Spatial Service Systems Modelled as Stochastic Integrals of Marked Point Processes

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Spatial Service Systems Modelled as Stochastic Integrals of Marked Point Processes

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To my family: Pat, Shiner, and Buddy.

I love you!

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF NOTATION

$\mathbb{1}(S)$	indicator function: $\mathbb{1}(S)$ is 1 or 0 if the statement S is true or false, respectively, p. 4.
$a \wedge b$	the minimum of a and b, p. 4.
$\mathcal{B}$	Borel σ-field on the real numbers, p. 26.
$\mathcal{B}_+$	Borel σ-field on the nonnegative real numbers, p. 27.
$b(x)$	probability a particle or customer at x is considered for deletion upon the next arrival, p. 42.
$\mathcal{C}_K^+(\mathbb{E})$	space of non-negative, continuous functions on $\mathbb{E}$ with compact support, p. 8.
c-**particle**	a particle that waits for service as a customer, p. 64.
$D(\mathbb{R}_+, \mathbb{E})$	Skorohod space of functions mapping $\mathbb{R}_+$ to $\mathbb{E}$ that are right continuous with left-hand limits, p. 4.
D_n	departure time of the nth particle or customer, p. 6.
$\delta_x(\cdot)$	Dirac measure: $\delta_x(A) = \mathbb{1}(x \in A)$, p. 8.
$\mathbb{E}$	space to which points of the input process are transformed, p. 17.
$\mathcal{E}$	Borel sigma-field on $\mathbb{E}$, p. 17.
$\mathbb{E}'$	mark space of the input process M, p. 17.

$\mathcal{E}'$	Borel sigma-field on $\mathbb{E}'$, p. 17.
$\mathbb{F}$	space of measurable functions mapping $[0,\infty] \times \mathbb{E}'$ into $\mathbb{E}$ such that for any $h \in \mathbb{F}$ and compact set $B \in \mathcal{E}$, $h^{-1}(B)$ is compact in $\mathbb{R}_+ \times \mathbb{E}'$., p. 17.
L_n	discrete lifetime of the nth particle or customer: the number of deletion attempts required to remove the nth particle or customer, p. 42.
λ	rate at which particles arrive from a Poisson process, p. 17.
$L_N f$	Laplace functional of the point process N for $f \in \mathcal{C}_K^+(\mathbb{E})$, p. 9.
$\mathcal{M}(\mathbb{E})$	set of finite counting measures on $\mathbb{E}$, p. 8.
$M(t)$	the number of particles arriving in $[0,t]$, p. 27.
$M^c(t)$	the number of c-type particles arriving in $[0,t]$, p. 66.
$M^s(t)$	the number of s-type particles arriving in $[0,t]$, p. 66.
$M_t(I \times \mathbb{E}')$	$M(t - I \times \mathbb{E}')$, p. 18.
$\mathbb{N}$	natural numbers: $\{1, 2, \ldots\}$, p. 7.
$\mathcal{N}$	Borel field on the natural numbers, p. 41.
$\mathbb{R}_+$	non-negative real numbers $[0,\infty)$, p. 2.
s-**particle**	a particle that triggers deletions as a server, p. 64.
supp(f)	the support of the function f, p. 20.
T_n	arrival time of the nth particle or customer, p. 1.

X_n	spatial location of the nth particle or customer, p. 41.
Y_n	mark or rank of the nth particle or customer, p. 1.

SUMMARY

This dissertation characterizes the equilibrium behavior of a class of stochastic particle systems, where particles (representing customers, jobs, animals, molecules, etc.) enter a space randomly through time, interact, and eventually leave. The results are useful for analyzing the dynamics of randomly evolving systems including spatial service systems, species populations, and chemical reactions. Such models with interactions arise in the study of species competitions and systems where customers compete for service (such as wireless networks).

The models we develop are space-time measure-valued Markov processes. Specifically, particles enter a space according to a space-time Poisson process and are assigned independent and identically distributed attributes. The attributes may determine their movement in the space, and whenever a new particle arrives, it randomly deletes particles from the system according to their attributes.

Our main result establishes that spatial Poisson processes are natural temporal limits for a large class of particle systems. Other results include the probability distributions of the sojourn times of particles in the systems, and probabilities of numbers of customers in spatial polling systems without Poisson limits.

CHAPTER I

PRELIMINARIES

In this study we characterize the limiting behavior of several time-varying transformations of marked point processes, and give applications to spatial service systems, species competitions, and particle systems. We begin this chapter with a brief introduction to marked point processes. Next, we present two examples of spatial service systems and describe many of the systems we will analyze in the later chapters. We end the chapter with point process terminology and notation.

1.1 Introduction

At its most basic level, a *point process* on the time axis $\mathbb{R}_+$ is a collection of random points T_n in $\mathbb{R}_+$. When additional information in the form of attributes is known about the points, say, the point T_n has been marked as Y_n, the sequence (T_n, Y_n) is said to form a *marked point process*. The elements T_n and Y_n belong to what are called the *ground space* and the *mark space*, respectively. The theory of point processes provides a very natural framework in which to model a number of stochastically evolving phenomena including chemical reactions, magnetism, weather, species competitions, and the spread of epidemics. The theory is important in operations research for modeling flows of items in inventory systems and jobs in service systems such as queueing networks.

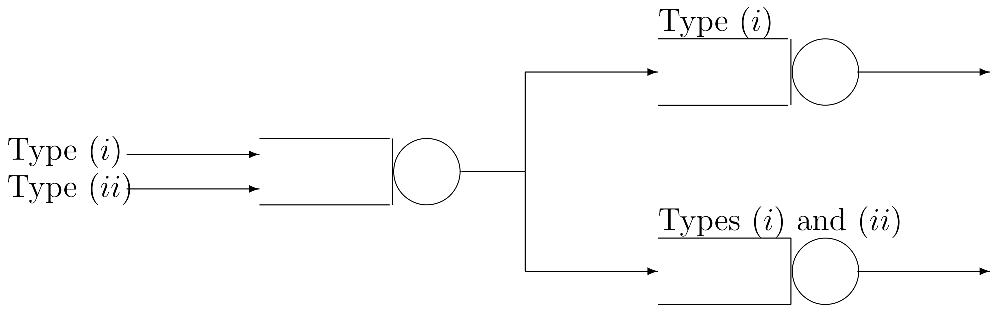

Figure 1: A three-node stochastic network with two item types.
Upon arrival, the nth item is marked with the attributes X_n, R_n, and V_n, representing its type, route through the network, and service times at each node, respectively. When departing node 1, each type (i) item is routed to node 2 with probability p, and to node 3 with probability $1-p$. Each type (ii) item is routed to node 3.

For instance, consider the three-node stochastic processing network in Figure 1. Imagine that $T_n \in \mathbb{R}_+$ represents the time at which the nth job enters a service system for processing. Upon its arrival, suppose we know the nth job's type $X_n \in \mathbb{E}$, the route it will take through the service system $R_n = (R_{n1}, R_{n2}, \ldots) \in \mathcal{R}$, and its service times at each station $V_n = (V_{n1}, V_{n2}, \ldots) \in \mathcal{V}$. These three, typically random, pieces of information are attributes associated with the nth arrival, and can be captured in the form of a vector (X_n, R_n, V_n), called a mark. Then the sequence $(T_n, (X_n, R_n, V_n))$ generates a marked point process M on the space $(\mathbb{R}_+ \times (\mathbb{E} \times \mathcal{R} \times \mathcal{V}))$ expressed as

$$M(I \times (A \times B \times C)) = \sum_n \mathbb{1}(T_n \in I, X_n \in A, R_n \in B, V_n \in C).$$

As another example, suppose T_n represents the time at which the nth person or animal enters a particular community $\mathbb{E}$. The mark associated with this person is (X_n, R_n, H_n), where X_n represents the location at which the nth person enters $\mathbb{E}$, $R_n \in [0,1]$ represents some level of natural resistance to a particular disease, and H_n tells us whether the nth person is initially well (w), sick (s), immune (i), or deceased (d). Then $(T_n, (X_n, R_n, H_n))$ generates a marked point process M on the space $(\mathbb{R}_+ \times (\mathbb{E} \times [0,1] \times \{w, s, i, d\}))$, where

$$M(I \times (A \times B \times C)) = \sum_n \mathbb{1}(T_n \in I, X_n \in A, R_n \in B, H_n \in C).$$

A marked point process is useful for describing the quality or performance of a system in space as well as time. For instance, in the service system example, at time t one would like to know how many jobs of each type are currently in the system, their present locations, and how many jobs of each type have been processed. In the epidemic model, one would certainly like to know how many well, sick, immune, and deceased individuals are in the community at time t. In each case, performance characteristics can be modelled as a random transformation of the initial marked point process M.

1.2 Spatial $M/G/\infty$ System

The following example motivates the general framework we develop in the third chapter.

Suppose particles enter the space $\mathbb{E}$ at times $0 < T_1 < T_2 < \ldots$ forming a Poisson process with rate λ. The nth customer arriving at time T_n moves independently according to a stochastic process $X_n \equiv \{X_n(t) : t \geq 0\}$ in $\mathbb{E}$ for a random time S_n and then exits the system. That is, the location of the nth particle at time $t > T_n$ is $X_n(t - T_n)$ provided $t - T_n < S_n$. Otherwise, the particle has departed and its last location was $X_n(S_n)$.

We assume (X_n, S_n) for $n \geq 1$ are independent and identically distributed with a known distribution, and that they are also independent of the T_n. Also, X_n is a random element of the Skorohod space of functions $D(\mathbb{R}_+, \mathbb{E})$ (see [12]), so that it is right continuous with left-hand limits. The data for this system is the family $\{(T_n, X_n, S_n), n \geq 1\}$, which generates a space-time Poisson point process M on $\mathbb{R}_+ \times D(\mathbb{R}_+, \mathbb{E}) \times \mathbb{R}_+$.

Suppose for each arrival to the system before time t we are interested in the time since its arrival, its location at time t, and whether or not it is still in the system. Then a natural and useful transformation of the data for this particle system at time t is

$$\phi_t(t - T_n, X_n, S_n) \equiv (t - T_n, X_n((t - T_n) \wedge S_n), \mathbb{1}(t - T_n < S_n)), \qquad T_n < t.$$

Here, $a \wedge b$ denotes the minimum of a and b. At time t, this transformation denotes for the nth particle the time since its arrival, its current location (or last location prior to departure), and whether or not it is still in the system, respectively. Here $\mathbb{1}(S)$ denotes the indicator function that is either 1 or 0 if the statement S is true or false, respectively. Thus the input process M along with the transformation function ϕ_t generate a point process N_t on $\mathbb{R}_+ \times \mathbb{E} \times \{0, 1\}$ by setting

$$N_t(I \times A \times B) = \sum_n \mathbb{1}(\phi_t(T_n, X_n, S_n) \in I \times A \times B), \qquad T_n < t$$

Here, $N_t(I \times A \times \{1\})$ counts the number of particles that arrive in the time interval $t - I$ with positions in A that are still in the system at time t. Similarly, $N_t(I \times A \times \{0\})$ counts the number of particles that arrive in the time interval $t - I$ with positions in A that have departed the system by time t. Our main concern is the distribution of N_t and its limiting distribution as $t \to \infty$.

For now, consider the case in which the particles do not move, but simply enter the system at their initial position X_n and stay there until they depart. We will return to the analysis of the more general model with moving particles in Chapter 6. That is, $X_n(t) \stackrel{d}{=} X_n$ for each t. This is called a spatial $M/G/\infty$ service system, where S_n is the service time of the nth customer that enters X_n at time T_n.

In this case, it is known that N_t, for fixed t, is a space-time Poisson process. Specifically, the number of customers in the set $A \subset \mathbb{E}$ at time t that entered the space $\mathbb{E}$ in the time interval $[t-u, t)$ has a Poisson distribution with mean

$$E[N_t((0,u] \times A \times \{1\})] = \lambda F_X(A) \int_{[t-u,t]} (1 - F_S(t-v))dv.$$

Here F_S and F_X are the distributions of the service times and locations, respectively. Similarly, the number of departures from A in $[t-u, t]$ has a Poisson distribution with mean

$$E[N_t((0,u] \times A \times \{0\})] = \lambda F_X(A) \int_{[t-u,t]} F_S(t-v)dv.$$

Furthermore, $N_t \xrightarrow{d} N$, where N is a space-time Poisson process. This convergence follows because the mean measure of N_t given converges to the mean measure of N, which is defined by

$$EN((0,u] \times B \times \{1\}) = \lambda F_X(B) \int_0^u (1 - F_S(v))dv.$$

1.3 General Spatial Service System

We will study the following example more in depth in Chapter 6.

Consider a system in which customers enter a Polish space $\mathbb{E}$ at times $0 < T_1 < T_2 < \ldots$. As in the previous section, the nth customer arrives at time T_n and moves in $\mathbb{E}$ according to a stochastic process $\{X_n(t) : t \geq 0\}$ in $D(\mathbb{R}_+, \mathbb{E})$ for some time and eventually exits the system. This customer has a sojourn time or service requirement denoted by S_n.

There is a service mechanism like a polling server that allocates service to the nth customer according to a nondecreasing stochastic process Z_n, which is a random element in $D(\mathbb{R}_+, \mathbb{R}_+)$. Specifically, during a time interval $(T_n, t]$, the service time S_n is decreased by the amount $Z_n(t - T_n)$. Then $D_n = \inf\{t : Z_n(t - T_n) > S_n\}$ is the departure time of the nth customer.

This system is driven by the data

$$(T_n, X_n, S_n, Z_n), \qquad n \geq 1,$$

which forms a marked point process M on $\mathbb{R}_+ \times D(\mathbb{R}_+, \mathbb{E}) \times \mathbb{R}_+ \times D(\mathbb{R}_+, \mathbb{R}_+)$. We are interested in the point process N_t on on $[0, t] \times \mathbb{E} \times \{0, 1\}$ which is generated by the set of points

$$\{(t - T_n, X_n((t - T_n) \wedge D_n), \mathbb{1}(Z_n(t - T_n) < S_n)) : T_n \leq t\},$$

which describes at time t how long each particle has been in the system, where each particle is or was upon its departure, and whether or not the particles remain in the system. We can also express N_t as an integral of the input process M. That is, for any $f \in \mathcal{C}_K^+$,

$$\begin{aligned} N_t f &\equiv \sum_{n=1}^{\infty} f(t - T_n, X_n((t - T_n) \wedge D_n), \mathbb{1}(Z_n(t - T_n) < S_n)) \\ &= \int_{[0,t] \times D(\mathbb{R}_+, \mathbb{E}) \times \mathbb{R}_+ \times D(\mathbb{R}_+, \mathbb{R}_+)} f(u, x(u \wedge \inf\{t : z(u) = s\}), \\ &\qquad \mathbb{1}(z(u) < s)) M_t(-du\, dx\, ds\, dz), \end{aligned}$$

where $M_t(I \times A) \equiv M(t - I \times A)$.

A variety of scenarios can be described by imposing various assumptions on the data. Here we describe several models which we will describe in later chapters:

1. Our main model, the so-called attribute-based thinning model[1], has an input process generated by the data (T_n, X_n, L_n) in the space $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{N}$. Here, $\mathbb{N}$ denotes the natural numbers $\{1, 2, \ldots\}$, and T_n represents the arrival time of the nth particle. We denote by X_n an attribute (or ranking) of the nth particle. Upon the nth arrival, this particle may consider other particles in the system for deletion if their ranks are strictly less than X_n. In this case L_n is the discrete number of arrivals required to remove the nth customer from the system (also called the discrete lifetime). The state of the system at time t is determined by the set of points

$$(t - T_n, X_n, \mathbb{1}(M((T_n, t] \times (X_n, \infty) \times \mathbb{N}) < L_n)),$$

for n such that $T_n \leq t$, which form a point process N_t on $\mathbb{R}_+ \times \mathbb{R} \times \{0, 1\}$. This information tells for the nth particle how far back in time it arrived, the position to which it arrived, and whether or not remains in the system.

2. An extension of the attribute-based thinning model is as follows. Suppose the nth arriving customer is assigned a rank $X_n \in \mathbb{R}_+$, as well as a position Y_n in Euclidean space $\mathbb{R}^d$. Denote by $B_r(y)$ the ball of radius r centered at y. Upon its arrival to (X_n, Y_n), the nth arrival may consider any customer for deletion whose

[1]This model is analyzed in Chapter 4.

position is within $B_r(Y_n)$ and whose rank is less than X_n. This system is driven by the data $(T_n, X_n, Y_n, S_n, Z_n)$ in the space $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{N} \times D(\mathbb{R}_+, \mathbb{N})$. In this case, S_n is still the discrete number of arrivals required to remove the nth customer from the system, and $Z_n(t) = M((T_n, T_n + t] \times (X_n, \infty) \times B_r(Y_n) \times \mathbb{N} \times D(\mathbb{R}_+, \mathbb{N}))$.

3. Particles enter a space according to a Poisson process, and take on random positions upon their arrival. Every time a new arrival occurs, a customer (or particle) with position x is served with probability $a(x)$. This is our so-called elastic polling model appearing in Chapter 5, where each arrival polls all particles in the system. Suppose instead that servers arrive to the system according to a Poisson process with rate γ, and that a customer (or particle) with position x is served with probability $a(x)$ upon arrival of a server. This is our so-called inelastic polling model, which appears in Chapter 5.

1.4 Point Process Notation

In this section we will introduce point processes and marked point processes, along with their standard terminology and notation. For excellent references on these subjects see [6], [12], and [15].

Let $(\mathbb{E}, \mathcal{E})$ be a Polish space, where $\mathcal{E}$ is the family of Borel sets of $\mathbb{E}$. Let $\mathcal{M}(\mathbb{E})$ set of finite counting measures on $\mathbb{E}$. A typical counting measure is $\mu = \sum_{k=1}^{n} \delta_{x_k}$, where $\delta_x(A) \equiv \mathbb{1}(x \in A)$ is the Dirac measure on $\mathbb{E}$ with unit mass at x and $n = \mu(\mathbb{E})$; when $n = 0$, $\mu = 0$ (the zero measure). For simplicity we will write $\mu = \sum_k \delta_{x_k}$ without the n. Let $\mathcal{C}_K^+(\mathbb{E})$ denote the set of nonnegative, continuous functions on $\mathbb{E}$

with compact support. We endow $\mathcal{M}$ with the vague topology (the smallest topology such that the mapping $\mu \to \mu f$ is continuous for any $f \in \mathcal{C}_K^+(\mathbb{E})$) so that it is a Polish space,[2] and let $\mathcal{B}(\mathcal{N})$ denote its Borel sets.

A point process N on $\mathbb{E}$ is a measurable function from a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ into $(\mathcal{N}, \mathcal{B}(\mathcal{N}))$. It has the form $N = \sum_k \delta_{X_k}$, where X_k are its point locations, and $N(B)$ denotes the number of points in a set $B \in \mathcal{E}$. We will frequently use the integral notation

$$Nf \equiv \int_{\mathbb{E}} f(x)N(dx) = \sum_k f(X_k).$$

As a classic example, a point process N is a *Poisson process with intensity measure* μ if N has independent increments and for any $B \in \mathcal{E}$, $N(B)$ has a Poisson distribution with mean $\mu(B)$, which is finite when B is bounded.

A point process N is defined by specifying the form or distribution of the random quantities $N(B)$ for all $B \in \mathcal{E}$. Equivalently, one can specify the form or distribution of the family of integrals Nf, for $f \in \mathcal{C}_K^+(\mathbb{E})$. That is, the distribution of a point process N is uniquely determined by the form of its Laplace functional L_N defined by

$$L_N(f) \equiv E[e^{-Nf}], \qquad f \in \mathcal{C}_K^+(\mathbb{E}). \tag{1}$$

As seen in (1), the Laplace functional plays an analogous role for point processes (and random measures) as Laplace transforms do for nonnegative random variables. As an example, the Laplace functional of a Poisson process N with intensity measure μ

[2] A measure space endowed with the vague topology is Polish. See Theorem A2.3 of [12].

is well known to be

$$L_N(f) = \exp\{-\int_{\mathbb{E}}(1 - e^{-f(x)})\mu(dx)\}, \qquad f \in \mathcal{C}_K^+ \tag{2}$$

(see [22]).

Given a point process M on a space $\mathbb{E}$, we may know some other additional information about the points. We call $\mathbb{E}$ the ground space. One may be interested in assigning to each point of T_n of M an attribute Y_n, which is an element of what is called the mark space $\mathbb{E}'$. Then we call $\{(T_n, Y_n) : T_n \in \mathbb{E}, Y_n \in \mathbb{E}'\}$ a *marked point process*.

The theory of point processes forms a sub-theory of that of random measures. Let $\mathcal{M}(\mathbb{E})$ denote a family of measures on $(\mathbb{E}, \mathcal{E})$, and let M be a measurable mapping from a probability space into $\mathcal{M}(\mathbb{E})$. Then M is a *random measure* on $\mathbb{E}$. Just as Laplace functionals completely determine the distributions of point processes, they also completely determine the distributions of random measures. Convergence in distribution of the random measures M_n to M is notated by $M_n \xrightarrow{d} M$ and defined by $M_n f \xrightarrow{d} Mf$, $f \in \mathcal{C}_K^+(\mathbb{E})$. In addition, convergence in distribution of a sequence of random measures is equivalent to the convergence of their Laplace functionals. That is, $M_n \xrightarrow{d} M$ if and only if $L_{M_n}(f) \to L_M(f)$, $f \in \mathcal{C}_K^+(\mathbb{E})$.

1.5 Outline of Thesis

The remainder of this thesis is organized as follows. Chapter 2 is a review of relevant literature on point processes and their applications. Here we discuss results from point process theory including thinnings and space-time Poisson models. We also discuss results from application areas, including particles systems, population models, and service systems.

We discuss space-time stochastic processes as marked point processes in Chapter 3. After this, we are able to provide notation and a general modelling framework, as well as a limit theorem that is useful for examining long-run (also called *limiting*) behavior of the systems in question. Specifically, this framework allows us to show convergence in distribution of the processes in which we are interested.

In chapters 4 through 6, we use our results to examine the limiting behavior of several specific models. The first of these is a space-time extension of an elementary species competition model that was first considered by Durrett and Limic in [8]. This model also has applications in the areas of spatial service systems like wireless networks.

We then analyze two spatial polling models in Chapter 5. The Laplace functionals of the limiting processes for these models contain infinite products with no closed-form expressions. However, we are able to get the probability generating functions for the numbers of particles in certain regions by making certain substitutions into the Laplace functionals.

In Chapter 6 we consider models where particles are allowed to move about the system. Again, the departures are triggered by arriving particles. The first model we consider in this chapter allows for particle movements, but no interactions. We then analyze a model where the particles move and interact.

We conclude with some possible areas of future research in Chapter 7.

CHAPTER II

LITERATURE REVIEW

In this chapter we discuss previous work in the areas of point processes, as well as their applications as they pertain to the results in this dissertation.

2.1 Point Processes

Point process theory has evolved over the last few centuries for building probability models of systems in science and engineering, such as those arising in population biology, epidemics, and queueing systems. See [6] for a brief history of point process development. The models we construct in this dissertation are marked point processes on the real line. Background material for marked point processes can be found in [6], [15], and [22].

An excellent introduction to space-time Poisson models is [22]. Specifically, the author discusses *p-transformations* of Poisson processes. Given a point process N, a p-transformation of N is found by mapping each point of N into a new space according to a probability kernel that is independent of the other points. The systems we consider in Chapters 4, 5, and 6 require different analysis because they are transformations of Poisson processes that do depend on the other points. That is, for our models we allow the probability kernels for the transformations to not only depend upon the initial attributes, but they will depend upon the future evolution of the process as well.

Thinnings are standard topics in point process literature, sometimes presented in the guise of disaggregation. Classical thinning results are addressed in most standard texts on stochastic processes, including [13], [19], and [21]. Boker and Serfozo consider the convergence of thinnings that are compositions of measures in [3]. The models we consider in Chapters 4, 5, and 6 can be viewed as thinnings that take place over time, and are very different from traditional thinnings. That is, in the models we consider, particles arrive to a system through time, and the particles collecting in the system are thinned as time passes.

In Chapter 4 we prove Proposition 10 which is concerned with the convergence in distribution of products of random variables. Results of this type are useful for evaluating Laplace functionals. Lemma 5.8 of [12] states that for a null array of constants $c_{ni} \geq 0$ and a constant $c \in [0, \infty]$ that the product $\prod_i (1 - c_{ni})$ converges to e^{-c} if and only if the sum $\sum_i c_{ni}$ converges to c. This result can also be extended to null arrays of random variables. However, given a null array of random variables ξ_{ni}, the ξ_{ni} are independent for each n. In order to analyze the main model in Chapter 4 we need Proposition 10, which does not require independence.

2.2 Particle Systems

Particle systems attempt to capture the random nature of many real-world phenomena involving customers, molecules, animals, plants, etc., that can be represented by particles. In such models, particles enter a system, possibly move about and interact, and eventually depart the system.

The phrase *interacting particle system* refers a class of stochastic spatial models that are restricted to evolve on a lattice or graph. The state of the system at time t is determined by the states of all the points on the lattice. Then the states of the points are allowed to change, typically at exponentially distributed rates, depending on the states of the surrounding points. Essential background and standard models are found in [16] and [17]. Models for particle movements in this area are called *exclusion processes*, where a particle jumps from a lattice point to an unoccupied lattice point at an exponential rate. Much of the study of interacting particle systems is devoted to finding invariant measures for the system states, as well as finding critical values for parameters that determine these invariant measures (like extinction). Our models are less intricate. It appears that each of our systems has only one invariant measure.

2.3 Population Models

The study of populations has been a constant application area of point processes since the seventeenth century. Some of the most popular tools for modelling populations have been branching processes and point processes. Branching processes were introduced in the nineteenth century to model the longevity of surnames of British nobility. See [1]. Since then the theory has evolved tremendously, and today there is a large amount of literature on measure-valued branching processes, which are well suited to capturing spatial aspects of population evolutions.

The work in this dissertation was originally motivated by an elementary toy species competition model considered by Durrett and Limic in [8]. In this model, particles arrive to the unit interval according to a Poisson process. Upon arrival, particles take on independent and identically distributed positions in the unit interval according to uniform $[0, 1]$ random variables. For all $y \in [0, 1]$, an arrival to y instantly removes any existing particle at $x < y$ with probability a. The authors first establish that the limiting process is equal in distribution to a unique stationary process. They then show the stationary process is a Poisson process with mean measure $\mu(dx) = dx/((1 - x)a)$. We generalize this model in Chapter 4 to a class of models we call attribute-based thinnings (ABTs).

2.4 Service Systems

Spatial service systems are popular for describing cellular telephone traffic and manufacturing systems. Spatial queueing models use point processes to generalize Jackson networks (see [11]). An introduction to spatial queueing systems is also given in [22].

Polling models describe service systems where servers arrive to a space and serve the customers. We present two models in Chapter 5 that resemble polling models. The first is what we call an *elastic polling model*, where all arrivals to the system are customers, but each customer serves a random number of customers in the system upon his arrival. The second is called an *inelastic polling model*. This case is more traditional in that customers and servers arrive to a service system, and upon arrival of a server, a random number of customers are serviced.

CHAPTER III

GENERAL FRAMEWORK

The focus of this dissertation is on modelling the evolution of particle systems by time-dependent random transformations of marked point processes. For instance, the marks of a point process may dictate how long particles are to remain in a system, or even how the particles are allowed to move about the system. A typical problem is to determine the point process describing the remaining particles at time t. We attack such a problem by performing a time-transformation of the marked point processes at time t.

The systems we consider are continuous-time, measure-valued Markov processes that are subordinated to time-homogeneous Poisson processes. Our main goal is to determine their limiting behavior. By representing the input data of such a system as a marked point process M, we define the spatial system N_t as a time transformation of M, and finally establish the convergence in distribution of N_t as t goes to infinity. For instance, $N_t(A)$ might represent the number of customers that are in the spatial region A at time t.

In this chapter we describe a very general particle system in which the transformation N_t of an input process is determined by an abstract random functional. We present limit theorems for this system, which we will use in our analysis of the upcoming models. We conclude with an illustrative example.

3.1 Time-Varying Point Processes as Integral Functionals

Consider a system in which particles (representing customers, animals, microbes, etc.) arrive to a Polish space $\mathbb{E}'$ according to a Poisson process with rate λ at the times $0 \leq T_1 \leq T_2 \leq \ldots$. The space $\mathbb{E}'$ is the mark space. Upon arrival, the nth particle is assigned the mark Y_n, which is a random element of the measure space $(\mathbb{E}', \mathcal{E}')$, where $\mathcal{E}'$ is the Borel σ-field on $\mathbb{E}'$. The Y_n are independent and identically distributed, and are independent of the arrival times T_n. The data $\{(T_n, Y_n) : n \geq 1\}$ generates the marked point process $M \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{E}')$ expressed as

$$M(A \times B) = \sum_n \mathbb{1}(T_n \in A, Y_n \in B).$$

The particles that arrive to the system prior to time t are transformed such that the nth point is mapped to a point in some measure space $(\mathbb{E}, \mathcal{E})$, and the points in $\mathbb{E}$ form a point process N_t on $\mathbb{E}$. Here, $\mathcal{E}$ is the Borel $\sigma-$field on $\mathbb{E}$. Specifically, the transformation of the nth point of M is a random function $\phi_t(t - T_n, Y_n)$ of the current time t, how far back in time the arrival occurred $t - T_n$, and its mark Y_n. This is a natural and very general transformation. To define this random function, let $\mathbb{F}$ denote a space of measurable functions mapping $[0, \infty] \times \mathbb{E}'$ into $\mathbb{E}$ such that for any $h \in \mathbb{F}$ and compact set $A \in \mathcal{E}$, $h^{-1}(A)$ is compact in $\mathbb{R}_+ \times \mathbb{E}'$. This will ensure that the transformed process is finite on compact regions of $\mathbb{E}$. We also assume there is a Polish topology on $\mathbb{F}$.

Our convention is that at any time t, the nth point of M is transformed into the point $\phi_t(t - T_n, Y_n) \in \mathbb{E}$, where ϕ_t is a random element of $\mathbb{F}$ defined on the same probability space as M. Thus the transformed points form a point process on $\mathbb{E}$ which we denote by N_t. Specifically, for any region $A \in \mathcal{E}$ and $t \geq 0$,

$$\begin{aligned} N_t(A) &= \sum_n \mathbb{1}(\phi_t(t - T_n, Y_n) \in A) \\ &= M(\{(s, y) : \phi_t(t - s, y) \in A\}). \end{aligned} \tag{3}$$

We will frequently characterize the transformed process N_t by its functional form $N_t f$ for any $f \in \mathcal{C}_K^+(\mathbb{E})$, which is

$$\begin{aligned} N_t f &= \sum_k f(\phi_t(t - T_k, Y_k)) \\ &= \int_{[0,t] \times \mathbb{E}'} f(\phi_t(t - s, y)) M(ds\, dy). \end{aligned}$$

A more convenient form of $N_t f$ is as follows. By making the change of variable $u = t - s$ in the above integral, we have

$$N_t f = \int_{[t,0] \times \mathbb{E}'} f(\phi_t(u, y)) M(t - du\, dy), \tag{4}$$

where $t - I = \{x : t - x \in I\}$. For each t, define the random element $M_t \in \mathcal{M}([0, t] \times \mathbb{E}')$ by

$$M_t(I \times A) = M(t - I \cap [0, t] \times A).$$

That is, given the point process M on the interval $[0, t]$ of the ground space, M_t counts the same points by going backward along the interval $[0, t]$ from t to 0. Thus we can write $N_t f$ as

$$N_t f = \int_{[0,t] \times \mathbb{E}'} f(\phi_t(u, y)) M_t(du\, dy). \tag{5}$$

Also, expression (3) can be written as

$$\begin{aligned} N_t(A) &= M(\{(t - u, y) : \phi_t(u, y) \in A\}) \\ &= M_t(\phi_t^{-1}(A)). \end{aligned} \tag{6}$$

We will use the representations in (5) and (6) to characterize the distribution of N_t and obtain its limit as $t \to \infty$. Because M is a time-homogeneous Poisson process, M_t is equal in distribution to M restricted to $[0, t] \times \mathbb{E}'$. This is an important property that we will exploit.

3.2 Main Limit Theorem

This section contains the main theorem that gives conditions under which the process N_t converges in distribution, and it describes the limit process.

Theorem 1 *Suppose there is a random element ϕ of $\mathbb{F}$ such that*

$$(\phi_t, M_t) \stackrel{d}{=} (\phi, M) \qquad \text{on } \mathbb{F} \times \mathcal{M}([0, t] \times \mathbb{E}'),\ t \geq 0. \tag{7}$$

Then $N_t \xrightarrow{d} N$, where N is defined by

$$Nf = \int_{\mathbb{R}_+ \times \mathbb{E}'} f(\phi(u, y)) M(du\, dy), \qquad f \in \mathcal{C}_K^+(\mathbb{E}). \tag{8}$$

The mean measure of N is

$$\mu_N(A) \equiv EN(A) = E[M(\phi^{-1}(A))], \qquad A \in \mathcal{E}.$$

If in addition $\mu_N(A) < \infty$ for every compact $A \in \mathcal{E}$, then

$$EN_t f \to ENf, \qquad f \in \mathcal{C}_K^+(\mathbb{E}). \tag{9}$$

Proof We prove $N_t \xrightarrow{d} N$ by proving the equivalent statement that $N_t f \xrightarrow{d} Nf$ for all $f \in \mathcal{C}_K^+$. From (5) and assumption (7),

$$N_t f \stackrel{d}{=} \int_{[0,t] \times \mathbb{E}'} f(\phi(u, y)) M(du\, dy)$$

Now as $t \to \infty$, this integral converges w.p.1 to $\int_{\mathbb{R}_+ \times \mathbb{E}'} f(\phi(u,y))M(du\,dy)$. Thus $N_t f \xrightarrow{d} Nf$.

Similar to expression (3), the N defined by (8) can be written as $N(A) = M(\phi^{-1}(A))$. Therefore, $EN(A) = E[M(\phi^{-1}(A))]$.

Finally, note that because f is bounded and has compact support,

$$N_t f \le cN_t(\text{supp}(f)) \le cM(\phi^{-1}(\text{supp}(f))),$$

where supp(f) denotes the support of f and c is an upper bound on f. Then $EN_t f \to ENf$ by dominated convergence because $N_t f \to Nf$ w.p.1. ∎

Remark 2 *Mean Measure.* In Theorem 1, the mean measure of the limiting process N can be expressed as

$$ENf = E\Big[\int_{\mathbb{R}_+ \times \mathbb{E}'} E[f(\phi(u,y))|M]M(du\,dy)\Big]. \tag{10}$$

This follows by conditioning on M.

Though elementary, (10) sheds light on the mean measure in certain cases. For instance, independence of ϕ and M implies

$$ENf = \int_{\mathbb{R}_+ \times \mathbb{E}'} E[f(\phi(u,y))]E[M(du\,dy)].$$

Furthermore, if M has mean measure given by $E[M(I \times A)] = \int_{I \times A} \lambda ds F(dy)$, then

$$ENf = \int_{\mathbb{R}_+ \times \mathbb{E}'} E[f(\phi(u,y))]\lambda\, ds\, F(dy). \tag{11}$$

The following known result is useful for identifying the limiting point process in Theorem 1.

Proposition 3 *In the context of Theorem 1, suppose ϕ is independent of the Poisson process M. Then the limiting process N is a Cox process with mean measure*

$$\mu_N(A) = E[\mu(\phi^{-1}(A))], \qquad A \in \mathcal{E}.$$

In particular, if ϕ is deterministic, then N is a Poisson process with mean measure

$$\mu_N(A) = \mu(\phi^{-1}(A)).$$

Proof Conditioning on ϕ, using (8) and the form of the Laplace functional of the Poisson process M given in (2), the Laplace functional of N can be written as

$$L_N f = E[E[e^{-Nf}|\phi]] = E\Big[\exp\Big\{-\int_{\mathbb{R}_+\times\mathbb{E}'}(1-e^{-f(\phi(u,y))})\mu(du\,dy)\Big\}\Big].$$

Then by the change of variable $x = \phi(u, y)$ in the integral and using the independence of ϕ and M to get $\mu_N(A) = EM(\phi^{-1}(A)) = \mu(\phi^{-1}(A))$,

$$L_N f = E\Big[\exp\Big\{-\int_{\mathbb{E}}(1-e^{-f(x)})\mu_N(dx)\Big\}\Big], \tag{12}$$

which is the Laplace functional of a Cox process. Thus, N is a Cox process with mean measure μ_N. When ϕ is deterministic, the expectation in (12) vanishes so that $L_N f$ is the Laplace functional of a Poisson process. ∎

We conclude this section with a few remarks.

Remark 4 *Convergence of Moments.* Under the assumption in Theorem 1 that μ_N is finite on compact sets, we have $EN_t(A) \to EN(A)$ for compact B from (9). We can also get convergence of other moments. For instance, for compact A,

$$E[N_t(A)^2] \to E[N(A)^2] = E[M(\phi^{-1}(A))^2]$$

by dominated convergence.

Remark 5 *An Extension.* Theorem 1 can be extended to a setting in which there exist ϕ and M such that $(\phi_t, M_t) \xrightarrow{d} (\phi, M)$. This would involve extra technical conditions regarding the convergence of integrals as addressed in [20]. We do not require this generality because all of the models we consider have input processes that are Poisson processes, implying that $(\phi_t, M_t) \stackrel{d}{=} (\phi, M)$. The proof that $N_t \xrightarrow{d} N$ in Theorem 1 is obvious when the input process is a time-homogeneous Poisson process.

Remark 6 *Traditional Queueing.* Unfortunately, traditional queueing processes are not appropriate for this modelling framework. The reason is that any ϕ_t function that describes a queueing process maps points from the input process as a function of the past of the process. In each of the models we consider, the ϕ_t function maps points from the input process as a function of the future of the process until time t.

3.3 Spatial $M/G/\infty$ System

In this section we revisit the spatial $M/G/\infty$ example from Section 1.2 to illustrate how to use the results of the previous section to get convergence results. Let M represent the Poisson input process to a service system in which the nth customer arrives at time T_n with the mark $Y_n = (X_n, V_n)$, representing its respective spatial location in a space $\mathcal{S}$ and sojourn time in the system. The arrivals occur according to a Poisson process with rate λ. Let F denote the distribution of a typical spatial location X_n, and G_x denote the conditional distribution of the service time V_n of the nth customer arriving to $x \in \mathcal{S}$. The (X_n, V_n) and T_n are independent, and the mean measure of M is given by

$$EM([a,b] \times A \times B) = \lambda |b-a| \int_{A \times B} F(dx) G_x(dy).$$

There are several quantities of interest surrounding the limiting process of this system. One might like to know (i) the limiting distribution of the number of customers that arrived in the interval $[t-u, t]$ that departed the system before time t, or (ii) the limiting distribution of the number of customers remaining in the system in a certain region of the mark space.

To this end, consider the transformed point process

$$N_t(I \times A \times \{j\}) = \sum_n \mathbb{1}\Big(t - T_n \in I, X_n \in A, \mathbb{1}(0 < t - T_n < V_n) = j\Big).$$

This process counts the number of arrivals in the time interval I and in the spatial region A that are still in the system or that have departed, depending on whether $j = 1$ or 0. The process N_t corresponds to a transformation of M under the function

$$\phi(u, x, v) = (u, x, \mathbb{1}(0 < u < v)),$$

which is non-random and independent of t. Therefore, as in (5), we can express N_t as

$$N_t f = \int_{\mathbb{R}_+ \times \mathcal{S} \times \mathbb{R}_+} f(\phi(u, x, v)) M_t(du\, dx\, dv). \tag{13}$$

Now the assumptions of Theorem 1 are satisfied because M is a time homogeneous Poisson process and ϕ is non-random and independent of t. Therefore, $N_t \xrightarrow{d} N$, where

$$Nf = \int_{\mathbb{R}_+ \times \mathcal{S} \times \mathbb{R}_+} f(\phi(u, x, v)) M(du\, dx\, dv).$$

Because ϕ is deterministic, by Proposition 3 it follows that N is a Poisson process with mean measure

$$\begin{aligned} EN(I \times A \times \{j\}) &= EM(\phi^{-1}(I \times A \times \{j\})) \\ &= \lambda \int_A F(dx)\Big[\mathbb{1}(j=0)\int_I G_x(y)dy \\ &\qquad +\mathbb{1}(j=1)\int_I (1-G_x(y))dy\Big]. \end{aligned} \tag{14}$$

As mentioned above, one might be interested in the limiting process of the number of customers that arrived to the system in the time interval $[t-u,t]$ that departed before time t. By the preceding remarks, this is a Poisson process with mean measure

$$EN(I \times A \times 0) = \lambda \int_A F(dx) \int_I G_x(y)dy.$$

The limiting process describing the total number of remaining customers in a spatial region over all time is also a Poisson process with mean measure

$$EM(\phi^{-1}(\mathbb{R}_+, A, \{1\})) = \lambda EV_1 \int_A F(dx).$$

The transformations of the previous example are tractable because the function ϕ is deterministic. While Theorem 1 is far from essential for analyzing this model, it sheds light on how one can prove limit theorems for marked point processes. Our main model appears in Section 4.2 and is more difficult to analyze because ϕ is random and dependent upon the input process M.

CHAPTER IV

ATTRIBUTE-BASED THINNINGS

In this chapter we will study a random time transformation of a marked point process called an *attribute-based thinning* (ABT). Not only is the system input data random, but the transformation function ϕ_t is random as well. The original motivation for this model was an elementary species competition model analyzed by Durrett and Limic in [8], which is discussed in Section 2.3.

Although we can establish the convergence in distribution of ABT processes, the limits are not always tractable. However, we are able to prove that the limiting process is Poisson for a certain generalization of the model of Durrett and Limic. This is the main result of the dissertation.

We begin this chapter by introducing the main result, along with discussing the exponential sojourn times. In Section 4.2 we prove three preliminary propositions that are not only of interest in their own right, but also facilitate the proof of the limit theorem for the main model. Section 4.3 consists entirely of the proof of the main result. We describe general ABTs and calculate their limiting mean measures in the fourth section. We conclude this chapter by showing the stationary Poisson process is consistent with the our limiting results, and we present some ABTs that do not have Poisson processes as stationary versions.

4.1 Main Result

Consider a system where particles arrive to the totally ordered attribute space $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-field on the real numbers. Each arrival considers for deletion all particles currently in the system that have strictly lower attributes. The attributes $X_n \in \mathbb{R}$ are independent continuous random variables with common distribution function F. Then the deletion region of a particle arriving to $y \in \mathbb{R}$ is defined by $D(y) \equiv \{x : x < y\}$, which is the set of points an arrival at y can consider for deletion. We write $a(x)$ to denote the probability that a particle with attribute x is removed from the system given an arrival to $y > x$. Then the probability that a particle at x is considered for deletion by the next arrival is $\overline{F}(x)$, and the probability it is deleted by the next arrival is $a(x)\overline{F}(x)$. Also, we require the condition

$$\int_A \frac{F(dx)}{a(x)^2 \overline{F}(x)^2} < \infty \tag{15}$$

for any compact set $A \in \mathcal{B}$.[1]

We write L_n to denote the *discrete lifetime* of the nth particle. That is, $L_n = \ell$ means the nth particle survives exactly $\ell - 1$ future deletion attempts, and exits the system upon the arrival of the particle indexed with $n + \ell$. The marks $(X_1, L_1), (X_2, L_2) \ldots \in \mathbb{E} \times \mathbb{N}$ are independent and identically distributed, and they are independent of $T_1, T_2, \ldots$, though X_n and L_n can be dependent on each other for each n. The nature of the deletion region is a key feature that leads to a Poisson limit.

[1] We require this condition in order to ensure the mean measure given in (18) below is finite on compacts. Moreover, it ensures the W_n in (36) below have finite second moments. Forcing the probability density function for X_n to be strictly non-negative on an open set $\mathbb{E} \subseteq \mathbb{R}$, 0 otherwise, and $a(x)$ to be bounded away from 0 on compact subsets of $\mathbb{E}$ satisfies this condition.

To illustrate the deletion mechanism, consider Figure 2 below. In (a), the time is just prior to the arrival time T_n of the nth particle. The particles currently remaining in the system are located at the positions X_k, X_j, X_i, and X_m. In (b), the time is T_n, and the nth particle is considering the ith, jth, and kth particles for deletion because $X_i, X_j, X_k < X_n$. The mth particle cannot be considered for deletion because $X_n < X_m$. In (c), the time is just after time T_n, and we see that particles i and k were removed by the nth arrival.

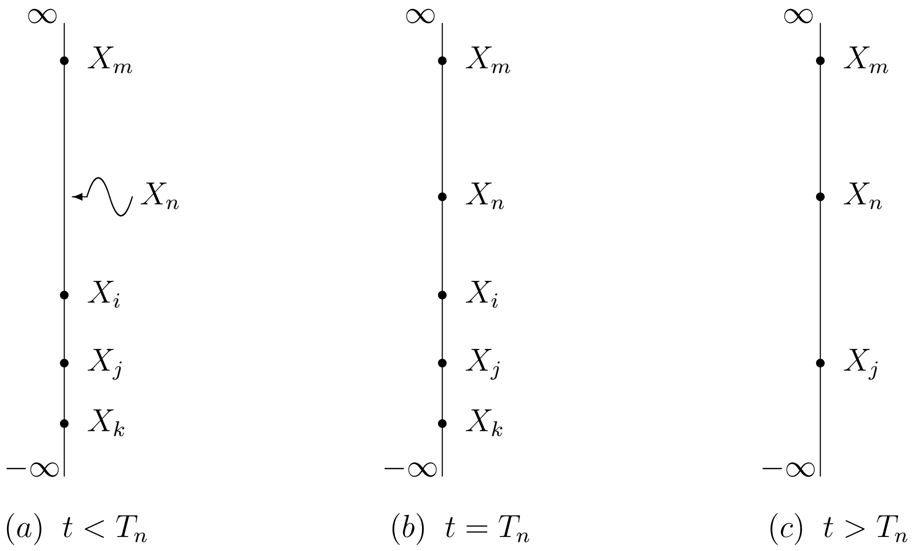

Figure 2: An attribute-based thinning model.

The input marked point process M on $\mathbb{R}_+ \times (\mathbb{E} \times \mathbb{N})$ given by

$$M(I \times A \times B) = \sum_n \mathbb{1}(T_n \in I, X_n \in A, L_n \in B), \qquad I \in \mathcal{B}_+, A \in \mathcal{B}, B \in \mathcal{N}, \quad (16)$$

counts the number of particles with discrete lifetime in B that enter region A in the time interval I. Here, $\mathcal{B}_+$ and $\mathcal{N}$ are the σ-fields on the nonnegative reals and the natural numbers, respectively. We will also write $M(t) = \sum_n \mathbb{1}(0 < T_n \leq t)$ to denote the total number of arrivals to the space in the time interval $[0, t]$.

Our interest is in the process

$$N_t(A) = N_0^t(A) + \sum_n \mathbb{1}(0 < T_n \leq t, X_n \in A, M((T_n, t] \times (X_n, \infty) \times \mathbb{N}) < L_n), \quad (17)$$

where N_0^t is the point process describing the numbers of particles initially in the system at time 0 that are still in the system at time t. Then $N_t(A)$ is the total number of particles with attributes in region A at time t. We are assuming the particles in the system at time 0 do not affect the arrivals after time 0. Therefore, N_t is a continuous-time Markov jump process. The transitions are at the arrival times of the particles.

An important performance measure of the system is the sojourn time of a typical particle. The following result says the sojourn time has an exponential distribution. However, it turns out that these times are highly dependent.

Proposition 7 (Sojourn Times.) *The sojourn time of a particle that enters* $\mathbb{R}$ *with attribute* x *has an exponential distribution with rate* $\lambda a(x)\overline{F}(x)$, *where* $a(x)\overline{F}(x)$ *is the probability that a particle with attribute* x *does not survive the next arrival.*

Proof A particle with attribute x at any time will be deleted by the next arrival with probability $a(x)\overline{F}(x)$, independently of everything else. This implies the number of new arrivals until the particle at x is deleted is a geometric random variable ν with parameter $a(x)\overline{F}(x)$. Thus the sojourn time of a particle at x is $\sum_{k=1}^{\nu} \tau_k$, where the τ_k represent independent exponential interarrival times with rate λ, which are independent of ν. It is well known that such a sum of a geometric number of i.i.d. exponential random variables is distributed exponentially with rate $\lambda a(x)\overline{F}(x)$. ∎

Our main result is the following theorem that says the limiting process for N_t is Poisson.

Theorem 8 *Under the preceding assumptions, $N_t \xrightarrow{d} N$ as $t \to \infty$, where N is a Poisson process that is independent of N_0^t, with mean measure*

$$\mu_N(A) = \int_A \frac{F(dx)}{a(x)\overline{F}(x)}, \qquad A \in \mathcal{B}. \tag{18}$$

Hence, the stationary distribution of the Markov process N_t is that of the Poisson process N.

Proof First note that by Proposition 7 each particle has a finite sojourn time w.p.1. Because N_0^t is a point process, there are finite numbers of particles in compact sets at time 0. Therefore $N_0^t \xrightarrow{d} 0$. Next, by Theorem 14 of Section 4.4, we will see that $N_t - N_0^t \xrightarrow{d} N$, where N is given by

$$Nf = \sum_n f(X_n)\mathbb{1}(M_n < L_n), \qquad f \in \mathcal{C}_K^+(\mathbb{R}), \tag{19}$$

and $M_n = M([0, T_n) \times (X_n, \infty) \times \mathbb{N})$. These observations prove $N_t \xrightarrow{d} N$ where N is given by (19). We complete the proof in Theorem 12 of Section 4.3 by establishing that N is a Poisson process with mean measure μ_N given by (18). We postpone the proof of Theorem 12 until Section 4.3 so that we can make use of the preliminary propositions in Section 4.2. ∎

As an aside, here is another way to think about the system. One can interpret the process $\{N_t, t \geq 0\}$ as a *measure-valued branching process with immigration*[2].

[2] On a related note, Branching processes have also been used to describe the busy period of queues. See for instance [9], [14], and [18].

In this context, at each time T_n a new particle immigrates into the space $\mathbb{R}$ with attribute y. At each of these times, a particle already located at x such that $x < y$ produces exactly one offspring with the same attribute with probability $1-a(x)$, then immediately dies. Otherwise, the particle at x produces no offspring and immediately dies. If $y \leq x$, then the particle at x produces exactly one particle and immediately dies. If at time 0 there are no particles in the system, then conditioned on the space-time coordinates of all $M(t)$ arrivals, $N_t(A)$ is an example of what has been referred to as a Poisson-Binomial random variable in [4] and [5].

4.2 Preliminaries for Poisson Limit

In this section we present three propositions that are of interest by themselves, and are required for the proof of Theorem 8 in the following section.

The first is a known result on the cumulative ranks of independent and identically distributed random variables; e.g. see page 52 of [7]. The proof below is a little different than this reference.

Proposition 9 (Cumulative Ranks.) *Let $Z_1, \ldots, Z_n$ be independent random variables with a common continuous distribution, and define*

$$R_j = \sum_{i=1}^{j} \mathbb{1}(Z_i \leq Z_j),$$

which is the rank of Z_j within the random sample $Z_1, \ldots, Z_j$. Then the cumulative ranks $R_1, \ldots, R_n$ are independent and

$$P(R_j = k) = j^{-1}, \qquad 1 \leq k \leq j,\ j = 1, \ldots, n. \tag{20}$$

Proof It is well known that the rank R_j of Z_j has the distribution given in (20) because Z_j is a member of the random sample $Z_1, \ldots, Z_j$. To prove the independence of the ranks, we will consider the way in which the R_j depend on the Z-values.

Because $Z_1, \ldots, Z_j$ are i.i.d. and continuous, Z_j is equally likely to be in any one of the j intervals

$$(-\infty, \tilde{Z}_1), (\tilde{Z}_1, \tilde{Z}_2), \ldots, (\tilde{Z}_{j-1}, \infty)$$

where $\tilde{Z}_1 < \tilde{Z}_2 < \ldots < \tilde{Z}_{j-1}$ are the ordered $Z_1, \ldots, Z_{j-1}$. That is,

$$P(R_j = k | \mathcal{F}_{j-1}) = j^{-1}, \qquad 1 \leq k \leq j, \tag{21}$$

where $\mathcal{F}_{j-1} = \sigma(Z_1, \ldots, Z_{j-1})$. Furthermore, because $\mathcal{F}_{j-1}$ contains $\sigma(R_1, \ldots, R_{j-1})$, we have

$$P(R_j = k | R_1, \ldots, R_{j-1}) = E[P(R_j = k | \mathcal{F}_{j-1}) | R_1, \ldots, R_{j-1}] = j^{-1}.$$

In light of this result, we have

$$\begin{aligned} P(R_1 = r_1, \ldots, R_n = r_n) &= \prod_{j=1}^{n} P(R_j = r_j | R_1 = r_1, \ldots, R_{j-1} = r_{j-1}) \\ &= \prod_{j=1}^{n} \frac{1}{j} = \frac{1}{n!}. \end{aligned}$$

Because this joint probability is the product of the marginal probabilities of the $R_1, \ldots, R_n$, the ranks are independent. ∎

Our next proposition gives a criterion for the convergence in distribution of products of dependent but identically distributed random variables. It is a generalization of Lemma 5.8 of [12] that states for a null array of constants $c_{ni} \geq 0$ and $c \in \mathbb{R}_+$,

$$\prod_i (1 - c_{ni}) \to e^{-c} \Longleftrightarrow \sum_i c_{ni} \to c.$$

A similar result applies to null arrays of independent random variables as well. However, the product we encounter in the proof of Theorem 8 is different since the random variables we have to deal with are not independent.

Proposition 10 *Let $W_{n1}\ldots,W_{nn}$ be an array of random variables in $[0,c]$, with $c<1$, that are identically distributed for each n but not necessarily independent. Suppose there are nonnegative iid random variables W_i such that (W_{ni},W_i) are identically distributed for $i\in\{1,\ldots,n\}$ and*

$$nW_{ni}\xrightarrow{L_1}W_i \qquad \text{as } n\to\infty \tag{22}$$

and

$$nW_{n1}^2\xrightarrow{L_1}0 \qquad \text{as } n\to\infty. \tag{23}$$

Then

$$\prod_{i=1}^{n}(1-W_{ni})\xrightarrow{d}e^{-EW_1}, \qquad \text{as } n\to\infty. \tag{24}$$

Proof Consider the expression

$$\prod_{i=1}^{n}(1-W_{ni})=\exp\{\sum_{i=1}^{n}\log(1-W_{ni})\}=\exp\{-n^{-1}\sum_{i=1}^{n}W_i+\aleph_n\},$$

where

$$\aleph_n=\sum_{i=1}^{n}\Big(n^{-1}W_i+\log(1-W_{ni})\Big).$$

We know $n^{-1}\sum_{i=1}^{n}W_i\to EW_1$ w.p.1 by the strong law of large numbers. Then the assertion in (24) will follow upon application of the continuous mapping theorem (see [2] and [12]) and showing that $\aleph_n\xrightarrow{d}0$.

Using the triangle inequality and the identically distributed assumptions,

$$\begin{aligned} E|\aleph_n| &\le E|W_1 + n\log(1-W_{n1})| \\ &= E\Big|W_1 - n\sum_{j=1}^{\infty}\frac{W_{n1}^j}{j}\Big| \\ &\le E|W_1 - nW_{n1}| + nE\Big|W_{n1}^2\sum_{j=0}^{\infty}\frac{W_{n1}^j}{2+j}\Big|. \end{aligned} \tag{25}$$

Assumption (22) ensures $E|W_1 - nW_{n1}| \to 0$. Because $W_{n1} \le c < 1$ for all n, the last term in (25) is bounded by

$$nE\Big[\frac{W_{n1}^2}{1-W_{n1}}\Big] \le n\frac{E\big[W_{n1}^2\big]}{1-c},$$

which goes to zero by (23). Applying this observation to (25) proves $\aleph_n \xrightarrow{d} 0$. ∎

The proof of Theorem 12 uses the following result for binomial random variables. Part (i) is used primarily to prove Part (ii), which can be thought of as an inverted law of large numbers for binomial random variables.

Proposition 11 *For $n \ge 1$, let S_n be a binomial random variable with parameters n and p. Then*

(i) *For each $k \ge 0$ and $f : \{0, 1, \ldots, n\} \longrightarrow \mathbb{R}$,*

$$E[f(S_n)] = p^{-k}E\Big[\mathbb{1}(S_{n+k} \ge k)f(S_{n+k} - k)\prod_{i=0}^{k-1}\frac{S_{n+k} - i}{n+k-i}\Big], \tag{26}$$

(ii) $n(S_n + 1)^{-1} \xrightarrow{L^2} 1/p$.

Proof To prove (i), we use the change of variable $\ell = j + k$ to obtain

$$E[f(S_n)] = \sum_{j=0}^{n} f(j)\binom{n}{j}p^j(1-p)^{n-j}$$

$$
\begin{aligned}
&= \sum_{\ell=k}^{n+k} f(\ell-k)\binom{n}{\ell-k}p^{\ell-k}(1-p)^{n+k-\ell} \\
&= p^{-k}\sum_{\ell=k}^{n+k} f(\ell-k)\frac{\ell!}{(\ell-k)!}\frac{n!}{(n+k)!}\binom{n+k}{\ell}p^{\ell}(1-p)^{n+k-\ell}.
\end{aligned}
$$

This is equal to the right-hand side of (26).

To prove (ii), we substitute

$$
f(S_n) = \Big(\frac{n}{S_n+1} - \frac{1}{p}\Big)^2,
$$

and $k = 2$ into (26) to get

$$
\begin{aligned}
E\Big[\Big(\frac{n}{S_n+1} - \frac{1}{p}\Big)^2\Big] &= p^{-2}E\Big[\mathbb{1}(S_{n+2} \geq 2)\Big(\frac{n^2 S_{n+2}}{(S_{n+2}-1)(n+2)(n+1)} \\
&\qquad - \frac{2nS_{n+2}}{p(n+2)(n+1)} + \frac{S_{n+2}(S_{n+2}-1)}{p^2(n+2)(n+1)}\Big)\Big].
\end{aligned}
$$

The expression inside the last expectation is bounded and converges to 0 w.p.1 as $n \to \infty$. Thus

$$
E\Big[\Big(\frac{n}{S_n+1} - \frac{1}{p}\Big)^2\Big] \to 0
$$

by the bounded convergence theorem, which proves (ii). ∎

4.3 Proof of Poisson Limit

In this section we prove the following theorem, which is required for the proof of Theorem 8. We will prove $N_t - N_0^t \xrightarrow{d} N$ in the following section.

Theorem 12 *The point process N defined by*

$$
Nf = \sum_n f(X_n)\mathbb{1}(M_n < L_n), \qquad f \in \mathcal{C}_K^+(\mathbb{R}),
$$

with $M_n = M([0,T_n) \times (X_n, \infty) \times \mathbb{N})$ *is a Poisson process with mean measure*

$$\mu_N(A) = \int_A \frac{F(dx)}{a(x)\overline{F}(x)}.$$

Proof We will prove this by showing the Laplace functional of N is that of a Poisson process. That is, we will show

$$E[e^{-Nf}] = \exp\{-\int_{\mathbb{R}} (1 - e^{-f(x)})\mu_N(dx)\}, \qquad f \in \mathcal{C}_K^+(\mathbb{R}). \tag{27}$$

For each $t > 0$, define the point process $\bar{N}_t$ on $\mathbb{R}$ by

$$\bar{N}_t f = \sum_n f(X_n)\mathbb{1}(M_n < L_n)\mathbb{1}(T_n \leq t). \tag{28}$$

Assume for the moment that the following statements are true:

(i) $\Phi_m \equiv E[e^{-\bar{N}_t f}|M(t) = m]$ is independent of t,

(ii) $\lim_{m\to\infty} \Phi_m = \Phi \equiv \exp\{-\int_{\mathbb{R}}(1 - e^{-f(x)})\mu_N(dx)\}$.

Then because $\bar{N}_t \uparrow N$ and $\Phi_{M(t)} \to \Phi$ w.p.1 as $t \to \infty$, it would follow by the bounded convergence theorem that

$$\begin{aligned} E[e^{-Nf}] &= \lim_{t\to\infty} E[e^{-\bar{N}_t f}] \\ &= \lim_{t\to\infty} E[E[e^{-\bar{N}_t f}|M(t)]] \\ &= \Phi. \end{aligned}$$

This means that expression (27) will follow upon proving statements 1 and 2 above.

At this point, we temporarily digress to prove the following lemma which verifies statement (i) above.

Lemma 13 *Define*

$$\nu_m(n) = \sum_{i=1}^{m} \mathbb{1}(X_i \geq X_n), \qquad n \in \{1, \ldots, m\}, \tag{29}$$

and let $\nu_m^{-1}(n)$ *be the index* k *such that* $\nu_m(k) = n$. *Then for each* $m \geq 1$, *the* $\nu_m^{-1}(1), \ldots, \nu_m^{-1}(m)$ *denotes the permutation of* $1, \ldots, m$ *such that* $X_{\nu_m^{-1}(1)} > X_{\nu_m^{-1}(2)} > \cdots > X_{\nu_m^{-1}(m)}$. *Define*

$$p_m(n) = \frac{1 - (1 - a(X_n))^{\nu_m(n)}}{\nu_m(n) a(X_n)}, \qquad n \in \{1, \ldots, m\}. \tag{30}$$

Then $E[e^{-\bar{N}_t f} | M(t) = m] = \Phi_m$ *is independent of* t, *where*

$$\Phi_m = E\Big[\prod_{n=1}^{m} [1 - p_m(n)(1 - e^{-f(X_n)})]\Big]. \tag{31}$$

Proof Conditioning on the σ-field $\mathcal{F}_m(t) = \sigma(M(t) = m, X_1, X_2, \ldots, X_m)$, we have

$$E[e^{-\bar{N}_t f} | M(t) = m] = E\Big[E\Big[\prod_{n=1}^{m} e^{-f(X_n)\mathbf{1}(M_n < L_n)} | \mathcal{F}_m(t)\Big]\Big| M(t) = m\Big], \tag{32}$$

where

$$M_n = \sum_{k=1}^{m} \mathbb{1}(T_k < T_n, X_k > X_n).$$

By a standard property of Poisson processes, we can assume the T_n are independent and identically distributed with the uniform distribution on $[0, t]$. Letting $T'_n = T_{\nu_m^{-1}(n)}$, we recognize that conditioned on $\mathcal{F}_m(t)$,

$$M_n = \sum_{k=1}^{\nu_m(n)-1} \mathbb{1}(T'_k < T'_{\nu_m(n)}) \tag{33}$$

is the rank of T'_n in the random sample $T'_1, \ldots, T'_n$. Then by Proposition 9, the cumulative ranks M_n are conditionally independent given $\mathcal{F}_m(t)$ and

$$P(M_n = k | \mathcal{F}_m(t)) = \frac{1}{\nu_m(n)}, \qquad 0 \leq k \leq \nu_m(n) - 1. \tag{34}$$

Next, note that $L_1, \ldots, L_m, M_1, \ldots, M_m$ are conditionally independent given $\mathcal{F}_m(t)$. This follows because the M_n are functions of the T'_n which are conditionally independent of the L_n given $\mathcal{F}_m(t)$. Note that conditioned on $\mathcal{F}_m(t)$ the L_n has the geometric $(a(X_n))$ distribution. This means

$$\begin{aligned} P(M_n < L_n|\mathcal{F}_m(t)) &= E[P(M_n < L_n|\mathcal{F}_m(t))|\mathcal{F}_m(t)] \\ &= E[(1-a(X_n))^{M_n}|\mathcal{F}_m(t)]. \end{aligned}$$

Then using the conditional distribution in (34) for M_n, it follows that

$$\begin{aligned} P(M_n < L_n|\mathcal{F}_m(t)) &= E\Big[E[(1-a(X_n))^{M_n}|\mathcal{F}_m(t), M_n]\Big|\mathcal{F}_m(t)\Big] \\ &= \sum_{k=0}^{\nu_m(n)-1} \frac{(1-a(X_n))^k}{\nu_m(n)} \\ &= \frac{1-(1-a(X_n))^{\nu_m(n)}}{\nu_m(n)a(X_n)} \\ &= p_m(n). \end{aligned}$$

Applying the preceding observations to (32) yields

$$\begin{aligned} E[e^{-\bar{N}_t f}|M(t)=m] &= E\Big[\prod_{n=1}^{m} E[e^{-f(X_n)\mathbb{1}(M_n<L_n)}|\mathcal{F}_m(t)]\Big|M(t)=m\Big] \\ &= E\Big[\prod_{n=1}^{m}\Big(e^{-f(X_n)}p_m(n)+1-p_m(n)\Big)\Big] \\ &= \Phi_m. \end{aligned}$$

This proves (31). ∎

We now return to the proof of Theorem 12. All that remains is to verify that statement (ii) holds, which is

$$\Phi_m = E\Big[\prod_{n=1}^{m}\Big(1-p_m(n)(1-e^{-f(X_n)})\Big)\Big]$$

$$\rightarrow \quad \Phi = \exp\{-\int_{\mathbb{R}} (1 - e^{-f(x)})\mu_N(dx)\}. \tag{35}$$

To do this, we will apply Proposition 10. Set

$$W_{mn} = p_m(n)(1 - e^{-f(X_n)}).$$

Note that for each m, the W_{mn} are not independent, but are identically distributed on $[0, c]$, where

$$c = 1 - \exp\Big\{ - \sup_{x \in \mathbb{R}} f(x) \Big\}.$$

Define the nonnegative random variables

$$W_n = \frac{1 - e^{-f(X_n)}}{a(X_n)\overline{F}(X_n)}. \tag{36}$$

Note the W_n are independent and identically distributed because they are functions of the X_n. Also note the (mW_{mn}, W_n) are identically distributed, and $E[W_n^2] < \infty$ by (15). Then in order to use Proposition 10 to show (35), we just need to show $mW_{mn} \xrightarrow{L^1} W_n$ and $mW_{m1}^2 \xrightarrow{L_1} 0$ as $m \to \infty$.

Convergence in L_2 implies convergence in L_1. Therefore showing

$$mW_{mn} \xrightarrow{L^2} W_n \tag{37}$$

implies $mW_{mn} \xrightarrow{L^1} W_n$. Next, by the triangle and Cauchy-Schwarz inequalities,

$$\begin{aligned} mE|W_{m1}^2| &= E|W_{m1}(mW_{m1} - W_1) + W_{m1}W_1| \\ &\leq E|W_{m1}(mW_{m1} - W_1)| + E|W_{m1}W_1| \\ &\leq \Big(E[W_{m1}^2]E[(mW_{m1} - W_1)^2]\Big)^{1/2} + \Big(E[W_{m1}^2]E[W_1^2]\Big)^{1/2}. \end{aligned}$$

The first term on the right goes to zero as $m \to \infty$ because $0 \le W_{m1} < 1$ and by (37). The second term on the right goes to zero because $E[W_1^2] < \infty$ and by (37) because

$$\begin{aligned} mW_{m1} \xrightarrow{L^2} W_1 \quad &\Rightarrow \quad m^2 E[W_{m1}^2] - 2mE[W_{m1}W_1] + E[W_1^2] \to 0 \\ &\Rightarrow \quad E[W_{m1}^2] - \frac{2}{m} E[W_{m1}W_1] + \frac{E[W_1^2]}{m^2} \to 0. \end{aligned}$$

Since $E[W_1^2] < \infty$, this implies $E[W_{m1}^2] \to 0$, so that $mW_{m1}^2 \xrightarrow{L_1} 0$. Therefore, it suffices to show (37).

Retaining the $\nu_m(n)$ from (29) and noting that $(1 - a(X_n))^{\nu_m(n)} \xrightarrow{L^2} 0$ as $m \to \infty$ because $\nu_m(n) \to \infty$ w.p.1, we see that showing (37) is equivalent to showing

$$E\Big[\Big(\frac{m}{\nu_m(n)} - \frac{1}{\overline{F}(X_n)}\Big)^2 \mathbb{1}(X_n \le x^*)\Big] \to 0, \tag{38}$$

where $x^* = \sup\{x : f(x) \ne 0\}$.

By the definition of $\nu_m(n)$, we know that

$$P(\nu_m(n) = k | X_n = x) = P(S_{m-1} = k - 1),$$

where S_{m-1} has a Binomial distribution with parameters $m - 1$ and $\overline{F}(x)$. Then conditioning on X_n,

$$E\Big[\Big(\frac{m}{\nu_m(n)} - \frac{1}{\overline{F}(X_n)}\Big)^2 \mathbb{1}(X_n \le x^*)\Big] = \int_{-\infty}^{x^*} h_m(x) F(dx), \tag{39}$$

where

$$h_m(x) = E\Big[\Big(\frac{m}{S_{m-1} + 1} - \frac{1}{\overline{F}(x)}\Big)^2\Big].$$

By part *(ii)* of Proposition 11, it follows that $h_m(x) \to 0$ as $m \to \infty$. Also, from the proof of this, it is clear that $|h_m(x)| \le C/\overline{F}(x^*)^2$, for some constant C. Then by the bounded convergence theorem, the expectation in (39) converges to 0, and this proves (38). Thus, the proof of Theorem 12 is complete. ∎

It appears our model results in a Poisson process in the limit because we can exploit Proposition 9 in the previous section. We can do this because of the special deletion rule based on the total ordering of the particle attributes. This proposition does not always hold for other deletion rules, and furthermore seems to be quite rare. In Section 4.5 we briefly examine some ABTs that are similar to the one described above that neither allow for the use of Proposition 9, nor produce Poisson limits. It is likely that a necessary criterion for Proposition 9 to be applied is for the deletion rule to follow a linear ordering.

Upon proving Theorem 8, it is evident that we can make some generalizations. First, the results immediately extend to certain cases of batch arrivals. Suppose arrivals come in batches consisting of a random number of particles, and each particle receives an independent and identically distributed attribute in $\mathbb{R}$ from the distribution F. Further, assume each arrival within each batch is ranked among the other particles within the batch, and considers for deletion those that arrived previously as well as those within the batch with a lower position. Theorem 8 prescribes the same limiting distribution for such cases, only the convergence is faster provided the batch distributions have means greater than one.

Our model can also be generalized to allow for multiple types of particles by simply allowing m independent ABT processes $N_t^1, \ldots, N_t^m$, each with its own parameters F_n and $a_n(\cdot)$, to take place on $\mathbb{R}$. Limiting and stationary results are easily achieved by straightforward applications of standard aggregation theorems for Poisson processes.

Letting N_t denote the entire process $N_t^1 + \cdots + N_t^m$, it follows that the limiting mean measure for such a system is given by

$$\mu_N = \sum_{n=1}^{m} \mu_{N^n},$$

where m is the number of particle types, $N_t^n \xrightarrow{d} N^n$ for each n, and

$$\mu_{N^n}(A) = \int_A \frac{F_n(dx)}{a_n(x)\overline{F}_n(x)}, \qquad A \in \mathcal{B}.$$

4.4 Limiting Distributions of ABTs

In this section we will consider a more general model than that of the previous section. We examine special cases in the next section. Now, instead of arriving to a totally ordered space, assume particles (representing customers, items, etc.) arrive to a Polish space $(\mathbb{E}, \mathcal{E})$ at times $0 < T_1 < T_2 < \ldots$, forming a time-homogeneous Poisson process on $\mathbb{R}_+$ with rate λ. Then nth arrival at time T_n will be assigned the mark (X_n, L_n) from the mark space $(\mathbb{E} \times \mathbb{N}, \mathcal{E} \otimes \mathcal{N})$, where $\mathcal{N}$ is the Borel field on the natural numbers, and L_n is the discrete lifetime as before. Here, X_n represents the attribute of the nth particle in $\mathbb{E}$ (which can now be a number, a vector, etc.), and we use F to denote its distribution. Then the input marked point process M on $\mathbb{R}_+ \times (\mathbb{E} \times \mathbb{N})$ given by

$$M(I \times A \times B) = \sum_n \mathbb{1}(T_n \in I, X_n \in A, L_n \in B), \qquad I \in \mathcal{R}_+, A \in \mathcal{E}, B \in \mathcal{N}, \quad (40)$$

counts the number of particles with discrete lifetime in B that enter region A in the time interval I. We will also write $M(t) = \sum_n \mathbb{1}(T_n \leq t)$ to denote the total number of arrivals to the space through time t.

The particles are again subject to deletion by future arrivals. Recall that $D(y)$ denotes the deletion region of y, meaning that upon an arrival to $y \in \mathbb{E}$, a particle with attribute $x \in \mathbb{E}$ is deleted with probability $a(x)$ if $x \in D(y)$, independently of everything else. We will also use the notation $D^{-1}(x) \equiv \{y : x \in D(y)\}$, and we assume $D^{-1}(x)$ to be a measurable set for all $x \in \mathbb{E}$. Here we do not specify the form of $D(y)$.

In order to illustrate the deletion mechanism, consider Figure 3. For the process depicted here, $X_n = (Y_n, Z_n)$, where Y_n is the actual position of the nth particle in the $\mathbb{R}^2$ plane, and Z_n is the rank[3] of the nth particle. In this figure, the nth particle has just arrived. Its deletion region is the set of all points located within the circle of radius r centered at Y_n that have lower ranks than Z_n. Because the ith and jth particles are the only ones within the circle, the ith particle will be removed from the system with probability $a(Y_i, Z_i)$ if $Z_i < Z_n$, and the jth particle will be removed from the system with probability $a(Y_j, Z_j)$ if $Z_j < Z_n$.

Let us define

$$b(x) \equiv F(D^{-1}(x)) = \int_{\mathbb{E}} \mathbb{1}(x \in D(y))F(dy), \qquad x \in \mathbb{E}',$$

which is the conditional probability that the next arrival to the system has the opportunity to consider a particle at x for deletion. Note that $a(x)b(x)$ is the probability that a particle located at x is removed from the system upon the next arrival. As before, L_n is conditionally geometric given X_n. That is,

$$P(L_n = \ell | X_n) = [1 - a(X_n)]^{\ell-1} a(X_n). \tag{41}$$

[3]The ranks here are not to be confused with the ranks of Proposition 9. Here, a rank is a value in the unit interval or $\mathbb{R}$.

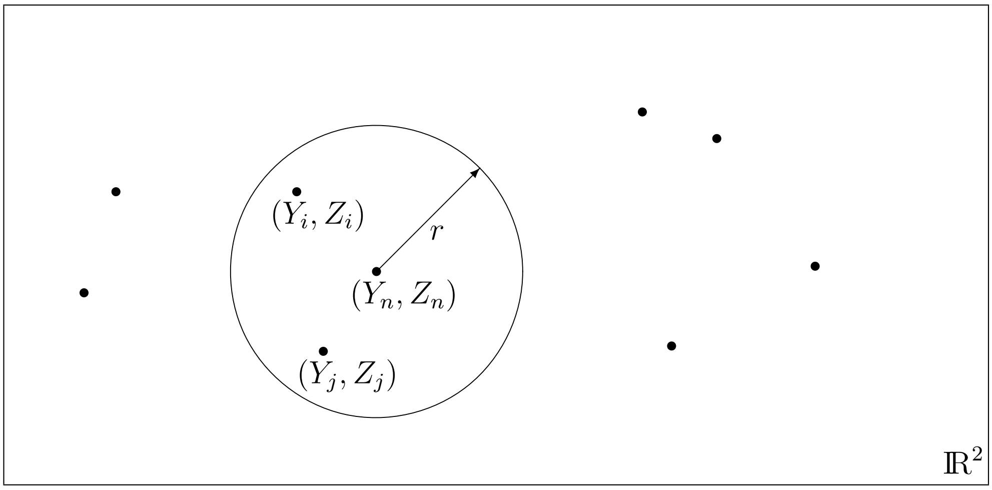

Figure 3: Attribute-based thinning with $\mathbb{E} = \mathbb{R}^2 \times \mathbb{R}$.

Under these assumptions, the input process M is a Poisson process. The mean measure of the Poisson process M is given by

$$E[M(I \times A \times B)] = \int_I \lambda ds \int_A F(dx) \sum_{\ell \in B} (1 - a(x))^{\ell - 1} a(x).$$

Our concern will be with the continuous-time stochastic point process N_t on the space $\mathbb{R}_+ \times \mathbb{E} \times \{0, 1\}$ defined by

$$\begin{aligned} N_t(I \times A \times \{i\}) \;&=\; \sum_n \mathbb{1}\Big(t - T_n \in I, X_n \in A, \\ &\qquad \mathbb{1}\Big(M\Big((T_n, t] \times D^{-1}(X_n) \times \mathbb{N}\Big) < L_n\Big) = i\Big). \qquad (42) \end{aligned}$$

In this section, we do not consider the process N_0^t because particles again have exponential sojourn times and $N_0^t \to 0$ on compact sets w.p.1. Note that N_t keeps track of more information than the previous model. In particular, $N_t([0, u] \times A \times \{1\})$ is the number of particles that arrived in the time interval $[t - u, t]$ that are retained in the set $A \in \mathcal{E}$ at time t. On the other hand, $N_t([0, u] \times A \times \{0\})$ is the number of particles that arrived in the time interval $[t - u, t]$ that have departed from the set

$A \in \mathcal{E}$ by time t. Note that N_t is a measure-valued Markov process because given $\{N_s : s \leq t\}$, the distribution of N_{t+h} for any $h > 0$ depends only upon N_t.

The following theorem describes the limiting distribution of the N_t process.

Theorem 14 *The process defined in (42) satisfies* $N_t \xrightarrow{d} N$*, where* N *is defined by*

$$Nf = \sum_n f(T_n, X_n, \mathbb{1}(M_n < L_n)), \qquad f \in \mathcal{C}_K^+(\mathbb{E}), \tag{43}$$

where $M_n = M([0, T_n) \times D^{-1}(X_n) \times \mathbb{N})$*. The mean measure of* N *is*

$$\mu_N(I \times B \times \{i\}) = \begin{cases} \lambda \int_B \int_I e^{-\lambda u a(x) b(x)} du\, F(dx), & i = 1, \\ \lambda \int_B \int_I (1 - e^{-\lambda u a(x) b(x)}) du\, F(dx), & i = 0. \end{cases} \tag{44}$$

In addition,

$$EN_t f \to ENf, \qquad f \in \mathcal{C}_K^+(\mathbb{R}). \tag{45}$$

Proof The process N_t defined by (42) can be expressed as

$$N_t f = \sum_n f(\phi_t(t - T_n, X_n, L_n)), \qquad f \in \mathcal{C}_K^+(\mathbb{E}),$$

where

$$\phi_t(u, x, \ell) = \Big(u, x, \mathbb{1}\Big(M([t - u, t] \times D^{-1}(x) \times \mathbb{N}) < \ell\Big)\Big),$$

and this is defined to be zero for all $u > t$. Then N_t is obtained from the input process M as discussed in Chapter 3 via the random transformation ϕ_t. By defining

$$M_t(I \times B \times C) \equiv M(t - I \times B \times C) \qquad \text{on } [0, t] \times \mathbb{E} \times \mathbb{N},$$

we can write

$$\phi_t(u, x, \ell) = \Big(u, x, \mathbb{1}\Big(M_t([0, u] \times D^{-1}(x) \times \mathbb{N}) < \ell\Big)\Big).$$

To prove that N_t converges in distribution, it suffices by Theorem 1 to show there exists a function $\phi \in \mathbb{F}$ such that $(\phi_t, M_t) \stackrel{d}{=} (\phi, M)$. To this end, define

$$\phi(u, x, \ell) = \Big(u, x, \mathbb{1}\Big(M([0, u] \times D^{-1}(x) \times \mathbb{N}) < \ell\Big)\Big).$$

Because M_t and M are time-homogeneous Poisson processes, they are equal in distribution on $[0, t] \times \mathbb{E} \times \mathbb{N}$. Then clearly $(\phi_t, M_t) \stackrel{d}{=} (\phi, M)$ on $\mathbb{F} \times \mathcal{M}([0, t] \times \mathbb{E} \times \mathbb{N})$, and the conditions of Theorem 1 are satisfied. Thus, $N_t \xrightarrow{d} N$, where N is defined by

$$Nf = \int_{\mathbb{R}_+ \times \mathbb{E}} \sum_{\ell=0}^{\infty} f(\phi(u, x, \ell)) M(du \times dx \times \{\ell\}),$$

which is equivalent to (43).

To prove the mean measure of N is given by μ_N, consider (43), where $\mathbb{1}(M_n < L_n) = 1$ or 0 if the nth particle survives T_n time units or not, respectively. Then we can write ENf as

$$\begin{aligned} ENf &= \sum_n Ef(T_n, X_n, \mathbb{1}(M_n < L_n)) \\ &= \sum_n E[f(T_n, X_n, 1) P(M_n < L_n | T_n, X_n) \\ &\qquad + f(T_n, X_n, 0) P(M_n \geq L_n | T_n, X_n)]. \end{aligned} \tag{46}$$

Because L_n given X_n is distributed geometric with parameter $a(X_n)$, and M_n given T_n and X_n is distributed Poisson with mean $\lambda T_n b(X_n)$, it follows that

$$\begin{aligned} P(M_n < L_n | T_n, X_n) &= E[P(M_n < L_n | T_n, X_n, M_n) | T_n, X_n] \\ &= E[(a(X_n))^{M_n} | T_n, X_n] \\ &= \exp\{-\lambda T_n a(X_n) b(X_n)\}. \end{aligned}$$

Substituting the above expression into (46) yields

$$
\begin{aligned}
ENf &= \sum_n E\Big[f(T_n, X_n, 1)\exp\{-\lambda T_n a(X_n) b(X_n)\} \\
&\qquad + f(T_n, X_n, 0)(1 - \exp\{-\lambda T_n a(X_n) b(X_n)\})\Big] \\
&= \lambda \int_{\mathbb{R}_+ \times \mathbb{E}} \Big(f(u, x, 1) e^{-\lambda u a(x) b(x)} \\
&\qquad + f(u, x, 0)(1 - e^{-\lambda u a(x) b(x)})\Big) du\, F(dx), \qquad (47)
\end{aligned}
$$

and the first part of the theorem is proved.

Finally, (45) follows by statement (9) of Theorem 1. ∎

4.5 Examples: Stationary Distributions and Non-Poisson Limits

In the previous section we studied a process where deletions were triggered by arrivals with higher ranks in a linearly ordered space. These assumptions yielded a limiting process that was a Poisson process. This is apparently a very special result, as the limiting process is very sensitive to the deletion rule.

In this section we will investigate the stationary distribution of the system N_t described in the previous sections, as well as the stationary distributions of some other similar systems by testing whether or not they they can be Poisson processes. We begin by considering the main model, and reprove that its stationary distribution is the limiting process proved in Theorem 8. Then we examine three similar processes with slightly different deletion rules, and show their stationary distributions cannot be Poisson processes.

Because each of the following processes is a Markov process subordinated to a Poisson process, in each case it will suffice to assume the process is stationary in time, and simply compare the forms of the Laplace functionals at times 0 and T_1, the time of the first arrival after time 0. That is, we know that if N_t is stationary in time, then $N_0 \stackrel{d}{=} N_{T_1}$ (in fact, $N_0 \stackrel{d}{=} N_t$ for all time t). Because Laplace functionals completely characterize point processes, this is equivalent to the statement

$$L_{N_0}(f) = L_{N_{T_1}}(f), \tag{48}$$

where $L_N(f)$ is the Laplace functional of the N process as defined in (1). Then in order to test whether or not a system has a Poisson process for a stationary version, we just need to verify (48) when N_0 and N_{T_1} are equal in distribution to a Poisson process with mean measure prescribed by (47):

$$EN(A) = \int_A \frac{F(dx)}{a(x)b(x)}. \tag{49}$$

Example 15 *Main Model.* Recall the model from the first section along with its notation. Assume the process N_t is stationary in time, and has the distribution of a Poisson process with mean measure (49). Then we know the Laplace functional of the process at these times must be given by

$$L_{N_0}f = L_{N_{T_1}}f = \exp\Big\{\int_{\mathbb{R}}(1 - e^{-f(x)})\mu_N(dx)\Big\}, \qquad f \in \mathcal{C}_K^+(\mathbb{R}). \tag{50}$$

In order to test this hypothesis, we begin by writing down the expression for the Laplace functional of N_{T_1} for $f \in \mathcal{C}_K^+$ by conditioning on the location of the first arrival X_1 after time 0:

$$L_{N_{T_1}}f \quad = \quad E[e^{-\int_{\mathbb{R}} f(x)N_{T_1}(dx)}]$$

$$= \int_{\mathbb{R}} E\Big[\exp\Big\{ -\int_{-\infty}^{y} f(x)(1-a(x))N_0(dx) - \int_{y}^{\infty} f(x)N_0(dx) - f(y)\Big\}\Big] F(dy),$$

which follows directly from the deletion rule of the N_t process. Next, because N_0 is a Poisson process, it has independent increments so that

$$\begin{aligned} L_{N_{T_1}} f &= \int_{\mathbb{R}} E\Big[\exp\Big\{ -\int_{-\infty}^{y} f(x)(1-a(x))N_0(dx)\Big\}\Big] \\ &\quad \times E\Big[\exp\Big\{ \int_{y}^{\infty} f(x)N_0(dx)\Big\}\Big] e^{-f(y)} F(dy). \end{aligned}$$

Because N_0 is a Poisson process with mean measure μ_N, we can use the known form of the Laplace functional for Poisson processes and standard thinning results to rewrite the integrand and get

$$\begin{aligned} L_{N_{T_1}} f &= \int_{\mathbb{R}} \exp\Big\{ -\int_{-\infty}^{y} (1-e^{-f(x)})(1-a(x))\mu_N(dx)\Big\} \\ &\quad \times \exp\Big\{ -\int_{y}^{\infty} (1-e^{-f(x)})\mu_N(dx)\Big\} e^{-f(y)} F(dy). \end{aligned}$$

Rearranging the terms in the exponents yields

$$\begin{aligned} L_{N_{T_1}} f &= \int_{\mathbb{R}} \exp\Big\{ -\int_{-\infty}^{\infty} (1-e^{-f(x)})\mu_N(dx)\Big\} \\ &\quad \times \exp\Big\{ \int_{-\infty}^{y} (1-e^{-f(x)})a(x)\mu_N(dx)\Big\} e^{-f(y)} F(dy) \\ &= \exp\Big\{ -\int_{-\infty}^{\infty} (1-e^{-f(x)})\mu_N(dx)\Big\} \\ &\quad \times \int_{\mathbb{R}} \exp\Big\{ \int_{-\infty}^{y} (1-e^{-f(x)})a(x)\mu_N(dx)\Big\} e^{-f(y)} F(dy). \end{aligned}$$

Because we assume (50), examining the expression above tells us we just need to verify

$$\Psi \equiv \int_{\mathbb{R}} \exp\Big\{ \int_{-\infty}^{y} (1-e^{-f(x)})a(x)\mu_N(dx)\Big\} e^{-f(y)} F(dy) = 1. \tag{51}$$

Substituting the expression for the mean measure $\mu_N(dx) = F(dx)a(x)^{-1}\overline{F}(x)^{-1}$ into the expression for Ψ yields

$$\begin{aligned}
\Psi &= \int_{\mathbb{R}} \exp\Big\{\int_{-\infty}^{y}(1-e^{-f(x)})\frac{F(dx)}{\overline{F}(x)} - f(y)\Big\}F(dy) \\
&= \int_{\mathbb{R}} \exp\Big\{\int_{-\infty}^{y}\frac{F(dx)}{\overline{F}(x)} - \int_{-\infty}^{y} e^{-f(x)}\frac{F(dx)}{\overline{F}(x)} - f(y)\Big\}F(dy) \\
&= \int_{\mathbb{R}} \exp\Big\{-\log\overline{F}(y) - \int_{-\infty}^{y} e^{-f(x)}\frac{F(dx)}{\overline{F}(x)} - f(y)\Big\}F(dy) \\
&= \int_{\mathbb{R}} \frac{1}{\overline{F}(y)}\exp\Big\{-\int_{-\infty}^{y} e^{-f(x)}\frac{F(dx)}{\overline{F}(x)} - f(y)\Big\}F(dy)
\end{aligned}$$

Letting $g(y) = \int_{-\infty}^{y} e^{-f(x)}/\overline{F}(x)F(dx)$ means Ψ can be written as

$$\Psi = \int_{\mathbb{R}} e^{-g(y)}dg(y),$$

which is indeed equal to 1. Thus (51) is established, which proves that the Poisson process N is a stationary distribution for N_t. This lengthy proof, which we include for illustrative purposes, is another way of showing what we already know is true.

The next process is examined more thoroughly in the next chapter under the heading of a spatial polling model. Here we simply show that the process cannot be a Poisson process in the limit.

Example 16 *Elastic Polling Model.* We will consider a model similar to the one presented in the previous section, except there is no discrimination of attributes. That is, each particle is considered for deletion at each future arrival. Specifically, suppose particles enter the space $\mathbb{R}$ according to a Poisson process with rate λ. Upon an arrival, each customer already in the system located at x is considered for service, and is so serviced (and exits) with probability $a(x) < 1$, independently of everything else.

We are interested in knowing whether the process N_t describing the numbers of customers in regions of the system is Poisson, assuming N_t is stationary. As in the previous example, we will compare the Laplace functionals of the system states N_0 at time 0 and N_{T_1} at the time of the first arrival. We will let μ_N denote the mean measure of N.

We proceed as in the previous example. By conditioning on the position of the first arrival X_1, using the form of the Laplace functional for Poisson processes given by (2), and using the deletion rule, we get for any $f \in \mathcal{C}_K^+(\mathbb{R})$

$$\begin{aligned}
L_{N_{T_1}} f &= E[e^{-\int_{\mathbb{R}} f(x) N_{T_1}(dx)}] \\
&= \int_{\mathbb{R}} \exp\Big\{-\int_{\mathbb{R}} (1 - e^{-f(x)})(1 - a(x))\mu_N(dx) - f(y)\Big\} F(dy) \\
&= \exp\Big\{-\int_{\mathbb{R}} (1 - e^{-f(x)})(1 - a(x))\mu_N(dx)\Big\} \int_{\mathbb{R}} e^{-f(y)} F(dy). \qquad (52)
\end{aligned}$$

Because the system is stationary at time 0, we know the expression (52) should be equal to the Laplace functional of the stationary process

$$L_{N_0} f = \exp\Big\{-\int_{\mathbb{R}} (1 - e^{-f(x)})\mu_N(dx)\Big\}. \qquad (53)$$

Setting (52) equal to (53) implies

$$\Psi \equiv \exp\Big\{\int_{\mathbb{R}} (1 - e^{-f(x)}) a(x) \mu_N(dx)\Big\} \int_{\mathbb{R}} e^{-f(y)} F(dy) = 1, \qquad (54)$$

which is what we have to check. We know from (47) that the mean measure for the N process is given by $\mu_N = F(dx)/a(x)$. Substituting μ_N into Ψ yields

$$\begin{aligned}
\Psi &= \exp\Big\{\int_{\mathbb{R}} (1 - e^{-f(x)}) F(dx)\Big\} \int_{\mathbb{R}} e^{-f(y)} F(dy) \\
&= \exp\Big\{1 - \int_{\mathbb{R}} e^{-f(x)} F(dx)\Big\} \int_{\mathbb{R}} e^{-f(y)} F(dy).
\end{aligned}$$

Taking the logarithm of Ψ yields

$$\log \Psi = 1 - \int_{\mathbb{R}} e^{-f(x)} F(dx) + \log \int_{\mathbb{R}} e^{-f(x)} F(dx).$$

But since $\Psi = 1$, we see that

$$\int_{\mathbb{R}} e^{-f(x)} F(dx) = 1,$$

which is not always true. For instance, if $f(x) = \mathbb{1}(x \in [0,1])$ and F is the cumulative distribution function for a uniform $[0,1]$ random variable, then the left-hand side of the above expression yields $e = 1$. Therefore, this system does not admit a Poisson process as a stationary distribution.

In retrospect, it may seem that the result in the previous example is intuitive. Standard thinning results for Poisson processes tell us that thinning a Poisson process with a certain intensity yields another Poisson process. If we then randomly add a point to the thinned process, we no longer have a Poisson process. However, this intuition seems to contradict the limiting result of the previous section. We will study the elastic polling model more in depth in the following chapter.

Example 17 *Discriminating Service System.* Suppose particles enter $\mathbb{R}$ according to a Poisson process with rate λ, and take on independent positions in $\mathbb{R}$ according to the distribution F. Assume the deletion mechanism is as follows. Upon an arrival to the position $y \in \mathbb{R}$, each existing particle at $x < y$ is deleted with probability $a_1(x)$, and each existing particle at $x' > y$ is deleted with probability $a_2(x')$, independently of everything. This simply cannot have a stationary Poisson process because setting $a_1(x) = a_2(x)$ for all $x \in \mathbb{R}$ yields the previous example.

Example 18 *Generalized Deletion Probabilities.* Suppose particles arrive to $\mathbb{R}$ according to a Poisson process with rate λ, taking on independent positions in $\mathbb{R}$ according to the distribution function F. Upon an arrival to y, each particle located at $x < y$ is deleted from the system with probability $a(x, y)$. Proceeding as we have in the previous examples, if the stationary distribution is that of a Poisson process with mean measure μ_N, we should be able to write for $f \in \mathcal{C}_K^+(\mathbb{R})$ the Laplace functional of the process N_{T_1} at the time of the first arrival. By conditioning on the location of the first arrival and using the deletion rule,

$$\begin{aligned} L_{N_{T_1}} f &= E[e^{-\int_{\mathbb{R}} f(x) N_{T_1}(dx)}] \\ &= \int_{\mathbb{R}} E\Big[\exp\Big\{-\int_{-\infty}^{y} f(x)(1 - a(x, y)) N_0(dx) \\ &\qquad - \int_{y}^{\infty} f(x) N_0(dx) - f(y)\Big\}\Big] F(dy). \end{aligned}$$

Using the independent increments of Poisson processes yields

$$\begin{aligned} L_{N_{T_1}} f &= \int_{\mathbb{R}} E\Big[\exp\Big\{-\int_{-\infty}^{y} f(x)(1 - a(x, y)) N_0(dx)\Big\}\Big] \\ &\qquad \times E\Big[\exp\Big\{-\int_{y}^{\infty} f(x) N_0(dx)\Big\}\Big] e^{-f(y)} F(dy). \end{aligned}$$

Recalling the known form of the Laplace functional for a Poisson process as given in expression (2),

$$\begin{aligned} L_{N_{T_1}} f &= \int_{\mathbb{R}} \exp\Big\{-\int_{-\infty}^{y} (1 - e^{-f(x)})(1 - a(x, y)) \mu_N(dx)\Big\} \\ &\qquad \times \exp\Big\{-\int_{y}^{\infty} (1 - e^{-f(x)}) \mu_N(dx)\Big\} e^{-f(y)} F(dy) \\ &= \exp\Big\{-\int_{\mathbb{R}} (1 - e^{-f(x)}) \mu_N(dx)\Big\} \\ &\qquad \times \int_{\mathbb{R}} \exp\Big\{\int_{-\infty}^{y} (1 - e^{-f(x)}) a(x, y) \mu_N(dx)\Big\} e^{-f(y)} F(dy). \end{aligned}$$

Because the system is stationary at time 0, we set $L_{N_{T_1}} f$ equal to the Laplace functional for the stationary Poisson process N given by

$$L_N f = \exp\Big\{ -\int_{\mathbb{R}} (1 - e^{-f(x)}) \mu_N(dx) \Big\}$$

to get

$$\Psi \equiv \int_{\mathbb{R}} \exp\Big\{ \int_{-\infty}^{y} (1 - e^{-f(x)}) a(x,y) \mu_N(dx) \Big\} e^{-f(y)} F(dy) = 1. \tag{55}$$

From (47), we know the limiting mean measure of the process must be

$$\mu_N = F(dx) / \int_x^\infty a(x,y) F(dy).$$

Then upon substituting μ_N into Ψ, it remains to show $\Psi = 1$. We have

$$\Psi = \int_{\mathbb{R}} \exp\Big\{ \int_{-\infty}^{y} \frac{(1 - e^{-f(x)}) a(x,y) F(dx)}{\int_x^\infty a(x,y) F(dy)} \Big\} e^{-f(y)} F(dy).$$

Now let

$$a(x,y) = \begin{cases} y & x \in [0,1],\ y \in [0,1], \\ 1 & \text{otherwise}, \end{cases}$$

let $f(x) = \mathbb{1}(x \in [0,1])$, and let F be the distribution function for a uniform $[0,1]$ random variable. Then

$$\begin{aligned}
\Psi &= \int_0^1 \exp\Big\{ \int_0^y \frac{(1 - e^{-1}) y dx}{\int_x^1 y dy} \Big\} e^{-1} dy \\
&= \int_0^1 \exp\Big\{ 2y(1 - e^{-1}) \int_0^y \frac{dx}{1 - x^2} \Big\} e^{-1} dy \\
&= \int_0^1 \exp\Big\{ 2y(1 - e^{-1}) \frac{1}{2} \log \frac{1+y}{1-y} \Big\} e^{-1} dy \\
&= \int_0^1 \frac{1+y}{1-y} \exp\Big\{ y(1 - e^{-1}) \Big\} e^{-1} dy \\
&> 1.
\end{aligned}$$

Thus the process described above cannot admit a Poisson process for a stationary distribution because (55) is not satisfied.

CHAPTER V

SPATIAL POLLING MODELS

Standard polling models describe service systems where a server serves several queues in some order. For example, n queues may be placed on the unit circle and a server moving clockwise on the circle may allocate a certain amount of time to each queue before proceeding to the next. This is also an example of processor sharing.

In this chapter we describe models where particles (customers, items, etc.) arrive to a service system and are selected for service by servers that arrive in the future. Upon the arrival of a server, each particle that is in the system is independently served immediately or not depending on the particle's location. After servicing the particles, if the servers remain in the system as particles, we call the model an *elastic polling model*. If the servers instantaneously depart the system after their arrival, we call the model an *inelastic polling model*. The motivation for these terms comes from elastic and inelastic collisions of particles in physics. A perfectly inelastic collision is one where momentum is conserved but kinetic energy is not. Both quantities are conserved in an elastic collision. We consider an elastic model in the first section and turn to an inelastic model in the following section.

As an example, consider a freshly poured glass of soda. Carbon dioxide bubbles form at the bottom of the glass, and subsequently stream to the surface where they form a bubble cluster. New bubbles arriving to the surface will annihilate some of the existing bubbles in the cluster upon contact. The new arrivals then remain in the bubble cluster at the surface and wait for arriving bubbles to annihilate them. Such a system can be modelled as an elastic polling model.

5.1 Elastic Spatial Polling Model

In this section we consider the following alteration of the ABT model of Section 4.1. As above, suppose particles arrive to a service system at times $0 < T_1 < T_2 < \dots$ that form a Poisson process with rate λ, and the nth particle arriving at time T_n takes on the position X_n in the space $\mathbb{E}$. Whenever a particle arrives, any particle already in $\mathbb{E}$ at some location x is removed from the system with probability $a(x)$, independently of everything else. That is, the deletion region $D(y)$ for an arrival to $y \in \mathbb{E}$ will be the entire space $\mathbb{E}$. New arrivals cannot delete themselves. This system can also be viewed as a spatial service system where customers are serviced in random order in binomially distributed batches.

To illustrate the process, consider Figure 4 below. In (a), the time t is just prior to T_n. The particles in the system are the mth, ith, jth, and kth arrivals. In (b) the time is exactly T_n. At this time each particle in the system is being considered for deletion according to its position, e.g., the mth particle will be deleted with probability $a(X_m)$, the jth particle will be deleted with probability $a(X_j)$, etc. In (c) we see the mth and the kth particles were deleted, and only the ith, jth, and nth particles remain.

The data that generates the input point process M is the set of points $\{(T_n, X_n, L_n) : n \geq 1\}$, where $T_n \in \mathbb{R}_+$ and $X_n \in \mathbb{E}$ denote the arrival time and location of the nth arriving customer. Here, the discrete lifetime $L_n \in \mathbb{N}$ denotes the number of particle

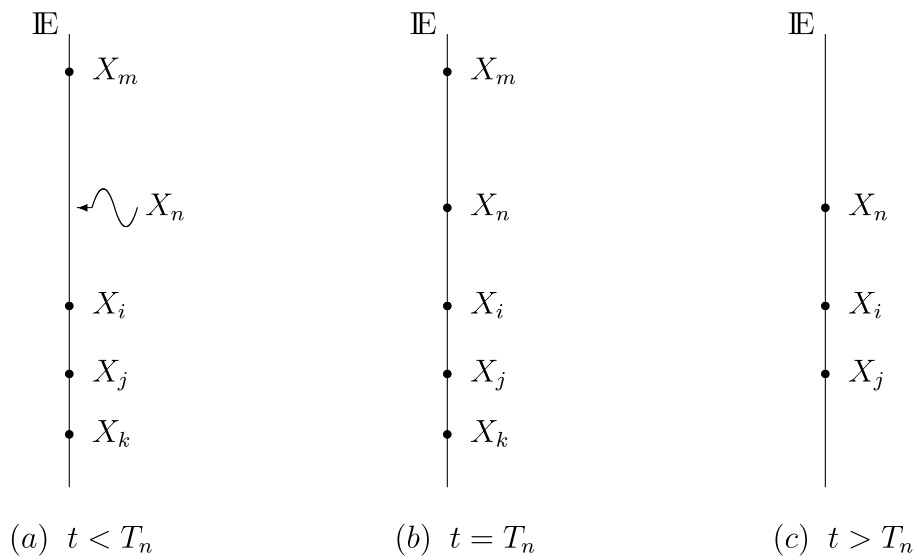

Figure 4: An elastic polling model.

arrivals required in order for the nth particle to depart the system. We assume the (X_n, L_n) are independent and identically distributed, independent of the arrival times T_n, and

$$P(X_n \in A, L_n > \ell) = \int_A (1 - a(x))^\ell F(dx),$$

where F is the distribution of X. Then

$$M(I \times A \times \mathbb{N}) = \sum_n \mathbb{1}(T_n \in I, X_n \in A)$$

counts the total number of particles that arrived in the time-space region $I \times A$. The M is a Poisson process, and its mean measure is

$$EM(I \times B \times C) = \lambda \int_I ds \int_B F(dx) \sum_{\ell \in C} [1 - a(X_n)]^{\ell - 1} a(X_n).$$

We will once again let

$$M(t) \equiv M([0, t] \times \mathbb{E} \times \mathbb{N}),$$

which is the number of arrivals up to time t.

5.1.1 Main Result

In this subsection we are concerned with finding the limiting distribution of the process N_t defined by

$$N_t(I, B, \{i\}) = \sum_n \mathbb{1}\Big(t - T_n \in I, X_n \in B, \mathbb{1}(M(t) - n < L_n) = i\Big). \tag{56}$$

In particular, $N_t([0,u] \times B \times \{1\})$ is the number of particles that arrived in the time interval $[t-u, t]$ that remain in the set B at time t.

The following theorem describes the limiting distribution of the N_t process.

Theorem 19 *The process defined above satisfies $N_t \xrightarrow{d} N$, where N is defined by*

$$Nf = \sum_n f(T_n, X_n, \mathbb{1}(M_n < L_n)), \qquad f \in \mathcal{C}_K^+(\mathbb{E}), \tag{57}$$

with $M_n = M([0, T_n) \times \mathbb{E} \times \mathbb{N})$. The mean measure of N is

$$\mu_N(I \times B \times \{i\}) = \begin{cases} \lambda \int_B \int_I e^{-\lambda u a(x)} du\, F(dx), & i = 1, \\ \lambda \int_B \int_I (1 - e^{-\lambda u a(x)}) du\, F(dx), & i = 0. \end{cases} \tag{58}$$

In addition,

$$EN_t f \to ENf, \qquad f \in \mathcal{C}_K^+(\mathbb{R}). \tag{59}$$

Proof As in Chapter 3, the process N_t defined by (56) is a transformation of M of the form

$$N_t f = \sum_n f(\phi_t(t - T_n, X_n, L_n)), \qquad f \in \mathcal{C}_K^+(\mathbb{R}),$$

where

$$\phi_t(u, x, \ell) = \Big(u, x, \mathbb{1}(M_t([0,u) \times \mathbb{E} \times \mathbb{N}) < \ell)\Big).$$

To prove that N_t converges in distribution, it suffices by Theorem 1 to show there exists a function $\phi \in \mathbb{F}$ such that $(\phi_t, M_t) \stackrel{d}{=} (\phi, M)$. To this end, define

$$\phi(u, x, \ell) = \Big(u, x, \mathbb{1}\Big(M(u) < \ell\Big)\Big).$$

Because M_t and M are time-homogeneous Poisson processes, they are equal in distribution. Then clearly $(\phi_t, M_t) \stackrel{d}{=} (\phi, M)$ on $\mathbb{F} \times \mathcal{M}([0,t] \times \mathbb{E} \times [0,1])$. Thus the conditions of Theorem 1 are satisfied so that $N_t \xrightarrow{d} N$, where N is defined by

$$Nf = \int_{\mathbb{R}_+ \times \mathbb{E}} \sum_{\ell=0}^{\infty} f(\phi(u, x, \ell)) M(du \times dx \times \{\ell\}),$$

and this representation is the same as (57).

To prove the mean measure of N is given by (58), we use the representation of N in (57) to write

$$\begin{aligned} ENf &= E\Big[\sum_{n=1}^{M(S(f))} Ef(T_n, X_n, \mathbb{1}(M_n < L_n))\Big] \\ &= E\Big[\sum_{n=1}^{M(S(f))} E[f(T_n, X_n, 1)P(M_n < L_n | T_n, X_n)] \\ &\qquad + f(T_n, X_n, 0)P(M_n \geq L_n | T_n, X_n)]\Big| S(f)\Big], \end{aligned}$$

where $M(S(f))$ denotes the number of arrivals in the support of f. Because L_n given X_n is distributed geometric with parameter $a(X_n)$, and M_n given T_n and X_n is distributed Poisson with mean λT_n, it follows that

$$\begin{aligned} P(M_n < L_n | T_n, X_n) &= E[P(M_n < L_n | T_n, X_n, M_n) | T_n, X_n] \\ &= E[(1 - a(X_n))^{M_n} | T_n, X_n] \\ &= \exp\{-\lambda T_n a(X_n)\}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
ENf &= E\Big[\sum_{n=1}^{M(S(f))} E[f(T_n, X_n, 1)\exp\{-\lambda T_n a(X_n)\} \\
&\qquad +f(T_n, X_n, 0)(1-\exp\{-\lambda T_n a(X_n)\})]\Big|M(S(f))\Big] \\
&= \lambda \int_{\mathbb{R}_+\times\mathbb{E}} \Big(f(u, x, 1)\exp\{-\lambda T_n a(X_n)\} \\
&\qquad +f(u, x, 0)(1-\exp\{-\lambda T_n a(X_n)\})\Big) du\, F(dx).
\end{aligned}
$$

Finally, the convergence $EN_t f \to ENf$ follows by statement (9) of Theorem 1. ∎

In the following section we will be concerned with the process that counts the total numbers of remaining particles in various regions. Unfortunately, the Laplace functional of this process does not have a known form from which we can obtain a closed form representation of the limiting process. However, we will derive the limiting distribution of the number of remaining particles in any region $A \in \mathbb{E}$ by using probability generating functions.

5.1.2 Limiting Distributions of Remaining Particles

In this subsection we will consider the process defined by (56) restricted to part of its space. Specifically, we examine the behavior of the N_t process defined by

$$N_t(A) = \sum_n \mathbb{1}(T_n \in [0, t], X_n \in A, M(t) - n < L_n), \tag{60}$$

where the M process is generated by the data $\{(T_n, X_n, L_n)\}$ as above. Here, $N_t(A)$ is the number of particles that are in the set A at time t. The $N_t(A)$ here is equal to $N_t(\mathbb{R}_+ \times A \times \{1\})$ for the process in (56).

Note that

$$N_t(A) = \sum_{n=1}^{M(t)} U_{tn} \mathbb{1}(X_n \in A), \qquad n \geq 1,$$

where $U_{tn} \equiv \mathbb{1}(M(t) - n < L_n)$ are conditionally independent given $M(t)$. Then given $M(t)$, the $N_t(A)$ has a binomial distribution with parameters $M(t)$ and $P(U_{tn} = 1)$. This is what has been referred to as a Poisson-Binomial random variable in [4].

Below is the main theorem for this subsection. We require the process to begin with finite numbers of particles in compact sets at time 0. Because each particle remains in the system for a finite amount of time, without loss of generality we may assume the process begins with no particles in the system. That is, $N_0(\mathbb{E}) = 0$.

Theorem 20 *The point process N_t above converges in distribution to the point process N, whose Laplace functional is given by*

$$L_N f = \prod_{n=0}^{\infty} (1 - a_n(f)), \qquad f \in \mathcal{C}_K^+(\mathbb{R}), \tag{61}$$

where $a_n(f) = \int_{\mathbb{R}} (1 - a(x))^n (1 - e^{-f(x)}) F(dx)$. Hence, the stationary distribution of N_t is that of the point process N.

Proof By Theorem 19, we know that $N_t \xrightarrow{d} N$, where

$$Nf = \sum_n f(X_n) \mathbb{1}(L_n \geq n), \qquad f \in \mathcal{C}_K^+(\mathbb{R})$$

because $M([0, T_n) \times \mathbb{E} \times \mathbb{N}) = n - 1$.

To finish the proof, it suffices to show the Laplace functional L_N of N is given by (61). Because the (X_n, L_n) are independent and identically distributed, we can write

$$L_N(f) \;=\; E\Big[e^{-\sum_{n=1}^{\infty} f(X_n) \mathbf{1}(L_n \geq n)}\Big]$$

$$
\begin{aligned}
&= \prod_{n=1}^{\infty} E[e^{-\mathbf{1}(L_n \geq n) f(X_n)}] \\
&= \prod_{n=1}^{\infty} E[(1-a(X_1))^{n-1} e^{-f(X_1)} + 1 - (1-a(X_1))^{n-1}] \\
&= \prod_{n=0}^{\infty} \Big(1 - E[(1-a(X_1))^n (1-e^{-f(X_1)})]\Big). \qquad (62)
\end{aligned}
$$

This proves (61). ∎

Unfortunately, a closed form expression for the product in (62) it not known. However, the Laplace functional of the limiting process N is still useful because we can determine distributions of the numbers of particles $N(A)$ remaining in various regions A of the system. That is, we can obtain the generating function $G_{N(A)}(s)$ of $N(A)$ by setting $f(x) = \mathbb{1}(x \in A)$ and by replacing e^{-1} with s in (61):

$$
G_{N(A)}(s) \equiv E[s^{N(A)}] = \prod_{n=0}^{\infty} \Big(1 - E[(1-a(X_1))^n (1 - s^{\mathbf{1}(X_1 \in A)})]\Big).
$$

Then the generating function for the limiting random variable $N(A)$ is given by

$$
G_{N(A)}(s) = e^{C(s)}, \qquad (63)
$$

where

$$
C(s) = \sum_{n=0}^{\infty} \log\Big(1 - (1-s)a_n\Big), \qquad (64)
$$

and

$$
a_n = E[(1-a(X_1))^n \mathbb{1}(X_1 \in A)].
$$

Using (63) we can write down the distribution of the process N, but first we need the following lemma.

Lemma 21 *Suppose the probability generating function* $G(s) = \sum_{n=0}^{\infty} p_n s^n$ *has the form* $G(s) = e^{C(s)}$ *where*

$$C(s) = \sum_{n=0}^{\infty} \log[1 - (1-s)a_n],$$

for some numbers $1 > a_0 > a_1 > \ldots > 0$*. Then*

$$\begin{aligned} p_0 &= \prod_{n=0}^{\infty}(1-a_n), \\ p_k &= k^{-1}\sum_{n=0}^{k-1}\frac{C^{(k-n)}(0)p_n}{(k-1-n)!}, \qquad k \geq 1, \end{aligned} \tag{65}$$

where

$$C^{(k)}(0) = (k-1)!\sum_{n=0}^{\infty}\frac{a_n^k}{(a_n-1)^k}.$$

Proof We prove the nth derivative of G is

$$G^{(n)}(s) = \sum_{i=0}^{n-1}\binom{n-1}{i}G^{(i)}(s)C^{(n-i)}(s) \tag{66}$$

by induction. Note that

$$G'(s) = G(s)C'(s).$$

Assume (66) is true for some n. Then we can write

$$\begin{aligned} G^{(n+1)}(s) &= \sum_{i=0}^{n-1}\binom{n-1}{i}[G^{(i+1)}(s)C^{(n-i)}(s) + G^{(i)}(s)C^{(n-i+1)}(s)] \\ &= G(s)C^{(n+1)}(s) + \sum_{i=0}^{n-1}\left(\binom{n-1}{i} + \binom{n-1}{i+1}\right)G^{(i+1)}(s)C^{(n-i)}(s) \\ &\quad + G^{(n)}(s)C'(s). \end{aligned}$$

Using the identity

$$\binom{n-1}{i} + \binom{n-1}{i+1} = \binom{n}{i+1},$$

we get

$$\begin{aligned}G^{(n+1)}(s) &= G(s)C^{(n+1)}(s) + \sum_{i=0}^{n-1}\binom{n}{i+1}G^{(i+1)}(s)C^{(n-i)}(s) + G^{(n)}(s)C'(s) \\ &= \sum_{i=0}^{n}\binom{n}{i}G^{(i)}(s)C^{(n-i+1)}(s).\end{aligned}$$

Then the distribution in (65) follows by noting $G^{(n)}(0) = n!p_n$. ∎

Now we are now ready to get the distribution of $N(A)$ for $A \in \mathcal{E}$.

Theorem 22 *The distribution of $N(A)$ for $A \in \mathcal{E}$ is given by*

$$\begin{aligned}P(N(A)=0) &= e^{C(0)}, \\ P(N(A)=k) &= k^{-1}\sum_{n=0}^{k-1}\frac{C^{(k-n)}(0)P(N(A)=n)}{(k-1-n)!}, \qquad k \geq 1,\end{aligned} \tag{67}$$

where $C(s)$ is

$$C(s) = \sum_{n=0}^{\infty}\log\Big(1-(1-s)a_n\Big),$$

and

$$a_n = \int_A (1-a(x))^n F(dx).$$

Proof The probability generating function in (63) has the form of $G(s)$ given in Lemma 21. Letting $a_n = \int_A (1-a(x))^n F(dx)$ in Lemma 21 yields the distribution given by (67). ∎

Remark 23 *Moments of $N(A)$.* We can use the expression for the derivative of $G(s)$ in (66) to get moments of the distribution of $N(A)$. For example, the first and second moments $EN(A)$ and $E[N(A)^2]$ are, respectively

$$E[N(A)] \quad = \quad G'_{N(A)}(1) = C'(1),$$

and

$$\begin{aligned} E[N(A)^2] &= G''_{N(A)}(1) - E[N(A)] \\ &= C''(1) + C'(1)^2 - C'(1), \end{aligned}$$

where

$$C^{(k)}(1) = \sum_{n=0}^{\infty}(k-1)!a_n^k.$$

5.2 Inelastic Spatial Polling Model

In this section we turn our attention to a slightly different model, where two different types of particles arrive to a space according to independent Poisson processes. Specifically, c-particles arrive according to a Poisson process with rate λ and s-particles arrive according to a Poisson process with rate γ.[1] Each c-particle takes on a position in $\mathbb{E}$ according to the distribution function F, independently of everything else. The deletions are triggered by the arrivals of the s-particles. That is, upon the arrival of an s-particle, a c-particle that is still in the system located at x is either removed from or retained in the system with probabilities $a(x)$ and $1-a(x)$, respectively. The s-particles never enter the system; their only purpose is to arrive, serve, and immediately leave. We call models of this type *inelastic polling models* because of the polling nature of the servicing, and because the s-particles do not remain in the system. As

[1] Here, the c stands for customer and s stands for server.

with the previous models, we model the particles in the system as a continuous-time Markov chain N_t on the space of counting measures on $\mathbb{E}$.

For example, consider Figure 5 below. In (a), the time is just prior to T_n (the arrival time of the nth particle), and the particles that remain in the system are the mth, ith, jth, and kth. In (b), the time is T_n. It turns out the nth particle is an s-particle. Thus the mth, ith, jth, and kth particles are independently deleted from the system with probabilities $a(X_m)$, $a(X_i)$, $a(X_j)$, and $a(X_k)$, respectively. In (c), the time is just after T_n. Here we see the nth particle has departed because it was an s-particle, and that only the mth and jth particles remain.

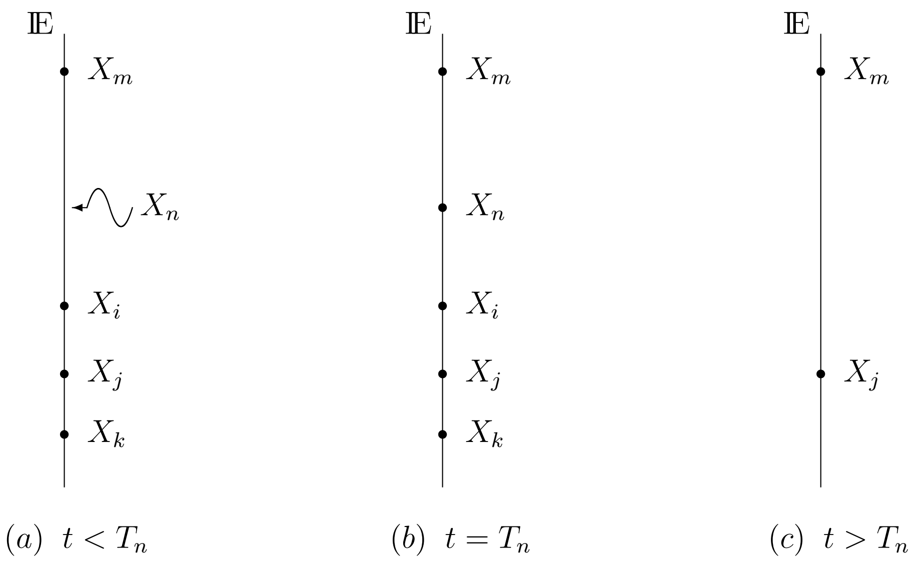

Figure 5: An inelastic polling model.

The data that generates the input process M for this system is the set of points $\{(T_n, X_n, Y_n, L_n) : n \geq 0\}$. As in the previous section, T_n and X_n denote the arrival time in $\mathbb{R}_+$ and the spatial location in $\mathbb{E}$. Here, $Y_n = \mathbb{1}(n\text{th particle is a } c\text{-particle})$. The discrete lifetime $L_n \in \mathbb{N}$ is the number of arriving s-particles required to remove the nth arrival. Because the s-particles do not actually enter the system, we will adopt the convention that $L_n = 0$ if and only if $Y_n = 0$. We will allow s-particles to receive a position coordinate in $\mathbb{E}$, even though they never really enter the system. The marks (X_n, Y_n, L_n) are independent and identically distributed, and independent of the arrival times, and we denote the distribution function of X_n by F. Then

$$M(I, A, \{1\}, \mathbb{N}) = \sum_n \mathbb{1}(T_n \in I, X_n \in A, Y_n = 1)$$

counts the total number of c-particles that arrived in the time-space region $I \times A$. We let $M^c(t)$ and $M^s(t)$ denote the total number of c-particles and the number of s-particles that arrived in the time interval $[0, t]$, respectively.

We see that L_n is conditionally geometric given $Y_n = 1$ and X_n so that

$$P(L_n = \ell | Y_n = 1, X_n) = [1 - a(X_n)]^{\ell-1} a(X_n).$$

Because M is a Poisson process, its mean measure is given by

$$EM(I \times A \times B \times C) = \int_I ds \int_A F(dx) \Big(\lambda \mathbb{1}(1 \in B) \sum_{\ell \in C} [1 - a(x)]^{\ell-1} a(x) + \gamma \mathbb{1}(0 \in B) \Big).$$

5.2.1 Main Result

In this subsection we are concerned with finding the limiting distribution of the process N_t defined by

$$N_t(I \times A \times B \times \{i\}) = \sum_n \mathbb{1}\Big(t - T_n \in I, X_n \in A, Y_n \in B, \zeta_n = i\Big), \qquad (68)$$

where $\zeta_n \equiv \mathbb{1}(M^s(t) - M^s(T_n) < L_n)$. In particular, $N_t(I \times A \times \{1\} \times \{1\})$ counts the number of c-particles that remain in the system at time t that arrived during the time interval I to the region A, and $N_t(I \times \mathbb{E} \times \{0\} \times \{0\})$ counts the total number of s-particles that arrived during the time interval I.

The following theorem describes the limiting distribution of the N_t process.

Theorem 24 *The N_t process described in (68) satisfies $N_t \xrightarrow{d} N$, where N is defined by*

$$Nf = \sum_n f(T_n, X_n, Y_n, \mathbb{1}(M^s(T_n) < L_n)), \qquad f \in \mathcal{C}_K^+(\mathbb{E}). \tag{69}$$

The mean measure of N is given by

$$\mu_N(I \times A \times \{i\} \times \{j\}) = \begin{cases} \lambda \int_I \int_A e^{-\gamma u a(x)} F(dx)\, du & i = j = 1, \\ \lambda \int_I \int_A (1 - e^{-\gamma u a(x)}) F(dx)\, du & i = 1, j = 0, \\ \gamma \int_I \int_A F(dx)\, du & i = j = 0. \end{cases}$$

In addition, if the measure μ_N is finite on compact sets, then

$$EN_t f \to ENf, \qquad f \in \mathcal{C}_K^+(\mathbb{E}). \tag{70}$$

Proof The process N_t defined by (68) is a transformation of M of the form

$$N_t f = \sum_n f(\phi_t(t - T_n, X_n, Y_n, L_n)), \qquad f \in \mathcal{C}_K^+(\mathbb{E}),$$

where

$$\phi_t(u, x, y, \ell) = (u, x, y, \mathbb{1}(y = 1, M^s(t) - M^s(t - u) < \ell))$$

is zero when $u > t$.

To prove $N_t \xrightarrow{d} N$, it suffices by Theorem 1 to show there exists a function $\phi \in \mathbb{F}$ such that $(\phi_t, M_t) \stackrel{d}{=} (\phi, M)$ where

$$M_t(I, A, B, C) \equiv M(t - I, A, B, C).$$

To this end, define

$$\phi(u,x,y,\ell) = (u,x,y,\mathbb{1}(y=1, M([0,u),\mathbb{E},\{0\},\{0\}) < \ell)).$$

Because the M_t and M processes are time-homogeneous Poisson processes, they are equal in distribution. Then the conditions of Theorem 1 are satisfied so that $N_t \xrightarrow{d} N$, where N is given by

$$Nf = \int_{\mathbb{R}\times\mathbb{E}} \sum_{y=0}^{1}\sum_{\ell=0}^{\infty} f(\phi(u,x,y,\ell))M(du \times dx \times \{y\} \times \{\ell\}).$$

This representation is the same as (69).

To prove the mean measure of N is given by μ_N, we use the representation of N in (69) to write

$$\begin{aligned}
ENf &= E\Big[\sum_{n=1}^{M(S(f))} Ef(T_n, X_n, Y_n, \mathbb{1}(M^s(T_n) < L_n))\Big] \\
&= E\Big[\sum_{n=1}^{M(S(f))} E[f(T_n, X_n, 1, 1)P(Y_n = 1, M^s(T_n) < L_n|T_n, X_n) \\
&\quad +f(T_n, X_n, 1, 0)P(Y_n = 1, M^s(T_n) \geq L_n|T_n, X_n) \\
&\quad +f(T_n, X_n, 0, 0)P(Y_n = 0|T_n, X_n)]\Big],
\end{aligned}$$

where $M(S(f))$ denotes the total number arrivals in the support of f. Because Y_n and $M^s(T_n)$ are independent, and L_n given X_n is distributed geometric with parameter $a(X_n)$, and $M^s(T_n)$ given T_n and X_n is distributed Poisson with mean γT_n, it follows that

$$\begin{aligned}
P(Y_n = 1, \quad & M^s(T_n) < L_n|T_n, X_n) \\
&= E[P(Y_n = 1, M^s(T_n) < L_n|T_n, X_n, M^s(T_n))|T_n, X_n] \\
&= \frac{\lambda}{\gamma+\lambda} E[P(M^s(T_n) < L_n|T_n, X_n, M^s(T_n))|T_n, X_n]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda}{\gamma+\lambda} E[(1-a(X_n))^{M^s(T_n)} | T_n, X_n] \\
&= \frac{\lambda}{\gamma+\lambda} e^{-\gamma a(X_n) T_n}.
\end{aligned}
$$

Similarly,

$$
P(Y_n = 1, M^s(T_n) \geq L_n | T_n, X_n) = \frac{\lambda}{\gamma+\lambda}(1 - e^{-\gamma a(X_n) T_n}).
$$

Thus,

$$
\begin{aligned}
ENf &= \sum_{n=1}^{M(S(f))} E\Big[f(T_n, X_n, 1, 1) \frac{\lambda}{\gamma+\lambda} e^{-\gamma a(X_n) T_n} \\
&\quad + f(T_n, X_n, 1, 0) \frac{\lambda}{\gamma+\lambda}(1 - e^{-\gamma a(X_n) T_n}) + f(T_n, X_n, 0, 0) \frac{\gamma}{\gamma+\lambda} \Big] \\
&= \int_{\mathbb{R}_+ \times \mathbb{E}} \Big(f(u, x, 1, 1) \lambda e^{-\gamma a(x) u} \\
&\quad + f(u, x, 1, 0) \lambda (1 - e^{-\gamma a(x) u}) + f(u, x, 0, 0) \gamma \Big) F(dx)\, du.
\end{aligned}
$$

Finally, the convergence of the mean measure in (70) follows by statement (9) of Theorem 1. ∎

5.2.2 Limiting Distributions of Remaining Particles

In this subsection, we will consider the process defined by (69) restricted to part of its space. Specifically, we examine the limiting behavior of the N_t process defined by

$$
N_t(A) = \sum_n \mathbb{1}(T_n \in [0, t], X_n \in A, Y_n = 1, M^s(t) - M^s(T_n) < L_n), \tag{71}
$$

where the M process is generated by the data $\{T - n, X_n, Y_n, L_n\}$ as above. That is, $N_t(A)$ counts the number of c-particles that are in the region A at time t. As in the previous section, the Laplace functional of the N process will prove to have no closed form. However, we can still use it to obtain the probability generating function of N and to prove the limiting distribution of the number of c-particles that remain in various regions of $\mathbb{E}$.

As usual, we will assume $N_0(\mathbb{E}) = 0$ w.p.1. The following theorem describes the convergence in distribution of the N_t process above.

Theorem 25 *The point process N_t above converges in distribution to the point process N whose Laplace functional is given by*

$$L_N f = \prod_{n=0}^{\infty} \frac{1}{1 + \lambda/\gamma E[(1 - a(X_1))^n (1 - e^{-f(X_1)})]}, \qquad f \in \mathcal{C}_K^+(\mathbb{E}),$$

Hence, the stationary distribution of the Markov process N_t is that of the point process N.

Proof By Theorem 24, the N_t process defined in (71) above converges in distribution to the process N defined by

$$Nf = \sum_n f(X_n) U_n, \qquad f \in \mathcal{C}_K^+(\mathbb{E}),$$

where $U_n = \mathbb{1}\Big(Y_n = 1, M_s(T_n) < L_n\Big)$. We just need to show the Laplace functional of N is given by 25.

To do so, note the first particles to arrive after time 0 form a sequence of c-particles, and the number of these plus 1 has the geometric$(\gamma/(\gamma + \lambda))$ distribution. After this initial sequence of c-particles, there is an $s-$particle, then another sequence of c-particles, an s particle, and so forth. To model the N process, note that at time 0 there are two point processes that compose the current state of the system. Let N^1 denote the process of particles remaining at time 0 that is generated by exactly those

c-particles that arrived between time 0 and the arrival time of the first s-particle. Let N^2 denote the process of particles remaining at time 0 that is generated by the arrivals begining with the first s-particle. Figure 6 below depicts the particle arrival times, as well as the arrivals that generate the N^1 and N^2 processes.

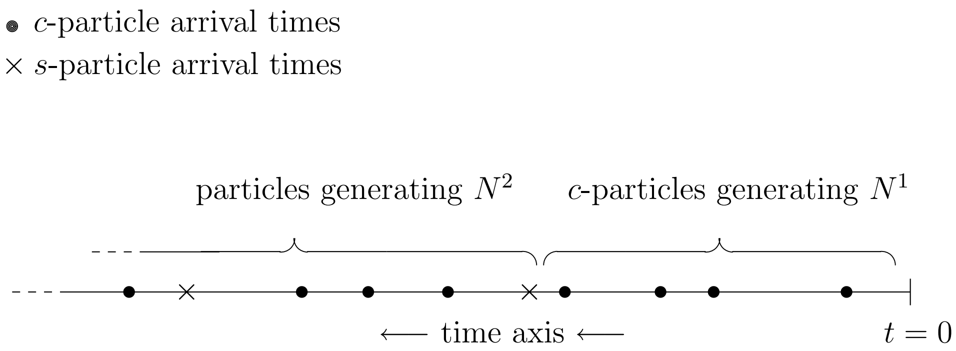

Figure 6: Arrival stream of the inelastic polling model.
The particles remaining at time 0 come from the aggregation of the N^1 and N^2 processes.

We now see that N^1 is generated by a geometric minus one number of c-particles with independent and identically distributed positions in $\mathbb{E}$. That is,

$$N^1(A) = \sum_{n=1}^{Z} \mathbb{1}(X_n \in A), \qquad A \subset \mathbb{E},$$

where $Z + 1$ is distributed geometric$(\gamma/(\lambda + \gamma))$. Also, we can represent N^2 as

$$N^2(A) = \sum_{n=1}^{\infty} \sum_{k=1}^{Z_n} U_{nk}.$$

Here, Z_n denotes the number of c-particles arriving between the nth and $(n + 1)$th s-particle, and the survival indicator functions U_{nk} are defined as

$$\begin{aligned} U_{nk} \;\equiv\; & \mathbb{1}(\text{the } k\text{th } c\text{-particle from the } n\text{th continuous sequence} \\ & \text{of } c\text{-particles survives the following } n \; s\text{-particles}). \end{aligned}$$

The Z_n are independent and identically distributed as geometric$(\lambda/(\lambda+\gamma))-1$ random variables. We will denote the position of the kth c-particle of the nth continuous sequence of c-particles by X_{nk}.

Then the Laplace functional of N^1 for $f \in \mathcal{C}_K^+(\mathbb{E})$ is given by

$$\begin{aligned} L_{N^1}(f) &= E[e^{-\sum_{n=1}^{N^1(\mathbb{E})} f(X_n)}] \\ &= E\Big[\prod_{n=1}^{N^1(\mathbb{E})} E[e^{-f(X_n)}|N^1(\mathbb{E})]\Big] \\ &= E\Big[E[e^{-f(X_1)}]^{N^1(\mathbb{E})}\Big] \\ &= \sum_{i=0}^{\infty} E[e^{-f(X_1)}]^i\Big(\frac{\lambda}{\lambda+\gamma}\Big)^i\frac{\gamma}{\lambda+\gamma} \\ &= \frac{\gamma}{\lambda+\gamma}\times\frac{1}{1-\lambda/(\lambda+\gamma)E[e^{-f(X_1)}]} \\ &= \frac{1}{1+\lambda/\gamma(1-E[e^{-f(X_1)}])}. \end{aligned} \tag{72}$$

Then noting the conditional independence of the U_{nk} and X_{nk} given the Z_n, the Laplace functional of the N^2 process is given by

$$\begin{aligned} L_{N^2}(f) &= E[e^{-\sum_{n=1}^{\infty}\sum_{k=1}^{Z_n} f(X_{nk})U_{nk}}] \\ &= E[E[e^{-\sum_{n=1}^{\infty}\sum_{k=1}^{Z_n} f(X_{nk})U_{nk}}|Z_1, Z_2, \ldots]] \\ &= E\Big[E[\prod_{n=1}^{\infty}\prod_{k=1}^{Z_n} e^{-f(X_{nk})U_{nk}}|Z_1, Z_2, \ldots]\Big] \\ &= \prod_{n=1}^{\infty} E\Big[\prod_{k=1}^{Z_n} E[e^{-f(X_{nk})U_{nk}}|Z_1, Z_2, \ldots]\Big] \\ &= \prod_{n=1}^{\infty} E\Big[\prod_{k=1}^{Z_n} E[e^{-f(X_1)U_{n1}}]\Big] \\ &= \prod_{n=1}^{\infty} E\Big[E[e^{-f(X_1)U_{n1}}]^{Z_n}\Big]. \end{aligned}$$

Noting that

$$U_{nk} = \begin{cases} 1 & w.p. \quad (1 - a(X_{nk}))^n, \\ 0 & w.p. \quad 1 - (1 - a(X_{nk}))^n, \end{cases}$$

we have

$$\begin{aligned} L_{N^2} f &= \prod_{n=1}^{\infty} E\Big[E[(1 - (1 - a(X_1))^n (1 - e^{-f(X_1)}))]^{Z_n}\Big] \\ &= \prod_{n=1}^{\infty} \frac{\gamma}{\lambda + \gamma} \times \frac{1}{1 - \lambda/(\lambda + \gamma) E[(1 - (1 - a(X_1))^n (1 - e^{-f(X_1)}))]} \\ &= \prod_{n=1}^{\infty} \frac{1}{1 + \lambda/\gamma E[(1 - a(X_1))^n (1 - e^{-f(X_1)})]}. \end{aligned} \tag{73}$$

The N process at time 0 is the aggregation of the two processes N^1 and N^2. Thus, the Laplace functional of the N process is the product of the Laplace functionals of the N^1 and N^2 processes. Multiplying (72) and (73) yields the Laplace functional of the process N in (25). ∎

Though there is no closed form for the product in (73), we can substitute $e^{-1} = s$ and $f(x) = \mathbb{1}(x \in A)$ into (25) to get the probability generating function $G_{N(A)}(s)$ for the number of remaining particles in the region $A \in \mathcal{E}$:

$$G_{N(A)}(s) = \prod_{n=0}^{\infty} \frac{1}{1 + (1 - s) a_n},$$

where

$$a_n = \lambda/\gamma E[(1 - a(X_1))^n \mathbb{1}(X_1 \in A)].$$

We can rewrite $G_{N(A)}(s)$ as

$$G_{N(A)}(s) = e^{-C(s)}, \tag{74}$$

where

$$C(s) = \sum_{n=0}^{\infty} \log\Big(1 + (1-s)a_n\Big).$$

Using Lemma 26 below, we can write down the distribution of $N(A)$.

Lemma 26 *Suppose the probability generating function* $G(s) = \sum_{n=0}^{\infty} p_n s^n$ *has the form* $G(s) = e^{-C(s)}$ *where*

$$C(s) = \sum_{n=0}^{\infty} \log[1 + (1-s)a_n],$$

for some numbers $1 > a_0 > a_1 > \ldots > 0$. *Then*

$$\begin{aligned} p_0 &= \prod_{n=0}^{\infty} \frac{1}{1+a_n}, \\ p_k &= -\frac{1}{k} \sum_{n=0}^{k-1} \frac{C^{(k-n)}(0) p_n}{(k-1-n)!}, \qquad k \geq 1, \end{aligned} \tag{75}$$

where

$$C^{(k)}(0) = -(k-1)! \sum_{n=0}^{\infty} \frac{a_n^k}{(1+a_n)^k}.$$

The proof of Lemma 26 is similar to that of Lemma 21. Now we can write down the distribution of $N(A)$.

Theorem 27 *The distribution of* $N(A)$ *for* $A \in \mathcal{E}$ *is given by*

$$\begin{aligned} P(N(A) = 0) &= \prod_{n=0}^{\infty} \frac{1}{1+a_n}, \\ P(N(A) = k) &= -\frac{1}{k} \sum_{n=0}^{k-1} \frac{C^{(k-n)}(0) P(N(A) = n)}{(k-1-n)!}, \qquad k \geq 1, \end{aligned} \tag{76}$$

where $C(s)$ *is*

$$C(s) = \sum_{n=0}^{\infty} \log\Big(1 + (1-s)a_n\Big) \tag{77}$$

and

$$a_n = \lambda/\gamma \int_A (1 - a(x))^n F(dx).$$

Proof The proof follows directly upon application of Lemma 26 to the generating function $G_{N(A)}(s)$ above. ∎

Remark 28 We can use the probability generating function to calculate moments of $N(A)$. For instance, the mean of $N(A)$ is given by

$$\begin{aligned} EN(A) &= G'_{N(A)}(1) \\ &= \lambda/\gamma E[a(X_1)^{-1} \mathbb{1}(X_1 \in A)]. \end{aligned}$$

The second moment of $N(A)$ can be found by taking the second derivative of $G_{N(A)}(s)$:

$$\begin{aligned} E[N(A)^2] &= G''_{N(A)}(1) + G'_{N(A)}(1) \\ &= -C''(1) + (EN(A))^2 + E[N(A)] \\ &= \sum_{n=0}^{\infty} \left(\lambda/\gamma \int_A (1 - a(x))^n F(dx)\right)^2 + (EN(A))^2 + E[N(A)]. \end{aligned}$$

CHAPTER VI

MODELS WITH PARTICLE MOVEMENTS

In this chapter we will focus on spatial systems where particles arrive to a space, are allowed to move about the space, and eventually depart. The departures may be triggered by predetermined service times, or upon the arrival of future particles. We describe these systems by random time transformations of marked point processes. Motivational systems for this material include stochastic networks, wireless networks, and mobile populations.

In the first section we return to Example 1.2 from section 1.2 regarding particle movements without interactions, and we obtain the limiting process. In the second section we provide a particle movement generalization of the extension of Durrett's and Limic's model from Section 4.2 where the deletions depend upon the initial attributes of the particles in the system.

6.1 Movements Without Interactions

Recall Example 1.2 from section 1.2 regarding the spatial $M/G/\infty$ system. Particles arrive to a Polish space $\mathbb{E}$ according to a Poisson process with rate λ at the times $0 < T_1 < T_2 < \ldots$. Upon arrival, each particle is assigned a location $X_n \in \mathbb{E}$ and a service time $V_n \in \mathbb{R}_+$. The locations are independent and identically distributed according to the distribution function F, and the service times are independent, but

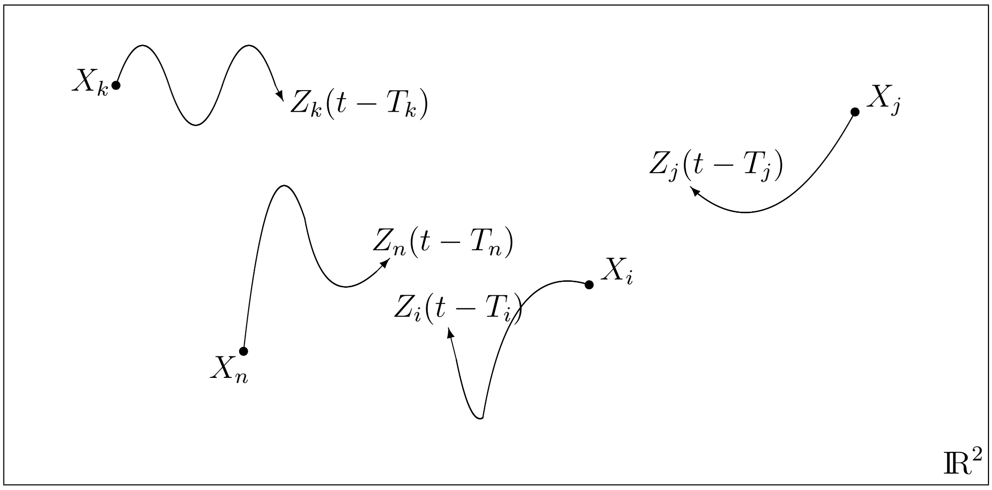

Figure 7: Particle movements without interactions.
Here, $\mathbb{E} = \mathbb{R}^2$ and the time is currently t. A particle is still in the system if its service time (not indicated in the figure) is larger than t minus its arrival time.

the distribution of V_n may depend on X_n. That is, given $X_n = x$, the nth service time is distributed according to the distribution function G_x. Upon arrival, the nth particle moves about $\mathbb{E}$ according to a stochastic process $\{Z_n(t), t \geq 0\} \in D(\mathbb{R}_+, \mathbb{E})$ such that $Z_n(0) = X_n$ w.p.1. that depends only upon the initial location X_n. Given $X_n = x$, the nth path Z_n is distributed according to the distribution function H_x on $D(\mathbb{R}_+, \mathbb{E})$. At time $T_n + V_n$, the nth particle exits the system. See Figure 7.

The data $\{(T_n, X_n, V_n, Z_n) : n \geq 1\}$ generates the input process M defined by

$$M(I \times A \times B \times C) = \sum_n \mathbb{1}(T_n \in I, X_n \in A, V_n \in B, Z_n \in C).$$

Because M is a Poisson process, its mean measure is given by

$$EM(I \times A \times B \times C) = \lambda \int_I \int_A \int_B \int_C H_x(dz)\, G_x(dv)\, F(dx)\, ds.$$

6.1.1 Main Results

We are interested in the N_t process defined by

$$\begin{aligned} N_t(I, A, B, C, \{i\}) &= \sum_n \mathbb{1}(t - T_n \in I, X_n \in A, \\ &\qquad V_n \in B, Z_n(t - T_n) \in C, \mathbb{1}(V_n > t - T_n) = 1). \end{aligned} \tag{78}$$

Specifically, if $i = 1$, N_t counts the number of particles that arrived in the time interval $t - I$ in the spatial region A with a service time in B that are still in the system at time t somewhere in region C.

The following theorem describes the limiting distribution of the N_t process.

Theorem 29 *The process defined above satisfies $N_t \xrightarrow{d} N$, where N is a Poisson process defined by*

$$Nf = \sum_n f(T_n, X_n, V_n, Z_n(T_n), \gamma_n), \qquad f \in \mathcal{C}_K^+(\mathbb{E}),$$

with $\gamma_n = \mathbb{1}(T_n < V_n)$. The mean measure of N is

$$\begin{aligned} \mu_N(I \times A \times B \times C \times \{i\}) &= \lambda \int_I \int_A \int_B \int_{\{h \in D(\mathbb{R}_+, \mathbb{E}): h(0) = x, h(u) \in C\}} g(u, x, i) \\ &\qquad \times H_x(dz)\, G_x(dv)\, F(dx)\, du, \end{aligned} \tag{79}$$

where

$$g(u, x, 0) = G_x(u), \qquad g(u, x, 1) = 1 - g(u, x, 0). \tag{80}$$

In addition,

$$EN_t f \to ENf, \qquad f \in \mathcal{C}_K^+(\mathbb{E}). \tag{81}$$

Proof The process N_t can be expressed as

$$N_t f = \sum_n f(\phi_t(t - T_n, X_n, V_n, Z_n)), \qquad f \in \mathcal{C}_K^+(\mathbb{E}),$$

where

$$\phi_t(u, x, v, z) = (u, x, v, z(u), 1\!\!1(0 < u < v)).$$

Thus N_t is obtained from the input process M via the random transformation ϕ_t as discussed in Chapter 3.

Let us define

$$M_t(I, A, B, C) \equiv M(t - I, A, B, C).$$

Then to prove N_t converges in distribution, it suffices by Theorem 3 to show there exists a function $\phi \in \mathbb{F}$ such that $(\phi_t, M_t) \stackrel{d}{=} (\phi, M)$. To this end, define ϕ by

$$\phi(u, x, v, z) = (u, x, v, z(u), 1\!\!1(0 < u < v).$$

Because M_t and M are time-homogeneous Poisson processes, they are equal in distribution. Thus the conditions of Theorem 3 are satisfied so that $N_t \xrightarrow{d} N$, where N is defined by

$$\begin{aligned} Nf &= \sum_n f(\phi(T_n, X_n, V_n, Z_n)) \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{E}} \int_{\mathbb{R}_+} \int_{D(\mathbb{R}_+, \mathbb{E})} f(u, x, v, z(u), 1\!\!1(0 < u < v) M(du\, dx\, dv\, dz). \end{aligned}$$

Then N is a Poisson process by Proposition 3 of Chapter 3.

To prove the mean measure of N is given by μ_N,

$$\begin{aligned} ENf &= E \int_{\mathbb{R}_+} \int_{\mathbb{E}} \int_{\mathbb{R}_+} \int_{D(\mathbb{R}_+, \mathbb{E})} f(u, x, v, z(u), 1\!\!1(0 < u < v) M(du\, dx\, dv, dz) \\ &= \lambda \int_{\mathbb{R}_+} \int_{\mathbb{E}} \int_{\mathbb{R}_+} \int_{D(\mathbb{R}_+, \mathbb{E})} \sum_{i=0}^{1} f(u, x, v, z(u), i) g(u, x, i) \\ &\qquad \times H_x(dz)\, G_x(dv)\, F(dx)\, du. \end{aligned}$$

Finally, the convergence $EN_tf \to ENf$ for $f \in \mathcal{C}_K^+$ follows from Theorem 1. ∎

In the following subsection we are concerned with the point process that counts the numbers of remaining particles in the system at time t, and we will prove the limiting process as $t \to \infty$ is also a Poisson process.

6.1.2 Limiting Process of Remaining Particles

For this subsection, define the process

$$N_t(A) = \sum_n \mathbb{1}(Z_n(t - T_n) \in A) U_{tn},$$

where the survival indicator function U_{tn} is defined by

$$U_{tn} = \mathbb{1}(0 < t - T_n < V_n).$$

Thus $N_t(A)$ is the number of particles that remain in the region $A \in \mathbb{E}$ at time t. The independence of the U_{tn} follows because there are no interactions. As usual, we suppose $N_0(\mathbb{E}) = 0$ w.p.1.

The main theorem of this subsection appears below.

Theorem 30 *The point process N_t converges in distribution to the Poisson process N with mean measure given by*

$$\mu_N(A) = \lambda \int_{\mathbb{R}_+} \int_{\mathbb{E}} \int_{\{h: h(0)=x, h(u) \in A\}} \int (1 - G_x(u)) H_x(dz) F(dx) du.$$

Hence, the stationary distribution of N_t is that of the Poisson process N.

Proof The proof follows by replacing B with $\mathbb{E}$, setting $I = [0, t]$ and letting $t \to \infty$ in Theorem 29. ∎

In the remaining sections we consider models where particles move about the system according to a Markov probability kernel and interact.

6.2 ABTs with Movements

The models we present in this section are the same as the main model in Chapter 4, except the particles are now allowed to move about the system. When the nth particle arrives to the system at time T_n, it is marked not only with an initial position X_n and discrete lifetime L_n as before, but it also receives a trajectory process Z_n in $D(\mathbb{R}_+ \times \mathbb{R})$ as in the previous section. That is, $Z_n(t - T_n)$ is the location of the nth particle at time t, provided it is still in the system at time t. We set $Z_n(0) = X_n$ for all n. The Z_n could be a continuous time Markov process, or a Brownian motion, for instance. We let H denote the distribution function on the set of paths.

As in the previous models, the marks are independent and identically distributed and independent of the T_n. Here the nth mark is (X_n, L_n, Z_n). We again denote by F the distribution function of X_n, and the discrete lifetimes L_n only depend only on the X_n. Once again, $a(x)$ is the probability that a particle that initially arrives to x will be deleted upon an arrival to $y > x$, and

$$P(L_n > k|X_n) = (1 - a(X_n))^k.$$

In order to describe the particle movements, we will use the probability kernels $P_t(x, A)$ defined by

$$P_t(x, A) = P(Z_n(t) \in A|X_n = x).$$

The data generates the input process M given by

$$M(I \times A \times B \times C) = \sum_n \mathbb{1}(T_n \in I, X_n \in A, L_n \in B, Z_n \in C).$$

Here, $M(I \times A \times B \times C)$ counts the number of particles that arrived in the time interval I that were initially in A with discrete lifetime in B that have a path in the set C. As before, we will let $M(t)$ denote the total number of particles that arrive to the system in the time interval $[0,t]$. Because M is a Poisson process, its mean measure is given by

$$EM(I \times A \times B \times C) = \lambda \int_I ds \int_A F(dx) \sum_{\ell \in B} (1 - a(x))^{\ell - 1} a(x) \int_C H(dz).$$

6.2.1 Convergence of Mean Measure

We are interested in the process N_t defined by

$$N_t(A) = \sum_n \mathbb{1}(Z_n(t - T_n) \in A)\mathbb{1}(L_n > M_n(t)), \tag{82}$$

where $M_n(t) \equiv M((T_n, t] \times (X_n, \infty) \times \mathbb{N} \times D(\mathbb{R}_+, \mathbb{R}))$ counts the number of particles that arrive before time t that could annihilate the nth particle.

The following proposition describes the limiting mean measure of N_t process.

Proposition 31 *For the N_t process defined above, we have*

$$EN_t f \to \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_{\mathbb{R}} f(z) P_s(x, dz) \lambda e^{-\lambda s \overline{F}(x) a(x)} ds\, F(dx), \qquad f \in \mathcal{C}_{\mathcal{K}}^+(\mathbb{E}).$$

Proof By conditioning on $M(t)$,

$$EN_t f = E[\sum_{n=1}^{M(t)} \Psi_{M(t),n}(t)], \tag{83}$$

where

$$\Psi_{mn}(t) = E[f(Z_n(t-T_n))\mathbb{1}(L_n > M_n(t))|M(t) = m].$$

Then by conditioning on T_n, X_n, $M_n(t)$, and $Z_n(t-T_n)$, we have

$$\begin{aligned}
\Psi_{mn}(t) &= \int_{\mathbb{R}}\int_0^t \sum_{k=0}^{m-1}\binom{m-1}{k}\Big(\frac{\overline{F}(x)s}{t}\Big)^k\Big(\frac{t-\overline{F}(x)s}{t}\Big)^{m-1-k} \\
&\qquad \times \int_{\mathbb{R}} f(z)P_s(x,dz)(1-a(x))^k t^{-1}dsF(dx) \\
&= \int_{\mathbb{R}}\int_0^t \Big(\frac{\overline{F}(x)s(1-a(x))+t-\overline{F}(x)s}{t}\Big)^{m-1} \\
&\qquad \times \int_{\mathbb{R}} f(z)P_s(x,dz)t^{-1}dsF(dx),
\end{aligned}$$

where the second step follows from the binomial theorem. Noting that $M(t)$ has the Poisson(λt) distribution, we can get an expression for EN_tf:

$$\begin{aligned}
EN_tf &= \sum_{m=0}^{\infty}\int_{\mathbb{R}}\int_0^t \frac{me^{-\lambda t}(\lambda t)^m}{m!}\Big(\frac{\overline{F}(x)s(1-a(x))+t-\overline{F}(x)s}{t}\Big)^{m-1} \\
&\qquad \times \int_{\mathbb{R}} f(z)P_s(x,dz)t^{-1}dsF(dx) \\
&= \int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}} f(z)P_s(x,dz)dsF(dx) \\
&\qquad \times \sum_{m=0}^{\infty}\frac{me^{-\lambda t}(\lambda)^m}{m!}\Big(\overline{F}(x)s(1-a(x))+t-\overline{F}(x)s\Big)^{m-1} \\
&= \int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}} f(z)P_s(x,dz)\lambda e^{-\lambda s\overline{F}(x)a(x)}ds\, F(dx).
\end{aligned}$$

Thus, as $t\to\infty$, we have

$$EN_tf \to \int_{\mathbb{R}}\int_{\mathbb{R}_+}\int_{\mathbb{R}} f(z)P_s(x,dz)\lambda e^{-\lambda s\overline{F}(x)a(x)}ds\, F(dx). \tag{84}$$

∎

CHAPTER VII

CONCLUSIONS AND PROPOSED FUTURE RESEARCH

The theme of this dissertation has been that one can achieve limiting results for certain space-time stochastic processes by modelling them as marked point processes and then taking a random time transformation. In this light, we have found the limiting behavior of several models for service systems and species competitions, focusing on models where arriving particles trigger departures from the system. Many more intricate models remain uninvestigated. We now discuss some of these.

7.1 Framework

All of the models we have considered have Poisson arrivals. Theorem 1 makes no such assumptions. As previously mentioned in Remark 5, in the context of Theorem 1 of Chapter 3, under certain conditions this theorem holds when there exists $\phi \in \mathbb{F}$ such that $(\phi_t, M_t) \xrightarrow{d} (\phi, M)$. Also, the theorem is obvious when the input process is a Poisson process. Therefore, models with more general input processes such as renewal processes could be constructed.

It seems that traditional queueing models do not fit within the framework laid down in Chapter 3. This is due to the fact that a customer that arrives to a queueing system at time T_n remains in the system at time $t > T_n$ depending on the service and arrival times of customers that arrived before time T_n. In the models we consider, a customer (or particle) that arrives at time T_n remains in the system at time $t > T_n$ depending on attributes of particles that arrive in the time interval $(T_n, t]$. Perhaps a more general framework exists in which one can well model systems of both types.

7.2 Service Systems

Generalizations of queueing models where departures may be triggered by an arriving customer should be considered. This would be like having a queueing system with traditional service times where waiting customers can depart upon the arrival of new customers. Even this description is vague. Suppose each customer is marked with a service time random variable. On one hand, customers may be allowed to depart only after a required number of deletion attempts have been made by arrivals and the service time is completed. On the other hand, customers may be allowed depart the system upon the minimum of their service time and the time until the arrival triggering their departure.

In the case of the spatial $M/G/\infty$ system with ABTs, it is not difficult to show the limiting mean measure of the number of particles remaining in the system is given by

$$\mu(dx) = \frac{F(dx)}{\gamma(x)/\lambda + a(x)D^{-1}(x)}.$$

Here, λ is the arrival rate of customers to the system, F is the distribution function determining the customers' positions, $\gamma(x)$ is the service rate of a customer at x, $D^{-1}(x)$ is the set of points where new arrivals can delete a particle at location x, and $a(x)$ is the deletion probability as before. In models similar to this with interactions, the analysis becomes more complicated because the process is no longer a Markov chain subordinated to a Poisson process; the transition rates due to services are not uniform.

An interesting generalization of the polling models we have considered is to allow the deletion probabilities $a(x)$ to change throughout time. That is, suppose the ith particle survives the nth arrival, and that at time T_n, the deletion probability of the ith particle is $a(X_{in})$. Then at the time of the arrival at T_{n+1}, the deletion probability of the ith particle is $a(X_{i,n+1})$. The changes in the deletion probabilities could be governed by Markov transition kernels that depend only on the current transition probability, or also upon the amount of time the particle has spent in the system so far. This setup models systems where particles become more or less resilient as time passes, depending on the transition kernel.

Capacity constraints are other interesting extensions. Suppose either a cap exists on the total number of particles in the system, or perhaps local capacity constraints exist for subsets of the space. Such models are important in service systems theory because real-life storage areas and buffers typically have finite sizes.

7.3 Particle Movements

Future models should include more intricate branching, movements, and interactions of the particles. It would be interesting to discover the limiting behavior of models where species are allowed to move about the system, reproduce, and meet their ends by natural causes or through interaction with other species already in the system. In such models, the departures may be triggered by particles already in the system. These models would perhaps require different framework because like traditional queueing models, the random time transformation that describes the process of interest would depend upon the past and future of the entire process.

It appears that subjecting the particles in the main model of Chapter 4 to Markovian movements that take place only at the arrival times yields a Poisson process in the limit with mean measure

$$\mu(dx) = \pi(dx) \int_{\mathbb{R}} \frac{F(dy)}{a(y)\overline{F}(y)}.$$

Here, π is the stationary distribution of an ergodic Markov chain that governs the particle movements, and F and a are as before. Although these conditions are rather restrictive, the limiting process appears tractable.

In the previous section, we allowed the particles to move about the system according to a path that was only dependent upon the initial locations and ranks of the one particle. However, more natural assumptions suggest environment constraints should be implemented. For example, the laws for particle movements could be allowed to depend on the state of the entire system at the current time. Such models could still be Markovian, yet much more complicated due to the dependence of the transition kernels upon the entire transformed process at time t, not just the attributes of a particular particle.

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INDEX