Strategic Capacity Investment with Partial Reversibility under Uncertain Economic Condition and Oligopolistic Competition

A Thesis
Presented to
The Academic Faculty

by

Hee Jung Sim

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
School of Industrial and Systems Engineering

Georgia Institute of Technology
January 2005

Strategic Capacity Investment with Partial Reversibility under Uncertain Economic Condition and Oligopolistic Competition

Approved by:

Dr. Shijie Deng, Advisor
School of Industrial and Systems Engineering
Georgia Institute of Technology

Dr. Paul M. Griffin
School of Industrial and Systems Engineering
Georgia Institute of Technology

Dr. Robert P. Kertz
School of Mathematics
Georgia Institute of Technology

Dr. Joel Sokol
School of Industrial and Systems Engineering
Georgia Institute of Technology

Dr. Qiong Wang
Bell Laboratories

Date Approved: 10 January 2005

ACKNOWLEDGEMENTS

During the preparation of this thesis, there have been many people whose prayer, support, and encouragement have contributed to its accomplishment.

First of all, I would like to thank my advisor, Professor Shijie Deng. His thoughtful comments, helpful suggestions, and consistent guidance made the preparation of this thesis an invaluable experience.

My sincere thanks go to Professor Robert P. Kerts. His kind suggestions and valuable comments contributed to this thesis in many ways. I would like to thank Professors Paul M. Griffin and Joel Sokol, and Dr. Qiong Wang for their much appreciated opinions and service on my thesis committee. I also wish to acknowledge financial support by the School of Industrial and Systems Engineering of the Georgia Institute of Technology.

I owe many thanks to my friends in South Korea and Atlanta for their friendship, prayers, and encouragement during my study. Finally, I dedicate all my work to my parents, brothers, sisters, and my husband. Their presence has always been my strength.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

SUMMARY

We consider the problem of capacity expansion in telecommunication networks by firms under uncertain economic conditions with various market structures. We assume that the price-demand functions for network capacity have constant price-elasticity and demand functions are parameterized by the index of a general economic condition that is modeled by a discrete time Markov process. We apply dynamic programming to obtain a state-dependent capacity expansion strategy that maximizes expected total discounted cash flow.

Firm's cost structure incorporates partial reversibility of investment by differentiating the purchasing cost and the salvage value of the capacity. This partial reversibility makes the value function non-differentiable and divides the solution space into BUY, KEEP, and SELL regions. In addition, with the non-differentiable value function, it is hard to obtain an analytical solution in general. By identifying certain structural properties of the optimal solution, we perform a series of sensitivity analyses of optimal investment decisions with respect to other market parameters. Under a typical condition in the telecommunications market, which states that the level of cost depreciation is larger than that of the downside movement of the economic condition in each time period, we are able to obtain analytical expression for the optimal level and reduce the multi-period investment decision problem into a single-period myopic problem. As a result, optimal capacity increment depends only on the current economic condition.

We study this problem in both the monopolistic and oligopolistic market structures. In particular, we investigate investment decisions of two firms in duopoly setting, assuming that firms follow Cournot competition behaviors. We prove the existence and the uniqueness of the Cournot equilibrium strategies in the duopolistic capacity investment problem. In addition, we show how competition between firms affects through total capacity in the

market in usage, capacity price, the consumer surplus, expected time to a certain level of price reduction, and the expected time to the first investment. We perform an empirical analysis to test a theoretical prediction obtained from our model through linear regression utilizing historical market data. By examining several alternative indicies as a proxy to the economic condition considered in our model, we show that the Civilian Employment is the best proxy to use in validating the linear relationship between capacity expansion and economic indicator.

CHAPTER I

INTRODUCTION

In this chapter, we present the background of our studies, including the motivation, related previous research results, features of the proposed model, and contributions.

1.1 Motivation

Capital investment planning is one of the key business decision-making processes for almost all firms. This investment decision includes initiating new projects, expanding existing business, shrinking or discontinuing current business, and so forth. In any case, decisions about the size and the timing of the investment is one of the most important assignments for a managerial team. Specifically, in the telecommunications market and the computer industry, the size of an expansion and the timing for the investment are crucial to the health and survival of a firm. For example, the usage of the bandwidth is exponentially increasing, so Internet service providers must respond to market demand qualitatively and quantitatively in a timely manner.

The telecommunications industry used to be a regulated monopoly. Without competition, a firm needed only to find a solution that satisfied the exogenously given demand at a fixed price. Therefore, relevant research focused on minimizing cost only. As deregulation occurred, firms began to control prices. Now, a firm must consider the effect of price on demand and at the same time, maximize profits. Thus, the challenge of the firm changes to maximizing profits by determining prices and corresponding capital/capacity expansion strategies from minimizing total cost through determining the capital/capacity expansion policies only.

Very few papers address the discrete time and multiple time investment decision problem. When an investment decision problem is solved in a continuous time setting, the resulting path of the capital/capacity movement might be difficult to determine in the real world. For example, some studies have modeled demand as a geometric Brownian motion in a continuous time frame, which results in a capital/capacity movement similar to the demand movement. In that case, the expansion size can be infinitesimal, and thus multiple expansions in a very short time interval are possible.

Considering the special characteristics of the telecommunications industry, we need to address the following issues: First, the investment decision should be made recursively to keep up with increasing demand, so we have to consider multiple-time investments, not a one-time investment decision; second, the investment decision should be discrete in time, which makes the size of the expansion lumpy; third, we should recognize that the unit cost for the capacity expansion depreciates rapidly due to improvements in technology; fourth, different from other industries, the telecommunications and computer industry demands that we incorporate trends in market price, which is decreasing and demand, which is increasing with time; a firm should consider the effect of competition between firms on its investment decision on its investment decision. Competition behavior among firms is addressed in several models: the Cournot model, the Bertrand model, the Stackelberg model, and others. We have adopted the Cournot model to investigate investment behavior in an oligopolistic market.

1.2 Literature Review

Numerous studies have dealt with the problem of the investment decision, which includes expanding current business, starting new projects, suspending current production lines for a certain period of time, and shutting down a company permanently. A traditional method that deals with such decisions is the min-cost approach. Smith[40] considered the decision of capacity expansion in terms of the timing of an investment by defining a cost rate and minimizing this quantity. He showed that the equal timing policy minimizes the present value of the investment cost. Using this cost rate, Ryan[39] established a heuristic method that determined expansion times and expansion amounts using the (s,

S) policy. Bean, Higle, and Smith[7] considered the optimal capacity expansion problem with stochastic demand. They showed that a stochastic problem could be changed to a deterministic one using a modified interest rate that was smaller than the original interest rate and approximately proportional to the uncertainty of demand.

Another big stream of investment-related research applied real option method to determine the timing of investments. This method applies the valuation method of financial options to estimate the opportunity cost of a investment. Adopting the concept of the exercise boundary of an American option, they try to pinpoint an optimal value of a project or prices of a product that must be reached before an investment decision is made. Here, the value of the project corresponds to the underlying security of the financial option, and the investment should be made at the early exercise boundary of the option. Dixie and Pindyck[11] analyzed the investment decision very exhaustively using this real option analysis. McDonald and Siegel[32], [31] formulated the value of the option to invest in an irreversible project and showed that this value could be as much as the investment cost for the reasonable parameter values. They also calculated the value of an option to shut down a production line temporarily at no cost when variable costs exceed operating revenues. Even though the real option concept creates a considerable challenge to the traditional Net Present Value (NPV) analysis, this method has a crucial weakness. When we consider multiple-time investment decision, the calculation becomes complicated and time consuming. Hubbard and Lehr[16] considered a two periods model of investment with real option method. Few papers solve for more than two periods.

The value to wait, which should be added to the opportunity cost, comes from the fact that the investment decision is totally/partially irreversible. This irreversibility should be included in the investment decision model so that it can be applied to real world investment problems. In Arrow[5], the aim of a firm is to maximize the sum of all the cash flow discounted at the market rate of interest. He analyzed the change in the investment path when he considered irreversibility in a deterministic and continuous time setting. Abel and Eberly[2] defined and calculated the user costs of capital associated with the purchase (C_u) and sale (C_l) of capital, given the purchase and sell price of unit capital. They found the optimality condition, which is, the marginal revenue product of capital should be in the interval $[C_l, C_u]$. Bertola and Caballero[8] proposed and solved a model of sequential irreversible investment and extended the result to the aggregated investment in a continuous time setting.

As deregulation spreads throughout the electricity and telecommunications industries the market is changing from monopolistic to duopolistic/oligopolistic. Many research papers have analyzed changes in optimal investment decisions when market conditions change. Some papers handle this issue of competitition by changing the price process from endogenous to exogenous. Others handle this by applying oligopolistic behavior to a market directly. In the latter case, they try to identify the Nash equilibrium point. Leahy[26] showed that the option-value thresholds of a monopolistic firm are the same as the free entry threshold of a competitive firm. Aguerrevere[4] studied the effect of competitive interactions on investment decisions and on the dynamics of the price of a non-storable commodity. White and Benson[44] illustrated the structure of the competitive electricity market. They also showed the trends in total market share and price in several electricity markets. Chuang, Wu, and Varaiya[9] explained the behavior of the firms in a generation expansion planning under Cournot, Cournot duopoly, and Cartel assumptions. Their results support the classic Cournot model. Hay and Liu[15] analyzed the behavior of a firm in fragmented, dominant firm, and dominant group sectors. They suggested that fragmented sectors are characterized by noncooperative investment behavior, dominant group sectors by cooperative behavior, and dominant firm sectors by competitive behavior.

1.3 Features in the Proposed Model

1. Market uncertainty in economic indicator:

   Some papers model market demand and/or market price as a stochastic process that is exogenously given. Abel[1] modeled the price of output as a geometric Brownian motion. Abel and Eberly[2] modeled demand as a stochastic process which depends on a random variable which evolves exogenously according to a geometric Brownian motion and in [3], they used Brownian motion to model randomness in technology, and/or randomness of the variable cost in demand. Aguerrevere[4] and d'Halluin, Forsyth, and Vetzal[10] modeled the demand as a geometric Brownian motion. But there are some research going on about network traffic which shows demand is not well represented by geometric Brownian motion. For more detailed explanation, refer to Riedi and Ribeiro[38], Ma[29], Nor, Yahya, and Ihsanto[36] and Yang[45]. Kou and Kou[22] specified two economic indicators: general economic conditions which are exogenously given, and sector/industry-specific economic conditions which can be endogenously calculated. They modeled the growth stocks using sector/industry-specific economic conditions, which can be represented by the total research and developement labor growth rate in such a sector/industry.

We model the general economic condition as a discrete Markov process, which is more general than (geometric) Brownian motion. Of course, by using a discrete Markov process, we can also approximate the geometric Brownian motion but with more flexibility.

2. Characteristics of demand:

   • Constant elasticity: We assume that the price-demand function has a constant price-elasticity of demand and is parameterized by the index of general economic conditions, which are modeled by discrete Markov processes. The iso-elastic demand function is frequently used in the investment model. Abel and Eberly[1], Kou and Kou[22] Mitra and Wang[35], and Kenyon and Cheliotis[20] also used this iso-elastic demand function in their papers. In addition, $\epsilon > 1$ is a common assumption in the telecommunications industry. Following the work of Lanning, Mitra, Wang, and Wright[25], we assume elasticity is between 1.28 and 2.84.

3. Market structures:

   • Monopolistic market: In a monopolistic market, a company is a price maker.

Therefore, it can control the demand by setting the price level. Also, it looks for the optimal capacity and price together through the price-demand function.

• Oligopolistic market: Firms in the market are non-cooperative competitive. We adopt the Cournot model to investigate the investment behavior of the firms in an oligopolistic market. We consider the cases of both of symmetric and asymmetric firms.

4. Capital/Capacity investment decision:

• Value function: The objective function is a firm's expected total discounted cash flow. We try to find an optimal capacity/price that maximizes this objective function. When the market structure is one of a regulated monopoly, the exogenous demand model and min-cost models were reasonable. However the recent change in the industry, featured by relaxed regulation and more competition, forces firms to consider their combined revenue and costs by influencing the demand through pricing.

• Partial reversibility of capacity investment: A considerable number of studies deal with this irreversibility. The real option approach is well known method that addresses irreversibility in a systematic way. Traditionally, it has been common sense for a firm to invest in a project when the present value of the project is greater than or equal to zero. However, recent studied have shown that this present value analysis can be wrong. When investment is irreversible, the firm should consider the option of delaying investment, and the option should be properly valued. This real option approach is simple and well applied when the investment is a one time decision. However, when a firm has to consider a series of investments, the real option approach is not appropriate because of exponentially increasing complexity. In addition, Grenadier[14] showed that option holders (in this case, project managers) have to consider that the competition between the market players and the value of waiting for the investment is not considerable in the competitive market and the classical NPV analysis is approximately accurate. Another way to incorporate irreversibility is to consider only incremental investment. In this case, reducing the present capital is not considered.

We incorporate this irreversibility of the investment by differentiating the purchasing price and the salvage value. With these two different values, we can consider a partially reversible investment decision. In addition, this model encompasses the cases of completely irreversible investment and costlessly reversible investment by setting the salvage value at zero, and by setting the salvage value to be the same as the purchase price respectively. By making the difference between the purchase price and the salvage value broad, we have a broad range of optimal capacity instead of one optimal point. In other words, we have minimum optimal capacity and maximum optimal capacity by differentiating the purchase price from the salvage value.

• Discrete and multiple-time expansion: We consider discrete and multiple-time investment. In other words, investment decisions can be made quarterly or monthly. Therefore, the resulting capital/capacity path takes a step function form, and the investment size is lumpy. Because an infinitesimal increment in capacity during an infinitesimal period is not realistic, particularly in the telecommunications industry, the discrete time model is more practical than the continuous time model.

1.4 Contributions

The purpose of this thesis is to investigate the investment behavior of a company under uncertain economic conditions. We formulate the problem as a discrete Markov decision problem. We obtain the optimal investment strategy, which depends on the states of the capacity of the previous period and current economic condition. By incorporating a broad range of state space, the model is applied to entry firms, well-established firms, and over-invested firms. Furthermore the trade-off between the lost revenue due to current insufficient capacity and the opportunity cost of a premature investment resulting from the foregone reduced investment cost due to technology advancement is examined.

We first examine the monopolistic market. We identify the linear relationship between the optimal solutions and the general economic condition. Using this linearity, we perform a series of sensitivity analyses. Under a typical parameter setting in a telecommunications industry, we obtain analytic expression of the optimal solution. Moreover, we investigate the price and demand trends in the market and then obtain analytical expressions for them. The result is consistent to the findings of the previous studies by Kenyon and Cheliotis[20], [21] and [19]. In addition, we perform a series of sensitivity analyses and show how the optimal investment decision changes with the parameters.

We then study the firm's investment behavior in an oligopolistic market. In particular, we investigate a duopoly model in which two firms follow the Cournot behavior. We show that the equilibrium point for the two-firm case exists and is unique analytically and numerically. We also investigate the effect of competition on market properties through total market capacity under usage, market price, consumer surplus, expected time of a certain price reduction, and expected time until the first capacity expansion.

Finally, using real market data, we try to show the validity of the proposed model.

1.5 Outline

The remaining dissertation is organized as follows. In chapter 2, we formulate the capacity investment problem in a monopolistic market [1]. We illustrate the structure of the solution and perform a series of sensitivity analyses. Using the structure of the solution, we find an analytic expression for the optimal capacity and the price under a typical condition in the telecommunications market. In addition, we show an optimal capacity trend and the corresponding price trend. Finally, we perform an experiment with a certain parameter set and show numerical results. In chapter 3, we consider the capacity investment problem in an oligopolistic market. By adopting the Cournot model, we illustrate investment behavior of the firms in a competitive market. We study the existence and the uniqueness of Cournot equilibrium point in the case of symmetric firms and asymmetric firms. How competition

[1] This section is based on the work at Bell Labs, where I was a summer intern 2002.

affects the market properties is explored through total market capacity, market price, consumer surplus, expected time of a certain price reduction, and expected time until the first capacity expansion. In chapter 4, using market data from telecommunications companies, we perform linear regression analysis and validate the proposed model.

CHAPTER II

THE INVESTMENT STRATEGY OF A MONOPOLISTIC FIRM

In this section, we study the capacity investment decisions of a monopolistic company. In a monopolistic market, a company is a price maker and thus controls market demand by changing the price. We address this investment decision problem in a discrete time framework with time horizon T. One period can be a year, a quarter, a month, a week, and so forth, depending on the problem characteristics. At the beginning of each period, say t period, the firm is given the capacity (x_{t-1}) of the previous period. It then needs to decide the optimal capacity (x_t^*) of the current period considering the general economic condition (ξ_t).

The increment (or decrement) $\hat{x}_t = x_t - x_{t-1}$. Depending on the sign of $\hat{x}_t$, a firm can choose three different actions, which are buy, sell, and maintain the capacity of the previous period. Figure 1 shows one possible path of the capacity evolution with time.

If the firm decides to invest in more capacity, it has to pay installation and increased maintenance costs, but it can collect more revenue due to the increased capacity. If the firm delays the investment decision and maintains the capacity of the previous period, it can take advantage of cost depreciation that comes from technology advancement. If the firm decides to retire some or all of the capacity of the previous period, it might get some revenue from the selling off of the excess capacity, but it will lose the revenue due to the reduced capacity. When the firm makes these investment decisions, it needs to consider the current and expected future economic conditions. The general economic condition is not deterministic, so we model it as a discrete Markov process.

By rephrasing the problem, it can be defined in the following way. Given two state variables (x_{t-1}, ξ_t) at the beginning of t period, we need to solve for the optimal capacity

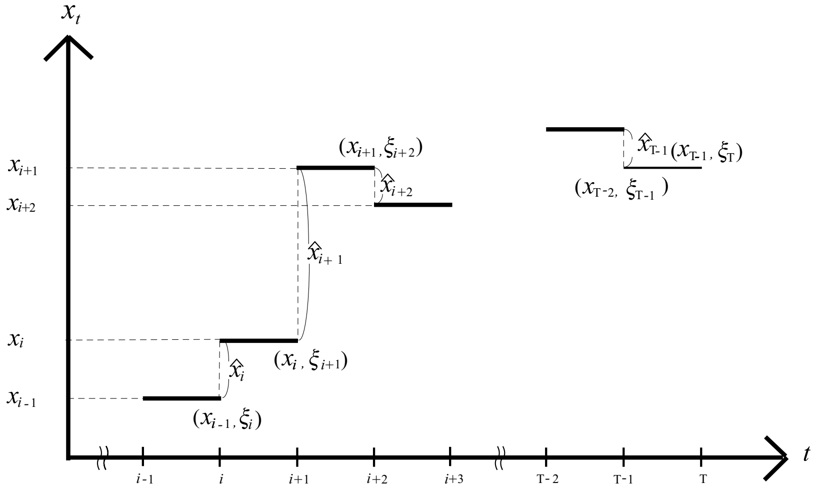

Figure 1: Possible Path of Capacity Movement with Time.

(x_t^*) of the current period by maximizing the company's expected total discounted cash flow from time t to time horizon T. This capacity will stay constant until the end of the t period. At the beginning of $t+1$ period, we need to determine again the optimal capacity (x_{t+1}^*) that will stay constant until the end of the $t+1$ period, and so on.

2.1 Model Formulation

We start this section by modeling the features of our problem.

First, we explain the notation we use throughout the chapter.

ξ_t : general economic condition at the beginning of period t

P_t: market price for period t

D_t: market demand for period t

x_t : capacity position at the beginning of period t

R_t: revenue during period t

C_t: total cost occurring during period t

F_T: terminal value function

$p_t(i,j) = \mathsf{P}\{\xi_{t+1} = \xi_j | \xi_t = \xi_i\}$: transition probability from ξ_i at time t

to ξ_j at time $t+1$

η_t : cost depreciation coefficient during period t

ϵ : elasticity of the price-demand function

T : time horizon

r : discount rate periodwise

0^+: infinitesimal, can be defined as $\lim_{n\uparrow\infty} \frac{1}{n}$.

1. Economic indicator.

   As we mentioned in section 1.3, we employ an economic indicator, ξ_t, to model the market uncertainty. Specifically, we model ξ_t as a discrete time Markov process with a transition probability $p_t(i,j)$ from state i at time t to state j at time $t+1$.

2. Price-demand function.

   We define the price-demand function as

$$P_t(D_t, \xi_t) = \left(\frac{\xi_t}{D_t}\right)^{\frac{1}{\epsilon}}, \tag{2.1.1}$$

   where P_t and D_t are price and demand at time t, respectively. The price-demand function is scaled by ξ_t, which reflects the general economic condition. Also ξ_t can be interpreted as the willingness to buy D_t when the price is 1. We call this a general economic condition or economic indicator afterward. The price-elasticity of demand is defined as

$$\epsilon = -\frac{dD/D}{dP/P}. \tag{2.1.2}$$

   Here we assume that the price-elasticity of demand is constant, which is a common assumption in telecommunications-related literature.

   Depending on the value of ϵ, the investment behavior changes.

   (a) If $\epsilon > 1$, a company can increase its revenue by lowering the price and taking advantage of increased demand.

   (b) If $\epsilon = 1$, the revenue of a company stays unchanged with respect to the price movement and the corresponding demand change.

(c) If $\epsilon < 1$, a company can increase its revenue from a higher price and decreased demand.

For more explanation of the relationship between ϵ and revenue, refer to Frank[12] Kou and Kou[22] used the constant elasticity function with $\epsilon > 1$. Kenyon and Cheliotis[20] used equation(2.1.1) as their price-demand function and showed that the trend of total market capacity is exponentially increasing and the price is exponentially decreasing with time in the telecommunications industry, which confirms that $\epsilon > 1$. Therefor, we assume $\epsilon > 1$ from now on.

3. Revenue.

$$R_t = P_t D_t = \left(\frac{\xi_t}{D_t}\right)^{\frac{1}{\epsilon}} D_t = \xi_t^{\frac{1}{\epsilon}} {D_t}^{1-\frac{1}{\epsilon}} = \xi_t^{\frac{1}{\epsilon}} {x_t}^{1-\frac{1}{\epsilon}},$$

where x_t is the firm's capacity at time t. We set the demand equal to the capacity level. In a monopolistic market, the firm can control the demand by changing the market price. When the demand is greater than the capacity level, which is possible under an improved economic condition, the firm can raise the price to decrease the demand to the capacity level, and vice versa.

4. Cost.

Cost consists of three parts: maintenance cost, expansion cost, and salvage value. We model these costs to be linear with capacity as follows.

$$C_t = b_t x_t + a_t(x_t - x_{t-1})_+ - \tilde{a}_t(x_{t-1} - x_t)_+,$$

$$\text{where } (\mathsf{A})_+ = \begin{cases} \mathsf{A} & \text{if } \mathsf{A} \geq 0, \\ 0 & \text{if } \mathsf{A} < 0. \end{cases} \tag{2.1.3}$$

b_t: the coefficient of the unit-maintenance cost at time t.

a_t: the coefficient of the unit-expansion cost at time t.

$\tilde{a}_t$: the coefficient of the unit salvage value at time t.

Bertola and Caballero[8] used linear cost structure similar to ours, and the cost coefficient is modeled by geometric Brownian motion and is given exogenously. Kenyon and Cheliotis[20] used a concave cost function considering economies of scale. Abel[1] used a convex cost function, which is used frequently in the manufacturing industry.

Remark 1. $b_t, a_t,$ and $\tilde{a}_t$ might be a function of the sector/industry-specific growth rate. As Kou and Kou[22] indicated, the growth rate of the growth stock can be represented by the industry-specific growth rate. Also, the growth rate is closely related to the growth rate of the labor force in the research and development (R&D) department. If the growth rate of the labor forth of the R&D department is large, the cost coefficient decreases, and if the labor force of the R&D department is small, then the cost coefficient will stay put with respect to time.

Here we assume that

(a) $b_t \geq b_{t+1}$, $a_t \geq a_{t+1}$ and $\tilde{a}_t \geq \tilde{a}_{t+1}$.

(b) $a_t \geq \tilde{a}_t$

The underlying reason of the assumption (a) is that the unit maintenance cost, unit installation cost, and the unit salvage value tend to decrease, which reflects technology improvement in the telecommunications market. Bertola and Caballero[8] used a negative value for the drift when they model the cost coefficient as a geometric Brownian motion. Kenyon and Cheliotis[20] assumed an exponentially decreasing price/cost trend and in their numerical experiment, the price/cost depreciates by half in two years.

The reasoning of assumption (b) is to incorporate irreversibility.

(a) If $a_t = \tilde{a}_t$, then the investment is totally reversible, which indicates that the unit purchasing price is the same as the unit salvage value.

(b) If $a_t > \tilde{a}_t$, then the investment is partially reversible, which is a dominant characteristic of the telecommunicationa and computer industry.

(c) If $\tilde{a}_t = 0$, then the investment is totally irreversible.

The cost of a period is directly dependent on the investment decision, which falls into one of the following three categories:

(a) SELL some part or all of the the capacity of the previous period. This will happen when the economic condition is very bad and/or if there is too much capacity from the previous period. In this case, we will get $\tilde{a}_t(x_{t-1} - x_t)$ as our selling off profit and still pay the maintenance cost $b_t x_t$ for the remaining capacity. Therefore, $C_t = b_t x_t - \tilde{a}_t(x_{t-1} - x_t)$.

(b) KEEP the previous period capacity. This will happen when the economic condition has not been changed much from the previous period, and we have enough capacity at hand. In this case, we need to pay the maintenance cost only. Therefore, $C_t = b_t x_t$.

(c) BUY some capacity and add to the capacity of the previous period. This case is dominant among all three cases in the telecommunications industry. This will happen when the economic condition improves, and we need to keep up with increased demand from the improved economic condition. It will also happen when the cost depreciation is steep, which results a drop in price and corresponding increases in demand. In this case, we need to pay the installation cost $a_t(x_t - x_{t-1})$ and the maintenance cost $b_t x_t$ as well. Therefore, $C_t = b_t x_t + a_t(x_t - x_{t-1})$.

5. Value Function.

The value function at time t is the maximum of the expected total discounted cash flow of a firm from time t to time horizon T.

$$
\begin{aligned}
V_t(x_{t-1}, \xi_t) &= Max_{(x_t, x_{t+1}, \cdots x_{T-1})} \mathsf{E}\left[\sum_{m=t}^{T-1} \left(e^{-r(m-t)} \left(e^{-r} R_t - C_t \right) \right) \right. \\
&+ \left. e^{-r(T-t)} F_T \left(x_{T-1}, \xi_T \right) \right],
\end{aligned}
$$

where r is the expected return in one period and $F_T\left(x_{T-1}, \xi_T\right)$ is the terminal value function. Here we assume that revenue is collected at the end of each period, and the cost is incurred at the beginning of each period, which explains e^{-r} in the front of R_t in the value function. We want to solve for a series of investment decisions from time t onwards. At the beginning of each period t, we are given a state (x_{t-1}, ξ_t), and we need to solve for the optimal capacity (x_t^*) of the current period to maximize the firm's expected total discounted cash flow from time t to time horizon T.

2.2 Modeling Assumptions

In this section, we summarize all the assumptions that we make in our model.

1. The price elasticity of demand, ϵ, is constant, and $\epsilon > 1$.

2. We assume that the market demand is the same as the capacity level of a firm.

3. $b_t \geq b_{t+1}$, $a_t \geq a_{t+1}$, and $\tilde{a}_t \geq \tilde{a}_{t+1}$, which reflects cost depreciation from improvements of technology.

4. $a_t \geq \tilde{a}_t$, which represents the partial reversibility of the investment.

5. η_t is the cost depreciation coefficient at time t with $\eta_t < 1$. In addition, maintenance cost, installation cost, and salvage value depreciate at the same rate as

$$(b_{t+1}, a_{t+1}, \tilde{a}_{t+1}) = \eta_t(b_t, a_t, \tilde{a}_t) \text{ with } \eta_t < 1 \text{ for all } t.$$

6. $F_T\left(x_{T-1}, \xi_T\right)$ is a concave function with respect to x_{T-1}, which guarantees the concavity of the value function at t. In addition, $F_T\left(x_{T-1}, \xi_T\right)$ is homogeneous, i.e.,

$$\forall \eta > 0, F_T\left(\eta x_{T-1}, \eta \xi_T\right) = \eta F_T\left(x_{T-1}, \xi_T\right).$$

With this assumption, the linearity between the optimal capacity and the economic condition is established.

2.3 The Structure of the Solution

The value function is the total expected discounted cash flow over time horizon T. Using Bellman's equation, the value function at t can be re-written as follows:

$$\begin{aligned} V_t(x_{t-1}, \xi_t) &= Max_{(x_t, x_{t+1}, \cdots x_{T-1})} \mathsf{E}\Bigg[\sum_{m=t}^{T-1} \Big(e^{-r(m-t)} \left(e^{-r} R_t - C_t\right)\Big) \\ &+ e^{-r(T-t)} F_T\left(x_{T-1}, \xi_T\right)\Bigg] \\ &= Max_{x_t}\Bigg[\bigg(e^{-r}\xi_t^{\frac{1}{\epsilon}} x_t^{1-\frac{1}{\epsilon}} - \left(b_t x_t + a_t(x_t - x_{t-1})_+ - \tilde{a}_t(x_{t-1} - x_t)_+\right)\bigg) \\ &+ e^{-r}\mathsf{E}\big[\left(V_{t+1}\left(x_t, \xi_{t+1} | \xi_t\right)\right]\Bigg] \end{aligned}$$

To solve the problem, at first, let us define the G_b and G_s functions[1] as a derivative of the value function of the BUY $x_{t-1} \leq x_t$ and SELL $x_{t-1} \geq x_t$ cases as

$$G_b(x_t, \xi_t) = \left(e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(1 - \frac{1}{\epsilon}\right) x_t^{-\frac{1}{\epsilon}} - (b_t + a_t)\right) + e^{-r}\frac{d\mathsf{E}\big[V_{t+1}\left(x_t, \xi_{t+1}|\xi_t\right)\big]}{dx_t}, \text{ and} \quad (2.3.1)$$

$$G_s(x_t, \xi_t) = \left(e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(1 - \frac{1}{\epsilon}\right) x_t^{-\frac{1}{\epsilon}} - (b_t + \tilde{a}_t)\right) + e^{-r}\frac{d\mathsf{E}\big[V_{t+1}\left(x_t, \xi_{t+1}|\xi_t\right)\big]}{dx_t}. \quad (2.3.2)$$

The solution to the above optimization problem belongs to one of the following three cases.

1. (BUY):

   Expand the current capacity. If $x^*_{t,L}$ is a solution for $G_b(x_t, \xi_t) = 0$ and if $x_{t-1} \leq x^*_{t,L}$, a firm needs to buy more capacity up to $x^*_{t,L}$.

2. (SELL)

   Cut off the excess capacity. If $x^*_{t,U}$ is a solution for $G_s(x_t, \xi_t) = 0$ and if $x_{t-1} \geq x^*_{t,U}$, a firm needs to sell the excess capacity down to $x^*_{t,U}$.

3. (KEEP)

   Continue with the capacity of the previous period. If $x^*_{t,L} \leq x_{t-1} \leq x^*_{t,U}$, then $x^*_{t,K} = x_{t-1}$.

[1] The subscript b in G_b means BUY and s in G_s means SELL.

Here we used the fact $x^*_{t,L} \leq x^*_{t,U}$, which will be explained in Lemma 2.3.2.

To prove that there is a solution to the above set of equations (2.3.1),(2.3.2) and that the solutions are unique, we first investigate the behavior of $\frac{d\mathsf{E}[V_{t+1}(x_t,\xi_{t+1}|\xi_t)]}{dx_t}$.

Theorem 2.3.1. *If $V_{t+1}(x_t, \xi_{t+1})$ is a concave function with respect to x_t, then $V_t(x_{t-1}, \xi_t)$ is a concave function with respect to x_{t-1}.*

Also,

$$\tilde{a}_{t+1} \leq \frac{d\mathsf{E}\left[V_{t+1}\left(x_t, \xi_{t+1}|\xi_t\right)\right]}{dx_t} \leq a_{t+1}.$$

Proof. See the proof in section 2.7. □

Abel[1], Abel and Eberly[2], and Grenadier[14] assumed that the revenue function is concave, which is consistent to our model. The intuition behind Theorem 2.3.1 is as follows: If the capacity of the previous period is greater than or equal to $x^*_{t+1,U}$, having one more capacity has an $\tilde{a}_{t+1}$ value. If the capacity of the previous period is less than $x^*_{t+1,L}$, having one more capacity has an a_{t+1} value. If the capacity of the previous period is between $x^*_{t+1,L}$ and $x^*_{t+1,U}$, then the value of the additional capacity in decreasing.

Aguerrevere[4] explained the underlying reason for Theorem 2.3.1 well using the real option concept. He interpreted $\frac{dV_{t+1}(x_t,\xi_{t+1})}{dx_t}$ as the value of a marginal unit of capacity and/or value of option to purchase an additional unit of capacity with a current capacity level of x_t. The value of the additional capacity is maximal when the current capacity is not adequate. If we have sufficient capacity, then the value of the additional capacity is minimal.

Lemma 2.3.2. *Solution exist for the equations (2.3.1) and (2.3.2) and they are unique. In addition, $x^*_{t,L} \leq x^*_{t,U}$ (equality is satisfied when $a_t = \tilde{a}_t$).*

Proof. Using the G_b and G_s functions, we can prove that solutions exist:

$$\lim_{x_t \downarrow 0} G_b(x_t, \xi_t) \to \infty$$
$$\lim_{x_t \uparrow \infty} G_b(x_t, \xi_t) \simeq -b_t - a_t + e^{-r}\tilde{a}_{t+1} < 0$$
$$\lim_{x_t \downarrow 0} G_s(x_t, \xi_t) \to \infty$$
$$\lim_{x_t \uparrow \infty} G_s(x_t, \xi_t) \simeq -b_t - \tilde{a}_t + e^{-r}\tilde{a}_{t+1} < 0$$

Therefore, solutions exist for the equation (2.3.1) and (2.3.2).

Proposition 2.3.3. *Here we use*

$$\lim_{x_t \uparrow \infty} \frac{d\mathsf{E}\big[V_{t+1}(x_t, \xi_{t+1}|\xi_t)\big]}{dx_t} = \tilde{a}_{t+1}. \tag{2.3.3}$$

The reason for the above equality is that as the capacity of the previous period goes to ∞, the decision of this period tends to be ***SELL****.*

On the other hand,

$$\lim_{x_t \downarrow 0} \frac{d\mathsf{E}\big[V_{t+1}(x_t, \xi_{t+1}|\xi_t)\big]}{dx_t} = a_{t+1}. \tag{2.3.4}$$

As the capacity of the previous period goes to zero, the decision of this period tends to be BUY.

Next, with large enough T, we can ignore the effect of the terminal value function F_T on our investment decision at the current period. Then we can choose any concave function for F_T, say $F_T(x_{T-1}, \xi_T|\xi_{T-1}) = 0$ or $F_T(x_{T-1}, \xi_T|\xi_{T-1}) = \tilde{a}_T x_{T-1}$. Then we can use theorem 2.3.1 to prove the uniqueness of the solution. If we take the derivative of the G_b and G_s functions, then we will have

$$\frac{dG_b}{dx_t} = \frac{dG_s}{dx_t} = -\frac{1}{\epsilon}\left(1 - \frac{1}{\epsilon}\right) e^{-r} \xi_t^{\frac{1}{\epsilon}} x_t^{-\frac{1}{\epsilon}-1} + \frac{d^2\mathsf{E}\big[V_{t+1}(x_t, \xi_{t+1}|\xi_t)\big]}{dx_t^2} < 0,$$

here we used the concavity of the value function. Therefore, $G_b(x_t, \xi_t)$ and $G_s(x_t, \xi_t)$ are monotonically decreasing functions, which guarantees the uniqueness of the solutions for the equations (2.3.1) and (2.3.2).

Finally, by using the fact that $G_b(x_t, \xi_t)$ and $G_s(x_t, \xi_t)$, are monotonically decreasing functions with respect to x_t, we can prove that $x^*_{t,L} < (=) x^*_{t,U}$ from the assumption $a_t > (=) \tilde{a}_t$. □

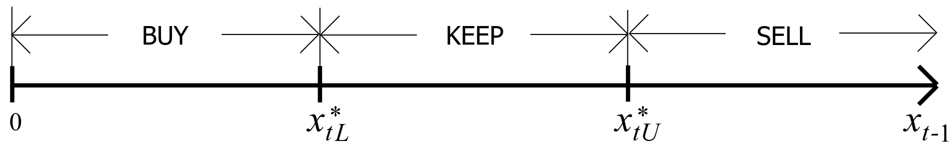

Figure 2: Solution space with respect to the capacity of the previous period

As we can see in Figure 2, we can divide the whole space of x_{t-1} into three pieces as follows:

1. $x_{t-1} \leq x^*_{t,L}$: at time t, the firm is in the BUY region, and it needs to increase the capacity up to $x^*_{t,L}$.
2. $x^*_{t,L} \leq x_{t-1} \leq x^*_{t,U}$: at time t, the firm is in the KEEP region, and it maintains the capacity of the previous period.
3. $x^*_{t,U} \leq x_{t-1}$: at time t, the firm is in the SELL region, and it needs to sell off the excess capacity down to $x^*_{t,U}$.

Remark 2. Assumption (b), which is ($a_t \geq \tilde{a}_t$), explains that the purchase price and the salvage value are different. This reflects a partially reversible investment. This partial reversibility divides our solution space into three pieces: BUY, KEEP, and SELL regions.

1. If we set $a_t > \tilde{a}_t \neq 0$, then the investment is partially reversible, and we will have all three regions.
2. If we set $a_t = \tilde{a}_t$, then the investment is totally reversible. In this case, there is no KEEP region in our solution space. Depending on the economic condition, we always have to buy or sell the capacity.
3. If we set $\tilde{a}_t = 0$, then we barely sell the current capacity. Therefore, our solution has no SELL region.

Theorem 2.3.4. *For $t = 1, ..., T$, suppose $V_{t+1}(x_t, \xi_{t+1})$ is homogeneous, i.e.,*

$$\forall \eta > 0, V_{t+1}(\eta x_t, \eta \xi_{t+1}) = \eta V_{t+1}(x_t, \xi_{t+1}).$$

Then

1. If the lower bound (upper bound) is $x^*_{t,L}(x^*_{t,U})$ *for* $\xi_t = \xi$*, then for* $\forall \eta > 0$*, the lower bound (upper bound) for* $\xi_t = \eta\xi$ *is* $\eta x^*_{t,L}(\eta x^*_{t,U})$*.*

2. Furthermore, $V_t(x_{t-1}, \xi_t)$ *is also homogeneous.*

Proof. See the proof in Baryshnicov, Sim, and Wang[6]. □

According to Theorem 2.3.4, $x^*_{t,L}$ and $x^*_{t,U}$ are linear functions of the economic indicator ξ_t. Therefore, we can write the optimal solutions as follows:

$$\begin{aligned} x^*_{t,L}(x_{t-1}, \xi_t) &= l_{t,L}\xi_t \\ x^*_{t,U}(x_{t-1}, \xi_t) &= l_{t,U}\xi_t, \end{aligned}$$

where $l_{t,L}$, $l_{t,U}$ are the functions of other parameters except the economic indicator (ξ_t).

Figure 3 illustrates a possible solution structure. In the left-hand graph, the x-axis represents economic indicator (ξ_t) at the beginning of period t, the y-axis represents the capacity (x_{t-1}) of the previous period and z-value represents optimal capacity $x^*_t(x_{t-1}, \xi_t)$ at the tth period. The right-hand graph is a projection of the left-hand graph. The figure shows a clear division of space into three regions: the BUY, KEEP, and SELL regions. Let us fix the economic indicator. Then, as capacity increases, the region shifts from BUY to KEEP and from KEEP to SELL. Next, let us fix the capacity. Then as economic indicator increases, the result changes from SELL to KEEP and from KEEP to BUY, which is intuitively very accurate.

Remark 3. If we determine the value of the slopes of these two boundary lines, BUY/KEEP, KEEP/SELL, the optimization problem is solved for every possible state of (x_{t-1}, ξ_t). Therefore, this optimization problem has been reduced to the problem of finding these two slopes.

Lemma 2.3.5. $l_{t,L}$ *and* $l_{t,U}$ *are the slopes for the boundaries of BUY/KEEP and KEEP/SELL,*

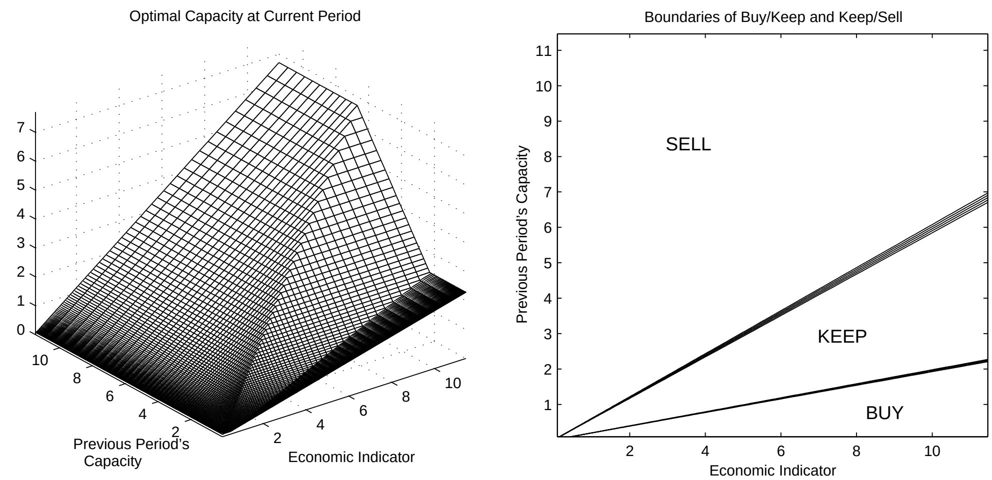

Figure 3: Linearity between Optimal Capacity and Economic Indicator

respectively, and these slopes are bounded.

$$\left(\frac{e^{-r}(1-\frac{1}{\epsilon})}{b_t+a_t-e^{-r}\tilde{a}_{t+1}}\right)^{\epsilon} \leq l_{t,L} \leq \left(\frac{e^{-r}(1-\frac{1}{\epsilon})}{b_t+a_t-e^{-r}a_{t+1}}\right)^{\epsilon}$$

$$\leq \left(\frac{e^{-r}(1-\frac{1}{\epsilon})}{b_t+\tilde{a}_t-e^{-r}\tilde{a}_{t+1}}\right)^{\epsilon} \leq l_{t,U} \leq \left(\frac{e^{-r}(1-\frac{1}{\epsilon})}{\max(b_t+\tilde{a}_t-e^{-r}a_{t+1},0^{+})}\right)^{\epsilon}$$

Proof. Using Theorem 2.3.1 with $a_t \geq \tilde{a}_t$ (buy-sell gap) and $(b_t, a_t, \tilde{a}_t) \geq (b_{t+1}, a_{t+1}, \tilde{a}_{t+1})$ (cost depreciation), the proof is straightforward. □

The idea behind the limits of the slopes for the boundaries are as follows:

1. When $l_{t,L}$ attains the lower bound: Regardless of the economic condition at $t+1$, the company should be in the BUY region at t and in the SELL region at $t+1$. However, this is very rare case and not very plausible.

2. When $l_{t,L}$ attains the upper bound: Regardless of the economic condition at $t+1$, the company should be in the BUY region at t and in the BUY region at $t+1$. In the telecommunications industry, the unit cost for capacity/capital depreciates quite rapidly. In this case, the company might purchase more capacity even though the expectation of the future general economic condition is bad.

3. When $l_{t,U}$ attains the lower bound: Regardless of the economic condition at $t+1$, the company should be in the SELL region at t and in the SELL region at $t+1$. If in some industries, the unit installation/maintenance cost has an increasing trend, the company tends to sell the current capacity to take advantage of the increased salvage value when the general economic condition is bad.

4. When $l_{t,U}$ attains the upper bound: Regardless of the economic condition at $t+1$, the company should be in the SELL region at t and in the BUY region at $t+1$. However, this is very rare case and not very plausible.

2.4 Sensitivity Analysis

From the previous section, we know that the lower and upper optimal capacities have a linear relationship with the current economic indicator. Therefore, if we find the analytical form of the slopes, then the problem is solved completely. However, even though there is no analytical solution for the slopes in general, we can investigate how the slopes change with respect to the parameters.

2.4.1 Cost Parameters vs. the Values of the Slopes

At first, we study the effect of the cost parameters on the values of the slopes. As we can expect, the investment decision becomes conservative as the cost parameter increases.

Lemma 2.4.1. *Let lower bound (upper bound) be $x^*_{t,L}(x^*_{t,U})$ for the cost parameters $(b_t, a_t, \tilde{a}_t)$.*

If we scale the cost parameters as $(\eta b_t, \eta a_t, \eta \tilde{a}_t)$ for all $t \in \{0, 1, ..., T\}$, then for $\forall \eta > 0$, the lower bound (upper bound) is $\eta^{-\epsilon} x^*_{t,L}(\eta^{-\epsilon} x^*_{t,U})$.

Proof. With the new cost parameters $(\eta b_t, \eta a_t, \eta \tilde{a}_t)$, the value function at time t is

$$\begin{aligned}
\bar{V}_t(x_{t-1}, \xi_t) &= Max_{x_t}\Bigg[\Bigg(e^{-r}\xi_t^{\frac{1}{\epsilon}} x_t^{1-\frac{1}{\epsilon}} - (\eta b_t x_t + \eta a_t(x_t - x_{t-1})_+ - \eta\tilde{a}_t(x_{t-1} - x_t)_+)\Bigg) \\
&+ e^{-r}\mathsf{E}\big[\bar{V}_{t+1}\left(x_t, \xi_{t+1}|\xi_t\right)\big]\Bigg] \\
&= Max_{x_t}\Bigg[\eta\Bigg(e^{-r}\left(\eta^{-\epsilon}\xi_t\right)^{\frac{1}{\epsilon}} x_t^{1-\frac{1}{\epsilon}} - (b_t x_t + a_t(x_t - x_{t-1})_+ - \tilde{a}_t(x_{t-1} - x_t)_+)\Bigg) \\
&+ e^{-r}\mathsf{P}\left\{\xi_{t+1}|\xi_t\right\}\bar{V}_{t+1}\left(x_t, \xi_{t+1}\right)\Bigg] \\
&= Max_{x_t}\Bigg[\eta\Bigg(e^{-r}\left(\eta^{-\epsilon}\xi_t\right)^{\frac{1}{\epsilon}} x_t^{1-\frac{1}{\epsilon}} - (b_t x_t + a_t(x_t - x_{t-1})_+ - \tilde{a}_t(x_{t-1} - x_t)_+)\Bigg) \\
&+ e^{-r}\eta\mathsf{P}\left\{\eta^{-\epsilon}\xi_{t+1}|\eta^{-\epsilon}\xi_t\right\}V_{t+1}\left(x_t, \eta^{-\epsilon}\xi_{t+1}\right)\Bigg] \\
&= \eta V_t\left(x_{t-1}, \eta^{-\epsilon}\xi_t\right).
\end{aligned}$$

If the lower (upper) optimal capacity is $x^*_{t,L}(x^*_{t,U})$ when the economic condition is ξ_t, then the lower/upper optimal capacity is $\eta^{-\epsilon}x^*_{t,L}(\eta^{-\epsilon}x^*_{t,U})$ when the economic condition is $\eta^{-\epsilon}\xi_t$ because of the linearity between the economic indicator and the optimal capacity. □

With $\eta < 1$, we have increased the lower and upper bounds and expanded KEEP region. In addition, the KEEP region is given by $(\eta^{-\epsilon}x^*_{t,L}, \eta^{-\epsilon}x^*_{t,U}) = \eta^{-\epsilon}(x^*_{t,L}, x^*_{t,U})$, which is broader than the original KEEP region with $\eta < 1$ and $\epsilon > 1$. In addition to the η effect, ϵ enforces this change. With large ϵ, the investment decision in more sensitive to the cost parameters.

Let us explain the intuition behind Lemma 2.4.1. The price is directly related to the cost parameters. Therefore, the lowered cost leads to the price reduction, which incurs more demand in the market. Moreover, customers respond more sensitively to the change of cost parameters with a larger value of ϵ. Therefore, the demand increases with smaller η and this increase is enlarged with ϵ.

Figure 4 shows the changes in slopes with respect to cost changes. In the example, a cost change is realized as a cost depreciation with time. We used the same cost depreciation factor $\eta_t = \eta$ for all $t \in \{0, 1, ..., T\}$. Therefore, the cost parameters at time t can be written as

$$(b_t, a_t, \tilde{a}_t) = \eta^t(b_0, a_0, \tilde{a}_0).$$

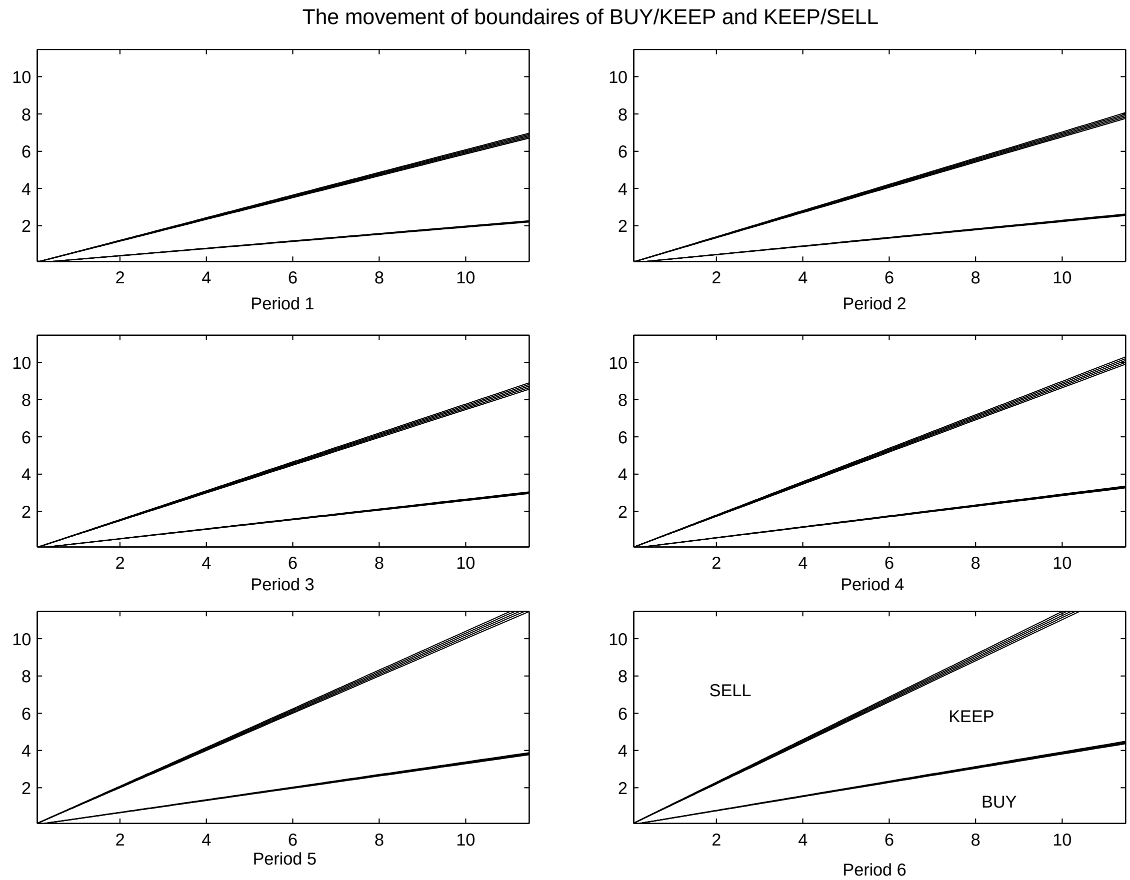

Figure 4: Movement of the boundaries of the BUY/KEEP and KEEP/SELL regions from $t = 1$ to $t = 6$. The parameters are $\eta = 0.5^{1/8}$ and $\epsilon = 1.5$, respectively

At time period 1, the decision is made with cost parameters $\eta(b_0, a_0, \tilde{a}_0)$ and at period 2, the decision is made with cost parameters $\eta^2(b_0, a_0, \tilde{a}_0)$, and so on. For the result, we used $\eta = 0.5^{\frac{1}{8}}$ and $\epsilon = 1.5$. The other parameters will be explained in subsection 2.6.2.

Table 1 lists the numerical results with the corresponding analytical values of the slopes. As will be explained in Theorem 2.5.1, in this case, we can obtain the analytic value of the slope for the boundary BUY/KEEP, but we do not have analytic expression for the slope for the boundary KEEP/SELL. Thus, assuming that $l_{1,U}$ is given by the numerical value, we calculate the slopes of upper bounds in the other periods. Even though small difference between numerical results and analytical values exists, which results from the discretization of the state space, these results validate the relationship between cost parameters and the slopes.

Table 1: Values of the slopes with cost depreciation

	Numerical Results		Analytic Results	
Period	BUY/KEEP	KEEP/SELL	BUY/KEEP	KEEP/SELL
1	0.1999	0.5847	0.2042	.
2	0.2314	0.6768	0.2325	0.6658
3	0.2678	0.7642	0.2648	0.7582
4	0.2953	0.8638	0.3015	0.8635
5	0.3418	1.0000	0.3433	0.9833
6	0.3957	1.1025	0.3910	1.1198

Remark 4. We use $\eta_t = \eta$ for all $t \in \{0, 1, ..., T\}$ and same cost depreciation rate to the installation cost, the maintenance cost, and the salvage value. This use might be true if the industry is stable and the corresponding industry indicator has upward trend with small volatility. However, if the growth rate of the industry is not stable but very volatile, η_t must be dependent on time.

2.4.2 Price Elasticity of Demand vs. the Values of the Slopes

Next, we study the effect of price elasticity of demand ϵ on the values of the slopes. The effect of ϵ cannot be determined in general. Its effect is correlated with cost parameters. Therefore, we can define several regions of cost parameters that provide different relationship between the values of slopes and ϵ. The slopes of the boundaries for $\mathsf{BUY/KEEP}$ and $\mathsf{KEEP/SELL}$ can be written approximately as

$$\begin{aligned} l_{t,L} &= \left(\mathsf{A}(t)\left(1 - \frac{1}{e}\right)\right)^{\epsilon} \\ l_{t,U} &= \left(\mathsf{B}(t)\left(1 - \frac{1}{e}\right)\right)^{\epsilon}. \end{aligned}$$

Lemma 2.4.2. *1. If $\mathsf{A}(t) \geq 1$ and $\mathsf{B}(t) \geq 1$ are satisfied, then the slopes are increasing functions with ϵ.*

2. If $\mathsf{A}(t) \geq 1$ and $\mathsf{B}(t) \geq 1$ are not satisfied, then there exists a maximum point where ϵ attains a maximum slope. Let us define a function

$$g(\epsilon) \equiv \frac{\epsilon e^{\frac{1}{1-\epsilon}}}{\epsilon - 1}.$$

Given $\mathsf{A}(t) < 1$, up to a point, which is given by $g^{-1}(\mathsf{A})$, the slope is increasing and then is decreasing thereafter.

3. *If we consider a possible range for ϵ, say (ϵ_L, ϵ_U) then,*

$$\mathsf{A}(t) \geq \frac{\epsilon_U e^{\frac{1}{1-\epsilon_U}}}{\epsilon_U - 1}$$

is the requited condition for the slopes to increase in the given range.

Proof. See the proof in section 2.7. □

When we confine $1 \leq \epsilon \leq 2.84$[2], then $\mathsf{A}(t) \geq 0.896$ is satisfactory for the slopes to increase with respect to ϵ.

Table 2 shows the numerical results of the relationship between the values of slopes and ϵ. In our cost parameter set, $\mathsf{A}(t) > 1$, so the results shows increasing trend only. For the values of the parameters, refer to subsection 2.6.2.

Table 2: Values of the slopes with ϵ

ϵ	BUY/KEEP	KEEP/SELL
1.2	0.1227	0.2812
1.4	0.1813	0.4810
1.6	0.2204	0.6768
1.8	0.2429	0.8638
2.0	0.2678	1.1025
2.2	0.2812	1.3401
2.4	0.2953	1.6289
2.6	0.3101	1.9799
2.8	0.3256	2.2920

2.4.3 Variance of the Economic Condition vs. the Values of the Slopes

In this section, we establish the relationship between the variance of the economic indicator and the values of the slopes. In doing so, we explore the effects of variance in the economic condition on the optimal capacity in two cases: with cost depreciation and with no cost depreciation. As we addressed in the previous section, with cost depreciation, we can

[2]for this values for ϵ, refer to Lanning, Mitra, Wang, and Wright[25]

delay our investment to take advantage of the reduced cost, but we should consider the revenue loss due to inadequate capacity in the current period. Therefore, the firm should decide the investment considering the trade off between cost depreciation and revenue and the variance of the future economic condition at the same time.

Table 3 shows how the slopes change with the variance in the economic condition. The parameters are

1.
$$\mathsf{E}\left(\frac{\xi_{t+1}}{\xi_t} - 1 \middle| \xi_t\right) = 1.025 \qquad \mathrm{Var}\left(\frac{\xi_{t+1}}{\xi_t} \middle| \xi_t\right) = 0.0506 \backsim 0.1552$$

2.
$$a_0 = 4 \qquad \tilde{a}_0 = 0.5 \cdot a_0 \qquad b_0 = 0.15 \cdot a_0 \qquad \eta = 0.5^{\frac{1}{8}}.$$

For the parameters used here, refer to subsection 2.6.2.

Table 3: Values of the slopes with a variance of economic indicator

Variance	With cost depreciation		Without cost depreciation	
	BUY/KEEP	KEEP/SELL	BUY/KEEP	KEEP/SELL
0.0506	0.1897	0.5789	0.2445	0.6224
0.0768	0.1716	0.5919	0.2345	0.6673
0.1029	0.1587	0.5986	0.2187	0.7155
0.1290	0.1501	0.6121	0.2011	0.7461
0.1552	0.1403	0.6189	0.1902	0.7780

In the derivatives of the value function, G_b and G_s, we have the following term:

$$f(x) \equiv \left.\frac{d\mathsf{E}\left[V_{t+1}\left(x_t, \xi_{t+1}|\xi_t\right)\right]}{dx_t}\right|_x. \tag{2.4.1}$$

Using Theorem 2.3.1, which addresses the concavity of the value function, we can explain the trend of the slopes with the variance of economic condition.

Under cost depreciation, the slope for the boundary BUY/KEEP has a smaller value for the larger variance, and the slope for the boundary SELL/KEEP does not have deterministic relationship with the variance of the economic indicator.

As the variance of ξ_{t+1} increases, function (2.4.1) is likely to decrease at $x = x^*_{t,L}$. To make up this change, the optimal capacity for BUY should shift to lower value. Therefore, the investment decision becomes more conservative for the BUY case.

Value of function (2.4.1) at $x = x^*_{t,U}$ does not provide any clear trend with the variance of economic indicator. Therefor, we can not determine the relationship between the optimal capacity and the variance of economic indicator for the SELL case in general. In this case, the relationship between the slopes of the boundary $\mathsf{SELL/KEEP}$ and the variance of the economic indicator is determined by how steep the cost depreciation is and the third derivative of the value function around the point $(x^*_{t,U}, \xi_t)$.

Under no cost depreciation, the slope for the boundary $\mathsf{BUY/KEEP}$ has a smaller value for the larger variance and the slope for the boundary $\mathsf{SELL/KEEP}$ has a larger value for the larger variance. In the BUY case, equation (2.4.1) has a lower value at $x = x^*_{t,L}$ and in the SELL case, equation (2.4.1) has a higher value at $x = x^*_{t,U}$. To make up these changes, the optimal capacity for BUY and SELL should shift to a lower value and a higher value, respectively. Therefore, as variance increases, the range of optimal capacity expands. In other words, the KEEP region expands, which means the investment decision becomes more conservative as the variance increases.

Figure 5 shows the derivatives of the value function. The left-hand side graph represents the case with cost depreciation and the right-hand side graph represents the case without cost depreciation. With cost depreciation, the optimal lower and upper bounds move to right with time, which explains the change of equation (2.4.1) with the variance of the economic condition. Without cost depreciation, the optimal lower and upper bounds stay at the same point, which also explains the change of equation (2.4.1) with the variance of the economic condition accurately.

Remark 5. Table 3 shows that the no depreciation case has larger slope values for $\mathsf{BUY/KEEP}$. With cost depreciation, the firm can take advantage of the reduced cost when it delay the investment. However, without cost depreciation, there is no cost reduction for an investment in the next period, which leads the firm to invest this period to make more

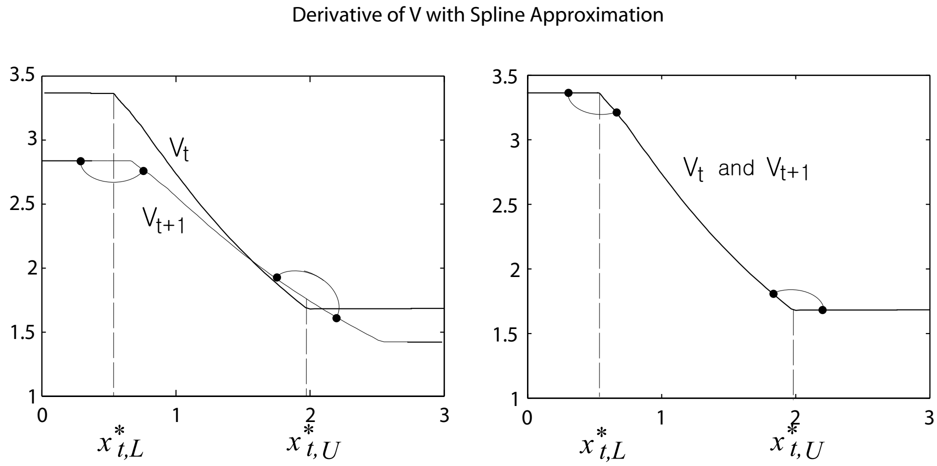

Figure 5: Derivative of value function with spline approximation

revenue. With cost depreciation, they want to sell more during this period to earn more profits and interest from the higher salvage value in this period than the lowered salvage value in the next period, which explains the smaller slope values for KEEP/SELL in the case with cost depreciation.

2.5 Incremental Investment with Cost Depreciation

In the telecommunications market, even if the future economic condition is expected to be bad, the use of bandwidth and the number of mobile phone subscribers might increase due to the reduced market price, because of improvements in technology, which lower unit costs. During the last few years, the economic condition has not improved all the time. However, the number of mobile phone and Internet users has continued to increase. In fact, according to the World Telecommunication Development Report [17], the number of mobile phone and Internet users has increased very rapidly worldwide. The data show this trend regardless of the economic condition. In addition, Kenyon and Cheliotis[21] stated that the Internet and network bandwidth have had periods of 100% growth every three or four months and during the past few years, the growth has slowed to 100% per year.

In this section, we investigate the causes of this trend in the market and the relationships between this trend and the investment behavior of firms.

At first, we examine the conditions under which incremental investment decisions are made.

Theorem 2.5.1. *Let us assume that the cost depreciates exponentially with time as* $b_{t+1} = \eta b_t$, $a_{t+1} = \eta a_t$, *and* $\tilde{a}_{t+1} = \eta \tilde{a}_t$ *for all* t, *with* $\eta < 1$. *If* $\eta^\epsilon \leq \frac{\xi_{t+1}}{\xi_t}$ *with probability 1, then*

1. *We can find an analytical function for the slope of the boundary between* BUY/KEEP, *which is*
$$l_{t,L} = \left(\frac{e^{-r}(1 - \frac{1}{\epsilon})}{b_t + a_t - e^{-r} a_{t+1}} \right)^\epsilon .$$

2. *If the firm is in the* BUY *region in the current period, the firm will be in the* BUY *region in the following period regardless of the future economic condition.*

3. *The investment decision of the current period is not dependent on the investment decision of the following period.*

Proof. The proof is in section 2.7. □

Remark 6. Under this assumption:

1. We have an analytic expression for the slope of the boundary of BUY/KEEP, which is
$$x^*_{t,L} = \left(\frac{e^{-r}\left(1 - \frac{1}{\epsilon}\right)}{a_t + b_t - e^{-r} a_{t+1}} \right)^\epsilon \xi_t = \mathsf{A}t)\xi_t,$$
where $\mathsf{A}(t)$ is deterministic. Therefore, $x^*_{t,L}$ is a stochastic process that is directly dependent on the movement of the general economic condition.

2. If $\eta^\epsilon \leq \frac{\xi_{t+1}}{\xi_t}$ with probability 1 is true for all t, the investment decision in one period is independent of the investment decision of the other periods. In this case, this multiple-time investment problem can be reduced to a one-time period investment problem.

3. The firm should increase capacity regardless of the future economic condition. In the extreme case, even if the future economic condition is likely to be bad, the firm still wants to buy. The increment is stochastic and is dependent only on the

current economic condition. Therefore, the increment is small if the current economic condition is bad and large if the current economic condition is good.

4. The driving force for buying comes from ϵ, the cost depreciation, and the economic condition of the following period. To satisfy the assumption, we need small η, large ϵ, and/or large $\frac{\xi_{t+1}}{\xi_t}$ for the worst case transition. Small η reduces the cost and forces the firm to buy more capacity. If ϵ is large, then the revenue increases with a lower price and increased capacity, and this effect accelerates with larger ϵ.

In the telecommunications market, technology improves very rapidly, which drives cost reductions year after year. Let $t_{\frac{1}{2}}$ be the time period during which unit costs reduced to half, η be the cost reduction in a period, and one period be a quarter,

$$\begin{aligned} \eta^{4 \cdot t_{\frac{1}{2}}} &= 0.5 \\ \eta &= 0.5^{\frac{1}{4 \cdot t_{\frac{1}{2}}}}. \end{aligned}$$

From [20] and [21], we set the possible range of $t_{1/2}$ between one and two years. From [35], the traditional elasticity estimate for voice traffic is approximately 1.05 and for data traffic, in the range of 1.3 - 1.7. Therefore,

$$0.7448 \leq \eta^{\epsilon} \leq 0.9130$$

Thus, if the worst possible transition of the general economic condition during one quarter is more than -9.7%, then the assumption is satisfied.

Table 4 shows numerical results. For the parameters used here, refer to subsection 2.6.2. We can separate the result into two cases: the first is when $\eta^{\epsilon} < \frac{\xi_{t+1}}{\xi_t}$ is satisfied, and the other is when the condition is not satisfied.

1. $\eta = (0.5)^{\frac{1}{4}}, (0.5)^{\frac{1}{6}}, (0.5)^{\frac{1}{8}}$ or $(0.5)^{\frac{1}{16}}$

   • We can observe that the investment decision is independent of the future economic condition. The values of the slope do not change with the probability of upward movement.

   • If the state is in the BUY region at period 1, then it is in the BUY region at period

Table 4: Slopes of the boundary of BUY/KEEP with the probability of upward movement

η	$(0.5)^{\frac{1}{4}}$	$(0.5)^{\frac{1}{6}}$	$(0.5)^{\frac{1}{8}}$	$(0.5)^{\frac{1}{16}}$	$(0.5)^{\frac{1}{32}}$	1
At Period 1						
p=0	0.1566	0.1813	0.1999	0.2429	0.2204	0.1904
p=0.5	0.1566	0.1813	0.1999	0.2429	0.2678	0.2678
p=1	0.1566	0.1813	0.1999	0.2429	0.2812	0.3101
Analytic Value	**0.1586**	**0.1849**	**0.2042**	**0.2462**	**.**	**.**
At Period 2						
p=0	0.2099	0.2204	0.2314	0.2678	0.2314	0.1904
p=0.5	0.2099	0.2204	0.2314	0.2678	0.2812	0.2678
p=1	0.2099	0.2204	0.2314	0.2678	0.2812	0.3101
Analytic Value	**0.2057**	**0.2199**	**0.2325**	**0.2628**	**.**	**.**

2, too, because $l_{1,L}\xi_1 < l_{2,L}\xi_2$ with the worst case of ξ_2, which is $\xi_1(1+\Delta)^{-1}$.

• In this case, the numerical result is compatible with analytic values with small discretization error.

• The slopes of periods 1 and 2 are independent and can be calculated separately. This independence will continue throughout the other periods.

2. $\eta = (0.5)^{\frac{1}{32}}$ or 1

   • The investment decision is very sensitive to future economic conditions.

   • For the case $\eta = 1$, no depreciation occurs and the slopes do not change with time.

The next two subsections will illustrate the actual movement of the optimal capacity and the resulting price under the assumption that the cost reduction is steep enough to satisfy $\eta^\epsilon < \frac{\xi_{t+1}}{\xi_t}$ for all possible transitions of the economic indicator.

2.5.1 Capacity Trend

Under the condition $\eta^\epsilon < \frac{\xi_{t+1}}{\xi_t}$ with probability 1 for all t, the capacity increment decisions fall into one of the following three categories:

1. The company starts with a small capacity at time 0 and increases the capacity to $\left(\frac{e^{-r}(1-\frac{1}{\epsilon})}{b_1+a_1-a_2}\right)^\epsilon \xi_1$ at time 1 and continues to increase capacity. The decision depends only on the realization of the uncertain economic condition with the deterministic

coefficient. The optimal capacity at time t is

$$x^*_{t,L} = \left(\frac{e^{-r}(1-\frac{1}{\epsilon})}{b_t + a_t - a_{t+1}}\right)^{\epsilon} \xi_t = \eta^{-(t-1)\epsilon}\left(\frac{e^{-r}(1-\frac{1}{\epsilon})}{b_1 + a_1 - a_2}\right)^{\epsilon} \xi_t.$$

2. The company starts with a moderate capacity at time 0 and stays in the KEEP region for a while. After some periods, it falls into the BUY region and keeps increasing the capacity from period to period. The length of the time of staying in the KEEP region depends on the actual movement of the economic indicator and the initial capacity at time 0.

3. The company starts with too much capacity at time 0 but cuts the capacity to $l_{1,U}\xi_1$ at time 1 and stays in the KEEP region for a while. After some period, it falls into the BUY region and continues to increase capacity from period to period. The length in the KEEP region depends on the actual movement of the economic indicator.

Figure 6 shows the actual movement of capacity and the price when $\eta^{\epsilon} < \frac{\xi_{t+1}}{\xi_t}$ with probability 1. The solid line represents the analytical value and the o represents numerical results.

1. According to the first graph, the firm starts lower than the lower optimal capacity. After one period, it jumps on the KEEP/BUY boundary and continues to follow the optimal capacity trend.

2. According to the second graph, the firm starts with higher than the lower optimal capacity but lower than upper optimal capacity. After the firm spends some periods in KEEP regions, it gets on the KEEP/BUY boundary and continues to follow the optimal capacity trend.

3. According to the third graph, the firm starts with more than the upper optimal capacity. After the selling off the excess capacity at period 1, it goes into the KEEP region. After going through several periods in KEEP region, the firm finally gets onto the KEEP/BUY boundary and continues to follow the optimal capacity trend.

2.5.2 Price Trend

Under the condition $\eta^\epsilon < \frac{\xi_{t+1}}{\xi_t}$ for all possible transitions of the economic indicator, we can obtain a trajectory of the price. Employing $x^*_{t,L}$ as our market capacity, the price function is given by

$$P_t(x^*_{t,L}, \xi_t) = \left(\frac{\xi_t}{x^*_{t,L}}\right)^{\frac{1}{\epsilon}} \text{ with } x^*_{t,L} = \left(\frac{e^{-r}(1-\frac{1}{\epsilon})}{b_t + a_t - e^{-r}a_{t+1}}\right)^{\epsilon} \xi_t.$$

Therefore,

$$\begin{aligned} P_t &= \left(\frac{b_t + a_t - e^{-r}a_{t+1}}{e^{-r}(1-\frac{1}{\epsilon})}\right), && (2.5.1) \\ & \quad \text{and with constant cost depreciation,} \\ P_{t+k} &= \eta^k \left(\frac{b_t + a_t - e^{-r}a_{t+1}}{e^{-r}(1-\frac{1}{\epsilon})}\right) \\ &= (\eta)^k P_t. && (2.5.2) \end{aligned}$$

The important thing is this price trend does not depend on the realization of the economic condition as contrast with the optimal capacity path which is dependent on the realization of the uncertain economic condition.

Figure 6 shows the actual movement of capacity and the price when $\eta^\epsilon < \frac{\xi_{t+1}}{\xi_t}$ with probability 1. After the capacity of a firm reaches the lower optimal capacity, the price follows equation 2.5.2.

2.6 Numerical Studies

In this section, we want to illustrate how we set up our numerical analysis.

2.6.1 Numerical model

We consider the investment decision of a monopolistic firm with time horizon T.
The set of possible states of the capacity is S, and

$$S_t = \{x_1, x_2, ..., x_{n_t}\} \text{ with } t = 0, 1, ..., T.$$

The set of possible states of the economic indicator is E, and

$$E_t = \{\xi_1, \xi_2, ..., \xi_{m_t}\} \text{ with } t = 0, 1, ..., T.$$

The number of possible states of the capacity and the economic indicator might depend on the time period.

The set of cost parameters at time 0 is $(b_0, a_0, \tilde{a}_0)$, which includes unit maintenance cost, unit installation cost and unit salvage value at time 0. These cost parameters depreciate with time, and the set of cost parameters at time t is $(\eta^t b_0, \eta^t a_0, \eta^t \tilde{a}_0)$, with $0 < \eta < 1$.

The value function at time 1, given the capacity of the previous period (x_0) and the economic condition of the current period (ξ_1), is

$$\begin{aligned} V_1(x_0, \xi_1) &= Max_{x_i}\Bigg[\bigg(e^{-r}\xi_1^{\frac{1}{\epsilon}} x_i^{1-\frac{1}{\epsilon}} - (b_1 x_i + a_1(x_i - x_0)_+ - \tilde{a}_1(x_0 - x_i)_+)\bigg) \\ &+ e^{-r}\mathsf{E}\big[V_2\left(x_i, \xi_2 | \xi_1\right)\big]\Bigg] \quad x_0 \in S_0, x_i \in S_1 \text{ , and } \xi_1 \in E_1, \end{aligned}$$

with

$$V_T(x_{T-1}, \xi_T) = F_T(x_{T-1}, \xi_T).$$

We solve for the optimal capacity at time 1, given all possible pairs of (x_0, ξ_1). Therefore, we consider all the possible kinds of firms, from the emerging company to the over-invested company and all the possible economic conditions.

Here, we need to define the transition probability

$$p_t(i,j) = \mathsf{P}\left\{\xi_{t+1} = \xi_j | \xi_t = \xi_i\right\} \text{ for } i = 1, 2, ..., m_t \text{ and } j = 1, 2, ..., m_{t+1},$$

with

$$\sum_{j=1}^{m(t+1)} p_t(i,j) = 1 \text{ for all } i.$$

To solve this problem, we use dynamic programming.

1. First, set the terminal value function $F_T(x_{T-1}, \xi_T)$. With a large enough T, $F_T(x_{T-1}, \xi_T)$ can be any simple function.

2. Then go back one period to $T-1$, and for every possible pair of (x_{T-2}, ξ_{T-1}), calculate

$$\begin{aligned} \tilde{V}(x_{T-2}, x_{T-1}, \xi_{T-1}) &= e^{-r}\xi_{T-1}^{\frac{1}{\epsilon}} x_{T-1}^{1-\frac{1}{\epsilon}} - \big(b_{T-1}x_{T-1} + a_{T-1}(x_{T-1} - x_{T-2})_+ \\ &- \tilde{a}_{T-1}(x_{T-2} - x_{T-1})_+\big) + e^{-r}\mathsf{E}\big[F_T\left(x_{T-1}, \xi_T | \xi_{T-1}\right)\big] \\ &\quad \text{with } x_{T-1} \in S_{T-1}. \end{aligned}$$

3. Choose x^*_{T-1} that maximize $\tilde{V}(x_{T-2}, x_{T-1}, \xi_{T-1})$ and set $V(x_{T-2}, \xi_{T-1}) = \tilde{V}(x_{T-2}, x^*_{T-1}, \xi_{T-1})$.

4. After calculating all the value functions for all the possible states, $\{(x_{T-2}, \xi_{T-1}) : x_{T-2} \in S_{T-2} \text{ and } \xi_{T-1} \in E_{T-1}\}$, go back one period to $T-2$.

5. Do the same calculation (2-4) until all the value functions for all the possible states at time 1 are found.

2.6.2 Numerical Example

The following set represents the parameter values that are used for our numerical results.

1. Time period and finite time horizon (T): in our model, one period corresponds to one quarter (3 months), and we set $T = 40$, which corresponds to ten years.

2. Expected return during a period: we expect a 2.5% return in a period and use the compounded return rate.

3. The possible states of capacity:

$$S(t) = \{x_0(1+\Delta)^i : i = -50, -49, ..., 0, 1, ..., 50, \ \Delta = 0.05 \text{ and } x_0 = 1\}$$

   With $\Delta = 0.05$, the minimum increment is 5% and decrement is 4.8% of the current capacity. This capacity set covers a very broad range of investment decisions. If we start with x_0 capacity, the possible investment increases to $x_0(1.05)^{50} = 11.46 \cdot x_0$ and decreases to $x_0(1.05)^{-50} = 0.0872 \cdot x_0$. The number of possible states is 101 for all t.

4. The possible states of the economic indicator:

$$E(t) = \{\xi_0(1+\Delta)^i : i = -n(t), -n(t)+1, ...n(t), \ \Delta = 0.05 \text{ and } \xi_0 = 1\},$$

   where $n(t) = 50 + 2 \cdot t$. The number of possible economic conditions increases with time.

5. The transition probabilities:

$$p_t(i,j) = \mathsf{P}\left\{\xi_{t+1} = (1+\Delta)^j | \xi_t = (1+\Delta)^i\right\} = \begin{cases} p & \text{if } j = i+1, \\ 1-p & \text{if } j = i-1. \end{cases}$$

We allow for two possible transitions (up and down). The expected return of the economic indicator is $\mathsf{E}\left(\frac{\xi_{t+1}}{\xi_t} - 1 \middle| \xi_t\right) = (1+\Delta)p + (1+\Delta)^{-1}q.$

6. We set the maintenance cost, installation cost, and the salvage value at time 0 as

$$a_0 = 4P_0 = 4\left(\frac{\xi_0}{x_0}\right)^{\frac{1}{\epsilon}}, \; b_0 = 0.15a_0, \text{ and } \tilde{a}_0 = 0.5a_0.$$

In our numerical model, one period corresponds to one quarter. By setting $a_0 = 4P_0$, the revenue of a firm in a year from an additional capacity is the same as the installation cost for the additional capacity. By $b_0 = 0.15a_0$, we assume the maintenance teams/companies charge 5% of the installation cost as their monthly maintenance fee. By $\tilde{a}_0 = 0.5a_0$, we assume that the salvage value of the capacity is the half of the purchase price.

7. We consider constant cost depreciation with time as

$$(b_t, a_t, \tilde{a}_t) = \eta^t(b_0, a_0, \tilde{a}_0) \text{ for all } t,$$

and we set $\eta = 0.5^{\frac{1}{8}}$, as we discussed before.

With this parameter set, Figures 6 and 7 show the numerical results. Figure 6 shows the actual movement of capacity and price. Figure 7 shows the average movement of capacity and price when $p = 0.5$. The first, second, and third rows correspond to cases 1, 2 and 3, in subsection 2.5.1 respectively. The only difference among them is the starting point. When the company starts with a small capacity (case 1), it will increase to the lower optimal capacity ($x^*_{1,L}$) at time 1 and continues to increase. When the company starts with a moderate capacity (case 2), it will stay in the KEEP region for a while and then fall into the BUY region, and continues to increase. When the company starts with too much capacity (case 3), it will reduce its capacity to the upper optimal capacity ($x^*_{1,U}$)

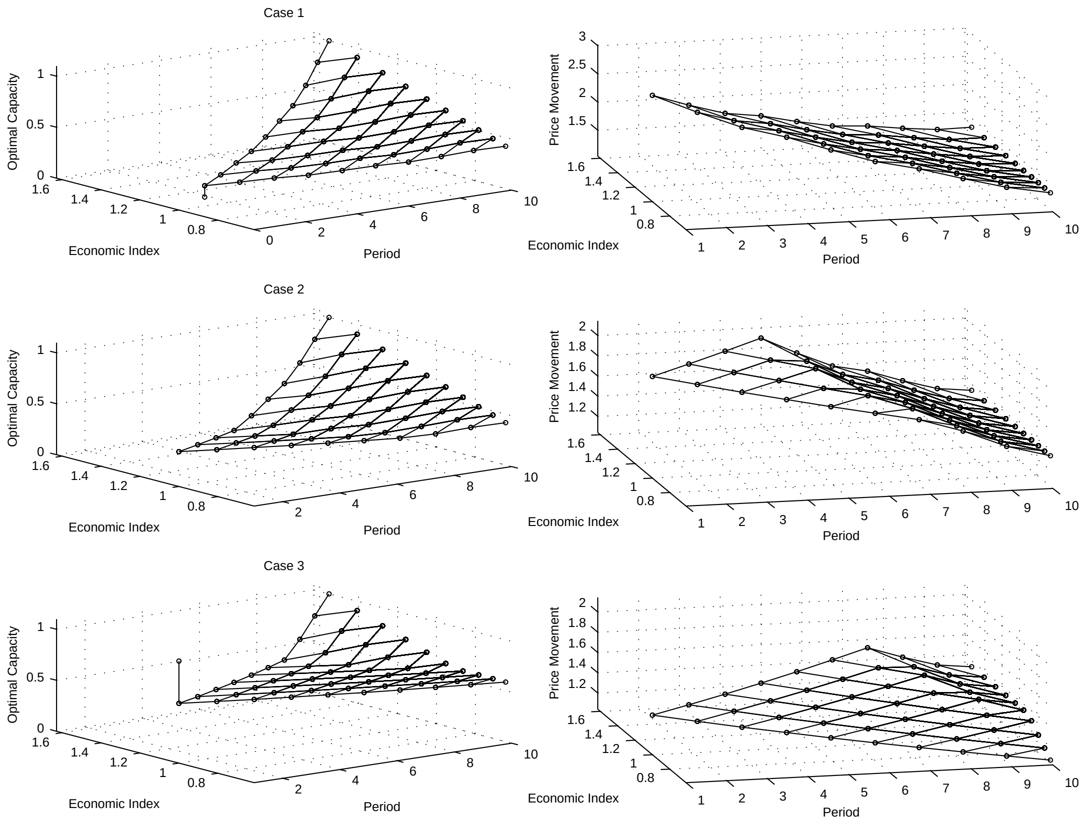

Figure 6: Actual movement of price and capacity when $\eta^{\epsilon} < \frac{\xi_{t+1}}{\xi_t}$ with probability 1

at time 1, stay in the KEEP region for a while, fall into the BUY region, and continues to increase.

Tables 5 and 6 are the analytic representation of the movement of capacity and price. For each case, we consider three different scenarios:

1. $p = 0.5$: the probabilities that the movement of the economic condition in any direction are equal.

2. $p = 1.0$: the probability of upward movement of the economic condition is 1; therefore, we expect the economic condition to improve during the following period by Δ, 5%. This corresponds to the upper rim of the triangle in Figure 6.

3. $p = 0.0$: the probability of downward movement of the economic condition is 1;

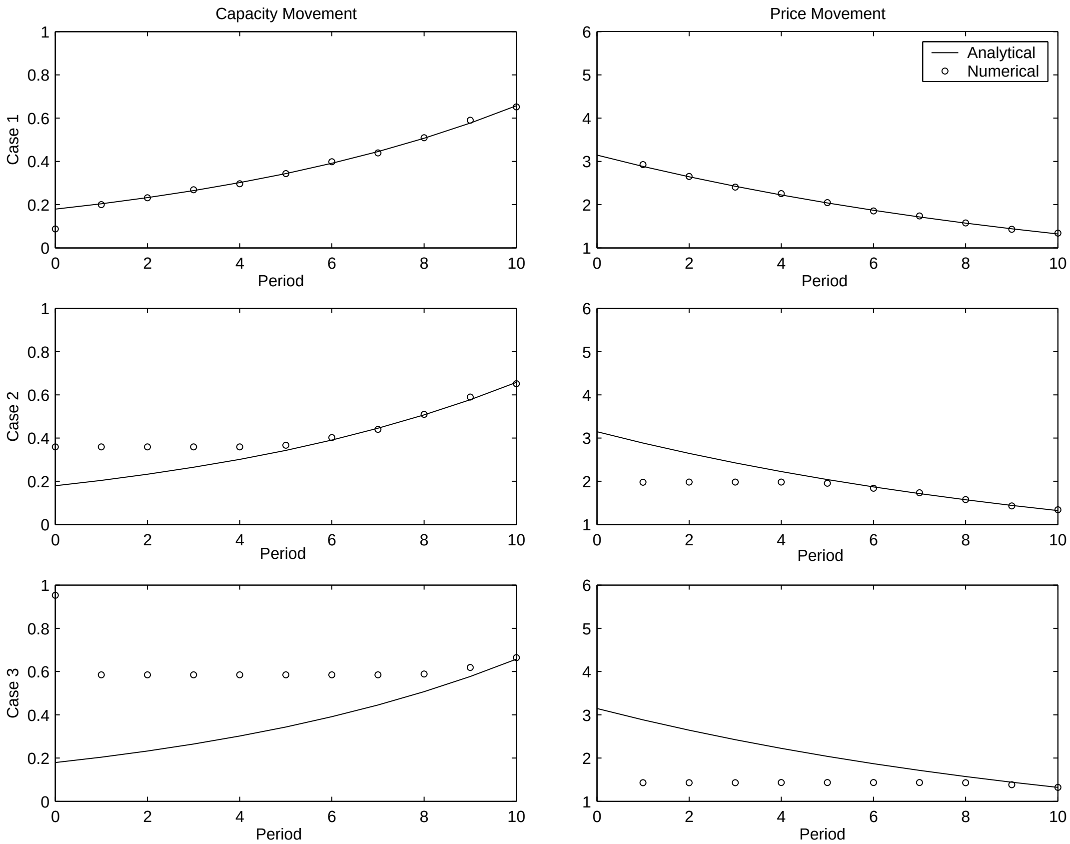

Figure 7: Average Movement of Price and Capacity when $\eta^{\epsilon} < \frac{\xi_{t+1}}{\xi_t}$ with Probability 1

therefore, we expect the economic condition to deteriorate during the following period by $1 - \frac{1}{1+\Delta}$, 4.8%. This corresponds to the lower rim of the triangle in Figure 6.

In the tables,

$$\mathsf{A} = \left(\frac{e^{-r}(1 - \frac{1}{\epsilon})}{b_0 + a_0 - e^{-r}\eta a_0} \right)^{\epsilon} \quad \text{and} \quad \mathsf{B} = \left(\frac{b_0 + a_0 - e^{-r}\eta a_0}{e^{-r}(1 - \frac{1}{\epsilon})} \right).$$

Notice that in all scenarios, the form of the analytical solution for the capacity in Table 5 is the same, but the real movement, which is illustrated in Figure 6 depends on the movement of the economic condition. This applies to the price movement with the same manner in Figure 6. However, for the price movement, after the firm falls into the BUY region, the price does not depend on the economic condition. The price is fixed to $\mathsf{B}\eta^t$ once the firm hits the KEEP/BUY boundary.

When we consider investment decision problem of a matured firm, the firm is likely to be in the BUY region and continue to increase its capacity. Therefore, the capacity movement will be pictured by the first row of the Figure 6, and the analytic expression will be the same as the first column of Table 5.

Table 5: Capacity movement with economic indicator and probability of upward movement

	Case 1(BUY)	Case 2(KEEP)	Case 3(SELL)
$p = 0.5$	$x_t = \mathsf{A}\left(\frac{1}{\eta^\epsilon}\right)^{\mathsf{t}}\xi_{\mathsf{t}}$	$x_t = \max\left(x_0,\, \mathsf{A}\left(\frac{1}{\eta^\epsilon}\right)^{\mathsf{t}}\xi_{\mathsf{t}}\right)$	$x_t = \max\left(x_1,\, \mathsf{A}\left(\frac{1}{\eta^\epsilon}\right)^{\mathsf{t}}\xi_{\mathsf{t}}\right)$
$p = 1.0$	$x_t = \mathsf{A}\left(\frac{1}{\eta^\epsilon}\right)^{\mathsf{t}}\xi_{\mathsf{t}}$	$x_t = \max\left(x_0,\, \mathsf{A}\left(\frac{1}{\eta^\epsilon}\right)^{\mathsf{t}}\xi_{\mathsf{t}}\right)$ $= \max\left(x_0,\, \mathsf{A}\left(\frac{1+\Delta}{\eta^\epsilon}\right)^{\mathsf{t}}\xi_0\right)$	$x_t = \max\left(x_1,\, \mathsf{A}\left(\frac{1}{\eta^\epsilon}\right)^{\mathsf{t}}\xi_{\mathsf{t}}\right)$ $= \max\left(x_1,\, \mathsf{A}\left(\frac{1+\Delta}{\eta^\epsilon}\right)^{\mathsf{t}}\xi_0\right)$
$p = 0.0$	$x_t = \mathsf{A}\left(\frac{1}{\eta^\epsilon}\right)^{\mathsf{t}}\xi_{\mathsf{t}}$	$x_t = \max\left(x_0,\, \mathsf{A}\left(\frac{1}{\eta^\epsilon}\right)^{\mathsf{t}}\xi_{\mathsf{t}}\right)$ $= \max\left(x_0,\, \mathsf{A}\left(\frac{1}{(1+\Delta)\eta^\epsilon}\right)^{\mathsf{t}}\xi_0\right)$	$x_t = \max\left(x_1,\, \mathsf{A}\left(\frac{1}{\eta^\epsilon}\right)^{\mathsf{t}}\xi_{\mathsf{t}}\right)$ $= \max\left(x_1,\, \mathsf{A}\left(\frac{1}{(1+\Delta)\eta^\epsilon}\right)^{\mathsf{t}}\xi_0\right)$

Table 6: Price movement with economic indicator and probability of upward movement

	Case 1(BUY)	Case 2 & 3(KEEP, SELL)
$p = 0.5$	$P_t = \mathsf{B}\cdot\eta^{\mathsf{t}}$	$P_t = \min\left(\mathsf{B}\cdot\eta^{\mathsf{t}},\, \left(\frac{\xi_{\mathsf{t}}}{\mathsf{x_t}}\right)^\epsilon\right)$
$p = 1.0$	$P_t = \mathsf{B}\cdot\eta^{\mathsf{t}}$	$P_t = \min\left(\mathsf{B}\cdot\eta^{\mathsf{t}},\, \left(\frac{\xi_{\mathsf{t}}}{\mathsf{x_t}}\right)^\epsilon\right)$ $= \min\left(\mathsf{B}\cdot\eta^{\mathsf{t}},\, \left(\frac{\xi_0(1+\Delta)^{\mathsf{t}}}{\mathsf{x_t}}\right)^\epsilon\right)$
$p = 0.0$	$P_t = \mathsf{B}\cdot\eta^{\mathsf{t}}$	$P_t = \min\left(\mathsf{B}\cdot\eta^{\mathsf{t}},\, \left(\frac{\xi_{\mathsf{t}}}{\mathsf{x_t}}\right)^\epsilon\right)$ $= \min\left(\mathsf{B}\cdot\eta^{\mathsf{t}},\, \left(\frac{\xi_0}{\mathsf{x_t}(1+\Delta)^{\mathsf{t}}}\right)^\epsilon\right)$

2.7 Proofs

Proof of Theorem 2.3.1. If $V_{t+1}(x_t, \xi_{t+1})$ is a concave function with respect to x_t, then $V_t(x_{t-1}, \xi_t)$ is a concave function with respect to x_{t-1}.

Also,

$$\tilde{a}_t \leq \frac{d\mathsf{E}\left[V_t(x_{t-1}, \xi_t|\xi_{t-1})\right]}{dx_{t-1}} \leq a_t.$$

First, the value function is

$$\begin{aligned} V_t(x_{t-1},\xi_t) &= Max_{x_t}\Bigg[\bigg(e^{-r}\xi_t^{\frac{1}{\epsilon}}x_t^{1-\frac{1}{\epsilon}} - (b_t x_t + a_t(x_t - x_{t-1})_+ - \tilde{a}_t(x_{t-1}-x_t)_+)\bigg) \\ &+ e^{-r}\mathsf{E}\big[V_{t+1}\left(x_t, \xi_{t+1}|\xi_t\right)\big]\Bigg]. \end{aligned}$$

1. (BUY): $x_{t-1} < x^*_{t,L}$ (BUY)

$$\begin{aligned} V_t(x_{t-1},\xi_t) &= \Bigg[\bigg(e^{-r}\xi_t^{\frac{1}{\epsilon}}(x^*_{t,L})^{1-\frac{1}{\epsilon}} - (b_t x^*_{t,L} + a_t(x^*_{t,L} - x_{t-1}))\bigg) \\ &+ e^{-r}\mathsf{E}\big[V_{t+1}\left(x^*_{t,L}, \xi_{t+1}|\xi_t\right)\big]\Bigg]. \end{aligned}$$

   For all x_{t-1} that is less than $x^*_{t,L}$, the optimal solution is $x^*_{t,L}$. From the above equation, we can see that smaller given capacity induces a larger investment cost. Therefore, as x_{t-1} increases $V_t(x_{t-1},\xi_t)$ increases by a_t, i.e., $\frac{dV_t(x_{t-1},\xi_t)}{dx_{t-1}} = a_t$.

2. (SELL): $x_{t-1} > x^*_{t,U}$

$$\begin{aligned} V_t(x_{t-1},\xi_t) &= \Bigg[\bigg(e^{-r}\xi_t^{\frac{1}{\epsilon}}(x^*_{t,U})^{1-\frac{1}{\epsilon}} - (b_t x^*_{t,U} - \tilde{a}_t(x_{t-1} - x^*_{t,U}))\bigg) \\ &+ e^{-r}\mathsf{E}\big[V_{t+1}\left(x^*_{t,U}, \xi_{t+1}|\xi_t\right)\big]\Bigg]. \end{aligned}$$

   For all x_{t-1} that is greater than $x^*_{t,U}$, the optimal solution is $x^*_{t,U}$. From the above equation, we can see a larger given capacity leads a larger salvage revenue. Therefore, as x_{t-1} increases, $V_t(x_{t-1},\xi_t)$ increases by $\tilde{a}_t$, i.e., $\frac{dV_t(x_{t-1},\xi_t)}{dx_{t-1}} = \tilde{a}_t$.

3. (KEEP): $x_{t-1} = x^*_{t,K}$

$$\begin{aligned} V_t(x_{t-1},\xi_t) &= \Bigg[\bigg(e^{-r}\xi_t^{\frac{1}{\epsilon}}(x^*_{t,K})^{1-\frac{1}{\epsilon}} - (b_t x^*_{t,K})\bigg) + e^{-r}\mathsf{E}\big[V_{t+1}\left(x^*_{t,K}, \xi_{t+1}|\xi_t\right)\big]\Bigg] \\ &= \Bigg[\bigg(e^{-r}\xi_t^{\frac{1}{\epsilon}}(x_{t-1})^{1-\frac{1}{\epsilon}} - (b_t x_{t-1})\bigg) + e^{-r}\mathsf{E}\big[V_{t+1}\left(x_{t-1}, \xi_{t+1}|\xi_t\right)\big]\Bigg]. \end{aligned}$$

   In this case, if $V_{t+1}(x_{t-1},\xi_{t+1})$ is concave with respect to x_{t-1}, then $V_t(x_{t-1},\xi_t)$ is concave, because the sum of the concave functions is concave.

   In addition, as $x_{t-1} \uparrow x^*_{t,U}$, $\frac{dV_t(x_{t-1},\xi_t)}{dx_{t-1}} \downarrow \tilde{a}_t$ and as $x_{t-1} \downarrow x^*_{t,L}$, $\frac{dV_t(x_{t-1},\xi_t)}{dx_{t-1}} \uparrow a_t$. If $\frac{dV_t(x_{t-1},\xi_t)}{dx_{t-1}} < \tilde{a}_t$ at some point x_{t-1}, which is less than $x^*_{t,U}$), then $x^*_{t,U}$ cannot be the optimal point and $x^*_{t,U}$ should shift down to the point where $\left.\frac{dV^*_t}{dx_{t-1}}\right|_{x^*_{t,U}} = \tilde{a}_t$ and this is similarly applied to $x^*_{t,L}$.

Therefore, for all cases, the value function is concave.

Second,

$$\begin{aligned}
& \frac{d\mathsf{E}\big[V_t\left(x_{t-1}, \xi_{t,i}|\xi_{t-1}\right)\big]}{dx_{t-1}} \\
= \quad & \frac{d}{dx_{t-1}}\left\{\sum_i p(\xi_{t-1}, \xi_{t,i}) V_t\left(x_{t-1}, \xi_{t,i}\right)\right\} \\
= \quad & \sum_i p(\xi_{t-1}, \xi_{t,i}) \frac{d}{dx_{t-1}}\left\{V_t\left(x_{t-1}, \xi_{t,i}\right)\right\}.
\end{aligned}$$

From the above proof, we know that for any i,

$$\tilde{a}_t \leq \frac{d}{dx_{t-1}} V_t\left(x_{t-1}, \xi_{t,i}\right) \leq a_t.$$

Therefore,

$$\tilde{a}_t \leq \frac{d\mathsf{E}\big[V_t\left(x_{t-1}, \xi_t|\xi_{t-1}\right)\big]}{dx_{t-1}} \leq a_t.$$

□

Proof of Theorem 2.5.1. Let us assume that cost depreciates as $b_{t+1} = \eta b_t, a_{t+1} = \eta a_t, \tilde{a}_{t+1} = \eta \tilde{a}_t$ for all t, then the optimal capacity is given by

$$\begin{aligned}
x_{t,L} &= l_{t,L}\xi_t \\
x_{t+1,L} &= l_{t+1,L}\xi_{t+1} = \eta^{-\epsilon} l_{t,L}\xi_{t+1} \\
x_{t+2,L} &= l_{t+2,L}\xi_{t+2} = \eta^{-2\epsilon} l_{t,L}\xi_{t+2} \qquad (2.7.1) \\
& \text{...etc.}
\end{aligned}$$

Here we used Theorem 2.3.4 of the linearity between the optimal capacity and economic condition and Lemma 2.4.1 of the relationship between the optimal capacity and scaler of cost parameters. This equation (2.7.1) satisfies regardless of the economic condition.

In order to satisfy $x_{t+1,L} \geq x_{t,L}$,

$$\begin{aligned}
\eta^{-\epsilon} l_{t,L}\xi_{t+1} &\geq l_{t,L}\xi_t \\
\eta^{-\epsilon}\xi_{t+1} &\geq \xi_t. \qquad (2.7.2)
\end{aligned}$$

If equation (2.7.2) is satisfied for the worst economic condition at $t+1$, it is satisfied for all possible economic conditions of next period, ξ_{t+1}.

In other words, if

$$\eta^{\epsilon} \leq \frac{\xi_{t+1}}{\xi_t} \text{ with probability 1,} \tag{2.7.3}$$

the company will increase the capacity position to $l_{t+1,L}\xi_{t+1}(= \eta^{-\epsilon} l_{t,L}\xi_{t+1})$ regardless of the future economic conditions ξ_{t+2}. □

Next, we want to prove that under condition (2.7.3) , the slope of the boundary between BUY/KEEP is given by $\left(\frac{e^{-r}(1-\frac{1}{\epsilon})}{b_t+a_t-e^{-r}a_{t+1}}\right)^{\epsilon}$.

Proof. As we have just shown, under condition (2.7.3), the company will be in the BUY region at time $t+1$. Therefore, the value function for BUY at time t is

$$\begin{aligned}
V_t(x_{t-1},\xi_t) \;&=\; Max_{x_t}\Bigg[\bigg(e^{-r}\xi_t^{\frac{1}{\epsilon}}x_t^{1-\frac{1}{\epsilon}} - (b_t x_t + a_t(x_t - x_{t-1})_+ - \tilde{a}_t(x_{t-1}-x_t)_+)\bigg)\\
&+\; e^{-r}\mathsf{E}\big[(V_{t+1}\left(x_t,\xi_{t+1}|\xi_t\right)\big]\Bigg]\\
&=\; Max_{x_t}\left[\bigg(e^{-r}\xi_t^{\frac{1}{\epsilon}}x_t^{1-\frac{1}{\epsilon}} - (b_t x_t + a_t(x_t - x_{t-1})\bigg) + e^{-r}\mathsf{E}\big[(V_{t+1}\left(x_t,\xi_{t+1}|\xi_t\right)\big]\right]\\
&=\; Max_{x_t}\Bigg[\bigg(e^{-r}\xi_t^{\frac{1}{\epsilon}}x_t^{1-\frac{1}{\epsilon}} - (b_t x_t + a_t(x_t - x_{t-1}))\bigg)\\
&+\; e^{-r}\sum_i p(\xi_t,\xi_{t+1,i})\Big\{Max_{x_{t+1}}(e^{-r}\xi_t^{\frac{1}{\epsilon}}x_{t+1}^{1-\frac{1}{\epsilon}} - (b_{t+1}x_{t+1} + a_{t+1}(x_{t+1}-x_t))\\
&+\; e^{-r}\mathsf{E}\big[V_{t+2}\left(x_{t+1},\xi_{t+2}|\xi_{t+1,i}\right)\big])\Big\}\Bigg]
\end{aligned}$$

Take derivative $V_t(x_{t-1},\xi_t)$ with respect to x_t and set it to zero,

$$\left(e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(1-\frac{1}{\epsilon}\right)(x_t)^{-\frac{1}{\epsilon}} - (b_t + a_t)\right) + e^{-r}\sum_i p(\xi_t,\xi_{t+1,i})a_{t+1} = 0.$$

Therefore, the lower optimal bound is

$$x^*_{t,L} = \left(\frac{e^{-r}(1-\frac{1}{\epsilon})}{b_t + a_t - e^{-r}a_{t+1}}\right)^{\epsilon}\xi_t.$$

□

Proof of Lemma 2.4.2. Let us define a function as:

$$f(\epsilon) \equiv \left(\mathsf{A}(t)\left(1-\frac{1}{\epsilon}\right)\right)^{\epsilon}.$$

If we take derivative to $f(\epsilon)$,

$$\frac{df(\epsilon)}{d\epsilon} = \left(\mathsf{A}(t)\left(1-\frac{1}{\epsilon}\right)\right)^{\epsilon}\left(\log\left(\mathsf{A}(t)\left(1-\frac{1}{\epsilon}\right)\right)+\frac{1}{\epsilon-1}\right)$$

If the following inequality satisfies, $f(\epsilon)$ is an increasing function with respect to ϵ.

$$\mathsf{A}(t) \geq \frac{\epsilon e^{\frac{1}{1-\epsilon}}}{\epsilon-1}.$$

Let us define a function again as:

$$g(\epsilon) = \frac{\epsilon e^{\frac{1}{1-\epsilon}}}{\epsilon-1},$$

then $\frac{dg(\epsilon)}{d\epsilon} > 0$, and $\lim_{\epsilon\uparrow\infty} g(\epsilon) = 1$.

Therefore,

1. If $\mathsf{A} \geq 1$ is satisfied, then $f(\epsilon)$ is an increasing function with ϵ.

2. If $\mathsf{A} \geq 1$ is not satisfied, $f(\epsilon)$ increases until ϵ reaches to $g^{-1}(\mathsf{A}(t))$ and then decreases thereafter, where g^{-1} is the inverse function of g.

$\square$

CHAPTER III

THE INVESTMENT BEHAVIOR OF A FIRM IN AN OLIGOPOLISTIC MARKET

In this section, we consider the capacity investment problem in an oligopolistic market. The International Telecommunication Union presented how reform from a public, monopolistic industry to a private, competitive market is taking place all over the world in the World Telecommunication Development Report, 2002. Eric Lie (2002) illustrated trends in telecommunication competition and gave guidelines for the telecommunication regulation and competition law. We want to support their work and assist with efforts to establish better guidelines by providing the firms' investment behavior in the competitive market. We find out if there exists an equilibrium point between firms' investment decisions at a given economic condition and prove the uniqueness of this equilibrium point. In addition, we study how the investment behavior of monopolistic firms changes when there are competitors in the market by investigating how firms in a market respond to the existence of the other firms. How competition affects the market properties is illustrated by exploring the relationship between the number of firms in the market and total market capacity, market price, consumer surplus, expected time to a certain price reduction and the expected time to the first investment decision.

Competition between firms in a non-cooperative competitive market can be modeled in three traditional ways: the Cournot model, the Bertrand model and the Stackelberg model. The main assumption of the Cournot model is that each firm in the market treats the output of other firms as a fixed number that will not respond to its own production decisions. The Bertrand model proposes that each firm chooses its price on the assumption that the prices of its competitors will remain fixed. In the Stackelberg model, one firm assumes that the rival is a naive Cournot duopolist. In our case, the Cournot and the Bertrand models procuce the same results, because the price is a function of demand scaled by the general economic condition. In addition, we treat the firms in the market equally. In other words, one firm cannot be superior to another in solving the investment problem. Therefore, among these three models, we choose the Cournot model to formulate and solve the investment decision problem. In addition, we are interested in the market structure where big players act simultaneously rather than sequentially. In a telecommunications industry, a firm's market share is important and firms in this industry more likely to choose investment strategies simultaneously rather than sequentially. The Cournot model is appropriate to study such a simultaneous investment decision between firms in a competitive market, which leads us to adopt the Cournot model to investigate the investment decision of a firm in an oligopolistic market.

A considerable number of studies have used the Cournot model to explain market behavior. One interesting application of the Cournot game is the decision making of information sharing. Li [27] investigated the equilibrium behavior of firms in exchange of their information in a Cournot oligopoly. It showed that (1) if there is uncertainty about a common parameter, referred to as " the true state of the world," no information sharing between firms is the unique Nash equilibrium; (2) if there is uncertainty about a firm-specific parameter (in this case, the constant marginal cost coefficient), complete information sharing is dominant over no information sharing. Kamien[18] introduced an interesting cake division game equivalent to the Cournot game. Using each player's reaction function, it showed equivalence between the cake division game and the Cournot game in several situations, specifically when the algebra to achieve an equilibrium in the Cournot game is complicated. Wen and David[43] used the Cournot oligopoly model to determine the equilibrium state of the electricity market. They introduced uncertainty into the information about the cost functions of competitors. They modeled this incomplete information in three cases: an estimated cost function, several estimated cost functions with probabilities for each estimate and an estimated distribution of a cost function for each of the other players. In addition, by providing an example for each case, they showed how market price, total output, and total payoff change. Maiorano, Song, and Trovato[30] used the Cournot model to explain the behavior of supplies in the electricity market. They focused on the effect of one firm's decision on other firms' profits by showing the best response curve of a firm with respect to other firms' actions. Grenadier[14] used the Cournot-Nash framework and the real option approach to derive equilibrium investment strategies. He addressed the lack of strategic interaction across option holders in the pre-existing literature and showed the value of waiting for the investment to converge to zero as the number of players in the market increases. In this case, the traditional NPV rule becomes approximately correct even for industries with a few competitors.

The Cournot equilibrium can be defined as follows:

Definition 3.0.1. We consider a market for a single homogeneous goods with inverse demand function $P(\cdot)$. There are n firms and firm $i \in \{1, 2, ..., n\}$ has a cost function $C_i(\cdot)$. Then $(x_1^*, x_2^*, ..., x_n^*) \in R_+^n$ is a Cournot equilibrium if

$$P\left(\sum_{j=1}^{n} x_j^*\right) x_i^* - C_i(x_i^*) \geq P\left(\sum_{j=1, j\neq i}^{n} x_j^* + x\right) x - C_i(x)$$

for all $x \geq 0$, for all $i \in \{1, 2, ..., n\}$.

The Nash equilibrium can be defined as follows:

Definition 3.0.2. If there is a set of strategies with the property that no player can benefit by changing his or her strategy while the other players do not change their strategies, then that set of strategies and the corresponding payoffs constitute the Nash Equilibrium.

From the above definitions, we know that if an optimal point is a Cournot equilibrium point, then the point is also a Nash Equilibrium point.

3.1 Problem Formulation in An Oligopolistic Market

In addition to the monopoly case, we study the problem of multi-period investments under an uncertain general economic condition in an oligopolistic market. In this section, we extend the monopolistic investment model to an oligopolistic investment model starting with the general formulation of the problem in an oligopolistic market. The oligopolistic market is assumed to be non-cooperatively competitive. We apply the Cournot model of oligopolistic behavior to find an optimal strategy for each firm in the market which has N firms in it.

At the beginning of each period, each firm makes an investment decision based on its capacity of previous period, the current economic condition, and the expectations of investment decisions of other firms.

We assume each firm shares all information with other firms, which includes the previous period capacities and cost structures of other firms and expectations for the future economic condition. In addition, we assume that all the firms follow the Cournot behavior for investment decisions.

At the beginning of period t:

1. Firm i has $x_{i,t-1}$ capacity and knows the current economic condition ξ_t and the previous period capacities of other firms, $\vec{X}_{-i,t-1}$, where $\vec{X}_{-i,t-1}$ is a vector which is defined by $(x_{1,t}, x_{2,t}, ..., x_{i-1,t}, x_{i+1,t}, ..., x_{N,t})$.

2. Firm i has to decide the optimal investment quantity $\hat{x}_{i,t}$, then $x^*_{i,t}(= \hat{x}_{i,t} + x_{i,t-1})$, which will continue from the beginning to the end of period t.

3. The firm i's cost parameter is given by $(b_{i,t}, a_{i,t}, \tilde{a}_{i,t})$ and the cost occurring in period t is $b_{i,t}x_{i,t} + a_{i,t}(x_{i,t} - x_{i,t-1})_+ - \tilde{a}_{i,t}(x_{i,t-1} - x_{i,t})_+$.

4. The price in period t is given by $\left(\frac{\xi_t}{x_{i,t}+x_{-i,t}}\right)^{\frac{1}{\epsilon}}$, where $x_{-i,t} = \sum_{j=1,j\neq i}^{N} x_j$. Therefore, $(x_{i,t} + x_{-i,t})$ is the total market capacity.

5. The temporary value function of a firm i is given by

$$\begin{aligned}\tilde{V}_{i,t}(x_{i,t-1}, \vec{X}_{-i,t}, \xi_t) \quad &= Max_{x_{i,t}}\Bigg\{e^{-r}\left(\frac{\xi_t}{x_{i,t}+x_{-i,t}}\right)^{\frac{1}{\epsilon}} x_{i,t} - \big(b_{i,t}x_{i,t} + a_{i,t}(x_{i,t} - x_{i,t-1})_+ \\ &\quad -\tilde{a}_{i,t}(x_{i,t-1} - x_{i,t})_+\big) + e^{-r}\mathsf{E}\big[V_{i,t+1}(x_{i,t}, x_{-i,t}, \xi_{t+1}|\xi_t)\big]\Bigg\} \\ &\qquad i = 1, 2, 3, ..., N.\end{aligned}$$

6. The optimal value function of a firm i is given by

$$\begin{aligned} V_{i,t}(x_{i,t-1}, \vec{X}_{-i,t-1}, \xi_t) = & \Big\{ e^{-r} \left(\tfrac{\xi_t}{x^*_{i,t} + x^*_{-i,t}} \right)^{\frac{1}{\epsilon}} x_{i,t} - \big(b_{i,t} x^*_{i,t} + a_{i,t}(x^*_{i,t} - x_{i,t-1})_+ \\ & - \ \tilde{a}_{i,t}(x_{i,t-1} - x^*_{i,t})_+ \big) + e^{-r} \mathsf{E}\big[V_{i,t+1}(x^*_{i,t}, x^*_{-i,t}, \xi_{t+1} | \xi_t) \big] \Big\} \\ & i = 1, 2, 3, ..., N. \end{aligned}$$

Notice that $\tilde{V}_{i,t}$ and $V_{i,t}$ differ. To get the optimal value function for each firm, we first need to define the temporary value function for each firm. By solving for $x^*_{i,t}$ that maximizes $\tilde{V}_{i,t}$ for all i, we can find a optimal capacity vector whose component is the optimal capacity of each firm as $\vec{X}^*_t = (x^*_{1,t}, x^*_{2,t}, ..., x^*_{N,t})$. Then we substitute this optimal capacity vector $\vec{X}^*_t$ to the temporary value function to get an optimal value function. In ordet to find the Cournot equilibrium point in the N-dimensional space, given the previous period capacity of each firm, we need to maximize the temporary value function of each firm simultaneously by using the iteration method referred as the "Round Robin Method" and the procedure of which follows:

1. We choose any value, say zero, of the terminal value function $F_{i,T}(x_{i,T-1}, \vec{X}_{-i,T-1}, \xi_T)$ for $i = 1, 2, ...N$.

2. At time $T-1$, given $(x_{1,T-2}, x_{2,T-2}, ..., x_{n,T-2}, \xi_{T-1})$, we start to determine the optimal capacity $x^*_{1,T-1}$ of firm 1 as follows:

   (a) we choose any initial capacity vector $\vec{X}_{-1,T-1} \equiv (x_{2,T-1}, x_{3,T-1}, ..., x_{n,T-1})$,

   (b) solve the temporary value function $\tilde{V}_{1,T-1}$ with the initial vector $\vec{X}_{-1,T-1}$ and get a temporary optimal capacity $\tilde{x}^*_{1,T-1}$.
   Update $\vec{X}_{T-1} = (\tilde{x}^*_{1,T-1}, x_{2,T-1}, ..., x_{n,T-1})$,

   (c) solve the temporary value function $\tilde{V}_{2,T-1}$ with the updated $\vec{X}_{-2,T-1}$ and get a temporary optimal capacity $\tilde{x}^*_{2,T-1}$. Update $\vec{X}_{T-1} = (\tilde{x}^*_{1,T-1}, \tilde{x}^*_{2,T-1}, ..., x_{n,T-1})$,

   (d) solve the temporary value function $\tilde{V}_{i,T-1}$ for $i = 3, 4, ..., n$ until we have all updated values for $\vec{X}_{T-1}$.

3. (2) We continue to iterate from (a) to (d) until we find an equilibrium point using the updated $\vec{X}_{T-1}$ at each time.

4. Finally we get $\{V_{i,T-1}(x_{i,T-2}, \vec{X}_{-i,T-2}, \xi_{T-1}), i = 1, 2, ..., N\}$ from $\{\tilde{V}_{i,T-1}(x_{i,T-2}, \vec{X}_{-i,T-1}, \xi_{T-1}), i = 1, 2, ..., N\}$ and find an optimal point $\vec{X}^*_{T-1} = (x^*_{1,T-1}, x^*_{2,T-1}, ..., x^*_{n,T-1})$ for one given state $(\vec{X}_{T-2}, \xi_{T-1})$.

5. We do the same thing (2-4) for the entire $(N+1)$-dimensional capacity and economic condition space.

6. We go one period back to $T-2$ and start loops (2-5) until we find the vector of optimal capacities, $\vec{X}^*_{T-2}$, for all possible states of $(\vec{X}_{T-3}, \xi_{T-2})$.

7. We do loops (2-5) until we return to period 1 and obtain the vector of optimal capacities $\vec{X}^*_1$, for all possible states of $(\vec{X}_0, \xi_1)$.

If we have m_1 possible states for the capacity for each firm and m_2 possible states for the economic condition, we need to do the loop (2-4) $N^{m_1} \cdot m_2$ times for one period calculation. Therefore, regardless of the rate of convergence from the temporary value function to the optimal value function, the calculation time would be huge.

In our proposed model, the cost function is not differentiable. Some other papers assume that the cost function is differentiable, which leads to this heavy calculation. In this case, the problem is reduced to finding a solution of a set of differentiable equations, and the resulting path does not show the KEEP region.

To reduce this calculation load, we first investigate the structure of the solution. By taking advantage of the structure of the solution, we can save considerable calculation time.

3.2 Modeling Assumptions

We assume the following throughout the analysis under oligopolistic market structure in chapter 3 and the experimental analysis in chapter 4.

1. The oligopolistic market is non-cooperatively competitive.

2. Each firm shares all information with other firms, which includes the previous period capacities and cost structures of other firms and expectations for the future economic condition.

3. The price elasticity of demand, ϵ, is constant, and $\epsilon > 1$.

4. We assume that the market demand is the same as the total of the capacity levels of all firms in the market.

5. $b_{i,t} \geq b_{i,t+1}$, $a_{i,t} \geq a_{i,t+1}$, and $\tilde{a}_{i,t} \geq \tilde{a}_{i,t+1}$, which reflects cost depreciation of firm i due to improvements of technology.

6. η_t is the cost depreciation coefficient at time t with $\eta_t < 1$. In addition, all firms have the same cost depreciation structure. Maintenance cost, installation cost, and salvage value depreciate at the same rate as

$$(b_{i,t+1}, a_{i,t+1}, \tilde{a}_{i,t+1}) = \eta_t(b_{i,t}, a_{i,t}, \tilde{a}_{i,t}) \text{ with } \eta_t < 1 \text{ for all } t \text{ and all } i.$$

7. $a_{i,t} \geq \tilde{a}_{i,t}$, which represents the partial reversibility of the investment.

8. $F_{i,T}\left(\vec{X}_{T-1}, \xi_T\right)$ is a concave function with respect to $x_{i,T-1}$ for all i, which guarantees the concavity of the value function at t. In addition, $F_{i,T}\left(\vec{X}_{T-1}, \xi_T\right)$ is homogeneous, i.e.,

$$\forall \eta > 0, F_{i,T}\left(\eta\vec{X}_{T-1}, \eta\xi_T\right) = \eta F_{i,T}\left(\vec{X}_{T-1}, \xi_T\right).$$

With this assumption, the linearity between the optimal capacity and the economic condition is established.

9. All firms have the same cost parameters in the symmetric case. Namely

$$(b_{i,t}, a_{i,t}, \tilde{a}_{i,t}) = (b_t, a_t, \tilde{a}_t) \text{ for all } i \in \{1, ..., N\}.$$

Each firm can have different cost parameters in the asymmetric case.

$$(b_{i,t}, a_{i,t}, \tilde{a}_{i,t}) \neq (b_{j,t}, a_{j,t}, \tilde{a}_{j,t}) \text{ for all different pair of } i, j \in \{1, ..., N\}.$$

10. We allow each firm to have different capacity in the previous period.

$$x_{i,t-1} \neq x_{j,t-1} \text{ for all different pair of } i,j \in \{1,...,N\}.$$

3.3 General Structure of the Solution

Similarly to the monopoly case, we can define the value function at every period as a vector. Each component of the vector is the optimal value function of each firm:

$$\vec{V}_t(x_{1,t-1}, x_{2,t-1}, ..., x_{n,t-1}, \xi_t) = \big(V_{1,t}(x_{1,t-1}, x_{-1,t-1}, \xi_t), ..., V_{N,t}(x_{N,t-1}, x_{-N,t-1}, \xi_t)\big).$$

The component of this vector corresponds to each firm's optimal value function. In other words, we have already found the optimal capacity expansion for all firms at time t and obtained the optimal capacity trajectory from time $t-1$ to t.

As we can see from the above vector of value functions, we need to find the solution given the information of the previous period. Given the capacities of all other firms in the previous period , a firm needs to decide the optimal capacity of the current period based on its capacity of previous period and the economic condition.

Using this vector of optimal capacity of each firm, we investigate if the solution structure of the monopolistic case still applies to the oligopolistic case.

□ Linear relationship between firms' optimal capacities and the economic condition.

As we illustrated in Theorem 2.3.4, economic condition and both the optimal lower and optimal upper bounds have a linear relationship in the case of monopolistic market. This theorem can be extended to the case of N firms in an oligopolistic market with a minor change in the proof.

Corollary 3.3.1. *For $t = 1, ..., T$, suppose $\vec{V}_{t+1}(x_{1,t}, x_{2,t}, ..., x_{N,t}, \xi_{t+1})$ is homogeneous, i.e., $\forall \eta > 0$,*

$$\begin{aligned}
& \vec{V}_{t+1}(\eta x_{1,t}, \eta x_{2,t}, ..., \eta x_{N,t}, \eta\xi_{t+1}) \\
= \;& (V_{1,t+1}(\eta x_{1,t}, \eta\xi_{t+1}), V_{2,t+1}(\eta x_{2,t}, \eta\xi_{t+1}), ..., V_{N,t+1}(\eta x_{N,t}, \eta\xi_{t+1})) \\
= \;& (\eta V_{1,t+1}(x_{1,t}, \xi_{t+1}), \eta V_{2,t+1}(x_{2,t}, \xi_{t+1}), ..., \eta V_{N,t+1}(x_{N,t}, \xi_{t+1})) \\
= \;& \eta V_{t+1}(x_{1,t}, x_{2,t}, ..., x_{N,t}, \xi_{t+1}).
\end{aligned}$$

Then

1. If the optimal policy at $(x_{1,t-1}, x_{2,t-1}, ..., x_{N,t-1}, \xi_t)$ *is* $(x^*_{1,t}, x^*_{2,t}, ..., x^*_{N,t})$*, then for* $\forall \eta > 0$*, the optimal policy at* $(\eta x_{1,t-1}, \eta x_{2,t-1}, ..., \eta x_{N,t-1}, \eta \xi_t)$ *is* $(\eta x^*_{1,t}, \eta x^*_{2,t}, ..., \eta x^*_{N,t})$*.*

2. Furthermore, $\vec{V}_t(x_{1,t-1}, x_{2,t-1}, ..., x_{N,t-1}, \xi_t)$ *is also homogeneous.*

Proof. See the proof in section 3.6. □

From Corollary 3.3.1, we know that a linear relationship between the economic condition and the optimal capacity is still satisfied in the case of N firms. When we scale the given capacities and the economic condition, the resulting optimal capacity for each company is scaled. Using this characteristic, we can reduce one dimension in our calculation. After getting solutions for a particular value of the current economic condition with all possible capacity vectors of the previous period, we can get a solution for the entire state space. Figure 8 shows one possible solution when we consider two firms in the market assuming a very simple solution. As we can see in Figure 8, once we have optimal capacity vectors for one fixed economic condition, we can get optimal capacity vectors for any other economic condition. The inside of the square cone is the (KEEP, KEEP) region for firms 1 and 2.

□ The relationship between firms optimal capacities and the cost parameters.

Corollary 3.3.2. *If the vector of optimal capacities of firms with the cost parameters* $(b_{i,t}, a_{i,t}, \tilde{a}_{i,t})$ *is* $\vec{X}^*_t$*, then for* $\forall \eta > 0$*, the vector of optimal capacities of firms for the cost parameters* $(\eta b_{i,t}, \eta a_{i,t}, \eta \tilde{a}_{i,t})$ *is* $\eta^{-\epsilon} \vec{X}^*_t$

Proof. The proof is similar to the proof of Lemma 2.4.1. □

3.4 Symmetric Firms in an Oligopolistic Market

In order to find the optimal investment strategy, we need to prove the existence of an equilibrium point beforehand. William [37] provided some general conditions for the existence of the equilibrium in the Cournot model. Long[28] provided sufficient conditions for the existence and the uniqueness of a Cournot equilibrium by applying the contraction mapping approach. However, the conditions in there two papers, specifically $P(x_t) = 0$ for some value of x_t, do not match those of our model. Thus we try to find the equilibrium point and prove its uniqueness in our model.

First, we study the investment decisions of a firm in an oligopolistic market when the firms in the market have same cost structures. For any firm i, the cost occurring at time t is

$$C_{i,t}(x_{i,t-1}, x_{i,t}) = b_{i,t}x_{i,t} + a_{i,t}(x_{i,t} - x_{i,t-1})_+ - \tilde{a}_{i,t}(x_{i,t-1} - x_{i,t})_+,$$

$$\text{where } (b_{i,t}, a_{i,t}, \tilde{a}_{i,t}) = (b_t, a_t, \tilde{a}_t) \text{ for all } i \in \{1, ..., N\}.$$

However, we allow each firm to have different capacity of the previous period. Therefore, at time t, we allow $x_{i,t-1} \neq x_{j,t-1}$ for all different pair of $i, j \in \{1, ..., N\}$.

In the next subsections, we will investigate the existence and the uniqueness of the equilibrium point. In subsection 3.4.1, we will consider a duopoly market and we will extend the results of it to the case of N symmetric firms in subsection 3.4.2.

3.4.1 Two Symmetric Firms in a Duopoly Market

In this subsection, we consider two symmetric firms in a duopoly market. Similar to those in the monopoly case, the possible decisions for each company are BUY, KEEP and SELL. Therefore, with two companies in the market, nine different decisions are possible. We will study the existence of optimal capacity and the uniqueness of the solution for each case. First, we explain the notations that will be used throughout this chapter. $x^*_{i,\mathsf{AB}}$ is the optimal capacity of firm i when firm 1 is in the A region and firm 2 is in the B region. For example, $x^*_{1,\mathsf{BB}}$ is the optimal solution of firm 1 when firm 1 is in the BUY region and firm 2 is in the BUY region. To make the notation simple, we ignore period subscript (t) and this can be well understood in the context.

Before starting to prove the existence and uniqueness of the Cournot equilibrium point, we define the $G_{i,b}$ and $G_{i,s}$ functions as a derivative of the temporary value functions of firm i for each case of BUY and SELL.

The temporary value function and the corresponding $G_{i,b}$ function of company i at time t for the BUY case are

$$\begin{aligned}\tilde{V}_{i,t}(x_{i,t-1}, x_{-i,t}, \xi_t) &= Max_{x_{i,t}}\Big\{e^{-r}\left(\frac{\xi_t}{x_{i,t}+x_{-i,t}}\right)^{\frac{1}{\epsilon}} x_{i,t} - \big(b_t x_{i,t} + a_t(x_{i,t}-x_{i,t-1})\big) \\ &+ e^{-r}\mathsf{E}\big[V_{i,t+1}(x_{i,t}, x_{-i,t}, \xi_{t+1}|\xi_t)\big]\Big\} \text{ for } i \in \{1,2\}, \text{ and}\end{aligned}$$

$$\begin{aligned}G_{i,b}(x_{i,t}, x_{-i,t}, \xi_t) &= e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x_{i,t}+x_{-i,t}}\right)^{\frac{1}{\epsilon}}\left(1-\frac{x_{i,t}}{\epsilon(x_{i,t}+x_{-i,t})}\right) - b_t - a_t \\ &+ e^{-r}\frac{d\mathsf{E}\big[V_{i,t+1}(x_{i,t}, x_{-i,t}, \xi_{t+1}|\xi_t)\big]}{dx_{i,t}} \text{ for } i \in \{1,2\}.\end{aligned}$$

The temporary value function and the corresponding $G_{i,s}$ function of company i at time t for the SELL case are

$$\begin{aligned}\tilde{V}_{i,t}(x_{i,t-1}, x_{-i,t}, \xi_t) &= Max_{x_{i,t}}\Big\{e^{-r}\left(\frac{\xi_t}{x_{i,t}+x_{-i,t}}\right)^{\frac{1}{\epsilon}} x_{i,t} - \big(b_t x_{i,t} - \tilde{a}_t(x_{i,t-1}-x_{i,t})\big) \\ &+ e^{-r}\mathsf{E}\big[(V_{i,t+1}(x_{i,t}, x_{-i,t+1}, \xi_{t+1}|\xi_t))\big] \text{ for } i \in \{1,2\}, \text{ and}\end{aligned}$$

$$\begin{aligned}G_{i,s}(x_{i,t}, x_{-i,t}, \xi_t) &= e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x_{i,t}+x_{-i,t}}\right)^{\frac{1}{\epsilon}}\left(1-\frac{x_{i,t}}{\epsilon(x_{i,t}+x_{-i,t})}\right) - b_t - \tilde{a}_t \\ &+ e^{-r}\frac{d\mathsf{E}\big[V_{i,t+1}(x_{i,t}, x_{-i,t+1}, \xi_{t+1}|\xi_t)\big]}{dx_{i,t}} \text{ for } i \in \{1,2\}.\end{aligned}$$

1. (BUY, BUY): Let $(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}}(= x^*_{1,\mathsf{BB}}))$ be the solution of a system of equations, $\{G_{i,b}(x_{1,t}, x_{2,t}, \xi_t) = 0, i = 1,2\}$. If $x_{1,t-1} \leq x^*_{1,\mathsf{BB}}$ and $x_{2,t-1} \leq x^*_{2,\mathsf{BB}}$, then both firms 1 and 2 increase their capacity to $x^*_{1,\mathsf{BB}}$ and $x^*_{2,\mathsf{BB}}$, respectively.

   ☐ Existence. Using the fact that $x^*_{1,\mathsf{BB}} = x_{2,\mathsf{BB}}$, we can only consider the case of $x_{1,t} = x_{2,t}$.

$$\begin{aligned}&G_{1,b}(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}}, \xi_t) = G_{2,b}(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}}, \xi_t) \\ &= e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{2x^*_{1,\mathsf{BB}}}\right)^{\frac{1}{\epsilon}}\left(1-\frac{1}{\epsilon}\right) - b_t - a_t + e^{-r}\frac{d\mathsf{E}\big[V_{1,t+1}(x_{1,t}, x_{2,t}, \xi_{t+1}|\xi_t)\big]}{dx_{1,t}}\end{aligned}$$

$$\begin{aligned}&\lim_{x^*_{1,\mathsf{BB}}\downarrow 0} G_{1,b}(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}}, \xi_t) = \infty, \text{ and} \\ &\lim_{x^*_{1,\mathsf{BB}}\uparrow\infty} G_{1,b}((x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}}, \xi_t) \simeq -b_t - a_t + e^{-r}\tilde{a}_{t+1} < 0,\end{aligned}$$

where we used Proposition 2.3.3 that will be used to prove the existence of the solution in other cases. Therefore, there is a solution $(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}})$ that satisfies $G_{1,b}(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}}, \xi_t) = 0$.

□ Uniqueness.

Lemma 3.4.1.

$$\frac{dG_{i,b}(x_{1,t}, x_{2,t}, \xi_t)}{dx_{i,t}} < 0 \text{ for } i \in \{1,2\},$$

and therefore, $G_{i,b}$ is a monotonically decreasing function with respect to $x_{i,t}$.

Proof. If we take derivative to $G_{1,b}(x_{1,t}, x_{2,t}, \xi_t)$ with respect to $x_{1,t}$,

$$\begin{aligned} \frac{dG_{1,b}(x_{1,t}, x_{2,t}, \xi_t)}{dx_{1,t}} &= -\frac{1}{\epsilon}(x_{1,t} + x_{2,t})^{-\frac{1}{\epsilon}-2}\left(\left(1-\frac{1}{\epsilon}\right)x_{1,t} + 2x_{2,t}\right) \\ &+ \frac{d^2\mathsf{E}\big[V_{1,t+1}(x_{1,t}, x_{2,t}, \xi_{t+1}|\xi_t)\big]}{dx^2_{1,t}}. \end{aligned}$$

Because $V_{1,t+1}(x_t, x_t, \xi_{t+1}|\xi_t)$ is concave with $x_{1,t}$, $\frac{d^2\mathsf{E}V_{1,t+1}(x_{1,t},x_{2,t},\xi_{t+1}|\xi_t)}{dx^2_{1,t}} < 0$. Therefore, $G_{1,b}(x_{1,t}, x_{2,t}, \xi_t)$ is a monotonically decreasing function in $x_{1,t}$. In addition, $\frac{dG_{2,b}(x_{1,t},x_{2,t},\xi_t)}{dx_{2,t}} < 0$ with the same calculation. Hence, $(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}})$ is a unique solution for the (BUY, BUY) case. □

2. (SELL, SELL): Let $(x^*_{1,\mathsf{SS}}, x^*_{2,\mathsf{SS}}(= x^*_{1,\mathsf{SS}}))$ be the solution of a system of equations, $\{G_{i,s}(x_{1,t}, x_{2,t}, \xi_t) = 0, i = 1,2\}$. If $x_{1,t-1} \geq x^*_{1,\mathsf{SS}}$ and $x_{2,t-1} \geq x^*_{2,\mathsf{SS}}$, then both of firms 1 and 2 reduce their capacity to $x^*_{1,\mathsf{SS}}$ and $x^*_{2,\mathsf{SS}}$, respectively.

   □ Existence. Using the fact that $x^*_{1,\mathsf{SS}} = x_{2,\mathsf{SS}}$, we can only consider the case of $x_{1,t} = x_{2,t}$

$$\begin{aligned} &G_{1,s}(x^*_{1,\mathsf{SS}}, x^*_{2,\mathsf{SS}}, \xi_t) = G_{2,s}(x^*_{1,\mathsf{SS}}, x^*_{2,\mathsf{SS}}, \xi_t) \\ &= e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{2x^*_{1,\mathsf{SS}}}\right)^{\frac{1}{\epsilon}}\left(1-\frac{1}{\epsilon}\right) - b_t - \tilde{a}_t + e^{-r}\frac{d\mathsf{E}\big[V_{1,t+1}(x_{1,t}, x_{2,t}, \xi_{t+1}|\xi_t)\big]}{dx_{1,t}} \end{aligned}$$

$$\begin{aligned} &\lim_{x^*_{1,\mathsf{SS}}\downarrow 0} G_{1,s}(x^*_{1,\mathsf{SS}}, x^*_{2,\mathsf{SS}}, \xi_t) = \infty \\ &\lim_{x^*_{1,\mathsf{SS}}\uparrow \infty} G_{1,s}(x^*_{1,\mathsf{SS}}, x^*_{2,\mathsf{SS}}, \xi_t) \simeq -b_t - \tilde{a}_t + e^{-r}\tilde{a}_{t+1} < 0. \end{aligned}$$

Therefore, there is a solution $(x^*_{1,\mathsf{SS}}, x^*_{2,\mathsf{SS}})$ that satisfies $G_{1,s}(x^*_{1,\mathsf{SS}}, x^*_{2,\mathsf{SS}}, \xi_t) = 0$.

□ Uniqueness:

$\frac{dG_{i,s}(x_{1,t},x_{2,t},\xi_t)}{dx_{i,t}} < 0$ for $i \in \{1,2\}$ guarantees the uniqueness of the solution.

3. (SELL, BUY): Let $(x^*_{1,\mathsf{SB}}, x^*_{2,\mathsf{SB}})$ be the solution for a system of equations, $\{G_{1,s}(x_{1,t}, x_{2,t}, \xi_t) = 0, G_{2,b}(x_{1,t}, x_{2,t}, \xi_t) = 0\}$. If $x_{1,t-1} \geq x^*_{1,\mathsf{SB}}$ and $x_{2,t-1} \leq x^*_{2,\mathsf{SB}}$, then firm 1 sells off the excess capacity down to $x^*_{1,\mathsf{SB}}$ and firm 2 buys more capacity up to $x^*_{2,\mathsf{SB}}$.

   □ Existence:

   From the symmetry of the two firms, we know $x^*_{1,\mathsf{SB}} > x^*_{2,\mathsf{SB}}$.

$$
\begin{aligned}
(1)\quad \lim_{x_{1,t}\downarrow x^*_{2,\mathsf{SB}}} G_{1,s} &= e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{2x^*_{2,\mathsf{SB}}}\right)^{\frac{1}{\epsilon}}\left(1-\frac{1}{2\epsilon}\right) \\
&\quad -b_t - \tilde{a}_t + e^{-r}\frac{d\mathsf{E}\big[V_{1,t+1}(x_{1,t}, x^*_{2,\mathsf{SB}}, \xi_{t+1}|\xi_t)\big]}{dx_{1,t}}\bigg|_{x^*_{2,\mathsf{SB}}} \underbrace{>}_{\text{want it to be}} 0. \\
(2)\quad \lim_{x_{1,t}\uparrow\infty} G_{1,s} &= -b_t - \tilde{a}_t + e^{-r}\tilde{a}_{t+1} < 0. \\
(3)\quad \lim_{x_{2,t}\downarrow 0} G_{2,b} &= e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x_{1,t}}\right)^{\frac{1}{\epsilon}} - b_t - a_t + e^{-r}a_{t+1} \underbrace{>}_{\text{want it to be}} 0. \\
(4)\quad \lim_{x_{2,t}\uparrow x^*_{1,\mathsf{SB}}} G_{2,b} &= e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{2x^*_{1,\mathsf{SB}}}\right)^{\frac{1}{\epsilon}}\left(1-\frac{1}{2\epsilon}\right) \\
&\quad -b_t - a_t + e^{-r}\frac{d\mathsf{E}\big[V_{1,t+1}(x^*_{1,\mathsf{SB}}, x_{2,t}, \xi_{t+1}|\xi_t)\big]}{dx_{2,t}}\bigg|_{x^*_{1,\mathsf{SB}}} \underbrace{<}_{\text{want it to be}} 0.
\end{aligned}
$$

   The positivity of equation (1) comes from the fact that $G_{1,s}$ is a decreasing function and $x^*_{2,\mathsf{SB}} < x^*_{2,\mathsf{SS}}$. Otherwise, $x^*_{1,\mathsf{SB}} > x^*_{2,\mathsf{SB}} > x^*_{2,\mathsf{SS}}$, which means $(x^*_{1,\mathsf{SS}}, x^*_{2,\mathsf{SS}})$ is not an optimal point for the (SELL, SELL) case.

   The negativity of equation (4)comes from the fact that $G_{2,b}$ is a decreasing function and $x^*_{1,\mathsf{SB}} > x^*_{1,\mathsf{BB}}$. Otherwise, $x^*_{1,\mathsf{BB}} > x^*_{1,\mathsf{SB}} > x^*_{2,\mathsf{SB}}$, which means $(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}})$ is not an optimal point for the (BUY, BUY) case.

   In order for equation (3) to be positive $x_{1,t}$ should satisfy the following equation:

$$x^*_{1,\mathsf{SB}} < \left(\frac{e^{-r}}{b_t + a_t - e^{-r}a_{t+1}}\right)^{\epsilon}\xi_t. \tag{3.4.1}$$

The right-hand side of equation 3.4.2 is the limiting value of the total market capacity when all firms in the market are in the BUY region[1]. Therefore, in order for firm 2 to be viable, the competitive output of firm 1 must not exceed the limiting value of the total market capacity when all firms in the market are in the BUY region.

If $x^*_{1,\mathsf{SB}}$ does not satisfy the above equation, then

$$x^*_{1,\mathsf{SB}} > \left(\frac{e^{-r}}{b_t + a_t - e^{-r}a_{t+1}}\right)^{\epsilon} \xi_t > \underbrace{\left(\frac{e^{-r}\left(1-\frac{1}{\epsilon}\right)}{b_t + a_t - e^{-r}a_{t+1}}\right)^{\epsilon} \xi_t}_{\text{monopolistic firm's optimal capacity in the BUY case.}} .$$

Hence, if the competitive output of firm 1 does not exceed the monopolistic firm's optimal capacity in the BUY case, then firm 2 is viable, . If $x_{1,t}$ does not satisfy equation (3.4.2), then $x_{2,t} = 0$ and the problem goes back to the monopolistic case. The implication of this inequality is that, if the optimal capacity of firm 1 is very large, then it is not profitable firm 2 to either enter or stay in the market. In addition, if the cost coefficients of firms are small, the right-hand side of equality 3.4.2 tends to be large, and the equality 3.4.2 has more possibility to satisfy. Therefore, we conclude that in an duopolistic market, firms are more likely to survive with efficient cost structure than with inefficient cost structure. □ Uniqueness:

$\frac{dG_{1,s}(x_{1,t},x_{2,t},\xi_t)}{dx_{1,t}} < 0$ and $\frac{dG_{2,b}(x_{1,t},x_{2,t},\xi_t)}{dx_{2,t}} < 0$ guarantee the uniqueness of the solution.

4. (KEEP, BUY): Starting from $(x_{1,t-1}, x_{2,t-1}) = (x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}})$, as $x_{1,t-1}$ increases beyond $x^*_{1,\mathsf{BB}}$, firm 1 goes into the KEEP region. The optimal capacity in this region is $(x^*_{1,t}, x^*_{2,t}) = (x_{1,t-1}, x^*_{2,\mathsf{KB}}) \equiv (x^*_{1,\mathsf{KB}}, x^*_{2,\mathsf{KB}})$. Therefore, if $x_{2,t-1} < x^*_{2,\mathsf{KB}}$, then firm 2 buys more capacity up to $x^*_{2,\mathsf{KB}}$ and firm 1 maintains its current capacity with $x^*_{1,\mathsf{BB}} < x_{1,t-1} < x^*_{1,\mathsf{SB}}$.

   □ Existence:

   At first, we know that $x^*_{1,\mathsf{KB}} > x^*_{1,\mathsf{BB}} (= x^*_{2,\mathsf{BB}})$ because $x^*_{1,\mathsf{KB}} > x^*_{2,\mathsf{KB}}$. Otherwise, $x^*_{2,\mathsf{KB}} > x^*_{1,\mathsf{KB}} > x^*_{2,\mathsf{BB}}$ and $x^*_{1,\mathsf{KB}} > x^*_{1,\mathsf{BB}}$, which means $(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}})$ is not an

[1]Refer to subsection 3.4.3 to get this limiting value of the total market capacity when all firms in the market are in the BUY region

optimal solution for the (BUY, BUY) case.

$$
\begin{aligned}
(1) \qquad \lim_{x_{2,t}\downarrow 0} G_{2,b} \quad &= e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x^*_{1,\mathsf{KB}}}\right)^{\frac{1}{\epsilon}} - b_t - a_t + e^{-r}a_{t+1} \underbrace{>}_{\text{want it to be}} 0. \\
(2) \quad \lim_{x_{2,t}\uparrow x^*_{1,\mathsf{KB}}} G_{2,b} \quad &= e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{2x^*_{1,\mathsf{KB}}}\right)^{\frac{1}{\epsilon}}\left(1-\frac{1}{2\epsilon}\right) \\
&\quad -b_t - a_t + e^{-r}\frac{d\mathsf{E}\left[V_{2,t+1}(x^*_{1,\mathsf{KB}}, x_{2,t}, \xi_{t+1}|\xi_t)\right]}{dx_{2,t}}\bigg|_{x^*_{1,\mathsf{KB}}} < 0.
\end{aligned}
$$

In order for the above equation (1) to be positive, $x^*_{1,\mathsf{KB}}$ should satisfy the following equation:

$$x^*_{1,\mathsf{KB}} < \left(\frac{e^{-r}}{b_t + a_t - e^{-r}a_{t+1}}\right)^{\epsilon}\xi_t. \tag{3.4.2}$$

This is the same condition for the existence condition for the (SELL, BUY) case; thus, if there is a solution for the case of (BUY, BUY), then there is a solution for the case of (SELL, BUY). If $x^*_{1,\mathsf{KB}}$ does not satisfy equation(3.4.2), then $x_{2,t} = 0$ and the problem goes back to the monopolistic case. In addition, the solution starts at the point, $(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}})$ and ends at the point, $(x^*_{1,\mathsf{SB}}, x^*_{2,\mathsf{SB}})$.

The negativity of equation (2) comes from the fact that

(a) $G_{2,b}(x_{1,t}, x_{2,t}, \xi_t)$ is a decreasing function with respect to $x_{2,t}$,

(b) $G_{2,b}(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}}, \xi_t) = 0$ and $x^*_{1,\mathsf{KB}} > x^*_{2,\mathsf{BB}}$, and

(c) equation (1) can be re-written as

$$\lim_{x_{2,t}\uparrow x^*_{1,\mathsf{KB}}} G_{2,b}(x^*_{1,\mathsf{KB}}, x_{2,t}, \xi_t) = G_{2,b}(x^*_{1,\mathsf{KB}}, x^*_{2,\mathsf{KB}}, \xi_t) < G_{2,b}(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}}, \xi_t) = 0.$$

□ Uniqueness:

$\frac{dG_{2,b}(x_{1,t},x_{2,t},\xi_t)}{dx_{2,t}} < 0$ guarantees the uniqueness of the solution.

□ Shape:

The shape of the (KEEP, BUY) line, which starts at $(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}})$ and ends at $(x^*_{1,\mathsf{SB}}, x^*_{2,\mathsf{SB}})$, is interesting.[2].

[2]Refer to Figure 9 for more accurate understanding

Under the condition that

$$\eta^{\epsilon} < \frac{\xi_{t+1}}{\xi_t} \quad \text{with probability 1,} \tag{3.4.3}$$

we can determine how $x^*_{2,\mathsf{KB}}$ behaves as $x^*_{1,\mathsf{KB}}$ increases from $x^*_{1,\mathsf{BB}}$.

Lemma 3.4.2. *In the (KEEP, BUY) case, $x^*_{2,\mathsf{KB}}$ decreases as $x^*_{1,\mathsf{KB}}$ increases. In addition, function $x^*_{2,\mathsf{KB}}(x^*_{1,\mathsf{KB}})$ is concave.*

Proof. The optimal solution $(x^*_{1,\mathsf{KB}}, x^*_{2,\mathsf{KB}})$ should satisfy the following equation:

$$\begin{aligned} G_{2,b}(x^*_{1,\mathsf{KB}}, x^*_{2,\mathsf{KB}}) &= e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x^*_{1,\mathsf{KB}} + x^*_{2,\mathsf{KB}}}\right)^{\frac{1}{\epsilon}}\left(1 - \frac{x^*_{2,\mathsf{KB}}}{\epsilon(x^*_{1,\mathsf{KB}} + x^*_{2,\mathsf{KB}})}\right) - b_t - a_t \\ &+ e^{-r}\frac{d\mathsf{E}\left[V_{2,t+1}(x^*_{1,\mathsf{KB}}, x_{2,t}, \xi_{t+1}|\xi_t)\right]}{dx_{2,t}}\bigg|_{x^*_{2,\mathsf{KB}}} = 0 \end{aligned} \tag{3.4.4}$$

Considering cost depreciation with time, we can set

$$\frac{d\mathsf{E}\left[V_{2,t+1}(x^*_{1,\mathsf{KB}}, x_{2,t}, \xi_{t+1}|\xi_t)\right]}{dx_{2,t}}\bigg|_{x^*_{2,\mathsf{KB}}} = a_{2,t+1}.$$

Then, the equation (3.4.4) changes to a simple function:

$$e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x^*_{1,\mathsf{KB}} + x^*_{2,\mathsf{KB}}}\right)^{\frac{1}{\epsilon}}\left(1 - \frac{x^*_{2,\mathsf{KB}}}{\epsilon(x^*_{1,\mathsf{KB}} + x^*_{2,\mathsf{KB}})}\right) - \left(b_t + a_t - e^{-r}a_{t+1}\right) = 0. \tag{3.4.5}$$

Taking the derivative of equation (3.4.5) with respect to $x^*_{1,t}$, we get

$$\frac{dx^*_{2,\mathsf{KB}}}{dx^*_{1,\mathsf{KB}}} = -\frac{x^*_{1,\mathsf{KB}} - \frac{x^*_{2,\mathsf{KB}}}{\epsilon}}{2x^*_{1,\mathsf{KB}} + \left(1 - \frac{1}{\epsilon}\right)x^*_{2,\mathsf{KB}}}, \tag{3.4.6}$$

which is always negative in the (KEEP, BUY) case, because $x^*_{1,\mathsf{KB}} > x^*_{2,\mathsf{KB}}$.

Taking the derivative of equation (3.4.6) again with respect to $x^*_{1,\mathsf{KB}}$,we get

$$\frac{d^2x^*_{2,\mathsf{KB}}}{d(x^*_{1,\mathsf{KB}})^2} = -\left(\frac{1}{\epsilon} + 1\right)\frac{x^{*2}_{1,\mathsf{KB}} + \left(2 - \frac{1}{\epsilon}\right)x^*_{1,\mathsf{KB}}x^*_{2,\mathsf{KB}} + \left(1 - \frac{1}{\epsilon}\right)x^{*2}_{2,\mathsf{KB}}}{\left(2x^*_{1,\mathsf{KB}} + \left(1 - \frac{1}{\epsilon}\right)x^*_{2,\mathsf{KB}}\right)^3}, \tag{3.4.7}$$

which is always negative.

Therefore, $x^*_{2,t}$ is a concave function with respect to $x^*_{1,t}$ and decreases. □

This can be explained from point $(x^*_{1,t}, x^*_{2,t}) = (x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}})$. From this point, if firm 1 increases capacity, firm 2 cannot buy much capacity, which might be disadvantageous to both firms. Therefore, the optimal capacity for firm 2 is less than $x^*_{2,\mathsf{BB}}$. In this case, the total market capacity is more than $2 \cdot x^*_{2,\mathsf{BB}}$ because equation (3.4.6) > -1.

Combining with the implication of equation 3.4.2 in the case of (SELL, BUY), we can explain the implication of the shape of the line of (KEEP, BUY). If firm 2 has a very big amount of capacity, then firm 1 cannot enter the market. However, as the amount of capacity of firm 2 decreases, firm 1 can enter the market and increases its capacity.

5. (SELL, KEEP): Starting from point $(x_{1,t-1}, x_{2,t-1}) = (x^*_{1,\mathsf{SS}}, x^*_{2,\mathsf{SS}})$, as $x_{2,t-1}$ decreases below $x^*_{2,\mathsf{SS}}$, firm 2 goes into the KEEP region. The optimal capacity in this region is $(x^*_{1,t}, x^*_{2,t}) = (x^*_{1,\mathsf{SK}}, x_{2,t-1}) \equiv (x^*_{1,\mathsf{SK}}, x^*_{2,\mathsf{SK}})$. Therefore, if $x_{1,t-1} > x^*_{1,\mathsf{SK}}$, then firm 1 sells the excess capacity down to $x^*_{1,\mathsf{SK}}$ and firm 2 maintains current capacity with $x^*_{2,\mathsf{SB}} < x_{2,t-1} < x^*_{2,\mathsf{SS}}$.

   ☐ Existence:

   First, we know that $x^*_{1,\mathsf{SK}} > x^*_{2,\mathsf{SK}}$ because $x^*_{2,\mathsf{SK}} < x^*_{2,\mathsf{SS}} (= x^*_{1,\mathsf{SS}})$. Otherwise, $x^*_{1,\mathsf{SK}} < x^*_{2,\mathsf{SK}} < x^*_{1,\mathsf{SS}}$ and $x^*_{2,\mathsf{SK}} < x^*_{2,\mathsf{SS}}$, which means $(x^*_{1,\mathsf{SS}}, x^*_{2,\mathsf{SS}})$ is not an optimal solution for the (SELL, SELL) case.

$$
\begin{aligned}
(1) \quad \lim_{x_{1,t} \downarrow x^*_{2,\mathsf{SK}}} G_{1,s} \quad &= e^{-r} \xi_t^{\frac{1}{\epsilon}} \left(\frac{1}{2x^*_{2,\mathsf{SK}}} \right)^{\frac{1}{\epsilon}} \left(1 - \frac{1}{2\epsilon} \right) \\
&\quad -b_t - \tilde{a}_t + e^{-r} \frac{d\mathsf{E}\big[V_{1,t+1}(x_{1,t}, x^*_{2,\mathsf{SK}}, \xi_{t+1} | \xi_t)\big]}{dx_{1,t}} \bigg|_{x^*_{2,\mathsf{SK}}} \underbrace{>}_{\text{want it to be}} 0, \\
(2) \quad \lim_{x_{1,t} \uparrow \infty} G_{1,s} \quad &= -b_t - \tilde{a}_t + e^{-r} \tilde{a}_{t+1} < 0.
\end{aligned}
$$

   The positivity of equation (1) comes from the fact that

   (a) $G_{1,s}(x_{1,t}, x_{2,t}, \xi_t)$ is a decreasing function with respect to $x_{1,t}$,

   (b) $G_{1,s}(x^*_{1,\mathsf{SS}}, x^*_{2,\mathsf{SS}}, \xi_t) = 0$ and $x^*_{2,\mathsf{SK}} < x^*_{2,\mathsf{SS}}$.

(c) equation (1) can be re-written as

$$\lim_{x_{1,t} \uparrow x^*_{2,\mathsf{SK}}} G_{1,s}(x_{1,t}, x^*_{2,\mathsf{SK}}, \xi_t) = G_{1,s}(x^*_{1,\mathsf{SK}}, x^*_{2,\mathsf{SK}}, \xi_t) > G_{1,s}(x^*_{1,\mathsf{SS}}, x^*_{2,\mathsf{SS}}, \xi_t) = 0.$$

□ Uniqueness:

$\frac{dG_{1,s}(x_{1,t}, x_{2,t}, \xi_t)}{dx_{1,t}} < 0$ guarantees the uniqueness of the solution.

6. (BUY, SELL): Same as the case of (SELL, BUY) by exchanging firm 1 with firm 2.

7. (BUY, KEEP): Same as the case of (KEEP, BUY)by exchanging firm 1 with firm 2.

8. (KEEP, SELL): Same as the case of (SELL, KEEP) by exchanging firm 1 with firm 2.

Figure 9 illustrates a solution of the case of two symmetric firms with a fixed economic condition. As explained in Lemma 3.4.2, the line for the (KEEP, BUY) shows decreasing trend as $x_{1,t-1}(= x^*_{1,\mathsf{KB}})$ increases. In addition, the line for the (KEEP, BUY) shows concavity, but some discretization error exists. This graph corresponds to the horizontal plain in Figure 8.

3.4.2 N Symmetric Firms in an Oligopolistic Market

In this subsection, we want to investigate the investment behavior of a firm when more than two firms occupy the market and study the existence of the Cournot equilibrium point and the uniqueness of the point. Let N be the number of firms in the market. Then for each firm i, three different investment decisions, BUY, KEEP, and SELL, can be reached, depending on the decision of other firms and the general economic condition.

Again, we consider BUY, KEEP, and SELL cases for a firm separately.

1. (BUY): Let $x^*_{i,\mathsf{B}}$ be the solution for $G_{i,b}(x_{i,t}, \vec{X}_{-i,t}, \xi_t) = 0$ given $\vec{X}_{-i,t}$. If $x_{i,t-1} \le x^*_{i,\mathsf{B}}$, then firm i buys more capacity up to $x^*_{i,\mathsf{B}}$.

□ Existence:

$$\lim_{x_{i,t} \downarrow 0} G_{i,b}(x_{i,t}, \vec{X}_{-i,t}, \xi_t) = e^{-r} \xi_t^{\frac{1}{\epsilon}} \left(\frac{1}{x_{-i,t}} \right)^{\frac{1}{\epsilon}} - b_t - a_t + e^{-r} a_{t+1} \underbrace{>}_{\text{want it to be}} 0.$$

$$\lim_{x_{i,t} \uparrow \infty} G_{i,b}(x_{i,t}, \vec{X}_{-i,t}, \xi_t) \simeq -b_t - a_t + e^{-r} \tilde{a}_{t+1} < 0.$$

In order for $\lim_{x_{i,t}\downarrow 0} G_{i,b}(x_{i,t}, x_{-i,t}, \xi_t)$ to be positive, $x_{-i,t}$ should satisfy the following inequality:

$$x_{-i,t} < \left(\frac{e^{-r}}{b_t + a_t - e^{-r}a_{t+1}}\right)^{\epsilon} \xi_t. \tag{3.4.8}$$

The right side of the inequality is the limiting value of total market capacity when all N firms are in the BUY region[3].

Therefore, if all the firms are in the BUY region, there is a solution for firm i. If it does not satisfy the inequality, then $x_{i,t} = 0$, and we have to solve the problem again with $N - 1$ firms. The implication of this inequality is that, if the total capacity of other firms is very large, then it is not profitable for firm i to either enter or stay in the market. In addition, if the cost parameters of firms are very small, then firm i can enter the market very easily. Therefore, we conclude that in an oligopolistic market, firms are more likely to survive with efficient cost structure than with inefficient cost structure.

2. (SELL): Let $x^*_{i,\mathsf{S}}$ be the solution for $G_{i,s}(x_{i,t}, x_{-i,t}, \xi_t) = 0$ given $x_{-i,t}$. If $x_{i,t-1} \geq x^*_{i,\mathsf{S}}$, then firm i sells off the excess capacity down to $x^*_{i,\mathsf{S}}$.

   □ Existence:

$$\lim_{x_{i,t}\downarrow 0} G_{i,s}(x_{i,t}, \vec{X}_{-i,t}, \xi_t) = e^{-r}\xi_t^{\frac{1}{\epsilon}} \left(\frac{1}{x_{-i,t}}\right)^{\frac{1}{\epsilon}} - b_t - \tilde{a}_t + e^{-r}a_{t+1} \underbrace{>}_{\text{want it to be}} 0.$$

$$\lim_{x_{i,t}\uparrow \infty} G_{i,s}(x_{i,t}, \vec{X}_{-i,t}, \xi_t) \simeq -b_t - \tilde{a}_t + e^{-r}\tilde{a}_{t+1} < 0.$$

In order for $\lim_{x_{i,t}\downarrow 0} G_{i,s}(x_{i,t}, \vec{X}_{-i,t}, \xi_t)$ to be positive, $x_{-i,t}$ should satisfy the following inequality:

$$x_{-i,t} < \left(\frac{e^{-r}}{\max(b_t + \tilde{a}_t - e^{-r}a_{t+1}, 0^+)}\right)^{\epsilon} \xi_t.$$

The right-hand side of this equality is larger than that of equality 3.4.8. Therefore, if there is a solution for the (BUY) case, the solution for the (SELL) case exists. In addition, the right side of the inequality is the limiting value of the upper bound for

[3]Refer to subsection 3.4.3 to get the total market capacity when all N firms are in the BUY region.

the total market capacity with all N firms. Therefore, the SELL case always has a solution.

3. (KEEP): $x^*_{i,\mathsf{K}} = x_{i,t-1}$ with $x^*_{i,\mathsf{B}} < x_{i,t-1} < x^*_{i,\mathsf{S}}$.

3.4.3 The Effect of Competition on Market Properties with Cost Depreciation

As in the monopolistic case, we consider incremental investment with cost depreciation. Using the linear relationship between the optimal capacity at each period and the economic condition, we can write

$$x^*_{i,t,L} = l_{i,t,L}\xi_t \qquad \text{for all } t \text{ and for all } i.$$

Here $x^*_{i,t,L}$ is the lower optimal capacity of the firm i at time t, $l_{i,t,L}$ is the slope of the boundary of $\mathsf{KEEP/BUY}$, and ξ_t is the economic condition at time t.

By assuming that cost depreciates exponentially with constant cost depreciation factor η as

$$(b_t, a_t, \tilde{a}_t) = \eta(b_{t-1}, a_{t-1}, \tilde{a}_{t-1}) \text{ with } \eta < 1,$$

we establish a relationship between the slope at time t ($l_{i,t,L}$) and the slope at time $t+1$ ($l_{i,t+1,L}$), which is

$$l_{i,t+1,L} = \eta^{-\epsilon} l_{i,t,L}.$$

Therefore, the sufficient condition for the incremental investment is

$$\eta^{\epsilon} < \frac{\xi_{t+1}}{\xi_t} \text{ with probability 1.} \tag{3.4.9}$$

Under this condition, we have $x^*_{i,t+1,L} > x^*_{i,t,L}$.

Using this condition for the incremental investment, the $G_{i,b}$ function changes to

$$G_{i,b}(x_{i,t}, \vec{X}_{-i,t}, \xi_t) = e^{-r}\xi_t^{\frac{1}{\epsilon}} \left(\frac{x_{i,t}}{x_{i,t} + x_{-i,t}}\right)^{\frac{1}{\epsilon}} \left(1 - \frac{1}{\epsilon(x_{i,t} + x_{-i,t})}\right) - \left(b_t + a_t - e^{-r}a_{t+1}\right).$$

By setting $G_{i,b} = 0$ for all i, we obtain the analytic expression of the lower optimal capacity and the slope, which are

$$x^*_{1,t,L} = x^*_{2,t,L} = ... = x^*_{N,t,L} = \left(\frac{e^{-r}\left(N^{-\frac{1}{\epsilon}-1}\left(N - \frac{1}{\epsilon}\right)\right)}{b_t + a_t - e^{-r}a_{t+1}}\right)^{\epsilon} \xi_t,$$

$$l_{i,t,L} = \left(\frac{e^{-r}\left(N^{-\frac{1}{\epsilon}-1}\left(N-\frac{1}{\epsilon}\right)\right)}{b_t + a_t - e^{-r}a_{t+1}} \right)^{\epsilon}.$$

Remark 7. The sufficient condition for the incremental investment (equation (3.4.9)) is irrelevant to the number of firms in the market.

Under the condition $\eta^{\epsilon} < \frac{\xi_{t+1}}{\xi_t}$ with probability 1, we can investigate the effect of competition on the market properties.

☐ Total market capacity vs. the number of firms in the market.

The optimal capacity of the firm i at time t is

$$x^*_{i,t,L} = \left(\frac{e^{-r}\left(N^{-\frac{1}{\epsilon}-1}\left(N-\frac{1}{\epsilon}\right)\right)}{b_t + a_t - e^{-r}a_{t+1}} \right)^{\epsilon} \xi_t, \text{ with } N \text{ firms in the market.}$$

The total market capacity at time t is

$$\begin{aligned} \sum_{i=1}^{N} x^*_{i,t,L} \equiv X_t(N) &= N \left(\frac{e^{-r}\left(N^{-\frac{1}{\epsilon}-1}\left(N-\frac{1}{\epsilon}\right)\right)}{b_t + a_t - e^{-r}a_{t+1}} \right)^{\epsilon} \xi_t \\ &= \left(\frac{e^{-r}\left(1-\frac{1}{N\epsilon}\right)}{b_t + a_t - e^{-r}a_{t+1}} \right)^{\epsilon} \xi_t. \end{aligned}$$

As the above equation shows, we can see the competition causes the total market capacity to increase.

We can compare the optimal capacity of a firm in a duopoly market with that in a monopolistic market as follows:

$$\left(\frac{e^{-r}2^{-\frac{1}{\epsilon}-1}\left(2-\frac{1}{\epsilon}\right)}{b_t + a_t - e^{-r}a_{t+1}} \right)^{\epsilon} \xi_t > \left(\frac{e^{-r}\left(1-\frac{1}{\epsilon}\right)}{b_t + a_t - e^{-r}a_{t+1}} \right)^{\epsilon} \xi_t,$$

Therefore, the total capacity in a duopoly market is more than double of that in a monopolistic market.

In addition, we obtain the limiting value of total market capacity.

As $N \uparrow \infty$, the total capacity in the market is

$$\begin{aligned}
\sum_{i=1}^{\infty} x_{i,t}^{*} = X_t &= \lim_{N\uparrow\infty} N \left(\frac{e^{-r}\left(N^{-\frac{1}{\epsilon}-1}\left(N-\frac{1}{\epsilon}\right)\right)}{b_t + a_t - e^{-r}a_{t+1}} \right)^{\epsilon} \xi_t \\
&= \left(\frac{e^{-r}}{b_t + a_t - e^{-r}a_{t+1}} \right)^{\epsilon} \xi_t \\
&= \left(1-\frac{1}{\epsilon}\right)^{-\epsilon} \underbrace{\left(\frac{e^{-r}\left(1-\frac{1}{\epsilon}\right)}{b_t + a_t - e^{-r}a_{t+1}} \right)^{\epsilon} \xi_t}_{\text{optimal capacity of monopolistic firm}} \quad .
\end{aligned}$$

As the number of firms in the market increases, the total market capacity increases up to the limiting point. With a small number of firms in the market, the total market capacity almost reaches this limiting value. Therefore, adding more firms to the market does not significantly change market property when several firms are already in the market. This limiting value is $\left(1-\frac{1}{\epsilon}\right)^{-\epsilon}$ times the optimal capacity of the monopolistic firm. $\left(1-\frac{1}{\epsilon}\right)^{-\epsilon}$ ranges from e (=2.7183 approximately) to ∞ as ϵ goes from 1 to ∞. With our assumption that ϵ ranges from 1.28 to 2.84,$\left(1-\frac{1}{\epsilon}\right)^{-\epsilon}$ ranges from 3.4304 to 6.9963.

In Figure 10, the graph at the upper right-hand corner illustrates the relationship between market capacity and the number of firms in the market with a limiting value. Notice that a dramatic change occurs when a monopolistic market changes to a duopoly market. □ Market price vs. number of firms in the market.
The market price with N firms and the limiting value are

$$P_t(N) = \frac{N}{N-\frac{1}{\epsilon}} e^{r}\left(b_t + a_t - e^{-r}a_{t+1}\right),$$

$$\lim_{N\uparrow\infty} P_t(N) = e^{r}(b_t + a_t - e^{-r}a_{t+1}).$$

As N increases, the total market capacity increases. The corresponding market price decreases and the limiting value for the market price depends on neither ϵ nor the number of firms in the market. In addition, the price depreciation trend of the price is the same as the depreciation trend of the cost. If the trend of the price is different from that of the cost, the firms in the market either take the huge price advantage or shut down their firm because the price might be lower than the cost. Therefore, the market price and the cost should eventually show the same trend.

☐ Consumer surplus and producer surplus vs. number of firms in the market.

Figure 11 shows consumer surplus and producer surplus in some cases. The first one (upper left-hand corner) illustrates the definition of consumer surplus and producer surplus with increasing marginal cost function and linearly decreasing price function. The second one (upper right-hand corner) depicts the consumer surplus of our model. The third one (lower left-hand corner) and the fourth one (lower right-hand corner) represent the consumer surplus and producer surplus of two other relevant cost structures.

In our model, the cost is linear with the capacity, which leads to a constant marginal cost. Therefore, the producer surplus is always zero. As the number of firms in the market increases, market price and cost factors decrease. Therefore, the horizontal line becomes lower as the number of firms increases, causing consumer surplus to increase. The analytic form is as follows:

$$CS_t(N) = \int_{P_t(N)}^{\infty} \frac{\xi_t}{P_t(N)^{\epsilon}} dP_t(N) = \xi_t \frac{1}{\epsilon - 1} P_t(N)^{-\epsilon+1}, \text{ and}$$

$$\lim_{N \uparrow \infty} CS_t(N) = \xi_t \frac{1}{\epsilon - 1} \left(e^r (b_t + a_t - e^{-r} a_{t+1})\right)^{-\epsilon+1}.$$

In addition to our constant marginal cost function, we considere two other relevant cost structures: quadratic and exponential forms, as many studies utilize these two cost structures.

This quadratic cost function is frequently used in manufacturing industry. For example, if the demand function and the supply function are given by

$$P_t(N) = \left(\frac{\xi_t}{D_t(N)}\right)^{\frac{1}{\epsilon}}, \quad D_t(N) = A_t P_t(N) \text{ with a function } A_t.$$

Then the consumer surplus and producer surplus are

$$\begin{aligned} CS_t(N) &= \int_{P_t(N)}^{\infty} \frac{\xi_t}{P_t(N)^{\epsilon}} dP_t(N) = \xi_t \frac{1}{\epsilon - 1} P_t(N)^{-\epsilon+1}, \text{ and} \\ PS_t(N) &= \frac{1}{2} A_t P_t(N) \cdot P_t(N) = \frac{1}{2} A_t P_t(N)^2. \end{aligned}$$

Therefore, in this case, as the number of firms increases, consumer surplus increases and producer surplus decreases.

The last one to consider is the exponential form[4] of the cost function. If a company can take advantage of the scale of economy, the cost function might be given as an exponential form. For example, if the cost function is

$$C_t(D_t(N)) = D_t(N)e^{-A_t D_t(N)} \text{ with a function } A_t,$$

then the supply function is

$$P_t(N) = (1 - A_t D_t(N))e^{-A_t D_t(N)}.$$

This supply function, which is a decreasing function with $D_t(N)$, is convex, which leads producer surplus to be negative. In this case, as the number of firms increases, consumer surplus increases and producer surplus decreases.

□ Time to x% price reduction vs. number of firms in the market.

The market price is

$$P_t(N) = \frac{N}{N - \frac{1}{\epsilon}} e^r \left(b_t + a_t - e^{-r} a_{t+1} \right),$$

and the price decreases as the number of firms in the market increases. In addition, we know that the cost factor depreciates and is reflected directly in market price. Then, how does the price depreciation relate to the number of firms in the market? We answer this question here. Let us assume that the cost factor depreciates exponentially as

$$(b_t, a_t, \tilde{a}_t) = \eta(b_{t-1}, a_{t-1}, \tilde{a}_{t-1}) \text{ with } \eta < 1 \text{ for all } t.$$

Then,

$$\begin{aligned} P_{t+1}(N) &= \frac{N}{N - \frac{1}{\epsilon}} e^r \left(b_{t+1} + a_{t+1} - e^{-r} a_{t+2} \right) \\ &= \frac{N}{N - \frac{1}{\epsilon}} e^r \eta \left(b_t + a_t - e^{-r} a_{t+1} \right) \\ &= \eta P_t(N). \end{aligned}$$

[4]Kenyon and Cheliotis[20] used an exponential form of the cost function to reflect of the scale of economy.

The above equation shows that price reduction is only dependent on η and not on the number of firms in the market. Therefore, the time taken until x% price reduction is

$$n = \inf_k \{x > \eta^k, k \in \mathbf{N}\}$$

The underlying reason for this phenomenon is the following. As we discussed when we considered the relationship between market price and the number of firms in the market, the cost is directly reflected in the market price. In our model, cost depreciation, which is driven by technology improvement, is given as a function of t, not as a function of number of firms in the market. Therefore, the market price reduction is not dependent on the number of firms in the market, and the resulting depreciation pattern of price should be the same as that of cost. If we model cost depreciation to be dependent on the number of firms in the market, the price reduction should also be dependent on the number of the firms in the market.

□ Expected number of periods until first expansion vs. number of firms in the market.

Let us assume that the firms in the market have $\frac{1}{N}K$ capacity at time 0 with constraints

$$x_{i,0,L} \leq \frac{1}{N}K \leq x_{i,0,U} \text{ for all of } i. \tag{3.4.10}$$

Under this constraint, the total market capacity is K, and each firm in the market has $\frac{1}{N}K$ market share. In addition, every firm is in the KEEP region.

Now we will establish how the number of firms in the market affects the expected time to first expansion of firms. Let $\tau(N, K)$ be the expected waiting time of the first expansion. Then,

$$\begin{aligned} \tau(N,K) &= \mathsf{E}\left[\inf_t \left(\frac{e^{-r}N^{-\frac{1}{\epsilon}-1}\left(N-\frac{1}{\epsilon}\right)}{b_t + a_t - e^{-r}a_{t+1}}\right)^{\epsilon} \xi_t > \frac{1}{N}K\right] \\ &= \mathsf{E}\left[\inf_t \eta^{-t\epsilon}\xi_t > \mathsf{C}\frac{K}{\left(1-\frac{1}{N\epsilon}\right)^{\epsilon}}\right], \end{aligned}$$

where $\mathsf{C} = e^r(b_0 + a_0 - e^{-r}a_1)^{\epsilon}$.

From the above equation, we know the expected number of periods until first expansion decreases as the number of firms in the market increases. However, t is integer valued and

K is any real number that satisfies equation (3.4.10). To the best of our knowledge, there is no analytical expression for the expected waiting time for the first expansion in this case. By approximating the Markov process as a geometric Brownian motion, we do have an analytic expression for the expected waiting time and thus have some idea about how the number of firms in the market affects the expected waiting time of the first expansion.

First, we define a Wiener process as

$$Y(t) = \mu t + \sigma B(t),$$

where $B(t)$ is a standard Brownian motion, μ is drift term, and σ is a standard deviation term. Then we define the approximated geometric Brownian motion as

$$\tilde{\xi}_t = \xi_0 e^{Y(t)}.$$

The first two moments of ξ_t and $\tilde{\xi}_t$ are

$$\begin{aligned} \mathsf{E}\left[\xi_t\right] &= \eta^{-t\epsilon}(pu + (1-p)d), \text{ and} \\ \mathsf{E}\left[\xi_t^2\right] &= \eta^{-2t\epsilon}(pu^2 + (1-p)d^2), \end{aligned}$$

and

$$\begin{aligned} \mathsf{E}\left[\tilde{\xi}_t\right] &= e^{\mu t + \sigma^2 t/2}, \text{ and} \\ \mathsf{E}\left[\tilde{\xi}_t^2\right] &= e^{2\mu t + 2\sigma^2 t}, \end{aligned}$$

where we used discrete Markov provess with upward movement and downward movement to model the economic condition as we modeled it in subsection 2.6.2. By matching the first two moments of ξ_t and $\tilde{\xi}_t$ and changing of the parameters as $\mu = \acute{\mu} - \epsilon \log \eta$, we obtain the following two equations.

$$\begin{aligned} e^{-t\epsilon \log \eta}(pu + (1-p)d) &= e^{\acute{\mu} - \epsilon \log \eta t + \sigma^2 t/2}, \text{ and} \\ e^{-2t\epsilon \log \eta}(pu^2 + (1-p)d^2) &= e^{2\acute{\mu} - \epsilon \log \eta + 2\sigma^2 t}, \end{aligned}$$

Using the above equations, we can easily get expressions for $\acute{\mu}$ and σ of p, u, and d. We can then approximate ξ_t as a geometric Brownian motion with drift ($\mu = \acute{\mu} - \epsilon \log \eta$) and variance ($\sigma^2$).

The approximated expected waiting time for the first expansion is

$$\tau(N,K) = \begin{cases} \frac{1}{\acute{\mu}-\epsilon\log\eta}\log\left(\mathsf{C}\frac{K}{\xi_0\left(1-\frac{1}{N\epsilon}\right)^{\epsilon}}\right) & \text{if } \acute{\mu}-\epsilon\log\eta > 0, \\ \infty & \text{if } \acute{\mu}-\epsilon\log\eta \leq 0. \end{cases}$$

For the calculation to obtain the above equation, see section 3.6.

As we can see from the above equation, the expected time of the first expansion decreases as the number of firms in the market increases. In addition, with $\eta < 1$, $\log\eta$ is negative, which shortens the expected time of the first expansion. Smaller η represents steeper cost depreciation, which boosts the firms in the market to invest in additional capacity.

3.5 Asymmetric Firms in an Oligopolistic Market

In this section, we study the investment decisions of a firm in an oligopolistic market when the firms in the market have different cost structures. We assume the basic cost structure is the same in all firms. In other words, for any firm i, the cost occurring at time t is

$$C_{i,t}(x_{i,t-1}, x_{i,t}) = b_{i,t}x_{i,t} + a_{i,t}(x_{i,t} - x_{i,t-1})_+ - \tilde{a}_{i,t}(x_{i,t-1} - x_{i,t})_+.$$

The cost function still has the same form as that in the case of the monopolistic firm and in the case of N symmetric oligopolistic firms case. However in this case, the assumption

$$(b_{i,t}, a_{i,t}, \tilde{a}_{i,t}) = (b_t, a_t, \tilde{a}_t) \text{ for all } i \in \{1, ..., N\}$$

is not made, and we allow

$$(b_{i,t}, a_{i,t}, \tilde{a}_{i,t}) \neq (b_{j,t}, a_{j,t}, \tilde{a}_{j,t}) \text{ for all different pair of } i, j \in \{1, ..., N\}.$$

More specifically, one firm might have a smaller maintenance cost but a larger installation cost (and/or salvage value) than the other firms. In addition, as we discussed before, $b_{i,t}$, $a_{i,t}$, and $\tilde{a}_{i,t}$ are functions of t, which might reflect cost depreciation and/or functions of the number of firms in the market. In addition, we allow each firm to have different capacity of the previous period. Therefore, at time t, we allow $x_{i,t-1} \neq x_{j,t-1}$ for all different pair of $i, j \in 1, ..., N$.

In the next subsections, we will investigate the existence of the equilibrium and the uniqueness of the equilibrium point. In section 3.5.1, we will consider a duopoly market and in section 3.5.2, we will extend the results of section 3.5.1 to the case of N asymmetric firms.

3.5.1 Two Asymmetric Firms in a Duopoly Market

In this subsection, we consider two firms in a duopoly market. We assume that they have different cost factors and different capacity position in the previous period. We will examine the existence of the Cournot equilibrium point and the uniqueness of the point. As in the monopoly case, the possible decisions for each company are BUY, KEEP, and SELL. Therefore, with two companies in the market, nine different decisions can be considered.

The $G_{i,b}$ and $G_{i,s}$ functions are derivatives of the temporary value functions for each case of BUY and SELL.

$$\begin{aligned} G_{i,b}(x_{i,t}, x_{-i,t}, \xi_t) &= e^{-r} \xi_t^{\frac{1}{\epsilon}} \left(\frac{1}{x_{i,t} + x_{-i,t}} \right)^{\frac{1}{\epsilon}} \left(1 - \frac{x_{i,t}}{\epsilon(x_{i,t} + x_{-i,t})} \right) - b_{i,t} - a_{i,t} \\ &+ e^{-r} \frac{d\mathsf{E}\big[V_{i,t+1}(x_{i,t}, x_{-i,t}, \xi_{t+1}|\xi_t)\big]}{dx_{i,t}} \text{ for } i \in \{1,2\}. \end{aligned}$$

$$\begin{aligned} G_{i,s}(x_{i,t}, x_{-i,t}, \xi_t) &= e^{-r} \xi_t^{\frac{1}{\epsilon}} \left(\frac{1}{x_{i,t} + x_{-i,t}} \right)^{\frac{1}{\epsilon}} \left(1 - \frac{x_{i,t}}{\epsilon(x_{i,t} + x_{-i,t})} \right) - b_{i,t} - \tilde{a}_{i,t} \\ &+ e^{-r} \frac{d\mathsf{E}\big[V_{i,t+1}(x_{i,t}, x_{-i,t+1}, \xi_{t+1}|\xi_t)\big]}{dx_{i,t}} \text{ for } i \in \{1,2\}. \end{aligned}$$

1. (BUY, BUY): Let $(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}})$ be the solution for a system of equations, $\{G_{i,b}(x_{1,t}, x_{2,t}, \xi_t) = 0, i = 1,2\}$. If $x_{1,t-1} \leq x^*_{1,\mathsf{BB}}$ and $x_{2,t-1} \leq x^*_{2,\mathsf{BB}}$, then both firms 1 and 2 increase their capacity to $x^*_{1,\mathsf{BB}}$ and $x^*_{2,\mathsf{BB}}$, respectively.

$\square$ Existence.

$$\lim_{x_{1,t}\downarrow 0} G_{1,b}(x_{1,t},x_{2,t},\xi_t) = e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x_{2,t}}\right)^{\frac{1}{\epsilon}} - b_{1,t} - a_{1,t} + e^{-r}a_{1,t+1} \underbrace{>}_{\text{want it to be}} 0.$$

$$\lim_{x_{1,t}\uparrow\infty} G_{1,b}(x_{1,t},x_{2,t},\xi_t) \simeq -b_{1,t} - a_{1,t} + e^{-r}\tilde{a}_{1,t+1} < 0.$$

$$\lim_{x_{2,t}\downarrow 0} G_{2,b}(x_{1,t},x_{2,t},\xi_t) = e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x_{1,t}}\right)^{\frac{1}{\epsilon}} - b_{2,t} - a_{2,t} + e^{-r}a_{2,t+1} \underbrace{>}_{\text{want it to be}} 0.$$

$$\lim_{x_{2,t}\uparrow\infty} G_{2,b}(x_{1,t},x_{2,t},\xi_t) \simeq -b_{2,t} - a_{2,t} + e^{-r}\tilde{a}_{2,t+1} < 0.$$

In order for $\lim_{x_{1,t}\downarrow 0} G_{1,b}(x_{1,t},x_{2,t},\xi_t)$ and $\lim_{x_{2,t}\downarrow 0} G_{2,b}(x_{1,t},x_{2,t},\xi_t)$ to be positive, $x_{1,t}$ and $x_{2,t}$ should satisfy the following inequalities.

$$x_{2,t} < \left(\frac{e^{-r}}{b_{1,t}+a_{1,t}-e^{-r}a_{1,t+1}}\right)^{\epsilon}\xi_t \tag{3.5.1}$$

and

$$x_{1,t} < \left(\frac{e^{-r}}{b_{2,t}+a_{2,t}-e^{-r}a_{2,t+1}}\right)^{\epsilon}\xi_t. \tag{3.5.2}$$

Otherwise,

$$x_{2,t} > \left(\frac{e^{-r}}{b_{1,t}+a_{1,t}-e^{-r}a_{1,t+1}}\right)^{\epsilon}\xi_t \left(> \underbrace{\left(\frac{e^{-r}\left(1-\frac{1}{\epsilon}\right)}{b_{2,t}+a_{2,t}-e^{-r}a_{2,t+1}}\right)^{\epsilon}\xi_t}_{\text{optimal capacity of a monopolisitc firm.}}\right)$$

$$x_{1,t} > \left(\frac{e^{-r}}{b_{2,t}+a_{2,t}-e^{-r}a_{2,t+1}}\right)^{\epsilon}\xi_t \left(> \underbrace{\left(\frac{e^{-r}\left(1-\frac{1}{\epsilon}\right)}{b_{1,t}+a_{1,t}-e^{-r}a_{1,t+1}}\right)^{\epsilon}\xi_t}_{\text{optimal capacity of a monopolisitc firm.}}\right).$$

The second inequality is satisfied because the following two inequalities are satisfied.[5].

$$\sum_{i=1}^{2}\left(b_{i,t}+a_{i,t}-e^{-r}a_{i,t+1}\right) - \left(b_{1,t}+a_{1,t}-e^{-r}a_{1,t+1}\right)\left(2-\frac{1}{\epsilon}\right) > 0, \text{ and}$$

$$\sum_{i=1}^{2}\left(b_{i,t}+a_{i,t}-e^{-r}a_{i,t+1}\right) - \left(b_{2,t}+a_{2,t}-e^{-r}a_{2,t+1}\right)\left(2-\frac{1}{\epsilon}\right) > 0$$

[5]See Lemma 3.5.1

Therefore, if $x_{2,t}$ does not satisfy the above inequality, then $x_{1,t} = 0$ and firm 2 becomes a monopolistic firm. Also, if $x_{1,t}$ does not satisfy the above inequality, then $x_{2,t} = 0$, and firm 1 becomes a monopolistic firm.

The implication of the above inequalities is that, if the optimal capacity of firm 1 (firm 2) is very large, firm 2 (firm 1) is not allowed to enter the market. In addition, if the cost coefficients of firm 1 (firm 2) is small, the right-hand side of equality 3.5.1(3.5.2) tends to be large, and the equality has more possibility to satisfy. Therefore, we conclude that if a firm in duopolistic market has efficient cost structures, the firm is more likely to survive.

2. (SELL, SELL): Let $(x^*_{1,\mathsf{SS}}, x^*_{2,\mathsf{SS}})$ be the solution for a system of equations, $\{G_{i,s}(x_{1,t}, x_{2,t}, \xi_t) = 0, i = 1, 2\}$. If $x_{1,t-1} \geq x^*_{1,\mathsf{SS}}$ and $x_{2,t-1} \geq x^*_{2,\mathsf{SS}}$, then both of firms 1 and 2 reduce their capacity down to $x^*_{1,\mathsf{SS}}$ and $x^*_{2,\mathsf{SS}}$, respectively.

   □ Existence.

$$\lim_{x_{1,t}\downarrow 0} G_{1,s}(x_{1,t}, x_{2,t}, \xi_t) = e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x_{2,t}}\right)^{\frac{1}{\epsilon}} - b_{1,t} - \tilde{a}_{1,t} + e^{-r}a_{1,t+1} \underbrace{>}_{\text{want it to be}} 0.$$
$$\lim_{x_{1,t}\uparrow\infty} G_{1,s}(x_{1,t}, x_{2,t}, \xi_t) \simeq -b_{1,t} - \tilde{a}_{1,t} + e^{-r}\tilde{a}_{1,t+1} < 0.$$
$$\lim_{x_{2,t}\downarrow 0} G_{2,s}(x_{1,t}, x_{2,t}, \xi_t) = e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x_{1,t}}\right)^{\frac{1}{\epsilon}} - b_{2,s} - \tilde{a}_{2,t} + e^{-r}a_{2,t+1} \underbrace{>}_{\text{want it to be}} 0.$$
$$\lim_{x_{2,t}\uparrow\infty} G_{2,b}(x_{1,t}, x_{2,t}, \xi_t) \simeq -b_{2,t} - \tilde{a}_{2,t} + e^{-r}\tilde{a}_{2,t+1} < 0.$$

In order for $\lim_{x_{1,t}\downarrow 0} G_{1,s}(x_{1,t}, x_{2,t}, \xi_t)$ and $\lim_{x_{2,t}\downarrow 0} G_{2,s}(x_{1,t}, x_{2,t}, \xi_t)$to be positive, $x_{1,t}$ and $x_{2,t}$ should satisfy the following inequalities:

$$x_{2,t} < \left(\frac{e^{-r}}{\max(b_{1,t} + \tilde{a}_{1,t} - e^{-r}a_{1,t+1}, 0^+)}\right)^{\epsilon}\xi_t \tag{3.5.3}$$

and

$$x_{1,t} < \left(\frac{e^{-r}}{\max(b_{2,t} + \tilde{a}_{2,t} - e^{-r}a_{2,t+1}, 0^+)}\right)^{\epsilon}\xi_t. \tag{3.5.4}$$

Otherwise,

$$x_{2,t} > \left(\frac{e^{-r}}{\max(b_{1,t}+\tilde{a}_{1,t}-e^{-r}a_{1,t+1},0^{+})}\right)^{\epsilon}\xi_t \left(> \underbrace{\left(\frac{e^{-r}}{b_{1,t}+a_{1,t}-e^{-r}a_{1,t+1}}\right)^{\epsilon}\xi_t}_{\text{constraint for (BUY, BUY) case.}}\right)$$

$$x_{1,t} > \left(\frac{e^{-r}}{\max(b_{2,t}+\tilde{a}_{2,t}-e^{-r}a_{2,t+1},0^{+})}\right)^{\epsilon}\xi_t \left(> \underbrace{\left(\frac{e^{-r}}{b_{2,t}+a_{2,t}-e^{-r}a_{2,t+1}}\right)^{\epsilon}\xi_t}_{\text{constraint for (BUY, BUY) case.}}\right).$$

Therefore, if there is a solution for the (BUY, BUY) case, then there is a solution for (SELL, SELL). If $x_{2,t}$ does not satisfy the above inequality, then $x_{1,t} = 0$, and firm 2 becomes a monopolistic firm. Also, if $x_{1,t}$ does not satisfy the above inequality, then $x_{2,t} = 0$, and firm 1 becomes a monopolistic firm.

3. (SELL, BUY): Let $(x^*_{1,\mathsf{SB}}, x^*_{2,\mathsf{SB}})$ be the solution for a system of equations, $\{G_{1,s}(x_{1,t}, x_{2,t}, \xi_t) = 0, G_{2,b}(x_{1,t}, x_{2,t}, \xi_t) = 0\}$. If $x_{1,t-1} \geq x^*_{1,\mathsf{SB}}$ and $x_{2,t-1} \leq x^*_{2,\mathsf{SB}}$, then firm 1 sells off the excess capacity down to $x^*_{1,\mathsf{SB}}$, and firm 2 buys more capacity up to $x^*_{2,\mathsf{SB}}$.

   □ Existence.

$$\lim_{x_{1,t}\downarrow 0} G_{1,s}(x_{1,t}, x_{2,t}, \xi_t) = e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x_{2,t}}\right)^{\frac{1}{\epsilon}} - b_{1,t} - \tilde{a}_{1,t} + e^{-r}a_{1,t+1} \underbrace{>}_{\text{want it to be}} 0.$$

$$\lim_{x_{1,t}\uparrow\infty} G_{1,s}(x_{1,t}, x_{2,t}, \xi_t) \simeq -b_{1,t} - \tilde{a}_{1,t} + e^{-r}\tilde{a}_{1,t+1} < 0.$$

$$\lim_{x_{2,t}\downarrow 0} G_{2,b}(x_{1,t}, x_{2,t}, \xi_t) = e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x_{1,t}}\right)^{\frac{1}{\epsilon}} - b_{2,t} - a_{2,t} + e^{-r}a_{2,t+1} \underbrace{>}_{\text{want it to be}} 0.$$

$$\lim_{x_{2,t}\uparrow\infty} G_{2,b}(x_{1,t}, x_{2,t}, \xi_t) \simeq -b_{2,t} - a_{2,t} + e^{-r}\tilde{a}_{2,t+1} < 0.$$

   The conditions for the existence of the solution are inequalities 3.5.3 and 3.5.2. Therefore, if there are solutions for the cases of (SELL, SELL) and (BUY, BUY), then the solution for (SELL, BUY) exists.

4. (KEEP, BUY): As $x_{1,t-1}$ increases beyond $x^*_{1,\mathsf{BB}}$, firm 1 goes into the KEEP region. The optimal capacity in this region is $(x^*_{1,t}, x^*_{2,t}) = (x_{1,t-1}, x^*_{2,\mathsf{KB}}) \equiv (x^*_{1,\mathsf{KB}}, x^*_{2,\mathsf{KB}})$.

Therefore, if $x_{2,t-1} < x^*_{2,\mathsf{KB}}$, then firm 2 buys more capacity up to $x^*_{2,\mathsf{KB}}$ and firm 1 maintains its current capacity with $x^*_{1,\mathsf{BB}} < x_{1,t-1} < x^*_{1,\mathsf{SB}}$.

□ Existence.

$$\lim_{x_{2,t}\downarrow 0} G_{2,b}(x_{1,t}, x_{2,t}, \xi_t) = e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x_{1,t}}\right)^{\frac{1}{\epsilon}} - b_{2,t} - a_{2,t} + e^{-r}a_{2,t+1} \underbrace{>}_{\text{want it to be}} 0.$$

$$\lim_{x_{2,t}\uparrow\infty} G_{2,b}(x_{1,t}, x_{2,t}, \xi_t) \simeq -b_{2,t} - a_{2,t} + e^{-r}\tilde{a}_{2,t+1} < 0.$$

The condition for the existence of the solution is inequality 3.5.2. Therefore, if there is a solution for the case of (BUY, BUY), the solution for (KEEP, BUY) exists.

□ Shape

Using Lemma 3.4.2, we can illustrate the shape of the solution $(x^*_{1,t}, x^*_{2,t})$. The only difference in the value functions between the case of identical firms and that of the different firms is the cost coefficient. However, after taking the second derivative of the value function, the cost coefficient disappears. Therefore the first and the second derivatives of the G function have the exact same form in the cases of both identical firms and different firms, which allows us to use the equations (3.4.6) and (3.4.7) with no changes. However, in this case, the optimal capacities of firms 1 and 2 of the (BUY, BUY) case differ, which, in turn, causes slight difference between the shape of the (KEEP, BUY) line of the identical firms and the different firms.

To investigate the shape of the (KEEP, BUY) line, we write the derivative again as follows:

$$\frac{dx^*_{2,t}}{dx^*_{1,t}} = -\frac{x^*_{1,t} - \frac{x^*_{2,t}}{\epsilon}}{2x^*_{1,t} + \left(1 - \frac{1}{\epsilon}\right)x^*_{2,t}}. \tag{3.5.5}$$

The numerator is $x^*_{1,t} - \frac{x^*_{2,t}}{\epsilon}$ which can be positive or negative depending on the solution pair $(x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}})$. Without loss of generality, we assume that the cost structure of firm 2 is more efficient than that of firm 1, which results in $x^*_{1,\mathsf{BB}} < x^*_{2,\mathsf{BB}}$. However, how much bigger $x^*_{2,\mathsf{BB}}$ is than $x^*_{1,\mathsf{BB}}$ depends on the cost difference between the two firms and ϵ.

(a) $x^*_{1,\mathsf{BB}} > \frac{x^*_{2,\mathsf{BB}}}{\epsilon}$, which is attained when the cost difference is small and/or when ϵ is large:

In this case, $x^*_{2,t}$ decreases as $x^*_{1,t}$ increases. Under this condition, $x^*_{1,t} - \frac{x^*_{2,t}}{\epsilon}$ is always positive and equation (3.5.5)is always negative. When the cost difference is not significant, the shape of the (KEEP, BUY) line is similar to that of the case of identical firms.

(b) $x^*_{1,\mathsf{BB}} < \frac{x^*_{2,\mathsf{BB}}}{\epsilon}$, which is attained when the cost difference is significant and/or ϵ is small:

In this case, $x^*_{2,t}$ increases at first and decreases later as $x^*_{1,t}$ increases. Under this condition, $x^*_{1,t} - \frac{x^*_{2,t}}{\epsilon}$ is negative at $(x^*_{1,t}, x^*_{2,t}) \simeq (x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}})$. As $x^*_{1,t}$ increases, $x^*_{1,t} - \frac{x^*_{2,t}}{\epsilon}$ becomes positive because $\left|\frac{dx^*_{2,t}}{dx^*_{1,t}}\right| < 1$, and equation (3.5.5) becomes negative.

These changes can be explained from point $(x^*_{1,t}, x^*_{2,t}) = (x^*_{1,\mathsf{BB}}, x^*_{2,\mathsf{BB}})$. From that point, if firm 1 increases the capacity, which is still less than $\frac{x^*_{2,t}}{\epsilon}$, then firm 2 increases its capacity. As firm 1 continues to increase the capacity which is larger than $\frac{x^*_{2,t}}{\epsilon}$, the firm 2 decreases its capacity. Therefore, we can conclude that if the initial capacity of firm 1 is small, firm 2 can increase its capacity taking cost advantage, but if the initial capacity of firm 1 is too large which indicates that the firm 1 has occupied a large portion of the market, then firm 2 cannot buy much capacity, which might be disadvantageous to both firms.

Figure 9 illustrates a solution of the case of two asymmetric firms with a fixed economic condition. The line for the (KEEP, BUY) shows increasing and then decreasing trend as $x_{1,t-1} (= x^*_{1,\mathsf{KB}})$ increases. In addition, the line for the (KEEP, BUY) shows concavity with a maximum point.

Remark 8. Notice that concavity is maintained in both cases. However, the shape in the case of (KEEP, BUY) has a pick point, but the shape in the case of (BUY, KEEP) does not.

5. (SELL,KEEP): As $x_{2,t-1}$ decreases below $x^*_{1,\mathsf{SS}}$, firm 2 goes into the KEEP region. Firm 1 sells the excess capacity and firm 2 maintains its given capacity. The optimal

capacity in this region is $(x^*_{1,t}, x^*_{2,t}) = (x^*_{1,\mathsf{SK}}, x_{2,t-1}) \equiv (x^*_{1,\mathsf{SK}}, x^*_{2,\mathsf{SK}})$. If $x_{1,t-1} \geq x^*_{1,\mathsf{SK}}$, then firm 1 sells off the excess capacity down to $x^*_{1,\mathsf{SB}}$ and firm 2 maintains its current capacity with $x^*_{2,\mathsf{SB}} < x_{2,t-1} < x^*_{2,\mathsf{SS}}$.

☐ Existence.

$$\lim_{x_{1,t}\downarrow 0} G_{1,s}(x_{1,t}, x_{2,t}, \xi_t) = e^{-r}\xi_t^{\frac{1}{\epsilon}}\left(\frac{1}{x_{2,t}}\right)^{\frac{1}{\epsilon}} - b_{1,t} - \tilde{a}_{1,t} + e^{-r}a_{1,t+1} \underbrace{>}_{\text{want it to be}} 0.$$

$$\lim_{x_{1,t}\uparrow\infty} G_{1,s}(x_{1,t}, x_{2,t}, \xi_t) \simeq -b_{1,t} - \tilde{a}_{1,t} + e^{-r}\tilde{a}_{1,t+1} < 0.$$

The conditions for the existence of the solution is inequality 3.5.3. Therefore, if there is a solution for the cases of (SELL, SELL), then the solution for (SELL, KEEP) exists.

6. (BUY, KEEP): Same to the case of (KEEP, BUY) by exchanging firm 1 with firm 2.

   ☐ Shape.

   $$\frac{dx^*_{1,t}}{dx^*_{2,t}} = -\frac{x^*_{2,t} - \frac{x^*_{1,t}}{\epsilon}}{2x^*_{2,t} + \left(1 - \frac{1}{\epsilon}\right)x^*_{1,t}}$$

   is always negative because $x^*_{2,\mathsf{BB}} \geq x^*_{1,\mathsf{BB}}$. Therefore, $x^*_{1,t}$ is a concave function with respect to $x^*_{2,t}$ but with no maximum point, and $x^*_{1,t}$ is a decreasing function with respect to $x^*_{2,t}$.

7. (BUY, SELL): Same to the case of (BUY, SELL) by exchanging firm 1 with firm 2.

8. (KEEP,SELL): Same to the case of (SELL, KEEP) by exchanging firm 1 with firm 2.

3.5.2 N Asymmetric Firms in an Oligopolistic Market

In this subsection, we want to investigate the investment behavior of a firm when more than two firms occupy the market and study the existence of the Cournot equilibrium point and the uniqueness of the point. Let N be the number of firms in the market. Then for each firm i, there can be three different investment decisions, BUY, KEEP, and SELL, can be reached, depending on the decision of the other firms and the general economic condition.

Again, we consider the BUY, KEEP, and SELL cases for a firm separately.

1. (BUY): Let $x^*_{i,\mathsf{B}}$ be the solution of $G_{i,b}(x_{i,t}, \vec{X}_{-i,t}, \xi_t) = 0$, given $x_{-i,t}$. If $x_{i,t-1} \le x^*_{i,\mathsf{B}}$, then firm i buys more capacity up to $x^*_{i,\mathsf{B}}$.

   □ Existence:

$$\lim_{x_{i,t}\downarrow 0} G_{i,b}(x_{i,t}, x_{-i,t}, \xi_t) = e^{-r}\xi_t^{\frac{1}{\epsilon}} \left(\frac{1}{x_{-i,t}}\right)^{\frac{1}{\epsilon}} - b_{i,t} - a_{i,t} + e^{-r}a_{i,t+1} \underbrace{>}_{\text{want it to be}} 0.$$

$$\lim_{x_{i,t}\uparrow \infty} G_{i,b}(x_{i,t}, x_{-i,t}, \xi_t) \simeq -b_{i,t} - a_{i,t} + e^{-r}\tilde{a}_{i,t+1} < 0.$$

   In order for $\lim_{x_{i,t}\downarrow 0} G_{i,b}(x_{i,t}, x_{-i,t}, \xi_t)$ to be positive, $x_{-i,t}$ should satisfy the following inequality:

$$x_{-i,t} < \left(\frac{e^{-r}}{b_{i,t} + a_{i,t} - e^{-r}a_{i,t+1}}\right)^{\epsilon} \xi_t.$$

   Otherwise,

$$x_{-i,t} > \left(\frac{e^{-r}}{b_{i,t} + a_{i,t} - e^{-r}a_{i,t+1}}\right)^{\epsilon} \xi_t \left(> \left(\frac{e^{-r}\left(N - 1 - \frac{1}{\epsilon}\right)}{\sum_{j=1,j\neq i}^{N}(b_{i,t} + a_{i,t} - e^{-r}a_{i,t+1})}\right)^{\epsilon} \xi_t\right).$$

   The inequality in parentheses is from Lemma 3.5.1 and the amount in the parentheses is the total market capacity when all $N-1$ firms are in the BUY region. Therefore, if all the firms are in the BUY region, there is a solution for firm i. If it does not satisfy the inequality, then $x_{i,t} = 0$ and we have to solve the problem again with $N-1$ firms. The implication of this inequality is that, if the total capacity of other firms is very large, firm i is not allowed to enter the market. In addition, if the cost parameters of firm i is very small, then firm i can enter the market very easily. Therefore, we conclude that if firms in oligopolistic market have efficient cost structures, all firms are more likely to survive.

2. (SELL): Let $x^*_{i,\mathsf{S}}$ be the solution of $G_{i,s}(x_{i,t}, x_{-i,t}, \xi_t) = 0$, given $x_{-i,t}$. If $x_{i,t-1} \ge x^*_{i,\mathsf{S}}$, then firm i sells off the excess capacity down to $x^*_{i,\mathsf{S}}$.

   □ Existence:

$$\lim_{x_{i,t}\downarrow 0} G_{i,s}(x_{i,t}, x_{-i,t}, \xi_t) = e^{-r}\xi_t^{\frac{1}{\epsilon}} \left(\frac{1}{x_{-i,t}}\right)^{\frac{1}{\epsilon}} - b_{i,t} - \tilde{a}_{i,t} + e^{-r}a_{i,t+1} \underbrace{>}_{\text{want it to be}} 0.$$

$$\lim_{x_{i,t}\uparrow \infty} G_{i,s}(x_{i,t}, x_{-i,t}, \xi_t) \simeq -b_{i,t} - \tilde{a}_{i,t} + e^{-r}\tilde{a}_{i,t+1} < 0.$$

In order for $\lim_{x_{i,t}\downarrow 0} G_{i,s}(x_{i,t}, x_{-i,t}, \xi_t)$ to be positive, $x_{-i,t}$ should satisfy the following inequality:

$$x_{-i,t} < \left(\frac{e^{-r}}{\max(b_{i,t} + \tilde{a}_{i,t} - e^{-r}a_{i,t+1}, 0^+)}\right)^{\epsilon} \xi_t.$$

Therefore, if there is a solution for the (BUY) case, the solution for the (SELL) case exists. In addition, the right side of the inequality is the limiting value of the upper bound for the total market capacity with all N firms. Therefore, the SELL case always has a solution.

3. KEEP: $x^*_{i,\mathsf{K}} = x_{i,t-1}$ with $x^*_{i,\mathsf{B}} < x_{i,t-1} < x^*_{i,\mathsf{S}}$.

3.5.3 The Effect of Competition on Market Properties with Cost Depreciation

As in the previous case, we consider incremental investment with cost depreciation. Adopting the reasoning in section 2.5, under the condition that $\eta^\epsilon \leq \frac{\xi_{t+1}}{\xi_t}$ for the worst economic condition transition, we can write $G_{i,b}$ as follows:

$$\left\{ G_{i,b} = e^{-r}\xi_t^{\frac{1}{\epsilon}} \left(\frac{1}{x_{i,t} + x_{-i,t}}\right)^{\frac{1}{\epsilon}} \left(1 - \frac{x_{i,t}}{\epsilon(x_{i,t} + x_{-i,t})}\right) - \left(b_{i,t} + a_{i,t} - e^{-r}a_{i,t+1}\right), i = 1, ..., N \right\}.$$

Then, we can obtain a solution vector for the above set of N equations. The i^{th} component of the solution vector is

$$x_{i,t} = \epsilon \left(1 - e^r \xi^{-\frac{1}{\epsilon}} \left(b_{i,t} + a_{i,t} - e^{-r}a_{i,t+1}\right) X_t^{\frac{1}{\epsilon}}\right) X_t,$$

with constraints $x_{i,t} > 0$ for all $i \in \{1, ..., N\}$,

and X_t is the total market capacity given by

$$\sum_{i=1}^{N} x_{i,t} = X_t(N) = \left(\frac{N - \frac{1}{\epsilon}}{e^r \sum_{i=1}^{N} \left(b_{i,t} + a_{i,t} - e^{-r}a_{i,t+1}\right)}\right)^{\epsilon} \xi_t.$$

If there is any i for which $x_{i,t} < 0$, then set $x_{i,t} = 0$ and take firm i out of consideration and solve a set of $(N-1)$ equations. The equivalent condition for $x_{i,t} > 0$ is as follows.

Lemma 3.5.1. *The equivalent condition for $x_{i,t} > 0$ for all $i \in \{1, ..., N\}$ is*

$$\sum_{j=1}^{N} \left(b_{j,t} + a_{j,t} - e^{-r}a_{j,t+1}\right) - \left(b_{i,t} + a_{i,t} - e^{-r}a_{i,t+1}\right)\left(N - \frac{1}{\epsilon}\right) > 0 \text{ for all } i \in \{1, ..., N\} \tag{3.5.6}$$

This condition is also satisfied componentwise.

Proof. The proof is straightforward. □

This condition indicates that any firm with a large cost structure will be out of the market. Therefore, if one company enters an oligopolistic market with very small cost coefficients, the incumbent company with largest cost might be out of the market; otherwise, the incumbent firms should consider the way to decrease the cost.

□ Total market capacity vs. the number of firms in the market.

Does any case in which adding an additional firm to the market causes the total market capacity to decrease exist? For such a situation occur, the following inequality must be satisfied.

$$\left(\frac{N-\frac{1}{\epsilon}}{e^r \sum_{i=1}^{N}\left(b_{i,t}+a_{i,t}-e^{-r}a_{i,t+1}\right)}\right)^{\epsilon}\xi_t > \left(\frac{N+1-\frac{1}{\epsilon}}{e^r \sum_{i=1}^{N+1}\left(b_{i,t}+a_{i,t}-e^{-r}a_{i,t+1}\right)}\right)^{\epsilon}\xi_t. \quad (3.5.7)$$

Proposition 3.5.2. *The equivalent condition for the inequality 3.5.7 is*

$$\sum_{j=1}^{N+1}\left(b_{j,t}+a_{j,t}-e^{-r}a_{j,t+1}\right)-\left(b_{N+1,t}+a_{N+1,t}-e^{-r}a_{N+1,t+1}\right)\left(N+1-\frac{1}{\epsilon}\right)<0.$$

This violates the necessary condition of $x_{N+1,t}>0$. Therefore, we conclude that the total market capacity increases as the number of firms in the market increases.

With sufficiently large N,

$$X_t(N)=\left(\frac{N}{e^r \sum_{i=1}^{N}\left(b_{i,t}+a_{i,t}-e^{-r}a_{i,t+1}\right)}\right)^{\epsilon}\xi_t\left(\simeq\left(\frac{e^{-r}}{b_t+a_t-e^{-r}a_{t+1}}\right)^{\epsilon}\xi_t\right).$$

Therefore the case of identical firms is a special case of different firms in which

$$\frac{1}{N}\sum_{i=1}^{N}\left(b_{i,t}+a_{i,t}-e^{-r}a_{i,t+1}\right)=\left(b_t+a_t-e^{-r}a_{t+1}\right).$$

□ Market price vs. the number of firms in the market.

$$\begin{aligned}P_t(N) &= \frac{N}{N-\frac{1}{\epsilon}}e^r\frac{1}{N}\sum_{i=1}^{N}\left(b_{i,t}+a_{i,t}-e^{-r}a_{i,t+1}\right)\\ &= e^r\frac{1}{N}\sum_{i=1}^{N}\left(b_{i,t}+a_{i,t}-e^{-r}a_{i,t+1}\right)\text{ with very large N.}\end{aligned}$$

Again, the case of symmetric firms is a special case of asymmetric firms. Because of increasing capacity, the market price decreases as the number of firms in the market increases.

□ Consumer surplus and producer surplus vs. number of firms in the market.

As discussed in subsection 3.4.3, the producer surplus is zero regardless of the number of firms in the market.

The consumer surplus is given by

$$CS_t(N) = \int_{P_t(N)}^{\infty} \frac{\xi_t}{P_t(N)^{\epsilon}} dP_t(N) = \xi_t \frac{1}{\epsilon - 1} P_t(N)^{-\epsilon+1}.$$

$CS_t(N)$ is an increasing function with respect to the number of firms in the market with a limiting value of

$$\lim_{N\uparrow\infty} CS_t(N) = \xi_t \frac{1}{\epsilon - 1} \left(e^r \bar{C}\right)^{-\epsilon+1},$$

where $\bar{C} = \lim_{N\uparrow\infty} \frac{1}{N} \sum_{i=1}^{N} (b_{i,t} + a_{i,t} - e^{-r} a_{i,t+1})$.

□ Time to an x% price reduction vs. number of firms in the market.

This is the same as the case of identical firms. We can write the relationship between the market price with N firms and $N+1$ firms as follows:

$$\begin{aligned}
P_t(N) &= \frac{N}{N - \frac{1}{\epsilon}} e^r \frac{1}{N} \sum_{i=1}^{N} \left(b_{i,t} + a_{i,t} - e^{-r} a_{i,t+1}\right), \\
P_{t+1}(N) &= \frac{N}{N - \frac{1}{\epsilon}} e^r \frac{1}{N} \sum_{i=1}^{N} \left(b_{i,t+1} + a_{i,t+1} - e^{-r} a_{i,t+2}\right) \\
&= \frac{N}{N - \frac{1}{\epsilon}} e^r \frac{1}{N} \sum_{i=1}^{N} \left(\eta(b_{i,t} + a_{i,t} - e^{-r} a_{i,t+1})\right) \\
&= \eta P_t(N) \text{ with cost depreciation factor } \eta.
\end{aligned}$$

Therefore, for an x% price reduction, we need at least n periods which satisfy,

$$n = \inf_k \{x > \eta^k, k \in \mathbf{N}\},$$

regardless of the number of firms in the market.

□ Expected number of periods until the first expansion vs. the number of firms in the market.

Let us assume that the firms in the market have $\frac{1}{N}K$ capacity at time 0 with

$$x_{i,0,L} \le \frac{1}{N}K \le x_{i,0,U} \text{ for all of } i. \tag{3.5.8}$$

Under this constraint, the total market capacity is K and each firm in the market has $\frac{1}{N}K$ market share. In addition, all firms are in the KEEP region.

We want to know how the number of firms in the market affects the expected time to the first expansion.

Let $\tau_i(N,K)$ be the expected waiting time of the first expansion of firm i. Then,

$$\tau_i(N,K) = \mathsf{E}\left[\inf_t \epsilon \left(1 - e^r \xi^{-\frac{1}{\epsilon}} \left(b_{i,t} + a_{i,t} - e^{-r} a_{i,t+1}\right) X_t^{\frac{1}{\epsilon}}\right) X_t > \frac{1}{N}K\right].$$

However, we cannot obtain an analytic form for $\tau_i(N,K)$ in this case. Unlike in the case of symmetric firms, the lower optimal capacity of one firm reaches $\frac{1}{N}K$ faster and that of the other firm reaches $\frac{1}{N}K$ later, which makes it hard to determine total market capacity (X_t) at time $\tau_i(N,K)$. In addition, $\tau_i(N,K)$ is a function of the cost coefficients of firm i as well as the cost coefficients of other firms, so it will behave differently when we add another firm in the market, depending on the cost structure of the ith firm, and the cost structure of the entering firm.

To have some idea of how the expected waiting time of the first expansion of firm i is related to the number of firms in the market, we investigate the behavior of the following function.

$$\left(1 - \frac{C_i(t)\left(N - \frac{1}{\epsilon}\right)}{e^r \sum_{i=1}^N \left(b_{i,t} + a_{i,t} - e^{-r} a_{i,t+1}\right)}\right) \left(\frac{N - \frac{1}{\epsilon}}{e^r \sum_{i=1}^N \left(b_{i,t} + a_{i,t} - e^{-r} a_{i,t+1}\right)}\right)^{\epsilon} \xi_t - \frac{1}{N}K \tag{3.5.9}$$

From the result of the case of the symmetric firms, we can expect that equation (3.5.9) is an increasing function with respect to N for all of $t \in \{1, 2, ..., T\}$.

Using $C_i(t) = b_{i,t} + a_{i,t} - e^{-r} a_{i,t+1} = \eta_0 \eta_1 \cdots \eta_{t-1} (b_{i,0} + a_{i,0} - e^{-r} a_{i,0})$, equation 3.5.9 can be written as

$$\left(1 - \frac{C_i(0)\left(N - \frac{1}{\epsilon}\right)}{e^r \sum_{i=1}^N \left(b_{i,0} + a_{i,0} - e^{-r} a_{i,1}\right)}\right) \left(\frac{N - \frac{1}{\epsilon}}{e^r \sum_{i=1}^N \left(b_{i,0} + a_{i,0} - e^{-r} a_{i,1}\right)}\right)^{\epsilon} \left(\prod_{m=0}^{t-1} \eta_m^{\epsilon}\right) \xi_t - \frac{1}{N}K, \tag{3.5.10}$$

and this equation is expected to increase with respect to N, regardless of the functional form η_t, because we assumed that deprecation applied to the all firms in the market in the same manner.

Nevertheless, unlike the result of the case of symmetric firms, the property of equation (3.5.10) depends on the cost factors of $(N+1)\,th$ firm.

The sufficient condition for equation (3.5.10) to increase with N is

$$\sum_{i=1}^{N+1} C_i(t) - C_j\left(N+1-\frac{1}{\epsilon}\right) > \sum_{i=1}^{N} C_i(t) - C_j\left(N-\frac{1}{\epsilon}\right).$$

Therefore,

1. if $C_{N+1}(t) > C_i(t)$ for all i, then the function (3.5.10) has more possibility to increase when we add $(N+1)\,th$ firm to the market, and

2. if $C_{N+1}(t) > C_i(t)$ for some i, then the function (3.5.10)has more possibility to increase only for those i when we add $(N+1)\,th$ firm to the market but it remains undetermined for the other firms.

3.6 Proofs

Proof of Theorem 3.3.1. The value function at the new point $(\eta x_{1,t-1}, \eta x_{2,t-1}, ..., \eta x_{n,t-1}, \eta\xi_t)$ is

$$\begin{aligned}
&V_{i,t}(\eta\vec{X}_{t-1}, \eta\xi_t) \\
&= \left\{ e^{-r}\left(\frac{\eta\xi_t}{\bar{x}^*_{i,t}+\bar{x}^*_{-i,t}}\right)^{\frac{1}{\epsilon}} \bar{x}^*_{i,t} - \left(b_{i,t}\bar{x}^*_{i,t} + a_{i,t}(\bar{x}^*_{i,t} - \eta x_{i,t-1})_+ - \tilde{a}_{i,t}(\eta x_{i,t-1} - \bar{x}^*_{i,t})_+\right)\right. \\
&\left. +e^{-r}\left(\sum p(\eta\xi_t, \eta\xi_{t+1}) V_{i,t+1}\left(\bar{x}^*_{i,t}, \eta\xi_{t+1}\right)\right)\right\} \text{ for all } i \in \{1,2,...,N\},
\end{aligned}$$

and the optimal vector $\vec{X}_t^* \equiv (\bar{x}_{1,t}^*, \bar{x}_{2,t}^*, ..., \bar{x}_{N,t}^*)$ is a solution for a set of equations, $\{\tilde{V}_{i,t}(\eta x_{i,t-1}, \eta \vec{X}_{-i,t}, \eta\xi_t), i = 1, 2, ..., N\}$. Then $V_{i,t}(\eta \vec{X}_{t-1}, \eta\xi_t)$ can be re-written as

$$\begin{aligned}
& V_{i,t}(\eta x_{i,t-1}, \eta\xi_t) \\
& = \eta \cdot \left\{ e^{-r} \left(\frac{\xi_t}{\frac{1}{\eta}\bar{x}_{i,t}^* + \frac{1}{\eta}\bar{x}_{-i,t}^*} \right)^{\frac{1}{\epsilon}} \frac{1}{\eta}\bar{x}_{i,t}^* \right. \\
& - \left(b_{i,t}\frac{1}{\eta}\bar{x}_{i,t}^* + a_{i,t}(\frac{1}{\eta}\bar{x}_{i,t}^* - x_{i,t-1})_+ - \tilde{a}_{i,t}(x_{i,t-1} - \frac{1}{\eta}\bar{x}_{i,t}^*)_+ \right) \\
& \left. + e^{-r} \left(\sum p(\xi_t, \xi_{t+1}) V_{i,t+1}\left(\frac{1}{\eta}\vec{X}_t^*, \xi_{t+1} \right) \right) \right\} \\
= \; & \eta V_{i,t}(x_{i,t-1}, \xi_t), \text{ with } \frac{1}{\eta}\bar{x}_{i,t}^* = x_{i,t}^* \text{ for all } i,
\end{aligned}$$

where we used the homogeneity of $V_{i,t+1}$ and $p(\eta\xi_t, \eta\xi_{t+1}) = p(\xi_t, \xi_{t+1})$. Therefore,

$$\bar{x}_{i,t}^* = \eta x_{i,t}^* \text{ for all } i \in \{1, 2, ..., N\}.$$

In addition, we get equation $V_{i,t}(\eta \vec{X}_{t-1}, \eta\xi_t) = \eta V_{i,t}(\vec{X}_{t-1}, \xi_t)$, which confirms the homogeneity of $V_{i,t}$. □

The waiting time for the geometric Brownian motion. Let $X(t)$ be a Brownian motion with drift parameter μ and variance parameter σ^2. Let us suppose that $X(0) = x$ and $a < x$ and $b > x$ are fixed quantities. In addition we define a hitting time as

$$T_{ab} = \min\{t \geq 0; X(t) = a \text{ or } X(t) = b\}$$

Then,

$$u(x) \equiv \mathsf{P}\{X(T_{ab}) = b | X(0) = x\} = \frac{e^{-2\mu x/\sigma^2} - e^{-2\mu a/\sigma^2}}{e^{-2\mu b/\sigma^2} - e^{-2\mu a/\sigma^2}}. \tag{3.6.1}$$

In addition,

$$\nu(x) \equiv \mathsf{E}\big[T_{ab} | X(0) = x\big] = \frac{1}{\mu}\left[u(x)(b-a) - (x-a)\right]. \tag{3.6.2}$$

In our case, $a = -\infty$, $b = \log\left(\mathsf{C}\frac{K}{\xi_0\left(1 - \frac{1}{N\epsilon}\right)^{\epsilon}} \right)$ and $x = 0$ with $\mu = \acute{\mu} - \epsilon \log \eta$. Therefore, the expected waiting time is,

$$\tau(N, K) = \begin{cases} \frac{1}{\acute{\mu} - \epsilon\log\eta} \log\left(\mathsf{C}\frac{K}{\xi_0\left(1 - \frac{1}{N\epsilon}\right)^{\epsilon}} \right) & \text{if } \acute{\mu} - \epsilon\log\eta > 0, \\ \infty & \text{if } \acute{\mu} - \epsilon\log\eta \leq 0. \end{cases}$$

Here we want to prove equation (3.6.1) and equation (3.6.2).

At time Δt,

$$X(\Delta t) = X(0) + \Delta X = x + \Delta X,$$

where we assume that Δt is very small so that exiting the interval (a, b) before Δt can be neglected.

The conditional probability of exiting at the upper point b is

$$u(x + \Delta X) = \mathsf{P}\{X(T_{ab} = b | X(\Delta t) = x + \Delta X\}$$

$$\begin{aligned} u(x) &= \mathsf{P}\{X(T_{ab}) = b | X(0) = x\} \\ &= \mathsf{E}\big[\mathsf{P}\{X(T_{ab}) = b | X(0) = x, X(\Delta t) = x + \Delta X\}\big] \\ &= \mathsf{E}\big[\mathsf{P}\{X(T_{ab}) = b | X(\Delta t) = x + \Delta X\}\big] \\ &= \mathsf{E}\big[u(x + \Delta X)\big] \end{aligned}$$

Next, expend $u(x + \Delta X)$ in a Taylor series,

$$u(x + \Delta X) = u(x) + u^{'}(x)\Delta X + \frac{1}{2}u^{''}(x)(\Delta X)^2 + o(\Delta X)^2$$

$$\begin{aligned} u(x) &= \mathsf{E}\big[u(x + \Delta X)\big] \\ &= u(x) + u^{'}(x)\mathsf{E}\big[\Delta X\big] + \frac{1}{2}u^{''}(x)\mathsf{E}\big[(\Delta X)^2\big] + \mathsf{E}\big[o(\Delta t)\big] \\ &= u(x) + u^{'}(x)\mu\Delta t + \frac{1}{2}u^{''}(x)\sigma^2\Delta t + \mathsf{E}\big[o(\Delta t)\big] \end{aligned}$$

$$0 = \mu u^{'}(x) + \frac{1}{2}\sigma^2 u^{''}(x) \tag{3.6.3}$$

The solution to equation (3.6.3) is

$$u(x) = Ae^{-2\mu x/\sigma^2} + B$$

Using boundary conditions $u(a) = 0$, $u(b) = 1$, we get the equation (3.6.1).

Next, let us prove the expression for $\nu(x)$.

At time Δt,

$$X(\Delta t) = X(0) + \Delta X = x + \Delta X,$$

where we again assume that Δt is very small so that exiting the interval (a, b) before Δt can be neglected.

The conditional mean time to exit the interval is now

$$\Delta t + \nu(\Delta t)$$

$$\begin{aligned}
\nu(x) &= \mathsf{E}\big[T_{ab}|X(0) = x\big] \\
&= \mathsf{E}\big[\Delta t + \nu(x + \Delta t)|X(\Delta t) = x + \Delta X\big] \\
&= \Delta t + \mathsf{E}\big[\nu(x + \Delta t)\big]
\end{aligned}$$

Next, expend $\nu(x + \Delta X)$ in a Taylor series,

$$\nu(x + \Delta X) = u(x) + \nu^{'}(x)\Delta X + \frac{1}{2}\nu^{''}(x)(\Delta X)^2 + o(\Delta X)^2$$

$$\begin{aligned}
\nu(x) &= \Delta t + \mathsf{E}\big[\nu(x + \Delta X)\big] \\
&= \Delta t + \nu(x) + u^{'}(x)\mathsf{E}\big[\Delta X\big] + \frac{1}{2}\nu^{''}(x)\mathsf{E}\big[(\Delta X)^2\big] + \mathsf{E}\big[o(\Delta X)^2\big] \\
&= \Delta t + \nu(x) + \nu^{'}(x)\mu\Delta t + \frac{1}{2}\nu^{''}(x)\sigma^2\Delta t + o(\Delta t)
\end{aligned}$$

$$-1 = \mu\nu^{'}(x) + \frac{1}{2}\sigma^2\nu^{''}(x) \tag{3.6.4}$$

The solution to equation (3.6.4) is

$$\nu(x) = Au(x) - \frac{x}{\mu} + B$$

Using boundary conditions $u(a) = 0$, $u(b) = 0$, we get the equation (3.6.2).

□

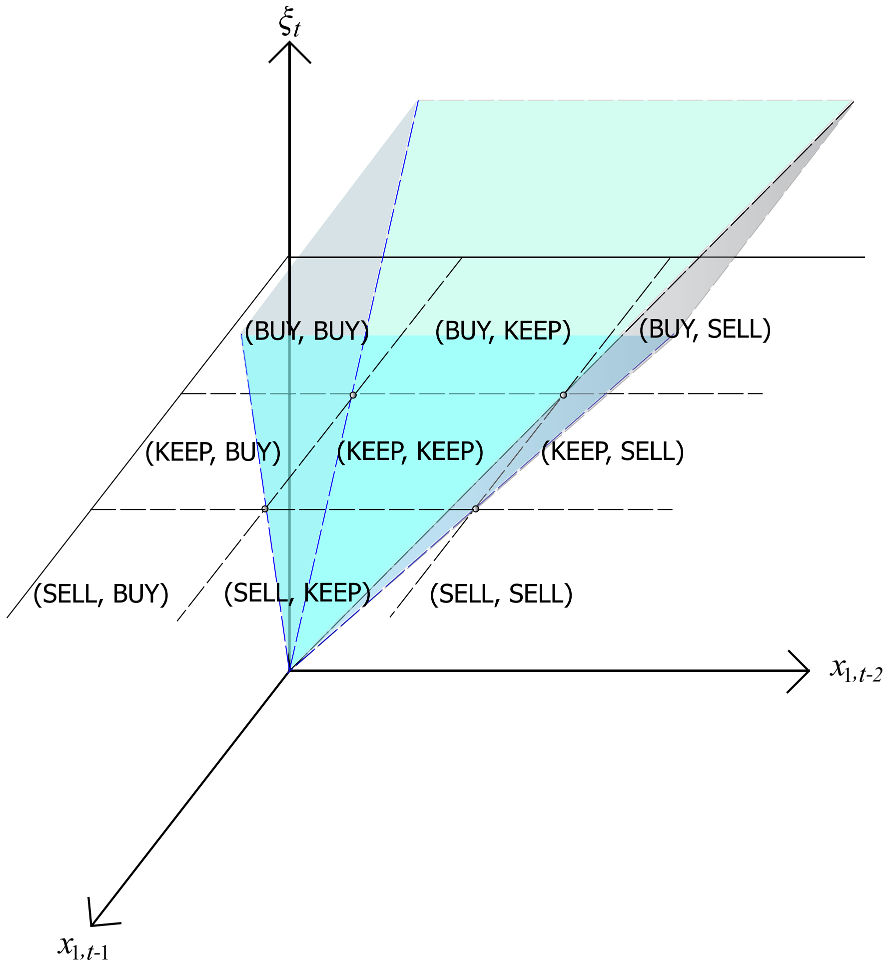

Figure 8: The linearity between the optimal capacities of two firms and the economic indicator in a duopolistic market

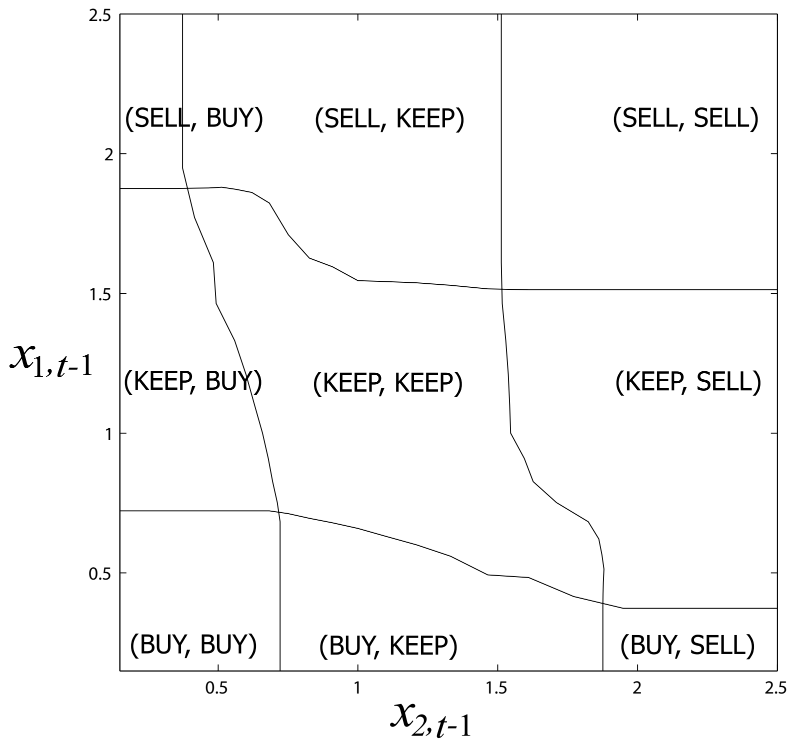

Figure 9: Optimal capacities of two symmetric firms with a fixed economic indicator

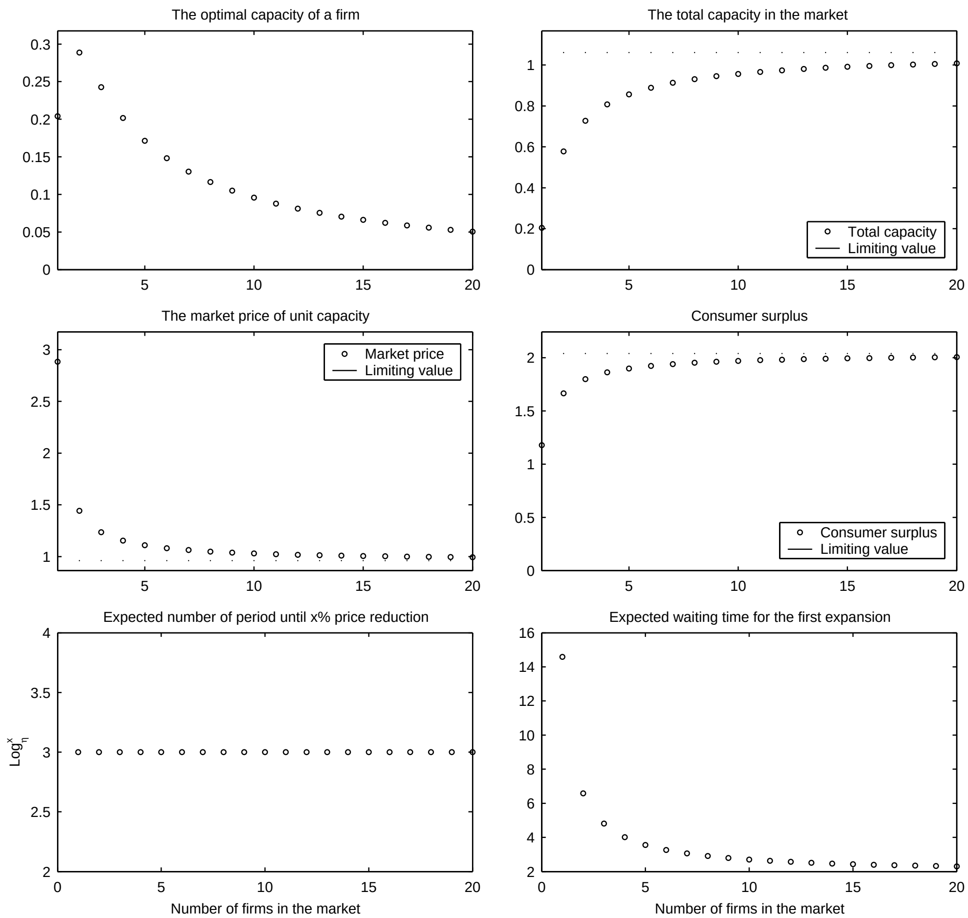

Figure 10: Optimal capacity of a firm, total market capacity, market price, consumer surplus, expected number of periods until certain price reduction and expected waiting time for the first expansion with the number of firms in the market

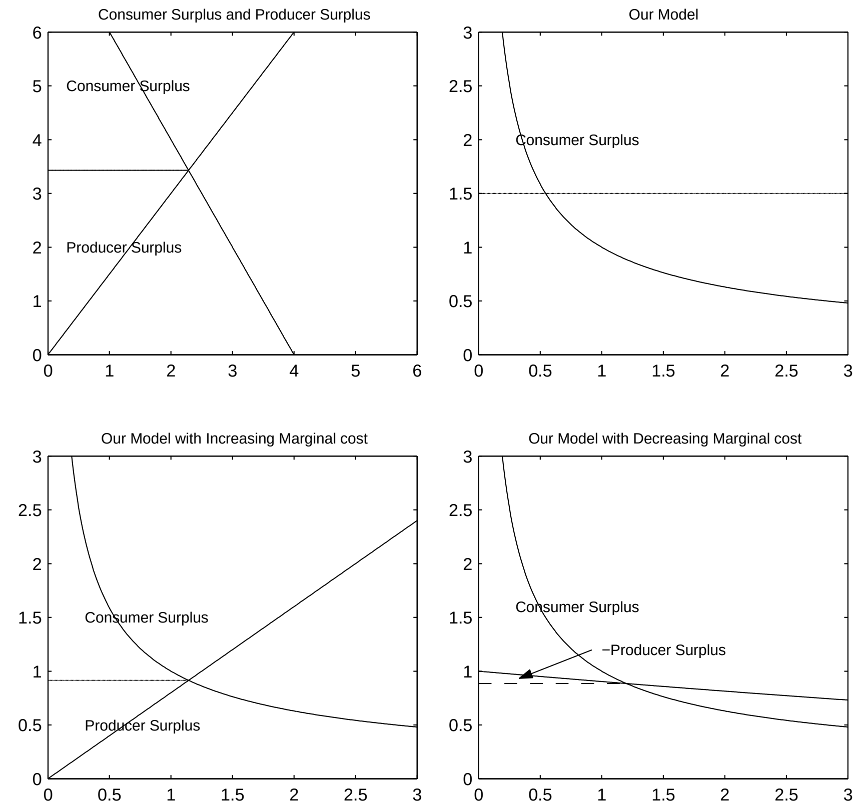

Figure 11: Consumer surplus and producer surplus with different cost structures

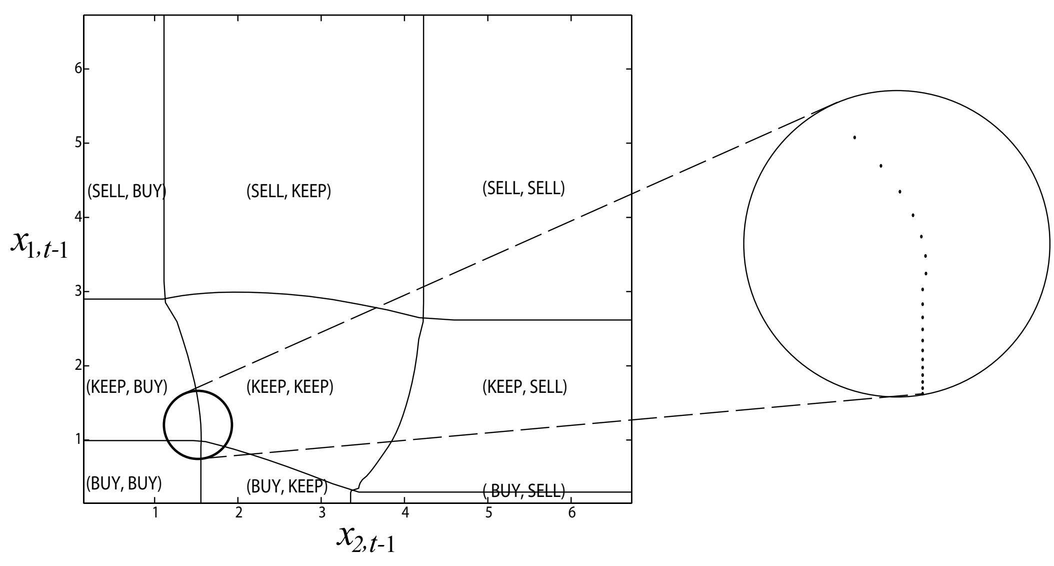

Figure 12: Optimal capacities of two asymmetric firms with a fixed economic indicator with concavity of the KEEP/BUY boundary

CHAPTER IV

AN EXPERIMENTAL ANALYSIS OF CAPACITY INVESTMENT IN AN OLIGOPOLISTIC MARKET

4.1 The Relationship Between The Optimal Capacity of a Firm and Economic Condition

In this section, our goal is to establish a relationship between the investment behaviors of firms in a market employing a set of real market data. From the previous section, we know the optimal capacity at time t is

$$x_{i,t} = \epsilon \left(1 - e^r \xi^{-\frac{1}{\epsilon}} \left(b_{i,t} + a_{i,t} - e^{-r} a_{i,t+1} \right) X_t^{\frac{1}{\epsilon}} \right) X_t \text{ for all } i \in \{1, ..., N\},$$

with X_t, the total market capacity, given by

$$\sum_{i=1}^{N} x_{i,t} = X_t = \left(\frac{N - \frac{1}{\epsilon}}{e^r \sum_{i=1}^{N} \left(b_{i,t} + a_{i,t} - e^{-r} a_{i,t+1} \right)} \right)^{\epsilon} \xi_t.$$

After some calculation, it yields

$$x_{i,t} = \epsilon \left(1 - C_i(t) \frac{N - \frac{1}{\epsilon}}{\sum_{j=1}^{N} C_j(t)} \right) \left(\frac{N - \frac{1}{\epsilon}}{e^r \sum_{j=1}^{N} C_j(t)} \right)^{\epsilon} \xi_t,$$

where $C_i(t) = b_{i,t} + a_{i,t} - e^{-r} a_{i,t+1}$. $C_i(t)$ might depend on the number of firms in the industry. With our assumption that the cost depreciates exponentially, we can write

$$C_i(t+1) = \eta_t C_i(t) \text{ for all } i \in \{1, ..., N\} \text{ and for all } t \in \{0, ..., T-1\}.$$

We assume that (1) all companies have the same cost depreciation structure, and (2) maintenance cost, installation cost, and salvage value depreciate at the same rate. As defined in section 2.5, η_t is a cost depreciation coefficient applicable to the entire industry, which is dependent on improvements of technology in corresponding sectors/industries, suggested by Kou and Kou[22].

The equation yields

$$
\begin{aligned}
x_{i,t+k} &= \epsilon\left(1 - C_i(t)\frac{N-\frac{1}{\epsilon}}{\sum_{j=1}^{N} C_j(t)}\right)\left(\frac{N-\frac{1}{\epsilon}}{e^r\sum_{j=1}^{N}\left(\prod_{m=t}^{t+k-1}\eta_m\right)C_j(t)}\right)^{\epsilon}\xi_{t+k} \\
&= \epsilon\left(1 - C_i(t)\frac{N-\frac{1}{\epsilon}}{\sum_{j=1}^{N} C_j(t)}\right)\left(\frac{N-\frac{1}{\epsilon}}{e^r\sum_{j=1}^{N} C_j(t)}\right)^{\epsilon}\left(\prod_{m=t}^{t+k-1}\eta_m\right)^{-\epsilon}\xi_{t+k}
\end{aligned}
$$

for all $k = 0, 1, ..., T-t$ and for all $i \in \{1, ..., N\}$.

Substituting

$$x_{i,t+k} = x_{i,t} + \sum_{m=1}^{k} \hat{x}_{i,t+m} \text{ for all } i \in \{1, ..., N\},$$

the above equation yields

$$\sum_{m=1}^{k} \hat{x}_{i,t+m} = -x_{i,t} + \epsilon\left(1 - C_i(t)\frac{N-\frac{1}{\epsilon}}{\sum_{j=1}^{N} C_j(t)}\right)\left(\frac{N-\frac{1}{\epsilon}}{e^r\sum_{j=1}^{N} C_j(t)}\right)^{\epsilon}\left(\prod_{m=t}^{t+k-1}\eta_m\right)^{-\epsilon}\xi_{t+k}.$$

Finally, we have an equation of a time series. If we fix a time t, then the equation looks like

$$\sum_{m=1}^{k} \hat{x}_{i,t+m} = -x_{i,t} + \mathsf{A}(t)\cdot\mathsf{B}(k)\xi_{t+k} \text{ with } k = 0, 1, ..., T-t,$$

where $\mathsf{B}(k) = (\eta_t\eta_{t+1}\cdots\eta_{t+k-1})^{\epsilon}$.

If we assume $\eta_m = \eta$ for all $m = t, t+1, ..., t+k-1$, the above equation changes to

$$\sum_{m=1}^{k} \hat{x}_{i,t+m} = -x_{i,t} + \mathsf{A}(t)\cdot\eta^{-k\epsilon}\xi_{t+k} \text{ with } k = 0, 1, ..., T-t. \tag{4.1.1}$$

In the next subsection, using market data, we establish a linear relationship between $\sum_{m=1}^{k}\hat{x}_{i,t+m}$ and $\eta^{-k\epsilon}\xi_{t+k}$, i.e., between the sum of capacity increments of a firm and economic indicators scaled by the cost depreciation factor, as time goes from t to $t+k$.

4.2 Experimental Analysis of Telecommunications Market Data

As our target is the telecommunications market, we choose "Communications Services" as our specific industry in "Services" sector. In addition, we consider only United States firms in forming the data set. We consider top-ranked firms in the communications services industry based on market capitalization.

To perform the empirical analysis, we require market data for capacity/capital increments for the chosen companies and the economic indicator.

4.2.1 Basic Statistics of the Capacity Expenditures

Capacity/capital purchase/sale contracts between firms are confidential and not publicly known. Therefore, we need a proxy for the capacity/capital increment.

Every firm has three financial statements, an Income Statement, a Balance Sheet, and a Cash Flow Statement that are available to the public. We choose "Capital Expenditures" in Cash Flow as our proxy for the capacity/capital investment.

In addition, we want to split telecommunications companies into wireless companies and wireline companies. If wireless companies invest in a previously no-coverage area, the actual demand can grow more than proportionally to the capacity increment because of the network effect. Therefore, we decide to apply our model to wireline-dedicated companies only. However, most telecommunications companies provide a combination of wireless and wireline services, we need to extract wireline-related investment from Capital Expenditures. Because capital investment for wireline capacity is not available, we have analyzed the revenue/profit of companies.

Under the following assumptions:

1. The revenue/profit ratio between wireline and other assets is the same as the capital expenditure ratio between the two.

2. Without any big changes in a company, it has continued to maintain this ratio constant for the past several years.

We obtain proxies of capital investment committed to wireline facilities and we used the quarterly data of Capital Expenditures for the following firms[1] from 4th quarter of 1999 to the 3rd quarter of 2004. Table 7 shows the names, market capitalization and wireline portion of the companies. The letters in parentheses represent the ticker symbols for each company. (We will use this ticker symbol instead of the full name of the firm from now on.) As we stated before, we need data for revenues/profits from wireline services

[1] We choose the top seven firms in the communications services industry. Out of these firms, we eliminate the Sprint Corp.(FON) due to an insufficient amount of data.

Table 7: Name, market cap., and wireline service portion of the companies considered

Company Name	Market Cap.	Wireline Portion
Verizon Communications (VZ)	114.16B	64.1 %
SBC Communications Inc. (SBC)	83.45B	75 %
BellSouth Corporation (BLS)	49.13B	approximately 100 %
Nextel Communications (NXTL)	31.63B	approximately 0 %
Alltel Corporation (AT)	17.22B	40.2 %
AT&T Corp (T)	14.56B	75% / 100 %
Qwest Communications International Inc.(Q)	7.26B	100 %

only. From S&P 500 corporation record[41], we can find similar data for some companies. The percentages refer to the revenue/profit portion that comes from the wireline services. AT&T includes a spin-off of AT&T wireless on July 9th 2001, which corresponds to 25% of the total capital. Therefore, until the 2nd quarter of 2001 , the wireline services reach 75% and from the 3rd quarter of 2001, these reach approximately 100%. For BLS and Q, we ignored information services and directory advertising & publishing, and wireless services which comprise less than 10% of the revenue/profit.

From the wireline and other asset data, we eliminate NXTL from this analysis, confining our study to VZ, SBC, BLS, AT, T, and Q.

First, we perform basic data analysis that includes descriptive statistics and bivariate correlation coefficients. Tables 8 and 9 show basic descriptive statistics of the capital

Table 8: Descriptive statistics of the capital expenditures of firms

	VZ	SBC	BLS	AT	T	Q
Mean	2270.87	1701.19	1186.95	199.30	1674.63	1414.92
Standard Deviation	642.99	904.29	462.51	308.43	1113.72	1477.75
Range	2165.94	3499.50	1467.00	1411.70	3711.00	6207.00
Minimum	1521.73	672.75	588.00	57.53	441.00	418.00
Maximum	3687.67	4172.25	2055.00	1469.23	4152.00	6625.00
Ratio(Range/Mean)	0.95	2.06	1.24	7.08	2.22	4.39
Ratio(Std./Mean)	0.28	0.53	0.39	1.55	0.67	1.04

expenditures of each firm and bivariate correlation coefficient between firms, respectively. AT has the largest std./mean ratio and largest range/mean ratio. Thus we can conclude

Table 9: Pairwise correlation coefficient between firms

		VZ	SBC	BLS	AT	T	Q
VZ	correlation	1.000	0.477	0.677	-0.186	0.404	0.139
	Sig. (2-tailed)	.	0.034	0.001	0.433	0.077	0.559
SBC	correlation	0.477	1.000	0.883	-0.080	0.856	0.849
	Sig. (2-tailed)	0.034	.	0.000	0.739	0.000	0.000
BLS	correlation	0.677	0.883	1.000	-0.136	0.760	0.686
	Sig. (2-tailed)	0.001	0.000	.	0.569	0.000	0.001
AT	correlation	-0.186	-0.080	-0.136	1.000	0.021	-0.141
	Sig. (2-tailed)	0.433	0.739	0.569	.	0.929	0.554
T	correlation	0.404	0.856	0.760	0.021	1.000	0.627
	Sig. (2-tailed)	0.077	0.000	0.000	0.929	.	0.003
Q	correlation	0.139	0.849	0.686	-0.141	0.627	1.000
	Sig. (2-tailed)	0.559	0.000	0.001	0.554	0.003	.

that AT is very volatile. The capital investment of AT is negatively correlated with that of VZ, SBC, BLS, and Q and it is positively correlated with that of T. However, none of the correlation coefficients of AT with other firms is significant. Therefore, we can assume that AT behaves independently of other firms, and the investment behavior of AT is very different from that of other firms. Thus, we eliminate AT from our experimental analysis.

Quite different from AT, VZ is a very stable firm and investment decisions are consistent throughout the period and its range/mean ratio of the capital expenditure is the smallest, indicating the amount of investment does not change much during the period.

In addition, Table 9 shows that VZ, SBC, and BLS are positively correlated with 95% confidence. SBC, BLS, T, and Q are positively correlated with 95%(99%) confidence. Therefore, we can assume that all the firms, except AT, exhibit similar investment behavior.

4.2.2 Basic Statistics of the Economic Indicators

In an effort to determine our economic indicator, we choose three candidates: Civil Employment, the Consumer Confidence Index, and the Nasdaq Telecommunications Index (IXUT). Again, IXUT is a ticker symbol for the Nasdaq Telecommunications Index. (We will use this ticker instead of the full name.) The first and second are general economic indicators and the third is a sector/industry-specific economic indicator.

From Economic Indicators [42], the indices on the Gross Domestic Product, the Real Gross Domestic Product and other factors[2] all serve as general economic indicators. Among these economic indicators, we choose Civilian Employment (Employment) as a possible proxy of an economic indicator for the telecommunications market. Our rationale is that as employment increases, both Internet usage at work and the number of mobile phone subscribers increase, and thus the demand for additional capacity increases. The reason we select the Consumer Confidence Index as a candidate as a proxy of an economic indicator is similar to the reason why we select Employment. As consumers often expect better future economic conditions, they increase their spending, which will affect Internet/mobile phone usage. For IXUT, we choose it because it is a strong indicator of the telecommunications market.

We take the quarterly data from the 3th quarter of 1996 to the 3rd quarter of 2004 and compute the basic statistics, and list them in Tables 10 and 11.

Figure 13 shows the trend of these three economic indicators. Employment shows clearly the increasing trend with small volatility of the three indicators. Consumer Confidence Index does not show any clear drift, but has a larger variation than Employment. IXUT does not show any trend, but clearly indicates when the Internet bubble existed. The graph shows that Internet bubble started around the beginning of 1998 and ended at the beginning of 2002. During this period, numerous telecommunications firms were established and then became defunct.

The trend is well explained in Table 10, which shows basic descriptive statistics. In addition to the basic statistics of the data, it shows the basic statistics of the return ($\frac{\xi_t}{xi_{t-1}} - 1$) of that period. In the prediction of future economic conditions, the value of return reflected in the economic indicator is more meaningful than the bare number.

Table 11 shows the bivariate correlation coefficients of economic indicators. When

[2]See Economic Indicators [42] for the full list of the economic indicators

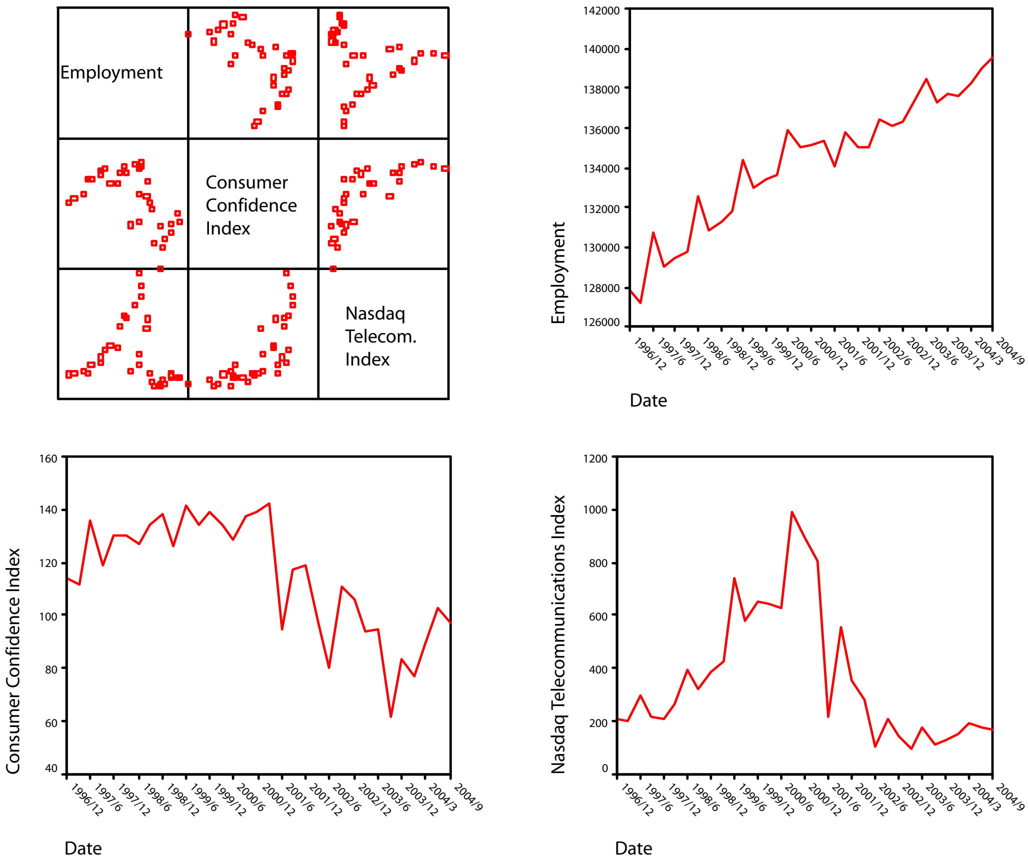

Figure 13: Trend of economic indicators and bivariate matrix graphs between economic indicators

we compare the return, with 95% confidence, we cannot determine if they are positively/negatively correlated. However, we can say Employment and the Consumer Confidence Index are negatively correlated and the Consumer Confidence Index and IXUT are positively correlated with 95% (99%) confidence. Therefore, we can expect a negatively correlated trend between Employment and the Consumer Confidence Index and a positively correlated trend between the Consumer Confidence Index and IXUT.

As shown in Figure 13, Employment and the Consumer Confidence Index are negatively correlated and the Consumer Confidence Index and IXUT are positively correlated. However, when we compare the returns, we find no clear relationship, but instead completely independent behavior.

The matrix scatter plot in Figure 13 illustrates this relationship very well. The trend between Employment and the Consumer Confidence Index is downward sloping and the trend between the Consumer Confidence Index and IXUT is curved and upward sloping. In addition, the relationship between Employment and IXUT appears to be random.

Table 10: Descriptive statistics of the economic indicators

	Employment		CCI		IXUT	
		return		return		return
Mean	134254.88	0.0029	114.72	0.0019	360.90	0.0133
Standard Deviation	3320.02	0.0044	22.0	0.1177	248.24	0.1914
Range	12232.00	0.0226	81.10	0.5991	890.89	0.8151
Minimum	127248.00	-0.0070	61.40	-0.2392	96.55	-0.3626
Maximum	139480.00	0.0156	142.50	0.3599	987.44	0.4525
Ratio (Std./Mean)	0.02473	1.5410	0.1919	61.2154	0.6878	14.3421

Table 11: Pairwise correlation coefficients of the economic indicators

					Return		
		Emt.	CCI	IXUT	Emt.	CCI	IXUT
Emt.	correlation	1.000	-0.541	-0.082	1.000	0.314	0.232
	Sig. (2-tailed)	.	0.001	0.650	.	0.080	0.200
CCI	correlation	-0.541	1.000	0.739	0.314	1.000	0.206
	Sig. (2-tailed)	0.001	.	0.000	0.080	.	0.259
IXUT	correlation	-0.082	0.739	1.000	0.232	0.206	1.000
	Sig. (2-tailed)	0.650	0.000	.	0.200	0.259	.

From this basic statistical analysis, we conclude that Employment is the most stable variable, which might lead us to choose Employment as our economic indicator. To confirm that Employment is best best economic indicator for our model, we perform linear regression on all three economic indicators and the capital increments of the firms in the next subsection.

4.2.3 Regression Analysis between the Capacity Expenditure of Firms and the Economic Indicators

From the previous basic analysis, we briefly examine the characteristics of our data. We expect that AT will behave differently from VZ, SBC, BLS, T, and Q, so we exclude AT from our analysis for this reason. In addition, we consider Employment is the most appropriate economic indicator. To confirm these two decisions, we perform basic linear regression analysis on all three economic indicators and the capital increments of the firms.

To perform a linear regression, we assume $\eta_i = (0.5)^{1/8}$ for all $i = t, t+1, ..., t+k$, $\epsilon = 1.5$. The results appear in Table 12.

Table 12: The results of regression analysis on capacity movement of firms and employment, the Consumer Confidence Index, and the IXUT

	R^2	Intercept	p-value	Slope	p-value
Employment					
VZ	0.9696	2687.5	0.2864	0.0677	0.0000
SBC	0.9130	9950.2	0.0012	0.0410	0.0000
BLS	0.9273	5451.1	0.0059	0.0306	0.0000
T	0.8673	12206.0	0.0009	0.0385	0.0000
Q	0.8134	14450	0.0000	0.0261	0.0000
Consumer Confidence Index					
VZ	0.8884	1117.9	0.8234	100.51	0.0000
SBC	0.8143	9395.2	0.0297	60.14	0.0000
BLS	0.8329	4961.4	0.0924	44.96	0.0000
T	0.7425	12246.0	0.0141	55.22	0.0000
Q	0.7079	14325	0.0005	37.77	0.0000
Nasdaq Telecommunications Index					
VZ	0.2783	9349.3	0.5894	35.97	0.0168
SBC	0.1856	17919.5	0.1301	18.36	0.0579
BLS	0.2054	10706.7	0.2107	14.28	0.0448
T	0.1370	21990.8	0.0648	15.16	0.1082
Q	0.0974	22597	0.0113	8.9599	0.1803

Of the three economic indicators, Employment has the best fit for linear regression. Employment and the Consumer Confidence Index shows similar results, with similar order of the firms in intercept, slope and respective p-values. Therefore, Employment and the Consumer Confidence Index show similar interpretation for the firms in cost structures and pre-existing capacities of the firms at the beginning of the period considered. As we explained before, the telecommunication companies can use Employment and/or the Consumer Confidence Index as their general economic indicators. Because telecommunication companies can expect more customers as Employment is increasing. The same if true for the Consumer Confidence Index. However, sector/industry-specific economic indicator, IXUT in our case, instead revealing how this industry does in the market, can be used to measure the performance of the industry.

The residual graphs show a trend. Figure 14 shows a significant trend in residuals with respect to Employment. The first half of the graph shows an upward trend and the last half of a the graph shows downward trend.

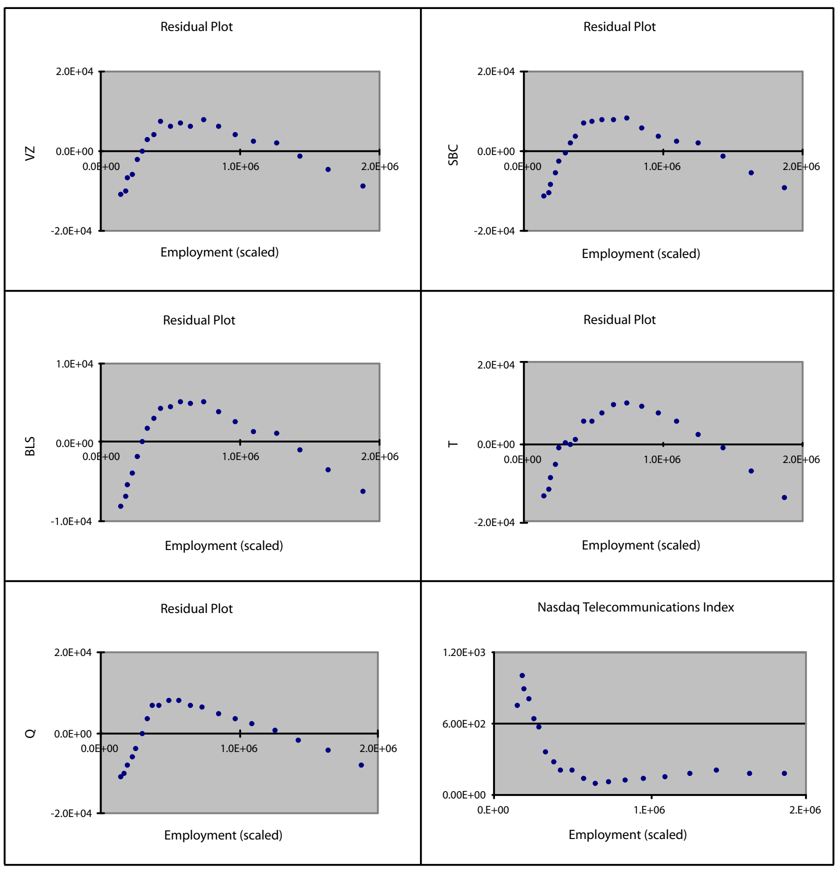

Figure 14: Residual graphs of the capacity movement of firms and Employment

The underlying reason for these findings can be found in the trend of the sector/industry-specific variable, IXUT. The movement of IXUT during this period is illustrated in Figure 14 for comparison.

Until the 4th quarter of 2001(or close to it), IXUT decreases very rapidly, and the residual graphs might reflect it. Therefore, we can infer from the residual graphs that the first half of the period reflects the Internet boom and over invested telecommunication and the last half of the period reflects the failure of some of the firms or the adjustments in the invested capital of the surviving firms.

In subsections 4.2.4 and 4.2.5, we will consider the Internet bubble phenomenon in our model and attempt to fix this curved residual graphs.

4.2.4 Regression Analysis between the Capacity Expenditures of Firms and the Economic Indicator by Incorporating of Industry-Specific Economic Conditions

One way to fix the residual trend can be to incorporate a sector/industry-specific variable to cost depreciation. As Kou and Kou[22] mentioned, the growth rate of an industry can be represented by the labor force of the R&D department, which might be directly related to technology improvement and cost depreciation. We change the cost depreciation factor $\eta(\cdot)$ as a function of the sector/industry-specific variable. Here, we used IXUT as our sector/industry-specific variable. When IXUT is high-valued, firms experience considerable cost depreciation and when IXUT is low-valued, firms have small cost depreciation. This concept is used to calculate the cost depreciation employing IXUT.

We use a Simple Moving Average (SMA) of the historical returns of IXUT. Moving average is widely used estimating the future trends of a financial asset. In addition, the return of IXUT in the period is so volatile that it cannot sufficiently explain the cost depreciation trend in the market. Also, when IXUT is high-valued, firms in the telecommunication industry do well and thus may invest more in research, which will lead to later improvements in technology.

We use a 12-period (3 years each) SMA to calculate η_t. We use

$$\eta_t = 0.5^{\frac{\frac{SMA_t}{\frac{1}{n}\sum_{j=1}^{n} SMA_j}}{16} + \frac{1}{16}} \tag{4.2.1}$$

as the cost depreciation for the period t. Here $\frac{1}{n}\sum_{j=1}^{n} SMA_j$ is the average of the SMAs during the considered period. By dividing the ratio of the value to the average by 16 and adding $\frac{1}{16}$, we lessen the effect of volatility. With SMA_t, which is the same as the mean of the SMAs, we have the same cost depreciation as in the case of the constant cost depreciation.

Table 13: The result of regression analysis on the capacity movement of firms and Employment with different cost depreciation during each period

	R^2	Intercept	p-value	Slope	p-value
VZ	0.9918	-1613	0.2855	0.0702	0.0000
SBC	0.9601	7202	0.0022	0.0431	0.0000
BLS	0.9689	3438	0.0178	0.0320	0.0000
T	0.9350	9239	0.0016	0.0412	0.0000
Q	0.8816	12716	0.0000	0.0276	0.0000

Table 13 presents the results, which show a better fit than the case with same cost depreciation for all the periods and improved residual graphs. The table reflects higher R^2 values for all firms than those of the same cost depreciation case. The intercept values are still positive, indicating that the residual graphs are still curved.

4.2.5 Regression Analysis between the Capacity Expenditures of Firms and the Economic Indicator with a Smaller Set of Data

As we explained in subsection 4.2.3, we can divide the data set into two parts. The first belongs to the Internet bubble and the last half corresponds to the adjustment period. We deleted the first half and perform linear analysis on the remaining data set.

Table 14, which shows the regression results for only the last half of the data, includes only 10 data sets from the 2nd quarter of 2002 to the 4th quarter of 2004. We performed linear regression with constant cost depreciation and with different cost depreciation for each period using the equation (4.2.1). The results show better fit in both cases, and the

Table 14: The result of regression analysis between the Capacity Movement of firms and Employment with the data set after the Internet bubble

With Constant Cost Depreciation Factor					
	R^2	Intercept	p-value	Slope	p-value
VZ	0.9971	-10346	0.0000	0.085	0.0000
SBC	0.9917	-4543	0.0000	0.043	0.0000
BLS	0.9949	-3657	0.0000	0.033	0.0000
T	0.9338	-887	0.3907	0.032	0.0000
Q	0.9725	-38493	0.0000	0.211	0.0000
With Different Cost Depreciation Factor					
VZ	0.9981	-15668	0.0000	0.1099	0.0000
SBC	0.9963	-7258	0.0000	0.0552	0.0000
BLS	0.9976	-5758	0.0000	0.0431	0.0000
T	0.9666	-3249	0.0044	0.0429	0.0000
Q	0.9975	-3907	0.0000	0.0280	0.0000

residuals appear to be random. Again, we obtain better result when we incorporate the sector/industry-specific variable in cost depreciation.

From the result, we can deduce some relationship between the cost structure and the capacity position of the firms at the beginning of the period. We analyze the results of the case with different cost depreciation.

First of all, let us compare the intercept of the regression analysis result. The negative value of the intercept represents the capacity position at the beginning of the period, which is, in this case, at the end of the 1st quarter of 2002. This negative value of the intercept initially decreases to T and slightly increases at Q. We do not have any available data set that provides the exact capacity position at the beginning of the period. On the Balance Sheet of each firm, the "Net Fixed Asset (Property Plant and Equipment)" provides a proxy for the capacity position during that period. By doing some calculations with the revenue/profit portion of the wireline, we obtain the data for the Net Fixed Asset dedicated to wireline services. Although the data in Table 15 do not match the intercept from the regression result exactly, they do not differ significantly. In addition, AT&T does not have publish, print and advertising services, but BLS and Q have that these services, which can translate into a higher value intercept in the regression analysis. Therefore,

Table 15: Capacity position at the end of 4th quarter of 2001

VZ	SBC	BLS	T	Q
47703	37370	24943	26803	29479

even though the negative value of the intercept of BLS has a larger value than that of T, it does not mean the capital/capacity position of the wireline of BLS is larger than that of AT&T. Therefore, we can conclude that the intercept accurately explains the initial capacity of the firms.

Next, let us compare the slopes of the regression result. The slope is the form of $(1 - C_i \cdot \mathsf{A})$ with constant A. According to Lemma 3.5.1, this value should be always positive. In addition, if a firm has an efficient cost structure, this slope has a large value, and if a firm has a poor cost structure, the slope has a small value. From VZ to Q, the slope shows a constantly decreasing trend. Possible interpretations of this result are that if the firm has larger assets, then it has a more efficient cost structure.

CHAPTER V

DISCUSSION AND FUTURE WORK

5.1 *Summary of the Theoretical and Numerical Findings and Strategic Implications*

We investigate the investment behavior of a firm in a telecommunications industry under uncertain economic condition. We formulate the investment decision problem as a discrete Markov process. By differentiating the purchase price and resale value of the capacity, we incorporate partial reversibility of an investment that presents a range of optimal capacity with a lower bound and an upper bound. In addition, the optimal strategy is one of the threshold type. We are able to characterize the optimal policy regions as buying new capacity, maintaining status quo, and divesting capacity.

By using the linearity between the optimal capacity and the economic condition, the problem has been reduced to finding the slopes of the boundaries of BUY/KEEP and KEEP/SELL. We perform a series of sensitivity analyses which show the relationship between the slopes of the boundaries of BUY/KEEP and KEEP/SELL and the parameters as follows. • As cost coefficients decrease, the values of the slopes increase. If a firm's cost parameters become smaller, the size of capacity investment increases. In addition, this investment behavior is amplified by large ϵ values because a large ϵ implies that the demand for capacity is more sensitive to the price.

• As the variance of future economic condition increases, the slope of the boundary of BUY/KEEP decreases in the case with cost depreciation. In the case without cost depreciation, an increase in the variance of future economic condition causes the slopes of BUY/KEEP and KEEP/SELL boundaries to decrease and increase, respectively. As the variance of future economic condition increases, a firm's investment decision becomes conservative, which means that the activity for buying new capacity and selling off the current capacity is reduced which is illustrated by the increased KEEP region in the solution space. Moreover, with cost depreciation, a firm tends to delay the investment decision for BUY to take advantage of the cost reduction in the next period, and it tends to sell more capacity to take advantage of a higher salvage value in this period.

• As the elasticity parameter, ϵ, increases, the slopes of the BUY/KEEP boundaries increase at first and then decrease. However, within a reasonable range of ϵ, the slopes tend to increase, a fact of which agrees with our intuition about a firm's investment behavior when the demand elasticity is large. With a large ϵ value, a firm can earn more revenue from increasing capacity by lowering price as stated in the general theory in microeconomics.

With cost depreciation and a dynamics of moderately changing economic condition, we find analytic expression for the BUY/KEEP boundaries, which establish independence between investment decisions in adjacent period. Therefore, the multiple-time investment problem is reduced to a one-time investment problem. In this case, the optimal increment is determined by the realization of uncertain economic conditions. Moreover, using this analytic expression between capacity investment and exogenous economic indicators, we obtain a capital investment trajectory by observing the realization of an uncertain economic indicator. In addition, the corresponding price trajectory is obtained through the price-demand function, given the capacity movement. The resulting trajectories of the capacity and price movement shows that market price decreases exponentially and market demand increases exponentially. These results are consistent with the findings in existing work by Kenyon and Cheliotis[20] and [21] and Lanning, Mitra, Wang, and Wright[25].

Next, we consider the investment behavior of a firm in an oligopolistic market. We formulated investment decision behavior within the Cournot framework. We consider this problem in two cases: one with symmetric firms and the other with asymmetric firms. For each case, we show the existence and the uniqueness of the Cournot equilibrium point, and we derive some conditions under which the solutions exist. In doing so, we investigate the investment behavior of a firm in a competitive market and obtain the following results:

• If a firm already had a very large capacity, then a new firm is hard to enter the market unless the new firm's cost structure is very efficient.

• If a firm already a considerable capacity position, then a new firm can enter the market with smaller capacity than the incumbent firm's capacity. However, the market share of the new firms increases with time to that of incumbent firm then they have the same cost structures. In the case of asymmetric firms, the new firms increases its market share with time, but the ration of the market share between the firms in the market is dependent on the cost structure of each firm.

In the case of asymmetric firms, we identify the condition under which a firm can enter the market: the new firm can not have large cost factors compared with other firms. In addition, if an incumbent firm has large cost factors compared with the new entry firm, the incumbent firm has to restructure the system to lower the cost; otherwise, the incumbent firm will be forced out of the market.

In both cases, we illustrate various aspects of market properties with the number of firms in the market as follows:

• Total market capacity in use, which increases with the number of firms in the market.

• Market price, which decreases with the number of firms in the market.

• Consumer surplus, which increases with the number of firms in the market.

• Time to certain percentage of price reduction, which does not depend on the number of firms in the market.

• Expected number of periods until first expansion, which decreases with the number of firms in the market.

We find an analytic expression for total market capacity, market price, consumer surplus, and time to a certain percentage of price reduction with limiting values as the number of firms goes to infinity. For the expected number of periods until the first expansion, we find an analytic expression by approximating to geometric Brownian motion. In addition, dramatic changes occur when the market structure changes from a monopoly to a duopoly. For example, total market capacity in use is more than doubled. However, when the number of firms in the market changes from two to more, little change takes place.

The results are reasonable in that:

• Competition increases the supply and demand of the capacity by lowering the price.

• competition increases consumer surplus and social welfare, which much of the literature addresses.

• Price reduction is directly related to cost depreciation, which is not modeled to be dependent on the number of firms in the market. Therefore, the price reduction is not dependent on the number of firms in the market.

• As competition increases, the time until the first expansion shortens.

Employing market data from the telecommunications service industry, we have carried out a series of empirical analyses. When we incorporate sector/industry-specific economic conditions into cost depreciation, we obtain a better fit from regression analysis than when we use constant cost depreciation. This result is consistent with the result of the previous work of Kou and Kou[22], who showed the growth rate of growth stock can be represented by the R&D labor force in the market. In our case, we use the Nasdaq Telecommunications Index as our sector/industry-specific economic condition. Its growth rate is positively related to the R&D labor force in the telecommunications market, which leads to technology improvement and corresponding cost depreciation. Therefore, utilizing the sector/industry-specific index to calculate cost depreciation provides a better understanding of the investment behavior of firms than simply adopting a constant cost depreciation factor throughout the period.

For our general economic indicator, we choose Civilian Employment (Employment). The regression result shows Employment should be a strong candidate for the economic indicator in the proposed model. Due to the clear increasing trend with moderate volatility of Employment, firms in the market continue to invest in capital/capacity. However, Employment is not directly related to the telecommunications industry, but indirectly related to it, as personal and work related Internet/mobile phone usage normally increases as employment increases.

When we consider the data set that belongs to after-Internet-bubble period, results accurately explain the cost structure and the capacity position of firms at the beginning of the period. However, the amount of data is very small, so we must continue to examine the accuracy of this result.

From this result, we can also conclude that firms have different cost structures which relate to the size of the firm with regard to market capitalization and assets. This relationship of cost structure to the size of the firm is intuitively correct, as larger firms are more likely to have an efficient cost structure. In addition, we obtain all positive values for the slopes from the regression analysis, confirming Lemma 3.5.1, which indicates that the cost structure of a firm is not much different from that of the other firms in the market.

5.2 Future Work

In our model, we did not consider depreciation of capital/capacity, which has been considered in other literature. We believe that depreciation of the capital/capacity can be incorporated in the proposed model without much effort. In addition, we did not consider the lead time of the facility, so our model can easily be applied to cases in which the lead time is very short. For example, an incumbent local phone company can use the copper line to provide ADSL service by deploying some equipment. In addition, an Internet service provider can purchase some fiber optic lines from a vender that already has dark fiber underground and sells it by lighting it up, but this would not apply to a firm that lays the new fiber optic lines and waits for the completion of the work.

Finally, we assumed that the cost depreciation can be a function of the sector/industry-specific parameter, but we ignored the fact that this cost depreciation can differ depending on the characteristics of a firm. If wireline service providers do not invest much in research fund or purchase the equipment in the market, this uniform cost depreciation assumption could prove to be true to all firms. However, if a firm spend invest a large amount of money in research and makes technological improvements within the firm, then the cost depreciation factor for the firm will differ from that of other firms.

If we consider the factors of depreciation, lead time lag, and firm-specific cost depreciation, the research will be more general and thus more applicable to real world investment problems.

REFERENCES

[1] Abel, A. B., Optimal Investment under Uncertainty, American Economic Review, 73, 1983, pp 228-233.

[2] Abel, A. B. and Eberly, J. C., Optimal Investment with Costly Reversibility, The Review of Economic Studies, 63, 1996, pp. 581-593.

[3] Abel, A. B. and Eberly, J. C., A Unified Model of Investment Under Uncertainty, American Economic Review., 84, 1994, pp 1369-1384.

[4] Aguerrevere, Felipe L. , Equilibrium Investment Strategies and Output Price Behavior: A Real-Options Approach, Working Paper, 2000.

[5] Arrow, Kenneth J., Optimal Capacity Policy with Irreversible Investment, Value, Capital and Growth, Essays in Honor of Sir John Hicks, ed. J. N. Wolfe, Edinburgh University Press, Edinburgh, Scotland, 1968.

[6] Baryshnicov, Y., Sim, H. J. and Wang, Q., Optimal Policy for Incremental Capacity Investment, Working paper Feb. 2003.

[7] Bean, James C., Higle, Julia L. and Smith, Robert L., Capacity Expansion Under Stochastic Demands, Operations Research, Vol. 40, Supp. No. 2, May-June 1992, pp S210-S216.

[8] Bertola, G. and R. J. Caballero, Irreversibility and Aggregate Investment, Review of Economic Studies, 61, 1994, pp 223-246.

[9] Chuang, A. S., Wu, F., and Varaiya, P., A Game-Theoretic Model for Generation Expansion Planning: Problem Formulation and Numerical Comparisons, IEEE Trans. Power Systems 16(4), Nov. 2001, pp 885-891.

[10] d'Halluin, Y., Forsyth, P. A., Vetzal, K. R., Managing Capacity for Telecommunications Networks Under Uncertainty, IEEE Volume 10 Issue 4, 2002, pp 579 - 587.

[11] Dixit, A. K. and Pindyck, R. S., Investment Under Uncertainty, Princeton, 1994.

[12] Frank, Robert H., Microeconomics and Behavior (4th edition), McGraw-Hill, 2000.

[13] Gilbert, R. J. and Lieberman, M.,Investment and Coordination in Oligopolistic Industries, The Rand Journal of Economics, Vol. 18, No. 1, 1987, pp 17-33.

[14] Grenadier, Steven R., Option Exercise Games: An Application to the Equilibrium Investment Strategies of Firms, The Review of Financial Studies, Summer 2002 Vol. 15, No. 3, pp 691-721.

[15] Hay, Donald A. and Liu, Guy S., The Investment Behavior of Firms in an Oligopolstic Setting, The Journal of Industrial Economics, Vol. XLVI, March 1998, No. 1, pp 79-99.

[16] Hubbard, R. G., and Lehr, W., Telecommunications, the Internet & the Cost of Capital, TPRC 27th, 1999.

[17] International Telecommunication Union, Telecommunication Development Report, March 2002.

[18] Kamien, M. I., A cake for Cournot, Economic Letters, 70, 2001, pp 349-355.

[19] Kenyon, C. and Cheliotis, G., European-Style Forward Derivatives for Telecom Commodities, IBM Research Report RZ3338, April 2001.

[20] Kenyon, C. and Cheliotis, G., Real Options and Bandwidth Markets: Dark Fiber Valuation, IBM Research Report RZ3355, July 2001.

[21] Kenyon, C. and Cheliotis, G., Stochastic Models for telecom commodity prices, Computer Networks, 36, 2001, pp 533-555.

[22] Kou, S. C. and Kou, S. G., A Tale of Two Growths: Modeling Stochastic Endogeneous Growth and Growth Stocks, Working Paper, Dec. 2002.

[23] Kou, S. C. and Kou, S. G., Modeling Growth Stocks via Birth-Death Processes, Advances in applied probability, Vol. 35, 2003, pp 641-664.

[24] Kou, S. C. and Kou, S. G., A Diffusion Model for Growth Stocks, Mathmatics of Operations Research, Vol. 29, No. 2, May 2004, pp 191-212.

[25] Lanning, S., Mitra, D., Wang, Q. and Wright, M., Optimal planning for optical transport networks, The Royal Networks, A(2000) 358, pp 2183-2196.

[26] Leahy, J. V., Investment in Competitive Equilibrium: The Optimality of Myopic Behavior, The Quarterly Journal of Economics, pp 1105-1133, Nov. 1993.

[27] Li, Lode, Cournot Oligopoly with Information Sharing, Rand Journal of Economics, Vol. 16, No. 4, Winter 1985, pp 521-536.

[28] Long, N. V., Existence and Uniqueness of Cournot Equilibrium: A Contraction Mapping Approach, CIRANO Working Papers, 1999.

[29] Ma, Sheng, Modeling Heterogeneous Network Traffic in Wavelet Domain, IEEE/ACM Transactions on Networking, Vol. 9, No. 5, October 2001, pp 634-649.

[30] Maiorano, A., Song, Y. H. and Trovato, M., Imperfect Competition: Modeling and Analysis of Oligopoly Electricity Markets, IEEE Power Engineerign Review, May 1999, pp 56-58.

[31] McDonald, R. and Siegel, D., Investment and the Valuation of Firms When There is an Option to Shut Down., International Economics Review Vol. 26, No. 2, June 1985, pp 331-349.

[32] McDonald, R. and Siegel, D., The Value of Waiting to Invest., The Quarterly Journal of Economics, November 1986, pp 707-727.

[33] Mergent's Handbool of Nasdaq Stocks, Fall 2002.

[34] Mergent's Handbool of Common Stocks, Fall 2002.

[35] Mitra, D. and Wang, Q., Network Economics and Traffic Engineering: Joint Optimization of Pricing and Provisioning in Multi-Service Networks, Working paper, 2003.

[36] Nor, Sulaman Mohd, Yahya, Zuraimy and Ihsanto, Eko, A Fractal Modeling for Network Traffic, IEEE Tencon 2000 Proceedings, Vol. 2, pp 472-477.

[37] Novshek, W., On the Existence of Cournot Equilibrium, Review of Economic Studies LII, 1985, pp 85-98.

[38] Riedi, Rudolf H. and Ribeiro, Vinay J., A Multifractal Wavelet Model with Application to Network Traffic, IEEE Transactions on Information Theory, Vol. 45, No. 3 April 1999, pp 992-1018.

[39] Ryan, S. M., Capacity Expansion for Random Exponential Demand Growth with Lead Times, Working Paper, 2002.

[40] Smith, R. L., Optimal Expansion Policies for the Deterministic Capacity Problem, The Engineering Economist, 25(3), 1980, pp 149-160.

[41] Standrard & Poor's, Standrad Corporation Descriptions, Vol. 4, 2002.

[42] United States Government Printing Office Washington, Economic Indicators 108th Congress 2d Session, Nov. 2004.

[43] Wen, F. S. and David, A. K., Oligopolistic Electricity Market Production under Incomplete Information, IEEE Power Engineerign Review, April 2001, pp 58-61.

[44] Tony White and Dresdner Kleinwort Benson, The New Power Markets: Corporate Strategies for Risk and Reward, Risk Books, 1999.

[45] Yang, Xueshi, The Extendedn Alternative Fractal Renewal Process for Modeling Traffic in High-Speed Communication Networks, IEEE Transactions on Networking, Vol. 49, No. 7, July 2001, pp 1349-1363.