TOPICS IN CONTRACT PRICING AND SPOT MARKETS

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Yi He

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TOPICS IN CONTRACT PRICING AND SPOT MARKETS

Approved by:

Dr. Anton Kleywegt,
Committee Chair
School of Industrial and Systems Engineering
Georgia Institute of Technology

Dr. Anton Kleywegt, Advisor
School of Industrial and Systems Engineering
Georgia Institute of Technology

Dr. George L. Nemhauser
School of Industrial and Systems Engineering
Georgia Institute of Technology

Dr. Ellis Johnson
School of Industrial and Systems Engineering
Georgia Institute of Technology

Dr. Pinar Keskinocak
School of Industrial and Systems Engineering
Georgia Institute of Technology

Dr. Dong Jun Wu
College of Management
Georgia Institute of Technology

Date Approved: 30 April 2008

To my parents,

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SUMMARY

This thesis studies two related topics in liner shipping. The first topic is the contract pricing problem for container carriers. The second part studies the interaction of the longer term contracts and the spot markets/exchanges for the same goods/services.

Most containerized freight is transported under the provisions of medium term contracts between ocean carriers and shippers. One of the biggest challenges for an ocean carrier is to find optimal ways to structure the prices in those contracts. In particular, an ocean carrier would like to set the prices such that the best match between supply and demand can be obtained to maximize its profit. We propose three optimization models as decision tools that carriers can use to plan the contract price structures, as well as the anticipated freight flows and empty container flows for the period covered by the contracts. Based on the models, we propose algorithms and build decision tools that generate the following output: optimal prices to be charged for the movement of freight, the anticipated freight flows and empty flows, containers to be leased, rented and purchased, and the additional voyage capacities to be procured. The first two models are deterministic and represent the problem at different levels of detail. In addition, a three-stage stochastic model is proposed to handle uncertainties in demand rates, costs, bookings and transit times on feeder arcs.

Recent developments in information technology and communication make spot transactions more economical and more convenient. Nevertheless, the incidental spot transactions still count for only a very small portion of freight transported both by the large carriers who are the leaders in implementing e-commerce and in the industry

as a whole. The second part of the thesis studies models to provide insight into the effect of spot market participation rates on various economic quantities. This may have implications for freight transportation industries, such as the sea cargo industry, in which longer term contracts are still prevalent. We focus our study on the following situation. Option contracts are signed before the demand is observed. As is common in liner shipping, sellers (carriers) also sell goods/services on the spot. Buyers (shippers) may or may not buy in the spot market as a matter of policy. We investigate the effects of spot market participation on the contract market and on the surpluses of all market players. It is found that the contract market shrinks as more and more buyers participate in the spot market. However, the effects on the surpluses of different market players are much more complicated and depend on the following factors: market structure, demand variation along time, demand variation among buyers and capacity level.

CHAPTER I

INTRODUCTION

1.1 Liner shipping

As a source of cheap transport, shipping is one of the crucial contributors to economic growth. It has provided access to the global market for almost every industry. U.S. ports and waterways handle over 2.5 billion tons of cargo annually, responsible for moving 99% of the country's overseas trade by volume and 61% by value [3]. The volume is projected to double in the next 15 years with the steady growth of U.S. international trade.

The shipping industry contains two subdivisions, liner and bulk shipping. Though liner companies and bulk companies belong to the same industry, they have very little in common [49]. A bulk shipping service moves cargo in large consignments at relatively low unit costs with flexible schedules, typically for large-scale industrial purposes. Compared to bulk shipping, a liner service promises fixed schedules and consolidates relatively smaller shipments from many shippers. The focus of the thesis is on liner shipping. The commitment to fixed schedules causes huge fixed costs for liner shipping. In contrast to bulk shipping, ships depart from predetermined ports on a service cycle at specified times regardless whether the ships are fully loaded. The number of ships needed to operate such a service is determined by the departure frequency and the time that a ship takes to complete the service cycle. For example, if it takes a ship approximately 6 weeks to complete the cycle, then 6 ships are needed to offer the service with weekly departures at each port. Since large container ships are very expensive, it becomes clear that for even the large carriers it would require a huge investment to establish a service. Also, as liner services handle many relatively small parcels, larger administrative overhead is involved.

Seasonality is very common in liner shipping. For many trades, shipping volumes fluctuate significantly during a year. Another widely known phenomenon in liner shipping is that the freight moved in one direction can be very different from that in the opposite direction. As a result, empty containers have to be moved from depots to depots to maintain balance. Since late 1997, the U.S. liner trades have experienced growing imbalance between the U.S. and Asia, Europe and Latin America. For example, in the transpacific trades, imports from Asia to the U.S. grew almost 20% per year, whereas exports increased at merely 5% annually [1]. Both seasonality and cargo imbalance have important consequences for shipping rates. As reported in [1], members of the Asia to U.S. Transpacific Stabilization Agreement applied a $900 per 40-foot equivalent unit (FEU) general rate increase for the high demand direction and a $300 per FEU peak season surcharge in 1999. On the other hand, in the U.S. to Asia trades carriers have constantly struggled with depressed demand and thus plunging rates.

Vast initial capital investment, fixed schedules, significant seasonality, and cargo imbalances make the pricing problem for ocean carriers important and complex. In liner shipping, the conference system had played an important role in dealing with pricing since the mid-1870s until very recently. Ocean carriers operating on the same trades formed conferences to fix prices. Early on, conferences had tight control over membership, capacity, cargo sharing, and prices. Because of anti-trust regulations, the conference system had evolved to a relatively loose form but still kept tight control over prices until the mid-1990s. In the United States, the conference system was further weakened by the implementation of the United States Ocean Shipping Reform Act (OSRA) of 1998. The key change of OSRA from the Shipping Act of 1984 was that OSRA allowed ocean carriers to enter into confidential contracts with shippers, aiming at promoting a more market-driven, efficient liner shipping industry.

Due to the flexibility and confidentiality of individual service contracting, the number of service contracts and amendments filed with the FMA as well as the volume of freight moving under those service contracts increased by 200% from May 1999 until September 2001 [2]. Most ocean carriers move 80% or more of their containerized freight under service contracts. An important problem for carriers is to structure the prices in those contracts.

Instead of entering service contracts in advance, freight transportation services can also be procured on the spot for a particular shipment at a price determined at the time of the transaction. Recent developments in information technology make such spot transactions more economical and more convenient. Prior to the passage of OSRA, e-commerce started appearing in the liner shipping industry. Originally, such e-commerce services focused on automated services tailored to the shipping industry and internet auctions. OSRA has created a more competitive, market-oriented environment and has arguably been a catalyst to emerging e-commence services. Right after the passage of OSRA, an explosion of such dot-com companies was observed in 1999. These companies underwent a rapid consolidation until early 2001. Many dot-com intermediaries labeled as internet auction sites went out of business. The surviving companies have shifted their focus drastically from auctions back to more fundamental cargo-based applications. The core capability of most current internet portals in liner shipping is the provision of track-and-trace systems. Most of the surviving internet portals have been founded by big carriers, for example, Inttra.com by Maersk Line, P&O Nedlloyd, Hapag-Lloyd, etc., CargoSmart.com by Orient Overseas Container Line, COSCO Container Line, Nippon Yusen Kaisha, etc., and ShipmentLink.com by Evergreen Line. These internet portals in the liner shipping industry are different from the well developed electronic marketplaces such as those in the energy sector. They provide services for shippers such as shipment booking, tracking and tracing, quotation of spot prices, and facilitation of spot transactions.

Most shippers still rely mostly on service contracts for a variety of reasons, including risk aversion and difficulties in obtaining spot rates from different carriers. As a consequence, spot transactions still account for only a small portion compared to transactions covered by contracts.

1.2 Contributions of the thesis

This thesis studies two closely related problems in liner shipping. The first part focuses on the contract pricing problem. The second part studies the interaction of the longer term service contracts and emerging spot exchanges.

As OSRA allows ocean carriers to enter confidential contracts with shippers, how to structure the prices in service contracts has become one of the most important problems for ocean carriers to achieve better revenue. In particular, an ocean carrier would like to set the prices such that the best match between supply and demand can be obtained, while maximizing its profit. The existing literature has not addressed this problem. Chapter 2 presents three models that can be used as decision tools by carriers before and during the contract negotiation season to plan the price structure, as well as the anticipated freight flows and empty container flows for the covered period. Based on the models, we propose algorithms and build decision tools that generate the following output: (1) prices to be charged for the movement of freight, as a function of the alternative paths allowed by the customer, the cargo class of the freight, and the time of the year, (2) the anticipated quantity of freight of each cargo class to be moved on each path at different times in the period, (3) the anticipated flows of freight on feeder legs as well as major legs on the transportation network, and (4) the anticipated flows of equipment, such as different types of containers and chassis, between different parts of the transportation network. The first two models are deterministic models that capture the problem at different levels of detail. Finally, we propose a three-stage stochastic model that incorporates uncertainties. Efficient algorithms to solve the large scale stochastic problem are proposed and compared. The pros and cons of the three models are discussed in terms of solution quality and computation cost.

Various issues are considered in those models, such as demand imbalance and seasonality, freight routing, container repositioning, procurement of extra voyage capacities, container leasing and rental, and container damage. Optimal solutions from the three models are evaluated by SimSea, a simulation model that simulates ocean carrier operations. The first model is called the steady state model. This model considers each season in the planning horizon separately, ignoring the initial conditions and flow changes between two seasons. It provides solutions with reasonable quality, requiring small computing overhead. The second model is called the time stamped model. The model integrates different seasons in the whole planning horizon into one bigger problem. Since the initial conditions and flow changes are captured, we can obtain significantly more revenue by using this model than the first one, at the cost of a small increase in computing time. The third model is the stochastic model. It captures the uncertainties in demand, travel times on feeder arcs, booking cancelations and various costs. The third model can obtain the largest revenue among the three. Although the model can obtain a little more revenue than the time stamped model, this model needs significantly more input data and more computation time. Based on our computational results, we conclude that the second model is likely the most practical of the three.

The second part of the thesis investigates how the introduction of spot markets changes the business of the liner shipping industry tightly bonded with longer term contracts. Though the motivation of the research originates from liner shipping, the models are quite general and can be applied to other industries with non-storable goods. Unlike most of the existing literature, the spot price is endogenous, which complicates the analysis significantly. As is present in practice, the spot market considered in our study is from sell side. Carriers can sell their remaining capacities on the spot markets after satisfying service contracts. It is common that a large portion of shippers still only use service contracts for a variety of reasons. This motivates us to model the buyers' participation in the spot market, which differs our study from other literature. In particular, we study the effects of the buyers' participation on the contract market, on the surpluses of all market players and on the total social welfare.

In Chapter 3, we first present a model of a single-seller single-buyer setting. Second, a model with a single seller and many buyers is considered. In that setting, every buyer has the same utility only depending on a random state of the market. Third, we consider the case where buyers have different utilities in addition to the state of the market. Numerical results are presented to illustrate the effects of the spot market participation. Fourth, a market with two sellers and a single buyer is studied. The last part of Chapter 3 presents a numerical study on a market with many sellers and many buyers for comparison purpose. In all settings, spot price is endogenous and the effect of capacity is also studied. It is found that as the spot market participation rate increases, the contract market shrinks under all market structures. For all the single-seller settings with large capacity, the seller's surplus increases in spot market participation. However, the effects of the spot market participation rate on the buyers' surplus and on the total social surplus are more complicated. Depending on the variation of the demand, an increase in the spot market participation rate may or may not benefit the buyers, thereby may or may not increase the total social surplus. For the undercapacity case, the surpluses of all players are invariant to spot market participation if all the buyers have the same utilities. If the buyers have different utilities, the results do not hold any more. Numerical results show that both the buyers and the seller are better off with higher participation rate. We also prove that all the players have higher surpluses with full participation in the spot market compared to the contract market only case when the capacity is tight. As the market structure moves from single seller to many sellers, it is observed that an increase in spot market participation always improves the total social welfare though it may hurt either the sellers or the buyers.

CHAPTER II

CONTRACT PLANNING MODELS FOR OCEAN CARRIERS

2.1 Introduction

Most of containerized freight transported by ocean carriers is transported under the provisions of medium term contracts between the ocean carriers and shippers. Many of the big ocean carriers do between 80% and 95% of their containerized freight transportation under these contracts. Most contracts between ocean carriers and shippers are negotiated once a year, typically one or two months before the peak season of the major trades covered by the contracts. For example, the peak season for the Trans-Pacific trade is approximately June through November, and most contracts involving Trans-Pacific movements are negotiated during April. The peak season for the Trans-Atlantic trade is approximately December through February, and most contracts involving Trans-Atlantic movements are negotiated during November. There seems to be a trend towards shorter term contracts, such as three month contracts, as well as towards contracts with more flexible stipulations.

A key parameter of a contract is the set of prices specified in the contract. The price charged for transporting a container depends on (1) the origin-destination pair and the alternative paths between them allowed by the customer, (2) the classification of the goods in the container (although this factor is often not taken into account with Intra-Asia transportation), and (3) the time of the year. The United States Ocean Shipping Reform Act (OSRA) of 1998 for the first time allows ocean carriers moving freight into and out of the United States to enter into confidential contracts with shippers, and to charge different shippers different prices. An important decision for an ocean carrier is how to structure these prices. Important considerations to be taken into account when making this decision are (1) the price structures of competitors, (2) the behavior of customer demand, (3) the available transportation capacities (capacities on voyages, and in some cases also the capacities of the domestic modes of transportation, such as truck, rail, and barge transportation), and (4) the availability and flow balance of equipment such as containers and truck chassis. An ocean carrier would like to structure the prices in such a way that the best match between supply and demand is obtained, with the objective to maximize profitability.

We propose three models as decision tools that carriers can use before and during the contract negotiation season to plan the contract price structure, as well as the anticipated freight flows and empty container flows for the period covered by the contracts. The output of the planning tool includes (1) prices to be charged for the movement of freight, as a function of the alternative paths allowed by the customer, the cargo class of the freight, and the time of the year, (2) the anticipated quantity of freight of each cargo class to be moved on each path at different times of the year, (3) the anticipated flows of freight on feeder legs as well as major legs of the transportation network, and (4) the anticipated flows of equipment, such as different types of containers and chassis, between different parts of the transportation network. The first two models are deterministic models, which capture the problem at different levels of detail. To deal with the uncertainty, we propose a three-stage stochastic model.

The rest of the chapter is organized as follows. Section 2.2 contains a brief review of the related literature. Section 2.3 describes a typical ocean carrier's transportation network and its operations. In Section 2.4, we present three different models for the contract planning problem. Solution algorithms are provided in Section 2.5. Section 2.6 presents computational results and Section 2.7 summarizes conclusions.

2.2 Literature review

A major part of existing literature related to ocean cargo transportation is focused on the empty container allocation problem. Florez [24] develops a deterministic network model for this problem. It can be solved by using standard network algorithms. Crainic and Gendreau et al. [17] develop empty container allocation models in a land distribution and transportation system. They propose two dynamic deterministic formulations for the single commodity and multi-commodity cases respectively. To deal with uncertain demands and supplies, they also provide a two-stage stochastic model for the single commodity case. In their models, empty containers received at ports and empty containers sent from ports via ocean transportation are not modeled as decision variables. Instead, they are modeled as external demands and supplies in addition to those from customers. Compared with the work of Crainic and Gendreau et al. [17], Cheung and Chen [14] propose a two-stage stochastic network model that is focused on ocean transportation system. In their model, the randomness arises from the demand and supply of empty containers and from the voyage capacities for empty containers. Stochastic linearization method and stochastic hybrid approximation method are used to solve the problem. They conduct implementations to compare the effectiveness of the approaches. Moreover, they compare the stochastic model with a deterministic model. Their results show that the stochastic model performs better but not significant.

Other literature related to revenue management topics in ocean cargo industry includes the follows. Wan and Levary [54] propose a negotiation procedure for a shipper contracting with ocean carriers. The procedure uses the results from a linear programming model with sensitivity analysis. The approach helps shippers to sign a contract with the lowest obtainable prices. Cao and Ang et al. [13] develop a two-stage mixed integer model for an ocean container carrier to select cargoes in order to maximize its profit for a particular trip. In their setting, the freight prices are fixed at the decision stage. The carrier can refuse or delay cargoes and select most profitable cargoes without violating the capacity constraints of the ship. Uncertain available empty containers at origin ports, ship capacity and costs are considered.

Another branch of literature on freight network closely related to the pricing problem of ocean cargo industry falls into a game theory framework. Friesz and Gottfried et al. [25] develop a sequential shipper-carrier model. Shippers first select commodity origins and carriers, which determines the transportation demands. Given the fixed demands, each carrier then routes freight over its own portion of the network to minimize the operation cost. The freight prices are not decision variables in this model. They are calculated from a function of the cost, the commodity price and other factors. Fisk [23] proposes a conceptual framework to formulate models for optimal transportation systems planning. The Nash equilibrium between a single supplier and its users and the equilibrium in an oligopolistic market are considered. Hurley and Petersen [28] use a nonlinear tariff to obtain an equilibrium solution for the freight network problem. They consider a system with multiple shippers and multiple carriers, each acting as profit maximizing agents. They show that if the carrier coalition uses vertically efficient nonlinear tariff schedule, then the problem can be reduced to maximize the joint profit of shippers and carriers. The distribution of the joint profit among the agents of the system is obtained by solving a linear problem. Smallwood and Mirchandani [47] propose a non-cooperative game model for the case where two carriers compete for a single customer. They consider a simple setting where the two carriers both provide transport of goods from Port O to Port D. The prices charged on the customer and terminal space to rent at the origin port (capacity) are both decision variables for the carriers. Interactions between the carriers and the customer are formulated by a bilevel programming. They show that a Nash Equilibrium is unable to obtain when both price and capacity are decision variables for the parameters they test. Brotcorne and Labbé et al. [10] propose a bilevel model for a freight tariff-setting problem in a single commodity case. They formulate the model in a game theory setting where the leader is a carrier of a group of competing carriers and the follower is a shipper. At the upper level, the leader maximizes its revenues by setting the optimal tariffs on the subset of arcs under its control. At the lower level, given the tariff schedule, the shipper minimizes its transportation cost. A class of heuristic procedures to solve this problem is provided. Numerical experiments are conducted to compare the efficiencies of these approaches with respect to exact optimal solutions on small problem instances. Also numerical results on large instances that could not be solved to optimal by exact method are presented.

Though the tariff-setting problem addressed in Brotcorne and Labbé et al. [10] in the previous paragraph are closely related, the contract planning problem we consider here has its unique properties. First, in practice, there are several major factors in determining freight prices. They include the paths required by customers, cargo classes, container types, the seasons of the year. Thus, from practical point of view, instead of setting prices to arcs, a better approach is to model prices according to those factors, which can be more easily implemented in practice. Second, the contract decisions must be made in a short period before the coming seasons and at a whole network level (usually a trade). The carrier must consider demand requests of different paths, cargoes and container types from different customers simultaneously. Therefore, it is difficult to model the demands as the outcome from each individual shipper to minimize its transportation costs. Third, the planning horizon is long and usually includes multiple seasons. Demand responses to prices are very different in the peak season than those in the off peak season. Thus, the optimal prices cannot be constant through out the entire planning horizon. At the same time, the freight flows change from one season to another. Fourth, empty container repositioning is an important issue in ocean cargo transportation industry. Empty containers need to be moved to keep container balance at depots. Therefore, we approach this problem differently by explicitly assuming the demand functions. That is the demands are modeled as a function of the prices charged by the carrier. Prices and demands are classified by the paths required by customer, cargo classes and container types. The problem is then reduced to a quadratic optimization problem. Different types of demands covering the entire network thus are integrated in one single model. Demand seasonality, empty container repositioning and other practical issues are also considered in the proposed models.

2.3 Ocean carrier operations

In this section we give a brief description of the aspects of ocean carrier operations that are important for modeling the contract planning problem.

2.3.1 Ships, voyages, and services

Ocean container carriers usually operate their ships on schedules that are planned months in advance. Each ship visits a set of ports in cyclical fashion. For example, suppose that the set of ports visited by a particular ship is denoted by $\{A, B, C\}$, and that the ship visits the ports in the sequence $A, B, C, A, B, C, A, \ldots$. Such a cycle is shown in Figure 1 with solid lines. Figure 1 also shows a cycle in which another ship visits ports $\{A, C\}$ in the sequence $A, C, A, C, A, \ldots$ with dotted lines.

A cyclical sequence of ports visited by one or more ships operated by a carrier is called a *service* (also called a *loop* or a *rotation* or a *service rotation*). The trip from a port in a cycle to the next port in the same cycle is called a *voyage*. Thus a cycle that visits m ports consists of m voyages. Each voyage belongs to a particular service. For example, if another service visits ports A,B,F,G, then the voyage AB in the first service and the voyage AB in the second service are regarded as different voyages.

It is common for a service to be scheduled with regular departures at each port

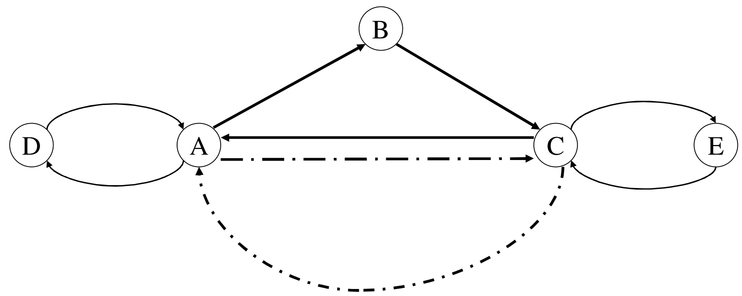

Figure 1: Example service network.

included in the cycle, typically weekly departures. Actual departure times may differ from scheduled departure times, but usually the deviation is not more than a day. To offer weekly departures at each port included in the cycle, the headway between successive ships traversing the cycle must be one week. In addition, if it takes a ship n weeks to complete one cycle, then n ships are needed to offer the service. For many services that visit ports in Asia and North America, and services that visit ports in Asia and Europe, it takes a ship approximately 6 weeks to complete one cycle, and thus 6 ships are needed to offer the service. Taking into account that large container ships are very expensive, it becomes clear that for even the large carriers it would require a huge investment to establish a service.

One way to introduce a service is for several carriers to enter into an alliance to offer the service. Many services that visit ports in Asia and North America, and services that visit ports in Asia and Europe, are offered by alliances between two carriers. Each carrier in the alliance provides one or more ships to be used for the service. The capacity on each ship is then allocated to all the alliance members, often in proportion to the capacity that the alliance member contributes to the service. For example, if carrier 1 contributes 2 ships and carrier 2 contributes 4 ships to the service, and all the ships in the service have the same capacity, then carrier 1 can use 1/3 of each ship's capacity, and carrier 2 can use 2/3 of each ship's capacity. That way, each carrier in the alliance can offer weekly departures at each port in the service even though it did not have enough ships by itself to do so. Provision is usually made for alliance members to obtain some of the capacity on a voyage that was allocated to another alliance member. For example, if carrier 1 needs space for 3 more containers on a voyage, and carrier 2 has enough surplus capacity on the voyage, then carrier 2 can sell the space to carrier 1 at an agreed price.

2.3.2 Freight flows, containers, and capacities

Containers filled with freight as well as empty containers are transported between many origins and destinations. The locations of origins and destinations are called *inland locations*. Origins and destinations are usually not at the ports, so that other transportation modes, such as truck, rail, and barge transportation, are used in addition to the ocean transportation, to move shipments from their origins to their destinations. Most often these other modes of transportation are used to move freight from its origin to a nearby port, and from another port to its destination. In these cases these other modes of transportation are called *feeder services*. Sometimes these other modes of transportation are used to move freight from one port to another port, as in the case of land bridge operations. In addition, when a customer uses a container provided by the carrier, the container is usually moved from a container depot to the consignor's facility where the container is loaded, and later the container is moved from the consignee's facility where the container is unloaded to another container depot. These movements usually take place by truck. To simplify the language, we will refer to all other modes of transportation as feeder services.

A freight shipment moves from its origin to its destination along a *path*. Some of the legs on a path are provided by feeder services, and in the applications that we consider here at least one leg is provided by a voyage. More than one leg in a path can be provided by voyages. For example, a shipment can be moved from origin O to port A, loaded onto a ship at port A, and remain on the ship during voyages AB and BC, after which it is unloaded at port C and then moved to destination D.

Different legs in a path can even be provided by voyages that belong to different services, in which case the shipment has to be transferred between ships, possibly with feeder services in between. For example, a shipment can be moved from origin O to port B, loaded onto a ship at port B, and remain on the ship during voyage BC, after which it is unloaded at port C, whereafter it is transferred by rail from port C to port F, loaded onto a ship at port F, remains on the ship during voyage FG, after which it is unloaded at port G and then moved to destination D.

There can be multiple paths from the same origin to the same destination. Different paths between the same origin-destination pair may use voyages belonging to the same or different services. For example, a shipment from Guangzhou to Atlanta can move by truck from Guangzhou to the Yantian port in Shenzhen, where it is loaded on a ship that stops thereafter at several other ports before the shipment is offloaded at the port of Long Beach on the US west coast. Thereafter the shipment is loaded on a train and is moved to Atlanta. Alternatively, the shipment can move by barge from Guangzhou to the port in Hong Kong, where it is loaded on a ship that operates on the same service mentioned in the previous example, that also stops thereafter at several other ports such as Yantian before the shipment is offloaded at the port of Long Beach on the US west coast. Thereafter the shipment is loaded on a truck and is moved to Atlanta. Alternatively, the shipment can move by barge from Guangzhou to the port in Hong Kong, where it is loaded on a ship that operates on a different service than the one mentioned in the previous two examples, that stops thereafter at several other ports before it moves through the Panama canal and the shipment is offloaded at the port of Savannah on the US east coast. Thereafter the shipment is loaded on a truck and is moved to Atlanta.

In this work we focus on containerized freight transportation. The containers that freight is transported in can be provided by one of several parties. Sometimes the customers provide their own containers. Sometimes the carrier rents containers from companies that specialize in container rentals. The carrier can rent containers for a long time so that the carrier can use a rented container for multiple shipments, or the carrier can rent a container for a single shipment. Most freight is transported in containers provided by ocean carriers. There are several different container types, including general purpose, refrigerated (reefer), high cube, reefer high cube, open top, flat top, hanger, and tank containers. There are also various container sizes, including 20-foot, 40-foot and 45-foot. The most common combinations of container type and container size are 20-foot general purpose, 40-foot general purpose, 40-foot high cube, 45-foot high cube, 20-foot reefer, and 40-foot reefer high cube containers.

It is a widely known phenomenon that usually the flow rates of loaded containers of each container type and size into and out of a location are not balanced. Often this results from the long run demand rate for loaded container movements in one direction being different from the long run demand rate in the opposite direction. For example, these days the demand rate for movements of loaded 40-foot general purpose containers from Asia to North America is much higher than the demand rate in the opposite direction. In addition, even if the long run demand rates were the same in both directions, short run fluctuations could cause significant temporary imbalances in loaded container flows. To provide a sufficient supply of empty containers at locations where they are needed, it is usually desired to maintain flow balance of each container type and size at each location over the longer run, and thus it is often necessary to move empty containers.

Both the voyage legs as well as the feeder service legs have limited capacities. There are constraints on the numbers of containers of different types that a container ship can carry. Container ships have special slots with power supply for refrigerated containers, as well as slots for 20-foot and 40-foot containers. Some slots can accommodate multiple container types; for example, a 40-foot general purpose container can be placed in a slot for a 40-foot refrigerated container, and in many cases two

20-foot general purpose containers can be placed in a single slot for a 40-foot general purpose container or a 40-foot refrigerated container. There are also constraints on the weight that a ship can carry, as well as the distribution of the weight, that affects the loading and unloading of the ship. When we mention the capacity of a ship or voyage, we refer to any combination of the constraints mentioned above. It is common practice to summarize the capacity of a ship in a single number, namely the number of twenty-foot equivalent units (TEU) that the ship can carry — one 40-foot container counts as 2 TEU. If a carrier operates a service by itself, then the capacity of each voyage in the service is the same as the capacity of the ship that makes the voyage. If a carrier operates a service as a member of an alliance, then each carrier in the alliance can use a specified amount of the capacity of each ship in the service, which determines the capacity of each voyage in the service from an individual carrier's point of view. In the latter case, an individual carrier can sometimes obtain additional capacity on one or more voyages from other alliance members, typically at an agreed price.

The capacities of feeder services are less predictable than the capacities of voyages, because feeder services are usually provided by several independent carriers that also carry freight for many other customers. At the same time, when several independent carriers provide feeder services, insufficient capacity on a particular day usually causes delays of at most a few days in the desired transportation time of freight. In contrast, if a voyage does not have sufficient capacity, it can cause a freight shipment to be delayed by a week or more.

2.3.3 Customers, demand, and contracts

Most customers of ocean carriers send multiple shipments from one or more origins to one or more destinations during a year. A customer can purchase transportation services on the spot market at the current price, or a customer can enter into a longer term contract with one or more ocean carriers. The prevalent practice depends on the market; for example, in markets involving freight flows to and/or from North America and/or Europe, most ocean freight (more than 80%) is transported under longer term contracts, whereas for intra-Asia freight flows it is more common to purchase transportation on the spot market. The duration of the contracts vary, with one year being a typical duration. For most of the contracts in the same market, the time periods covered by the contracts are the same, and the contract negotiations take place within a short time period before the time periods covered by the contracts. For example, in the trans-Pacific market between North America and East Asia, most of the contracts are negotiated once a year around April, and in the intra-Asia market, most of the contracts are negotiated quarterly.

The most important parameters specified by a contract are the prices that will be charged for various transportation services. The specified prices depend on several factors. The most important factor is the origin-destination pair, or more specifically, the path or set of paths to be used for the freight movement. In some cases, customers specify not only the origin and destination of a shipment, but also require a particular path or a set of acceptable paths. A customer may prefer one path over other paths for the same origin-destination pair because different paths have different transit times. A customer may also want the shipment to clear customs at a particular port, in which case the customer may be indifferent among all paths that clear customs at the preferred port as long as the transit times of the paths are not significantly different. In such a case, the customer lets the carrier choose the path among the set of acceptable paths, and the customer pays the same price specified in the contract for any of the acceptable paths. The contract prices also depend on the container type and container size, even for the same path; for example, prices are typically lower for general purpose containers than for refrigerated containers, and lower for 20-foot containers than for 40-foot containers. If the container used for a shipment is provided by the carrier, the carrier may provide financial incentives in an attempt to balance container flows; for example, the carrier may charge a lower price for 40-foot refrigerated containers than for 40-foot general purpose containers for particular origin-destination pairs, in an attempt to get the refrigerated containers back to the locations where the demand for them is relatively high. In some markets the contract prices also depend on the freight classes even for the same path and in the same containers; for example, it is quite common to charge higher prices to transport more valuable freight. In addition, contract prices also depend on the time of the year that the shipment is made; such price variations are usually specified in the contract as peak period surcharges. There are also many other surcharges and fees, such as a "bunker adjustment factor" that compensates the carrier for fuel prices that are higher than a specified amount, and terminal handling charges.

Another parameter specified in freight transportation contracts that are sometimes of interest is a "minimum freight guarantee". This specifies that the customer promises to ship at least a certain amount of freight with the carrier during the time period covered by the contract. Such a parameter has several shortcomings. First, often contracts do not specify the consequences that would result if the customer failed to ship the promised amount of freight, and even when the contract specifies penalties, based on anecdotal evidence such penalties are hardly ever enforced. Second, besides the amount of freight, the timing of freight shipments are also important to the carrier. In fact, if too much of the freight is offered in a short time period, the carrier would be unable to handle it. Some contracts also specify the maximum amount of freight that the carrier can accept from the customer per week, and some contracts specify higher prices if the amount of freight offered in a week exceeds a specified amount.

Many customers who enter into contracts with ocean carriers do so with several ocean carriers. Thus, when a customer wants to make a reservation for a freight shipment, the customer often has a choice between more than one carrier. The customer's choice can be influenced by the prices specified in the contracts with the different carriers, the carriers' availability of capacity, and other measures of service quality. All other factors being the same, a carrier can expect higher demand if the prices specified in its contracts are lower.

2.4 Contract planning models

An individual carrier's contract planning problem can be summarized as follows. For a given market, such as the trans-Pacific market, a given set of services in the market with its associated scheduled voyages and capacities, and a given time period covered by the contracts in the market, choose the prices in the contracts to obtain a portfolio of contracts to maximize the carrier's expected total profit over the time period.

The problem stated above can be modeled in many different ways. Below, in Sections 2.4.2–2.4.4 we introduce three models in increasing order of complexity. First we introduce common notation and assumptions in Section 2.4.1.

2.4.1 Basic assumptions and model description

The models are formulated towards maximizing an individual carrier's profit under the following assumptions:

1. The demand arrival rate of a certain type is a function of the charged price.

2. Prices depend on cargo classes, time and alternative paths allowed by the customers.

3. The containers can be owned by the carrier, or rented by the carrier, or provided by the customers.

4. The carrier may join alliance. In this case, extra voyage capacities can be purchased from other carriers.

5. Transshipment between different service rotations is not allowed in the stochastic model.

The proposed models can be used dynamically with updated inputs during negotiation seasons. For the deterministic model, the last assumption is not needed. It will be discussed in the next section.

Let $\mathcal{H}$ denote the set of ports, let $\mathcal{O}$ denote the set of inland locations that act as origins for both freight flows and empty container flows, and let $\mathcal{D}$ denote the set of inland locations that act as destinations for freight flows and empty container flows. One may choose $\mathcal{O} = \mathcal{D}$; nevertheless, it is convenient to use distinct notation for origins and destinations. A natural choice for $\mathcal{O}$ and $\mathcal{D}$ is the set of container depots used by the carrier; all consignors served from a container depot are associated with the corresponding point in $\mathcal{O}$, and all consignees close to a container depot are associated with the corresponding point in $\mathcal{D}$. Let $\mathcal{V}$ denote the set of voyages. Recall that a voyage is a single leg of a service or cycle operated by the ocean carrier (or alliance) under consideration, and thus a voyage $v \in \mathcal{V}$ is specified not only by the start port and end port of the voyage, but also by the service that it belongs to.

Recall that a path l from an origin to a destination consists of a sequence of feeder service legs and voyage legs, and that in some cases customers not only specify the origin and destination of a shipment, but also require a particular path or one path of a particular set L of paths to be used. Let $\overline{L}$ denote the set of all paths l in the model, and let $\mathcal{L}$ denote the collection of all path sets L required by customers.

Hereafter, we use the term "container class" to refer to a combination of container type and container size. Let $\mathcal{B}$ denote the set of all container classes. We use the term "cargo class" to make a distinction between different types of freight on any basis that is relevant for pricing purposes. For example, if the carrier sets different prices for different freight classes, such as consumer electronics, furniture, and textiles, then different cargo classes are used to distinguish these different freight classes. If the carrier sets different prices for different customer classes, then different cargo classes are used to distinguish these different customer classes. Different cargo classes are also used to distinguish different container classes, and thus each cargo class is transported in a unique container class. Let $\mathcal{C}$ denote the set of cargo classes. Set $\mathcal{C}$ is partitioned into two subsets, $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$, such that $\mathcal{C}'$ denotes the cargo classes for which the carrier provides the container (the carrier may own the container, or lease the container for multiple shipments, or rent the container for a single shipment — these are the same from the customer's point of view), and $\mathcal{C}''$ denotes the cargo classes for which the customer provides the container. For each cargo class $c \in \mathcal{C}'$, let $b(c) \in \mathcal{B}$ denote the container class used to transport cargo class c, and for each container class $b \in \mathcal{B}$, let $\mathcal{C}(b) \subset \mathcal{C}'$ denote the set of cargo classes that is transported in class b containers.

In our models, there are various types of decisions. First, there are the pricing decisions, denoted by π. The demands during the time horizon depend on the prices. In the first two models, the demands depend deterministically on the prices, and in the third model the demands are random variables that depend on the prices. Second, there are decisions regarding loaded container flows. If a customer's demand for transportation of a shipment can be satisfied by using one of several paths, then there is a decision regarding which path to use. For the cargo classes $\mathcal{C}'$ for which the carrier provides the containers, a distinction is made between flows of loaded containers that belong to the carrier or that are leased by the carrier for a long term (more than one shipment), denoted by x_o; and flows of loaded containers that are rented by the carrier for a single shipment, denoted by x_r. The distinction is needed because the carrier is responsible for the flow balance of some containers (the containers owned by the carrier or leased by the carrier for a long term), but not for the flow balance of containers rented for a single shipment. For the cargo classes $\mathcal{C}''$ for which the customer provides the containers, the flows of the containers are denoted by x_c. In the stochastic model, a distinction is made between planned flows and actual flows, the difference being due to cancelations of shipment bookings and shipments that arrive too late for a voyage that it was planned to take, due to delays at the consignor or delays with the feeder service. There are also decisions regarding empty container flows of containers owned by the carrier or leased for a long term, denoted by x_e. If additional containers can be purchased or obtained with a long term lease, then there are purchase decisions or lease decisions that are denoted by x_p and x_l respectively. (Flows of containers rented for a single shipment, denoted by x_r, implies a decision to rent x_r containers for the path that is indicated by the notation introduced for each particular model.) If the carrier under consideration can obtain additional capacity on voyages from other alliance members, then there are decisions, denoted by y_a, regarding how much additional capacity to obtain.

For each voyage $v \in \mathcal{V}$, let $u(v)$ denote the capacity that the carrier under consideration can use on voyage v. If the carrier is a member of an alliance, then the carrier may also be able to obtain additional capacities on the voyages, because the alliance members have capacities on the same voyages. For each voyage $v \in \mathcal{V}$, let $u_a(v)$ denote the maximum additional amount of capacity that the carrier can obtain on voyage v, and let $\psi_a(v)$ denote the cost per unit of additional capacity obtained on voyage v. At the planning stage the values $u_a(v)$ and $\psi_a(v)$ may be unknown, and in such a case point estimates of the values are used in the deterministic models.

2.4.2 Steady state deterministic optimization model

In this section, we present the simplest of the three models. For this model, it is assumed that all input parameters are time-invariant, and that the system has reached a steady state. Therefore, it is sufficient to express process parameters such as demand, and decision variables such as flows, as rates per unit time. Initial conditions, such as locations of containers and flows at the beginning of the planning period, do not occur in the model. Similarly, terminal conditions, such as locations of containers and flows at the end of the planning period, do not occur in the model. These assumptions simplify the steady state model considerably relative to the models that follow later. As explained in Section 2.6, for the computational results, if the planning period contains multiple seasons with different values for input parameters, then the steady state model is applied separately for each season to determine the prices.

Decision variable $\pi(c, L)$ denotes the contract price for cargo class $c \in \mathcal{C}$ to be transported on any path l chosen by the carrier such that $l \in L \in \mathcal{L}$, and π denotes the vector of all contract prices. Let $q(c, L, \pi)$ denote the demand rate of transportation requests for cargo class $c \in \mathcal{C}$ per unit time on the path set L if the vector of contract prices is π, measured in the same units as the transportation capacities. To facilitate the description, we assume that the demand rate is expressed in number of containers (or TEU) per week. As mentioned above, in the steady state model, $\pi(c, L)$ and $q(c, L, \pi)$ are not functions of time. In the model, we assume that the demand is a linear function of the price vector π, as follows:

$$q(c, L, \pi) \;=\; \alpha(c, L) + \sum_{c' \in \mathcal{C}} \sum_{L' \in \mathcal{L}} \beta(c, L, c', L')\pi(c', L') \qquad \text{forall } c \in \mathcal{C}, L \in \mathcal{L} \tag{2.4.1}$$

where $\alpha(c, L)$ and $\beta(c, L, c', L')$ are input parameters.

As mentioned before, for the cargo classes $\mathcal{C}'$ for which the carrier provides the containers, the model distinguishes between freight moved in containers owned or leased by the carrier under consideration for a long term; and freight moved in containers rented by the carrier for a single shipment. Decision variable $x_o(c, l, L)$ denotes the amount of flow per week (generated by demand requests with rate $q(c, L, \pi)$) in containers owned or leased for a long term by the carrier, that move on path $l \in L$ loaded with cargo class $c \in \mathcal{C}'$. The unit cost of such a movement is denoted by $\phi_o(c, l)$ and includes feeder service cost, container storage cost, and loading and unloading cost.

Similarly, decision variable $x_r(c, l, L)$ denotes the amount of flow per week in containers rented by the carrier for the particular shipment, that move on path $l \in L$ loaded with cargo class $c \in \mathcal{C}'$. The unit cost of such a movement is denoted by $\phi_r(c, l)$ and includes feeder service cost, container rental cost, container storage cost, and loading and unloading cost. For the cargo classes $\mathcal{C}''$, decision variable $x_c(c, l, L)$ denotes the amount of flow per week in containers provided by customers, that move on path $l \in L$ loaded with cargo class $c \in \mathcal{C}''$. The unit cost of such a movement is denoted by $\phi_c(c, l)$ and includes feeder service cost, container storage cost, and loading and unloading cost. (The reason for including path sets L as arguments of the decision variables becomes clear when the demand satisfaction constraints are considered.)

The demand satisfaction constraints are formulated as follows. In steady state, the intensity of demand for transportation for each cargo class c and path set L must equal the freight flow per unit time for the cargo class and path set:

$$\sum_{l \in L} [x_o(c, l, L) + x_r(c, l, L)] = \alpha(c, L) + \sum_{c' \in \mathcal{C}} \sum_{L' \in \mathcal{L}} \beta(c, L, c', L') \pi(c', L') \quad \text{for all } c \in \mathcal{C}', L \in \mathcal{L} \tag{2.4.2}$$

$$\sum_{l \in L} x_c(c, l, L) = \alpha(c, L) + \sum_{c' \in \mathcal{C}} \sum_{L' \in \mathcal{L}} \beta(c, L, c', L') \pi(c', L') \quad \text{for all } c \in \mathcal{C}'', L \in \mathcal{L} \tag{2.4.3}$$

Containers are damaged at a rate proportional to their use, and to replace damaged containers, new containers can be purchased or leased at specified locations. Let $\mu(c, l)$ denote the fraction of containers moving on path $l \in \overline{L}$ loaded with cargo class $c \in \mathcal{C}'$ that arrive at the destination of path l too damaged to be used again. Similar damage rates can be defined for empty container flows, but are omitted here to reduce notation. Decision variables $x_p(b, o)$ and $x_l(b, o)$ denote the number of containers of class $b \in \mathcal{B}$ respectively purchased or long term leased per week with the container supplied at location $o \in \mathcal{O}$ by the container supplier. Also to reduce notation, we assume here that long term leased containers are leased for the duration of the time period covered by the model. The unit cost of a class b container purchased or leased at location o is denoted by $\phi_p(b,o)$ and $\phi_l(b,o)$ respectively.

Decision variable $x_e(b,l)$ denotes the amount of empty container flows per week of container class $b \in \mathcal{B}$ on path $l \in \overline{L}$. The unit cost of such a movement is denoted by $\phi_e(b,l)$, and includes feeder service cost, container storage cost, and loading and unloading cost.

For any path $l \in \overline{L}$, let $\mathcal{L}(l) := \{L \in \mathcal{L} \ : \ l \in L\}$ denote the collection of path sets that contain l.

The container flow balance constraints are formulated as follows. For any origin $o' \in \mathcal{O}$, let $L_o(o')$ denote the set of all paths starting at origin o'; and for any destination $d' \in \mathcal{D}$, let $L_d(d')$ denote the set of all paths ending at destination d'. At each location o, for each container class b owned or leased by the carrier, the rate of container flow into the location is equal to the rate of container flow out of the location:

$$\begin{aligned}&\sum_{c\in\mathcal{C}(b)}\sum_{l\in L_o(o)}\sum_{L\in\mathcal{L}(l)} x_o(c,l,L) + \sum_{l\in\ L_o(o)} x_e(b,l)\\ =\quad &\sum_{c\in\mathcal{C}(b)}\sum_{l\in L_d(o)}\sum_{L\in\mathcal{L}(l)} (1-\mu(c,l))x_o(c,l,L) + \sum_{l\in L_d(o)} x_e(b,l) + x_p(b,o) + x_l(b,o)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for all } b\in\mathcal{B}, o\in\mathcal{O} \quad (2.4.4)\end{aligned}$$

Note that the above container flow balance constraints do not apply to the containers rented by the carrier or the containers provided by the customers.

Decision variable $y_a(v)$ denotes the amount of additional capacity obtained on voyage $v \in \mathcal{V}$ from alliance members per week, and $\psi_a(v)$ denotes the unit cost for such additional voyage capacity.

The voyage capacity constraints are formulated as follows. For any voyage $v' \in \mathcal{V}$, let $L_v(v')$ denote the set of all paths that contain voyage v'. For each voyage $v \in \mathcal{V}$, the total rate of loaded container flows and empty container flows that use the voyage per unit time must be less than or equal to the voyage capacity:

$$\begin{aligned}\sum_{c\in\mathcal{C}'}\sum_{l\in L_v(v)}\sum_{L\in\mathcal{L}(l)}[x_o(c,l,L)+x_r(c,l,L)]+\sum_{c\in\mathcal{C}''}\sum_{l\in L_v(v)}\sum_{L\in\mathcal{L}(l)}x_c(c,l,L)+\sum_{b\in\mathcal{B}}\sum_{l\in L_v(v)}x_e(b,l)\\ \leq \quad u(v)+y_a(v) \qquad\qquad \text{for all } v\in\mathcal{V} \quad (2.4.5)\end{aligned}$$

The individual decision variables also have bounds, such as

$$\begin{aligned}
\underline{\pi}(c,L) &\leq \pi(c,L) \leq \overline{\pi}(c,L) && \text{for all } c\in\mathcal{C}, L\in\mathcal{L} && (2.4.6)\\
0 &\leq x_o(c,l,L) \leq u_o(c,l,L) && \text{for all } c\in\mathcal{C}', L\in\mathcal{L}, l\in L && (2.4.7)\\
0 &\leq x_r(c,l,L) \leq u_r(c,l,L) && \text{for all } c\in\mathcal{C}', L\in\mathcal{L}, l\in L && (2.4.8)\\
0 &\leq x_c(c,l,L) \leq u_c(c,l,L) && \text{for all } c\in\mathcal{C}'', L\in\mathcal{L}, l\in L && (2.4.9)\\
0 &\leq x_e(b,l) \leq u_e(b,l) && \text{for all } b\in\mathcal{B}, l\in \overline{L} && (2.4.10)\\
0 &\leq x_p(b,o) \leq u_p(b,o) && \text{for all } b\in\mathcal{B}, o\in\mathcal{O} && (2.4.11)\\
0 &\leq x_l(b,o) \leq u_l(b,o) && \text{for all } b\in\mathcal{B}, o\in\mathcal{O} && (2.4.12)\\
0 &\leq y_a(v) \leq u_a(v) && \text{for all } v\in\mathcal{V} && (2.4.13)
\end{aligned}$$

The total revenue per unit time (per week) is given by

$$\begin{aligned}&\sum_{c\in\mathcal{C}}\sum_{L\in\mathcal{L}}\pi(c,L)q(c,L,\pi)\\
&-\sum_{c\in\mathcal{C}'}\sum_{L\in\mathcal{L}}\sum_{l\in L}[\phi_o(c,l)x_o(c,l,L)+\phi_r(c,l)x_r(c,l,L)]-\sum_{c\in\mathcal{C}''}\sum_{L\in\mathcal{L}}\sum_{l\in L}\phi_c(c,l)x_c(c,l,L)\\
&-\sum_{b\in\mathcal{B}}\sum_{l\in\overline{L}}\phi_e(b,l)x_e(b,l)-\sum_{v\in\mathcal{V}}\psi_a(v)y_a(v)-\sum_{b\in\mathcal{B}}\sum_{o\in\mathcal{O}}[\phi_p(b,o)x_p(b,o)+\phi_l(b,o)x_l(b,o)]\end{aligned} \qquad (2.4.14)$$

In summary, this model is to optimize objective function (2.4.14) subject to constraints (2.4.1)–(2.4.13). The decision variables include prices $\pi(c,L)$, freight flow rates $x_o(c,l,L)$, $x_r(c,l,L)$, $x_c(c,l,L)$, empty container flow rates $x_e(b,l)$, numbers

$x_p(b,o)$ and $x_l(b,o)$ of containers added at each location per week, and purchased extra voyage capacities per week $y_a(v)$. The input parameters of the model include the coefficients $\alpha(c,L)$ and $\beta(c,L,c',L')$ defining the demand functions, the costs $\phi_o(c,l)$, $\phi_r(c,l)$, $\phi_c(c,l)$, $\phi_e(b,l)$ of movements on the paths, the costs $\phi_p(b,o)$ and $\phi_l(b,o)$ to obtain additional containers, the costs $\psi_a(v)$ of extra voyage capacities, container damage rates $\mu(c,l)$, voyage capacities $u(v)$, and the bounds of all the variables.

Except the bounds for the prices and the voyage capacities, all the parameters are not known at the planning stage. Thus point estimates are used in this model. As mentioned before, the planning horizon may be partitioned into multiple time intervals such that the demand rates and costs do not vary too much with respect to time within each time interval. In this case, the steady state model can be applied to each time interval. The initial conditions and the transitions from one time interval to the next are not captured in the model. This simplifies the model, but also sacrifices model accuracy. To overcome this shortcoming, we propose the time stamped model next.

2.4.3 Time stamped deterministic optimization model

In this model, for each cargo class c and path set $L \in \mathcal{L}$, the planning horizon $[0,T]$ is partitioned into several time intervals, and within each time interval, the demand rates for that cargo class c and path set L are modeled as constant over time. For each cargo class c and path set $L \in \mathcal{L}$, let $\mathcal{T}(c,L)$ denote the collection of time intervals. The contract prices are allowed to be a function of the cargo class c, the path set L, and the time interval $\tau \in \mathcal{T}(c,L)$ in which the shipment originates. This is consistent with current practice, in which contract prices are varied with time by specifying a peak period surcharge. The quantities of freight flows are allowed to vary with time on an even smaller time scale. Specifically, the flows on a path in one week can be different from the flows on the same path in another week, i.e., the flows are functions of time even within the same time interval τ. Thereby, the impact of initial conditions and the flow changes from one time interval to the next can be captured more accurately. As mentioned before, the schedules of the service rotations are model input.

In this model we have to make a distinction between a voyage, that is, a single leg of a service rotation in which a ship moves from a port (eg. port A) to the next port (eg. port B) on the service rotation, and a time stamped voyage, in which a ship moves from a port to the next port at a particular time (eg. from port A to port B starting at port A on the second day of week 3). For example, if a service rotation maintains weekly departures, then for each voyage in the service rotation, a new time stamped copy of the voyage occurs each week. Similarly, we have to make a distinction between paths and time stamped paths. For example, if all service rotations maintain weekly departures, then for each path, a new time stamped copy of the path occurs each week. Here, we ignore the fact that the movements of feeder services, such as by trucks, are often not scheduled with fixed headways, and often can be more frequent than once per week, because the frequency of a path is determined by the path leg with the lowest frequency, which is typically the long-haul ocean transportation by the carrier under consideration. Time stamped voyages will be called timed-voyages and time stamped paths will be called timed-paths for short.

The *start time* and *end time* of timed-path r are denoted by $t_o(r)$ and $t_d(r)$ respectively. The start time $t_o(r)$ is a cut-off time — it is the latest time at which a transportation request should be received to enable the freight to be moved on timed-path r, allowing sufficient time for an empty container to be moved from the appropriate container depot to the consignor's facility, time for the consignor to load the container, time for the feeder service to move the loaded container from the consignor's facility to the first port on timed-path r, and the processing time of the container at the first port, such that the container will be on time for the scheduled start time of the first timed-voyage on timed-path r. The end time $t_d(r)$ is determined similarly, i.e., by the arrival time of the last timed-voyage on timed-path r, the processing time of the container at the last port, the travel time of the feeder service from the last port to the consignee's facility, the time allowed the consignee to unload the container, and the travel time of the empty container from the consignee's facility to the appropriate container depot. Both timed-paths r with $t_o(r) \in [0, T]$ as well as timed-paths r with $t_o(r) < 0$ and $t_d(r) > 0$ are included in the model. The flows on the timed-paths r with $t_o(r) \in [0, T]$ are modelled as decision variables, but the flows on the timed-paths r with $t_o(r) < 0$ and $t_d(r) > 0$ are input as initial conditions.

A set of timed-paths r with $t_o(r) \in [0, T]$ is denoted by R, and a set of timed-paths r with $t_o(r) < 0$ and $t_d(r) > 0$ is denoted by R^0. The set of all timed-paths r with $t_o(r) \in [0, T]$ is denoted by $\overline{R}$, and the set of all timed-paths r with $t_o(r) < 0$ and $t_d(r) > 0$ is denoted by $\overline{R}^0$. For any path $l \in \overline{L}$, let $R(l)$ denote the set of all timed-paths r along path l with $t_o(r) \in [0, T]$; and for any path set $L \in \mathcal{L}$, let $R(L)$ denote the set of all timed-paths r along a path $l \in L$ with $t_o(r) \in [0, T]$. The path associated with the timed-path r is denoted by $l(r)$.

Decision variable $\pi(c, L, \tau)$ denotes the contract price for cargo class $c \in \mathcal{C}$ to be transported on a timed-path r chosen by the carrier such that $l(r) \in L \in \mathcal{L}$ and $t_o(r) \in \tau \in \mathcal{T}(c, L)$, and π denotes the vector of contract prices. Let $q(c, L, \tau, \pi)$ denote the demand rate of transportation requests for cargo class $c \in \mathcal{C}$ and path set L per unit time during time interval $\tau \in \mathcal{T}(c, L)$ if the vector of contract prices is π, measured in the same units as the transportation capacities. As before, we assume that the demand intensity $q(c, L, \tau, \pi)$ is a linear function of the price vector π, as follows:

$$q(c, L, \tau, \pi) = \alpha(c, L, \tau) - \sum_{c' \in \mathcal{C}} \sum_{L' \in \mathcal{L}} \sum_{\tau' \in \mathcal{T}(c', L')} \beta(c, L, \tau, c', L', \tau') \pi(c', L', \tau') \quad \text{for all } c \in \mathcal{C}, L \in \mathcal{L}, \tau \in \mathcal{T} \qquad (2.4.15)$$

where $\alpha(c, L, \tau)$ and $\beta(c, L, \tau, c', L', \tau')$ are input parameters.

The flow variables are defined as follows. Decision variable $x_o(c, r, L)$ denotes the amount of flow (generated by demand requests with rate $q(c, L, \tau, \pi)$, with $t_o(r) \in \tau$) in containers owned or leased for a long term by the carrier, that move on timed-path $r \in R(L)$ loaded with cargo class $c \in \mathcal{C}'$. The unit cost of such a movement is denoted by $\phi_o(c, r)$ and includes feeder service cost, container storage cost, and loading and unloading cost. Similarly, decision variable $x_r(c, r, L)$ denotes the amount of flow in containers rented by the carrier for the particular shipment, that move on timed-path $r \in R(L)$ loaded with cargo class $c \in \mathcal{C}'$. The unit cost of such a movement is denoted by $\phi_r(c, r)$ and includes feeder service cost, container rental cost, container storage cost, and loading and unloading cost. Decision variable $x_c(c, r, L)$ denotes the amount of flow in containers provided by customers, that move on timed-path $r \in R(L)$ loaded with cargo class $c \in \mathcal{C}''$. The unit cost of such a movement is denoted by $\phi_c(c, r)$ and includes feeder service cost, container storage cost, and loading and unloading cost.

Constraints have to be formulated that relate demand intensities to flow quantities, to ensure that the freight flow demands are satisfied along acceptable paths and within acceptable times. Clearly, no transportation requests received after time $t_o(r)$ can be satisfied with freight flows on timed-path r. However, transportation requests for path set L received before time $t_o(r)$ may be satisfied with freight flows on timed-path $r \in R(L)$ or on a later timed-path $r' \in R(L)$ with $t_o(r) < t_o(r')$, as long as the delay between the transportation request and time $t_o(r')$ is acceptable. (In this model, the start times of the timed-paths are used to determine whether a timed-path can satisfy demand that appears at a particular time within acceptable time — recall that all the paths in the set L are regarded as acceptable by the customer, and thus have durations that are acceptable to the customer, and hence an acceptable start time of a timed-path $r \in R(L)$ should imply an acceptable delivery time for timed-path r.

One may also use the end times of the timed-paths in a similar way to determine the set of timed-paths with acceptable delivery times.) In this model, acceptable timed-paths for satisfying demand are determined as follows. Consider any path set $L \in \mathcal{L}$ and the associated set $R(L)$ of timed-paths, and denote the number of associated timed-paths by $|R(L)|$. Index the timed-paths $r_1, \ldots, r_{|R(L)|} \in R(L)$ in increasing order of their start times, i.e., $t_o(r_1) \leq t_o(r_2) \leq \cdots \leq t_o(r_{|R(L)|})$. Let input parameter $w(c, L)$ denote the maximum time that transportation demand for cargo class c and path set L can be delayed from the time the transportation request is made to a later timed-path if it can also be moved on an earlier timed-path (if demand for cargo class c and path set L originates at time t and there are no timed paths $r \in R(L)$ with $t_o(r) \in [t, t+w(c, L)]$, then the model allows the freight to move on the first timed-path $r \in R(L)$ with $t_o(r) \geq t$). That is, all the transportation demand for cargo class c and path set L that originates in time interval $[0,\ \min\{t_o(r_1),\ \max\{t_o(r_2) - w(c, L),\ 0\}\}]$ has to be moved on timed-path r_1, all the transportation demand for cargo class c and path set L that originates in time interval $[0,\ \min\{t_o(r_2),\ \max\{t_o(r_3) - w(c, L),\ 0\}\}]$ has to be moved on timed-path r_1 or timed-path r_2, and so on. At the same time, the maximum amount of freight of cargo class c and path set L that can be moved on timed-path r_1 is the transportation demand originating in time interval $[0, t_o(r_1)]$, the maximum amount of freight of cargo class c and path set L that can be moved on timed-paths r_1 and r_2 is the transportation demand originating in time interval $[0, t_o(r_2)]$, and so on. The remaining task is to calculate the transportation demand originating in each time interval.

The transportation demand for cargo class c and path set L that originates in any time interval $[t_1, t_2]$ depends on the chosen prices. Thus, it depends on the intersection of $[t_1, t_2]$ with the pricing time intervals in $\mathcal{T}(c, L)$. For any time interval $\tau \in \mathcal{T}(c, L)$, the demand for cargo class c and path set L that occurs in $\tau \cap [t_1, t_2]$ is given by $|\tau \cap [t_1, t_2]|\, q(c, L, \tau, \pi)$, where $|\tau \cap [t_1, t_2]|$ denotes the length of interval $\tau \cap [t_1, t_2]$.

(Usually, $[t_1, t_2]$ is small compared with the intervals in $\mathcal{T}(c, L)$, and thus $\tau \cap [t_1, t_2]$ is typically nonempty for at most two intervals $\tau \in \mathcal{T}(c, L)$.) Thus, the transportation demand for cargo class c and path set L that originates in time interval $[t_1, t_2]$ is given by $\sum_{\tau \in \mathcal{T}(c,L)} |\tau \cap [t_1, t_2]|\, q(c, L, \tau, \pi)$.

The two types of demand satisfaction constraints for each cargo class c and path set L can now be written as follows. Recall that the timed-paths in $R(L)$ are indexed in increasing order of their start times. To satisfy demand within acceptable times, the cumulative flow on all timed-paths in $R(L)$ up to $r_k \in R(L)$ must be greater than or equal to the demand that has arrived until that time point and which cannot be delayed any more:

$$\begin{aligned}
&\sum_{\{r \in R(L)\,:\, t_o(r) \leq t_o(r_k)\}} [x_o(c, r, L) + x_r(c, r, L)] \\
\geq\ & \sum_{\tau \in \mathcal{T}(c,L)} \left| \tau \cap \left[0,\ \min\left\{ t_o(r_k),\ \max\left\{ t_o(r_{k+1}) - w(c, L),\ 0 \right\} \right\} \right] \right| \\
&\qquad \times \left[\alpha(c, L, \tau) - \sum_{c' \in \mathcal{C}} \sum_{L' \in \mathcal{L}} \sum_{\tau' \in \mathcal{T}(c',L')} \beta(c, L, \tau, c', L', \tau') \pi(c', L', \tau') \right] \\
&\qquad\qquad \text{for all } c \in \mathcal{C}', L \in \mathcal{L}, r_k \in R(L) \qquad (2.4.16)
\end{aligned}$$

$$\begin{aligned}
&\sum_{\{r \in R(L)\,:\, t_o(r) \leq t_o(r_k)\}} x_c(c, r, L) \\
\geq\ & \sum_{\tau \in \mathcal{T}(c,L)} \left| \tau \cap \left[0,\ \min\left\{ t_o(r_k),\ \max\left\{ t_o(r_{k+1}) - w(c, L),\ 0 \right\} \right\} \right] \right| \\
&\qquad \times \left[\alpha(c, L, \tau) - \sum_{c' \in \mathcal{C}} \sum_{L' \in \mathcal{L}} \sum_{\tau' \in \mathcal{T}(c',L')} \beta(c, L, \tau, c', L', \tau') \pi(c', L', \tau') \right] \\
&\qquad\qquad \text{for all } c \in \mathcal{C}'', L \in \mathcal{L}, r_k \in R(L) \qquad (2.4.17)
\end{aligned}$$

Also, the cumulative flow on all timed-paths in $R(L)$ up to $r_k \in R(L)$ must be less than or equal to the demand that has arrived until that time point:

$$\begin{aligned}
&\sum_{\{r \in R(L)\,:\, t_o(r) \leq t_o(r_k)\}} [x_o(c, r, L) + x_r(c, r, L)] \\
\leq\ & \sum_{\tau \in \mathcal{T}(c,L)} \left| \tau \cap \left[0,\ t_o(r_k)\right] \right|
\end{aligned}$$

$$\times \left[\alpha(c, L, \tau) - \sum_{c' \in \mathcal{C}} \sum_{L' \in \mathcal{L}} \sum_{\tau' \in \mathcal{T}(c', L')} \beta(c, L, \tau, c', L', \tau') \pi(c', L', \tau') \right]$$

$$\text{for all } c \in \mathcal{C}', L \in \mathcal{L}, r_k \in R(L) \qquad (2.4.18)$$

$$\sum_{\{r \in R(L) \,:\, t_o(r) \leq t_o(r_k)\}} x_c(c, r, L)$$

$$\leq \sum_{\tau \in \mathcal{T}(c, L)} \left| \tau \cap \left[0,\ t_o(r_k)\right] \right|$$

$$\times \left[\alpha(c, L, \tau) - \sum_{c' \in \mathcal{C}} \sum_{L' \in \mathcal{L}} \sum_{\tau' \in \mathcal{T}(c', L')} \beta(c, L, \tau, c', L', \tau') \pi(c', L', \tau') \right]$$

$$\text{for all } c \in \mathcal{C}'', L \in \mathcal{L}, r_k \in R(L) \qquad (2.4.19)$$

Initial conditions, including initial container inventories and initial container flows, are included in the model. As mentioned before, the flows on the timed-paths that start before time 0 but end after time 0 are input as part of the initial conditions. The initial flows of cargo class c on timed-path r with $t_o(r) < 0$ and $t_d(r) > 0$ moved in the carrier's own or long term leased containers, in containers rented by the carrier for the particular shipment, and in containers provided by the customer, are denoted by $x_o^0(c, r)$, $x_r^0(c, r)$, and $x_c^0(c, r)$, respectively. Let $x_e^0(b, r)$ denote the initial flows of empty class b containers on timed-path r. The initial number of class b containers at location o is denoted by $s(b, o)$.

For any origin $o' \in \mathcal{O}$, let $R_o(o')$ denote the set of all timed-paths r with origin o' and $t_o(r) \in [0, T]$; for any destination $d' \in \mathcal{D}$, let $R_d(d')$ denote the set of all timed-paths r with destination d' and with $t_o(r) \in [0, T]$; and for any destination $d' \in \mathcal{D}$, let $R_d^0(d')$ denote the set of all timed-paths r with destination d' and with $t_o(r) < 0$ and $t_d(r) > 0$.

Let $\mu(c, r)$ denote the fraction of containers moving on timed-path $r \in \overline{R} \cup \overline{R}^0$ loaded with cargo class $c \in \mathcal{C}'$ that arrive at the destination of timed-path r too damaged to be used again. Damage rates can also be defined for empty container flows, but are again omitted to reduce notation. Decision variables $x_p(b, o, r)$ and

$x_l(b,o,r)$ denote the cumulative number of containers of type $b \in \mathcal{B}$ respectively purchased or leased for a long term until time $t_o(r)$ with the container supplied at location $o \in \mathcal{O}$ by the container supplier. In this model we assume that long term leased containers are leased for the remainder of the time period covered by the model. The unit cost of a class b container purchased or leased at location o is denoted by $\phi_p(b,o)$ and $\phi_l(b,o)$ respectively.

Decision variable $x_e(b,r)$ denotes the number of class $b \in \mathcal{B}$ empty containers moved on timed-path $r \in \overline{R}$. The unit cost of such a movement is denoted by $\phi_e(b,r)$, and includes feeder service cost, container storage cost, and loading and unloading cost.

For any timed-path $r \in \overline{R}$, let $\mathcal{L}(r) := \{L \in \mathcal{L} \ : \ l(r) \in L\}$ denote the collection of path sets that contain $l(r)$.

The container flow balance constraints are formulated as follows. For each container class b, each location o, and each time t, the cumulative outflow from o of class b containers up to time t is less than or equal to the cumulative inflow into o of class b containers up to time t. It is sufficient to consider only times $t = t_o(r)$ for timed-paths $r \in R_o(o)$:

$$
\begin{aligned}
& \sum_{\{r' \in R_o(o) \,:\, t_o(r') \le t_o(r)\}} \left[\sum_{c \in \mathcal{C}(b)} \sum_{L \in \mathcal{L}(r')} x_o(c,r',L) + x_e(b,r') \right] \\
\le \; & \sum_{c \in \mathcal{C}(b)} \left[\sum_{\{r' \in R_d(o) \,:\, t_d(r') \le t_o(r)\}} \sum_{L \in \mathcal{L}(r')} (1-\mu(c,r')) x_o(c,r',L) \right. \\
& \qquad \left. + \sum_{\{r' \in R_d^0(o) \,:\, t_d(r') \le t_o(r)\}} (1-\mu(c,r')) x_o^0(c,r') \right] \\
& + \sum_{\{r' \in R_d(o) \,:\, t_d(r') \le t_o(r)\}} x_e(b,r') \\
& + \sum_{\{r' \in R_d^0(o) \,:\, t_d(r') \le t_o(r)\}} x_e^0(b,r') + s(b,o) + x_p(b,o,r) + x_l(b,o,r) \\
& \qquad\qquad\qquad \text{for all } b \in \mathcal{B}, o \in \mathcal{O}, r \in R_o(o) \qquad (2.4.20)
\end{aligned}
$$

In this model, $\mathcal{V}$ denotes the set of timed-voyages. Decision variable $y_a(v)$ denotes the amount of additional capacity obtained on timed-voyage $v \in \mathcal{V}$ from alliance members, and $\psi_a(v)$ denotes the unit cost for such additional voyage capacity.

For any timed-voyage $v' \in \mathcal{V}$, let $R_v(v')$ denote the set of all timed-paths r with $t_o(r) \in [0, T]$ and that contain timed-voyage v', and let $R_v^0(v')$ denote the set of all timed-paths r with $t_o(r) < 0$ and $t_d(r) > 0$, and that contain timed-voyage v'.

The capacity constraints on the timed-voyages are formulated as follows:

$$\begin{aligned}
&\sum_{c \in \mathcal{C}'} \left[\sum_{r \in R_v(v)} \sum_{L \in \mathcal{L}(r)} [x_o(c,r,L) + x_r(c,r,L)] + \sum_{r \in R_v^0(v)} \left[x_o^0(c,r) + x_r^0(c,r)\right] \right] \\
&+ \sum_{c \in \mathcal{C}''} \left[\sum_{r \in R_v(v)} \sum_{L \in \mathcal{L}(r)} x_c(c,r,L) + \sum_{r \in R_v^0(v)} x_c^0(c,r) \right] \\
&+ \sum_{b \in \mathcal{B}} \left[\sum_{r \in R_v(v)} x_e(b,r) + \sum_{r \in R_v^0} x_e^0(b,r) \right] \quad \leq \quad u(v) + y_a(v) \qquad \text{for all } v \in \mathcal{V}
\end{aligned} \tag{2.4.21}$$

Recall that decision variables $x_p(b,o,r)$ and $x_l(b,o,r)$ denote the cumulative number of containers of type $b \in \mathcal{B}$ respectively purchased or leased for a long term until time $t_o(r)$ with the container supplied at location $o \in \mathcal{O}$ by the container supplier. For each location $o \in \mathcal{O}$, index the timed-paths $r_1, \ldots, r_{|R_o(o)|} \in R_o(o)$ in increasing order of their start times, i.e., $t_o(r_1) \leq t_o(r_2) \leq \cdots \leq t_o(r_{|R_o(o)|})$. Thus, the following constraints must hold:

$$x_p(b,o,r_1) \quad \leq \quad x_p(b,o,r_2) \quad \leq \quad \cdots \quad \leq \quad x_p(b,o,r_{|R_o(o)|}) \qquad \text{for all } b \in \mathcal{B}, o \in \mathcal{O} \tag{2.4.22}$$

$$x_l(b,o,r_1) \quad \leq \quad x_l(b,o,r_2) \quad \leq \quad \cdots \quad \leq \quad x_l(b,o,r_{|R_o(o)|}) \qquad \text{for all } b \in \mathcal{B}, o \in \mathcal{O} \tag{2.4.23}$$

The individual decision variables also have bounds as follows.

$$\underline{\pi}(c,L,\tau) \quad \leq \quad \pi(c,L,\tau) \quad \leq \quad \overline{\pi}(c,L,\tau)$$

$$\text{for all } c \in \mathcal{C}, L \in \mathcal{L}, \tau \in \mathcal{T}(c, L) \quad (2.4.24)$$

$$0 \leq x_o(c, r, L) \leq u_o(c, r, L)$$

$$\text{for all } c \in \mathcal{C}', r \in \overline{R}, L \in \mathcal{L}(r) \quad (2.4.25)$$

$$0 \leq x_r(c, r, L) \leq u_r(c, r, L)$$

$$\text{for all } c \in \mathcal{C}', r \in \overline{R}, L \in \mathcal{L}(r) \quad (2.4.26)$$

$$0 \leq x_c(c, r, L) \leq u_c(c, r, L)$$

$$\text{for all } c \in \mathcal{C}'', r \in \overline{R}, L \in \mathcal{L}(r) \quad (2.4.27)$$

$$0 \leq x_e(b, r) \leq u_e(b, r)$$

$$\text{for all } b \in \mathcal{B}, r \in \overline{R} \quad (2.4.28)$$

$$0 \leq x_p(b, o, r) \leq u_p(b, o, r)$$

$$\text{for all } b \in \mathcal{B}, o \in \mathcal{O}, r \in R_o(o) \quad (2.4.29)$$

$$0 \leq x_l(b, o, r) \leq u_l(b, o, r)$$

$$\text{for all } b \in \mathcal{B}, o \in \mathcal{O}, r \in R_o(o) \quad (2.4.30)$$

$$0 \leq y_a(v) \leq u_a(v)$$

$$\text{for all } v \in \mathcal{V} \quad (2.4.31)$$

The total revenue over the time horizon $[0, T]$ is given by

$$\begin{aligned}
&\sum_{c\in\mathcal{C}} \sum_{L\in\mathcal{L}} \sum_{\tau\in\mathcal{T}(c,L)} |\tau| \; \pi(c, L, \tau) q(c, L, \tau, \pi) \\
&- \sum_{c\in\mathcal{C}'} \sum_{r\in\overline{R}} \sum_{L\in\mathcal{L}(r)} [\phi_o(c, r) x_o(c, r, L) + \phi_r(c, r) x_r(c, r, L)] - \sum_{c\in\mathcal{C}''} \sum_{r\in\overline{R}} \sum_{L\in\mathcal{L}(r)} \phi_c(c, r) x_c(c, r, L) \\
&- \sum_{b\in\mathcal{B}} \sum_{r\in\overline{R}} \phi_e(b, r) x_e(b, r) - \sum_{v\in\mathcal{V}} \psi_a(v) y_a(v) \\
&- \sum_{b\in\mathcal{B}} \sum_{o\in\mathcal{O}} \left[\phi_p(b, o) x_p(b, o, r_{|R_o(o)|}) + \phi_l(b, o) x_l(b, o, r_{|R_o(o)|})\right] \qquad (2.4.32)
\end{aligned}$$

where $r_{|R_o(o)|}$ is defined by indexing, for each location $o \in \mathcal{O}$, the timed-paths $r_1, \ldots, r_{|R_o(o)|} \in R_o(o)$ in increasing order of their start times.

In summary, this model is to optimize objective function (2.4.32) subject to constraints (2.4.15)–(2.4.31). The decision variables in this model include prices $\pi(c, L, \tau)$, freight flows $x_o(c, r, L)$, $x_r(c, r, L)$, $x_c(c, r, L)$, empty container flows $x_e(b, r)$, numbers $x_p(b, o, r)$ and $x_l(b, o, r)$ of containers added at each location, and purchased extra voyage capacities $y_a(v)$. The input parameters of the model include the coefficients $\alpha(c, L, \tau)$ and $\beta(c, L, \tau, c', L', \tau')$ defining the demand functions, the costs $\phi_o(c, r)$, $\phi_r(c, r)$, $\phi_c(c, r)$, $\phi_e(b, r)$ of movements on the timed-paths, the costs $\phi_p(b, o)$ and $\phi_l(b, o)$ to obtain additional containers, the costs $\psi_a(v)$ of extra voyage capacities, initial conditions $x_o^0(c, r)$, $x_r^0(c, r)$, $x_c^0(c, r)$, $x_e^0(b, r)$, $s(b, o)$, container damage rates $\mu(c, r)$, voyage capacities $u(v)$, and the bounds of all the variables.

Similar to the steady state model, except for the bounds for the prices and the voyage capacities, all the parameters are not known at the planning stage, and point estimates are used in this model. One way to pursue solutions that are robust against variations in the unknown parameters values, is to formulate and solve a stochastic optimization model. Such a model is formulated in the next section.

2.4.4 Stochastic optimization model

The following model parameters are usually uncertain in applications:

1. The demand rate $q(c, L, \pi)$ as a function of cargo class c, path set L, and price π.

2. The booking cancellations.

3. The travel times on the domestic arcs.

4. The maximum additional amount of capacity that the carrier can obtain on each timed-voyage $u_a(v)$.

5. The maximum quantity of containers that can be rented $u_r(c, r, L)$.

6. The costs of freight flows moved in the carrier's own containers $\phi_o(c, r, L)$, the costs of freight flows moved in the containers rented by the carrier $\phi_r(c, r, L)$, the

costs of freight flows moved in the containers provided by customers $\phi_c(c, r, L)$, the costs of empty container flows $\phi_e(b, r)$, the cost of purchasing a type b box at each origin $\phi_p(b, o)$, the cost of leasing a type b box at each origin $\phi_l(b, o)$, and the cost of obtaining a unit of extra voyage capacity $\psi_a(v)$.

7. The container damage ratios $\mu(c, r)$.

8. The initial conditions $x_o^0(c, r)$, $x_r^0(c, r)$, $x_c^0(c, r)$, $x_e^0(b, r)$ and $s(b, o)$.

The uncertain parameters are modeled as random variables. It is assumed that the decision maker has a joint probability distribution for these random variables at the planning stage.

One can formulate a multistage stochastic optimization model (Markov decision process) that models how decisions are to be made over time using the information that is available when each decision is to be made. In such a model, the contract pricing decisions would be made initially, and thereafter operational decisions such as the routing of freight, the repositioning of empty containers, the rental of containers, the procurement of containers, and the acquisition of additional capacity on voyages would be made over time as the values of the random variables become known. Here we formulate a three-stage stochastic optimization model, in which the pricing decisions are made in the first stage when the values of the random variables are still unknown. At the second stage, given the demand rates, the freight flows and empty container flows are booked. At the third stage, the actual flows take place. A multistage stochastic optimization model with more than three stages may be more realistic, but a three-stage model is chosen for the following reasons. First, such a multistage stochastic optimization problem is extremely hard to solve, whereas the three-stage problem presented in this section is reasonably tractable. Second, the purpose of the optimization model is to provide decision support to the carrier for negotiating contracts with potential customers, and not to control operations, and thus it seems that a very complex multistage problem that models operational decisions in great detail is unnecessary.

In this model, we assume that the schedules of the service rotations are fixed and ignore the uncertainties in the travel time on voyages. Though the real travel times on domestic arcs are random, the start time and end time of routes $t_o(r)$ and $t_d(r)$ are assumed to be deterministic according to the service schedule. The expected values are used. Let ω_1 denote a realization of the second stage random input parameters and let ω_2 denote a realization of the third stage random input parameters conditional on ω_1.

The first stage decision variables are the prices $\pi(c, L, \tau)$ for each $c \in \mathcal{C}$, $L \in \mathcal{L}$, and $\tau \in \mathcal{T}(c, L)$. At the end of the first stage, the demand is observed. Denote mathematical expectation with respect to ω_1 as E_{ω_1}.

The first stage problem is:

$$\text{maximize} E_{\omega_1}\left[Q_1(\pi, \omega_1)\right] \tag{2.4.33}$$

Subject to the following constraints:

$$\underline{\pi}(c, L, \tau) \quad \leq \quad \pi(c, L, \tau) \quad \leq \quad \overline{\pi}(c, L, \tau)$$
$$\text{for all } c \in \mathcal{C}, L \in \mathcal{L}, \tau \in \mathcal{T}(c, L) \tag{2.4.34}$$

where $\underline{\pi}$ ($\overline{\pi}$) denote the lower (upper) bounds of the prices and $Q_1(\pi, \omega_1)$ denotes the optimal value of the second stage problem for each ω_1 given π.

At the second stage, the freight flows and empty container flows are booked along each route. For each second stage realization ω_1, we again assume that each demand rate is a linear function of the price. The coefficients $\alpha(c, L, \tau, \omega_1)$ and $\beta(c, L, \tau, c', L', \tau', \omega_1)$ in the linear function are assumed as second stage parameters, which become known at the beginning of the stage.

$$q(c, L, \tau, \pi, \omega_1) \quad = \quad \alpha(c, L, \tau, \omega_1) - \sum_{c' \in \mathcal{C}} \sum_{L' \in \mathcal{L}} \sum_{\tau' \in \mathcal{T}(c', L')} \beta(c, L, \tau, c', L', \tau', \omega_1) \pi(c', L', \tau')$$

$$\text{for all } c \in \mathcal{C}, L \in \mathcal{L}, \tau \in \mathcal{T} \quad (2.4.35)$$

Let second stage decision variable $x_o(c, r, L, \omega_1)$ denote the booked flow (generated by demand request with rate $q(c, L, \tau, \pi, \omega_1)$ with $t_0(r) \in \tau$)in the carrier's own or long term leased containers that move on timed-path r loaded with cargo class $c \in \mathcal{C}'$. Similarly, decision variable $x_r(c, r, L, \omega_1)$ denotes the booked flow in containers rented by the carrier for the particular shipment. Decision variable $x_c(c, r, L, \omega_1)$ denotes the booked flow in containers provided by the customer that move on timed-path r loaded with cargo class $c \in \mathcal{C}''$. Unlike the previous deterministic models, not all demand requests must be accepted. In some cases the demand is very high, the carrier may not take all of it. Therefore, we use second stage decision variable $x_{rj}(c, r, L, \omega_1)$ to indicate the cumulative amount of demand (with rate $q(c, L, \tau, \omega_1)$, $t_o(r) \in \tau$) rejected until time $t_o(r)$. Let $\phi_{rj}(c, r, \omega_1)$ be the associated unit cost. For each path set L, index the timed-paths $r_1, \ldots, r_{|R(L)|} \in R(L)$ in increasing order of their start times, i.e., $t_o(r_1) \leq t_o(r_2) \leq \cdots \leq t_o(r_{|R(L)|})$. Thus, the cumulative flow on all timed-paths in $R(L)$ up to $r_k \in R(L)$ must be greater than or equal to the demand that has been accepted until that time point and which cannot be delayed any more:

$$\begin{aligned}
&\sum_{\{r \in\ R(L)\,:\, t_o(r) \leq t_o(r_k)\}} [x_o(c, r, L, \omega_1) + x_r(c, r, L, \omega_1)] + x_{rj}(c, r_k, L, \omega_1) \\
\geq\ & \sum_{\tau \in \mathcal{T}(c,L)} \left| \tau \cap \left[0,\ \min\left\{t_o(r_k),\ \max\left\{t_o(r_{k+1}) - w(c, L),\ 0\right\}\right\}\right] \right| \\
& \qquad \times \left[\alpha(c, L, \tau, \omega_1) - \sum_{c' \in \mathcal{C}} \sum_{L' \in \mathcal{L}} \sum_{\tau' \in \mathcal{T}(c', L')} \beta(c, L, \tau, c', L', \tau', \omega_1) \pi(c', L', \tau')\right] \\
& \qquad\qquad \text{for all } c \in \mathcal{C}', L \in \mathcal{L}, \text{ and } r_k \in R(L) \quad (2.4.36)
\end{aligned}$$

$$\begin{aligned}
&\sum_{\{r \in\ R(L)\,:\, t_o(r) \leq t_o(r_k)\}} x_c(c, r, L, \omega_1) + x_{rj}(c, r_k, L, \omega_1) \\
\geq\ & \sum_{\tau \in \mathcal{T}(c,L)} \left| \tau \cap \left[0,\ \min\left\{t_o(r_k),\ \max\left\{t_o(r_{k+1}) - w(c, L),\ 0\right\}\right\}\right] \right|
\end{aligned}$$

$$\times \left[\alpha(c,L,\tau,\omega_1) - \sum_{c'\in\mathcal{C}}\sum_{L'\in\mathcal{L}}\sum_{\tau'\in\mathcal{T}(c',L')} \beta(c,L,\tau,c',L',\tau',\omega_1)\pi(c',L',\tau')\right]$$

$$\text{for all } c \in \mathcal{C}'', L \in \mathcal{L}, \text{ and } r_k \in R(L) \quad (2.4.37)$$

Also, the cumulative flows on all timed-paths in $R(L)$ up to $r_k \in R(L)$ is less or equal to the demand accepted until the time point $t_o(r_k)$:

$$\begin{aligned}
& \sum_{\{r\in\ R(L)\,:\, t_o(r)\le t_o(r_k)\}} [x_o(c,r,L,\omega_1) + x_r(c,r,L,\omega_1)] + x_{rj}(c,r_k,L,\omega_1) \\
\le \quad & \sum_{\tau\in\mathcal{T}(c,L)} |\tau \cap [0,\ t_o(r_k)]| \\
& \times \left[\alpha(c,L,\tau,\omega_1) - \sum_{c'\in\mathcal{C}}\sum_{L'\in\mathcal{L}}\sum_{\tau'\in\mathcal{T}(c',L')} \beta(c,L,\tau,c',L',\tau',\omega_1)\pi(c',L',\tau')\right] \\
& \text{for all } c \in \mathcal{C}', L \in \mathcal{L}, \text{ and } r_k \in R(L) \quad (2.4.38)
\end{aligned}$$

$$\begin{aligned}
& \sum_{\{r\in\ R(L)\,:\, t_o(r)\le t_o(r_k)\}} x_c(c,r,L,\omega_1) + x_{rj}(c,r_k,L,\omega_1) \\
\le \quad & \sum_{\tau\in\mathcal{T}(c,L)} |\tau \cap [0,\ t_o(r_k)]| \\
& \times \left[\alpha(c,L,\tau,\omega_1) - \sum_{c'\in\mathcal{C}}\sum_{L'\in\mathcal{L}}\sum_{\tau'\in\mathcal{T}(c',L')} \beta(c,L,\tau,c',L',\tau',\omega_1)\pi(c',L',\tau')\right] \\
& \text{for all } c \in \mathcal{C}'', L \in \mathcal{L}, \text{ and } r_k \in R(L) \quad (2.4.39)
\end{aligned}$$

It is assumed that the rejected demand cannot be recovered. We then have the following constrains:

$$x_{rj}(c,r_1,L,\omega_1) \;\le\; x_{rj}(c,r_2,L,\omega_1) \;\le\; \cdots \;\le\; x_{rj}(c,r_{|R(L)|},L,\omega_1)$$

$$\text{for all } c \in \mathcal{C}, L \in \mathcal{L} \quad (2.4.40)$$

Second stage decision variable $x_e(b,r,\omega_1)$ denotes the flows of class b empty containers on timed-path r. Let second stage decision variables $x_p(b,o,r,\omega_1)$ and $x_l(b,o,r,\omega_1)$ be the cumulative number of containers of type b respectively purchased or leased for a long term until time $t_o(r)$ at location $o \in \mathcal{O}$. Denote the associated unit costs as $\phi_p(b, o, \omega_1)$ and $\phi_l(b, o, \omega_1)$ respectively.

The initial conditions and container damage rate are modelled as second stage random variables that are observed at the beginning of the second stage. Denote the initial flows of cargo class c on timed-path r with $t_o(r) < 0$ and $t_d(r) > 0$ moved in the carrier's own or long term leased containers, in containers rented by the carrier, and in containers provided by the customer as $x_o^0(c, r, \omega_1)$, $x_r^0(c, r, \omega_1)$ and $x_c^0(c, r, \omega_1)$ respectively. Similarly, the initial flows of empty class b containers on timed-path r is denoted as $x_e^0(b, r, \omega_1)$. Let the initial number of class b empty containers at location o be $s(b, o, \omega_1)$. Random variable $\mu(c, r, \omega_1)$ denotes the fraction of containers that move on timed-path $r \in \overline{R} \bigcup \overline{R}^0$ loaded with cargo class c too damaged to be used again after arriving the destination of r.

For each $o \in \mathcal{O}$, $b \in \mathcal{B}$ and each ω_1, the cumulative outflow must be less or equal to the cumulative inflow.

$$\sum_{\{r' \in R_o(o)\,:\, t_o(r') \leq t_o(r)\}} \left[\sum_{c \in \mathcal{C}(b)} \sum_{L \in \mathcal{L}(r')} x_o(c, r', L, \omega_1) + x_e(b, r', \omega_1) \right] \leq$$
$$\sum_{c \in \mathcal{C}(b)} \left[\sum_{\{r' \in\ R_d(o)\,:\, t_d(r') < t_o(r)\}} \sum_{L \in \mathcal{L}(r')} (1 - \mu(c, r', \omega_1)) x_o(c, r', L, \omega_1) \right.$$
$$\left. + \sum_{\{r' \in R_d^0(o)\,:\, t_d(r') < t_o(r)\}} (1 - \mu(c, r', \omega_1)) x_o^0(c, r', \omega_1) \right]$$
$$+ \sum_{\{r' \in R_d(o)\,:\, t_d(r') < t_o(r)\}} x_e(b, r', \omega_1) + \sum_{\{r' \in R_d^0(o)\,:\, t_d(r') < t_o(r)\}} x_e^0(b, r', \omega_1)$$
$$+ s(b, o, \omega_1) + x_p(b, o, r, \omega_1) + x_l(b, o, r, \omega_1)$$
$$\text{for all } b \in \mathcal{B}, o \in \mathcal{O}, \text{ and } r \in\ R_o(o) \quad (2.4.41)$$

Similar to the time stamped model, we assume that the added containers at the origins can not be returned. Index the routes $r \in R_o(o)$ such that $t_o(r_1) \leq t_o(r_2) \leq$

$\cdots \leq t_o(r_{|R_o(o)|})$. Thus, we have the following constrains:

$$\begin{aligned} x_p(b,o,r_1,\omega_1) \;\leq\; x_p(b,o,r_2,\omega_1) \;\leq\; \cdots \;\leq\; x_p(b,o,r_{|R_o(o)|},\omega_1) & \\ \text{for all } b \in \mathcal{B}, o \in \mathcal{O} & \quad (2.4.42) \\ x_l(b,o,r_1,\omega_1) \;\leq\; x_l(b,o,r_2,\omega_1) \;\leq\; \cdots \;\leq\; x_l(b,o,r_{|R_o(o)|},\omega_1) & \\ \text{for all } b \in \mathcal{B}, o \in \mathcal{O} & \quad (2.4.43) \end{aligned}$$

The second stage variables have bounds, which are assumed as second stage random variables.

$$\begin{aligned} 0 \;\leq\; x_o(c,r,L,\omega_1) \;\leq\; u_o(c,r,L,\omega_1) & \\ \text{for all } c \in \mathcal{C}', r \in \overline{R}, L \in \mathcal{L}(r) & \quad (2.4.44) \\ 0 \;\leq\; x_r(c,r,L,\omega_1) \;\leq\; u_r(c,r,L,\omega_1) & \\ \text{for all } c \in \mathcal{C}', r \in \overline{R}, L \in \mathcal{L}(r) & \quad (2.4.45) \\ 0 \;\leq\; x_c(c,r,L,\omega_1) \;\leq\; u_c(c,r,L,\omega_1) & \\ \text{for all } c \in \mathcal{C}'', r \in \overline{R}, L \in \mathcal{L}(r) & \quad (2.4.46) \\ 0 \;\leq\; x_{rj}(c,r,L,\omega_1) \;\leq\; u_{rj}(c,r,L,\omega_1) & \\ \text{for all } c \in \mathcal{C}, r \in \overline{R}, L \in \mathcal{L}(r) & \quad (2.4.47) \\ 0 \;\leq\; x_e(b,r,\omega_1) \;\leq\; u_e(b,r,\omega_1) & \\ \text{for all } b \in \mathcal{B}, r \in \overline{R} & \quad (2.4.48) \\ 0 \;\leq\; x_p(b,o,r,\omega_1) \;\leq\; u_p(b,o,r,\omega_1) & \\ \text{for all } b \in \mathcal{B}, o \in \mathcal{O}, r \in R_o(o) & \quad (2.4.49) \\ 0 \;\leq\; x_l(b,o,r,\omega_1) \;\leq\; u_l(b,o,r,\omega_1) & \\ \text{for all } b \in \mathcal{B}, o \in \mathcal{O}, r \in R_o(o) & \quad (2.4.50) \\ 0 \;\leq\; y_a(v,\omega_1) \;\leq\; u_v(v,\omega_1) & \\ \text{for all } v \in \mathcal{V} & \quad (2.4.51) \end{aligned}$$

Part of the booked flows may be canceled before actual flows take place. Let $\rho(c, L, \tau, \omega_2)$ be the cancellation rate, which is realized at beginning of the third stage, for the freight of cargo class c on path set L at time τ. Let $E_{\omega_2|\omega_1}$ be the expectation with respect to ω_2 conditional on ω_1. Second stage decision variables $y_a(v, \omega_1)$ denotes the extra capacity procured on the timed-voyage v. The associated unit cost is denoted as $\psi(v, \omega_1)$. Let vector x denote all the second stage decision variables. Denote the optimal objective value at the third stage for each scenario ω_2 given x as $Q_2(x, \omega_2)$. The objective function of the second stage is:

$$\begin{aligned} G_1(x, \omega_1) \quad = \quad & E_{\omega_2|\omega_1}\left[Q_2(x, \omega_2)\right] \\ & + \sum_{c \in \mathcal{C}} \sum_{L \in \mathcal{L}} \sum_{\tau \in \mathcal{T}(c,L)} |\tau| \, \pi(c, L, \tau) q(c, L, \tau, \pi, \omega_1)(1 - E_{\omega_2|\omega_1}[\rho(c, L, \tau, \omega_2)]) \\ & - \sum_{v \in \mathcal{V}} \psi_a(v, \omega_1) y_a(v, \omega_1) - \sum_{c \in \mathcal{C}} \sum_{L \in \mathcal{L}} \phi_{rj}(c, r, \omega_1) x_{rj}(c, r_{|R(L)|}, L, \omega_1) \\ & - \sum_{b \in \mathcal{B}} \sum_{o \in \mathcal{O}} \phi_p(b, o, \omega_1) x_p(b, o, r_{|R_o(o)|}, \omega_1) \\ & - \sum_{b \in \mathcal{B}} \sum_{o \in \mathcal{O}} \phi_l(b, o, \omega_1) x_l(b, o, r_{|R_o(o)|}, \omega_1) \end{aligned} \tag{2.4.52}$$

where $r_{|R(L)|}$ is defined by indexing the timed-paths in $R(L)$ in increasing order of their start times and $r_{|R_o(o)|}$ is defined by indexing the timed-paths in $R_o(o)$ in increasing order of their start times. Thus, the optimal value of the second stage is

$$Q_1(\pi, \omega_1) \quad = \quad \text{maximize} \quad G_1(x, \omega_1) \tag{2.4.53}$$

subject to constraints (2.4.35)–(2.4.51).

At the third stage, actual freight flows and empty container flows take place. At this stage, some of scheduled flows are allowed to be canceled. The travel times on the domestic arcs from origins to ports also involves uncertainties. In this model, we assume that no transshipment is involved, i.e., the containers are moved on only one service rotation along each path. On the domestic arc between the origin and the first port, not all the booked flows can arrive the port on time. Let third stage random variable $\lambda(c, r, \omega_2)$ denote the fraction of the freight flow with cargo class $c \in \mathcal{C}$ scheduled on timed-path r that departs from o can arrive before the departure time of the ship conditional on the capacity on the domestic leg being sufficient. Let $y_o(c, r, L, \omega_2)$ be the actual flow (associated with scheduled flow $x_o(c, r, L, \omega_1)$) of cargo class $c \in \mathcal{C}'$ on timed-path r moved in containers owned by the carrier. The unit cost of such a movement is denoted as $\phi_o(c, r, \omega_2)$. Similarly, let $y_r(c, r, L, \omega_2)$ be the actual flow of $c \in \mathcal{C}'$ on timed-path r moved in the containers rented by the carrier. The unit cost of such a movement is denoted as $\phi_r(c, r, \omega_2)$. Decision variable $y_c(c, r, L, \omega_2)$ denotes the amount of actual flow of cargo class $c \in \mathcal{C}''$ in containers provided by the customer. The unit cost of such a movement is denoted as $\phi_c(c, r, \omega_2)$. The remaining fraction of freight flow that is late for the ship is stored at the port and rolled to the next ship.

The amount of flow, which is late for the ship and needs to be loaded on the next timed-path after r, of freight type $c \in \mathcal{C}'$ in the carrier's own or long term leased containers and of demand path set L is denoted as $z_o(c, r, L, \omega_2)$. The unit cost of such delay is denoted as $\psi_o(c, r, \omega_2)$. Similarly, let $z_r(c, r, L, \omega_2)$ denote the amount of delayed freight flow of cargo class $c \in \mathcal{C}'$ in containers rented by the carrier. The unit cost of such delay is denoted as $\psi_r(c, r, \omega_2)$. Decision variable $z_c(c, r, L, \omega_2)$ denotes the amount of flow with cargo class $c \in \mathcal{C}''$ in the customer's own containers delayed at the port. The associated unit cost is denoted as $\psi_c(c, r, \omega_2)$. Let $y_e(b, r, \omega_2)$ denote the actual empty flow of type b containers on timed-path r. The associated unit cost is denoted as $\phi_e(b, r, \omega_2)$. The delayed empty container flow of type b containers, which needs to be moved on the timed-path next to r, is denoted as $z_e(b, r, \omega_2)$. The associated unit cost is denoted as and $\psi_e(b, r, \omega_2)$.

The flow balance between the booked flows and actual flows in containers owned or long term leased by the carrier is formulated as the following.

$$y_o(c, r_k, L, \omega_2) + z_o(c, r_k, L, \omega_2)$$

$$\begin{aligned} &= z_o(c, r_{k-1}, L, \omega_2) + \lambda(c, r_k, \omega_2) x_o(c, r_k, L, \omega_1)(1 - \rho(c, L, t_o(r_k), \omega_2)) \\ &\quad +(1 - \lambda(c, r_{k-1}, \omega_2)) x_o(c, r_{k-1}, L, \omega_1)(1 - \rho(c, L, t_o(r_{k-1}), \omega_2)) \\ &\qquad \text{for all } c \in \mathcal{C}', L \in \mathcal{L}, l \in L, r_k \in R(l) \end{aligned} \tag{2.4.54}$$

where r_k is defined by indexing all the timed-paths in $R(l)$ in increasing order of their start times. The RHS of equation (2.4.54) represents the inflow at the first port on the path l, including the flow booked on the timed-path r_k arriving on time, the delayed flow from the last timed-path r_{k-1}, and the cumulative delayed freight stored at the port. The LHS of equation (2.4.54) is outflow at the first port and is equal to the summation of the actual flow on timed-path r_k and the delayed flow stored at the port after the ship tied to the timed-path r_k sails. Similarly, this must also hold on other types of flows:

$$\begin{aligned} & y_r(c, r_k, L, \omega_2) + z_r(c, r_k, L, \omega_2) \\ &= z_r(c, r_{k-1}, L, \omega_2) + \lambda(c, r_k, \omega_2) x_r(c, r_k, L, \omega_1)(1 - \rho(c, L, t_o(r_k), \omega_2) \\ &\quad +(1 - \lambda(c, r_{k-1}, \omega_2)) x_r(c, r_{k-1}, L, \omega_1)(1 - \rho(c, L, t_o(r_{k-1}), \omega_2)) \\ &\qquad \text{for all } c \in \mathcal{C}', L \in \mathcal{L}, l \in L, r_k \in R(l) \end{aligned} \tag{2.4.55}$$

$$\begin{aligned} & y_c(c, r_k, L, \omega_2) + z_c(c, r_k, L, \omega_2) \\ &= z_c(c, r_{k-1}, L, \omega_2) + \lambda(c, r_k, \omega_2) x_c(c, r_k, L, \omega_1)(1 - \rho(c, L, t_o(r_k), \omega_2)) \\ &\quad +(1 - \lambda(c, r_{k-1}, \omega_2)) x_c(c, r_{k-1}, L, \omega_1)(1 - \rho(c, L, t_o(r_{k-1}), \omega_2)) \\ &\qquad \text{for all } c \in \mathcal{C}'', L \in \mathcal{L}, l \in L, r_k \in R(l) \end{aligned} \tag{2.4.56}$$

$$\begin{aligned} & y_e(b, r_k, \omega_2) + z_e(b, r_k, \omega_2) \\ &= z_e(b, r_{k-1}, \omega_2) + \lambda(b, r_k, \omega_2) x_e(b, r_k, \omega_1) + (1 - \lambda(b, r_{k-1}, \omega_2)) x_e(b, r_{k-1}, \omega_2) \\ &\qquad \text{for all } b \in \mathcal{B}, l \in \overline{L}, r_k \in R(l) \end{aligned} \tag{2.4.57}$$

The following constraints ensure that the capacities of all timed-voyages can not exceeded. Recall that the procured voyage capacities are second stage decision variables and can not be changed after the end of second stage.

$$\sum_{c\in\mathcal{C}'}\left[\sum_{r\in R_v(v)}\sum_{L\in\mathcal{L}(r)}[y_o(c,r,L,\omega_2)+y_r(c,r,L,\omega_2)]+\sum_{r\in R_v^0(v)}[x_o^0(c,r,\omega_1)+x_r^0(c,r,\omega_1)]\right]$$
$$+\sum_{c\in\mathcal{C}''}\left[\sum_{r\in R_v(v)}\sum_{L\in\mathcal{L}(r)}y_c(c,r,L,\omega_2)+\sum_{r\in R_v^0(v)}x_c^0(c,r,\omega_1)\right]$$
$$+\sum_{b\in\mathcal{B}}\left[\sum_{r\in R_v(v)}y_e(b,r,\omega_2)+\sum_{r\in R_v^0(v)}x_e^0(b,r,\omega_1)\right] \;\le\; u(v,\omega_1)+y_a(v,\omega_1)$$
$$\text{for all } v\in\mathcal{V} \quad (2.4.58)$$

The third stage variables are bounded for each ω_2.

$$0 \le y_o(c,r,L,\omega_2) \le u_o(c,r,L,\omega_2)$$
$$\text{for all } c\in\mathcal{C}', r\in\overline{R}, L\in\mathcal{L} \quad (2.4.59)$$

$$0 \le y_r(c,r,L,\omega_2) \le u_r(c,r,L,\omega_2)$$
$$\text{for all } c\in\mathcal{C}', r\in\overline{R}, L\in\mathcal{L} \quad (2.4.60)$$

$$0 \le y_c(c,r,L,\omega_2) \le u_c(c,r,L,\omega_2)$$
$$\text{for all } c\in\mathcal{C}'', r\in\overline{R}, L\in\mathcal{L} \quad (2.4.61)$$

$$0 \le y_e(b,r,\omega_2) \le u_e(b,r,\omega_2)$$
$$\text{for all } b\in\mathcal{B}, r\in\overline{R} \quad (2.4.62)$$

$$0 \le z_o(c,r,L,\omega_2) \le u_o(c,r,L,\omega_2)$$
$$\text{for all } c\in\mathcal{C}', r\in\overline{R}, L\in\mathcal{L} \quad (2.4.63)$$

$$0 \le z_r(c,r,L_r,\omega_2) \le u_r(c,r,L,\omega_2)$$
$$\text{for all } c\in\mathcal{C}', r\in\overline{R}, L\in\mathcal{L} \quad (2.4.64)$$

$$0 \le z_c(c,r,L_r,\omega_2) \le u_c(c,r,L,\omega_2)$$
$$\text{for all } c\in\mathcal{C}'', r\in\overline{R}, L\in\mathcal{L} \quad (2.4.65)$$

$$0 \leq z_e(b,r,\omega_2) \leq u_e(b,r,\omega_2)$$

$$\text{for all } b \in \mathcal{B}, r \in \overline{R} \qquad (2.4.66)$$

Let y be the vector containing all the third stage decision variables. The objective function of the third stage is:

$$\begin{aligned} G_2(y,\omega_2) = & -\sum_{c\in\mathcal{C}'}\sum_{r\in\overline{R}}\sum_{L\in\mathcal{L}}[\phi_o(c,r,\omega_2)y_o(c,r,L,\omega_2)+\phi_r(c,r,\omega_2)y_r(c,r,L,\omega_2)] \\ & -\sum_{c\in\mathcal{C}''}\sum_{r\in\overline{R}}\sum_{L\in\mathcal{L}}\phi_c(c,r,\omega_2)y_c(c,r,L,\omega_2) \\ & -\sum_{c\in\mathcal{C}'}\sum_{r\in\overline{R}}\sum_{L\in\mathcal{L}}[\psi_o(c,r,\omega_2)z_o(c,r,L,\omega_2)+\psi_r(c,r,\omega_2)z_r(c,r,L,\omega_2)] \\ & -\sum_{c\in\mathcal{C}''}\sum_{r\in\overline{R}}\sum_{L\in\mathcal{L}}\psi_c(c,r,\omega_2)z_c(c,r,L,\omega_2) \\ & -\sum_{b\in\mathcal{B}}\sum_{r\in\overline{R}}[\phi_e(b,r,\omega_2)y_e(b,r,\omega_2)+\psi_e(b,r,\omega_2)z_e(b,r,\omega_2)] \qquad (2.4.67) \end{aligned}$$

Thus, the optimal value of the second stage is

$$Q_2(x,\omega_2) = \text{maximize} \quad G_2(y,\omega_2) \qquad (2.4.68)$$

subject to (2.4.54)–(2.4.66).

2.5 Solution algorithms

2.5.1 Algorithms for the deterministic models

The two deterministic models are quadratic problems with linear constraints. Any algorithm in standard software such as CPLEX can effectively solve the problems.

2.5.2 Algorithms for the stochastic model

In this section, we propose several approaches to solve the stochastic problem. First, we simplify the notation for the demand. Recall the demand rate is

$$q(c,L,\pi,\tau,\omega_1) = \alpha(c,L,\tau,\omega_1)+\sum_{c'\in\mathcal{C}}\sum_{L'\in\mathcal{L}}\sum_{\tau'\in\mathcal{T}(c',L')}\beta(c,L,\tau,c',L',\tau',\omega_1)\pi(c',L',\tau')$$

$$\text{for all } c\in\mathcal{C}, L\in\mathcal{L}, \tau\in\mathcal{T}(c,L)$$

over the region given by

$$\underline{\pi}(c, L, \tau) \quad \leq \quad \pi(c, L, \tau) \quad \leq \quad \overline{\pi}(c, L, \tau)$$

Let π denote the column vector with entries $\pi(c, L, \tau)$, let a denote the column vector with entry $|\tau| E_{\omega_1} \left[\alpha(c, L, \tau, \omega_1)(1 - E_{\omega_2|\omega_1}[\rho(c, L, \tau, \omega_2)])\right]$ in the column corresponding to π, and let B denote the symmetric matrix with entry

$$\begin{aligned}&|\tau| E_{\omega_1} \left[\beta(c, L, \tau, c', L', \tau', \omega_1)(1 - E_{\omega_2|\omega_1}[\rho(c, L, \tau, \omega_2)])\right. \\ &\quad \left. + \beta(c', L', \tau', c, L, \tau, \omega_1)(1 - E_{\omega_2|\omega_1}[\rho(c', L', \tau', \omega_2)])\right] / 2\end{aligned}$$

in the column corresponding to (c, L, τ) and the row corresponding to (c', L', τ'). Note that the expected total revenue is given by

$$\begin{aligned}
& E_{\omega_1} \left[\sum_{c \in \mathcal{C}} \sum_{L \in \mathcal{L}} \sum_{\tau \in \mathcal{T}(c,L)} |\tau| \; \pi(c, L, \tau) q(c, L, \tau, \pi, \omega_1)(1 - E_{\omega_2|\omega_1}[\rho(c, L, \tau, \omega_2)])\right] \\
= \; & \sum_{c \in \mathcal{C}} \sum_{L \in \mathcal{L}} \sum_{\tau \in \mathcal{T}(c,L)} |\tau| \; \pi(c, L, \tau) E_{\omega_1} \left[\alpha(c, L, \tau, \omega_1)(1 - E_{\omega_2|\omega_1}[\rho(c, L, \tau, \omega_2)])\right] \\
& + \sum_{c \in \mathcal{C}} \sum_{L \in \mathcal{L}} \sum_{\tau \in \mathcal{T}(c,L)} |\tau| \; \pi(c, L, \tau) \\
& \qquad \times \sum_{c' \in \mathcal{C}} \sum_{L' \in \mathcal{L}} \sum_{\tau' \in \mathcal{T}(c',L')} \{\pi(c', L', \tau') \\
& \qquad \qquad \times E_{\omega_1} \left[\beta(c, L, \tau, c', L', \tau', \omega_1)(1 - E_{\omega_2|\omega_1}[\rho(c, L, \tau, \omega_2)])\right]\} \\
= \; & a^T \pi + \pi^T B \pi \qquad\qquad (2.5.1)
\end{aligned}$$

We assume that B is negative semidefinite.

We propose to use the sample average approximation (SAA) method [33] to solve the stochastic problem. The sample average problem is a three-stage stochastic program with a much smaller number of scenarios than the original problem. We solve it as a two-stage problem by combining the second stage and the third stage together. The outline of the SAA method is as follows.

We use Bender's decomposition to solve the sample average problem described in

Algorithm 1: SAA Algorithm

1. Choose initial second stage sample size N_1, N_1', initial third stage sample size N_2 and N_2' and the number of replications M;

for $m = 1, ..., M$ **do**

2.1 Generate $N_1 \times N_2$ samples and formulate the stochastic problem with those samples. Solve the problem and obtain the optimal value $\hat{f}^m$ and the optimal solution $\hat{\pi}^m$;

2.2 Generate another $N1' \times N2'$ random samples. Evaluate the expected value with those samples as the lower bound of the objective function value $\hat{f}^m_{lb}$;

2.3 Calculate the upper bound of the objective function value $\hat{f}_{ub} = \frac{\sum_{m=1}^{M} \hat{f}^m}{M}$;

2.4 Estimate the optimality gap $\hat{f}^m_{\Delta} = \hat{f}_{ub} - \hat{f}^m_{lb}$ and the variance of the gap estimator;

end

3. If the optimal gap or the variance of the gap estimator are large, increase the sample sizes $N_i, N_i', i = 1, 2$ or the number of replications M and go to STEP 1.

4. Choose one $\hat{\pi}^i$ out of the M candidate solutions by a screening and selection procedure and stop.

STEP 2.1 denoted as $(PSAA)$.

$$
\begin{aligned}
(PSAA) \qquad \max_{\pi} \quad & f(\pi) = \pi^T B \pi + a^T \pi + \frac{1}{N_1} \sum_{n=1}^{N_1} Q_1(\pi, \omega_1^n) \\
\text{s.t.} \quad & \underline{\pi} \leq \pi \leq \overline{\pi}
\end{aligned}
$$

where $Q_1(\pi, \omega_1)$ is the optimal value of the second stage problem of the sample average problem:

$$
\begin{aligned}
Q_1(x, \omega_1) = \max \Bigg[& \frac{1}{N_2} \sum_{n=1}^{N_2} G_2(y, \omega_2^n) \\
& - \sum_{v \in \mathcal{V}} \psi_a(v, \omega_1) y_a(v, \omega_1) - \sum_{c \in \mathcal{C}} \sum_{L \in \mathcal{L}} \phi_{rj}(c, r, \omega_1) x_{rj}(c, r_{|R(L)|}, L, \omega_1) \\
& - \sum_{b \in \mathcal{B}} \sum_{o \in \mathcal{O}} \phi_p(b, o, \omega_1) x_p(b, o, r_{|R_o(o)|}, \omega_1) \\
& - \sum_{b \in \mathcal{B}} \sum_{o \in \mathcal{O}} \phi_l(b, o, \omega_1) x_l(b, o, r_{|R_o(o)|}, \omega_1) \Bigg]
\end{aligned} \tag{2.5.2}
$$

subject to the all the second stage constraints (2.4.35)–(2.4.51) and N_2 sets of third stage constraints (2.4.54)–(2.4.66).

Above problem is usually referred as the L-shaped problem. For each ω_1, $Q_1(\pi, \omega_1)$ is a non-smooth concave function of π. Such problem can be solved by cutting plane algorithms. A generic cutting plane method is summarized here. The upper bound and lower bound of the objective value of problem $(PSAA)$ are denoted as f_{ub} and f_{lb}. Let $SG_k = \{g_0, g_1 ...\}$ be the set of subgradients at iteration k. Denote the best solution till iteration k as π_k^*. Let the objective value at π be $f(\pi)$. Denote the optimal solution as π^* and the optimal value as f^*.

Algorithm 2: Cutting Plane Algorithm

Initialization: Let $\Pi_0 = \{\pi_0\}$ and $f_{ub} = \infty$. Compute the subgradient g_0 at π_0 and let $SG_0 = \{g_0\}$. Compute $f_{lb} = f(\pi_0)$. Choose tolerance ε of the optimal gap ;

while $f_{ub} - f_{lb} > \varepsilon$ **do**
  Compute the subgradient g_k at π_k, $SG_k \leftarrow SG_{k-1} \bigcup \{g_k\}$;
  Compute $f(\pi_k)$.
  if $f(\pi_k) > f_{lb}$ **then**
    $f_{lb} \leftarrow f(\pi_k)$, $\pi_k^* \leftarrow \pi_k$
  end
  Solve the relaxed problem (RP). Let f_{ub} be the optimal objective value;
  Solve a candidate problem (CP). Let π_{k+1} be the optimal solution of that problem, $\Pi_{k+1} \leftarrow \Pi_k \bigcup \{\pi_{k+1}\}$;
  $k \leftarrow k + 1$
end
Let $\pi^* \leftarrow \pi_k^*$, $f^* \leftarrow f(\pi_k^*)$. Stop.

To construct the candidate problem (CP) and the relaxed problem (RP), there are several approaches that we ill discuss in detail next.

Trust Region Method: The trust region method can be found in [43]. Let $h(\pi) = \frac{1}{N} \sum_{n=1}^{N_1} Q_1(\pi, \omega_1^n)$. To efficiently solve this problem, we keep the quadratic term and only approximate $h(\pi)$ by cutting planes. In this case, the subgradients in SG_k are corresponding to $h(\pi)$ instead of $f(\pi)$. The relaxed problem (RP) is defined as follows.

$$
\begin{array}{llrcll}
(RQP) & \max\limits_{\pi,\theta} & \multicolumn{3}{l}{\pi^T B \pi + a^T \pi + \theta} & \\
 & \text{s.t.} & \theta & \leq & h(\pi_i) + g_i^T(\pi - \pi_i) & \forall g_i \in SG_k, \pi_i \in \Pi_k
\end{array}
$$

$$\underline{\pi} \leq \pi \leq \overline{\pi}$$

In this approach, the candidate problem is the relaxed problem with trust region constraints to stabilize the iterates, denoted as (TR). Choose some positive value κ as the trust region size.

$$\begin{array}{llll} (TR) & \max\limits_{\pi,\theta} & \pi^T B\pi + a^T\pi + \theta & \\ & \text{s.t.} & \theta \leq h(\pi_i) + g_i^T(\pi - \pi_i) & \forall g_i \in SG_k, \pi_i \in \Pi_k \\ & & \underline{\pi} \leq \pi \leq \overline{\pi} & \\ & & \|\pi - \pi_k^*\|_\infty \leq \kappa & \end{array}$$

In this method, the trust region size κ can be adjusted according to the quality of the solution of problem (TR). The details can be found in [43] and are not included here.

Standard Bundle Level Method: We use a standard bundle level method in [30] and [38] to approximate the concave function $f(\pi)$ with cutting planes. Denote this approach as BundleLp because the projection problems solved in this method are linear. In this case, the subgradients in SG_k are corresponding to function $f(\pi)$. The relaxed problem (RLP) is defined as follows.

$$\begin{array}{llll} (RLP) & \max\limits_{\pi,\theta} & \theta & \\ & \text{s.t.} & \theta \leq f(\pi_i) + g_i^T(\pi - \pi_i) & \forall g_i \in SG_k, \pi_i \in \Pi_k \\ & & \underline{\pi} \leq \pi \leq \overline{\pi} & \end{array}$$

In this approach, the candidate problem is a projection problem defined as follows. Let π_k^* be the projection center π_c. Choose some $\nu \in (0, 1)$ and let $f_\ell = \nu f_{lb} + (1-\nu) f_{ub}$ be the projection level. The projection problem is a liner program.

$$\begin{array}{llll} (PRJLP) & \min\limits_{\pi} & (\pi - \pi_c)^T(\pi - \pi_c) & \\ & \text{s.t.} & f(\pi_i) + g_i^T(\pi - \pi_i) \geq f_\ell & \forall g_i \in SG_k, \pi_i \in \Pi_k \end{array}$$

$$\underline{\pi} \leq \pi \leq \overline{\pi}$$

Note the the projection level f_ℓ can also be tuned dynamically.

Nonstandard Bundle Level Method: A (nonstandard) bundle level method that only approximates $h(\pi)$ by cutting planes as in the trust region approach. The subgradients in SG_k are corresponding to $h(\pi)$ instead of $f(\pi)$. The relaxed problem is (RQP). The projection problem becomes

$$\begin{array}{llll} (PRJQCP) & \min\limits_{\pi} & (\pi - \pi_c)^T(\pi - \pi_c) & \\ & \text{s.t.} & \pi^T B\pi + a^T\pi + h(\pi_i) + g_i^T(\pi - \pi_i) \ \geq \ f_\ell & \forall g_i \in SG_k, \pi_i \in \Pi_k \\ & & \underline{\pi} \ \leq \ \pi \ \leq \ \overline{\pi} & \end{array}$$

The complicating aspect is that the constraint $\pi^T B\pi + a^T\pi + h(\pi_i) + g_i^T(\pi - \pi_i) \ \geq \ f_\ell$ is nonlinear, specifically, it is quadratic. There are various approaches for solving such a problem:

BundleQcp Approach: Solve the problem by using a barrier method, for example the quadratic constrained problem (QCP) solver in CPLEX 9.0. We use BundleQcp to denote this bundle level approach.

BundleDual Approach: Use a Lagrangian dual approach as follows. We refer this bundle level approach as BundleDual. For simplicity, let $h(\pi_i) - g_i^T\pi_i = a_i$ and assume $|SG_k| = m$. The projection problem $(PRJQCP)$ can be rewritten as follows.

$$\begin{array}{lll} \min\limits_{\pi} & (\pi - \pi_c)^T(\pi - \pi_c) & \\ \text{s.t.} & \pi^T B\pi + a^T\pi + a_i + g_i^T\pi \ \ \geq \ \ f_\ell & \text{for all } i = 1, \ldots, m \\ & \underline{\pi} \ \leq \ \pi \ \leq \ \overline{\pi} & \end{array}$$

As the target level f_ℓ is chosen in a way such that $f_\ell < f_{ub}$, the optimal solution of problem (RQP), denoted as $(\tilde{\pi}, \tilde{\theta})$, is feasible in problem $(PRJQCP)$ and satisfies

$$\tilde{\pi}^T B\tilde{\pi} + a^T\tilde{\pi} + a_i + g_i^T\tilde{\pi} \ \ > \ \ f_\ell \qquad \text{for all } i = 1, \ldots, m$$

Therefore, Slater condition is satisfied and strong duality holds [27]. Solving the Lagrangian dual problem can obtain the optimal solution of problem $(PRJQCP)$. Let

$$
\begin{aligned}
L^*(\lambda) \quad = \quad \min_{\pi} \quad & (\pi - \pi_c)^T(\pi - \pi_c) + \sum_{i=1}^{m} \lambda_i \left[g_\ell - \left(\pi^T B \pi + a^T \pi + a_i + g_i^T \pi \right) \right] \\
\text{s.t.} \quad & \underline{\pi} \leq \pi \leq \overline{\pi}
\end{aligned}
$$

Consider the Lagrangian dual problem

$$
\begin{aligned}
\max_{\lambda} \{L^*(\lambda) : \lambda \geq 0\} \quad &= \quad \max_{\lambda,\mu} \{L^*(\lambda) : \lambda \geq 0, \sum_{i=1}^{m} \lambda_i = \mu\} \\
&= \quad \max_{\lambda,\mu} \min_{\pi} \{ \quad (\pi - \pi_c)^T(\pi - \pi_c) + \sum_{i=1}^{m} \lambda_i \left[f_\ell \right. \\
&\qquad \left. - \left(\pi^T B \pi + a^T \pi + a_i + g_i^T \pi \right) \right] \ : \ \underline{\pi} \leq \pi \leq \overline{\pi} \} \\
&\qquad \text{s.t.} \quad \sum_{i=1}^{m} \lambda_i = \mu \\
&\qquad \qquad \lambda \geq 0 \\
&= \quad \max_{\mu} \min_{\pi} \max_{\lambda} \quad (\pi - \pi_c)^T(\pi - \pi_c) \\
&\qquad + \sum_{i=1}^{m} \lambda_i \left[f_\ell - \left(\pi^T B \pi + a^T \pi + a_i + g_i^T \pi \right) \right] \\
&\qquad \qquad \qquad \text{s.t.} \quad \sum_{i=1}^{m} \lambda_i = \mu \\
&\qquad \qquad \qquad \qquad \lambda \geq 0 \\
&\qquad \qquad \text{s.t.} \quad \underline{\pi} \leq \pi \leq \overline{\pi} \\
&\qquad \text{s.t.} \quad \mu \geq 0
\end{aligned}
$$

Consider the inner optimization problem for any given $\mu \geq 0$ and given π:

$$
\begin{aligned}
\max_{\lambda} \quad & \sum_{i=1}^{m} \lambda_i \left[f_\ell - \left(\pi^T B \pi + a^T \pi + a_i + g_i^T \pi \right) \right] \\
\text{s.t.} \quad & \sum_{i=1}^{m} \lambda_i = \mu \\
& \lambda \geq 0
\end{aligned}
$$

Note that the optimal objective value of the inner optimization problem is

$$\mu\left[f_\ell - (\pi^T B\pi + a^T\pi) - \min\{\left(a_i + g_i^T\pi\right) \,:\, i = 1, \ldots, m\}\right]$$

As a result, for given $\mu \geq 0$, the middle optimization problem is the following quadratic program:

$$\begin{aligned} M(\mu) \quad = \quad \min_{\pi,\theta} \quad & (\pi - \pi_c)^T(\pi - \pi_c) + \mu\left[f_\ell - (\pi^T B\pi + a^T\pi + \theta)\right] \\ = \quad & \pi^T(I - \mu B)\pi - (2\pi_c + \mu a)^T\pi - \mu\theta + \pi_c^T\pi_c + \mu f_\ell \\ \text{s.t.} \quad & \theta \;\leq\; a_i + g_i^T\pi \qquad\qquad \text{for all } i = 1, \ldots, m \\ & \underline{\pi} \;\leq\; \pi \;\leq\; \overline{\pi} \end{aligned}$$

In summary, the dual problem of the projection problem (DP) is

$$\begin{aligned} (DP) \qquad\qquad & \max_{\mu} \quad M(\mu) \\ & \text{s.t.} \quad \mu \;\geq\; 0 \end{aligned}$$

The optimal value of μ can be found with a line search method, such as bisection search, if μ is bounded from above.

Now, we propose the method to get the upper bound of the optimal value of μ. For a given $\mu \geq 0$, let $\pi^*(\mu)$, $\theta^*(\mu)$ denote an optimal solution of the problem $M(\mu)$. Denote the optimal solution of the projection problem (DP) as $(\mu^*, \pi^*(\mu^*), \theta^*(\mu^*))$. Let $h(\mu, \pi, \theta) = (\pi - \pi_c)^T(\pi - \pi_c) + \mu\left[f_\ell - \pi^T B\pi - a^T\pi - \theta\right]$. Let $\mu = 0$ and compute the optimal objective value of the middle optimization problem $M(0)$. Note that $\underline{\pi} \leq \pi_c \leq \overline{\pi}$, thus, $M(0) = 0$ and $\pi^*(0) = \pi_c$. For μ^* and a feasible solution to the middle optimization problem, $(\pi, \theta) \in \{(\pi, \theta) \,:\, \theta \leq a_i + g_i^T\pi \text{ for all } i = 1, \ldots, m; \underline{\pi} \leq \pi \leq \overline{\pi}\}$, we need $h(\mu^*, \pi, \theta) \geq M(0)$. Otherwise $g(\mu^*) \leq h(\mu^*, \pi, \theta) < M(0)$ and μ^* can not be optimal.

Note the optimal solution of the relaxed problem (RQP), $(\tilde{\pi}, \tilde{\theta})$, is also a feasible solution to the middle optimization problem $M(\pi)$. Thus,

$$\begin{aligned} (\tilde{\pi} - \pi_c)^T(\tilde{\pi} - \pi_c) + \mu^* \left[f_\ell - (\tilde{\pi}^T B \tilde{\pi} + a^T \tilde{\pi} + \tilde{\theta}) \right] &\geq 0 \\ \Rightarrow \quad \mu^* &\leq \frac{(\tilde{\pi} - \pi_c)^T(\tilde{\pi} - \pi_c)}{\tilde{\pi}^T B \tilde{\pi} + a^T \tilde{\pi} + \tilde{\theta} - f_\ell} \end{aligned}$$

The last step is by $f_\ell - (\tilde{\pi}^T B \tilde{\pi} + a^T \tilde{\pi} + \tilde{\theta}) < 0$.

The two bundle level methods, BundleLp and BundleQcp, that we implemented are in an improved version presented in [30]. In this approach, the relaxed problem is solved to update f_{ub} only when the projection problem becomes infeasible. By doing so, the computing time can be reduced. For details, please refer to [30]. For the last bundle level method, BundleDual, we still use the original version since we need the projection problem to be feasible thus the strong duality holds. In fact, the most computional cost lies in solving the second stage subproblems (combined with the third stage problems) to construct the subgradients at each iterate. Skipping the relaxed problem in some iterates does not make a big difference in our case.

2.6 Computational results

In this section, we present computational results of the three models on several instances. Running time and solution quality of the three models are reported and compared. In addition, we evaluate the performance of the algorithms in solving the stochastic problems.

2.6.1 Input data

All the instances used here have the same service rotations, sailing schedules, and paths, which are constructed based on two service rotations of Orient Overseas Container Line (OOCL), one of the world's largest integrated international container transportation companies. The demand parameters and costs are not real data. In these instances, the two service rotations are South China Express (SCX) and Super

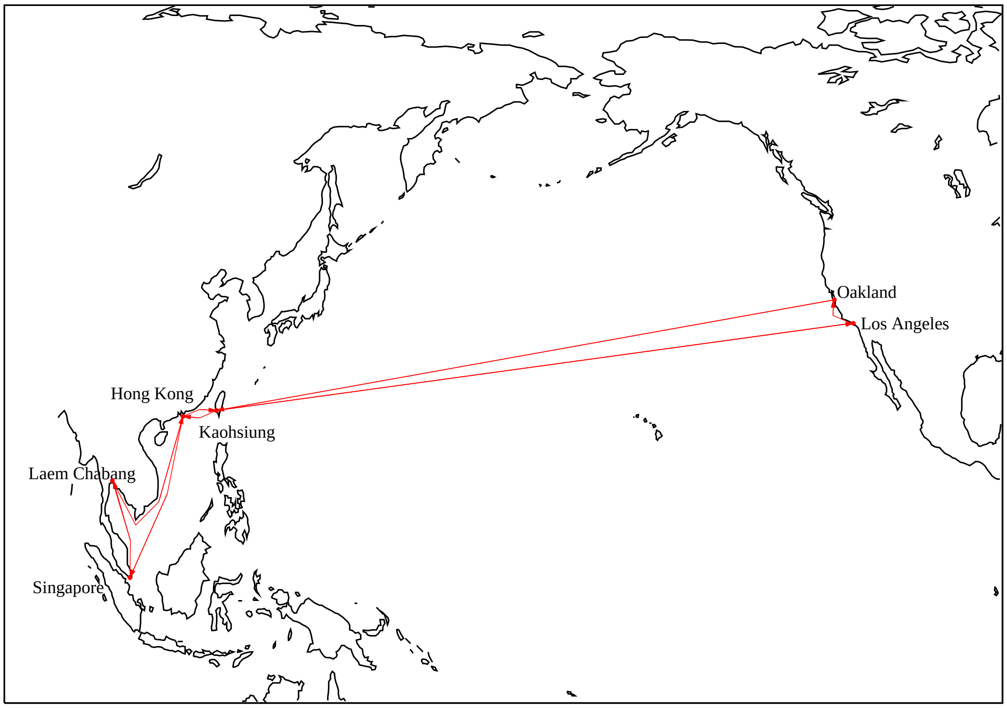

Figure 2: The South China Express (SCX) service rotation.

Shuttle Express (SSX), which transport freight between west coast of North America and Southeast Asia.

Total 17 different voyages and 28 Origin-Destination pairs are involved. Four of the pairs have multiple alternative paths connecting the origin to the destination. There are 10 ports, 15 inland locations, 32 paths in the network. The following computations incorporate one type of container, one type of cargo class and 40 types of demand requests. Half of the paths are from North America to Asia and the others are of the opposite direction. The parameters in the demand functions, costs, time horizon and other inputs are different for each instance. The time horizon of instance A1, B1 and C1 is of three months with demand rates vary from month to month. Instance A2, B2 and C2 are of 6 months. The peak season includes the first three months and the off-peak season includes the last three months. Demand rates vary from season to season. Instance A3–A5, B3–B5 and C3–C5 have one year time horizon. Instance A3, B3 and C3 have two seasons with the first half year as season

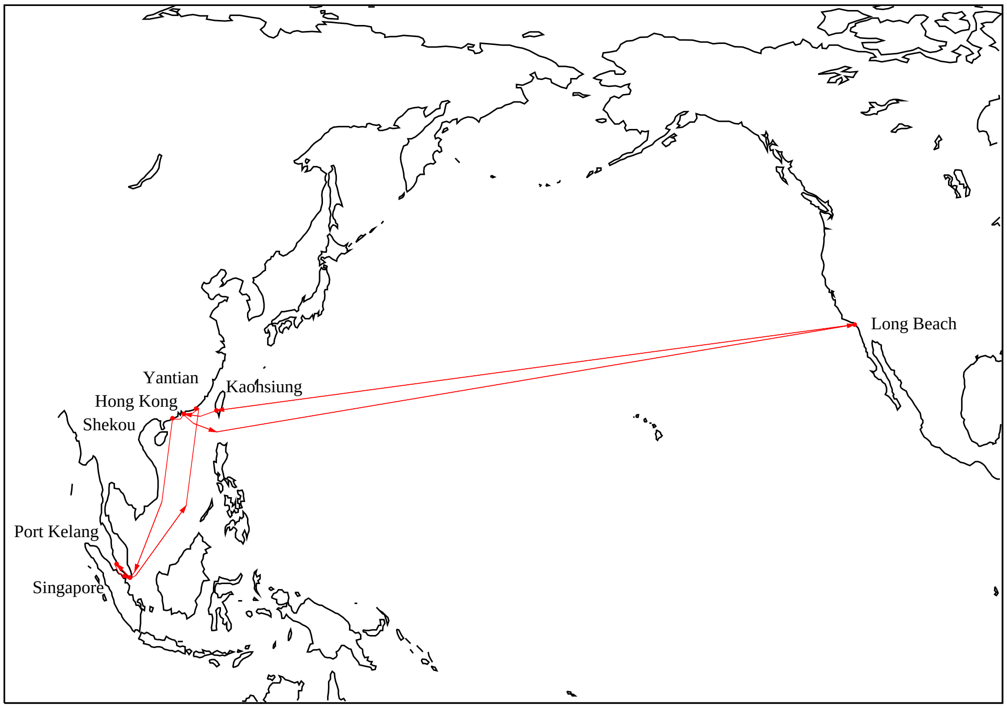

Figure 3: The Super Shuttle Express (SSX) service rotation.

one and the other half as season two. Similarly, demand rates are different in the first season from those in the second season. Instance A4, B4, and C4 have four seasons with different demand rates in each quarter. Instance A5, B5 and C5 have different demand arrival rates each month. The time window of each demand type is randomly chosen from one to six weeks. The solutions of instance B1–B5 and C1–C5 are tested in a simulation software SimSea, which is a computer simulation model of ocean container carrier operations. Those instances do not have extra voyage capacities and damaged containers that are not incorporated in the current simulation model. Instance A1–A5 contain all the features captured by the models. Thus, we could not test those instances by SimSea. The prices are allowed from 500 to 3500 dollars per TEU and the costs are chosen from 400 to 1200 dollars per TEU. The demand parameters (α and β) are randomly generated in a way such that for a given price set, the demand arrival rates from Asia to North America are bigger than those of the opposite direction, which is consistent with the current situation in the industry. Also

Table 1: Solution time in seconds for the models

Instance	SS	TS	STO	Instance	SS	TS	STO	Instance	SS	TS	STO
A1	0	2	144	B1	0	2	63	C1	0	1	173
A2	0	22	2103	B2	0	9	689	C2	0	8	1269
A3	1	84	8838	B3	0	50	7131	C3	1	40	4609
A4	0	81	9430	B4	1	47	5696	C4	0	39	5188
A5	0	81	9800	B5	4	42	5584	C5	0	36	5277

for a given price set, the arrival rates in the peak seasons are much higher than those in the off-peak seasons. The cancellation rates (ρ) vary from 0 to 0.3. The ratios(λ) are in $[0.7, 1.0]$ and the damage rates are in $[0, 0.1]$. For all the instances, all the random parameters are assumed to follow uniform distributions. Point estimators are used in the deterministic models. For the stochastic models, the random parameters are positively correlated.

2.6.2 Comparison among the models

The quadratic solver of CPLEX 9.0 is used to solve the deterministic models. Both the quadratic and linear solvers of CPLEX 9.0 are iteratively called to solve the subproblems for the stochastic model. All computations are done on a Linux workstation with dual 2.4 GHz Intel Xeon processors and 2 GB RAM. To solve the stochastic model, the solution from the time stamped model is used as the starting point. For all the instances in the Sample Average Approximation approach, the estimated gap of the objective function values is less 1.5% and the standard deviation of the gap is less than 1.6%. The solution time to solve the models are presented in Table 1, where SS, TS, STO denote the stable status model, the time stamped model and the stochastic model respectively.

Let π^{ss}, π^{ts} and π^{sto} denote the optimal price sets from the stable status model (SS), the time stamped model (TS), and the stochastic model (STO) respectively.

Table 2: Performance comparison of the deterministic models

Instance	$\Delta F_{TS}(\pi^{ts})$	Instance	$\Delta F_{TS}(\pi^{ts})$	Instance	$\Delta F_{TS}(\pi^{ts})$
A1	7.9%	B1	7.3%	C1	13.6%
A2	7.9%	B2	8.3%	C2	5.2%
A3	2.8%	B3	3.8%	C3	3.3%
A4	5.8%	B4	7.0%	C4	4.7%
A5	15.8%	B5	16.8%	C5	14.6%

And let FSS and FTS be the optimal revenues of the stable status model and the time stamped model. For a given price set π, the revenue estimated in the time stamped model is denoted as $F_{TS}(\pi)$. To compare the quality of the solutions from the two deterministic models, the stable status solution π^{ss} is estimated in the more accurate and detailed time stamped model. The relative revenue difference $\Delta F_{TS}(\pi^{ts}) = [F_{TS}(\pi^{ts}) - F_{TS}(\pi^{ss})]/F_{TS}(\pi^{ss})$ is listed in Table 2.

To compare the quality of the three solutions, we also evaluate them in the stochastic models. A new set of samples is generated with second stage sample size 80, third stage sample size 25 for the smaller problems (A1, A2, B1, B2, C1, C2) and 20 for bigger problems (A3–A5, B3–B5, C3–C5). The three solutions are evaluated in the stochastic model with those samples. Common random numbers are used in the estimation. Let $\tilde{F}_{STO}(\pi)$ be the expected revenue evaluated in the stochastic model with price set π for each replication. For each replication, compute $\Delta_{ts,ss} = \frac{\tilde{F}_{STO}(\pi^{ts}) - \tilde{F}_{STO}(\pi^{ss})}{\tilde{F}_{STO}(\pi^{ss})}$, $\Delta_{sto,ss} = \frac{\tilde{F}_{STO}(\pi^{sto}) - \tilde{F}_{STO}(\pi^{ss})}{\tilde{F}_{STO}(\pi^{ss})}$ and $\Delta_{sto,ts} = \frac{\tilde{F}_{STO}(\pi^{sto}) - \tilde{F}_{STO}(\pi^{ts})}{\tilde{F}_{STO}(\pi^{ts})}$. Let $\hat{\Delta}_{ts,ss}$, $\hat{\Delta}_{sto,ss}$ and $\hat{\Delta}_{sto,ts}$ be the estimated mean of $\Delta_{ts,ss}$, $\Delta_{sto,ss}$ and $\Delta_{sto,ts}$. Denote the estimated standard deviation of $\hat{\Delta}_{ts,ss}$, $\hat{\Delta}_{sto,ss}$ and $\hat{\Delta}_{sto,ts}$ as $\sigma_{ts,ss}$, $\sigma_{sto,ss}$ and $\sigma_{sto,ts}$ respectively. Table 3 summarizes the results.

The optimal solutions of instance B1–B5 and C1–C5 are also evaluated by SimSea. The results are reported in Table 4 with the same notation as in Table 3. Instead of evaluating the solutions in the stochastic model, the expected revenues are generated

Table 3: Performance comparison of all models by STO

Instance	$\hat{\Delta}_{ts,ss}$	$\sigma_{ts,ss}$	$\hat{\Delta}_{sto,ss}$	$\sigma_{sto,ss}$	$\hat{\Delta}_{sto,ts}$	$\sigma_{sto,ts}$
A1	1.2%	0.1%	1.2%	0.1%	0.0%	0.0%
A2	2.8%	0.1%	2.9%	0.1%	0.1%	0.0%
A3	0.8%	0.1%	1.6%	0.1%	0.7%	0.1%
A4	1.4%	0.1%	1.8%	0.1%	0.4%	0.1%
A5	1.8%	0.1%	2.3%	0.1%	0.4%	0.1%
B1	0.5%	0.0%	0.5%	0.0%	0.0%	0.0%
B2	2.6%	0.2%	2.7%	0.1%	0.1%	0.0%
B3	1.0%	0.1%	1.4%	0.2%	0.4%	0.1%
B4	1.9%	0.1%	2.1%	0.1%	0.2%	0.1%
B5	1.9%	0.1%	2.1%	0.1%	0.1 %	0.0%
C1	2.1%	0.1%	2.3%	0.1%	0.2%	0.0%
C2	0.9%	0.0%	1.7%	0.1%	0.8%	0.1%
C3	1.0%	0.1%	1.7%	0.1%	0.7%	0.1%
C4	0.5%	0.0%	1.0%	0.1%	0.6%	0.1%
C5	0.4%	0.0%	1.2%	0.1%	0.7 %	0.1 %

by SimSea. Common random numbers are also used in the comparison.

From the above results, we can see the solutions from the time stamped model are much better than those from the stable status model and are fairly close to the optimal solutions from the stochastic model for all the instances.

2.6.3 Comparison among the algorithms

In this section, we present the numerical results to investigate the performance of the approaches to solve the stochastic L-shaped problem. The optimal solution of the time stamped model is used as the starting point for each problem. The average number of major iterations and average running time for solving one stochastic problem are listed in Table 5. The solution time is the average time used in solving the M stochastic problems and $\overline{N}$ denotes the average number of iterations for the M replications with $M = 5$.

It is found that the BundleLp approach performances poorly in solving those instances. That is because the number of cutting planes needed is much more than

Table 4: Performance comparison of all models by SimSea

Instance	$\hat{\Delta}_{ts,ss}$	$\sigma_{ts,ss}$	$\hat{\Delta}_{sto,ss}$	$\sigma_{sto,ss}$	$\hat{\Delta}_{sto,ts}$	$\sigma_{sto,ts}$
B1	0.3%	0.1%	0.3%	0.1%	0.0%	0.0%
B2	3.3%	0.2%	3.3%	0.1%	0.0%	0.0%
B3	0.9%	0.1%	0.9%	0.1%	0.1%	0.1%
B4	0.7%	0.1%	0.6%	0.1%	0.0%	0.0%
B5	1.0%	0.1%	0.8%	0.1%	-0.2%	0.0 %
C1	2.2%	0.1%	1.9%	0.1%	-0.3%	0.1%
C2	0.6%	0.1%	0.3%	0.1%	-0.2%	0.1%
C3	0.7%	0.1%	0.4%	0.1%	-0.4%	0.1%
C4	0.1%	0.0%	-0.5%	0.1%	-0.5%	0.1%
C5	0.5%	0.0%	1.7%	0.1%	1.1%	0.1 %

Table 5: Performance comparison of the algorithms

Instance	BundleDual		BundleQcp		Trust Region	
	Solution Time (s)	$\overline{N}$	Solution Time (s)	$\overline{N}$	Solution Time (s)	$\overline{N}$
A1	161	1	164	1	174	1
A2	1008	2	1072	2	1019	2
A3	4458	3	-	-	6782	3
A4	4586	3	-	-	7065	3
A5	4376	3	-	-	6705	3
B1	66	1	66	1	67	1
B2	535	2	793	2	537	2
B3	2419	3	-	-	5269	3
B4	3554	3	-	-	5724	3
B5	2540	3	-	-	5646	3

the other methods, at least as many as the dimension of the first stage problem. Even for the smallest problem A1, it takes 12423 seconds and 122 iterations. Thus, we only compare the performances of the other three methods. For instance A3–A5 and B3–B5 with the BundleQcp approach, the quadratic constrained projection problems experience numerical difficulties when using the QCP solver of CPLEX 9.0. Even we scale those problems carefully, the solver is not able to get reasonable solutions. Thus, this approach fails in solving those six problems. The performances of the BundleDual method and the Trust Region method are comparably good. The BundleDual method slightly outperforms the Trust Region Method for most problems.

2.7 Concluding remarks

In this chapter, we develop three models for the contract planning problem. Various issues are considered in the models such as demand imbalance and seasonality, freight routing, container balance at each origin, procurement of extra voyage capacities, and container leasing, rental and procurement. The first model is called the steady state model. This model considers each season in the planning horizon separately, ignoring the initial conditions and flow changes between two seasons. It provides solutions with reasonable quality, requiring small computing time. The second model is called the time stamped model. The model integrates all the seasons in the planning horizon into one bigger problem. Since the initial conditions and flow changes are captured, we can obtain significantly more revenue by using this model than the first one, at the cost of a small increase in computing time. The third model is called the stochastic model. The third model captures the uncertainties in demand arrival rates, travel times on feeder arcs, booking cancelations and costs. The third model can obtain the largest revenue among the three. Although the model can bring a little more revenue than the second model, this model needs significantly more computing time. Based on our computational results, we conclude that the second model is likely the most practical of the three. All of the models can serve as decision tools for ocean carriers to structure the optimal prices in service contracts and develop optimal negotiation strategies.

CHAPTER III

MODELS OF SPOT MARKETS AND LONGER TERM CONTRACTS

3.1 Introduction

This work was originally motivated by the following observations in freight transportation markets. Freight transportation services can be procured on the spot, that is, when it is decided to send a shipment, the service of a carrier is procured for the particular shipment at a price determined at the time of the transaction. Most passenger transportation services are procured in this fashion. Freight transportation services can also be procured by entering into longer term contracts with one or more carriers. Such contracts usually apply to a specified time period, and the prices and other conditions specified in the contracts apply to multiple shipments.

In many parts of the world, most freight is transported under the provisions of longer term contracts. Most of the trucking and almost all rail freight in the United States are transported under longer term contracts. Most ocean freight is also transported in this fashion. Many of the big ocean carriers do 80% or more of their containerized freight transportation under these contracts [2]. Many contracts between ocean carriers and shippers are negotiated once a year, typically one or two months before the peak season of the major trades covered by the contracts.

Recent developments in information technology and communication make spot transactions more economical and more convenient. Electronic spot marketplaces can reduce search cost and facilitate spot transactions in a timely manner. In the ocean cargo industry, a survey conducted by Penaloza et al. [46] shows that more and more ocean carriers are beginning to view electronic spot markets as a strategic driver for increasing their profitability. Nevertheless, the implementation of e-commerce has been slow. A report by Bakker et al. [7] shows that until 2001, only 23% of the 66 large ocean carriers worldwide who participated in the survey were at the stage of implementing e-commerce and providing internet systems to integrate contracts and spot exchanges. Almost all of the current internet portals in this industry are supported by one or several of those large carriers, e.g., Inttra.com supported by Maersk Line, MSC Mediterranean Shipping Company, CMA CGM, etc., CargoSmart.com supported by Orient Overseas Container Line Limited, COSCO Container Line Limited, Nippon Yusen Kaisha, etc., and ShipmentLink.com supported by Evergreen Line. The main purpose of such internet portals is to serve the transactions covered by longer term contracts. Their functionalities include facilitating online booking, cargo tracking, and Bill of Lading (B/L) process, as well as publishing sailing schedules and spot rates. For the transactions not covered by longer term contracts, spot rates are applied. Such spot rates usually are higher than those in the longer term contracts. Incidental spot transactions still count for only a very small portion both in the large carriers who are the leaders in implementing e-commerce and in the industry as a whole.

In freight transportation industries, demand uncertainties are typical. For example, many shippers are freight forwarders who enter into longer term contracts with carriers before knowing the actual amounts of freight to be shipped by their own customers. Therefore, on one hand, those contracts, which fix prices for the covered periods, reduce price fluctuation for both carriers and shippers. On the other hand, fixing contract terms before the demand is revealed adds rigidity to the market. In contrast, spot markets may make better use of up-to-date information and may facilitate better dynamic matching between supply and demand.

In this chapter, we study the problem of how participation in spot markets may change the business of the freight transportation industries which are dominated by longer term contracts. We consider settings with different market structures. The seller/sellers sells/sell products under contracts and in the spot market. As is currently typical in many freight transportation industries, the buyers may or may not participate in the spot market as a matter of policy. Reasons for not participating in the spot market can be risk aversion and difficulties in arranging last minute transactions. Contracts are signed before the demand is observed. Spot market participation is modeled as the fraction of all buyers who consider spot transactions, denoted as λ.

In this study, the effect of capacity is also considered. We first study the case in which the seller's capacity is large. In other words, the seller can satisfy any level of demand. Then we consider the case in which the capacity is small. We investigate how the results differ from those in the former case.

The contracts are modeled as option contracts. The Black-Scholes paradigm is widely used for modeling the pricing of financial and real options. In that framework, the option transactions do not affect the prices of the underlying securities or products. In our models, however, both option prices and spot prices are set by the seller. Therefore, the spot prices depend on how many option contracts have been sold in advance. The reason we consider option contracts instead of forward contracts is because of the nature of most freight contracts currently used in practice. In most sea cargo service contracts, the prices of movements are specified as a function of paths and cargo classes. In addition, a minimum quantity guarantee is specified to which the shipper agrees to commit during the time period covered by the contract. It is not uncommon that the minimum quantity is relatively small compared with the actual amount of freight shipped. Moreover, shippers are seldom penalized if the minimum quantity guarantee is not met because carriers are reluctant to damage the relationships. Thus those contracts resemble free options in some sense.

First we start with the single-seller single-buyer setting. Second, a model of a single seller and many buyers is considered. In that setting, every buyer has the same utility only depending on the state of the market. Third, we consider the case when the buyers have different utilities in addition to a random state of the market. Last, the settings with multiple sellers are also studied. In all settings, spot prices are endogenous and the effect of capacity is also considered.

In the first two settings with large capacity, it is found that as the spot market participation rate increases, the contract market shrinks, i.e., the quantity of contracts transacted decreases. In the single-buyer setting, this quantity remains positive even when $\lambda = 1$. In the many-buyer setting, the quantity of contracts transacted decreases to zero as λ increases to 1. Under both market structures, the seller's surplus increases as spot market participation increases. However, the effects of the spot market participation rate on the buyers' surplus and on the total social surplus are more complicated. Depending on the variation of the demand, an increase in the spot market participation rate may or may not benefit the buyers, thereby may or may not increase the total social surplus. On the other hand, in the undercapacity case, it is found that the seller's surplus and the buyer's/buyers' total surplus are invariant with respect to the participation rate.

For the setting with a single seller and a continuum of buyers with different utilities, the results on the the contract market and on the seller's surplus still hold. Though we are not able to obtain analytical solution for any value of λ, numerical results indicate the buyers' total surplus also increases as spot market participation increases. We also prove that if the seller's capacity is small, both the seller and the buyers are better off in the case with full spot market participation ($\lambda = 1$) compared to the contract market only case ($\lambda = 0$).

We also consider the setting where there are two sellers and a single buyer. If the sellers' capacities are small, it is found that the sellers' total surplus and the buyer's surplus are constant regardless of the spot market participation, which is consistent with the results in the single-seller undercapacity setting. In the last section of this chapter, we also consider a market where there is a continuum of sellers and a continuum of buyers with different utilities. Numerical results indicate that the total quantity of contracts transacted decreases as spot market participation increases. Though both the sellers and the buyers may be worse off, it seems the total social welfare is always improved as spot market participation increases.

The rest of the chapter is organized as follows. Section 3.2 contains a brief review of the relevant literature. A brief model description is contained in Section 3.3. Section 3.4 presents our first model with a single seller and a single buyer. Section 3.5 extends that model with a continuum of buyers with the same utility. Section 3.6 considers the case when different buyers have different random utilities. In Section 3.7, a market with two sellers and a single buyers is studied. For comparison purpose, a market with many sellers and many buyers is also studied in Section 3.8. A numerical study is conducted to investigate the effects of the spot market participation. The results are summarized in Section 3.9 and the proofs are contained in the appendix.

3.2 Literature review

The contracts used in our models are in the form of call options that provide contract holders with the right to purchase the underlying products at a fixed price. The literature on financial options includes Bachelier [6] and Black and Scholes [9]. The literature on real options includes Dixit and Pindyck [19], Majd and Pindyck [41], Triantis and Hodder [50] and Trigeorgis [51]. In their models, the price of the underline securities or products follows a stochastic process, which is independent from the transactions of the options. In contrast, our model is different from them in that we take the spot price as endogenous. Thus, the transaction of option contracts in advance may alter the spot price of the underlying product.

In the economics literature, many papers address reasons for the existence of long term contracts. One approach focuses on transaction costs to explain the existence of such contracts between purchasers and suppliers. Literature along this line includes Coase [15], Williamson [56], Klein [31], Williams [55], Laffont and Tirole [35].

Allaz and Vila [4] develop a model with an oligopolistic market structure and explain the strategic reasons behind forward contracting. Green [26] considers an electricity market with two sellers and many buyers. He shows that forward contracts may hedge well their output in the spot market and remove much of the incentive to use their market power.

There is also literature in operations management on the use of forward contracts and option contracts in supply chain management. Eppen and Iyer [21] investigate the impact of backup agreements between a fashion merchandise buyer and upstream sellers. In such contracts, the buyer commits to a certain backup quantity of products which he requests the seller to hold before demand is revealed. After the demand is observed, the buyer can order up to that quantity at the original purchase cost with quick delivery and will pay a penalty cost for the leftover units. They show that the backup agreements have a substantial impact and may increase both the buyer and the sellers' revenue. Donohue [20] develops a two-stage newsvendor model between a seller and a buyer under demand uncertainty. She shows that supply contracts with predetermined wholesale prices and return price can coordinate the buyer and the seller to achieve better performance of the channel. Barnes-Schuster et al. [8] study the role of option contracts in supply chain performance. They consider a two-period model with correlated demand. In their model, there is a single seller and a single buyer who sells products to end consumers in the two periods. They demonstrate how the option contracts improve channel performance by providing the buyer with flexibility and increasing profits of both the buyer and the seller. Taylor (2002) develops a two-period model with contracts between sellers and buyers. Such contracts provide the buyers with price protection and rebates for the return of unsold inventory. Such contracts may guarantee both channel coordination and win-win outcome. Kamrad and Siddique [29] analyze and value supply contracts with uncertainty in exchange rates between sellers and a single buyer. Burnetas and Ritchken [11] focus their study on the effect of option contracts in a supply chain with a seller and a buyer. The seller sells option contracts that provide the buyer with the right to reorder or return products at a fixed price after the demand is revealed. They show that option contracts may make the buyer either better off or worse off, depending on the level of the demand uncertainty. Excellent reviews of this branch of literature can be found in Anupindi and Bassok [5], Lariviere [36], Tsay et al. [52], and Cachon [12].

There is an emerging literature that focuses on the interaction between longer term contracts and spot markets/exchanges. An excellent review on this topic can be found in Kleindorfer and Wu [32]. The following is some of the literature along this line. Within this type of literature, another important issue studied is the reason for the existence of longer term contracts.

Cohen and Agrawal [16] compare the tradeoff between the flexibility provided by short term contracts and price certainty offered by long term contracts. They show that long term contracting is not always an optimal strategy and discuss conditions under which short term contracts perform better.

Lee and Whang [37] develop a two-period model with a single upstream seller and many downstream buyers who re-sell the products to end consumers. The buyers order products from the seller at the beginning of the first period. Then the first period retail sales are observed. Before the second period begins, the buyers can trade their inventories among themselves in a secondary spot market. They endogenously derive the exchange price, the optimal decisions for the buyers and investigate the impact of the secondary market on the quantity sold by the seller and on the supply chain performance. The total sales volume for the seller may increase or decrease. However, the secondary market always benefits the buyers and improves the supply chain allocative efficiency by increasing sales to end consumers and decreasing stockouts and leftover stock. The combined effect on the welfare of the supply chain is unclear.

Peleg et al. [45] investigate the difference among three procurement strategies, relational contracts, online search and the combination of the two. In their model, the buyer's decisions are solely driven by the expected cost. They derive the conditions under which each strategy outperforms others and show that no strategy is always the best. In their second part, they relax the number of suppliers to be contacted in the online search as a decision variable. A numerical analysis is provided to compare the three alternative strategies.

Wu et al. [58] consider a capital-intensive and non-storable good or service that can be sold by a single seller under option contracts in advance or in a backup spot market. The source of the uncertainty is the spot market price which is assumed to be distributed according to an exogenous distribution. They show that the seller's optimal strategy is to set the strike price at the marginal cost and to extract the margin from the buyer only using option prices. The imperfection of the spot market is modeled by the probability that the seller can successfully sell its residual output in the spot market. They show that if the seller can sell all its residual output in the spot market probability 1, then no contracts are transacted. Wu and Kleindorfer [57] extend the results to a setting with multiple sellers. Existence and structure of market equilibria are characterized.

Deng and Wu [18] extend the two-period model in [58] to continuous time trading. The spot price is assumed to follow a stochastic process. Between the contracting period and the spot market period, the option contracts are traded continuously between the single seller and single buyer. They find that contract market and spot market coexist in this setting.

Spinler et al. [48] extend the results of Wu et al. [58] to a setting where the buyer's willingness to pay (WTP) function depends on the state of the world and find the main results in Wu et al. [58] still hold. They derive the buyer's optimal contracting strategy and the seller's optimal price that reflect the correlation of the buyer's demand and the spot price.

Mendelson and Tunca [42] derive a three-stage model with a single seller and multiple buyers for an intermediate industrial good. The buyers sell the end products in the consumer market. In their model, forward contracts are employed and a spot exchange among the seller and the buyers takes place after the contracting stage. Between the contracting stage and the spot exchange, the seller receives private information of the realization of her costs and the buyers receives a signal of the realization of consumer demand. They find that spot trading reduces prices, increases the quantities produced, and improves supply chain profits and consumer surplus. However, spot trading may make the seller or the buyers worse off. In addition, they find that contracting is persistent. Only when the number of buyers goes to infinity, the contracted quantities converge to zero.

Levi et al. [39] consider a buyer's decision to source intermediate products from multiple sellers either by signing long term contracts or via spot markets. They study the tradeoff between demand uncertainty and additional costs if the parties transact on spot markets. They show that high additional cost in spot markets pushes the buyer to relational long term contracts.

Tunca and Zenios [53] investigate the competition between two procurement mechanisms, long term contracts and online auction that serves the role of spot markets. They study a supply chain where an industrial part with nonverifiable attributes is sold. The procurement of high-quality parts relies on relational contracts whereas the procurement of low-quality parts relies on auctions. Conditions under which the two procurement mechanisms coexist and conditions under which one drives the other out of the market are characterized. They consider a different reason for contracts: Relational contract is incentive to provide higher quality because of promise of future sales.

Erhun et al. [22] develop a two-period model and compare it with two single period models in an environment where a buyer procures capacity from a capacitated seller. Equilibria in all capacity regions are characterized. They investigate the impact of additional information and trading periods on both players' welfare. The supplier's optimal capacity decision is also studied. Their two-period model in specific capacity regions is the same as ours in the case in which 100 % participation in the spot market. Instead of focusing on the impact of additional trading periods, we study the efficiency of the spot markets that is modeled by the buyer's participation rate. We investigate the effects of the buyer's spot market participation rate on all players' welfare.

Our research follows the same framework. Unlike most of the existing literature, we do not take the spot price as given or distributed according to an exogenous distribution but endogenously derive the spot price, which complicates the analysis significantly. Another aspect differs our study from other literature is on the effects of the spot market participation from the buy side. As is present in practice, carriers can sell their remaining capacities on the spot after satisfying service contracts. Thus, the spot market considered in our study is a sell side market. It is common that a large portion of shippers still only use service contracts for a variety of reasons. This motivates us to model the buyers' participation in the spot market. In particular, we study the effects of the buyers' participation on the contract market, on the surpluses of all market players and on the total social welfare.

3.3 Model description

This section gives a brief description of some of the features applied to all models through out this chapter.

We model the contracts in the form of call options that are widely used in financial markets. Call options provide option holders with the right to purchase the underlying securities or products at specified prices. Options have values and are sold at some positive prices. As mentioned before, the reason that we use option contracts instead of other forms of contracts is the special property of most contracts used in freight transportation industries.

It is assumed that the seller has a constant marginal cost c. In the sea cargo industry, it is common that freight contracts always provide lower prices than the published spot rates. Thus, in our model, we set the the strike price equal to the marginal cost and use the option price to exact all the margin. Under this assumption, the buyers always rely on contracts first and use the spot market to purchase extra products if necessary. Similar assumption is used in Burnetas and Ritchken [11]. Wu et al. [58] prove that setting the strike price at the marginal cost is optimal if the spot price is exogenous. It should be pointed out, setting strike price at the marginal cost may not always be optimal to the seller in general if the spot price is endogenous. Relaxing this assumption complicates the analysis significantly and doesn't represent the current practice in the sea cargo industry.

Assumption 3.3.1. *The strike price of the option contract is equal to the marginal cost.*

For the single-seller single-buyer model, the sequence of decisions is as follows. Contracts are signed in Period 1 when the buyer's future demand in Period 2 is not observed. Those contracts provide the buyer with the right to purchase products at a fixed price, which is the marginal cost, in Period 2. Before the beginning of Period 2, the buyer's demand is revealed. Given the number of contracts purchased in Period 1, the buyer decides the quantity to purchase under contracts. In financial economics literature, those contracts are called exercised. If the seller still has remaining capacity and the buyer's policy is to participate in the spot market, the seller sets the spot price. The buyer then decides the additional quantity to purchase from the spot market according to the spot price. In the following context, we also refer Period 1 as the contracting period and Period 2 as the spot market period. The timeline for all the other models is almost the same. For the two-seller model and the model with a continuum of sellers, sellers simultaneously choose the option price and the spot price.

In freight transportation industries, the spot markets are not as well developed as those in some other industries, such as energy and electronic markets. In the sea cargo industry, though some large carriers do support spot transactions as a less important functionality of their internet portals, most shippers still prefer longer term contracts due to various reasons. Incidental spot transactions count only for a small potion of the total transactions. This motivates us to model the spot market participation from buy side. In the models with a continuum of buyers, a fraction λ of the buyers participate in the spot market and the remaining $1-\lambda$ of the buyers do not as a matter of policy. For comparison purpose, we also include the counterpart under the single-buyer market structure with λ interpreted as the buyer's participation probability. With probability λ, the buyer transacts in the spot market. With probability $1-\lambda$, as a matter of policy, the buyer doesn't.

3.4 Single seller, single buyer

This section presents a two-period model with a market where there is a single seller and a single buyer. The sole source of the uncertainty is the buyer's demand that depends on the state of the market in the spot market period. Assume the buyer's normal utility function is quadratic and as follows:

$$U(q) \quad = \quad -\frac{q^2}{2\beta} + \frac{\alpha q}{\beta}$$

where q is the units of products purchased. This utility function is the basis to derive normal demand function. The demand at a given price s is the optimal solution to maximize $U(q) - qs$. Thus, the corresponding normal demand function is $s = (\alpha - q)/\beta$, i.e., $q = \alpha - \beta s$. We assume that the uncertainty in the demand curve is represented by the random variable α having probability distribution P. In particular, P is a Bernoulli distribution,

$$\alpha = \begin{cases} \alpha_h & \text{with probability } p, \\ \alpha_l & \text{with probability } 1-p. \end{cases}$$

To avoid degeneracy, it is assumed that

Assumption 3.4.1. *The parameters satisfy* $\alpha_h > \alpha_l$, $0 < p < 1$.

Denote the expectation of α as $\mathbb{E}(\alpha)$ and the variance of α as σ^2. It holds that

$$\begin{aligned} \mathbb{E}(\alpha) &= p\alpha_h + (1-p)\alpha_l \\ \sigma^2 &= p(1-p)(\alpha_h - \alpha_l)^2 \end{aligned}$$

We also assume that even at the low demand state, the buyer has nonnegative demand if the price is at the marginal cost.

Assumption 3.4.2. *If the demand state is low and the price is at the marginal cost, then the buyer's normal demand satisfies* $q = \alpha_l - \beta c \geq 0$.

As mentioned in the previous section, whether the buyer participates in the spot market or not depends on many factors. Therefore, in this model, we assume the buyer's policy on participating in the spot market is exogenous and is not a consequence of the game. Two cornerstone models of the two policies are presented in this section, which correspond to $\lambda = 0$ and $\lambda = 1$. As a benchmark to the models with a continuum of buyers in the latter sections, we also include the case $\lambda \in (0, 1)$ with λ interpreted as the buyer's spot market participation probablity unrevealed to the seller in Period 1. With probability λ, the buyer transacts in the spot market. With probability $1 - \lambda$, as a matter of policy, the buyer doesn't. At the end of the first period, as the buyer's contracting decision is revealed, the seller can deduce the buyer's policy on the spot market. In this model, we also assume the buyer doesn't take this information asymmetry into consideration because of consistency for comparison. Therefore, the buyer's decision in Period 1 is suboptimal to his problem.

3.4.1 Large capacity case

This subsection considers the case when the seller's capacity is large, i.e., the seller can satisfy any of the buyer's demand. Therefore, the capacity constraint needs not to be considered at all in this case.

This section is organized as follows. First, we consider the case when the buyer only transacts in the contract market, i.e., $\lambda = 0$. Second, we consider the case when the buyer participates in both the contract market and the spot market, i.e., $\lambda = 1$. Third, we consider the case $\lambda \in (0, 1)$. Last, we investigate the effects of the spot market participation on the quantity of contracts transacted, as well as on the surpluses of the seller and the buyer.

3.4.1.1 Contract market only

This subsection considers that the buyer only transacts on the contract market, i.e., $\lambda = 0$. The sequence of the events is as follows. In Period 1, the seller sets the option contract price, denoted as π. The option contracts give the buyer the right to purchase products at strike price c in Period 2. Depending on the option price, the buyer decides the quantity of contracts, Q, to purchase. In Period 2, based on Q and the realization of the demand, the buyer decides how many products to purchase under contracts. There are total three decision stages indexed forward in time. Stage 1 is the seller's decision stage in Period 1. The buyer's decision stage in Period 1 is referred as stage 2 and his decision stage in Period 2 is referred as stage 3. The timeline of this model is shown in Figure 4. We model this situation as a Stackelberg game and use backward induction to characterize the market equilibrium.

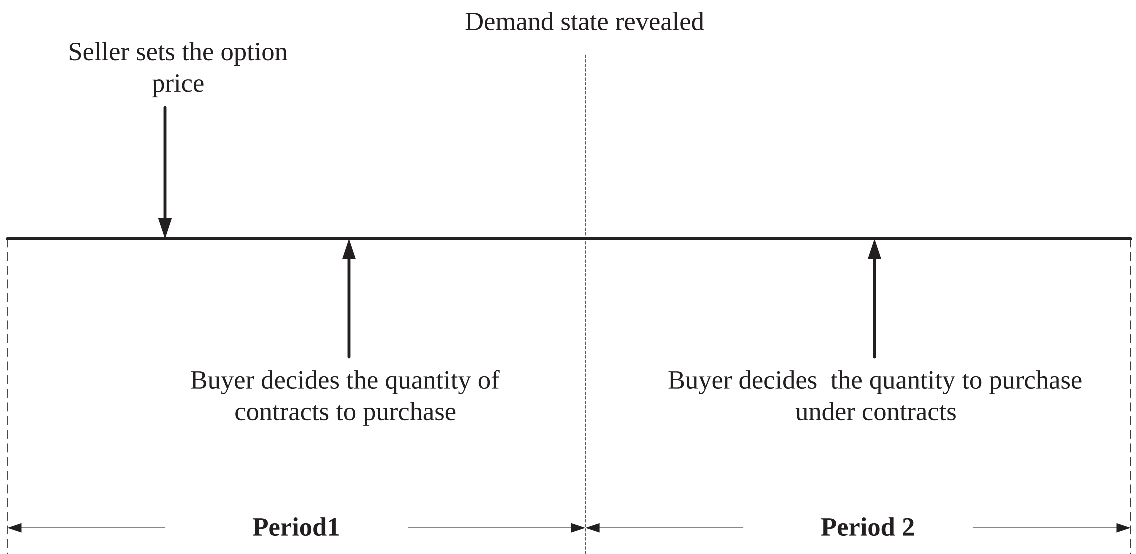

Figure 4: The timeline for the contract market only model

Stage 3 – buyer's problem

Stage 3 is in Period 2. At this stage, the demand is observed. Given the number of contracts bought at Period 1, Q, the buyer decides the actual quantity of products to transact under those contracts. Let q_c be the quantity of products that the buyer decides to purchase under the contracts. Denote the return of the buyer at Period 2 as $r_2(q_c|\alpha)$. The arguments q_c and α indicate that the buyer's return depends on both the buyer's decision q_c and the realization of α. The buyer's problem at this stage is

$$\begin{aligned} g_2(Q,\alpha) \;\; &= \;\; \max_{q_c} \quad r_2(q_c|\alpha) \\ &\quad\;\; \text{s.t.} \quad\;\; 0 \;\leq\; q_c \;\leq\; Q \end{aligned} \tag{3.4.1}$$

where $r_2(q_c|\alpha) = -\frac{q_c^2}{2\beta} + \frac{\alpha q_c}{\beta} - q_c c$. Let q_c^* be the optimal decision for the buyer.

Lemma 3.4.1. *The buyer's optimal decision in Period 2 is as follows:*

1. *If $Q > \alpha - \beta c$, then $q_c^* = \alpha - \beta c$.*

2. *If $Q \leq \alpha - \beta c$, then $q_c^* = Q$.*

The intuition behind Lemma 3.4.1 is clear. In this setting, the buyer only purchases products under contracts that have strike price at c. The quantity of products the buyer can transact is limited by the contracting quantity Q. If there is no such restriction, the buyer's demand is $\alpha - \beta c$. If Q is smaller than that demand, the buyer purchases up to Q. On the other hand, if Q is greater than the demand, the buyer only purchases $\alpha - \beta c$.

Stage 2 – buyer's problem

Stage 2 is in Period 1. At this stage, the option contract price π is given. The buyer anticipates the return in Period 2 and determines how many option contracts to sign. Let Q be the buyer's decision and $r_1(Q|\pi)$ be the buyer's return at this stage. Note we use $g_2(Q, \alpha)$ to denote the buyer's optimal return in Period 2, which depends on Q and the realization of α. The buyer's decision problem is

$$\begin{aligned} g_1(\pi) \quad = \quad & \max_{Q} \quad r_1(Q|\pi) \\ & \text{s.t.} \quad\quad Q \;\geq\; 0 \end{aligned} \tag{3.4.2}$$

where $r_1(Q|\pi) = -\pi Q + \mathbb{E}[g_2(Q, \alpha)] = -\pi Q + p g_2(Q, \alpha_h) + (1 - p) g_2(Q, \alpha_l)$. The first term in $r_1(Q|\pi)$ is the cost of purchasing the contracts. The second term represents the expected optimal return as a function of Q at stage 3. Note that the seller never chooses $\pi < 0$. The buyer's optimal contracting decision Q^* is as follows.

Lemma 3.4.2. *The Buyer's optimal contracting decision $Q^*(\pi)$ is a continuous function of π for any $\pi > 0$ and is as follows:*

1. *If $\pi = 0$, then any $Q \in [\alpha_h - \beta c, +\infty)$ is optimal.*

2. *If* $\pi \in \left(0, \frac{p(\alpha_h - \alpha_l)}{\beta}\right]$*, then* $Q^* = \alpha_h - \beta c - \frac{\beta\pi}{p}$.

3. *If* $\pi \in \left(\frac{p(\alpha_h - \alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha) - \beta c}{\beta}\right]$*, then* $Q^* = \mathbb{E}(\alpha) - \beta(c + \pi)$.

4. *If* $\pi > \frac{\mathbb{E}(\alpha) - \beta c}{\beta}$*, then* $Q^* = 0$.

It should be noted that if $\pi \in (0, p(\alpha_h - \alpha_l)/\beta]$, then $Q^* = \alpha_h - \beta c - \beta\pi/p \in [\alpha_l - \beta c, \alpha_h - \beta c)$. If $\pi \in (p(\alpha_h - \alpha_l)/\beta, (\mathbb{E}(\alpha) - \beta c)/\beta]$, then $Q^* = \mathbb{E}(\alpha) - \beta(c+\pi) \in [0, \alpha_l - \beta c)$. Lemma 3.4.2 indicates that the buyer's optimal decision $Q^*(\pi)$ is a continuous function of π and is unique for any $\pi > 0$. If the option price π is higher than $(\mathbb{E}(\alpha) - \beta c)/\beta$, the buyer never enters contracts. On the other hand, if the price is small enough, i.e., $0 < \pi \leq p(\alpha_h - \alpha_l)/\beta$, buyer's optimal decision Q^* is dominated by the demand from the high state and $Q^* \geq \alpha_l - \beta c$. Thus, the buyer will bear the risk that he may not utilize all the contracts when the demand state turns out to be low. If the price is fairly high, i.e., $p(\alpha_h - \alpha_l)/\beta < \pi \leq (\mathbb{E}(\alpha) - \beta c)/\beta$, the buyer buys a moderate quantity of contracts with $Q^* < \alpha_l - \beta c$. In this case, the buyer uses all the contracts in Period 2 no matter the demand state turns out to be high or low. Note that $Q^*(0)$ is not unique. In this case, the seller doesn't make any profit. Since the seller's sole objective is to maximize her profit, the seller does not chose $\pi = 0$ and this never takes place.

<u>Stage 1 – seller's problem</u>

Stage 1 is in Period 1. Henceforth, we use Q to represent the buyer's optimal contracting decision and omit the superscript "*". Since the strike price is at the marginal cost and there is no transaction on the spot market, the seller's profit is generated only by selling the option contracts. The seller's first stage maximization problem is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) &= \pi Q(\pi) \\ \text{s.t.} \qquad \qquad \pi &\geq 0 \end{aligned} \tag{3.4.3}$$

Denote the optimal option price as π^*.

Theorem 3.4.1. *The seller's optimal decision in Period 1 is as follows:*

1. *If* $\alpha_h - \alpha_l < \frac{\alpha_l - \beta c}{\sqrt{p}}$*, then* $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$.

2. *If* $\alpha_h - \alpha_l > \frac{\alpha_l - \beta c}{\sqrt{p}}$*, then* $\pi^* = \frac{p(\alpha_h-\beta c)}{2\beta}$.

3. *If* $\alpha_h - \alpha_l = \frac{\alpha_l - \beta c}{\sqrt{p}}$*, then both* π_l *and* π_r *are optimal, where* $\pi_l = \frac{p(\alpha_h-\beta c)}{2\beta}$ *and* $\pi_r = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$.

Note that if $\alpha_h - \alpha_l < (\alpha_l - \beta c)/\sqrt{p}$, then $\pi^* \in (p(\alpha_h - \alpha_l)/\beta, (\mathbb{E}(\alpha) - \beta c)/\beta)$ and $Q(\pi^*) \in (0, \alpha_l - \beta c)$. If $\alpha_h - \alpha_l > (\alpha_l - \beta c)/\sqrt{p}$, then $\pi^* \in (0, p(\alpha_h - \alpha_l)/\beta)$ and $Q(\pi^*) \in (\alpha_l - \beta c, \alpha_h - \beta c)$. If $\alpha_h - \alpha_l = (\alpha_l - \beta c)/\sqrt{p}$, then $\pi^* \in \{\pi_l, \pi_r\}$. Note $R_1(\pi_l) = R_1(\pi_r)$, $\pi_l \in (0, p(\alpha_h - \alpha_l)/\beta)$, $Q(\pi_l) \in (\alpha_l - \beta c, \alpha_h - \beta c)$, $\pi_r \in (p(\alpha_h - \alpha_l)/\beta, (\mathbb{E}(\alpha) - \beta c)/\beta)$ and $Q(\pi_r) \in (0, \alpha_l - \beta c)$.

It should be noted that $R_1(\pi)$ is a continuous function of π on $[0, +\infty)$. If $\pi > (\mathbb{E}(\alpha) - \beta c)/\beta$, $R_1(\pi) = 0$. Therefore, $\pi^* \in [0, (\mathbb{E}(\alpha) - \beta c)/\beta]$. The objective function $R_1(\pi)$ is piecewise concave on $[0, p(\alpha_h - \alpha_l)/\beta)$ and $[p(\alpha_h - \alpha_l)/\beta, (\mathbb{E}(\alpha) - \beta c)/\beta]$ respectively. At the breakpoint $\pi = p(\alpha_h - \alpha_l)/\beta$, the right derivative is greater or equal to the left derivative. Whether the optimal solution falls in the first interval or the second interval depends on the parameters in the model. Theorem 3.4.1 explicitly characterizes the market equilibrium under different conditions. If the shift of demand $\alpha_h - \alpha_l$ is bigger than the threshold $(\alpha_l - \beta c)/\sqrt{p}$, the effect of the high demand state dominates. Hence, the seller's optimal decision relies only on the high demand state. By Lemma 3.4.2, at this option price, the buyer purchases a large number of contracts with $Q(\pi^*) > \alpha_l - \beta c$. Not all those contracts are used if the demand turns out to be low in Period 2. If the shift is not large, i.e., $\alpha_h - \alpha_l < (\alpha_l - \beta c)/\sqrt{p}$, the seller's decision reflects both the high demand state and the low demand state. Given this price, the buyer signs a moderate quantity of contracts such that $Q(\pi^*) < \alpha_l - \beta c$.

All of those contracts are used in Period 2 regardless of the state of the market. If $\alpha_h - \alpha_l = (\alpha_l - \beta c)/\sqrt{p}$, the optimal option price is not unique.

3.4.1.2 Contract market and spot market with full participation

This subsection considers the case that buyer's policy is to participate in both markets, i.e., $\lambda = 1$. Similar to the previous case, the sales horizon is divided into two periods, the contracting period (Period 1) and the spot market period (Period 2). The sequence of events is as follows. In the contracting period, the seller determines the option price π with strike price at the marginal cost c. Given this information, the buyer decides the quantity of contracts to purchase Q. Contracting period ends. The demand is realized and the spot market period begins. Based on the number of contracts the buyer reserves, the seller sets the spot price. The buyer determines the quantity of products to purchase under those contracts and the quantity to purchase from the spot market. Let s be the spot price set by the seller in Period 2. Denote the spot price at the high(low) demand state as $s_h(s_l)$. Compared to the contract market only setting, there are total four decision stages. The extra one is the seller's decision stage in Period 2. We also index the four decision stages moving forward in time. Stage 1 refers to the seller's decision stage in Period 1 and stage 2 refers to the buyer's decision stage in Period 1. Stage 3 and stage 4 refer to the decision stages of the seller and the buyer in Period 2 respectively. The timeline of this model is illustrated in Figure 5.

<u>Stage 4 – buyer's problem</u>

At stage 4, the number of contracts purchased in Period 1 Q and the spot price s are given. In addition, the random parameter α in the demand function is realized. Let q_c be the quantity that the buyer decides to purchase under contracts and q_s be the quantity that the buyer decides to purchase from the spot market. Denote the

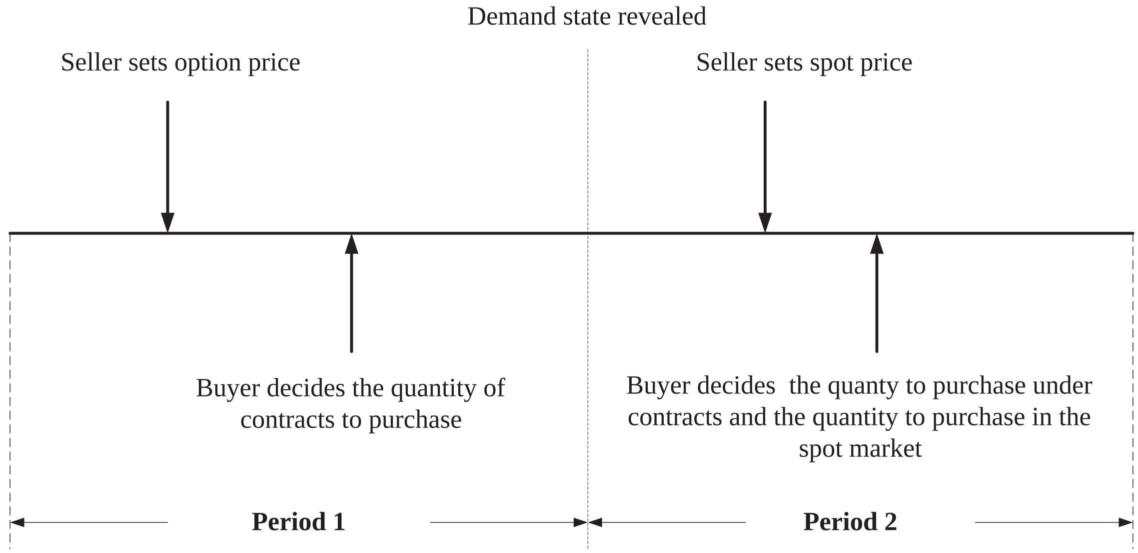

Figure 5: The timeline for the contract and spot market model

buyer's return in Period 2 as $r_2(q_c, q_s|s, \alpha)$. The buyer's problem is

$$\begin{aligned} g_2(Q, s, \alpha) &= \max_{q_c, q_s} \quad r_2(q_c, q_s|s, \alpha) \\ &\quad \text{s.t.} \ \ 0 \ \leq \ q_c \ \leq \ Q \\ &\qquad\qquad q_s \ \geq \ 0 \end{aligned} \tag{3.4.4}$$

where $r_2(q_c, q_s|s, \alpha) = U(q_c + q_s) - q_c c - q_s s = -\frac{(q_c+q_s)^2}{2\beta} + \frac{\alpha(q_c+q_s)}{\beta} - q_c c - q_s s$. It is assumed that the seller never sells at a price less than the marginal cost, i.e., the $s \geq c$. Denote the optimal solution as (q_c^*, q_s^*).

Lemma 3.4.3. *The buyer's optimal decision in Period 2 is as follows:*

1. *If $s = c$, then any q_c^* and q_s^* satisfying the following conditions are optimal:*

$$\begin{aligned} q_c^* + q_s^* &= \alpha - \beta c \\ 0 \leq q_c^* &\leq Q \\ q_s^* &\geq 0 \end{aligned}$$

2. *If $Q \geq \alpha - \beta c$ and $s > c$, then $q_c^* = \alpha - \beta c$ and $q_s^* = 0$.*

3. *If* $Q < \alpha - \beta c$ *and* $s > c$*, then* $q_c^* = Q$ *and* $q_s^* = (\alpha - \beta s - Q)^+$*, where* $(\alpha - \beta s - Q)^+ = \max\{\alpha - \beta s - Q, 0\}$*.*

Lemma 3.4.3 indicates that the buyer's demand in Period 2 is fulfilled by two sources, contracts and the spot market. If $s = c$, the buyer's optimal decision is not unique. In this case, the seller doesn't make any profit on the spot market. If $s > c$, the optimal solution (q_c^*, q_s^*) is unique. If the quantity Q is bigger than the demand at the marginal cost, the buyer only uses contracts to fulfill his demand. If Q is smaller than that threshold, the buyer uses spot market to fulfil his residual demand according to the spot price. The buyer's residual demand function is $D' = (\alpha - \beta s - Q)^+$. If the spot price is less than $(\alpha - Q)/\beta$, the residual demand at s is positive, thus the seller purchases additional products from the spot market. Otherwise, the residual demand is zero and the buyer doesn't purchase from the spot market at all.

Stage 3 – seller's problem

Stage 3 is in Period 2. At this stage, the seller observes the buyer's demand. Based on the number of contracts bought by the buyer Q, the seller sets the spot price s. Denote the seller's revenue in Period 2 as $R_2(s|Q, \alpha)$. Henceforth, we use $q_c(Q, s, \alpha)$ and $q_s(Q, s, \alpha)$ to denote the buyer's optimal response to a given set of (Q, s, α) at stage 4. The problem for the seller is

$$\begin{aligned} G_2(Q, \alpha) &= \max_s \quad R_2(s|Q, \alpha) \\ &\text{s.t.} \quad s \geq c \end{aligned} \tag{3.4.5}$$

where $R_2(s|Q, \alpha) = q_s(Q, s, \alpha)(s - c)$. Denote the optimal spot price as s^*.

Lemma 3.4.4. *The seller's optimal spot price is as follows:*

1. *If* $Q < \alpha - \beta c$*, then* $s^* = (\alpha + \beta c - Q)/(2\beta)$*.*

2. *If* $Q \geq \alpha - \beta c$*, then any* $s \in [c, +\infty)$ *is optimal.*

Note if $Q \geq \alpha - \beta c$, then $R_2(s^*|Q, \alpha) = 0$. The above results are intuitively clear. As the strike price is set at the marginal cost, whether the buyer uses spot market or not depends on the relationship between the number of contracts Q and the demand at the marginal cost $\alpha - \beta c$. By Lemma 3.4.3, if $Q \geq \alpha - \beta c$, the seller's profit from the spot market is zero. Thus, any $s \geq c$ is optimal. If Q is small, i.e., $Q < \alpha - \beta c$, the buyer uses the spot market to fulfil his residual demand. The seller sets spot price to maximize her profit. The buyer's residual demand function on the spot market is $D' = (\alpha - \beta s - Q)^+$. Therefore, the seller sets the spot price at the monopoly price $s = (\alpha + \beta c - Q)/(2\beta)$ with respect to such residual demand function. Note that in this case, the spot price s decreases in Q. Therefore, as long as $Q < \alpha - \beta c$, increasing Q forces the seller to lower the spot price.

Stage 2 – buyer's problem

Stage 2 is in Period 1. At this stage, given the option price π, the buyer anticipates his return in Period 2 and decides how many contracts to purchase. Without confusion, denote the the seller's best response to a given Q as $s(Q, \alpha)$, which also depends on the realization of α. For simplicity, we use the abbreviation s for $s(Q, \alpha)$. Let the buyer's return in Period 1 be $r_1(Q|\pi)$. Thus, the first period problem of the buyer is

$$\begin{aligned} g_1(\pi) \quad = \quad \max_Q \quad r_1(Q|\pi) \quad &= \quad -\pi Q + \mathbb{E}[g_2(Q, s, \alpha)] \\ \text{s.t.} \qquad\qquad\qquad\qquad\quad Q \quad &\geq \quad 0 \end{aligned} \tag{3.4.6}$$

where $\mathbb{E}[g_2(Q, s, \alpha)] = pg_2(Q, s, \alpha_h) + (1 - p)g_2(Q, s, \alpha_l)$. Let the buyer's optimal decision at stage 2 be Q^* .

Lemma 3.4.5. *The buyer's optimal contracting decision $Q^*(\pi)$ is a continuous function of π for any $\pi > 0$ and is as follows:*

1. *If $\pi = 0$, then any $Q \in [\alpha_h - \beta c, +\infty)$ is optimal.*

2. *If* $\pi \in \left(0, \frac{3p(\alpha_h - \alpha_l)}{4\beta}\right]$, *then* $Q^* = \alpha_h - \beta c - \frac{4\beta\pi}{3p}$.

3. *If* $\pi \in \left(\frac{3p(\alpha_h - \alpha_l)}{4\beta}, \frac{3(\mathbb{E}(\alpha) - \beta c)}{4\beta}\right]$, *then* $Q^* = \mathbb{E}(\alpha) - \beta c - \frac{4\beta\pi}{3}$.

4. *If* $\pi > \frac{3(\mathbb{E}(\alpha) - \beta c)}{4\beta}$, *then* $Q^* = 0$.

Note if $\pi \in (0, 3p(\alpha_h - \alpha_l)/(4\beta)]$, then $Q^* \in [\alpha_l - \beta c, \alpha_h - \beta c)$. On the other hand, if $\pi \in (3p(\alpha_h - \alpha_l)/(4\beta),\ 3(\mathbb{E}(\alpha) - \beta c)/(4\beta)]$, then $Q^* \in [0, \alpha_l - \beta c)$. Lemma 3.4.5 shows similar results to Lemma 3.4.2. If π is small, i.e., $0 \leq \pi < 3p(\alpha_h - \alpha_l)/(4\beta)$, the buyer's decision depends only on the high demand state with $Q^* \geq \alpha_l - \beta c$. In this case, not all of the contracts are used if the demand turns out to be low. If π is large, i.e., $3p(\alpha_h - \alpha_l)/(4\beta) \leq \pi \leq 3(\mathbb{E}(\alpha) - \beta c)/(4\beta)$, then the buyer purchases a relatively small quantity of contracts and uses all of them in both demand states. Compared to the results in Lemma 3.4.2, for any given $\pi > 0$, the buyer's response $Q(\pi)$ is smaller.

Stage 1 – seller's problem

Decision stage 1 is in Period 1. At this stage, the buyer's demand is not observed. Anticipating her return in Period 2 and the buyer's response, the seller chooses the option price π to maximize her revenue. There are two parts in her total revenue. One part is from the option contracts and the other part is from the spot market. Let $Q(\pi)$ be the buyer's best response to π. The seller's first stage problem is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) = \quad & \pi Q(\pi) + \mathbb{E}[G_2(Q(\pi), \alpha)] \\ \text{s.t.} \quad & \pi \geq 0 \end{aligned} \tag{3.4.7}$$

where $\mathbb{E}[G_2(Q(\pi), \alpha)] = pG_2(Q(\pi), \alpha_h) + (1-p)G_2(Q(\pi), \alpha_l)$.

It should be noted that function $R_1(\pi)$ is continuous but not concave on $[0, \infty)$, which is similar to that in the contract market only setting. Instead, $R_1(\pi)$ is piecewise concave on $[0, 3p(\alpha_h - \alpha_l)/(4\beta))$ and $[3p(\alpha_h - \alpha_l)/(4\beta), 3(\mathbb{E}(\alpha) - \beta c)/(4\beta)]$. If $\pi > 3(\mathbb{E}(\alpha) - \beta c)/(4\beta)$, by Lemma 3.4.5, $Q(\pi) = 0$, thus $R_1(\pi)$ is constant and doesn't depend on π. In that case, all the revenue is obtained from the spot market. The equilibrium price π^* is explicitly stated in the following Theorem.

Theorem 3.4.2. *The seller's optimal decision in Period 1 is as follows:*

1. *If* $\alpha_h - \alpha_l < \frac{3(\alpha_l-\beta c)}{\sqrt{p}}$, *then* $\pi^* = \frac{9(\mathbb{E}(\alpha)-\beta c)}{16\beta}$.

2. *If* $\alpha_h - \alpha_l > \frac{3(\alpha_l-\beta c)}{\sqrt{p}}$, *then* $\pi^* = \frac{9p(\alpha_h-\beta c)}{16\beta}$.

3. *If* $\alpha_h - \alpha_l = \frac{3(\alpha_l-\beta c)}{\sqrt{p}}$, *then both* π_l *and* π_r *are optimal, where* $\pi_l = \frac{9p(\alpha_h-\beta c)}{16\beta}$ *and* $\pi_r = \frac{9(\mathbb{E}(\alpha)-\beta c)}{16\beta}$.

Note if $\alpha_h - \alpha_l < 3(\alpha_l - \beta c)/\sqrt{p}$, then $\pi^* \in (3p(\alpha_h - \alpha_l)/(4\beta), 3(\mathbb{E}(\alpha) - \beta c)/(4\beta))$ and $Q(\pi^*) \in (0, \alpha_l - \beta c)$. If $\alpha_h - \alpha_l > 3(\alpha_l - \beta c)/\sqrt{p}$, then $\pi^* \in (0, 3p(\alpha_h - \alpha_l)/(4\beta))$ and $Q(\pi^*) \in (\alpha_l - \beta c, \alpha_h - \beta c)$. If $\alpha_h - \alpha_l = 3(\alpha_l - \beta c)/\sqrt{p}$, then $\pi^* \in \{\pi_l, \pi_r\}$. Note $R_1(\pi_l) = R_1(\pi_r)$, $\pi_l \in (0, 3p(\alpha_h - \alpha_l)/(4\beta))$, $Q(\pi_l) \in (\alpha_l - \beta c, \alpha_h - \beta c)$, $\pi_r \in (3p(\alpha_h - \alpha_l)/(4\beta), 3(\mathbb{E}(\alpha) - \beta c)/(4\beta))$ and $Q(\pi_r) \in (0, \alpha_l - \beta c)$

From above results, we can see similar property in the contract market only setting with the threshold for the shift of the demand $\alpha_h - \alpha_l$ three times bigger. For a given set of parameters $\alpha_h, \alpha_l, \beta, c$ and p, the equilibrium price here π^* is bigger than that in the contract market only setting and thereby Q^* is smaller, which indicates the spot market affects the transactions of contracts. Note $Q(\pi^*)$ is always positive, i.e., at market equilibrium, the buyer buys a positive quantity of contracts regardless of the parameter values.

Corollary 3.4.1. *The buyer's contracting decision in equilibrium satisfies* $Q(\pi^*) > 0$.

The intuition behind Corollary 3.4.1 is as follows. In this setting, the seller also has exclusive power on the spot market. Since the the spot price in Period 2 decreases in Q (Lemma 3.4.4), holding a positive number of contracts helps the buyer get lower spot prices. Hence, even the buyer considers to transact on the spot market, the buyer still enters contracts in Period 1.

Corollary 3.4.2. *The relationship of the option price and the spot prices is as follows:*

1. *If* $\pi \in \left[\frac{3p(\alpha_h-\alpha_l)}{4\beta}, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right]$*, then the spot prices in both the high demand state and the low demand state increases as* π *increases.*

2. *If* $\pi \in \left(0, \frac{3p(\alpha_h-\alpha_l)}{4\beta}\right)$*, then the spot price in the high demand state increases as* π *increases and no transaction takes place in the spot market in the low demand state.*

If the option price is large, i.e, $\pi \in \left[\frac{3p(\alpha_h-\alpha_l)}{4\beta}, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right]$, then the buyer's contracting quantity Q is smaller than $\alpha_l - \beta c$. Thereby the buyer will purchase from the spot market in both demand states. An increase in option price leads lower contracting quantity and in turn results in higher spot prices. If the option price is small, i.e., $\pi \in \left(0, \frac{3p(\alpha_h-\alpha_l)}{4\beta}\right)$, the relationship of the option price and the spot price in the high demand state is the same. As in this case $Q > \alpha_l - \beta c$, there is no spot transaction in the low demand state and the corresponding spot price has no practical meanings.

3.4.1.3 Contract market and spot market with partial participation

This section considers a spot market with partial participation, i.e. the buyer participates in the spot market with probability $\lambda \in (0, 1)$ for comparison. In Period 1, the seller sets the option contract price. At this stage, the seller doesn't know whether the buyer transacts in the spot market or not. The seller only knows that the buyer will participate in the spot market with probability λ. If the buyer's policy is to transact in the spot market, his best response is indicated in Lemma 3.4.5. If he doesn't transact in the spot market, his best response is stated in Lemma 3.4.2. Given π, the buyer decides the quantity of contracts to purchase according to his policy on the spot market. Before the beginning of Period 2, the buyer's policy and demand state are revealed. If the buyer's policy is to participate in the spot market, the seller sets the spot price and the buyer makes decision accordingly. If the buyer doesn't purchase in the spot market, he only decides the quantity of products to transact under the contracts he already has. In this section, we focus our study on the role of the spot market. Specifically, we investigate the effects of the participation rate λ on the quantity of contracts purchased and on the surpluses of the seller and the buyer.

Let $Q_A(\pi)$ be the buyer's best response to π if the buyer transacts in the contract market and $Q_A(\pi)$ is already characterized in Section 3.4.1.1. If the buyer's policy is to participate in the spot market, let $Q_B(\pi)$ be the buyer's best response characterized in Section 3.4.1.2. Denote the seller's optimal revenue in Period 2 as $G_2(Q_B(\pi), \alpha)$. For a given $\lambda \in (0, 1)$, The seller's decision problem in Period 1 is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) &= (1-\lambda)\pi Q_A(\pi) + \lambda\{\pi Q_B(\pi) + \mathbb{E}[G_2(Q_B(\pi), \alpha)]\} \\ \text{s.t.} \quad \pi &\geq 0 \end{aligned} \tag{3.4.8}$$

where $\mathbb{E}[G_2(Q_B(\pi), \alpha)] = pG_2(Q_B(\pi), \alpha_h) + (1-p)G_2(Q_B(\pi), \alpha_l)$. The first term in $R_1(\pi)$ is the return in the scenario when the buyer only participates in the contract market. The second term is the return when the buyer participates in both markets. For a given λ, denote the optimal solution to above problem as $\pi^*(\lambda)$. Let $Q^*(\lambda) = (1-\lambda)Q_A(\pi^*(\lambda)) + \lambda Q_B(\pi^*(\lambda))$ be the expected quantity of contracts transacted in equilibrium. Denote the seller's surplus as $G(\lambda)$ and buyer's surplus as $V(\lambda) = g_1(\pi^*(\lambda))$ in equilibrium. Let $W(\lambda) = G(\lambda) + V(\lambda)$ be the total social surplus. We now investigate the effects of spot market participation rate λ on $Q^*(\lambda)$, $G(\lambda)$, $V(\lambda)$ and $W(\lambda)$.

Note that for a fixed λ, since $Q_A(\pi^*) \geq Q_B(\pi^*)$, an increase in λ leads a decrease in Q^* if the equilibrium price π^* doesn't change. However, the equilibrium price π^* also depends on λ. The following Lemma shows that π^* increases in λ.

Lemma 3.4.6. *As λ increases, the seller's optimal option price $\pi^*(\lambda)$ increases.*

Theorem 3.4.3. *(Effect on Contract Market)*

As λ increases, the expected quantity of contracts transacted $Q^*(\lambda)$ decreases.

Theorem 3.4.3 indicates that as λ increases, the expected quantity of contracts bought by the buyer decreases. In other words, as the buyer's spot market participation rate increases, the contract market shrinks. Note that from previous analysis, even if λ increases to 1, Q^* is still positive.

Presumably, spot markets may facilitate better dynamic matching of supply and demand. In spot markets, players can make better use of up-to-date information to adjust their decisions. Especially, we expect that a higher spot market participation rate results in a higher surplus of the seller who is the Stackelberg leader in both markets. Though the buyer also has more information in the spot market, as the seller has exclusive power on both the contract market and the spot market, it is unclear how the participation rate affects the buyer's surplus.

Theorem 3.4.4. *(Effect on Seller's Surplus)*

The seller's surplus $G(\lambda)$ increases as λ increases.

Theorem 3.4.4 states that an increase in λ always benefits the seller. The rationale behind this theorem is that as λ increases, the seller has more flexibility by controlling both the option price and the spot prices. As the buyer is more willing to transact on the spot market, the seller can make better decision with dynamically updated information.

In contrast to the monotonicity shown above, the impact of λ on the buyer's surplus and on the total social surplus is indeterminate, depending on the parameters in the model.

Theorem 3.4.5. *(Effects on Buyer's Surplus and Total Social Surplus)*

The effects of the buyer's participation rate λ on the buyer's total surplus and on the total social welfare are as follows.

1. *If $\alpha_h - \alpha_l \geq 3(\alpha_l - \beta c)/p$, then the buyer's surplus $V(\lambda)$ and the total social surplus $W(\lambda)$ increase as λ increases.*

2. *If $\alpha_h - \alpha_l < 3(\alpha_l - \beta c)/p$, then the buyer's surplus $V(\lambda)$ and the total social surplus $W(\lambda)$ may increase or decrease as λ increases.*

Theorem 3.4.5 states in a market with a monopolist seller, spot market participation may or may not benefit the buyers, therefore may or may not increase the total social surplus. The total social surplus is equal to the buyer's expected utility minus the manufacturing cost. Since the seller has exclusive market power on the spot market, she sets the spot price at the monopoly price with respect to the residual demand, which is relatively high. As the buyer is more willing to transact in the spot market, the total quantity of products transacted might decrease. Therefore, the total social surplus might decrease. Example 1 and 2 show that as λ increases, $V(\lambda)$ and $W(\lambda)$ may increase or decrease. Example 3 indicates that even if $\lambda = 1$, the total social surplus and the buyer's surplus may be less than those in the contract market only setting $\lambda = 0$. In Example 2 and 3, the contracting quantity $Q^*(\lambda)$, the buyer's surplus $V(\lambda)$ and the total social surplus $W(\lambda)$ are discontinuous at two values of λ. This is because that the seller's objective function is piecewise concave in π. As λ increases, the optimal option price increases and can jump from one interval to another on the right. At such jump, the seller is indifferent in charging two different option prices that result in the same profit. Those two prices are the two local maximizers in two different intervals. Though both of the prices lead to the same profit to the seller, the buyer's surplus and the total social surplus are different. If the seller chooses the lower price, more option contracts are sold and the residual demand in the spot market is smaller. Thus, the seller charges a lower spot price with respect to such residual demand. In this case, the seller sells more products overall. Therefore, the total social surplus is higher and the buyer is also better off. If the seller chooses the higher option price, the buyer is worse off and the total social surplus becomes lower. As λ keeps increasing from such value, the optimal price stays in the same interval and the buyer's surplus and total social surplus become unique again.

Example 1: $\alpha_h = 8$, $\alpha_l = 4$, $\beta = 1$, $c = 1$, and $p = 0.5$.

Example 2: $\alpha_h = 12$, $\alpha_l = 9$, $\beta = 2$, $c = 4$, and $p = 0.3$.

Example 3: $\alpha_h = 5.2$, $\alpha_l = 2$, $\beta = 1$, $c = 1$, and $p = 0.1$.

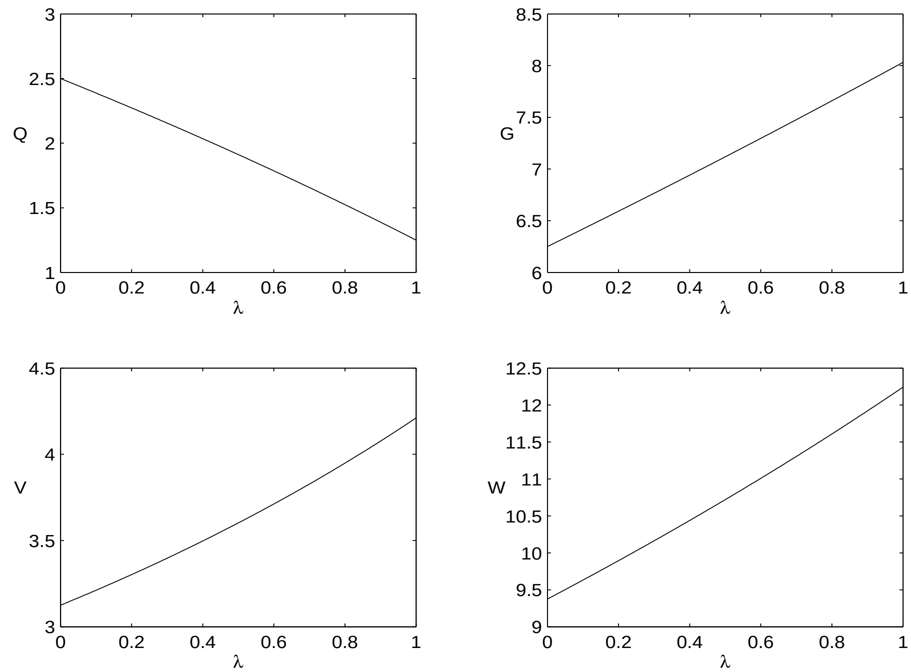

Figure 6: Effects of spot market participation for Example 1

In the models discussed in this section, the seller has large capacity. Denote the seller's capacity as C. It is easy to show that all the results in this section hold if $C \geq \alpha_h - \beta c$. As the capacity decreases, the analysis of the market equilibrium divides into a number of cases depending on the capacity level. To investigate the impact of the capacity factor, we limit our study in the next section to another extreme case where the capacity is very small, which we expect the the impact is the strongest. We investigate how the results under this condition deviate from those in the large (unlimited) capacity case.

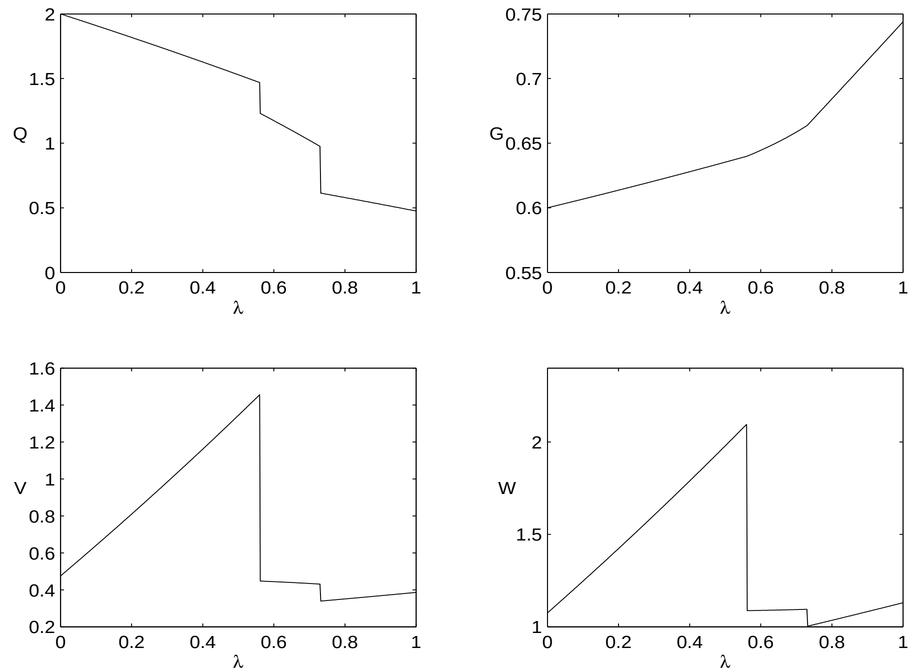

Figure 7: Effects of spot market participation for Example 2

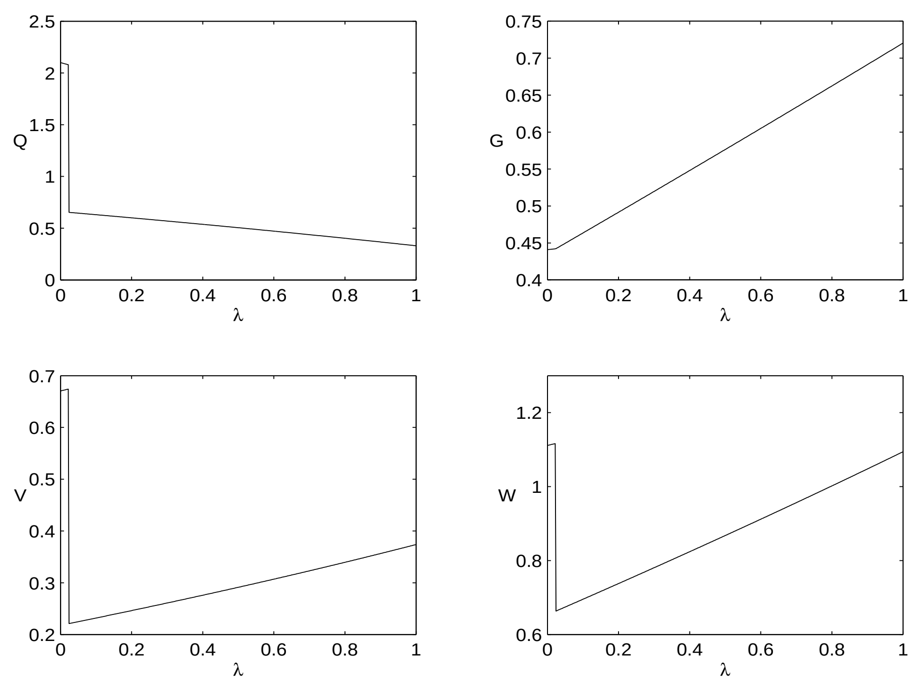

Figure 8: Effects of spot market participation for Example 3

3.4.2 Small capacity case

We still use the same notation as before. It is assumed that the capacity level C is common knowledge. Assume that the number of contracts sold by the seller in Period 1 can not exceed the capacity, i.e., $Q \leq C$. As the models are very similar to those in the previous section, we only present the last model for the case $\lambda \in (0,1)$.

Similar to Section 3.4.1.3, the seller's stage 1 problem is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) &= (1-\lambda)\pi Q_A(\pi) + \lambda\{\pi Q_B(\pi) + \mathbb{E}[G_2(Q_B(\pi),\alpha)]\} \\ \text{s.t.} \quad Q_A(\pi) &\leq C \\ Q_B(\pi) &\leq C \\ \pi &\geq 0 \end{aligned} \tag{3.4.9}$$

where $\mathbb{E}[G_2(Q_B(\pi),\alpha)] = pG_2(Q_B(\pi),\alpha_h) + (1-p)G_2(Q_B(\pi),\alpha_l)$.

Let $C_a = (\alpha_l - \beta c)/2$. For any $C \leq C_a$, Lemma 3.4.7, Theorem 3.4.6, Theorem 3.4.7 and Theorem 3.4.8 hold. The effect of the spot market participation rate on the quantity of contracts transacted is characterized in Theorem 3.4.6. The effects of the spot market participation rate on the seller's surplus, on the buyer's surplus and on the total social surplus are characterized in Theorem 3.4.7 and Theorem 3.4.8.

Lemma 3.4.7. *The seller's optimal decision in Period 1 satisfies* $\pi^* = (\mathbb{E}(\alpha) - \beta c - C)/\beta$ *for all* $\lambda \in [0,1]$.

Theorem 3.4.6. *(Effect on Contract Market)*

The expected number of contracts transacted $Q^*(\lambda)$ *decreases as* λ *increases.*

Theorem 3.4.7. *(Effect on Seller's Surplus)*

The seller's surplus $G(\lambda)$ *is constant for any* $\lambda \in [0,1]$, $G(\lambda) = \frac{C(\mathbb{E}(\alpha)-\beta c-C)}{\beta}$.

Theorem 3.4.8. *(Effects on Buyer's Surplus and Total Social Surplus)*

The total social surplus $W(\lambda)$ *and the buyer's surplus* $V(\lambda)$ *are constant for any* $\lambda \in [0,1]$, $V(\lambda) = \frac{C^2}{2\beta}$ *and* $W(\lambda) = \frac{C(\mathbb{E}(\alpha)-\beta c-C/2)}{\beta}$.

Theorem 3.4.6 states the same property as that in the uncapacitated case: the expected quantity of contracts transacted decreases in λ. The different results are Theorem 3.4.7 and Theorem 3.4.8. They state that as λ increases, both the seller's surplus and the buyer's surplus remain constant. In other words, if the capacity is small enough such that $C \leq C_a$, the participation of the spot market doesn't affect the surpluses of the market participants at all. Therefore, both the seller and the buyer are indifferent to transact in the spot market or under contracts. From Theorem 3.4.7 and Theorem 3.4.8, we can see the seller always extract more surplus than the buyer. Since $C \leq C_a$, the ratio of the seller's expected surplus to the buyer's surplus is $2(\mathbb{E}(\alpha) - \beta c - C)/C \geq 2$.

Corollary 3.4.3. *For $\lambda \in (0, 1]$, the relationship of the option price and the spot prices is as follows:*

1. *For $\pi > (\mathbb{E}(\alpha) - \beta c - C)/\beta$, the expected contracting quantity $Q(\pi) < C$ and the spot prices in the subgame in both the high demand state and the low demand state are constant for any π, $s_h = (\alpha_h - C)/\beta$ and $s_l = (\alpha_l - C)/\beta$.*

2. *For $\pi = (\mathbb{E}(\alpha) - \beta c - C)/\beta$, $Q_A(\pi) = C$ and $Q_B(\pi)$ can be any value in $[0, C]$. If $Q_B(\pi) < C$, then the spot prices in the subgame in both the high demand state and the low demand state are constant for any π, $s_h = (\alpha_h - C)/\beta$ and $s_l = (\alpha_l - C)/\beta$. Otherwise, no transaction takes place in the spot market.*

3. *For $\pi < (\mathbb{E}(\alpha) - \beta c - C)/\beta$, $Q(\pi) = C$ and no transaction takes place in the spot market.*

Compared to Corollary 3.4.2, Corollary 3.4.3 says whenever there is remaining capacity on the spot market, the seller always sets the spot price at $(\alpha - C)/\beta$ such that all the capacity is sold.

3.5 Single seller, a continuum of buyers

This section extends the single buyer model to a setting with a continuum of buyers indexed by $\mathcal{B}^\infty \equiv [0, N]$. In this setting, there are a "very large" number of buyers such that a single buyer's effect is "infinitesimal" relative to the market as a whole.

The sequence of the decisions is almost the same as that in the single-seller single-buyer model except that all buyers move simultaneously at the buyer's decision stages in Period 1 and Period 2.

A single small buyer's normal utility function is assumed to be quadratic as in Section 3.4. It is assumed that every buyer's realization α in Period 2 is the same, which only depends on the state of the market. Thus, a single buyer's demand at price s is

$$q(s)d\mu = (\alpha - \beta s)d\mu \tag{3.5.1}$$

where $d\mu$ represents a single buyer's mass. The aggregated demand is the integral over the whole population of the buyers.

$$\int_0^N q(s)d\mu = N(\alpha - \beta s) \tag{3.5.2}$$

Assumption 3.3.1, 3.4.1 and 3.4.2 are also applied to this section.

The spot market participation is modeled by the buyers' participation rate, λ. A fraction λ of buyers participate in both the contract market and the spot market. The remaining $1 - \lambda$ buyers only enter contracts according to their policy. If $\lambda = 0$, all buyers only enter contracts. If $\lambda = 1$, all buyers use both contracts and the spot market.

In this section, the formulation of the model is very similar to Section 3.4. The same notation is used in this section. The buyers' decision variables q_c and q_s refer to the quantity of contracts to exercise and the quantity to purchase from the spot market for per unit of buyers. Similarly, Q represents the contracting quantity per unit of buyers in Period 1. The organization of this section is the same as Section

3.4. First, we study the large capacity case. Then, we consider the case with small capacity.

3.5.1 Large capacity case

This subsection considers the case with large capacity. In all the analysis, the capacity constraint needs not be considered.

3.5.1.1 Contract market only

All the results in this subsection can be derived directly from Section 3.4.1.1. A single buyer's decision problems at stage 2 and stage 3 are the same as those in Section 3.4.1.1. The results of Lemma 3.4.1 and Lemma 3.4.2 hold.

The seller's return at the first stage is the profit from selling option contracts to all buyers. Since every small buyer has the same utility in Period 2, all buyers have the same decision problems in both periods. Therefore, the best contracting quantity is the same for every buyer. Denote the best response from a single buyer as $Q(\pi)$ for a given π. Thus, the seller's maximization problem in Period 1 is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) \quad &= \quad \int_0^N \pi Q(\pi) d\mu \\ \text{s.t.} \qquad\qquad \pi \quad &\geq \quad 0 \end{aligned} \tag{3.5.3}$$

Note that $\int_0^N \pi Q(\pi) d\mu = N\pi Q(\pi)$. The above problem is basically the same as the seller's problem in Section 3.4.1.1. Hence, Theorem 3.4.1 follows. This subsection says that the seller facing a continuum of buyers is essentially the same as the seller facing one big aggregated buyer. Since the seller has no action in Period 2, a single small buyer's stage 2 decision is exactly the same as that in the single buyer setting indicated in Lemma 3.4.2. In the next subsection, we will show that this doesn't hold anymore when the buyers also participate in the spot market.

3.5.1.2 Contract market and spot market with full participation

This subsection considers a spot market with $\lambda = 1$. In addition to longer term contracts, all buyers participate in the spot market. The sequence of the events is as follows. In Period 1, the seller sets the option contract price. Each buyer decides how many contracts to purchase simultaneously. The demand is revealed. The seller sets the spot price. Again, the buyers simultaneously decide the quantity to transact under contracts and the quantity to transact via spot market.

At stage 4, a single buyer's problem is the same as that in Section 3.4.1.2. The optimal quantity of contracts to exercise q_c^* and the optimal additional quantity to purchase on the spot market q_s^* per unit of buyers are characterized in Lemma 3.4.3.

Decision stage 3 is in Period 2. At this stage, the quantity of contracts sold to the buyers in Period 1 is given. Also, each buyer's demand is revealed. Let $q_c(Q, s, \alpha)$ and $q_s(Q, s, \alpha)$ be per buyer's best response in stage 4. The seller sets the spot price to maximize her profit from spot market. The maximization problem is

$$\begin{aligned} G_2(Q, \alpha) \quad = \quad \max_{s} \quad & R_2(s|Q, \alpha) \\ \text{s.t.} \quad & s \;\geq\; c \end{aligned} \tag{3.5.4}$$

where $R_2(s|Q, \alpha) = (s - c)\int_0^N q_s(s, Q, \alpha)d\mu = N(s - c)q_s(s, Q, \alpha)$. Note this problem is essentially the same as problem (3.4.5). Therefore, the results in Lemma 3.4.4 hold.

In Period 1, given the option price π, each buyer anticipates his return in Period 2 and decides how many contracts to enter. As a single buyer's influence is "negligible", each buyer doesn't consider the spot price in Period 2 as an outcome of his decision Q. Therefore, each buyer takes the spot price as given. Let $r_1(Q|\pi)$ be a single buyer's optimal return in Period 1. Each buyer's problem at this stage is

$$\begin{aligned} g_1(\pi) \quad = \quad \max_{Q} \quad & r_1(Q|\pi) \\ \text{s.t.} \quad & Q \geq 0 \end{aligned} \tag{3.5.5}$$

where $r_1(Q, \pi) = -\pi Q + \mathbb{E}[g_2(Q, s, \alpha)] = -\pi Q + pg_2(Q, s, \alpha_h) + (1 - p)g_2(Q, s, \alpha_l)$.

Lemma 3.5.1. *The buyer's optimal contracting decision $Q^*(\pi)$ is a continuous function of π for any $\pi > 0$ and is as follows:*

1. *If $\pi = 0$, then any $Q \in [\alpha_h - \beta c, +\infty)$ is optimal.*
2. *If $\pi \in \left(0, \frac{p(\alpha_h - \alpha_l)}{2\beta}\right]$, then $Q^* = \alpha_h - \beta c - 2\beta\pi/p$.*
3. *If $\pi \in \left(\frac{p(\alpha_h - \alpha_l)}{2\beta}, \frac{\mathbb{E}(\alpha) - \beta c}{2\beta}\right]$, then $Q^* = \mathbb{E}(\alpha) - \beta c - 2\beta\pi$.*
4. *If $\pi > \frac{\mathbb{E}(\alpha) - \beta c}{2\beta}$, then $Q^* = 0$.*

Note that if $\pi \in \left(0, \frac{p(\alpha_h - \alpha_l)}{2\beta}\right]$, then $Q^* \in [\alpha_l - \beta c, \alpha_h - \beta c)$. If $\pi \in \left(\frac{p(\alpha_h - \alpha_l)}{2\beta}, \frac{\mathbb{E}(\alpha) - \beta c}{2\beta}\right]$, then $Q^* \in [0, \alpha_l - \beta c)$. Comparing the results in Lemma 3.5.1 to Lemma 3.4.5, for a given option price π, the optimal Q^* in this setting is smaller. The intuition is as follows. Lemma 3.4.4 shows that an increase in Q leads the seller to lower the spot price in the second period as long as $Q \leq \alpha - \beta c$. When there is a continuum of buyers, each small buyer is price taker of the spot prices and thereby it has no impact on the buyers first period decision. However, in the single buyer setting, the monopolist buyer considers this effect, which results in higher contracting quantity.

Anticipating buyers' decisions and her own response in spot market, the seller determines the optimal option price π^* to maximize her profit. Denote a single buyer's best response to a given π as $Q(\pi)$. Let $R_1(\pi)$ be the seller's revenue in Period 1. The seller's decision problem at this stage is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) \quad &= \int_0^N \pi Q(\pi) d\mu + \mathbb{E}[G_2(Q(\pi), \alpha)] \\ \text{s.t.} \qquad\qquad \pi \quad &\geq \quad 0 \end{aligned} \tag{3.5.6}$$

where $\mathbb{E}[G_2(Q(\pi), \alpha)] = pG_2(Q(\pi), \alpha_h) + (1-p)G_2(Q(\pi), \alpha_l)$ and $\int_0^N \pi Q(\pi) d\mu = N\pi Q(\pi)$.

Theorem 3.5.1. *Any option price $\pi \in \left[\frac{\mathbb{E}(\alpha) - \beta c}{2\beta}, \infty\right)$ is optimal to the seller.*

Note that $Q(\pi^*) = 0$. Lemma 3.5.1 states in equilibrium, the seller sets the option price high enough such that no buyer is willing to enter contracts. Therefore, there is no contract market at all when the spot market participation is 1. It is different from the single buyer setting. In that case, even $\lambda = 1$, there still exists a positive quantity of transacted contracts. The relationship of π and the spot prices has the same property as indicated by Corollary 3.4.2 with the two intervals for π become $\left[\frac{p(\alpha_h-\alpha_l)}{2\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}\right]$ and $\left(0, \frac{p(\alpha_h-\alpha_l)}{2\beta}\right)$.

3.5.1.3 Contract market and spot market with partial participation

Based on the above two models, this section investigates the effects of participation rate λ on the quantity of transacted contracts, on the seller's surplus and on the buyers' total surplus. In Period 1, the seller sets the option price, knowing that only a fraction λ of buyers transact on the spot market and the remaining $1 - \lambda$ buyers don't. Given the price, all buyers decide how many contracts to purchase. The demand state is then revealed. In Period 2, the seller sets the spot price. The buyers who do not participate in the spot market only decide the quantity of products to purchase under contracts. The buyers who consider to transact on the spot market decide how many to purchase under contracts and how many to purchase from the spot market. Denote the former buyers as type "A" buyers and the latter buyers as type "B" buyers. For a given option contract price π, let a single type "A" buyer's best response be $Q_A(\pi)$, which is characterized in Lemma 3.4.2; let a single type "B" buyer's best response be $Q_B(\pi)$, which is characterized in Lemma 3.5.1. Denote the seller's optimal aggregated revenue in Period 2 as $G_2(Q_B(\pi), \alpha)$. The seller's problem at stage 1 is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) &= N[(1-\lambda)\pi Q_A(\pi) + \lambda\pi Q_B(\pi)] + \lambda\mathbb{E}(G_2(Q_B(\pi), \alpha)) \\ \text{s.t.} \quad \pi &\geq 0 \end{aligned} \tag{3.5.7}$$

For a given $\lambda \in [0, 1]$, let $\pi^*(\lambda)$ be the optimal solution to the above decision problem. Note that when $\lambda = 0$, the optimal option price $\pi^*(\lambda)$ has been characterized in Theorem 3.4.1; when $\lambda = 1$, the optimal option price $\pi^*(\lambda)$ is indicated in Theorem 3.5.1. Let $Q^*(\lambda)$ be the total quantity of transacted contracts in equilibrium, i.e.,

$$Q^*(\lambda) \quad = \quad N[(1-\lambda)Q_A(\pi^*) + \lambda Q_B(\pi^*)] \tag{3.5.8}$$

In the following part, we investigate how the spot market participation rate λ affects the total contracts transacted $Q^*(\lambda)$, the seller's surplus $G(\lambda)$, the buyers' total surplus $V(\lambda)$ and the total social surplus $W(\lambda)$.

Lemma 3.5.2. *As λ increases, the optimal option price $\pi^*(\lambda)$ increases.*

Theorem 3.5.2. *(Effect on Contract Market)*

The total quantity of contracts transacted $Q^*(\lambda)$ decreases as λ increases.

Theorem 3.5.2 states that as more and more buyers participate in the spot market, i.e., λ increases, the total number of contracts transacted Q^* decreases. When λ increases to 1, Q^* decreases to 0. The rationale behind this is as follows. The buyers who consider to transact on the spot market are more willing to postpone their decision to Period 2 after their demand is realized. Though a decrease in Q allows the seller to increase the spot prices, each small buyer who is price-taker and doesn't take this effect into consideration. Therefore, when all buyers transact on the spot market, i.e. $\lambda = 1$, Q^* decreases to 0.

Theorem 3.5.3. *(Effect on Seller's Surplus)*

The seller's surplus $G(\lambda)$ increases in as λ increases.

Theorem 3.5.3 indicates that an increase in the participation rate λ always benefits the seller. This is consistent with the results indicated in Theorem 3.4.4 in the single buyer setting.

Theorem 3.5.4. *(Effects on Buyers' Total Surplus and Total Social Surplus)*

The effects of the buyers's participation rate λ on the buyers' total surplus and on the total social welfare are as follows.

1. *If $\alpha_h - \alpha_l \geq \frac{\alpha_l - \beta c}{p}$, both the buyers' total surplus $V(\lambda)$ and total social surplus $W(\lambda)$ decrease as λ increases.*

2. *If $\alpha_h - \alpha_l < \frac{\alpha_l - \beta c}{\sqrt{p}}$, both the buyers' total surplus $V(\lambda)$ and total social surplus $W(\lambda)$ increase as λ increases.*

3. *If $\frac{\alpha_l - \beta c}{\sqrt{p}} < \alpha_h - \alpha_l < \frac{\alpha_l - \beta c}{p}$, the buyers' total surplus $V(\lambda)$ and total social surplus $W(\lambda)$ may increase or decrease as λ increases.*

Theorem 3.5.4 indicates similar results in the single buyer setting. An increase in the spot market participation rate λ may or may not increase the buyers' total surplus, thereby may or may not improve the total social surplus. Only when the demand variation is small, an increase in λ benefits the buyers. Compared to Theorem 3.4.5, the thresholds of the demand variation are different due to the different market structures. In the single buyer case, the buyer has more market power. However in the many-buyer case, each small buyer has very small influence.

The effects of the participation rate on Q, G, V and W are illustrated in Example 4 -6.

Example 4: $\alpha_h = 12$, $\alpha_l = 4$, $\beta = 1$, $c = 1$, and $p = 0.5$. In this example $\alpha_h - \alpha_l \geq \frac{\alpha_l - \beta c}{p}$, both $V(\lambda)$ and $W(\lambda)$ decrease in λ.

Example 5: $\alpha_h = 7$, $\alpha_l = 4$, $\beta = 1$, $c = 1$, and $p = 0.5$. In this example $\alpha_h - \alpha_l < \frac{\alpha_l - \beta c}{\sqrt{p}}$, both $V(\lambda)$ and $W(\lambda)$ increase in λ.

Example 6: $\alpha_h = 9$, $\alpha_l = 4$, $\beta = 1$, $c = 1$, and $p = 0.5$. In this example, $\frac{\alpha_l - \beta c}{\sqrt{p}} < \alpha_h - \alpha_l < \frac{\alpha_l - \beta c}{p}$, both $V(\lambda)$ and $W(\lambda)$ may increase or decrease in λ

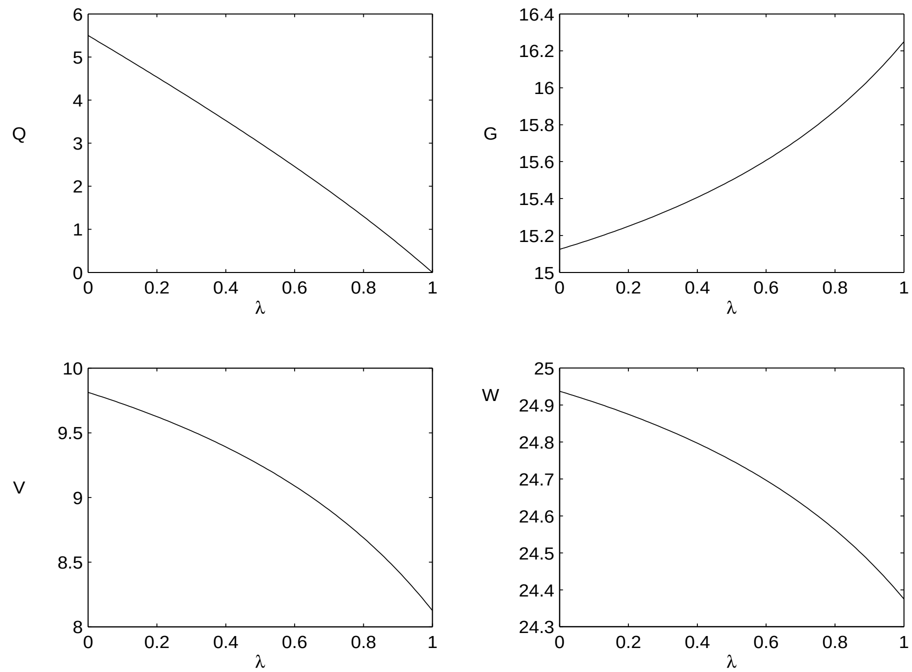

Figure 9: Effects of spot market participation for Example 4

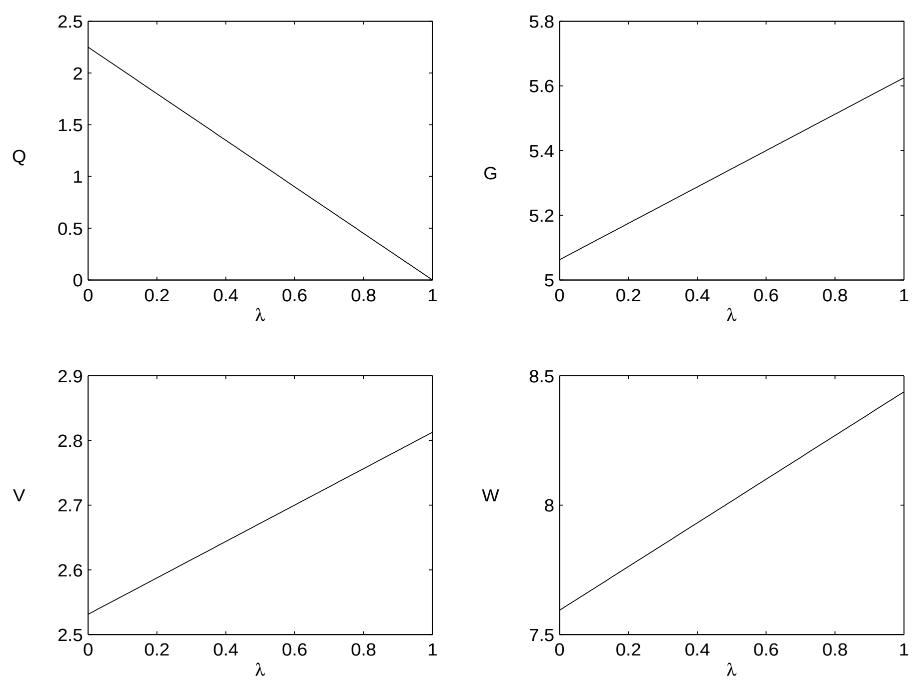

Figure 10: Effects of spot market participation for Example 5

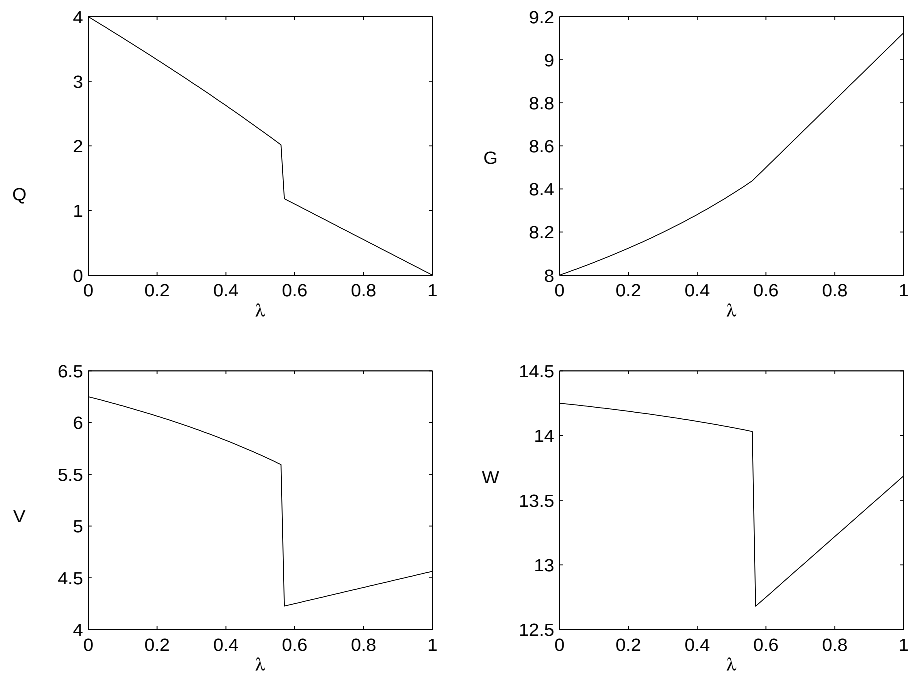

Figure 11: Effects of spot market participation for Example 6

3.5.2 Small capacity case

Similar to Section 3.4.2, this section considers another extreme case when the capacity is very small such that the spot prices are solely determined by the capacity in Period 2. We also assume the quantity of available contracts equal to the capacity.

If all buyers only transact under longer term contracts, i.e., $\lambda = 0$, the decision problems at stage 2 and stage 3 are exactly the same as those in the large capacity case. We will show that at stage 1, the seller's optimal decision is to set the option price such that all the contracts sold.

However, for the case $\lambda \in (0, 1]$, additional assumptions are needed to make our model complete. Consider the following situation. For some option price π, the aggregated contracting quantity submitted by all buyers exceeds the available contracts. How to allocate the limited contracts among the buyers will affect the spot prices in the later stage and thereby will influence the contracting decision for those buyers participating in the spot market.

There are different ways used in practice when the aggregated demand is larger than the capacity. One common rationing scheme is first-come first-serve. The seller can also restrict each buyer's contracting quantity no larger than the capacity divided by the market size. Another allocation rule used in practice is to apply the same fraction to all buyers. Different allocation rules eventually affect the seller's total revenue. The first rationing scheme is natural. Suppose the seller chooses the first-come first-serve rationing rule, only a fraction of the buyers' contracting decision can be satisfied if the total demand exceeds the capacity. From a single buyer's point view, his request is fulfilled with some probability in this situation. Since a buyer is an infinitesimal relative to the market as a whole, he will take this probability as given. Therefore, each buyer will honestly submit the quantity to optimize his return if his demand is satisfied. The second rule is easy to applied but has flaws. If all buyers are identical, then it makes sense to do so. However, in our case $\lambda \in (0, 1)$, applying the same restriction to the two different types of buyers seems unsatisfactory. Though the third rule is also used in practice, it also has some drawbacks. The third rule is to apply the same fraction to all buyers. If this is the case, buyers will ask more than what they really want and thereby inflate the market. It should be pointed out that all the allocation rules may not be optimal to the seller. Allowing the allocation rule as the seller's decision complicates the problem significantly. In this section, we only study the first two rules.

It should be noted that in the case $\lambda \in (0, 1)$, the capacity constraint of the seller's problem in stage 3 makes the problem much more complicated no matter which allocation rule is used. The surprising result is that in equilibrium, the seller's optimal option price as well as the effects of λ on contracting quantity, the seller's surplus and the buyers' total surplus are the same under both rationing rules.

This section is organized as follows. We start with the first rationing scheme, first-come first-serve. Under this assumption, we study the case in which all buyers participate in the spot market, i.e., $\lambda = 1$. Then we investigate the case in which all the buyers only participate in the contract market, i.e., $\lambda = 0$. Finally, we consider part of buyers only participate in the contract market and part of the buyers participate in both market, $0 < \lambda < 1$. The second rationing scheme is also discussed. Both rationing schemes reach the same results. Since the models and proofs under the latter assumption are very similar to the first, we will only highlight the differences.

3.5.2.1 Rationing scheme 1 – first-come first-serve

A. Contract market and spot market with full participation

The decision problem for the buyers at stage 4 is the same as that in the large capacity case.

If the aggregated contracting quantity QN is larger than C, the seller can only satisfy a fraction γ of the buyers, where $\gamma = C/(QN)$. If $QN \leq C$, then all the buyers' demand is satisfied, i.e., $\gamma = 1$. Let $q_c(Q)$ denote the quantity of contracts to exercise in the spot market for the buyers having Q contracts. For those buyers, let $q_s(Q, s)$ denote the extra quantity the buyers decide to purchase from the spot market at spot price s. For the buyers without any contract, let $q_s(0, s)$ denote the quantity they determine to purchase from the spot market at spot price s. The seller's problem at stage 3 is as follows.

$$\begin{aligned} G_2(Q, \alpha) &= \max_s \quad R_2(s|Q, \alpha) \\ &\quad \text{s.t.} \quad N\left[\gamma(q_c(Q) + q_s(Q, s)) + (1-\gamma)q_s(0, s)\right] \leq C \end{aligned} \tag{3.5.9}$$

where $R_2(s|Q, \alpha) = (s - c)N\left[\gamma q_s(Q, s) + (1-\gamma)q_s(0, s)\right]$. If $\gamma = 1$, the problem is reduced to

$$\begin{aligned} G_2(Q, \alpha) &= \max_s \quad (s-c)q_s(Q, s)N \\ &\quad \text{s.t.} \quad N\left[q_c(Q) + q_s(Q, s)\right] \leq C \end{aligned} \tag{3.5.10}$$

Lemma 3.5.3. *If the seller's capacity $C \le NC_a = N(\alpha_l - \beta c)/2$, then the optimal spot prices are as follows.*

1. *For $Q \in [0, C/N]$, the optimal spot price in the high demand state is $s_h^* = \frac{\alpha_h - C/N}{\beta}$ and the optimal price in the low demand state is $s_l^* = \frac{\alpha_l - C/N}{\beta}$.*

2. *For $Q \in (C/N, \alpha_l - \beta c]$, $\gamma < 1$ and all the products are sold under contracts in both demand states. No transaction takes place on the spot market.*

3. *For $Q \in (\alpha_l - \beta c, \alpha_h - \beta c]$, $\gamma < 1$, all the products are sold only under contracts in the high demand state. In the low demand state, the optimal price $s_l^* = \frac{\gamma(\alpha_l - \beta c) + (1-\gamma)\alpha_l - C/N}{\beta(1-\gamma)}$.*

Lemma 3.5.3 states if each buyer's contracting quantity $Q \le C/N$, then the decision problem for the seller in Period 2 is essentially the same as facing an aggregated big buyer with size N. If $Q \in (C/N, \alpha_l - \beta c]$, all the products are consumed by the buyers under contracts in both demand states. If $Q \in (\alpha_l - \beta c, \alpha_h - \beta c]$, fraction γ of all buyers only exercise part of their contracts ($\alpha_l - \beta c$ out of Q) in the low demand state. The remaining capacity, $C - \gamma N(\alpha_l - \beta c)$, is consumed by the other part of buyers ($1 - \gamma$ of all buyers) having no contract. It should be noted that in this case, the seller sets the spot price s_l^* at which all the remaining capacity is sold.

A single buyer is very small comparing to the population of all buyers. The influence of an individual buyer's decision is negligible and doesn't affect the spot prices as well as the probability with which his contracting demand to be satisfied. Therefore, in a single buyer's decision problem, he takes the spot prices and γ as given. With probability γ, the buyer's demand request Q is fully satisfied. With probability $1 - \gamma$, the buyer's demand request is rejected. Let $s = (s_h, s_l)$ be the corresponding prices. For any given (π, γ, s_h, s_l), the buyer's decision problem at stage 2 is as follows.

$$g_1(\pi) \quad = \quad \max \ r_1(Q|\pi) \quad = \quad \gamma \left\{-\pi Q + \mathbb{E}[g_2(Q, s, \alpha)]\right\} + (1-\gamma)\mathbb{E}[g_2(0, s, \alpha)]$$

$$\text{s.t.} \qquad Q \geq 0 \tag{3.5.11}$$

where

$$\mathbb{E}[g_2(Q,s,\alpha)] = pg_2(Q,s_h,\alpha_h) + (1-p)g_2(Q,s_l,\alpha_l) \tag{3.5.12}$$

$$\mathbb{E}[g_2(0,s,\alpha)] = pg_2(0,s_h,\alpha_h) + (1-p)g_2(0,s_l,\alpha_l) \tag{3.5.13}$$

Since s is given, $(1-\gamma)\mathbb{E}[g_2(0,s,\alpha)]$ is constant. The decision problem can be reduced to

$$\begin{aligned} g_1(\pi) \;=\; & \max \quad -\pi Q + \mathbb{E}[g_2(Q,s,\alpha)] \\ & \text{s.t.} \quad Q \geq 0 \end{aligned} \tag{3.5.14}$$

Denote the optimal decision as Q^*, which is a function of the input parameters (π,γ,s_h,s_l). Without confusion, let's temporarily drop π and denote best response function as $\widetilde{Q}(\gamma,s_h,s_l)$. For a given Q, γ is implied by the following function.

$$\widetilde{\gamma}(Q) = \begin{cases} 1 & \text{if } Q \leq C/N \\ C/(NQ) & \text{if } Q > C/N \end{cases} \tag{3.5.15}$$

Given Q and $\gamma = \widetilde{\gamma}(Q)$, the optimal spot prices are determined by Lemma 3.5.3. Denote such best response function as $\widetilde{s}_h(Q)$ and $\widetilde{s}_l(Q)$. A set $(Q^*,s_h^*,s_l^*,\gamma^*)$ is an equilibrium for any given π in the subgame must satisfy the following system:

$$Q^* = \widetilde{Q}(\gamma^*,s_h^*,s_l^*) \tag{3.5.16}$$

$$s_h^* = \widetilde{s}_h(Q^*) \tag{3.5.17}$$

$$s_l^* = \widetilde{s}_l(Q^*) \tag{3.5.18}$$

$$\gamma^* = \widetilde{\gamma}(Q^*) \tag{3.5.19}$$

The optimal contracting quantity Q^* is a fixed point of function $f(Q) = \widetilde{Q}(\widetilde{\gamma}(Q),\widetilde{s}_h(Q),\widetilde{s}_l(Q))$, i.e.,

$$Q^* = f(Q^*) = \widetilde{Q}(\widetilde{\gamma}(Q^*),\widetilde{s}_h(Q^*),\widetilde{s}_l(Q^*)) \tag{3.5.20}$$

Lemma 3.5.4. *If the seller's capacity $C \leq NC_a$, each buyer's optimal contracting quantity is as follows:*

1. *If $\pi \in \left[0, \frac{p(\alpha_h - \alpha_l)}{\beta}\right)$, then $Q^* = \alpha_h - \beta c - \beta\pi/p$.*

2. *If $\pi \in \left[\frac{p(\alpha_h - \alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}\right)$, then $Q^* = \mathbb{E}(\alpha) - \beta c - \beta\pi$.*

3. *If $\pi = \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$, then any $Q \in [0, C/N]$ is optimal.*

4. *If $\pi > \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$, then $Q^* = 0$.*

Note that the price $\frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$ is equal to the expected spot price minus the marginal cost under the condition $Q = 0$, i.e, there is no contracting period. Lemma 3.5.4 says that if the option price is lower than the critical price $\frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$, then the buyer's optimal contracting quantity is the same as if the buyers only entering contracts. If the option price is equal to the critical price, then the buyers are indifferent between entering contracts now or procure from the spot market later since the expected spot price is the same as the critical price plus the marginal cost. If the option price is higher than the critical price, then the buyers will not enter contracts at all. Note the optimal quantity is discontinuous at the critical price. However, the seller's revenue at this price is still continuous as we will show next.

For a given π, the buyers' best response $Q(\pi)$, which is characterized in Lemma 3.5.4. If $Q(\pi) > C/N$, the seller applies first-come first-serve rationing scheme. The seller's problem at the first stage is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) \quad &= \quad \pi\widetilde{\gamma}(Q(\pi))NQ(\pi) + \mathbb{E}[G_2(Q(\pi), \alpha)] \\ \text{s.t.} \quad & \widetilde{\gamma}(Q(\pi))NQ(\pi) \quad \leq \quad C \\ & \pi \quad \geq \quad 0 \end{aligned} \tag{3.5.21}$$

where $\mathbb{E}[G_2(Q(\pi), \alpha)] = pG_2(Q(\pi), \alpha_h) + (1 - p)G_2(Q(\pi), \alpha_l)$ denotes the optimal expected return from the spot market.

Theorem 3.5.5. *If the seller's capacity $C \leq NC_a$, any $\pi \in \left[\frac{\mathbb{E}(\alpha)-\beta c - C/N}{\beta}, \infty\right)$ is optimal.*

Theorem 3.5.5 basically says if the seller's capacity is small, then the best strategy for the seller is to set the price such that the total revenue is the same as putting all the capacity only on the spot market. This result is consistent with that in the single-buyer setting.

B. Contract market only

Henceforth, we only consider the case when $C \leq NC_a$ in this section. If the buyers only participate in the contract market, the decision problems for the buyers in Period 1 and Period 2 are the same as those in Section 3.5.1.1. At the first stage, the seller's decision problem with first-come first-serve rationing scheme is

$$\begin{aligned} \max \quad R_1(\pi) &= \pi\widetilde{\gamma}(Q(\pi))NQ(\pi) \\ \text{s.t.} \quad \widetilde{\gamma}(Q(\pi))NQ(\pi) &\leq C \\ \pi &\geq 0 \end{aligned} \tag{3.5.22}$$

Theorem 3.5.6. *If the seller's capacity $C \leq NC_a$, the optimal option price for the seller $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c - C/N}{\beta}$.*

C. Contract market and spot market with partial participation

In this section, we study the case in which $\lambda \in (0, 1)$ and we still limit our study to the small capacity case $C \leq NC_a$. As in the large capacity case, type A buyers, who count for $1 - \lambda$ of all buyers, only participate in the contract market. The remaining λ of the buyers participate in both contract and spot markets, referred as type B buyers.

At stage 4, the contracting quantity for type A buyers Q_A and the contracting quantity for type B buyers Q_B are given. Note that Q_A and Q_B imply a unique γ by the following function.

$$\widetilde{\gamma}(Q_A, Q_B) = \begin{cases} 1 & \text{if } (1-\lambda)Q_A + \lambda Q_B \le \frac{C}{N} \\ \frac{C}{N[(1-\lambda)Q_A+\lambda Q_B]} & \text{if } (1-\lambda)Q_A + \lambda Q_B > \frac{C}{N} \end{cases} \tag{3.5.23}$$

The decision problem for the seller at this stage is

$$\begin{aligned} G_2(Q_A, Q_B, \alpha) &= \max_s \quad R_2(s|Q_A, Q_B, \alpha) \\ &\quad \text{s.t.} \quad \overline{D}(Q_A, Q_B, s) \le C \end{aligned} \tag{3.5.24}$$

where $\overline{D}(Q_A, Q_B, s)$ denotes the total quantity of products to be transacted for any given (Q_A, Q_B, s). Let $q_{c,A}(Q_A)$ denote the number of contracts to be exercised for a type A buyer who has Q_A contracts. Similarly, let $q_{c,B}(Q_B)$ denote the optimal quantity to be exercised for a type B buyer who has Q_B contracts. The quantity of products to be purchased on the spot market is denoted as $q_{s,B}(Q_B, s)$ for a type B buyers with Q_B contracts. Let $q_{s,B}(0, s)$ denote the quantity of products to purchased from the spot market per type B buyer having no contracts. Thus, the total quantity is

$$\begin{aligned} \overline{D}(Q_A, Q_B, s) &= N\{(1-\lambda)\gamma q_{c,A}(Q_A) + \lambda\gamma[q_{c,B}(Q_B) + q_{s,B}(Q_B, s)] \\ &\quad + \lambda(1-\gamma)q_{s,B}(0, s)\} \end{aligned} \tag{3.5.25}$$

and the objective function is

$$R_2(s|Q, \alpha) = N(s-c)\{\lambda\gamma q_{s,B}(Q_B, s)) + \lambda(1-\gamma)q_{s,B}(0, s)\} \tag{3.5.26}$$

Denote the optimal price for problem (3.5.24) in the high demand state and the low demand state as a function of the input parameters (Q_A, Q_B): $\widetilde{s}_h(Q_A, Q_B)$ and $\widetilde{s}_l(Q_A, Q_B)$.

At stage 2, type A buyers' decision problem and optimal strategy have been investigated in the previous subsection. For any given (s_h, s_l, γ), the formulation of the decision problem for type B buyers is same as problem (3.5.14). Let $\widetilde{Q}_B(\gamma, s_h, s_l)$

be the optimal solution as a function of (s_h, s_l, γ). For a given option price π, a set of values $(Q_A^*, Q_B^*, s_h^*, s_l^*, \gamma^*)$ is an equilibrium in the subgame must satisfy the following system, with Q_A^* determined by Lemma 3.4.1.

$$Q_B^* = \widetilde{Q}_B(\gamma^*, s_h^*, s_l^*) \tag{3.5.27}$$

$$s_h^* = \widetilde{s}_h(Q_A^*, Q_B^*) \tag{3.5.28}$$

$$s_l^* = \widetilde{s}_l(Q_A^*, Q_B^*) \tag{3.5.29}$$

$$\gamma^* = \widetilde{\gamma}(Q_A^*, Q_B^*) \tag{3.5.30}$$

Type B buyers' optimal contracting policy is a fixed point of function

$$f(Q_B) = \widetilde{Q}_B(\widetilde{\gamma}(Q_A^*, Q_B), \widetilde{s}_h(Q_A^*, Q_B), \widetilde{s}_l(Q_A^*, Q_B)) \tag{3.5.31}$$

Lemma 3.5.5. *If seller's capacity $C \leq NC_a$, each type B buyer's optimal contracting quantity is as follows.*

1. *If $\pi \in \left[0, \frac{p(\alpha_h - \alpha_l)}{\beta}\right)$, then $Q_B^* = \alpha_h - \beta c - \beta\pi/p$.*
2. *If $\pi \in \left[\frac{p(\alpha_h - \alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}\right)$, then $Q_B^* = \mathbb{E}(\alpha) - \beta c - \beta\pi$.*
3. *If $\pi = \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$, then any $Q_B \in [0, C/N]$ is optimal.*
4. *If $\pi > \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$, then $Q_B^* = 0$.*

The results in Lemma 3.5.5 are exactly the same as Lemma 3.5.4. However, the proof of Lemma 3.5.5 is much more complicated because the capacity constraint in stage 3. Lemma 3.5.5 states that type B buyers' best contracting policy for any $\lambda \in (0, 1)$ is the same as the case $\lambda = 1$.

At stage 1, the decision problem for the seller is

$$\begin{aligned} \max_{\pi} \quad & R_1(\pi) \\ \text{s.t.} \quad & \widetilde{\gamma}(Q_A, Q_B) N \{(1-\lambda)Q_A(\pi) + \lambda Q_B(\pi)\} \leq C \end{aligned} \tag{3.5.32}$$

$$\pi \geq 0$$

where

$$R_1(\pi) = N\widetilde{\gamma}(Q_A, Q_B)[(1-\lambda)\pi Q_A(\pi) + \lambda\pi Q_B(\pi)] + \mathbb{E}[G_2(Q_A(\pi), Q_B(\pi), \alpha)]$$

Note that $G_2(Q_A(\pi), Q_B(\pi), \alpha)$ is the optimal return for the seller in Period 2 for a given $(Q_A(\pi), Q_B(\pi), \alpha)$.

Theorem 3.5.7. *If $C \leq NC_a$, for any $\lambda \in (0,1)$, the optimal option price for the seller $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$.*

Note at the equilibrium price $\frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$, $Q_A^* = C/N$, and Q_B^* can be any value in $[0, C/N]$. The spot prices are $\frac{\alpha_h-C/N}{\beta}$ and $\frac{\alpha_l-C/N}{\beta}$. At this price, all the capacity is sold. Theorem 3.5.7 says regardless of the value of λ, the optimal option price is constant, $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$. At this price, a single type A buyer's contracting quantity is C/N. A single type B buyer's contracting quantity can be any value in $[0, C/N]$, since the expected spot price is equal to the option price plus the marginal cost. Under this condition, all the capacity is sold in Period 2. Let $Q^* = (1-\lambda)Q_A^* + \lambda Q_B^*$.

Theorem 3.5.8. *If $C \leq NC_a$, the following results hold:*

1. *The expected total number of transacted contracts $Q^*(\lambda)$ decreases in λ.*
2. *As λ increases, the seller's total surplus $G(\lambda)$ doesn't change, $G(\lambda) = \frac{C(\mathbb{E}(\alpha)-\beta c-C/N)}{\beta}$.*
3. *As λ increases, the buyers' total surplus $V(\lambda)$ and the total social surplus $W(\lambda)$ do not change, $V(\lambda) = \frac{C^2}{2N\beta}$ and $W(\lambda) = \frac{C(\mathbb{E}(\alpha)-\beta c-C/(2N))}{\beta}$.*

Theorem 3.5.8 is a direct consequence of Theorem 3.5.7. It states the same properties as in the single-buyer setting. In the undercapacity case, contracts transacted decreases as more and more buyers participate in both markets. However, the surplus of the seller, the total surplus of the buyers, and the total social surplus do not change.

3.5.2.2 Rationing scheme 2 – limiting contracting quantity per buyer

The second possible way is to restrict every buyer's purchase of contracts no larger than C/N.

If there are only type A buyers in the market, i.e., $\lambda = 0$, it is easy to show that equilibrium option price is still $\frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$. In the case $\lambda = 1$, the results in Theorem 3.5.5 hold. For $\lambda \in (0, 1)$, though the model formulation is slightly different, Theorem 3.5.7 still hold. The effects of the buyers' participation rate on the contracts transacted, on the seller's surplus, on the buyers' total surplus and on the total social surplus are exactly the same as Theorem 3.5.8.

Surprisingly, although the two rationing rules are very different, the results in equilibrium are the same if the seller's capacity is small. Since the second rationing scheme does not appear to introduce any particularly interesting new phenomena, the model and the proofs are skipped here.

3.6 Single seller, a continuum of buyers with different utility

This is section considers a market in which there are a single seller and a continuum of buyers with market size N, which is similar to Section 3.5. In addition to each buyer's random utility depending on the state of the market, we also considers the uncertainty depending on each individual buyer. Define a single buyer's utility function as

$$U(q) = -\frac{q^2}{2\beta} + \frac{(\alpha+\phi)q}{\beta} \tag{3.6.1}$$

where ϕ and α are independent random variables realized after the contracting period. Parameter α represents the uncertainty depending on the state of the market, which is the same as that in previous sections. With probability p, the state is high and with probability $1-p$, the state is low. To be consistent with the notation in pervious sections, we still use σ^2 to represent the variance of α. The other parameter, ϕ, models each individual buyer's random utility. This is an approximation to the transportation industry in which there are buyers including different individual shippers and forwarders from different industry sectors. Thus, besides the influence of the market environment, each buyer has his own different utility. In this model, ϕ of every small buyer is assumed to be an i.i.d random variable and uniformly distributed on $[-\overline{\phi}, \overline{\phi}]$. Denote the distribution function as F. Thus, in Period 2, after ϕ is revealed, the mass of the buyers with realization of ϕ below any given $\phi_0 \in [-\overline{\phi}, \overline{\phi}]$ is

$$N \int_{-\overline{\phi}}^{\phi_0} dF(\phi) \quad = \quad \frac{N(\phi_0 + \overline{\phi})}{2\overline{\phi}} \tag{3.6.2}$$

Assumption 3.3.1, 3.4.1 and 3.4.2 are also used here. To emphasize different utility among the buyers, it is also assumed that $\overline{\phi}$ is big.

Assumption 3.6.1. *The variation of the utility among different buyers is large,* $\overline{\phi} \geq \alpha_h$.

Based on the utility function, each single buyer's normal demand function is $s = -\frac{q}{\beta} + \frac{\alpha+\phi}{\beta}$, i.e., $q = \alpha + \phi - \beta s$, where s denotes the price. It should be noted that as $\overline{\phi} \geq \alpha_h$, if the realization ϕ is small such that $-\overline{\phi} \leq \phi < -\alpha + \beta c$, then $\alpha + \phi - \beta c < 0$. That is some buyers do not purchase at all either through contracts or from the spot market. For a given price s, the aggregated market demand $\overline{q}$ is as follows. Denote a single buyer with realization ϕ has demand $q(\phi)$. Depending on the price s, there are two cases.

1. For $s > \frac{\alpha+\overline{\phi}}{\beta}$, it holds that $\alpha + \phi - \beta s < 0$ for any $\phi \in [-\overline{\phi}, \overline{\phi}]$. Thus, $q(\phi) = 0$ for any $\phi \in [-\overline{\phi}, \overline{\phi}]$ and the aggregated market demand is

$$\overline{q} \quad = \quad \frac{N}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} q(\phi) d\phi \quad = \quad 0 \tag{3.6.3}$$

2. For $s \in \left[0, \frac{\alpha+\overline{\phi}}{\beta}\right]$,it holds that $q(\phi) = 0$ for $\phi < -\alpha + \beta s$ and $q(\phi) = \alpha + \phi - \beta s$ for $\phi \geq -\alpha + \beta s$. Note $-\overline{\phi} \leq -\alpha + \beta s \leq \overline{\phi}$. Thus,

$$\overline{q} \quad = \quad \frac{N}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} q(\phi) d\phi$$

$$
\begin{aligned}
&= \frac{N}{2\overline{\phi}}\left[\int_{-\overline{\phi}}^{-\alpha+\beta s} 0 d\phi + \int_{-\alpha+\beta s}^{\overline{\phi}} (\alpha+\phi-\beta s) d\phi\right] \\
&= \frac{N(\alpha+\overline{\phi}-\beta s)^2}{4\overline{\phi}} \qquad (3.6.4)
\end{aligned}
$$

Presumably, the spot market can provide an opportunity to better allocate resource according to buyers' different utility that is only observed after the contracting period. We term this effect the allocation effect of the spot market. Especially, when the seller has a very limited capacity, the allocation effect will be strong. Like previous sections, we first study the large capacity case. Then, the model is extended to the small capacity case.

3.6.1 Large capacity case

3.6.1.1 Contract market only

This subsection studies the setting when the buyers only transact on the contract market. At stage 3 in Period 2, the number of contracts purchased by a single buyer, Q, is given. Also, the random parameters in the utility function, α and ϕ are observed. Let q_c be the quantity of products that each buyer decides to purchase under contracts. Denote each buyer's return in Period 2 as $r_2(q_c|\alpha, \phi)$. A single buyer's decision problem is

$$
\begin{aligned}
g_2(Q, \alpha, \phi) \;=\; & \max_{q_c} \quad r_2(q_c|\alpha, \phi) \\
& \text{s.t.} \quad 0 \leq q_c \leq Q \qquad (3.6.5)
\end{aligned}
$$

where $r_2(q_c|\alpha, \phi) = -\frac{q_c^2}{2\beta} + \frac{(\alpha+\phi)q_c}{\beta} - q_c c$. Let q_c^* be the optimal solution to above problem.

Lemma 3.6.1. *The buyers' optimal decision in Period 2 is as follows.*

1. *If* $\alpha + \phi - \beta c \leq 0$, *then* $q_c^* = 0$.

2. *If* $0 < \alpha + \phi - \beta c < Q$, *then* $q_c^* = \alpha + \phi - \beta c$.

3. If $Q \leq \alpha + \phi - \beta c$, then $q_c^* = Q$.

Lemma 3.6.1 is similar to Lemma 3.4.1, by replacing α with $\alpha + \phi$. The only difference is that the unconstrained optimizer $\alpha + \phi - \beta c$ could be negative if ϕ is small enough. In that case, the buyer does not purchase any products at all.

At stage 2, both α and ϕ are not observed. Given the option contract price π, each buyer anticipates his return in Period 2 and decides how many contracts to purchase. We use $q_c(Q, \alpha, \phi)$ to denote the buyers' best response for a given set of Q, α and ϕ, which is characterized in Lemma 3.6.1. And let $q_c(Q, \alpha_h, \phi)$ and $q_c(Q, \alpha_l, \phi)$ denote each buyer's best decision in high market state and low market state respectively. A single buyer's decision problem is

$$\begin{aligned} g_1(\pi) \quad = \quad & \max_{Q} \quad r_1(Q|\pi) \\ & \text{s.t.} \quad\ \ Q \ \geq \ 0 \end{aligned} \tag{3.6.6}$$

where

$$\begin{aligned} r_1(Q|\pi) &= -\pi Q + \mathbb{E}[g_2(Q, \alpha, \phi)] \\ &= -\pi Q + \frac{p}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} \left[-\frac{q_c(Q, \alpha_h, \phi)^2}{2\beta} + \frac{(\alpha_h + \phi) q_c(Q, \alpha_h, \phi)}{\beta} - q_c(Q, \alpha_h, \phi) c \right] d\phi \\ &\quad + \frac{1-p}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} \left[-\frac{q_c(Q, \alpha_l, \phi)^2}{2\beta} + \frac{(\alpha_l + \phi) q_c(Q, \alpha_l, \phi)}{\beta} - q_c(Q, \alpha_l, \phi) c \right] d\phi \end{aligned} \tag{3.6.7}$$

Let the buyers' best response be Q^*.

Lemma 3.6.2. *The buyers' optimal contracting decision Q^* is as follows.*

1. *If $\pi \in \left[0, \frac{p(\alpha_h - \alpha_l)^2}{4\beta\overline{\phi}}\right)$, then $Q^* = \alpha_h + \overline{\phi} - \beta c - \sqrt{\frac{4\beta\overline{\phi}\pi}{p}}$.*

2. *If $\pi \in \left[\frac{p(\alpha_h - \alpha_l)^2}{4\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha + \overline{\phi} - \beta c)^2}{4\beta\overline{\phi}}\right]$, then $Q^* = \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{4\beta\overline{\phi}\pi - \sigma^2}$.*

3. *If $\pi > \frac{\mathbb{E}(\alpha + \overline{\phi} - \beta c)^2}{4\beta\phi}$, then $Q^* = 0$.*

Note if $\pi \in \left[0, \frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}\right)$, then $Q^* \in (\alpha_l + \overline{\phi} - \beta c, \alpha_h + \overline{\phi} - \beta c]$. If $\pi \in \left[\frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}\right]$, then $Q^* \in [0, \alpha_l + \overline{\phi} - \beta c]$. Also, at the breakpoint $\frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}$, $Q^*(\pi)$ is continuous. Lemma 3.6.2 indicates that if the option price π is small, $\pi < \frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}$, then each buyer purchases a large number of contracts such that $Q^* > \alpha_l + \overline{\phi} - \beta c$. If the state of the market turns out to be low in Period 2, every single buyer has unused contracts regardless of the value of ϕ. If the option price is higher than $\frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}$, then each buyer purchases a moderate quantity of contracts. Some buyers with high ϕ realization use all the contracts and some buyers with low ϕ realization only use part of the contracts. If the option price is high enough, $\pi > \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}$, then all of the buyers don't enter contracts at all.

In Period 1, the seller chooses the option price π to maximize her total revenue. Note that every buyer's decision problem at stage 2 is the same, thereby the best response $Q^*(\pi)$ is the same. Hence, the aggregated response is $NQ^*(\pi)$. To simplify the notation, we use $Q(\pi)$ to represent $Q^*(\pi)$ and omit the superscript "*". The seller's optimization problem at this stage is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) &= N[\pi Q(\pi)] \\ \text{s.t.} \qquad\qquad \pi &\geq 0 \end{aligned} \tag{3.6.8}$$

Let the optimal option price be π^*.

Theorem 3.6.1. *The optimal option price is unique and is determined by the following conditions.*

1. $\pi^* \in \left[\frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha+\overline{\phi}+\beta c)^2}{4\beta\overline{\phi}}\right]$.

2. $\mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{4\beta\overline{\phi}\pi^* - \sigma^2} - \frac{2\beta\overline{\phi}\pi^*}{\sqrt{4\beta\overline{\phi}\pi^*-\sigma^2}} = 0.$

Theorem 3.6.1 indicates that in market equilibrium, the seller sets the option contract price such that some of the buyers always use all the contracts no matter

Period 2 turns to be high market state or low market state. Intuitively, by Assumption 3.6.1, the seller's optimal decision should not only reflect the variation of α but also should reflect the variation of ϕ.

3.6.1.2 Contract market and spot market with full participation

This section studies the case $\lambda = 1$, i.e., all buyers participate in both the contract market and the spot market. At stage 4, the number of contracts purchased in Period 1, Q, is given and the random variable α are observed by all the buyers. Parameter ϕ representing each individual buyer's random utility is also observed by every buyer. Given the spot price s buyers decide how many products to purchase under contracts and how many to purchase from the spot market. Let q_c be the quantity of products to be transacted under contracts and q_s be the quantity of products to be purchased from the spot market. Denote each buyer's return in Period 2 as $r_2(q_c, q_s|s, \alpha, \phi)$. The decision problem for a single buyer is as follows.

$$
\begin{aligned}
g_2(Q, s, \alpha, \phi) \;=\; & \max_{q_c, q_s} \quad r_2(q_c, q_s, |s, \alpha, \phi) \\
& \text{s.t.} \quad 0 \;\leq\; q_c \;\leq\; Q \\
& \qquad\quad q_s \;\geq\; 0
\end{aligned} \tag{3.6.9}
$$

where $r_2(q_c, q_s|s, \alpha, \phi) = -\frac{(q_c+q_s)^2}{2\beta} + \frac{(\alpha+\phi)(q_c+q_s)}{\beta} - q_c c - q_s s$.

Lemma 3.6.3. *The buyers' optimal decision in Period 2 is as follows.*

1. *If $\alpha + \phi - \beta c \leq 0$, then $q_c^* = q_s^* = 0$*

2. *If $\alpha+\phi-\beta c > 0$ and $s = c$, then any q_c^* and q_s^* satisfying the following conditions are optimal:*

$$
\begin{aligned}
q_c^* + q_s^* &= \alpha + \phi - \beta c \\
0 \leq\; & q_c^* \leq Q \\
q_s^* &\geq 0
\end{aligned}
$$

3. *If* $0 \le \alpha + \phi - \beta c \le Q$ *and* $s > c$*, then* $q_c^* = \alpha + \phi - \beta c$ *and* $q_s^* = 0$.

4. *If* $0 \le Q \le \alpha + \phi - \beta c$ *and* $s > c$*, then* $q_c^* = Q$ *and* $q_s^* = (\alpha + \phi - \beta s - Q)^+$.

Lemma 3.6.3 is similar to Lemma 3.4.3, by replacing α with $\alpha + \phi$. The difference is that the unconstrained optimizer $\alpha + \phi - \beta c$ can be negative. In that case, buyers don't purchase products at all.

At stage 3, α is revealed. In addition, every single buyer's random utility parameter ϕ is observed. Depending on the number of contracts each buyer has, the seller sets the spot price to maximize her return from the spot market. Since each small buyer's utility parameter ϕ is independent and uniformly distributed on $[-\overline{\phi}, \overline{\phi}]$. Therefore, the mass of buyers with ϕ below any given $\phi_0 \in [-\overline{\phi}, \overline{\phi}]$ is $\frac{N(\phi_0 + \overline{\phi})}{2\overline{\phi}}$. In other words, the seller observes a continuum of buyers with ϕ evenly from $-\overline{\phi}$ to $\overline{\phi}$. Let $q_c(Q, s, \alpha, \phi)$ and $q_s(Q, s, \alpha, \phi)$ be a single buyer's best response in stage 4 given Q, s, α and ϕ. Denote the seller's revenue in Period 2 as $R_2(s|Q, \alpha)$. The seller's decision problem is

$$\begin{aligned} G_2(Q, \alpha) \quad = \quad & \max_s \quad R_2(s|Q, \alpha) \\ & \text{s.t.} \quad s \ge c \end{aligned} \tag{3.6.10}$$

where

$$\begin{aligned} R_2(s|Q, \alpha) &= (s - c)N \int_{-\overline{\phi}}^{\overline{\phi}} q_s(Q, s, \alpha, \phi) dF(\phi) \\ &= \frac{N(s - c)}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} q_s(Q, s, \alpha, \phi) d\phi \end{aligned} \tag{3.6.11}$$

Lemma 3.6.4. *The seller's optimal spot price is as follows.*

1. *If* $Q > \alpha + \overline{\phi} - \beta c$*, then any* $s \ge c$ *is optimal.*

2. *If* $Q \le \alpha + \overline{\phi} - \beta c$*, then the seller's optimal price* $s^* = \frac{\alpha + \overline{\phi} + 2\beta c - Q}{3\beta}$.

The rationale behind Lemma 3.6.4 is as follows. Since the strike price is at the marginal cost, the buyer with maximum demand is $\alpha_h + \overline{\phi} - \beta c$. If the number of contracts that each buyer has is bigger than that quantity, every buyer only transacts under contracts with the lowest possible unit price c. In this case, there is no transaction at all in the spot market. If Q is smaller than that quantity, then the buyers with high utility will purchase from the spot market to fulfill their residual demand. Therefore, the seller sets the spot price to maximize her revenue from the spot market according to the aggregated residual demand.

Given the option price π, each buyer anticipates his expected return in Period 2 and decides how many contracts to purchase at stage 2. As each buyer is infinitesimal, a single buyer takes the spot price s as given and doesn't consider it is a consequence of his decision Q. The seller's best response in Period 2 at the high market state and low market state are denoted as s_h and s_h for a given Q. Let $q_c(Q, s, \alpha, \phi)$ and $q_s(Q, s, \alpha, \phi)$ be a single buyer's corresponding best response in stage 4 for a given set of Q, s, α, and ϕ. Let $r_1(Q|\pi)$ be a single buyer's expected return in Period 1. A single buyer's decision problem at stage 2 in Period 1 is

$$
\begin{aligned}
g_1(\pi) \;=\; \max_{Q} \quad r_1(Q|\pi) \;&=\; -\pi Q + \mathbb{E}[g_2(Q, s, \alpha, \phi)] \\
\text{s.t.} \qquad\qquad\qquad Q \;&\geq\; 0
\end{aligned}
\tag{3.6.12}
$$

where

$$
\begin{aligned}
\mathbb{E}[g_2(Q, s, \alpha, \phi)] \;=\; & \frac{p}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} \left[\frac{(q_c(Q, s, \alpha_h, \phi) + q_s(Q, s, \alpha_h, \phi))^2}{2\beta} + \right. \\
& \frac{(\alpha_h + \phi)(q_c(Q, s, \alpha_h, \phi) + q_s(Q, s, \alpha_h, \phi))}{\beta} \\
& \left. - \frac{q_c(Q, s, \alpha_h, \phi)\beta c + q_s(Q, s, \alpha_h, \phi)\beta s_h}{\beta} \right] d\phi + \\
& \frac{1-p}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} \left[\frac{(q_c(Q, s, \alpha_l, \phi) + q_s(Q, s, \alpha_l, \phi))^2}{2\beta} + \right. \\
& \frac{(\alpha_l + \phi)(q_c(Q, s, \alpha_l, \phi) + q_s(Q, s, \alpha_l, \phi))}{\beta}
\end{aligned}
$$

$$-\frac{q_c(Q,s,\alpha_l,\phi)\beta c + q_s(Q,s,\alpha_l,\phi)\beta s_l}{\beta}\Bigg] d\phi \qquad (3.6.13)$$

Let Q^* be the buyers' optimal decision.

Lemma 3.6.5. *The buyers' optimal contracting quantity is as follows.*

1. *If* $\pi \in \left[0, \frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}\right)$, *then* $Q^* = \alpha_h + \overline{\phi} - \beta c - \sqrt{\frac{36\beta\overline{\phi}\pi}{5p}}$.
2. *If* $\pi \in \left[\frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}, \frac{5\left[\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2\right]}{36\beta\overline{\phi}}\right]$, *then* $Q^* = \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{\frac{36\beta\overline{\phi}\pi}{5} - \sigma^2}$.
3. *If* $\pi > \frac{5\left[\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2\right]}{36\beta\overline{\phi}}$, *then* $Q^* = 0$.

Note in part 1 of Lemma 3.6.5, $Q^* \in (\alpha_l + \overline{\phi} - \beta c, \alpha_h + \overline{\phi} - \beta c]$ and in part 2, $Q^* \in [0, \alpha_l + \overline{\phi} - \beta c]$. Lemma 3.6.5 indicates that if the option price is small, i.e., $\pi \leq \frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}$, each buyer's best response Q^* is large. If the state of the market turns out to be low in Period 2, all buyers only transact under contracts. If the option price is bigger, i.e., $\frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}} \leq \pi \leq \frac{5\left[\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2\right]}{36\beta\overline{\phi}}$, then each buyer purchase a smaller number of contracts. In that case, some buyers with high utility always purchase on the spot market in either high state or low state. Compared to the results in Lemma 3.6.2, for a given price π, each buyer purchases fewer contracts and relies more on the spot market to fulfill his residual demand.

At stage 1, the seller chooses an optimal option price, anticipating her return in Period 2. Denote the seller's total revenue as $R_1(\pi)$ and denote each buyer's best response at stage 2 as $Q(\pi)$. The seller's problem at stage 1 is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) &= N\pi Q(\pi) + \mathbb{E}[G_2(Q(\pi),\alpha)] \\ \text{s.t.} \qquad \pi &\geq 0 \end{aligned} \qquad (3.6.14)$$

where $\mathbb{E}[G_2(Q(\pi),\alpha)] = pG_2(Q(\pi),\alpha_h) + (1-p)G_2(Q(\pi),\alpha_l)$. Denote the buyers' optimal price as π^*.

Theorem 3.6.2. *Seller's optimal option price is unique and is determined by the following conditions:*

1. $\pi^* \in \left[\frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}, \frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}}\right]$.
2. $\mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{\frac{36\beta\overline{\phi}\pi^*}{5} - \sigma^2} - \frac{18\beta\overline{\phi}\pi^*}{25\sqrt{\frac{36\beta\overline{\phi}\pi^*}{5}-\sigma^2}} = 0$

Similar to the results in Theorem 3.6.1, in equilibrium, the seller sets the option price fairly high such that some buyers always purchase on the spot market no matter the state is high or low.

3.6.1.3 Contract market and spot market with partial participation

This subsection studies the case when only part of the buyers buy in the spot market. The previous results on the total contracts transacted and on the seller's surplus still hold.

Theorem 3.6.3. *(Effect on Contract Market)*

The total quantity of contracts transacted $Q^*(\lambda)$ decreases as λ increases.

Theorem 3.6.4. *(Effect on Seller's Surplus)*

The seller's surplus $G(\lambda)$ increases as λ increases.

A numerical study is conducted to investigate the effects of the spot market participation on the buyers' total surplus and the total social surplus. We have tested a variety of data. It is observed that both the buyers' total surplus and the total social surplus always increases as λ increases. This is due to the demand variation among the buyers. Since the buyers' utilities are observed in the spot market period, the buyers with higher utilities can purchase additional products from the spot market. As the demand variation is large, this effect becomes significant and outperforms the factor that the spot market is dominated by the monopolist seller. Figure 12 presents the results for one of the examples.

Example 7: $\alpha_h = 15$, $\alpha_l = 3$, $\overline{\phi} = 15$, $\beta = 1$, $c = 2$, and $p = 0.5$.

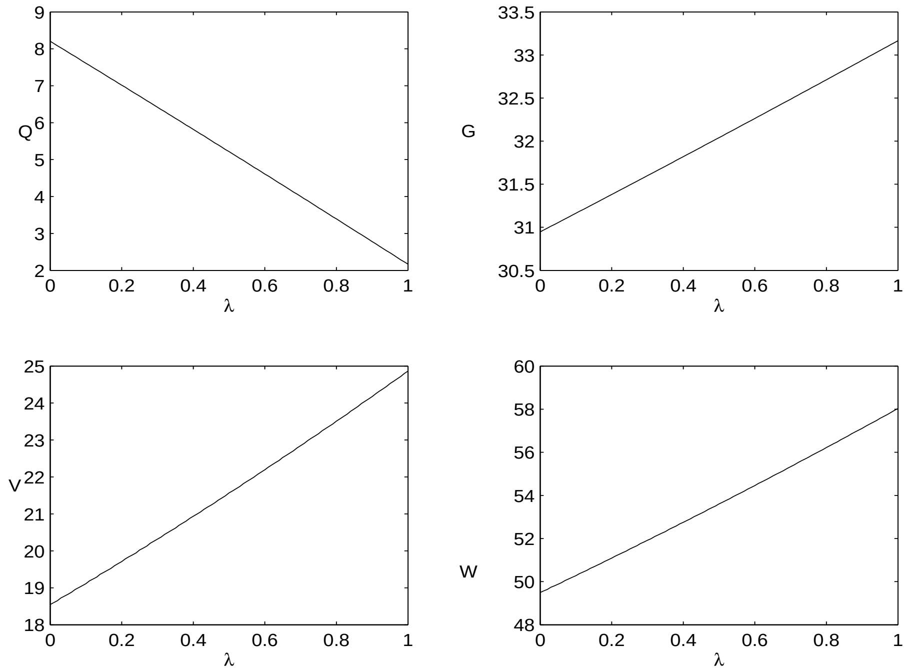

Figure 12: Effects of spot market participation for Example 7

3.6.2 Small capacity case

In Section 3.5, we have shown that if all buyers have the same utilities and the capacity is large, an increase in spot market spot market participation rate always benefits the seller but may or may not benefit the buyers. If the capacity is small enough, then an increase in spot market participation rate doesn't change the buyers' surplus. In this subsection, we extend the model of many buyers with different utilities to the small capacity case. Similar to Section 3.5.2, the two rationing schemes are studied. For the first-come first-serve rationing scheme, a numerical study is conducted to investigate the effects of the buyers' spot market participation rate. For the second rationing rule, analytical results are obtained for $\lambda = 0$ and $\lambda = 1$. We show that if the seller's capacity is smaller than a threshold, it is better for both the seller and the buyers in the latter case ($\lambda = 1$), than in the former case ($\lambda = 0$). It is also assumed that the total number of contracts sold to the buyers can not be larger than C.

3.6.2.1 *Rationing scheme 1 – first-come first-serve*

In this section, we first study the case when all buyers participate only in the contract market. Then we extend the model to the case $\lambda \in (0, 1)$. Since the formulation of the case $\lambda = 1$ is similar, it is not included here.

A. Contract Market Only

This subsection studies the case when all buyers only participate in the contract market, i.e., $\lambda = 0$. At this stage 3 in Period 2, the contracting quantity Q is given. Also, the random parameters, α and ϕ, in each buyer's utility function are observed. A single buyer's problem is the same as problem (3.6.5) and Lemma 3.6.1 holds. Buyers' stage 2 problem is the same as problem (3.6.6) with the objective function scaled by γ, which is determined by function (3.5.15) in terms of Q. The buyers' optimal contracting quantity is characterized by Lemma 3.6.2.

At the first stage, the seller chooses an option price to maximize her total revenue. Denote each buyer's best response as $Q(\pi)$. The seller's problem is

$$\begin{array}{llcl} \max\limits_{\pi} & R_1(\pi) & = & \pi\widetilde{\gamma}(Q)NQ(\pi) \\ \text{s.t.} & & & 0 \;\leq\; \widetilde{\gamma}(Q)NQ(\pi) \;\leq\; C \\ & & & \pi \;\geq\; 0 \end{array} \tag{3.6.15}$$

where $\widetilde{\gamma}(Q)$ is defined as function (3.5.15).

B. Contract market and Spot Market with full participation

The stage 4 decision problem for a type B buyers is the same as problem (3.6.10). For each type A buyer, the decision problem is formulated as problem (3.6.5). Lemma 3.6.1 and Lemma 3.6.3 hold for type A buyers and type B buyers respectively.

At stage 3, the seller chooses the spot price to maximize her return from the spot prices. At this stage, (Q_A, Q_B) is given, which implies γ by function (3.5.23) under the first-come first-serve scheme. The formulation of the decision problem is the same as problem (3.5.24) with $R_2(s|Q_A, Q_B, \alpha)$ and $\overline{D}(Q_A, Q_B, s)$ defined as follows. Let

$D_{c,A}(Q_A)$ denote the total number of contracts exercised by type A buyers. The best decision for a single type A buyer, who has Q_A contracts, is characterized in Lemma 3.6.5 and is denoted as $q_{c,A}(Q_A, \phi)$. Fraction γ of the type B buyers have Q_B contracts each. For those buyers, denote the best quantity to purchase under contracts and from the spot market in stage 4 as $q_{c,B}(Q_B, \phi)$ and $q_{s,B1}(Q_B, s, \phi)$, which are characterized by Lemma 3.6.3. Denote the aggregated quantity transacted under contracts from those buyers as $D_{c,B}(Q_B)$ and the aggregated quantity transacted from the spot market as $D_{s,B1}(Q_B, s)$. Because of the first-come first-serve rationing scheme, fraction $1 - \gamma$ of the type B buyers are not able to purchase any contract in Period 1. For those buyers, denote the best quantity to purchase from the spot market in stage 4 per buyer as $q_{s,B2}(0, s, \phi)$ and the aggregated quantity as $D_{s,B2}(0, s)$. The total aggregated demand for any (Q_A, Q_B, s) is

$$\overline{D}(Q_A, Q_B, s) = D_{c,A}(Q_A) + D_{c,B}(Q_B) + D_{s,B1}(Q_B, s) + D_{s,B2}(0, s) \tag{3.6.16}$$

where

$$D_{c,A}(Q_A) = \frac{N\gamma(1-\lambda)}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} q_{c,A}(Q_A, \phi)d\phi \tag{3.6.17}$$

$$D_{c,B}(Q_B) = \frac{N\gamma\lambda}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} q_{c,B}(Q_B, \phi)d\phi \tag{3.6.18}$$

$$D_{s,B1}(Q_B, s) = \frac{N\gamma\lambda}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} q_{s,B1}(Q_B, s, \phi)d\phi \tag{3.6.19}$$

$$D_{s,B2}(0, s) = \frac{N(1-\gamma)\lambda}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} q_{s,B2}(0, s, \phi)d\phi \tag{3.6.20}$$

The objective function for this problem is

$$R_2(s|Q_A, Q_B, \alpha) = (s - c)[D_{s,B1}(Q_B, s) + D_{s,B2}(0, s)] \tag{3.6.21}$$

Denote the optimal price function in terms of (Q_A, Q_B) of the above problem in the high demand state and low demand state as $\widetilde{s}_h(Q_A, Q_B)$ and $\widetilde{s}_l(Q_A, Q_B)$. The following lemma characterizes the condition for the uniqueness of the optimal price s^* at stage 3.

Lemma 3.6.6. *The optimal spot price s^* is as follows.*

1. *If $Q_B \geq \alpha + \overline{\phi} - \beta c$ and $\gamma = 1$, then no buyer purchases from the spot market.*

2. *If $Q_B < \alpha + \overline{\phi} - \beta c$ or $\gamma < 1$, the following results hold. If equation (3.6.22) has two roots s_1 and s_2 with $s_1 < s_2$, $s_{1,2} \in \left[\frac{\alpha+\overline{\phi}+2\beta c - Q_B}{3\beta}, \frac{\alpha+\overline{\phi}-Q_B}{\beta}\right]$ and s_1 satisfies equation (3.6.23), then both s_1 and $\frac{\alpha+\overline{\phi}+2\beta c}{3\beta}$ are optimal. Otherwise, the optimal price s^* is unique.*

$$\begin{aligned}\gamma(\alpha + \overline{\phi} - \beta s - Q_B)(\alpha + \overline{\phi} + 2\beta c - 3\beta s - Q_B) \\ + (1-\gamma)(\alpha + \overline{\phi} - \beta s)(\alpha + \overline{\phi} + 2\beta c - 3\beta s) &= 0 \qquad (3.6.22)\end{aligned}$$

$$\begin{aligned}\frac{(s-c)\left[\gamma(\alpha + \overline{\phi} - \beta s - Q_B)^2 + (1-\gamma)(\alpha + \overline{\phi} - \beta s)^2\right]}{4} &= \\ \frac{(1-\gamma)(\alpha + \overline{\phi} - \beta c)^3}{27\beta} \qquad (3.6.23)\end{aligned}$$

The conditions in Lemma 3.6.6 under which there are multiple optimal spot prices are fairly strong. In most cases, the optimal spot price is unique.

By similar arguments in Section 3.5.2, at stage 2, type B buyers' decision problem is formulated the same as problem (3.6.12). Let $\widetilde{Q}_B(\gamma, s_h, s_l)$ be the optimal solution as a function of (s_h, s_l, γ). For a given option price π, a set $(Q_A^*, Q_B^*, s_h^*, s_l^*, \gamma^*)$ is an equilibrium in the subgame must satisfy the equations (3.5.27)–(3.5.30), with Q_A^* determined by Lemma 3.6.2, $\widetilde{\gamma}(Q_A, Q_B)$ defined by function (3.5.23), $\widetilde{s}_h(Q_A, Q_B)$ and $\widetilde{s}_l(Q_A, Q_B)$ defined in the previous paragraph. Though it seems there might be multiple equilibriums in the subgame for a given option price π, such unpredictable situation never takes place in all the numerical examples tested in the following subsection.

The formulation of the stage 1 decision for the seller is the same as problem (3.5.32), with $Q_B(\pi)$ and $Q_A(\pi)$ denoting the best contracting policies defined in this context.

C. Numerical Study

In this subsection, we conduct a numerical study to investigate the effects of the participation rate on the total contracts transacted, on the seller's surplus and on the buyers' total surplus. Though the results might not hold in general, they still provide valuable insights in this setting and provide comparison to the results in the previous sections.

The input parameter values are $\alpha_h = 10$, $\alpha_l = 5$, $\beta = 3$, $c = 1$, $\overline{\phi} = 10$, $p = 0.5$. We compare the results at different capacity levels. From Figure 13 to 16, the seller's capacity decreases.

Similar to the previous results, the quantity of the total contracts transacted always decreases as λ increases from 0 to 1. Compared to the contracts, the spot market provides an opportunity to better allocate the capacity to the buyers according to their different utilities that are only observed in Period 2. Thus, the total social surplus always increases as more and more buyers participate in the spot market (Figure 13–16). If the capacity is not too small, this also benefits the buyers (Figure 13 and Figure 14). However, if the capacity is considerably small and there are enough type B buyers in the market, the seller can push the market equilibrium to the spot market period, i.e., no buyer enters into contracts in advance. The seller can extract a large amount of profit from the buyers. In that case, the surplus of the buyers deceases in λ again (Figure 15 and Figure 16).

3.6.2.2 Rationing scheme 2 – limiting contracting quantity per buyer

In this section, we prove that if the seller's capacity is smaller than a certain threshold, then both the seller and the buyers are better off in the case $\lambda = 1$ compared to $\lambda = 0$ under the second rationing scheme. We characterize the threshold with which the

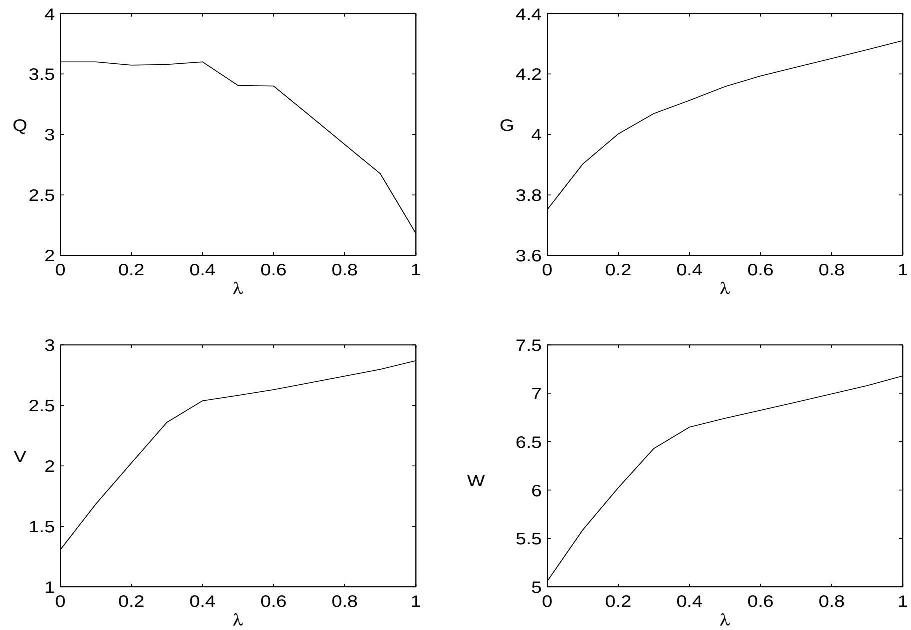

Figure 13: Effects of spot market participation for Example 8, capacity = 3.6

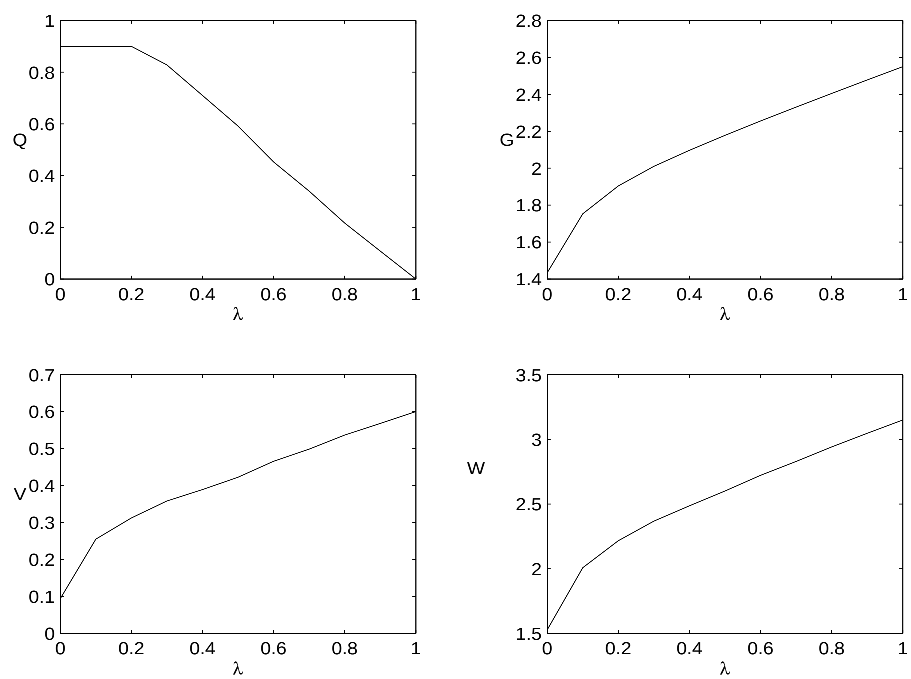

Figure 14: Effects of spot market participation for Example 9, capacity = 0.9

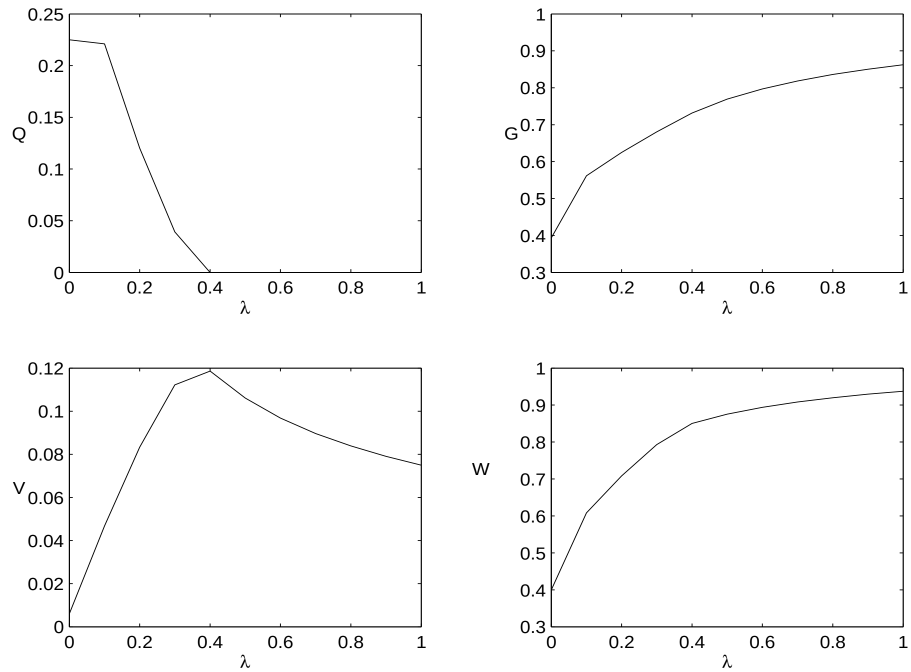

Figure 15: Effects of spot market participation for Example 10, capacity = 0.225

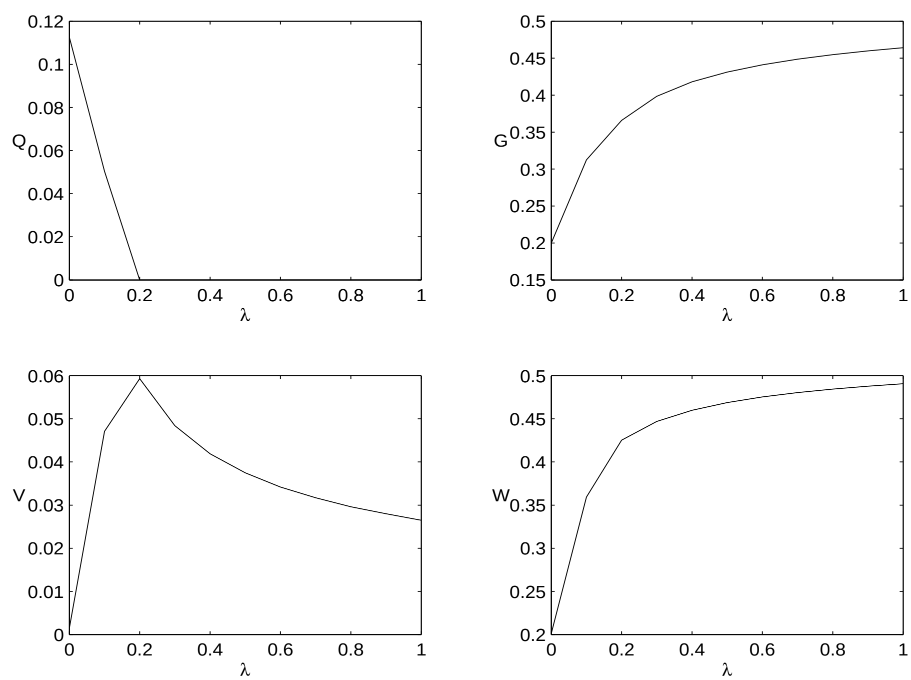

Figure 16: Effects of spot market participation for Example 11, capacity = 0.1125

above conclusion holds. Let C be the seller's capacity. It is assumed that each buyer's contracting quantity can not exceed C/N. In this section, we use the same notation in Section 3.6.

A. Contract market and spot market with full participation

Here we consider the seller's capacity is small and all buyers participate in the spot market. At stage 4, the decision problem for the buyers is the same as that in the large capacity case. The optimal decision is characterized in Lemma 3.6.3.

At stage 3, the seller sets the spot price such that her revenue on the spot market is maximized and the buyers' aggregated demand doesn't exceed the capacity constraint. Note that the buyers' demand contains two parts: one part of the demand is fulfilled by contracts and the other part is fulfilled via spot market. Denote the buyer's best response in stage 4 as $q_c(Q, s, \alpha, \phi)$ and $q_s(Q, s, \alpha, \phi)$. The seller's problem is

$$\begin{aligned} G_2(Q, \alpha) \quad = \quad \max_s \quad & R_2(s|Q, \alpha) \\ \text{s.t.} \quad & N \int_{-\overline{\phi}}^{\overline{\phi}} [q_c(Q, s, \alpha, \phi) + q_s(Q, s, \alpha, \phi)] dF(\phi) \quad \leq \quad C \\ & s \quad \geq \quad c \end{aligned} \tag{3.6.24}$$

where $R_2(s|Q, \alpha) = \frac{N(s-c)}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} q_s(Q, s, \alpha, \phi) d\phi$.

Lemma 3.6.7. *If capacity* $C \leq \frac{N(\alpha_l + \overline{\phi} - \beta c)^2}{9\overline{\phi}} = C_b$, *then the seller's optimal price is* $s^* = \frac{1}{\beta}\left[\alpha + \overline{\phi} - Q - \sqrt{(\alpha + \overline{\phi} - \beta c - Q)^2 - (\alpha + \overline{\phi} - \beta c)^2 + \frac{4\overline{\phi} C}{N}}\right]$.

Note the seller sells all of the remaining capacity in the spot market for any $Q \geq 0$ at the optimal spot price. Under this condition, the optimal solution s^* is always greater or equal to the optimal spot price when there is no capacity constraint. In contrast to the previous results, an increase in Q results in a higher spot price. The intuition is as follows. If there is no capacity constraint, the optimal spot price is the monopoly price with respect to the aggregated residual demand. Since the capacity is small, the seller keeps increasing the spot price till the aggregated residual demand equal to the leftover capacity in Period 2. As Q increases, the remaining capacity in Period 2 becomes smaller, which drives the spot price higher.

Based on the seller's rule, the number of contacts purchased by each buyer Q can not exceed C/N. Let $q_c(Q,s,\alpha_h,\phi)$ and $q_s(Q,s,\alpha_h,\phi)$ be the buyers' best response in the spot market when the market state is high. Let $q_c(Q,s,\alpha_l,\phi)$ and $q_s(Q,s,\alpha_l,\phi)$ be the buyers' best response in the spot market when the market state is low. Denote a single buyer's expected return in Period 1 as $r_1(Q|\pi)$. Denote the seller's corresponding equilibrium spot prices as s_h and s_l. Each buyer's decision problem at stage 2 is

$$\begin{aligned} g_1(\pi) \quad &= \quad \max_Q \quad r_1(Q|\pi) \quad = \quad -\pi Q + \mathbb{E}[g_2(Q,s,\alpha)] \\ &\qquad \text{s.t.} \qquad\qquad\quad 0 \quad \leq \quad Q \quad \leq \quad \frac{C}{N} \end{aligned} \tag{3.6.25}$$

where

$$\begin{aligned} \mathbb{E}[g_2(Q,s,\alpha)] \quad = \quad & \frac{p}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} \Bigg[-\frac{(q_c(Q,s,\alpha_h,\phi)+q_s(Q,s,\alpha_h,\phi))^2}{2\beta} + \\ & \frac{(\alpha_h+\phi)(q_c(Q,s,\alpha_h,\phi)+q_s(Q,s,\alpha_h,\phi))}{\beta} \\ & -\frac{q_c(Q,s,\alpha_h,\phi)\beta c+q_s(Q,s,\alpha_h,\phi)\beta s_h}{\beta} \Bigg] d\phi + \\ & \frac{1-p}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} \Bigg[-\frac{(q_c(Q,s,\alpha_l,\phi)+q_s(Q,s,\alpha_l,\phi))^2}{2\beta} + \\ & \frac{(\alpha_l+\phi)(q_c(Q,s,\alpha_l,\phi)+q_s(Q,s,\alpha_l,\phi))}{\beta} \\ & -\frac{q_c(Q,s,\alpha_l,\phi)\beta c+q_s(Q,s,\alpha_l,\phi)\beta s_l}{\beta} \Bigg] d\phi \end{aligned} \tag{3.6.26}$$

Similar to Section 3.5 and 3.6, each buyer doesn't take s_h and s_l as a function of his decision Q.

Lemma 3.6.8. *If capacity $C \leq C_b$, then buyers' optimal contracting policy is as follows:*

1. *If $\pi < \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2-4\overline{\phi}C/N}{4\beta\overline{\phi}}$, then $Q^* = C/N$.*

2. If $\pi > \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2-4\overline{\phi}C/N}{4\beta\overline{\phi}}$*, then* $Q^* = 0$*.*

3. If $\pi = \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2-4\overline{\phi}C/N}{4\beta\overline{\phi}}$*, then any* $Q^* \in [0, C/N]$ *is optimal.*

Lemma 3.6.8 indicates that if the option contract price is less then the threshold $\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2-4\overline{\phi}C/N}{4\beta\overline{\phi}}$, then a single buyer purchases up to C/N. If the option contract price is greater than the threshold, each buyer only purchase products on the spot market.

At stage 1, the seller anticipates her expected return in Period 2 and chooses an option price to maximize her total revenue. Denote each buyer's best response as $Q(\pi)$. Let the seller's expected return be $R_1(\pi)$. The seller's decision problem is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) \quad &= \quad N\pi Q(\pi) + \mathbb{E}[G_2(Q(\pi), \alpha)] \\ \text{s.t.} \quad & \quad 0 \;\leq\; Q(\pi) \;\leq\; \frac{C}{N} \\ & \quad \pi \;\geq\; 0 \end{aligned} \tag{3.6.27}$$

where $\mathbb{E}[G_2(Q, s, \alpha)] = pG_2(Q(\pi), \alpha_h) + (1-p)G(Q(\pi), \alpha_l)$. Let $C_c = \frac{N(\mathbb{E}(\alpha)+\overline{\phi}-\beta c)^2}{32\overline{\phi}}$.

Theorem 3.6.5. *If capacity* $C \leq \min\{C_b, C_c\}$*, then any option price in* $\left(\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2-4\overline{\phi}C/N}{4\beta\overline{\phi}}, +\infty\right)$ *is optimal .*

Corollary 3.6.1. *No contract is sold in equilibrium,* $Q(\pi^*) = 0$*.*

Corollary 3.6.1 states that if the capacity is small enough, i.e., $C \leq \min\{C_b, C_c\}$, then in market equilibrium, the seller sets the price high enough such that all the buyers only purchase products from the spot market.

B. Contract market only

This subsection studies the case when all buyers only participate in the contract market $\lambda = 0$ under the condition $C \leq \min\{C_b, C_c\}$.

At stage 3 in Period 2, the decision problem and optimal decision are the same as those in the large capacity case. At stage 2, given the option price π, a single buyer anticipates his return in Period 2 and decides how many contracts to enter. Denote each buyer's best response in Period 2 as $q_c(Q, \alpha, \phi)$. The decision problem is

$$\begin{aligned} g_1(\pi) &= \max_{Q} \quad r_1(Q|\pi) \\ &\text{s.t.} \quad 0 \leq Q \leq \frac{C}{N} \end{aligned} \tag{3.6.28}$$

where

$$r_1(Q|\pi) = -\pi Q + \mathbb{E}[g_2(Q, \alpha, \phi)]$$

and

$$\begin{aligned} \mathbb{E}[g_2(Q,\alpha,\phi)] = {} & \frac{p}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} \left[-\frac{q_c(Q,\alpha_h,\phi)^2}{2\beta} + \frac{(\alpha_h + \phi) q_c(Q,\alpha_h,\phi)}{\beta} - q_c(Q,\alpha_h,\phi)c \right] d\phi + \\ & \frac{1-p}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} \left[-\frac{q_c(Q,\alpha_l,\phi)^2}{2\beta} + \frac{(\alpha_l + \phi) q_c(Q,\alpha_l,\phi)}{\beta} - q_c(Q,\alpha_l,\phi)c \right] d\phi \end{aligned}$$

Lemma 3.6.9. *Buyers' optimal contracting decision is as follows.*

1. *If* $\pi \leq \frac{\mathbb{E}(\alpha+\phi-\beta c-C/N)^2}{4\beta\overline{\phi}}$, *then* $Q^* = C/N$.
2. *If* $\pi \in \left[\frac{\mathbb{E}(\alpha+\phi-\beta c-C/N)^2}{4\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha+\phi-\beta c)^2}{4\beta\overline{\phi}}\right]$, *then* $Q^* = \mathbb{E}(\alpha) + \overline{\phi} - \beta c \sqrt{4\beta\overline{\phi}\pi - \sigma^2}$.
3. *If* $\pi > \frac{\mathbb{E}(\alpha+\phi-\beta c)^2}{4\beta\overline{\phi}}$, *then* $Q^* = 0$.

Lemma 3.6.9 indicates that if the option contract price π is less or equal to the threshold $\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}$, then each buyer purchases up to the limit C/N. If the option price is greater than $\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}$, then buyers do not enter contracts at all. If the option price falls in between, then each buyer's best response is a decreasing function of the option price.

At stage 1, the seller chooses an option price to maximize her total revenue. Denote each buyer's best response as $Q(\pi)$. The seller's problem is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) &= N[\pi Q(\pi)] \\ \text{s.t.} \quad & 0 \leq Q(\pi) \leq \frac{C}{N} \\ & \pi \geq 0 \end{aligned} \tag{3.6.29}$$

Theorem 3.6.6. *Seller's optimal option price* $\pi^* = \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}$.

Note $Q(\pi^*) = C/N$. Theorem 3.6.6 states that if the capacity is small, in market equilibrium, the seller sets the option contract price such that each buyer's best response is equal to the capacity equally distributed among all buyers. The surplus of the seller, the total surplus of the buyers and the total social surplus with $\lambda = 1$ are compared to those with $\lambda = 0$.

Theorem 3.6.7. *If the seller's capacity* $C \leq \min\{C_b, C_c\}$, *then both the seller and the buyers are better off when* $\lambda = 1$ *compared to* $\lambda = 0$, *i.e.,* $G(1) > G(0)$, $V(1) > V(0)$ *and* $W(1) > W(0)$.

Theorem 3.6.7 shows that when the capacity is small, both the seller and the buyers are better off with $\lambda = 1$ compared to $\lambda = 0$. The two thresholds C_b and C_c are from the case when all buyers participate in the spot market. The first threshold C_b comes from the seller's problem at stage 3. If $C \leq C_b$, it is the undercapacity case in both demand states and the optimal spot price is determined solely by the capacity constraint. The second threshold C_c comes from the seller's problem at stage 1. If the capacity is smaller than C_c and C_b, then the best strategy for the seller is to set the option price high enough such that no buyer enters into contracts at all. The rationale behind Theorem 3.6.7 is clear. If all buyers only participate in the contract market, all buyers purchase C/N contracts in Period 1. As the buyers don't transact on the spot market, the limited products can't be efficiently reallocated among the buyers according to their different utilities in Period 2. Some of the buyers with higher utility could not purchase extra products and some of them with lower utility hold unused contracts. On the other hand, the spot market provides an opportunity for the seller to sell the remaining capacity to the buyers with higher demand. Therefore, total spot market participation increases both the buyers' total surplus and the seller's surplus, thereby the total social surplus.

3.7 Two sellers, single buyer

This section studies a market where there are two sellers and a single buyer. We assume the two sellers have identical capacity C and the same marginal cost c.

The sequence of the decisions is as follows. In the first period, the two sellers choose the option prices simultaneously. Given the option prices, the buyer decides how many contracts to enter with each seller. In period 2, demand state is observed. The two sellers choose the spot prices. The buyer then decides the quantity of contracts to exercise with each seller. If the buyer's policy is to participate the spot market, he also determines the quantity of products to purchase from each seller on the spot market depending on the spot prices.

The buyer's normal utility as a function of the total quantity purchased from both sellers is assumed the same as that in Section 3.4. In order to make the model complete, we will introduce assumptions on buyer's normal demand (without contracting stage) in the two-seller setting. In particular, we need to define the demand and profit for each seller for any given price pair (s_i, s_j) (whenever i and j appear in the same expression, index i is not equal to j). One widely used assumption in literature, such as in [40], [34] and [44], is as follows. Suppose the two sellers' available capacities are C_1 and C_2 respectively (C_1 and C_2 can be different). If the two sellers charge different prices, $s_i < s_j$, then the buyer first purchases from seller i at the lower price. When seller i's capacity is exhausted, the buyer then turns to seller j. Under this assumption, the residual demand for the higher price seller j is the minimum of his available capacity and $(\alpha - \beta s_j - C_i)^+$. Note this residual demand for seller j does not depend s_i anymore. If the two sellers choose the same price, the buyer splits the total demand between the two sellers proportionally to the sellers' capacities.

Assumption 3.7.1. *For any price pair (s_i, s_j), we assume that the demand for seller*

i is as follows:

$$D_i(s_i, s_j) = \begin{cases} \min\left\{C_i, (\alpha - \beta s_i)^+\right\} & \text{if } s_i < s_j, \\ \min\left\{C_i, \frac{C_i}{C_i + Cj}(\alpha - \beta s_i)^+\right\} & \text{if } s_i = s_j, \\ \min\left\{C_i, (\alpha - \beta s_i - C_j)^+\right\} & \text{if } s_i > s_j, \end{cases} \tag{3.7.1}$$

Note the above assumption includes the case $C_1 \neq C_2$, which is necessary to our model. Even the two sellers have the same capacity, in the two-period problem, the buyer can purchase and exercise different quantity of contracts with each seller. Thus, the remaining capacities that the sellers can put on the spot markets might be different.

For our two-period model, another additional assumption is needed. Let Q_i and Q_j be the contracting quantities signed with seller i and seller j respectively. Suppose in Period 2, the total quantity to purchase under the contracts, denoted as q_c, is less than $Q_i + Q_j$ (in this case, there is no transaction on the spot market at all), then q_c is divided between the two sellers according to the following assumption. Denote $q_{c,i}$ and $q_{c,j}$ as the quantities to exercise with seller i and seller j respectively.

Assumption 3.7.2. *If the total quantity of contracts the buyer decides to exercise is less than the total contracting quantity Q_i+Q_j, then the quantity of products purchased from each seller is proportional to the contracting levels,*

$$q_{c,i} = \frac{q_c Q_i}{Q_i + Qj} \tag{3.7.2}$$

Note that Assumption 3.7.1 and 3.7.2 do not affect the buyer's payoff. However, they do affect the way in which the profit splitting between the sellers.

As in previous sections, we will consider the three cases, $\lambda = 0$, $\lambda = 1$ and $0 < \lambda < 1$. If the sellers' capacities are large enough, it can be shown that in all three cases, the game reaches at the competitive equilibrium. The unique equilibrium is both sellers charge zero for options and choose the marginal cost as the spot prices if the buyer participates in the spot markets. In this study, we only investigate if the two sellers' capacities smaller than a certain threshold, a pure strategy equilibrium exits. We first start with $\lambda = 1$ and explicitly characterize such threshold.

3.7.1 Contract market and spot market with full participation

In this section, we present our model with $\lambda = 1$.

3.7.1.1 Stage 4 – buyer's problem

Let s_1 and s_2 denote the spot prices charged by the two sellers. Denote the quantities of contracts to exercise with each seller as $q_{c,1}$ and $q_{c,1}$. Let $q_{s,2}$ and $q_{s,2}$ be the quantities of products the buyer decides to purchase on the spot market from the two sellers. Given the contracting level pair (Q_1, Q_2), the buyer's problem is

$$
\begin{aligned}
g(Q_1, Q_2, s_1, s_2, \alpha) \quad = \quad & \max_{q_{c,1}, q_{c,2}, q_{s,1}, q_{s,2}} \quad r_2(q_{c,1}, q_{c,2}, q_{s,1}, q_{s,2} | \alpha, s_1, s_2) \\
& \text{s.t.} \quad 0 \leq q_{c,1} \leq Q_1 \\
& \qquad 0 \leq q_{c,2} \leq Q_2 \\
& \qquad 0 \leq q_{s,1} \leq C - Q_1 \\
& \qquad 0 \leq q_{s,2} \leq C - Q_2
\end{aligned}
\tag{3.7.3}
$$

where

$$
\begin{aligned}
r_2(q_{c,1}, q_{c,2}, q_{s,1}, q_{s,2} | \alpha, s_1, s_2) \quad = \quad & -\frac{(q_{c,1} + q_{c,2} + q_{s,1} + q_{s,2})^2}{2\beta} + \frac{\alpha(q_{c,1} + q_{c,2} + q_{s,1} + q_{s,2})}{\beta} \\
& -(q_{c,1} + q_{c,2})c - q_{s,1}s_1 - q_{s,2}s_2
\end{aligned}
\tag{3.7.4}
$$

It is easy to see that only when $\alpha - \beta c > Q_1 + Q_2$, the buyer purchases from the spot markets. Based on the single-seller single-buyer model, we limit to the case $C \leq C_a = (\alpha_l - \beta c)/2$. Then $Q_1 + Q_2 \leq \alpha - \beta c$ for both high and low states. For the single-seller model, the total residual demand on the spot market is $(\alpha-\beta s-Q_1-Q_2)^+$. Under Assumption 3.7.1, $q^*_{s,i}$, $i=1,2$ can be determined in the two-seller case.

Lemma 3.7.1. *If each seller's capacity $C \leq C_a$, the buyer's optimal decision in Period 2, $q^*_{c,i}, q^*_{s,i}$ for $i=1,2$, is as follows.*

1. *All the contracts are exercised, i.e., $q^*_{c,i} = Q_i$, $i=1,2$.*

2. *For the quantities to be purchased from the spot market,*

$$q^*_{s,i} = \begin{cases} \min\left\{C - Q_i, (\alpha - \beta s_i - Q_i - Q_j)^+\right\} & \text{if } s_i < s_j, \\ \min\left\{C - Q_i, \frac{C-Q_i}{2C-Q_i-Q_j}(\alpha - \beta s_i - Q_i - Q_j)^+\right\} & \text{if } s_i = s_j, \\ \min\left\{C - Q_i, (\alpha - \beta s_i - C - Q_i)^+\right\} & \text{if } s_i > s_j, \end{cases} \tag{3.7.5}$$

3.7.1.2 Stage 3 – sellers' problem

At this stage, seller i's return on the spot market for a pair of prices (s_i, s_j) is

$$R_{2,i}(s_i, s_j | Q_i, Q_j, \alpha) = (s_i - c) q_{s,i}(Q_i, Q_j, s_i, s_j, \alpha) \tag{3.7.6}$$

where $q_{s,i}(Q_i, Q_j, s_i, s_j, \alpha)$ denotes the best response characterized in Lemma 3.7.1. Let s_0 be the spot price, such that

$$\alpha - \beta s_0 - Q_1 - Q_2 = 2C - Q_1 - Q_2 \tag{3.7.7}$$

That is if both sellers choose price $s_0 = (\alpha - 2C)/\beta$, all the remaining capacities are sold. We will show that if C is less or equal to a threshold, (s_0, s_0) is the unique pure strategy equilibrium for any $Q_i \in [0, C]$, $i=1,2$.

Lemma 3.7.2. *If $C \leq C_d = (\alpha_l - \beta c)/3$, then (s_0, s_0) is the only pure equilibrium in the subgame.*

3.7.1.3 Stage 2 – buyer's problem

At stage 2, the buyer decides the best contracting quantities with each seller. The buyer's problem is

$$\begin{aligned} g_1(\pi_1, \pi_2) \quad &= \quad \max_{Q_1, Q_2} \; r_1(Q_1, Q_2 | \pi_1, \pi_2) \quad = \quad -\pi_1 Q_1 - \pi_2 Q_2 + \mathbb{E}[g_2(Q_1, Q_2, s_1, s_2, \alpha)] \\ &\quad\;\; \text{s.t.} \qquad\qquad 0 \;\le\; Q_1 \;\le\; C \qquad\qquad\qquad\qquad\qquad\qquad (3.7.8) \\ &\qquad\qquad\qquad\qquad\;\; 0 \;\le\; Q_2 \;\le\; C \end{aligned}$$

By Lemma 3.7.1, for any $Q_1, Q_2 \in [0, C]$, it holds that

$$\begin{aligned} \mathbb{E}[g_2(Q_1, Q_2, s_1, s_2, \alpha)] \;&=\; p\left[-\frac{(2C)^2}{2\beta} + \frac{2C\alpha_h}{\beta} - (Q_1 + Q_2)c - (2C - Q_1 - Q_2)s_0^h\right] \\ &+ (1-p)\left[-\frac{(2C)^2}{2\beta} + \frac{2C\alpha_l}{\beta} - (Q_1 + Q_2)c - (2C - Q_1 - Q_2)s_0^l\right] \quad (3.7.9) \end{aligned}$$

where $s_0^h = (\alpha_h - 2C)/\beta$ and $s_0^l = (\alpha_l - 2C)/\beta$.

Lemma 3.7.3. *If $C \le C_d$, the buyer's optimal contracting quantities $Q_i^*(\pi)$, $i = 1, 2$ are as follows:*

1. *If $\pi_i = \frac{\mathbb{E}(\alpha) - \beta c - 2C}{\beta}$, then any $Q_i \in [0, C]$ is optimal.*

2. *If $\pi_i > \frac{\mathbb{E}(\alpha) - \beta c - 2C}{\beta}$, then $Q_i^* = 0$.*

3. *If $\pi_i < \frac{\mathbb{E}(\alpha) - \beta c - 2C}{\beta}$, then $Q_i^* = C$.*

From Lemma 3.7.3, we can see that the buyer's optimal contracting demand $Q_i^*(\pi_i, \pi_j)$ only depends on π_i but not on seller j's option price π_j. It is because the buyer knows no matter how many contracts enter now, all of the remaining capacities are going to be sold at the expected spot price $(\mathbb{E}(\alpha) - 2C)/\beta$ as a consequence of the competition of the two sellers later. Therefore, the buyer only need to compare the option prices to the expected spot price minus the marginal cost c.

3.7.1.4 stage 1 – sellers' problem

At this stage, seller i's problem for a given π_j is

$$\begin{aligned} \max_{\pi_i} \quad R_{1,i}(\pi_i, \pi_j) &= \pi_i Q_i(\pi_i, \pi_j)\mathbb{E}[G_{2,i}(Q_i, Q_j, \alpha)] \\ \text{s.t.} \qquad\qquad \pi_i &\geq 0 \end{aligned} \tag{3.7.10}$$

where Q_i and Q_j are characterized in Lemma 3.7.3 and $\mathbb{E}[G_{2,i}(Q_i, Q_j, \alpha)]$ denotes the the expected return from the spot market in equilibrium in the subgame.

Theorem 3.7.1. *If both sellers' capacities are small such that $C \leq C_d$, for each seller, any option price $\pi \geq (\mathbb{E}(\alpha) - \beta c - 2C)/\beta$ is optimal. Each seller's optimal revenue is $C(\mathbb{E}(\alpha) - \beta c - 2C)/\beta$.*

3.7.2 Contract market only

Henceforth, we restrict our study to the case $C \leq C_d$. If the buyer only participates in the contract market, in Period 2, the buyer's decision problem is

$$\begin{aligned} g_2(Q_1, Q_2, \alpha) = \max_{q_{c,1}, q_{c,2}} \quad & r_2(q_{c,1}, q_{c,2}|\alpha) \\ \text{s.t.} \quad & 0 \leq q_{c,1} \leq Q_1 \\ & 0 \leq q_{c,2} \leq Q_2 \end{aligned} \tag{3.7.11}$$

where

$$r_2(q_{c,1}, q_{c,2}|\alpha) = -\frac{(q_{c,1} + q_{c,2})^2}{2\beta} + \frac{\alpha(q_{c,1} + q_{c,2})}{\beta} - (q_{c,1} + q_{c,2})c \tag{3.7.12}$$

Since $C \leq C_d$, $Q_i \leq C \leq C_d$. Thus, $Q_1 + Q_2 \leq 2(\alpha_l - \beta c)/3 \leq \alpha_l - \beta c$. All the contracts are exercised in both demand states.

Lemma 3.7.4. *If $C \leq C_d$, the best strategy for the buyer is to exercise all the contracts, i.e., $q^*_{c,1} = Q_1$ and $q^*_{c,2} = Q_2$.*

In Period 1, the buyer's problem is

$$
\begin{aligned}
g_1(\pi_1, \pi_2) \quad = \quad \max_{Q_1, Q_2} r_1(Q_1, Q_2 | \pi_1 \pi_2) \quad &= \quad -\pi_1 Q_1 - \pi_2 Q_2 + \mathbb{E}[g_2(Q_1, Q_2, \alpha)] \\
\text{s.t.} \quad 0 \quad \leq \quad Q_1 \quad &\leq \quad C \\
0 \quad \leq \quad Q_2 \quad &\leq \quad C
\end{aligned}
$$

By Lemma 3.7.4, for any $Q_1, Q_2 \in [0, C]$, it holds that

$$
\begin{aligned}
\mathbb{E}[g_2(Q_1, Q_2, \alpha)] \quad = \quad & p \left[-\frac{(Q_1 + Q_2)^2}{2\beta} - \frac{\alpha_h (Q_1 + Q_2)}{\beta} - (Q_1 + Q_2) c \right] \\
& + (1 - p) \left[-\frac{(Q_1 + Q_2)^2}{2\beta} - \frac{\alpha_l (Q_1 + Q_2)}{\beta} - (Q_1 + Q_2) c \right] \qquad (3.7.13)
\end{aligned}
$$

Define function $D_c(\pi)$ as follows, which is the contracting strategy in the single-seller case.

$$
D_c(\pi) = \begin{cases}
\alpha_h - \beta c - \beta \pi / p & \text{if } \pi \in [0, p(\alpha_h - \alpha_l)/\beta), \\
\mathbb{E}(\alpha) - \beta c - \beta \pi & \text{if } \pi \in [p(\alpha_h - \alpha_l)/\beta, (\mathbb{E}(\alpha) - \beta c)/\beta], \\
0 & \text{if } \pi > (\mathbb{E}(\alpha) - \beta c)/\beta
\end{cases} \qquad (3.7.14)
$$

Lemma 3.7.5. *If $C \leq C_d$, the buyer's best contracting strategy is as follows:*

$$
Q_i^* = \begin{cases}
\min \{C, D_c(\pi_i)\} & \text{if } \pi_i < \pi_j, \\
\min \{C, D_c(\pi_i)/2\} & \text{if } \pi_i = \pi_j, \\
\min \{C, (D_c(\pi_j) - C)^+\} & \text{if } \pi_i > \pi_j
\end{cases} \qquad (3.7.15)
$$

In Period 1, for every pair of option prices (π_i, π_j), the total revenue for seller i is

$$
R_{1,i}(\pi_i, \pi_j) \quad = \quad \pi_i Q_i(\pi_i, \pi_j) \qquad (3.7.16)
$$

where $Q_i(\pi_i, \pi_j)$ is determined by Lemma 3.7.5.

Theorem 3.7.2. *If both sellers' capacities are small such that $C \leq C_d$, option price pair (π^*, π^*) is the unique equilibrium where $\pi^* = (\mathbb{E}(\alpha) - \beta c - 2C)/\beta$. Each seller's optimal revenue is $C(\mathbb{E}(\alpha) - \beta c - 2C)/\beta$.*

3.7.3 Contract market and spot market with partial participation

This section considers the case in which the buyer participates in the spot market with some probability $\lambda \in (0,1)$ for comparison purpose. At stage 1, the sellers don't know the buyer's contracting policy. The decision problem for seller i is

$$\begin{aligned} \max_{\pi_i} \quad R_{1,i}(\pi_i,\pi_j) &= (1-\lambda)R_{1,i,A}(\pi_i,\pi_j) + \lambda R_{1,i,B}(\pi_i,\pi_j) \\ \text{s.t.} \quad \pi_i &\geq 0 \end{aligned} \tag{3.7.17}$$

where $R_{1,i,A}(\pi_i,\pi_j)$ denotes the return if the buyer only participates in the contract market and $R_{1,i,B}(\pi_i,\pi_j)$ denotes the return if the buyer participates in both markets, which have been studied in the previous subsections. Based on the previous results, we can see that option price pair (π^*,π^*), where $\pi^* = (\mathbb{E}(\alpha) - \beta c - 2C)/\beta$, is the unique equilibrium if the buyer doesn't participate in the spot market and it is also an equilibrium in the case when the buyer participates in both markets. Therefore, if the sellers do not know whether the buyer participates in the spot market or not, option price pair (π^*,π^*) is the only equilibrium.

Theorem 3.7.3. *If both sellers' capacities are small, such that $C \leq C_d$, the following results hold:*

1. *The option price pair (π^*,π^*) is the unique equilibrium, where $\pi^* = (\mathbb{E}(\alpha) - \beta c - 2C)/\beta$.*

2. *The expected number of contracts transacted $Q^*(\lambda)$ decreases in λ.*

3. *As λ increases, the sellers' total surplus $G(\lambda)$ does not change, $G(\lambda) = 2C(\mathbb{E}(\alpha) - \beta c - 2C)/\beta$.*

4. *As λ increases, the buyer's surplus $V(\lambda)$ does't change, $V(\lambda) = 2C^2/\beta$. Thus, the total social surplus $W(\lambda)$ does not change, $W(\lambda) = 2C(\mathbb{E}(\alpha) - \beta c - C)/\beta$.*

Theorem 3.7.3 is a direct consequence of Theorem 3.7.1 and 3.7.2. Theorem 3.7.3 says that if the sellers' capacities are small, the spot market participation does not change the sellers' total surplus, the buyer's surplus and the total social welfare. The results are consistent with those in the single-seller setting.

3.8 A continuum of sellers, a continuum of buyers

This section extends the model to a market where there are many sellers and many buyers for comparison purpose. Let $\mathcal{S}^\infty = [0, 1]$ denote the continuum of such sellers and let $\mathcal{B}^\infty = [0, N]$ denote the continuum of such buyers. Each seller is very small relative to the market as a whole and has capacity $Cd\xi$. The aggregated capacity is denoted as $\overline{C}$,

$$\overline{C} = \int C d\xi \tag{3.8.1}$$

It is also assumed that each small seller is relative large compared to the buyers. If the aggregated contracting quantity is large, a seller applies first-come first-serve rationing scheme until all of her contracts are sold. The utility function of the buyers is assumed the same as that in Section 3.6. All sellers move simultaneously in both periods. The sequence of the decisions of the sellers and the buyers is the same as before. In this section, we only investigate the first-come first-serve rationing scheme. A numerical study is conducted to investigate how the previous results change under this market structure.

The organization of this section is as follows. First, we consider the case all the buyers only participate in the contract market. Second, we consider the case some of buyers participate in both markets, i.e., $\lambda \in (0, 1)$. The formulation of the model for the case $\lambda = 1$ is an analogy to $\lambda \in (0, 1)$ and is not repeated here. Last, a numerical study is presented to illustrate the effects of the participation rate moving from 0 to 1. The results are compared to those in the corresponding single seller market.

3.8.1 Contract market only

In this setting, the decision problems at stage 3 and stage 2 are the same as those in Section 3.6. All the results in Lemma 3.6.1 and Lemma 3.6.2 apply in this case. At stage 1, each seller chooses option prices to maximize her own revenue. An assumption used here is that if all sellers charge the same price while one of the sellers increases the price, the buyers do not purchase from that seller at all since a single seller is very small compared to the market as a whole. The buyers only turn to the other sellers with lower price.

Theorem 3.8.1. *The sellers' equilibrium prices are as follows.*

1. *If $\overline{C}/N > \alpha_h + \overline{\phi} - \beta c$, all sellers charge the same option price $\pi^* = 0$.*

2. *If $\overline{C}/N < \alpha_h + \overline{\phi} - \beta c$, all sellers charge the same option price such that all the contracts sold. That is for $\overline{C}/N \in [0, \alpha_l + \overline{\phi} - \beta c)$, $\pi^* = \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-\overline{C}/N)^2}{4\beta\overline{\phi}}$, where $\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-\overline{C}/N)^2}{4\beta\overline{\phi}} = \frac{p(\alpha_h+\overline{\phi}-\beta c-\overline{C}/N)^2+(1-p)(\alpha_l+\overline{\phi}-\beta c-\overline{C}/N)^2}{4\beta\overline{\phi}}$. For $\overline{C}/N \in [\alpha_l + \overline{\phi} - \beta c, \alpha_h + \overline{\phi} - \beta c]$, $\pi^* = \frac{p(\alpha_h+\overline{\phi}-\beta c-\overline{C}/N)^2}{4\beta\overline{\phi}}$.*

Theorem 3.8.1 is a direct result from the above assumption under this market structure. It states that if there are many sellers and the aggregated capacity is not too large, then in equilibrium all the sellers charge the same option price such that all the contracts sold. Otherwise, they can only charge zero.

3.8.2 Contract market and spot market with partial participation

In this section, we present the model in the case $\lambda \in (0, 1)$.

3.8.2.1 Stage 4 – buyers' problem

The type A buyers' problem at this stage is the same as that in Section 3.6. The optimal decision for any Q_A is given by Lemma 3.6.1. The type B buyers' problem is the same as that in Section 3.6 and all the results of Lemma 3.6.3 hold.

3.8.2.2 Stage 3 – sellers' problem

At the stage, the sellers choose the spot prices simultaneously, given the buyers' contracting quantity Q_A and Q_B. Follow the same notation in Section 3.6.2.1. If $Q_B \geq \alpha+\overline{\phi}-\beta c$ and $\gamma = 1$, then no buyer purchases from the spot market. Otherwise, let $\hat{s}$ be the spot price at which all the remaining capacity is sold:

$$D_{s,B1}(Q_B, \hat{s}) + D_{s,B2}(0, \hat{s}) \quad = \quad \overline{C} - D_{c,A}(Q_A) - D_{c,B}(Q_B) \tag{3.8.2}$$

Since $D_{s,B1}(Q_B, s) + D_{s,B2}(0, s)$ strictly decrease in s, $\hat{s}$ is unique.

Lemma 3.8.1. *If $Q_B \geq \alpha + \overline{\phi} - \beta c$ and $\gamma = 1$, then no buyer purchases from the spot market. Otherwise, the sellers charge a unique spot price $s^* = \max\{\hat{s}, c\}$ in equilibrium.*

Lemma 3.8.1 states that in equilibrium, all sellers charge the same spot price. If the remaining capacity compared to the residual demand is small such that $\hat{s} \geq c$, then all sellers charge $\hat{s}$. Otherwise, they charge the marginal cost c. This lemma is also a direct consequence of the previous assumption.

3.8.2.3 Stage 2 – buyers' problem

The decision problem and the optimal decision for type A buyers are same as those in Section 3.6 and all the results in Lemma 3.6.2 hold. The formulation of the type B buyers' problem is the same as that for the first-come first-serve rationing rule in Section 3.6.2, with $\widetilde{s_h}(Q_A, Q_B)$ and $\widetilde{s_l}(Q_A, Q_B)$ defined as the spot prices from Lemma 3.8.1.

3.8.2.4 Stage 1– sellers' problem

At this stage, each seller chooses the option price to maximize her own revenue. We first start with the equilibrium such that all of the sellers charge the same option price π^* from which none of the sellers has incentive to deviate. Denote the first stage revenue for seller i as $R_{1,i}(\pi_i, \pi_{-i})$, where π_i denotes the price charged by seller i and π_{-i} denotes option price charged by all the sellers other than seller i. Then π^* must satisfy

$$R_{1,i}(\pi^*, \pi^*) \quad \geq \quad R_{1,i}(\pi, \pi^*), \qquad \forall \pi > 0, \pi \neq \pi^*, \quad \forall i \tag{3.8.3}$$

If one of the sellers raises the option price, then none of her contracts will be sold. That seller will put all the capacity on the spot market. If the seller lowers her option price, all of her contracts are sold under the first-come first-serve scheme. In both cases, the spot prices in the second period do not change since the influence of a single seller is negligible.

Denote the equilibrium spot prices in the subgame as s_h and s_l for the high demand state and the low demand state respectively when all sellers charge the same π. If seller i decreases the price to π', $\pi' < \pi$, then her total revenue is formulated as follows. Since a single seller is an infinitesimal and other sellers still stay at the same price, the spot prices do not change. All the buyers try to purchase the contracts from this particular seller at the lower price first. If their requests are not satisfied, they will turn to other sellers. Let $Q'_A = Q_A(\pi')$ and $Q'_B = Q_B(\pi')$ denote the best response with respect to option price π'. Seller i can only satisfy a very small fraction of the buyers:

$$\widetilde{\gamma}' \quad = \quad \frac{C d\xi}{N[(1-\lambda)Q'_A + \lambda Q'_B]} \tag{3.8.4}$$

Let $q^h_{c,A}(Q'_A, \phi)$, $q^l_{c,A}(Q'_A, \phi)$, $q^h_{c,B}(Q'_B, \phi)$ and $q^l_{c,B}(Q'_B, \phi)$ denote the corresponding optimal quantities to purchase under contracts per buyer at stage 4 for the two types of buyers in the high and low demand states. Define

$$\overline{q}^h_{c,A}(Q'_A) \quad = \quad \frac{1}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} q^h_{c,A}(Q'_A, \phi) d\phi \tag{3.8.5}$$

$$\overline{q}^l_{c,A}(Q'_A) \quad = \quad \frac{1}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} q^l_{c,A}(Q'_A, \phi) d\phi \tag{3.8.6}$$

$$\overline{q}^h_{c,B}(Q'_B) = \frac{1}{2\overline{\phi}}\int_{-\overline{\phi}}^{\overline{\phi}} q^h_{c,B}(Q'_B,\phi)d\phi \tag{3.8.7}$$

$$\overline{q}^l_{c,B}(Q'_B) = \frac{1}{2\overline{\phi}}\int_{-\overline{\phi}}^{\overline{\phi}} q^l_{c,B}(Q'_B,\phi)d\phi \tag{3.8.8}$$

Among those buyers satisfied by seller i, the total contracts exercised by type A buyers in the high demand state and the low demand state are $\widetilde{\gamma}'(1-\lambda)N\overline{q}^h_{c,A}(Q'_A)$ and $\widetilde{\gamma}'(1-\lambda)N\overline{q}^l_{c,A}(Q'_A)$. The total contracts exercised by type B buyers in the two demand states are $\widetilde{\gamma}'\lambda N\overline{q}^h_{c,B}(Q'_B)$ and $\widetilde{\gamma}'\lambda N\overline{q}^l_{c,B}(Q'_B)$. It follows that revenue for seller i at the first stage is

$$\begin{aligned} R_{1,i}(\pi',\pi) &= \pi' C d\xi + p(s_h - c)Cd\xi\left[1 - \frac{(1-\lambda)\overline{q}^h_{c,A}(Q'_A) + \lambda\overline{q}^h_{c,B}(Q'_B)}{(1-\lambda)Q'_A + \lambda Q'_B}\right] \\ &\quad + (1-p)(s_l - c)Cd\xi\left[1 - \frac{(1-\lambda)\overline{q}^l_{c,A}(Q'_A) + \lambda\overline{q}^l_{c,B}(Q'_B)}{(1-\lambda)Q'_A + \lambda Q'_B}\right] \end{aligned} \tag{3.8.9}$$

If seller i increases the option price to π'' with $\pi'' > \pi$, then the total revenue for that seller is

$$R_{1,i}(\pi'',\pi) = Cd\xi[p(s_h - c) + (1-p)(s_l - c)] \tag{3.8.10}$$

Therefore, the option price π^* in equilibrium must satisfy

$$R_{1,i}(\pi^*,\pi^*) \geq R_{1,i}(\pi',\pi^*), \qquad \forall \pi > 0, \pi' < \pi^*, \quad \forall i \tag{3.8.11}$$

$$R_{1,i}(\pi^*,\pi^*) \geq R_{1,i}(\pi'',\pi^*), \qquad \forall \pi > 0, \pi'' > \pi^*, \quad \forall i \tag{3.8.12}$$

3.8.3 Numerical study

In this section, a numerical study is used to investigate the effects of the buyers' participation rate of the spot market. As discussed in the previous section, we need to consider the uniqueness of the equilibrium in the study. Otherwise, the outcome of the game is unpredictable. Though it seems there might be multiple equilibriums, such unpredictable situation never takes place in all the numerical examples tested here.

The input parameters are the same as those in Section 3.6.2. The results are shown in Figure 17–20.

Observation 1 As more and more buyers participate in the spot market, the aggregated contracting quantity decreases.

Observation 2 As more and more buyers participate in the spot market, the total social surplus always increases.

Observation 3 As more and more buyers participate in the spot markets, the sellers' total surplus may increase/decrease and the buyers' total surplus may increase/decrease.

As the spot market can make better allocation to the buyers with different utilities, higher total social surplus can be obtained as more and more buyers participate in the spot market (Figure 17–20). Compared to the single seller setting, the competition among the sellers can hurt the sellers if the capacity is large and one type of buyers dominates the market. However, the balance between the two types of the buyers can make the sellers better off. Figure 17 shows that the surplus of the sellers increases in λ on $[0.2, 0.4]$, where the sizes of the two types of buyers become close to each other. As type B buyers dominate the market, the total surplus of the sellers decreases again. From Figure 18 to Figure 20, the results approach the single seller case. This is due to the tight capacity. In that case, the sellers can drive the equilibrium to the spot market period and exact more profit from the buyers.

Figure 21–24 compare the results to those from the corresponding single seller setting. The solid lines represent the results of the single seller market from Section 3.6.2 and the dotted lines indicate the result of the market with a continuum of sellers. The results approach to the singe seller setting as the sellers' total capacity decreases. Due to the competition among the sellers, the market with many sellers always achieves higher total social surplus and higher buyers' total surplus compared to the single seller market.

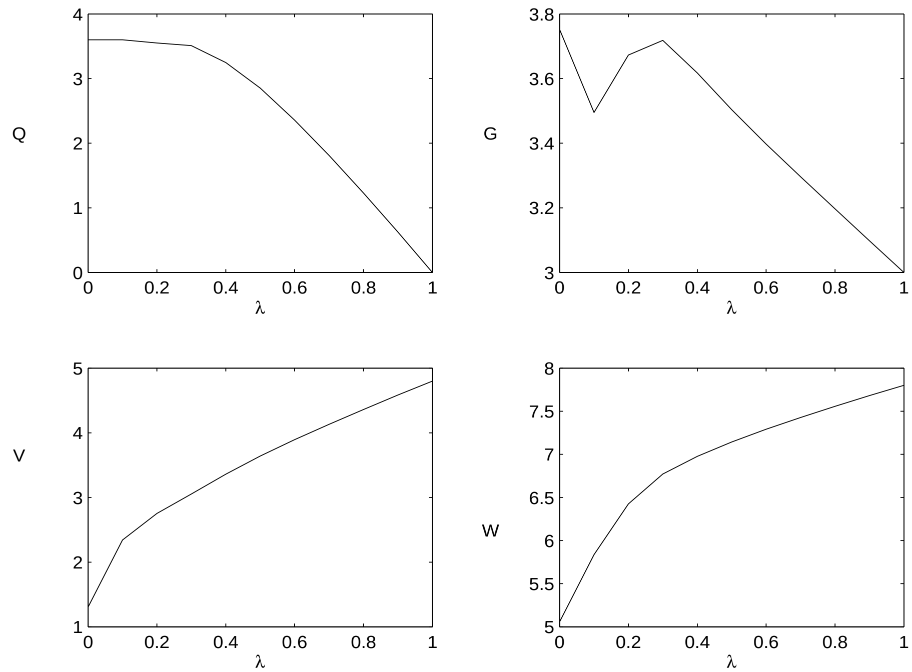

Figure 17: Effects of spot market participation for Example 12, capacity = 3.6

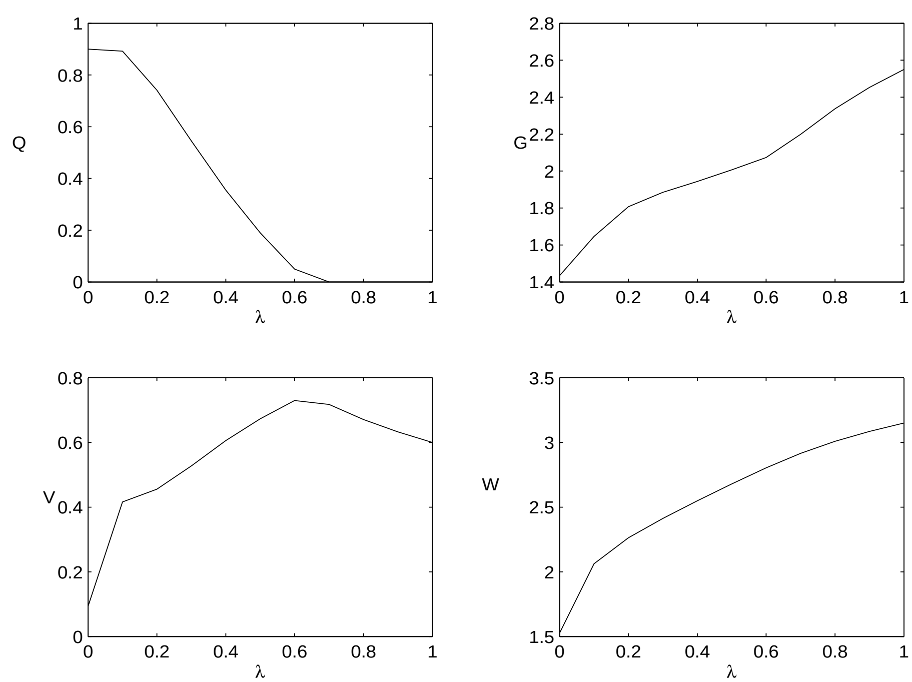

Figure 18: Effects of spot market participation for Example 13, capacity = 0.9

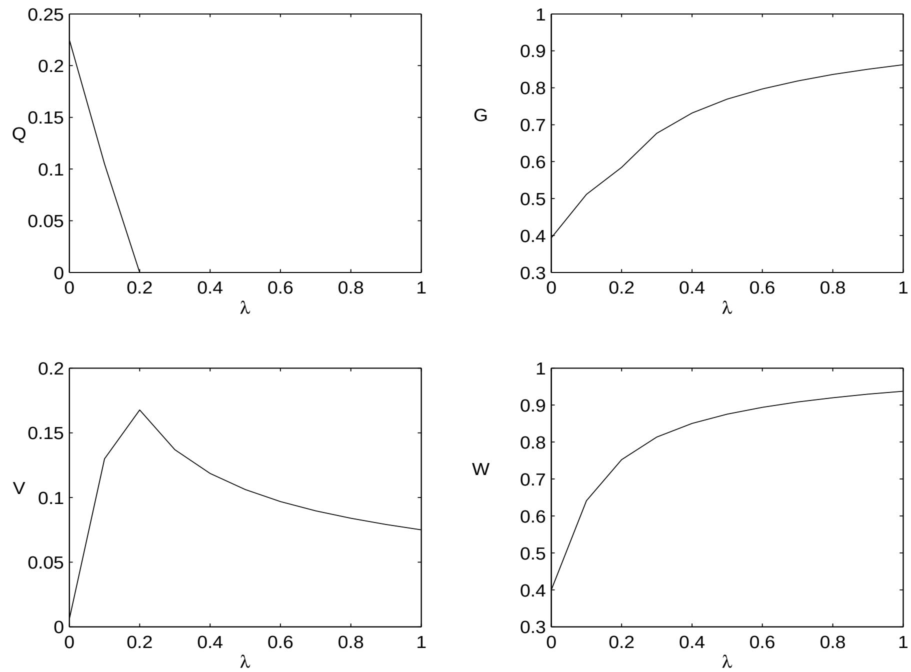

Figure 19: Effects of spot market participation for Example 14, capacity $= 0.225$

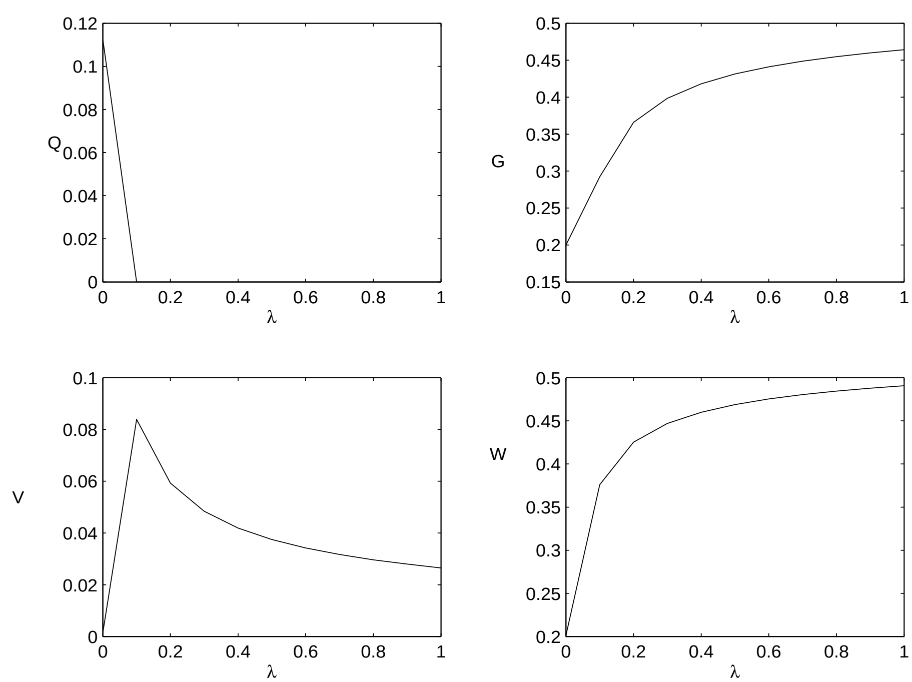

Figure 20: Effects of spot market participation for Example 15, capacity $= 0.1125$

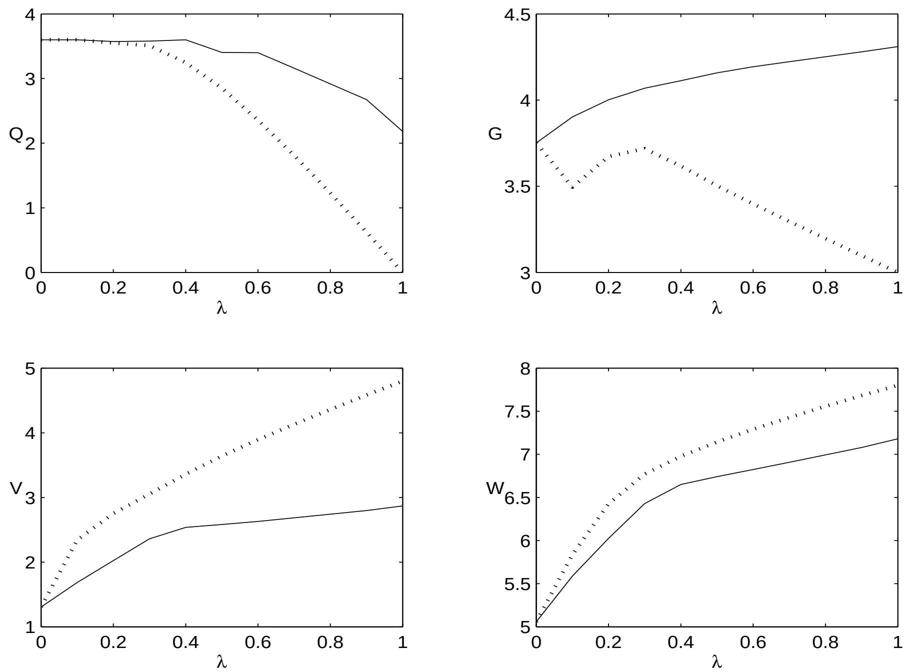

Figure 21: Comparison of Example 8 and 12, capacity = 3.6

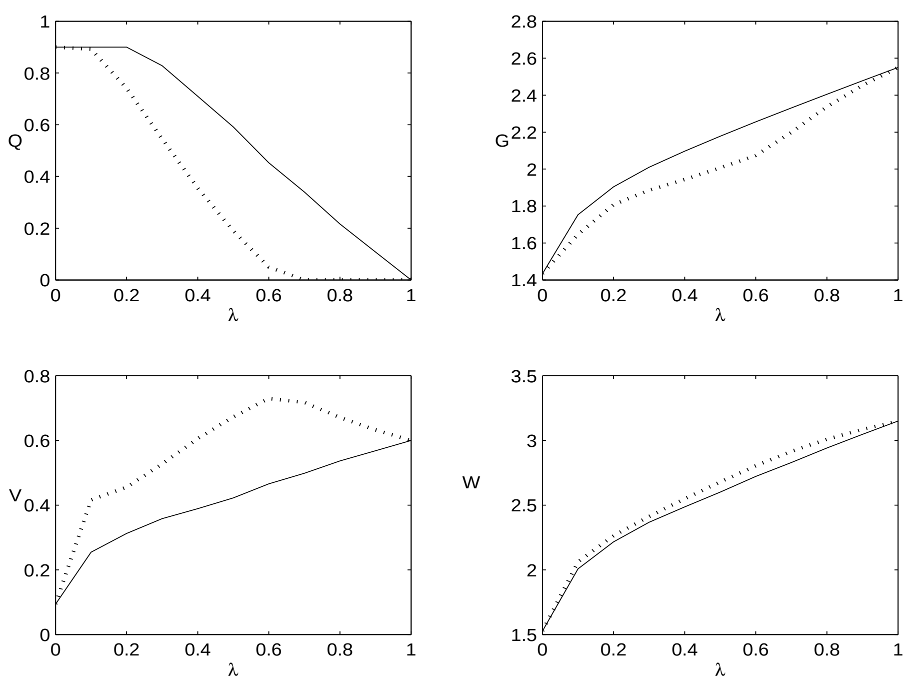

Figure 22: Comparison of Example 9 and 13, capacity = 0.9

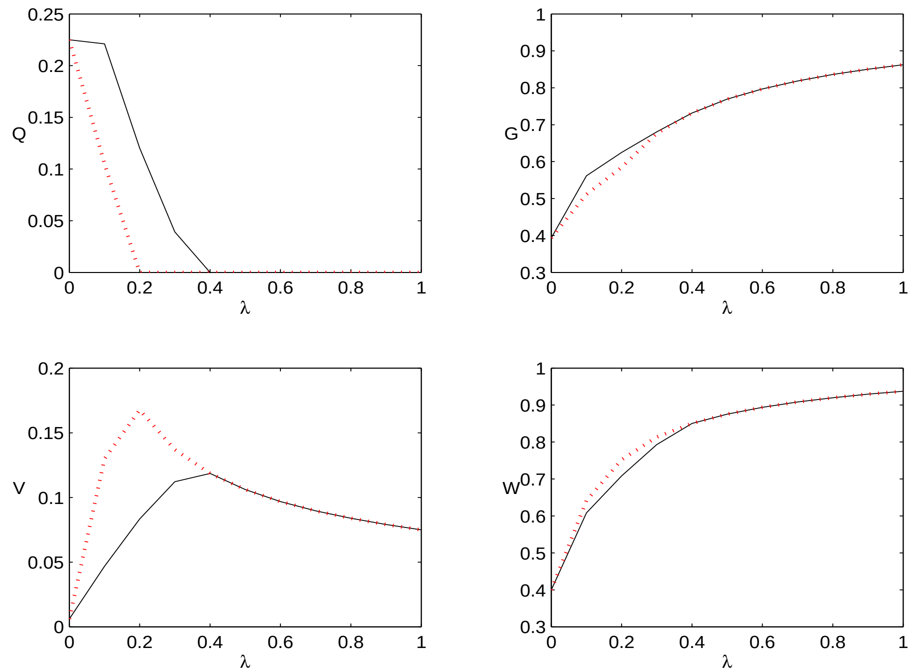

Figure 23: Comparison of Example 10 and 14, capacity = 0.225

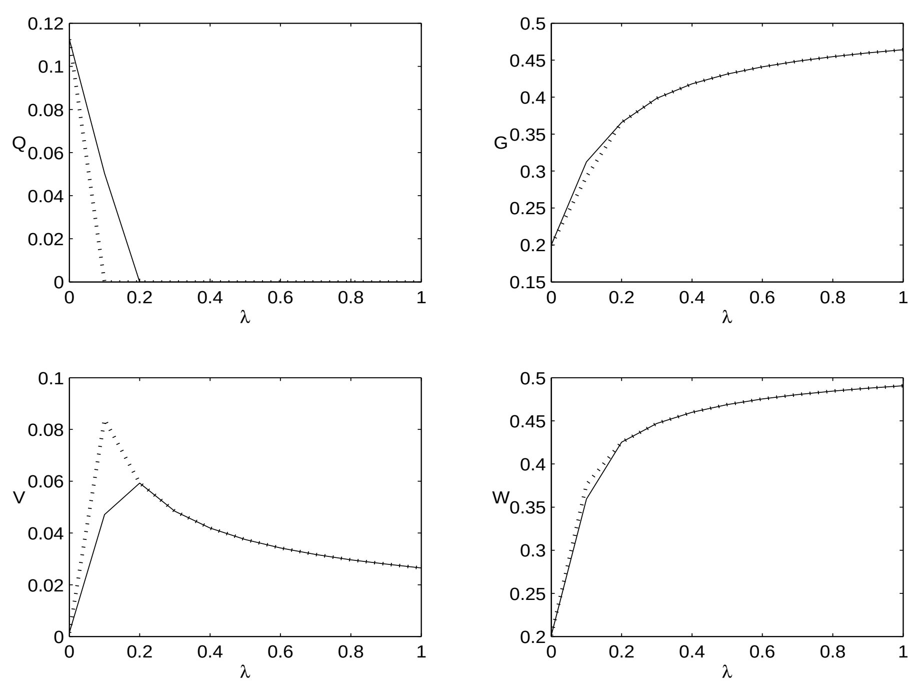

Figure 24: Comparison of Example 11 and 15, capacity = 0.1125

3.9 Concluding remarks

Though the original motivation of this research comes from the freight transportation industries, the models are general and applicable to other industries with non-storable products or service facing demand uncertainty. A key factor considered is the buyers' participation in the spot market. Our focus is to investigate the role of the spot market. Specifically, we study the effects of the spot market participation rate on the quantity of contracts transacted, as well as on the surpluses of all players and on the total social surplus. As mentioned in previous sections, some literature also addresses the friction in spot markets from different points of view. Wu et al. [58] consider the risk factor m for the spot market, which represents the percentage ($0 \leq m \leq 1$) of the residual output that the seller can sell on the spot market. In their setting, the contract market exists only if $m < 1$. In contrast to the results of the single-seller single-buyer model in [58], the contract market always exists even when $\lambda = 1$ in our single-seller single-buyer setting due to the buyer's strategic reason. In [42], a comparable factor studied is the cost of spot trading. It refers to the fact that each unit a market participant trades drives the price against her. Due to such cost of trading, the spot market does not push the contracted quantities to zero either.

The effect of the capacity level is also considered. It is assumed that available contracts do not exceed the capacity level. First, we derive models for the case where the capacity is large. In this case, the capacity constraint need not be considered at all. Second, we extend the results to the case where the capacity is small. In the latter case, all of the capacity is sold no matter the demand state turns out high or low.

We start with the single-seller single-buyer setting. For comparison purpose, we also study the case with $\lambda \in (0, 1)$, where λ is interpreted as the buyer's participation probability of the spot market. It is found that as the spot market participation rate increases, the contract market shrinks. Such result holds for both of the capacity levels considered. In the large capacity case, the quantity of contracts transacted remains positive even when $\lambda = 1$. As the spot market participation rate increases, the seller's surplus strictly increases. The effects of the spot market participation rate on the buyer's surplus and on the total social surplus are more complicated. For the large capacity case, an increase in the spot market participation rate may or may not benefit the buyer, therefore may or may not increase the total social surplus. Even when the buyer participates in the spot market with probability 1, the buyer can be worse off compared to the contract market only setting ($\lambda = 0$). One interesting observation is the relationship between the option price and the spot price. It is found that as long as the contracting quantity is not too large, an increase in option price also leads an increase in the spot price. The results for the small capacity case are fairly different. Since the capacity is small, all the products are sold in Period 2 regardless of the state of the market. The buyer decides whether to enter contracts only by comparing the option price to the expected spot price. All values of λ eventually give the same outcome. That is the surpluses of the seller and the buyer do not change as the spot market participation rate increases.

Second, we extends the single-buyer model to a continuum of buyers who have the same utility. We model the spot market participation as a fraction of buyers transacting on the spot market. The results from the single-buyer setting hold in general in the large capacity case. However, under this setting, the quantity of contracts transacted decreases to zero as all buyers transact on the spot market. In the small capacity case, one important issue is the allocation of the limited quantity of contracts among the buyers. We study two rationing schemes that are used in practice, first-come first-serve and limiting the contracting quantity per buyer. It is found that the surplus of the seller and the total surplus of the buyers are invariant with respect to the spot market participation regardless which rationing scheme is applied. The results are surprisingly the same as those from the single-buyer model.

Third, we study a market where there are a single seller and a continuum of buyers. In addition to the random state of the market, we also consider the demand uncertainty dependent on each individual buyer. A numerical study is conducted to investigate the effects of the spot market participation. Though those results may not hold in general, they still provide valuable insights. The numerical analysis indicates that the effects of the spot market participation on the contract market and on the seller's surplus still hold. However, the buyers' total surplus always increases in λ. We also prove that the buyers are always better off when $\lambda = 1$ than when $\lambda = 0$ in the undercapacity case with the second rationing rule.

Fourth, we consider a market with two sellers and a single buyer. The undercapacity case is studied. It is found that all the results in the undercapacity case of the single-seller single-buyer setting hold. This again confirms that the outcome does not change for all values of λ if the capacity level is low enough.

In the last section of this chapter, we extend the model to a market with a continuum of sellers and a continuum of buyers with different utilities for comparison purpose. A numerical study is conducted to investigate how the results in this setting differ from those under other market structures. It is found the quantity of contracts transacted in equilibrium decreases in λ and the buyers' total surplus may increase/decrease in λ. In contrast to the single-seller setting, the sellers can be worse off as more buyers participate in the spot market because of the competition among the sellers. The total social welfare is observed always increasing in spot market participation.

From the results under different market structures, we can see that the contract market always shrinks as spot market participation increases. However, it does't guarantee a higher total social surplus. Under different demand variation and different market structure, the effects of the spot market participation on the surpluses of all market players may be different.

APPENDIX A

PROOFS FOR CHAPTER 3

Proof of Lemma 3.4.1 Since the objective function is concave in q_c, the Karush-Kuhn-Tucker (KKT) condition is necessary and sufficient for the optimality of the buyer's problem. The derivative of the objective function is $\frac{dr_2(q_c|\alpha)}{dq_c} = -\frac{q_c}{\beta} + \frac{\alpha}{\beta} - c$. Since $\frac{dr_2(0|\alpha)}{dq_c} = \frac{\alpha}{\beta} - c \geq 0$, $q_c^* \geq 0$. Depending on whether q_c^* is active at the constraint $q_c \leq Q$ or not, there are two cases.

Case 1: If q_c^* is not tight at the constraint, i.e., $q_c^* < Q$, then the optimal solution q_c^* satisfies $\frac{dr_2(q_c|\alpha)}{dq_c} = 0$. Thus, $q_c^* = \alpha - \beta c < Q$.

Case 2: If q_c^* is tight at the constraint, i.e., $q_c^* = Q$, then the optimal solution q_c^* satisfies $\frac{dr_2(q_c|\alpha)}{dq_c} \geq 0$. Thus, $q_c^* = Q \leq \alpha - \beta c$.

Therefore, if $Q > \alpha - \beta c$, then $q_c^* = \alpha - \beta c$. If $Q \leq \alpha - \beta c$, then, $q_c^* = Q$. $\square$

Proof of Lemma 3.4.2 Given the option price π, the buyer's objective function is as follows.

For $Q < \alpha_l - \beta c$, it holds that

$$r_1(Q|\pi) = -\pi Q + p\left[-\frac{Q^2}{2\beta} + \frac{\alpha_h Q}{\beta} - Qc\right] + (1-p)\left[-\frac{Q^2}{2\beta} + \frac{\alpha_l Q}{\beta} - Qc\right]$$

$$\frac{dr_1(Q|\pi)}{dQ} = -\pi - \frac{Q}{\beta} + \frac{\mathbb{E}(\alpha)}{\beta} - c$$

For $\alpha_l - \beta c \leq Q < \alpha_h - \beta c$, it holds that

$$r_1(Q|\pi) = -\pi Q + p\left[-\frac{Q^2}{2\beta} + \frac{\alpha_h Q}{\beta} - Qc\right] + \frac{(1-p)(\alpha_l - \beta c)^2}{2\beta}$$

$$\frac{dr_1(Q|\pi)}{dQ} = -\pi + p\left(-\frac{Q}{\beta} + \frac{\alpha_h}{\beta} - c\right)$$

For $Q \geq \alpha_h - \beta c$, it holds that

$$r_1(Q|\pi) = -\pi Q + \frac{p(\alpha_h - \beta c)^2}{2\beta} + \frac{(1-p)(\alpha_l - \beta c)^2}{2\beta} \text{ and } \frac{dr_1(Q|\pi)}{dQ} = -\pi \leq 0$$

Note that the objective function $r_1(Q|\pi)$ is continuous at the breakpoints $Q = \alpha_l - \beta c$ and $Q = \alpha_h - \beta c$. Also note that $r_1(Q|\pi)$ is concave on $[0, \alpha_l - \beta c)$, $[\alpha_l - \beta c, \alpha_h - \beta c)$

and $[\alpha_h - \beta c, +\infty)$ respectively. Let $r'_{1+}(Q|\pi)$ be the right derivative at Q and let $r'_{1-}(Q|\pi)$ be the left derivative at Q. The derivatives at the breakpoints are as follows.
At $Q = 0$, $\frac{dr_1(0|\pi)}{dQ} = \frac{\mathbb{E}(\alpha)}{\beta} - \pi - c$.
At $Q = \alpha_l - \beta c$, $r'_{1-}(\alpha_l - \beta c|\pi) = \frac{p(\alpha_h - \alpha_l)}{\beta} - \pi = r'_{1+}(\alpha_l - \beta c|\pi)$.
At $Q = \alpha_h - \beta c$, $r'_{1-}(\alpha_h - \beta c|\pi) = -\pi = r'_{1+}(\alpha_h - \beta c|\pi)$.
Since at $Q = \alpha_l - \beta c$ and $Q = \alpha_h - \beta c$, $r'_{1+}(Q|\pi) = r'_{1-}(Q|\pi)$, the objective function $r_1(Q|\pi)$ is concave on $[0, +\infty)$, i.e., $\frac{dr_1(Q|\pi)}{dQ}$ is a monotonically decreasing function of Q. Therefore, the buyer's optimal decision is as follows.

1. If $\frac{dr_1(0|\pi)}{dQ} \leq 0$ ($\Leftrightarrow \pi \geq \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$), then $Q^* = 0$.

2. If $\frac{dr_1(0|\pi)}{dQ} > 0$ and $\frac{dr_1(\alpha_l-\beta c|\pi)}{dQ} \leq 0$ ($\Leftrightarrow \frac{p(\alpha_h-\alpha_l)}{\beta} \leq \pi < \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$), then $Q^* = \mathbb{E}(\alpha) - \beta(c + \pi)$. Note that $Q^* \in (0, \alpha_l - \beta c]$.

3. If $\frac{dr_1(\alpha_l-\beta c|\pi)}{dQ} > 0$ and $\frac{dr_1(\alpha_h-\beta c|\pi)}{dQ} < 0$ ($\Leftrightarrow 0 < \pi < \frac{p(\alpha_h-\alpha_l)}{\beta}$), then $Q^* = \alpha_h - \beta c - \beta\pi/p$. Note that $Q^* \in (\alpha_l - \beta c, \alpha_h - \beta c)$.

4. If $\frac{dr_1(\alpha_h-\beta c|\pi)}{dQ} = 0$ ($\Leftrightarrow \pi = 0$), then any $Q \in [\alpha_h - \beta c, \infty)$ is optimal.

It should be noted that $Q^*(\pi)$ is a continuous function of π for any $\pi > 0$. If $\pi^* > 0$, $Q^*(\pi)$ is unique. However, if $\pi = 0$, any $Q \geq \alpha_h - \beta c$ is optimal. $\square$

Proof of Theorem 3.4.1 Based on the results of Lemma 3.4.2, $Q(\pi)$ is a continuous function of π. Therefore, the seller's objective function $R_1(\pi) = \pi Q(\pi)$ is also continuous in π. For $\pi = 0$, $R_1(\pi) = 0$. For $0 < \pi < \frac{p(\alpha_h-\alpha_l)}{\beta}$, $Q(\pi) = \alpha_h - \beta c - \beta\pi/p$, $R_1(\pi) = \pi\left(\alpha_h - \beta c - \beta\pi/p\right)$ and $\frac{dR_1(\pi)}{d\pi} = \alpha_h - \beta c - 2\beta\pi/p$. For $\frac{p(\alpha_h-\alpha_l)}{\beta} \leq \pi < \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$, $Q(\pi) = \mathbb{E}(\alpha) - \beta c - \beta\pi$, $R_1(\pi) = \pi(\mathbb{E}(\alpha) - \beta c - \pi\beta)$, and $\frac{dR_1(\pi)}{d\pi} = \mathbb{E}(\alpha) - \beta c - 2\beta\pi$. For $\pi > \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$, $Q(\pi) = 0$ and $R_1(\pi) = 0$. Note $R_1(\pi)$ is concave on $\left[0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$, $\left[\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right)$ and $\left[\frac{\mathbb{E}(\alpha)-\beta c}{\beta}, +\infty\right)$ respectively. The derivatives at the breakpoints are as follows.
At $\pi = 0$, $\frac{dR_1(0)}{d\pi} = \alpha_h - \beta c > 0$

At $\pi = \frac{p(\alpha_h-\alpha_l)}{\beta}$, $R'_{1-}(\pi) = -(\alpha_h - \alpha_l) + \alpha_l - \beta c$ and $R'_{1+}(\pi) = -p(\alpha_h - \alpha_l) + \alpha_l - \beta c$. Since $p \in (0,1)$ (assumption 3.4.1), $R'_{1-}(\pi) < R'_{1+}(\pi)$.

At $\pi = \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$, $R'_{1-}(\pi) = -(\mathbb{E}(\alpha) - \beta c) < 0$ and $R'_{1+}(\pi) = 0$.

Thus, the optimal price $\pi^* \in \left(0, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right)$. Since at $\frac{p(\alpha_h-\alpha_l)}{\beta}$ and $\frac{\mathbb{E}(\alpha)-\beta c}{\beta}$, the left derivative is less than the right derivative, the seller's objective function $R_1(\pi)$ is not concave. There are three cases:

Case 1: If $R'_{1+}(\frac{p(\alpha_h-\alpha_l)}{\beta}) \leq 0$, i.e., $\alpha_h - \alpha_l \geq \frac{\alpha_l - \beta c}{p}$, then $\pi^* = \frac{p(\alpha_h-\beta c)}{2\beta} \in \left(0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$.

Case 2: If $R'_{1-}(\frac{p(\alpha_h-\alpha_l)}{\beta}) \geq 0$, i.e., $\alpha_h - \alpha_l \leq \alpha_l - \beta c$, then $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta} \in \left(\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right)$.

Case 3: If $R'_{1+}(\frac{p(\alpha_h-\alpha_l)}{\beta}) > 0$ and $R'_{1-}(\frac{p(\alpha_h-\alpha_l)}{\beta}) < 0$, i.e., $\alpha_l - \beta c < \alpha_h - \alpha_l < \frac{\alpha_l-\beta c}{p}$, the optimal price π^* may fall in the first interval $\left(0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$ or the second interval $\left(\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right)$. Let $\pi_l = \frac{p(\alpha_h-\beta c)}{2\beta}$ and $\pi_r = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$, $\pi^* = \text{argmax}\{R_1(\pi_l), R_1(\pi_r)\}$, with $R_1(\pi_l) = \frac{p(\alpha_h-\beta c)^2}{4\beta}$ and $R_1(\pi_r) = \frac{(\mathbb{E}(\alpha)-\beta c)^2}{4\beta}$. Thus, if $\frac{\alpha_l-\beta c}{\sqrt{p}} < \alpha_h - \alpha_l < \frac{\alpha_l-\beta c}{p}$, $R_1(\pi_l) > R_r(\pi_r)$, then $\pi^* = \frac{p(\alpha_h-\beta c)}{2\beta}$. Note that $\pi^* \in \left(0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$. If $\alpha_l - \beta c < \alpha_h - \alpha_l < \frac{\alpha_l-\beta c}{\sqrt{p}}$, $R_1(\pi_r) > R_1(\pi_l)$, then $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$ with $\pi^* \in \left(\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right)$. If $\alpha_h - \alpha_l = \frac{\alpha_l-\beta c}{\sqrt{p}}$, then both π_l and π_r are optimal.

The results can be summarized as follows.

1. If $\alpha_h - \alpha_l > \frac{\alpha_l-\beta c}{\sqrt{p}}$, then $\pi^* = \frac{p(\alpha_h-\beta c)}{2\beta}$. Note $\pi^* \in \left(0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$.

2. If $\alpha_h - \alpha_l < \frac{\alpha_l-\beta c}{\sqrt{p}}$, then $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$. Note $\pi^* \in \left(\frac{p(\alpha_h-\alpha_l)}{\beta} \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right)$.

3. If $\alpha_h - \alpha_l = \frac{\alpha_l-\beta c}{\sqrt{p}}$, then $\pi^* \in \{\pi_l, \pi_r\}$ with $\pi_l = \frac{p(\alpha_h-\beta c)}{2\beta} \in \left(0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$ and $\pi_r = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta} \in \left(\frac{p(\alpha_h-\alpha_l)}{\beta} \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right)$.

$\square$

Proof of Lemma 3.4.3 The buyer's problem is

$$\begin{aligned} \max_{q_c, q_s} \quad r_2(q_c, q_s|s, \alpha) \quad &= \quad U(q_c + q_s) - cq_c - sq_s \\ s.t. \quad &\quad q_c - Q \leq 0 \end{aligned}$$

$$-q_c \leq 0$$

$$-q_s \leq 0$$

Since the objective function is concave and the all the constraints are linear, the Karush-Kuhn-Tucker condition is necessary and sufficient for the optimality. Let ξ_1, ξ_2 and ξ_3 be the lagrange multipliers with the three constraints in the order defined above. The Karush-Kuhn-Tucker condition is

$$\begin{aligned}
-\frac{(q_c+q_s)}{\beta} + \frac{\alpha}{\beta} - c - \xi_1 + \xi_2 &= 0 \\
-\frac{(q_c+q_s)}{\beta} + \frac{\alpha}{\beta} - s + \xi_3 &= 0 \\
\xi_1(q_c - Q) &= 0 \\
\xi_2 q_c &= 0 \\
\xi_3 q_s &= 0 \\
q_c - Q &\leq 0 \\
-q_c &\leq 0 \\
-q_s &\leq 0 \\
\xi_1, \xi_2, \xi_3 &\geq 0
\end{aligned}$$

Note that if $s = c$, then the buyer is indifferent in purchasing under contracts or from the spot market, and the seller's return doesn't change in either case. Therefore, if $s = c$, any solution satisfying the conditions in (1) is optimal. Let's consider the case $s > c$. If constraint $q_c \leq Q$ is not active, then $\xi_1^* = 0$. Since $s > c$ and $\xi_2 \geq 0$, $\xi_3^* > 0$ and $q_s^* = 0$. Therefore, if $q_c^* < Q$, $q_s^* = 0$. And the problem can be reduced to an optimization problem with one variable, $D = q_c + q_s$. For $D < Q$, $q_c = D$ and $q_s = 0$. For $D \geq Q$, $q_c = Q$ and $q_s = D - Q$. Let $f(D|s,\alpha) = r_2(q_c, q_s|s,\alpha)$.
For $D < Q$, $D = q_c$, $f(D|s,\alpha) = -\frac{D^2}{2\beta} + \frac{\alpha D}{\beta} - Dc$ and $\frac{df(D|s,\alpha)}{dD} = -\frac{D}{\beta} + \frac{\alpha}{\beta} - c$.
For $D > Q$, then $q_c = Q$, $D = Q + q_s$, $f(D|s,\alpha) = \frac{D^2}{2\beta} + \frac{\alpha D}{\beta} - Qc - (D-Q)s$, $\frac{df(D|s,\alpha)}{dD} = -\frac{D}{\beta} + \frac{\alpha}{\beta} - s$.

Note that the $f(D|s,\alpha)$ is continuous at Q. Since $s > c$, $f'_+(Q|s,\alpha) \le f'_-(Q|s,\alpha)$, $f(D|s,\alpha)$ is a continuous concave function of D. The results for $s > c$ are summarized as follows.

1. If $Q \ge \alpha - \beta c$, i.e. $f'_-(Q|s,\alpha) \le 0$, then $D^* = q_c^* = \alpha - \beta c$ and $q_s^* = 0$.

2. If $Q < \alpha - \beta c$ and $s \le (\alpha - Q)/\beta$, i.e. $f'_+(Q|s,\alpha) \ge 0$, then $D^* = \alpha - \beta s$ with $q_c^* = Q$ and $q_s^* = \alpha - \beta s - Q$.

3. If $Q < \alpha - \beta c$ and $s > (\alpha - Q)/\beta$, i.e. $f'_-(Q|s,\alpha) \ge 0$ and $f'_+(Q|s,\alpha) < 0$, then $D^* = q_c^* = Q$ and $q_s^* = 0$.

$\square$

Proof of Lemma 3.4.4 The value of the objective function depends on Q. If $Q \ge \alpha - \beta c$, $R(s|Q,\alpha) = 0$ for any $s \ge c$. Thus, any $s \ge c$ is optimal. If $Q < \alpha - \beta c$, then $q_s(Q,s,\alpha) = (\alpha - \beta s - Q)^+$. Note for $s \ge \frac{\alpha - Q}{\beta}$ and $s = c$, $R(s|Q,\alpha) = 0$. Therefore, only $s \in \left[c, \frac{\alpha-Q}{\beta}\right]$ is interesting, where $q_s(Q,s,\alpha) = \alpha - \beta s - Q$. Under this condition, it holds that $R_2(s|Q,\alpha) = (\alpha - \beta s - Q)(s - c)$ and $\frac{dR_2(s|Q,\alpha)}{ds} = \beta(c - 2s) + \alpha - Q$. Therefore, $s^* = \frac{\alpha + \beta c - Q}{2\beta}$. Note that $s^* \in \left(c, \frac{\alpha - Q}{\beta}\right)$. $\square$

Proof of Lemma 3.4.5 Note that only when $\pi = 0$, $Q(\pi) > \alpha_h - \beta c$. In this case, $R_1(\pi) = 0$. Therefore, the seller will never set $\pi = 0$ and he buyer will never choose $Q > \alpha_h - \beta c$. Based on this argument, only $Q \in [0, \alpha_h - \beta c]$ need to be considered. For $Q \in [0, \alpha_l - \beta c)$, by Lemma 3.4.4, we have the seller's best response $s(Q,\alpha_h) = \frac{\alpha_h + \beta c - Q}{2\beta}$ and $s(Q,\alpha_l) = \frac{\alpha_l + \beta c - Q}{2\beta}$. The objective function and its derivative are

$$\begin{aligned} r_1(Q|\pi) = -\pi Q + p &\left[-\frac{(\alpha_h - \beta c + Q)^2}{8\beta} + \frac{\alpha_h(\alpha_h - \beta c + Q)}{2\beta} - Qc - \frac{(\alpha_h - \beta c - Q)(\alpha_h + \beta c - Q)}{4\beta}\right] \\ +(1-p) &\left[-\frac{(\alpha_l - \beta c + Q)^2}{8\beta} + \frac{\alpha_l(\alpha_l - \beta c + Q)}{2\beta} - Qc - \frac{(\alpha_l - \beta c - Q)(\alpha_l + \beta c - Q)}{4\beta}\right] \end{aligned}$$

$$\frac{dr_1(Q|\pi)}{dQ} = -\pi + \frac{3(\mathbb{E}(\alpha) - \beta c - Q)}{4\beta}$$

For $Q \in [\alpha_l - \beta c, \alpha_h - \beta c]$, we have $s(Q,\alpha_h) = \frac{\alpha_h + \beta c - Q}{2\beta}$ and $s_l(Q,\alpha_h) \in [c, +\infty)$. The objective function and its derivative are

$r_1(Q|\pi) = -\pi Q + p\left[-\frac{(\alpha_h-\beta c+Q)^2}{8\beta} + \frac{\alpha_h(\alpha_h-\beta c+Q)}{2\beta} - Qc - \frac{(\alpha_h-\beta c-Q)(\alpha_h+\beta c-Q)}{4\beta}\right] + \frac{(1-p)(\alpha_l-\beta c)^2}{2\beta}$

$\frac{dr_1(Q|\pi)}{dQ} = -\pi + \frac{3p(\alpha_h-\beta c-Q)}{4\beta}$

Note both $r_1(Q|\pi)$ and $\frac{dr_1(Q|\pi)}{dQ}$ are continuous at $\alpha_l - \beta c$. Thus, $r_1(Q|\pi)$ is a continuous concave function of Q on $[0, \alpha_h - \beta c]$. Depending on the value of π, the optimal Q^* can fall in either of the two intervals.

If $\frac{dr_1(0|\pi)}{dQ} \leq 0$, then $Q^* = 0$. The condition $\frac{dr_1(0|\pi)}{dQ} \leq 0$ holds if and only if $\pi \geq \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}$.

If $\frac{dr_1(0|\pi)}{dQ} > 0$ and $\frac{dr_1(\alpha_l-\beta c|\pi)}{dQ} \leq 0$, then $Q^* = \mathbb{E}(\alpha) - \beta c - \frac{4\beta\pi}{3}$. Conditions $\frac{dr_1(0|\pi)}{dQ} > 0$ and $\frac{dr_1(\alpha_l-\beta c|\pi)}{dQ} \leq 0$ hold if and only if $\frac{3p(\alpha_h-\alpha_l)}{4\beta} \leq \pi < \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}$.

If $\frac{dr_1(\alpha_l-\beta c|\pi)}{dQ} > 0$, $Q^* = \alpha_h - \beta c - \frac{4\beta\pi}{3p}$. Condition $\frac{dr_1(\alpha_l-\beta c|\pi)}{dQ} > 0$ holds if and only if $\pi < \frac{3p(\alpha_h-\alpha_l)}{4\beta}$.

In summary, if $\pi = 0$, any $Q \in [\alpha_h - \beta c, +\infty)$ is optimal. As mentioned before, this will never take place. If $\pi \in \left(0, \frac{3p(\alpha_h-\alpha_l)}{4\beta}\right)$, then $Q^* = \alpha_h - \beta c - \frac{4\beta\pi}{3p}$. Note $Q^* \in (\alpha_l - \beta c, \alpha_h - \beta c)$. If $\pi \in \left[\frac{3p(\alpha_h-\alpha_l)}{4\beta}, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right]$, then $Q^* = \mathbb{E}(\alpha) - \beta c - \frac{4\beta\pi}{3}$ with $Q^* \in [0, \alpha_l - \beta c]$. If $\pi > \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}$, $Q^* = 0$. It should be noted that $Q^*(\pi)$ is a continuous function of π on $(0, +\infty)$. For any given $\pi \in (0, +\infty)$, $Q^*(\pi)$ is unique. □

Proof of Theorem 3.4.2 By Lemma 3.4.5, the buyer's best response $Q(\pi)$ is a piecewise function. Denote the corresponding seller's best response in stage 3 as $s(Q, \alpha)$. Let $q_c(Q, s, \alpha)$ and $q_s(Q, s, \alpha)$ be the buyer's corresponding best response in stage 4. For $\pi = 0$, $R_1(\pi) = 0$.

For $\pi \in \left(0, \frac{3p(\alpha_h-\alpha_l)}{4\beta}\right)$, $Q(\pi) = \alpha_h - \beta c - \frac{4\beta\pi}{3p}$. Note $Q(\pi) \in (\alpha_l - \beta c, \alpha_h - \beta c)$. In the high demand state, it holds that $s(Q, \alpha_h) = \frac{\alpha_h+\beta c-Q}{2\beta}$, $q_c(Q, s, \alpha_h) = Q$, $q_s(Q, s, \alpha_h) = \frac{\alpha_h-\beta c-Q}{2}$. In the low demand state, no transaction takes place in the spot market $q_c(Q, s, \alpha_l) = \alpha_l - \beta c$ and $q_s(Q, s, \alpha_l) = 0$. Therefore, $R_1(\pi) = \pi(\alpha_h - \beta c - \frac{4\beta\pi}{3p}) + \frac{4\beta\pi^2}{9p}$ and $\frac{dR_1(\pi)}{d\pi} = \alpha_h - \beta c - \frac{16\beta\pi}{9p}$. Let $\pi_0 = \frac{3p(\alpha_h-\alpha_l)}{4\beta}$. It holds $R'_{1-}(\pi_0) = -\frac{\alpha_h-\alpha_l}{3} + \alpha_l - \beta c$ and $\frac{dR_1(0)}{d\pi} = \alpha_h - \beta c \geq 0$.

For $\pi \in \left[\frac{3p(\alpha_h-\alpha_l)}{4\beta}, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right]$, $Q(\pi) = \mathbb{E}(\alpha) - \beta c - \frac{4\beta\pi}{3} \in [0, \alpha_l - \beta c]$. It holds that

$s(Q,\alpha_h) = \frac{\alpha_h+\beta c-Q}{2\beta}$, $s(Q,\alpha_l) = \frac{\alpha_l+\beta c-Q}{2\beta}$, $q_c(Q,s,\alpha_h) = q_c(Q,s,\alpha_l) = Q$, $q_s(Q,s,\alpha_h) = \frac{\alpha_h-\beta c-Q}{2}$ and $q_s(Q,s,\alpha_l) = \frac{\alpha_l-\beta c-Q}{2}$. Thus, the objective function and derivative are $R_1(\pi) = \pi\left(\mathbb{E}(\alpha) - \beta c - \frac{4\beta\pi}{3}\right) + \frac{p(\alpha_h-\mathbb{E}(\alpha)+4\beta\pi/3)^2}{4\beta} + \frac{(1-p)(\alpha_l-\mathbb{E}(\alpha)+4\beta\pi/3)^2}{4\beta}$ and
$\frac{dR_1(\pi)}{d\pi} = \mathbb{E}(\alpha) - \beta c - \frac{16\beta\pi}{9}$

It should be noted that $R_1(\pi)$ is continuous on $[0,+\infty)$. At the breakpoints π_0 and $\pi_1 = \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}$, $R'_{1+}(\pi_0) = -\frac{p(\alpha_h-\alpha_l)}{3} + \alpha_l - \beta c > R'_{1-}(\pi_0)$, $R'_{1-}(\pi_1) = -\frac{\mathbb{E}(\alpha)-\beta c}{4} < 0 = R'_{1+}(\pi_1)$. Thus, $R_1(\pi)$ is not concave on $\pi \in \left[0, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right]$. Instead, $R_1(\pi)$ is a piecewise concave function on $[0,+\infty)$. For $\pi > \pi_1$, $R_1(\pi) = R_1(\pi_1)$. Depending on the values of the derivatives at the breakpoints, there are three cases.

Case 1: If $R'_{1-}(\pi_0) = -\frac{\alpha_h-\alpha_l}{3} + \alpha_l - \beta c \geq 0$, i.e., $\frac{\alpha_h-\alpha_l}{3} \leq \alpha_l - \beta c$, then $\pi^* = \frac{9(\mathbb{E}(\alpha)-\beta c)}{16\beta}$ and $R_1^* = \frac{9(\mathbb{E}(\alpha)-\beta c)^2}{64\beta} + \frac{p}{4\beta}\left(\alpha_h - \frac{\mathbb{E}(\alpha)}{4} - \frac{3\beta c}{4}\right)^2 + \frac{(1-p)}{4\beta}\left(\alpha_l - \frac{\mathbb{E}(\alpha)}{4} - \frac{3\beta c}{4}\right)^2$.

Case 2: If $R'_{1+}(\pi_0) = -\frac{p(\alpha_h-\alpha_l)}{3} + \alpha_l - \beta c \leq 0$, i.e., $\frac{(\alpha_h-\alpha_l)}{3} \geq \frac{\alpha_l-\beta c}{p}$, then $\pi^* = \frac{9p(\alpha_h-\beta c)}{16\beta}$ and $R_1^* = \frac{9p(\alpha_h-\beta c)^2}{32\beta}$.

Case 3: If $R'_{1+}(\pi_0) > 0$ and $R'_{1+}(\pi_0) < 0$, then the optimal π^* can be either of the two local maximizers, $\pi_l = \frac{9p(\alpha_h-\beta c)}{16\beta}$ and $\pi_r = \frac{9(\mathbb{E}(\alpha)-\beta c)}{16\beta}$. Let $R_1^* = \max\{R_1(\pi_r), R_1(\pi_l)\}$. Let $y = \frac{\alpha_h-\alpha_l}{3}$ and $x = \alpha_l - \beta c$. Note $R'_{1+}(\pi_0) > 0$ and $R'_{1+}(\pi_0) < 0 \Leftrightarrow x < y < \frac{x}{p}$.

$R_1(\pi_r) - R_1(\pi_0) = \int_{\pi_0}^{\pi_r} \frac{dR_1(\pi)}{d\pi} d\pi = \int_{\pi_0}^{\pi_r}\left(\mathbb{E}(\alpha) - \beta c - \frac{16\beta\pi}{9}\right) d\pi = \frac{9}{16\beta}\left(\frac{p^2y^2}{2} - pxy + \frac{x^2}{2}\right)$

$R_1(\pi_l) - R_1(\pi_0) = \int_{\pi_0}^{\pi_l} \frac{dR_1(\pi)}{d\pi} d\pi = \int_{\pi_0}^{\pi_l}\left(\alpha_h - \beta c - \frac{16\beta\pi}{9p}\right) d\pi = \frac{9}{16\beta}\left(\frac{py^2}{2} - pxy + \frac{px^2}{2}\right)$

$R_1(\pi_r) - R_1(\pi_l) = \frac{9(1-p)(x^2-y^2p)}{32\beta}$

Thus, if $x < y < \frac{x}{\sqrt{p}}$, $\pi^* = \pi_r$. If $\frac{x}{\sqrt{p}} < y < \frac{x}{p}$, then $\pi^* = \pi_l$. If $y = \frac{x}{\sqrt{p}}$, both π_l and π_r are optimal. In summary,

1. if $\frac{\alpha_h-\alpha_l}{3} < \frac{\alpha_l-\beta c}{\sqrt{p}}$, then $\pi^* = \frac{9(\mathbb{E}(\alpha)-\beta c)}{16\beta}$ and
   $R_1^* = \frac{9(\mathbb{E}(\alpha)-\beta c)^2}{64\beta} + \frac{p}{4\beta}\left(\alpha_h - \frac{\mathbb{E}(\alpha)}{4} - \frac{3\beta c}{4}\right)^2 + \frac{(1-p)}{4\beta}\left(\alpha_l - \frac{\mathbb{E}(\alpha)}{4} - \frac{3\beta c}{4}\right)^2$;

2. if $\frac{\alpha_h-\alpha_l}{3} > \frac{\alpha_l-\beta c}{\sqrt{p}}$, then $\pi^* = \frac{9p(\alpha_h-\beta c)}{16\beta}$ and $R_1^* = \frac{9p(\alpha_h-\beta c)^2}{32\beta}$;

3. if $\frac{\alpha_h-\alpha_l}{3} = \frac{\alpha_l-\beta c}{\sqrt{p}}$, both π_l and π_r are optimal, where $\pi_l = \frac{9p(\alpha_h-\beta c)}{16\beta}$, $\pi_r = \frac{9(\mathbb{E}(\alpha)-\beta c)}{16\beta}$.

$\square$

Proof of Corollary 3.4.1 By Theorem 3.4.2, if $\alpha_h - \alpha_l < \frac{3(\alpha_l - \beta c)}{\sqrt{p}}$, $\pi^* = \frac{9(\mathbb{E}(\alpha)-\beta c)}{16\beta}$. Note $\pi^* \in \left[\frac{3p(\alpha_h-\alpha_l)}{4\beta}, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right]$. By Lemma 3.4.5, $Q(\pi^*) = \mathbb{E}(\alpha) - \beta c - \frac{4\beta\pi^*}{3} = \frac{\mathbb{E}(\alpha)-\beta c}{4} > 0$. If $\alpha_h - \alpha_l > \frac{3(\alpha_l-\beta c)}{\sqrt{p}}$, $\pi^* = \frac{9p(\alpha_h-\beta c)}{16\beta}$. Note that $\pi^* \in \left[0, \frac{3p(\alpha_h-\alpha_l)}{4\beta}\right]$. By Lemma 3.4.5, $Q(\pi^*) = \alpha_h - \beta c - \frac{4\beta\pi^*}{3p} = \frac{\alpha_h - \beta c}{4} > 0$. If $\alpha_h - \alpha_l = \frac{3(\alpha_l-\beta c)}{\sqrt{p}}$, $Q(\pi^*) > 0$ holds based on the above arguments. □

Proof of Lemma 3.4.6. The seller's problem is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) &= (1-\lambda)\pi Q_A(\pi) + \lambda\{\pi Q_B(\pi) + \mathbb{E}[G_2(Q_B(\pi),\alpha)]\} \\ \text{s.t.} \quad\quad \pi &\geq 0 \end{aligned}$$

Let $R_A(\pi) = \pi Q_A(\pi)$ and $R_B(\pi) = \pi Q_B(\pi) + \mathbb{E}[G_2(Q_B(\pi),\alpha)]$. Note $R_A(\pi)$ is the seller's objective function at stage 1 in Section 3.4.1.1 and $R_B(\pi)$ is the objective function of the seller's stage 1 problem in Section 3.4.1.2. Recall that $R_A(\pi)$ is a piecewise function of π on intervals $\left[0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$ and $\left[\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right]$. For $\pi \geq \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$, $R_A(\pi) = 0$. Similarly, function $R_B(\pi)$ is a piecewise function of π on intervals $\left[0, \frac{3p(\alpha_h-\alpha_l)}{4\beta}\right)$ and $\left[\frac{3p(\alpha_h-\alpha_l)}{4\beta}, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right]$. For $\pi \geq \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}$, $R_B(\pi) = R_B\left(\frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right)$. Both $R_A(\pi)$ and $R_B(\pi)$ are continuous on $[0, +\infty)$. Depending on the values of the breakpoints, there are two cases.

Case 1: $\frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta} \leq \frac{p(\alpha_h-\alpha_l)}{\beta}$, i.e. $\alpha_h - \alpha_l \geq \frac{3(\alpha_l-\beta c)}{p}$.

For $\pi \in \left[0, \frac{3p(\alpha_h-\alpha_l)}{4\beta}\right]$, it holds that

$R_1(\pi) = (1-\lambda)\pi\left(\alpha_h - \beta c - \frac{\beta\pi}{p}\right) + \lambda\pi\left(\alpha_h - \beta c - \frac{8\beta\pi}{9p}\right)$

$\frac{dR_1(\pi)}{d\pi} = (1-\lambda)\left(\alpha_h - \beta c - \frac{2\beta\pi}{p}\right) + \lambda\left(\alpha_h - \beta c - \frac{16\beta\pi}{9p}\right)$

Note that $\frac{dR_1(0)}{d\pi} = \alpha_h - \beta c > 0$.

For $\pi \in \left[\frac{3p(\alpha_h-\alpha_l)}{4\beta}, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right]$, it holds that

$R_1(\pi) = (1-\lambda)\pi\left(\alpha_h - \beta c - \frac{\beta\pi}{p}\right) + \lambda\left[\pi\left(\mathbb{E}(\alpha) - \beta c - \frac{4\beta}{3}\pi\right) + \frac{16\beta^2\pi^2/9+\sigma^2}{4\beta}\right]$

$\frac{dR_1(\pi)}{d\pi} = (1-\lambda)\left(\alpha_h - \beta c - \frac{2\beta\pi}{p}\right) + \lambda\left(\mathbb{E}(\alpha) - \beta c - \frac{16\beta\pi}{9}\right)$

$$R'_{1-}\left(\frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right) = (1-\lambda)\left(\alpha_h - \beta c - \frac{3(\mathbb{E}(\alpha)-\beta c)}{2p}\right) - \frac{\lambda(\mathbb{E}(\alpha)-\beta c)}{3}$$

$$
\begin{aligned}
&= (1-\lambda)\left(-\frac{\alpha_h-\beta c}{2}-\frac{3(1-p)(\alpha_l-\beta c)}{2p}\right)-\frac{\lambda(\mathbb{E}(\alpha)-\beta c)}{3} \\
&< 0 \\
R'_{1+}\left(\frac{3p(\alpha_h-\alpha_l)}{4\beta}\right) &= (\alpha_l-\beta c)-(\alpha_h-\alpha_l)\left(\frac{1}{2}-\frac{\lambda}{2}+\frac{p\lambda}{3}\right) \\
&\le (\alpha_l-\beta c)-\frac{p(\alpha_h-\alpha_l)}{3} \quad \le \quad 0
\end{aligned}
$$

For $\pi \in \left[\frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}, \frac{p(\alpha_h-\alpha_l)}{\beta}\right]$, it holds that
$R_1(\pi) = (1-\lambda)\pi\left(\alpha_h-\beta c-\frac{\beta\pi}{p}\right)+\lambda R_B\left(\frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right)$,
$\frac{dR_1(\pi)}{d\pi} = (1-\lambda)\left(\alpha_h-\beta c-\frac{2\beta\pi}{p}\right)$ and
$R'_{1+}\left(\frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right) = (1-\lambda)\left(\alpha_h-\beta c-\frac{3(\mathbb{E}(\alpha)-\beta c)}{2p}\right) < 0.$
For $\pi \in \left[\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right]$, it holds that
$R_1(\pi) = (1-\lambda)\pi\left(\mathbb{E}(\alpha)-\beta c-\beta\pi\right)+\lambda R_B\left(\frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right)$
$\frac{dR_1(\pi)}{d\pi} = (1-\lambda)\left(\mathbb{E}(\alpha)-\beta c-2\beta\pi\right)$
$R'_{1+}\left(\frac{p(\alpha_h-\alpha_l)}{\beta}\right) = (1-\lambda)\left(-p(\alpha_h-\alpha_l)+\alpha_l-\beta c\right) \le 0$
For $\pi \in \left[\frac{\mathbb{E}(\alpha)-\beta c}{\beta}, +\infty\right)$, it holds that $R_1(\pi) = \lambda R_B\left(\frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right)$.

Thus, the optimal price $\pi^* \in \left[0, \frac{3p(\alpha_h-\alpha_l)}{4\beta}\right]$ and satisfies $\frac{dR_1(\pi^*)}{d\pi} = 0 \Rightarrow \pi^* = \frac{p(\alpha_h-\beta c)}{2\beta(1-\frac{\lambda}{9})}$. It holds that $\frac{d\pi^*}{d\lambda} = \frac{p(\alpha_h-\beta c)}{18\beta(1-\lambda/9)^2} > 0$.

Case 2: $\frac{p(\alpha_h-\alpha_l)}{\beta} \le \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}$, i.e. $\alpha_h-\alpha_l \le \frac{3(\alpha_l-\beta c)}{p}$.

As $R_1(0)=0$, $\pi=0$ is not optimal for any λ. Note $R_A(\pi)$ and $R_B(\pi)$ are continuous and piecewise concave on $(0,+\infty)$. Therefore, $R_1(\pi)$ is also continuous and piecewise concave on $(0,+\infty)$. Let $I_1 = \left(0, \frac{3p(\alpha_h-\alpha_l)}{4\beta}\right]$, $I_2 = \left(\frac{3p(\alpha_h-\alpha_l)}{4\beta}, \frac{p(\alpha_h-\alpha_l)}{\beta}\right]$, $I_3 = \left(\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right]$, $I_4 = \left(\frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right]$, and $I_5 = \left(\frac{\mathbb{E}(\alpha)-\beta c}{\beta}, +\infty\right)$. Based on the property of $R_A(\pi)$ and $R_B(\pi)$, it holds that $R'_{1-}(\pi) \le R'_{1+}(\pi)$ at the breakpoints $\pi = \frac{3p(\alpha_h-\alpha_l)}{4\beta}, \frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}$ and $\frac{\mathbb{E}(\alpha)-\beta c}{\beta}$. Since $R'_{1+}\left(\frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right) = \frac{-(1-\lambda)(\mathbb{E}(\alpha)-\beta c)}{2} < 0$ and $R_1(\pi) = R_1\left(\frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right)$ for all $\pi \ge \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$, we only need to consider $\left(0, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right)$. In other words, $\pi^*(\lambda) \in I_1$, I_2 or I_3 for any $\lambda \in [0,1]$. Define π_1, π_2 and π_3 as the local maximizer on the intervals I_1, I_2 and I_3 respectively. First, we prove the following claims.

Claim A.0.1. *The seller's optimal surplus $G(\lambda)$ is a continuous function of λ on $[0,1]$.*

Proof of Claim A.0.1. It holds that $G(\lambda) = R_1(\pi^*(\lambda)) = \max\{R_1(\pi_i)|i = 1,2,3\}$. Since π_i is a continuous function of λ, $R_1(\pi_i)$ is also a continuous function of λ for all $i = 1,2,3$. Hence $R_1(\pi^*(\lambda))$ and $G(\lambda)$ are also continuous in λ.

Note that $R_1(\pi)$ also depends on λ. Thus, we can use $R_1(\pi,\lambda)$ instead. Let $f(\pi,\lambda) = \frac{\partial R_1(\pi,\lambda)}{\partial \pi}$. Let $\pi_a = \frac{3p(\alpha_h-\alpha_l)}{4\beta}$ and $\pi_b = \frac{p(\alpha_h-\alpha_l)}{\beta}$. Define open intervals $\tilde{I}_1 = \left(0, \frac{3p(\alpha_h-\alpha_l)}{4\beta}\right)$, $\tilde{I}_2 = \left(\frac{3p(\alpha_h-\alpha_l)}{4\beta}, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$ and $\tilde{I}_3 = \left(\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right)$. The next claim says for any fixed $\pi \in \tilde{I}_1 \bigcup \tilde{I}_2 \bigcup \tilde{I}_3$, the derivative of the objective function increases in λ. At the breakpoints, both the right and the left derivative increases as well.

Claim A.0.2. *1. For any fixed $\pi \in \tilde{I}_i$, $i = 1,2,3$, $\frac{\partial f(\pi,\lambda)}{\partial \lambda} > 0$ for any $\lambda \in [0,1]$.*

2. For $\pi = \pi_a(\pi_b)$, $\left.\frac{\partial f(\pi,\lambda)}{\partial \lambda}\right|_{\pi^+} > 0$ and $\left.\frac{\partial f(\pi,\lambda)}{\partial \lambda}\right|_{\pi^-} > 0$ for any $\lambda \in [0,1]$.

Proof of Claim A.0.2. For $\pi \in \tilde{I}_1$, $\frac{\partial f(\pi,\lambda)}{\partial \lambda} = \frac{2\beta\pi}{9p} > 0$.
For $\pi \in \tilde{I}_2$, $\frac{\partial f(\pi,\lambda)}{\partial \lambda} = 2\beta\pi\left(\frac{1}{p} - \frac{8}{9}\right) - (1-p)(\alpha_h-\alpha_l) > 2\beta\pi_a\left(\frac{1}{p} - \frac{8}{9}\right) - (1-p)(\alpha_h-\alpha_l) = (\alpha_h - \alpha_l)\left(\frac{1}{2} - \frac{p}{3}\right) > 0$
For $\pi \in \tilde{I}_3$, $\frac{\partial f(\pi,\lambda)}{\partial \lambda} = \frac{2\beta\pi}{9} > 0$.
For $\pi = \pi_a$, $\left.\frac{\partial f(\pi,\lambda)}{\partial \lambda}\right|_{\pi^-} = \frac{2\beta\pi_a}{9p} > 0$ and $\left.\frac{\partial f(\pi,\lambda)}{\partial \lambda}\right|_{\pi^+} = 2\beta\pi_a\left(\frac{1}{p} - \frac{8}{9}\right) - (1-p)(\alpha_h-\alpha_l) > 0$.
For $\pi = \pi_b$, $\left.\frac{\partial f(\pi,\lambda)}{\partial \lambda}\right|_{\pi^-} = 2\beta\pi_b\left(\frac{1}{p} - \frac{8}{9}\right) - (1-p)(\alpha_h-\alpha_l) > 0$ and $\left.\frac{\partial f(\pi,\lambda)}{\partial \lambda}\right|_{\pi^+} = \frac{2\beta\pi_b}{9} > 0$.

The next claim states that the optimal solution can not be the breakpoints, π_a and π_b.

Claim A.0.3. *For any $\lambda \in (0,1)$, the optimal solution $\pi^*(\lambda) \neq \pi_a(\pi_b)$.*

Proof of Claim A.0.3. If $\pi^*(\lambda) = \pi_a(\pi_b)$, $R'_{1-}(\pi^*) \geq 0$. Note $R'_{1-}(\pi^*) < R'_{1+}(\pi^*)$ for any $\lambda \in (0,1)$. Therefore, $R'_{1+}(\pi^*) > 0$ and $\pi_a(\pi_b)$ can not be optimal.

Claim A.0.4. *If $\pi^*(\lambda_0) \in \tilde{I}_i$, $i = 1, 2$ or 3, is the unique optimal solution for some $\lambda_0 \in (0, 1)$, then $\exists \delta > 0$ such that $\pi^*(\lambda) \in \tilde{I}_i$ for any $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$.*

Proof of Claim A.0.4. Note $\pi^*(\lambda) = \text{argmax}\{R_1(\pi_i(\lambda))|i = 1, 2, 3\}$, where $\pi_i(\lambda)$ is the local optimizer in I_i. Suppose $\pi^*(\lambda_0) \in \tilde{I}_k$. It holds that $R_1(\pi_k(\lambda_0)) > R_1(\pi_j(\lambda_0))$, $j \neq k$. Let $\varepsilon = \min\{R_1(\pi_k(\lambda_0)) - R_1(\pi_j(\lambda_0))|j \neq k\}$. By the continuity of $R_1(\pi_i(\lambda))$, $i = 1, 2, 3$, there exist δ_1 such that $|R_1(\pi_k(\lambda)) - R_1(\pi_k(\lambda_0))| < \frac{\varepsilon}{2}$ and $\pi(\lambda) \in \tilde{I}_k$ for any $\lambda \in (\lambda_0 - \delta_1, \lambda_0 + \delta_1)$. Similarly, there exists δ_2 such that $|R_1(\pi_j(\lambda)) - R_1(\pi_j(\lambda_0))| < \frac{\varepsilon}{2}$, $j \neq k$, for any $\lambda \in (\lambda_0 - \delta_2, \lambda_0 + \delta_2)$. Let $\delta = \min\{\delta_1, \delta_2\}$. Therefore,
$R_1(\pi_k(\lambda)) > R_1(\pi_j(\lambda))$ for any $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$.

Based on above claims, we first prove π^* increases in λ for any $\lambda \in (0, 1)$. By Claim A.0.3, $\pi^*(\lambda) \in \tilde{I}_i$, $i = 1, 2, 3$. Since $R_1(\pi)$ is strictly concave on I_i, $i = 1, 2, 3$, there are only two cases. Either $\pi^*(\lambda) \in \tilde{I}_i$ is the unique optimal solution or $\pi^*(\lambda) \in \{\pi_j|j \in J\}$ with $J \subseteq \{1, 2, 3\}$ and $|J| > 1$. For the former case, by Claim A.0.4, if λ increases a small amount, π^* remains in the same interval. We will prove π^* increases in that interval as λ increases a small amount. For the latter case, we will prove that if π^* jumps from one interval to another when λ increases, π^* can only jump to an interval on the right. Last, the case $\lambda = 0, 1$ is addressed.

Claim A.0.5. *If $\pi^*(\lambda)$ is the unique optimal solution at some $\lambda \in (0, 1)$, then $\pi^*(\lambda)$ increases as λ increases a small amount.*

Proof of Claim A.0.5. If $\pi^* \in \tilde{I}_1$, from above results, we can see $\pi^* = \frac{p(\alpha_h - \beta c)}{2\beta(1 - \frac{\lambda}{9})}$, $\frac{d\pi^*}{d\lambda} = \frac{p(\alpha_h - \beta c)}{18\beta(1 - \lambda/9)^2} > 0$.

If $\pi^* \in \tilde{I}_2$, π^* satisfies $\frac{dR_1(\pi^*)}{d\pi} = 0$. In this interval, it holds
$\frac{dR_1(\pi)}{d\pi} = (1 - \lambda)\left(\alpha_h - \beta c - \frac{2\beta\pi}{p}\right) + \lambda\left(\mathbb{E}(\alpha) - \beta c - \frac{16\beta\pi}{9}\right)$
Thus $\pi^* = \frac{(1-\lambda)(\alpha_h - \beta c) + \lambda(\mathbb{E}(\alpha) - \beta c)}{2\beta\left(\frac{1-\lambda}{p} + \frac{8\lambda}{9}\right)}$ and $\frac{d\pi^*}{d\lambda} = \frac{(\alpha_l - \beta c)(\frac{9}{p} - 8) + (\alpha_h - \alpha_l)}{18\beta(\frac{1-\lambda}{p} + \frac{8\lambda}{9})^2} > 0$.

If $\pi^* \in \tilde{I}_3$, π^* satisfies $\frac{dR_1(\pi^*)}{d\pi} = 0$. In this interval, it holds that
$\frac{dR_1(\pi)}{d\pi} = \mathbb{E}(\alpha) - \beta c - 2\beta\pi\left(1 - \frac{\lambda}{9}\right)$. Thus, $\pi^* = \frac{\mathbb{E}(\alpha) - \beta c}{2\beta(1 - \frac{\lambda}{9})}$ and $\frac{d\pi^*}{d\lambda} = \frac{\mathbb{E}(\alpha) - \beta c}{18\beta(1 - \frac{\lambda}{9})^2} > 0$.

Hence, if π^* moves in the same interval when λ increases, then π^* increases for any $\lambda \in (0, 1)$.

Second, we will prove that as λ increases from 0 to 1 and if π^* jumps from one interval to another, π^* can only jump to the right. Suppose π^* falls in $\tilde{I}_2$ at some $\lambda_0 < 1$. Suppose λ_1 is the smallest value such that $\lambda_1 > \lambda_0$, $R_1(\pi_1(\lambda_1)) \geq R_1(\pi_2(\lambda_1))$. This could happen only if at $\pi = \pi_a$, the left derivative is negative and the right derivative is positive for all $\lambda \in [\lambda_0, \lambda_1]$. In this case $\pi_1(\lambda) \in \tilde{I}_1$ and $\pi_2(\lambda) \in \tilde{I}_2$ for any $\lambda \in [\lambda_0, \lambda_1]$.

At λ_1, it holds that

$R_1(\pi_1(\lambda_1)) - R_1(\pi_2(\lambda_1)) = \int_{\pi_1(\lambda_1)}^{\pi_a} \left| \frac{\partial R_1(\pi, \lambda_1)}{\partial \pi} \right| d\pi - \int_{\pi_a}^{\pi_2(\lambda_1)} \frac{\partial R_1(\pi, \lambda_1)}{\partial \pi} d\pi \geq 0$

At λ_0, it holds that

$R_1(\pi_2(\lambda_0)) - R_1(\pi_1(\lambda_0)) = \int_{\pi_a}^{\pi_2(\lambda_0)} \frac{\partial R_1(\pi, \lambda_0)}{\partial \pi} d\pi - \int_{\pi_1(\lambda_0)}^{\pi_a} \left| \frac{\partial R_1(\pi, \lambda_0)}{\partial \pi} \right| d\pi \geq 0$

As π_1 and π_2 strictly increases in λ, we know that $|\pi_1(\lambda_1) - \pi_a| < |\pi_1(\lambda_0) - \pi_a|$ and $|\pi_2(\lambda_1) - \pi_a| > |\pi_1(\lambda_0) - \pi_a|$. By Claim A.0.2, $\left| \frac{\partial R_1(\pi, \lambda_1)}{\partial \pi} \right| < \left| \frac{\partial R_1(\pi, \lambda_0)}{\partial \pi} \right|$ for any fixed π on $[\pi_1(\lambda_0), \pi_a]$ and $\frac{\partial R_1(\pi, \lambda_1)}{\partial \pi} > \frac{\partial R_1(\pi, \lambda_0)}{\partial \pi}$ for any fixed π on $[\pi_a, \pi_2(\lambda_1)]$. Thus, $R_1(\pi_2(\lambda_1)) - R_1(\pi_1(\lambda_1)) > R_1(\pi_2(\lambda_0)) - R_1(\pi_1(\lambda_0)) \geq 0$, which contradicts the statement.

By similar arguments, we can see that if $\pi^*(\lambda_0)$ falls in $\tilde{I}_3$ for some $\lambda_0 > 0$, then $\pi^*(\lambda)$ can not fall in $\tilde{I}_2$, $\forall \lambda > \lambda_0$. Next we will prove $\pi^*(\lambda)$ can not fall in $\tilde{I}_1$ also. Suppose λ_1 is the smallest value such that $\lambda_1 > \lambda_0$, $R_1(\pi_1(\lambda_1)) \geq R_1(\pi_3(\lambda_1))$. This holds only if $R'_{1-}(\pi_a) < 0$ and $R'_{1+}(\pi_a) > 0$ for any $\lambda \in (\lambda_0, \lambda_1)$.

At λ_1, it holds that $R_1(\pi_1(\lambda_1)) - R_1(\pi_3(\lambda_1)) = \int_{\pi_1(\lambda_1)}^{\pi_a} \left| \frac{\partial R_1(\pi, \lambda_1)}{\partial \pi} \right| d\pi - \int_{\pi_a}^{\pi_b} \frac{\partial R_1(\pi, \lambda_1)}{\partial \pi} d\pi - \int_{\pi_b}^{\pi_3(\lambda_1)} \frac{\partial R_1(\pi, \lambda_1)}{\partial \pi} d\pi \geq 0$

At λ_0, it holds that $R_1(\pi_3(\lambda_0)) - R_1(\pi_1(\lambda_0)) = -\int_{\pi_1(\lambda_0)}^{\pi_a} \left| \frac{\partial R_1(\pi, \lambda_0)}{\partial \pi} \right| d\pi + \int_{\pi_a}^{\pi_b} \frac{\partial R_1(\pi, \lambda_1)}{\partial \pi} d\pi + \int_{\pi_b}^{\pi_3(\lambda_0)} \frac{\partial R_1(\pi, \lambda_0)}{\partial \pi} d\pi \geq 0$

As π_1 and π_3 strictly increases in λ, we know that $|\pi_1(\lambda_1) - \pi_a| < |\pi_1(\lambda_0) - \pi_a|$ and $|\pi_3(\lambda_1) - \pi_b| > |\pi_1(\lambda_0) - \pi_b|$. By Claim A.0.2, $\left| \frac{\partial R_1(\pi, \lambda_1)}{\partial \pi} \right| < \left| \frac{\partial R_1(\pi, \lambda_0)}{\partial \pi} \right|$ for any fixed

π on $[\pi_1(\lambda_0), \pi_a]$, $\frac{\partial R_1(\pi,\lambda_1)}{\partial \pi} > \frac{\partial R_1(\pi,\lambda_0)}{\partial \pi}$ for any fixed π on $[\pi_b, \pi_3(\lambda_1)]$ and $\frac{\partial R_1(\pi,\lambda_1)}{\partial \pi} > \frac{\partial R_1(\pi,\lambda_0)}{\partial \pi}$ for any fixed π on (π_a, π_b) . Thus, $R_1(\pi_3(\lambda_1)) - R_1(\pi_1(\lambda_1)) > R_1(\pi_3(\lambda_0)) - R_1(\pi_1(\lambda_0)) \geq 0$, which contradicts the statement.

Finally, we show that at $\lambda = 0$ or 1, π^* increases in λ. For $\lambda = 0$, by Theorem 3.4.1, $\pi^* \neq \pi_b$. If $\pi^* \in \tilde{I}_i$, $i = 1, 2, 3$, the statement holds directly from the above arguments. If $\pi^* = \pi_a$, $R'_{1-}(\pi_a) = R'_{1+}(\pi_a)$ at $\lambda = 0$. By Claim A.0.2, as λ increases, both $R'_{1-}(\pi_a)$ and $R'_{1+}(\pi_a)$ increase. Therefore, π^* moves to the right.

For $\lambda = 1$, by Theorem 3.4.2, $\pi^* \neq \pi_a$. If $\pi^* \in \tilde{I}_i$, $i = 1, 2, 3$, the statement holds directly from the above arguments. If $\pi^* = \pi_b$, similar arguments hold.

Combining the results from Case 1 and Case 2, we conclude that $\pi^*(\lambda)$ strictly increases as λ increases. □

Proof of Theorem 3.4.3. In the proof of Lemma 3.4.6, we have showed that depending the values of the breakpoints, there are two cases.

Case 1: $\frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta} \leq \frac{p(\alpha_h-\alpha_l)}{\beta}$, i.e. $\alpha_h - \alpha_l \geq \frac{3(\alpha_l-\beta c)}{p}$. In this case $\pi^*(\lambda) = \frac{p(\alpha_h-\beta c)}{2\beta(1-\frac{\lambda}{9})}$ and $\frac{d\pi^*(\lambda)}{d\lambda} = \frac{p(\alpha_h-\beta c)}{18\beta(1-\frac{\lambda}{9})^2} > 0$. It holds that
$Q^*(\lambda) = \alpha_h - \beta c - \frac{\beta\pi^*}{p}\left(1 + \frac{\lambda}{3}\right)$ and $\frac{dQ^*(\lambda)}{d\lambda} = -\frac{\beta\pi^*}{3p} - \frac{\beta}{p}\left(1 + \frac{\lambda}{3}\right)\frac{d\pi^*(\lambda)}{d\lambda} < 0$
Therefore, as λ increases, the price of the option increases and the quantity of options bought decreases in this interval.

Case 2: $\frac{p(\alpha_h-\alpha_l)}{\beta} < \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}$, i.e. $\alpha_h - \alpha_l < \frac{3(\alpha_l-\beta c)}{p}$. By Lemma 3.4.6, we have showed that if π^* stays in the same interval as λ increases, then π^* increases. And if there is a jump in π^*, π^* can only jump from one interval to another to the right and can never jump back. We will first show that in the former case, as λ increases, Q^* decreases. Then we will show that a jump in π^* results in a jump down in Q^*.

Case 2.a: If $\pi^* \in \left[0, \frac{3p(\alpha_h-\alpha_l)}{4\beta}\right)$, $\frac{dQ^*(\lambda)}{d\lambda} \leq 0$ directly from Case 1.

Case 2.b: If $\pi^* \in \left[\frac{3p(\alpha_h-\alpha_l)}{4\beta}, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$, $\pi^*(\lambda) = \frac{(\alpha_h-\beta c)-\lambda(1-p)(\alpha_h-\alpha_l)}{2\beta(\frac{1-\lambda}{p}+\frac{8\lambda}{9})}$ and

$\frac{d\pi^*(\lambda)}{d\lambda} = \frac{(\alpha_l-\beta c)(\frac{9}{p}-8)+(\alpha_h-\alpha_l)}{18\beta(\frac{1-\lambda}{p}+\frac{8\lambda}{9})^2} > 0$. It holds that $Q^*(\lambda) = \frac{(1-\lambda)\beta\pi^*}{p} + \frac{\lambda 4\beta\pi^*}{9}$

$$\begin{aligned}\frac{dQ^*(\lambda)}{d\lambda} &= \frac{-\frac{4(\alpha_l-\beta c)}{9p} + (1-\lambda(1-p))\left(\frac{1-\lambda}{p}+\frac{8\lambda}{9}\right)\left(-\frac{1}{p}+\frac{4}{9}\right)(\alpha_h-\alpha_l)}{2\left(\frac{1-\lambda}{p}+\frac{8\lambda}{9}\right)^2} \\ &< -\frac{4(\alpha_l-\beta c)}{18p\left(\frac{1-\lambda}{p}+\frac{8\lambda}{9}\right)^2} \quad \leq \quad 0\end{aligned}$$

Therefore, if the optimal price falls in this interval, then as λ increases, π^* increases and Q^* decreases.

Case 2.c: If $\pi^* \in \left[\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right]$, $\pi^*(\lambda) = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta(1-\frac{\lambda}{9})}$ and $\frac{d\pi^*(\lambda)}{d\lambda} = \frac{\mathbb{E}(\alpha)-\beta c}{18\beta(1-\frac{\lambda}{9})^2} > 0$. It holds that $Q^*(\lambda) = \mathbb{E}(\alpha) - \beta c - \beta\pi^*\left(1+\frac{\lambda}{3}\right)$ and $\frac{dQ^*(\lambda)}{d\lambda} = -\frac{\beta\pi^*}{3} - \beta\left(1+\frac{\lambda}{3}\right)\frac{d\pi^*(\lambda)}{d\lambda} < 0$ Thus, if the optimal price falls in this interval, the statement still holds.

In the proof of Lemma 3.4.6, we have shown that π^* may moves from one interval to another on the right. That is at some λ, $\pi^*(\lambda) \in \{\pi_i, \pi_j\}$ with $\pi_i < \pi_j$, where $\pi_{i(j)}$ is the local maximizer in $\tilde{I}_{i(j)}$ and $i < j$. In this case both $Q_A(\pi_i) > Q_A(\pi_j)$ and $Q_B(\pi_i) \geq Q_B(\pi_j)$. Therefore, Q^* strictly decreases in λ. □

Proof of Theorem 3.4.4. In the proof of Lemma 3.4.6, we have shown that depending the values of the breakpoints, there are two cases.

Case 1: $\frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta} \leq \frac{p(\alpha_h-\alpha_l)}{\beta}$, i.e. $\alpha_h - \alpha_l \geq \frac{3(\alpha_l-\beta c)}{p}$. In the proof of Lemma 3.4.6, we have already shown that $\pi^*(\lambda) \in \left(0, \frac{3p(\alpha_h-\alpha_l)}{4\beta}\right)$, $\pi^*(\lambda) = \frac{p(\alpha_h-\beta c)}{2\beta(1-\lambda/9)}$
$G(\lambda) = (1-\lambda)\pi^*\left(\alpha_h - \beta c - \frac{\beta\pi^*}{p}\right) + \lambda\pi^*\left(\alpha_h - \beta c - \frac{8\beta\pi^*}{9p}\right)$. Let $F(\lambda, \pi^*(\lambda)) = G(\lambda)$. Note that $\frac{\partial F(\lambda,\pi^*(\lambda))}{\partial\pi^*} = \frac{dR_1(\pi^*)}{d\pi} = 0$. Hence,
$\frac{dG(\lambda)}{d\lambda} = \frac{\partial F(\lambda,\pi^*(\lambda))}{\partial\lambda} + \frac{\partial F(\lambda,\pi^*(\lambda))}{\partial\pi^*}\frac{d\pi^*}{d\lambda} = \frac{\beta(\pi^*)^2}{9p} > 0$ Therefore, $G(\lambda)$ increases in λ.

Case 2: $\frac{p(\alpha_h-\alpha_l)}{\beta} < \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}$, i.e. $\alpha_h - \alpha_l < \frac{3(\alpha_l-\beta c)}{p}$. It has been shown in the proof of Lemma 3.4.6 that $\pi^*(\lambda) \in \left(0, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right)$ for any $\lambda \in (0,1)$. Depending on the values of the parameters, for a given $\lambda \in (0,1)$, the optimal option price $\pi^*(\lambda)$ can fall in any of the intervals $\tilde{I}_1$, $\tilde{I}_2$ or $\tilde{I}_3$ defined in the proof of Lemma 3.4.6. Let $\pi_a = \frac{3p(\alpha_h-\alpha_l)}{4\beta}$ and $\pi_b = \frac{p(\alpha_h-\alpha_l)}{\beta}$. Note that only if $\lambda = 0$, $\pi^* = \pi_a$ may hold. Only if $\lambda = 1$, $\pi^* = \pi_b$ may hold. In both cases, $\frac{dR_1(\pi^*)}{d\pi} = 0$ holds. In summary, for any

$\lambda \in [0,1]$, π^* may fall in $\tilde{I}_1$, $\hat{I}_2 = \tilde{I}_2 \cup \{\pi_a\}$ or $\hat{I}_3 = \tilde{I}_3 \cup \{\pi_b\}$ and $\frac{dR_1(\pi^*)}{d\pi} = 0$ holds.

As shown in the proof of Lemma 3.4.6, when λ increases, π^* increases from left to right and may jump from an interval to another interval on the right. We first prove that in every interval, when λ increases and π^* stays in the same interval, the conclusion holds. Also, if there is a jump of π^* among intervals, G stays the same before and after the jump because of the continuity of $G(\lambda)$.

Case 2.a: If $\pi^*(\lambda) \in \tilde{I}_1$, then the proof is the same as that in Case 1.

Case 2.b: If $\pi^*(\lambda) \in \hat{I}_2$, then $\pi^* = \frac{(1-\lambda)(\alpha_h-\beta c)+\lambda(\mathbb{E}(\alpha)-\beta c)}{2\beta(\frac{1-\lambda}{p}+\frac{8\lambda}{9})}$. Let $\sigma = \sqrt{p(1-p)(\alpha_h-\alpha_l)^2}$. It holds that

$$
\begin{aligned}
G(\lambda) &= \frac{[(1-\lambda)(\alpha_h-\beta c)+\lambda(\mathbb{E}(\alpha)-\beta c)]^2}{4\beta\left(\frac{1-\lambda}{p}+\frac{8\lambda}{9}\right)} + \frac{\lambda\sigma^2}{4\beta} \\
\frac{dG(\lambda)}{d\lambda} &= \frac{[-(\alpha_h-\beta c)+(\mathbb{E}(\alpha)-\beta c)][(1-\lambda)(\alpha_h-\beta c)+\lambda(\mathbb{E}(\alpha)-\beta c)]}{2\beta\left(\frac{1-\lambda}{p}+\frac{8\lambda}{9}\right)} \\
&\quad + \frac{\left(\frac{1}{p}-\frac{8}{9}\right)[(1-\lambda)(\alpha_h-\beta c)+\lambda(\mathbb{E}(\alpha)-\beta c)]^2}{4\beta\left(\frac{1-\lambda}{p}+\frac{8\lambda}{9}\right)^2} + \frac{\sigma^2}{4\beta} \\
&= \frac{l_1}{4\beta(\frac{1-\lambda}{p}+\frac{8\lambda}{9})^2}
\end{aligned}
$$

where

$$
\begin{aligned}
l_1 &= (\alpha_h-\alpha_l)^2\left[\frac{(1-\lambda+\lambda p)-\lambda(1-p)(1-\lambda+8\lambda p/9)}{9}\right] \\
&\quad +2(\alpha_h-\alpha_l)(\alpha_l-\beta c)(1-\lambda+\lambda p)\left[\frac{1}{9}+\lambda\left(\frac{1}{p}+\frac{8p}{9}-\frac{17}{9}\right)\right] \\
&\quad +(\alpha_l-\beta c)^2\left(\frac{1}{p}-\frac{8}{9}\right)
\end{aligned}
$$

Since $\left[\frac{(1-\lambda+\lambda p)-\lambda(1-p)(1-\lambda+8\lambda p/9)}{9}\right] > 0$ and $\left[\frac{1}{9}+\lambda\left(\frac{1}{p}+\frac{8p}{9}-\frac{17}{9}\right)\right] \geq 0$, $\frac{dG(\lambda)}{d\lambda} > 0$.

Case 2.c: If $\pi^*(\lambda) \in \hat{I}_3$, then

$G(\lambda) = (1-\lambda)\pi^*\left(\mathbb{E}(\alpha)-\beta c-\beta\pi^*\right) + \lambda\left[\pi^*\left(\mathbb{E}(\alpha)-\beta c-\frac{8\beta\pi^*}{9}\right)+\frac{\sigma^2}{4\beta}\right]$

Let $F(\lambda,\pi^*(\lambda)) = G(\lambda)$. Since $\frac{\partial F^*(\lambda,\pi^*(\lambda))}{\partial\pi^*} = \frac{dR_1(\pi^*)}{d\pi} = 0$,

$$
\frac{dG(\lambda)}{d\lambda} = \frac{\partial F^*(\lambda,\pi^*(\lambda))}{\partial\lambda} + \frac{\partial F^*(\lambda,\pi^*(\lambda))}{\partial\pi^*}\frac{d\pi^*(\lambda)}{d\lambda}
$$

$$
\begin{aligned}
&= -\pi^*(\mathbb{E}(\alpha) - \beta c - \beta\pi^*) + \pi^*\left(\mathbb{E}(\alpha) - \beta c - \frac{8\beta\pi^*}{9}\right) + \frac{\sigma^2}{4\beta} \\
&= \frac{\beta(\pi^*)^2}{9} + \frac{\sigma^2}{4\beta} \quad > \quad 0
\end{aligned}
$$

Therefore, $\frac{dG(\lambda)}{d\lambda} > 0$ for all $\lambda \in [0, 1]$. $\square$

Proof of Theorem 3.4.5. Let $\pi^*(\lambda)$ be the optimal option price for a given $\lambda \in [0, 1]$. Let $V(\lambda) = g_1(\pi^*(\lambda))$ and $W = G(\lambda) + V(\lambda)$. Base on the proof of Theorem 3.4.4, there are two cases as follows.

Case 1: $\frac{p(\alpha_h-\alpha_l)}{\beta} \geq \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}$, i.e., $\alpha_h - \alpha_l \geq \frac{3(\alpha_l-\beta c)}{p}$.

In this case, it follows that $\pi^*(\lambda) = \frac{p(\alpha_h-\beta c)}{2\beta\left(1-\frac{\lambda}{9}\right)}$. Note that $\pi^*(\lambda) < \frac{3p(\alpha_h-\alpha_l)}{4\beta} = \pi_a$. Therefore, $Q_A(\pi^*) = \alpha_h - \beta c - \frac{\beta\pi^*}{p}$, $Q_B(\pi^*) = \alpha_h - \beta c - \frac{4\beta\pi^*}{3p}$ and $\alpha_h - \beta s_h = \alpha_h - \beta c - \frac{2\beta\pi^*}{3p}$. It holds that

$$
\begin{aligned}
W(\lambda) &= (1-p)\left[-\frac{(\alpha_l - \beta c)^2}{2\beta} + \frac{(\alpha_l - \beta c)\alpha_h}{\beta} - (\alpha_l - \beta c)c\right] \\
&\quad +(1-\lambda)p\left(-\frac{Q_A(\pi^*)^2}{2\beta} + \frac{\alpha_h Q_A(\pi^*)}{\beta} - Q_A(\pi^*)c\right) \\
&\quad +\lambda p\left[-\frac{(\alpha_h - \beta s_h)^2}{2\beta} + \frac{(\alpha_h - \beta s_h)\alpha_h}{\beta} - (\alpha_h - \beta s_h)c\right] \\
&= l_1 + l_2 + l_3
\end{aligned}
$$

where $l_1 = (1-p)\left[-\frac{(\alpha_l-\beta c)^2}{2\beta} + \frac{(\alpha_l-\beta c)\alpha_l}{\beta} - (\alpha_l - \beta c)c\right]$,
$l_2 = p\left[-\frac{(\alpha_h-\beta c-\beta\pi^*/p)^2}{2\beta} + \frac{(\alpha_h-\beta c)(\alpha_h-\beta c-\beta\pi^*/p)}{\beta}\right]$ and $l_3 = \frac{\lambda 5\beta(\pi^*)^2}{18p}$.

It holds that $\frac{dl_1}{d\lambda} = 0$, $\frac{dl_2}{d\lambda} = -\frac{\beta\pi^*}{p}\frac{d\pi^*}{d\lambda}$ and $\frac{dl_3}{d\lambda} = \frac{5\beta(\pi^*)^2}{18p} + \frac{\lambda 5\beta\pi^*}{9p}\frac{d\pi^*}{d\lambda}$. Since $\frac{\lambda 5\beta\pi^*}{9p}\frac{d\pi^*}{d\lambda} \geq 0$ and $\frac{\beta\pi^*}{p}\frac{d\pi^*}{d\lambda} = \frac{\pi^*(\alpha_h-\beta c)}{18\left(1-\frac{\lambda}{9}\right)^2}$, $\frac{5\beta(\pi^*)^2}{18p} = \frac{\pi^*(\alpha_h-\beta c)}{18\left(1-\frac{\lambda}{9}\right)^2} \times \frac{5\left(1-\frac{\lambda}{9}\right)}{2} > \frac{\beta\pi^*}{p}\frac{d\pi^*}{d\lambda}$, we can conclude $\frac{dW(\lambda)}{d\lambda} = \frac{dl_1}{d\lambda} + \frac{dl_2}{d\lambda} + \frac{dl_3}{d\lambda} > 0$.

Note $W(\lambda) = G(\lambda) + V(\lambda)$ and $\frac{dG(\lambda)}{d\lambda} = \frac{\beta(\pi^*)^2}{9p}$. It holds
$\frac{dV(\lambda)}{d\lambda} = \frac{dW(\lambda)}{d\lambda} - \frac{dG(\lambda)}{d\lambda} = -\frac{\beta\pi^*}{p}\frac{d\pi^*}{d\lambda} + \frac{3\beta(\pi^*)^2}{18p} + \frac{\lambda 5\beta\pi^*}{9p}\frac{d\pi^*}{d\lambda} > 0$

Case 2: $\frac{p(\alpha_h-\alpha_l)}{\beta} < \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}$, i.e., $\alpha_h - \alpha_l < \frac{3(\alpha_l-\beta c)}{p}$

The optimal solution can fall in any of the three intervals, $\left[0, \frac{3p(\alpha_h-\alpha_l)}{4\beta}\right)$, $\left[\frac{3p(\alpha_h-\alpha_l)}{4\beta}, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$ and $\left[\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{3(\mathbb{E}(\alpha)-\beta c)}{4\beta}\right]$. If the optimal solution π^* jumps from one interval to another at some λ, by Lemma 3.4.6, total purchased products drops. Therefore, total social surplus and buyer's surplus may decrease.

If $\pi^*(\lambda)$ for some $\lambda \in I \subset I_1$, it has been shown that $\frac{dW(\lambda)}{d\lambda} \geq 0$ and $\frac{dV(\lambda)}{d\lambda} \geq 0$.

Therefore, as λ increases, $W(\lambda)$ and $V(\lambda)$ may increase or decrease. Such property is demonstrated in Example 1 and Example 2. $\square$

Proof of Lemma 3.4.7 Let $R_A(\pi) = \pi Q_A(\pi)$ and $R_B(\pi) = \pi Q_B(\pi) + \mathbb{E}[G_2(Q_B(\pi), \alpha)]$. Since $Q_A \leq C \leq \alpha_l - \beta c$, all Q_A will be exercised in both demand states in Period 2. By simple analysis, we can get $Q_A(\pi) = C$, $\forall \pi \in \left[0, \frac{\mathbb{E}(\alpha)-\beta c-C}{\beta}\right)$ and $Q_A(\pi) = \mathbb{E}(\alpha) - \beta(c+\pi)$, $\forall \pi \in \left[\frac{\mathbb{E}(\alpha)-\beta c-C}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right]$. It follows that the unique optimal solution of $R_A(\pi)$ is $\frac{\mathbb{E}(\alpha)-\beta c-C}{\beta}$.

Similarly, $Q_B \leq C \leq \alpha_l - \beta c$, all Q_B will be exercised in both demand states in Period 2. For any $0 \leq Q_B < C$, the optimal spot prices in the subgame are $\frac{\alpha_h - C}{\beta}$ and $\frac{\alpha_l - C}{\beta}$. It follows that for $\pi > \frac{\mathbb{E}(\alpha)-\beta c-C}{\beta}$, $Q_B(\pi) = 0$; for $\pi < \frac{\mathbb{E}(\alpha)-\beta c-C}{\beta}$, $Q_B(\pi) = C$; for $\pi = \frac{\mathbb{E}(\alpha)-\beta c-C}{\beta}$, any $Q_B(\pi) \in [0, C]$ is optimal to the buyer. By simple analysis, we can see any $\pi \geq \frac{\mathbb{E}(\alpha)-\beta c-C}{\beta}$ is optimal to $R_B(\pi)$.

Therefore, $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c-C}{\beta}$ is the solution for all $\lambda \in [0, 1]$. $\square$

Proof of Theorem 3.4.6 By Lemma 3.4.7, $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c-C}{\beta}$ for all $\lambda \in [0, 1]$. And $Q^*(\lambda) = (1-\lambda)Q_A(\pi^*) + \lambda Q_B(\pi^*)$. Note that $Q_A(\pi^*) = C$ and $Q_B(\pi^*) \leq C$. Therefore, $\frac{dQ^*(\lambda)}{d\lambda} = Q_B(\pi^*) - Q_A(\pi^*) \leq 0$. $\square$

Proof of Theorem 3.4.7 By Lemma 3.4.7, $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c-C}{\beta}$ for all $\lambda \in [0, 1]$. The seller's surplus $G(\lambda) = (1-\lambda)R_A(\pi^*) + \lambda R_B(\pi^*)$. Note that $R_A(\pi^*) = R_B(\pi^*) = \pi^* C$. Therefore, $G(\lambda) = \pi^* C = \frac{C(\mathbb{E}(\alpha)-\beta c-C)}{\beta}$ for all $\lambda \in [0, 1]$. $\square$

Proof of Theorem 3.4.8 By Lemma 3.4.7, $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c-C}{\beta}$ for all $\lambda \in [0, 1]$. Let $V_A(\pi^*)$ be the buyer's surplus if he doesn't transact on the spot market and let $V_B(\pi^*)$ be his surplus if he does. The buyer's expected surplus $V(\lambda) = (1-\lambda)V_A(\pi^*) + \lambda V_B(\pi^*)$. Note that $V_A(\pi^*) = V_B(\pi^*) = \frac{C^2}{2\beta}$ for all $\lambda \in [0, 1]$. Therefore, $V(\lambda) = \frac{C^2}{2\beta}$ for all $\lambda \in [0, 1]$. The total social surplus $W(\lambda) = \frac{C(\mathbb{E}(\alpha)-\beta c-C^2/2)}{\beta}$ for all $\lambda \in [0, 1]$. $\square$

Proof of Lemma 3.5.1 Since no buyer will purchase more than $\alpha_h - \beta c$, only $Q \in [0, \alpha_h - \beta c]$ is interesting. The outline of the proof is as follows. First, we prove that for any given π, $s_h \geq c$ and $s_l \geq c$, the objective function of the buyers' problem, $r_1(Q|\pi)$, is concave in Q on $[0, \alpha_h - \beta c]$, thus the first order condition is necessary and sufficient for optimality. For any given Q, the optimal spot prices in Period 2, s_h and s_l, are characterized in Lemma 3.4.4. Combining those results, a set of values (Q, s_h, s_l) is an equilibrium in the subgame satisfy the optimal condition for the buyers' problem and the seller's problem at stage 3.

Let $g_2(Q, s, \alpha)$ denotes the best return for each buyer in Period 2. The argument s denotes the spot price that each buyer takes as given and not a function of Q. We use s_h and s_l to denote spot prices for the high demand state and low demand state respectively in this proof. The objective function in the buyers' problem at stage 2 is

$$r_1(Q|\pi) \quad = \quad -\pi Q + pg_2(Q, s_h, \alpha_h) + (1-p)g_2(Q, s_l, \alpha_l)$$

Since the first term $-\pi Q$ is linear, it is concave. Next, we will show $g_2(Q, s_h, \alpha_h)$ and $g_2(Q, s_l, \alpha_l)$ are both concave in Q.

Denote each buyer's corresponding best response at stage 4 as $q_c(Q, s, \alpha)$ and $q_s(Q, s, \alpha)$. It holds that

$$\begin{aligned} g_2(Q, s_h, \alpha_h) \quad = \quad & -\frac{(q_c(Q, s_h, \alpha_h) + q_s(Q, s_h, \alpha_h)))^2}{2\beta} \\ & + \frac{\alpha_h(q_c(Q, s_h, \alpha_h) + q_s(Q, s, \alpha_h))}{\beta} - q_c(Q, s_h, \alpha_h)c - q_s(Q, s_h, \alpha_h)s_h \end{aligned}$$

Depending on the value of $\alpha_h - \beta s_h$, there are two cases.

Case 1-H: $\alpha_h - \beta s_h < 0$. For $Q \in [0, \alpha_h - \beta c)$, $g_2(Q, s_h, \alpha_h) = -\frac{Q^2}{2\beta} + \frac{\alpha_h Q}{\beta} - Qc$. Therefore, it is concave.

Case 2-H: $0 \leq \alpha_h - \beta s_h \leq \alpha_h - \beta c$. For $Q \in [0, \alpha_h - \beta s_h)$, it holds that

$$\begin{aligned} g_2(Q, s_h, \alpha_h) \quad &= \quad -\frac{(\alpha_h - \beta s_h)^2}{2\beta} + \frac{\alpha_h(\alpha_h - \beta s_h)}{\beta} - Qc - (\alpha_h - \beta s_h - Q)s_h \\ \frac{d(g_2(Q, s, \alpha_h))}{dQ} \quad &= \quad s_h - c \end{aligned}$$

For $Q \in [\alpha_h - \beta s_h, \alpha_h - \beta c]$, it holds that

$$\begin{aligned} g_2(Q, s_h, \alpha_h) &= -\frac{Q^2}{2\beta} + \frac{\alpha_h Q}{\beta} - Qc \\ \frac{d(g_2(Q, s_h, \alpha_h))}{dQ} &= \frac{\alpha_h - \beta c - Q}{\beta} \end{aligned}$$

At the breakpoint $Q = \alpha_h - \beta s_h$, $g_2(Q, s_h, \alpha_h)$ is continuous and the left derivative is equal to the right derivative.

$$g'_{2-}(Q, s_h, \alpha_h) = g'_{2+}(Q, s_h, \alpha_h) = \frac{\alpha_h - \beta c - Q}{\beta}$$

Combining both cases, $g_2(Q, s_h, \alpha_h)$ is concave in Q on $[0, \alpha_h - \beta c]$.

Similarly, we can prove $g_2(Q, s_l, \alpha_l)$ is also concave on $[0, \alpha_h - \beta c]$. Note

$$\begin{aligned} g_2(Q, s_l, \alpha_l) = & -\frac{(q_c(Q, s_l, \alpha_l) + q_s(Q, s_l, \alpha_l)))^2}{2\beta} \\ & + \frac{\alpha_l(q_c(Q, s_l, \alpha_l) + q_s(Q, s, \alpha_l))}{\beta} - q_c(Q, s_l, \alpha_l)c - q_s(Q, s_l, \alpha_l)s_l \end{aligned}$$

Depending on the values of $\alpha_l - \beta s_l$, there are also two cases, $\alpha_l - \beta s_l < 0$ and $0 \le \alpha_l - \beta s_l \le \alpha_l - \beta c$. The proof of the concavity of this term is as follows.

Case 1-L: $\alpha_l - \beta s_l < 0$. For $Q \in [0, \alpha_l - \beta c]$, $g_2(Q, s_l, \alpha_l) = -\frac{Q^2}{2\beta} + \frac{\alpha_l Q}{\beta} - Qc$. Thus, it is concave. For $Q \in [\alpha_l - \beta c, \alpha_h - \beta c]$, $g_2(Q, s_l, \alpha_l) = \frac{(\alpha_l - \beta c)^2}{2\beta}$. Note at $Q = \alpha_l - \beta c$, $g_2(Q, s_l, \alpha_l)$ is also continuous, and the right derivative and left derivative are both equal to zero.

Case 2-L: $0 \le \alpha_l - \beta s_l \le \alpha_l - \beta c$. For $Q \in [0, \alpha_l - \beta s_l)$, it holds that

$$\begin{aligned} g_2(Q, s_l, \alpha_l) &= -\frac{(\alpha_l - \beta s_l)^2}{2\beta} + \frac{\alpha_l(\alpha_l - \beta s_l)}{\beta} - Qc - (\alpha_l - \beta s_l - Q)s_l \\ \frac{dg_2(Q, s_l, \alpha_l)}{dQ} &= s_l - c \end{aligned}$$

For $Q \in [\alpha_l - \beta s_l, \alpha_l - \beta c)$, it holds that $g_2(Q, s_l, \alpha_l) = -\frac{Q^2}{2\beta} + \frac{\alpha_l Q}{\beta} - Qc$ and $\frac{dg_2(Q, s_l, \alpha_l)}{dQ} = \frac{\alpha_l - \beta c - Q}{\beta}$. For $Q \in [\alpha_l - \beta c, \alpha_h - \beta c]$, it holds that $g_2(Q, s_l, \alpha_l) = \frac{(\alpha_l - \beta c)^2}{2\beta}$.

Note at $Q = \alpha_l - \beta s_l$, $g_2(Q, s_l, \alpha_l)$ is continuous and

$$g'_{2-}(Q, s_l, \alpha_l) = g'_{2+}(Q, s_l, \alpha_l) = \frac{\alpha_l - \beta c - Q}{\beta}$$

Also at $Q = \alpha_l - \beta c$, $g_2(Q, s_l, \alpha_l)$ is continuous and

$$g'_{2-}(Q, s_l, \alpha_l) \quad = \quad g'_{2+}(Q, s_l, \alpha_l) \quad = \quad 0$$

Thus, $g_2(Q, s_l, \alpha_l)$, is concave on $[0, \alpha_h - \beta c]$.

Because the objective function is concave and the constraints are linear , the Karush-Kuhn-Tucker condition is necessary and sufficient for the optimality in this problem. Now, let's consider the seller's best response for a given Q. We also denote the best response of the seller in the spot market as s_h and s_l in the high and low market states respectively. By Lemma 3.4.4, for any $Q \in [0, \alpha_l - \beta c]$, $s_h = \frac{\alpha_h + \beta c - Q}{2\beta}$ and $s_l = \frac{\alpha_l + \beta c - Q}{2\beta}$. If $Q \in [\alpha_l - \beta c, \alpha_h - \beta c]$, $s_h = \frac{\alpha + \beta c - Q}{2\beta}$ and $s_l \geq c$. Let (Q^*, s_h^*, s_l^*) be an equilibrium, it must satisfy the optimality conditions for both the seller's problem and the buyer's problem.

If $Q^* \in (0, \alpha_l - \beta c]$, the conditions are

$$s_h^* \quad = \quad \frac{\alpha_h + \beta c - Q^*}{2\beta} \tag{A.0.1}$$

$$s_l^* \quad = \quad \frac{\alpha_l + \beta c - Q^*}{2\beta} \tag{A.0.2}$$

$$\frac{dr_1(Q^*|\pi)}{dQ} \quad = \quad 0 \tag{A.0.3}$$

Equations (A.0.1) and (A.0.2) imply $Q^* \leq \alpha_l - \beta s_l^* \leq \alpha_h - \beta s_h^*$. Equation (A.0.3) implies $\frac{dr_1(Q^*|\pi)}{dQ} = -\pi + p(s_h^* - c) + (1-p)(s_l^* - c) = 0$. Thus, the system (A.0.1), (A.0.2) and (A.0.3) requires, for a given $\pi \geq 0$, a market equilibrium $Q^* \in (0, \alpha_l - \beta c]$ must satisfy $-\pi + \frac{\mathbb{E}(\alpha) - \beta c - Q^*}{2\beta} = 0$. At the endpoints $Q = 0$ and $Q = \alpha_l - \beta c$, $r'_{1+}(0, \pi) = -\pi + \frac{\mathbb{E}(\alpha) - \beta c}{2\beta} > 0$ and $r'_{1-}(\alpha_l - \beta c, \pi) = -\pi + \frac{p(\alpha_h - \alpha_l)}{2\beta} \leq 0$. Such condition holds if and only if $\pi \in \left[\frac{p(\alpha_h - \alpha_l)}{2\beta}, \frac{\mathbb{E}(\alpha) - \beta c}{2\beta}\right)$, $Q^* = \mathbb{E}(\alpha) - \beta c - 2\beta\pi$.

If $Q^* \in (\alpha_l - \beta c, \alpha_h - \beta c]$, the conditions are

$$s_h^* \quad = \quad \frac{\alpha_h + \beta c - Q^*}{2\beta} \tag{A.0.4}$$

$$s_l^* \quad \geq \quad c \tag{A.0.5}$$

$$\frac{dr_1(Q^*|\pi)}{dQ} \quad = \quad 0 \tag{A.0.6}$$

Equations (A.0.4) and (A.0.5) imply $Q^* \le \alpha_h - \beta s_h^*$. Equation (A.0.6) is $\frac{dr_1(Q^*|\pi)}{dQ} = -\pi + p(s_h^* - c) = 0$. Thus, the system (A.0.4), (A.0.5) and (A.0.6) implies, for a given $\pi \ge 0$, a market equilibrium $Q^* \in (\alpha_l - \beta c, \alpha_h - \beta c]$ must satisfy $-\pi + \frac{p(\alpha_h - \beta c - Q^*)}{2\beta} = 0$. At the endpoints $Q = \alpha_l - \beta c$ and $Q = \alpha_h - \beta c$, $r'_{1+}(\alpha_l - \beta c|\pi) = -\pi + \frac{p(\alpha_h - \alpha_l)}{2\beta} > 0$ and $r_{1+}(\alpha_h - \beta c|\pi) = -\pi \le 0$. Such condition holds if and only if $\pi \in \left[0, \frac{p(\alpha_h - \alpha_l)}{2\beta}\right)$, $Q^* = \alpha_h - \beta c - \beta\pi/p$.

If $Q^* = 0$, the conditions are

$$s_h^* = \frac{\alpha_h + \beta c - Q^*}{2\beta} \tag{A.0.7}$$

$$s_l^* = \frac{\alpha_l + \beta c - Q^*}{2\beta} \tag{A.0.8}$$

$$\frac{dr_1(Q^*|\pi)}{dQ} \quad \le \quad 0 \tag{A.0.9}$$

Such condition holds if and only if $\pi \ge \frac{\mathbb{E}(\alpha) - \beta c}{2\beta}$.

Based on the above analysis, we summarize the results as follows.

1. If $\pi \in \left(0, \frac{p(\alpha_h - \alpha_l)}{2\beta}\right]$, then the equilibrium (Q^*, s_h^*, s_l^*) in the subgame is $Q^* = \alpha_h - \beta c - \frac{2\beta\pi}{p}$, $s_h^* = \frac{\alpha_h + \beta c - Q^*}{2\beta}$, and $s_l^* \ge c$. Note $Q^* \in [\alpha_l - \beta c, \alpha_h - \beta c)$.

2. If $\pi \in \left(\frac{p(\alpha_h - \alpha_l)}{2\beta}, \frac{\mathbb{E}(\alpha) - \beta c}{2\beta}\right]$, then the unique equilibrium in the subgame is (Q^*, s_h^*, s_l^*), where $Q^* = \mathbb{E}(\alpha) - \beta c - 2\beta\pi$, $s_h^* = \frac{\alpha_h + \beta c - Q^*}{2\beta}$, and $s_l^* = \frac{\alpha_h + \beta c - Q^*}{2\beta}$. Note $Q^* \in [0, \alpha_l - \beta c)$.

3. If $\pi > \frac{\mathbb{E}(\alpha) - \beta c}{2\beta}$, then the unique equilibrium is (Q^*, s_h^*, s_l^*), where $Q^* = 0$, $s_h^* = \frac{\alpha_h + \beta c}{2\beta}$, and $s_l^* = \frac{\alpha_h + \beta c}{2\beta}$.

4. If $\pi = 0$, then (Q^*, s_h^*, s_l^*) satisfies $Q^* \ge \alpha_h - \beta c$, $s_h^* \ge c$, and $s_l^* \ge c$.

Note, at the breakpoint $\pi = \frac{p(\alpha_h - \alpha_l)}{2\beta}$, $Q^* = \mathbb{E}(\alpha) - \beta c - 2\beta\pi = \alpha_h - \beta c - \frac{2\beta\pi}{p}$. Also, there are no overlaps among the intervals defined above. Therefore, for any given $\pi \in (0, +\infty)$, Q^* is unique. $\square$

Proof of Theorem 3.5.1 Let $\hat{R}_1(\pi) = R_1(\pi)/N$. Since $N > 0$, the optimization problem is equivalent to maximize $\hat{R}_1(\pi)$ with the nonnegative constraint. Without confusion, we use$R_1(\pi)$ as $\hat{R}_1(\pi)$ in the following proof for simplicity.

For $\pi \in \left[0, \frac{p(\alpha_h-\alpha_l)}{2\beta}\right)$, by Lemma 3.5.1, $Q = \alpha_h - \beta c - \frac{2\beta\pi}{p}$. The corresponding spot prices $s_h = \frac{\alpha_h+\beta c-Q}{2\beta}$ and s_l can be any value no smaller than the marginal cost. It holds that

$$\begin{aligned} R_1(\pi) &= \pi\left(\alpha_h - \beta c - \frac{2\beta\pi}{p}\right) + \frac{p(\alpha_h - \beta c - Q)^2}{4\beta} \\ &= \pi\left(\alpha_h - \beta c - \frac{\beta\pi}{p}\right) \end{aligned}$$

The derivative of function $R_1(\pi)$ in this interval is $\frac{dR_1(\pi)}{d\pi} = \alpha_h - \beta c - \frac{2\beta\pi}{p}$. At $\pi = 0$, $\frac{dR_1(0)}{d\pi} = \alpha_h - \beta c > 0$. At $\pi = \frac{p(\alpha_h-\alpha_l)}{2\beta}$, $R'_{1-}(\pi) = \alpha_l - \beta c \geq 0$. Therefore, the optimal solution is not in this interval.

For $\pi \in \left[\frac{p(\alpha_h-\alpha_l)}{2\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}\right]$, $Q = \mathbb{E}(\alpha) - \beta c - 2\beta\pi$. It holds that

$$\begin{aligned} R_1(\pi) &= \pi(\mathbb{E}(\alpha) - \beta c - 2\beta\pi) + \frac{p(\alpha_h - \beta c - Q)^2 + (1-p)(\alpha_l - \beta c - Q)^2}{4\beta} \\ &= \pi(\mathbb{E}(\alpha) - \beta c - \beta\pi) + \frac{\sigma^2}{4\beta} \end{aligned}$$

The derivative is $\frac{dR_1(\pi)}{d\pi} = \mathbb{E}(\alpha) - \beta c - 2\beta\pi$. At $\pi = \frac{p(\alpha_h-\alpha_l)}{2\beta}$, function $R_1(\pi)$ is continuous and $R'_{1-}(\pi) = R'_{1+}(\pi) = \alpha_l - \beta c \geq 0$. At the end point $\pi = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$, $Q = 0$ and $\frac{dR_1(\pi)}{d\pi} = 0$.

For $\pi > \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$, $Q = 0$ and $R_1(\pi) = R_1\left(\frac{\mathbb{E}(\alpha)-\beta c}{2\beta}\right)$. Therefore, function $R_1(\pi)$ is a continuous concave function on $[0, +\infty)$. Since at the endpoint $\pi = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$, $\frac{dR_1(\pi)}{d\pi} = 0$, the optimal option price can be any value in $\left[\frac{\mathbb{E}(\alpha)-\beta c}{2\beta}, \infty\right)$. In this price range, the result is the same as if the buyers do not enter into contracts at all. □

Proof of Lemma 3.5.2 To simplify the notation, we first divide $R_1(\pi)$ by N without changing the problem. Let $R_A(\pi) = \pi Q_A(\pi)$ and $R_B(\pi) = \pi Q_B(\pi) + \mathbb{E}(G_2(Q_B(\pi), \alpha))/N$. Note that from the proof of Theorem 3.4.1, $R_A(\pi)$ is a piecewise function on $\left[0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$ and $\left[\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right]$. For $\pi > \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$, $R_A(\pi) =$

$R_A\left(\frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right) = 0$. Similarly, from Theorem 3.5.1, $R_B(\pi)$ is a piecewise function on $\left[0, \frac{p(\alpha_h-\alpha_l)}{2\beta}\right)$ and $\left[\frac{p(\alpha_h-\alpha_l)}{2\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}\right]$. For $\pi > \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$, $R_B(\pi) = R_B\left(\frac{\mathbb{E}(\alpha)-\beta c}{2\beta}\right)$. Therefore, based on the positions of the breakpoints, there are two cases.

Case 1: $\frac{p(\alpha_h-\alpha_l)}{\beta} \geq \frac{\mathbb{E}(\alpha)-\beta c}{2\beta} \Leftrightarrow \alpha_h - \alpha_l \geq \frac{\alpha_l-\beta c}{p}$.

Let the optimal option price in the contract market only setting be π_A^* and the optimal price in the contract market and spot market with full participation setting be π_B^*. By Theorem 3.4.1, $\pi_A^* = \frac{p(\alpha_h-\beta c)}{2\beta}$. By Theorem 3.5.1, π_B^* can be any value larger or equal to $\frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$. Note $\pi_A^* < \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$, the optimal solution π^* should only fall in $\left[\frac{p(\alpha_h-\beta c)}{2\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}\right]$. Since both $R_A(\pi)$ and $R_B(\pi)$ are concave on $\left[\frac{p(\alpha_h-\beta c)}{2\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}\right]$, $R_1(\pi)$ is also concave in this interval. Thus, the first order condition is necessary and sufficient for optimality.

Since $\frac{p(\alpha_h-\alpha_l)}{2\beta} \leq \frac{p(\alpha_h-\beta c)}{2\beta}$, $Q_B(\pi) = \mathbb{E}(\alpha) - \beta c - 2\beta\pi$ for any $\pi \in \left[\frac{p(\alpha_h-\beta c)}{2\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}\right]$. In this interval, it holds that

$$R_1(\pi) = (1-\lambda)\pi\left(\alpha_h - \beta c - \frac{\beta\pi}{p}\right) + \lambda\left[\pi(\mathbb{E}(\alpha) - \beta c - \beta\pi) + \frac{\sigma^2}{4\beta}\right]$$

The derivative of the objective function is

$$\frac{dR_1(\pi)}{d\pi} = (1-\lambda)\left(\alpha_h - \beta c - \frac{2\beta\pi}{p}\right) + \lambda(\mathbb{E}(\alpha) - \beta c - 2\beta\pi)$$

The optimal price π^* must satisfy $\frac{dR_1(\pi^*)}{d\pi} = 0$. Therefore, $\pi^* = \frac{(1-\lambda)(\alpha_h-\beta c)+\lambda(\mathbb{E}(\alpha)-\beta c)}{2\beta\left[\frac{1-\lambda}{p}+\lambda\right]}$ and $\frac{d\pi^*(\lambda)}{d\lambda} = \frac{(1-p)(\alpha_l-\beta c)}{2\beta p\left(\frac{1-\lambda}{p}+\lambda\right)^2} \geq 0$. Hence, in this case, the equilibrium π^* increases in λ.

Case 2: $\frac{p(\alpha_h-\alpha_l)}{\beta} < \frac{\mathbb{E}(\alpha)-\beta c}{2\beta} \Leftrightarrow \alpha_h - \alpha_l < \frac{\alpha_l-\beta c}{p}$.

Case 2.a: $\alpha_h - \alpha_l \leq \frac{\alpha_l-\beta c}{\sqrt{p}}$. In this case, $\pi_A^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$, $\pi_B^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$, thus $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$ and does not change as λ changes.

Case 2.b: $\frac{\alpha_l-\beta c}{\sqrt{p}} < \alpha_h - \alpha_l < \frac{\alpha_l-\beta c}{p}$. In this case $\pi_A^* = \frac{\alpha_h-\beta c}{2\beta}$. Let $\pi_B^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$, which is also a local maximizer of $R_A(\pi)$ in the interval $\left[\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right]$. Denote the intervals $\tilde{I}_1 = \left(\frac{p(\alpha_h-\beta c)}{2\beta}, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$ and $\tilde{I}_2 = \left(\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}\right)$.

If $\pi^* \in \tilde{I}_1$, $\pi^* = \frac{(1-\lambda)(\alpha_h-\beta c)+\lambda(E(\lambda)-\beta c)}{2\beta\left[\frac{1-\lambda}{p}+\lambda\right]}$. From the results in Case 1, we can get that π^* increases in λ if π^* moves in this interval. If $\pi^* \in \tilde{I}_2$, $\pi_A^* = \pi_B^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$.

Thus, $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$. Note π^* can't be the breakpoint $\frac{p(\alpha_h-\alpha_l)}{\beta}$, since at this point the right derivative is strictly larger than the left derivative. Note for $\lambda = 0$, $\pi^* = \frac{\alpha_h-\beta c}{2\beta}$ and for $\lambda = 1$, $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$. Thus, it is easy to see the result holds.

Next we will show that that when λ increases from 0 to 1, the optimal solution $\pi^*(\lambda)$ can only jump from one interval to the other on the right and can never jump to the left. The proof is similar to that in Lemma 3.4.6. First, we will show at for a fixed π, as λ increases, $\frac{dR_1(\pi)}{d\pi}$ increases. Without confusion, let $f(\pi,\lambda) = \frac{dR_1(\pi)}{d\pi}$. For $\pi \in \tilde{I}_1$, it holds that

$$\begin{aligned} f(\pi,\lambda) &= (1-\lambda)\left(\alpha_h - \beta c - \frac{2\beta\pi}{p}\right) + \lambda(\mathbb{E}(\alpha) - \beta c - 2\beta\pi) \\ \frac{\partial f(\pi,\lambda)}{\partial \lambda} &= \frac{2\beta\pi}{p} - 2\beta\pi \quad > \quad 0 \end{aligned}$$

For $\pi \in \tilde{I}_2$, it holds that

$$\begin{aligned} f(\pi,\lambda) &= \mathbb{E}(\alpha) - \beta c - 2\beta\pi \\ \frac{\partial f(\pi,\lambda)}{\partial \lambda} &= 0 \end{aligned}$$

Let $\pi_l(\lambda)$ and $\pi_r(\lambda)$ be the local maximizers in $\tilde{I}_1$ and $\tilde{I}_2$ respectively. For a given λ, $\pi^* = \mathrm{argmax}\{R_1(\pi_l), R_1(\pi_r)\}$. Suppose at λ_0, $\pi^* = \pi_r$. We will prove that as λ_0 increases to 1, π^* can never fall back to $\tilde{I}_1$. Suppose at the smallest λ_1 such that $\lambda_1 > \lambda_0$ and the optimal price becomes π_l again. This can happen only if from λ_0 to λ_1, $R'_{1+}\left(\frac{p(\alpha_h-\alpha_l)}{\beta}\right) > 0$ and $R'_{1-}\left(\frac{p(\alpha_h-\alpha_l)}{\beta}\right) < 0$ always hold. Note that

$$R_1(\pi_l(\lambda)) - R_l(\pi_r(\lambda)) = \int_{\pi_l(\lambda)}^{\frac{p(\alpha_h-\alpha_l)}{\beta}} \left|\frac{dR_1(\pi)}{d\pi}\right| d\pi - \int_{\frac{p(\alpha_h-\alpha_l)}{\beta}}^{\pi_r(\lambda)} \frac{dR_1(\pi)}{d\pi} d\pi$$

decreases in λ. Thus, this provides a contradiction. Therefore, we can see that as λ increases from 0 to 1, $\pi^*(\lambda)$ increases from π_A^* to π_B^*. □

Proof of Theorem 3.5.2 To simplify the notation, we first divide $Q^*(\lambda)$ by N without changing the problem. Based on the proof of Lemma 3.5.2, we can divide this proof into two cases according to values of the breakpoints.

Case 1: $\frac{p(\alpha_h-\alpha_l)}{\beta} \geq \frac{\mathbb{E}(\alpha)-\beta c}{2\beta} \Leftrightarrow \alpha_h - \alpha_l \geq \frac{\alpha_l - \beta c}{p}$.

In this case, the equilibrium price is $\pi^* = \frac{(1-\lambda)(\alpha_h-\beta c)+\lambda(\mathbb{E}(\alpha)-\beta c)}{2\beta\left[\frac{1-\lambda}{p}+\lambda\right]}$. The total contracting quantity in equilibrium is $Q^* = \frac{(1-\lambda)\beta\pi^*}{p}$. It holds that

$$\begin{aligned}\frac{dQ^*(\lambda,\pi^*(\lambda))}{d\lambda} &= \frac{\partial Q^*(\lambda,\pi^*(\lambda))}{\partial\lambda} + \frac{\partial Q^*(\lambda,\pi^*(\lambda))}{\partial\pi^*}\frac{d\pi^*(\lambda)}{d\lambda} \\ &= -\frac{\beta\pi^*}{p} + (1-\lambda)\frac{\beta}{p}\frac{d\pi^*(\lambda)}{d\lambda}\end{aligned}$$

where $\frac{d\pi^*(\lambda)}{d\lambda} = \frac{(1-p)(\alpha_l-\beta c)}{2\beta p\left[\frac{1-\lambda}{p}+\lambda\right]^2}$. Substitute $\frac{d\pi^*(\lambda)}{d\lambda}$ into above equation, we can get

$$\begin{aligned}\frac{dQ^*(\lambda)}{d\lambda} &= -\frac{(1-\lambda)(\alpha_h-\beta c)+\lambda(\mathbb{E}(\alpha)-\beta c)}{2\left(1-\lambda+p\lambda\right)} + \frac{(1-p)(\alpha_l-\beta c)(1-\lambda)}{2\left(1-\lambda+p\lambda\right)^2} \\ &< \frac{(\alpha_l-\beta c)[(1-\lambda)(1-p)-(1-\lambda+p\lambda)]}{2\left(1-\lambda+p\lambda\right)^2} \\ &\leq 0\end{aligned}$$

Therefore, in this case, the total number of contracts transacted decreases as λ increases.

Case 2: $\frac{p(\alpha_h-\alpha_l)}{\beta} < \frac{\mathbb{E}(\alpha)-\beta c}{2\beta} \Leftrightarrow \alpha_h - \alpha_l < \frac{\alpha_l - \beta c}{p}$.

Case 2.a: $\alpha_h - \alpha_l \leq \frac{\alpha_l-\beta c}{\sqrt{p}}$. In this case, $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$ and

$$\begin{aligned}Q^* &= (1-\lambda)(E(\alpha-\beta c-\beta\pi^*) + \lambda(E(\alpha-\beta c-2\beta\pi^*) \quad = \quad (1-\lambda)\beta\pi^* \\ \frac{dQ^*}{d\lambda} &= -\beta\pi^* = -\frac{\mathbb{E}(\alpha)-\beta c}{2} < 0\end{aligned}$$

Thus, Q^* decreases in λ.

Case 2.b: $\frac{\alpha_l-\beta c}{\sqrt{p}} < \alpha_h - \alpha_l < \frac{\alpha_l-\beta c}{p}$. In this case, $\pi^* \in \left[\frac{p(\alpha_h-\beta c)}{2\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right]$.

If $\pi^* \in \left[\frac{p(\alpha_h-\beta c)}{2\beta}, \frac{p(\alpha_h-\alpha_l)}{\beta}\right]$, then using the results in Case 1, we can get that Q^* decreases in λ if π^* moves in this interval. If $\pi^* \in \left[\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}\right]$, $\pi_A^* = \pi_B^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$ and $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$. By using the results in Case 2.a, we can see that Q^* decreases in λ in this interval. When π^* jumps from interval $\tilde{I}_1$ to $\tilde{I}_2$, both $Q_A(\pi^*)$ and $Q_B(\pi^*)$ decrease. Therefore, Q^* also decreases as λ increases. □

Proof of Theorem 3.5.3 To simplify the notation, we first divide $G(\lambda)$ by N without changing the problem. Based on the proof of Lemma 3.5.2, we can divide this proof into two cases according to values of the breakpoints.

Case 1: $\frac{p(\alpha_h-\alpha_l)}{\beta} \geq \frac{\mathbb{E}(\alpha)-\beta c}{2\beta} \Leftrightarrow \alpha_h - \alpha_l \geq \frac{\alpha_l - \beta c}{p}$. It holds that

$$G(\lambda) = \frac{p(\alpha_h - \beta c - \lambda(1-p)(\alpha_h - \alpha_l))^2}{4\beta(1-\lambda+\lambda p)} + \frac{\lambda p(1-p)(\alpha_h-\alpha_l)^2}{4\beta}$$

Let $x = \alpha_h - \beta c$, $y = \alpha_h - \alpha_l$ and $z = \alpha_l - \beta c$. Note $x = z + y$.

$$\frac{dG(\lambda)}{d\lambda} = \frac{p(1-p)[(x-\lambda(1-p)y)(x-(2-\lambda+\lambda p)y) - (1-\lambda+\lambda p)^2 y^2]}{4\beta(1-\lambda+\lambda p)^2}$$

Substitute $x = z + y$ into above equation, we get $\frac{dG(\lambda)}{d\lambda} = \frac{p(1-p)z^2}{4\beta(1-\lambda+\lambda p)^2} \geq 0$.

Case 2: $\frac{p(\alpha_h-\alpha_l)}{\beta} < \frac{\mathbb{E}(\alpha)-\beta c}{2\beta} \Leftrightarrow \alpha_h - \alpha_l < \frac{\alpha_l - \beta c}{p}$.

Case 2.a: $\alpha_h - \alpha_l \leq \frac{\alpha_l - \beta c}{\sqrt{p}}$. It holds that $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$ and

$$\begin{aligned} G(\lambda) &= \frac{(\mathbb{E}(\alpha)-\beta c)^2}{4\beta} + \frac{\lambda p(1-p)(\alpha_h-\alpha_l)^2}{4\beta} \\ \frac{dG(\lambda)}{d\lambda} &= \frac{p(1-p)(\alpha_h-\alpha_l)^2}{4\beta} = \frac{\sigma^2}{4\beta} > 0 \end{aligned}$$

Case 2.b: $\frac{\alpha_l-\beta c}{\sqrt{p}} < \alpha_h - \alpha_l < \frac{\alpha_l-\beta c}{p}$. By similar arguments in the proof of Lemma 3.4.6, $G(\lambda)$ is a continuous function of λ. The optimal option price π^* may move from the local maximizer in one interval to another. In this case, $G(\lambda)$ is the same at both local maximizer. Also, Lemma 3.5.2 says π^* increases as λ increases. Therefore, we only need to show that $G(\lambda)$ increases in λ if π^* moves in each interval. For $\pi^* \in \left[0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$, the results in Case 1 hold. For $\pi^* \in \left[\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}\right]$, the proof is the same as Case 2a.

Therefore, $G(\lambda)$ increases as λ increases for all cases. □

Proof of Theorem 3.5.4 To simplify the notation, we first divide $V(\lambda)$ and $W(\lambda)$ by N without changing the problem. This proof is also based on the proofs of Lemma 3.5.2 and Theorem3.5.2. Consider the following two cases.

Case 1: $\frac{p(\alpha_h-\alpha_l)}{\beta} \geq \frac{\mathbb{E}(\alpha)-\beta c}{2\beta} \Leftrightarrow \alpha_h - \alpha_l \geq \frac{\alpha_l-\beta c}{p}$. Let $\pi^*(\lambda)$ be the option price in equilibrium. It holds that

$$W(\lambda) = \frac{p(\alpha_h-\beta c)^2 + (1-p)(\alpha_l-\beta c)^2}{2\beta} - \frac{1}{2}\left[\frac{\beta \pi^{*2}(1-\lambda+\lambda p)}{p} + \frac{\lambda\sigma^2}{4\beta}\right]$$

Note that $\frac{\beta\pi^{*2}(1-\lambda+\lambda p)}{p} + \frac{\lambda\sigma^2}{4\beta} = G(\lambda)$. Since by Theorem 3.5.3, $\frac{dG(\lambda)}{d\lambda} \geq 0$, it follows $\frac{dW(\lambda)}{d\lambda} = -\frac{1}{2}\frac{dG(\lambda)}{d\lambda} \leq 0$. Therefore, $V(\lambda)$ must decrease in λ.

Case 2: $\frac{p(\alpha_h-\alpha_l)}{\beta} < \frac{\mathbb{E}(\alpha)-\beta c}{2\beta} \Leftrightarrow \alpha_h - \alpha_l < \frac{\alpha_l-\beta c}{p}$.

Case 2.a: $\alpha_h - \alpha_l \leq \frac{\alpha_l-\beta c}{\sqrt{p}}$. It holds that $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c}{2\beta}$ and

$$W(\lambda) = \frac{p(\alpha_h-\beta c)^2 + (1-p)(\alpha_l-\beta c)^2}{2\beta} - \frac{1}{2}\left[\beta\pi^{*2} + \frac{\sigma^2(1-3\lambda/4)}{\beta}\right]$$

Thus, $\frac{dW(\lambda)}{d\lambda} = \frac{3\sigma^2}{8\beta} > 0$. Since $\frac{dG(\lambda)}{d\lambda} = \frac{\sigma^2}{4\beta}$, $\frac{dV(\lambda)}{d\lambda} = \frac{dW(\lambda)}{d\lambda} - \frac{dG(\lambda)}{d\lambda} = \frac{\sigma^2}{8\beta} > 0$.

Case 2.b: $\frac{\alpha_l-\beta c}{\sqrt{p}} < \alpha_h - \alpha_l < \frac{\alpha_l-\beta c}{p}$. If π^* jumps from one interval to another, both $W(\lambda)$ and $V(\lambda)$ decrease. Therefore, monotonicity doesn't hold. □

Proof of Lemma 3.5.3 Consider $Q \in [0, C/N]$, which implies $\gamma = 1$. It is easy to show that if the capacity is less or equal to C_a, the optimal solution is determined by the capacity constraint, $s_h^* = \frac{\alpha_h - C/N}{\beta}$ and $s_l^* = \frac{\alpha_l - C/N}{\beta}$.

In the high demand state, for $Q \in [C/N, \alpha_h - \beta c]$, $\gamma = C/(NQ) < 1$ and all the contracts are exercised since $Q \leq C/N \leq \alpha_h - \beta c$. There is no transaction on the spot market at all.

In the low demand state, for $Q \in [C/N, \alpha_l - \beta c]$, $\gamma = C/(QN) < 1$, no transaction takes place on the spot market by similar arguments. For $Q \in [\alpha_l - \beta c, \alpha_h - \beta c]$, fraction γ of all buyers have Q contracts and exercise only $\alpha_l - \beta c$. The remaining fraction $1-\gamma$ of the buyers have no contract and will purchase from the spot market according to the spot price. The seller's problem is reduced to

$$\begin{aligned} \max_{s_l} \quad & R_2(s_l|Q,\alpha_l) = (s_l - c)N(1-\gamma)(\alpha_l - \beta s_l)^+ \\ \text{s.t.} \quad & N\left[\gamma(\alpha_l - \beta c) + (1-\gamma)(\alpha_l - \beta s_l)^+\right] \leq C \end{aligned}$$

It can be shown the unconstrained optimizer $(\alpha_l + \beta c)/2$ is not feasible. Thus, the optimal spot price is determined by the capacity constraint $s_l^* = \frac{\gamma(\alpha_l-\beta c)+(1-\gamma)\alpha_l - C/N}{\beta(1-\gamma)}$.

□

Proof of Lemma 3.5.4 Let s_h and s_l denote the spot prices in the hight demand state and the low demand state respectively. In the proof of the Lemma 3.5.1, it has been shown that the objective function $\hat{r}_1(Q|\pi) = r_1(Q|\pi)/\gamma$ is concave in Q for any given $s_h \geq c$, $s_l \geq c$ and π. Therefore, the Karush-Kuhn-Tucker condition is necessary and sufficient for optimality. Suppose $(s_h^*, s_l^*, Q^*, \gamma^*)$ is an equilibrium in the subgame for a given π. It must satisfy equation (3.5.16) – (3.5.19).

For $Q \in [0, C/N]$, $\gamma = 1$, $\widetilde{s}_h = \frac{\alpha_h - C/N}{\beta}$ and $\widetilde{s}_l = \frac{\alpha_l - C/N}{\beta}$. Thus, $Q \leq \alpha_h - \beta s_h = C/N$ and $Q \leq \alpha_l - \beta s_l = C/N$. If $Q^* = 0$, Equation (3.5.16) - (3.5.19) are

$$\frac{d\hat{r}_1(0|\pi)}{dQ} = -\pi + p(s_h^* - c) + (1-p)(s_l^* - c) \leq 0 \tag{A.0.10}$$

$$s_h^* = \frac{\alpha_h - C/N}{\beta} \tag{A.0.11}$$

$$s_l^* = \frac{\alpha_l - C/N}{\beta} \tag{A.0.12}$$

Such condition is satisfied if and only $\pi \geq \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$. If $Q^* \in (0, C/N]$, then it must hold that

$$\frac{d\hat{r}_1(Q^*|\pi)}{dQ} = -\pi + p(s_h^* - c) + (1-p)(s_l^* - c) = 0 \tag{A.0.13}$$

$$s_h^* = \frac{\alpha_h - C/N}{\beta} \tag{A.0.14}$$

$$s_l^* = \frac{\alpha_l - C/N}{\beta} \tag{A.0.15}$$

which implies $\pi = \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$.

For any $Q^* \in (C/N, \alpha_l - \beta c]$, there is no transaction on the spot market. It must hold that

$$\frac{d\hat{r}_1(Q^*|\pi)}{dQ} = 0 \tag{A.0.16}$$

$$s_h^* \geq \frac{\alpha_h - Q^*}{\beta} \tag{A.0.17}$$

$$s_l^* \geq \frac{\alpha_l - Q^*}{\beta} \tag{A.0.18}$$

With above spot prices, $\frac{d\hat{r}_1(Q^*|\pi)}{dQ} = -\pi + (\mathbb{E}(\alpha) - \beta c - Q^*)/\beta$. It implies $p(\alpha_h - \alpha_l)/\beta \leq \pi < (\mathbb{E}(\alpha) - \beta c - C/N)/\beta$. Under this condition, $Q^* = \mathbb{E}(\alpha) - \beta c - \beta\pi$ and $\gamma^* < 1$.

Similarly, for $Q^* \in (\alpha_l - \beta c, \alpha_h - \beta c]$, $(Q^*, s_h^*, s_l^*, \gamma^*)$ must satisfy

$$\frac{d\hat{r}_1(Q^*|\pi)}{dQ} = 0 \tag{A.0.19}$$

$$s_h^* \geq \frac{\alpha_h - Q^*}{\beta} \tag{A.0.20}$$

$$s_l^* = \frac{\gamma^*(\alpha_l - \beta c) + (1-\gamma^*)\alpha_l - C/N}{\beta(1-\gamma^*)}. \tag{A.0.21}$$

In the low demand state, since $Q > \alpha_l - \beta c$, the buyer doesn't purchase on the spot market if he has Q contracts already. Therefore, $\frac{d\hat{r}_1(Q^*|\pi)}{dQ} = -\pi + \frac{p(\alpha_h - \beta c - Q^*)}{\beta}$. It implies $0 \leq \pi < \frac{p(\alpha_h - \alpha_l)}{\beta}$. Under this condition, $Q^* = \alpha_h - \beta c - \beta\pi/p$ with $\gamma^* < 1$.

The results are summarized as follows.

1. If $\pi \in \left[0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$, then $Q^* = \alpha_h - \beta c - \beta\pi/p$ and $\gamma^* < 1$.

2. If $\pi \in \left[\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c - C/N}{\beta}\right)$, then $Q^* = \mathbb{E}(\alpha) - \beta c - \beta\pi$ and $\gamma^* < 1$.

3. If $\pi = \frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$, then any $Q \in [0, C/N]$ is optimal and $\gamma^* = 1$.

4. If $\pi > \frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$, then $Q^* = 0$ and $\gamma^* = 1$.

□

Proof of Theorem 3.5.5 For $\pi \geq \frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$, $R_1(\pi) = \frac{C(\mathbb{E}(\alpha)-\beta c-C/N)}{\beta}$. For $\pi \in \left[\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}\right)$, $\mathbb{E}[G_2(Q(\pi), \alpha)] = 0$. Thus, $R_1(\pi) < \frac{C(\mathbb{E}(\alpha)-\beta c-C/N)}{\beta}$ and the optimal option price π^* can't fall in this interval. For $\pi \in \left[0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$, $R_1(\pi) = \pi C + \mathbb{E}[G_2(Q(\pi), \alpha)]$. By Lemma 3.5.4 and Lemma 3.5.3, it holds that

$$\begin{aligned}
\mathbb{E}[G_2(Q(\pi), \alpha)] &= (1-p)(s_l - c)(1-\gamma)(\alpha_l - \beta s_l)N \\
&= \frac{(1-p)(\alpha_l - \beta c - C/N)[C/N - \gamma(\alpha_l - \beta c)]N}{\beta(1-\gamma)} \\
&< \frac{C(\alpha_l - \beta c - C/N)}{\beta}
\end{aligned}$$

The last step follows from $\alpha_l - \beta c > C/N$ and $1 - p < 1$. Since $\pi < \frac{p(\alpha_h-\alpha_l)}{\beta}$, $R_1(\pi) = \pi C + \mathbb{E}[G_2(Q(\pi), \alpha)] < \frac{C[p(\alpha_h-\alpha_l)+\alpha_l-\beta c-C/N]}{\beta} = \frac{C(\mathbb{E}(\alpha)-\beta c-C/N)}{\beta}$.

Therefore, the optimal option price can be any value in $\left[\frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta},\infty\right)$ and $R_1(\pi^*) = \frac{C(\mathbb{E}(\alpha)-\beta c-C/N)}{\beta}$. $\square$

Proof of Theorem 3.5.6 For $\pi \le \frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$, $NQ(\pi) \ge C$. Therefore, $\gamma \le 1$ and all contracts are exhausted. In this case, $R_1(\pi) = \pi C$ and $R_1(\pi)$ strictly increases in π to $\pi = \frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$. For $\pi \ge \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$, $Q(\pi) = 0$ and $R_1(\pi) = 0$. For $\pi \in \left[\frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right)$, $Q(\pi) = \mathbb{E}(\alpha) - \beta c - \beta\pi$ and $R_1(\pi) = N\pi Q(\pi) = N\pi(\mathbb{E}(\alpha) - \beta c - \beta\pi)$. Note at $\frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$ and $\frac{\mathbb{E}(\alpha)-\beta c}{\beta}$, $R_1(\pi)$ is continuous. For $\pi \in \left[\frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right)$, it holds that

$$\frac{dR_1(\pi)}{d\pi} = N(\mathbb{E}(\alpha) - \beta c - 2\beta\pi) \tag{A.0.22}$$

At $\pi = \frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$, the right derivative $R'_{1+}(\pi) = 2C/N-(\mathbb{E}(\alpha)-\beta c) < 0$. Therefore, $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$. $\square$

Proof of Lemma 3.5.5 For a given option price π, let $(Q_A^*, Q_B^*, s_h^*, s_l^*)$ be an equilibrium in the subgame, which must satisfy Equation (3.5.27) – (3.5.30). This proof proceeds as follows. First, based on function $Q_A(\pi)$, we divide our proof into two major cases. Second, according to the value of λ, each case is divided further into several subcases. In the end, we will show that for a given π, the best response for each type B buyer $Q_B^*(\pi)$ is constant for all $\lambda \in (0, 1)$.

Noting type A buyers do not participate in the spot market, the optimal contracting quantity as a function of π for type A buyers is the same as contract market only case, which is characterized in Lemma 3.4.2. Function $Q_A(\pi)$ is a piecewise function of π on $[0, +\infty)$. For $\pi \in \left[0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$, $Q_A(\pi) = \alpha_h - \beta c - \beta\pi/p > \alpha_l - \beta c$. For $\pi \in \left[\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right)$, $Q_A(\pi) = \mathbb{E}(\alpha) - \beta c - \beta\pi \le \alpha_l - \beta c$. For $\pi > \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$, $Q_A(\pi) = 0$. Based on this, we can divide the proof into two cases.

Case 1: $\pi \in \left[0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$.

In this case, $Q_A(\pi) = \alpha_h - \beta c - \beta\pi/p > \alpha_l - \beta c$. For $\gamma = 1$, the the optimal spot prices in Period 2 can be the unconstrained optimizer if Q_B is small or determined by the capacity constraint if Q_B is large. Define Q_B^0, Q_B^1 and Q_B^2 as follows. Let Q_B^2 be the critical value that γ becomes less than 1 for any $Q_B > Q_B^2$, i.e.,

$$(1-\lambda)Q_A + \lambda Q_B^2 \quad = \quad C/N$$

If $Q_B^2 > 0$, the following situation could happen. For small value of Q_B, the unconstrained optimizer $s_h = \frac{\alpha_h+\beta c-Q_B}{2\beta}$ is feasible in the high demand state. As Q_B increases, the capacity constraint becomes tight and the optimal price is solely determined by the capacity constraint. As long as $\gamma = 1$, $s_h = [(1-\lambda)Q_A + \lambda\alpha_h - C/N]/(\beta\lambda)$, which doesn't depends on Q_B. Similarly property holds in the low demand state. Let $Q_B^0 \leq Q_B^2$ be the breakpoint such that for any $Q_B \leq Q_B^0$, $\gamma = 1$ and the optimal spot price in the high state is the unconstrained optimizer, $s_h = \frac{\alpha_h+\beta c-Q_B}{2\beta}$. Similarly, let Q_B^1 be the breakpoint for the low state. Noting type A buyers only exercise $\alpha_l - \beta c < Q_A$ contracts, we have

$$\begin{aligned}
(1-\lambda)Q_A + \lambda\left(\frac{\alpha_h - \beta c + Q_B^0}{2}\right) &= \frac{C}{N} \\
\Leftrightarrow \qquad Q_B^0 &= \frac{2}{\lambda}\left[\frac{C}{N} - (1-\lambda)Q_A - \frac{\lambda(\alpha_h - \beta c)}{2}\right]
\end{aligned}$$

$$\begin{aligned}
(1-\lambda)(\alpha_l - \beta c) + \lambda\left(\frac{\alpha_l - \beta c + Q_B^1}{2}\right) &= \frac{C}{N} \\
\Leftrightarrow \qquad Q_B^1 &= \frac{2}{\lambda}\left[\frac{C}{N} - (1-\lambda)(\alpha_l - \beta c) - \frac{\lambda(\alpha_l - \beta c)}{2}\right]
\end{aligned}$$

Since $C/N \leq (\alpha_l - \beta c)/2$ and $Q_A > \alpha_l - \beta c$, $Q_B^0 < 0$ and $Q_B^1 < 0$. Note $Q_B^2 \geq 0$ if and only if $(1-\lambda)Q_A \leq C/N$, which implies $\pi \geq \frac{p}{\beta}\left(\alpha_h - \beta c - \frac{C}{N(1-\lambda)}\right) = \pi_1$. Since $C/N \leq (\alpha_l - \beta c)/2$ and $Q_A > \alpha_l - \beta c$, $Q_B^2 \leq \alpha_l - \beta c$. Price $\pi_1 \leq p(\alpha_h - \alpha_l)/\beta$ if only if $\lambda \geq 1 - \frac{C}{N(\alpha_l - \beta c)}$. Thus, according to λ, we can further divide Case 1 into two subcases.

Case 1.a: $\lambda \geq 1 - \frac{C}{N(\alpha_l-\beta c)} \Leftrightarrow \pi_1 \in \left[0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$.

Case 1.a.1: $\pi \in \left(\pi_1, \frac{p(\alpha_h-\alpha_l)}{\beta}\right]$. Under this condition, $Q_B^2 > 0$. For any $Q_B \in (0, Q_B^2]$, $s_h = \frac{(1-\lambda)Q_A+\lambda\alpha_h-C/N}{\beta\lambda}$ and $s_l = \frac{(1-\lambda)(\alpha_l-\beta c)+\lambda\alpha_l-C/N}{\beta\lambda}$. If $Q^* \in (0, Q_B^2]$ ($\gamma = 1$), it must satisfy $\frac{dr_1(Q_B^*|\pi)}{Q_B} = 0$. If $Q_B^* = 0$, then $\frac{dr_1(0|\pi)}{Q_B} < 0$ must hold. Note

$$\begin{aligned}\frac{dr_1(Q_B|\pi)}{dQ_B} &= -\pi + p(s_h - c) + (1-p)(s_l - c) \\ &> -\pi + \frac{p(\alpha_h - \alpha_l)}{\beta} \\ &> 0\end{aligned}$$

Therefore, $Q_B^* \notin [0, Q_B^2]$.

For $Q_B \in (Q_B^2, \alpha_l - \beta c]$, $\gamma < 1$. In the high demand state, all the capacity is exhausted under contracts. Therefore, $Q_B \geq \alpha_h - \beta s_h$ for all Q_B in this interval. However, the low demand state is different depending on the value of s_l. Noting $s_l \geq c$ and $Q_B \geq \alpha_l - \beta c$, we obtain for $s_l \in [c, (\alpha_l - Q_B)/\beta]$,

$$\begin{aligned}\frac{dr_1(Q_B|\pi)}{dQ_B} &= \gamma\left[-\pi + \frac{p}{\beta}(\alpha_h - \beta c - Q_B) + (1-p)(s_l - c)\right] \\ &\geq \gamma\left[-\pi + \frac{p(\alpha_h - \alpha_l)}{\beta}\right] \\ &> 0\end{aligned}$$

For $s_l > (\alpha_l - Q_B)/\beta$,

$$\frac{dr_1(Q_B|\pi)}{dQ_B} = \gamma\left[-\pi + \frac{p}{\beta}(\alpha_h - \beta c - Q_B) + \frac{1-p}{\beta}(\alpha_l - \beta c - Q_B)\right] > 0$$

Thus, $Q_B^* \notin (Q_B^2, \alpha_l - \beta c]$.

For $Q_B \in (\alpha_l - \beta c, \alpha_h - \beta c]$, $\gamma < 1$.

$$\frac{dr_1(Q_B|\pi)}{dQ_B} = \gamma\left[-\pi + \frac{p}{\beta}(\alpha_h - \beta c - Q_B)\right]$$

Therefore, the optimal contracting quantity can only fall in this interval, $Q_B^* = \alpha_h - \beta c - \beta\pi/p = Q_A(\pi)$.

Case 1.a.2: $\pi \in [0, \pi_1]$. Under this condition, $Q_B^2 \leq 0$. Follow the same arguments, we have $Q_B^* = \alpha_h - \beta c - \beta\pi/p = Q_A(\pi)$. Note that at Q_B^* is continuous on $\left[0, \frac{p(\alpha_h - \alpha_l}{\beta}\right)$.

Case 1.b: $\lambda < 1 - \frac{C}{N(\alpha_l - \beta c)} \Leftrightarrow \pi_1 \leq 0$. For any $\pi \in \left[0, \frac{p(\alpha_h - \alpha_l}{\beta}\right)$, $Q_A(\pi)(1-\lambda) > C/N$ and $Q_B^2 < 0$. By previous results, we have $Q_B^* = \alpha_h - \beta c - \beta\pi/p = Q_A(\pi)$.

Case 2: $\pi \in \left(\frac{p(\alpha_h-\alpha_l)}{\beta}\frac{\mathbb{E}(\alpha)-\beta c)}{\beta}\right]$

In this case, $Q_A(\pi) = \mathbb{E}(\alpha) - \beta c - \beta\pi \leq \alpha_l - \beta c$. Define Q_B^0, Q_B^1 and Q_B^2 the same as those in Case 1. Since $Q_A(\pi) \leq \alpha_l - \beta c$, all the $Q_A(\pi)$ contracts are exercised in both the high demand state and the low demand state. Thus, Q_B^1 is different from that in Case 1:

$$Q_B^1 \quad = \quad \frac{2}{\lambda}\left[\frac{C}{N} - (1-\lambda)Q_A(\pi) - \frac{\lambda(\alpha_l - \beta c)}{2}\right]$$

Substituting $Q_A(\pi) = \mathbb{E}(\alpha) - \beta c - \beta\pi$, we obtain

$$\begin{aligned}
Q_B^0 &= \frac{2}{\lambda}\left[\frac{C}{N} - (1-\lambda)(\mathbb{E}(\alpha) - \beta c - \beta\pi) - \frac{\lambda(\alpha_h - \beta c)}{2}\right] \\
Q_B^1 &= \frac{2}{\lambda}\left[\frac{C}{N} - (1-\lambda)(\mathbb{E}(\alpha) - \beta c - \beta\pi) - \frac{\lambda(\alpha_l - \beta c)}{2}\right] \\
Q_B^2 &= \frac{1}{\lambda}\left[\frac{C}{N} - (1-\lambda)(\mathbb{E}(\alpha) - \beta c - \beta)\pi\right]
\end{aligned}$$

Note that $Q_B^0 < Q_B^1$. According to the positions of Q_B^0, Q_B^1 and Q_B^2 on $[0, \alpha_h - \beta c)$, we can find the following breakpoints of π:

$$\begin{aligned}
\pi_a &= \frac{1}{(1-\lambda)\beta}\left[(1-\lambda)(\mathbb{E}(\alpha) - \beta c) - \frac{C}{N}\right] \\
\pi_b &= \frac{1}{(1-\lambda)\beta}\left[\frac{\lambda}{2}(\alpha_l - \beta c) + (1-\lambda)(\mathbb{E}(\alpha) - \beta c) - \frac{C}{N}\right] \\
\pi_c &= \frac{1}{(1-\lambda)\beta}\left[\frac{\lambda}{2}(\alpha_h - \beta c) + (1-\lambda)(\mathbb{E}(\alpha) - \beta c) - \frac{C}{N}\right] \\
\pi_d &= \frac{1}{(1-\lambda)\beta}\left[\lambda(\alpha_l - \beta c) + (1-\lambda)(\mathbb{E}(\alpha) - \beta c) - \frac{C}{N}\right] \\
\pi_e &= \frac{1}{(1-\lambda)\beta}\left[\frac{\lambda}{2}(\alpha_l - 2\beta c + \alpha_h) + (1-\lambda)(\mathbb{E}(\alpha) - \beta c) - \frac{C}{N}\right] \\
\pi_f &= \frac{1}{(1-\lambda)\beta}\left[\lambda(\alpha_h - \beta c) + (1-\lambda)(\mathbb{E}(\alpha) - \beta c) - \frac{C}{N}\right]
\end{aligned}$$

Depending the value of α_h and α_l, there are two cases, $\pi_a < \pi_b < \pi_c \leq \pi_d < \pi_e < \pi_f$ and $\pi_a < \pi_b < \pi_d \leq \pi_c < \pi_e < \pi_f$. Compare those breakpoints to $\frac{p(\alpha_h-\alpha_l)}{\beta}$ and $\frac{\mathbb{E}(\alpha)-\beta c}{\beta}$, we can find the breakpoints for λ. Define $\lambda_a = 1 - \frac{C}{N(\alpha_l-\beta c)}$, $\lambda_b = \frac{2C}{N(\alpha_l-\beta c)}$, $\lambda_c = \frac{2C}{N(\alpha_h-\beta c)}$, $\lambda_d = \frac{C}{N(\alpha_l-\beta c)}$, $\lambda_e = \frac{2C}{N(\alpha_l+\alpha_h-2\beta c)}$, and $\lambda_f = \frac{C}{N(\alpha_h-\beta c)}$. The relationship between the breakpoints in Q_B and π is as follows.

$$\begin{aligned}
Q_B^2 &\geq 0 \Leftrightarrow \pi \geq \pi_a \\
Q_B^1 &\geq 0 \Leftrightarrow \pi \geq \pi_b \\
Q_B^0 &\geq 0 \Leftrightarrow \pi \geq \pi_c \\
Q_B^2 &\leq \alpha_l - \beta c \Leftrightarrow \pi \leq \pi_d \\
Q_B^2 &\leq \alpha_h - \beta c \Leftrightarrow \pi \leq \pi_f \\
Q_B^1 &\leq \alpha_l - \beta c \Leftrightarrow \pi \leq \pi_d \\
Q_B^1 &\leq \alpha_h - \beta c \Leftrightarrow \pi \leq \pi_e \\
Q_B^0 &\leq \alpha_l - \beta c \Leftrightarrow \pi \leq \pi_e \\
Q_B^0 &\leq \alpha_h - \beta c \Leftrightarrow \pi \leq \pi_f
\end{aligned}$$

Since $C \leq C_a$, $\pi_i \geq \frac{p(\alpha_h-\alpha_l)}{\beta}$ for all $i \in \{a,b,c,d,e,f\}$. Also note $\pi_a \leq \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$. Thus, we obtain the following relationship between π and λ.

$$\begin{aligned}
\pi_a &\geq \frac{p(\alpha_h-\alpha_l)}{\beta} \Leftrightarrow \lambda \leq \lambda_a \\
\pi_b &\leq \frac{\mathbb{E}(\alpha)-\beta c}{\beta} \Leftrightarrow \lambda \leq \lambda_b \\
\pi_c &\leq \frac{\mathbb{E}(\alpha)-\beta c}{\beta} \Leftrightarrow \lambda \leq \lambda_c \\
\pi_d &\leq \frac{\mathbb{E}(\alpha)-\beta c}{\beta} \Leftrightarrow \lambda \leq \lambda_d \\
\pi_e &\leq \frac{\mathbb{E}(\alpha)-\beta c}{\beta} \Leftrightarrow \lambda \leq \lambda_e \\
\pi_f &\leq \frac{\mathbb{E}(\alpha)-\beta c}{\beta} \Leftrightarrow \lambda \leq \lambda_f
\end{aligned}$$

According to above breakpoints, we divide Case 2 into 5 subcases:

Case 2.a: $\lambda_f < \lambda_e < \lambda_d \leq \lambda_c < \lambda_b \leq \lambda_a$.

Case 2.b: $\lambda_f < \lambda_e < \lambda_d \leq \lambda_c \leq \lambda_a \leq \lambda_b$.

Case 2.c: $\lambda_f < \lambda_e < \lambda_d \leq \lambda_a \leq \lambda_c < \lambda_b$.

Case 2.d: $\lambda_f < \lambda_e < \lambda_c \leq \lambda_d < \lambda_b \leq \lambda_a$.

Case 2.e: $\lambda_f < \lambda_e < \lambda_c \le \lambda_d \le \lambda_a \le \lambda_b$.

We only present the proof for Case 2.a. here. The proof for the other cases is similar. Consider Case 2.a,

For $\lambda \in [0, \lambda_f]$, $\frac{p(\alpha_h-\alpha_l)}{\beta} < \pi_a < \pi_b < \pi_c \le \pi_d < \pi_e < \pi_f \le \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$ or $\frac{p(\alpha_h-\alpha_l)}{\beta} < \pi_a < \pi_b < \pi_d \le \pi_c < \pi_e < \pi_f \le \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$ holds. Consider the former case $\frac{p(\alpha_h-\alpha_l)}{\beta} < \pi_a < \pi_b < \pi_c \le \pi_d < \pi_e < \pi_f \le \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$. For $\pi \in \left[\pi_f, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right]$, it holds that $Q_B^k \ge \alpha_h - \beta c$, $k = 0, 1, 2$. Note we only need to consider $Q_B \in [0, \alpha_h - \beta c]$. For any given (s_h, s_l, γ), $\frac{dr_1(Q_B|\pi)}{dQB}$ is a concave function of Q_B. Let $\frac{dr_1(Q_B|\pi)}{dQB} = h(Q_B, s_h, s_l, \gamma)$. Substituting the best response $\widetilde{s}_h(Q_B)$, $\widetilde{s}_(Q_B)$ and function $\widetilde{\gamma}(Q_B)$ into to $h(Q_B, s_h, s_l, \gamma)$ and define $\hat{h}(Q_B) = h(Q_B, \widetilde{s}_h(Q_B), \widetilde{s}_l(Q_B), \widetilde{\gamma}(Q_B))$. It is easy to see that $\hat{h}(Q_B) = h(Q_B, \widetilde{s}_h(Q_B), \widetilde{s}_l(Q_B), \widetilde{\gamma}(Q_B))$ is a decreasing function in Q_B on $[0, \alpha_h - \beta c]$. For $[0, \alpha_l - \beta c]$, $\hat{h}(Q_B) = -\pi + \frac{\mathbb{E}(\alpha)-\beta c-Q_B}{2\beta}$. For $Q_B \in [\alpha_l - \beta c, \alpha_h - \beta c]$, $\hat{h}(Q_B) = -\pi + \frac{p(\alpha_h-\beta c-Q_B)}{2\beta}$. Since $\pi \ge \pi_f$, $\hat{h}(Q_B) < 0$ for any $Q_B \in [0, \alpha_h - \beta c]$. Therefore, $Q_B^* = 0$ for $\pi \in \left[\pi_f, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right]$. For $\pi \in [\pi_e, \pi_f)$, it holds that $0 \le \alpha_l - \beta c \le Q_B^0 \le Q_B^2 \le \alpha_h - \beta c \le Q_B^1$. Since only $Q_B \in [0, \alpha_h - \beta c]$ is interesting, we do not need to consider Q_B^1. It is easy to see that $\hat{h}(Q_B) = h(Q_B, \widetilde{s}_h(Q_B), \widetilde{s}_l(Q_B), \widetilde{\gamma}(Q_B))$ is a decreasing function in Q_B on $[0, Q_B^2]$. For $[0, \alpha_l - \beta c]$, $\hat{h}(Q_B) = -\pi + \frac{\mathbb{E}(\alpha)-\beta c-Q_B}{2\beta}$. For $Q_B \in [\alpha_l - \beta c, Q_B^2]$, $\hat{h}(Q_B) = -\pi + \frac{p(\alpha_h-\beta c-Q_B)}{2\beta}$. For $Q_B \in [Q_B^2, \alpha_h - \beta c]$, $\hat{h}(Q_B) = \frac{C}{N[(1-\lambda)Q_A+\lambda Q_B]}\left[-\pi + \frac{p(\alpha_h-\beta c-Q_B)}{\beta}\right]$. Note $\hat{h}(Q_B)$ is continuous at Q_B^2 and $\alpha_l - \beta c$. Since $\pi \ge \pi_e$, $\hat{h}(Q_B) < 0$ for any $Q_B \in [0, \alpha_h - \beta c]$. Therefore, $Q_B^* = 0$ for $\pi \in [\pi_e, \pi_f)$. Follow the same arguments for π in the other intervals, we can obtain the results. The proof for the latter case $\frac{p(\alpha_h-\alpha_l)}{\beta} < \pi_a < \pi_b < \pi_d \le \pi_c < \pi_e < \pi_f \le \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$ is an analogy.

The proof for λ in the other intervals proceeds in the same way: From the value of λ, we can obtain the breakpoints of π. Considering π in the different intervals, we can compare the breakpoints Q_B^0, Q_B^1 and Q_B^2 to 0, $\alpha_l - \beta c$ and $\alpha_h - \beta c$. By investigating $\hat{h}(Q_B)$ on $[0, \alpha_h - \beta c]$, the equilibrium Q_B^* can be obtained.

The proofs for Case 2.b – 2.e are similar and are not included here. The results are summarized as follows.

1. If $\pi \in \left[0, \frac{p(\alpha_h - \alpha_l)}{\beta}\right)$, then $Q_B^* = \alpha_h - \beta c - \beta\pi/p$. Note $\gamma < 1$ in this case.

2. If $\pi \in \left[\frac{p(\alpha_h - \alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}\right)$, then $Q_B^* = \mathbb{E}(\alpha) - \beta c - \beta\pi$. Note $\gamma < 1$ in this case.

3. If $\pi = \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$, then any $Q_B \in \left[0, \frac{C}{N}\right]$ is optimal. In this case, $\gamma = 1$.

4. If $\pi > \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$, then $Q_B^* = 0$. In this case, $\gamma = 1$.

□

Proof of Theorem 3.5.7 The proof proceeds as follows. First, we will show that for $\pi \in \left[\frac{p(\alpha_h - \alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha) - \beta c}{\beta}\right]$, $\pi^* = \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$ is the local maximizer. Second, we will show that for all $\pi \in \left[0, \frac{p(\alpha_h - \alpha_l}{\beta}\right)$, $R_1(\pi) < R_1(\pi^*)$.

It has been shown in Lemma 3.5.5, at $\pi = \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$, $Q_B \in [0, C/N]$. $R_1(\pi) = \frac{C(\mathbb{E}(\alpha) - \beta c - C/N)}{\beta}$. For $\pi \geq \frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$, $\gamma = 1$. As π slightly increases from $\frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}$, it is shown in the proof of Lemma 3.5.5, the optimal spot prices first active at the capacity constraints in both demand states. For large λ, the capacity constraints are always active at stage 3. For smaller λ, the optimal spot prices in the low demand state and the high demand state may become inactive as π increases. After that, the return in Period 2 stays constant. Note that $R_1(\pi)$ is continuous in π.

Using the same notation in the proof of Lemma 3.5.5, let π_b and π_c be the smallest option prices such that the optimal spot price becomes the unconstrained optimizers at the low demand and the high demand state respectively.

Henceforth, we divide $R_1(\pi)$ by N to simplify the notation. For large λ, $\pi_b \geq \frac{\mathbb{E}(\alpha) - \beta c}{\beta}$ and $\pi_c \geq \frac{\mathbb{E}(\alpha) - \beta c}{\beta}$. Then $R_1(\pi)$ is a strictly concave function on $\left(\frac{\mathbb{E}(\alpha) - \beta c - C/N}{\beta}, \frac{\mathbb{E}(\alpha) - \beta c}{\beta}\right]$. It holds that

$$R_1(\pi) \quad = \quad (1-\lambda)\pi(\mathbb{E}(\alpha) - \beta c - \beta\pi) + \frac{1}{\lambda\beta}\left[\frac{C}{N} - (1-\lambda)(\mathbb{E}(\alpha) - \beta c - \beta\pi)\right]$$

$$\times\left[(1-\lambda)(\mathbb{E}(\alpha)-\beta c-\beta\pi)-\frac{C}{N}+\lambda(\mathbb{E}(\alpha)-\beta c)\right]$$

It follows that for all $\pi \geq \frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$ and $\lambda \in (0,1)$,

$$\frac{dR_1(\pi)}{d\pi} = \frac{2(1-\lambda)(\mathbb{E}(\alpha)-\beta c-\beta\pi-C/N)}{\lambda} \leq 0$$

For moderate λ such that $\pi_b \leq \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$ and $\pi_c > \frac{\mathbb{E}(\alpha)-\beta c}{\beta}$. It holds that for $\pi \in \left[\frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}, \pi_b\right]$, $R_1(\pi)$ and $\frac{dR_1(\pi)}{d\pi}$ are the same as above. For $\pi \in \left(\pi_b, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right]$, it follows

$$\begin{aligned} R_1(\pi) &= (1-\lambda)\pi(\mathbb{E}(\alpha)-\beta c-\beta\pi)+\frac{p}{\lambda\beta}\left[\frac{C}{N}-(1-\lambda)(\mathbb{E}(\alpha)-\beta c-\beta\pi)\right] \\ &\quad \times\left[(1-\lambda)(\mathbb{E}(\alpha)-\beta c-\beta\pi)-\frac{C}{N}+\lambda(\alpha_h-\beta c)\right]+\frac{(1-p)\lambda(\alpha_l-\beta c)^2}{4\beta} \end{aligned}$$

$$\frac{dR_1(\pi)}{d\pi} = \frac{(1-\lambda)[2(\mathbb{E}(\alpha)-\beta c-\beta\pi-C/N)-(1-p)(\alpha_l-\beta c)]}{\lambda} < 0$$

For small λ, $\pi_b \leq (\mathbb{E}(\alpha)-\beta c)/\beta$, $\pi_c \leq (\mathbb{E}(\alpha)-\beta c)/\beta$ and $\pi_b < \pi_c$. It holds that $R_1(\pi)$ strictly decreases in the two intervals $\left(\frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}, \pi_b\right]$ and $(\pi_b, \pi_c]$, and becomes constant on $\left(\pi_c, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right]$.

Therefore, π^* is the only maximizer in $\left[\frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c}{\beta}\right]$.

For $\pi \in \left(\frac{p(\alpha_h-\alpha_l)}{\beta}, \frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}\right]$, $Q_A(\pi)=Q_B(\pi)=\mathbb{E}(\alpha)-\beta c-\beta\pi$ and $\gamma<1$. Therefore, $R_1(\pi)=C\pi$ and $R_1(\pi)$ strictly increases in π to $\frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$.

For $\pi \in \left[0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$, $Q_A(\pi)=Q_B(\pi)=\alpha-\beta c-\beta\pi/p$. This is equivalent to the case $\lambda=1$ and $\pi \in \left[0, \frac{p(\alpha_h-\alpha_l)}{\beta}\right)$. It has been shown in the proof of Lemma 3.5.5 that $R_1(\pi) < \frac{C(\mathbb{E}(\alpha)-\beta c-C/N)}{\beta}$.

Thus, $\pi^* = \frac{\mathbb{E}(\alpha)-\beta c-C/N}{\beta}$ is the optimal option price for any $\lambda \in (0,1)$. □

Proof of Lemma 3.6.1 Since the objective function is concave in q_c, the KKT condition is necessary and sufficient for the optimality of this problem. Take the derivative of the objective function, $\frac{dr_2(q_c|\alpha,\phi)}{dq_c} = -\frac{q_c}{\beta}+\frac{\alpha+\phi}{\beta}-c$. At the endpoints $q_c=0$ and $q_c=Q$, it holds that $\frac{dr_2(0|\alpha,\phi)}{dq_c}=\frac{\alpha+\phi}{\beta}-c$ and $\frac{dr_2(Q|\alpha,\phi)}{dq_c}=-\frac{Q}{\beta}+\frac{\alpha+\phi}{\beta}-c$.

Note that constraints $q_c \leq Q$ and $q_c \geq 0$ can not be active at the same time. Depending on whether q_c^* is active at the constraints or not, there are three cases.

Case 1: If q_c^* is not tight at any constraint, i.e., $0 < q_c^* < Q$, then q_c^* must satisfy $\frac{dr_2(q_c^*|\alpha,\phi)}{dq_c} = 0$. Such condition holds if and only if $q_c^* = \alpha + \phi - \beta c$ and $0 < \alpha + \phi - \beta c < Q$.

Case 2: If $q_c^* = 0$, i.e., q_c^* is tight at the constraint $q_c \geq 0$, then $\frac{dr_2(0|\alpha,\phi)}{dq_c} \leq 0$. Thus, $\alpha + \phi - \beta c \leq 0$. Note that $\overline{\phi} \geq \alpha_h$, $\alpha + \phi - \beta c \leq 0$ is possible.

Case 3: If $q_c^* = Q$, i.e., q_c^* is tight at the constraint $q_c \leq Q$, then $\frac{dr_2(Q|\alpha,\phi)}{dq_c} \geq 0$, which implies $0 \leq Q \leq \alpha + \phi - \beta c$.

The results are summarized as follows.

1. If $\alpha + \phi - \beta c \leq 0$, then $q_c^* = 0$.

2. If $0 < \alpha + \phi - \beta c < Q$, then $q_c^* = \alpha + \phi - \beta c$.

3. If $Q \leq \alpha + \phi - \beta c$, then $q_c^* = Q$.

$\square$

Proof of Lemma 3.6.2 By Lemma 3.6.1, it is easy to see that each buyer never purchases more than $\alpha_h + \overline{\phi} - \beta c$. Thus, only $Q \in [0, \alpha_h + \overline{\phi} - \beta c]$ is interesting.

First, we will prove that $r_1(Q|\pi)$ is a concave function of Q on $[0, \alpha_h + \overline{\phi} - \beta c]$. The first term $-\pi Q$ is linear, thus it is concave. For the second term, we first calculate the two integrals. The first integral is

$$\int_{-\overline{\phi}}^{\overline{\phi}} \left[-\frac{q_c(Q,\alpha_h,\phi)^2}{2\beta} + \frac{(\alpha_h + \phi) q_c(Q,\alpha_h,\phi)}{\beta} - q_c(Q,\alpha_h,\phi)c \right] d\phi \quad \text{(A.0.23)}$$

Note that $q_c(Q,\alpha_h,\phi)$ depends not only on Q, but also on the realization of ϕ as follows.

If $\alpha_h + \phi - \beta c \leq 0$, i.e., $\phi \leq -\alpha_h + \beta c$, then $q_c(Q,\alpha_h,\phi) = 0$. Since $\alpha_h \leq \overline{\phi}$ and $\alpha_h - \beta c \geq 0$, $-\overline{\phi} \leq -\alpha_h + \beta c \leq 0 \leq \overline{\phi}$.

If $0 \leq \alpha_h+\phi-\beta c \leq Q$, i.e., $-\alpha_h+\beta c \leq \phi \leq Q-\alpha_h+\beta c$, then $q_c(Q,\alpha_h,\phi) = \alpha_h+\phi-\beta c$. Since $Q \leq \alpha_h + \overline{\phi} - \beta c$ and $Q \geq 0$, $-\overline{\phi} \leq Q - \alpha_h + \beta c \leq \overline{\phi}$.

If $\alpha_h + \phi - \beta c \geq Q$, i.e., $\phi \geq Q - \alpha_h + \beta c$, then $q_c(Q,\alpha_h,\phi) = Q$.

Hence,

$$\begin{aligned}
(A.0.23) &= \int_{-\overline{\phi}}^{-\alpha_h+\beta c} 0 d\phi + \int_{-\alpha_h+\beta c}^{Q-\alpha_h+\beta c} \left[\frac{(\alpha_h+\phi-\beta c)^2}{2\beta}\right] d\phi \\
&\quad + \int_{Q-\alpha_h+\beta c}^{\overline{\phi}} \left[-\frac{Q^2}{2\beta} + \frac{(\alpha_h+\phi)Q}{\beta} - Qc\right] d\phi \\
&= \frac{Q^3}{6\beta} + \frac{Q}{2\beta}(\alpha_h+\overline{\phi}-\beta c)(\alpha_h+\overline{\phi}-\beta c - Q)
\end{aligned}$$

The second integral is

$$\int_{-\overline{\phi}}^{\overline{\phi}} \left[-\frac{q_c(Q,\alpha_l,\phi)^2}{2\beta} + \frac{(\alpha_l+\phi)q_c(Q,\alpha_l,\phi)}{\beta} - q_c(Q,\alpha_l,\phi)c\right] d\phi \quad \text{(A.0.24)}$$

To calculate the integral, we divide the interval $[0, \alpha_h + \overline{\phi} - \beta c]$ into two pieces, $[0, \alpha_l + \overline{\phi} - \beta c)$ and $[\alpha_l + \overline{\phi} - \beta c, \alpha_h + \overline{\phi} - \beta c]$. For $Q \in [0, \alpha_l + \overline{\phi} - \beta c)$, by the same arguments as above, we obtain

$$(A.0.24) = \frac{Q^3}{6\beta} + \frac{Q}{2\beta}(\alpha_l+\overline{\phi}-\beta c)(\alpha_l+\overline{\phi}-\beta c - Q)$$

For $Q \in [\alpha_l + \overline{\phi} - \beta c, \alpha_h + \overline{\phi} - \beta c]$, note $Q - \alpha_l + \beta c \geq \overline{\phi}$. It holds that

$$\begin{aligned}
(A.0.24) &= \int_{-\overline{\phi}}^{-\alpha_l+\beta c} 0 d\phi + \int_{-\alpha_l+\beta c}^{\overline{\phi}} \left[\frac{(\alpha_l+\phi-\beta c)^2}{2\beta}\right] d\phi \\
&= \frac{(\alpha_l+\overline{\phi}-\beta c)^3}{6\beta}
\end{aligned}$$

Therefore, for $Q \in [0, \alpha_l + \overline{\phi} - \beta c)$, it holds that

$$\begin{aligned}
r_1(Q|\pi) &= -\pi Q + \frac{p}{2\overline{\phi}}\left[\frac{Q^3}{6\beta} + \frac{Q}{2\beta}(\alpha_h+\overline{\phi}-\beta c)(\alpha_h+\overline{\phi}-\beta c - Q)\right] \\
&\quad + \frac{1-p}{2\overline{\phi}}\left[\frac{Q^3}{6\beta} + \frac{Q}{2\beta}(\alpha_l+\overline{\phi}-\beta c)(\alpha_l+\overline{\phi}-\beta c - Q)\right] \\
\frac{dr_1(Q|\pi)}{dQ} &= -\pi + \frac{p(\alpha_h+\overline{\phi}-\beta c-Q)^2}{4\beta\overline{\phi}} + \frac{(1-p)(\alpha_l+\overline{\phi}-\beta c-Q)^2}{4\beta\overline{\phi}}
\end{aligned}$$

In this interval, $\frac{dr_1(Q|\pi)}{dQ}$ decreases in Q. Thus, $r_1(Q|\pi)$ is a concave function of Q.

For $Q \in [\alpha_l + \overline{\phi} - \beta c, \alpha_h + \overline{\phi} - \beta c]$, it holds that

$$\begin{aligned} r_1(Q|\pi) &= -\pi Q + \frac{p}{2\overline{\phi}} \left[\frac{Q^3}{6\beta} + \frac{Q}{2\beta}(\alpha_h + \overline{\phi} - \beta c)(\alpha_h + \overline{\phi} - \beta c - Q) \right] \\ &\quad + \frac{1-p}{2\overline{\phi}} \left[\frac{(\alpha_l + \overline{\phi} - \beta c)^3}{6\beta} \right] \\ \frac{dr_1(Q|\pi)}{dQ} &= -\pi + \frac{p(\alpha_h + \overline{\phi} - \beta c - Q)^2}{4\beta\overline{\phi}} \end{aligned}$$

Also, in this interval, $\frac{dr_1(Q|\pi)}{dQ}$ decreases in Q. Hence, $r_1(Q|\pi)$ is a concave function of Q.

Note that $r_1(Q|\pi)$ is continuous at the breakpoint $Q_0 = \alpha_l + \overline{\phi} - \beta c$ and left derivative is equal to the right derivative,

$$r'_{1-}(Q_0|\pi) = r'_{1+}(Q_0|\pi) = -\pi + \frac{p(\alpha_h - \alpha_l)^2}{4\beta\overline{\phi}}$$

Therefore, $r_1(Q|\pi)$ is a concave function of Q on $[0, \alpha_h + \overline{\phi} - \beta c]$ and the Karush-Kuhn-Tucker condition is necessary and sufficient for the optimality of this problem. The derivatives at the three breakpoints are $\frac{dr_1(\alpha_h + \overline{\phi} - \beta c|\pi)}{dQ} = -\pi$, $\frac{dr_1(\alpha_l + \overline{\phi} - \beta c|\pi)}{dQ} = -\pi + \frac{p(\alpha_h - \alpha_l)^2}{4\beta\overline{\phi}}$ and $\frac{dr_1(0|\pi)}{dQ} = -\pi + \frac{\mathbb{E}(\alpha + \overline{\phi} - \beta c)^2}{4\beta\phi}$, where $\mathbb{E}(\alpha + \overline{\phi} - \beta c)^2 = p(\alpha_h + \overline{\phi} - \beta c)^2 + (1-p)(\alpha_l + \overline{\phi} - \beta c)^2$.

For $\pi \in \left[0, \frac{p(\alpha_h - \alpha_l)^2}{4\beta\overline{\phi}}\right)$, $\frac{dr_1(\alpha_l + \overline{\phi} - \beta c|\pi)}{dQ} \geq 0$ and $\frac{dr_1(\alpha_h + \overline{\phi} - \beta c|\pi)}{dQ} \leq 0$. Thus, $Q^* \in (\alpha_l + \overline{\phi} - \beta c, \alpha_h + \overline{\phi} - \beta c]$ and must satisfy

$$\begin{aligned} \frac{dr_1(Q^*|\pi)}{dQ} &= -\pi + \frac{p(\alpha_h + \overline{\phi} - \beta c - Q^*)^2}{4\beta\overline{\phi}} = 0 \\ \Leftrightarrow \quad Q^* &= \alpha_h + \overline{\phi} - \beta c - \sqrt{\frac{4\beta\overline{\phi}\pi}{p}} \end{aligned}$$

For $\pi \in \left[\frac{p(\alpha_h - \alpha_l)^2}{4\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha + \overline{\phi} - \beta c)^2}{4\beta\overline{\phi}}\right]$, $\frac{dr_1(\alpha_l + \overline{\phi} - \beta c|\pi)}{dQ} \leq 0$ and $\frac{dr_1(0|\pi)}{dQ} \geq 0$. Thus, $Q^* \in [0, \alpha_l + \overline{\phi} - \beta c]$ and must satisfy

$$\begin{aligned} \frac{dr_1(Q^*|\pi)}{dQ} &= -\pi + \frac{p(\alpha_h + \overline{\phi} - \beta c - Q^*)^2}{4\beta\overline{\phi}} + \frac{(1-p)(\alpha_l + \overline{\phi} - \beta c - Q^*)^2}{4\beta\overline{\phi}} = 0 \\ \Leftrightarrow \quad Q^* &= \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{4\beta\overline{\phi}\pi - \sigma^2} \end{aligned}$$

For $\pi > \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}$, $\frac{dr_1(0|\pi)}{dQ} < 0$. Therefore, $Q^* = 0$. $\square$

Proof of Theorem 3.6.1 To simplify the notation, we first divide $R_1(\pi)$ by N without changing the problem. By Lemma 3.6.2, each buyer's best response at stage 2 $Q(\pi)$ is a piecewise continuous function of π. Thus, the objective function of the seller's decision problem is a piecewise function of π. The only breakpoint is $\pi_0 = \frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}$. At π_0, $Q(\pi_0^-) = Q(\pi_0^+)$. Thus, $R_1(\pi)$ is also continuous at π_0.

For $\pi \in \left[0, \frac{p(\alpha_h-\alpha_l)^2}{4\beta\phi}\right)$, $Q(\pi) = \alpha_h + \overline{\phi} - \beta c - \sqrt{\frac{4\beta\overline{\phi}\pi}{p}}$. It holds that

$$\begin{aligned}
R_1(\pi) &= \pi\left[\alpha_h + \overline{\phi} - \beta c - \sqrt{\frac{4\beta\overline{\phi}\pi}{p}}\right] \\
\frac{dR_1(\pi)}{d\pi} &= \alpha_h + \overline{\phi} - \beta c - 3\sqrt{\frac{\beta\overline{\phi}\pi}{p}}
\end{aligned}$$

In this interval, $R_1(\pi)$ is a concave function of π. At $\pi = \pi_0$, the left derivative $R'_{1-}(\pi_0) = \frac{3}{2}\alpha_l - \frac{1}{2}\alpha_h + \overline{\phi} - \beta c$. Since $\alpha_h \leq \overline{\phi}$, $R'_{1-}(\pi_0) > 0$. Therefore, $\pi^* \geq \pi_0$.

For $\pi > \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\phi}$, $Q(\pi) = 0$. It holds that $R_1(\pi) = 0$. Thus $\pi^* \leq \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\phi}$.

For $\pi \in \left[\frac{p(\alpha_h-\alpha_l)^2}{4\beta\phi}, \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}\right]$, $Q(\pi) = \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{4\beta\overline{\phi}\pi - \sigma^2}$. It holds that

$$\begin{aligned}
R_1(\pi) &= \pi\left[\mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{4\beta\overline{\phi}\pi - \sigma^2}\right] \\
\frac{dR_1(\pi)}{d\pi} &= \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{4\beta\overline{\phi}\pi - \sigma^2} - \frac{2\beta\overline{\phi}\pi}{\sqrt{4\beta\overline{\phi}\pi - \sigma^2}} \\
\frac{d^2R(\pi)}{d\pi^2} &= \frac{4\beta\overline{\phi}(-3\beta\overline{\phi}\pi + \sigma^2)}{(4\beta\overline{\phi}\pi - \sigma^2)^{3/2}}
\end{aligned}$$

At the breakpoint π_0, $R'_{1-}(\pi_0) = R'_{1+}(\pi_0) > 0$. Also at $\pi = \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}$, $R'_{1-}(\pi) = -\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{2(\mathbb{E}(\alpha)+\overline{\phi}-\beta c)} < 0$. We have already shown that the optimal π^* can only fall in this interval. We now characterize the optimal condition. Note that if $\pi \geq \frac{\sigma^2}{3\beta\overline{\phi}}$, then $\frac{d^2R_1(\pi)}{d\pi^2} \leq 0$ and function $R_1(\pi)$ is concave. Then, the Karush-Kuhn-Tucker condition is necessary and sufficient for optimality. If $\pi < \frac{\sigma^2}{3\beta\overline{\phi}}$, then $\frac{d^2R_1(\pi)}{d\pi^2} > 0$ and function $R_1(\pi)$ is convex. Depending the value of the new breakpoint $\pi_1 = \frac{\sigma^2}{3\beta\overline{\phi}}$, there are two cases. Note $\pi_1 < \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}$.

Case 1: $\frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}} \geq \frac{\sigma^2}{3\beta\overline{\phi}}$. In this case, $R_1(\pi)$ is concave on the whole interval $\left[\frac{p(\alpha_h-\alpha_l)^2}{4\beta\phi}, \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}\right]$.

Case 2: $\frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}} < \frac{\sigma^2}{3\beta\overline{\phi}}$. In this case, the interval is divided into two pieces, $\left[\frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}, \frac{\sigma^2}{3\beta\overline{\phi}}\right)$ and $\left[\frac{\sigma^2}{3\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}\right]$. We will show that the optimal π^* falls in the second interval, where $R_1(\pi)$ is concave. Thus, the Karush-Kuhn-Tucker condition is necessary and sufficient for optimality. If $\pi \in \left[\frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}, \frac{\sigma^2}{3\overline{\phi}\pi}\right)$, then $R_1(\pi)$ is convex. Thus, $\frac{dR_1(\pi)}{d\pi}$ increases in π. Since $R'_{1+}(\pi_0) > 0$, $\frac{dR_1(\pi)}{d\pi} > 0$ for all $\pi \in \left[\frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}, \frac{\sigma^2}{3\overline{\phi}\pi}\right)$. Hence, $\pi^* \geq \frac{\sigma^2}{3\beta\overline{\phi}}$.

Since in both cases, π^* falls in an interval on which $R_1(\pi)$ is strictly concave, π^* is unique. Therefore, $\pi^* \in \left[\frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}\right]$ and must satisfy the Karush-Kuhn-Tucker condition:

$$\frac{dR_1(\pi^*)}{d\pi} \quad = \quad \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{4\beta\overline{\phi}\pi^* - \sigma^2} - \frac{2\beta\overline{\phi}\pi^*}{\sqrt{4\beta\overline{\phi}\pi^* - \sigma^2}} \quad = \quad 0$$

$\square$

Proof of Lemma 3.6.3 Let $q = q_c + q_s$. Denote the optimal solution as (q_c^*, q_s^*) and $q^* = q_c^* + q_s^*$. First, we reformulate the objective function by considering the constraint $q_c \leq Q$.

If $q^* < Q$ and $s > c$, then the only optimal solution is $q_c^* = q^*$ and $q_s^* = 0$. Suppose $q_c^* < q^*$ and $q_s^* > 0$, then $r_2(q_c^* + q_s^*, 0|s, \alpha, \phi) > r_2(q_c^*, q_s^*|s, \alpha, \phi)$. Thus, q_c^* and q_s^* can not be optimal. If $q^* < Q$ and $s = c$, then there exists an optimal solution such that $q_c^* = q^*$ and $q_s^* = 0$. Note in this case since $s = c$, any (q_c^*, q_s^*) satisfying $q_c^* + q_s^* = q^*$, $q_c^* \in [0, Q]$ and $q_s^* \geq 0$ is optimal.

By similar arguments, if $q^* \geq Q$ and $s > c$, then $q_c^* = Q$ and $q_s^* = q^* - Q$. Suppose $q_c^* < Q$ and $q_s^* = q - q_c^* > 0$, then $r_2(Q, q^* - Q|s, \alpha, \phi) > r_2(q_c^*, q_s^*|s, \alpha, \phi)$. Thus, q_c^* and q_s^* can not be optimal. If $q^* \geq Q$ and $s = c$, then there exists an optimal solution such that $q_c^* = Q$ and $q_s^* = q^* - Q$. Also since $s = c$, any (q_c^*, q_s^*) satisfying $q_c^* + q_s^* = q^*$, $q_c^* \in [0, Q]$ and $q_s^* \geq 0$ is optimal.

Therefore, the problem can be reformulated as follows.

$$\begin{aligned} \max_q \quad & \hat{r}_2(q|s,\alpha,\phi) \\ \text{s.t.} \quad & q \geq 0 \end{aligned}$$

where $\hat{r}_2(q|s,\alpha,\phi)$ is defined as follows. For $q < Q$, $\hat{r}_2(q|s,\alpha,\phi) = -\frac{q^2}{2\beta} + \frac{(\alpha+\phi)q}{\beta} - qc$. For $q \geq Q$, $\hat{r}_2(q|s,\alpha,\phi) = -\frac{q^2}{2\beta} + \frac{(\alpha+\phi)q}{\beta} - Qc - (q-Q)s$. Function $\hat{r}_2(q|s,\alpha,\phi)$ is concave on the two intervals and is continuous at $q = Q$. In addition, at the breakpoint, $q = Q$, the right derivative is less or equal to the left derivative because $s \geq c$. Hence, $\hat{r}_2(q|s,\alpha,\phi)$ is a concave function of q and the Karush-Kuhn-Tucker condition is necessary and sufficient for optimality. Solving above problem, we obtain the following results.

If $\alpha + \phi - \beta c \leq 0$, then $\frac{d\hat{r}_2(0|s,\alpha,\phi)}{dq} \leq 0$. Hence $q^* = 0$, i.e., $q_c^* = q_s^* = 0$. Note that $-\overline{\phi} \leq -\alpha_h$, $\alpha + \phi - \beta c \leq 0$ is possible.

If $\alpha + \phi - \beta c > 0$ and $s = c$, then any q_c^* and q_s^* satisfying $q_c^* + q_s^* = \alpha + \phi - \beta c$, $q_c^* \in [0, Q]$ and $q_s^* \geq 0$ is optimal.

If $0 < \alpha+\phi-\beta c \leq Q$ and $s > c$, then $\frac{d\hat{r}_2(0|s,\alpha,\phi)}{dq} \geq 0$ and $\hat{r}'_{2-}(Q|s,\alpha,\phi) \leq 0$. Thus, $q^* \in (0, Q]$ and must satisfy $\frac{d\hat{r}_2(q^*|s,\alpha,\phi)}{dq} = 0$. We obtain $q^* = q_c^* = \alpha + \phi - \beta c \geq 0$ and $q_s^* = 0$.

If $0 < Q \leq \alpha + \phi - \beta c$, $Q \geq \alpha + \phi - \beta s$ and $s > c$, then $\hat{r}'_{2-}(Q|s,\alpha,\phi) \geq 0$ and $\hat{r}'_{2+}(Q|s,\alpha,\phi) \leq 0$. Thus $q^* = q_c^* = Q$ and $q_s^* = 0$.

If $0 < Q \leq \alpha + \phi - \beta s \leq \alpha + \phi - \beta c$ and $s > c$, then $\hat{r}'_{2+}(Q|s,\alpha,\phi) \geq 0$. Thus, $q^* \geq Q$ and must satisfy $\frac{d\hat{r}_2(q^*|s,\alpha,\phi)}{dq} = 0$. We obtain $q^* = \alpha + \phi - \beta s$, $q_c^* = Q$ and $q_s^* = \alpha + \phi - \beta s - Q$. □

Proof of Lemma 3.6.4 The objective function is

$$\begin{aligned} R_2(s|Q,\alpha) &= \frac{N(s-c)}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} q_s(Q,\alpha,\phi) d\phi \\ &= \frac{N(s-c)}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} (\alpha + \phi - \beta s - Q)^+ d\phi \end{aligned}$$

Note that if $s \geq \frac{\alpha+\overline{\phi}-Q}{\beta}$, $R_2(s|Q,\alpha) = 0$. Also, at $s = c$, $R_2(c|Q,\alpha) = 0$. Thus, only $s \in \left[c, \frac{\alpha+\overline{\phi}-Q}{\beta}\right]$ is interesting. Depending on the value of Q, there are two cases.

Case 1: $Q > \alpha + \overline{\phi} - \beta c$, $q_s(Q,\alpha,\phi) = (\alpha + \phi - \beta s - Q)^+ = 0$ for any $\phi \in [-\overline{\phi}, \overline{\phi}]$. Thus, $R_2(s|Q,\alpha) = 0$ for any $s \geq c$.

Case 2: $Q \leq \alpha + \overline{\phi} - \beta c$. Since $s \leq \frac{\alpha+\overline{\phi}-Q}{\beta}$, $Q - \alpha + \beta s \leq \overline{\phi}$. Also $-\alpha \geq -\overline{\phi}$, $Q - \alpha + \beta s \geq -\overline{\phi}$. Thus, $Q - \alpha + \beta s \in [-\overline{\phi}, \overline{\phi}]$. For $\phi \in [-\overline{\phi}, Q - \alpha + \beta s]$, $q_s(Q,\alpha,\phi) = (\alpha + \phi - \beta s - Q)^+ = 0$. For $\phi \in [Q - \alpha + \beta s, \overline{\phi}]$, $q_s(Q,\alpha,\phi) = (\alpha + \phi - \beta s - Q)^+ = \alpha + \phi - \beta s - Q$. Therefore, it holds that

$$\begin{aligned}
R_2(s|Q,\alpha) &= \frac{N(s-c)}{2\overline{\phi}}\left[\int_{-\overline{\phi}}^{Q-\alpha+\beta s} 0 d\phi + \int_{Q-\alpha+\beta s}^{\overline{\phi}} (\alpha + \phi - \beta s - Q) d\phi\right] \\
&= \frac{N}{4\overline{\phi}}(s-c)(\alpha + \overline{\phi} - \beta s - Q)^2 \\
\frac{dR_2(s|Q,\alpha)}{ds} &= \frac{N}{4\overline{\phi}}(\alpha + \overline{\phi} - Q - \beta s)(\alpha + \overline{\phi} - Q + 2\beta c - 3\beta s)
\end{aligned}$$

Solving $\frac{dR_2(s|Q,\alpha)}{ds} = 0$, we obtain $s_1 = \frac{\alpha+\overline{\phi}+2\beta c-Q}{3\beta}$ and $s_2 = \frac{\alpha+\overline{\phi}-Q}{\beta}$. Note $c < s_1 \leq s_2$. For $s \in [c, s_1]$, $\frac{dR_2(s|Q,\alpha)}{ds} \geq 0$. For $s \in [s_1, s_2]$, $\frac{dR_2(s|Q,\alpha)}{ds} \leq 0$. At $s = c$ and $s_2 = \frac{\alpha+\overline{\phi}-Q}{\beta}$, $R_2(s|Q,\alpha) = 0$. Therefore, from c to s_1, $R_2(s|Q,\alpha)$ increases in s to its maximum. Then, $R_2(s|Q,\alpha)$ decreases in s to 0 at s_2. Hence, $s^* = s_1 = \frac{\alpha+\overline{\phi}+2\beta c-Q}{3\beta}$. $\square$

Proof of Lemma 3.6.5 As there are infinitely many buyers, each small buyer's decision doesn't influence the seller's decision in Period 2. Thus, a single buyer takes the spot prices as given. Denote the spot price at high market state as s_h and the spot price at low market state as s_l. Denote each buyer's best response at stage 4 as $q_c(Q,s,\alpha,\phi)$ and $q_s(Q,s,\alpha,\phi)$.

First, we prove that for any given π, $s_h \geq c$ and $s_l \geq c$, $r_1(Q|\pi)$ is a concave function of Q, $\forall Q \geq 0$. Note that, each buyer will never choose $Q > \alpha_h + \overline{\phi} - \beta c$. Thus, only values of Q on $[0, \alpha_h + \overline{\phi} - \beta c]$ are interesting. We prove the concavity of $r_1(Q|\pi)$ term by term. The first term $-\pi Q$ is linear, thus it is concave. The second term is the expected return from Period 2. First, let's calculate the following integral:

$$\begin{aligned}
g_h(Q) &= \int_{-\overline{\phi}}^{\overline{\phi}} \left[-\frac{(q_c(Q,s,\alpha_h,\phi)+q_s(Q,s,\alpha_h,\phi))^2}{2\beta} \right. \\
&\quad + \frac{(\alpha_h+\phi)(q_c(Q,s,\alpha_h,\phi)+q_s(Q,s,\alpha_h,\phi))}{\beta} \\
&\quad \left. - \frac{q_c(Q,s,\alpha_h,\phi)c+q_s(Q,s,\alpha_h,\phi)s_h}{\beta} \right] d\phi
\end{aligned}$$

Note that $c \leq s_h$. Depending on the value of $\alpha_h + \overline{\phi} - \beta s_h$, there are two cases.

Case 1-H: $0 \leq \alpha_h + \overline{\phi} - \beta s_h \leq \alpha_h + \overline{\phi} - \beta c$. For $Q \in [0, \alpha_h + \overline{\phi} - \beta s_h)$, it holds that

$$\begin{aligned}
g_h(Q) &= \int_{-\overline{\phi}}^{-\alpha_h+\beta c} 0 d\phi + \int_{-\alpha_h+\beta c}^{Q-\alpha_h+\beta c} \left[-\frac{(\alpha_h+\phi-\beta c)^2}{2\beta} \right. \\
&\quad \left. + \frac{(\alpha_h+\phi)(\alpha_h+\phi-\beta c)}{\beta} - c(\alpha_h+\phi-\beta c) \right] d\phi \\
&\quad + \int_{Q-\alpha_h+\beta c}^{Q-\alpha_h+\beta s_h} \left[-\frac{Q^2}{2\beta} + \frac{(\alpha_h+\phi)Q}{\beta} - Qc \right] d\phi \\
&\quad + \int_{Q-\alpha_h+\beta s_h}^{\overline{\phi}} \left[-\frac{(\alpha_h+\phi-\beta s_h)^2}{2\beta} \right. \\
&\quad \left. + \frac{(\alpha_h+\phi)(\alpha_h+\phi-\beta s_h)}{\beta} - Qc - (\alpha_h+\phi-\beta s_h - Q)s_h \right] d\phi \\
&= Q(s_h - c)\left(\alpha_h + \overline{\phi} - \frac{1}{2}\beta s_h - \frac{1}{2}\beta c - \frac{1}{2}Q \right) + \frac{(\alpha_h+\overline{\phi}-\beta s_h)^3}{6\beta}
\end{aligned}$$

This term is a concave function of Q in this interval. For $Q \in [\alpha_h+\overline{\phi}-\beta s_h, \alpha_h+\overline{\phi}-\beta c]$, it holds that

$$\begin{aligned}
g_h(Q) &= \int_{-\overline{\phi}}^{-\alpha_h+\beta c} 0 d\phi + \int_{-\alpha_h+\beta c}^{Q-\alpha_h+\beta c} \left[-\frac{(\alpha_h+\phi-\beta c)^2}{2\beta} \right. \\
&\quad \left. + \frac{(\alpha_h+\phi)(\alpha_h+\phi-\beta c)}{\beta} - c(\alpha_h+\phi-\beta c) \right] d\phi \\
&\quad + \int_{Q-\alpha_h+\beta c}^{\overline{\phi}} \left[-\frac{Q^2}{2\beta} + \frac{(\alpha_h+\phi)Q}{\beta} - Qc \right] d\phi \\
&= \frac{Q^3}{6\beta} + \frac{Q}{2\beta}(\alpha_h+\overline{\phi}-\beta c)(\alpha_h+\overline{\phi}-\beta c - Q) \\
\frac{dg_h(Q)}{dQ} &= \frac{(\alpha_h+\overline{\phi}-\beta c - Q)^2}{2\beta}
\end{aligned}$$

Since the buyer will never choose $Q > \alpha_h + \overline{\phi} - \beta c$, $\frac{dg_h(Q)}{dQ}$ deceases in Q on $[\alpha_h + \overline{\phi} - \beta s_h, \alpha_h + \overline{\phi} - \beta c]$. Therefore, $g_h(Q)$ is also concave on this interval. At the breakpoint

$Q = \alpha_h + \overline{\phi} - \beta s_h$, $g_h(Q)$ is continuous and it holds that

$$g'_{h-}(Q) \;=\; g'_{h+}(Q) \;=\; \frac{\beta(s_h - c)^2}{2}$$

Thus, $g_h(Q)$ is concave on $[0, \alpha_h + \overline{\phi} - \beta c]$

Case 2-H: $\alpha_h + \overline{\phi} - \beta s_h < 0$. Following similar arguments, we can show that $g_h(Q)$ is also concave on $[0, \alpha_h + \overline{\phi} - \beta c]$.

Calculate the integral for the low demand state:

$$\begin{aligned} g_l(Q) \;=\; & \int_{-\overline{\phi}}^{\overline{\phi}} \Big[-\frac{(q_c(Q,s,\alpha_l,\phi) + q_s(Q,s,\alpha_l,\phi))^2}{2\beta} \\ & + \frac{(\alpha_l + \phi)(q_c(Q,s,\alpha_l,\phi) + q_s(Q,s,\alpha_l,\phi))}{\beta} \\ & - \frac{q_c(Q,s,\alpha_l,\phi)\beta c + q_s(Q,s,\alpha_l,\phi)\beta s_l}{\beta} \Big] d\phi \end{aligned}$$

Since $s_l \geq c$, $\alpha_l + \overline{\phi} - \beta s_l \leq \alpha_l + \overline{\phi} - \beta c$. Similarly, there are two cases.

Case 1-L: $0 \leq \alpha_l + \overline{\phi} - \beta s_l \leq \alpha_l + \overline{\phi} - \beta c$. For $Q \in [0, \alpha_l + \overline{\phi} - \beta s_l)$, it holds that

$$\begin{aligned} g_l(Q) \;=\; & \int_{-\overline{\phi}}^{-\alpha_l + \beta c} 0 d\phi + \int_{-\alpha_l + \beta c}^{Q - \alpha_l + \beta c} \Big[-\frac{(\alpha_l + \phi - \beta c)^2}{2\beta} \\ & + \frac{(\alpha_l + \phi)(\alpha_l + \phi - \beta c)}{\beta} - c(\alpha_l + \phi - \beta c) \Big] d\phi \\ & + \int_{Q - \alpha_h + \beta c}^{Q - \alpha_l + \beta s_l} \Big[-\frac{Q^2}{2\beta} + \frac{(\alpha_l + \phi)Q}{\beta} - Qc \Big] d\phi \\ & + \int_{Q - \alpha_l + \beta s_l}^{\overline{\phi}} \Big[-\frac{(\alpha_l + \phi - \beta s_l)^2}{2\beta} \\ & + \frac{(\alpha_l + \phi)(\alpha_l + \phi - \beta s_l)}{\beta} - Qc - (\alpha_l + \phi - \beta s_l - Q)s_l \Big] d\phi \\ \;=\; & Q(s_l - c)\left(\alpha_l + \overline{\phi} - \frac{1}{2}\beta s_l - \frac{1}{2}\beta c - \frac{1}{2}Q\right) + \frac{(\alpha_l + \overline{\phi} - \beta s_l)^3}{6\beta} \end{aligned}$$

Note $g_l(Q)$ is concave on this interval. For $Q \in [\alpha_l + \overline{\phi} - \beta s_l, \alpha_l + \overline{\phi} - \beta c)$, it holds that

$$\begin{aligned} g_l(Q) \;=\; & \int_{-\overline{\phi}}^{-\alpha_l + \beta c} 0 d\phi + \int_{-\alpha_l + \beta c}^{Q - \alpha_l + \beta c} \Big[-\frac{(\alpha_l + \phi - \beta c)^2}{2\beta} \\ & + \frac{(\alpha_l + \phi)(\alpha_l + \phi - \beta c)}{\beta} - c(\alpha_l + \phi - \beta c) \Big] d\phi \end{aligned}$$

$$
\begin{aligned}
&+ \int_{Q-\alpha_l+\beta c}^{\overline{\phi}} \left[-\frac{Q^2}{2\beta} + \frac{(\alpha_l+\phi)Q}{\beta} - Qc\right] d\phi \\
&= \frac{Q^3}{6\beta} + \frac{Q}{2\beta}(\alpha_l+\overline{\phi}-\beta c)(\alpha_l+\overline{\phi}-\beta c - Q)
\end{aligned}
$$

$$
\frac{dg_l(Q)}{dQ} = \frac{(\alpha_l+\overline{\phi}-\beta c-Q)^2}{2\beta}
$$

Since $Q \leq \alpha_l + \overline{\phi} - \beta c$, this term is also concave on $[\alpha_l + \overline{\phi} - \beta s_l, \alpha_l + \overline{\phi} - \beta c]$.

For $Q \in [\alpha_l + \overline{\phi} - \beta c, \alpha_h + \overline{\phi} - \beta c]$, it holds that

$$
\begin{aligned}
g_l(Q) &= \int_{-\overline{\phi}}^{-\alpha_l+\beta c} 0 d\phi + \int_{-\alpha_l+\beta c}^{\overline{\phi}} \left[\frac{(\alpha_l+\phi-\beta c)^2}{2\beta}\right] d\phi \\
&= \frac{(\alpha_l+\overline{\phi}-\beta c)^3}{6\beta} \\
\frac{dg_l(Q)}{dQ} &= 0
\end{aligned}
$$

Note $g_l(Q)$ is concave on each interval. At the breakpoints $Q_0 = \alpha_l + \overline{\phi} - \beta s_l$ and $Q_1 = \alpha_l + \overline{\phi} - \beta c$, $g_l(Q)$ is continuous and

$$
\begin{aligned}
g'_{l-}(Q_0) &= g'_{l+}(Q_0) = \frac{\beta(s_l-c)^2}{2} \\
g'_{l-}(Q_1) &= g'_{l+}(Q_1) = 0
\end{aligned}
$$

Thus, $g_l(Q)$ is concave for any $Q \in [0, \alpha_h + \overline{\phi} - \beta c]$.

Case 2-L: $\alpha_l + \overline{\phi} - \beta s_l < 0$. Following similar arguments, it can be shown that $g_l(Q)$ is concave on $[0, \alpha_h + \overline{\phi} - \beta c]$.

Therefore, $r_1(Q|\pi)$ is a concave function of Q on $[0, \alpha_h + \overline{\phi} - \beta c]$ and the Karush-Kuhn-Tucker condition is necessary and sufficient for the optimality of this problem. Let (Q^*, s_h^*, s_l^*) be an equilibrium, it must satisfy the optimality conditions for both the seller's problem in Period 2 and the buyers' problem. Note the buyers' problem has linear constraints, $0 \leq Q \leq \alpha_h + \overline{\phi} - \beta c$.

If $Q^* \in (0, \alpha_l + \overline{\phi} - \beta c)$, the equilibrium conditions in the subgame are

$$
s_h^* = \frac{\alpha_h + \overline{\phi} + 2\beta c - Q^*}{3\beta} \tag{A.0.25}
$$

$$s_l^* = \frac{\alpha_l + \overline{\phi} + 2\beta c - Q^*}{3\beta} \tag{A.0.26}$$

$$\frac{dr_1(Q^*|\pi)}{dQ} = 0 \tag{A.0.27}$$

Equation (A.0.25) and (A.0.26) imply $Q^* - \alpha_h + \beta s_h^* \leq \overline{\phi}$ and $Q^* - \alpha_l + \beta s_l^* \leq \overline{\phi}$. Thus,

$$\begin{aligned}\frac{dr_1(Q^*|\pi)}{dQ} &= -\pi + \frac{p}{2\overline{\phi}}(s_h^* - c)\left(\alpha_h + \overline{\phi} - Q^* - \frac{1}{2}\beta s_h^* - \frac{1}{2}\beta c\right) \\ &\quad + \frac{1-p}{2\overline{\phi}}(s_l^* - c)\left(\alpha_l + \overline{\phi} - Q^* - \frac{1}{2}\beta s_h^* - \frac{1}{2}\beta c\right)\end{aligned}$$

Substitute (A.0.25)(A.0.26) into (A.0.27). The system is reduced to

$$\begin{aligned}\frac{dr_1(Q^*|\pi)}{dQ} &= -\pi + \frac{5\mathbb{E}(\alpha + \overline{\phi} - \beta c - Q^*)^2}{36\beta\overline{\phi}} = 0 \\ \Rightarrow \quad Q^* &= \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{\frac{36\beta\overline{\phi}\pi}{5} - \sigma^2}\end{aligned}$$

At the endpoints $Q = 0$ and $Q = \alpha_l + \overline{\phi} - \beta c$,

$$\frac{dr_1(0|\pi)}{dQ} = -\pi + \frac{5\mathbb{E}(\alpha + \overline{\phi} - \beta c)^2}{36\beta\overline{\phi}}$$

where $\mathbb{E}(\alpha + \overline{\phi} - \beta c)^2 = p(\alpha_h + \overline{\phi} - \beta c)^2 + (1-p)(\alpha_l + \overline{\phi} - \beta c)^2$,

$$r'_{1-}(\alpha_l + \overline{\phi} - \beta c|\pi) = -\pi + \frac{5p(\alpha_h - \alpha_l)^2}{36\beta\overline{\phi}}$$

Note if $\pi < \frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}$, then $\frac{dr_1(Q|\pi)}{dQ} > 0$, $\forall Q \in (0, \alpha_l + \overline{\phi} - \beta c)$. If $\pi > \frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}}$, $\frac{dr_1(Q|\pi)}{dQ} < 0$, $\forall Q \in (0, \alpha_l + \overline{\phi} - \beta c)$. Thus, to satisfy the equilibrium condition, $\pi \in \left[\frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}, \frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}}\right]$.

If $Q^* \in [\alpha_l + \overline{\phi} - \beta c, \alpha_h + \overline{\phi} - \beta c]$, the conditions are

$$s_h^* = \frac{\alpha_h + \overline{\phi} + 2\beta c - Q^*}{3\beta} \tag{A.0.28}$$

$$s_l^* \geq c \tag{A.0.29}$$

$$\frac{dr_1(Q^*|\pi)}{dQ} = 0 \tag{A.0.30}$$

Equations (A.0.28) implies $Q^* - \alpha_h + \beta s_h^* \leq \overline{\phi}$. Thus,

$$\frac{dr_1(Q^*|\pi)}{dQ} = -\pi + \frac{p}{2\overline{\phi}}(s_h^* - c)(\alpha_h + \overline{\phi} - Q^* - \frac{1}{2}\beta s_h^* - \frac{1}{2}\beta c)$$

The system is reduced to

$$\begin{aligned} \frac{dr_1(Q^*|\pi)}{dQ} &= -\pi + \frac{5p}{36\beta\overline{\phi}}(\alpha_h + \overline{\phi} - \beta c - Q^*)^2 = 0 \\ \Rightarrow Q^* &= \alpha_h + \overline{\phi} - \beta c - \sqrt{\frac{36\beta\overline{\phi}\pi}{5p}} \end{aligned}$$

At the endpoints $Q = \alpha_l + \overline{\phi} - \beta c$ and $Q = \alpha_h + \overline{\phi} - \beta c$,

$$\begin{aligned} r'_{1+}(\alpha_l + \overline{\phi} - \beta c|\pi) &= -\pi + \frac{5p(\alpha_h - \alpha_l)^2}{36\beta\overline{\phi}} \\ r'_{1-}(\alpha_h + \overline{\phi} - \beta c|\pi) &= -\pi \end{aligned}$$

Note if $\pi > \frac{5p(\alpha_h - \alpha_l)^2}{36\beta\overline{\phi}}$, $\frac{dr_1(Q|\pi)}{dQ} < 0$, $\forall Q \in [\alpha_l + \overline{\phi} - \beta c, \alpha_h + \overline{\phi} - \beta c]$. Thus, to satisfy the equilibrium condition, $\pi \in \left[0, \frac{5p(\alpha_h - \alpha_l)^2}{36\beta\overline{\phi}}\right]$.

If $Q^* = 0$, the conditions are

$$s_h^* = \frac{\alpha_h + \overline{\phi} + 2\beta c - Q^*}{3\beta} \quad \text{(A.0.31)}$$

$$s_l^* = \frac{\alpha_l + \overline{\phi} + 2\beta c - Q^*}{3\beta} \quad \text{(A.0.32)}$$

$$\frac{dr_1(0|\pi)}{dQ} \leq 0 \quad \text{(A.0.33)}$$

The system implies

$$\begin{aligned} -\pi + \frac{5\mathbb{E}(\alpha + \overline{\phi} - \beta c)^2}{36\beta\overline{\phi}} &\leq 0 \\ \Rightarrow \pi &\geq \frac{5\mathbb{E}(\alpha + \overline{\phi} - \beta c)^2}{36\beta\overline{\phi}} \end{aligned}$$

If $Q^* = \alpha_h + \overline{\phi} - \beta c$, the conditions are

$$s_h^* \geq c \quad \text{(A.0.34)}$$

$$s_l^* \geq c \quad \text{(A.0.35)}$$

$$\frac{dr_1(\alpha_h + \overline{\phi} - \beta c|\pi)}{dQ} \geq 0 \tag{A.0.36}$$

The system implies $-\pi \geq 0 \Rightarrow \pi = 0$.

The results are summarized as follows.

1. If $\pi \in \left(0, \frac{5(\alpha_h - \alpha_l)^2}{36\beta\overline{\phi}}\right)$, an equilibrium is (Q^*, s_h^*, s_l^*), where $Q^* = \alpha_h + \overline{\phi} - \beta c - \sqrt{\frac{36\beta\overline{\phi}\pi}{5p}}$, $s_h^* = \frac{\alpha_h + \overline{\phi} + 2\beta c - Q^*}{3\beta}$ and $s_l^* \geq c$. Note that $Q^* \in (\alpha_l + \overline{\phi} - \beta c, \alpha_h + \overline{\phi} - \beta c)$.

2. If $\pi \in \left[\frac{5(\alpha_h - \alpha_l)^2}{36\beta\overline{\phi}}, \frac{5\mathbb{E}(\alpha + \overline{\phi} - \beta c)^2}{36\beta\overline{\phi}}\right)$, the unique equilibrium is (Q^*, s_h^*, s_l^*), where $Q^* = \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{\frac{36\beta\overline{\phi}\pi}{5} - \sigma^2}$, $s_h^* = \frac{\alpha_h + \overline{\phi} + 2\beta c - Q^*}{3\beta}$ and $s_l^* = \frac{\alpha_l + \overline{\phi} + 2\beta c - Q^*}{3\beta}$. Note that $Q^* \in (0, \alpha_l + \overline{\phi} - \beta c]$.

3. If $\pi \geq \frac{5\mathbb{E}(\alpha + \overline{\phi} - \beta c)^2}{36\beta\overline{\phi}}$, the unique equilibrium is (Q^*, s_h^*, s_l^*), where, $Q^* = 0$, $s_h^* = \frac{\alpha_h + \overline{\phi} + 2\beta c}{3\beta}$ and $s_l^* = \frac{\alpha_l + \overline{\phi} + 2\beta c}{3\beta}$.

4. If $\pi = 0$, an equilibrium is (Q^*, s_h^*, s_l^*), where, $Q^* \geq \alpha_h + \overline{\phi} - \beta c$, $s_h^* \geq c$ and $s_l^* \geq c$.

It should be noted that for any given $\pi > 0$, Q^* is unique. $\square$

Proof of Theorem 3.6.2 To simplify the notation, we first divide $R_1(\pi)$ by N without changing the problem. For a given π, denote a single buyer's best response at stage 4 as $q_c(Q, \alpha, \phi)$ and $q_s(Q, \alpha, \phi)$. Let the equilibrium spot prices in the subgame be s_h and s_l for the high demand state and low demand state respectively. The seller's problem is

$$\begin{aligned} \max_{\pi} \quad R_1(\pi) = & \quad \pi Q(\pi) + \mathbb{E}[G_2(Q(\pi), \alpha)]/N \\ s.t. \quad & \quad \pi \geq 0 \end{aligned}$$

where

$$\mathbb{E}[G_2(Q(\pi), \alpha)] = N\left[\frac{p(s_h - c)}{2\overline{\phi}} \int_{\underline{\phi}}^{\overline{\phi}} q_s(Q, \alpha_h, \phi)d\phi + \frac{(1-p)(s_l - c)}{2\overline{\phi}} \int_{\underline{\phi}}^{\overline{\phi}} q_s(Q, \alpha_l, \phi)d\phi\right]$$

For $\pi \in \left[0, \frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}\right)$, it holds that

$$\begin{aligned}
R_1(\pi) &= \pi Q(\pi) + \frac{p(\alpha_h + \overline{\phi} - Q(\pi) - \beta c)^3}{27\beta\overline{\phi}} \\
Q(\pi) &= \alpha_h + \overline{\phi} - \beta c - \sqrt{\frac{36\beta\overline{\phi}\pi}{5p}} \\
\frac{dR_1(\pi)}{d\pi} &= \alpha_h + \overline{\phi} - \beta c - \frac{11}{10}\sqrt{\frac{36\beta\overline{\phi}\pi}{5p}}
\end{aligned}$$

Thus, $R_1(\pi)$ is concave on this interval. At the endpoints $\pi = 0$ and $\pi = \frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}$,

$$\begin{aligned}
\frac{dR_1(0)}{d\pi} &= \alpha_h + \overline{\phi} - \beta c \quad > \quad 0 \\
R'_{1-}\left(\frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}\right) &= \alpha_l + \overline{\phi} - \beta c - \frac{1}{10}(\alpha_h - \alpha_l) \quad > \quad 0
\end{aligned}$$

Therefore, $\pi^* \geq \frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}$.

For $\pi \in \left[\frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}, \frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}}\right]$, it holds that

$$\begin{aligned}
R_1(\pi) &= \pi Q(\pi) + \frac{p(\alpha_h + \overline{\phi} - Q(\pi) - \beta c)^3 + (1-p)(\alpha_l + \overline{\phi} - Q(\pi) - \beta c)^3}{27\beta\overline{\phi}} \\
Q(\pi) &= \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{\frac{36\beta\overline{\phi}\pi}{5} - \sigma^2} \\
\frac{dR_1(\pi)}{d\pi} &= \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{\frac{36\beta\overline{\phi}\pi}{5} - \sigma^2} - \frac{18\beta\overline{\phi}\pi}{25\sqrt{\frac{36\beta\overline{\phi}\pi}{5} - \sigma^2}} \\
\frac{dR_1^2(\pi)}{d\pi^2} &= \frac{108\beta\overline{\phi}\left(25\sigma^2 - 33\beta\overline{\phi}\pi\right)}{25\left(\frac{36\beta\overline{\phi}\pi}{5} - \sigma^2\right)^{3/2}}
\end{aligned}$$

At the endpoints $\pi = \frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}$ and $\pi = \frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}}$,

$$\begin{aligned}
R'_{1-}\left(\frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}\right) &= R'_{1+}\left(\frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}\right) = \alpha_l + \overline{\phi} - \beta c - \frac{1}{10}(\alpha_h - \alpha_l) \\
R'_{1-}\left(\frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}}\right) &= -\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{10(\mathbb{E}(\alpha)+\overline{\phi}-\beta c)} \quad < \quad 0
\end{aligned}$$

Also, note that $R_1(\pi)$ is continuous at $\pi = \frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}$ and $\pi = \frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}}$. Thus, $\frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}} \leq \pi^* < \frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}}$. If $\pi \leq \frac{5\sigma^2}{33\beta\overline{\phi}}$, $\frac{d^2R_2(\pi)}{d\pi^2} \geq 0$. Depending on the value of $\frac{5\sigma^2}{33\beta\overline{\phi}}$, there are two cases.

Case 1: $\frac{5\sigma^2}{33\beta\overline{\phi}} \leq \frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}$. Then $R_1(\pi)$ is concave on this interval.

Case 2: $\frac{5\sigma^2}{33\beta\overline{\phi}} > \frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}$. The interval is divided into two pieces, $\left[\frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}, \frac{5\sigma^2}{33\beta\overline{\phi}}\right)$ and $\left[\frac{5\sigma^2}{33\beta\overline{\phi}}, \frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}}\right]$. Since on $\left[\frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}, \frac{5\sigma^2}{33\beta\overline{\phi}}\right)$ $R_1(\pi)$ is convex, $\frac{dR_1(\pi)}{d\pi}$ increases in π. Therefore at $\pi = \frac{5\sigma^2}{33\beta\overline{\phi}}$, $\frac{dR_1(\pi)}{d\pi} > 0$. The optimal price π^* falls in the second piece on which $R_1(\pi)$ is concave.

Since in both cases, π^* is in an interval on which $R_1(\pi)$ is strictly concave, π^* is unique. Therefore, $\pi^* \in \left[\frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}, \frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}}\right]$, such that

$$\begin{aligned} \frac{dR_1(\pi^*)}{d\pi} &= 0 \\ \Leftrightarrow\ \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{\frac{36\beta\overline{\phi}\pi^*}{5} - \sigma^2} - \frac{18\beta\overline{\phi}\pi^*}{25\sqrt{\frac{36\beta\overline{\phi}\pi^*}{5} - \sigma^2}} &= 0 \end{aligned}$$

□

Proof of Theorem 3.6.3 Similarly as before, we refer the buyers only purchase under contracts as type A buyers and the buyers participate in both markets as type B buyers. It holds that $R_1(\pi) = \lambda N R_A(\pi) + (1-\lambda) N R_B(\pi)$, where $R_A(\pi)$ and $R_B(\pi)$ represent the profit from per type A buyer and per type B buyer respectively. In the proof of Theorem 3.6.1, we have shown that $R_A(\pi)$ is piecewise concave on $\left[0, \frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}\right)$ and $\left[\frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}\right]$. In addition, $R_B(\pi)$ is piecewise concave on $\left[0, \frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}\right)$ and $\left[\frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}, \frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}}\right]$ as shown in the proof of Theorem 3.6.2.

First we will show that the optimal option price $\pi^* > \frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}$. Let $\pi_0 = \frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}$. In the proof of Theorem 3.6.1, it has been shown that $\frac{dR_A(\pi)}{d\pi} > 0$ at π_0. For $\frac{dR_B(\pi)}{d\pi}$, it holds that

$$\begin{aligned} \frac{dR_B(\pi_0)}{d\pi} &= \mathbb{E}(\alpha) + \overline{\phi} - \beta c - (\alpha_h - \alpha_l)\left[\sqrt{p(0.8+p)} + \frac{9p}{50\sqrt{p(0.8+p)}}\right] \\ &\geq (1+p)(\alpha_h-\alpha_l) - (\alpha_h - \alpha_l)\left[\sqrt{p(0.8+p)} + \frac{9p}{50\sqrt{p(0.8+p)}}\right] \end{aligned}$$

Since $\sqrt{p(0.8+p)} < 0.8+p$ and $\frac{9p}{50\sqrt{p(0.8+p)}} < 0.18$, $\frac{dR_B(\pi_0)}{d\pi} > (\alpha_h-\alpha_l)[1+p-(0.8+p+0.18)] > 0$. Thus, $\pi^* > \pi_0$.

Next we will show $\pi^* < \frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}}$. Let $\frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}} = \pi_1$. The proof of Theorem 3.6.2 has shown $\frac{dR_B(\pi_1)}{d\pi} < 0$. For $\frac{dR_A(\pi)}{d\pi}$, it holds that

$$\begin{aligned}
\frac{dR_A(\pi_1)}{d\pi} &= \mathbb{E}(\alpha)+\overline{\phi}-\beta c - \frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2-6\sigma^2}{2\sqrt{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2-9\sigma^2}} \\
&\le \mathbb{E}(\alpha)+\overline{\phi}-\beta c - \frac{5(\mathbb{E}(\alpha)+\overline{\phi}-\beta c)^2-\sigma^2}{2\sqrt{5}(\mathbb{E}(\alpha)+\overline{\phi}-\beta c)} \\
&= \mathbb{E}(\alpha)+\overline{\phi}-\beta c - \frac{\sqrt{5}}{2}(\mathbb{E}(\alpha)+\overline{\phi}-\beta c) + \frac{\sigma^2}{2\sqrt{5}(\mathbb{E}(\alpha)+\overline{\phi}-\beta c)}
\end{aligned}$$

Since $\sigma^2 \le (\mathbb{E}(\alpha)+\overline{\phi}-\beta c)^2/4$, $\frac{dR_A(\pi_1)}{d\pi} \le (\mathbb{E}(\alpha)+\overline{\phi}-\beta c)(1-\sqrt{5}/2-1/(8\sqrt{5})) < 0$. Thus, $\pi^* < \pi_1$.

Consider the following cases:

Case 1: $\frac{5\sigma^2}{33\beta\overline{\phi}} \ge \frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}} \Leftrightarrow p \le 1/12$.

Case 2: $\frac{5\sigma^2}{33\beta\overline{\phi}} \le \frac{5p(\alpha_h-\alpha_l)^2}{36\beta\overline{\phi}}$ and $\frac{\sigma^2}{3\beta\overline{\phi}} \ge \frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}$. This holds if and only if $1/12 \le p \le 1/4$.

Case 3: $\frac{\sigma^2}{3\beta\overline{\phi}} \le \frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}} \Leftrightarrow p \ge 1/4$.

We will show $\pi^* > \frac{\sigma^2}{3\beta\overline{\phi}}$ for all cases. Since $\frac{\sigma^2}{3\beta\overline{\phi}} \le \frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}$ in Case 3, the result follows. For Case 1 and Case 2, it has been shown that $\frac{dR_A(\pi)}{d\pi} \ge 0$ at $\frac{\sigma^2}{3\beta\overline{\phi}}$. For $\frac{dR_B(\pi)}{d\pi}$, it holds that

$$\begin{aligned}
\frac{dR_B(\pi)}{d\pi} &= \mathbb{E}(\alpha)+\overline{\phi}-\beta c-(\alpha_h-\alpha_l)\sqrt{p(1-p)}\left(\sqrt{1.4}+6/\sqrt{7}\right) \\
&\ge (\alpha_h-\alpha_l)\left[1-\left(\sqrt{1.4}+6/\sqrt{7}\right)/4\right] \quad > \quad 0
\end{aligned}$$

Therefore, $\pi^* > \frac{\sigma^2}{3\beta\overline{\phi}}$. Let $\pi_2 = \max\left\{\pi_0, \frac{\sigma^2}{3\beta\overline{\phi}}\right\}$. It holds that $\pi^* \in (\pi_2, \pi_1)$, on which $R_1(\pi)$ is concave. The KKT condition is necessary and sufficient for optimality.

Next we will show that $\frac{dR_B(\pi)}{d\pi} \ge \frac{dR_A(\pi)}{d\pi}$ for any $\pi \in (\pi_2, \pi_1)$. Let $x = \beta\overline{\phi}\pi$

$$\begin{aligned}
& \frac{dR_B(\pi)}{d\pi} \ge \frac{dR_A(\pi)}{d\pi} \\
\Leftrightarrow \quad & \frac{6x-\sigma^2}{\sqrt{4x-\sigma^2}} - \frac{7.92x-\sigma^2}{\sqrt{7.2x-\sigma^2}} \quad \ge \quad 0 \\
\Leftrightarrow \quad & 8.2944x^3+3.6864x^2\sigma^2-0.64x\sigma^4 \quad \ge \quad 0
\end{aligned}$$

Note $x > 0$. Solve $8.2944x^3 + 3.6864x^2\sigma^2 - 0.64x\sigma^4 = 0$. We obtain two roots x_1 and x_2 with $x_2 < 0$ and $x_1 \approx 0.1335$. Since $\beta\overline{\phi}\pi_2 > x_1$, $\frac{dR_B(\pi)}{d\pi} \geq \frac{dR_A(\pi)}{d\pi}$ for any $\pi \in (\pi_2, \pi_1)$. On this interval, for a fixed π, $\frac{dR_1(\pi)}{d\pi}$ increases in λ. It follows that $\pi^*(\lambda)$ increases in λ.

Note $Q^*(\lambda) = N[\lambda Q_B(\pi^*(\lambda)) + (1-\lambda)Q_A(\pi^*(\lambda))]$, where Q_A and Q_B denote the contracting quantity per type A buyer and per type B buyer respectively. Without confusion, we divide $Q^*(\lambda)$ by N to simplify the notation, which doesn't change the problem. It holds that

$$\begin{aligned}
Q_A(\pi^*(\lambda)) &= \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{4\beta\overline{\phi}\pi^* - \sigma^2} \\
Q_B(\pi^*(\lambda)) &= \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{7.2\beta\overline{\phi}\pi^* - \sigma^2} \\
\frac{dQ^*(\lambda)}{d\lambda} &= Q_B(\pi^*(\lambda)) - Q_A(\pi^*(\lambda)) + \lambda\frac{dQ_B(\pi^*(\lambda))}{d\pi^*}\frac{d\pi^*(\lambda)}{d\lambda} \\
&\quad (1-\lambda)\frac{dQ_A(\pi^*(\lambda))}{d\pi^*}\frac{d\pi^*(\lambda)}{d\lambda} \\
&= \sqrt{4\beta\overline{\phi}\pi^* - \sigma^2} - \sqrt{7.2\beta\overline{\phi}\pi^* - \sigma^2} + \lambda\frac{dQ_B(\pi^*(\lambda))}{d\pi^*}\frac{d\pi^*(\lambda)}{d\lambda} + \\
&\quad (1-\lambda)\frac{dQ_A(\pi^*(\lambda))}{d\pi^*}\frac{d\pi^*(\lambda)}{d\lambda} \\
&= \quad < \quad 0
\end{aligned}$$

The last step follows by $\frac{dQ_A(\pi^*(\lambda))}{d\pi^*} < 0$, $\frac{dQ_B(\pi^*(\lambda))}{d\pi^*} < 0$ and $\frac{d\pi^*(\lambda)}{d\lambda} > 0$. $\square$

Proof of Theorem 3.6.4 This proof is based on the proof of Theorem 3.6.3. It holds that

$$\begin{aligned}
R_A(\pi^*(\lambda)) &= \pi^*\left(\mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{4\beta\overline{\phi}\pi^* - \sigma^2}\right) \\
R_B(\pi^*(\lambda)) &= \pi^*\left(\mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{7.2\beta\overline{\phi}\pi^* - \sigma^2}\right) + \\
&\quad \frac{p}{27\beta\overline{\phi}}\left[(1-p)(\alpha_h - \alpha_l) + \sqrt{7.2\beta\overline{\phi}\pi^* - \sigma^2}\right]^3 + \\
&\quad \frac{1-p}{27\beta\overline{\phi}}\left[-p(\alpha_h - \alpha_l) + \sqrt{7.2\beta\overline{\phi}\pi^* - \sigma^2}\right]^3
\end{aligned}$$

To simplify the notation, we divide the seller's surplus $G(\lambda)$ by N, which doesn't change the problem.

$$
\begin{aligned}
G(\lambda) &= (1-\lambda)R_A(\pi^*(\lambda)) + \lambda R_B(\pi^*(\lambda)) \\
\frac{dG(\lambda)}{d\pi} &= R_B(\pi^*(\lambda)) - R_A(\pi^*(\lambda)) + \frac{d((1-\lambda)R_A(\pi^*(\lambda)) + \lambda R_B(\pi^*(\lambda))}{d\pi^*}\frac{d\pi^*}{d\lambda} \\
&= R_B(\pi^*(\lambda)) - R_A(\pi^*(\lambda))
\end{aligned}
$$

The last steps follows by $\frac{d((1-\lambda)R_A(\pi^*(\lambda))+\lambda R_B(\pi^*(\lambda))}{d\pi^*} = 0$.

It has been shown in the previous proof that $\frac{dR_B(\pi)}{d\pi} - \frac{dR_A(\pi)}{d\pi} > 0$ for all $\pi \in (\pi_2, \pi_1)$, where $\pi_1 = \frac{5\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{36\beta\overline{\phi}}$ and $\pi_2 = \max\left\{\pi_0, \frac{\sigma^2}{3\beta\overline{\phi}}\right\}$. Note $\pi^* \in (\pi_2, \pi_1)$. For Case 1 and Case 2, $\pi_2 = \frac{\sigma^2}{3\beta\overline{\phi}}$. We will show that $R_B\left(\frac{\sigma^2}{3\beta\overline{\phi}}\right) - R_A\left(\frac{\sigma^2}{3\beta\overline{\phi}}\right) \geq 0$. Since $\pi^* > \frac{\sigma^2}{3\beta\overline{\phi}}$ and $\frac{dR_B(\pi)}{d\pi} - \frac{dR_A(\pi)}{d\pi} > 0$, $R_B(\pi^*(\lambda)) - R_A(\pi^*(\lambda)) > R_B\left(\frac{\sigma^2}{3\beta\overline{\phi}}\right) - R_A\left(\frac{\sigma^2}{3\beta\overline{\phi}}\right) \geq 0$. The result follows. For Case 3, $\pi_2 = \pi_0$. Similarly we will show $R_B(\pi_0) - R_A(\pi_0) \geq 0$, which completes the proof.

Case 1 and Case 2: In this case, $p \leq 1/4$ and $\pi_2 = \frac{\sigma^2}{3\beta\overline{\phi}}$. Let $\pi_3 = \frac{\sigma^2}{3\beta\overline{\phi}}$, it holds that

$$
\begin{aligned}
R_A(\pi_3) &= \frac{\sigma^2}{3\beta\overline{\phi}}\left[\mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{\sigma^2/3}\right] \\
R_B(\pi_3) &= \frac{\sigma^2}{3\beta\overline{\phi}}\left[\mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{1.4\sigma^2}\right] \\
&\quad + \frac{\sigma^3}{27\beta\overline{\phi}}\left[\frac{(1-p)^{1.5}}{p^{0.5}} - \frac{p^{1.5}}{(1-p)^{0.5}} + 3\sqrt{1.4} + 1.4^{1.5}\right] \\
R_B(\pi_3) - R_A(\pi_3) &= \frac{\sigma^2}{3\beta\overline{\phi}}\left[-\sqrt{1.4} + \sqrt{1/3}\right] + \\
&\quad \frac{\sigma^2}{27\beta\overline{\phi}}\left[\frac{(1-p)^{1.5}}{p^{0.5}} - \frac{p^{1.5}}{(1-p)^{0.5}} + 3\sqrt{1.4} + 1.4^{1.5}\right] \\
&\geq \frac{\sigma^2}{3\beta\overline{\phi}}\left[-\sqrt{1.4} + \sqrt{1/3}\right] + \\
&\quad \frac{\sigma^2}{27\beta\overline{\phi}}\left[\frac{(1-1/4)^{1.5}}{(1/4)^{0.5}} - \frac{(1/4)^{1.5}}{(1-1/4)^{0.5}} + 3\sqrt{1.4} + 1.4^{1.5}\right] \\
&\approx \frac{0.0336\sigma^2}{3\beta\overline{\phi}} \quad > \quad 0
\end{aligned}
$$

Case 3: In this case $\pi_2 = \pi_0$. At π_0, it holds that

$$
R_A(\pi_0) = \frac{p(\alpha_h - \alpha_l)^2}{4\beta\overline{\phi}}\left[\mathbb{E}(\alpha) + \overline{\phi} - \beta c - p(\alpha_h - \alpha_l)\right]
$$

$$
\begin{aligned}
R_B(\pi_0) &= \frac{p(\alpha_h-\alpha_l)^2}{4\beta\overline{\phi}}\left[\mathbb{E}(\alpha)+\overline{\phi}-\beta c-\sqrt{p(0.8+p)}(\alpha_h-\alpha_l)\right]+\\
&\quad \frac{(\alpha_h-\alpha_l)^3}{27\beta\overline{\phi}}\left[p(1-p)(1-2p)+3p(1-p)\sqrt{p(0.8+p)}+(p(0.8+p))^{1.5}\right]\\
R_B(\pi_0)-R_A(\pi_0) &\approx \frac{p(\alpha_h-\alpha_l)^3}{108\beta\overline{\phi}}\left[-\sqrt{p(0.8+p)}(8p+11.8)+8p^2+15p+4\right]
\end{aligned}
$$

Since $-\sqrt{p(0.8+p)}(8p+11.8)+8p^2+15p+4>0$ for all $p\in(0,1)$, $R_B(\pi_0)-R_A(\pi_0)>0$.

Therefore, $\frac{dG(\lambda)}{d\pi}=R_B(\pi^*(\lambda))-R_A(\pi^*(\lambda))>R_B(\pi_2)-R_A(\pi_2)>0$ for all cases. □

Proof of Lemma 3.6.6 The result in (1) is obvious. We only prove (2). To simplify the notation, we first divide $R_2(s|Q_A,Q_B,\alpha)$ by N and drop the arguments Q_A, Q_B and α. Let s_0 be the optimal price without considering the capacity constraint and $\hat{s}$ be the price at which the capacity constraint is tight. Since $\overline{D}(Q_A,Q_B,s)$ strictly decrease in s, $\hat{s}$ is unique. Let $\underline{s}=\max\{c,\hat{s}\}$.

If $Q_B>\alpha+\overline{\phi}-\beta c$, then $R_2(s)=\frac{\lambda(1-\gamma)(s-c)(\alpha+\overline{\phi}-\beta s)^2}{4\overline{\phi}}$. The optimal price $s^*=\max\left\{\frac{\alpha+\overline{\phi}+2\beta c}{3\beta},\underline{s}\right\}$ is unique.

Now we prove the case $Q_B\le\alpha+\overline{\phi}-\beta c$. Note $R_2(s)$ is a piecewise function. For $s\le\frac{\alpha+\overline{\phi}-Q_B}{\beta}$,

$$
R_2(s) = \frac{\lambda(s-c)\left[\gamma(\alpha+\overline{\phi}-\beta s-Q_B)^2+(1-\gamma)(\alpha+\overline{\phi}-\beta s)^2\right]}{4\overline{\phi}}
$$

For $s>\frac{\alpha+\overline{\phi}-Q_B}{\beta}$,

$$
R_2(s) = \frac{\lambda(1-\gamma)(s-c)(\alpha+\overline{\phi}-\beta s)^2}{4\overline{\phi}}
$$

Case 1: $\frac{\alpha+\overline{\phi}+2\beta c}{3\beta}\le\frac{\alpha+\overline{\phi}-Q_B}{\beta}$. In this case, $s_0\in\left[\frac{\alpha+\overline{\phi}+2\beta c-Q_B}{3\beta},\frac{\alpha+\overline{\phi}+2\beta c}{3\beta}\right]$. Since $\frac{dR_2(s)}{ds}<0$ at $\frac{\alpha+\overline{\phi}-Q_B}{\beta}$, $\frac{dR_2(s)}{ds}>0$ at $\frac{\alpha+\overline{\phi}+2\beta c-Q_B}{3\beta}$ and $\frac{dR_2(s)}{ds}$ is quadratic, equation $\frac{dR_2(s)}{ds}=0$ only has one root in this interval. Such root is s_0. Thus, $s^*=\{s_0,\underline{s}\}$ and is unique.

Case 2: $\frac{\alpha+\overline{\phi}+2\beta c}{3\beta}>\frac{\alpha+\overline{\phi}-Q_B}{\beta}$. For $s>\frac{\alpha+\phi+2\beta c}{\beta}$, $\frac{dR_2(s)}{ds}<0$. For $s<\frac{\alpha+\overline{\phi}+2\beta c-Q_B}{3\beta}$, $\frac{dR_2(s)}{ds}>0$. Hence, $s_0\in\left(\frac{\alpha+\overline{\phi}+2\beta c-Q_B}{3\beta},\frac{\alpha+\phi+2\beta c}{3\beta}\right)$. In addition, $\frac{dR_2(s)}{ds}\ge0$ for all $s\in\left[\frac{\alpha+\overline{\phi}-Q_B}{\beta},\frac{\alpha+\overline{\phi}+2\beta c}{3\beta}\right]$.

Case 2.a: For $s \in \left[\frac{\alpha+\overline{\phi}+2\beta c-Q_B}{3\beta}, \frac{\alpha+\overline{\phi}+2\beta c}{3\beta}\right]$, $\frac{dR_2(s)}{ds} \geq 0$. In this case $s_0 = \frac{\alpha+\overline{\phi}+2\beta c}{3\beta}$. Thus, $s^* = \{s_0, \underline{s}\}$ and is unique.

Case 2.b: For some $s \in \left[\frac{\alpha+\overline{\phi}+2\beta c-Q_B}{3\beta}, \frac{\alpha+\overline{\phi}-Q_B}{\beta}\right]$, $\frac{dR_2(s)}{ds} < 0$. Since $\frac{dR_2(s)}{ds}$ is quadratic and $\frac{dR_2(s)}{ds} > 0$ at $\frac{\alpha+\overline{\phi}+2\beta c-Q_B}{3\beta}$ and $\frac{\alpha+\overline{\phi}-Q_B}{\beta}$, equation $\frac{dR_2(s)}{ds} = 0$ has two roots s_1 and s_2 with $s_1 < s_2$, $s_{1,2} \in \left[\frac{\alpha+\overline{\phi}+2\beta c-Q_B}{3\beta}, \frac{\alpha+\overline{\phi}-Q_B}{\beta}\right]$.

Case 2.b.1: $\underline{s} > \frac{\alpha+\overline{\phi}+2\beta c}{3\beta}$. In this case, $s^* = \underline{s}$ and is unique.

Case 2.b.2: $\underline{s} \in \left(s_1, \frac{\alpha+\overline{\phi}+2\beta c}{3\beta}\right]$. In this case, $s^* = \frac{\alpha+\overline{\phi}+2\beta c}{3\beta}$ and is unique.

Case 2.b.3: $\underline{s} \leq s_1$. In this case, $s^* = \text{argmax}\left\{R_2\left(\frac{\alpha+\overline{\phi}+2\beta c}{3\beta}\right), R_2(s_1)\right\}$. Only if $R_2\left(\frac{\alpha+\overline{\phi}+2\beta c}{3\beta}\right) = R_2(s_1)$, then both s_1 and $\frac{\alpha+\overline{\phi}+2\beta c}{3\beta}$ can be s^*. Simply $R_2\left(\frac{\alpha+\overline{\phi}+2\beta c}{3\beta}\right) = R_2(s_1)$. We obtain equation (3.6.23). □

Proof of Lemma 3.6.7 To simplify the notation, we first divide $R_2(s|Q,\alpha)$ by N without changing the problem. First, recall that by Lemma 3.6.4, the optimal price to the problem without the capacity constraint is $s_0 = \frac{\alpha+\overline{\phi}+2\beta c-Q}{3\beta}$. The objective function is

$$\begin{aligned}
R_2(s|Q,\alpha) &= (s-c)\int_{-\overline{\phi}}^{\overline{\phi}} q_s(Q,s,\alpha,\phi)dF(\phi) \\
&= \frac{1}{4\overline{\phi}}(s-c)(\alpha+\overline{\phi}-Q-\beta s)^2 \\
\frac{dR_2(s|Q,\alpha)}{ds} &= \frac{1}{4\overline{\phi}}\left(\alpha+\overline{\phi}-Q+2\beta c-3\beta s\right)\left(\alpha+\overline{\phi}-Q-\beta s\right)
\end{aligned}$$

If there is no capacity constraint, the optimal solution $s_0 \in \left[c, \frac{\alpha+\overline{\phi}-Q}{\beta}\right]$. At $s = c$ and $s = \frac{\alpha+\overline{\phi}-Q}{\beta}$, $R_2(s|Q,\alpha) = 0$. From $s = c$, $R_2(s|Q,\alpha)$ keeps increasing to s_0, where it reaches its maximum. Then, $R_2(s|Q,\alpha)$ decreases to 0 at $s = \frac{\alpha+\overline{\phi}-Q}{\beta}$.

Now let's look at the capacity constraint:

$$N\int_{-\overline{\phi}}^{\overline{\phi}}[q_c(Q,s,\alpha,\phi)+q_s(Q,s,\alpha,\phi)]dF(\phi) \leq C \qquad \text{(A.0.37)}$$

The LHS of the capacity constraint

$$\frac{N}{2\overline{\phi}}\left[\int_{-\overline{\phi}}^{\overline{\phi}}(q_c(Q,s,\alpha,\phi)+q_s(Q,s,\alpha,\phi))d\phi\right]$$

$$
\begin{aligned}
&= \frac{N}{2\overline{\phi}}\left[\int_{-\overline{\phi}}^{-\alpha+\beta c} 0 d\phi + \int_{-\alpha+\beta c}^{Q-\alpha+\beta c} (\alpha+\phi-\beta c) d\phi \right. \\
&\quad \left. + \int_{Q-\alpha+\beta c}^{Q-\alpha+\beta s} Q d\phi + \int_{Q-\alpha+\beta s}^{\overline{\phi}} (\alpha+\phi-\beta s) d\phi \right] \\
&= \frac{N}{4\overline{\phi}}\left[\beta^2 s^2 - 2\beta s(\alpha+\overline{\phi}-Q) + (\alpha+\overline{\phi})^2 - 2Q\beta c\right] \qquad \text{(A.0.38)}
\end{aligned}
$$

If this constraint is tight at s, then

$$
\frac{N}{4\overline{\phi}}\left[\beta^2 s^2 - 2\beta s(\alpha+\overline{\phi}-Q) + (\alpha+\overline{\phi})^2 - 2Q\beta c\right] = C
$$

Solve above equation. We get

$$
s_{1,2} = \frac{1}{\beta}\left[\alpha+\overline{\phi}-Q \mp \sqrt{(\alpha+\overline{\phi}-\beta c-Q)^2 - (\alpha+\overline{\phi}-\beta c)^2 + 4\overline{\phi}C/N}\right]
$$

Note that $s_2 = \frac{1}{\beta}\left[\alpha+\overline{\phi}-Q + \sqrt{(\alpha+\overline{\phi}-\beta c-Q)^2 - (\alpha+\overline{\phi}-\beta c)^2 + 4\overline{\phi}C/N}\right] > \frac{\alpha+\overline{\phi}-Q}{\beta}$, thus $R_2(s|Q,\alpha) = 0$. Also, LHS of (A.0.37) decreases in s, if $s \leq s_1$. Let's consider the case when the capacity is small, such that at s_0, the capacity constraint might be violated. Instead of s_0, the optimal solution is determined by the capacity constraint such that $s^* \geq s_0$ and at s^* constraint (A.0.37) is active. For $s \in (s_0, \frac{\alpha+\overline{\phi}-Q}{\beta})$, $\frac{dR_2(s|Q,\alpha)}{ds} < 0$. Therefore, $s^* = s_1$. We now characterize the capacity threshold, C_1, such that the condition holds.

$$
\begin{aligned}
& s_1 \geq s_0 \\
\Leftrightarrow\ & \frac{5(\alpha+\overline{\phi}-\beta c-Q)^2}{9} \leq (\alpha+\overline{\phi}-\beta c)^2 - \frac{4\overline{\phi}C}{N} \qquad \text{(A.0.39)}
\end{aligned}
$$

Let $Q = 0$, we get if $C \leq \frac{N(\alpha+\overline{\phi}-\beta c)^2}{9\overline{\phi}}$, then $s_1 \geq s_0$ for any $Q \geq 0$. Let $C \leq C_b = \frac{N(\alpha_l+\overline{\phi}-\beta c)^2}{9\overline{\phi}}$, then the condition holds in both high demand state and low demand state, i.e., both $s_1(Q,\alpha_h) \geq s_0(Q,\alpha_h)$ and $s_1(Q,\alpha_l) \geq s_0(Q,\alpha_l)$, $\forall Q \geq 0$. □

Proof of Lemma 3.6.8 As there are infinitely many small buyers, each buyer doesn't take the seller's decision in Period 2 as the consequence of his decision in Period 1. Thus, in each buyer's problem, the spot prices s_h and s_l are given. In Lemma 3.6.5, we have proved that for any given π, $s_h \geq c$ and $s_l \geq c$, $r_1(Q|\pi)$ is a concave function of Q, $\forall Q \in [0, \alpha_h + \overline{\phi} - \beta c]$. Note since $C \leq \frac{N(\alpha_l + \overline{\phi} - \beta c)^2}{9\overline{\phi}}$ and $\overline{\phi} \geq \alpha_h$, $C/N \leq \alpha_l + \overline{\phi} - \beta c$. It follows $r_1(Q|\pi)$ is a concave function of Q, $\forall Q \in [0, C/N]$. Therefore, the Karush-Kuhn-Tucker condition is necessary and sufficient condition for optimality.

Let (Q^*, s_h^*, s_l^*) denote an equilibrium for a given π. By Lemma 3.6.7,

$$\begin{aligned}
s_h^* &= \frac{1}{\beta}\left[\alpha_h + \overline{\phi} - Q^* - \sqrt{(\alpha_h + \overline{\phi} - Q^* - \beta c)^2 - (\alpha_h + \overline{\phi} - \beta c)^2 + 4\overline{\phi}C/N}\right] \\
s_l^* &= \frac{1}{\beta}\left[\alpha_l + \overline{\phi} - Q^* - \sqrt{(\alpha_l + \overline{\phi} - Q^* - \beta c)^2 - (\alpha_l + \overline{\phi} - \beta c)^2 + 4\overline{\phi}C/N}\right]
\end{aligned}$$

Note that $s_h^* \leq \frac{\alpha_h + \overline{\phi} - Q^*}{\beta}$, i.e. $Q^* - \alpha_h - \beta s_h \leq \overline{\phi}$ and $s_l^* \leq \frac{\alpha_l + \overline{\phi} - Q^*}{\beta}$, i.e. $Q^* - \alpha_l - \beta s_h \leq \overline{\phi}$. The market equilibrium (Q^*, s_h^*, s_l^*) for a given π must satisfy the following conditions.

If $Q^* \in (0, \frac{C}{N})$, then

$$\begin{aligned}
s_h^* &= \frac{1}{\beta}\left[\alpha_h + \overline{\phi} - Q^* - \sqrt{(\alpha_h + \overline{\phi} - Q^* - \beta c)^2 - (\alpha_h + \overline{\phi} - \beta c)^2 + 4\overline{\phi}C/N}\right] \\
s_l^* &= \frac{1}{\beta}\left[\alpha_l + \overline{\phi} - Q^* - \sqrt{(\alpha_l + \overline{\phi} - Q^* - \beta c)^2 - (\alpha_l + \overline{\phi} - \beta c)^2 + 4\overline{\phi}C/N}\right] \\
\frac{dr_1(Q^*|\pi)}{dQ} &= 0
\end{aligned}$$

where

$$\begin{aligned}
\frac{dr_1(Q^*|\pi)}{dQ} &= -\pi + \frac{p}{2\overline{\phi}}(s_h^* - c)\left(\alpha_h + \overline{\phi} - Q^* - \frac{1}{2}\beta s_h^* - \frac{1}{2}\beta c\right) \\
&\quad + \frac{1-p}{2\overline{\phi}}(s_l^* - c)\left(\alpha_l + \overline{\phi} - Q^* - \frac{1}{2}\beta s_l - \frac{1}{2}\beta c\right)
\end{aligned}$$

Above system is reduced to

$$-4\beta\overline{\phi}\pi + \mathbb{E}(\alpha + \overline{\phi} - \beta c)^2 - \frac{4\overline{\phi}C}{N} = 0$$

If $Q^* = 0$, then

$$s_h^* = \frac{1}{\beta}\left[\alpha_h + \overline{\phi} - Q^* - \sqrt{(\alpha_h + \overline{\phi} - Q^* - \beta c)^2 - (\alpha_h + \overline{\phi} - \beta c)^2 + 4\overline{\phi}C/N}\right]$$

$$
\begin{aligned}
s_l^* &= \frac{1}{\beta}\left[\alpha_l + \overline{\phi} - Q^* - \sqrt{(\alpha_l + \overline{\phi} - Q^* - \beta c)^2 - (\alpha_l + \overline{\phi} - \beta c)^2 + 4\overline{\phi}C/N}\right] \\
\frac{dr_1(0|\pi)}{dQ} &\leq 0
\end{aligned}
$$

Above system is reduced to

$$
-4\beta\overline{\phi}\pi + \mathbb{E}(\alpha + \overline{\phi} - \beta c)^2 - \frac{4\overline{\phi}C}{N} \leq 0
$$

If $Q^* = C/N$, then

$$
\begin{aligned}
s_h^* &= \frac{1}{\beta}\left[\alpha_h + \overline{\phi} - Q^* - \sqrt{(\alpha_h + \overline{\phi} - Q^* - \beta c)^2 - (\alpha_h + \overline{\phi} - \beta c)^2 + 4\overline{\phi}C/N}\right] \\
s_l^* &= \frac{1}{\beta}\left[\alpha_l + \overline{\phi} - Q^* - \sqrt{(\alpha_l + \overline{\phi} - Q^* - \beta c)^2 - (\alpha_l + \overline{\phi} - \beta c)^2 + 4\overline{\phi}C/N}\right] \\
\frac{dr_1(C/N|\pi)}{dQ} &\geq 0
\end{aligned}
$$

Above system is reduced to

$$
-4\beta\overline{\phi}\pi + \mathbb{E}(\alpha + \overline{\phi} - \beta c)^2 - \frac{4\overline{\phi}C}{N} \geq 0
$$

In summary, for any given π, buyers' optimal contracting policy is as follows.

If $\pi < \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2 - 4\overline{\phi}C/N}{4\beta\overline{\phi}}$, then $Q^* = C/N$.

If $\pi > \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2 - 4\overline{\phi}C/N}{4\beta\overline{\phi}}$, then $Q^* = 0$.

If $\pi = \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2 - 4\overline{\phi}C/N}{4\beta\overline{\phi}}$, then any $Q \in [0, C/N]$ is optimal. □

Proof of Theorem 3.6.5 To simplify the notation, we first divide $R_2(s|Q,\alpha)$ by N without changing the problem. In addition, denote the equilibrium spot prices in the subgame for a given π as s_h and s_l. And denote a single buyer's corresponding equilibrium response at stage 2 as $Q(\pi)$ and at stage 4 as $q_c(Q(\pi), \alpha, \phi)$, $q_s(Q(\pi), \alpha, \phi)$. The seller's problem is

$$
\begin{aligned}
\max_{\pi} \quad & R_1(\pi) \\
\text{s.t.} \quad & 0 \leq Q(\pi) \leq C/N \\
& \pi \geq 0
\end{aligned}
\tag{A.0.40}
$$

where

$$R_1(\pi) = \pi Q(\pi) + \frac{p(s_h - c)}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} q_s(Q(\pi), \alpha_h, \phi) d\phi + \frac{(1-p)(s_l - c)}{2\overline{\phi}} \int_{-\overline{\phi}}^{\overline{\phi}} q_s(Q(\pi), \alpha_l, \phi) d\phi \tag{A.0.41}$$

By Lemma 3.6.3, Lemma 3.6.7 and Lemma 3.6.8,

$$\begin{aligned} R_1(\pi) &= \pi Q(\pi) + \frac{p}{4\overline{\phi}}(s_h - c)(\alpha_h + \overline{\phi} - Q(\pi) - \beta s_h)^2 \\ &\quad + \frac{p}{4\overline{\phi}}(s_l - c)(\alpha_l + \overline{\phi} - Q(\pi) - \beta s_l)^2 \end{aligned}$$

Let $\pi_0 = \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2 - 4\overline{\phi}C/N}{4\beta\overline{\phi}}$. If $\pi < \pi_0$, it holds that

$$\begin{aligned} Q(\pi) &= C/N \\ s_h &= \frac{1}{\beta}\left[\alpha_h + \overline{\phi} - Q - \sqrt{(\alpha_h + \overline{\phi} - Q - \beta c)^2 - (\alpha_h + \overline{\phi} - \beta c)^2 + 4\overline{\phi}C/N}\right] \\ s_l &= \frac{1}{\beta}\left[\alpha_l + \overline{\phi} - Q - \sqrt{(\alpha_l + \overline{\phi} - Q - \beta c)^2 - (\alpha_l + \overline{\phi} - \beta c)^2 + 4\overline{\phi}C/N}\right] \\ \frac{dR_1(\pi)}{d\pi} &= C/N \end{aligned}$$

For $\pi > \pi_0$,

$$\begin{aligned} Q(\pi) &= 0 \\ s_h &= \frac{1}{\beta}\left[\alpha_h + \overline{\phi} - \sqrt{4\overline{\phi}C/N}\right] \\ s_l &= \frac{1}{\beta}\left[\alpha_l + \overline{\phi} - \sqrt{4\overline{\phi}C/N}\right] \\ \frac{dR_1(\pi)}{d\pi} &= 0 \end{aligned}$$

At the breakpoint π_0, $Q(\pi) \in [0, C/N]$. For C is small enough, it can be shown that $R_1(\pi_0)$ strictly increases as $Q(\pi)$ decreases. Let

$$\begin{aligned} y_h &= \sqrt{(\alpha_h + \overline{\phi} - \beta c - Q)^2 - (\alpha_h + \overline{\phi} - \beta c)^2 + 4\overline{\phi}C/N} \\ y_l &= \sqrt{(\alpha_l + \overline{\phi} - \beta c - Q)^2 - (\alpha_l + \overline{\phi} - \beta c)^2 + 4\overline{\phi}C/N} \end{aligned}$$

Take the derivative of $R_1(\pi_0)$ with respective to Q at π_0. We obtain

$$\begin{aligned}\frac{dR_1(\pi_0)}{dQ} &= \frac{p}{4\overline{\phi}\beta}\left(-2{y_h}^2+(\alpha_h+\overline{\phi}-\beta c-Q)(y_h-2s_h+2c+(\alpha_h+\overline{\phi}-\beta c-Q))\right)+\\ &\quad \frac{1-p}{4\overline{\phi}\beta}\left(-2{y_l}^2+(\alpha_l+\overline{\phi}-\beta c-Q)(y_l-2s_l+2c+(\alpha_l+\overline{\phi}-\beta c-Q))\right)\end{aligned}$$

A sufficient condition for $\frac{dR_1(\pi_0)}{dQ}\leq 0$ is $C\leq \frac{N(\mathbb{E}(\alpha)+\overline{\phi}-\beta c)^2}{32\overline{\phi}}=C_c$.

Therefore, if $C\leq C_c$, $R_1(\pi_0^-)\leq R_1(\pi_0)\leq R_1(\pi)$, for all $\pi>\pi_0$. Thus, any option price higher than π_0 is optimal. □

Proof of Lemma 3.6.9 Since $Q\leq C/N\leq \alpha_l+\overline{\phi}-\beta c$,

$$\begin{aligned}&\int_{-\overline{\phi}}^{\overline{\phi}}\left[-\frac{q_c(Q,\alpha_h,\phi)^2}{2\beta}+\frac{(\alpha_h+\phi)q_c(Q,\alpha_h,\phi)}{\beta}-q_c(Q,\alpha_h,\phi)c\right]d\phi\\ =\ &\int_{-\overline{\phi}}^{-\alpha_h+\beta c}0d\phi+\int_{-\alpha_h+\beta c}^{Q-\alpha_h+\beta c}\frac{(\alpha_h+\phi-\beta c)^2}{2\beta}d\phi+\int_{Q-\alpha_h+\beta c}^{\overline{\phi}}\left[-\frac{Q^2}{2\beta}+\frac{(\alpha_h+\phi)Q}{\beta}-Qc\right]\\ =\ &\frac{Q^3}{6\beta}+\frac{Q}{2\beta}(\alpha_h+\overline{\phi}-\beta c)(\alpha_h+\overline{\phi}-\beta c-Q)\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_{-\overline{\phi}}^{\overline{\phi}}\left[-\frac{q_c(Q,\alpha_l,\phi)^2}{2\beta}+\frac{(\alpha_l+\phi)q_c(Q,\alpha_l,\phi)}{\beta}-q_c(Q,\alpha_l,\phi)c\right]d\phi\\ =\ &\frac{Q^3}{6\beta}+\frac{Q}{2\beta}(\alpha_l+\overline{\phi}-\beta c)(\alpha_l+\overline{\phi}-\beta c-Q)\end{aligned}$$

Therefore,

$$\begin{aligned}r_1(Q|\pi) &= -\pi Q+\frac{p}{2\overline{\phi}}\left[\frac{Q^3}{6\beta}+\frac{Q}{2\beta}(\alpha_h+\overline{\phi}-\beta c)(\alpha_h+\overline{\phi}-\beta c-Q)\right]\\ &\quad +\frac{1-p}{2\overline{\phi}}\left[\frac{Q^3}{6\beta}+\frac{Q}{2\beta}(\alpha_l+\overline{\phi}-\beta c)(\alpha_l+\overline{\phi}-\beta c-Q)\right]\end{aligned}$$

Differentiate r_1,

$$\begin{aligned}\frac{dr_1(Q|\pi)}{dQ} &= -\pi+\frac{1}{4\beta\overline{\phi}}\left[p(\alpha_h+\overline{\phi}-\beta c-Q)^2+(1-p)(\alpha_l+\overline{\phi}-\beta c-Q)^2\right]\\ &= -\pi+\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-Q)^2}{4\beta\overline{\phi}}\end{aligned}$$

Since $Q\leq C/N\leq \alpha_l+\overline{\phi}-\beta c$, $\frac{dr_1(Q|\pi)}{dQ}$ decreases in Q. Hence, r_1 is a concave function of Q on $[0,C/N]$. The Karush-Kuhn-Tucker condition is necessary and sufficient for optimality.

The derivative at the end points $Q = 0$ and $Q = C/N$ is

$$\frac{dr_1(0|\pi)}{dQ} = -\pi + \frac{\mathbb{E}(\alpha + \overline{\phi} - \beta c)^2}{4\beta\overline{\phi}}$$

$$\frac{dr_1(C/N|\pi)}{dQ} = -\pi + \frac{\mathbb{E}(\alpha + \overline{\phi} - \beta c - C/N)^2}{4\beta\overline{\phi}}$$

Therefore,

1. If $\pi \geq \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}$, then $Q^* = 0$.

2. If $\pi \leq \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}$, then $Q^* = C/N$.

3. If $\pi \in [\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}]$, then $\frac{dr_1(Q^*|\pi)}{dQ} = 0$, thus, $Q^* = \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{4\beta\overline{\phi}\pi - \sigma^2}$.

□

Proof of Theorem 3.6.6 To simplify the notation, we first divide $R_1(\pi)$ by N without changing the problem. By Lemma 3.6.9, $R_1(\pi)$ is a piecewise function.

For $\pi < \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}$, $Q(\pi) = C/N$, $R_1(\pi) = \pi C/N$ and $\frac{dR_1(\pi)}{d\pi} = C/N$.

For $\pi > \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}$, $Q(\pi) = 0$, $R_1(\pi) = 0$.

For $\pi \in \left[\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}\right]$, $Q(\pi) = \mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{4\beta\overline{\phi}\pi - \sigma^2}$.

$$R(\pi) = \pi\left[\mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{4\beta\overline{\phi}\pi - \sigma^2}\right] \tag{A.0.42}$$

At the breakpoints $\pi_0 = \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}$ and $\pi_1 = \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}$,

$$\begin{aligned} Q(\pi_0^+) &= Q(\pi_0^-) \\ \Rightarrow R_1(\pi_0^+) &= R_1(\pi_0^-) \end{aligned}$$

$$\begin{aligned} Q(\pi_1^+) &= Q(\pi_1^-) \\ \Rightarrow R_1(\pi_1^+) &= R_1(\pi_1^-) \end{aligned}$$

For $\pi < \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}$, $\frac{dR_1(\pi)}{d\pi} = C/N$, $\pi^* \geq \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}$. In addition, for $\pi > \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}$, $R_1(\pi) = 0$. Therefore, the optimal solution $\pi^* \in \left[\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}\right]$. We will show that if C/N is small, then $\pi^* = \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}$.

For $\pi \in \left[\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}\right]$,

$$\frac{dR_1(\pi)}{d\pi} = \frac{1}{\sqrt{4\beta\overline{\phi}\pi - \sigma^2}}\left[(\mathbb{E}(\alpha) + \overline{\phi} - \beta c)\sqrt{4\beta\overline{\phi}\pi - \sigma^2} - (6\beta\overline{\phi}\pi - \sigma^2)\right]$$

Let $\sqrt{4\beta\overline{\phi}\pi - \sigma^2} = x$. Since $\pi \geq \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}$, $x > 0$. Let $\frac{dR_1(\pi)}{d\pi} = 0$.

$$\begin{aligned} \frac{dR_1(\pi)}{d\pi} &= 0 \\ \Leftrightarrow \quad -\frac{3x^2}{2} + (\mathbb{E}(\alpha) + \overline{\phi} - \beta c)x - \frac{\sigma^2}{2} &= 0 \end{aligned} \tag{A.0.43}$$

Solve equation (A.0.43). We obtain the two roots:

$$x_{1,2} = \frac{1}{3}\left[\mathbb{E}(\alpha) + \overline{\phi} - \beta c \pm \sqrt{(\mathbb{E}(\alpha) + \overline{\phi} - \beta c)^2 - 3\sigma^2}\right] \tag{A.0.44}$$

Note that $\sigma^2 = p(1-p)(\alpha_h - \alpha_l)^2 \leq \frac{(\alpha_h-\alpha_l)^2}{4}$. Thus, $3\sigma^2 \leq \frac{3(\alpha_h-\alpha_l)^2}{4} < (\mathbb{E}(\alpha) + \overline{\phi} - \beta c)^2$, $x_{1,2}$ are well defined. Since $\pi \in \left[\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}\right]$, the corresponding $x \in [\mathbb{E}(\alpha) + \overline{\phi} - \beta c - C/N, \mathbb{E}(\alpha) + \overline{\phi} - \beta c]$. If $C/N \leq \frac{1}{3}(\mathbb{E}(\alpha) + \overline{\phi} - \beta c)$, then the corresponding $x \in [\frac{2}{3}(\mathbb{E}(\alpha) + \overline{\phi} - \beta c), \mathbb{E}(\alpha) + \overline{\phi} - \beta c]$. Since $C \leq C_b$, $C/N \leq \frac{1}{3}(\mathbb{E}(\alpha) + \overline{\phi} - \beta c)$ holds. Note $x_{1,2} < \frac{2}{3}(\mathbb{E}(\alpha) + \overline{\phi} - \beta c)$. Therefore, $\frac{dR_1(\pi)}{d\pi} < 0$, $\forall \pi \in \left[\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}, \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c)^2}{4\beta\overline{\phi}}\right]$. Hence, $\pi^* = \frac{\mathbb{E}(\alpha+\overline{\phi}-\beta c-C/N)^2}{4\beta\overline{\phi}}$. $\square$

Proof of Theorem 3.6.7 First, we prove $G(0) < G(1)$. Note that

$$\begin{aligned} G(0) &= \frac{C}{4\beta\overline{\phi}}\left[\mathbb{E}(\alpha + \overline{\phi} - \beta c - C/N)^2\right] \\ G(1) &= \frac{C}{4\beta\overline{\phi}}\left[4\overline{\phi}(\mathbb{E}(\alpha) + \overline{\phi} - \beta c - \sqrt{4\overline{\phi}C/N})\right] \end{aligned}$$

Since $C \leq \min\{C_b, C_c\}$, $C \leq \frac{N(\mathbb{E}(\alpha)+\overline{\phi}-\beta c)^2}{16\overline{\phi}}$. Thus, $\sqrt{4\overline{\phi}C/N} \leq \frac{\mathbb{E}(\alpha)+\overline{\phi}-\beta c}{2}$ and

$$\begin{aligned} G(1) &\geq \frac{C}{4\beta\overline{\phi}}\left[2\overline{\phi}(\mathbb{E}(\alpha) + \overline{\phi} - \beta c)\right] \\ G(0) &= \frac{C}{4\beta\overline{\phi}}\left[\mathbb{E}(\alpha + \overline{\phi} - \beta c - C/N)^2\right] \end{aligned}$$

$$
\begin{aligned}
&< \frac{C}{4\beta\overline{\phi}}\left[2\overline{\phi}p(\alpha_h+\overline{\phi}-\beta c)+2\overline{\phi}(1-p)(\alpha_l+\overline{\phi}-\beta c)\right] \\
&= \frac{C}{4\beta\overline{\phi}}\left[2\overline{\phi}(\mathbb{E}(\alpha)+\overline{\phi}-\beta c)\right]
\end{aligned}
$$

Therefore, $G(0) < G(1)$.

Next, we will prove $V(0) < V(1)$. It holds that

$$
\begin{aligned}
V(0) &= C\left[-\frac{\mathbb{E}(\alpha+\overline{\phi}-\beta-C/N)^2}{4\beta\overline{\phi}}+\frac{C^2}{12\beta\overline{\phi}N^2}\right. \\
&\quad +\frac{p}{4\beta\overline{\phi}}(\alpha_h+\overline{\phi}-\beta c)(\alpha_h+\overline{\phi}-\beta c-C/N) \\
&\quad \left.+\frac{(1-p)}{4\beta\overline{\phi}}(\alpha_l+\overline{\phi}-\beta c)(\alpha_l+\overline{\phi}-\beta c-C/N)\right] \\
&= C\left[\frac{C}{4\beta\overline{\phi}N}(\mathbb{E}(\alpha)+\overline{\phi}-\beta c-C/N)+\frac{C^2}{12\beta\overline{\phi}N^2}\right] \\
&= \frac{C^2\left[3(\mathbb{E}(\alpha)+\overline{\phi}-\beta c)-2C/N\right]}{12\beta\overline{\phi}N}
\end{aligned}
$$

$$
V(1) = \frac{N}{12\beta\overline{\phi}}\left[p(\alpha_h+\overline{\phi}-\beta s_h)^3+(1-p)(\alpha_l+\overline{\phi}-\beta s_l)^3\right]
$$

where $s_h = \frac{1}{\beta}\left[\alpha_h+\overline{\phi}-\sqrt{\frac{4\overline{\phi}C}{N}}\right]$ and $s_l = \frac{1}{\beta}\left[\alpha_l+\overline{\phi}-\sqrt{\frac{4\overline{\phi}C}{N}}\right]$. Thus,

$$
V(1) = \frac{N}{12\beta\overline{\phi}}\left[4\overline{\phi}C/N\right]^{3/2}
$$

$$
V(1)-V(0) = \frac{C^{3/2}}{12\beta\overline{\phi}N^{1/2}}\left[8\overline{\phi}^{3/2}-\sqrt{\frac{C}{N}}\left[3(\mathbb{E}(\alpha)+\overline{\phi}-\beta c)-2C/N\right]\right]
$$

Note that $C/N \leq C_b/N = \frac{(\alpha_l+\overline{\phi}-\beta c)^2}{9\overline{\phi}}$, $C/N \leq \frac{4\overline{\phi}}{9} < \overline{\phi}$. Also, $\alpha_h \leq \overline{\phi}$. Thus

$$
\begin{aligned}
3(\mathbb{E}(\alpha)+\overline{\phi}-\beta c)-2C/N &< 8\overline{\phi} \\
\sqrt{\frac{C}{N}} &< \overline{\phi}^{0.5} \\
\Rightarrow V(1)-V(0) &> 0
\end{aligned}
$$

Since both $G(1) > G(0)$ and $V(1) > V(0)$, total social surplus $W(1) > W(0)$. □

Proof of Lemma3.7.2 First we will show that (s_0, s_0) is an equilibrium. Suppose seller 1 lowers the price to $s < s_0$, then her return at this stage is $R_{2,1}(s, s_0|Q_1, Q_2, \alpha) = (s - c)(C - Q_1) < R_{2,1}(s_0, s_0|Q_1, Q_2, \alpha)$, which can not be optimal. If the Seller increases the price to $s > s_0$, then for $s \leq (\alpha - C - Q_1)/\beta$,

$$R_{2,1}(s, s_0|Q_1, Q_2, \alpha) = (s - c)(\alpha - \beta s - C - Q_1) \tag{A.0.45}$$

$$\frac{dR_{2,1}(s, s_0|Q_1, Q_2, \alpha)}{ds} = \alpha - 2\beta s + \beta c - C - Q_1 \tag{A.0.46}$$

where $R_{2,1}(s, s_0|Q_1, Q_2, \alpha)$ is strictly concave on the interval. For $s > (\alpha - C - Q_1)/\beta$, $R_{2,1}(s, s_0|Q_1, Q_2, \alpha) = 0$. Note $R_{2,1}(s, s_0|Q_1, Q_2, \alpha)$ is continuous on $[s_0, +\infty)$. For $C \leq (\alpha - \beta c)/3$, $\frac{dR_{2,1}(s,s_0|Q_1,Q_2,\alpha)}{ds} \leq 0$ for $s \in (s_0, (\alpha - C - Q_1)/\beta]$ and any $Q_i \in [0, C]$, $i = 1, 2$. Therefore, $R_{2,1}(s, s_0|Q_1, Q_2, \alpha) < R_{2,1}(s_0, s_0|Q_1, Q_2, \alpha)$ in this case either. Hence if $C \leq C_d$, (s_0, s_0) is an equilibrium.

Next we will show (s_0, s_0) is the unique equilibrium in the subgame. Suppose the equilibrium price is (s_i, s_j) other than (s_0, s_0) with $s_i \leq s_j$. Clearly s_i and s_j can't be less than s_0. Suppose $s_0 < s_i \leq s_j$, seller j will lower the price slightly below s_i, which would result in a better revenue for her. The two sellers would undercut each other and both reach s_0 eventually. □

Proof of Lemma3.7.3 Since all the contracts are exercised in stage 4, it holds that

$$\frac{\partial r_1(Q_1, Q_2|\pi_1, \pi_2)}{\partial Q_1} = -\pi_1 + \frac{\mathbb{E}(\alpha) - \beta c - 2C}{\beta} \tag{A.0.47}$$

$$\frac{\partial r_1(Q_1, Q_2|\pi, \pi_2)}{\partial Q_2} = -\pi_2 + \frac{\mathbb{E}(\alpha) - \beta c - 2C}{\beta} \tag{A.0.48}$$

Therefore, the results follow. □

Proof of Theorem 3.7.1 By previous results, the buyer's contracting quantity for one seller is independent from the other seller's option price. Therefore, if $\pi_i < (\mathbb{E}(\alpha) - \beta c - 2C)/\beta$, $R_{1,i}(\pi_i, \pi_j) = \pi_i C < C(\mathbb{E}(\alpha) - \beta c - 2C)/\beta$. For $\pi_i > (\mathbb{E}(\alpha) - \beta c - 2C)/\beta$, $Q_i(\pi_i, \pi_j) = 0$ and $R_{1,i}(\pi_i, \pi_j) = C(\mathbb{E}(\alpha) - \beta c - 2C)/\beta$. For $\pi_i = (\mathbb{E}(\alpha) - \beta c - 2C)/\beta$, $Q_i(\pi_i, \pi_j) \in [0, C]$. It still holds that $R_{1,i}(\pi_i, \pi_j) = C(\mathbb{E}(\alpha) - \beta c - 2C)/\beta$. Therefore, any option price $\pi \geq C(\mathbb{E}(\alpha) - \beta c - 2C)/\beta$ is optimal. □

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