi)^{2\alpha} \cdot \int_{\mathbb{R}^n} \frac{|g(p')|^2}{|p|^{2\alpha}} dp = (2\pi)^{2\alpha} \cdot \pi^{\frac{j}{2}} \cdot \frac{\Gamma\left(\frac{2\alpha - j}{2}\right)}{\Gamma(\alpha)} \cdot \int_{\mathbb{R}^{n-j}} \frac{|g(p')|^2}{|p'|^{2\alpha - j}} dp'$$

which leads to

$$\left\langle \tau_j f, (-\Delta)^{\alpha - \frac{j}{2}} \tau_j f \right\rangle = \frac{\Gamma\left(\frac{2\alpha - j}{2}\right)}{2^j \cdot \pi^{\frac{j}{2}} \cdot \Gamma(\alpha)} \cdot \langle f, (-\Delta)^{\alpha} f\rangle$$

□

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