SENSOR-BASED PROGNOSTICS AND STRUCTURED MAINTENANCE POLICIES FOR COMPONENTS WITH COMPLEX DEGRADATION

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SENSOR-BASED PROGNOSTICS AND STRUCTURED MAINTENANCE POLICIES FOR COMPONENTS WITH COMPLEX DEGRADATION

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To My Parents
Prof. Hamdy Elwany and Mrs. Suzy Abou Sabe

For Always Being Inspirational and Supportive

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ACRONYM AND NOTATION

ACRONYM

ALT	Accelerated Life Testing
BM	Brownian motion
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BPF	Ball-Passing Frequency
BSF	Ball Spin Frequency
CBM	Condition Based Maintenance
CDF	Cumulative distribution function
CI	Confidence Interval
CLP	Control Limit Policy
CM	Condition Monitoring
DAQ	Data acquisition
FFT	Fast Fourier Transform
FTF	Fundamental Train Frequency
GUI	Graphical User Interface
IG	Inverse Gaussian
iid	Independent and Identically Distributed
MDP	Markov Decision Process
PDF	Probability density function
PHM	Proportional Hazard Model
POMDP	Partially Observable Markov Decision Process
RLD	Remaining Life Distribution
RMS	Root-mean-square
RPM	Revolutions per Minute
RTD	Replacement Time Distribution
SLLN	Strong Law of Large Numbers

NOTATION

k	Observation/Decision epoch
t_k	Observation time
S_k	Signal value (amplitude) at epoch k
t	Time between two epochs
$h(\cdot)$	Functional form of the degradation model
ϕ	Deterministic model parameter
θ', β'	Stochastic model parameters
$\pi(\theta')$, $\pi(\beta')$	Prior distributions of θ', β'
$\varepsilon(t_k)$	Error term at time t_k

ξ	Failure threshold
σ^2	Variance parameter of Brownian motion
Δ_k	Increment of the degradation signal logarithm
ℓ_k	Logarithm of the degradation signal
$\Gamma(t_k + \tau)$	Future logarithm of the degradation signal
T	Remaining life
$W(t)$	Standard Brownian Motion
$X(t)$	Brownian Motion with positive drift
δ	Drift of Brownian Motion
ν	Location parameter of inverse Gaussian distribution
γ	Shape parameter of inverse Gaussian distribution
$\Phi(\cdot)$	Standard normal CDF
t_r	Replacement time
t_o	Inventory ordering time
C_r	Long-run average replacement cost per cycle
c_1	Planned replacement cost
c_2	Failure replacement cost
C_o	Long-run average inventory cost per cycle
k_h	Inventory holding cost per unit time
k_s	Inventory shortage cost per unit time
$\mathbf{W}$	State space of the replacement MDP model
P	Set of real numbers
$V(k, \ell_k)$	Total expected cost-to-go
λ	Discount factor
ℓ_k^*	Replacement signal control limit
k_ℓ^*	Replacement age threshold
$\tilde{k}$	Maximum allowable system age
ϑ_k^*	Inventory signal control limit

SUMMARY

We propose a mathematical framework that integrates low-level sensory signals from monitoring engineering systems and their components with high-level decision models for maintenance optimization. Our objective is to derive optimal adaptive maintenance strategies that capitalize on condition monitoring information to update maintenance actions based upon the current state of health of the system. We refer to this sensor-based decision methodology as “sense-and-respond logistics”.

As a first step, we develop and extend degradation models to compute and periodically update the remaining life distribution of fielded components using *in situ* degradation signals. Next, we integrate these sensory updated remaining life distributions with maintenance decision models to; (1) determine, in real-time, the optimal time to replace a component such that the lost opportunity costs due to early replacements are minimized and system utilization is increased, and (2) sequentially determine the optimal time to order a spare part such that inventory holding costs are minimized while preventing stock outs.

Lastly, we integrate the proposed degradation model with Markov process models to derive structured replacement and spare parts ordering policies. In particular, we show that the optimal maintenance policy for our problem setting is a monotonically non-decreasing control limit type policy. We validate our methodology using real-world data from monitoring a piece of rotating machinery using vibration accelerometers. We also demonstrate that the proposed sense-and-respond decision methodology results in better decisions and reduced costs compared to other traditional approaches.

CHAPTER 1
INTRODUCTION

Unexpected failures of engineering systems can have a significant impact on manufacturing and service applications, national infrastructure (nuclear power plants and civil structures), healthcare applications, and military operations. For example, unexpected failures in Just-In-Time production systems, such as production lines in the automotive industry, can immediately halt delivery schedules because there is not enough work-in-process to buffer any interruptions. Similar arguments can be made about numerous applications in the service and logistics sector. The reliability of transportation fleets, such as airlines and railways, is necessary to ensure that personnel, raw materials, finished goods, and spare parts are delivered on schedule to eliminate waste of perishable goods and prevent other costly stock outs. In healthcare applications, sudden failures of medical equipment, such as pacemakers, operation room equipment, and health monitoring systems can have fatal repercussions. In homeland security applications, the reliability of sensor systems is necessary to ensure fidelity of the data streams communicated by this technology. More seriously, the sudden failure of civil infrastructure (e.g. bridges and buildings) can result in human fatalities. Motivated by previous discussion, a large body of the literature has been targeted towards predicting unexpected failures in an effort to eliminate or minimize their negative impacts.

Accurately predicting unexpected failures is a very challenging problem due to two primary components, (1) our limited understanding of the physics-of-failure, and (2) the high level of uncertainty associated with the degradation processes that occur prior to failure. Numerous research efforts have developed methodologies that address these challenges and attempt to predict sudden failures accurately. The next step is to use this information to determine good and economical maintenance strategies. This represents another significant research challenge. The costs of maintenance operations can represent up to 15-60% of the cost of produced goods in manufacturing and production plants [1]. These significant costs primarily arise due to ineffective maintenance planning. For example, early replacement of a piece of equipment reduces its utilization and results in lost opportunity costs. On the other hand, delayed replacement involves the risk of expensive catastrophic failures. Consequently, extensive investigation of decision models for optimizing maintenance decisions has been studied in the literature. A plethora of approaches have been followed to devise efficient maintenance strategies for operating components and systems. Condition-based maintenance (CBM) is one of these approaches whereby the maintenance actions are triggered by accurate assessments of the system's condition/health. Recent advances in sensor technology now enable us to continuously monitor the health of operating systems and components using dedicated sensors. These sensors provide rich streams of real-time signals that are typically correlated with the severity of the system's underlying degradation process, and can be used to determine cost-effective maintenance policies.

In this dissertation, we focus on developing a mathematical framework that integrates low-level sensor-based information from the condition monitoring of engineering systems with high-level mathematical models for maintenance optimization. Our objective is to determine adaptive maintenance policies, with a special emphasis on equipment replacement and spare parts inventory policies.

1.1 Reliability and Failure Time Prediction

Estimating component failure times has been extensively studied in the field of reliability theory. Historically, reliability models have focused on evaluating failure measures for a population of components, primarily, by collecting and analyzing failure data [2-6]. One approach is to utilize actual in-field performance data through warranty reports of failed components to estimate time-to-failure [2, 10]. In other cases, traditional reliability testing is used to derive generalized failure distributions for a population of components [4, 11]. Increased product lifecycles and improved reliability have made traditional reliability testing more difficult. Consequently, there has been an increased interest on pursuing accelerated reliability/degradation testing [12, 13].

The uncertainty associated with degradation and failure processes is usually characterized by parametric and empirical failure distributions. One drawback of this approach is that it does not pay much attention to the fact that even the failure of identical components can differ drastically due to many factors including inherent material in-homogeneity, variations in processing technology, and differences in environmental and operating conditions. Since failure time distributions are unaffected by the underlying physical process, they do not distinguish between the different degradation characteristics of individual components and do not capture the unit-to-unit variability among identical components. Research efforts on condition monitoring and degradation modeling address this aspect as discussed next.

1.2 Condition Monitoring and Degradation Modeling

Degradation processes involve a gradual accumulation of damage, which eventually leads to failure. In some applications, the physical transitions that occur during degradation can be directly observed, e.g., the length of a crack. However, in most engineering systems, it is only possible to make indirect observations by monitoring some manifestations of the degradation process, such as temperature changes, vibration levels, etc. [14]. In many applications, these manifestations can be monitored directly or indirectly using dedicated sensors (e.g. temperature thermocouples and vibration accelerometers). Condition Monitoring techniques utilize real-time sensor-based information to evaluate the health of a component during operation [8, 9, 15]. The magnitude (or amplitude) of these condition-based signals can be used to assess the severity of the degradation state of the component that is being monitored and can be used to prevent unexpected failures. One of the main shortcomings associated with condition monitoring is that it relies solely on real-time sensory information from the individual component and does not capture the failure characteristics of a component's population. In many cases, slight deviations in the sensory signals may be classified as failures although they may be originating from minor changes in the operational or environmental conditions.

The measures described above are usually correlated with the physical degradation process that evolve over time, and are known as degradation signals. Many components exhibit characteristic patterns in their degradation signals that evolve with as the component's degradation progresses [15]. Degradation modeling focuses on mathematically modeling these degradation signals to predict their future evolution. Different tools have been used to model the evolutionary paths of the degradation signals such as random coefficients models [16-20], Brownian motion [3, 13, 21, 22], Gamma processes [22-27], Markov processes [28], and semi-Markov processes [29]. Very few works, however, focused on integrating the failure characteristics of component populations from reliability testing with degradation rates of individual components from condition monitoring.

In this research, we present a degradation model that combines the two aspects discussed above. In particular, we perform degradation testing on a sample of components to determine the failure time distribution of the population. Next, we utilize sensory signals acquired from monitoring fielded components to revise their RLD based on their current health state.

1.3 Decision Models for Equipment Replacement and Spare Parts Inventory

Maintenance optimization models have been studied extensively in the literature. Generally, most of the maintenance decision models focus on establishing inspection [30, 31, 32], repair [34, 35] or replacement policies [33, 36, 37]. General surveys on mathematical optimization models used in maintenance applications can be found in [38-41]. For the purpose of this research, we focus our attention on two primary decision policies, equipment replacement and spare parts inventory models.

The uncertainty of failure processes is among the primary sources of difficulty encountered in developing efficient and accurate maintenance-related replacement and spare parts inventory models. Precise reliability assessment is key to making sound maintenance decisions. For instance, deciding which component to replace and when, requires a careful balance between the cost associated with premature replacement and the cost of unexpected failure. Furthermore, the ordering time of spare parts and their stocking quantities need to be planned such that holding costs are kept to a minimum while avoiding stock outs.

Some of the existing models are age dependent, and use failure time distributions to evaluate replacement and spare parts inventory policies. For example, the renewal theoretic age replacement model utilizes the probability of failure to strike a balance between the costs associated with premature replacement and the costs of unexpected failure during the component's life cycles [42]. Similar models also exist for evaluating optimal ordering times of a spare part in a single buffer inventory system [43]. In this class of models, decisions are solely based on the failure time distributions of the component's population and do not account for the degradation state of the individual components. Other replacement and inventory models make the simplifying assumption that failures are random and can be modeled, for example, as a Poisson Process [44-49]. Such assumptions often compromise the accuracy of the decision making process, especially since they do not account for variations in the degradation processes.

Another class of maintenance optimization models assumes that the component or system can be in some state of degradation from a set of possible states. We refer to these models as "state models". Numerous approaches have been used to characterize the transitions between the degradation states, such as Markov processes [36,50, 51] and semi-Markov processes [30, 33, 52]. This category primarily focuses on establishing special appealing structures of the optimal policies, such as control limit (threshold) replacement policies.

Other research efforts focus on determining optimal maintenance policies based upon the condition of the system that can be captured using condition monitoring techniques. This class is known as condition-based maintenance (CBM) [53-55]. In our research, we integrate the benefits of the two previous categories by leveraging the benefits of real-time sensory-signals from condition monitoring with Markov decision processes (MDPs) to determine optimal structured replacement policies.

1.4 Research Tasks

Our research objective is to develop a mathematical framework for integrating real-time condition-based signals with maintenance related operational and logistical decisions. The research tasks involved in achieving this objective are summarized below.

1.4.1 Sensor-Based Degradation Modeling

In this task, we present and extend stochastic degradation models for computing and updating the remaining life distributions (RLDs) of partially degraded components/systems using real-time condition monitoring sensory-signals. The failure characteristics of component's populations derived from reliability and degradation testing are first used to estimate preliminary failure time distributions. Next, real-time signals are used to update the RLD of individual components that degrade differently using Bayesian techniques. Consequently, the updated RLDs evolve according to the latest degradation state of the component being monitored.

We focus on a base-case random-coefficients degradation model with an exponential functional form and Brownian error terms, and extend it in two different directions. First, we study the performance of the model under different assumptions. More specifically, we study the effect of assuming dependent stochastic model parameters versus independent parameters in the original model, and assess the prediction accuracy under this dependency assumption. Second, we study the computational challenges associated with computing the RLDs and their moments. In the original model, the sensory-updated RLDs cannot be characterized using parametric distributions and their moments do not exist. Such difficulties hinder the implementation of this sensor-based framework, especially from the standpoint of computational efficiency of embedded algorithms. We identify a procedure by which we can compute a conservative mean of the sensory-updated RLDs and express the mean and variance using easy to evaluate closed-form expressions. This is accomplished using the first passage time of Brownian motion with positive drift, which follows an Inverse Gaussian (IG) distribution, as an approximation of the RLD. We show that the mean of the IG distribution is a conservative lower bound of the mean remaining life using Jensen's inequality. The approach is validated using real-world vibration-based degradation data.

1.4.2 Renewal-based Replacement and Spare Parts Ordering Policies

This task focuses on integrating the sensory-updated RLDs with replacement and spare parts inventory decisions models. We consider a conventional renewal theoretic age replacement and spare part ordering model [42, 43], and develop a heuristic decision model wherein the failure time distributions used in the conventional approach are replaced with the dynamically evolving RLDs. Each time we acquire a signal from monitoring the component, we use it to update the RLD and, in turn, update previous replacement or inventory decisions. We refer to this methodology as "sense-and-respond logistics", which is summarized in Figure 1 .

Using a simulation case study, we demonstrate that the sense-and-respond logistics policy results in better decision policies and reduced total maintenance costs, compared to the traditional reliability-based approach. This is primarily due to the improved failure prediction accuracy offered by the degradation model.

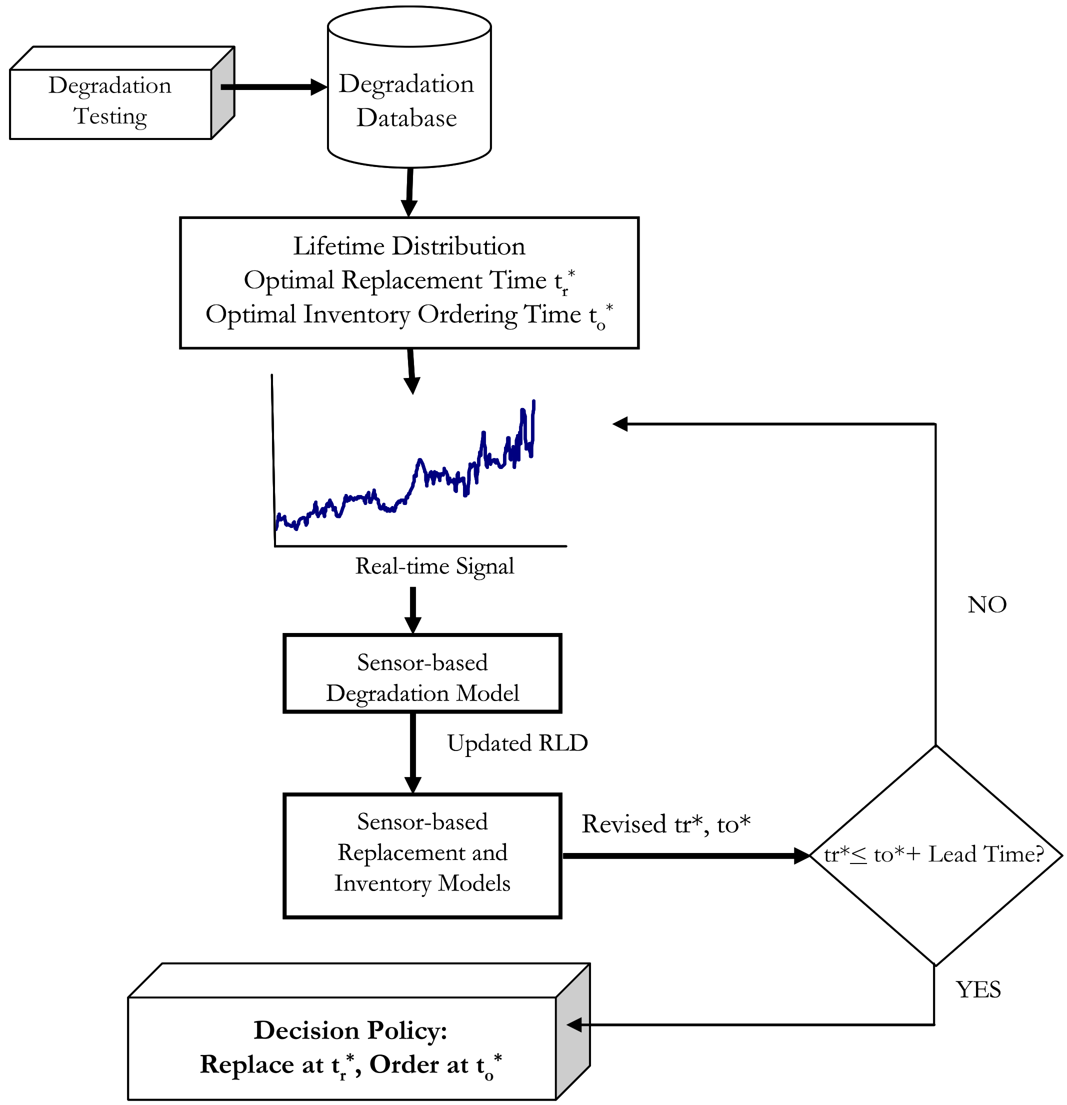

Figure 1 Summary of the sense-and-respond logistics framework

This approach is a heuristic in the sense that updating epochs do not explicitly represent regeneration points of the system. In the following task, we determine exact sensor-based replacement and inventory policies.

1.4.3 Structured Replacement and Spare Parts Ordering Policies

We present a generic single-unit replacement problem, and develop a Markov decision process (MDP) model that utilizes sensory signals to determine the optimal replacement time. We use the degradation modeling framework developed in task 1.4.1 to compute and update the predictive distribution of the degradation signal, as opposed to other state models that commonly assume arbitrary transition probabilities between the system states.

We show that the optimal replacement policy under the infinite horizon expected discounted cost criterion is a monotonically non-decreasing control limit policy that optimally balances the cost of failure, the cost of preventive replacement, and the cost of observing sensor data. This result might seem counterintuitive, since one would typically expect a monotonically non-increasing control limit policy, or in other words, that the urgency to preventively replace the system increases as the system ages. We provide explanation and provide mathematical proofs of this counterintuitive result.

Under a monotonic control limit policy, the system is kept operating until the observed signal exceeds a certain control limit, which is monotone in the system's age. In the event that the decision at some decision epoch, given an observed signal, is to continue operation, the monotonic replacement control limit can be used to determine the optimal time to order a spare part. This methodology is illustrated in Figure 2. Finally, we present a case study based on real-world vibration data from rotating machinery and study the performance of the policy under different cost settings.

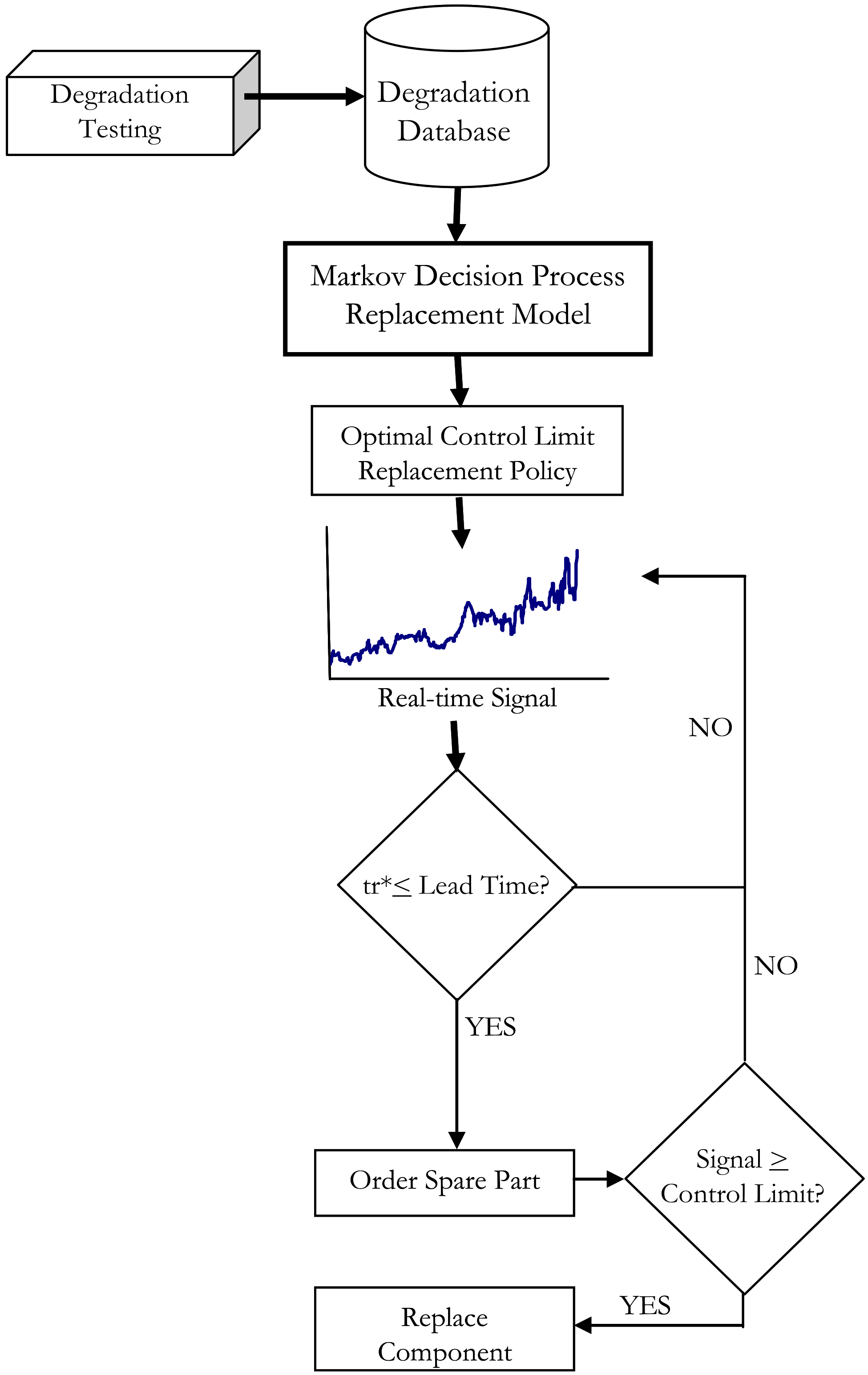

Figure 2 Summary of control limit replacement and spare parts provisioning policies

1.5 Thesis Organization

Chapter 2 surveys some of the related literature on engineering reliability and failure time prediction, condition monitoring, degradation modeling, and decision models for equipment replacement and spare parts inventory policies, both in systems with single and multiple components. Chapter 3 presents the sensor-based degradation modeling framework that use real-time sensory signals to compute and update the RLD of partially degraded components. We present in detail a base-case random-coefficients degradation model with an exponential functional form and Brownian error terms. Next, we re-derive the expressions for the posterior distributions of the stochastic model parameters under the dependency assumption. We also provide closed-form expressions for a conservative lower bound of the mean remaining life that are easy to compute using the first passage time of a Brownian motion with positive drift. Chapter 4 presents an approximate renewal-based decision methodology for computing the optimal replacement and spare parts ordering times based on the observed sensory signals. In Chapter 5, we present exact structured replacement and inventory policies using Markov decision processes (MDP). Finally conclusions and future extensions are outlined in Chapter 6.

CHAPTER 2
LITERATURE REVIEW

The proposed research integrates established research disciplines in a single framework. In particular, these disciplines include failure time prediction of engineering systems and maintenance optimization models. This chapter surveys the relevant literature on these research areas.

2.1 Failure Time Prediction in Engineering Systems

There is a large body of the literature dedicated to estimating the failure time of engineering systems and their components. This literature can be generally categorized into: (1) conventional reliability approaches that focus on collecting and analyzing failure data, and (2) condition monitoring approaches that focus on collecting sensory information from operating systems to assess their state of health and predict their remaining life.

2.1.1 Conventional Reliability Approach

Early foundations of Reliability were found in actuarial concepts used in the insurance industry [56]. It was not until World War II that reliability became a subject of study [57]. In classical reliability, system and component failures are treated as random occurrences. The focus is on modeling the pattern of these random processes by some probability distribution. Several parametric distributions were in use during the 1950s and 1960s to model failure times, such as the exponential, Weibull, normal, and gamma distributions [58-61]. It is also common to encounter cases where no theoretical distribution adequately fits the data. The focus in this case is on using nonparametric (empirical methods) that derive the failure distribution directly from the failure data (see for example [62-66]).

After collecting relevant failure data, statistical inferences are used to provide generalized reliability characteristics and failure time distributions of component families. One approach to collect necessary data is to utilize actual in-field performance data. Hu and Lawless [10] consider nonparametric estimation of lifetime distributions using automobile failure data from warranty reports. Bharatendra and Singh [67] discuss parameter estimation methods that address data incompleteness using incomplete and unclean warranty data. Kalbfleisch and Lawless [2] suggest a procedure for the collection of field failure data and use a regression model to estimate lifetime distributions from this data. In other cases, traditional reliability testing is used to collected the data and derive generalized failure distributions. Coit and Jin [4] derive maximum likelihood estimators for the gamma distribution based on data records from reliability testing to model diverse-to-failure behavior. With increased product lifecycles and improved reliability (e.g. airplane engines and other aviation components), traditional reliability testing is more difficult, time consuming, and sometimes infeasible. Consequently, there has been an increased interest on pursuing accelerated lifetime/degradation testing (ALT). Doksum and Hoyland [12] use reliability information obtained from a series of accelerated reliability tests to model the degradation of cable insulation under variable stress levels as an inverse Gaussian process. Whitmore and Schnkelberg [13] model the degradation of self-regulating heat cables subject to high stress reliability testing. Zhang and Meeker [68] propose Bayesian methods for planning accelerated life tests satisfying practical constraints. Xu and Fei

[69] present guidelines for planning two-factor step stress accelerated life tests with no interaction between the factors. Zhao and Elsayed [70] present a general likelihood formulation for step-stress accelerated life tests that use the Weibull and the lognormal distribution.

As mentioned earlier, the conventional reliability approach treats failure as a random shocks process rather than a process of evolution across degradation states. Hence, these models do not consider the condition of individual components that typically degrade differently due to natural variations and different operating conditions. To address different degradation characteristics of identical components from the same population, research efforts resort to condition monitoring discussed next.

2.1.2 Condition Monitoring

Condition monitoring is the process of collecting real-time sensory information from a functioning device to determine its state of health. The focus in condition monitoring is on diagnosing faults and health classification rather than explicitly predicting the failure time. (Degradation modeling, discussed in the next section, uses condition monitoring information to predict the remaining life of operating components).

Condition monitoring is used in numerous applications including a wide variety of different components such as bearings [71-73], machine tools [8, 74, 75], gears [76, 77], engines [78], and generators [79, 80], among others. In condition monitoring, various condition phenomena such as temperature [83], degree of wear [84, 85], and vibration [76, 81, 82] that are directly or indirectly associated with degradation are captured using sensors.

The evolution of these phenomena is characterized by degradation signals, which are commonly used to assess the extent of component degradation, diagnose faults, or as triggers for maintenance activities. Christer *et al.* [86] use the Conductance Ratio as a measure to assess the erosion condition of the inductors in a furnace. Christer and Wang [87] monitor the cumulative wear of components as a basis for determining its maintenance policy. Vlok *et al*. [88] use condition monitoring to measure the vibration level of a circulating pump in a mining application and utilize this information to determine the optimal replacement policy using the proportional hazards model (PHM). Pedregal and Carnero [89] present a state space model to forecast the state of a turbine using vibration monitoring data. The authors use recursive Kalman-Filter algorithms to estimate the probability of failure, then present a cost model for making preventive replacement decisions.

One major drawback of condition monitoring techniques is that they might give "false alarms". In other words, these techniques can interpret deviations from normal running conditions due to slight changes in operational/environmental conditions as failure. Banjevic *et al.* [90] develop a software tool (EXAKT) for optimizing predictive maintenance based on condition monitoring oil debris and vibration data. The authors discuss and report experience with collecting, preprocessing, and using real-world data, and the common problems associated with it. Another drawback of condition monitoring is that it focuses on the degradation characteristics of the individual component being monitored and pays little attention to the failure behavior and characteristics of the component's population. Furthermore, classical condition monitoring is useful for diagnostic purposes and identifying faults. In contrast, degradation modeling uses sensory-data from condition monitoring for prognostic purposes and estimating the remaining lives of fielded components.

2.1.3 Degradation Modeling

The measures captured by condition monitoring sensors are usually correlated with the physical degradation process of components. These measures commonly evolve following characteristic patterns known as degradation signals [15]. Degradation modeling focuses on mathematically modeling these degradation signals to predict their future evolution. The remaining life of the component is defined as the time elapsed until the degradation signal crosses a pre-specified failure threshold. Different tools are used to model the evolutionary paths of the degradation signals. Grall *et al.* [55] model a single-unit gradually deteriorating system using a Gamma process, and then develop a cost model to optimize the predictive maintenance of the system. Liao *et al.* [26] use a Gamma process to model continuously degrading systems under continuous condition monitoring. The authors then derive maintenance policies that achieve the maximum availability level of the system. Some other research efforts use Markov chains to model degradation signals. Kharoufeh and Cox [28] present a hybrid approach for estimating the residual life distribution of a single-unit system. The degradation rate of the system is assumed to be dependent on external factors in the operating environment. The evolution of these factors is modeled as a Continuous-time Markov Chain. Their hybrid approach includes two models: In the first, real sensor data provide information on the degradation state of the system, and in the second, information on the cumulative degradation up to some point in time is provided. Brownian motion (BM), also known is Wiener process, is also commonly used to model the evolution of degradation signals. Doksum and

Hoyland [12] model the accumulated decay of cable insulation Wiener process with drift and diffusion. The associated failure time follows the inverse Gaussian (IG) distribution. Whitmore and Schnkelberg [13] model the degradation of self-regulating heat cables subject to high stress reliability testing as a Wiener process with a constant rate of degradation. Whitmore [91] describes a statistical model that takes inherent randomness of degradation and measurement errors created by imperfect instruments, procedures, and environments into account. The degradation processes are modeled using a Wiener diffusion process. Modeling approaches other than random coefficients models were also used to model degradation using condition monitoring data.

Other modeling approaches are also used in the literature. Goode *et al.* [55] develop an exponential model for the growth of the vibration signal of a hot strip mill. Yang and Jeang [97] present a statistical model to model cutting tool wear in metal cutting processes by monitoring surface roughness of manufactured products. Tseng and Hamada [98] present a study to measure and improve the lifetime of fluorescent lamps by monitoring their luminous intensity.

Random coefficients model is a common tool used in many research efforts. We focus on this class of models to model the path of degradation signals in this research. A useful review on the use of random coefficients models is provided by Johnson [94]. Lu and Meeker [16] develop a random coefficients model to estimate the failure time distribution based on degradation data from a population of components. Robinson and Crowder [92] model fatigue crack growth as a nonlinear regression model with random coefficients. Two-stage least-squares, maximum likelihood principles, and Bayesian approaches are used to estimate the model parameters. Mallet [95] presents a general method and uses maximum likelihood estimation to estimate the parameters of the random coefficients in this type of models. Chen and Zheng [93] use degradation data to obtain predictive intervals of components' lifetimes using random a coefficients model. Bae and Kvam [99] use a nonlinear random coefficients model with a nonparametric degradation path to capture the burn-in characteristics of vacuum fluorescent displays. Su *et al.* [100] present a random coefficients model with random sample sizes, and shows the influence of the sample size on estimating the model parameters in a semiconductor application.

We notice that very few research efforts target the integration of reliability information with condition-based sensory signals to predict the remaining life of components. Gebraeel *et al.* [19] use a Bayesian updating methodology to predict and continuously update the RLD of individual components using a random coefficients model. The authors model the degradation signal as a continuous time stochastic process having an exponential functional form plus measurement noise error terms. The authors consider error terms that are independent and identically distributed (*iid*), and Brownian error terms. The stochastic model parameters on the other hand are assumed to be independent. In [20], Gebraeel tests the effect of assuming dependence of the stochastic parameters for the exponential model with *iid* error terms. Approximate expressions for the RLD are provided. Elwany and Gebraeel [101] further extend these works by providing closed-form easy to compute expressions of the mean remaining life using the first passage time of Brownian motion with positive drift. Gebraeel *et al.* [102] consider the case where no prior degradation knowledge is available. The authors use historical failure data and assume that it follows the Bernstein distribution to estimate the prior parameters of the degradation models.

2.2 Maintenance Optimization Models

Predicting the lifetime of engineering systems is the first and necessary step towards minimizing the impact of unexpected system failures. There still exists the need to develop mathematical decision models to determine cost-effective maintenance strategies. Maintenance optimization models are studied extensively in the literature. Maintenance decision models focus on establishing economical policies for inspection [30-32], repair [34, 35] replacement [33, 36, 37], and spare parts inventory [103-105]. Wang [38] provides the most recent review of maintenance models for deteriorating systems. Valdez-Flores and Feldman’s [39] survey focuses on maintenance optimization models for single-unit systems. They consider models for repair, replacement, and inspection. Other general surveys on mathematical optimization models used in maintenance applications can be found in Dekker [40] and Scarf [41].

For the sake of relevance to the current research, we categorize the works on maintenance optimization into; (1) Reliability-based models that use the failure time distribution of components and systems to determine optimal maintenance strategies, and (2) State models that characterize degrading systems using a set of states and model the transition between these states using tools such as Markov processes and semi-Markov processes. We focus our attention on maintenance decision models for determining optimal equipment replacement and spare parts provisioning policies.

2.2.1 Reliability-based Models

This class of maintenance optimization models relies on the failure time distribution of the system to derive optimal maintenance strategies. The objective in these models is typically to determine maintenance policies that minimize the long-run average maintenance costs.

2.2.1.1 Models for Single-unit Systems

This segment of the literature focuses on systems with only one operating unit. Aronis *et al.* [44] present an approach to determine the parameter of an (S, S-1) inventory policy for spare parts of electronic equipment. The demand is considered to be generated by random failures of the component assumed to follow a Poisson process. The authors determine the demand distribution using a Bayesian approach based on the prior distribution and historical data on failure rates. The demand distribution is then used to determine the optimal inventory policy for the spare parts. In other works, such as Vaughan [106] the demand is generated both from random failures and regular scheduled preventive maintenance. The proposed model is used to determine the optimal inventory policy for spare parts. Belzunce *et al.* [107] use the failure time distribution of components to determine planned replacement policies. They consider both age and block replacement policies and compare the resulting expected failure times of the components upon the implementation of either policy with the case where no planned replacements are made. Pongpech and Murthy [108] derive periodic preventive replacement policies for leased equipment that achieve a tradeoff between penalty and maintenance costs based on the failure time distribution of the equipment. Wang and Zhang [109] derive an optimal replacement policy for a simple deteriorating system with repair. The operating time of the system after repair is assumed to follow a stochastically decreasing geometric process. An explicit expression for the average cost rate of the system is derived.

Some studies focus on determining replacement policies and spare parts provisioning policies simultaneously rather than separately. In certain cases under specific assumptions, it is shown that sequential or joint policies can result in better decisions and reduced costs. Armstrong and Atkins [42] consider age replacement and inventory ordering decisions for a simple system with one component subject to random failures. The authors propose cycle-based replacement and inventory cost functions, with the objective of computing optimal replacement and inventory ordering times that minimize these cost functions. Two alternative models are proposed; in the first model, the replacement and inventory decisions are made sequentially, whereas the second model considers jointly optimized replacement and inventory decisions. The authors later extend their model to incorporate variable lead time [43]. Cheung and Hausman [110] formulate an analytical model for the joint implementation of preventive replacement and spare parts safety stocks in unreliable environments. The objective is to minimize the total expected costs of the system, which is subject to random failures. Aka *et al.* [111] address the joint optimization of replacement and spare part inventory decisions for systems with machines operating in parallel. Mine and Kawai [112] determine joint ordering and replacement policies for a 1-unit system with a component that degrades. The objective is to minimize the average cost rate over an infinite time horizon. Other works that consider sequential or joint optimization of both decisions can be found in [113-117].

2.2.1.2 Models for Multiple-unit Systems

Although the focus of the current research is on single-unit systems, the sensor-based decision making framework represents a very promising potential to be extended to more complex systems with more than one unit operating in series, in parallel, or a combination of both. The literature is rich with models that consider multiple-unit systems. The literature on multiple-component replacement can generally be categorized into: asset replacement problems, age/block replacement policies, and opportunistic replacement policies.

In asset replacement problems, multiple assets (equipment) operate either in series or in parallel and it is required to find replacement policies for these assets that would minimize total discounted costs over a finite planning horizon. Assets typically have economic interdependence in this class of replacement problems. Although these problems are deterministic in the sense that it is known with certainty how many assets are required at each time period, they are difficult to analyze due to the combinatorial nature of replacing components of a group of assets. Hartman [118] presents an asset replacement problem for a system that has multiple stations operating in series, each of which has a set of assets operating in parallel. The author presented an integer program with the objective of minimizing asset purchase, operating, maintenance, and storage costs, less salvage values over a finite horizon. Constraints represented the capacity, budget, and balance constraints of the system. Jones *et al.* [119] study the parallel machine replacement problem. The decision rules include replacing whole clusters of machines based on their ages, and a proof of their optimality is provided.

We note that asset replacement problems do not take into account the probability of equipment failure. It is assumed that the machines operate without failure, and just experience a decline in their book values with time due to depreciation. Block replacement policies determine replacement decision rules for multiple components taking into account the failure probability obtained from failure time distributions. In such policies, the decision rule is to determine regular time intervals at which a whole subset of components is replaced simultaneously. This is a generalization of the single unit age replacement policy where a component is replaced upon reaching a threshold age T or upon failure, whichever occurs first. Jhang and Sheu-Huei [120] develop block replacement decision rules for multiple unreliable components from a mining application. The authors derive an expression for the long-run average cost and determine the replacement policy that minimizes this expression. Sheu-Huei [121] considers a modified block replacement policy where the system is either replaced or minimally repaired at regular time intervals. The author obtained the expected cost rate of the system using renewal reward theory. Yoo *et al.* [122] determine joint optimal block replacement and spare parts inventory policies for multiple component systems by minimizing their long run average cost. Acharya *et el.* [123] also determine joint block replacement and spare parts ordering policies for multiple component systems. Specifically, the authors develop decision policies to determine the optimal preventive block replacement interval, the optimal level up to which spares are to be ordered, and the optimal ordering interval that minimizes the total cost of replacement, spares procurement, and inventory carrying cost rate.

One disadvantage of block replacement policies is the potential loss due to replacing as-good-as-new components at regular intervals regardless of their state or age. In opportunistic replacement, on the other hand, preventive replacement of a component can be performed at any opportunity created by another component's failure, another component's planned preventive replacement, or the individual component's planned replacement time. Works in the literature demonstrate that the cost of replacing several components simultaneously can be less than the costs of performing many separate replacements. Zheng and Fard [124] propose an opportunistic replacement policy for a system with multiple components characterized by increasing hazard rates. The authors determine a threshold hazard rate L and a hazard tolerance u. A unit is replaced upon failure if its hazard rate lies in the interval $[0, L-u]$ or replaced preventively when its hazard rate reaches the threshold L. Whenever a component is preventively replaced, all other components that have hazard rates between $[L-u, L]$ are opportunistically replaced. The parameters L and u are computed such that the total maintenance cost rate of the system is minimized.

2.2.2 State Models

In state models, the degradation of the components and systems is modeled as a set of states that denote different deterioration stages. For example, a discrete set of states, $S = \{0, 1, \ldots, N\}$, can be used to denote the set of possible degradation states that the system can assume, with 0 being the "good-as-new" state and N the "failed" state. Different modeling tools are used to model the transition between these states as discussed next.

2.2.2.1 Markov Process Models

In this class of models, the transition between the system states is modeled as a Markovian process. Ross [36] considers the problem of sequentially scheduling inspections for a Markovian deteriorating production system. On performing an inspection, the decision maker can also decide whether or not to replace the system, restoring it to the good-as-new condition. The author provids a general framework to handle problems of this nature, and then discusses in detail a 2-state version. Albin and Chao [50] consider a multi-component system with failure dependence. One of the components is assumed to deteriorate according to a continuous-time Markov process, and its degradation affects the life distribution of the other components in the system. An efficient algorithm that exploits the system structure is proposed to determine optimal replacement policies. Chan and Asgarpooh [125] find an optimum maintenance policy for a component that experiences both random failure and failure due to deterioration. The state probabilities are calculated using Markov processes and the optimal value of the mean time to preventive maintenance is determined by maximizing the availability of the single component with respect to the mean time to minimal preventive maintenance.

A lot of works focus on establishing special structures of the maintenance policies for degrading systems under specific assumptions. These structures can sometimes be useful for many practical purposes. They can be exploited to develop computationally efficient algorithms. They also facilitate the implementation of the optimal policy from a practical standpoint. For example, under a control limit policy (CLP), the system is kept operating until the signal exceeds a certain state, known as the control limit. If the system state exceeds this control limit, a preventive maintenance action is performed.

Control limit policies are widely discussed in the literature. Kolesar [51] discusses the optimality of control limit replacement rules for equipment subject to a stationary Markovian degradation process, and shows that linear programming is an efficient method for computing optimal policies. Wood [126] investigates optimal repair policies for constantly monitoring systems deteriorating according to a continuous-time Markov process. The author shows that under certain conditions, the optimal restoration policy is a control limit rule, and presents other several situations in which a control limit rule is counter-intuitively not optimal. Douer and Yechiali [34] derive a generalized control limit rule for the optimal repair and replacement of Markovian deteriorating systems under the total expected discounted cost and long-run average cost criteria. Anderson [128] formulates maintenance models for machines operating under Markovian deterioration as continuous time Markov decision processes under the infinite horizon expected discounted reward. The author presents conditions under which the optimal maintenance policy exhibits monotonic control limit structures for replacement, and the preventive repair prior to replacement is a non-increasing function of the machine state. Ohnishi *et al.* [129] investigate a system that deteriorates according to a continuous-time Markov process. It is assumed that the system state can only be identified through inspection. The paper derives an optimal policy minimizing the expected total long-run average cost per unit time. Under reasonable assumptions reflecting the practical meaning of deterioration, it is shown a control limit rule holds for replacement, and the time interval between successive inspections decreases as the degree of deterioration increases.

Maintenance policies under Markovian deterioration has been studied in numerous other research efforts (see for example [130-133]). Under the Markovian formulation, the deterioration of the system is only indicated by the changes of states but not the age. This assumption may not be valid in practice, and has been studied primarily because of its attractive mathematical tractability. To model aging and cumulative damage in degrading systems, semi-Markov processes formulation is more commonly used.

2.2.2.2 Semi-Markov Process Models

In real-life systems, the behavior of the degrading system typically changes as the system ages. For example, the transition probabilities or associated costs may exhibit characteristic behaviors depending on the cumulative age of the system. Many research efforts resort to semi-Markov process models to capture this practical feature. Care needs to be taken in using the term age in the context of semi-Markov processes. Beynamini and Yechiali [33] state that "while the sojourn time is measured from the last transition or maintenance action, true age accumulates through the system's life cycle". The outcomes of this cumulative aging may or may not be reversed through maintenance actions.

As in the case with Markov process models, semi-Markov process models focus on establishing structured maintenance policies. Kao [52] uses a discrete-time semi-Markov process to model a deteriorating system. State-dependent and age-dependent replacement rules are proposed. The system operating costs and replacement costs are functions of the underlying state. Sufficient conditions for the existence of optimal control limit state dependent replacement rules are derived. Feldman [134] derives optimal control limit replacement rules for a system subject to random shocks. A semi-

Markov process formulation is used to allow for both the time between shocks and the damage due to the next shock to be dependent on the present cumulative damage level. Gottlieb [135] extends this study to the case where the failure rate of the system need not be increasing. So [127] uses a parametric analysis to establish sufficient conditions for the optimality of control limit replacement policies under semi-Markovian deterioration. Lam and Yeh [136] present algorithms for deriving optimal maintenance policies to minimize the mean long-run cost-rate for continuous-time semi-Markov deteriorating systems. The degree of deterioration is known only through inspection, and the durations of inspection and replacement are assumed to be non-negligible. The authors consider five maintenance strategies and develop iterative algorithms to derive the optimal policy for each strategy. Sufficient conditions are established for the optimality of structural polices. Benyamini and Yechiali [33] determine conditions under which control limit policies are optimal for a non-stationary age dependent degradation process. The authors discuss both "replacement-only" models, whereby the only possible maintenance action is to preventively replace the system, and "repair-replacement models". Yeh [30] derives optimal replacement and inspection policies for semi-Markovian deteriorating systems. The authors approximate the distribution of the sojourn time for the semi-Markovian model by acyclic phase-type distributions, and use this approximation to transform it to a Markovian maintenance model. Dimitrakos and Kyriakidis [137] consider a system consisting of a deteriorating installation that transfers raw material to a production unit, and a buffer built between the installation and the production to account for unexpected failures. The authors consider the problem of optimal preventive maintenance of the installation, under the assumption that the repair times follow some known continuous distributions. An efficient semi-Markov decision algorithm is proposed, which operates on the class of control limit policies. Chen and Trivedi [138] build a semi-Markov decision process for the maintenance policy optimization and present an approach for the joint optimization of inspection rate and maintenance policy. The authors describe a special case where the optimal policy is a dynamic threshold-type scheme with threshold value depending on the inspection rate.

Most of the state models assume that the degradation state of the system is fully observable through perfect inspection or monitoring. In other words, they assume that inspection and/or monitoring reveal the underlying state of the system with certainty. In some practical real-world cases, it is reasonable to consider the case were inspection only partially reflects this underlying state. In such cases, there exist probability distributions on the underlying degradation state of the system after taking observations. A common approach to handle this class of problems is to use partially observable Markov Decision Processes (POMDP).

2.2.2.3 Partially Observable Markov Process Models

In this class of models, maintenance policies are only made on the basis of noise-corrupted observations. Sources of such noise can include, for example, human error due to an inexperienced inspector, or an unreliable sensor. In a POMDP, there exist a set of probability distributions over the underlying deterioration states as stated by Monohan [139]. Makis and Jiang [140] formulate the replacement problem for a technical system which can be in one of *N* operational states or a failure state as an optimal stopping problem with partial observation. The problem is transferred to one with complete information by applying the projection theorem to a smooth semi-martingale process in the objective function. Afterwards, the dynamic equation is derived and analyzed in the piecewise deterministic Markov process stopping framework. This is used to find a replacement policy minimizing the long run expected average cost per unit time. White [141] studies the problem of optimally controlling a production process with countable state space. At each discrete time epoch, the three available decisions are: produce, inspect while producing, or repair the process. It is assumed that imperfect observations are received at both times of production and inspection. The author determines several results associated with the two-state case, which are sufficient for simple characterizations of an optimal policy. In [142], White derives bounds on the optimal cost function for a partially observable replacement problem. This is done by characterizing the structure of optimal strategies for completely observed case and the completely unobserved case as having a generalized control limit form. Sinuany-Stern *et al.* [143] suggest a simple heuristic decision rule to handle replacement type problems of large size based on the Howard solution of the fully observable version of the problem. The authors present a simulation experimental design to compare the performance of this heuristic relative to generic POMDP solution algorithms. Kuo [144] studies the joint machine maintenance and product quality control problem with an unobservable state of the production system. The authors include the timing of the sampling action and the sample size directly in the action space of the POMDP and derive some properties of the optimal value function that facilitates searching for the optimal maintenance and quality control problem. Rosenfield [145] presents a model of a deteriorating process with imperfect information. With each decision epoch, the decision maker decides whether to repair, inspect at a cost, or do nothing. The author shows that the optimal policy is an at-most-four-regions type policy. Maillart [146] examines the problem of adaptively scheduling perfect observations, imperfect observations, and maintenance actions for a multistate Markovian deteriorating system with obvious failures. Structural properties of the perfect observation case are established, and used in a heuristic to solve the imperfect observation problem. Smallwood and Sondik [147] discuss finite horizon POMDPs and present an algorithm to calculate the optimal control policy and payoff function. The results are illustrated by an example for the machine-maintenance problem. Sondik [148] later extends this work to the infinite horizon discounted cost criterion. Hopp and Kuo [149] formulate the maintenance problem of aircraft engine components subject to stress as a POMDP. The authors use this approach because cracks that characterize the deterioration of the components are not easily observable. Consequently, decisions are made on the basis of stress information collected via sensors. Optimal structured maintenance schedules are derives for the problem. Albright [150] examines conditions that guarantee monotonicity of the reward function and optimal actions in two-states POMDPs, and provides examples of maintenance systems where the results hold. Excellent reviews on POMDPS, applications and solution algorithms, are provided by Fernandez-Gaucherand *et al.* [152], White [151], and Monahan [139].

2.2.3 Condition Based Maintenance (CBM)

Reliability-based models and state models usually pay little attention to sensory information from condition monitoring. Reliability-based models rely on failure time distributions derived from reliability testing of components' populations. On the other hand, in most cases, state models model the transition between the degradation states of the system using arbitrary transition probability matrices. The literature on CBM utilizes condition monitoring information to determine optimal maintenance policies. Bloch and Geitner [153] state that the majority of machine failures are preceded by certain signs, conditions, or indicators that a failure was going to occur. Mann Jr. *et al.* [154] explore the benefits of CBM versus traditional reliability-based maintenance, and provide a brief review of research efforts in this area. Jardine *et al.* [155] provide an up-to-date and excellent review on the recent research and developments in diagnostics and prognostics of mechanical systems implementing CBM. Emphasis is put on models, algorithms, and technologies for data processing and maintenance decision making.

One approach of CBM is to trigger maintenance actions when the system's condition reaches some alarm or failure threshold. Christer and Wang [156] address a cost-effective condition monitoring policy of a production plant. A condition monitoring test is assumed in which the state of wear of a component is recorded as a (0,1) signal depending upon whether or not the wear is below a critical level. Aven [157] considers a unit subject to random deterioration, and bases the replacement of the system upon measurements of wear characteristics and damage inflicted on the system. Coolen *et al.* [158] presents a basic model for the economic evaluation and optimization of the interval between successive condition measurements when measurement are costly and cannot be made continuously.

Another approach involves using Proportional Hazard Models (PHM). Makis *et. al.* [159] blend PHM with vibration and oil debris analysis to model condition-based replacement decision problems. Koomsap *et al.* [160] integrate process control and CBM of manufacturing systems using a Weibull hazard function. In [161], Koomsap *et al.* use condition monitoring along with the Weibull hazard function to estimate the lifetime of a

CO_2 laser. Lin *et al.* [162] proposes the application of principal components proportional hazards regression model in condition-based maintenance. Principal component analysis is applied to the PHM covariates to reduce the number of variables in the model and eliminate possible colinearity between the covariates. Jardine *et al.* [163] discuss a work completed at Cardinal River Coals in Canada to improve the existing oil analysis condition monitoring program. The PHM is used to find the key condition variables relating to failures from among the 19 elements monitored. Jardine *et al.* [164] report the development of an optimal maintenance program for critical bearings on machinery in the food industry subject to vibration monitoring. The PHM is used to identify the risk curve, and then cost data is blended with risk to identify the optimal maintenance program.

Some CBM research efforts also focus on establishing structured policies based on condition monitoring information. Makis and Jardine [165] establish control limit replacement policies for deteriorating systems subject to condition monitoring using the proportional hazard model (PHM). Makis *et al.* [166] propose a CBM model when only partial information is available through a signal process with no monotone behavior. The optimal replacement policy is shown to be a control limit policy and an algorithm is developed to find the limit for an $\varepsilon-$optimal policy. Barbera *et al.* [167] discuss a condition based maintenance model for a two-unit system in series with exponential failures and fixed inspection intervals. A condition indicator variable for each unit is used to decide whether to repair the unit or to overhaul the whole system. After a maintenance action is performed, the condition indicator takes on its initial value. Lu *et al.* [168] investigate the structure of a discrete-time Markov deteriorating system monitored by multiple dependent monitors. The authors find that the expected optimal cost function over an infinite horizon is monotone given that the transition probability matrix is totally positive of order 2 and the conditional probability of the monitors having the property of weak multivariate monotone likelihood ratio. The optimal policy is shown to have an at-most-four-region structure.

In this research, we propose a mathematical framework that integrates various aspects of the discussed literature. In particular, we use condition monitoring sensory information to determine, and update, optimal replacement and spare parts ordering policies. We utilize reliability-based approaches to determine approximate decision policies. We also model the deterioration of single-unit systems as a non-stationary Markov process whose transition probability matrix is based upon sensory information from condition monitoring. This is used to establish monotone control limit replacement and spare parts ordering policies. The next chapters discuss these methodologies in detail. We start by presenting sensor-based degradation models for exponentially degrading components in the following chapter.

CHAPTER 3
SENSOR-BASED DEGRADATION MODELING

In this chapter, we present a degradation modeling framework for computing and updating the RLD of degrading components using real-time sensory signals. Our approach is to model the evolutionary path of the sensory-signals using a random coefficients model with deterministic and stochastic model parameters, and superimposed error terms. The stochastic parameters are assumed to follow some prior distribution, whose parameters can be estimated from a database of degradation signals acquired from accelerated degradation testing of an initial sample of components. This prior distribution can be used to compute an initial failure time distribution of the population. Real-time signals from monitoring each individual component are then used to update the predictive distribution of the future degradation signal using Bayesian techniques. This predictive distribution can then be used to revise the RLD of the component by evaluating the distribution of the time taken by the future degradation signal to cross a pre-determined failure threshold.

3.1 Degradation Model Development

Consider a component that degrades during its operation. Dedicated sensors are used to observe real-time signals at deterministic and equally spaces time epochs k, $k = 0,1,\ldots$, with t denoting the time between two consecutive observations. Define $\mathbf{S} = \{S_k, k \geq 0\}$ as a stochastic process, where S_k denotes the value (amplitude) of the degradation signal at time $t_k = t \cdot k, k = 0,1,\ldots$. This signal evolves according to a parametric model $S_k = h(t_k; \phi, \Theta) + \varepsilon(t_k)$. The term $h(\cdot)$ represents the functional form characterizing the evolution of the signal, S_k. The choice of this functional form depends on the type of component being modeled and can take a variety of forms (e.g. linear, exponential, etc.) depending on the type of component being modeled. The parameter ϕ is deterministic that captures constant degradation characteristics over the components' population, such as a fixed initial level of degradation. The parameter Θ is a stochastic parameter (or vector of parameters) that captures unit-to-unit variability, i.e. the random degradation characteristics of the individual components being monitored. The term $\varepsilon(t_k)$ is an error term used to model measurement noise and signal fluctuations.

In applications where preliminary degradation stages do not accelerate the degradation process, a linear functional form may be suitable. The wear of break pads on automotive wheels is a good example. Initial wear does not speed up subsequent wear of the brake pads. In contrast, the exponential functional form is more suitable for applications where the initial degradation level increases the rate of subsequent degradation. This type of degradation process is common in many mechanical systems, especially rotating machinery and rolling elements bearing applications. In bearings, the formation of spalls (pits) on the surface of bearing raceways is an initial form of degradation. These spalls become weak points and increase the rate of subsequent degradation [19, 97, 102].

Most previous research efforts assumed the error terms to be *iid* normal with mean zero and variance σ^2 across the population of components [19, 20, 47]. Some other research efforts used Brownian motion (BM) to model the error terms [12, 13, 21]. Brownian error terms are more appropriate for applications where successive error fluctuations in sensor readings are correlated. In this research, we focus on modeling components whose degradation signals evolve according to an exponential functional form, and error terms that follow a Brownian motion (BM) with mean zero and variance parameter σ^2. Brownian motion is a stochastic process $\{ W(t), t \geq 0 \}$ with the following properties [170]:

(1) If $t_0 < t_1 < \ldots < t_n$, then $W(t_0)$, $W(t_1) - W(t_0), \ldots, W(t_n) - W(t_{n-1})$ are mutually independent.

(2) If $s, t \geq 0$, then $W(s+t) - W(s) \in A$ (real values) and its probability is given by:

$$\Pr \{ W(s+t) - W(s) \} = \int_A (2\pi\, t)^{-1/2}\, \mathrm{e}^{-x^2/2t}\, dx \,.$$

(3) With probability one, $t \rightarrow W(t)$.

Properties 1 and 2 state that the process $W(t)$ has independent increments and that these increments are normally distributed with mean zero and variance t. Property 3 states that $W(t)$, $t \geq 0$, almost surely has continuous paths [170].

3.1.1 Random-Coefficients Degradation Model with Exponential Functional Form and Brownian Error Terms

Using the exponential functional form, the degradation signal is expressed as follows:

$$S_k = \phi + \theta \exp\left(\beta \cdot t_k + \varepsilon(t_k) - \frac{\sigma^2}{2} \cdot t_k \right)$$

$$= \phi + \left(\theta\, e^{\beta \cdot t_k} \right) \left(e^{\varepsilon(t_k) - \frac{\sigma^2}{2} \cdot t_k} \right) \tag{1}$$

where ϕ is a known constant, θ is a lognormal random variable, i.e., $\theta' = \ln(\theta)$ is normal with mean μ_0 and variance σ_0^2, β is a normal random variable, independent of θ, with mean μ_1 and variance σ_1^2, and $\varepsilon(t_k)$ is the error term that follows a BM with mean 0 and variance σ^2. For mathematical convenience, we work with the logarithm of the degradation signal. We define ℓ_k as the logarithm of degradation signal amplitude:

$$\begin{aligned}\ell_k &= \ln(S_k - \phi) = \ln\theta + \beta \cdot t_k + \varepsilon(t_k) - \frac{\sigma^2}{2} \cdot t_k \\ &= \theta' + \beta' \cdot t_k + \varepsilon(t_k) \qquad (2)\end{aligned}$$

where $\beta' = \beta - \frac{\sigma^2}{2}$. In expression (2), θ' and β' are independent and follow prior normal distributions $\pi(\theta')$ with mean μ_0 and variance σ_0^2, and $\pi(\beta')$ with mean μ_1' and variance σ_1^2, respectively. Note that $\mu_1' = \mu_1 - \frac{\sigma^2}{2}$. In practice, prior information about these distributions can be gathered by performing degradation testing on a sample of identical components. The resulting database of degradation signals are then used to estimate the parameters of the prior distributions. These prior distributions can be used to compute the failure time distribution of the component's population by evaluating the distribution of time at which the degradation signal crosses a predetermined failure threshold. Real-time condition monitoring signals are then used to update the parameters of the prior distributions and, subsequently, revise the RLD of the individual components. We discuss next a Bayesian methodology to update the prior distributions parameters given observed sensory signals.

3.1.2 Bayesian Updating of the Prior Distributions Parameters

We start by defining $\Delta_k \equiv \ell_k - \ell_{k-1}$, $k = 1,2,\ldots$, as the difference between the value of the degradation signal logarithm at times t_k and t_{k-1}, with $\Delta_0 = \ell_0$. Based on the properties of BM, Δ_k is normally distributed with mean 0 and variance $\sigma^2 t$. The prior distributions of the stochastic parameters are updated by computing the joint distribution of θ' and β' given the observed data, $\Pr(\theta', \beta' \mid \Delta_0, \ldots, \Delta_k)$. From Bayes theorem we have:

$$\Pr(\theta', \beta' \mid \Delta_0, \ldots, \Delta_k) \propto f(\Delta_0, \ldots, \Delta_k \mid \theta', \beta') \cdot \pi(\theta') \cdot \pi(\beta') \tag{3}$$

where $f(\Delta_0, \ldots, \Delta_k \mid \theta', \beta')$ is the likelihood function of $\Delta_0, \ldots, \Delta_k$ given θ' and β'. By the properties of BM, the error increments $\varepsilon(t_k) - \varepsilon(t_{k-1})$, are independent normal random variables. Therefore $f(\Delta_0, \ldots, \Delta_k \mid \theta', \beta')$ can be expressed as:

$$f(\Delta_0, \ldots, \Delta_k \mid \theta', \beta') = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{\mathrm{k}} \times \exp\left\{-\left(\frac{(\Delta_0 - \theta' - \beta' t_0)^2}{2\sigma^2 t_0} + \sum_{i=1}^{k}\left(\frac{(\Delta_i - \beta'(t_i - t_{i-1}))^2}{2\sigma^2 (t_i - t_{i-1})}\right)\right)\right\}. \tag{4}$$

Generally, however, θ' and β' are unknown. The posterior (updated) distribution of θ' and β' can be found by evaluating $\Pr(\theta', \beta' \mid \Delta_0, \ldots, \Delta_k)$ given the observed signals.

Proposition 1. The posterior distribution of the stochastic parameters θ' and β' given the observed data $\Delta_0, \ldots, \Delta_k$, $\psi(\theta', \beta')$, is bivariate normal with the following parameters:

$$\mu_{\theta'}(k, \ell_k) = \frac{(\ell_0 \sigma_0^2 + \mu_0 \sigma^2 t_0)(\sigma_1^2 kt + \sigma^2) - \sigma_0^2 t_0 (\sigma_1^2 \ell_k + \mu_1' \sigma^2)}{(\sigma_0^2 + \sigma^2 t)(\sigma_1^2 kt + \sigma^2) - t\sigma_0^2 \sigma_1^2}, \tag{5}$$

$$\mu_{\beta'}(k, \ell_k) = \frac{(\sigma_1^2 \ell_k + \mu_1' \sigma^2)(\sigma_0^2 + \sigma^2 t) - \sigma_1^2 (\ell_o \sigma_0^2 + t\mu_0 \sigma^2)}{(\sigma_0^2 + t\sigma^2)(\sigma_1^2 kt + \sigma^2) - t\sigma_0^2 \sigma_1^2}, \tag{6}$$

$$\sigma_{\theta'}^2(k) = \frac{\sigma^2 \sigma_0^2 t \left(\sigma_1^2 kt + \sigma^2\right)}{\left(\sigma_0^2 + t\sigma^2\right)\left(\sigma_1^2 kt + \sigma^2\right) - t\sigma_0^2\sigma_1^2}, \tag{7}$$

$$\sigma_{\beta'}^2(k) = \frac{\sigma^2 \sigma_1^2 \left(\sigma_0^2 + t_0\sigma^2\right)}{\left(\sigma_0^2 + t\sigma^2\right)\left(\sigma_1^2 kt + \sigma^2\right) - t\sigma_0^2\sigma_1^2}, \tag{8}$$

$$\rho(k) = \frac{-\sigma_0\sigma_1\sqrt{t}}{\sqrt{\left(\sigma_0^2 + t\sigma^2\right)\left(\sigma_1^2 kt + \sigma^2\right)}}. \tag{9}$$

Proof. Given the prior distributions $\pi(\theta')$ and $\pi(\beta')$, we can find the posterior distribution $\psi(\theta', \beta')$ as follows:

$$\Pr(\theta', \beta' \mid \Delta_0, \ldots, \Delta_k) \propto f(\Delta_0, \ldots, \Delta_k \mid \theta', \beta') \cdot \pi(\theta') \cdot \pi(\beta')$$

$$\propto \exp\left\{-\frac{(\Delta_0 - \theta' - \beta' t_0)^2}{2\sigma^2 t_0} - \sum_{i=1}^{k} \frac{(\Delta_i - \beta'(t_i - t_{i-1}))^2}{2\sigma^2 (t_i - t_{i-1})}\right\} \cdot \exp\left\{-\frac{(\theta' - \mu_0)^2}{2\sigma_0^2}\right\} \cdot \exp\left\{-\frac{(\beta' - \mu_1')^2}{2\sigma_1^2}\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\theta'^2\left(\frac{\sigma_0^2 + \sigma^2 t_0}{\sigma^2\sigma_0^2 t_0}\right) + \beta'^2\left(\frac{\sigma_1^2 t_k + \sigma^2}{\sigma^2\sigma_1^2}\right) - 2\theta'\left(\frac{\Delta_0 + \mu_0\sigma^2 t_0}{\sigma^2\sigma_0^2 t_0}\right) - 2\beta'\left(\frac{\sigma_1^2 \sum_{i=0}^{k}\Delta_i + \mu_1'\sigma^2}{\sigma^2\sigma_1^2}\right) + 2\theta'\beta'\left(\frac{1}{\sigma^2}\right)\right]\right\}. \tag{10}$$

Since the parameters θ' and β' follow a bivariate posterior normal distribution with means $\left(\mu_{\theta'}(k, \ell_k), \mu_{\beta'}(k, \ell_k)\right)$, variances $\left(\sigma_{\theta'}^2(k), \sigma_{\beta'}^2(k)\right)$, and correlation coefficient $\rho(k)$, defined in Proposition 1 above, then $\Pr(\theta', \beta' \mid \Delta_0, \ldots, \Delta_k)$ takes the form:

$$\Pr(\theta', \beta' \mid \Delta_0, \ldots, \Delta_k) = \frac{1}{2\pi\sigma_{\theta'}^2(k)\sigma_{\beta'}^2(k)(1-\rho^2)} \cdot \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(\theta' - \mu_{\theta''}(k, \ell_k))^2}{\sigma_{\theta''}^2} \right.\right.$$
$$\left.\left. - \frac{2\rho(k)(\theta' - \mu_{\theta''}(k, \ell_k))(\beta' - \mu_{\beta'}(k, \ell_k))}{\sigma_{\theta'}\sigma_{\beta'}} + \frac{(\beta' - \mu_{\beta'}(k, \ell_k))^2}{\sigma_{\beta'}^2} \right] \right\}. \tag{11}$$

By comparing coefficients in equations (10) and (11), we get the expressions of the posterior parameters of $\pi(\theta', \beta')$ in Proposition 1, where $\sum_{i=0}^{k} \Delta_k = \ell_k$, ■

3.1.3 Computing the Remaining Life Distribution

The updated posterior parameters in Proposition 1 can be used to update the predictive distribution of the future degradation signal. We define the random variable $\Gamma(t_k + \tau)$ as the logarithm of the degradation signal after τ time units. The predictive distribution of $\Gamma(t_k + \tau)$ is given in the following Proposition.

Proposition 2. The predictive distribution of the random variable $\Gamma(t_k + \tau)$ is normal with the following mean and variance:

$$\tilde{\mu}(t_k + \tau) = \ell_k + \mu_{\beta'}(k, \ell_k)\tau \tag{12}$$

$$\tilde{\sigma}^2(t_k + \tau) = \sigma_{\beta'}^2(k)\tau^2 + \sigma^2\tau. \tag{13}$$

Proof. From equation (2), we can express the difference increment $\Gamma(t_k + \tau) - \ell_k$ as:

$$\Gamma(t_k + \tau) - \ell_k = \beta'\tau + \varepsilon(t_k + \tau) - \varepsilon(\tau)$$

where, $\ell_k = \sum_{i=0}^{k} \Delta_k$. Therefore, given $\Delta_0, \ldots, \Delta_k$, the random variable $\Gamma(t_k + \tau)$ is follows a normal distribution with mean and variance:

$$\tilde{\mu}(t_k + \tau) = \ell_k + \mathrm{E}[\beta']\tau = \ell_k + \mu_{\beta'}(k, \ell_k)\tau$$

$$\tilde{\sigma}^2(t_k + \tau) = \tau^2\ \mathrm{V}[\beta'] + \mathrm{V}[\varepsilon(t + t_k) - \varepsilon(t_k)] = \sigma_{\beta'}^2(k)\tau^2 + \sigma^2\tau. \qquad ■$$

We can now evaluate the RLD by computing the distribution of the time until the degradation signals reaches the failure threshold ξ. Let T be a random variable denoting the remaining life of a partially degraded component given that we have observed $\Delta_0, \ldots, \Delta_k$. Therefore, its distribution is given by:

$$
\begin{aligned}
\Pr\left(T \le \tau \mid \Delta_0, \ldots, \Delta_k\right) &= \Pr\left(\Gamma\left(t_k + \tau\right) \ge \xi \mid \Delta_0, \ldots, \Delta_k\right), \\
&= 1 - \Pr\left(\Gamma\left(t_k + \tau\right) \le \xi \mid \Delta_0, \ldots, \Delta_k\right) \\
&= 1 - \Pr\left(Z \le \frac{\xi - \tilde{\mu}\left(t_k + \tau\right)}{\tilde{\sigma}\left(t_k + \tau\right)}\right) \\
&= \Phi\left(\frac{\tilde{\mu}\left(t_k + \tau\right) - \xi}{\tilde{\sigma}\left(t_k + \tau\right)}\right) \qquad (14)
\end{aligned}
$$

where $\Phi(\cdot)$ is the standard normal CDF. The conditional distribution of $\Gamma\left(t_k + \tau\right)$ can be used to show that $\lim_{t \to 0} F_{T|\Delta_0,\ldots,\Delta_k}(\tau) = 0$, therefore, the domain of T is $(0, \infty)$. Notice that, in reality, this is an approximation because it is possible that the fluctuations of the signal may have already crossed the failure threshold, signifying failure, prior to predicting the remaining life. We address this issue in Section 3.2.

3.1.4 Exponential Degradation Model with Dependent Stochastic Parameters

Although the authors in [19] assumed that the prior distributions of the stochastic parameters are independent, the updated (posterior) distribution was assumed to be bivariate normal. To avoid this inconsistency, we consider that case where θ' and β' are assumed to follow a prior bivariate normal distribution, $\pi(\theta', \beta')$ with means (μ_0, μ_1'), variances $\left(\sigma_0^2, \sigma_1^2\right)$ and correlation coefficient, ρ_0. This assumption results in different expressions for the updated parameters of the posterior distribution $\psi(\theta', \beta')$.

Proposition 3. Assume that the prior distribution of θ' and β' is a bivariate normal distribution $\pi(\theta', \beta')$. Then their posterior distribution given the observed data $\Delta_0, \ldots, \Delta_k$, $\psi(\theta', \beta')$, is bivariate normal with the following parameters:

$$\mu_{\theta'}(k, \ell_k) = \frac{\left(t_k + \mathrm{B}\sigma^2\right)\left(\dfrac{\ell_0}{t_0} + \mathrm{C}\sigma^2\right) - \left(\ell_k + \mathrm{D}\sigma^2\right)\left(1 + \mathrm{E}\sigma^2\right)}{\left(\dfrac{1}{t_0} + \mathrm{A}\sigma^2\right)\left(t_k + \mathrm{B}\sigma^2\right) - \left(\dfrac{1}{\sigma^2} + \mathrm{E}\left(\mathrm{E}\sigma^2 + 2\right)\right)}, \tag{15}$$

$$\mu_{\beta'}(k, \ell_k) = \frac{\left(\ell_k + \mathrm{D}\sigma^2\right)\left[\left(\dfrac{1}{t_0} + \mathrm{A}\sigma^2\right) - \left(1 + \mathrm{E}\sigma^2\right)\right]}{\left(\dfrac{1}{t_0} + \mathrm{A}\sigma^2\right)\left(t_k + \mathrm{B}\sigma^2\right) - \left(\dfrac{1}{\sigma^2} + \mathrm{E}\left(\mathrm{E}\sigma^2 + 2\right)\right)}, \tag{16}$$

$$\sigma_{\theta'}^2(k) = \frac{\left[t_k \sigma_1^2\left(1 - \rho_o^2\right) + \sigma^2\right]\left(\sigma^2 \sigma_o^2 t_0\right)}{\sigma^2\left(\sigma_o^2 + \sigma_1^2 t_k t_0 + \sigma^2 t_0 + 2\sigma_o \sigma_1 \rho_o t_0\right) - \left[\sigma_o^2 \sigma_1^2\left(1 - \rho_o^2\right)\left(t_0 - t_k\right)\right]}, \tag{17}$$

$$\sigma_{\beta'}^2(k) = \frac{\sigma_1^2 \sigma^2\left[\sigma_o^2\left(1 - \rho_o^2\right) + \sigma^2 t_0\right]}{\sigma^2\left(\sigma_o^2 + \sigma_1^2 t_k t_0 + \sigma^2 t_0 + 2\sigma_o \sigma_1 \rho_o t_0\right) - \left[\sigma_o^2 \sigma_1^2\left(1 - \rho_o^2\right)\left(t_0 - t_k\right)\right]}, \tag{18}$$

$$\rho(k) = \frac{\sigma^2 \rho_1 - \sigma_0\, \sigma_1\left(1 - \rho_o^2\right)\sqrt{t_0}}{\sqrt{\left[\left(\sigma_0^2\left(1 - \rho_o^2\right) + \sigma^2 t_0\right)\left(t_k\, \sigma_1^2\left(1 - \rho_o^2\right) + \sigma^2\right)\right]}}, \tag{19}$$

where $A = \dfrac{1}{\sigma_o^2\left(1 - \rho_o^2\right)}$, $B = \dfrac{1}{\sigma_1^2\left(1 - \rho_o^2\right)}$, $C = \dfrac{\mu_o}{\sigma_o^2\left(1 - \rho_o^2\right)} - \dfrac{\mu_1' \rho_o}{\sigma_o \sigma_1\left(1 - \rho_o^2\right)}$,

$D = \dfrac{\mu_1'}{\sigma_1^2\left(1 - \rho_o^2\right)} - \dfrac{\mu_o \rho_o}{\sigma_o \sigma_1\left(1 - \rho_o^2\right)}$, and $E = -\dfrac{\rho_o}{\sigma_o \sigma_1\left(1 - \rho_o^2\right)}$.

Proof. The proof goes along the same steps followed to establish Proposition 1. The likelihood function of $\Delta_0,\ldots,\Delta_k$ given θ' and β' is given by equation (4). The posterior (updated) distribution of θ' and β' is found by evaluating $\Pr(\theta',\beta' \mid \Delta_0,\ldots,\Delta_k)$ given the observed signals:

$$\Pr(\theta',\beta' \mid \Delta_0,\ldots,\Delta_k) \propto f(\Delta_0,\ldots,\Delta_k \mid \theta',\beta')\cdot \pi(\theta',\beta')$$

$$\propto \exp\left\{-\frac{\Delta_0-\theta'-\beta' t_0}{2\sigma^2 t_0} - \sum_{i=1}^{k}\frac{(\Delta_i-\beta'(t_i-t_{i-1}))^2}{2\sigma^2(t_i-t_{i-1})}\right\}$$
$$\times\left\{-\frac{1}{2(1-\rho_o^2)}\left[\frac{(\theta'-\mu_o)^2}{\sigma_o^2}-\frac{2\rho(\theta'-\mu_o)(\beta'-\mu_1')}{\sigma_o\sigma_1}+\frac{(\beta'-\mu_1')^2}{\sigma_1^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma^2 t_0}+A\right)\theta'^2+\left(\frac{t_k}{\sigma^2}+B\right)\beta'^2-2\theta'\left(\frac{\Delta_0}{\sigma^2 t_0}+C\right)\right.\right.$$
$$\left.\left.-2\beta'\left(\frac{\sum_{i=0}^{k}\Delta_i}{\sigma^2}+D\right)+2\theta'\beta'\left(\frac{1}{\sigma^2}+E\right)\right]\right\} \tag{20}$$

where the terms A, B, C, D, and E are given in Proposition 3 above. Since θ' and β' follow a bivariate posterior normal distribution with means $(\mu_{\theta'}(k,\ell_k),\mu_{\beta'}(k,\ell_k))$, variances $(\sigma_{\theta'}^2(k),\sigma_{\beta'}^2(k))$, and correlation coefficient $\rho(k)$, then $\Pr(\theta',\beta' \mid \Delta_0,\ldots,\Delta_k)$ takes the form given in equation (11). By comparing coefficients in (11) and (20), we obtain the expressions for the posterior parameters given above. ■

The predictive distribution of $\Gamma(t_k+\tau)$ for this model is similar to the model with independent parameters, and is normal with mean and variance given in Proposition 2. Hence, the RLD of the component can be computed using equation (14).

3.2 Conservative Lower Bound on the Mean Remaining Life

The remaining life distributions computed using the methods presented in the previous Section are approximations that work well if the variance of the error terms are small [19]. We notice that the RLD was estimated as the distribution of the time it takes the trajectory of the degradation signal to cross the failure threshold given an observed degradation signal. As mentioned earlier, in reality this is an approximation. The RLD computed in equation (14) does not represent the first passage time distribution of the degradation signal to the failure threshold. In cases where the variance of the error terms is large, the signal fluctuations are also large, and these approximations can be significantly imprecise.

Furthermore, the RLDs expressed in equation (14) are similar to the Bernstein distribution [171]. Tere do not exist closed form expressions for their moments. These moments, such as the expected remaining life and other parameters like the variance, are of great importance in developing sensor-based decision methodologies for maintenance management and spare parts logistics as will be discussed in the coming chapters. The need arises to input these quantities into other algorithms for maintenance optimization, and the absence of closed form expressions for them causes significant difficulties, especially from the standpoint of computational efficiency. For example, Elwany and Gebraeel [37, 101] estimated the remaining life using the median of the RLD that had to be computed numerically.

In this section, we identify closed-form expressions for a conservative lower-bound of the mean remaining life. To do this, we use the properties of the Brownian error terms, and show that the Inverse Gaussian (IG) distribution provides a good approximation of the RLD, and its mean represents a lower bound on the mean remaining life. The IG distribution is widely used to model the failure time of components and systems. It characterizes the first passage time distribution of BM with positive drift, which is used to model degradation in many research efforts. For example, Doksum and Hoyland [12] model the accumulated decay of cable insulation for units subject to accelerated testing under variable stress levels. Accumulated decay is modeled as a BM with drift and the resulting failure time follows the IG distribution. Whitmore and Schnkelberg [13] model the degradation of self-regulating heat cables subject to high stress reliability testing as a BM with constant rate of degradation. Pettit and Young [172] use a combination of failure time and degradation data from testing a sample of components to obtain the failure time distribution of components' populations. Brownian motion with drift and diffusion is used to model degradation. A Bayesian approach is used to compute the posterior densities of the drift and diffusion given data sets obtained from reliability testing for a fixed duration. Elsayed and Liao [3] propose a Geometric BM degradation rate model that utilizes field data to estimate reliability. Other works that model degradation using BM and the IG distribution include [173-177]. Excellent reviews on the IG distribution, its applications, properties, and estimation of parameters is provided by Chhikara *et al.* [178] and Seshdari [179].

3.2.1 First Passage Time Distribution of Brownian motion with Positive Drift

Let $\{X(t),\ t > 0\}$ be a BM process with drift $\delta > 0$ and variance parameter σ^2, having the following form:

$$X(t) = \delta\ t + W(t) \tag{21}$$

where $W(t)$ is a standard BM with mean zero and variance parameter σ^2. Then $X(t)$ has the following properties:

1. $X(0)=0$.

2. The increments $X(t_j)-X(t_i)$, $t_j > t_i$ are independent and identically normally distributed with mean $\delta(t_j - t_i)$ and variance $\sigma^2(t_j - t_i)$.

Given some critical level ξ, let $\nu = \xi/\delta$, $\gamma = \xi^2/\sigma^2$, and define the random variable T to be the first passage time of $X(t)$ to ξ. Then T has an IG distribution with mean ν and shape parameter γ, $IG(\nu,\gamma)$, with the following PDF [178]:

$$f_T(t;\nu,\gamma) = \sqrt{\frac{\gamma}{2\pi t^3}} \exp\left\{-\frac{\gamma}{2\nu^2}\frac{(t-\nu)^2}{t}\right\} \qquad t,\nu,\gamma > 0 \tag{22}$$

The IG has very useful properties such as the existence of closed form expressions for the probability density function (PDF), cumulative distribution function (CDF), moment generating function (MGF), and well defined statistics such as the population mean, mode, and variance. Table 1 summarizes these properties of the inverse Gaussian Distribution [178, 179]. We show next how to use the mean of the IG as a conservative lower bound on the mean remaining life for the exponential degradation model with Brownian errors.

Table 1 Summary of the IG distribution properties

Parameters	$\nu > 0$ Location Parameter $\gamma > 0$ Shape Parameter
Support	$t \in (0, \infty)$
PDF	$\sqrt{\frac{\gamma}{2\pi t^3}} \exp\left\{-\frac{\gamma}{2\nu^2}\frac{(t-\nu)^2}{t}\right\}$
CDF	$\Phi\left(\sqrt{\frac{\gamma}{t}}\left(\frac{t}{\nu}-1\right)\right)$ $+\exp\left\{\frac{2\gamma}{\nu}\right\}\Phi\left(-\sqrt{\frac{\gamma}{t}}\left(\frac{t}{\nu}+1\right)\right)$ Where $\Phi(\cdot)$is the CDF of a standard normal random variable
Mean	ν
Variance	$\frac{\nu^3}{\gamma}$
Mode	$\nu\left[\left(1+\frac{9\nu^2}{4\gamma^2}\right)^{\frac{1}{2}}-\frac{3\nu}{2\gamma}\right]$
MGF	$\exp\left\{\left(\frac{\gamma}{\nu}\right)\left[1-\sqrt{1-\frac{2\nu^2 t}{\gamma}}\right]\right\}$

3.2.2 Conservative Lower-Bound on the Mean Remaining Life

We start by re-writing the logged degradation signal model in equation (2) as:

$$\ell_k - \theta' = \beta' t_k + \varepsilon(t_k) \tag{23}$$

and noticing the similarity between expression (23) and BM with positive drift in (21). Hence, at each updating epoch k, we can approximate the first passage distribution of the degradation signal to the failure threshold ξ with an IG distribution with mean $\eta(k)/\omega(k)$ and shape parameter $\eta(k)/\sigma^2$ defined as:

$$\eta(k) = \begin{cases} \ln(\xi) - \mu_o & k = 0, \\ \ln(\xi) - \ell_k & k = 1,2,\ldots\ldots \end{cases}, \tag{24}$$

$$\omega(k) = \begin{cases} \mu_1' & k = 0, \\ \mu_{\beta'}(k, \ell_k) & k = 1,2,\ldots\ldots \end{cases}. \tag{25}$$

In other words, the mean of the first passage time distribution to the failure threshold ξ, which is IG, is expressed as $(\ln(\xi) - \ell_k)/\beta'$, where $(\ln(\xi) - \ell_k)$ represents the critical level at time t_k and β' is the drift of the Brownian motion at time t_k. Since β' is a random variable, we use its expectation, $\mu_{\beta'}(k, \ell_k)$, to estimate the updated drift at each updating epoch. We state the following Proposition:

Proposition 4. The mean of the IG distribution, $\eta(k)/\omega(k)$, is a conservative lower bound on the mean remaining life.

Proof. We know that for any positive random variable, $Y > 0$, the following inequality holds (Jensen's inequality [180]);

$$E\left[\frac{1}{Y}\right] \geq \frac{1}{E[Y]} \tag{26}$$

therefore, for each updating epoch, we can write:

$$E\left[\frac{1}{\beta'}\right] \geq \frac{1}{E[\beta']} \Rightarrow E\left[\frac{\ln(\xi) - \ell_k}{\beta'}\right] \geq \frac{\ln(\xi) - \ell_k}{E[\beta']} = \frac{\ln(\xi) - \ell_k}{\mu_{\beta'}(k, \ell_k)} = \frac{\eta(k)}{\omega(k)} \tag{27}$$

■

3.3 Case Study for Rotating Machinery Application

We present a case study to validate the proposed degradation modeling framework and assess its performance in predicting failures accurately. In this case study, real-world vibration-based degradation signals acquired from a rotating machinery application are used to demonstrate our degradation model. An experimental testing rig is used to perform accelerated degradation tests on several rolling element thrust bearings. The bearings are installed into the testing chamber and are run until failure. During each test, the bearing's degradation state is monitored using vibration signals.

There are two primary reasons for choosing bearings to implement and validate the degradation model; first the exponential functional form is well suited for modeling bearing degradation. This assumption is supported by existing literature on bearing condition monitoring and degradation modeling, such as Harris [182], Shao and Nezu in [181] and Gebraeel *et al.* in [19]. Second, the relatively low cost of thrust bearings provide the capability of high volume destructive testing for validation purposes.

3.3.1 Degradation of Rolling Elements Thrust Bearings

Rolling bearings are used to support rotating machine components. They are composed of two hardened steel raceways. One of the races is usually connected to the rotating part of the machine (the inner race in radial bearings and upper race in thrust bearings). The second race is fixed to a stationary housing (outer race in radial bearings and lower race in thrust bearings). Rolling elements (usually steel balls, cylinder, and needles) found in between the two races facilitate the relative rotational motion. Although there are several physical phenomena that characterize bearing degradation, we focus our attention on bearing vibration.

3.3.2 Experimental Test Rig for the Degradation Testing of Bearings

Because bearing degradation is time-consuming, investigating the natural failure of several test bearings is a challenging task. Several researchers have resorted to introducing artificial defects as one way to accelerate the failure process. However, we believe that artificially induced defects may interfere with the bearing's natural degradation process. As a result, an experimental testing rig is specifically designed and fabricated to perform accelerated degradation tests on thrust ball bearings (Figure 3).

Each test bearing is placed in a testing chamber where the lower race of the bearing is fixed to a stationary housing and its upper race is fastened to a rotating shaft. Each bearing is run until failure at a constant rotational speed of 2200 rpm and is subjected to a constant load of 200 lbs. The testing chamber is connected to a lubricating system that provides continuous cooling and lubrication during each test run. Bearings degradation begins with the creation of subsurface cracks within the steel raceways and culminates in the formation of spalls (pits) on the surface of the races. The passage of rolling elements (steel balls) over the spalls excites defective vibration frequencies unique to the bearing. These frequencies are computed using physical formulas that are functions of the bearing geometry, rotational speed, and number of rolling elements [182].

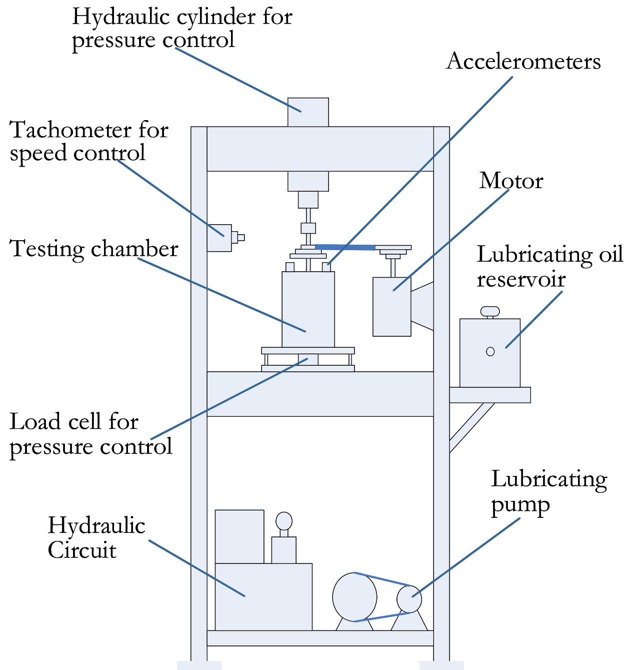

Figure 3 Experimental test rig for monitoring bearing degradation.

For the thrust ball bearings used in this case study, the formulas reduce to the following expressions:

1. The Ball-Passing Frequency (BPF) is the fundamental defective frequency and is excited by a defect on the surface of the bearing's raceways:

$$BPF = \frac{1}{2} \times z \times \frac{RPM}{60} \tag{28}$$

2. The Fundamental Train Frequency (FTF) is related to the formation of cracks in the bearing's cage that holds the rolling elements in place:

$$FTF = \frac{1}{2}\frac{RPM}{60} \tag{29}$$

3. The Ball Spin Frequency (BSF) is excited due to a defective rolling element (chipped steel ball):

$$\mathrm{BSF} = \frac{1}{2} \times \frac{\mathrm{RPM}}{60} \times \frac{d_c}{d_r} \tag{30}$$

where, d_r is the diameter of the rolling elements; d_c is the diameter of the cage, (d_{out} + d_{in})/2, z is the number of rolling elements, and RPM is the rotational speed in revolutions per minute. Usually, the first signs of bearing degradation manifest themselves through the excitation of the BPF. As degradation progresses over time, harmonic frequencies, integer multiples of the BPF, begin to appear in the vibration spectrum. Accelerometers attached to the testing chamber are used to capture the vibration signals created during each test run. Time varying vibration signals are acquired using a data acquisition program designed in LABVIEW. The time domain signals are transformed into frequency domain using standard Fast Fourier Transform (FFT). Figure 4 shows an example of the evolution of a sample of vibration spectra from the beginning of a test until bearing failure.

The amplitude of the defective frequency and its harmonics are correlated with the severity of the bearing's degradation. This is clear by observing the evolution of the vibration spectra along the "Percentage of Life Accomplished" axis in Figure 4, where 0% corresponds to a brand new bearing and 100% corresponds to a bearing's vibration spectrum at the point of failure. This observation was used to construct a vibration-based degradation signal. There are several possible ways to develop a degradation signal. In this paper, we use the average amplitude of the BPF and its first six harmonic frequencies to develop a vibration-based degradation signal. Higher order harmonics were observed to behave irregularly; consequently we limited ourselves to the first six harmonics.

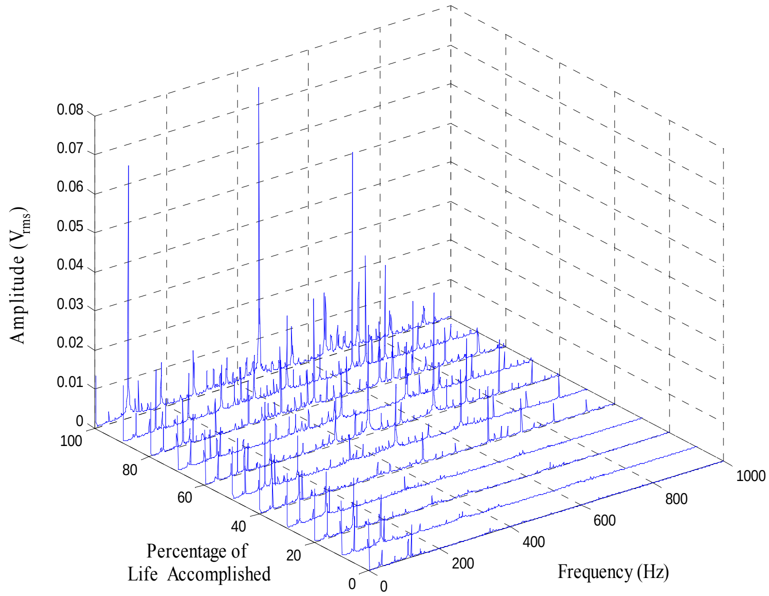

Figure 4 Evolution of the vibration spectra of a degrading bearing over its service life.

Figure 5 presents an example of the vibration-based signal for one of the test bearings. We would like to emphasize that each point on the signal corresponds to an aggregation of the information in one single vibration spectrum. Recall that a vibration spectrum for a single test bearing is acquired every 2 minutes for the entire duration of the test run, until failure. The degradation signal used in this work is similar to the degradation signal used in [19, 20, 101]. It consists of two distinct parts. The first is characterized by a flat region (Phase-I), which typically corresponds to non-defective bearing operation. This phase extends from the beginning of the test until the first instance of spall formation. At the instance of spall formation, a noticeable spike in the amplitude of the degradation signal. The time associated with this spike is referred to as the “first defect time”. The second part of the degradation signal, phase-II, begins from the first defect time, and continues until bearing failure, i.e., until the signal reaches the pre-specified failure threshold. This phase is characterized by an increasing trend in the degradation signal significant due to the signals noise associated with the randomness of the degradation process. This phase corresponds to a partially degraded system that has not failed yet.

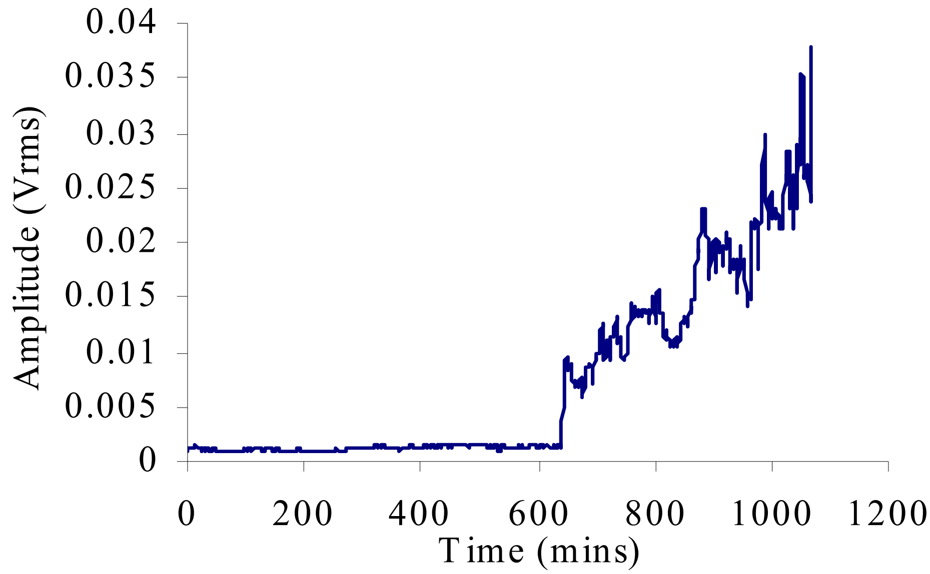

Figure 5 Vibration-based degradation signal.

The failure threshold is defined using the root mean square (RMS) of the overall vibration acceleration. According to industrial standards for machinery vibration, ISO 2372, 2.0-2.2 Gs represents a 'vibration-based danger level' for applications involving general-purpose machinery. For the degradation signal developed in this paper, we define the failure threshold as the amplitude of the degradation signal corresponding to 2.2 Gs of overall vibration. After observing several degradation signals, the failure threshold of the degradation signals was identified as 0.025 V_{rms} (root mean Square volts). Figure 6 shows two sample thrust ball bearings. The bearing on the right is new, while the one on the left is a failed bearing. We would like to point out that at a threshold of 0.025 V_{rms}, the groove was observed to have spread along the entire surface of the bearing's raceway.

Figure 6 A new bearing (right) and a failed bearing (left).

3.3.3 Model Implementation

The experimental test rig was used to test 50 bearings to failure. The first set of 25 bearings was used to build a database of degradation signals. This database was used to estimate the parameters of the prior distribution. The second set of 25 bearings was used for validation purposes as described next.

3.3.3.1 Estimating the Prior Distributions Parameters

The assumption that the error terms follow a BM requires that $\varepsilon(t_0)=0$, thus the value of the initial degradation is given by $\ell_0=\theta'$. To estimate β', we utilize the property of BM that the error increments are independent. We define the random variables R_k as follows:

$$R_k=\frac{\ell_k-\ell_{k-1}}{t_k-t_{k-1}} \quad k=0,12,\ldots,\frac{t_f}{2} \tag{31}$$

where t_k-t_{k-1} is 2 minutes, for this example, and t_f is the observed failure time of the bearing. The random variables R_k are independent *iid* with mean β'. We use $\overline{R}$ to estimate β' for each bearing. The values of θ' and β' for the 25 test bearing are then used to estimate the prior means, variances, correlation coefficient, and the variance parameter of the Brownian error terms. The computed values of these parameters are $\mu_0=-6.031$, $\mu_1'=8.061\times10^{-3}$, $\sigma_0^2=0.346$, $\sigma_1^2=1.034\times10^{-5}$, $\rho_0=-0.3464$, and $\sigma^2=0.0073$. Recall that the constant deterministic parameter ϕ represents a fixed initial level of degradation for the whole population, and was observed to be $\approx 0.002\mathrm{V_{rms}}$. We set ϕ equal to zero, without loss of generality, for simplification.

3.3.3.2 Testing the Model Assumptions

Recall that we modify the assumption in the original model by Gebraeel *et al.* [19] that θ' and β' follow independent prior distributions. Assuming dependence between θ' and β' is justified by the prior correlation coefficient, $\rho_0=-0.3464$. There is no explicit goodness-of-fit test for the bivariate normal distribution. However, there are consequences that must be satisfied in order not to reject the assumption [183]:

1. The marginal distribution of each random variable has to follow a univariate normal distribution.

2. Given that the data consists of pairs of observations, (θ'_i, β'_j), where $i, j = 1, \ldots, 25$, roughly 50% of the sample observations $\mathbf{x} = [\theta'_i, \beta'_j]$ must lie within the contour of an ellipse defined by:

$$(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \le \chi_2^2(0.5) \tag{32}$$

where μ is estimated by $\bar{x}$, and Σ^{-1} is estimated by S^{-1} with S being the sample covariance matrix.

3. A plot of the squared generalized distances given by equation (33) versus the corresponding chi-square percentiles has to follow a straight line.

$$d_j^2 = (\mathbf{x}_j - \bar{\mathbf{x}})^T S^{-1} (\mathbf{x}_j - \bar{\mathbf{x}}), \quad j = 1,2,\ldots,n \tag{33}$$

The three consequences were evaluated and found to hold true for the values of θ' and β'. Consequently, the assumption that θ' and β' follow a bivariate normal distribution was not rejected.

The error term, $\varepsilon(t_k)$, is assumed to follow a Brownian motion with mean zero and variance parameter σ^2. By properties of BM, the increments $\varepsilon(t_k) - \varepsilon(t_{k-1})$ are independent and normally distributed with mean zero and variance $\sigma^2 t$. Figure 7 shows a plot of the computed error increments. The mean and of these increments was computed to be equal to $9.48 \times 10^{-6} \approx 0$, justifying the assumption.

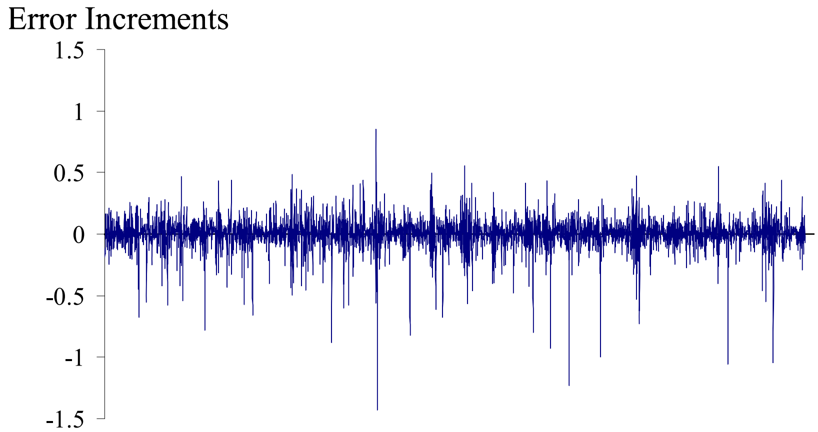

Figure 7 Plot of the computed error increments.

3.3.3.3 Updating the RLD

The second group of 25 validation bearings, 26 to 50, was used to evaluate the prediction accuracy of the model. The bearings were run-to-failure under the same loading and operating conditions. Every two minutes a vibration acquisition was performed, and the degradation signal evaluated and used to update the RLD and predict the remaining life of the bearing. We evaluated the RLD using two methods:

1. Method I – Approximate expressions with independent stochastic parameters: we use the exponential degradation model with independent stochastic parameters presented in Section 3.1.2. Each time a signal is acquired, the parameters of the prior distributions are updated using equations (5-9), and the RLD is computed using equation (14).
2. Method II – Closed-form expressions with dependent stochastic parameters: we use the exponential degradation model with dependent stochastic parameters

presented in Section 3.2. Each time a signal is acquired, the parameters of the prior distributions are updated using equations (15-19), and the RLD is approximated by the IG distribution. At each updating epoch, the updated posterior mean, $\mu_{\beta'}(k, \ell_k)$, was taken to be the updated degradation signal's trajectory (drift).

For demonstration, Figure 8 and Figure 9 display the updated CDF and PDF, respectively, of the RLD at various degradation percentiles of Bearing #50 using Method II.

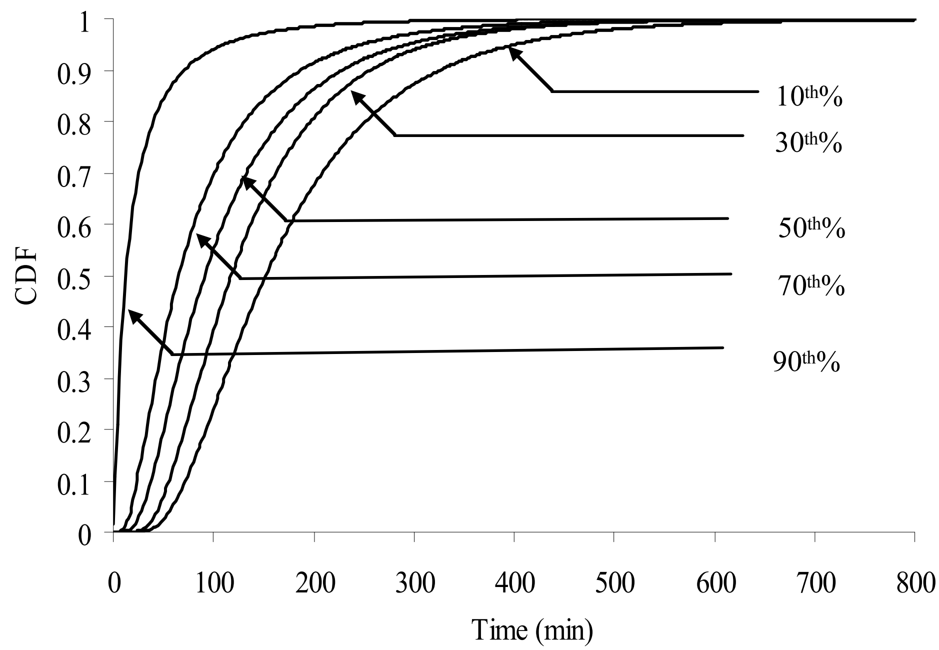

Figure 8 Updated CDF of Bearing #50 at different degradation percentiles.

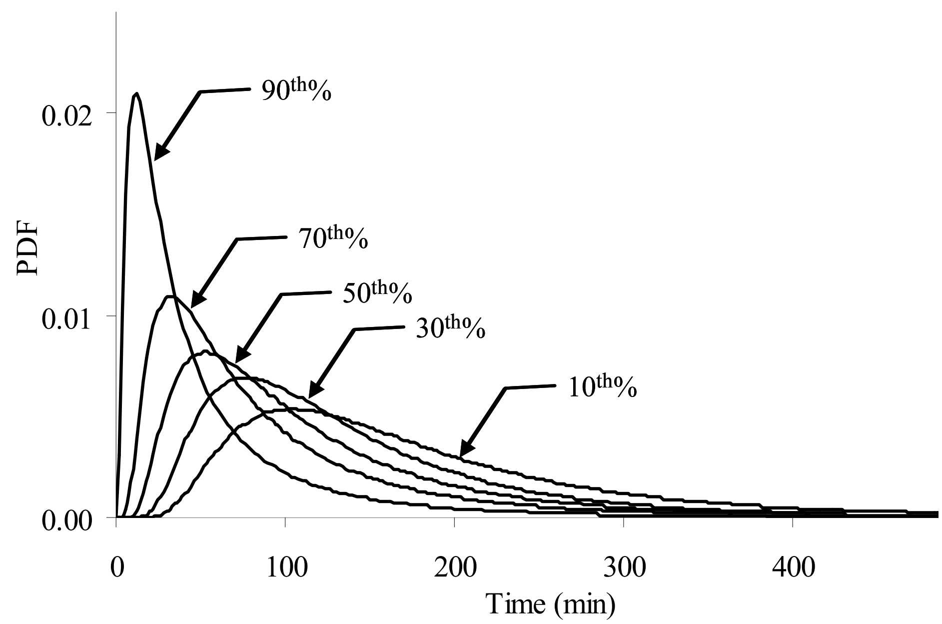

Figure 9 Updated PDF of Bearing #50 at different degradation percentiles.

3.3.3.4 Implementation Results

After the failure time was observed for a given bearing, we computed a prediction error for each predicted remaining life evaluated at each updating epoch using:

$$D_k^j = \frac{\left|\left(p_k^j + \tilde{t}_k^{\,j}\right) - t_N^j\right|}{t_N^j} \times 100 \tag{34}$$

where, D_k^j is the percentage prediction error associated with bearing j, computed at epoch k; t_N^j is the actual observed failure time of bearing j; p_k^j is the current total operating time of bearing j at epoch k. This is obtained by summing the predicted remaining life and the total time the bearing has been in operation, where $\tilde{t}_k^{\,j}$ is the remaining life estimator at the k^{th} updating epoch. The prediction errors are then grouped by degradation percentile. We define a degradation percentile as the percentage duration in the degradation phase, given that we have observed the bearing's failure time.

For Method I, the median of the RLD is used to estimate the remaining life, $\tilde{t}_k^{\,j}$, at each updating epoch. This is because the RLD computed in Method I does not have closed-form moments as discussed previously. Using the median of the RLD as an estimate of the remaining life is common practice [57], especially in cases where the RLD is skewed, as demonstrated in Figure 9. The median is computed numerically as the value of the remaining life at which the CDF is equal to 0.5. Figure 10 displays the 95% CI of the prediction errors for the 25 validation bearings (bearings 26 to 50) using Method I.

For Method II, the mean remaining life is used to estimate the remaining life, $\tilde{t}_k^{\,j}$. At each updating epoch, the conservative lower bound of the mean remaining life is computed as $\left(\ln(\xi) - \ell_k\right) / \mu_{\beta'}(k, \ell_k)$. For consistency, when comparing the results with Method I, we perform the error analysis for Method II using the median to estimate the remaining life. Figure 10 and Figure 11 display the 95% CI of the prediction errors for the same set of validation bearings using Method II. The error analyses are based on the mean remaining life and the median remaining life, respectively.

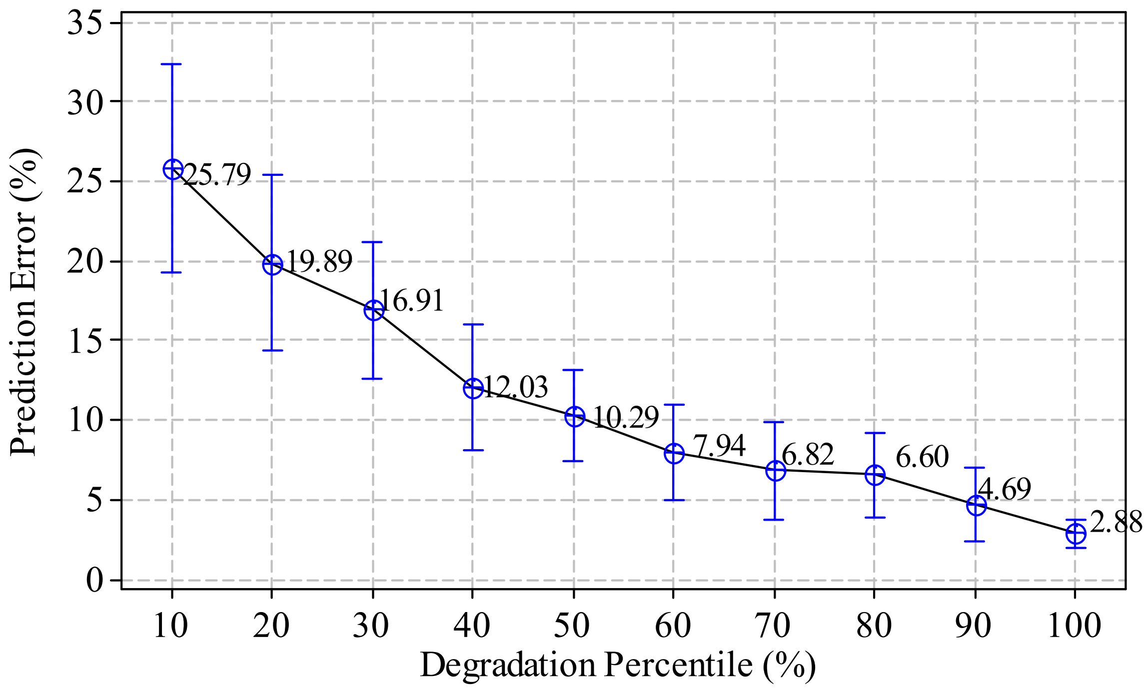

Figure 10 Method I - 95% CI for the prediction error bas on the median.

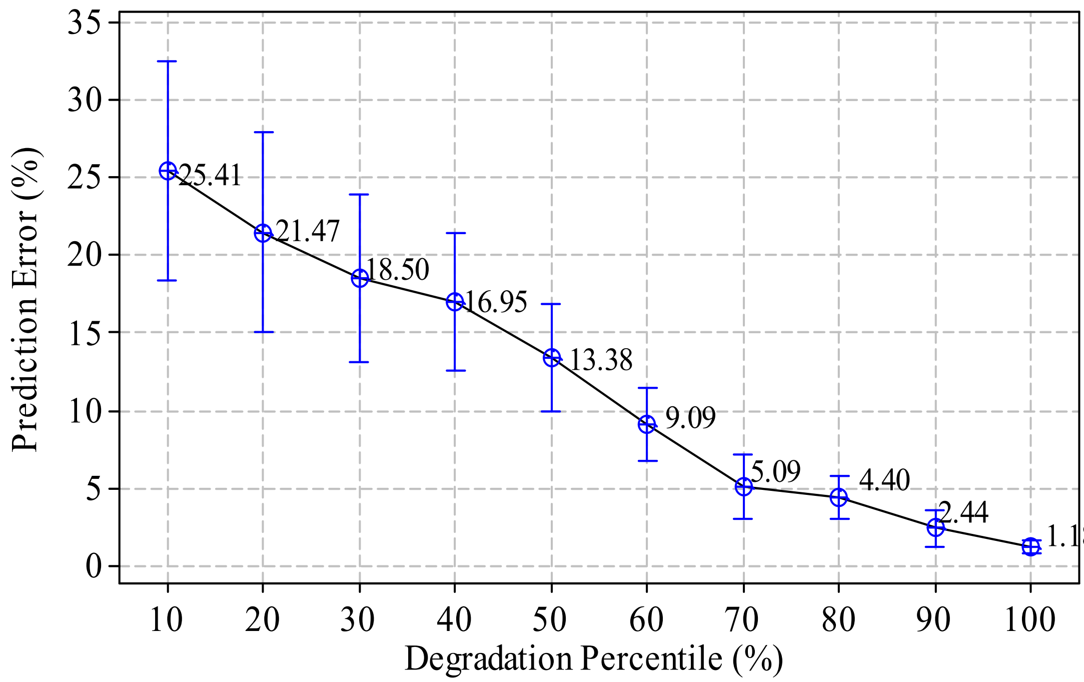

Figure 11 Method II - 95% CI for the prediction error based on the median.

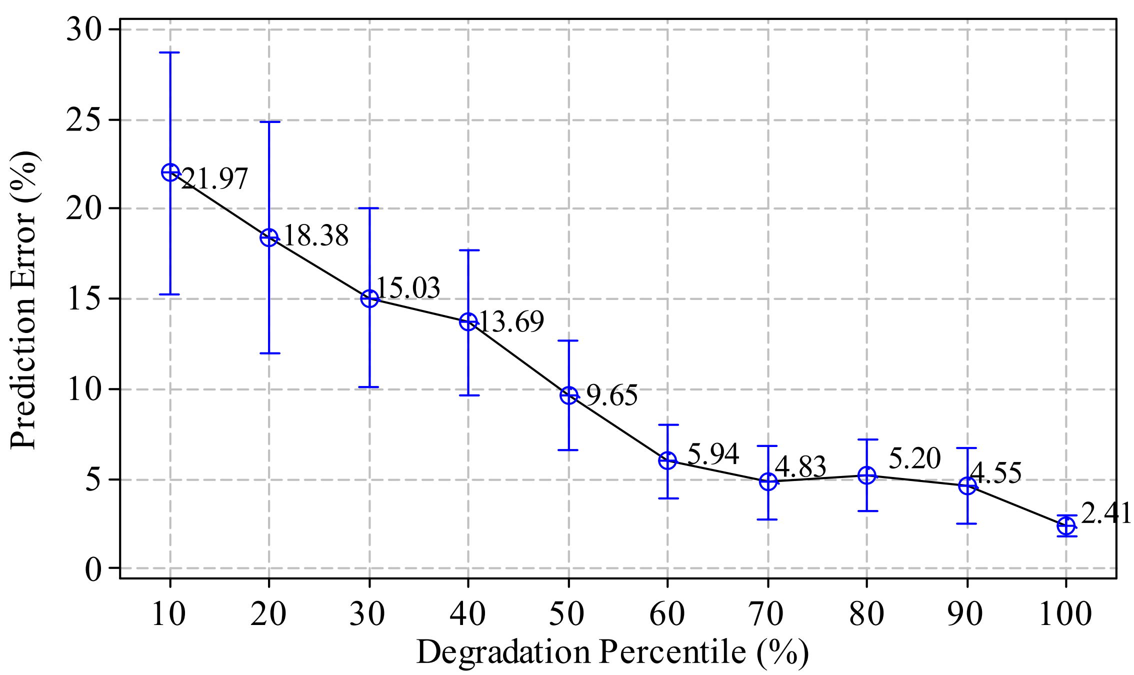

Figure 12 Method II - 95% CI for the prediction error based on the mean.

The prediction accuracy of both methods is comparable. We notice that Method II, in which the stochastic model parameters are assumed to be dependent, provides better prediction accuracy at late degradation percentiles (70th percentile onwards) compared to Method I. This is favorable because the risk of failure is expected to be high at late degradation percentile after degradation has progressed.

We also notice that, for Method II, using the median as an estimate of the remaining life results in better prediction accuracy compared to using the mean. The benefit of using the mean is the existence of a closed-form expression to compute it, unlike the median. This facilitates computations and is suitable for real-time decision making applications.

3.3.3.5 Benchmarking

Our prediction results are benchmarked against two other procedures for predicting failure:

1. Benchmark I is based on the work done by Gebraeel [20]. The degradation model assumes that the stochastic model parameters are jointly distributed and follow a bivariate normal distribution. However, the error terms are assumed to be *iid* normal with mean 0 and variance σ^2 rather than to follow a BM.
2. Benchmark II is based on the degradation model proposed by Lu and Meeker [16]. This benchmark demonstrates the importance of sensory-updating using real-time degradation information. In [16], the path of the degradation signal is still assumed to follow an exponential functional form. However, no updating of the remaining life distribution is performed based on condition monitoring information. Thus, the

degradation model is based entirely on the degradation characteristics of the sample bearings (bearings 1 to 25) used to estimate the prior parameters.

The 95% CI for prediction errors of the same set of 25 validation bearing are outlined in Figure 13 and Figure 14 for Benchmark I and Benchmark II, respectively. It is evident that the methods presented in this research provide considerably better prediction accuracy. This offers a powerful tool for optimizing maintenance policies. In the next chapters, we integrate the sensor-based degradation modeling framework presented in this chapter with maintenance decision models.

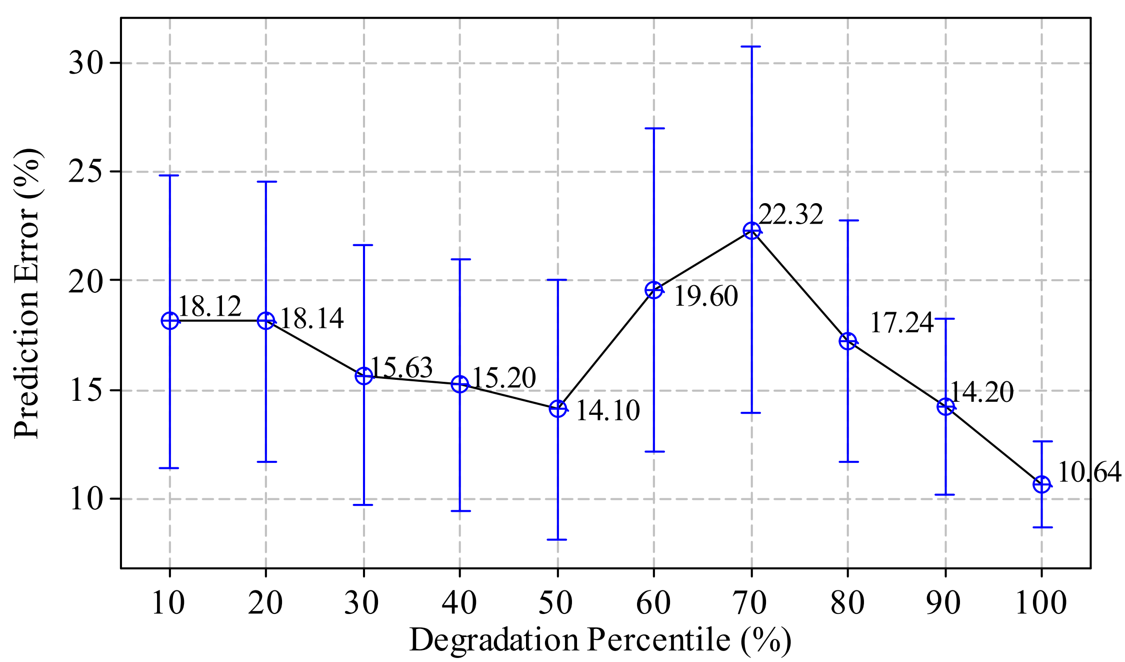

Figure 13 Benchmark I (Gebraeel [20]) - 95% CI for the prediction error.

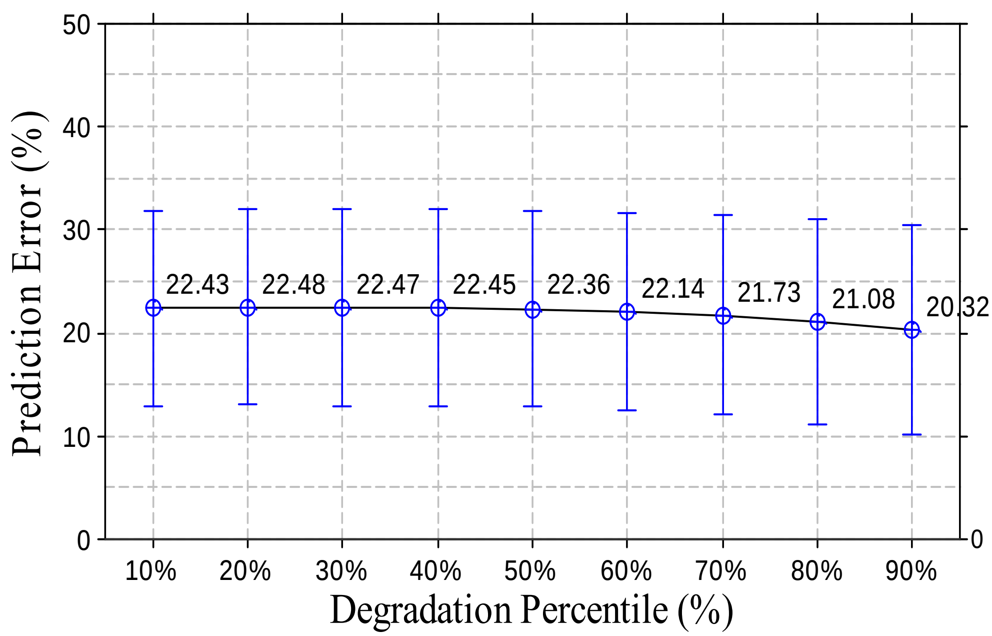

Figure 14 Benchmark II (Lu and Meeker [16]) - 95% CI for the prediction error.

CHAPTER 4
SENSE-AND-RESPOND LOGISTICS FRAMEWORK

In this chapter, we develop sensor-driven prognostic models for supporting component replacement and spare parts inventory decision making. This is achieved through presenting a mathematical framework for integrating degradation-based sensory data streams from condition monitoring with high-level logistical decision models. The integration provides an effective “Sense and Respond” architecture for decision making. In particular, the sensory-updated RLDs computed using sensor-based degradation models discussed in Chapter 3 are integrated with replacement and spare parts inventory models in place of the traditional failure time distributions. Each time we monitor a component, the signals are used to update the remaining life and, in turn, update any previous replacement or inventory decisions.

4.1 Traditional Reliability-based Replacement and Spare Parts Inventory Model

We start by reviewing the work by Armstrong and Atkins [42], which represents a good example of the traditional approach for determining replacement and spare parts inventory policies. The authors consider a single-unit system with room to store only one spare part. The component is subject to random failure with a failure time probability density function (PDF) $f(\cdot)$ and cumulative density function (CDF) $F(\cdot)$. Each time the component fails, the system incurs a failure cost. This cost is usually high and includes cost of corrective maintenance, labor, lost production, etc. In the event that the component is replaced according to a planned schedule, the system incurs a planned replacement cost, which is the regular cost of preventive maintenance, labor, and cost of the part itself. Typically, the planned replacement cost is less than the cost incurred due to sudden failure. Whether planned or failure replacements occur, it is necessary to have a spare part available in stock in order to perform the replacement action. The system incurs a holding cost per unit time to store. If the spare part is unavailable at the required replacement time, the system incurs a shortage cost per unit time. The objective is to determine the optimal planned replacement time, t_r^*, and the optimal spare part ordering time, t_o^*, such that the total cost rate of the system is minimized. The lead time, LT, is assumed to be fixed, thus, $t_o + LT \leq t_r$ for all values of t_r and t_o. When the component is replaced, the system is restored to its initial, as good as new, condition. In other words, planned and unplanned replacements are considered to be regeneration points of the system. The objective is to minimize the system's long-run average cost per unit time per cycle, where a cycle is defined by the random time between two successive planned or unplanned replacements.

4.1.1 Replacement Policy

Given the failure time distribution of a component, the objective of the replacement model is to find the optimum planned replacement time, t_r^*. The optimal replacement time is value of the replacement time, $t_r \in [0,\infty)$, that minimizes the expected costs of preventive replacement and failure replacement. The long-run average cost per cycle is expressed as:

$$C_r = \frac{c_1 \overline{F}(t_r) + c_2 \overline{F}(t_r)}{\int_0^{t_r} \overline{F}(x).dx} \tag{35}$$

where C_r is the expected long-run replacement cost, c_1 is the planned replacement cost, c_2 is the failure replacement cost, and $\overline{F}(x)=1-F(x)$, where $F(x)$ is the CDF of the component's failure time evaluated at $x \in [0,\infty)$. The numerator represents the expected cost per cycle and the denominator represents the expected cycle length.

4.1.2 Inventory Ordering Policy

The authors in [42] consider a sequential decision making process where the optimal replacement time is first evaluated followed by the optimal ordering time. Once the optimum replacement time, t_r^*, has been computed, it is then used to decide when to order the spare part. Due to the assumption of single unit storage capacity, the order quantity is always a single unit. The optimal ordering time, t_o^*, is the value of the ordering time, $t_o \in [0,\infty)$,that minimizes the holding cost of spare parts and the cost of stock outs. The long-run average inventory cost per cycle is expressed as:

$$C_o = \frac{k_h \int_{t_o}^{t_o+LT} F(x).dx + k_s \int_{t_o+LT}^{t_r} \overline{F}(x).dx}{\int_{t_o}^{t_o+LT} F(x).dx + \int_{0}^{t_r} \overline{F}(x).dx} \tag{36}$$

where C_o is the expected long-run ordering cost, k_h is the holding cost per unit time, k_s is the shortage cost per unit time, and LT is the fixed lead time elapsed from the moment of placing the order up till order receipt. We note that the expected cycle length is not the same for the replacement policy case due to the possibility of stock-outs occurring and resulting in a longer cycle.

4.2 Sensor-based Replacement and Spare Parts Inventory Policies

In this research, we extend the traditional approach discussed above for determining replacement and inventory policies. Rather than using the failure time distribution of the components' population, we incorporate sensory-updated remaining life distributions (RLDs) obtained using the degradation modeling framework discussed in Chapter 3. These updated RLDs capture the underlying state of degradation of the components using real- time sensory signals. The impact of doing this is twofold; first the increased accuracy of failure prediction results in more sound decision policies and less costs. Second, it allows for dynamically updating the decision policies based on the health of the component.

Next we consider online sensory signals obtained at a specific updating time t_k. Each time a signal is acquired, the RLD of the degraded component is updated. The updated RLD is then used to compute the optimal replacement and the optimal spare part ordering times. The long-run average replacement and inventory costs can now be expressed in terms of the updated RLDs as follows,

$$C_r^k = \frac{c_1 \overline{F}^k\left(t_r^k\right) + c_2 \overline{F}^k\left(t_r^k\right)}{\int_0^{t_r^k} \overline{F}(x).dx + t_k} \tag{37}$$

$$C_o^k = \frac{k_h \int_{t_o^k}^{t_o^k+LT} F^k(x).dx + k_s \int_{t_o^k+LT}^{t_r^k} \overline{F}^k(x).dx}{\int_{t_o^k}^{t_o^k+LT} F^k(x).dx + \int_0^{t_r^k} \overline{F}^k(x).dx + t_k} \tag{38}$$

where C_r^k and C_o^k are the replacement and inventory ordering cost rates per cycle, respectively, at updating time t_k. $F^k(x)$ is the updated CDF of the remaining life at the updating time t_k, evaluated at $x \in [0, \infty)$. In other words, given that the component has survived up to time t_k, and that we have observed a partial degradation signal up to time t_k, $F^k(x)$ is the cumulative probability that the component fails after an additional $x \in [0, \infty)$ time units. The terms $t_r^{k^*}$ and $t_o^{k^*}$ are the optimal replacement and inventory ordering times, respectively, at the given updating epoch. Note that t_k, the updating time, has been added in the denominator to the cycle time. Each cycle is now composed of two components, a fixed term given by the time up to which the component has survived and a random component given by integrals of the remaining life distribution.

An important note to make is that we use renewal theory as a heuristic to approximate the replacement and inventory decisions. In reality, updating epochs do not explicitly represent regeneration points of the system. In the next chapter, we discuss and propose exact structured replacement and inventory policies.

4.2.1 Model Implementation

We present a case study based on the real-world data used to validate the sensor-based degradation model in Chapter 3. The case study illustrates utilizing the sensory-updated RLDs to determine and dynamically update replacement and inventory decisions using the proposed sensor-based methodology.

During monitoring an individual bearing and acquiring its degradation signal, each time the RLD is updated, the updated CDF, $F^k(x)$, is used to compute the optimal replacement time, $t_r^{k^*}$, at time t_k using equation (37). This quantity is, in turn, used to compute the optimal inventory ordering time, $t_o^{k^*}$, using equation (38). The following data was used: c_1 = \$ 25, c_2 = \$ 100, k_h = \$0.10/unit time, k_s = \$350/unit time, and LT = 4 time units. Figure 15 shows the evolution of the updated optimal replacement and inventory ordering times at different degradation percentiles, and Table 2 displays numerical results.

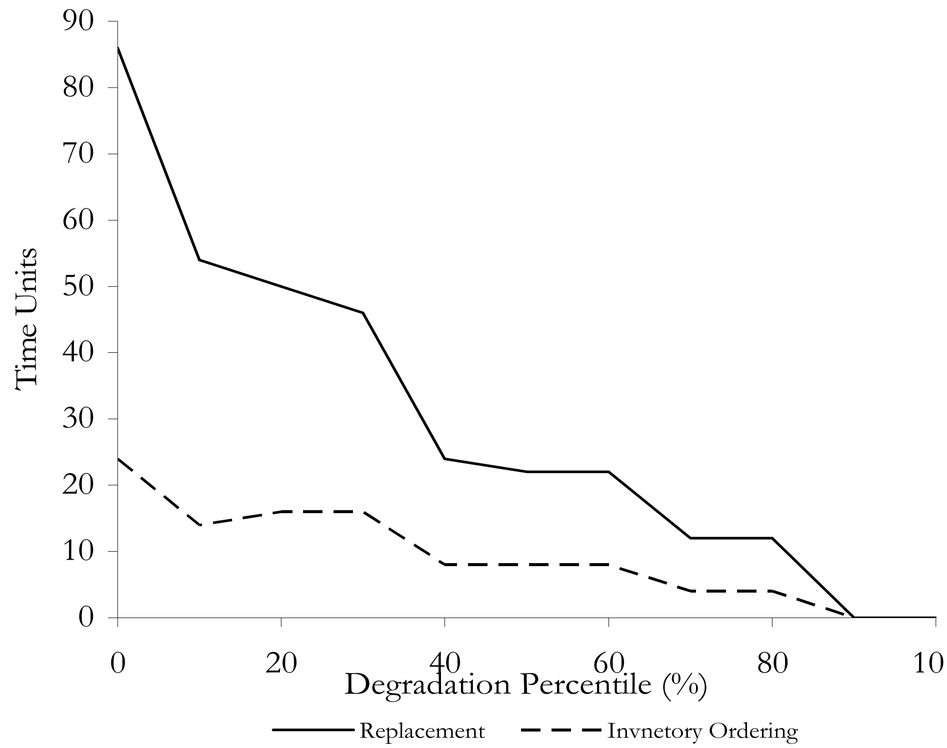

Figure 15 Updated optimal replacement and spare part inventory ordering times.

We note from the results that the constraint imposed on the inventory ordering time, $t_o + LT \leq t_r$, is satisfied for all degradation percentiles. Figure 16 below summarizes the proposed methodology.

Table 2 Numerical results of the case study.

Degradation Percentile (%)	t_r	t_o
0	86	24
10	54	14
20	50	16
30	46	16
40	24	8
50	22	8
60	22	8
70	12	4
80	12	4
90	0	0
100	0	0

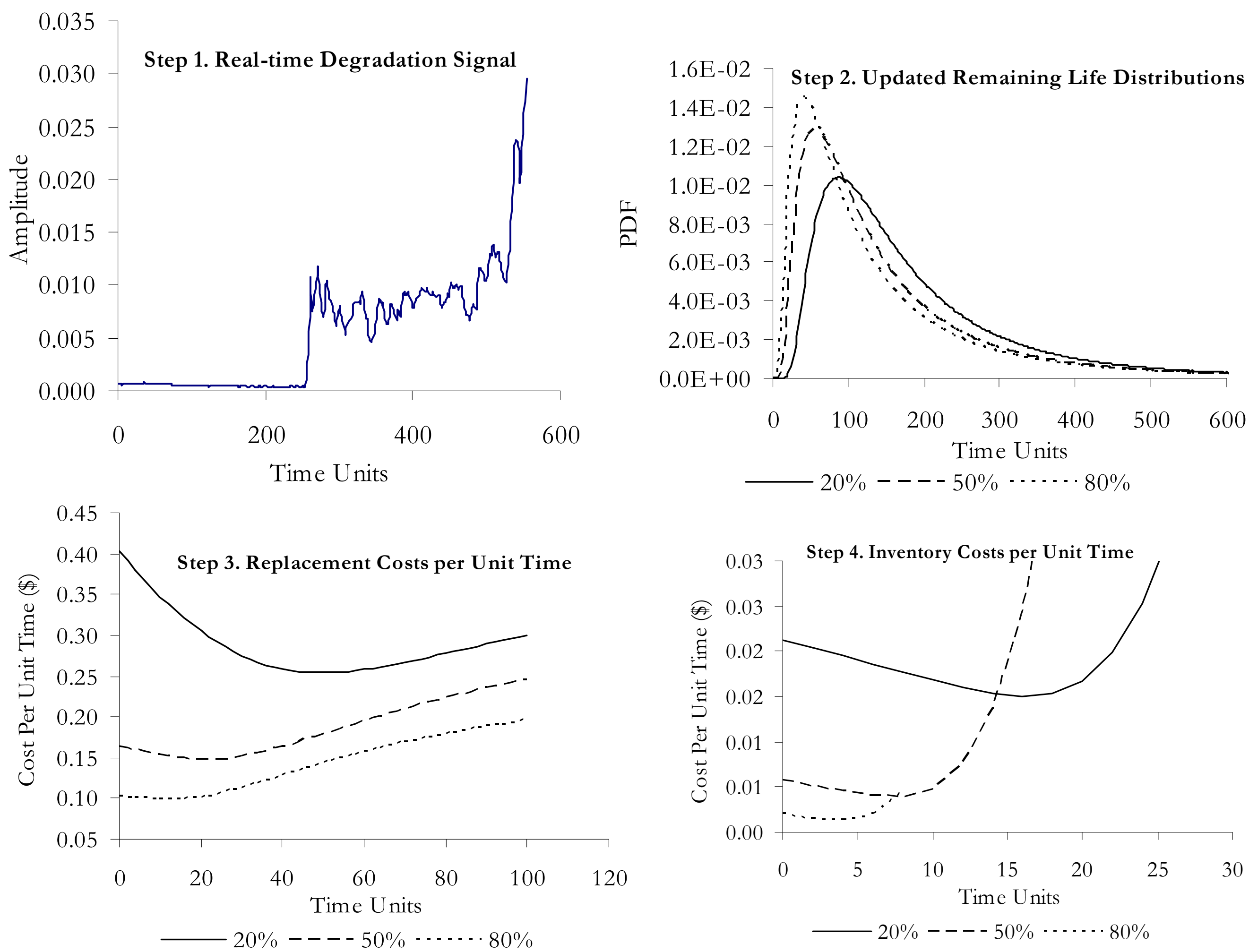

Figure 16 Summary of the sense-and-respond decision making framework.

The first step is to observe real-time degradation signal from the component during operation. Each time we acquire a signal, the RLD is updated using the degradation modeling framework, as shown in step two. The type of degradation model to be used (functional form and assumptions regarding the model parameters and error terms) is chosen according to the application. At each updating epoch, the updated RLD is used to compute optimal replacement and inventory ordering times. Steps three and four show replacement and inventory cost rate curves, respectively, for three different degradation percentiles. When the replacement and inventory ordering policies are jointly optimized, steps three and four are performed simultaneously.

4.2.2 Sensor-based Versus Reliability-based Decision Policies

We highlight the advantages of the senor-based decision methodology versus the traditional reliability-based approach. This is achieved by demonstrating, through two case studies, that implementing these sensor-based policies can result in reduced maintenance costs, primarily due to the better decisions resulting from improved failure prediction.

4.2.2.1 Case Study 1 – Single-unit System

The bearing degradation data from our experimental test rig is used for this case study. We evaluate the system costs associated with implementing traditional policies over a finite planning horizon and compare these costs to the costs incurred when sensor-based policies are implemented to the same population of components over the same horizon.

- <u>Traditional decision policy</u>: First, a Weibull distribution was fit to the actual failure times of the set of 25 bearings used to estimate the prior distribution parameters.

The Weibull distribution is very commonly used to characterize failure times of component populations. The failure data yielded a Weibull distribution with scale parameter $\varphi = 797.47$ and shape parameter $\gamma = 2.65$. This failure time distribution was used to compute optimal replacement and inventory ordering times using equations (35-36) based on the same cost data used in the previous section. The computed optimal times were $t_r^* = 440$ time units and $t_o^* = 60$ time units.

Next, 9 random failure times were generated from the given distribution and the decision policy computed above was implemented. The costs associated with this decision policy over the 9 failure cycles were then computed and found to be $C_{tot} = \$500.4$. Table 3 below summarizes the results

- <u>Sensor-based decision policy:</u> The same generated failure times were considered, but dynamic sensor-based decision policies were implemented using the data from 9 validation bearings. The remaining life distribution of the bearing, replacement time, and inventory ordering times are updated every 2 minutes. The process continues until the updated optimal replacement time is less than the inventory lead time. This results in a planned replacement. Otherwise, if failure occurs prior to a planned replacement, a failure replacement is performed at a higher cost. For the 9 bearings, over the same planning horizon, the total costs associated were computed and found to be $C_{tot} = \$228.6$. Following the traditional decision policy resulted in 3 failures over the 9 cycles, whereas the sensor-based decision policy resulted in no failures due to the higher prediction accuracy and the dynamic updating of replacement and inventory ordering times.

Table 3 Results summary for the traditional decision policy.

Cycle	Failure Time (Units)	Failure/Planned	Replacement Time (Units)	Inventory Ordering Time (Units)	Cycle Cost ($)
1	919.62	P	440	380	30.6
2	552.82	P	880	820	30.6
3	430.49	F	1310.49	1250.49	105.6
4	290.58	F	1601.07	1541.07	105.6
5	697.89	P	2041.07	1981.07	30.6
6	432.67	F	2473.74	2413.74	105.6
7	1070.82	P	2913.74	2853.74	30.6
8	448.84	P	3353.74	3293.74	30.6
9	1017.82	P	3793.74	3733.74	30.6
Total					500.4

4.2.2.2 Case Study 2 – Simulated Manufacturing System

Case study 1 demonstrated highlighted that the sensor-based decision methodology outperforms the traditional approach in a single unit system. In this section, we present a case study that extends this to a manufacturing system with more than one component. shows the series/parallel configuration of the system used in this study. Failure of any of the three series work cells results in system failure. Note that in order for work cells A or C to fail, both workstations in the work cells must fail due to redundancy.

Pre-processed parts are assumed to arrive at work cell A in accordance to a Poisson process with mean 4 parts/minute. Upon arrival, the part is processed on either workstations 1 or 2, whichever is available. Next, it enters workstation 3, after which it finally gets processed on either workstations 4 or 5. Upon completion, the part goes to the shipping department for packaging and shipping. Processing times on the workstations in work cells A, C are assumed to follow a triangular distribution with parameters (4.25, 4.75, and 5.25 minutes), while those in work cell B are assumed to follow a triangular distribution with parameters (2.25, 2.75, and 3.00 minutes).

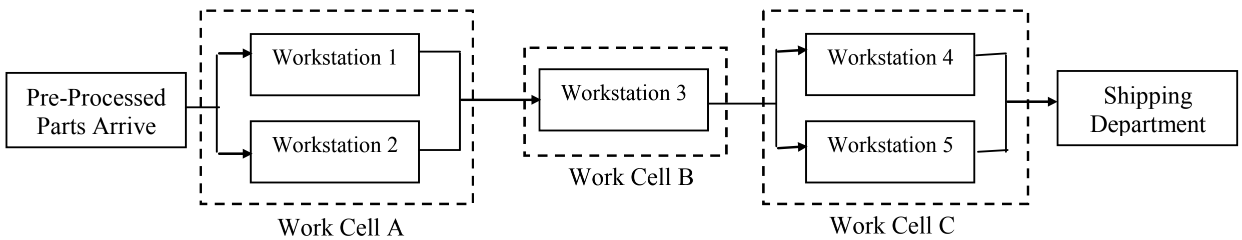

Figure 17 Configuration of the simulated manufacturing system.

The manufacturing system becomes unavailable if a random system failure occurs or a planned system replacement is performed. Downtime resulting from a system failure is assumed to be random and follows a normal distribution with mean 300 minutes and variance 30 minutes. Downtime resulting from a planned replacement routine is assumed to be random and follow a Normal distribution with mean 30 minutes and variance 5 minutes. The downtime resulting from an unplanned system failure is assumed to be greater since the demand for replacement parts and maintenance personnel is unexpected. Furthermore, we assume that each workstation degrades gradually until it fails. Degradation of the workstations is represented by the evolution of the vibration signals from monitoring the bearings.

In this study, maintenance decisions are performed based on the condition of the entire manufacturing system. In other words, given the optimal replacement and ordering times of each workstation, a single planned maintenance routine is scheduled for the entire manufacturing system. Similarly, failure of one of the work cells results in an unexpected failure of the entire system, at which time the entire system would be shut down for maintenance.

- Traditional Maintenance Policy: Workstations are assumed subject to random failures with Weibull distributed failure times. The parameters of this distribution were estimated using the same initial set of 25 bearings. Next, the Weibull CDF was used to compute optimal system replacement and spare parts ordering times, and the decision policies are implemented.

- Sensor-Based Maintenance Policy: The underlying assumption for this decision policy is that condition monitoring is used to acquire data every 2 minutes. Although these degradation signals are associated with individual bearings, for the purpose of this research they were used to simulate the degradation of the entire workstations. As we continue to monitor the manufacturing system during operation, we update the RLDs use them to compute and dynamically update optimal replacement and inventory ordering times for the system. This updating process continues until a stopping criterion is satisfied, after which we stop updating and implement the most recently updated decision policies. The stopping criterion we used is to stop updating once $t_r^{system} \leq t_o^{system} + LT$. In this expression, $t_r^{system} = \min\{\max\{t_r^1, t_r^2\}, t_r^3, \max\{t_r^4, t_r^5\}\}$, where t_r^j is the computed optimal replacement time of workstation $j, j = 1, \ldots, 5$. The system replacement time is then used to compute the system ordering time, t_o^{system}. This stopping rule attempts to eliminate spare part holding time and ensure just-in-time spare part delivery.

Arena simulation was used to simulate the continuous operation of the manufacturing system. Each simulation consists of five runs, each running for 365-days. Separate runs were performed for each decision policy. We used the following data: $LT = 20$ minutes, $c_1 = \$30$, $c_2 = \$400$, $k_h = \$0.10$/minute, $k_h = \$10$/minute. Workstation utilization and throughput were used to assess the performance of each maintenance policy. Table 4 summarizes the results, showing that the sensor-based policy outperforms the traditional policy. We also note that the Sensor-based policy results in a lower number of failure replacements.

Table 4 Summary of Simulation study results.

Policy	**No. of Failure Replacements N_f**		**No. of Planned Replacements N_p**		**Utilization**
	Mean	St'd Deviation	Mean	St'd Deviation	
Traditional	322.0	55.9	8141.3	104.4	0.8664
Sensor-Based	193.0	21.7	5140.7	155.0	0.9163

The total maintenance costs were also used to measure the performance of each policy, where the total maintenance costs are the sum of total inventory and total replacement costs. Figure 18 and Table 5 show average inventory cost, replacement cost, and total cost for each decision policy. The total maintenance costs for the sensor-based decision policy results in 35% saving in total maintenance costs than the traditional approach.

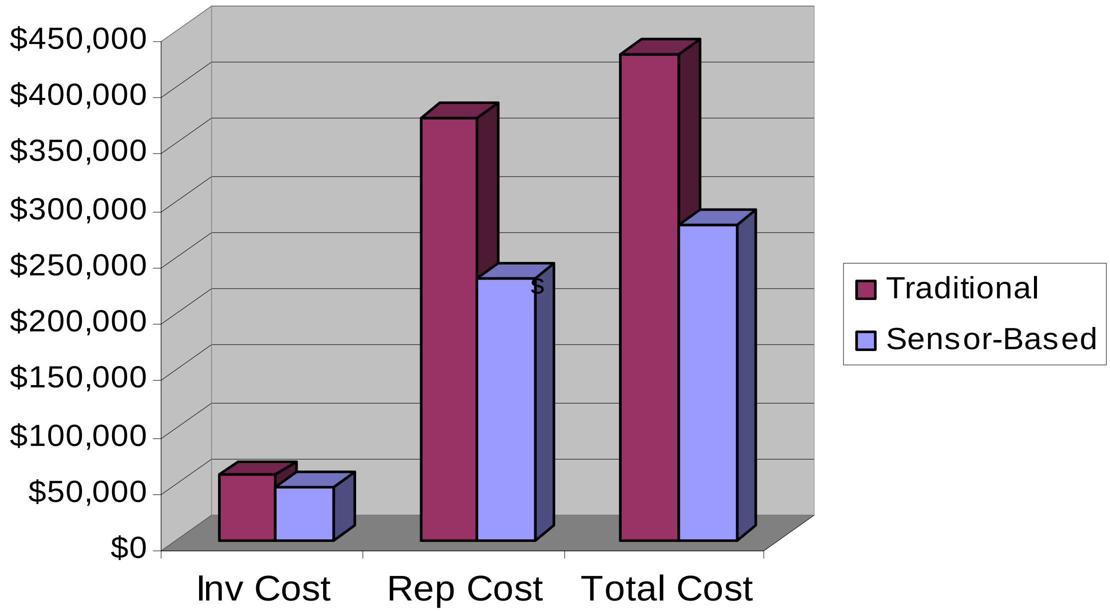

Figure 18 Maintenance Costs for the two policies

Table 5 Summary of the maintenance costs for the two policies

	Inv Cost		Rep Cost		Total Cost	
Policy	**Mean**	**St'd Dev**	**Mean**	**St'd Dev**	**Mean**	**St'd Dev**
Traditional	$57,174	$9,266	$372,907	$17,463	$430,080	$26,729
Sensor-Based	$47,700	$5,565	$231,420	$9,147	$279,120	$14,717

4.3 User Interface for Sense-and-Respond Logistics Framework

To facilitate real-world application, we have built a graphical user interface (GUI) to implement the sense-and-respond logistics methodology. The user interface is linked to the experimental test rig discussed in Section 3.3.2. Two photographs of the test rig are displayed in Figure 19 and Figure 20.

Figure 19 Photograph of the experimental test rig.

Figure 20 Experimental test rig linked to the data acquisition computer.

The user interface was developed using LABVIEW. This user interface links the test rig to the computer for two primary purposes; (1) to acquire the vibration-based degradation signals captured by the accelerometers through a DAQ (data acquisition) board, and (2) to provide closed loop control of the shaft rotational speed and the load applied by the hydraulic unit. Shows a snap shot of the GUI screen in which the parameters of one degradation test are set up. In particular, values for the shaft rotational speed (RPM), load applied to the bearing (lb), and the time between successive acquisition epochs (minutes) are entered in this screen.

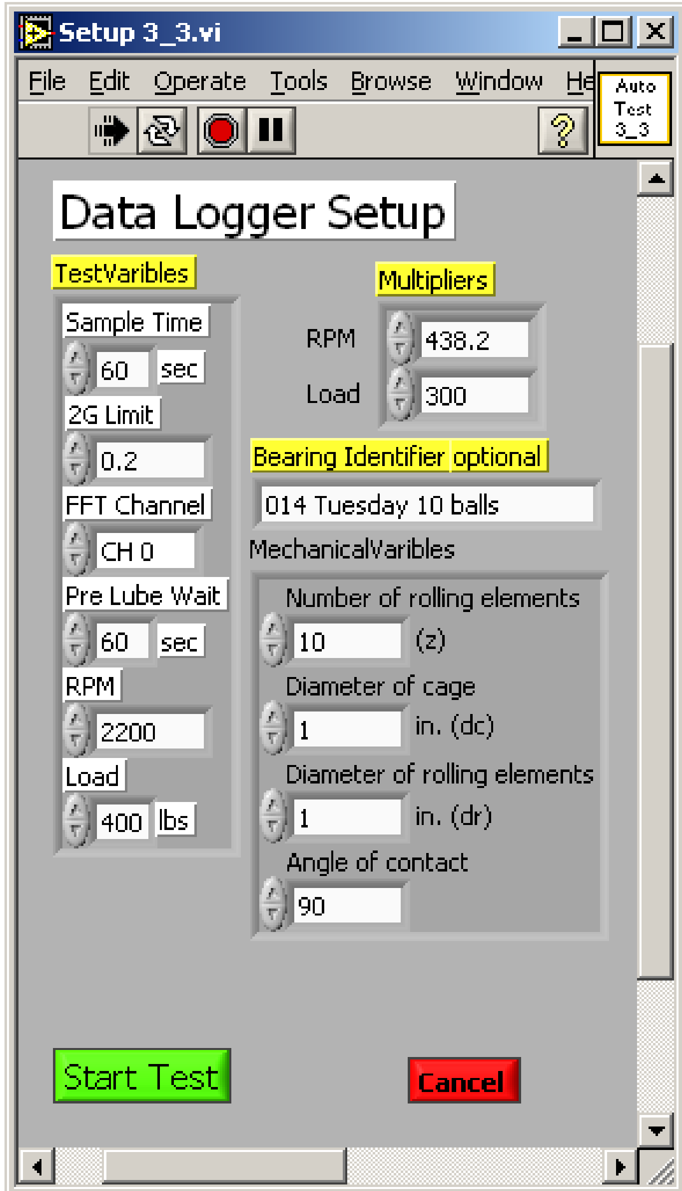

Figure 21 Snapshot of the GUI screen used to enter the parameters of an experiment.

When an experiment is started, the data logger subroutine automatically runs. This portion of the GUI has many modules. We display two modules of interest; first, the GUI screen that displays the vibration spectrum (in the frequency domain) in Figure 22.

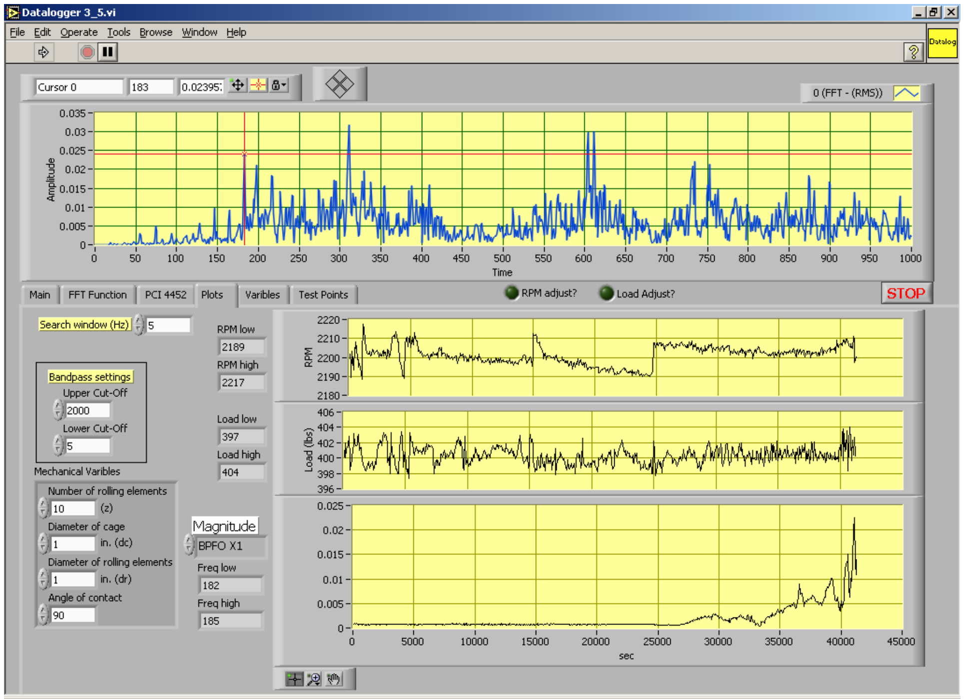

Figure 22 Snapshot of the GUI screen that captures the degradation signal and the experiment parameters.

This screen displays the degradation signal (lower plot), and tracks the value of the shaft rotational speed and load applied on the bearing to assure proper control of these parameters. The second module is the sense-respond-logistics screen shown in Figure 23. This screen plots the RLD, replacement cost curve, inventory cost curve, optimal replacement and spare parts ordering times, and the mean remaining life. These curves are computed using a MATLAB code embedded within the LABVIEW block diagram.

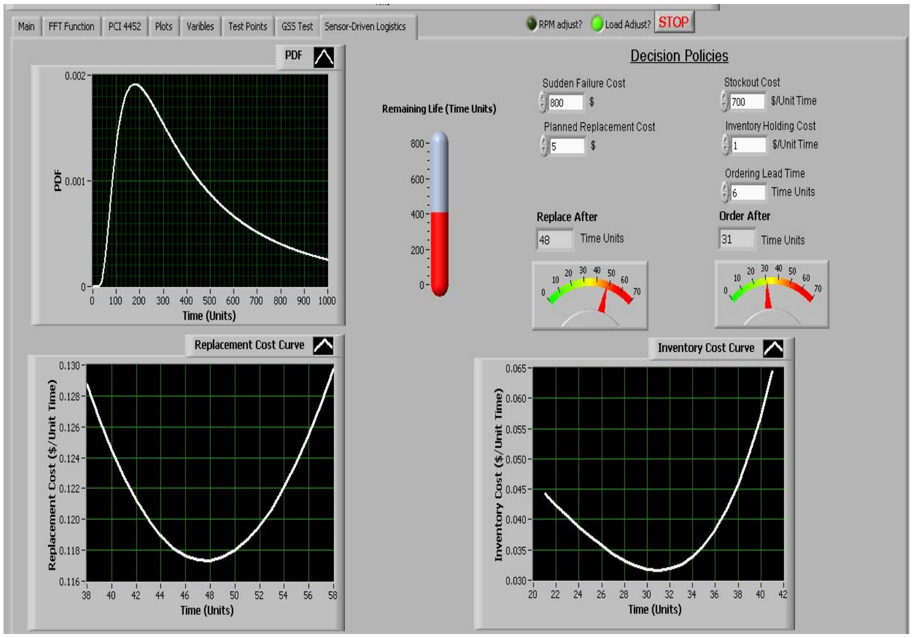

Figure 23 Snapshot of the sense-and-respond logistics GUI screen.

As discussed earlier in this chapter, we use the renewal-reward equation to determine optimal replacement and inventory ordering times in the sensor-based decision model as a heuristic to approximate decision policies. This is a heuristic approximation since updating epochs do not explicitly represent regeneration points of the system like the events of failure or replacement. In the next chapter, we present exact structured maintenance policies for the single-unit sensor-based problem.

CHAPTER 5
SENSOR-BASED STRUCTURED MAINTENANCE POLICIES

In this chapter, we focus on deriving structured maintenance policies for single-unit degrading systems whose degradation signal grows exponentially. First, we present a somewhat generic, single-unit semi-Markov decision process model that utilizes sensory signals to determine the optimal replacement time. Second, we model the amplitude (or level) of a degradation signal using the exponential degradation model presented in Chapter 3, wherein real-time signals are used to update the predictive distribution of the degradation signal using Bayesian techniques.

Next, we integrate the predictive distribution of the degradation signal with the semi-Markov decision process model to derive the optimal sensor-based replacement policy for a single unit system. We show that the optimal policy under the infinite horizon expected discounted cost criterion is a monotonically non-decreasing control limit policy that optimally balances the cost of failure, the cost of preventive replacement, and the cost of observing sensor data. This result might seem counterintuitive. One would typically expect a monotonically non-increasing control limit policy, or in other words, that the urgency to preventively replace the system increases as the system ages. We provide explanation and provide mathematical proofs of this result. We present a case study based on the real-world vibration data used to validate the models discussed in previous chapters, and study the performance of the policy under different cost settings.

Finally, we utilize the optimal replacement policy to sequentially derive a structured policy for spare parts ordering. In particular, at each decision epoch, we compute the expected replacement time at the next decision epoch. In other words, we compute the distribution of the time it takes the signal trajectory to cross the monotonically non-decreasing replacement control limit. To achieve this, we discuss computing the first passage time distribution of a BM to a time-varying piecewise linear threshold. Our ordering criterion is to order a spare part when the expected replacement time at the next decision epoch is less than the ordering lead time. This ordering criterion attempts to eliminate spare part holding time and ensure just-in-time spare part delivery. We show that the optimal spare part ordering policy also has a monotonic non-decreasing control limit structure.

5.1 Sensor-based Replacement Problem

We start by presenting a single-unit replacement problem. First, we formulate the problem as an MDP, then we establish structural properties of the optimal replacement policy.

5.1.1 Model Formulation

Consider a single-unit system that degrades during its operation. Dedicated sensors are used to observe real-time signals at equally spaced discrete time epochs k, $k = 0,1,\ldots$, with t being the constant time between two consecutive observations. Each time we observe the signal, if it does not exceed the failure threshold, then our goal is to decide whether to instantaneously perform a preventive replacement of the system at cost c_1, or continue to the next observation time. If we decide to wait, we incur an observation cost, c_3, and run the risk of sudden failure (within the t time units until the next observation), in which case we will be forced to perform a failure (reactive) replacement at a cost $c_2 > c_1$. As discussed before, ordering the costs in this manner reflects the fact that reactive replacements may include additional costs such as lost production and overtime labor. In the automotive industry for instance, an unexpected failure can result in stopping an assembly line, resulting in very high lost production costs.

Let **W** = (**K,** P) be the state space, where $\mathbf{K} = \{0,1,\ldots\}$ represents the set of observation numbers. In other words, at observation $k \in \mathbf{K}$ the system is $t_k = kt$ time units old. Let $S_k \in$ P$^+$ represent the amplitude or level of the degradation signal at observation $k \in \mathbf{K}$. We use the signal logarithm, ℓ_k, to define the degradation states for mathematical convenience. Hence, ℓ_k takes on any value in the set of all real numbers P. We will refer to ℓ_k as the "observed signal" here forth.

Let $0 < \lambda < 1$ be a discount factor, and let $V(k, \ell_k)$ be the total expected infinite horizon discounted cost when the system starts in state $(k, \ell_k) \in \mathbf{W}$. The optimality equations can be expressed as follows:

$$V(k,\ell_k) = \begin{cases} c_2 + V(0,\ell_0), & \ell_k > \xi' \\ \min\{c_1 + V(0,\ell_0), \lambda\,(c_3 + E[V(k+1,L)])\} & \ell_k \le \xi' \end{cases}, \quad \forall (k,\ell_k) \in \mathbf{W} \tag{39}$$

for all $(k, \ell_k) \in \mathbf{W}$, where ℓ_0 is the initial observation for each identical component, assumed to be a known constant, and $\xi' = \ln(\xi)$ where ξ is a pre-determined failure threshold as defined previously. We assume that if the observed signal crosses the threshold ξ' between observations and then returns to a value below this threshold at the next observation time, this does not constitute a failure.

Equation (39) follows from the logic that if the observed signal at the current decision epoch, ℓ_k, exceeds the failure threshold ξ', we perform an instantaneous failure replacement at cost c_2 and restart with a new component. On the other hand, if the observed signal is below the failure threshold, we can either choose to preventively replace the component with an identical new component, or do nothing and continue to observe a new signal at the next observation time. If we choose to preventively replace the system, we incur a cost c_1 and restart with a new system. Otherwise, we continue and observe a new signal value with an expected cost-to-go $E[V(k+1, L)]$, while incurring an observation cost c_3. The term L is a random variable denoting the observed signal at the next observation time; i.e. $L = \Gamma(t_k + t)$. We suppress L's dependence on k for notational convenience. The exponential degradation modeling framework with independent stochastic parameters, presented in Chapter 3 is used to characterize the evolution of the system's degradation, and will be utilized to determine the form of the expected cost-to-go, $E[V(k+1, L)]$. We now present two important Propositions that will be used to establish the structural properties of the optimal replacement policy.

Proposition 5. The random variable L is stochastically increasing in ℓ_k.

Proof. Consider two different observed signals, ℓ_k^+ and ℓ_k^-, at the same epoch k, such that $\ell_k^+ \geq \ell_k^-$. Let the random variables L^+ and L^- denote the signal logarithm at the next observation time given the two observed signals ℓ_k^+ and ℓ_k^-, respectively. We need to show that $F_{L^+}(x) \leq F_{L^-}(x)\ \forall x$, where $F_X(\cdot)$ is the CDF of the random variable X. Recall from Proposition 2 that the distribution of L can be expressed as follows:

$$\begin{aligned} F_L(x) &= \Phi\left(\frac{x - \tilde{\mu}(k, \ell_k)}{\tilde{\sigma}(k)}\right) \\ &= \Phi(g(k, \ell_k)). \end{aligned} \tag{40}$$

where $g(k,\ell_k)=\dfrac{x-\tilde{\mu}(k,\ell_k)}{\tilde{\sigma}(k)}$ and $\Phi(\cdot)$, and notice that in computing $\tilde{\mu}(k,\ell_k)$ and $\tilde{\sigma}(k)$ using equations (12) and (13), respectively, we set $\tau=t$, since we are interested in the next observation epoch, $k+1$.

Since $\Phi(\cdot)$ is a monotonically non-decreasing function, it suffices to show that $g(k,\ell_k^+)\le g(k,\ell_k^-)$ to establish that $F_{L^+}(x)\le F_{L^-}(x)$. To do so, we show that $\tilde{\mu}(k,\ell_k^+)\ge\tilde{\mu}(k,\ell_k^-)$ and that the denominator of (40), $\tilde{\sigma}^2(k)$, is constant in ℓ_k, which implies the intended result.

We start by showing that $\tilde{\mu}(k,\ell_k^+)\ge\tilde{\mu}(k,\ell_k^-)$. Since $\ell_k^+\ge\ell_k^-$, from equation (5) we have:

$$\mu_{\beta'}(k,\ell_k^+)-\mu_{\beta'}(k,\ell_k^-)=\frac{\left(\sigma_1^2\ell_k^++\mu_1'\sigma^2\right)\left(\sigma_0^2+\sigma^2t\right)-\sigma_1^2\left(\ell_0\sigma_0^2+t\mu_0\sigma^2\right)}{\left(\sigma_0^2+t\sigma^2\right)\left(\sigma_1^2kt+\sigma^2\right)-t\sigma_0^2\sigma_1^2}$$

$$-\frac{\left(\sigma_1^2\ell_k^-+\mu_1'\sigma^2\right)\left(\sigma_0^2+\sigma^2t\right)-\sigma_1^2\left(\ell_0\sigma_0^2+t\mu_0\sigma^2\right)}{\left(\sigma_0^2+t\sigma^2\right)\left(\sigma_1^2kt+\sigma^2\right)-t\sigma_0^2\sigma_1^2}$$

$$=\frac{\sigma_1^2\left(\ell_k^+-\ell_k^-\right)\left(\sigma_0^2+\sigma^2t\right)}{\left(\sigma_0^2+t\sigma^2\right)\left(\sigma_1^2kt+\sigma^2\right)-t\sigma_0^2\sigma_1^2}\ge 0$$

$$\Rightarrow \mu_{\beta'}(k,\ell_k^+)\ge\mu_{\beta'}(k,\ell_k^-).$$

By inspecting equation (13), we notice that $\tilde{\sigma}^2(k)$ does not depend on the signal amplitude, hence the denominator of $F_{L^+}(x)$ is equal to that of $F_{L^-}(x)$. We have, then, established that: $g(k,\ell_k^+)\le g(k,\ell_k^-)\Rightarrow F_{L^+}(x)\le F_{L^-}(x)\ \forall x$. ■

Proposition 6. The random variable L is stochastically decreasing in k.

Proof. Consider two observed signals, ℓ_{k^+} and ℓ_{k^-}, at different observation epochs k^+ and k^-, such that $k^+ \geq k^-$ and $\ell_{k^+} = \ell_{k^-} = \ell$. Let the random variables L^+ and L^- denote the signal logarithm at the next observation times for the two observed signal increments ℓ_{k^+} and ℓ_{k^-}, respectively. To establish the result, it suffices to show that $g(k^-, \ell_{k^-}) \leq g(k^+, \ell_{k^+})$. We start by expressing the numerator of $g(k^-, \ell_{k^-})$ as follows for any fixed x:

$$x - \tilde{\mu}(k^-, \ell_{k^-}) = x - \ell_{k^-} - \mu_{\beta'}(k^-, \ell_{k^-})$$

$$= x - \ell_{k^-} - \frac{\left(\sigma_1^2 \ell_k + \mu_1' \sigma^2\right)\left(\sigma_0^2 + \sigma^2 t\right) - \sigma_1^2 \left(\ell_0 \sigma_0^2 + t\mu_0 \sigma^2\right)}{\left(\sigma_0^2 + t\sigma^2\right)\left(\sigma_1^2 k^- t + \sigma^2\right) - t\sigma_0^2 \sigma_1^2} = B_{k^-} - \frac{A_{k^-}}{C_{k^-}}.$$

Similarly, we express $\tilde{\sigma}^2(k^-)$ as follows:

$$\tilde{\sigma}^2(k^-) = \sigma_{\beta'}^2(k^-) t^2 + \sigma^2 t$$

$$= \frac{\sigma^2 \sigma_0^2 \left(\sigma_1^2 + t\sigma^2\right)}{\left(\sigma_0^2 + t\sigma^2\right)\left(\sigma_1^2 k^- t + \sigma^2\right) - t\sigma_0^2 \sigma_1^2} t^2 + \sigma^2 t = \frac{E}{C_{k^-}} + F.$$

Note that the terms A_{k^-} and B_{k^-} depend only on the observed signal, ℓ_{k^-}. Thus, for any two observations such that $\ell_{k^+} = \ell_{k^-} = \ell$, we can define $A_{k^-} = A_{k^+} = A$ and $B_{k^-} = B_{k^+} = B$. Next, we express the difference between $g(k^-, \ell_{k^-})$ and $g(k^+, \ell_{k^+})$ as follows:

$$g(k^-, \ell_{k^-}) - g(k^+, \ell_{k^+}) = \frac{x - \tilde{\mu}(k^-, \ell_{k^-})}{\sqrt{\tilde{\sigma}^2(k^-)}} - \frac{x - \tilde{\mu}(k^+, \ell_{k^+})}{\sqrt{\tilde{\sigma}^2(k^+)}}$$

$$= \frac{\sqrt{\tilde{\sigma}^2(k^+)} \cdot \left(x - \tilde{\mu}(k^-, \ell_{k^-})\right) - \sqrt{\tilde{\sigma}^2(k^-)} \cdot \left(x - \tilde{\mu}(k^+, \ell_{k^+})\right)}{\sqrt{\tilde{\sigma}^2(k^-)} \cdot \sqrt{\tilde{\sigma}^2(k^+)}}.$$

It is clear from the above expression that the sign of the difference depends only on the sign of the numerator. We now examine the numerator:

$$\sqrt{\tilde{\sigma}^2(k^+)}\cdot\left(x-\tilde{\mu}(k^-,\ell_{k^-})\right)-\sqrt{\tilde{\sigma}^2(k^-)}\cdot\left(x-\tilde{\mu}(k^+,\ell_{k^+})\right)$$

$$=\sqrt{\frac{E}{C_{k^+}}+F}\cdot\left(B-\frac{A}{C_{k^-}}\right)-\sqrt{\frac{E}{C_{k^-}}+F}\cdot\left(B-\frac{A}{C_{k^+}}\right)$$

$$=\frac{A}{C_{k^+}C_{k^-}}\left(\sqrt{EC_{k^-}+FC_{k^-}^2}-\sqrt{EC_{k^+}+FC_{k^+}^2}\right)$$

$$-B\left(\sqrt{\frac{E}{C_{k^+}}+F}-\sqrt{\frac{E}{C_{k^-}}+F}\right)$$

Since $k^+\ge k^-\Rightarrow C_{k^+}\ge C_{k^-}$, then:

$\sqrt{EC_{k^-}+FC_{k^-}^2}-\sqrt{EC_{k^+}+FC_{k^+}^2}\le 0\;\Rightarrow\sqrt{\frac{E}{C_{k^+}}+F}-\sqrt{\frac{E}{C_{k^-}}+F}\ge 0$. Therefore:

$g(k^-,\ell_{k^-})\le g(k^+,\ell_{k^+})\Rightarrow F_{L^-}(x)\le F_{L^+}(x)\quad\forall x$. ■

In the next section, the results presented in Propositions 5 and 6 are used to derive structural properties of the optimal policy for the sensor-based single-unit replacement model formulated above.

5.1.2 Structural Properties of the Optimal Policy

In some cases, it is possible to show that the optimal policy for a sequential decision problem has a certain structure. These structures can sometimes be useful for many practical purposes. For example, they can be exploited to develop computationally efficient algorithms. They also facilitate the implementation of the optimal policy from a practical standpoint.

We will focus our attention on control limit policies (CLPs). Under a monotonic control limit policy, the system is kept operating until the signal exceeds a certain control limit, ℓ_k^*, which is monotone in k. If the system state exceeds this control limit, a preventive replacement is performed. Control limit policies have been widely discussed in the literature [51, 34, 33, 184]. We start by stating two theorems necessary for establishing a control limit structure for the optimal replacement policy.

Theorem 1. The value function, $V(k, \ell_k)$, is non-decreasing in ℓ_k for all $k \in \mathbf{K}$.

Proof. Let $V^n(k, \ell_k)$ denote the value function at the n^{th} iteration of the value iteration algorithm. We start by defining the initial value for $n = 0$:

$$V^0(k, \ell_k) = 0 \quad \forall\ \ell_k \in \mathrm{P}$$

which is non-decreasing in ℓ_k. Next, assume that $V^n(k, \ell_k)$ is non-decreasing in ℓ_k, then from equation (39):

$$V^{n+1}(k, \ell_k) = \begin{cases} c_2 + V^n(0, \ell_0), & \ell_k > \xi' \\ \min\{c_1 + V^n(0, \ell_0), \lambda(c_3 + E[V^n(k+1, L)])\}, & \ell_k \le \xi'. \end{cases} \tag{41}$$

Recall from Proposition 2 that the random variable L is stochastically increasing in ℓ_k. Since $V^n(k, \ell_k)$ is non-decreasing by the induction hypothesis, $E[V^n(k+1, L)]$ is also non-decreasing by the standard result in Ross [185] (Proposition 9.1.2, p. 405). Notice that since all of the terms on the right hand side of equation (41) are non-decreasing in ℓ_k, then $V^{n+1}(k, \ell_k)$ is also no-decreasing, which establishes the result. ■

Theorem 2. The value function, $V(k, \ell_k)$, is non-increasing in k for all ℓ_k.

Proof. This proof is also done by induction, similar to Theorem. We define the initial value for $n = 0$ as

$$V^0(k, \ell_k) = 0 \quad \forall\, k \in \mathbf{K}.$$

Next, assume that $V^n(k, \ell_k)$ is non-increasing in ℓ_k, then:

$$V^{n+1}(k, \ell_k) = \begin{cases} c_2 + V^n(0, \ell_0), & \ell_k > \xi' \\ \min\{c_1 + V^n(0, \ell_0), \lambda(c_3 + E[V^n(k+1, L)])\}, & \ell_k \le \xi'. \end{cases}$$

By similar reasoning to the proof of Theorem1, by Proposition 6 and the induction hypothesis $E[V^n(k+1, L)]$ is non-increasing in k. Therefore $V^{n+1}(k, \ell_k)$ is also non-increasing, which completes the proof. ■

Next, we show in Theorem that for the infinite horizon sensor-based replacement problem, under the discounted cost criterion, the optimal policy is a monotonically non-increasing control limit policy.

Theorem 3. For all decision epochs $k \in \mathbf{K}$, $\exists\ \ell_k^* \le \xi'$ such that the optimal decision is to preventively replace if and only if $\ell_k \ge \ell_k^*$. The control limit ℓ_k^* is monotonically non-decreasing in k.

Proof. Assume that the result holds and consider the inequality

$$c_1 + V(0, \ell_0) \le \lambda(c_3 + E[V(k+1, L)]). \qquad (42)$$

The left hand side of inequality (42) is constant in ℓ_k. On the other hand, by Theorem 1 and Proposition 5, the right hand side is non-decreasing in ℓ_k. Therefore, inequality (42) holds for any pair $(k, \ell_k) \in \mathbf{W}$ such that $\ell_k \ge \ell_k^*$. In other words, given that the optimal decision in state (k, ℓ_k^*) is to replace, the optimal decision for any pair

$(k,\ell_k)\in \mathbf{W}$ such that $\ell_k \geq \ell_k^*$ is also to replace. Therefore the optimal replacement policy is a control limit policy with control limit ℓ_k^*.

Similarly, the right hand side of inequality (42) is non-increasing in k by Proposition 6 and Theorem 2. Hence, for each observed signal $\ell \in \mathrm{P}$, there exists a threshold age k_ℓ^* such that the optimal decision for any pair $(k,\ell_k)\in \mathbf{W}$ with $k \leq k_\ell^*$ is to preventively replace. By the existence a control limit ℓ_k^* for each age k and a threshold age k_ℓ^* for each observed signal ℓ, the control limit ℓ_k^* is non-decreasing in age k. ■

Theorem 3 may seem counterintuitive. One might intuitively expect the control limit to be monotonically non-increasing in the sense that the urgency for preventive replacement increases as the system ages, i.e., replacements are triggered by smaller signal values in older systems. This counterintuitive structure of the optimal policy is not uncommon. For example, Benyamini and Yechiali state on p. 57 in [33] that "certain two-dimensional bisections may not be as simple or intuitive as one would expect of a control limit policy". The structure of the optimal policy in Theorem 3 can be justified as follows: notice from equation (8) that $\sigma_{\beta'}^2(k)$ only depends on the age k and other given constants. Furthermore, since k is in the denominator of $\sigma_{\beta'}^2(k)$, we conclude that $\sigma_{\beta'}^2(k)$ is non-increasing in k. By inspecting equation (13), we conclude that the variance of the future signal logarithm, L, is also non-increasing in k. In other words, there is increased accuracy (less variability) in predicting the future signal value, which results in the optimal policy being more tolerant of large signal values in older systems.

5.1.3 Computing the Optimal Policy

The optimal replacement policy can be computed using one of many existing algorithms such as policy iteration. However, the specific form of the problem and the control limit structure of the optimal policy can be exploited to decrease the computational burden associated with such problems, especially when the state space becomes large. We use Puterman's monotone policy iteration (Puterman [186], p. 428) to take advantage of the results established above. As a first step, we define $\tilde{k}t$ to be the maximum allowable system age for some integer $\tilde{k} < \infty$. If the system reaches this age, we enforce a preventive replacement at cost c_1. The idea is to set $\tilde{k}$ to be very large so as not to influence the optimal policy, and then truncate the state space prior to this limit when reporting the optimal policy.

Next, we discretize the set of possible signal values by defining $\Lambda = \{\Delta, 2\Delta, \ldots, \xi - \Delta, \xi\}$, where $\Delta = \xi/m$ for some positive integer $m < \infty$. Our state space becomes $\mathbf{W} = \{\mathbf{K} \times \Lambda\} \cup \{(0, \ell_0)\}$ where $\mathbf{K} = \{1, \ldots, \tilde{k}\}$ and ℓ_o is the initial observation, assumed to be a known constant. The detailed algorithm is outlined below.

Let $\upsilon = (\ell_1^*, \ldots, \ell_{\tilde{k}}^*)$ be an arbitrary replacement policy, where $\ell_k^* \le \xi'$ is the control limit when the system is kt units old, $k \in \mathbf{K}$. This representation of the policy results in a $\tilde{k}$-dimensional vector rather than a $\tilde{k} \times m$- dimensional vector of actions for each state. Next, let $\mathbf{P}_\upsilon$ and $\mathbf{r}_\upsilon$ be the transition probability matrix and vector of rewards, respectively, corresponding to policy υ. Furthermore, define $\mathbf{P}_\upsilon^i$ as the row corresponding to the i^{th} state in $\mathbf{P}_\upsilon$, $i \in \Lambda$. The following algorithm can be used to find the optimal policy:

1. **Initialization:** set $j = 0$. Start with an arbitrary initial policy υ.

2. **Policy Evaluation:** evaluate current policy by solving: $\mathbf{v}_{\upsilon^j} = \mathbf{r}_{\upsilon^j} + \lambda \mathbf{P}_{\upsilon^j} \mathbf{v}_{\upsilon^j}$.

3. **Policy Improvement:** check whether shifting the control limit ℓ_j^* up or down results in an improvement:

If $c_3 + \lambda \mathbf{P}_{\upsilon^j}^{\ell_k^*} \mathbf{v}_{\upsilon^j} < c_1 + v_{\upsilon^j}(0, \ell_0)$,

Set $\ell_j^* = \ell_j^* + \Delta$,

Go to step 2.

If $c_1 + v_{\upsilon^j}(0, \ell_0) < c_3 + \lambda \mathbf{P}_{\upsilon^j}^{\ell_k^* - \Delta} \mathbf{v}_{\upsilon^j}$,

Set $\ell_j^* = \ell_j^* - \Delta$

Go to step 2.

Otherwise go to step 4.

4. **Check for Optimality:** if $j = \tilde{k}$, Stop. Set optimal policy $\upsilon^* = \upsilon$. Otherwise, increment j by 1, go to step 2.

Notice that in the policy improvement step, step 3, we only need to explore a small subset of the state space represented by one step above or below the current control limit. Moreover, if some shift is found to be improving for some $k \in \mathbf{K}$ in one iteration, we only need to check for improvement in the same direction in subsequent iterations. The computational saving compared to the standard policy iteration algorithm is evident.

Another advantage of this algorithm is in the policy evaluation step, step 2. Policy evaluation includes solving a system of linear equations, one corresponding to each state. We notice in our algorithm that if a shift is found to be improving in step 3, we need to re-evaluate the improved policy by solving a new system of linear equations.

However, this new system differs only very slightly from the previous iteration. We can improve the computational efficiency by updating the solution to the new system of equations given the solution to the previous system. This can be done, for example, using QR-decomposition [187].

An issue that typically arises in solving large instances of Markov decision processes is the limited capacity of computer memory allocated to store different variables and parameters. The specific form of the replacement problem can also be useful for improving memory allocation. Specifically, recall that when the system is in some state (k, ℓ_k), the system is kt units old. Hence, it can only make a transition in the next decision epoch to the subset of the state space where the system age is equal to $(k+1)t$. In other words, the transition probability from any state (k, ℓ_k) to (j, ℓ_j) is non-zero if and only if $j = k+1$, otherwise it is equal to zero. Therefore, the transition probability matrix corresponding to every stationary policy is a sparse matrix. This can be utilized for better computer memory utilization, where only non-zero elements need to be stored. In general, the total number of elements in the transition probability matrix corresponding to each stationary policy is $(m \times \tilde{k}) \times (m \times \tilde{k}) = m^2 \tilde{k}^2$, whereas the number of non-zero elements that need to be stored is equal to $(m \times m) \times (\tilde{k}) = m^2 \tilde{k}$.

5.1.4 Case Study

We present a numerical case study based on real-world data acquired from degradation testing of bearings using our experimental test rig. Recall that that the failure threshold $\xi = 0.025$ and the values of the prior parameters are $\mu_0 = -6.031$, $\mu_1' = 8.061 \times 10^{-3}$, $\sigma_0^2 = 0.346$, $\sigma_1^2 = 1.034 \times 10^{-5}$, and $\sigma^2 = 0.0073$. Next, we solve for the optimal replacement policy given the planned replacement cost, c_1, the failure replacement cost, c_2, and the observation cost, c_3. Figure 24 shows a schematic of the optimal control policy and how it relates to the degradation signal. The stepped line represents the control limit at various ages of the system. If the observed signal falls below this line (white area), the optimal decision is to continue observing. Otherwise, if it falls above the stepped line (gray area) the optimal decision is to preventively replace.

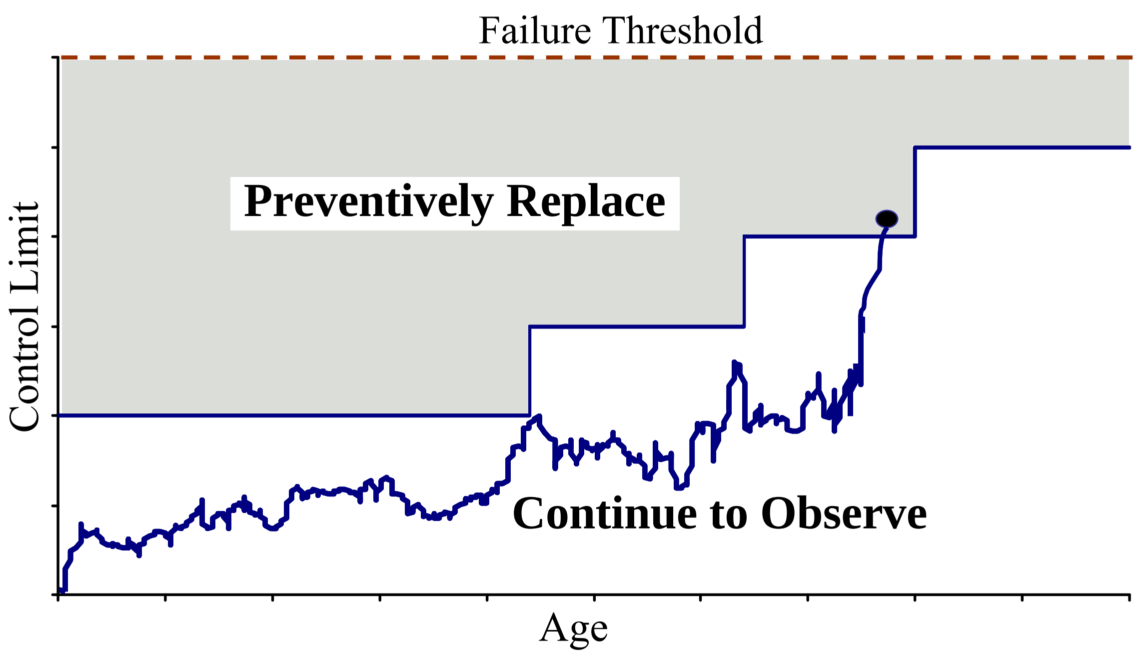

Figure 24 Schematic representation of a monotonically non-decreasing control limit replacement policy.

We use a test instance with the following cost data: $c_1 = \$3.00$, $c_2 = \$12.00$, and $c_3 = \$0.005$. The initial signal was set to $\ell_0 = \log(0.001) = -6.9078$. To determine a suitable value for the maximum age $\tilde{k}$, we examine our data to determine the time at which the degradation signal is estimated to cross the failure threshold. Notice from equation (23) that the logged degradation signal model is similar to a Brownian motion with positive drift β' and starting at θ'. We generated 10^3 realizations of β' using its prior distribution $\pi(\beta')$. For each realization, we simulated 10^3 signals as a Brownian motion with drift β', using μ_0 as a fixed intercept. A histogram of the first passage times for the 10^6 simulated signals is shown in Figure 25.

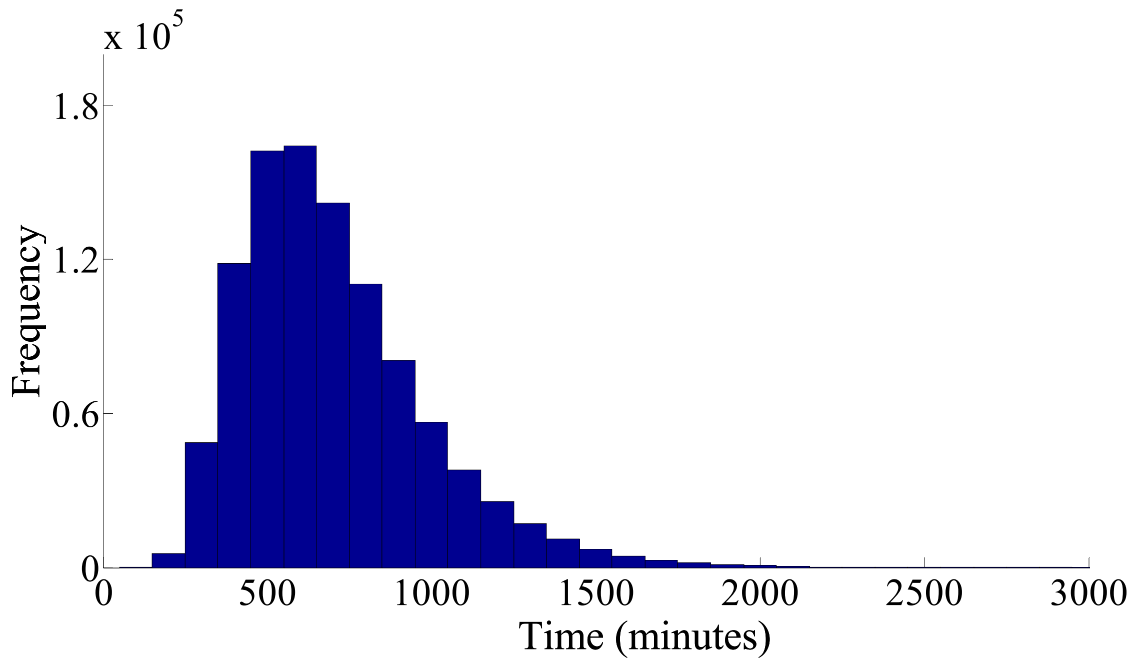

Figure 25 Histogram of the first passage time for the 10^6 simulated signals.

The maximum age was set equal to the 99th percentile, $\tilde{k} = 2500$. A discretization scheme with $m = 20$ and a discount factor of 0.99 were used. The optimal policy is shown in Figure 26 (middle curve), with a computed optimal value function $V^*(0, \ell_0) = 38.024$.

If the cost ratio c_2/c_1 is small (i.e., the cost of failure replacement, c_2, is close to the cost of planned replacement, c_1), one would expect a policy that is relatively tolerant to sudden failures of system. In other words, we would expect that the control limit would be closer to the failure threshold compared to another scenario where the cost of failure replacement is much higher than that of the planned replacement. To validate this claim, we set c_2 at \$4.00 and \$50.00, respectively, then solve for the optimal policy. In both cases the cost of planned replacement was kept at $c_1 = \$3.00$. The resulting control limits are displayed in Figure 26 for $c_2 = \$4.00$ (upper curve) and $c_2 = \$50.00$ (lower curve). We can verify that having a smaller ratio between the costs of planned replacement and failure replacement results in a less conservative control limit structure. This is apparent in the higher control limit over the system's lifetime (i.e., closer to the failure threshold).

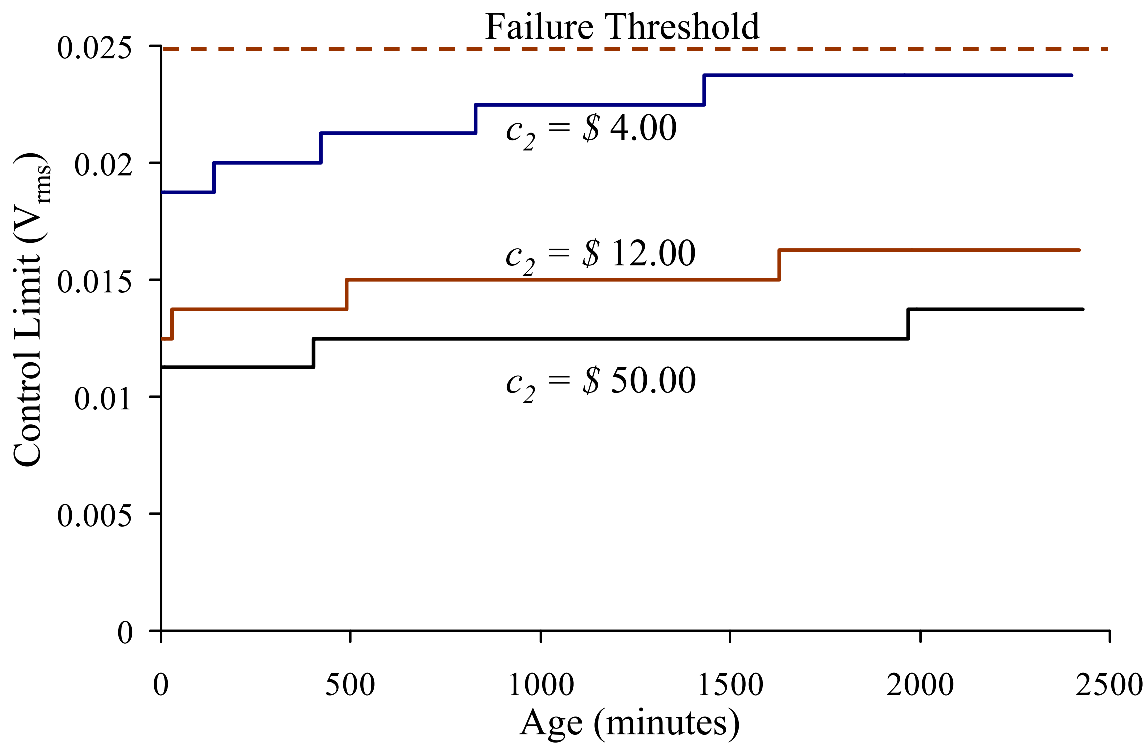

Figure 26 Optimal replacement policies.

5.2 Spare Parts Provisioning Policy

Our objective now is to determine a policy for ordering spare parts such that a part will be in stock when the need to replace the system arises. The monotonic control limit replacement policy established in Theorem 3, dictates that the system should be replaced whenever the observed signal, ℓ_k, crosses the replacement control limit, ℓ^*_j, at epoch any k. We can treat the non-decreasing replacement control limit (such as the one shown in Figure 24) as a "replacement threshold", as opposed to a "failure threshold". This replacement threshold can be used to get the distribution of the time taken by the signal to cross it. We refer to this distribution as the "replacement time distribution" (RTD). This RTD is then used to derive a spare parts provisioning policy.

5.2.1 Spare Part Ordering Criterion

Our approach is to use the RTD, at each epoch k, to compute the expected replacement time at the next epoch $k+1$. Let $T_R(k)$ be a random variable denoting the replacement time at epoch k. We can then define the following:

Ordering Criterion. At each decision epoch, $k \in \mathbf{K}$, order a spare part if the following holds:

$$E\left[T_R(k+1) \mid \ell_k = \ell\right] \leq LT \tag{43}$$

where LT is the constant ordering lead time. The term on the left hand side of inequality (43) represents the conditional expectation of the replacement time at the next decision epoch, given that the current observed signal, ℓ_k, is equal to some value ℓ. As discussed before, this ordering criterion attempts to eliminate spare part holding time and ensure just-in-time spare part delivery.

5.2.1.1 Structural Properties of the Spare Parts Provisioning Policy

We start by presenting two important Lemmas and Propositions that will be used to establish the structural properties of the spare parts provisioning policy proposed above.

Lemma 1. The random variable $T_R(k)$ is stochastically decreasing in ℓ_k, given a constant threshold ξ.

Proof. We notice from equation (14) that the CDF of $T_R(k)$, can be expressed as:

$$\Pr\left(T_R(k) \leq \tau \mid \Delta_0, \ldots, \Delta_k\right) = \Phi\left(\frac{\tilde{\mu}(t_k + \tau) - \xi}{\tilde{\sigma}(t_k + \tau)}\right) = \Phi\left(g'(k, \ell_k)\right),$$

where $g'(k, \ell_k) = \frac{\tilde{\mu}(t_k + \tau) - \xi}{\tilde{\sigma}(t_k + \tau)}$. Consider two different observed signals, ℓ_k^+ and ℓ_k^-, at the same epoch k, such that $\ell_k^+ \geq \ell_k^-$, and let $T_R(k)^+$ and $T_R(k)^-$ be random variables denoting the replacement time given the observed signals ℓ_k^+ and ℓ_k^-, respectively. By comparing $g'(k, \ell_k)$ to $g(k, \ell_k)$ in the proof of Proposition 5, it follows directly that $g'(k, \ell_k^+) \geq g'(k, \ell_k^-) \Rightarrow F_{T_R(k)^+}(x) \geq F_{T_R(k)^-}(x) \ \forall x$. ■

Lemma 2. The random variable $T_R(k)$ is stochastically increasing in k, given a constant threshold ξ.

Proof. Following a similar argument to the proof of Lemma 1, the result holds directly by Proposition 6. ■

Lemmas 1 and 2 establish stochastic ordering results for $T_R(k)$ given a constant threshold. We now use these Lemmas to establish similar results for $T_R(k)$ given a monotonically increasing threshold.

Proposition 7. The random variable $T_R(k)$ is stochastically decreasing in ℓ_k, given a monotonically non-decreasing piecewise linear threshold.

Proof. Consider two different observed signals, ℓ_k^+ and ℓ_k^-, at the same epoch k, such that $\ell_k^+ \geq \ell_k^-$, and let $T_R(k)^+$ and $T_R(k)^-$ be random variables denoting the replacement time given the observed signals ℓ_k^+ and ℓ_k^-, respectively, and the threshold ξ. Furthermore, let $T_R(k)'$ be a random variable denoting the replacement time at epoch k given ℓ_k^- and threshold $\xi^+ \geq \xi$. By Lemma 1, we know that:

$$F_{T_R(k)^+}(x) \geq F_{T_R(k)^-}(x) \qquad \forall x. \tag{44}$$

Recall that the distribution of $T_R(k)$ is given by $\Phi(g'(k,\ell_k))$. Since $\xi^+ \geq \xi$ by assumption, then we can write:

$$F_{T_R(k)^-}(x) \geq F_{T_R(k)'}(x) \qquad \forall x, \tag{45}$$

which can be combined with inequality (44) to establish that:

$$F_{T_R(k)^+}(x) \geq F_{T_R(k)'}(x) \qquad \forall x\ , \tag{46}$$

and hence, the result is established. ■

A similar argument can be followed to prove the following proposition that we state without proof.

Proposition 8. The random variable $T_R(k)$ is stochastically increasing in k, given a monotonically non-decreasing piecewise linear threshold.

The above results can be used to establish a monotonically non-decreasing control limit for the proposed spare parts provisioning policy as shown in the next theorem.

Theorem 4. For all decision epochs $k \in \mathbf{K}$, $\exists\ \vartheta_k^* \leq \ell_k^*$ such that the optimal decision is to order if and only if $\ell_k \geq \vartheta_k^*$. The control limit ϑ_k^* is monotonically non-decreasing in k.

Proof. The conditional expectation $E[T_R(k+1) \mid \ell_k = \ell]$ can be expressed as:

$$E[T_R(k+1) \mid \ell_k = \ell] = \int_0^{\ell_{k+1}^*} E[T_R(k+1)] f_{L|\ell_k}(\ell) d\ell\ , \tag{47}$$

where $f_{L|\ell_k}(\ell)$ is the updated density of the predictive distribution of the random variable L at the current epoch k, given an observed signal ℓ_k. In equation (47) above, $E[T_R(k+1)]$ is non-increasing in ℓ_k, since $T_R(k+1)$ is stochastically decreasing in ℓ_k by Proposition 7. Furthermore, since $f_{L|\ell_k}(\ell)$ is stochastically non-decreasing in ℓ_k, by

Proposition 5, then $E[T_R(k+1) \mid \ell_k = \ell]$ is non-increasing in ℓ_k. Now we assume that the result in Theorem 4 holds, and consider the inequality:

$$E[T_R(k+1) \mid \ell_k = \ell] \leq LT \tag{48}$$

The right hand side of this inequality is constant by assumption, and the left hand side is non-increasing in ℓ_k. Therefore, the inequality for any pair $(k, \ell_k) \in \mathbf{W}$ such that $\ell_k \geq \vartheta_k^*$. In other words, given that the optimal decision when the observed signal is ϑ_k^* is to order a spare part, the optimal decision for any $\ell_k \geq \vartheta_k^*$ is also to order. Therefore the proposed spare parts provisioning policy is a control limit policy with control limit ϑ_k^*.

Similarly, the left hand side of inequality (48) is non-decreasing in k by Propositions 6 and 8. Hence, for each observed signal $\ell \in \mathrm{P}$, there exists a threshold age k'_ℓ such that the optimal decision for any $k \leq k'_\ell$ is to order. By the existence a control limit ϑ_k^* for each age k and a threshold age k'_ℓ for each observed signal ℓ, the control limit ϑ_k^* is non-decreasing in age k. ■

5.2.2 Optimal Time to Order

As we have discussed previously in Chapter 3, the expression for determining the distribution of $T_R(k)$ does not have closed form moments. This imposes difficulties in implementing the spare part ordering criterion. We resort to the methodology proposed in Section 3.2 to obtain a closed form expression for the mean replacement time.

5.2.2.1 Computing the Mean Replacement Time

We start by re-visiting the discussion in Section 3.2 for determining a closed-form lower bound on the mean remaining life/replacement time given a constant threshold.

We have shown in Proposition 4 that when we use the updated posterior mean of the stochastic parameter β', $\mu_{\beta'}(k, \ell_k)$, to estimate the constant drift, we obtain a conservative lower bound on the mean replacement time. This method can be used within the spare part ordering framework discussed in this chapter. Another possible approach is to notice that the RTD follows an IG distribution for a fixed value of β'. In other words, $f_{T_R}(\tau \mid \beta')$ is IG. Therefore, the RTD can be determined by computing:

$$\int_{\beta'} f_{T_R}(\tau \mid \beta').d\beta' \tag{49}$$

This integral can only be computed numerically. We choose to use Monte Carlo methods to estimate the mean replacement time. That is, at each updating epoch, we generate n realizations of β' from its posterior distribution. For each realization, we compute the mean of the resulting IG distribution, $f_{T_R}(\tau \mid \beta')$. When n is sufficiently large, the average of the mean replacement times computed for each realization of β' will give a good estimate of the mean replacement time by the Strong Law of Large Number (SLLN).

This method for estimating the mean replacement time works in the case of a constant threshold. Recall that the replacement threshold is a monotonically non-decreasing piecewise linear threshold. Therefore, we need to consider the first passage time distribution of BM to this type of thresholds.

5.2.2.2 First Passage Time Distribution of Brownian motion to Monotonically Non-Decreasing Piecewise Linear Threshold

The problem of determining the first passage time to a time-varying threshold for BM is relevant in many applications, such as biology [188], economics [191], epidemiology [190-193], statistics [194-198], genetics [199], and mathematical finance [189]. Despite its importance and wide applications, explicit analytic solutions to the first passage time do no exist except for very instances. Abundo [200-202] discussed the distribution of the first passage time of BM to a piecewise linear threshold as follows:

Define the piecewise linear threshold, $u(t)$:

$$u(t) = \begin{cases} a_1 + b_1 t & , t < T_1 \\ a_2 + b_2 t & , T_1 \le t < T_2 \\ . \\ . \\ a_{n+1} + b_{n+1} t & , T_n \le t < T_f \ . \end{cases}$$

Then the first passage time distribution of the Brownian motion, $W(t)$, to $u(t)$ is given by:

$$\Pr\left(\sup_{[0,T_f]}\left(W_t < u(t)\right)\right) = \int_{-\infty}^{k_1} \gamma_n\left(T_1, \ldots, T_f\right) \cdot$$

$$\Pr\left(\cap_{[0,T_1]}\left(W_t < a_1 + b_1 t \mid W_{T_1} = c_1\right)\right) e^{-c_1^2/2T_1} \Big/ \sqrt{2\pi t_1} \cdot dc_1 \tag{50}$$

where $\gamma_n\left(T_1, \ldots, T_f\right)$ is defined inductively by:

$$\gamma_n(T_1,\ldots,T_f)=\int_{-\infty}^{k_2}\gamma_{n-1}(T_2-T_1,T_3-T_1,\ldots,T_n-T_1,T_f-T_1)\cdot$$

$$\Pr\left(\cap_{[0,T_2-T_2]}\left(W_t<a_2+b_2T_1-c_1+b_2t\,|\,W_{T_2-T_1}=c_2\right)\right)\cdot$$

$$\frac{\exp\left(-c_2^2/2(T_2-T_1)\right)}{\sqrt{2\pi(T_2-T_1)}}\cdot dc_2 \tag{51}$$

and $k_1=\min(a_1+b_1T_1,a_2+b_2T_1)$, $k_2=\min(a_2+b_2T_2-c_1,a_3+b_3T_2-c_1)$.

It is evident from equations (50) and (51) that computing this analytical expression is computationally burdensome, especially if the piecewise linear threshold has a large number of jumps. We propose next heuristic approaches to compute the mean replacement time to a piecewise linear threshold that make computations easier to provide the capability for real-time decision making.

5.2.2.3 Heuristic Approaches to Compute the Mean Replacement Time

The following heuristic approaches can be to compute the mean replacement time with less computational burden than the analytical method presented in Section 5.2.2.2:

1. Heuristic I:
    a. Approximate the piecewise linear replacement control limit by a straight line with slope s, as depicted in .
    b. Rotate the degradation signal data using:

$$S'_k=S_k-b\cdot kt\quad\forall k \tag{52}$$

    c. At each updating epoch k, generate n realizations of β'; for each realization, compute the mean replacement time as the mean of the first passage distribution of S'_k to a constant threshold equal to the intercept of the straight line at $k = 0$. (Notice that the first passage time distribution is IG).

d. Estimate the mean replacement time as the average of the n replacement times for each value of β'.

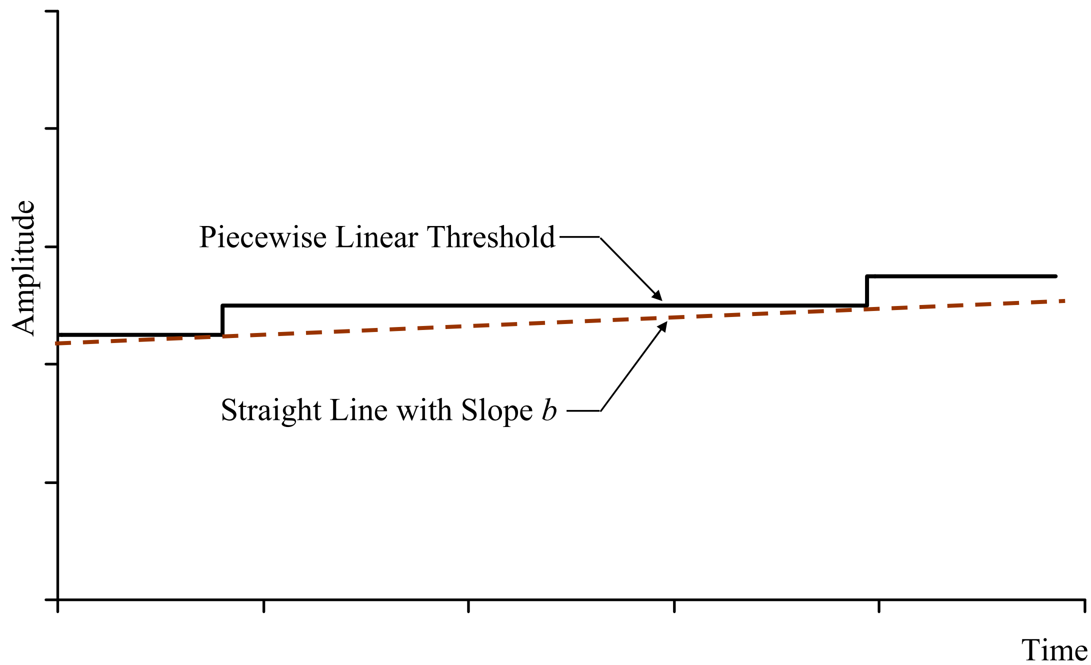

Figure 27 Schematic representation of Heuristic I

2. Heuristic I:

a. At each updating epoch k, take the replacement control limit, ℓ_k^*, to be the constant threshold from epoch k onwards.

b. Generate n realizations of β'; for each realization, compute the mean replacement time as the mean of the first passage distribution of S_k to the constant threshold from step (a). (Notice that the first passage time distribution is IG).

c. Estimate the mean replacement time as the average of the n replacement times for each value of β'.

3. Heuristic III:

a. At each updating epoch k, generate n realizations of β'. For each realizations, simulate m degradation signals as BM with drift β'.

b. For each simulated signal, observe the first passage time to the piecewise linear replacement threshold.

c. Estimate the mean replacement time as the average of the $m \times n$ replacement times for each simulated signal.

Heuristics I and II are only useful when the jumps of the piecewise replacement threshold are small. If these jumps are large, the estimate of the mean replacement time becomes inaccurate. We assess the performance of these heuristics now using our bearing degradation signals.

First, we compute the optimal replacement policy using the cost data $c_1 = \$2.50$, $c_2 = \$25.00$, and $c_3 = \$0.002$. The optimal policy is shown in Figure 28. Next, we consider one of the validation bearings, Bearing # 47, and compute the expected replacement time at each updating epoch using the three heuristic approaches discussed above, and the analytical method discussed in Section 5.2.2.2. The data for Bearing # 47 is displayed in Figure 29.

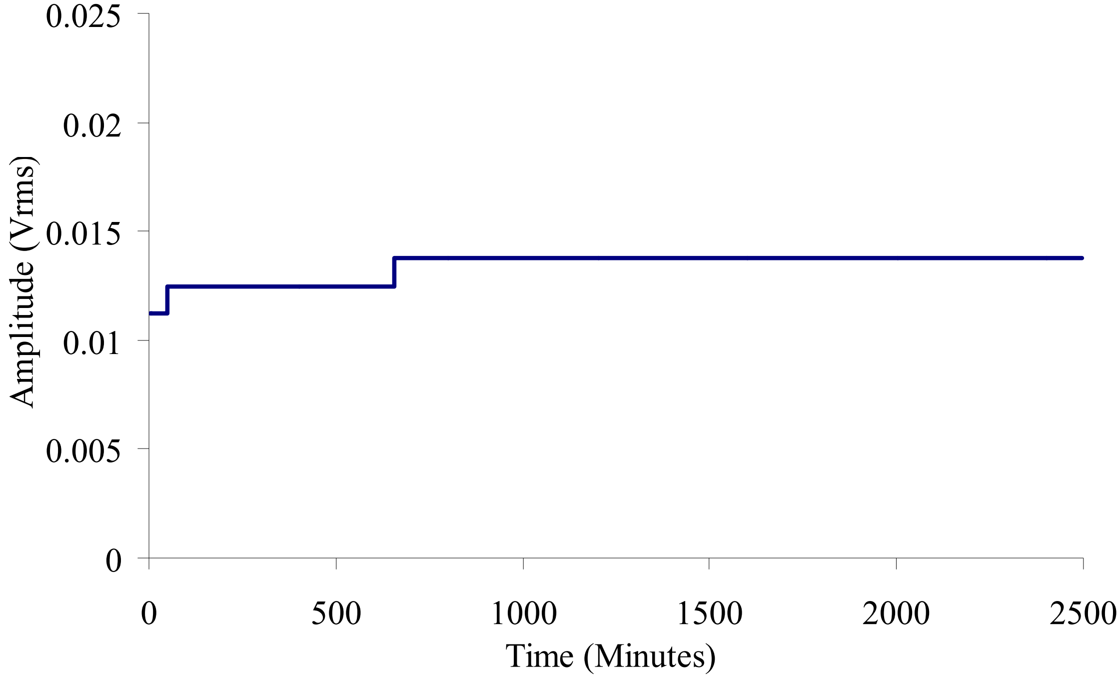

Figure 28 Optimal replacement policy.

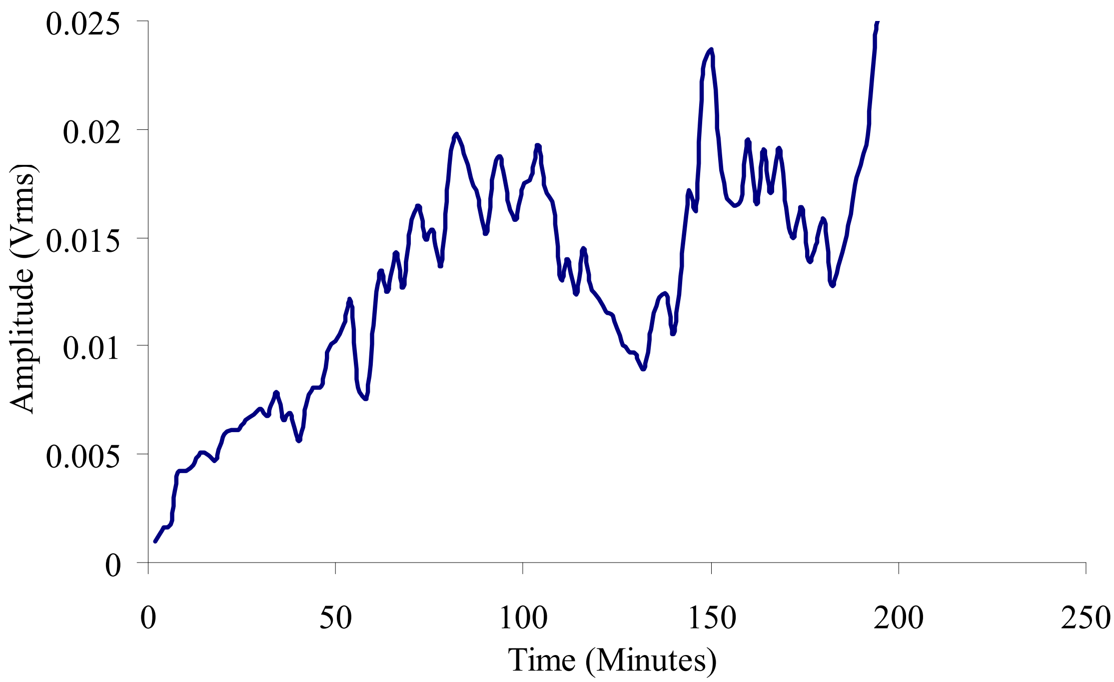

Figure 29 Degradation data for Bearing # 47.

Figure 30, Figure 31, and Figure 32 show the prediction results using Heuristic I, Heuristic II, and Heuristic III, respectively, for different numbers of β' realizations and simulated degradation signals at each epoch. Figure 33 benchmarks the results of Heuristic II using 10,000 β' realization and Heuristic III using 40,000 simulated signals at each epoch against the analytical approach.

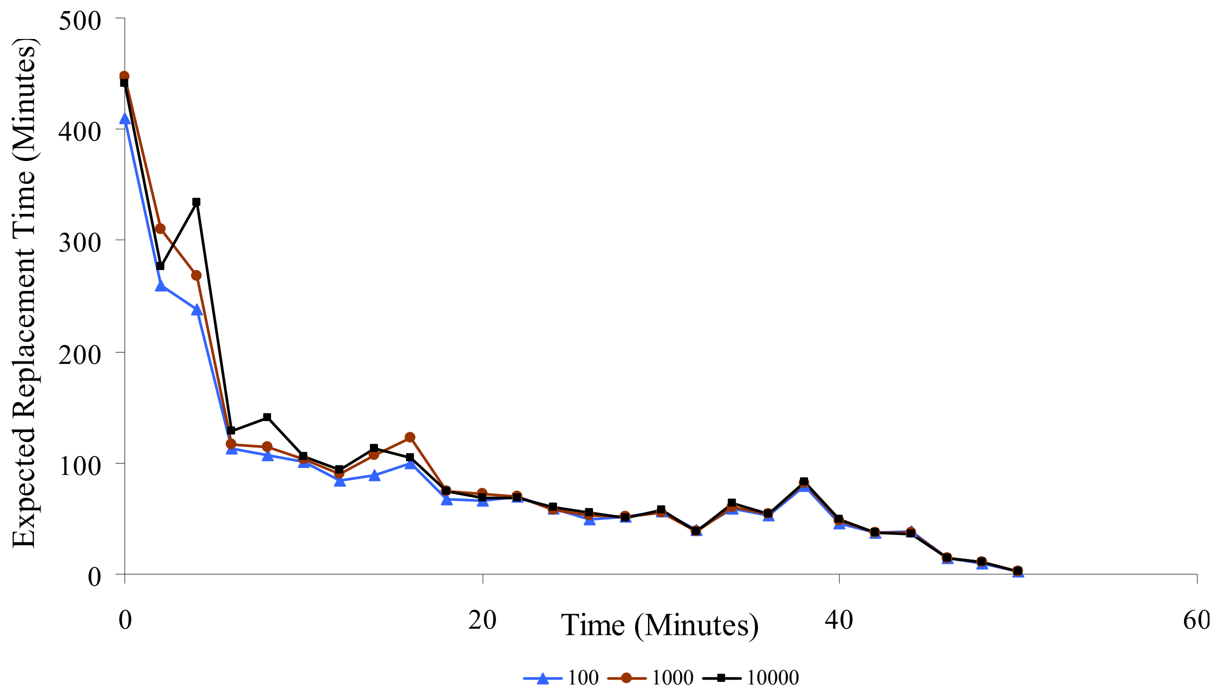

Figure 30 Prediction results using Heuristic I.

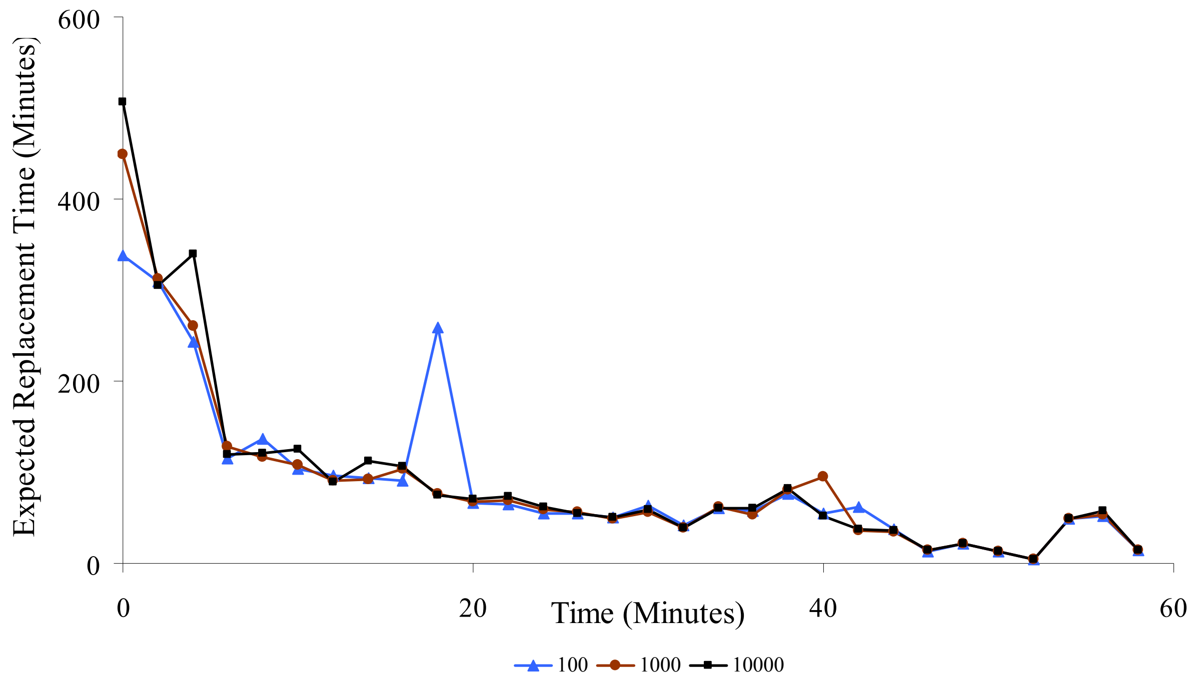

Figure 31 Prediction results using Heuristic II.

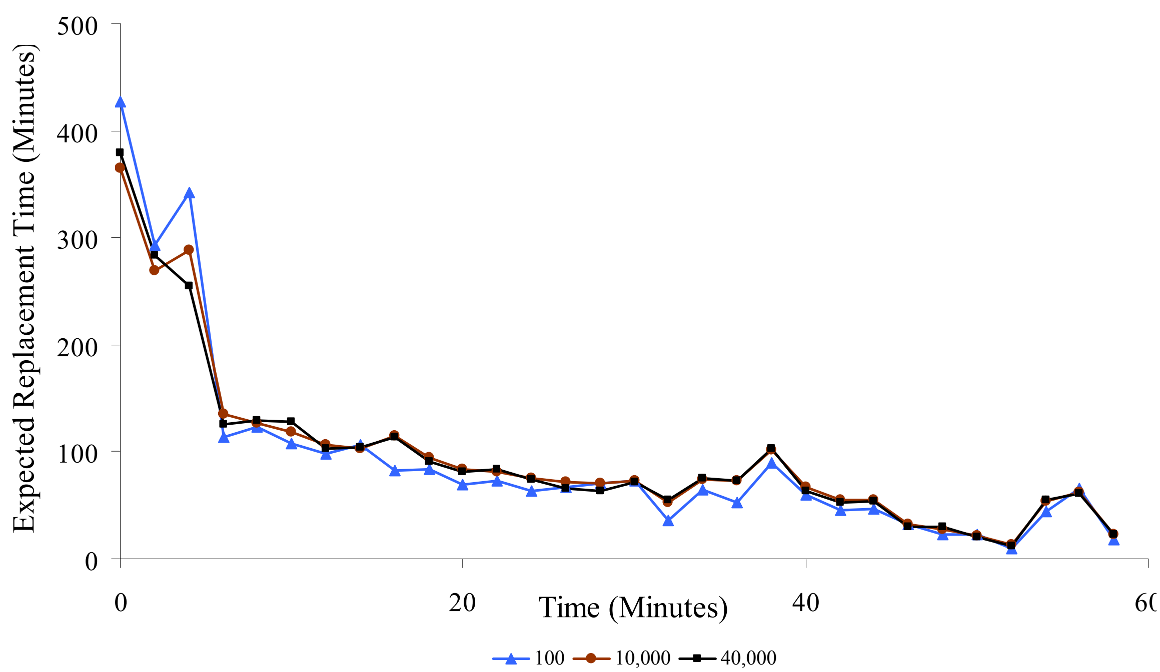

Figure 32 Prediction results using Heuristic III.

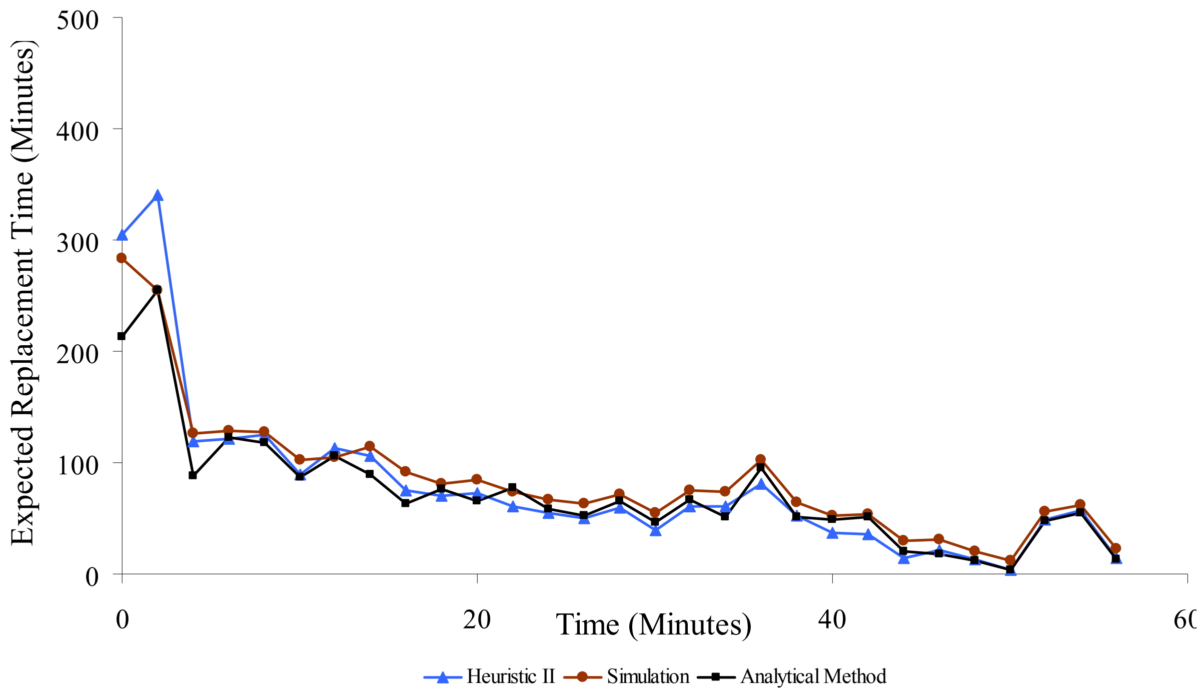

Figure 33 Benchmarking against analytical method.

The following conclusions can be made by inspecting the above results:

1. The proposed Heuristic approaches yield good results in comparison to the analytical method. This is represented in Figure 33, which shows the predictions made using Heuristics II and III being sufficiently close to the predictions made using the analytical method. Hence, these Heuristics can be used for real-world applications to enhance real-time decision making. We stress on the fact that Heuristics I and II work satisfactorily for this example because of the small jumps in the piecewise replacement threshold shown in Figure 28.
2. The number of β' realizations and simulated degradation signals in using Heuristics I, II, and III, respectively, need not be excessively large to yield

good results. Based on the results shown above, an order of 100 realizations/simulations would suffice. This helps achieve further reduction in the computation time, which is attractive for practical purposes. Summarizes the computation times for different approaches for the precious case study, emphasizing the computational savings offered by the heuristic approaches.

Table 6 Summary of computations times for different approaches

Method	Number Of β' Realizations/ Simulated Signals	Elapsed Time (Seconds)
Heuristic I	100	0.68
	1000	2.22
	10,000	21.68
Heuristic II	100	0.45
	1000	2.56
	10,000	24.93
Heuristic III	100	1.15
	10,000	93.39
	40,000	371.31
Analytical Method	10	76.55
	100	750.45

Finally, a one-step-look-ahead implementation of the spare part ordering policy was implemented for this example, setting the lead time $LT = 20$ minutes. For Bearing #47, at each updating epoch and given the observed signal, the expected replacement time at the next updating epoch was computed using Heuristics II and III. The optimal time to order the spare part was computed to be after 34 minutes. In summary, for Bearing #47, given the cost data and lead time for this example, the optimal replacement policy is shown in Figure 28. The optimal decisions are to replace the component after 62 minutes, at which the observed signal exceeds the replacement control limit, and order a spare part after 34 minutes.

In the next chapter, we outline the conclusions and contributions of this research and discuss future extensions.

CHAPTER 6
CONCLUSIONS AND FUTURE EXTENSIONS

In this thesis, we developed a mathematical framework that capitalizes on real-time *in situ* signals from monitoring fielded components to derive adaptive structured maintenance policies for single-unit systems. We outline the thesis contribution next.

6.1 Thesis Contributions

Within the existing literature on maintenance optimization, there is a lack of research efforts that target utilizing real-time condition monitoring data streams with maintenance decision models for real-time adaptive decision making. The contributions of this research can be categorized into the following items:

1. Degradation Modeling:
    a. Extending existing exponential degradation models by assuming the joint distribution of the stochastic model parameters and assessing the effect of this assumption on the model prediction accuracy.
    b. Deriving closed-form and easy to compute expressions for the lower bound on the mean remaining life.
    c. Providing a computationally efficient method to compute the remaining life distribution and its moments using the IG distribution.
2. Maintenance Decision Models:
    a. Developing heuristic replacement and spare parts ordering policies through integrating the proposed degradation modeling framework with traditional reliability-based decision models.

b. Integrating the degradation models with semi-Markov process models to determine optimal structured sensor-based replacement and spare parts ordering policies.

c. Validating both the proposed degradation models and sensor-based decision methodology using real-world data from rotating machinery, and developing a user friendly GUI interface for implementing the "sense-and-respond" logistics framework.

Validation results show that using sensory signals from fielded components result in better maintenance decisions than traditional reliability-based approaches and state models, through reduced costs and better system utilization.

6.2 Directions for Future Research

This thesis contributes in laying the foundations of sensor-based maintenance decision making. The findings of the current research motivate future extensions that include, but are not limited to, the following directions:

1. Structured maintenance policies under partial observations:

   In our proposed sensor-based maintenance policies, we assume that observed sensory signals fully capture the underlying degradation state of the system. This might not be the case in some real-world applications, where the acquired signals only reveal some partial information about the underlying system state. Partially observable Markov decision process models (POMDP) have been used for this class of problems. In the basic POMDP model, there is a set $\Theta = \{1, \ldots, N\}$ that describes the underlying system states. The stochastic process governing the transitions between these states is referred to as the core process, and is not directly observable; that is, its realizations are not determined with certainty. Alternatively, there is a finite message space, $\Omega = \{1, \ldots, M\}$, governed by a stochastic process referred to as the observation process. The core process and observation process are related through an $N \times M$ information matrix of probabilities, Q. There is a plethora of works on POMDP applications and solution algorithms. However, most of these efforts assume arbitrary probabilities for both the core and observation processes, and the information matrix. The main challenge in our case is to determine these probabilities based on *in situ* degradation signals, like we have demonstrated for the perfect observation case in Chapter 5. This research direction is attractive and has lot of potential practical significance. For example, one can derive bounds on the optimal cost function for a partially observable replacement problem by characterizing the structure of optimal strategies for completely observed and the completely unobserved cases, such as in the work done by White [142]. Also, special structures of the optimal sensor-based maintenance policies under partial observation need to be investigated.

2. Repair as a possible maintenance action:

   In our model, we only considered that the actions available at each decision epoch are to preventively replace the system with a new one, or do nothing and continue to observe. We plan to provide a new formulation of the MDP replacement model where the "repair" action is included in the action space. This action does not necessarily result in moving the system to the good-as-new state, but can possibly result in moving the system to some better degradation state. We will investigate the structure of the optimal policy for the new model and carefully investigate the necessary conditions for this structure to hold.

3. Sensor-based Maintenance Policies for Systems with Multiple Components:

   The proposed maintenance policies address single-unit systems. Further investigation is needed to extend our sensor-based decision methodology to systems with multiple components. Some challenges in this research direction include: (1) deriving expressions for the updated system remaining life distribution, (2) interpreting large data streams and using this sensor based information to identify priority clusters of components for performing replacement, and (3) developing special algorithms for solving the resulting complex models in real time.

   Systematic approaches need to be used to identify economical multi-component sensor-driven replacement models and their appropriate applications. In particular, sensor-based block replacement and opportunistic replacement policies can be revisited. In opportunistic replacement, for example, preventive replacement of a component can be performed at any opportunity created by another component's failure, another component's planned preventive replacement, or the individual component's planned replacement time. Also, inventory models with space to store more than one unit (single buffer) need to be considered.

4. New sensor-based decision models:

   This task includes developing new decision models that consider:

   - Different objective functions; such as system availability.

- Constraints; such as warehouse capacity, customer service level, and budget constraints.
- Assumptions; such as assuming variable inventory lead time.

5. Large-Scale Condition-Based Supply Chain Management:

The proposed research is a starting point towards establishing the foundation of real-time condition-based decision making methodologies for supply chain management. There is good potential to capitalize on the benefits of emerging trends in sensor technology and advanced monitoring systems to create dynamic decision making strategies in which decisions are based on real-time information (for example, integrating condition-based spare parts inventory policies with distribution decisions and vehicle dispatching). This will have positive impact both in industries with emphasis on manufacturing systems, such as the automotive industry and aerospace industry, and other application domains, such as health care applications and military operations, among others.

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