Matching structure and Pfaffian orientations of graphs

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Matching structure and Pfaffian orientations of graphs

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To my mother

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TABLE OF CONTENTS

LIST OF FIGURES

SUMMARY

The first result of this thesis is a generation theorem for bricks. A brick is a 3-connected graph such that the graph obtained from it by deleting any two distinct vertices has a perfect matching. The importance of bricks stems from the fact that they are building blocks of a decomposition procedure of Kotzig, and Lovász and Plummer. We prove that every brick except for the Petersen graph can be generated from K_4 or the prism by repeatedly applying certain operations in such a way that all the intermediate graphs are bricks. We use this theorem to prove an exact upper bound on the number of edges in a minimal brick with given number of vertices and to prove that every minimal brick has at least three vertices of degree three.

The second half of the thesis is devoted to an investigation of graphs that admit Pfaffian orientations. We prove that a graph admits a Pfaffian orientation if and only if it can be drawn in the plane in such a way that every perfect matching crosses itself even number of times. Using similar techniques, we give a new proof of a theorem of Kleitman on the parity of crossings in drawings of K_{2j+1} and $K_{2j+1,2k+1}$ and develop a new approach to Turan's problem of estimating crossing number of complete bipartite graphs.

We further extend our methods to study k-Pfaffian graphs and generalize a theorem by Gallucio, Loebl and Tessler. Finally, we relate Pfaffian orientations and signs of edge-colorings and prove a conjecture of Goddyn that every k-edge-colorable k-regular Pfaffian graph is k-list-edge-colorable. This generalizes a theorem of Ellingham and Goddyn for planar graphs.

CHAPTER I

INTRODUCTION

In this chapter we give an overview of the problems we will be addressing and of the main results of this dissertation. Terminology and notation are introduced.

1.1 Perfect Matchings and Bricks

We adopt the definition of a graph from the book by Bondy and Murty [5], except we do not allow loops. A *graph* G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a finite non-empty set $V(G)$ of *vertices*, a finite set $E(G)$, disjoint from $V(G)$, of *edges*, and an incidence function ψ_G that associates with each edge an unordered pair of distinct vertices of G. We say that a graph is *simple* if ψ_G is injective.

If $e \in E(G)$ and $\psi_G(e) = \{u, v\}$ we say that the edge e *joins* vertices u and v and frequently write $e = uv$. We say that vertices u and v are *adjacent* if they are joined by an edge. We say that an edge e is *incident* with v and that e *covers* v.

The graph theoretical terminology we will be using is fairly standard and to avoid getting bogged down in the definitions we refer the reader to the book of Diestel [11] for the definitions of connectivity, paths, cycles, components, etc. We use "$\backslash$" for edge and vertex deletion and "$-$" for set-theoretic difference. The cardinality of a set X is denoted by $|X|$, and the symmetric difference of sets X and Y is denoted by $X \triangle Y$.

A *perfect matching* is a set of edges in a graph that covers every vertex exactly once. Let $\mathcal{M}(G)$ or $\mathcal{M}$, if the graph is understood from the context, denote the set of all perfect matchings of a graph G. Properties of this set are the main focus of this dissertation. A brief exposition of the known results in this area follows. See [28] for a comprehensive overview of matching theory.

The fundamental result of Tutte [53] describes the graphs for which $\mathcal{M}$ is non-empty.

Theorem 1.1.1. *A graph G has a perfect matching if and only if for every $S \subseteq V(G)$ the* *number of odd components in $G \setminus S$ is less then or equal to $|S|$.*

In graphs with perfect matchings one can define different objects that describe the structure of $\mathcal{M}$ from different points of view. Let the perfect matching polytope of a graph PM(G) be defined as the convex hull of incidence vectors of perfect matchings. A famous theorem of Edmonds [13] gives a description of this polytope. To state this result we need to give another definition.

A *cut* in a graph G is a set $\delta(S)$ of all edges joining vertices of S to vertices of $V(G) - S$ for some non-empty $S \subsetneq V(G)$. We say that a cut is *odd* if S and $V(G) - S$ have odd cardinality and we say that a cut is *trivial* if S or $V(G) - S$ contains only one vertex.

Theorem 1.1.2. *Let G be a graph with $V(G)$ even. A vector $x \in \mathbb{R}^{E(G)}$ lies in PM(G) if and only if it satisfies the following constraints:*

(i) $x \geq 0$*;*

(ii) $x(C) = 1$ *for every trivial cut C;*

(iii) $x(C) \geq 1$ *for every odd cut C.*

We say that an odd cut C in a graph G is *tight* if every perfect matching of G contains exactly one edge in it. Clearly every trivial cut is tight. Let us denote by lin($\mathcal{M}$) the linear hull of the incidence vectors of perfect matchings. Then $x(C) = x(D)$ for every $x \in \text{lin}(\mathcal{M})$ and two tight cuts C and D in G. It turns out that there are no other constraints.

Theorem 1.1.3. [31] *Let G be a graph with $V(G)$ even. A vector $x \in \mathbb{R}^{E(G)}$ lies in lin($\mathcal{M}$) if and only if $x(C) = x(D)$ for every two tight cuts C and D in G.*

When investigating the structure of $\mathcal{M}$ it suffices to restrict our attention to graphs in which every edge belongs to a perfect matching. We say that such graphs are *matching-covered* or 1-*extendable*.

One might naturally be interested in the dimension of PM(G) or lin($\mathcal{M}$) for a graph G. Tight cuts play an important role in this problem. Using notation from [28] let us denote, for $S \subseteq V(G)$, by $G \times S$ the graph obtained from G by contracting S to a single point.

Lemma 1.1.4. [28] *Let G be a matching-covered graph. Let (S_1, S_2) be a partition of $V(G)$ such that $C = \delta(S_1) = \delta(S_2)$ is a tight cut. Then*

$$dimPM(G \times S_1) + dimPM(G \times S_2) = dimPM(G) + |C| - 1.$$

It turns out that many important properties of $\mathcal{M}(G)$ can be read off from $\mathcal{M}(G_1)$ and $\mathcal{M}(G_2)$, where $G_1 = G \times S$, $G_2 = G \times (V(G) - S)$ and $C = \delta(S)$ is a tight cut. If C is non-trivial then $|V(G_1)| < |V(G)|$ and $|V(G_2)| < |V(G)|$. We say that G *decomposes* along C into G_1 and G_2. We can apply this decomposition procedure repeatedly until all the resulting graphs have no non-trivial tight cuts and reconstruct properties of $\mathcal{M}(G)$ (such as dim PM(G)) from the corresponding properties of these resulting graphs. This *tight cut decomposition procedure* is due to Kotzig, and Lovász and Plummer [28].

This motivates the study of the graphs that have no non-trivial tight cuts. There are two such classes of graphs. A *brick* is a 3-connected bicritical graph, where a graph G is *bicritical* if $G\backslash u\backslash v$ has a perfect matching for every two distinct vertices $u, v \in V(G)$. A *brace* is a connected bipartite graph such that every matching of size at most two is contained in a perfect matching.

Theorem 1.1.5. [12, 29] *A matching covered graph has no non-trivial tight cuts if and only if it is either a brick or a brace.*

Therefore the tight cut decomposition procedure decomposes every matching-covered graph into bricks and braces. Moreover, except for parallel edges, the result of such a decomposition does not depend on our choice of non-trivial tight cuts during the process.

Theorem 1.1.6. [29] *The result of any two tight cut decomposition procedures of the same graph is the same list of bricks and braces, up to multiplicity of edges.*

Let us now return our attention to the dimension of perfect matching polytope.

Theorem 1.1.7. [12] *The dimension of the perfect matching polytope of a connected matching-covered bipartite graph G is $|E(G)| - |V(G)| + 1$.*

The dimension of the perfect matching polytope of a brick G is $|E(G)| - |V(G)|$.

Finally, the next theorem gives the dimension of the perfect matching polytope of a general matching-covered graph.

Theorem 1.1.8. [12] *The dimension of the perfect matching polytope of a matching-covered graph G is*

$$|E(G)| - |V(G)| + 1 - b,$$

where b is the number of bricks in the tight cut decomposition of G.

Note that Theorem 1.1.8 is an immediate corollary of Lemma 1.1.4 and Theorem 1.1.7. The dimension of lin($\mathcal{M}$) can be calculated in a similar fashion.

Theorem 1.1.9. [12] *The dimension of the linear hull of perfect matchings in a matching covered graph G is $|E(G)| - |V(G)| + 2 - b$.*

Let the *matching lattice*, lat($\mathcal{M}$), be the set of all linear combinations with integer coefficients of the incidence vectors of perfect matchings. The problem of describing lat($\mathcal{M}$) remained unsolved longer than the analogous problems for PM(G) and lin($\mathcal{M}$). The analogue of Lemma 1.1.4 holds, and the problem reduces to describing lat($\mathcal{M}$) for bricks and braces. In [27] Lovász resolved the problem for braces.

Theorem 1.1.10. *If G is a brace, then lat($\mathcal{M}$) consists of all vectors $x \in \mathbb{Z}^{E(G)}$ such that $x(\delta(v)) = x(\delta(v'))$ for every two vertices $v, v' \in V(G)$.*

Theorem 1.1.10 implies that for braces (and therefore for all bipartite graphs) lat($\mathcal{M}$) = lin($\mathcal{M}$) $\cap\, \mathbb{Z}^{E(G)}$. This is a natural characterization, but, unfortunately, it does not hold for bricks. The Petersen graph is a counterexample. Consider the incidence vector x of the edge set of two vertex disjoint cycles of length five in the Petersen graph (see Figure 1). Then $x \in$lin($\mathcal{M}$) $\cap\, \mathbb{Z}^{E(G)}$ by Theorem 1.1.3, but $x \notin$ lat($\mathcal{M}$) as every perfect matching in the Petersen graph has even number of edges in common with every cycle of length five. However, Lovász [29] proved the following deep result.

Theorem 1.1.11. *Let G be a brick other than the Petersen graph. Then lat($\mathcal{M}$) consists precisely of all vectors $x \in \mathbb{Z}^{E(G)}$ such that $x(\delta(v)) = x(\delta(v'))$ for every two vertices $v, v' \in V(G)$.*

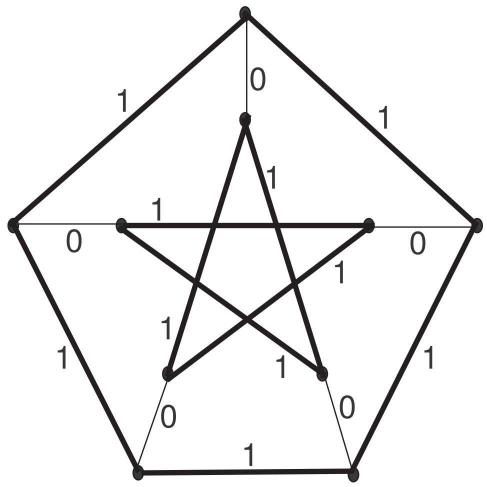


Figure 1: An integer valued vector that lies in $\mathrm{lin}(\mathcal{M})$, but not in $\mathrm{lat}(\mathcal{M})$

Many problems of interest concerning perfect matchings are reduced to bricks in similar fashion via the tight cut decomposition procedure. One such problem is described in detail in the next section. Thus understanding the properties of bricks is important.

In Chapter 2 we prove a "splitter theorem" for bricks. More precisely, we show that if a simple brick H is a "matching minor" of a simple brick G, then, except for a few well-described exceptions, a graph isomorphic to H can be obtained from G by repeatedly applying a certain operation in such a way that all the intermediate graphs are bricks and have no parallel edges. The operation is as follows: first delete an edge, and for every vertex of degree two that results contract both edges incident with it. Precise definitions and statements are given in Section 2.1. This theorem generalizes a recent result of de Carvalho, Lucchesi and Murty [7], and immediately implies Theorem 1.1.11.

In Chapter 3 we apply the results of Chapter 2 to prove a generating theorem for minimal bricks. Such a theorem is preferable to theorems in Chapter 2 for computational purposes. We also prove two corollaries of this generating theorem. The first one is an exact upper bound for the number of edges in a minimal brick. The second one establishes the fact that every minimal brick has at least three vertices of degree three. The conjecture of Lovász that every minimal brick has at least one such vertex was recently settled by de Carvalho, Lucchesi and Murty [7].

Our motivation for generating bricks came from Pfaffian orientations: another matching-related problem that can be reduced to bricks. An overview of the history of the problem and our results occupies the remainder of the introduction.

1.2 Pfaffian Orientations: History

Another property of $\mathcal{M}$ one might want to compute is its cardinality. Unfortunately the problem of counting perfect matchings in a bipartite graphs is equivalent to the problem of computing the permanent of a $(0,1)$-matrix, which is known to be $\sharp P$-complete [56]. However, if a graph has an orientation satisfying certain properties described below then the problem can be solved in polynomial time.

In a directed graph we denote by uv an edge directed from u to v. A *labeled graph* or *digraph* is a graph or digraph with vertex-set $\{1, 2, \ldots, n\}$ for some n. Let G be a directed labeled graph and let $M = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be a perfect matching of G. Define the *sign* of M to be the sign of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_k & v_k \end{pmatrix}.$$

Note that the sign of a perfect matching is well-defined as it does not depend on the order in which the edges are listed. We say that a labeled graph G is *Pfaffian* if there exists an orientation D of G such that the signs of all perfect matchings in D are positive, in which case we say that D is a *Pfaffian orientation* of G. An unlabeled graph G is *Pfaffian* if it is isomorphic to a labeled Pfaffian graph. It is well-known and easy to verify that in this case every labeling of G is Pfaffian. Pfaffian orientations have been introduced by Kasteleyn [19, 20, 21], who demonstrated that one can enumerate perfect matchings in a Pfaffian graph in polynomial time. He proved the following.

Theorem 1.2.1. [21] *Every planar graph is Pfaffian.*

We say that an $n \times n$ matrix $A(D) = (a_{ij})$ is a *skew adjacency matrix* of a directed labeled graph D with n vertices if

$$a_{ij} = \begin{cases} 1 & \text{if } ij \in E(D), \\ -1 & \text{if } ji \in E(D), \\ 0 & \text{otherwise.} \end{cases}$$

Let A be a skew-symmetric $2n \times 2n$ matrix. For each partition

$$P = \{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_n, j_n\}\}$$

of the set $\{1, 2, \ldots, 2n\}$ into pairs, define

$$a_P = sgn \begin{pmatrix} 1 & 2 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & \ldots & i_n & j_n \end{pmatrix} a_{i_1 j_1} \ldots a_{i_n j_n}.$$

Note that a_P is well defined as it does not depend on the order of the pairs in the partitions nor on the order in which the pairs are listed. The *Pfaffian* of the matrix A is defined by

$$Pf(A) = \sum_P a_P,$$

where the sum is taken over all partitions P of the set $\{1, 2, \ldots, 2n\}$ into pairs. Note that if D is a Pfaffian orientation of a labeled graph G then $Pf(A(D))$ is equal to the number of perfect matchings in G. One can evaluate the Pfaffian efficiently using the following identity from linear algebra: for a skew-symmetric matrix A

$$det(A) = (Pf(A))^2.$$

Thus the number of perfect matchings, and more generally the generating function of perfect matchings of a Pfaffian graph, can be computed in polynomial time.

The problem of recognizing Pfaffian bipartite graphs is equivalent to many problems of interest outside graph theory, eg. the Pólya permanent problem [39], the even circuit problem for directed graphs [57], or the problem of determining which real square matrices are sign non-singular [22], where the latter has applications in economics [44]. Two satisfactory solutions of this problem are known.

The complete bipartite graph $K_{3,3}$ is not Pfaffian. Each edge of $K_{3,3}$ belongs to exactly two perfect matchings and therefore changing an orientation of any edge does not change

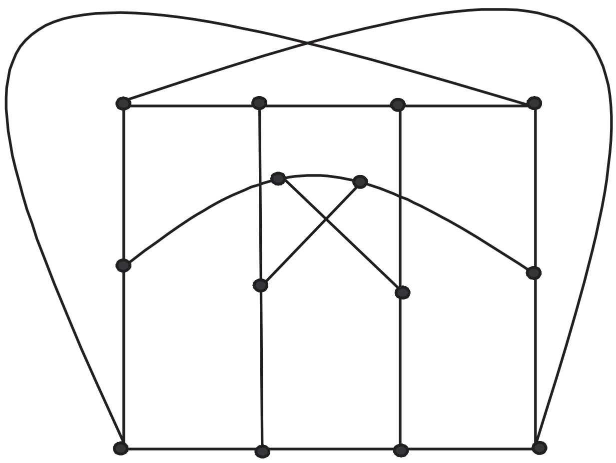


Figure 2: The Heawood graph

the parity of the number of perfect matchings with negative sign. One can easily verify that for some (and therefore for every) orientation of $K_{3,3}$ this number is odd.

We say that a graph G' is an *even subdivision* of a graph G if G' is obtained from G by repeatedly replacing edges of G by paths of odd length. A graph is Pfaffian if and only if every even subdivision of it is Pfaffian. We say that J is a *central subgraph* of G if $G \backslash V(J)$ has a perfect matching. The property of being Pfaffian is closed under taking central subgraphs. Little [24] gave the following characterization of Pfaffian bipartite graphs.

Theorem 1.2.2. *A bipartite graph is Pfaffian if and only if it does not contain an even subdivision of $K_{3,3}$ as a central subgraph.*

Recently several shorter proofs of Theorem 1.2.2 were obtained. See [9, 35, 48].

Another, structural characterization of Pfaffian bipartite graphs was given by Robertson, Seymour and Thomas [41] and independently by McCuaig [30]. To state this result we need to introduce some definitions from [41]. Let G_0 be a graph, let C be a central cycle in G_0 of length four, and let G_1, G_2 be two subgraphs of G_0 such that $G_1 \cup G_2 = G_0$, $G_1 \cap G_2 = C$, $V(G_1) - V(G_2) \neq \emptyset$ and $V(G_2) - V(G_1) \neq \emptyset$. Let G be obtained from G_0 by deleting some (possibly none) edges of C. Then we say that G is a 4-sum of G_1 and G_2. The *Heawood graph* is the bipartite graph which is the incidence graph of the Fano plane (see Figure 2).

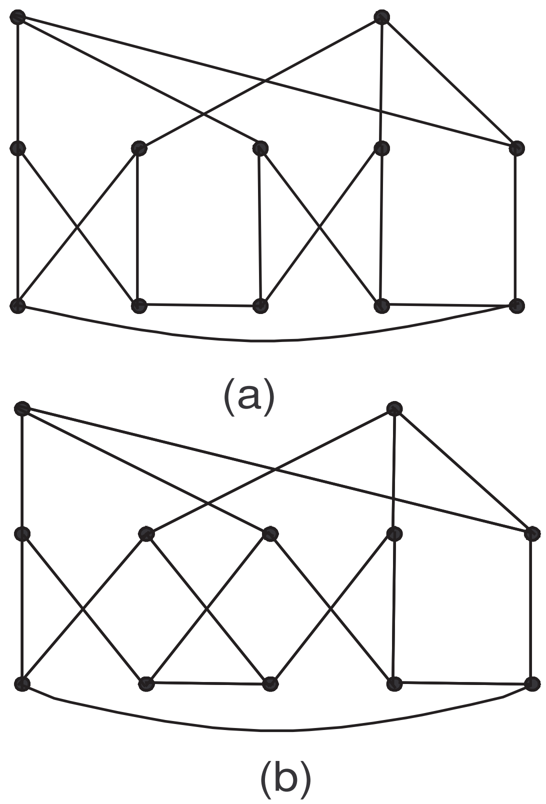


Figure 3: (a) Cubeplex, (b) Twinplex.

Theorem 1.2.3. *A brace is Pfaffian if and only if either it is isomorphic to the Heawood graph, or it can be obtained from planar braces by repeated application of the* 4*-sum operation.*

One of the more important aspects of Theorem 1.2.3 is the fact that Robertson, Seymour and Thomas [41] use it to design a polynomial-time algorithm to decide if a bipartite graph is Pfaffian.

In [15] Fischer and Little extend Theorem 1.2.2 to characterize near-bipartite Pfaffian graphs. A matching-covered non-bipartite graph G is *near-bipartite* if there exist $e, f \in E(G)$ such that $G \backslash \{e, f\}$ is matching-covered and bipartite. A graph G is said to be *reducible* to a graph H if H can be obtained from G by a sequence of odd cycle contractions. It is shown in [26] that the property of being Pfaffian is closed under such reductions. Cubeplex and twinplex are particular graphs on 12 vertices (see Figure 3).

Theorem 1.2.4. [15] *A near-bipartite graph is Pfaffian if and only if it contains no central*

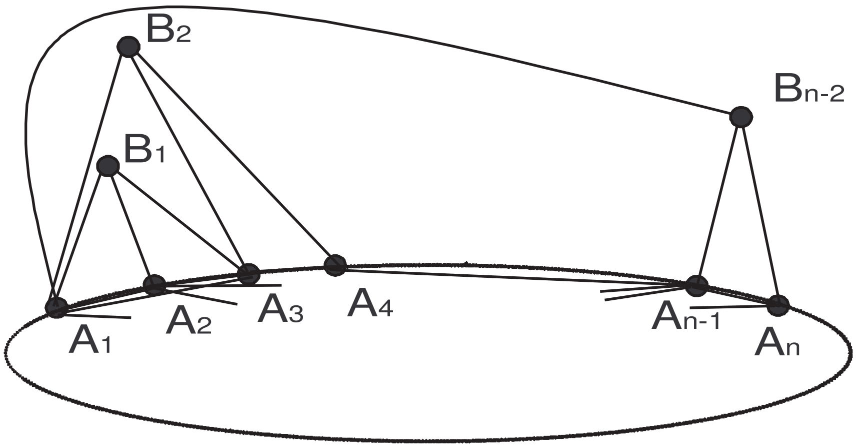


Figure 4: Dense Pfaffian brick

subgraph reducible to an even subdivision of $K_{3,3}$, cubeplex or twinplex.

The problem of characterizing general Pfaffian graphs is wide open. We will return to it.

1.3 Pfaffian Orientations: Our Results

By [25, 57] a graph G is Pfaffian if and only if every brick and every brace in its tight cut decomposition is Pfaffian. This fact motivated our research on generating bricks.

However, we have discovered substantial obstructions to implementing both the structural approach of Theorem 1.2.3 and the "excluded minor" approach of Theorems 1.2.2 and 1.2.4.

We have found a family of Pfaffian bricks H_n on $2n-2$ vertices and $(n^2+5n-12)/2$ edges that have K_n as a subgraph, which we will now describe. Let $V(H_n) = \{A_1, A_2, \ldots$ $A_n, B_1, B_2, \ldots, B_{n-2}\}$. Let the vertices $A_1, A_2, \ldots, A_n$ form a clique and let B_i be joined by an edge to A_1, A_{i+1} and A_{i+2} for every $1 \le i \le n-2$ (see Figure 4). It is easy to see that H_n is a brick and we will soon see that H_n is Pfaffian.

The existence of these examples implies that most likely there is no structural characterization of Pfaffian bricks similar to Theorem 1.2.3, because such a characterization would imply a linear upper bound on the number of edges in Pfaffian bricks.

We have also found an infinite family of bricks, which are minimally non-Pfaffian with respect to taking central subgraphs and reductions. In fact, this family contains exponentially many elements with given number of vertices. This family obstructs the approach suggested by Theorems 1.2.2 and 1.2.4.

In Chapter 4 we obtain a characterization of Pfaffian graphs in terms of their drawings (with crossings) in the plane. Drawings and crossings are formally defined in Section 4.1, but the definitions are fairly intuitive.

Theorem 1.3.1. *A graph is Pfaffian if and only if it can be drawn in the plane in such a way that every perfect matching crosses itself an even number of times.*

It is easy to see using Theorem 1.3.1 that the bricks H_n described above are Pfaffian. Indeed, no two edges that belong to the same perfect matching cross in the drawing depicted on Figure 4 (vertices $B_1, B_2, \ldots, B_{n-2}$ form an independent set and therefore every perfect matching contains exactly one edge joining two of the vertices $A_1, A_2, \ldots, A_n$).

We prove Theorem 1.3.1 as a corollary of a general, but technical result about parities of crossings in "T-joins" in different drawings of the same graph. In Section 4.4 we take a slight detour from our main objective and demonstrate other applications of this result. We give a new proof of a theorem of Kleitman [23] on the parity of crossings in drawings of K_{2j+1} and $K_{2j+1,2k+1}$. We also give a purely combinatorial reformulation of the famous Turan's brickyard problem [52] and prove the uniqueness of the drawing of the Petersen graph which minimizes the number of crossings.

There are several ways to generalize Pfaffian graphs and Theorem 1.3.1.

In Chapter 5 we consider k-Pfaffian graphs. For a labeled graph G, an orientation D of G and a perfect matching M of G, denote the sign of M in the directed graph corresponding to D by $D(M)$. We say that a labeled graph G is *k-Pfaffian* if there exist orientations $D_1, D_2, \ldots, D_k$ of G and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, such that for every perfect matching M of G

$$\sum_{i=1}^{k} \alpha_i D_i(M) = 1.$$

Clearly a graph is Pfaffian if and only if it is 1-Pfaffian. The number of perfect matchings in a k-Pfaffian graph is equal to $\sum_{i=1}^{k} \alpha_i P_i$, where P_i is the Pfaffian of the skew-symmetric matrix associated with D_i. Therefore if $D_1, D_2, \ldots, D_k$ and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are given one can enumerate perfect matchings in a k-Pfaffian graph efficiently.

The following result was mentioned by Kasteleyn [20] and proved by Galluccio and Loebl [16] and independently by Tesler [51].

Theorem 1.3.2. *Every graph that can be embedded on an orientable surface of genus g is 4^g-Pfaffian.*

We prove that a graph is 4-Pfaffian if and only if it can be drawn on the torus in such a way that every perfect matching crosses itself an even number of times. We also prove that 3-Pfaffian graphs are Pfaffian and that 5-Pfaffian graphs are 4-Pfaffian. For $k > 5$ we prove partial results (including a generalization of Theorem 1.3.2) and state conjectures.

Another way to generalize Pfaffian orientations is motivated by the list-edge coloring conjecture. A graph G is called *k-list-colorable* if for every set system $\{S_v : v \in V(G)\}$ such that $|S_v| = k$ there exists a proper vertex coloring c with $c(v) \in S_v$ for every $v \in V(G)$. Not every k-colorable graph is k-list colorable. A classic example is $K_{3,3}$ with bipartition (A, B) and $\{S_v : v \in A\} = \{S_v : v \in B\} = \{\{1,2\}, \{1,3\}, \{2,3\}\}$.

A graph is called *k-list-edge-colorable* if for every set system $\{S_e : e \in E(G)\}$ such that $|S_e| = k$ there exists a proper edge coloring c with $c(e) \in S_e$ for every $e \in E(G)$. The following famous list-edge-coloring conjecture was suggested independently by various researchers and first appeared in print in [4].

Conjecture 1.3.3. *Every k-edge-colorable graph is k-list-edge-colorable.*

In a k-regular graph there is a way to define a sign for every k-edge coloring. By a beautiful and powerful argument of Alon and Tarsi [1], the list-edge-coloring conjecture holds for all k-regular k-edge-colorable graphs in which all the k-edge colorings have the same sign. Ellingham and Goddyn [14] proved using this technique that the list-edge-coloring conjecture holds for every k-regular k-edge-colorable planar graph. Note that our definition of graphs allows multiple edges and therefore there exist k-regular planar graphs with $k > 5$.

Goddyn conjectured that this result generalizes to Pfaffian graphs. In Chapter 6 we generalize Pfaffian orientations to Pfaffian labelings and prove that Goddyn's conjecture holds for those graphs that admit Pfaffian labelings. We also give two characterizations of graphs that admit a Pfaffian labeling: one in terms of bricks and braces in their tight cut decomposition, and another in terms of their drawings in the projective plane.

Finally, in Chapter 7 we discuss the possibilities of a structural characterization of Pfaffian graphs and a polynomial time recognition algorithm. We outline possible directions of further work.

CHAPTER II

GENERATING BRICKS

In this chapter we prove a generating theorem for bricks. The material presented in this chapter will also appear in [36].

All the graphs considered in this chapter are simple.

2.1 Introduction

The following well-known theorem of Tutte [54] describes how to generate all 3-connected graphs, but first a definition. Let v be a vertex of a graph H, and let N_1, N_2 be a partition of the neighbors of v into two disjoint sets, each of size at least two. Let G be obtained from $H \backslash v$ by adding two vertices v_1 and v_2, where v_i has neighbors $N_i \cup \{v_{3-i}\}$. We say that G was obtained from H by *splitting a vertex.* Thus for 3-connected graphs splitting a vertex is the inverse of contracting an edge that belongs to no triangle. A *wheel* is a graph obtained from a cycle by adding a vertex joined to every vertex of the cycle.

Theorem 2.1.1. *Every* 3*-connected graph can be obtained from a wheel by repeatedly applying the operations of adding an edge between two nonadjacent vertices and splitting a vertex.*

A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. Seymour [46] extended Theorem 2.1.1 as follows.

Theorem 2.1.2. *Let H be a* 3*-connected minor of a* 3*-connected graph G such that H is not isomorphic to K_4 and G is not a wheel. Then a graph isomorphic to G can be obtained from H by repeatedly applying the operations of adding an edge between two nonadjacent vertices and splitting a vertex.*

Our objective is to prove an analogous theorem for bricks.

We need a few definitions before we can describe it. Let G be a graph, and let v_0 be a vertex of G of degree two incident with the edges $e_1 = v_0v_1$ and $e_2 = v_0v_2$. Let H be obtained from G by contracting both e_1 and e_2 and deleting all resulting parallel edges. We say that H was obtained from G by *bicontracting* or *bicontracting the vertex* v_0, and write $H = G/v_0$. Let us say that a graph H is a *reduction* of a graph G if H can be obtained from G by deleting an edge and bicontracting all resulting vertices of degree two. By a *prism* we mean the unique 3-regular planar graph on six vertices. The following is a generation theorem of de Carvalho, Lucchesi and Murty [7].

Theorem 2.1.3. *If G is a brick other than K_4, the prism, and the Petersen graph, then some reduction of G is a brick other than the Petersen graph.*

Thus if a brick G is not the Petersen graph, then the reduction operation can be repeated until we reach K_4 or the prism. By reversing the process Theorem 2.1.3 can be viewed as a generation theorem. It is routine to verify that Theorem 2.1.3 implies Theorem 1.1.11, and that demonstrates the usefulness of Theorem 2.1.3. Our main theorem strengthens Theorem 2.1.3 in two respects. (We have obtained our result independently of [7], but later. We are indebted to the authors of [7] for bringing their work to our attention.) The first strengthening is that the generation procedure can start at graphs other than K_4 or the prism, as we explain next. Let a graph J be a subgraph of a graph G. We say that a graph H is a *matching minor* of G if H can be obtained from a central subgraph of G by repeatedly bicontracting vertices of degree two. Thus if H can be obtained from G by repeatedly taking reductions, then H is isomorphic to a matching minor of G. We will denote the fact that G has a matching minor isomorphic to H by writing $H \hookrightarrow G$. This is consistent with our notation for embeddings, to be introduced in Section 2.4. Since every brick has a matching minor isomorphic to K_4 or the prism by [28, Theorem 5.4.11], , the following implies Theorem 2.1.3.

Theorem 2.1.4. *Let G be a brick other than the Petersen graph, and let H be a brick that is a matching minor of G. Then a graph isomorphic to H can be obtained from G by repeatedly taking reductions in such a way that all the intermediate graphs are bricks not* *isomorphic to the Petersen graph.*

We say that a graph H is a *proper reduction* of a graph G if it is a reduction in such a way that the bicontractions involved do not produce parallel edges. We would like to further strengthen Theorem 2.1.4 by replacing reductions by proper reductions; such an improvement is worthwhile, because in applications it reduces the number of cases that need to be examined. Unfortunately, Theorem 2.1.4 does not hold for proper reductions, but all the exceptions can be conveniently described. Let us do that now.

Let C_1 and C_2 be two vertex-disjoint cycles of length $n \geq 3$ with vertex-sets $\{u_1, u_2,\ldots, u_n\}$ and $\{v_1, v_2,\ldots, v_n\}$ (in order), respectively, and let G_1 be the graph obtained from the union of C_1 and C_2 by adding an edge joining u_i and v_i for each $i = 1, 2, \ldots, n$. We say that G_1 is a *planar ladder*. Let G_2 be the graph consisting of a cycle C with vertex-set $\{u_1, u_2,\ldots,u_{2n}\}$ (in order), where $n \geq 2$ is an integer, and n edges with ends u_i and u_{n+i} for $i = 1, 2,\ldots, n$. We say that G_2 is a *Möbius ladder*. A *ladder* is a planar ladder or a Möbius ladder. Let G_1 be a planar ladder as above on at least six vertices, and let G_3 be obtained from G_1 by deleting the edge u_1u_2 and contracting the edges u_1v_1 and u_2v_2. We say that G_3 is a *staircase*. Let $t \geq 2$ be an integer, and let P be a path with vertices $v_1, v_2, \ldots, v_t$ in order. Let G_4 be obtained from P by adding two distinct vertices x, y and edges xv_i and yv_j for $i = 1, t$ and all even $i \in \{1, 2, \ldots, t\}$ and $j = 1, t$ and all odd $j \in \{1, 2, \ldots, t\}$. Let G_5 be obtained from G_4 by adding the edge xy. We say that G_5 is an *upper prismoid*, and if $t \geq 4$, then we say that G_4 is a *lower prismoid*. A *prismoid* is a lower prismoid or an upper prismoid. We are now ready to state our main theorem.

Theorem 2.1.5. *Let H, G be bricks, where H is isomorphic to a matching minor of G. Assume that H is not isomorphic to K_4 or the prism, and G is not a ladder, wheel, staircase or prismoid. Then a graph isomorphic to H can be obtained from G by repeatedly taking proper reductions in such a way that all the intermediate graphs are bricks not isomorphic to the Petersen graph.*

If H is a brick isomorphic to a matching minor of a brick G and G is a ladder, wheel, staircase or prismoid, then H itself is a ladder, wheel, staircase or prismoid, and can be obtained from a graph isomorphic to G by taking (improper) reductions in such a way that all intermediate graphs are bricks. Thus Theorem 2.1.5 implies Theorem 2.1.4. (Well, this is not immediately clear if the graph H from Theorem 2.1.4 is a K_4 or a prism, but in those cases the implication follows with the aid of the next theorem.)

As a counterpart to Theorem 2.1.5 we should describe the starting graphs for the generation process of Theorem 2.1.5. Notice that K_4 is a wheel, a Möbius ladder, a staircase and an upper prismoid, and that the prism is a planar ladder, a staircase and a lower prismoid. Later in this section we show

Theorem 2.1.6. *Let G be a brick not isomorphic to K_4, the prism or the Petersen graph. Then G has a matching minor isomorphic to one of the following seven graphs: the graph obtained from the prism by adding an edge, the lower prismoid on eight vertices, the staircase on eight vertices, the staircase on ten vertices, the planar ladder on ten vertices, the wheel on six vertices, and the Möbius ladder on eight vertices.*

McCuaig [30] proved an analogue of Theorem 2.1.5 for braces. To state his result we need another exceptional class of graphs. Let C be an even cycle with vertex-set $v_1, v_2, \ldots, v_{2t}$ in order, where $t \geq 2$is an integer and let G_6 be obtained from C by adding vertices v_{2t+1} and v_{2t+2} and edges joining v_{2t+1} to the vertices of C with odd indices and v_{2t+2} to the vertices of C with even indices. Let G_7 be obtained from G_6 by adding an edge $v_{2t+1}v_{2t+2}$. We say that G_7 is an *upper biwheel*, and if $t \geq 3$ we say that G_6 is a *lower biwheel*. A *biwheel* is a lower biwheel or an upper biwheel. McCuaig's result is as follows.

Theorem 2.1.7. *Let H, G be braces, where H is isomorphic to a matching minor of G. Assume that if H is a planar ladder, then it is the largest planar ladder matching minor of G, and similarly for Möbius ladders, lower biwheels and upper biwheels. Then a graph isomorphic to H can be obtained from G by repeatedly taking proper reductions in such a way that all the intermediate graphs are braces.*

Actually, Theorem 2.1.7 follows from a version of our theorem stated in Section 2.11.

Let us now introduce terminology that we will be using in the rest of the thesis. Let $H, G, v_0, v_1, v_2, e_1, e_2$ be as in the definition of bicontraction. Assume that both v_1 and v_2

have degree at least three and that they have no common neighbors except v_0; then no parallel edges are produced during the contraction of e_1 and e_2. Let v be the new vertex that resulted from the contraction. If both v_1 and v_2 have degree at least three, then we say that G was obtained from H by *bisplitting the vertex* v. We call v_0 the *new inner vertex* and v_1 and v_2 the *new outer vertices.*

Let H be a graph. We wish to define a new graph H'' and two vertices of H''. Either $H'' = H$ and u, v are two nonadjacent vertices of H, or H'' is obtained from H by bisplitting a vertex, u is the new inner vertex of H'' and $v \in V(H'')$ is not adjacent to u, or H'' is obtained by bisplitting a vertex of a graph obtained from H by bisplitting a vertex, and u and v are the two new inner vertices of H''. Finally, let $H' = H'' + (u, v)$. We say that H' is a *linear extension* of H. By *the cube* we mean the graph of the 1-skeleton of the 3-dimensional cube. Notice that the cube and $K_{3,3}$ are bipartite, and hence are not bricks. Using this terminology Theorem 2.1.5 can be restated in a mildly stronger form. It is easy to check that if G' is obtained from a brick G by bisplitting a vertex into new outer vertices v_1 and v_2, then $\{v_1, v_2\}$ is the only set $X \subseteq V(G')$ such that $|X| \geq 2$ and $G' \backslash X$ has at least $|X|$ odd components. Thus a linear extension of a brick is a brick, and hence Theorem 2.1.8 implies Theorem 2.1.5.

Theorem 2.1.8. *Let G be a brick other than the Petersen graph, and let H be a 3-connected matching minor of G not isomorphic to K_4, the prism, the cube, or $K_{3,3}$. If G is not isomorphic to H and G is not a ladder, wheel, biwheel, staircase or prismoid, then a linear extension of H is isomorphic to a matching minor of G.*

The main step in the proof of Theorem 2.1.8 is the following.

Theorem 2.1.9. *Let G be a brick other than the Petersen graph, and let H be a 3-connected matching minor of G. Assume that if H is a planar ladder, then there is no strictly larger planar ladder L with $H \hookrightarrow L \hookrightarrow G$, and similarly for Möbius ladders, wheels, lower biwheels, upper biwheels, staircases, lower prismoids and upper prismoids. If H is not isomorphic to G, then some matching minor of G is isomorphic to a linear extension of H.*

It is routine to verify that if G is a ladder, wheel, biwheel, staircase or prismoid, G' is a linear extension of G, and H is a 3-connected matching minor of G not isomorphic to K_4, the prism, the cube, or $K_{3,3}$, then G' has a matching minor isomorphic to a linear extension of H. Thus Theorem 2.1.9 implies Theorem 2.1.8, and we omit the details. The proof of Theorem 2.1.9 will occupy the rest of the chapter. However, assuming Theorem 2.1.9 we can now deduce Theorem 2.1.6.

Proof of Theorem 2.1.6, *assuming* Theorem 2.1.9. Let G be a brick not isomorphic to K_4, the prism or the Petersen graph. By [28], Theorem 5.4.11, G has a matching minor M isomorphic to K_4 or the prism. Since M is not bipartite, it is not a biwheel, a planar ladder on $4k$ vertices, or a Möbius ladder on $4k+2$ vertices. Thus if a prismoid, wheel, ladder or staircase larger than M is isomorphic to a matching minor of G, then G has a matching minor as required for Theorem 2.1.6. Thus we may assume that the hypothesis of Theorem 2.1.9 is satisfied, and hence a matching minor of G is isomorphic to a linear extension of M. But K_4 does not have any linear extensions, and the prism has, up to isomorphism, exactly one, namely the graph obtained from it by adding an edge. This proves Theorem 2.1.6. $\square$

Here is an outline of the chapter. First we need to develop some machinery; that is be done in Sections 2.2, 2.3, and 2.4. In Section 2.5 we prove a first major step toward Theorem 2.1.9, namely that the theorem holds provided a graph obtained from H by bisplitting a vertex is isomorphic to a matching minor of G. Then in Section 2.6 we reformulate our key lemma in a form that is easier to apply, and introduce several different types of extensions. In Section 2.7 we use the 3-connectivity of G to show that at least one of those extensions of H is isomorphic to a matching minor of G, and in Sections 2.8– 2.10 we gradually eliminate all the additional extensions. Theorem Theorem 2.1.9 is proved in Section 2.10. Finally, in Section 2.11 we state a strengthening of Theorem 2.1.9 that can be obtained by following the proof of Theorem 2.1.9 with minimal changes. We delegate the strengthening to the last section, because the statement is somewhat cumbersome and perhaps of lesser interest. Its applications include Theorem 2.1.9, Theorem 2.1.7 and a generation theorem for a subclass of factor-critical graphs.

A word about notation. If H is a graph, and $u, v \in V(H)$ are distinct vertices, then $H + (u, v)$ or $H + uv$ denotes the graph obtained from H by adding an edge with ends u and v. If u and v are adjacent then $H + uv = H$. Now let $u, v \in V(H)$ be adjacent. By *bisubdividing* the edge uv we mean replacing the edge by a path of length three, say a path with vertices u, x, y, v, in order. Let H' be obtained from H by this operation. We say that x, y (in that order) are the *new vertices*. Thus y, x are the new vertices resulting from subdividing the edge vu (we are conveniently exploiting the notational asymmetry for edges). Now if $w \in V(H) - \{u\}$, then by $H + (w, uv)$ we mean the graph $H' + (w, x)$. Notice that the graphs $H + (w, uv)$ and $H + (w, vu)$ are different. In the same spirit, if $a, b \in V(H)$ are adjacent vertices of H with $\{u, v\} \neq \{a, b\}$, then we define $H + (uv, ab)$ to be the graph $H' + (x, ab)$.

2.2 Octopi and Frames

Let H be a graph with a perfect matching, and let $X \subseteq V(H)$ be a set of size k. If $H \backslash X$ has at least k odd components, then X is called a *barrier* in H. The following is easy and well-known.

Lemma 2.2.1. *A brick has no barrier of size at least two.*

Now if H and X are as above and H is a subgraph of a brick G, then X cannot be a barrier in G. If H is a central subgraph of G, then we get the following useful outcome.

Lemma 2.2.2. *Let G be a brick and let H be a subgraph of G. Let M be a perfect matching of $G \backslash V(H)$ and let $V(H)$ be a disjoint union of X, $R_1, R_2, \ldots, R_k$, where $k \geq 2$, $|X| \leq k$ and $|R_i|$ is odd for every $i \in \{1, 2, \ldots, k\}$. Then there exist distinct integers $i, j \in \{1, 2, \ldots, k\}$ and an M-alternating path joining a vertex in R_i to a vertex in R_j.*

Proof. Suppose for a contradiction that the lemma is false, and let H be a maximal subgraph of G that satisfies the hypothesis of the lemma, but not the conclusion.

By Lemma 2.2.1 there exists an edge $e_1 \in E(G)$ with one end $v \in R_i$ for some $i \in \{1, 2, \ldots, k\}$ and the other end $u \in V(G) - R_i - X$. Without loss of generality we may assume that $i = 1$. If $u \in V(H)$ then the path with edge-set $\{e_1\}$ is as required. Thus

$u \notin V(H)$, and hence u is incident with an edge $e_2 \in M$. Let w be the other end of e_2; then clearly $w \notin V(H)$. Let $X' = X \cup \{u\}$, $R_{k+1} = \{w\}$, $M' = M - \{e_2\}$ and construct H' by adding the vertices u and w and edges e_1 and e_2 to H. By the maximality of H the graph H', matching M' and sets $X', R_1, R_2, \ldots, R_{k+1}$ satisfy the conclusion of the lemma. Thus for some distinct integers $i, j \in \{1, 2, \ldots, k+1\}$ there exists an M'-alternating path P joining a vertex in R_i to a vertex in R_j. Since H does not satisfy the conclusion of the lemma we may assume that $j = k+1$. Let P' be the graph obtained from P by adding the edges e_1 and e_2. If $i > 1$, then P' is a path and satisfies the conclusion of the lemma.

Thus we may assume that $i = 1$. Let $H'' = H \cup P'$, $M'' = M - E(P')$ and $R_1' = R_1 \cup V(P')$. Then the graph H'', matching M'' and sets $X, R_1', R_2, \ldots, R_k$ also satisfy the conclusion of the lemma by the maximality of H. Thus we may assume that there is an M''-alternating path Q joining a vertex in R_1' to a vertex in R_j for some $j \in \{2, 3, \ldots, k\}$. If neither of the ends of Q lies in $V(P')$ then Q is a required path for H. If one of them, say x, is in $V(P')$, we add to Q one of the subpaths of P' with end x to obtain a required path. $\square$

In applications we will need a stronger conclusion and H will have a special structure, which we now introduce. Let H be a graph, let C be a subgraph of H with an odd number of vertices, and let $P_1, P_2, ..., P_k$ be odd paths in H. For $i = 1, 2, ..., k$ let u_i and v_i be the ends of P_i. If for $i = 1, 2, ..., k$ we have $u_i \in V(C)$ and $V(P_i) \cap (V(C) \cup \bigcup_{j \neq i} V(P_j)) = \{u_i\}$ we say that $\Omega = (C, P_1, P_2, \ldots, P_k)$ is an *octopus* in H. We say that the paths $P_1, P_2, ..., P_k$ are the *tentacles* of Ω, C is the *head* of Ω and v_i are the *ends* of Ω. We define the *graph of* Ω to be $C \cup P_1 \cup P_2 \cup \cdots \cup P_k$, and by abusing notation slightly we will denote this graph also by Ω. We say that a matching M in G is Ω-*compatible* if every tentacle is M-alternating and no vertex of C is incident to an edge of M. See Figure 5.

Let G be a graph, and let $k \geq 1$ be an integer. We say that the pair $(\mathcal{F}, X)$ is a *frame* in G if $X \subseteq V(G)$ and $\mathcal{F} = \{\Omega_1, \Omega_2, \ldots, \Omega_k\}$ satisfy

(1) $\Omega_1, \Omega_2, \ldots, \Omega_k$ are octopi,

(2) for $i = 1, 2, \ldots, k$ the ends and only the ends of Ω_i belong to X,

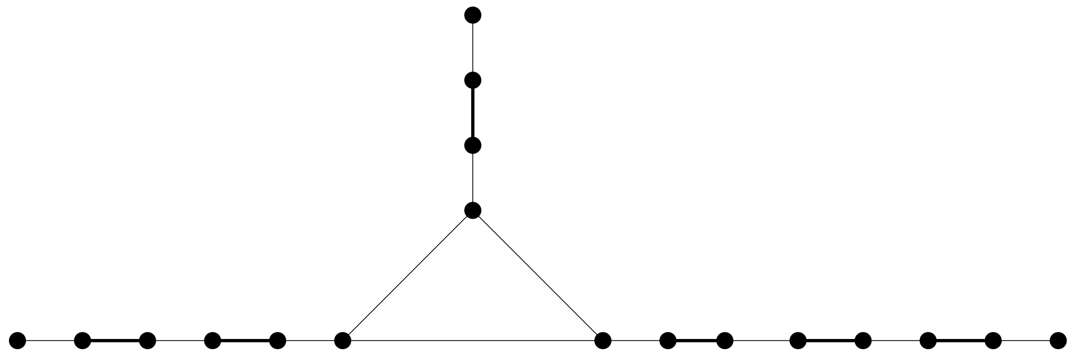


Figure 5: An octopus Ω and an Ω-compatible matching

(3) for distinct $i, j \in \{1, 2, \ldots, k\}$, $V(\Omega_i) \cap V(\Omega_j) \subseteq X$,

(4) $|X| \leq k$.

We say that $\Omega_1, \Omega_2, \ldots, \Omega_k$ are the *components* of $(\mathcal{F}, X)$. We define the graph of $(\mathcal{F}, X)$ to be $\Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_k$, and denote it by $\mathcal{F}$, again abusing notation. The following is the main result of this section. We say that a graph H is *M-covered* if a subset of M is a perfect matching of H.

Theorem 2.2.3. *Let G be a brick, let M be a matching in G, and let $(\mathcal{F}, X)$ be a frame in G such that $G \setminus (V(\mathcal{F}) \cup X)$ is M-covered and M is Ω-compatible for each $\Omega \in \mathcal{F}$. Then there exists an M-alternating path P joining vertices of the heads of two different components Ω_1, Ω_2 of $(\mathcal{F}, X)$. Moreover, there is an edge $e \in E(P) - M$ such that the two components of $P \setminus e$ can be numbered P_1 and P_2 in such a way that $V(P_i) \cap V(\mathcal{F}) \subseteq V(\Omega_i)$ for $i = 1, 2$.*

Proof. We say that a subpath Q of a path P is an *$\mathcal{F}$-jump* in P if the ends of Q belong to different components of $\mathcal{F}$ and Q is otherwise disjoint from $\mathcal{F}$. Let $\mathcal{F} = \{\Omega_1, \Omega_2, \ldots, \Omega_k\}$ and let C_i denote the vertex-set of the head of Ω_i. By Lemma 2.2.2 applied to $X, C_1, C_2, \ldots, C_k$ there exists an M-alternating path joining vertices of the heads of two different components of $(\mathcal{F}, X)$. Choose such path P with the minimal number of $\mathcal{F}$-jumps in it. We prove that P satisfies the requirements of the theorem.

Let $v_1 \in C_1$ and $v_2 \in C_2$ be the ends of P. Since P is M-alternating and M is Ω_i-compatible for all $i = 1, 2, \ldots, k$, it follows that no internal vertex of P belongs to C_i. Suppose that $P \cap T \neq \emptyset$ for some tentacle T of Ω_i, where $i \geq 3$. Let $\{v_0\} = V(T) \cap C_i$ and let $v \in V(P) \cap V(T)$ be chosen so that $T[v, v_0]$ is minimal. For some $j \in \{1, 2\}$ the path $P[v_j, v] \cup T[v, v_0]$ is M-alternating and contradicts the choice of P. Thus $V(P) \cap V(\mathcal{F}) \subseteq V(\Omega_1) \cup V(\Omega_2)$.

Define a linear order on $V(P)$ so that $v \succ v'$ if and only if $v' \in P[v_1, v]$. Let P_0 be an $\mathcal{F}$-jump in P with ends $u_1 \in V(\Omega_1)$ and $u_2 \in V(\Omega_2)$ chosen so that $u_1 \succ u_2$ and $P[v_1, u_2]$ is minimal. Equivalently we can define P_0 as a second $\mathcal{F}$-jump we encounter if we traverse P from v_1 to v_2. If such an $\mathcal{F}$-jump P_0 in P does not exist then P contains a unique $\mathcal{F}$-jump. Let $e \notin M$ be an edge of this unique $\mathcal{F}$-jump; then P and e satisfy the requirements of the theorem. Therefore we may assume the existence of P_0.

For $i \in \{1, 2\}$ let T_i be the tentacle of Ω_i such that $u_i \in V(T_i)$ and let $\{w_i\} = V(T_i) \cap C_i$. Let $s_1 \in V(T_1) \cap V(P)$ be chosen so that $s_1 \succ u_1$ and $T_1[s_1, w_1]$ is minimal. Note that $s_1 \neq w_1$, because the only vertex in $V(P) \cap C_1$ is v_1 and $s_1 \succ u_1 \succ v_1$. Let s_1t_1 be the edge of M incident to s_1. We have $s_1t_1 \in E(T_1 \cap P)$ as both T_1 and P are M-alternating, $s_1 \in T_1[t_1, w_1]$ by the choice of s_1 and $s_1 \succ t_1$ as otherwise the path $T_1[w_1, s_1] \cup P[s_1, v_2]$ contradicts the choice of P. Let $s_2 \in V(T_2) \cap V(P)$ be chosen so that $s_2 \prec s_1$ and $T_2[s_2, w_2]$ is minimal. Let s_2t_2 be the edge of M incident to s_2. We again have $s_2t_2 \in T_2 \cap P$, $s_2 \in T_2[t_2, w_2]$ and $s_2 \prec t_2$, as otherwise the path $P[v_1, s_2] \cup T_2[s_2, w_2]$ contradicts the choice of P.

Consider $P' = P[s_2, s_1]$. By the choice of s_1 we have $V(P[u_2, s_1]) \cap V(T_1[s_1, w_1]) = \{s_1\}$. Also if $s_2 \prec u_2$ we have $V(P[s_2, u_2] \cap V(\Omega_1) = \emptyset$ by the choice of P_0. It follows that $V(P') \cap V(T_1[s_1, w_1]) = \{s_1\}$. By the choice of s_2 we have $V(P') \cap V(T_2[s_2, w_2]) = \{s_2\}$. Therefore $T_2[w_2, s_2] \cup P' \cup T_1[w_1, s_1]$ is an M-alternating path contradicting the choice of P. $\square$

2.3 Two Paths Meeting

In this section we study the following problem. Let G be a graph, let M be a matching, and let P_1 and P_2 be two M-alternating paths. In the applications we will be permitted to replace the matching M by a matching M' saturating the same set of vertices, and to replace the paths P_1 and P_2 by a pair of M'-alternating paths with the same ends. Thus we are interested in graphs that are minimal in the sense that there is no replacement as above upon which an edge of G may be deleted. The main result of this section, Theorem 2.3.3 below, asserts that there are exactly four types of minimally intersecting pairs of M-alternating paths, three of which are depicted in Figure 2.3. We start with two auxiliary lemmas.

Lemma 2.3.1. *Let M be a matching in G, let P be an M-alternating path with ends x and y, let C be an M-alternating cycle such that x and y have degree at most two in $P \cup C$ and let $M' = M \triangle E(C)$. Then there exists an M'-alternating path Q with ends x and y satisfying $E(Q) \subseteq E(P) \triangle E(C)$.*

Proof. Let H be the subgraph of G with vertex-set $V(G)$ and edge-set $E(P) \triangle E(C)$. Then x, y have degree one in H, every other vertex of H has degree zero or two, and if it has degree two, then it is incident with an edge of M'. Thus some component of H is an M'-alternating path joining x and y, as desired. □

Lemma 2.3.2. *Let M be a matching in G, let P be an M-alternating path with ends w and v, and let R be a path with ends v and z such that $R \setminus v$ is M-covered, v is incident with no edge of M, and $w \notin V(R)$. Let $M' = M \triangle E(R)$. Then there exists an M'-alternating path Q with ends w and z satisfying $E(Q) \subseteq E(P) \triangle E(R)$.*

Proof. This follows similarly as Lemma 2.3.1 by considering the graph with edge-set $E(P) \triangle E(R)$. □

Let G be a graph, let M be a matching in G, and let P and Q be two M-alternating paths in G. For the purpose of this definition let a *segment* be a maximal subpath of $P \cap Q$, and let an *arc* be a maximal subpath of Q with no internal vertex or edge in P. We say that P and Q *intersect transversally* if either they are vertex-disjoint, or there exist vertices $q_0, q_1, \ldots, q_7 \in V(Q)$ such

(1) $q_0, q_1, \ldots, q_7$ occur on Q in the order listed, and q_0 and q_7 are the ends of Q,

(2) $q_2, q_1, q_3, q_4, q_6, q_5$ all belong to P and occur on P in the order listed,

(3) if $q_0 \in V(P)$, then $q_0 = q_1 = q_2 = q_3$, and otherwise $Q[q_0, q_1]$ is an arc,

(4) if $q_7 \in V(P)$, then $q_7 = q_6 = q_5 = q_4$, and otherwise $Q[q_6, q_7]$ is an arc,

(5) $Q[q_3, q_4]$ is a segment,

(6) either $q_1 = q_2 = q_3$, or q_1, q_2, q_3 are pairwise distinct, $Q[q_1, q_2]$ is a segment, $Q[q_2, q_3]$ is an arc and q_2 is not an end of P, and

(7) either $q_4 = q_5 = q_6$, or q_4, q_5, q_6 are pairwise distinct, $Q[q_5, q_6]$ is a segment, $Q[q_4, q_5]$ is an arc and q_5 is not an end of P.

It follows that the definition is symmetric in P and Q. There are four cases of transversal intersection depending on the number of components of $P \cap Q$; the three cases when P and Q intersect are depicted in Figure 2.3, where matching edges are drawn thicker. We shall prove the following lemma.

Lemma 2.3.3. *Let M be a matching in a graph G and let P_1 and P_2 be two M-alternating paths, where P_i has ends s_i and t_i. Assume that s_1, s_2, t_1 and t_2 have degree at most two in $P_1 \cup P_2$. Then there exist a matching M' saturating the same set of vertices as M and two M'-alternating paths Q_1 and Q_2 such that $M \triangle M' \subset E(P_1) \cup E(P_2)$, Q_i has ends s_i and t_i and Q_1 and Q_2 intersect transversally.*

Unfortunately, for later application we need a more general, but less nice result, the following. Please notice that it immediately implies Lemma 2.3.3 on taking $r = t_2$.

Theorem 2.3.4. *Let M be a matching in a graph G and let P_1 and P_2 be two M-alternating paths, where P_i has ends s_i and t_i. Assume that s_1, s_2, t_1 and t_2 have degree at most two in $P_1 \cup P_2$. Let $r \in V(P_2)$, and let $P_2' = P_2[s_2, r]$. Then one of the following conditions hold:*

(1) *There exist a matching M' saturating the same set of vertices as M and two M'-alternating paths Q_1 and Q_2 such that Q_i has ends s_i and t_i, $M \triangle M' \subseteq E(P_1) \cup E(P_2')$, Q_i has ends s_i and t_i, $Q_1 \subseteq P_1 \cup P_2'$, and $Q_1 \cup Q_2$ is a proper subgraph of $P_1 \cup P_2$,*

(2) *$r \neq t_2$, and there exists an M-alternating path $R \subseteq P_1 \cup P_2'$ with ends s_2 and t_1 such that R and P_1 intersect transversally,*

(3) *P_2' intersects P_1 transversally.*

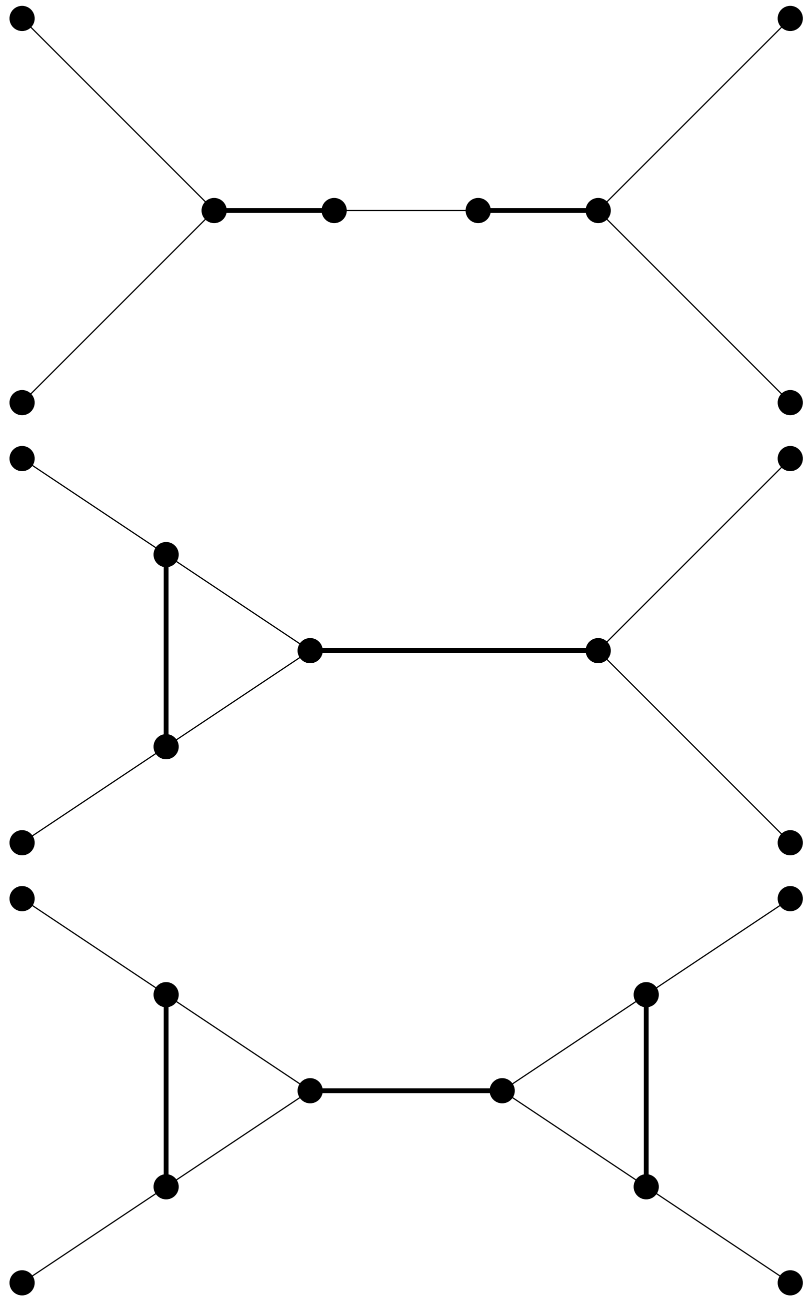


Figure 6: Three cases of transversal intersection.

Proof. We may assume that $G = P_1 \cup P_2$ and (1) does not hold. We shall refer to this as the *minimality of* G.

We claim that $P_1 \cup P_2'$ contains no M-alternating cycles. Suppose for a contradiction there exists an M-alternating cycle $C \subseteq P_1 \cup P_2'$. Let $M' = M \triangle E(C)$ and let Q_1, Q_2 be the two M'-alternating paths obtained by applying Lemma 2.3.1 to P_1 and P_2, respectively. Since P_1 and P_2 are M-alternating and their union includes C, they either share an edge of $M \cap E(C)$, say e, or P_1 and P_2 have the same ends. In the later case replacing P_2 by P_1 contradicts the minimality of G, and so we may assume the former. Now $Q_1 \subseteq P_1 \cup P_2'$ and $Q_1 \cup Q_2$ is a subgraph of $(P_1 \cup P_2) \setminus e$, contradicting the minimality of G.

For the purpose of this proof let us define an *arc* as a maximal subpath of P_2' that has at least one edge or contains an end of P_2' and has no internal vertex or edge in P_1. Define *segment* as a maximal subpath of $P_1 \cap P_2$. We say that two vertices of P_1 have the *same biparity* if their distance on P_1 is even, and otherwise we say they have *opposite biparity*. We claim that the ends of every arc have the same biparity. To see that, let $P_2'[s,t]$ be an arc with ends of opposite biparity. There are two cases. Either both end-edges of $P_1[s,t]$ belong to M, or both of them do not. If they do, then $P_1[s,t] \cup P_2'[s,t]$ is an M-alternating cycle, and if they do not, then P_1', P_2 contradict the minimality of G, where P_1' is obtained from P_1 by replacing the interior of $P_1[s,t]$ by $P_2'[s,t]$. (Notice that the edge of $P_1[s,t]$ incident with s does not belong to P_1' or P_2.) This proves our claim that the ends of every arc have the same biparity.

We may assume that there is an arc with both ends on P_1, for otherwise (3) holds. Let $P_2'[u_0, v_0]$ be such an arc. Since u_0, v_0 have the same biparity, exactly one end-edge of $P_1[u_0, v_0]$ belongs to M, say the one incident with u_0. Then the unique segment incident with u_0, say $P_1[u_0, v_1] = P_2'[u_0, v_1]$ has the property that v_1 lies between u_0 and v_0 on P_1. Let $P_2'[v_1, u_1]$ be the unique arc incident with v_1. Then either u_1 is an end of P_2', or u_1, v_1 have the same biparity, opposite to the biparity of u_0, v_0.

We claim that either u_1 is an end of P_2', or u_1 lies between v_1 and v_0 on P_1. To prove this claim we need to prove that neither u_0 nor v_0 lie between u_1 and v_1 on P_1. To this end suppose first that u_0 lies between u_1 and v_1 on P_1. Then P_1'' and P_2 contradict the minimality of G, where P_1'' is obtained from P_1 by replacing the interior of $P_1[u_1, v_0]$ by $P_2'[u_1, v_0]$ (the edge of $P_1[v_1, v_0]$ incident with v_1 does not belong to $P_1'' \cup P_2$). Suppose now that v_0 lies between u_1 and v_1 on P_1. Then $P_2'[v_0, u_1] \cup P_1[u_1, v_0]$ is an M-alternating cycle, a contradiction. This proves that either u_1 is an end of P_2', or u_1 lies between v_1 and v_0 on P_1.

Now assume that $P_2'[u_0, v_0]$ is chosen so that $P_1[u_0, v_0]$ is maximal, and let u_1, v_1 be as in the previous paragraph. If u_1 is an end of P_2' we stop, and so assume that it is not. Recall that u_1, v_1 have opposite biparity from u_0, v_0. Thus the unique segment incident with u_1, say $P_1[u_1, v_2] = P_2'[u_1, v_2]$ has the property that v_2 lies between v_1 and u_1 on P_1. Now let $P_2'[v_2, u_2]$ be the unique arc incident with v_2. By the result of the previous paragraph either u_2 is an end of P_2', or u_2 lies between v_2 and v_1 on P_1. By arguing in this manner we arrive at a sequence of vertices $u_0, v_0, \ldots, u_{k+1}, v_{k+1}$ such that

(i)$u_0, v_1, u_2, v_3, \ldots, v_{k+1}, \ldots, u_3, v_2, u_1, v_0$ occur on P_1 in the order listed,

(ii)u_{k+1} is an end of P_2',

(iii)$P_2'[u_i, v_i]$ are arcs for $i = 0, 1, \ldots, k+1$, and

(iv)$P_1[u_i, v_{i+1}]$ are segments for $i = 0, 1, \ldots, k$.

It follows that u_i, v_i have the same biparity and that their biparity depends on the parity of i. Let $P_1[v_0, v_0']$ be the unique segment incident with v_0. Then v_0 lies between v_0' and u_0 on P_1. Let $P_2'[v_0', u_0']$ be the unique arc incident with v_0'. The maximality of $P_1[u_0, v_0]$ and the result of the previous paragraph imply that either u_0' is an end of P_2', or that u_0, v_0, v_0', u_0' occur on P_1 in the order listed. In the latter case by an analogous argument there exists a sequence of vertices $u_0', v_0', \ldots, u_{k'+1}', v_{k'+1}'$ such that

(i)$u_0', v_1', u_2', v_3', \ldots, v_{k'+1}', \ldots, u_3', v_2', u_1', v_0'$ occur on P_1 in the order listed,

(ii)$u_{k'+1}'$ is an end of P_2,

(iii)$P_2'[u_i', v_i']$ are arcs for $i = 0, 1, \ldots, k'+1$, and

(iv)$P_1[u_i', v_{i+1}']$ are segments for $i = 0, 1, \ldots, k'$.

Suppose $r = t_2$. Then $k = 0$, for otherwise P_1 and and the path obtained from P_2 by replacing the interior of $P_2[v_0, u_1]$ by $P_1[v_0, u_1]$ contradict the minimality of G. Similarly, either u'_0 is an end of P_2 or $k' = 0$. Thus (3) holds.

Therefore we may assume $r \neq t_2$. Suppose $s_2 \neq u'_0$. Then without loss of generality we assume $s_2 = u_{k+1}$. We define $R_1 = P'_2[s_2, u_k] \cup P_1[u_k, t_1]$ and $R_2 = P'_2[s_2, v_k] \cup P_1[v_k, t_1]$. For some $i \in \{1, 2\}$ $R_i \subseteq P_1 \cup P'_2$ is an M-alternating path with ends s_2 and t_1 such that R_i and P_1 intersect transversally. Thus (2) holds.

It remains to consider the case when $s_2 = u'_0$ and $u_{k+1} = r$. Suppose $k \geq 1$. We claim that $E(P_1[v_{k+1}, v_k] \cap P_2) = \emptyset$. Suppose for a contradiction $P_2[x, y] \subseteq P_1[v_{k+1}, v_k]$ is a segment, and let $P_2[x, y]$ be chosen so that $P_2[y, t_2]$ is minimal. If $x \in V(P_1[v_k, y])$ define $Q_2 = P_2[s_2, v_k] \cup P_1[v_k, x] \cup P_2[x, t_2]$, and otherwise define $Q_2 = P_2[s_2, v_{k+1}] \cup P_1[v_{k+1}, x] \cup P_2[x, t_2]$. As $E(P_1[v_{k+1}, v_k] \cap P'_2) = \emptyset$ we see that Q_2 is an M-alternating path. We replace P_2 with Q_2 to contradict the minimality of G.

Now we claim $E(P_1[v_{k-1}, u_k] \cap P_2) = \emptyset$. Again suppose $P_2[x, y] \subseteq P_1[v_{k-1}, u_k]$ is a segment, and let $P_2[x, y]$ be chosen so that $P_2[y, t_2]$ is minimal. If $x \in V(P_1[v_{k-1}, y])$ define $Q_2 = P_2[s_2, v_{k-1}] \cup P_1[v_{k-1}, x] \cup P_2[x, t_2]$, and otherwise define $Q_2 = P_2[s_2, v_k] \cup P_1[v_k, x] \cup P_2[x, t_2]$. As $E(P_1[v_{k+1}, v_k] \cap P_2) = \emptyset$ we see that Q_2 is an M-alternating path. Again we replace P_2 with Q_2 to contradict the minimality of G.

Now let $Q_2 = P_2[s_2, v_{k-1}] \cup P_1[v_{k-1}, u_k] \cup P_2[u_k, t_2]$. As $E(P_1[v_{k-1}, u_k] \cap P_2) = \emptyset$ we see that Q_2 is an M-alternating path and replacing P_2 with Q_2 we once again contradict the minimality of G. Thus $k = 0$ and (3) holds. □

We deduce several corollaries of Theorem 2.3.4. Let Ω be an octopus in a graph G, where Ω consists of two tentacles and a head C with $V(C) = \{v\}$. Then the graph of Ω is a path. We say that Ω is a *path octopus* with head v. The head of a path octopus can be moved along Ω in the sense that if $v' \in V(\Omega)$ is at even distance from v in Ω, then there is another path octopus with the same graph and head v'. The next lemma will use this fact.

Lemma 2.3.5. *Let G be a graph, let Ω be a path octopus in G with head v and ends v_1 and v_2, let z be the neighbor of v_1 in Ω, let M be an Ω-compatible matching, and let P be an M-alternating path in $G \backslash v_1 \backslash v_2$ with ends v and $w \notin V(\Omega)$. Then there exist a path octopus*

Ω' with head z and ends v_1 and v_2, an Ω'-compatible matching M', and a path P' with ends z and w such that $E(\Omega') \subseteq E(\Omega \cup P)$, $zv_1 \in E(\Omega')$, $v_1 \notin V(P')$, M coincides with M' on $G \setminus (V(P) \cup V(\Omega))$, $\Omega \cup P \setminus V(\Omega' \cup P')$ is M'-covered, and P' intersects $\Omega' \backslash v_1$ transversally.

Proof. Since M is Ω-compatible, v is incident with no edge of M. Let $R = \Omega[z, v]$, let $M' = M \triangle E(R)$, and let Ω' be the octopus with graph Ω and head z. Then M' is an Ω'-compatible matching. By Lemma 2.3.2 there exists an M'-alternating path P' with ends z and w such that $E(P') \subseteq E(P) \triangle E(R)$. By Lemma 2.3.3 we may assume, by replacing the tentacle $\Omega'[z, v_2]$ and path P', that P' intersects $\Omega' \backslash v_1 = \Omega'[z, v_2]$ transversally, as desired. □

Let P_1, P_2, P_3 be odd paths in a graph H. For $i = 1, 2, 3$ let u_i and v_i be the ends of P_i. If $u_1 = u_2 = u_3$ and otherwise P_1, P_2, P_3 are pairwise disjoint, then we say that the octopus with tentacles P_1, P_2 and P_3 and a head the graph with vertex-set $\{u_1\}$ is a *triad* in H. Assume now that P_1, P_2, P_3 are pairwise disjoint, and let Q_1, Q_2, Q_3 be three odd paths such that for $\{i, j, k\} = \{1, 2, 3\}$ the ends of Q_k are u_i and u_j. Assume further that $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are pairwise disjoint, except for common ends in the set $\{u_1, u_2, u_3\}$. In those circumstances we say that an octopus with tentacles P_1, P_2 and P_3 and head $Q_1 \cup Q_2 \cup Q_3$ is a *tripod* in H.

Lemma 2.3.6. *Let G be a graph. Let T be a triad or tripod in G with ends v_1, v_2 and v_3. Let M be a T-compatible matching, and let P be an M-alternating path in $G \setminus v_1 \setminus v_2$ with one end in the head of T and another end $w \notin V(T)$. Assume that the edge of P incident with w does not belong to M. Then there exists a triad or tripod $T' \subseteq T \cup P$ with ends v_1, v_2 and w and a T'-compatible matching M' such that M is identical to M' on $G \setminus V(P \cup T)$ and $(T \cup P) \setminus V(T')$ is M'-covered.*

Proof. If T is triad then the result follows immediately from Lemma 2.3.5. If T is a tripod, then for $i \in \{1, 2, 3\}$ let P_i, Q_i, u_i, v_i be as in the definition of tripod. Extend M to Q_1, Q_2 and Q_3 in such a way that $Q_1 \cup Q_2 \cup Q_3 \setminus u_1$ is M-covered. Let T'' be the path octopus with tentacles P_1 and $P_2 \cup Q_1 \cup Q_2$. Extend P along $Q_1 \cup Q_2 \cup Q_3$ to a path P'' so that P''

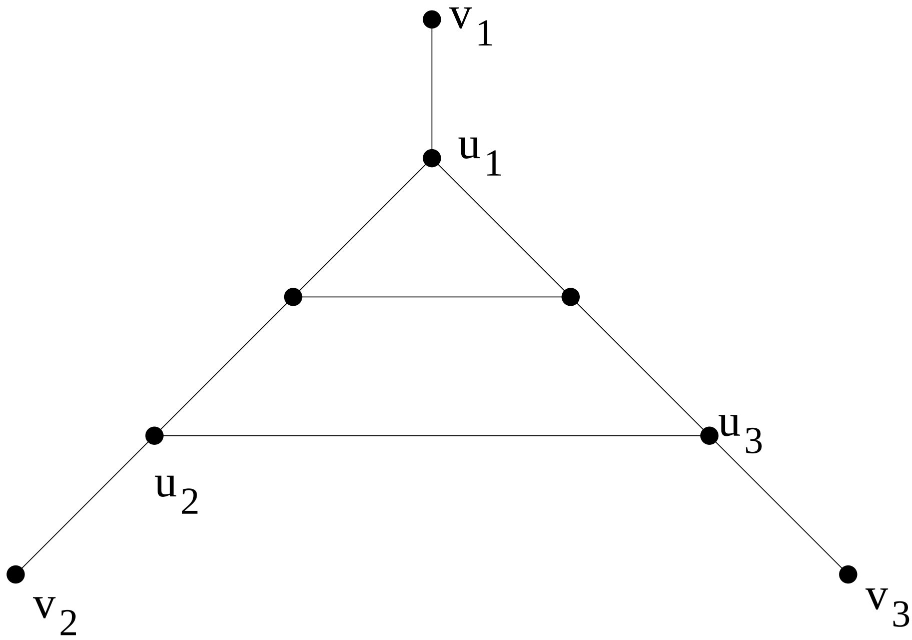


Figure 7: A quasi-tripod.

is an M-alternating path with ends w and u_1. It remains to apply Lemma 2.3.5 to P and T''. □

Let Q be an even path with ends u_1 and u_3, let $u_2 = u_1$ and $u_4 = u_3$, and for $i = 1, 2, 3, 4$ let P_i be an odd path with ends u_i and v_i, disjoint from Q except for u_i, and such that the paths P_i are pairwise disjoint, except that P_1 and P_2 share a common end $u_1 = u_2$ and P_3 and P_4 share a common end $u_3 = u_4$. In those circumstances we say that the octopus with head Q and tentacles P_1, P_2, P_3, P_4 is a *quadropod*.

Now let $P_1, P_2, P_3, Q_1, Q_2, Q_3$ be as in the definition of tripod, except that Q_2 and Q_3 are allowed to intersect beyond the vertex u_1. Suppose there exists a perfect matching M of $Q_2 \cup Q_3 \setminus u_1 \setminus u_2 \setminus u_3$ such that Q_2 and Q_3 are M-alternating and intersect transversally. Then we say that the octopus Ω with tentacles P_1, P_2 and P_3 and a head $Q_1 \cup Q_2 \cup Q_3$ is a *quasi-tripod* in H. Clearly every tripod is a quasi-tripod. It follows from the definition of transversal intersection that $Q_2 \cap Q_3$ consists of one or two paths, one of which contains the vertex u_1. By shortening both Q_2 and Q_3 and extending P_1 we may assume that one of the components of $Q_2 \cap Q_3$ has vertex-set $\{u_1\}$. If that is the only component of $Q_2 \cap Q_3$, then Ω is a tripod; otherwise Ω looks as depicted in Figure 2.3.

Lemma 2.3.7. *Let G be a graph. Let T be a triad or tripod in G with ends v_1, v_2 and v_3. Let M be a T-compatible matching, and let P be an M-alternating path in $G \setminus \{v_1, v_2, v_3\}$ with one end in the head of T and another end $w \notin V(T)$. Assume that the edge of P incident with w does not belong to M. Then there exists an octopus $T' \subseteq T \cup P$ and a T'-compatible matching M', such that M is identical to M' on $G \setminus V(P \cup T)$, the graph $(T \cup P) \setminus V(T')$ is M'-covered and either T' is a quasi-tripod with ends v_i, v_j and w, for some distinct indices $i, j \in \{1, 2, 3\}$, or T' is a quadropod with ends v_1, v_2, v_3 and w.*

Proof. We may assume that $G = T \cup P$ and that there do not exist a triad or tripod T' with ends v_1, v_2 and v_3, a T'-compatible matching M' and an M'-alternating path P' in $G \setminus \{v_1, v_2, v_3\}$ with one end in the head of T' and the other end w such that $w \notin V(T')$, $(T \cup P) \setminus V(T' \cup P')$ is M'-covered and $P' \cup T'$ is a proper subgraph of G. We refer to this as the *minimality of* G.

Let the tentacles of T be P_1, P_2, P_3, where P_i has one end v_i, and let u_i be the other end of P_i. If T is a tripod, then let Q_i be as in the definition of tripod, and otherwise let Q_i be the null graph. We say that a vertex v of P_i is *inbound* if $P_i[v, u_i]$ is even and we say that v is *outbound* otherwise.

Let $u_0 \in V(P \cap T)$ be chosen to minimize $P[w, u_0]$. If T is a triad and u_0 is inbound, then $T \cup P[w, u_0]$ is a required quadropod. If T is a tripod and $u_0 \in V(P_i)$ is inbound then by replacing $P_i[v_i, u_0]$ by $P[w, u_0]$ in T we obtain a required quasi-tripod. If T is a tripod and $u_0 \in V(Q_i)$, then we may assume from the symmetry that $Q_i[u_0, u_j]$ is even, in which case by replacing P_j by $P[w, u_0]$ we obtain a required quasi-tripod.

Therefore for the rest of the proof we may assume that $u_0 \in V(P_1)$ and that u_0 is outbound. Let $r \in V(T) \cap V(P) - V(P_1)$ be chosen to minimize $P[w, r]$ and if no such r exists let $r \neq w$ be the end of P. Apply Theorem 2.3.4 to P_1 and P with $s_1 = v_1$, $t_1 = u_1$ and $s_2 = w$. Outcome of Theorem 2.3.4(1) does not hold by the minimality of G. If outcome (2) holds, then by considering the path guaranteed therein we obtain a desired quasi-tripod or quadropod. Thus we may assume outcome (3) holds, and hence P_1 intersects $P[w, r]$ transversally.

Let v_0 be such that $P[v_0, u_0]$ is a component of $P \cap P_1$, and let u be such that $P[v_0, u]$ is a maximal path with no internal vertex or edge in T. If $u \in V(P_1)$, then by the definition of transversal intersection the vertices v_1, v_0, u_0, u, u_1 occur on P_1 in the order listed and u is inbound. By considering $T \cup P[w, u]$ and deleting $P_3 \backslash u_3$ and the interior of Q_3 we obtain a required quasi-tripod. Thus we may assume that $u \notin V(P_1)$, and hence $u = r$. If r is not outbound, then a similar argument gives a required quasi-tripod.

It follows that for the remainder of the proof we may assume that $r \in V(P_2)$, and that r is outbound. Let M_1 be the unique perfect matching of $Q_1 \cup Q_2 \cup Q_3 \backslash u_1$, and let $M^+ = M \cup M_1$. We can extend P along $Q_1 \cup Q_2 \cup Q_3$ to an M^+-alternating path P^+ so that u_1 is an end of P^+. Apply Lemma 2.3.2 to P^+ and $P_1[v_0, u_1]$ to produce an M'-alternating path P' with ends w and v_0, where $M' = M^+ \triangle P_1[u_1, v_0]$. Let T' be obtained from $T \cup P[v_0, r]$ by deleting the interiors of $P_2[r, u_2]$ and Q_2; then T' is a triad with ends v_1, v_2, v_3. But now T' and P' contradict the minimality of G. $\square$

2.4 Embeddings and Main Lemma

In this section we first formalize the notion of a matching minor by introducing the concept of an embedding, and show in Lemma 2.4.2 below that a graph H has a matching minor isomorphic to a graph G if and only if there is an embedding $H \hookrightarrow G$. Then we study the following question. Suppose that $\eta : H \hookrightarrow G$ is an embedding, G is a brick, and $v_0 \in V(H)$ has degree two. Since bricks have no vertices of degree two, there is a subgraph of G that "fixes" this violation of being a brick. What can we say about this subgraph? The answer leads to the notion of v_0-augmentation of η. We define this concept formally, and then prove two results about its existence. The second, Lemma 2.4.4, will be used when some graph obtained from H by bisplitting a vertex is isomorphic to a matching minor of G; otherwise we will use Lemma 2.4.3, the first of these results. Finally, we classify all "minimal" v_0-augmentations into one of four types.

Let T' be a tree, and let T be obtained from T' by subdividing every edge an odd number of times. Then $V(T') \subseteq V(T)$. The vertices of T that belong to $V(T')$ will be called *old* and the vertices of $V(T) - V(T')$ will be called *new*. We say that T is a *barycentric tree*. Please note that the partition into old and new vertices depends on T' (there is an ambiguity concerning vertices of degree two). We shall assume that each barycentric tree has a fixed partition into new and old vertices. By a *branch* of a barycentric tree T we mean a subpath of T with ends old vertices and all internal vertices new.

We need to formalize the concept of matching minor. Let H and G be graphs. A *weak embedding of H to G* is a mapping η with domain $V(H) \cup E(H)$ such that for $v, v' \in V(H)$ and $e, e' \in E(H)$

(1) $\eta(v)$ is a barycentric subtree in G,

(2) if $v \neq v'$, then $\eta(v)$ and $\eta(v')$ are vertex-disjoint,

(3) $\eta(e)$ is an odd path with no internal vertex in any $\eta(v)$ or $\eta(e')$ for $e' \neq e$,

(4) if $e = u_1u_2$, then the ends of $\eta(e)$ can be denoted by x_1, x_2 in such a way that x_i is an old vertex of $\eta(u_i)$, and

(5) $G \setminus \bigcup_{x \in V(H) \cup E(H)} V(\eta(x))$ has a perfect matching.

The next lemma will show that H is isomorphic to a matching minor of G if and only if there is a weak embedding of H to G. Then we will show that such a weak embedding can be chosen with two additional properties. Thus we say that a weak embedding from H to G is an *embedding* if, in addition, it satisfies

(6)if v has degree one then $\eta(v)$ has exactly one vertex,

(7) if $v \in V(H)$ has degree two and e_1, e_2 are its incident edges, then $\eta(v)$ is an even path with ends x_1, x_2, say, and $\eta(e_1)$, $\eta(e_2)$ both have length one, one has x_1 as its end and the other has x_2 as its end, and

(8)if v has degree at least three and x is an old vertex of $\eta(v)$ of degree d, then x is an end of $\eta(e)$ for at least $3 - d$ distinct edges e.

For every subgraph H' of H define $\eta(H') = \bigcup_{x \in V(H) \cup E(H)} \eta(x)$. We denote the fact that η is an embedding of H into G by writing $\eta : H \hookrightarrow G$.

Let $T \subseteq H$ be a barycentric tree, and let (X, Y) be the unique partition of $V(T)$ into two independent sets with X including all the old vertices. The vertices of X will be called *protected* and the vertices of Y will be called *exposed*.

Lemma 2.4.1. *Let H and G be graphs. There exists a weak embedding of H to G if and only if H is isomorphic to a matching minor of G.*

Proof. If $\eta : H \hookrightarrow G$ then a graph isomorphic to H can be obtained from the central subgraph $\eta(H)$ of G by repeatedly bicontracting exposed vertices of $\eta(v)$ and internal vertices of $\eta(e)$ for $v \in V(H)$ and $e \in E(H)$. Thus H is a matching minor of G.

To prove the converse we may assume that H is a matching minor of G. Thus there exist graphs $H_1, H_2, \ldots, H_k$ such that $H_1 = H$, H_k is a central subgraph of G, and for $i = 2, 3, \ldots, k$ the graph H_{i-1} is obtained from H_i by bicontracting a vertex. We define $\eta_k : H_k \hookrightarrow G$ by saying that if $v \in V(H_k)$, then $\eta_k(v)$ is the graph with vertex-set $\{v\}$, and if $e \in E(H_k)$, then $\eta_k(e)$ is the graph consisting of e and its ends. It is clear that η_k satisfies (1)-(7). We now construct a sequence of mappings satisfying (1)-(7). Assuming that η_i has been defined we define η_{i-1} as follows. Let v be the vertex of H_i whose bicontraction produces H_{i-1}, let x, y be the neighbors of v, and let w be the the new vertex of H_{i-1}. For $z \in V(H_{i-1}) \cup E(H_{i-1}) - \{w\}$ let $\eta_{i-1}(z) = \eta_i(z)$, and let $\eta_{i-1}(w) = \eta_i(x) \cup \eta_i(y) \cup \eta_i(v) \cup \eta_i(xv) \cup \eta_i(yv)$. This completes the construction. It is clear that η_1 satisfies (1)-(5). $\square$

We now show that if there is weak embedding of H to G, then there is an embedding of H to G.

Lemma 2.4.2. *Let H and G be graphs. There exists an embedding of H to G if and only if H is isomorphic to a matching minor of G.*

Proof. By Lemma 2.4.1 it suffices to show that if η is a weak embedding of H to G, then there exists an embedding of H to G.

It is easy to modify η so that it satisfies conditions (6) and (7). Thus we may choose a mapping η with domain $V(H) \cup E(H)$ satisfying (1)-(7) such that the total number of old vertices in $\eta(v)$ over all vertices $v \in V(H)$ of degree at least three is minimum. We claim that η satisfies (8) as well.

To prove that η satisfies (8) let $v \in V(H)$ have degree at least three, let x be an old vertex of $\eta(v)$, and let d be the degree of x in $\eta(v)$. If $d = 2$ and x is not an end of $\eta(e)$

for any $e \in E(G)$, then we change the barycentric structure of $\eta(v)$ by declaring x to be a new vertex. The new embedding thus obtained contradicts the minimality of η. If $d = 0$, then x is the unique vertex of $\eta(v)$, and it is an end of $\eta(e)$ for all the (at least three) edges e incident with v by (4). Thus we may assume that $d = 1$. If x is not an end of any $\eta(e)$, then we remove from $\eta(v)$ the vertex x and all internal vertices of Q, where Q is the unique subpath of $\eta(v)$ between x and the nearest old vertex. Then set of vertices removed has a perfect matching, because Q is even by the definition of barycentric subdivision, and hence the new embedding satisfies (5). Thus the new embedding contradicts the minimality of η. To complete the proof we may therefore suppose for a contradiction that x is incident with $\eta(e)$ for exactly one $e \in E(H)$. By (4) one end of e is v; let u be the other end. If u has degree at most two, then we define a new embedding by moving x and the internal vertices of Q from $\eta(v)$ to $\eta(u)$, and changing $\eta(e)$ accordingly. If u has degree at least three, then we move x and all internal vertices of Q from $\eta(v)$ to $\eta(e)$. In either case the new embedding contradicts the minimality of η. Thus η satisfies (8), and hence it is an embedding $H \hookrightarrow G$, as desired. $\square$

Let T be an even subpath of a graph H, and let T be regarded as a barycentric tree, with its ends designated as old and all internal vertices designated as new. Let us recall that the notions of protected and exposed were defined prior to Lemma 2.4.1. Let P be a path with one end, say v, in the interior of T and no other vertex in T. If v is exposed, then let Q be the null graph, and if v is protected, then let Q be a path with ends exposed vertices $q_1, q_2 \in V(T)$ and otherwise disjoint from $H \cup P$ such that v lies on T between q_1 and q_2. In those circumstances we say that *Q is a cap for P at v with respect to T and H.*

Let $\eta : H \hookrightarrow G$. For every edge $e = uv \in E(H)$ the path $\eta(e)$ is odd. Let P_e denote its interior (that is, the path obtained by deleting the ends), and let M_e be the unique perfect matching of P_e (possibly $M_e = \emptyset$). We define $M(\eta)$ to be the union of M_e over all $e \in E(H)$.

Now let $v_0 \in V(H)$ have degree two, and let v_1, v_2 be its neighbors. For $i = 1, 2$ let E_i be the set of edges of H incident with v_i, except for the edge v_0v_i, and let $E_1 \cap E_2 = \emptyset$. Let M_1 be a perfect matching of $G \backslash V(\eta(H))$, and let $M = M_1 \cup M(\eta)$. Let P be an M-alternating path with one end $x \in V(\eta(v_0))$ and the other end u in $\bigcup\{\eta(v) : v \in V(H) - \{v_0, v_1, v_2\}\}$ with the property that if P intersects $\eta(e)$ for some $e \in E(H)$ not incident with v_0, v_1, or v_2, then P and $\eta(e)$ intersect in a path and have a common end. Let S denote the path $\eta(v_0) \cup \eta(v_0v_1) \cup \eta(v_0v_2)$; then S is obtained from $\eta(v_0)$ by appending two edges, one at each end. Let Q be an M_1-alternating cap for P at x with respect to S and $\eta(H)$. We say that the pair (P, Q) is a *v_0-augmentation of η*. It follows that P and Q have no internal vertices in $\bigcup_{v \in V(H)} \eta(v)$. We say that x is the *origin* and u is the *terminus* of P.

Our first result about augmentations is the following.

Lemma 2.4.3. *Let H be a graph on at least four vertices, let v_0 be a vertex of H that has exactly two neighbors v_1 and v_2, and let v_1 and v_2 be not adjacent. Let G be a brick and let $\eta : H \hookrightarrow G$ be an embedding such that both $\eta(v_1)$ and $\eta(v_2)$ have exactly one vertex. Then there exists an embedding $\eta' : H \hookrightarrow G$ and a v_0-augmentation of η'.*

Proof. Define E_1, E_2 and M as in the definition of v_0-augmentation. The path $\eta(v_0) \cup \eta(v_0v_1) \cup \eta(v_0v_1)$ is even and can therefore be regarded as path octopus, which we denote by Ω_1. Let Ω_2 be the octopus with the set of tentacles $\{\eta(e) : e \in E_1 \cup E_2\}$ and head $\eta(H \backslash v_0 \backslash v_1 \backslash v_2)$. The head of Ω_2 is non-null, because H has at least four vertices. We can convert M to a matching M^+ so that M^+ is Ω_i-compatible for $i = 1, 2$. We apply Theorem 2.2.3 to the frame $(\{\Omega_1, \Omega_2\}, V(\eta(v_1)) \cup V(\eta(v_2)))$ and denote the resulting path by R. Let R have ends $r_1 \in V(\Omega_1)$ and $r_2 \in V(\Omega_2)$ and let $e \in E(R)$ be such that each of the components $R_i = R[s_i, r_i]$ of $R \backslash e$ intersects only one of the octopi Ω_1 and Ω_2.

By Lemma 2.3.5 we may assume, by changing M^+, R_1, and $\eta(v_0)$, that there exists an M^+-alternating path P_1 with ends $p_1 \in V(\eta(v_0))$ and s_1, and an M^+-alternating cap Q_1 for P_1 at p_1 with respect to Ω_1 and $\eta(H)$ such that $P_1 \cup Q_1 \subseteq R_1$. We may also assume that r_2 is the only vertex of R in the head of Ω_2. If $r_2 \in \eta(v)$ for some $v \in V(H)$, then let R_2' be the null graph, and if $r_2 \in \eta(e)$ for some $e \in E(H)$, then let R_2' be an M^+-alternating subpath of $\eta(e)$ with one end r_2 and the other in $\eta(v)$ for some $v \in V(H)$. Then $(P_1 \cup R_2 \cup R_2', Q_1)$ is a desired v_0-augmentation of η. $\square$

In the next section we will need the following lemma.

Lemma 2.4.4. *Let H be a graph, and let v be a vertex of H of degree at least four, let G be a brick, and let $\eta : H \hookrightarrow G$ be such that $\eta(v)$ has at least two vertices. Then either*

(1) there exists a graph H_1 obtained from H by bisplitting v, an embedding $\eta_1 : H_1 \hookrightarrow G$ and a v_0-augmentation of η_1, where v_0 is the new inner vertex of H_1, or

(2) there exists an embedding $\eta_2 : H \hookrightarrow G$, a path P with ends p_1 and p_2 in the interiors of different branches, say B_1 and B_2, of $\eta_2(v)$ and otherwise disjoint from $\eta_2(H)$ and for $i = 1, 2$ there exists a cap Q_i for P at p_i with respect to B_i and $\eta_2(H)$ such that Q_1 and Q_2 are disjoint.

Proof. Denote the branches of $\eta(v)$ by $B_1, B_2, \ldots, B_n$. They can be considered as octopi, which we denote by $\Omega_1, \Omega_2, \ldots, \Omega_n$, respectively. Let Ω_0 be the octopus with the set of tentacles $\{\eta(e) : e$ is incident to $v\}$ and head $\eta(H\backslash v)$, let X be the set of old vertices of $\eta(v)$, and let $\mathcal{F} = \{\Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_n\}$. We can extend a perfect matching of $G \setminus \eta(H)$ to a matching M so that M is Ω-compatible for every $\Omega \in \mathcal{F}$. Clearly $|X| = n + 1$. Therefore $(\mathcal{F}, X)$ is a frame. We apply Theorem 2.2.3 to it and denote the resulting path by R. Furthermore, there is an edge $e \in E(R)$ such that each of the components $R_i = R[s_i, r_i]$ of $R\backslash e$ intersects only one of the octopi of $\mathcal{F}$.

If for some $i \in \{1, 2\}$ the path R_i intersects Ω_j for $j \geq 1$ we may assume, by changing M, and Ω_j, that there exists an M-alternating path P_i with ends $p_i \in V(B_j)$ and s_i, and an M-alternating cap Q_i for P_i at p_i with respect to B_j and $\eta(H)$ such that $P_i \cup Q_i \subseteq R_i \cup B_j$. If this happens for both R_1 and R_2 define $P = P_1 \cup P_2 + e$ and outcome (2) holds.

Therefore we may assume that R_2 intersects Ω_0 and R_1 intersects Ω_j for some $j \geq 1$, and furthermore that r_2 is the only vertex of R in the head of Ω_0. If $r_2 \in \eta(v)$ for some $v \in V(H)$, then let R_2' be the null graph, and if $r_2 \in \eta(e)$ for some $e \in E(H)$, then let R_2' be an M-alternating subpath of $\eta(e)$ with one end r_2 and the other in $\eta(v)$ for some $v \in V(H)$.

Let T_1 and T_2 be the two components of the graph obtained from $\eta(v)$ by removing the internal vertices of B_j. Let H_1 be obtained from H by splitting v into new outer vertices v_1 and v_2 and new inner vertex v_0 in such a way that v_i is adjacent to a neighbor u of v in H if $\eta(uv_i)$ has an end in T_i. Let $\eta_1(v_i) = T_i$, let $\eta_1(v_0)$ be B_1 minus its ends, let $\eta_1(v_1v_0)$ and

$\eta_1(v_2v_0)$ be the two end-edges of B_1 and let $\eta_1(x) = \eta(x)$ for all other $x \in V(H_1) \cup E(H_1)$. Then $(P_1 \cup R_2' \cup \{e\}, Q_1)$ is a v_0-augmentation of η_1 and outcome (1) holds. $\square$

Let H and G be graphs, let $\eta : H \hookrightarrow G$, let v_0 be a vertex of H of degree two, and let (P, Q) be a v_0-augmentation of η. We say that η is *minimal* if there exists no embedding $\eta' : H \hookrightarrow G$ and a v_0-augmentation (P', Q') of η' such that $\eta'(H) \cup P' \cup Q'$ is a proper subgraph of $\eta(H) \cup P \cup Q$. In applications we may assume that our v_0-augmentations are minimal. The next lemma will classify minimal augmentations into four types, which we now introduce.

Let $\eta : H \hookrightarrow G$, let $v_0 \in V(H)$ have degree two, let $v_1, v_2 \in V(H)$ be its neighbors, and let E_1, E_2 be as in the definition of v_0-augmentation. Let $i \in \{1, 2\}$ and $e \in E_i$. Let x_e be the end of $\eta(e)$ that belongs to $V(\eta(v_i))$. We say that an internal vertex $x \in V(\eta(e))$ is an *inbound vertex* if it is at even distance from x_e in $\eta(e)$, and otherwise we say that it is an *outbound vertex*.

Let M be a matching containing $M(\eta)$, let P be an M-alternating path with ends x_0 and x_5, and let the vertices $x_0, x_1, x_2, x_3, x_4, x_5$ appear on P in the order listed. Assume that $P[x_1, x_2]$ and $P[x_3, x_4]$ are subpaths of $\eta(e)$, and that otherwise P is disjoint from $\bigcup\{\eta(e) : e \in E_1 \cup E_2\}$. Assume also that x_1 is an inbound vertex of $\eta(e)$, that x_2 and x_3 are outbound, and that either $x_2 = x_3 = x_4$, or x_1, x_2, x_4, x_3, x_e are pairwise distinct and occur on $\eta(e)$ in the order listed. In those circumstances we say that P intersects $\eta(e)$ *regularly from x_0 to x_5*.

Let (P, Q) be a v_0-augmentation of η and let P have ends a and b where $a \in V(\eta(v_0))$. We say that (P, Q) is of type A if whenever P intersects $\eta(e)$ for some $e \in E_1 \cup E_2$, then P and $\eta(e)$ intersect in a path whose one end is a common end of P and $\eta(e)$. Thus P intersects at most one $\eta(e)$, because the common end must be b, and b does not belong to $\eta(v_1) \cup \eta(v_2)$. See Figure 2.4.

We say that (P, Q) is of type B if there exist a vertex $x \in V(P)$, an index $i \in \{1, 2\}$, and an edge $e \in E_i$ such that the vertex v_i has degree at most three, the path $P[a, x]$ intersects $\eta(e)$ regularly from a to x, and if $P[x, b]\backslash x$ intersects $\eta(e')$ for some $e' \in E(H)$, then $P[x, b]\backslash x$

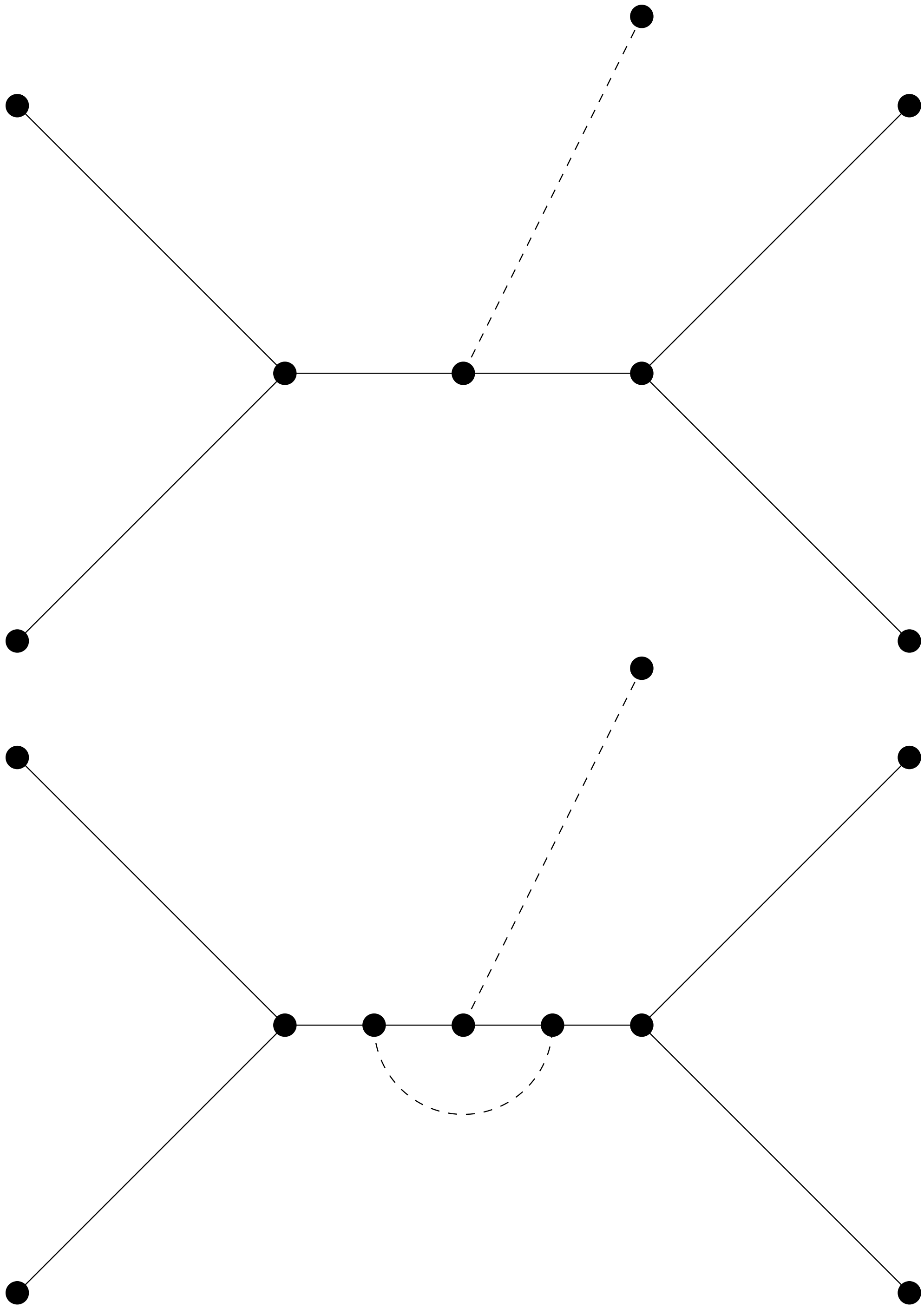


Figure 8: Augmentations of type A.

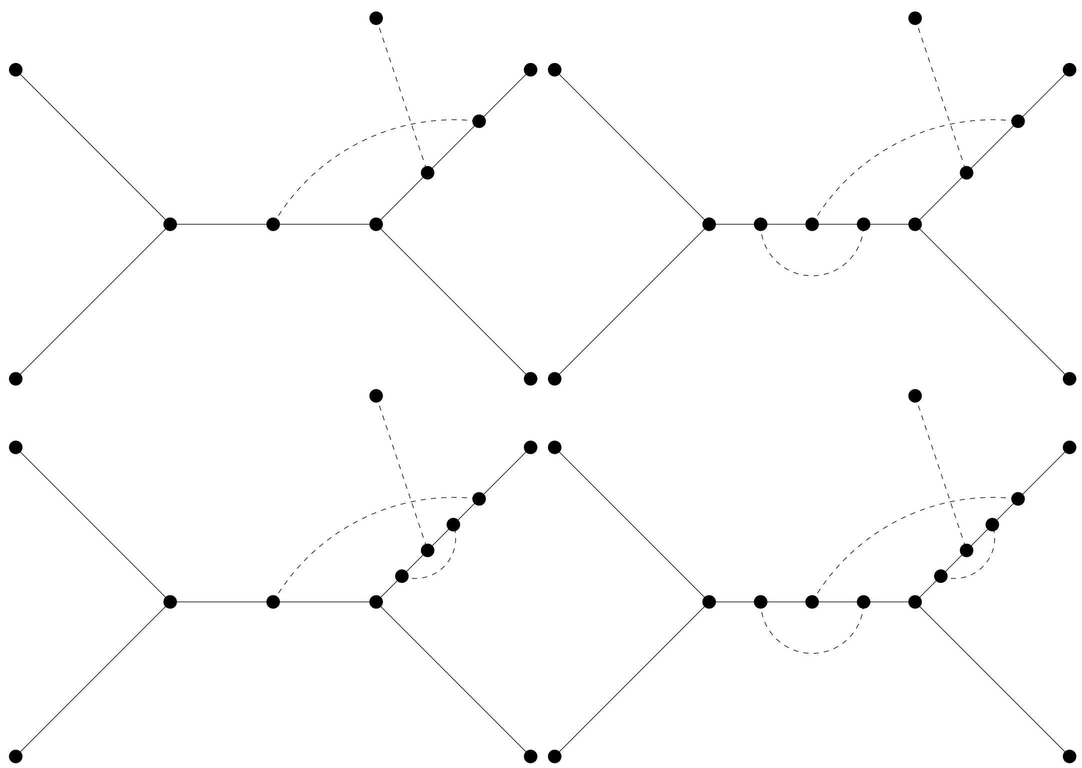


Figure 9: Augmentations of type B.

and $\eta(e')$ intersect in a path and have a common end. Moreover, if $e = e'$, then we require that $P[a, x] \cap \eta(e)$ be a path. We say that (P, Q) *crosses* $\eta(e)$. See Figure 2.4.

We say that (P, Q) is of type C if there exist vertices $x_1, x_2 \in V(P)$ such that a, x_1, x_2, b occur on P in the order listed, and there exist distinct edges e_1, e_2, one in E_1 and the other in E_2, such that the end of e_1 in $\{v_1, v_2\}$ has degree at most three, $P[a, x_1]$ intersects $\eta(e_1)$ regularly from a to x_1, $P[x_1, x_2]$ has no internal vertices in $\eta(H)$ and x_2 is an inbound vertex of $\eta(e_2)$. We say that (P, Q) *crosses* $\eta(e_1)$. See Figure 2.4.

We say that (P, Q) is of type D if for some $i \in \{1, 2\}$ and some $e \in E_i$ the vertex v_i has degree at least four and there exists an inbound vertex x of $\eta(e)$ such that $x \in V(P)$ and $P[a, x]$ has no internal vertex in $\eta(H)$.

The following classification of minimal v_0-augmentations is the third main result of this section.

Lemma 2.4.5. *Let H and G be graphs, and let $\eta : H \hookrightarrow G$. Let $v_0 \in V(H)$ have degree two, and let v_1, v_2 be its neighbors. Assume that v_1 is not adjacent to v_2. Then every minimal*

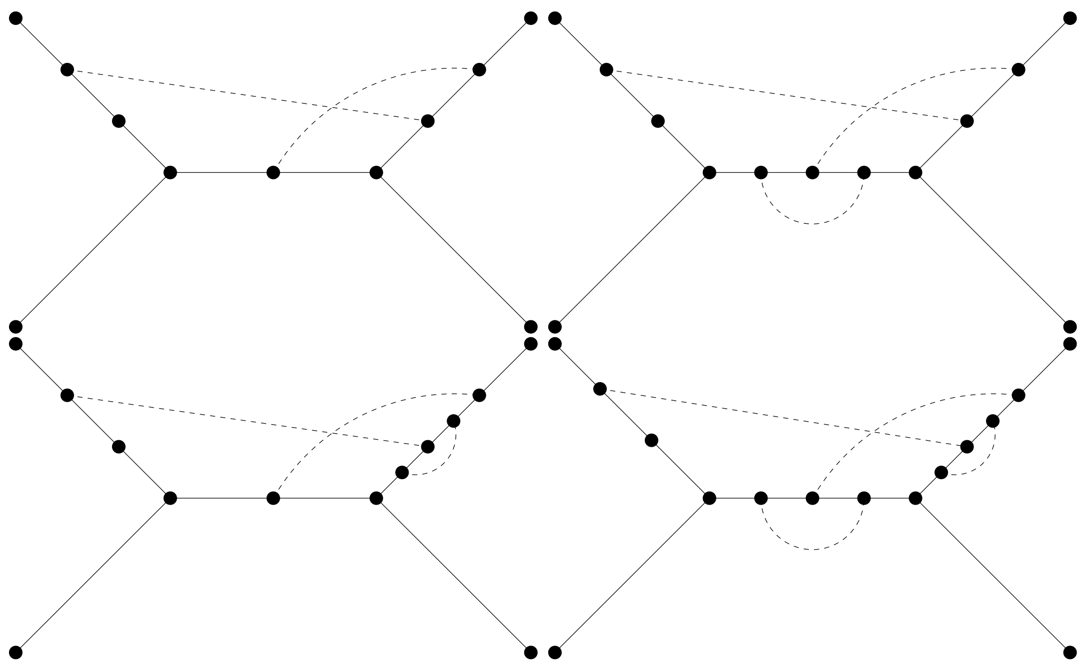


Figure 10: Augmentations of type C.

v_0-augmentation of η is of type A, B, C, or D.

Proof. Let (P, Q) be a minimal v_0-augmentation of η, let x_0 be the end of P in $\eta(v_0)$, and let b be the other end of P. We wish to think of P as being directed away from x_0; thus language such as "the first vertex of P in a set Z" will mean the vertex of $V(P) \cap Z$ that is closest to x_0 on P. Let E_1 and E_2 be as in the definition of v_0-augmentation.

Let us assume for a moment that P includes an internal vertex of some $\eta(e)$, where $e \in E(H)$ is not incident with v_0, v_1, or v_2. Let z be the first such vertex on P. The vertex z divides $\eta(e)$ into two subpaths, one even and one odd. Let R be the even one. Then $(P[x_0, z] \cup R, Q)$ is a v_0-augmentation, and hence the minimality of (P, Q) implies that $R = P[z, b]$. If $e \in E_1 \cup E_2$ and z is an outbound vertex, then the same conclusion holds. This will be later referred to as the *confluence property of P*.

If P includes an internal vertex of $\eta(e_1)$ for no $e_1 \in E_1 \cup E_2$, then (P, Q) is of type A. Thus we may assume that P includes such a vertex, and let x_1 be the first such vertex on P. From the symmetry we may assume that $e_1 = v_1v_3 \in E_1$. If x_1 is an outbound vertex, then the confluence property of P implies that (P, Q) is of type A. Thus we may assume that x_1 is inbound. If v_1 has degree at least four, then (P, Q) is of type D, and so we may assume that v_1 has degree at most three. It follows from axiom (7) in the definition of an embedding that v_1 has degree exactly three.

Let x_2 be the first vertex on P that belongs to $\eta(z)$ for some $z \in V(H) \cup E(H)$ not equal, incident or adjacent to v_0 and not equal to e_1. Then x_1 lies on P between x_0 and x_2. Let $P_1 = \eta(e_1)$. By Theorem 2.3.4 applied to P_1, $P_2 = P$, $r = x_2$, $s_2 = x_0$, $t_2 = b$ and the ends of P_1 numbered so that $s_1 \in V(\eta(v_0))$ and $t_1 \in V(\eta(v_3))$ we deduce that (1), (2), or (3) of Theorem 2.3.4 holds. But (1) does not hold by the minimality of (P, Q), and if (2) holds, then (R, Q) is a v_0-augmentation of type A or B. Thus we may assume that (3) of Theorem 2.3.4 holds. Since x_1 is an inbound vertex, this implies that either there exist vertices $y_1, y_2 \in V(P_1)$, such that y_1 and y_2 are outbound, $P[x_1, y_1] \subseteq P_1$, $x_1 \in P_1[y_1, y_2]$ and $P[y_1, y_2]$ has no internal vertices in $\eta(H)$, or $P[x_0, x_2] \setminus x_2$ intersects $\eta(e_1)$ regularly from x_0 to x_2. In the former case $(P[x_0, y_2] \cup P_1[y_2, t_1], Q)$ is a v_0-augmentation of η of type B, and hence we may assume that the latter case holds. Thus $P[x_0, x_2] \setminus x_2$ intersects $\eta(e_1)$ regularly from x_0 to x_2, and if $x_2 = t_2$, then $P[x_0, x_2] \setminus x_2$ intersects $\eta(e_1)$ in a path.

If $x_2 \in \bigcup\{V(\eta(v)) : v \in V(H) - \{v_0, v_1, v_2\}\}$, then (P, Q) is of type B. Therefore we may assume that $x_2 \in V(\eta(e_2))$ for some $e_2 \in E(H \setminus v_0) - \{e_1\}$. By the confluence property of P we may assume that $e_2 \in E_1 \cup E_2$ and that x_2 is inbound, for otherwise (P, Q) is of type B.

If $e_2 \in E_2$, then (P, Q) is of type C, and the lemma holds. Thus we may assume that $e_2 \in E_1 - \{e_1\}$. Let y be such that $\eta(e_2)[x_2, y]$ is a component of $\eta(e_2) \cap P$. For simplicity of notation assume that Q is empty. The argument in the other case is similar. As v_1 has degree three, axiom (8) in the definition of an embedding implies that the tree $\eta(v_1)$ consists a single vertex, say u_1. Since x_2 is inbound it follows that y lies between u_1 and x_2 in $\eta(e_2)$. Let C be the cycle $P[x_0, y] \cup \eta(e_2)[y, u_1] \cup S$, where $S = \eta(v_0)[x_0, u_1]$. The subgraph of G with edge-set $E(P) \triangle E(C)$ includes a path with ends x_0 and b, say P'. Let f be the edge of $P[x_0, x_1]$ incident to x_1. We define a new embedding $\eta' : H \hookrightarrow G$ by $\eta'(e_1) = \eta(e_1)[x_1, t_1]$, $\eta'(e_2) = P[x_1, x_2] \cup \eta(e_2)[x_2, z]$ (where $z \neq u_1$ is the other end of $\eta(e_2)$), $\eta'(v_1)$ is the graph with vertex-set $\{x_1\}$, we define $\eta'(v_0 v_1)$ to be the path with edge-set $\{f\}$, we define $\eta'(v_0)$ to be the path obtained from $\eta(v_0)$ by replacing $\eta(v_0)[x_0, u_1]$ by $P[x_0, x_1] \setminus x_1$, and we define

$\eta'(x) = \eta(x)$ for all other $x \in V(H) \cup E(H)$. It follows that (P', Q) is a v_0-augmentation of η', contrary to the minimality of (P, Q), because $P' \cup Q \cup \eta'(H)$ does not include the edge of $\eta(e_2)[y, x_2]$ incident with x_2. □

2.5 Disposition Of Bisplits

The purpose of this section is to prove Theorem 2.1.9 under the additional hypothesis that a graph, say H', obtained from H by bisplitting some vertex is isomorphic to a matching minor of G. If that is the case we apply Lemma 2.4.4 and Lemma 2.4.5. We handle the four possible outcomes of Lemma 2.4.5 separately.

Lemma 2.5.1. *Let H and G be graphs, where H has minimum degree at least three. Let H' be obtained from H by bisplitting a vertex v, and let v_0 be the new inner vertex. Let $\eta : H' \hookrightarrow G$, and assume that there exists a v_0-augmentation of η of type A. Then a linear extension of H is isomorphic to a matching minor of G.*

Proof. Let v_1 and v_2 be the new outer vertices of H', let (P, Q) be a v_0-augmentation of η of type A, and let a and b be the ends of P, where $a \in V(\eta(v_0))$. Let $b \in \eta(u)$, where $u \in V(H) - \{v_0, v_1, v_2\}$. Let us assume first that b is protected. If Q is null, then $H' + (v_0, u)$ is isomorphic to a matching minor of G, and otherwise (by ignoring Q and bicontracting its ends) we see that that $H + (v, u)$ is isomorphic to a matching minor of G and is a linear extension of H unless $vu \in E(H)$. If $vu \in E(H)$ we assume without loss of generality that $uv_1 \in E(H')$. Then $\eta(H' \setminus uv_1) \cup P \cup Q$ is isomorphic to a bisubdivision of a linear extension of H.

Now let us assume that b is exposed. Let T_1, T_2 be the two components of $\eta(u)\backslash b$. For each neighbor w of u in H the path $\eta(uw)$ has exactly one end in $\eta(u)$; that end is an old vertex by axiom (4) in the definition of an embedding, and hence belongs to either T_1 or T_2. For $i = 1, 2$ let N_i be the set of all neighbors w of u such that the end of $\eta(uw)$ in $\eta(u)$ belongs to T_i. Let H_1 be obtained from H by bisplitting u so that one of the new outer vertices is adjacent to every vertex of N_1, and the other new outer vertex is adjacent to every vertex of N_2. (Here we use that u has degree at least three.) Let u_0 be the new inner vertex of H_1. Let H_1' be defined similarly, but starting from H' rather than H, and let the new inner vertex be also u_0. If Q is null, then $H_1' + (v_0, u_0)$ is isomorphic to a matching minor of G; otherwise $H_1 + (v, u_0)$ is isomorphic to a matching minor of G, as desired. $\square$

Lemma 2.5.2. *Let H and G be graphs, let $\eta : H \hookrightarrow G$ be an embedding, let v_0 be vertex of H of degree two, and let v_1 be a neighbor of v_0 of degree three with neighbors v_0, v_1', v_1''. Let (P, Q) be a v_0-augmentation of η of type B or C that crosses $\eta(v_1v_1')$. Then there exists an embedding $\eta' : H \hookrightarrow G$ and a v_0-augmentation (P', Q') of η' of the same type as (P, Q) that crosses $\eta'(v_1v_1'')$ such that $\eta'(H) \cup P' \cup Q' \subseteq \eta(H) \cup P \cup Q$ and P and P' have the same terminus.*

Proof. We first define η'. Let x_0 be the end of P in $\eta(v_0)$, let x_6 be the other end of P, let $x_5 \in V(P)$, and let $x_0, x_1, \ldots, x_5$ be as in the definition of regular intersection, witnessing that $P[x_0, x_5]$ intersects $\eta(v_1v_1')$ regularly from x_0 to x_5. We define $\eta'(v_1) = x_1$, we define $\eta'(v_1v_1')$ to be the subpath of $\eta(v_1v_1')$ with one end x_1 and the other end in $\eta(v_1')$, we define $\eta'(v_1v_1'')$ to be the union of the complementary subpath of $\eta(v_1v_1')$ and $\eta(v_1v_1'')$, we define $\eta'(v_0)$ to be a suitable subgraph of $\eta(v_0) \cup P \cup Q$, define $\eta'(v_0v_1)$ to be the edge of $P[x_0, x_1]$ incident with x_1, and we define $\eta'(x) = \eta(x)$ for all other $x \in V(H) \cup E(H)$. Then $\eta' : H \hookrightarrow G$.

It is now easy to find subpaths Q' and P'' of $\eta(v_0) \cup \eta(v_0v_1) \cup \eta(v_1v_1') \cup P \cup Q$ such that $(P'' \cup P[x_4, x_6], Q')$ is the desired v_0-augmentation of η'. $\square$

Lemma 2.5.3. *Let H and G be graphs, where H has minimum degree at least three. Let H' be obtained from H by bisplitting a vertex v, and let v_0 be the new inner vertex. Let $\eta : H' \hookrightarrow G$, and assume that there exists a v_0-augmentation of η of type B. Then a linear extension of H is isomorphic to a matching minor of G.*

Proof. Let v_1 and v_2 be the new outer vertices of H'. Let (P, Q) be a v_0-augmentation of η of type B, let x_0, x_6 be the ends of P, where $x_0 \in V(\eta(v_0))$ and $x_6 \in V(\eta(u))$, and let P cross $e_1 = v_1v_1'$, where $v_1' \neq v_0$ is a neighbor of v_1 in H'. Let $x_5 \in V(P)$ be such that $P[x_0, x_5]$ intersects $\eta(e_1)$ regularly from x_0 to x_5, and let the vertices $x_0, x_1, x_2, x_3, x_4, x_5$

be as in the definition of regular intersection. Notice that v_1 has degree three; thus $\eta(v_1)$ consists of a unique vertex by condition (8) in the definition of embedding. Let v_1'' be the third neighbor of v_1. By Lemma 2.5.2 we may assume that $u \neq v_1'$.

Assume first that x_2, x_3, x_4 are pairwise distinct. The path $P[x_4, x_6]$ proves that a linear extension of H is isomorphic to a matching minor of G, unless x_6 is a protected vertex of $\eta(u)$ and u is adjacent to v in H. Let $i \in \{1, 2\}$ be such that u is adjacent to v_i in H'. Consider the graph obtained from $\eta(H) \cup P[x_4, x_0]$ by deleting the interior of $\eta(v_i u)$; the path $P[x_2, x_3]$ proves that the linear extension $H'' + (v_0', v_1')$ of H is isomorphic to a matching minor of G, where H'' is obtained from H by bisplitting of the vertex v so that one of the new outer vertices is adjacent to v_1' and u, the other outer vertex is adjacent to all other neighbors of v and v_0' is the new inner vertex.

Thus we may assume that $x_2 = x_3 = x_4$. Again the path $P[x_4, x_6]$ proves that a linear extension of H is isomorphic to a matching minor of G, unless x_6 is a protected vertex of $\eta(u)$ and u is adjacent to v_1' in H. Thus we may assume that x_6 is a protected vertex of $\eta(u)$ and u is adjacent to v_1' in H. If v_1' has degree at least four, then let H'' be obtained from H' by bisplitting v_1' in such a way that one of the new vertices is adjacent to v_1 and u, and let z be the new vertex. Then $H'' + (v_0, z)$ is a linear extension of H and is clearly isomorphic to a matching minor of G. If v_1' has degree three we replace $\eta(v_1' u)$ by $P[x_4, x_6]$ and notice that (P, Q) can be easily converted to a v_0-augmentation (P', Q') of type A of the embedding thus obtained. (Notice that the terminus of P' does not belong to $\eta(v_2)$, because H' is obtained from H by bisplitting v.) Hence the theorem follows from Lemma 2.5.1. $\square$

For the next lemma we need the following generalization of v_0-augmentations. Let $v_0 \in V(H)$ have degree two, and let v_1, v_2 be its neighbors. For $i = 1, 2$ let E_i be the set of edges of H incident with v_i, except for the edge $v_0 v_i$, and let $E_1 \cap E_2 = \emptyset$. Let R be the interior of $\eta(v_0)$, let M_1 be a perfect matching of $G \backslash V(\eta(H))$, let $x \in V(R)$, let M_2 be a perfect matching of $R \backslash x$, and let $M = M_1 \cup M_2 \cup M(\eta)$. Let P be an M-alternating path with one end x and the other end u in $\bigcup\{\eta(v) : v \in V(H) - \{v_0, v_1, v_2\}\}$. We say that P is a *weak v_0-augmentation of η*. It follows that P has no internal vertex in $\bigcup_{v \in V(H) - \{v_0\}} \eta(v)$.

This is indeed a generalization of v_0-augmentation. For let (P, Q) be a v_0-augmentation of η. If Q is null, then P is a weak v_0-augmentation of η, and otherwise $Q \cup S \cup P$ is a weak v_0-augmentation of η, where S is a subpath of $\eta(v_0)$ with one end the end of P and the other end an end of Q.

Lemma 2.5.4. *Let H, G be graphs, let $\eta : H \hookrightarrow G$ be an embedding, let v_0 be a vertex of H of degree two belonging to no triangle of H, and let R be a weak v_0-augmentation of η. Then there exist an embedding $\eta' : H \hookrightarrow G$ and a v_0-augmentation (P, Q) of η' such that $P \cup Q \cup \eta'(H) \subseteq R \cup \eta(H)$.*

Proof. We may assume that R is minimal in the sense that there does not exist an embedding $\eta' : H \hookrightarrow G$ and a weak v_0-augmentation R' of η' such that $R' \cup \eta'(H)$ is a proper subgraph of $R \cup \eta(H)$. It follows that R has the confluence property introduced in the proof of Lemma 2.4.5. Let v_1, v_2 be the neighbors of v_0, and let E be the set of all edges of H incident with a neighbor of v_0, but not with v_0 itself.

Let a, b be the ends of R, where $a \in V(\eta(v_0))$ and let z_1, z_2 be the ends of $\eta(v_0)$. Assume first that R has a vertex x such that $R[a, x]$ includes an internal vertex of $\eta(e)$ for no edge $e \in E$, and $R[x, b]$ includes no vertex of $\eta(v_0)$. Let Ω be a path octopus with head a and graph $\eta(v_0)$. We apply Lemma 2.3.5 to Ω and $R[a, x]$ to produce a path octopus Ω' with head z and ends z_1 and z_2 and a path R' with ends z and x. Define η' so that $\eta'(v_0)$ is the graph of Ω' and otherwise η' coincides with η. Let P be a maximal subpath of $R' \cup R[x, b]$ with no internal vertex in $\eta'(v_0)$ containing b and let Q be a maximal non-empty subpath of $R' \backslash V(P)$ with no internal vertex in $\eta'(v_0)$ if such a path exists, and otherwise let Q be the null graph. It is easy to check that (P, Q) is a v_0-augmentation of η'.

Thus we may assume that the assumption of the previous paragraph does not hold. Thus there exists an edge $e \in E$ such that when following R starting from a at some point we encounter an internal vertex of $\eta(e)$, and later an internal vertex of $\eta(v_0)$, say t. Let T be the component of $R \cap \eta(v_0)$ containing t. Let the ends of e be v_1 and v_1', where v_1 is adjacent to v_0, and let the ends of $\eta(e)$ be u_1 and u_1', where u_1 belongs to $\eta(v_1)$ and u_1' belongs to $\eta(v_1')$. Let S be the component of $R[a, t] \cap \eta(e)$ that is closest to u_1' on $\eta(e)$.

Let t_1, t_2 be the ends of T, where a, t_1, t_2, b occur on R in the order listed, and let s_1, s_2 be the ends of S chosen similarly. If t_2 lies at an even distance from a on $\eta(v_0)$, then $R[t_2, b]$ is a weak v_0-augmentation of η, contrary to the minimality of R. Thus t_1 lies at an even distance from a on $\eta(v_0)$. It follows from the confluence property that s_1 is an inbound vertex of $\eta(e)$ (that is, its distance from u_1 on $\eta(e)$ is even). Thus s_2 is an outbound vertex, and hence $R[t_1, s_2] \cup \eta(e)[s_2, u_1']$ is a weak v_0-augmentation of η, contrary to the minimality of R. □

Let H and G be graphs, let H' be obtained from H by bisplitting a vertex v, and let v_0 be the new inner vertex. Let $\eta : H' \hookrightarrow G$, and let (P, Q) be a v_0-augmentation of η. We say that (P, Q) is *strongly minimal* if there exists no graph H'' obtained from H by bisplitting v, an embedding $\eta'' : H'' \hookrightarrow G$ and (letting v_0'' denote the new inner vertex of H_0'') a v_0''-augmentation (P'', Q'') of η'' such that $\eta''(H'') \cup P'' \cup Q''$ is a proper subgraph of $\eta(H') \cup P \cup Q$.

Lemma 2.5.5. *Let H and G be graphs. Let H' be obtained from H by bisplitting a vertex v, let v_0 be the new inner vertex, and let $\eta : H' \hookrightarrow G$. Then no v_0-augmentation of η of type C is strongly minimal.*

Proof. Let v_1, v_2 be the new outer vertices of H', let (P, Q) be a v_0-augmentation of η of type C, let a, b be the ends of P with $a \in V(\eta(v_0))$, and let x_1, x_2, e_1, e_2 be as in the definition of augmentation of type C. The vertex v_1 has degree three; let $e_1' \notin \{e_1, v_1v_0\}$ be the third incident edge. Let H'' be obtained from H by bisplitting v into new outer vertices v_1'', v_2'' and new inner vertex v_0'', where v_1'' is incident with e_1 and e_2, and v_2'' is incident with all the remaining edges of H incident with v. The embedding η can be modified to produce an embedding $\eta'' : H'' \hookrightarrow G$ with $\eta''(H) \subseteq P \cup \eta(H)$ by defining $\eta''(v_2'') = \eta(v_2)$, by defining $\eta''(v_1'')$ to be the graph with vertex set $\{x_2\}$, by letting $\eta''(e_2)$ be a subpath of $\eta(e_2)$ with end x_2, by letting $\eta''(e_1)$ be the union of a subpath of $P[x_2, a]$ with a suitable subpath of $\eta(e_1)$, and by letting $\eta''(e_1') = \eta(v_0) \cup \eta(v_1v_0) \cup \eta(v_2v_0) \cup \eta(e_1')$. Now $P[x_2, b] \backslash x_2$ is a weak v_0''-augmentation of η''. By Lemma 2.5.4 there exists an embedding $\xi : H'' \hookrightarrow G$ and a

v_0''-augmentation (P'', Q'') of η'' such that

$$P'' \cup Q'' \cup \xi(H'') \subseteq P[x_2, b] \cup \eta''(H'') \subseteq P \cup \eta(H),$$

but $P'' \cup Q'' \cup \xi(H'')$ does not use the edge of P incident with a, contrary to the weak minimality of (P, Q). □

Lemma 2.5.6. *Let H and G be graphs, let H' be obtained from H by bisplitting a vertex v, let v_0 be the new inner vertex, and let $\eta : H' \hookrightarrow G$. Then no v_0-augmentation of η of type D is strongly minimal.*

Proof. Let v_1, v_2 be the new outer vertices of H', let (P, Q) be a v_0-augmentation of η of type D, let a, b be the ends of P with $a \in V(\eta(v_0))$, and let i, e, x be as in the definition of augmentation of type D. We may assume that $i = 1$. Let H'' be obtained from H by bisplitting v into new outer vertices v_1'', v_2'' and new inner vertex v_0'', where v_1'' is incident with all the edges of H incident with v_1 in H' except e (note that $v_0v_1 \notin E(H)$), and v_2'' is incident with all the remaining edges of H incident with v. The embedding η can be modified to produce an embedding $\eta'' : H'' \hookrightarrow G$ with $\eta''(H) \subseteq P \cup \eta(H)$ by defining $\eta''(v_1'') = \eta(v_1')$ and letting $\eta''(v_2'')$ be a suitable subgraph of $\eta(v_2) \cup \eta(v_0) \cup \eta(v_0v_2) \cup P[a, x] \cup Q$. Now $P[x, b] \backslash x$ includes a weak v_0''-augmentation of η''. By Lemma 2.5.4 there exists an embedding $\xi : H'' \hookrightarrow G$ and a v_0''-augmentation (P'', Q'') of η'' such that

$$P'' \cup Q'' \cup \xi(H'') \subseteq P[x, b] \cup \eta''(H'') \subseteq P \cup \eta(H),$$

but $P'' \cup Q'' \cup \xi(H'')$ does not use one of the edges of $\eta(v_0)$ incident with a, contrary to the weak minimality of (P, Q). □

We summarize Lemmas 2.5.1, 2.5.3, 2.5.5, and 2.5.6 into the following.

Lemma 2.5.7. *Let H and G be graphs, where H has minimum degree at least three, let H' be obtained from H by bisplitting a vertex v, let v_0 be the new inner vertex, let $\eta : H' \hookrightarrow G$ be an embedding and assume that there exists a v_0-augmentation of η. Then a linear extension is isomorphic to a matching minor of H.*

Proof. We may assume that the v_0-augmentation is strongly minimal. By Lemma 2.4.5 it is of type A, B, C, or D. By Lemmas 2.5.5 and 2.5.6 it is of type A or B, and so the result holds by Lemmas 2.5.1 and 2.5.3. $\square$

We say that an embedding $\eta : H \hookrightarrow G$ is a *homeomorphic embedding* if $\eta(v)$ has exactly one vertex for every $v \in V(H)$ of degree at least three. The next lemma motivates this definition.

Lemma 2.5.8. *Let H and G be graphs. Then there exists an embedding $\eta : H \hookrightarrow G$ which is not a homeomorphic embedding if and only if a graph obtained from H by bisplitting a vertex is isomorphic to a matching minor of G.*

Proof. Suppose that $\eta : H \hookrightarrow G$ and that for some vertex $v \in V(H)$ of degree at least three its image $\eta(v)$ has more than one vertex. Then there exists a branch B of $\eta(v)$ with length greater than zero. The argument from the last paragraph of the proof of Lemma 2.4.4 applied to $\eta(v)$ and B, provides us with an embedding into G of a graph H_1 obtained from H by bisplitting v and therefore by Lemma 2.4.2 the graph H_1 is isomorphic to a matching minor of G.

On the other hand let a graph H', obtained from H by bisplitting some vertex v into new outer vertices v_1 and v_2 and new inner vertex v_0, be isomorphic to a matching minor of G. Then by Lemma 2.4.2 there exists an embedding $\eta' : H' \hookrightarrow G$. Let J be the subgraph of H induced by $\{v_0, v_1, v_2\}$. Define an embedding $\eta : H \hookrightarrow G$ by saying that $\eta(v) = \eta'(J)$, $\eta(vu) = \eta'(v_i u)$ for $i \in \{1, 2\}$ and all neighbors $u \neq v_0$ of v_i, and otherwise η coincides with η'. Clearly $\eta(v)$ has more than one vertex and therefore η is not a homeomorphic embedding. $\square$

The following theorem and its corollary are the main results of this section.

Theorem 2.5.9. *Let G be a brick, let H be a graph of minimum degree at least three, and let $\eta : H \hookrightarrow G$. If η is not a homeomorphic embedding, then a linear extension of H is isomorphic to a matching minor of G.*

Proof. Let v be a vertex of H of degree at least three such that $\eta(v)$ has at least two vertices. By axiom (8) in the definition of an embedding the vertex v has degree at least four. We apply Lemma 2.4.4 to H, G, η and v. If outcome (1) of Lemma 2.4.4 holds then the theorem holds by Lemma 2.5.7.

Therefore we may assume that (2) of Lemma 2.4.4 holds, and let η_2, P, p_1, p_2, B_1, B_2, Q_1 and Q_2 be as in Lemma 2.4.4. Let G' be the graph obtained from $\eta_2(H) \cup P \cup Q_1 \cup Q_2$ by bicontracting all exposed vertices, except those in $B_1 \cup B_2$. Note that G' is a matching minor of G and therefore it suffices to prove that a linear extension of H is isomorphic to a matching minor of G'. If both Q_1 and Q_2 are null, then the graph G' is isomorphic to a bisubdivision of a graph obtained from H by two bisplits and adding an edge joining the two new inner vertices. Thus a linear extension of H is isomorphic to a matching minor of G.

Therefore we may assume that Q_2 is not null. Let u be the common end of B_1 and B_2 in G' and let u_1 and u_2 be the other ends of B_1 and B_2 correspondingly. If Q_1 is not null, denote its ends by q and q' so that $q \in B_1[p_1, u_1]$ and let $q = q' = p_1$ otherwise. If u has degree at least four in G' then the graph G'' obtained from G' by deleting the interiors of $B_1[u, q']$, $B_1[p_1, q]$ and Q_2 can be bicontracted to a graph obtained from H by two bisplits and Q_2 can be bicontracted to an edge joining the two new inner vertices. Thus again a linear extension of H is isomorphic to a matching minor of G.

Therefore we may assume that u has degree three in G'. Hence there exists a unique vertex $w \in V(H)$ such that $u \in \eta_2(vw)$. Now G'' can be bicontracted to a graph obtained from H by bisplitting v and Q_2 can be bicontracted to an edge joining the new inner vertex to w. We deduce that a linear extension of H is isomorphic to a matching minor of G, as desired. $\square$

The next result follows immediately from Lemma 2.5.8 and Theorem 2.5.9.

Theorem 2.5.10. *Let G be a brick, let H be a graph of minimum degree at least three, and assume that a graph obtained from H by bisplitting a vertex is isomorphic to a matching minor of G. Then a linear extension of H is isomorphic to a matching minor of G.*

2.6 The Hierarchy of Extensions

For the sake of exposition let us define a *split extension* of a graph H to be any graph obtained from H by bisplitting a vertex. We have seen in the previous section that if a split extension of H is isomorphic to a matching minor of G, then the conclusion of Theorem 2.1.9 holds. The purpose of this short section is to define other types of extensions and to give an ordering on these extensions, and to reformulate Lemma 2.4.5. The ordering reflects the order in which these extensions will be dealt with. We will be proving theorems of the form "if such an such extension is isomorphic to a matching minor of G, then an extension that is higher on our list of priorities is also isomorpchic to a matching minor of G". Of course, the highest priority extensions are linear extensions.

Let us begin the definitions. The lowest on our list will be the following. Let H be a graph, let $v \in V(H)$ be a vertex of degree at least three, and let v_1, v_2 be two distinct neighbors of v in H. Let H' be obtained from H by bisubdividing the edge vv_1, and let x, y be the new vertices numbered so that x is adjacent to v. We say that the graph $H + (y, v_2v)$ is a *vertex-parallel extension of* H. We say that $H + (y, v_2)$ is an *edge-parallel extension of* H.

Let v be a vertex of degree 3 in a graph H and let v_1, v_2 and v_3 be its neighbors. We say that K is obtained from H by *replacing* v *by a triangle* if K is obtained from H by deleting the vertex v and adding the vertices u_1, u_2, u_3 and edges $u_1u_2, u_2u_3, u_3u_1, u_1v_1, u_2v_2$ and u_3v_3.

Let H be a graph, let v be a vertex of H of degree at least three in a graph H, and let v_1 and v_2 be two neighbors of v. Let K be obtained from H by bisubdividing the edges v, v_1 and v, v_2 and let x_1, y_1, x_2, y_2 be the new vertices numbered so that $v_1y_1x_1vx_2y_2$ is a path in K. Let $K' = K + (x_1, y_2) + (x_2y_1)$, and let $J = K'$, or let J be obtained from K' by replacing one or both of the vertices x_1, x_2 by triangles. We say that J is a *cross extension* of H, and that v is its *hub*. See Figure 11.

Let u be a vertex of H of degree three and let u_1,u_2 and u_3 be its neighbors. Let H_0 be obtained from H by bisubdividing each of the edges uu_1, uu_2 and uu_3. Let the new vertices be y_1, y_2, y_3 and z_1, z_2, z_3 in such a way that $u_1y_1z_3u$, $u_2y_2z_1u$ and $u_3y_3z_2u$ are paths. Let

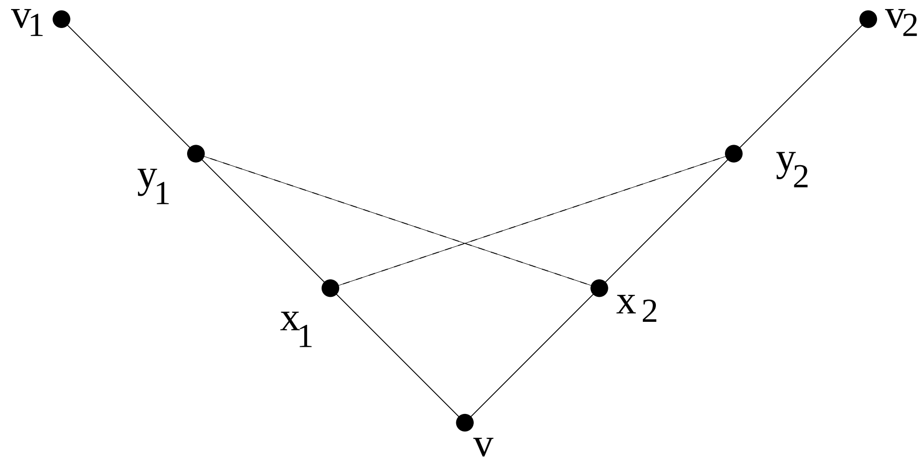


Figure 11: A cross extension.

$H_1 := H_0 + (y_1, z_2) + (y_2, z_3) + (y_3, z_1)$, let H_2 be obtained from H_1 by replacing z_1 by a triangle, let H_3 be obtained from H_2 by replacing z_2 by a triangle, and let H_4 be obtained from H_3 by replacing z_3 by a triangle. Then each of the graphs H_1, H_2, H_3, H_4 is called a *cube extension of* H. See Figure 12.

Let H be a graph, let $uv \in E(H)$, and let H' be obtained from H by bisubdividing uv, where the new vertices x, y are such that x is adjacent to u and y. Let $x' \in V(H) - \{u\}$ and $y' \in V(H) - \{v\}$ be not necessarily distinct vertices such that not both belong to $\{u, v\}$. In those circumstances we say that $H' + (x, x') + (y, y')$ is a *quadratic extension of* H. We say that uv is *the base* of this quadratic extension. Now let $ab \in E(H) - \{uv\}$ be such that $a \neq v$ and $u \neq b$, let H'' be obtained from H' by bisubdividing ab, and let x', y' be the new vertices. Then the graph $H'' + (x, x') + (y, y')$ is called a *quartic extension of* H. We say that uv, ab are *the bases* of this quartic extension.

We are now ready to define the promised linear order on extensions. We define that linear extensions are better than quartic extensions, quartic extensions are better than quadratic extensions, which in turn are better than cross extensions, which are better than cube extensions, which are better than edge-parallel extensions, and those are better than vertex-parallel extensions.

For later convenience we reformulate Lemma 2.4.5 in a form more suitable for applications. To do so we will need a definition, but before we can state it, we need to introduce

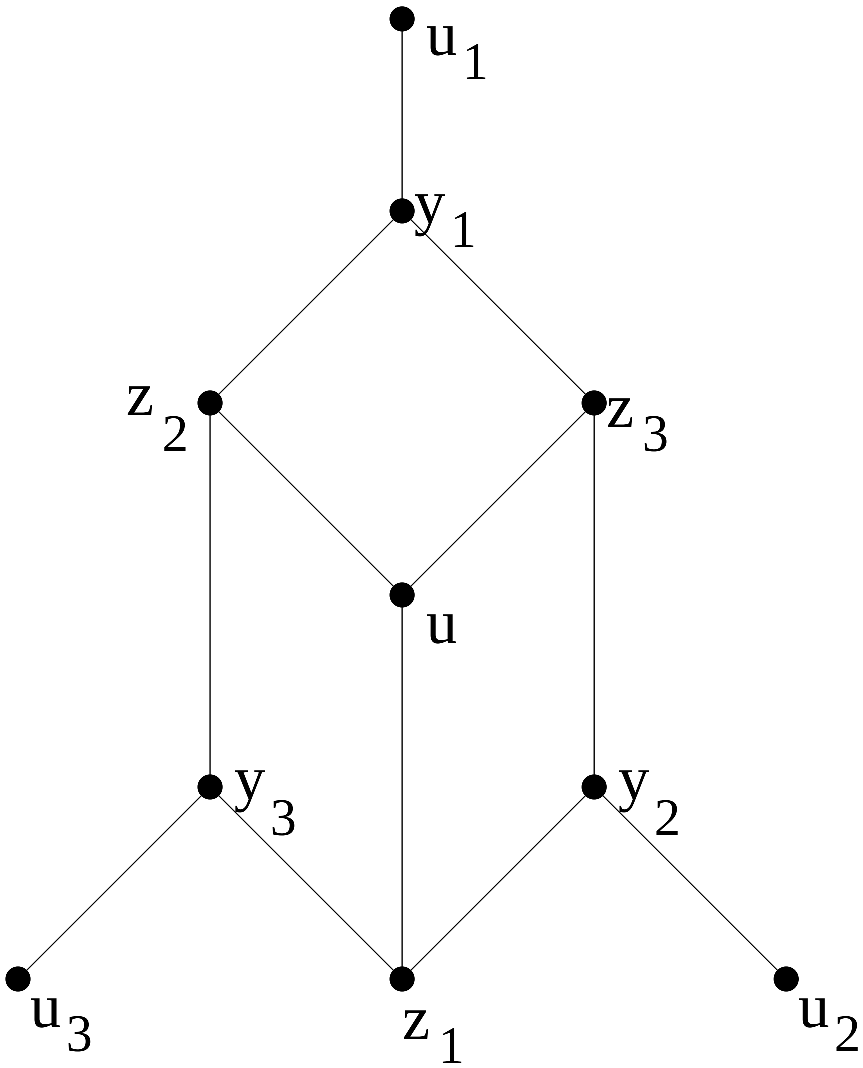


Figure 12: A cube extension.

a convention. Let G be a graph, let $w \in V(G)$, and let uv be an edge of G not incident with w. Then the graph $G' = G + (w, uv)$ has two new vertices, and it will be convenient to have a default notation for them. We shall use τ_1 and τ_2 to denote the new vertices, so that τ_1 is adjacent to u, w and τ_2 in G'. We shall extend this convention naturally to more complicated scenarios, as exemplified by the following illustration. For instance, if $ab \in E(G) - \{uv\}$, then $G'' = G + (w, uv) + (\tau_2, ab)$ means the graph $G' + (\tau_2, ab)$, and its new vertices are denoted by τ_3 and τ_4 so that τ_3 is adjacent to a, τ_2 and τ_4 in G''. In general, the new vertices will be numbered $\tau_1, \tau_2, \tau_3, \ldots$ in the order they arise as the input graph is read from left to right. Sometimes we will use $\rho_1, \rho_2, \ldots$ rather than $\tau_1, \tau_2, \ldots$ in order to avoid confusion.

Now we are ready for the definition. Let J, G be graphs, let v_0 be a vertex of J of degree two, and let v_1, v_2 be the neighbors of v_0. We wish to reformulate the outcomes of Lemma 2.4.5. Let $v \in V(J) - \{v_0, v_1, v_2\}$, let $i \in \{1, 2\}$, and for $j = 1, 2$ let v'_j be a neighbor of v_j other than v_0. We define the following graphs:

- $A_1(v) = J + (v_0, v)$,
- $A_2(v) = J + (v_0, v_1v_0) + (\tau_2, v)$,
- $B_1(v'_iv_i, v) = J + (v_0, v'_iv_i) + (\tau_2, v)$,
- $B_2(v'_iv_i, v) = J + (v_0, v'_iv_i) + (\tau_2, v_i\tau_2) + (\tau_4, v)$,
- $B_3(v'_iv_i, v) = J + (v_0, v_iv_0) + (\tau_2, v'_iv_i) + (\tau_4, v)$,
- $B_4(v'_iv_i, v) = J + (v_0, v_iv_0) + (\tau_2, v'_iv_i) + (\tau_4, v_i\tau_4) + (\tau_6, v)$,
- $C_1(v'_iv_i, v'_{3-i}v_{3-i}) = J + (v_0, v'_iv_i) + (\tau_2, v'_{3-i}v_{3-i})$,
- $C_2(v'_iv_i, v'_{3-i}v_{3-i}) = J + (v_0, v'_iv_i) + (\tau_2, v_i\tau_2) + (\tau_4, v'_{3-i}v_{3-i})$,
- $C_3(v'_iv_i, v'_{3-i}v_{3-i}) = J + (v_0, v_iv_0) + (\tau_2, v'_iv_i) + (\tau_4, v'_{3-i}v_{3-i})$,
- $C_4(v'_iv_i, v'_{3-i}v_{3-i}) = J + (v_0, v_iv_0) + (\tau_2, v'_iv_i) + (\tau_4, v_i\tau_4) + (\tau_6, v'_{3-i}v_{3-i})$.

Sometimes we will omit the arguments when they will be clear from the context and write, e.g., B_3 instead of $B_3(v'_iv_i, v)$. The following lemma gives the promised reformulation of the outcomes of Lemma 2.4.5.

Lemma 2.6.1. *Let J be a graph, let G be a brick, let v_0 be a vertex of J of degree two, let v_1, v_2 be the neighbors of v_0, for $i = 1, 2$ let $v_i' \neq v_0$ be a neighbor of v_i, assume that v_1 is not adjacent to v_2, assume that every vertex $v \in V(J) - \{v_0\}$ has a neighbor in $V(J) - \{v_1, v_2\}$, and assume that there exists an embedding $J \hookrightarrow G$. Then one of the following holds:*

(A) *there exists a vertex $v \in V(J) - \{v_0, v_1, v_2\}$ such that $A_1(v) \hookrightarrow G$ or $A_2(v) \hookrightarrow G$,*

(B) *there exist a vertex $v \in V(J) - \{v_0, v_1, v_2\}$ and indices $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\}$ such that v_i has degree three and $B_j(v_i'v_i, v) \hookrightarrow G$,*

(C) *there exist indices $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\}$ such that v_1, v_2 have degree three and $C_j(v_i'v_i, v_{3-i}'v_{3-i}) \hookrightarrow G$,*

(D) *some split extension of J is isomorphic to a matching minor of G.*

Proof. Let $\eta : J \hookrightarrow G$. We may assume that η is a homeomorphic embedding, for otherwise (D) holds by Lemma 2.5.8. By changing η we may assume that $\eta(v_1)$ and $\eta(v_2)$ each have exactly one vertex, even if v_1 or v_2 has degree less than three. By Lemma 2.4.4 there exists an embedding $\eta' : J \hookrightarrow G$ and a v_0-augmentation (P, Q) of η'. We may assume that (P, Q) is minimal, and hence by Lemma 2.4.5 it is of type A, B, C or D. Similarly as above, we may assume that η' is a homeomorphic embedding. Let P have origin $a \in V(\eta'(v_0))$ and terminus $b \in V(\eta'(u))$. We say that (P, Q) is *good* if either u has degree not equal to two, or u has degree two and b is at even distance from either end of $\eta'(u)$ (recall that $\eta'(u)$ is an even path when u has degree two, and otherwise $\eta'(u)$ has exactly one vertex).

Suppose (P, Q) is not good. Then u has degree two and b is at odd distance from the ends of $\eta(u)$. Let u' be a neighbor of u in $V(J) - \{v_1, v_2\}$ and let b' and b'' be the ends of $\eta(u)$, such that $b' \in V(\eta(uu'))$. Let G' be obtained from $\eta(H) \cup P \cup Q$ by contracting the even path $\eta(u)[b, b'] \cup \eta(uu')$. Define $\eta' : J \hookrightarrow G'$ as follows. Let $\eta'(u) = \eta(u)[b'', b]\backslash b$, $\eta'(uu')$ is a length one subpath of $\eta(u)[b, b'']$ with one end at b and η' is otherwise equal to η. Note that (P, Q) is a good augmentation of η'. Note also that G' is a matching minor of G.

Therefore we may assume that (P, Q) is a good augmentation of η of type A, B, C or D. Now if (P, Q) is of type A, then outcome (A) holds, and similarly for type D, and, by

Lemma 2.5.2, for type B. Thus we may assume that (P, Q) is of type C. From the symmetry we may assume that P crosses an edge incident with v_1, and by Lemma 2.5.2 we may assume that it crosses the edge v_1v_1'. In particular, v_1 has degree at most three. But it has degree exactly three by axiom (7) in the definition of an embedding, because $\eta(v_1v_1')$ has at least one internal vertex. The existence of (P, Q) implies, by the same argument as above, that there is an integer $j \in \{1, 2, 3, 4\}$ such that $C_j(v_1'v_1, v_2''v_2) \hookrightarrow G$ for some neighbor v_2'' of v_2 other than v_0. Let $L = C_1(v_1'v_1, v_2''v_2)\backslash v_0v_2\backslash \tau_1\tau_2$ if $j = 1$, and let it be defined analogously for $j \geq 2$. If v_2 has degree at least four, then L is isomorphic to a bisubdivision of a split extension of H, and hence the lemma holds. Thus we may assume that v_2 has degree at most three, but it has degree exactly three by the same reason as v_1. If $v_2' = v_2''$, then (C) holds, and so we may assume not. If $j = 1$, then by considering L and the edges $\tau_1\tau_2$ and v_0v_2 we deduce that $C_1(v_1'v_1, v_2'v_2) \hookrightarrow G$. An analogous argument works for $j = 4$, while for $j \in \{2, 3\}$ the analogous argument proves that $C_{5-j}(v_1'v_1, v_2'v_2) \hookrightarrow G$. Thus (C) holds, as desired. □

2.7 Using 3-Connectivity

Recall that, a graph G is *matching covered* if every edge of G belongs to a perfect matching of G. Thus every brick is matching covered.

Lemma 2.7.1. *Let H and G be graphs such that H has minimum degree at least three, G is connected and matching covered, and H is isomorphic to a matching minor of G. If H is not isomorphic to G, then either a linear or split extension of H is isomorphic to a matching minor of G, or there exists a homeomorphic embedding $\eta' : H \hookrightarrow G$ such that $\eta'(e)$ has at least three edges for some $e \in E(H)$.*

Proof. By Lemma 2.4.2 there exists an embedding $\eta : H \hookrightarrow G$. We may assume that η is a homeomorphic embedding, for otherwise the lemma holds by Lemma 2.5.8. We may also assume that $\eta(e)$ has exactly one edge for each $e \in E(H)$. Thus $\eta(H)$ is isomorphic to H. But G is not isomorphic to H, and hence there exists an edge e of G with exactly one end in $\eta(H)$. Let M_1 be a perfect matching of G containing e, and let M_2 be a perfect matching of $G\backslash V(\eta(H))$. (This exists, because $\eta(H)$ is a central subgraph of G.) The component of $M_1 \triangle M_2$ containing e is a path with both ends in $\eta(H)$; let $u, v \in V(H)$ be such that P has one end in $\eta(v)$ and the other end in $\eta(u)$. If u and v are not adjacent in H, then by letting $\eta(uv) = P$ the embedding η can be extended to an embedding $H + uv \hookrightarrow G$, and hence a linear extension of H is isomorphic to a matching minor of G. On the other hand, if u and v are adjacent in H, then P has at least three edges, because in that case the unique edge of G between $\eta(u)$ and $\eta(v)$ belongs to $\eta(uv)$. Thus we obtain the desired homeomorphic embedding by replacing $\eta(uv)$ by P. $\square$

Let G be a graph, let $A, B \subseteq V(G)$, let M be a perfect matching of $G\backslash(A \cup B)$, and let $k \geq 0$ be an integer. We say that the sequence of paths $(P, Q_1, Q_2, \ldots, Q_k)$ is an (A, B)-*hook with respect to* M if the following conditions hold:

(1)P has ends $p_0 \in A - B$ and $p_{k+1} \in B - A$, and is otherwise disjoint from $A \cup B$,

(2)for $i = 1, 2, \ldots, k$, Q_i is an even path with ends $p_i \in V(P) - \{p_0, p_{k+1}\}$ and $q_i \in A \cap B$, and is otherwise disjoint from $A \cup B \cup V(P)$,

(3)$V(Q_i) \cap V(Q_j) \subseteq \{q_i, q_j\}$ for all distinct indices $i, j \in \{1, 2, \ldots, k\}$,

(4)the graph $P \cup Q_1 \cup Q_2 \cup \ldots \cup Q_k\backslash(A \cup B)$ is M-covered, and

(5)the vertices $p_0, p_1, p_2, \ldots, p_k, p_{k+1}$ are distinct and occur on P in the order listed.

Lemma 2.7.2. *Let G be a matching covered graph, let $A, B \subseteq V(G)$, and let M be a perfect matching of $G\backslash(A \cup B)$. If $A - B$ and $B - A$ are both nonempty and belong to the same component of $G\backslash(A \cap B)$, then there exists an (A, B)-hook with respect to M.*

Proof. Suppose for a contradiction that the graph G does not satisfy the lemma, and choose (A, B) violating the lemma with $A \cup B$ maximum. Let e be an edge of G with one end in $A - B$ and the other end in $V(G) - A$. Let M' be a perfect matching of G containing e, and let P_0 be the component of $M \triangle M'$ containing e. Then P_0 is a path with one end in $A - B$, the other end in $A \cup B$, and otherwise disjoint from $A \cup B$. If the other end of P_0 is in $B - A$, then the sequence with sole term P_0 is a required (A, B)-hook, and so we may assume that the other end of P_0 is in A. Let $A' := A \cup V(P_0)$. Then $A' \cap B = A \cap B$. By the maximality of $A \cup B$ there exists an (A', B)-hook $h = (P, Q_1, Q_2, \ldots, Q_k)$.

Let $p_0 \in A'$ be an end of P. If $p_0 \in A$, then h is an (A, B)-hook, and the lemma holds. Thus we may assume that p_0 is an internal vertex of P_0. Let P_1 and P_2 be the two subpaths of P_0 with common end p_0 and union P_0. Exactly one of them, say P_1, has the property that $P_1 \cup P \cup Q_1 \cup Q_2 \cup \ldots \cup Q_k \backslash (A \cup B)$ is M-covered. If the other end of P_1 is in $A - B$, then $(P \cup P_1, Q_1, Q_2, \ldots, Q_k)$ is a desired (A, B)-hook. Thus we may assume that P_1 has one end in $A \cap B$, in which case $(P \cup P_2, P_1, Q_1, Q_2, \ldots, Q_k)$ is a desired (A, B)-hook. $\square$

Theorem 2.7.3. *Let H and G be graphs, where H has minimum degree at least three and is isomorphic to a matching minor of G and G is a brick. If H is not isomorphic to G, then a vertex-parallel, edge-parallel or a linear extension of H is isomorphic to a matching minor of G.*

Proof. By Lemma 2.4.2 and Theorem 2.5.9 we may assume that there exists a homeomorphic embedding $\eta : H \hookrightarrow G$. By Lemma 2.7.1 we may assume that there exists an edge $uv \in E(H)$ such that $\eta(uv)$ has at least three edges. Let $A = V(\eta(uv))$ and let B consist of $V(\eta(H))$, except the internal vertices of $\eta(uv)$. Then $A - B$ and $B - A$ are nonempty and $|A \cap B| = 2$. Thus $A - B$ and $B - A$ belong to the same component of $G \backslash (A \cap B)$, because G is 3-connected. We have $A \cup B = V(\eta(H))$, and hence $G \backslash (A \cup B)$ has a perfect matching, say M, because $\eta(H)$ is a central subgraph of G. By Lemma 2.7.2 there exists an (A, B)-hook $h = (P, Q_1, Q_2, \ldots, Q_k)$ with respect to M. We may choose η, uv and h so that k is minimum. If $k = 0$, then by considering the path P we conclude that a required extension is isomorphic to a matching minor of G.

Thus we may assume that $k > 0$. Let the notation be as in the definition of (A, B)-hook. Thus p_0 is an internal vertex of $\eta(uv)$, and from the symmetry we may assume it is located at even distance from $\eta(v)$ on $\eta(uv)$. We have $q_i \in \{\eta(u), \eta(v)\}$ for all $i = 1, 2, \ldots, k$. We properly two-color the graph $\eta(uv) \cup P$ using the colors black and white so that $\eta(v)$ is black and $\eta(u)$ is white. For convenience let $q_0 := p_0$. We will show that $q_0, q_1, q_2, ..., q_k$ all have the same color. Indeed, suppose for some $i \in \{0, 1, \ldots, k-1\}$ the vertices q_i and

q_{i+1} have different color. We replace $\eta(uv)[q_i, q_{i+1}]$ by $Q_i \cup P[p_i, p_{i+1}] \cup Q_{i+1}$ to obtain an embedding $\eta'' : H \hookrightarrow G$. Then the sequence $h' = (P[p_{i+1}, p_{k+1}], Q_{i+2}, Q_{i+3}, \ldots, Q_k)$ is an (A', B')-hook, where A' and B' are defined in the same way as A and B but relative to η''. But h' contradicts the minimality of k. This proves our claim that $q_0, q_1, q_2, ..., q_k$ all have the same color; in particular, $q_1 = q_2 = \cdots = q_k = \eta(v)$.

The graph $\eta(H) \cup Q_k \cup P[p_k, p_{k+1}]$ has a matching minor isomorphic to a desired extension of H, unless p_{k+1} belongs to $\eta(vw)$ for some neighbor w of v other than u. By using the argument of the previous paragraph we deduce that p_{k+1} is an internal vertex of $\eta(vw)$ located at even distance from $\eta(v)$ on $\eta(vw)$ for some neighbor $w \neq u$ of v. Let J be obtained from H as follows. First we bisubdivide the edges uv and vw; let the new vertices be p'_0,r_0 and p'_{k+1},r_{k+1} correspondingly, where p'_0 is adjacent to u and p'_{k+1} is adjacent to w. Denote resulting graph by H'. Then we add new vertices $p'_1, p'_2, \ldots, p'_k$ and $r_1, r_2, \ldots, r_k$ in such a way that $p'_0 p'_1 \ldots p'_k p'_{k+1}$ is a path, and $p'_i r_i v$ is a path for all $i = 1, 2, \ldots, k$, and there are no other edges incident with the new vertices. This completes the definition of J. Now η can be converted to an embedding $\eta' : J \hookrightarrow G$ in a natural way; thus, for instance, $\eta'(p'_i)$ is the graph with vertex-set $\{p_i\}$.

We apply Lemma 2.6.1 to the graphs J and G and the vertex r_0; let J' be the resulting graph, and let $\eta'' : J' \hookrightarrow G$. Suppose outcome (D) of Lemma 2.6.1 holds. Then either a split extension of H is isomorphic to a matching minor of G, in which case the desired result follows from Theorem 2.5.9, or J' is obtained from J by splitting v. Let v_1 and v_2 be the new outer vertices and v_0 the new inner vertex. As we may assume that no split extension of H is isomorphic to a matching minor of G, we have that $|N_{J'}(v_i) \cap N_{H'}(v)| \geq 2$ for at most one $i \in \{1, 2\}$, where $N_{J'}(v_i)$ and $N_{H'}(v)$ denote the neighborhoods of v_i and v in J' and H' correspondingly. Without loss of generality let $|N_{J'}(v_1) \cap N_{H'}(v)| \leq 1$. Assume first $N(v_1) \cap N_{H'}(v) = \emptyset$, then we can choose $1 \leq i < i' \leq k$ such that $r_i, r_{i'} \in N(v_1)$ and $r_j \notin N(v_1)$ for every j such that $1 \leq j < i$ or $i' < j \leq k$. The image of the hook $h' = (p'_0 p'_1 \ldots p'_i r_i v_1 r_{i'} p'_{i'} p'_{i'+1} \ldots p'_{k+1}, p'_1 r_1 v_2, \ldots, p'_{i-1} r_{i-1} v_2, v_1 v_0 v_2, p'_{i'+1} r_{i'+1} v_2, \ldots)$ under η'' contradicts the minimality of k. Assume now $|N_{J'}(v_1) \cap N_{H'}(v)| = 1$. From the symmetry between r_0 and r_{k+1} we may assume $r_0 \in N(v_2)$. Let i be minimal such that $r_i \in N(v_1)$

then $i \le k$ and the image of the hook $h' = (p'_0 p'_1 \ldots p'_i r_i v_1, p'_1 r_1 v_2, \ldots, p'_{i-1} r_{i-1} v_2)$ under η'' contradicts the minimality of k. We assume now that one of the outcomes (A),(B) or (C) of Lemma 2.6.1 holds.

Throughout the rest of the proof let $z \in V(J) - \{v, p'_0, r_0\}$. Outcome (C) cannot hold, because v has degree at least four in J.

Assume next that either $J' = A_1(z)$, in which case we put $\tau_1 = \tau_2 = v_0$, or that $J' = A_2(z) = J + (r_0, p'_0 r_0) + (\tau_2, z)$, in which case τ_1 and τ_2 have their usual meaning. If $z \in (V(H) - \{u\}) \cup \{r_{k+1}, p'_{k+1}\}$, then $J + (\tau_2, z)$ is isomorphic to a bisubdivision of a suitable extension of H. If $z = u$ we replace $\eta(uv)$ by $\eta''(u\tau_2 r_0 \tau_1 p'_0 p'_1 r_1 v)$ and the hook $h' = (P[p_1, p_{k+1}], Q_2, Q_3, \ldots, Q_k)$ contradicts the minimality of k. If $z = r_i$ for some $1 \le i \le k$ then the hook $h' = (\eta''(\tau_2 r_i p'_i) \cup P[p_i, p_{k+1}], Q_{i+1}, Q_{i+2}, \ldots, Q_k)$ contradicts the minimality of k. Finally if $z = p'_i$ for some $1 \le i \le k$ we replace $\eta(uv)$ by $\eta''(up'_0 \tau_1 r_0 \tau_2 p'_i r_i v)$ and the hook $h' = (P[p_i, p_{k+1}], Q_{i+1}, Q_{i+2}, \ldots, Q_k)$ contradicts the minimality of k. This completes the case $J' = A_i$.

Since v has degree at least four in J we assume that $J' = B_i(p'_1 p'_0, z)$ for some $i \in \{1, 2, 3, 4\}$. Note that J' contains $J'' = A_j(z) \backslash p'_1 p'_0$ as a matching minor for some $j \in \{1, 2\}$, and unless $z = u$ the argument from the previous paragraph provides us with a suitable extension or a contradiction. If $z = u$ we replace $\eta(uv)$ by $\eta'(u\tau\tau'\tau'' p'_1 r_1 v)$, where $\tau = \tau' = \tau'' = \tau_{2i}$ if $i \in \{1, 2\}$ and $\tau = \tau_{2i-2}$, $\tau' = \tau_{2i-3}$, $\tau'' = \tau_{2i-4}$ if $i \in \{3, 4\}$. The hook $h' = (P[p_1, p_{k+1}], Q_2, Q_3, \ldots, Q_k)$ now contradicts the minimality of k. $\square$

2.8 Vertex-Parallel and Edge-Parallel Extensions

The purpose of this section is to replace vertex-parallel and edge-parallel extensions in the statement of Theorem 2.7.3 by extensions that are closer to linear extensions. Our first goal is to prove that if a brick G has a matching minor isomorphic to a vertex-parallel extension of a 2-connected graph H, then it also has a matching minor isomorphic to a better extension of H. We will proceed in two steps; in the intermediate step we will produce a better extension or one that is "almost better", the following. Let H be a graph, let u be a vertex of degree at least three, let u_1 and u_2 be distinct neighbors of u, and let

$H' = H + (u_1, uu_1) + (\tau_2, u_2u)$. We say that H' is a *semi-edge-parallel* extension of H.

Lemma 2.8.1. *Let H be a graph of minimum degree at least three, and let G be a brick. If a vertex-parallel extension of H is isomorphic to a matching minor of G, then an edge-parallel, a semi-edge-parallel, a linear, a cross, or a split extension of H is isomorphic to a matching minor of G.*

Proof. Let u_0 be the vertex of H with neighbors u_1 and u_2 such that the graph H_2 defined below is isomorphic to a matching minor of G. Let H_1 be obtained from H by bisubdividing the edges u_0u_1 and u_0u_2 exactly once, and let x_1, y_1, x_2, y_2 be the new vertices, numbered so that $u_2y_2x_2u_0x_1y_1u_1$ is a path. The graph H_2 is defined as $H_1 + (y_1, y_2)$. By Lemma 2.6.1 applied to $J = H_2$ and the vertex x_1 there exists a graph $J' \hookrightarrow G$ satisfying (A), (B), (C) or (D) of that lemma. If J' is a split extension of J, then the graph obtained from $J' \backslash y_1y_2$ by bicontracting y_1 and y_2 is a split extension of H. Thus if (D) holds, then the theorem holds, and so we may assume that (A), (B) or (C) holds. Throughout this proof let $v \in V(J) - \{u_0, x_1, y_1\}$. The symbols $\tau_1, \tau_2, \ldots$ will refer to the new vertices of J' according to the convention introduced prior to Lemma 2.6.1.

Assume first that $J' = A_1 = J + (x_1, v)$. If $v = u_1$, then J' is isomorphic to a semi-edge-parallel extension of H. If $v = x_2$, then $H + (u_1, u_0u_2) \hookrightarrow G$; if $v = y_2$, then $H + (u_1, u_2u_0) \hookrightarrow G$; and in all other cases $H + (v, u_0u_1) \hookrightarrow G$. In the last case, if v is not adjacent to u_1, then $H + (v, u_1)$ is a linear extension of H, and otherwise $H + (v, u_0u_1)$ is an edge-parallel extension of H. The same argument will be used later. We will also use later the fact that the inclusions above did not use the edge y_1y_2. This completes the case $J' = A_1$.

Now we assume that $J' = A_2 = J + (x_1, x_1u_0) + (\tau_2, v)$. If $v = x_2$, then $H + (u_2, u_1u_0) \hookrightarrow G$; if $v = y_2$, then by deleting the edge y_1y_2 and bicontracting y_1 we see that a semi-edge-parallel extension of H is isomorphic to a minor of G; if $v = u_1$, then the graph $A_1 \backslash x_1 \backslash y_1$ is isomorphic to a bisubdivision of H, and by considering the path $y_2y_1x_1\tau_1$ we deduce that $H + (u_1, u_2u_0) \hookrightarrow G$; and in all other cases $H + (v, u_1u_0) \hookrightarrow G$. This completes the case $J' = A_2$.

Let $j \in \{1, 2, 3, 4\}$ and let $J' = B_j(y_2y_1, v)$. We have $A_1(v) \backslash y_1y_2 \hookrightarrow B_j(y_2y_1, v)$ for

$j = 1, 2$ and $A_2(v)\backslash y_1y_2 \hookrightarrow B_j(y_2y_1, v)$ for $j = 3, 4$ (if $j = 1$ we delete the edges $y_2\tau_1$ and $y_1\tau_2$ and analogously for $j \geq 2$). Since the arguments of the previous two paragraphs did not use the edge y_1y_2, except for the cases of $A_1(u_1)$ and $A_2(u_1)$, we may assume that $J' = B_j(y_2y_1, u_1)$, for some $j \in \{1, 2, 3, 4\}$. But $H + (u_1, u_2u_0) \hookrightarrow B_j(y_2y_1, u_1)$ (consider the path $u_1\tau_2\tau_1y_2$ when $j = 1$). This completes the cases $J' = B_j(y_2y_1, v)$.

Our next step is to handle the cases $J' = B_j(x_2u_0, v)$ and $J' = C_j(x_2u_0, y_2y_1)$. If $j \leq 2$, then $H + (u_1, u_2u_0) \hookrightarrow G$, and if $j \geq 3$, then $H_1 + (x_1, x_1u_0) + (\rho_2, x_2u_0) \hookrightarrow G$ and $H_1 + (x_1, x_1u_0) + (\rho_2, x_2u_0)$ after bicontraction of y_1 and y_2 becomes isomorphic to a semi-edge parallel extension of H. (We are using "ρ" instead of "τ", because the "τ" notation is reserved for vertices of J'.)

Thus the only remaining cases are $J' = C_j(y_2y_1, x_2u_0)$. If $j = 1$, then by considering the path $x_1\tau_1\tau_2\tau_3$ we deduce that $H + (u_1, u_2u_0) \hookrightarrow G$; for $j = 2$ the argument is analogous. For $j = 3$ notice that $C_3(y_2y_1, x_2u_0)\backslash\tau_1y_1\backslash y_2\tau_3\backslash x_1\tau_2\backslash\tau_4\tau_5$ is isomorphic to a bisubdivision of H. By considering the edge $\tau_4\tau_5$ we see that $H + (u_1, u_2u_0) \hookrightarrow G$. Finally, $C_4(y_2y_1, x_2u_0)$ has a matching minor isomorphic to a semi-edge-parallel extension of H. To see that, consider the edge $x_1\tau_1$ and path $\tau_2\tau_3\tau_4\tau_5\tau_6\tau_7$. (The last argument applies to $j = 3$ as well, but for the sake of the next proof we wish to avoid semi-parallel extensions as much as possible.) $\square$

Lemma 2.8.2. *Let H be a 2-connected graph of minimum degree at least three, and let G be a brick. If a semi-edge-parallel extension of H is isomorphic to a matching minor of G, then an edge-parallel, a linear, a cross, a cube or a split extension of H is isomorphic to a matching minor of G, unless H is isomorphic to K_4 and G has a matching minor isomorphic to the Petersen graph.*

Proof. By hypothesis there exists a vertex u_0 of H with distinct neighbors u_1 and u_2 such that the graph H_3 is isomorphic to a matching minor of G, where $H_1, H_2, x_1, y_1, x_2, y_2$ are defined as in the proof of Lemma 2.8.1, and $H_3 = H_2 + (x_2, u_2)$. We may assume that u_0 has degree exactly three, for otherwise $H_3\backslash u_2y_2\backslash x_2u_0$ is isomorphic to a bisubdivision of a split extension of H, and hence a split extension of H is isomorphic to a matching minor of G. Let u_3 be the third neighbor of u_0. Since $H_3 \hookrightarrow G$, either a split extension of H

is isomorphic to a matching minor of G, or one of the graphs H_3, $H_4 = H_2 + (x_2, y_2u_2)$, $H_5 = H_2 + (x_2, u_2'u_2)$, where $u_2' \neq u_0$ is a neighbor of u_2, has a homeomorphic embedding into G. Let J denote that graph, and let it be chosen so that $J \neq H_3$, if possible. This choice implies that if a split extension of J is isomorphic to a matching minor of G, then so is a split extension of H. Let x_2', y_2' be the new vertices of H_4 and H_5. We apply Lemma 2.6.1 to J and the vertex x_1, and so we may assume that (A), (B), or (C) holds, for otherwise the theorem holds. Let J' be the graph satisfying (A), (B) or (C). The symbols $\tau_1, \tau_2, \ldots$ will again refer to the new vertices of J'.

Let us assume first that either $J = H_3$, or that y_2' has degree two in J'. Then by deleting the edge x_2u_2 (and bicontracting y_2' if $J \neq H_3$) we may use the proof of Lemma 2.8.1. By that argument the theorem holds, unless $J' = A_1(u_1)$, $J' = A_2(y_2)$, $J' = B_j(y_2y_1, y_2)$, $J' = B_j(x_2u_0, v)$, $J' = C_j(x_2u_0, y_2y_1)$ or $J' = C_4(y_2y_1, x_2u_0)$ for some $j \in \{3, 4\}$ and $v \in V(J) - \{x_1, y_1, u_0\}$.

If $J' = A_1(u_1)$, then $J' \backslash u_1y_1 \backslash x_1u_0 \backslash x_2u_2$ is isomorphic to a bisubdivision of H, and by considering the edge u_2x_2 we deduce that $H + (u_2, u_0u_3) \hookrightarrow G$. If $J' = A_2(y_2)$ we delete the edge y_1y_2, bicontract the vertex y_1 and apply the previous argument.

Next, let $J' = B_3(y_2y_1, y_2)$. The graph obtained from J' by deleting the edges $y_1\tau_4$ and τ_3y_2 and bicontracting the vertices y_1 and τ_4 is isomorphic to $A_2(y_2)$. Thus $H + (u_2, u_0u_3) \hookrightarrow G$. Similarly if $J' = B_4(y_2y_1, y_2)$ we delete the edges $y_1\tau_5$, $\tau_4\tau_6$ and τ_3y_2 and bicontract the vertices y_1, τ_4 and τ_6 to demonstrate that $H + (u_2, u_0u_3) \hookrightarrow G$.

Our next step is to handle the cases $J' = B_j(x_2u_0, v)$ and $J' = C_j(x_2u_0, y_2y_1)$. Assume first that $j = 3$. If $v \notin \{u_2, x_2, y_2\}$, then by considering the edge τ_4v we deduce that $H + (v, u_0u_2) \hookrightarrow B_3(x_2u_0, v) \hookrightarrow G$, and similarly $H + (u_2, u_1u_0) \hookrightarrow C_3(x_2u_0, y_2y_1) \hookrightarrow G$. For the cases $v \in \{u_2, x_2, y_2\}$ let $L_3 = B_3(x_2u_0, v) \backslash y_1y_2 \backslash x_1\tau_2 \backslash \tau_1u_0 \backslash \tau_4v \backslash x_2u_2$. By considering the edge τ_4v we deduce that $H + (u_2, u_0u_3) \hookrightarrow G$ if $v \in \{u_2, x_2\}$ and $H + (u_3, u_2u_0) \hookrightarrow G$ if $v = y_2$. Now assume $j = 4$. If $v \notin \{u_2, x_2, y_2\}$, then by considering the edge τ_6v we deduce that $H + (v, u_2u_0) \hookrightarrow B_4(x_2u_0, v) \hookrightarrow G$. If $v = u_2$ then let $L_4 = B_4(x_2u_0, u_2) \backslash x_2 \backslash y_2 \backslash x_1\tau_1 \backslash \tau_4\tau_6 \backslash \tau_5u_0$. By considering the edge $x_1\tau_1$ we deduce that $H + (u_3, u_0u_1) \hookrightarrow G$. If $v = x_2$ we get the same result by the graph obtained from L_4

by adding the path $x_2y_2u_2$, and if $v = y_2$ we add the path $y_2x_2u_2$ instead. The graph $C_4(x_2u_0, y_2y_1)$ has a matching minor isomorphic to a cross extension of H (delete the edges τ_7y_2 and x_2u_2; the cross extension has two vertices replaced by triangles). This concludes the cases $J' = B_j(x_2u_0, v)$ and $J' = C_j(x_2u_0, y_2y_1)$.

The graph $C_4(y_2y_1, x_2u_0)$ also has a matching minor isomorphic to a cross extension of H. To see that, delete the edges u_2y_2 and $x_2\tau_7$; the cross extension has two vertices replaced by triangles.

We may therefore assume that $J = H_4$ or $J = H_5$, and that y_2' has degree three in J'. Thus $J' = A_j(y_2')$ or $J' = B_j(y_2y_1, y_2')$ or $J' = B_j(x_2u_0, y_2')$ for some j. Assume first that $J' = A_j(y_2')$. If $J = H_4$, then J' is isomorphic to a cross extension of H (with one or two vertices replaced by triangles depending on the value of j), and so we may assume that $J = H_5$. If $j = 2$, then by considering the edge τ_2y_2' we deduce that $H+(u_0, u_2u_2') \hookrightarrow G$, and so we may assume that $j = 1$. We may assume that $u_2' = u_1$, for otherwise by considering the edge x_1y_2' we deduce that $H+(u_1, u_2u_2') \hookrightarrow G$. Now there is symmetry among u_0, u_1, u_2, and since we could assume u_0 had degree three, we may also assume u_1 and u_2 have degree three in H. The graph $K := J' \backslash u_0x_1 \backslash x_2y_2 \backslash u_2y_2'$ is isomorphic to a bisubdivision of H. If u_2 is not adjacent to u_3, then let u_2'' be the third neighbor of u_2; by considering K and the edge x_2y_2 we see that $H + (u_3, u_2u_2'') \hookrightarrow G$, as desired. Thus we may assume that u_2 is adjacent to u_3, and by symmetry we may also assume that u_1 is adjacent to u_3. But H is 2-connected, and hence u_3 is not a cutvertex; thus H is isomorphic to K_4. It follows that J' is isomorphic to the Petersen graph, as desired. This completes the case $J' = A_j(y_2')$.

Now let $J' = B_j(y_2y_1, y_2')$ or $J' = B_j(x_2u_0, y_2')$. If $J = H_4$, then J' is isomorphic to a cube extension of H, and so we may assume that $J = H_5$. If $J' = B_j(y_2y_1, y_2')$ and $j = 1$, then by considering the path $y_2\tau_1\tau_2y_2'$ we deduce that $H + (u_2', u_2u_0) \hookrightarrow G$. The argument for $j > 1$ is analogous. Thus we may assume that $J' = B_j(x_2u_0, y_2')$. If $j = 1$, then by considering the path τ_2y_2' we deduce that $H + (u_2', u_0u_2) \hookrightarrow G$. The argument is analogous for $j > 1$ with the proviso that when j is even the conclusion is $H + (u_2', u_2u_0) \hookrightarrow G$. □

We now turn our attention to edge-parallel extensions. Let us recall that G/v denotes the graph obtained from the graph G by bicontracting the vertex v.

Lemma 2.8.3. *Let H be a graph of minimum degree at least three, and let G be a brick. If an edge-parallel extension of H is isomorphic to a matching minor of G, then a cross, cube, linear, quadratic, quartic or split extension of H is isomorphic to a matching minor of G.*

Proof. By hypothesis there exists a vertex $u_0 \in V(H)$ of degree at least three with neighbors u_1 and u_2 such that the graph $H_2 := H + (u_2, u_1u_0)$ is isomorphic to a matching minor of G. Let y_1, x_1 be the new vertices of H_2; thus $u_0x_1y_1u_1$ is a path of H_2. Let $H_1 := H_2\backslash u_2y_1$. Since $H_2 \hookrightarrow G$, either a split extension of H is isomorphic to a matching minor of G, or one of the graphs H_2, $H_3 = H_1 + (y_1, u_0u_2)$, $H_4 = H_1 + (y_1, u_2'u_2)$, where $u_2' \neq u_0$ is a neighbor of u_2, has a homeomorphic embedding into G. Let J denote that graph, and let it be chosen so that $J \neq H_2$, if possible. This choice implies that if a split extension of J is isomorphic to a matching minor of G, then so is a split extension of H. Let x_2, y_2 be the new vertices of H_3 and H_4. If $J = H_2$ let $x_2 := u_2$ and let y_2 be undefined. We apply Lemma 2.6.1 to J and the vertex x_1, and so we may assume that (A), (B), or (C) holds, for otherwise the theorem holds. Let J' be the graph satisfying (A), (B) or (C). Throughout this proof let $v \in V(J) - \{x_1, y_1, u_0\}$ and once again the symbols $\tau_1, \tau_2, \ldots$ will again refer to the new vertices of J'.

We first notice that if u_0 has degree at least four, then $H_2\backslash u_0u_2$ is isomorphic to a split extension of H, and so we may and will assume that u_0 has degree three. Let u_3 be the third neighbor of u_0. We now show that we may assume that if $J = H_4$, then u_2 has degree three. Indeed, if $J = H_4$ and u_2 has degree at least four then $H_4\backslash u_0u_2/x_1$ is isomorphic to a split extension of H. So in the case $J = H_4$ let u_2'' be the third neighbor of u_2. Let L be obtained from J' by deleting u_0u_2 and all the "new" edges. Thus, for instance, if $J' = A_2(v)$, then $L = J'\backslash u_0u_2\backslash x_1\tau_1\backslash\tau_2v$. Then $L/u_0/y_2$ is isomorphic to H.

Assume first that $J' = A_1(v) = J + (x_1, v)$. If $v = y_2$, then $J \in \{H_3, H_4\}$, and J' is a cross extension of H if $J = H_3$, and a quartic or cross extension of H if $J = H_4$. Thus we may assume that $v \neq y_2$, and hence we may assume (by bicontracting y_2) that $J = H_2$. It follows that J' is a quadratic extension of H, as desired. This completes the case $J' = A_1$.

Next we assume that $J' = A_2(v) = J + (x_1, u_0x_1) + (\tau_2, v)$. Assume first that $v = y_2$. If $J = H_3$, then J' is a cross extension of H, and so we may assume that $J = H_4$. But then $J'\backslash x_1\tau_1/x_1/\tau_1$ is isomorphic to a quadratic extension of H.

Thus we may assume that $v \neq y_2$, and hence, by bicontracting y_2, we may assume that $J = H_2$. If $v \neq u_1$, then $J'\backslash y_1u_2/y_1$ is a quadratic extension of H, and so we may assume that $v = u_1$. But then by considering the graph L/u_0 and edges $x_1\tau_1$ and τ_2u_1 we deduce that a quadratic extension of H is isomorphic to a matching minor of G. This completes the case $J' = A_2$.

Next we handle the cases $J' = B_j(x_2y_1, v)$. We start by assuming that $v = y_2$. If $J = H_3$, then J' is isomorphic to a cube extension of H, and so we may assume that $J = H_4$. Recall the definiton of L and that u_2 has degree three. If $j = 1$, then by considering L and the edges $x_1\tau_1$ and τ_2y_2 we deduce that a quadratic extension of H, namely $H + (u_2'', u_0u_2) + (\rho_2, u_3)$, is isomorphic to a matching minor of G. If $j = 2$, then by considering the edges $\tau_2\tau_3$ and τ_4y_2 we deduce that the quadratic extension $H + (u_2'', u_2u_0) + (\rho_2, u_2)$ is isomorphic to a matching minor of G. An analogous argument applies when $j = 4$. If $j = 3$ then by deleting the edge $x_1\tau_1$ and bicontracting x_1 and τ_1 we deduce that $H + (u_2'', u_0u_2) + (\rho_2, u_0) \hookrightarrow G$, as desired. Thus we may assume that $v \neq y_2$, and hence, by bicontracting y_2, we may assume that $J = H_2$. If $j = 1$, then by considering L and the edges $x_1\tau_1$ and τ_2v we deduce that the quadratic extension $H + (u_3, u_2u_0) + (\rho_2, v)$ is isomorphic to a matching minor of G. Let $j = 2$. If $v \neq u_2$, then by considering L and the edges $\tau_2\tau_3$ and τ_4v we deduce that the quadratic extension $H + (v, u_2u_0) + (\rho_2, u_2)$ is isomorphic to a matching minor of G. If $v = u_2$ then by considering the graph obtained from L by replacing the edge x_1y_1 by τ_1x_1 and considering the edges $\tau_2\tau_3$ and τ_4u_2 we deduce that the quadratic extension $H + (u_2, u_1u_0) + (\rho_2, u_1)$ is isomorphic to a matching minor of G. Thus we may assume $j \in \{3, 4\}$. Let us assume that $v = u_1$. Then we may assume that u_1 is adjacent to u_2, for otherwise $H + (u_1, u_2) \hookrightarrow G$ (consider the path $u_1\tau_4\tau_3u_2$ when $j = 3$ and the analogous path for $j = 4$). If $j = 3$, then by replacing the edge u_1u_2 by the path $u_1\tau_4\tau_3u_2$ we obtain a graph isomorphic to a bisubdivision of H, and by considering the edges $y_1\tau_4$ and $\tau_2\tau_3$ we deduce that a quadratic extension of H, namely $H + (u_0, u_2u_1) + (\rho_2, u_0)$, is isomorphic to a matching minor of G. If $j = 4$ then by replacing the edge u_1u_2 by the path $u_1\tau_6\tau_5\tau_4\tau_3u_2$, by considering the edges $\tau_4\tau_6$ and $y_1\tau_5$ and by bicontracting x_1 and τ_3 we deduce that a quadratic extension of H, namely $H + (u_0, u_2u_1) + (\rho_2, u_2)$, is isomorphic to a matching minor of G. Thus we may assume that $v \neq u_1$. If $j = 3$, then by considering the edge $x_1\tau_1$ and path $\tau_2\tau_3\tau_4v$ we see that the quadratic extension $H + (v, u_1u_0) + (\rho_2, u_1)$ is isomorphic to a matching minor of G; an analogous argument gives the same conclusion when $j = 4$.

The cases $J' = B_j(u_2u_0, v)$ can be reduced to the cases just handled by noticing that $J \backslash u_0u_2$ is isomorphic to a bisubdivision of H, and hence J is isomorphic to the edge-parallel extension $H + (u_2, u_3u_0)$. Similarly the cases $J' = C_j(u_2y_1, u_2u_0)$ can be reduced to $J' = C_j(u_2u_0, u_2y_1)$, and so it remains to handle the cases $J' = C_j(u_2u_0, u_2y_1)$. But in all four of those cases a cross extension of H is isomorphic to a matching minor of G. $\square$

The results of this section allow us to strengthen Theorem 2.7.3 as follows.

Theorem 2.8.4. *Let H and G be graphs, where H is 2-connected, has minimum degree at least three and is isomorphic to a matching minor of G, and G is a brick. Assume that if H is isomorphic to K_4, then G has no matching minor isomorphic to the Petersen graph. If H is not isomorphic to G, then a cross, cube, linear, quadratic or quartic extension of H is isomorphic to a matching minor of G.*

Proof. By Theorem 2.7.3 we may assume that a vertex-parallel or an edge-parallel extension of H is isomorphic to a matching minor of G. Thus the result follows from Lemmas 2.8.1, 2.8.2 and 2.8.3. $\square$

2.9 Cube and Cross Extensions

In this section we strengthen 2.8.4 by eliminating cube and cross extensions from the conclusion.

Lemma 2.9.1. *Let H be a graph, let u be a vertex of H of degree three, and let u_1 and u_2 be two neighbors of u. Let H_1 be obtained from H by bisubdividing the edges uu_1 and uu_2 once, and let x_1, y_1, x_2, y_2 be the new vertices so that $u_1y_1x_1ux_2y_2u_2$ is a path. Let*

$H_2 := H_1 + (x_2, y_2x_2) + (\tau_2, x_1)$, let $H_3 := H_1 + (x_2, y_2x_2) + (\tau_2, x_1y_1) + (\tau_4, x_1)$, and let H_4 be obtained from H_2 or H_3 by replacing exactly one of the vertices x_2, τ_1, τ_2 by a triangle. Then each of H_2, H_3, H_4 has a matching minor isomorphic to an alpha or prism extension of H.

Proof. Throughout this proof let τ_1, τ_2 denote the new vertices of H_2, and let $\tau_1, \tau_2, \tau_3, \tau_4$ denote the new vertices of H_3 with the usual numbering convention. We can naturally embed H into H_2. By bicontracting y_1 and y_2 and considering edges $x_2\tau_1$ and $x_1\tau_2$, we see that H_2 is isomorphic to a bisubdivision of a prism extension of H. The graph $H_3\backslash\tau_1x_2\backslash x_1u\backslash\tau_3\tau_4$ is isomorphic to a bisubdivision of H and by bicontracting y_1, τ_3 and τ_4 and considering edges τ_1x_2 and x_1u we deduce that H_3 has a matching minor isomorphic to an alpha extension of H. This completes the proof for H_2 and H_3.

Suppose H_4 is obtained from H_2 by replacing τ_2 with a triangle, then $H_4\backslash x_2\tau_1/x_2/\tau_1/y_1$ is isomorphic to an alpha extension $H + (u_1, uu_2) + (\rho_2, u)$ of H. Similarly if H_4 is obtained from H_2 by replacing x_2 or τ_1 with a triangle then $H_4\backslash x_1u/x_1/u/y_1$ is isomorphic to an alpha extension of H.

It remains to consider the case when H_4 be obtained from H_3 by replacing exactly one of the vertices x_2, τ_1, τ_2 by a triangle. We need to make the following easy observation. If a graph G_1 is obtained from a graph G by replacing a vertex $t \in V(G)$ of degree three with a triangle T and G_2 is obtained from G_1 by replacing one of the vertices of T by a triangle, then G is isomorphic to a matching minor of G_2. Let $H_2' = H_1 + (x_1, y_1x_1) + (\rho_2, x_2)$. Clearly a graph obtained from H_3 by contracting a triangle with vertex set $\{x_2, \tau_1, \tau_2\}$ is isomorphic to H_2'. Therefore, by the observation above, H_4 contains H_2' as a matching minor and $H_2'/y_1/y_2$ is isomorphic to a quadratic extension of H. □

Lemma 2.9.2. *Let H be a graph of minimum degree at least three, and let G be a brick. If a cube extension of H is isomorphic to a matching minor of G, then a linear, cross or quadratic extension of H is isomorphic to a matching minor of G.*

Proof. Let u be a vertex of H of degree three and let u_1,u_2 and u_3 be its neighbors. Let H_0 be obtained from H by bisubdividing each of the edges uu_1, uu_2 and uu_3. Let the new vertices be y_1, y_2, y_3 and z_1, z_2, z_3 in such a way that $u_1y_1z_3u$, $u_2y_2z_1u$ and $u_3y_3z_2u$ are paths. Let $H_1 := H_0 + (y_1, z_2) + (y_2, z_3) + (y_3, z_1)$, and let J be obtained from H_1 by replacing a subset of $\{z_1, z_2, z_3\}$ by triangles. If z_i is replaced by a triangle, then let the triangle be Z_i; otherwise, let Z_i denote the graph with vertex-set $\{z_i\}$. By hypothesis the vertex u and graph J may be selected so that J is isomorphic to a matching minor of G. Let $\eta : J \hookrightarrow G$. We may assume that η is a homeomorphic embedding, for otherwise a split extension of H is isomorphic to a matching minor of G and the result holds by Theorem 2.5.9.

When $v \in V(J)$ we will abuse notation and use $\eta(v)$ to denote the unique vertex of the graph $\eta(v)$. With that in mind let $J' = \eta(J)$, let $u'_i = \eta(u_i)$, $u' = \eta(u)$ and $z'_i = \eta(z_i)$. For $i = 1, 2, 3$ let P_i denote the path $\eta(u_iy_i)$. We may assume that J and η are chosen so that $|V(P_1)| + |V(P_2)| + |V(P_3)|$ is minimum.

Let Ω_1 be the octopus with head $\eta(Z_1)$ and tentacles the paths of $\eta(J)$ joining u', y'_2 and y'_3 to Z_1, and let Ω_2 and Ω_3 be defined analogously. Let Ω_4 be the octopus with head $\eta(J\backslash V(Z_1)\backslash V(Z_2)\backslash V(Z_3)\backslash\{y_1, y_2, y_3, u\})$ and tentacles P_1, P_2, P_3, let $\mathcal{F} = \{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$, and let $Y' = \{y'_1, y'_2, y'_3, u'\}$. Then $(\mathcal{F}, Y')$ is a frame in G. Let M be a perfect matching of $G\backslash V(\eta(J))$; then M has a unique extension to a matching M' that is Ω_i-compatible for all $i = 1, 2, 3, 4$. By Theorem 2.2.3 there exist distinct integers $i, j \in \{1, 2, 3, 4\}$ and an M'-alternating path S joining vertices v_i and v_j, where v_i belongs to the head of Ω_i and v_j belongs to the head of Ω_j, such that for some edge $e \in E(S)\backslash M'$ the two components of $S\backslash e$ may be denoted by S_i and S_j so that $V(S_i) \cap V(\mathcal{F}) \subseteq V(\Omega_i)$ and $V(S_j) \cap V(\mathcal{F}) \subseteq V(\Omega_j)$.

Assume first that $j = 4$. Then from the symmetry we may assume that $i = 2$. In this case it will be convenient to allow v_4 to be an internal vertex of a tentacle of Ω_4. By doing so we may assume (by replacing S by its subpath) that v_4 is the only vertex of $S \cap \Omega_4$. If for some $l \in \{1, 2, 3\}$ we have $v_4 \in V(P_l)$ and $P_l[u'_l, v_4]$ is even, then let $v = u_l$; if $v_4 \in V(P_l)$ and $P_l[u'_l, v_4]$ is odd, then v is undefined. If v_4 belongs to $V(\eta(z))$ for some $z \in V(J)$, then let $v = z$. Finally, if $v \in V(\eta(zz'))$ for some edge $zz' \in E(H \setminus u)$, then v_4 is at even distance on $\eta(zz')$ from exactly one of $\eta(z)$, $\eta(z')$, say from $\eta(z)$. In that case we put $v = z$. Notice that if v is defined, then $v \in V(H) - \{u\}$. From the symmetry we may assume $v \neq u_1$ and

$v_4 \notin V(P_1)$. By Lemma 2.3.6 the graph $\Omega_2 \cup S_2 + e$ includes a triad or tripod T with ends y_1', u', v_4.

We claim that if v_4 belongs to P_3, then the path $P_i[v_4, u_3']$ is even. Indeed, otherwise by making use of T, Ω_1 and Ω_3 we obtain contradiction to the minimality of $|V(P_1)|+|V(P_2)|+|V(P_3)|$. This proves that if v is undefined then $v_4 \in V(P_2)$. In that case by deleting the path of $\eta(J)$ joining y_2' and Z_1 and by considering the path of $\eta(J)$ joining y_1' and Z_3 and using T we deduce that a cross extension of H is isomorphic to a matching minor of G. If v is defined, then one of the following graphs is isomorphic to a matching minor of G:

- $H + (v, uu_1) + (\tau_2, uu_2)$, if T is a triad and $Z_3 = \{z_3\}$,
- $H + (v, uu_1) + (\tau_2, u_2u)$, if T is a triad and Z_3 is a triangle,
- $H + (v, u_1u) + (\tau_2, \tau_1u_1)$, if T is a tripod.

But each of the above graphs has a matching minor isomorphic to a quadratic extension of H. This completes the case $j = 4$.

Thus we may assume that $i = 1$ and $j = 2$. By Lemma 2.3.6 $\Omega_1 \cup S_1 + e$ includes a triad or tripod T_1 with ends y_3', u', s_2 and $\Omega_2 \cup S_2 + e$ includes a triad or tripod T_2 with ends y_1', u', s_1, where $s_1 \in V(S_1), s_2 \in V(S_2)$ are the ends of e. If either T_1 or T_2 is a tripod then the required result follows from Lemma 2.9.1 by deleting the path of $\eta(J)$ joining y_1' and Z_3 and making use of T_1 and T_2. If both T_1 and T_2 are triads then one of the following graphs is isomorphic to a matching minor of G:

- $H + (uu_3, uu_1) + (\tau_4, uu_2)$, if Z_3 is not a triangle,
- $H + (uu_3, uu_1) + (\tau_4, u_2u)$, if Z_3 is a triangle.

Both of these graphs have matching minors isomorphic to quadratic extensions of H. □

Lemma 2.9.3. *Let H be a graph, let J be a cross extension of H and let v be the hub of J. If the degree of v in H is at least four then a split extension of H is isomorphic to a matching minor of J.*

Proof. Let x_1, y_1, x_2, y_2 and K' be as in the definition of cross extension. If $J = K'$ then $J\backslash vx_1\backslash x_2y_1/x_1$ is isomorphic to a split extension of H. If $J \neq K'$ the argument is analogous.

□

Lemma 2.9.4. *Let H be a graph of minimum degree at least three, and let G be a brick. If a cross extension of H is isomorphic to a matching minor of G, then a linear or quadratic extension of H is isomorphic to a matching minor of G.*

Proof. Let u be a vertex of H of degree three and let u_1,u_2 and u_3 be its neighbors. Let H_1 be a cross extension of H obtained by deleting the vertex u and adding the vertices x_1, x_2, y_1, y_2, y_3 and edges $y_j u_j$ and $y_j x_i$ for all $i = 1, 2$ and $j = 1, 2, 3$. Let H_2 be obtained from H_1 by replacing x_1 by the triangle X_1, and let H_3 be obtained from H_2 by replacing x_2 by the triangle X_2. Let the vertices of X_1 be a_1, a_2, a_3 such that a_i is adjacent to y_i, and let the vertices of X_2 be b_1, b_2, b_3 such that b_i is adjacent to y_i. By hypothesis, Lemma 2.9.3 and Theorem 2.5.10 we may assume that there exist a vertex u of H of degree three, a graph $J \in \{H_1, H_2, H_3\}$, and an embedding $\eta : J \hookrightarrow G$. If $J \neq H_3$ we define X_2 to be the subgraph of J with vertex-set $\{x_2\}$ and let $b_1 = b_2 = b_3 = x_2$, and if $J = H_1$ we define X_1 to be the subgraph of J with vertex-set $\{x_1\}$ and let $a_1 = a_2 = a_3 = x_1$. By Theorem 2.5.9 we may assume that η is a homeomorphic embedding. Let $J' = \eta(J)$, let $u_i' = \eta(u_i)$, and $y_i' = \eta(y_i)$. Let P_i denote the path $\eta(u_i y_i)$. We may assume that J and η are chosen so that $|V(P_1)| + |V(P_2)| + |V(P_3)|$ is minimum.

Let Ω_1 be the octopus with head $\eta(X_1)$ and tentacles $\eta(a_j y_j)$, where $j = 1, 2, 3$, and let Ω_2 be defined analogously. Let Ω_3 be the octopus with head $\eta(J \backslash V(X_1) \backslash V(X_2) \backslash \{y_1, y_2, y_3\})$ and tentacles P_1, P_2, P_3, let $\mathcal{F} = \{\Omega_1, \Omega_2, \Omega_3\}$, and let $Y' = \{y_1', y_2', y_3'\}$. Then $(\mathcal{F}, Y')$ is a frame in G. Let M be a perfect matching of $G \backslash V(\eta(J))$; then M has a unique extension to a matching M' that is Ω_i-compatible for all $i = 1, 2, 3$. By Theorem 2.2.3 there exist distinct integers $i, j \in \{1, 2, 3\}$ and an M'-alternating path S joining vertices v_i and v_j, where v_i belongs to the head of Ω_i and v_j belongs to the head of Ω_j, and an edge $e \in E(S) \backslash M'$ such that the components of $S \backslash e$ may be denoted by S_i and S_j so that $V(S_i) \cap V(\mathcal{F}) \subseteq V(\Omega_i)$ and $V(S_j) \cap V(\mathcal{F}) \subseteq V(\Omega_j)$.

Assume first that $j = 3$. In this case it will be convenient to allow v_3 to be an internal vertex of a tentacle of Ω_3. By doing so we may assume (by replacing S by its subpath) that

v_3 is the only vertex of $S \cap \Omega_3$. If $v_3 \in V(P_i)$, then let $v = u_i$. If v_3 belongs to $V(\eta(z))$ for some $z \in V(J)$, then let $v := z$. Finally, if $v \in V(\eta(zz'))$ for some edge $zz' \in E(J)$, then v_3 is at even distance on $\eta(zz')$ from exactly one of $\eta(z)$, $\eta(z')$, say from $\eta(z)$. In that case we put $v := z$. We may assume that $v \in V(H) - \{u, u_1, u_2\}$, and that if $v_3 \in V(P_1 \cup P_2 \cup P_3)$ then $v_3 \in V(P_3)$. By Lemma 2.3.6 we may assume that $S \cup \Omega_i$ includes a triad or tripod T with ends y_1', y_2', v_3. We claim that if v_3 belongs to P_3, then the path $P_3[v_3, u_3']$ is even. Indeed, otherwise by making use of T and Ω_{3-i} we obtain contradiction to the minimality of $|V(P_1)| + |V(P_2)| + |V(P_3)|$. We deduce that of the following graphs is isomorphic to a matching minor of G:

- $H_1 \backslash x_1y_3 + (x_1, v)$,
- $H_2 \backslash x_2y_3 + (x_2, v)$,
- $H_2 \backslash a_3y_3 + (a_3, v)$,
- $H_3 \backslash a_3y_3 + (a_3, v)$.

But each of the above graphs has a matching minor isomorphic to a suitable extension of H. In the first case we get a prism extension (bicontract y_3 and consider the edges x_1v and y_1x_2), and in the other cases we get alpha extensions. In the second case delete a_2a_3, bicontract its ends and consider the edges y_1a_1 and x_2v; in the third case delete y_1x_2, bicontract its ends, and consider the edges a_1a_2 and a_3v; and in the fourth case consider the same two edges, delete y_1b_1 and b_2b_3 and bicontract their ends. This completes the case $j = 3$.

Thus we may assume that $i = 1$ and $j = 2$. Let $s_1 \in V(S_1)$ and $s_2 \in V(S_2)$ be the ends of e. We apply Lemma 2.3.7 to $S_2 \cup \Omega_2$ to conclude that $\Omega_2 \cup S_2 + e$ has a central subgraph T_2 such that T_2 is either a quadropod with ends y_1', y_2', y_3', s_1, or a quasi-tripod, in which case we may assume by symmetry that its ends are y_1', y_2', s_1. By Lemma 2.3.6 the graph $\Omega_1 \cup S_1 + e$ has a central subgraph T_1 that is a triad or tripod with ends y_1', y_3', s_2. If T_2 is a quasi-tripod then the theorem holds by Lemma 2.9.1. If T_2 is a quadropod with ends y_1', y_2', y_3', s_1, then one of the following graphs is isomorphic to a matching minor of G:

- $H_1 \backslash x_1y_2 + (x_1, x_2)$,

- $H_2 \backslash a_2y_2 + (a_2, x_2)$.

Both of these graphs have a matching minor isomorphic to a suitable extension of H. In the first case we get a prism extension by bicontracting y_2 and considering the edges x_2y_1 and x_2x_1. In the second case we get an alpha extension by deleting x_2y_1, bicontracting y_1 and y_2 and considering the edges x_2a_2 and a_1a_3. □

Using Lemma 2.9.2 and Lemma 2.9.4 we can upgrade Theorem 2.8.4 to the following statement.

Theorem 2.9.5. *Let H and G be graphs, where H is 2-connected and has minimum degree at least three, G is a brick and H is isomorphic to a matching minor of G. Assume that if H is isomorphic to K_4, then G has no matching minor isomorphic to the Petersen graph. If H is not isomorphic to G, then a linear, quadratic or quartic extension of H is isomorphic to a matching minor of G.*

Proof. This follows immediately from Theorem 2.8.4 and Lemmas 2.9.2 and 2.9.4. □

2.10 Exceptional Families

We now handle quadratic extensions. The next lemma will show that a quadratic extension gives rise to a linear extension, unless it is of one of the following two types. Let $H, u, v, x, y, x', y', H'$ be as in the definition of quadratic extension; that is, H is a graph, $uv \in E(H)$, H' is obtained from H by bisubdividing uv, where the new vertices x, y are such that x is adjacent to u and y. Further, $x' \in V(H) - \{u\}$ and $y' \in V(H) - \{v\}$ do not both belong to $\{u, v\}$. Let $H_1 = H' + (x, x') + (y, y')$ be a quadratic extension of H. If $y' = u$, x' is adjacent to v, and v has degree three, then we say that H_1 is an *alpha extension of H*. If $x', y' \in V(H) - \{u, v\}$, x' is adjacent to v, y' is adjacent to u and both u and v have degree three, then we say that H_1 is a *prism extension of H.*

Lemma 2.10.1. *Let H be a graph of minimum degree at least three, and let K be a quadratic extension of H. Then K has a matching minor isomorphic to a linear, alpha or prism extension of H. Furthermore, if $H, u, v, x, y, x', y', H'$ are as in the definition of quadratic* *extension and $x', y' \in V(H) - \{u, v\}$, then K has a matching minor isomorphic to a linear or prism extension of H.*

Proof. Let $H, u, v, x, y, x', y', H'$ be as in the definition of quadratic extension, and let $K = H' + (x, x') + (y, y')$ be a quadratic extension of H. By symmetry we may assume that $y' \neq u$. If y' is not adjacent to u, then $H + (u, y') \hookrightarrow K$, as desired. Thus we may assume that y' is adjacent to u. If u has degree at least four, then $K \backslash uy'$ is isomorphic to a linear extension of H, as desired. Thus we may assume that u has degree three. If $x' \neq v$, then by symmetry K is a prism extension of H, and if $x' = v$, then K is an alpha extension of H, as desired. $\square$

Lemma 2.10.2. *Let K be an alpha extension of a graph H of minimum degree at least three. Then K has a matching minor isomorphic to a linear or prism extension of H.*

Proof. Let $H, u, v, x, y, x', y', H'$ be as in the definition of quadratic extension, and let $K = H' + (x, x') + (y, y')$ be an alpha extension of H, where $y' = u$. Thus v has degree three and is adjacent to x'. There exists a homeomorphic embedding $\eta : H \hookrightarrow K$ with $\eta(v) = x$ and $\eta(z) = z$ for $z \in V(H) - \{v\}$, and by considering $\eta(H)$ and the edges vx' and uy we deduce that K is isomorphic to a quadratic extension of H that satisfies the second statement of Lemma 2.10.1. Thus the lemma holds by that statement. $\square$

Let H be a graph. By a *fan* in H we mean a sequence of vertices $(x, y, u_1, u_2, \ldots, u_k)$ such that these vertices are pairwise distinct, except that possibly $x = y$, and further $k \geq 2$, $u_1, u_2, \ldots, u_k$ all have degree three and form a path in H in the order listed, and for $i = 1, 2, \ldots, k$ the vertex u_i is adjacent to x if i is even, and otherwise it is adjacent to y.

Lemma 2.10.3. *Let K be a prism extension of a 3-connected graph H. If K is not a prismoid, a wheel or a biwheel, then K has a matching minor isomorphic to a linear extension of H.*

Proof. By hypothesis there exists a fan (x, y, u_1, u_2) in H such that $K = H + (x, u_1u_2) + (y, \tau_2)$. Let t_1, t_2 denote the new vertices τ_1, τ_2 of K, respectively. Let us choose a maximum integer k such that H has a fan $(x, y, u_1, u_2, \ldots, u_k)$ such that $H + (x, u_1u_2) + (y, \tau_2) \hookrightarrow K$.

Let u_0 be the neighbor of u_1 other than u_2 and y. Now $u_0 \neq u_k$, for otherwise H is a wheel or a biwheel (depending on whether x and y are distinct or not). Assume first that $u_0 \neq x$. There exists an embedding $\eta : H \hookrightarrow K$ such that $\eta(u_1) = t_2$. By considering the edges u_1y and xt_1 we deduce that $H + (y, u_0u_1) + (x, \tau_2) \hookrightarrow K$, and by using the proof of Lemma 2.10.1 we deduce that either a linear extension of H is isomorphic to a matching minor of K, or that x is adjacent to u_0 and that u_0 has degree three. But then the fan $(y, x, u_0, u_1, \ldots, u_k)$ contradicts the maximality of k. Thus we may assume that $u_0 = x$, and by symmetry we may assume that u_k is adjacent to both x and y. It follows from the 3-connectivity of H that K is a prismoid, as desired. □

We now turn to quartic extensions. Again, we will show that a quartic extension gives rise to a linear extension, unless it is of two special types, the following ones. Let H be a graph, and let u, v, H', x, y, a, b be as in the definition of a quartic extension. That is, $uv \in E(H)$, H' is obtained from H by bisubdividing uv, where the new vertices are x, y numbered so that x is adjacent to u and y, and let $K = H + (x, ab) + (\tau_2, y)$ be a quartic extension of H. If $b = v$ and the vertices u and a are adjacent and both have degree three, then we say that K is a *staircase extension of* H. If a, b, u, v are pairwise distinct, all have degree three, a is adjacent to u and b is adjacent to v, then we say that K is a *ladder extension of* H. We also say that the extension is *based at* u, v, b, a (in that order).

Lemma 2.10.4. *Let H be a graph of minimum degree at least three, and let K be a quartic extension of H. Then K has a matching minor isomorphic to a linear, staircase or ladder extension of H.*

Proof. If a and u are not equal or adjacent, then $H + au \hookrightarrow K$ (delete $x\tau_1$ and bicontract its ends), and hence the theorem holds. Assume now that a and u are adjacent. If both u and a have degree at least four, then $K \backslash au$ is a linear extension of H. If exactly one of a, u has degree three, say a does, then the graph obtained from $K \backslash au$ by bicontracting a is isomorphic to a linear extension of H. Thus if $a \neq u$, and either they are not adjacent or one of them has degree at least four, then a linear extension of H is isomorphic to a matching minor of K. By symmetry the same conclusion holds about the vertices v and b, and the lemma follows. $\square$

Lemma 2.10.5. *Let K be a staircase extension of a 3-connected graph H. If H has at least five vertices, then a linear or ladder extension of H is isomorphic to a matching minor of K.*

Proof. Let $K = H' + x_1x_2 + y_1y_2$, where H' is obtained from H by bisubdividing the edges vv_1 and vv_2 so that $v_1y_1x_1vx_2y_2v_2$ is a path of H', and assume that v_1, v_2 have degree three and are adjacent to each other. Let v_1', v_2' be the third neighbors of v_1 and v_2, respectively. If v_1' and v_2' are not equal or adjacent, then $H + v_1'v_2' \hookrightarrow K$ (bicontract v_1 and v_2 in $K \backslash v_1v_2$), and so the lemma holds. If v_1' and v_2' are adjacent, then K can be regarded as a ladder extension of H, and if $v_1' = v_2'$, then the 3-connectivity of H implies that it is isomorphic to K_4, contrary to hypothesis. $\square$

A *fence* in a graph H is a sequence $(u_1, v_1, u_2, v_2, \ldots, u_k, v_k)$ of distinct vertices of H such that $k \geq 2$, each of theses vertices has degree three, $u_1u_2 \ldots u_k$ and $v_1v_2 \ldots v_k$ are paths and u_i is adjacent to v_i for all $i = 1, 2, \ldots, k$.

Lemma 2.10.6. *Let K be a ladder extension of a 3-connected graph H on an even number of vertices. If K is not a ladder or a staircase, then K has a matching minor isomorphic to a linear extension of H.*

Proof. By hypothesis there exists a fence $(u_1, v_1, u_2, v_2, \ldots, u_k, v_k)$ in H such that $K = H' + x_1y_1 + x_2y_2$, where H' is obtained from H by bisubdividing u_1u_2 and v_1v_2 and x_1, x_2, y_1, y_2 are the new vertices numbered so that $u_1x_1x_2u_2v_2y_2y_1v_1$ is a cycle in H'. We may assume that the fence is chosen with k maximum. Let u_0, v_0 be the third neighbors of u_1, v_1, respectively. Assume first that $u_0 \neq v_0$. Since the quartic extension of H based at u_0, u_1, v_1, v_0 is isomorphic to K, the argument in the proof of Lemma 2.10.4 shows that either a linear extension of H is isomorphic to a matching minor of K, or that u_0 and v_0 are adjacent and both have degree three. We may assume the latter, for otherwise the lemma holds. By the maximality of k the sequence $(u_0, v_0, u_1, v_1, \ldots, u_k, v_k)$ is not a fence in H, and hence we may assume that $u_0 = u_k$ or $u_0 = v_k$. But H is 3-connected, and so in the former case K is a planar ladder, and in the latter case it is a Möbius ladder. Thus we may assume that $u_0 = v_0$. The ladder extension of H based at $u_{k-1}u_kv_kv_{k-1}$ is clearly isomorphic to K, and hence the above argument shows that we may assume that the third neighbors of u_k and v_k are equal. Since H is 3-connected and has an even number of vertices, it is a staircase. $\square$

The following result summarizes the previous lemmas.

Theorem 2.10.7. *Let K be a quadratic or quartic extension of a 3-connected graph H on an even number of vertices, and assume that K is not a prismoid, wheel, biwheel, ladder or staircase. Then a linear extension of H is isomorphic to a matching minor of K.*

Proof. If H is isomorphic to K_4, then K is not a staircase extension of H, because K is not a staircase. Thus the lemma follows from the results of this section. $\square$

We are now ready to prove Theorem 2.1.9. Let H and G be as stated therein, and assume that they are not isomorphic. Assume first that either H is not isomorphic to K_4, or G has no matching minor isomorphic to the Petersen graph. By Theorem 2.9.5 we may assume that a quadratic or quartic extension K of H is isomorphic to a matching minor of G. It follows from the hypothesis of Theorem 2.1.9 that K is not a prismoid, wheel, biwheel, ladder or staircase. Thus K has a matching minor isomorphic to a linear extension of H by Theorem 2.10.7, and hence so does G, as desired. Thus we may assume that H is isomorphic to K_4 and G has a matching minor isomorphic to the Petersen graph. But G is not isomorphic to the Petersen graph by hypothesis. Since we have already shown that Theorem 2.1.9 holds when H is the Petersen graph, we may now apply it to deduce that G has a matching minor isomorphic to a linear extension of the Petersen graph. The Petersen graph has, up to isomorphism, a unique linear extension, and this linear extension has a matching minor isomorphic to the staircase on eight vertices. But the latter graph has a matching minor isomorphic to K_4, the staircase on four vertices, contrary to hypothesis. $\square$

2.11 A Generalization

In this section we state a generalization of Theorem 2.1.9, and point out how it follows from the theory that we developed. Let G be a graph with a perfect matching. Let us recall that a barrier in G is a set $X \subseteq V(G)$ such that $G \backslash X$ has at least $|X|$ odd components, and that bricks are 3-connected graphs with perfect matchings and no barriers of size at least two. Braces almost have no barriers, either, for if X is a barrier in a brace and X has at least two elements, then X is one of the two color classes of G. We use this fact to weaken the condition on bricks. Let $s \geq 0$ be an integer. We say that a set $X \subseteq V(G)$ is an s-barrier in G if $G \backslash X$ has $|X| - 1$ odd components such that the union of the remaining components of $G \backslash X$ has at least s vertices. We say that a graph is an *s-brick* if it is 3-connected and has no s-barrier of size at least two. Thus bricks are 1-bricks and braces are 2-bricks. Our proof of Theorem 2.1.9 actually proves the following more general theorem. A *pinched staircase* is a graph obtained from a staircase by contracting the edge v_1v_2, where the vertices v_1 and v_2 are as in the definition of a staircase.

Theorem 2.11.1. *Let $s \geq 0$ be an integer, G be an s-brick other than the Petersen graph, and let H be a 3-connected matching minor of G on at least $s+1$ vertices. Assume that if H is a planar ladder, then there is no strictly larger planar ladder L with $H \hookrightarrow L \hookrightarrow G$, and similarly for Möbius ladders, wheels, lower biwheels, upper biwheels, staircases, pinched staircases, lower prismoids and upper prismoids. If H is not isomorphic to G, then some matching minor of G is isomorphic to a linear extension of H.*

Proof. The proof follows the proof of Theorem 2.1.9, with the following minor modifications. In Lemma 2.2.2 the set R_k is not required to be odd, but instead must have at least s vertices. The proof goes through with the obvious changes. Then the definition of octopus needs to be changed to permit heads with even number of vertices, and in the definition of frame we need to add a condition guaranteeing that the heads of $\Omega_1, \Omega_2, \ldots, \Omega_{k-1}$ are odd and that the head of Ω_k has at least s vertices. The assumption that H has at least $s+1$ vertices will guarantee that this additional condition is satisfied whenever Theorem 2.2.3 is applied. Finally, in Lemma 2.10.6 the assumption that H has an even number of vertices can be replaced by assuming that K is not a pinched staircase. $\square$

Clearly Theorem 2.11.1 implies Theorem 2.1.9 on taking $s = 1$. Let us now turn to braces. Let L be a linear extension of a brace H. Then L need not be a brace, but if L is bipartite, then it is a brace. Furthermore, if L is isomorphic to a matching minor of a bipartite graph, then L itself is bipartite. Thus Theorem 2.11.1 implies Theorem 2.1.7 by taking $s = 2$. The third application of Theorem 2.11.1 is to factor-critical graphs. A graph G is *factor-critical* if $G\backslash v$ has a perfect matching for every vertex $v \in V(G)$. It is easy to see that every 1-brick on an odd number of vertices is factor-critical. Thus the following immediate corollary of Theorem 2.11.1 gives a generation theorem for a subclass of factor-critical graphs.

Corollary 2.11.2. *Let G be a 1-brick on an odd number of vertices, and let H be a 3-connected matching minor of G. Assume that if H is a wheel, then there is no strictly larger wheel W with $H \hookrightarrow W \hookrightarrow G$, and similarly for pinched staircases, lower prismoids and upper prismoids. If H is not isomorphic to G, then some matching minor of G is isomorphic to a linear extension of H.*

Unfortunately, a linear extension of a 1-brick need not be a 1-brick.

CHAPTER III

MINIMAL BRICKS

In this chapter we utilize the results of Chapter 2 to prove a generating theorem for minimal bricks.

The first advantage of such a theorem is computational. The theorem was used by Robin Thomas in a program that generates all bricks up to a certain number of vertices. The program allows one to input a starting graph (the graph all the resulting bricks will contain as a matching minor) and several excluded graphs which the resulting graphs are not allowed to have as matching minors. The program can test whether the resulting graphs allow Pfaffian orientation.

This program was tremendously helpful in testing our conjectures and producing interesting examples. The use of generating procedure suggested by Theorem 2.1.5 in such a program is less efficient. We do not quantify this statement.

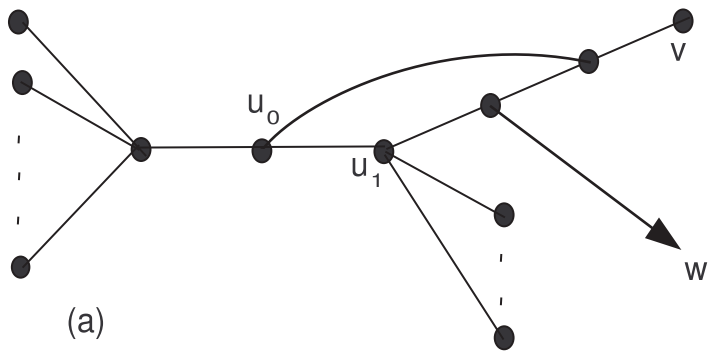
Secondly, the corollaries of this generation theorem shed light on the structure of minimal bricks. We prove an exact upper bound for the number of edges in a minimal brick. We also prove that every minimal brick has at least three vertices of degree three, generalizing a recent result of de Carvalho, Lucchesi and Murty [7], which settles a conjecture of Lovász.

The material presented in this chapter will also appear in [37]. All the graphs considered in this chapter are simple.

3.1 Generating Theorem for Minimal Bricks

A brick G is *minimal* if for every $e \in E(G)$ the graph $G \setminus e$ is not a brick. In this chapter we derive a generating theorem for minimal bricks from the results of the previous chapter. We also prove two corollaries: for $n \geq 4$, a minimal brick on $2n$ vertices has at most $5n - 7$ edges; every minimal brick has at least three vertices of degree three.

We need to define new types of extensions that will be used in this generating theorem.



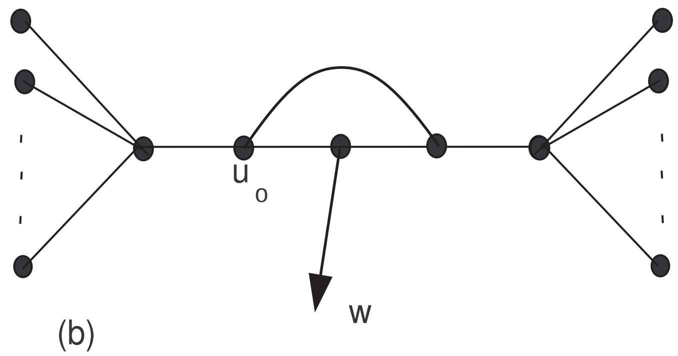


Figure 13: (a) Bilinear extension, (b) Pseudolinear extension

We say that a linear extension H' of a graph H is *strict* if $|V(H')| > |V(H)|$. Let u, v, w be pairwise distinct vertices of H, let H' be obtained from H by bisplitting u and let u_0 be the new inner vertex and u_1 a new outer vertex. If $u_1v \in E(H')$ and $vw \notin E(H)$ then the graph $H' + (u_0, vu_1) + (\tau_2, w)$ is called a *bilinear* extension of H. If $uw \notin E(H)$ then the graph $H' + (u_0, u_1u_0) + (\tau_2, w)$ is called a *pseudolinear* extension of H. See Figure 13.

Let $u, v \in V(H)$ be distinct. We say that H' is a *quasiquadratic* extension of H if H' is a quadratic extension of $H + uv$ with base uv. Similarly, if $u, v, a, b \in V(H)$ are such that $\{u, v\} \neq \{a, b\}$ we say that a quartic extension of $H + uv + ab$ is a *quasiquartic* extension of H with bases uv and ab. Recall our convention that if u and v are adjacent in H, then $H + uv = H$. Thus quadratic extensions are quasiquadratic and quartic are quasiquartic. Note also that quasiquadratic, quasiquartic, bilinear and pseudolinear extensions of bricks are bricks. Finally, we say that H' is a *strict* extension of H if H' is a quasiquadratic, quasiquartic, bilinear, pseudolinear or strict linear extension of H.

Theorem 3.1.1. *Let G be a brick other than the Petersen graph, and let H be a 3-connected matching minor of G, such that $|V(H)| < |V(G)|$. Then some matching minor of G is isomorphic to a strict extension of H.*

Proof. Let a graph H' be chosen so that H is a spanning subgraph of H', $H' \hookrightarrow G$ and $|E(H')|$ is maximal.

Suppose first that H' is a planar ladder and there exists a planar ladder L with $H' \hookrightarrow L \hookrightarrow G$ and $|V(L)| > |V(H')|$. Then clearly $H' = H$, L is a quartic extension of H and therefore the theorem holds. Therefore we can assume that if H' is a planar ladder, then there is no strictly larger planar ladder L with $H \hookrightarrow L \hookrightarrow G$, and similarly for Möbius ladders, wheels, lower biwheels, upper biwheels, staircases, lower prismoids and upper prismoids. By Theorem 2.1.9 and the choice of H' there exists a strict linear extension K of H' such that $K \hookrightarrow G$. We denote $E(H') - E(H)$ by E'. We break the analysis into cases depending on the type of strict linear extension.

Suppose first that $K = K' + uv$, where K' is obtained from H' by bisplitting a vertex, v is the new inner vertex of K' and $u \in V(H')$. Let v_1 and v_2 be the new outer vertices. We have $E(H') \subseteq E(K')$, in a natural way. For $i = 1, 2$ let d_i be the number of edges of $E(H)$ that are incident with v_i in K' (or K). We assume without loss of generality that $d_1 \geq d_2$. Note that $d_1 + d_2 \geq 3$, because v has degree at least three in H.

If $d_2 \geq 2$ then $K \setminus E'$ is a strict linear extension of H. If $d_2 = 1$ let $f \in E'$ be an edge incident with v_2; then $K \setminus (E' - \{f\})$ is a quadratic extension of H. Finally, if $d_2 = 0$ and $f_1, f_2 \in E'$ are incident with v_2 then $K \setminus (E' - \{f_1, f_2\})$ is a quasiquadratic extension of H.

Now suppose $K = K' + u_1u_2$, where K' is obtained by bisplitting a vertex of a graph obtained from H' by bisplitting a vertex, and u_1 and u_2 are the two new inner vertices of K'. Let v_1 and v_2, v_3 and v_4, respectively, be the corresponding new outer vertices. Let d_1, d_2, d_3 and d_4 be defined analogously as above. We start by assuming that v_1, v_2, v_3 and v_4 are pairwise distinct and without loss of generality assume $d_1 \geq d_2$, $d_3 \geq d_4 \geq d_2$.

If $d_2 \geq 2$ then $K \setminus E'$ is a strict linear extension of H. If $d_2 = 1, d_4 \geq 2$ then $K \setminus E'/v_2$ is isomorphic to a strict linear extension of H unless the edge of H incident with v_2 is incident also with one of the vertices v_3 and v_4. In this case $K \setminus (E' - \{f\})$ is a bilinear extension of H, for every $f \in E'$ incident with v_2. If $d_2 = d_4 = 1$ for $i \in \{1, 2\}$ let e_i denote the unique edge in $E(H)$ incident with d_{2i} and let f_i denote some edge in E' incident with d_{2i}. If $e_1 = e_2$ then $K \setminus (E' - \{f_1, f_2\})$ is a quasiquartic extension of H. Otherwise, without loss of generality we assume that e_2 is not incident with v_1 and deduce that $K \setminus (E' - \{f_1\})/v_4$ is a quadratic extension of H with base e_1.

It remains to consider the subcase when $d_2 = 0$. Let $f, f' \in E'$ be incident with v_2 such that f has no end in $\{v_3, v_4\}$. If $d_4 \geq 2$ then $K \setminus (E' - \{f\}) \setminus u_1v_1/u_1$ is a strict linear extension of H. If $d_4 = 1$ let e denote the unique edge in $E(H)$ incident with d_4. If e is not incident with v_1 then $K \setminus (E' - \{f, f'\})/v_4$ is a quasiquadratic extension of H if f' is not incident with v_4 and $K \setminus (E' - \{f, f'\})$ is a quasiquartic extension of H if f' is incident with v_4. If on the other hand e is incident with v_1 then $K \setminus (E' - \{f, f''\}) \setminus u_1v_1/u_1$ is a quadratic extension of H, where f'' is any edge in E' incident with v_4. Finally, if $d_4 = 0$ let $f^* \in E'$ be incident with v_4 and have no end in $\{v_1, v_2\}$. Then $K \setminus (E' - \{f, f', f^*\}) \setminus u_2v_3/u_2$ is a quasiquadratic extension of H. This completes the case when v_1, v_2, v_3 and v_4 are pairwise distinct.

We now assume without loss of generality that $v_1 = v_4$. Then v_1, v_2 and v_3 are pairwise distinct and we assume $d_2 \geq d_3$, again without loss of generality. Suppose first $d_1 = 0$. If $d_3 \geq 2$ then $K \setminus (E' - \{g\})$ is a pseudolinear extension of H, where $g \in E'$ is incident with v_1; if $d_3 = 1$ then $K \setminus (E' - \{g\})/v_3$ is a quadratic extension of H and if $d_3 = 0$ then $K \setminus (E' - \{f, g\})/v_3$ is a quasiquadratic extension of H, where f is an edge in E' incident with v_3 and not adjacent to g. Therefore we may assume $d_1 \geq 1$. If $d_2 \geq 2$ and $d_3 \geq 1$ then $K \setminus E'$ or $K \setminus E'/v_3$ is a strict linear extension of H. If $d_2 \geq 2$ and $d_3 = 0$ then $K \setminus (E' \setminus f)/v_3$ is a quadratic extension of H, where f is as above. If, finally, $d_2 \leq 1$ then let E'' be obtained from E' by deleting $2 - d_2$ edges of E' incident with v_2 and $1 - d_3$ edges incident with v_3; $K \setminus (E'')/u_2$ is a quasiquadratic extension of H.

This completes the case analysis. $\square$

Theorem 3.1.1 implies the following generating theorem for minimal bricks.

Theorem 3.1.2. *Let G be a minimal brick other than the Petersen graph. Then G can be obtained from K_4 or the prism by taking strict extensions, in such a way that all the intermediate graphs are minimal bricks not isomorphic to the Petersen graph.*

Proof. Suppose the statement of the theorem is false and let G be a counterexample with $|V(G)|$ minimum.

Let a minimal brick $H \hookrightarrow G$ be chosen such that H can be obtained from K_4 or the prism by taking strict extensions and $|V(H)|$ is maximum. By [28], Theorem 5.4.11, G has a matching minor M isomorphic to K_4 or the prism and therefore such choice is possible. If $|V(H)| = |V(G)|$ then H is isomorphic to G by the minimality of G. If, on the other hand, $|V(H)| < |V(G)|$ then by Theorem 3.1.1 there exists a strict extension $H' \hookrightarrow G$ of H. Let $H'' \hookrightarrow H'$ be a minimal brick with $|V(H'')| = |V(H')|$; then $H'' \hookrightarrow G$. It follows that H'' is not isomorphic to G, for otherwise so is H', contrary to our assumption that G is a counterexample to the theorem. By the minimality of G the graph H'' can be obtained from K_4 or the prism by taking strict extensions, contrary to the choice of H. $\square$

Note that there exist bricks obtained from K_4 or the prism by a sequence of strict extensions, that are not minimal. A simple example follows.

Let G be the prism, $V(G) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$, the vertices v_1,v_2,v_3 are pairwise adjacent and so are the vertices u_1, u_2, u_3, and u_i is adjacent to v_i for $i \in \{1, 2, 3\}$. Let $G' = G + u_1v_2$ and let $G'' = G' + (u_2, u_1v_2) + v_1\tau_2$. Then G'' is a quasiquadratic extension of G and $G'' \setminus u_1v_1$ is a brick, which can be obtained from a prism by a quadratic extension or a sequence of two linear extensions.

3.2 Edge Bound for Minimal Bricks

In [28], Corollary 5.4.16 an exact upper edge bound for minimal bicritical graphs is given.

Theorem 3.2.1. *If G is a minimal bicritical graph with $n \geq 6$ vertices, then $|E(G)| \leq 5(n-2)/2$.*

We use Theorem 3.1.1 to prove a similar bound for bricks.

Theorem 3.2.2. *Let G be a minimal brick on $2n$ vertices. Then $|E(G)| \leq 5n - 7$, unless G is the prism or the wheel on four, six or eight vertices.*

Proof. The theorem holds for the Petersen graph, so from now on we assume that G is not the Petersen graph, the prism or the wheel on six or eight vertices. Denote the last three graphs by R_6, W_6 and W_8, respectively.

Note that a strict linear extension increases the number of vertices in a graph by 2 or 4 and the number of edges by 3 or 5, respectively. Similarly, a quasiquadratic extension increases the number of vertices by 2 and the number of edges by at most 5, while quasi-quadric, bilinear and pseudolinear extensions increase the number of vertices by 4 and the number of edges by at most 8.

We say that a brick H is *sparse* if $|E(H)| \leq \frac{5}{2}|V(H)| - 7$ and we say that H is dense otherwise. We claim that any minimal brick that contains a sparse minor is sparse. Suppose G_1 and G_2 are bricks, $G_1 \hookrightarrow G_2$, G_1 is sparse and G_2 is minimal. Let the sparse brick $H \hookrightarrow G_2$ be chosen such that $|V(H)|$ maximal. From Theorem 3.1.1 we deduce that either $|V(H)| = |V(G_2)|$ or some strict extension H' of H is a matching minor of G_2. In the latter case, by the calculations above, H' is sparse in contradiction with the choice of H. Therefore $|V(H)| = |V(G_2)|$ and G_2 is isomorphic to H by the minimality of G_2. The claim follows.

Suppose G is dense. By Theorem 2.1.6 G has a matching minor isomorphic to one of the following four graphs: R_6, W_6, the staircase on eight vertices, and the Möbius ladder on eight vertices. Among these graphs only two are dense: R_6 and W_6 .

Assume first that G contains R_6 as a matching minor. By Theorem 3.1.1 there exists a strict extension H of the prism such that $H \hookrightarrow G$. By the calculations above H is sparse, unless H is a quadratic extension of $R_6 + uv$ with the base uv, where $uv \notin E(R_6)$. We will show that there exists $e \in E(H)$ such that $H \setminus e$ is a brick. Note that $H \setminus e$ is sparse. Therefore it follows that any minimal brick containing the prism as a matching minor and not equal to it is sparse. We prove the existence of e by listing all possible quasiquadratic extensions of R_6 with 14 edges on Figure 14. An edge e that satisfies the conditions above is indicated by a cross. A spanning bisubdivision or bisplit of R_6 or W_6 in $H \setminus e$ is indicated by bold lines and allows the reader to easily verify that the claim holds in each of the cases.

Therefore we may assume that G contains W_6 as a matching minor and does not contain R_6. By Theorem 2.1.5 G is a wheel or G contains a linear extension of W_6 as a matching minor. All the wheels on at least ten vertices and all strict linear extensions of W_6 are sparse and therefore G must contain a graph obtained from W_6 by an edge addition. Such graph is unique up to isomorphism and contains R_6 as a spanning subgraph, in contradiction with our assumptions. $\square$

The bound given in Theorem 3.2.2 is tight for every $n \geq 4$. An example of a minimal brick G_n on $2n+4$ vertices with $5n+3$ edges follows for $n \geq 2$. Let $V(G_n) = \{x, y, z, t, v_1, u_1, v_2, u_2, \ldots, v_n, u_n\}$. For every $i \in \{1, 2, \ldots, n\}$ let xt, yt, zt, xu_i,yu_i, yv_i,zv_i and u_iv_i be the edges of G_n. Then for every $e \in E(G_n)$ the graph $G_n \setminus e$ contains a vertex of degree two and is not a brick. It remains to show that G_n is a brick for every n. Note that G_k is a quasiquadratic extension of G_{k-1} for every $k > 2$. Therefore it suffices to show that G_1 is a brick. The graph $G_2 \setminus u_1y \setminus v_1y$ is isomorphic to the prism with one of its edges bisubdivided and consequently G_2 can be obtained from the prism by a quadratic extension.

3.3 Three Cubic Vertices

The conjecture that every minimal brick contains a vertex of degree three is attributed to Lovász . This conjecture was verified in [7]. Below we prove a strengthening of this conjecture.

Theorem 3.3.1. *Every minimal brick has at least three vertices of degree three.*

Proof. Let a minimal brick G that has at most two vertices of degree three be chosen with $|V(G)|$ minimal. By Theorem 3.1.2 there exists a minimal brick $H \hookrightarrow G$ with at least three vertices of degree three, such that G is isomorphic to a strict extension of H.

Note that if a strict linear extension is used to obtain G from H then the degree of at most one vertex of H increased and at least one vertex in $V(G) - V(H)$ has degree three.

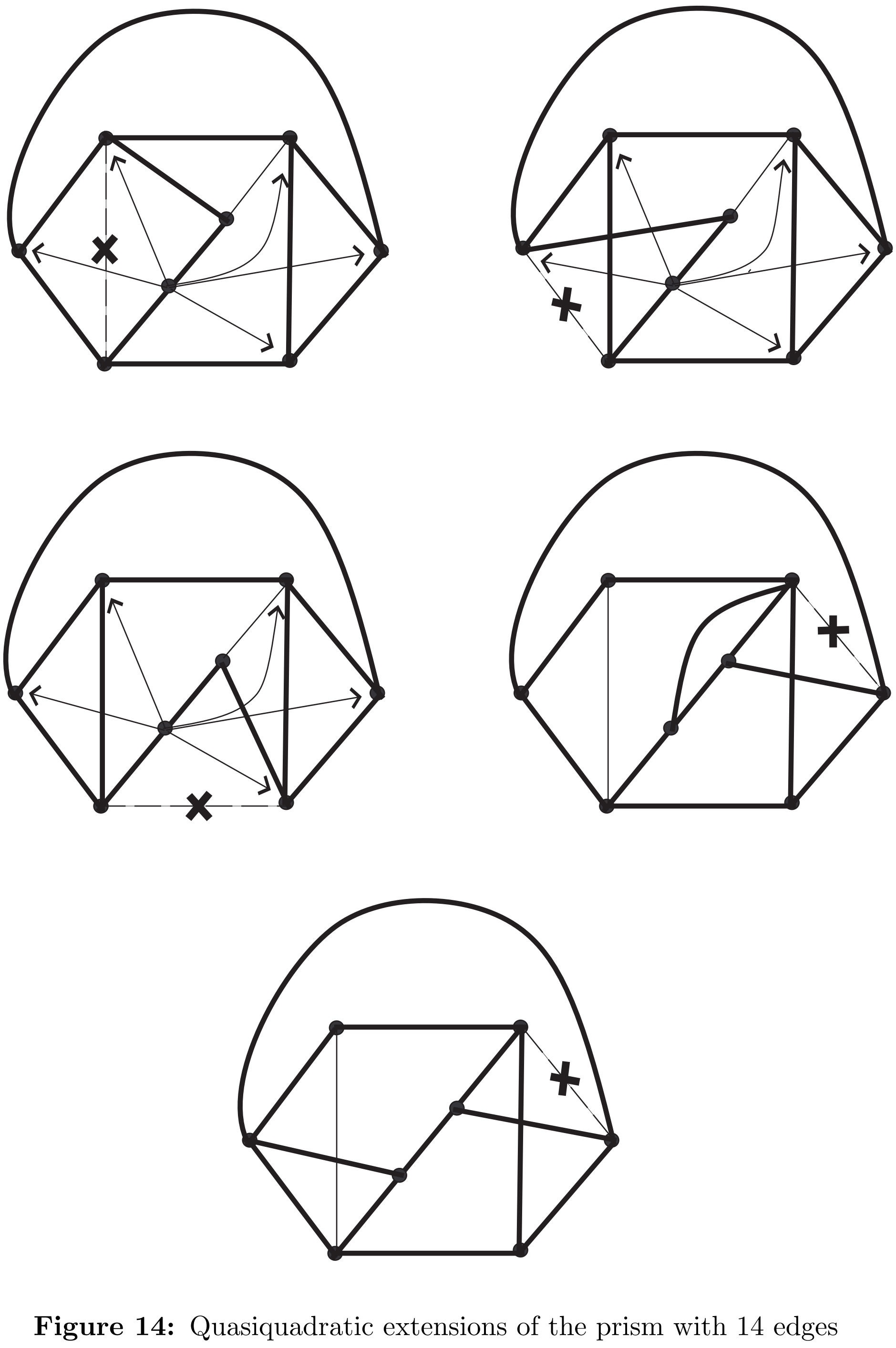


Figure 14: Quasiquadratic extensions of the prism with 14 edges

If a quasiquartic, bilinear or pseudolinear extension is used to obtain G then $V(G) - V(H)$ contains at least three vertices of degree three. Therefore G is isomorphic to a quasiquadratic extension of H.

We assume without loss of generality that $V(G) - V(H) = \{u_1, u_2\}$ and there exist $v_1, v_2, v_3, v_4 \in V(H)$ such that $E(G) - E(H) = \{u_1v_1,\ u_1v_2,\ u_2v_3,\ u_2v_4, u_1v_1\}$, at least three of the vertices v_1, v_2, v_3, v_4 are distinct, $v_1 \neq v_2$ and $v_3 \neq v_4$. Note that vertices of degree three in H must form a subset of $\{v_1, v_2, v_3, v_4\}$ and that $v_1v_3, v_2v_3, v_2v_4, v_1v_4 \notin E(H)$, for the deletion of such an edge results in a quadratic extension of H, contrary to the fact that G is a minimal brick.

By Theorem 2.1.5 either H is a ladder, wheel, staircase or prismoid or some proper reduction of H is a brick. If H is a ladder, wheel, staircase or prismoid distinct from K_4 then H has at least 5 vertices of degree three, and consequently G has at least three vertices of degree three. If $H = K_4$ then G is not minimal, by an observation in the previous paragraph.

Therefore there exists $e \in E(H)$ such that $H \setminus e$ becomes a brick H_1 after possible bicontractions of vertices of degree two and no parallel edges are created by these bicontractions. Note that H is minimal and therefore at least one end of e is a vertex of degree three in H. Assume first that exactly one end of e has degree three in H. Without loss of generality this end is v_1. The graph $G \setminus e$ is a brick, because it can be obtained by a linear extension (first bisplit to produce $H \backslash e$, then add edge v_1v_3) followed by a quadratic extension with base v_1v_3. Recall that v_1 is not adjacent to v_3 in H.

It remains to consider the case when both of the ends of e have degree three in H. Without loss of generality we assume that $e = v_1v_2$ and v_1, v_2, v_3 and v_4 are pairwise distinct. It follows that $G \setminus e$ is a strict linear extension of $H + v_1v_3 + v_1v_4$ and is again a brick. This completes the case analysis. $\square$

We conjecture the following strengthening of Theorem 3.3.1.

Conjecture 3.3.2. *There exists $\alpha > 0$ such that every minimal brick G has at least $\alpha|V(G)|$ vertices of degree three.*

Even a much weaker strengthening, namely, a conjecture that every brick has at least four vertices of degree three, seems to require new ideas or a substantial refinement of our techniques.

CHAPTER IV

PFAFFIAN GRAPHS, T-JOINS AND CROSSING NUMBERS

In this chapter we prove a technical theorem about the numbers of crossings in T-joins in different drawings of a fixed graph.

As a main corollary we characterize Pfaffian graphs in terms of their drawings in the plane. We also consider applications of this theorem to the theory of crossing numbers. We give a new proof of a theorem of Kleitman [23] on the parity of crossings in drawings of K_{2j+1} and $K_{2j+1,2k+1}$, which in turn gives a new proof of the Hanani-Tutte theorem [17, 55]. We state a hypergraph conjecture, which if true implies Zarankiewicz's conjecture on crossing number of $K_{n,n}$ and prove a uniqueness of the drawing of the Petersen which minimizes the number of crossings.

Further applications of the method appear in Chapters 5 and 6. The material presented in this chapter will also appear in [33].

4.1 Introduction

A pair (G,T) consisting of a graph G and a set $T \subseteq V(G)$ of even cardinality is called *a graft*. *A T-join* is a subset $J \subseteq E(G)$ such that every vertex $v \in V(G)$ is incident with an odd number of edges in J if and only if $v \in T$.

T-joins were first introduced in relation to the Chinese Postman problem, which can be reformulated as follows: find the minimum set of edges in a graph whose doubling results in an Eulerian graph. Note that such set of edges is a T-join, where T is the set of all vertices of odd degree. A perfect matching is another example of a T-join, where $T = V(G)$. Since their introduction, T-joins have been extensively studied (see for example [47], [28, sections 6.5 and 6.6], [16], [6, section 2]).

By *a drawing* Γ of a graph G we mean an immersion of G in the plane such that edges are represented by homeomorphic images of $[0,1]$, not containing vertices in their interiors. Edges are permitted to intersect, but there are only finitely many intersections and each intersection is a crossing. For edges e, f of a graph G drawn in the plane let $cr(e,f)$ denote the number of times the edges e and f cross. For a set $J \subseteq E(G)$ let $cr(J,\Gamma)$, or $cr(J)$ if the drawing is understood from context, denote $\sum cr(e,f)$, where the sum is taken over all unordered pairs of distinct edges $e, f \in J$.

We say that an unordered pair $\{e, f\}$ of adjacent edges in G is *an angle*. We denote the set of all edges and angles in a graph G by $Æ(G)$. If $J \subseteq E(G)$ we say that $e \in E(G)$ *lies* in J if $e \in J$, and we say that an angle $\{e, f\}$ *lies* in J if $e, f \in J$. For $J \subseteq E(G)$ and $S \subseteq Æ(G)$ we denote by $J \sqcap S$ the set of elements of S which lie in J.

The following theorem is the main result of this chapter. While the theorem itself is rather technical, it has a number of interesting applications.

Theorem 4.1.1. *Let (G,T) be a graft and let Γ_1 and Γ_2 be two drawings of G in the plane. Then there exists $S = S(T,\Gamma_1,\Gamma_2) \subseteq Æ(G)$ such that for every T-join $J \subseteq E(G)$ the following identity holds modulo 2*

$$cr(J,\Gamma_1) = cr(J,\Gamma_2) + |J \sqcap S|, \tag{1}$$

and if $T = \emptyset$ then S contains no edges.

We prove Theorem 4.1.1 in Section 2. In Section 3 Theorem 4.1.1 is used to characterize Pfaffian graphs in terms of their drawings in the plane.

In Section 4 we consider several applications of Theorem 4.1.1 to the theory of crossing numbers. We give a new proof of a result of Kleitman on the parity of the number of crossings in a graph. A well-known theorem of Hanani and Tutte follows as a corollary. We develop an approach to the problem of estimating the crossing number of complete bipartite graphs, also known as the Turán's brickyard problem. Finally, we characterize the drawings of the Petersen graph that minimize the number of crossings.

4.2 Proof of The Main Theorem

Throughout this section all integer identities are modulo 2.

For any n and any two sequences $(a_1, a_2, .., a_n)$ and $(b_1, b_2, .., b_n)$ of pairwise distinct points in the plane, there clearly exists a homeomorphic transformation of the plane that takes a_i to b_i for all $1 \leq i \leq n$. Therefore without loss of generality we assume that the vertices of G are represented by the same points in the plane in both Γ_1 and Γ_2.

We say that the drawings Γ_1 and Γ_2 are *adjacent* if they differ only in the position of a single edge $e = u_1u_2$. We start by proving Theorem 4.1.1 for adjacent drawings.

Let e_1 and e_2 denote the images of e in Γ_1 and Γ_2 correspondingly. By changing these images within the regions of $\Gamma_1 \setminus e_1$ we can assume that e_1 and e_2 have finitely many intersections and each intersection is a crossing. Define $C = e_1 \cup e_2$. The closed curve C separates its complement into two sets P_1 and P_2 with the property that every simple curve with ends $a \in P_i$ and $b \in P_j$ crosses C an even number of times if and only if $i = j$.

For $x \in (V(G) \cup E(G)) \setminus \{e\}$ we will not distinguish between x and its representation in Γ_1 and Γ_2. Define F_i to be the set of all edges $f \in E(G) \setminus \{e\}$ such that f is adjacent to u_j for some $j \in \{1, 2\}$ and $f \cap U \subseteq P_i \cup \{u_j\}$ for every some neighborhood U of u_j in the plane. Define

$$S = \{\{e, f\} | f \in F_1\}$$

if $|T \cap P_1|$ is even, and

$$S = \{\{e, f\} | f \in F_1\} \cup \{e\}$$

if $|T \cap P_1|$ is odd. If $T = \emptyset$ then S contains no edges.

If $e \notin J$ then $cr(J, \Gamma_1) = cr(J, \Gamma_2)$ and (1) trivially holds, so we assume $e \in J$. We have

$$\begin{aligned} cr(J, \Gamma_1) + cr(J, \Gamma_2) &= 2 \sum_{\{f,g\} \subseteq J \setminus \{e\}} cr(f, g) + \sum_{f \in J \setminus \{e\}} (cr(f, e_1) + cr(f, e_2)) \\ &= \sum_{f \in J \setminus \{e\}} cr(f, C) \end{aligned}$$

Therefore it suffices to prove that

$$|J \sqcap S| = \sum_{f \in J \setminus \{e\}} cr(f, C),$$

or equivalently that

$$|J \cap F_1| + |T \cap P_1| = \sum_{f \in J \setminus \{e\}} cr(f, C). \tag{2}$$

From the definition of T-join we can deduce that for any $X \subseteq V(G)$

$$|T \cap X| = |\{uv \in J | u \in X, v \notin X\}|.$$

In particular

$$|T \cap P_1| = |\{uv \in J | u \in P_1, v \notin P_1\}| = |\{uv \in J | u \in P_1, v \in P_2\}| + \\ + |\{uv \in J | u \in P_1, v \in \{u_1, u_2\}\}|. \quad (3)$$

Let $J_1 = \{uv \in J \cap F_2 | u \in P_1\}$ and $J_2 = \{uv \in J \cap F_1 | u \in P_2\}$. Note that

$$(J \cap F_1) \triangle \{uv \in J | u \in P_1, v \in \{u_1, u_2\}\} = J_1 \cup J_2,$$

and therefore

$$|J \cap F_1| + |\{uv \in J | u \in P_1, v \in \{u_1, u_2\}\}| = |J_1 \cup J_2|. \quad (4)$$

Let $J_3 = \{uv \in J | u \in P_1, v \in P_2\}$. The sets J_1, J_2 and J_3 are pairwise disjoint. From (3) and (4) we have

$$|J \cap F_1| + |T \cap P_1| = |J_1 \cup J_2 \cup J_3|. \quad (5)$$

But $J_1 \cup J_2 \cup J_3$ is exactly the set of those edges $f \in J \setminus \{e\}$ which cross C an odd number of times. Therefore (2) follows from (5) and the proof of Theorem 4.1.1 for adjacent drawings is complete.

For two arbitrary drawings Γ_1 and Γ_2 of G there always exist an integer n and a sequence of drawings $\Gamma_1 = \Gamma'_1, \Gamma'_2, \ldots, \Gamma'_n = \Gamma_2$ of G such that Γ'_i is adjacent to Γ'_{i+1} for all $i \in \{1, 2, \ldots, n-1\}$. We have proved that there exist sets $S_i \subseteq Æ(G)$ for all $i \in \{1, 2, \ldots, n-1\}$ such that

$$cr(J, \Gamma'_i) = cr(J, \Gamma'_{i+1}) + |J \sqcap S_i| \quad (6)$$

for all T-joins J. Let $S = S_1 \triangle S_2 \triangle \ldots \triangle S_{n-1}$. Summing up (6) over all $i \in \{1, 2, \ldots, n-1\}$ we get (1), thereby completing the proof of Theorem 4.1.1 for arbitrary drawings.

4.3 Drawing Pfaffian Graphs

The following theorem is the main result of this section.

Theorem 4.3.1. *A graph G is Pfaffian if and only if there exists a drawing of G in the plane such that $cr(M)$ is even for every $M \in \mathcal{M}(G)$.*

The "if" part of this theorem was known to Kasteleyn [21] and was proved by Tesler [51]; however our proof of this part is different. A self-contained proof of Theorem 4.3.1 has recently appeared in [34].

We derive Theorem 4.3.1 from a more general theorem. To state it we need a definition. Recall that $\mathcal{M}(G)$ or $\mathcal{M}$ if the graph is understood from the context denotes the set of all perfect matchings of a graph G.

Let Γ be a drawing of a graph G and let $s : \mathcal{M} \to \{-1, 1\}$. We say that $S \subseteq E(G)$ is an *s-marking* of Γ if

$$s(M) = (-1)^{cr(M)+|M \cap S|}$$

for every $M \in \mathcal{M}$.

Theorem 4.3.2. *Let G be a labeled graph and let $s : \mathcal{M} \to \{-1, 1\}$. Then the following are equivalent:*

(a) there exists an orientation D of G such that for every $M \in \mathcal{M}$ its sign in the corresponding directed graph is equal to $s(M)$;

(b) some drawing of G in the plane has an s-marking;

(c) every drawing of G in the plane has an s-marking;

(d) there exists a drawing of G in the plane such that for every $M \in \mathcal{M}$

$$s(M) = (-1)^{cr(M)}.$$

We say that Γ is *a standard drawing* of a labeled graph G if the vertices of Γ are arranged on a circle in order and every edge of Γ is drawn as a straight line.

The equivalence of conditions (a), (b) and (c) of Theorem 4.3.2 immediately follows from the next two lemmas.

Lemma 4.3.3. *Let G be a labeled graph, let Γ be a standard drawing of G and let $s : \mathcal{M} \to \{-1, 1\}$. Then the following are equivalent:*

(1)there exists an orientation D of G such that for every $M \in \mathcal{M}$ its sign in the corresponding directed graph is equal to $s(M)$;

(2)there exists an s-marking S of Γ.

Proof. Let D be an orientation of G. Let $M = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be a perfect matching of D. The sign of M is the sign of the permutation

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_k & v_k \end{pmatrix}.$$

Let $i(P)$ denote the number of inversions in P. We have

$$\begin{aligned} sgn(M) = sgn(P) = (-1)^{i(P)} = \prod_{1\le i<j\le 2k} sgn(P(j) - P(i)) = \\ = \prod_{1\le i<j\le k} sgn((u_j - u_i)(v_j - u_i)(u_j - v_i)(v_j - v_i)) \times \\ \times \prod_{1\le i\le k} sgn(v_i - u_i). \end{aligned} \tag{7}$$

In Γ edges u_iv_i and u_jv_j cross if and only if, in the circle containing the vertices of Γ, each of the two arcs with ends u_i and v_i contains one of the vertices u_j and v_j, in other words if and only if

$$sgn((u_j - u_i)(v_j - u_i)(u_j - v_i)(v_j - v_i)) = -1.$$

Define $S_D = \{uv \in E(D) | u > v\}$. From (7) we deduce that

$$sgn(M) = (-1)^{cr(M)} \times (-1)^{|M\cap S_D|}.$$

It follows that $sgn(M) = s(M)$ in D if and only if S_D is an s-marking of Γ. □

Notice that we have in fact shown that there exists a one-to-one correspondence between Pfaffian orientations of a labeled graph and markings of its standard drawing.

Lemma 4.3.4. *Let G be a labeled graph and let $s : \mathcal{M} \to \{-1, 1\}$. Let Γ_1 and Γ_2 be two drawings of a labeled graph G in the plane. Then Γ_1 has an s-marking if and only if Γ_2 has one.*

Proof. Perfect matchings are T-joins in the graft $(G, V(G))$. Therefore by Theorem 4.1.1 there exists $S \subseteq Æ(G)$ such that for every $M \in \mathcal{M}$ we have

$$cr(M, \Gamma_1) = cr(M, \Gamma_2) + |M \sqcap S|.$$

modulo 2. Let $S' = S \cap E(G)$. As no perfect matching contains an angle we have

$$cr(M, \Gamma_1) = cr(M, \Gamma_2) + |M \cap S'|$$

modulo 2 for every $M \in \mathcal{M}$. Let S_1 be an s-marking of Γ_1. Then

$$s(M) = (-1)^{cr(M,\Gamma_1)+|M \cap S|} = (-1)^{cr(M,\Gamma_2)+|M \cap S'|+|M \cap S'|} = (-1)^{cr(M,\Gamma_2)+|M \cap (S' \triangle S_1)|}$$

for every $M \in \mathcal{M}$. Therefore $S' \triangle S_1$ is an s-marking of Γ_2. □

Since clearly (d) implies (b), to finish the proof of Theorem 4.3.2 it remains to show that (b) implies (d). Suppose G satisfies (b) and consider a drawing of G in the plane with an s-marking S. Suppose there exists $e \in S$. We change the way e is drawn, so that the closed curve C which is composed from the old and the new drawing of e separates one vertex of G from the rest. From the proof of Theorem 4.1.1 it follows that $S \setminus \{e\}$ is a marking in the new drawing. By repeating the procedure we produce a drawing of G such that the empty set is an s-marking, therefore demonstrating that G satisfies condition (d) of Theorem 4.3.2.

4.4 Applications to Crossing Numbers

We say that a set $\mathcal{J}$ of T-joins in a graft (G, T) is *nice* if every $x \in Æ(G)$ lies in an even number of elements of $\mathcal{J}$.

Lemma 4.4.1. *Let $\mathcal{J}$ be a nice set of T-joins in a graft (G, T). Then the parity of*

$$\sum_{J \in \mathcal{J}} cr(J, \Gamma) \tag{8}$$

is independent of the choice of a drawing Γ of G in the plane.

Proof. By Theorem 4.1.1 it suffices to prove that

$$\sum_{J \in \mathcal{J}} |J \sqcap S|$$

is even for any $S \subseteq Æ(G)$. This is true by the definition of a nice set of T-joins. $\square$

We derive the next theorem from Lemma 4.4.1.

Theorem 4.4.2. (Kleitman [23]) *Let $G = K_{2j+1}$ or $G = K_{2j+1,2k+1}$ for some positive integers j and k. Then the parity of the total number of crossings of non-adjacent edges is independent of the choice of a drawing of G in the plane.*

Proof. By Lemma 4.4.1 it suffices to find $T \subseteq V(G)$ and a nice set $\mathcal{J}$ of T-joins such that

$$|\{J \in \mathcal{J} | \{e, f\} \subseteq J\}|$$

is odd for every two non-adjacent edges e, f of G. (By the definition of a nice set, $|\{J \in \mathcal{J} | \{e, f\} \subseteq J\}|$ is even for every angle $\{e, f\}$.)

For $G = K_{2j+1,2k+1}$ we choose $T = \emptyset$ and we choose $\mathcal{J}$ to be the set of all cycles of length 4 in G.

For $G = K_{2j+1}$ the construction is slightly more complicated. Choose $v \in V(G)$ and let $T = V(G) \setminus \{v\}$. Let $\mathcal{J}_1$ be the set of all perfect matchings of $G \setminus \{v\}$. For distinct vertices $u_1, u_2 \in T$ let

$$J_{u_1 u_2} = \{vw | w \in T \setminus \{u_1, u_2\}\} \cup \{u_1 u_2\}$$

and let $\mathcal{J}_2 = \{J_{u_1 u_2} | \{u_1, u_2\} \subseteq T, u_1 \neq u_2\}$. Let $J_3 = \{vw | w \in T\}$. Finally, if j is odd let $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2$ and if j is even let $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \{J_3\}$.

In both cases by straightforward counting we can check that $\mathcal{J}$ is as required. $\square$

Kuratowki's theorem states that every non-planar graph has a subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$. One can therefore easily deduce the following well-known theorem from Theorem 4.4.2 and Kuratowski's theorem.

Theorem 4.4.3. (Hanani [17], Tutte [55]) *Let Γ be a drawing of a non-planar graph G in the plane. Then there exist distinct non-adjacent edges $e, f \in E(G)$ such that $cr(e, f)$ is odd.*

One of the oldest and the most widely known problems in crossing number theory is the problem of estimating the crossing number of the complete bipartite graph $K_{m,n}$ also

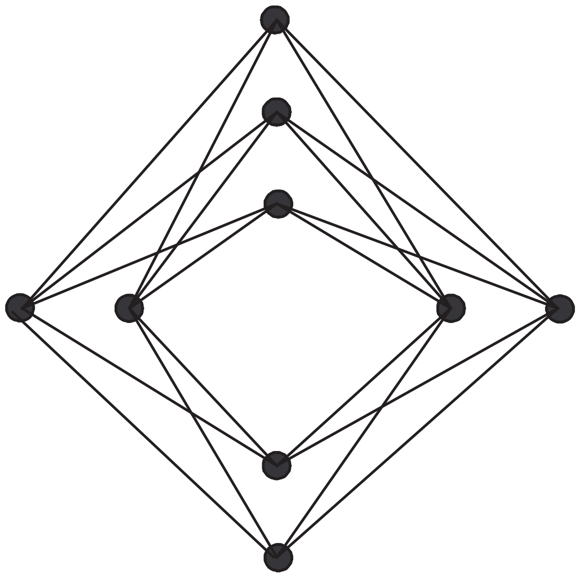


Figure 15: A Drawing of $K_{4,5}$ with $Z(4,5)$ crossings

known as Turán's brickyard problem [52]. For a graph G the *crossing number* $\mathrm{CR}(G)$ is equal to the minimum of $\sum_{\{e,f\}\subseteq E(G), e\neq f} cr(e,f)$ taken over all drawings of G in the plane. It has been long conjectured that $\mathrm{CR}(K_{m,n})$ equals the Zarankiewicz's number

$$Z(m,n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor.$$

This conjecture is known as Zarankiewicz's conjecture. A natural straight line drawing shows that $Z(m,n) \geq \mathrm{CR}(K_{m,n})$ for every m and n. An example for $m = 5$ and $n = 4$ is shown on Figure 15. The following best known bound for $\mathrm{CR}(K_{m,n})$ has been recently proved by de Klerk et al. [10].

Theorem 4.4.4. (i) *$lim_{n\longrightarrow\infty}\mathrm{CR}(K_{m,n})/Z(m,n) \geq 0.83m/(m-1)$ for each fixed $m \geq 9$;*

(ii) *$lim_{n\longrightarrow\infty}\mathrm{CR}(K_{n,n})/Z(n,n) \geq 0.83$.*

There are other possible ways to define the crossing number of a graph. We adopt the definition first implicit in the paper by Tutte [55] and formalized by Székely in [49]. For a graph G the *independent odd crossing number* $\mathrm{CR\text{-}IODD}(G)$ is equal to the minimum number of unordered pairs of non-adjacent edges that cross each other odd number of times taken over all drawings of G in the plane. Clearly $\mathrm{CR\text{-}IODD}(G) \leq \mathrm{CR}(G)$ for every G.

We will state a conjecture about hypergraphs that if true will imply that CR-IODD(G)$\geq$ $Z(m,n)$ for every $m,n \in \mathbb{Z}_+$ and therefore implies Zarankiewicz's conjecture. The purpose of such reformulation is to eliminate the geometrical aspect of the problem.

Let $A(m,n)$ be the set of angles in $K_{m,n}$. Let $\mathcal{G}(m,n)$ be a 4-uniform hypergraph with $V(\mathcal{G}(m,n)) = A(m,n)$ and $E(\mathcal{G}(m,n))$ equal to the set of the sets of angles lying in cycles of length four in $K_{m,n}$. We say that $C \subseteq E(\mathcal{G})$ is *a circuit* if it covers every vertex even number of times. Then the following lemma holds.

Lemma 4.4.5. *There exists a subhypergraph $\mathcal{G}'$ of $\mathcal{G}(m,n)$ with no odd circuits such that*

$$|E(\mathcal{G}')| \geq |E(\mathcal{G}(m,n))| - \text{CR-IODD}(K_{m,n}).$$

Proof. Let Γ_1 be a straight line drawing of $K_{m,n}$ in the plane with the vertices of parts of $K_{m,n}$ mapped to two parallel lines. Then $cr(C,\Gamma_1) = 1$ for every cycle C of length four in $K_{m,n}$. Let Γ_2 be the drawing of $K_{m,n}$ in the plane that achieves CR-IODD($K_{m,n}$). For an angle $A = \{e,f\} \in A(m,n)$ define $cr'(A) = cr_{\Gamma_2}(e,f)$ and let $\mathcal{A} = \{A \in A(m,n) \mid cr'(A)$ is odd$\}$. For a cycle $C = u_1v_1u_2v_2u_1$ in $K_{m,n}$ define

$$cr'(C) = cr_{\Gamma_2}(u_1v_1, u_2v_2) + cr_{\Gamma_2}(u_2v_1, u_1v_2).$$

Note that

$$cr(C,\Gamma_2) = cr'(C) + \sum_{A \in A(m,n),\ A \text{ lies in } C} cr'(A).$$

By Theorem 4.1.1 there exists $S \subseteq A(m,n)$ such that for every cycle C of length four

$$cr(C,\Gamma_2) = cr(C,\Gamma_1) + |C \sqcap S|$$

modulo 2. Let $S' = S \triangle \mathcal{A}$. Then the following identities hold modulo 2.

$$cr'(C) = cr(C,\Gamma_2) - \sum_{A \in A(m,n),\ A \text{ lies in } C} cr'(A) = cr(C,\Gamma_1) + |C \sqcap S| - |C \sqcap \mathcal{A}| =$$

$$= 1 + |C \sqcap S'|.$$

Let $\mathcal{C}$ be the set of all the cycles C in $K_{m,n}$ of length four with even $cr'(C)$. Consider the subhypergraph of $\mathcal{G}'$ of $\mathcal{G}(m,n)$ that has only the edges that correspond to the elements of $\mathcal{C}$. We have

$$|E(\mathcal{G}')| \geq |E(\mathcal{G}(m,n))| - \text{CR-IODD}(K_{m,n}),$$

as every cycle C of length four with odd $cr'(C)$ contains a pair of non-adjacent edges that cross odd number of times in Γ_2, and each pair of non-adjacent edges belongs to the unique cycle of length four in $K_{m,n}$. Therefore it only remains to show that $\mathcal{G}'$ has no odd circuits. Suppose the hyperedges corresponding to $C_1, C_2, \dots C_k$ form an odd circuit of $\mathcal{G}'$. Note that $cr'(C_i)$ is even and therefore $|C_i \sqcap S'|$ is odd for every $1 \leq i \leq k$. It follows that $\sum_{i=1}^{k} |C_i \sqcap S'|$ is odd. This is a contradiction as every element of S' lies in even number of cycles $C_1, C_2, \dots C_k$ by the definition of a circuit. □

Now we are ready to state our conjecture that implies Zarankiewicz's conjecture by Lemma 4.4.5. Note that $|E(\mathcal{G}(m,n))| = \frac{m(m-1)}{2}\frac{n(n-1)}{2}$.

Conjecture 4.4.6. *For every subhypergraph $\mathcal{G}'$ of $\mathcal{G}(m,n)$ with no odd circuits*

$$|E(\mathcal{G}')| \leq \frac{m(m-1)}{2}\frac{n(n-1)}{2} - \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor.$$

The last application of our method that we would like to demonstrate focuses on the crossing number of the Petersen graph P_{10}. It is well-known that $\text{CR}(P_{10}) = 2$.

Lemma 4.4.7. *The drawing Γ of the Petersen graph in the sphere that achieves $CR(P_{10}) =$ 2 is unique up to a homeomorphism of the sphere and an isomorphism of the Petersen graph.*

Proof. The Petersen graph P_{10} has 6 distinct perfect matchings. See Figure 16. We assume that the vertice s of P_{10} are labeled $v_1, \dots, v_{10}$ as shown on Figure 16. For $e, f \in E(P_{10})$ denote by $d(e,f)$ the length of the shortest (possibly trivial) path in P_{10} that joins an end of e to an end of f.

We find the following properties of the Petersen graph useful:

(i) every two distinct perfect matchings of the Petersen graph share exactly one edge edge;

(ii) $e, f \in E(P_{10})$ belong to a common perfect matching if and only if $d(e,f) = 1$;

(iii) if $e_1, f_1, e_2, f_2 \in E(P_{10})$ then there exists an isomorphism of the Petersen graph that maps e_1 to e_2 and f_1 to f_2 if and only if $d(e_1, f_1) = d(e_2, f_2)$;

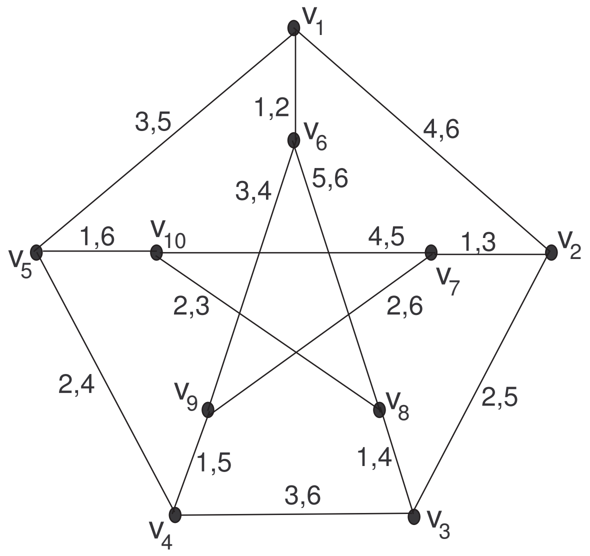


Figure 16: Six perfect matchings of the Petersen graph

(iv) $d(e, f) \leq 2$ for every $e, f \in E(P_{10})$;

(v) $P_{10} \setminus v$ is non-planar for every $v \in V(P_{10})$.

A standard and simple argument shows that in Γ no two adjacent edges cross (see for example [49]). Let $e_1, f_1, e_2, f_2 \in E(P_{10})$ be such that the image of e_i crosses the image f_i in Γ for $i = 1, 2$ and no other pair of edges of P_{10} cross in Γ. Then e_1, f_1, e_2 and f_2 form a matching in P_{10}. Indeed if two edges among e_1, f_1, e_2 and f_2 share a vertex v then Γ includes a drawing of $P_{10} \setminus v$ with no crossings in contradiction with (v). Moreover, by applying Lemma 4.4.1 to a nice family $\mathcal{M}(P_{10})$ and by (i) and (ii) we have

$$\sum_{\{e,f\} \subset E(P_{10}),\, d(e,f)=1} cr_\Gamma(e, f) = 1$$

modulo 2. It follows from (iii) and (iv) that we may assume that $e_1 = v_4v_9, f_1 = v_3v_8$ and $d(e_2, f_2) = 2$. These conditions also determine $\{e_2, f_2\}$ uniquely, as $G \setminus \{v_3, v_4, v_8, v_9\}$ is a cycle of length five with a pendant edge. Therefore we may assume $e_2 = v_1v_6, f_2 = v_7v_{10}$. Let G be an auxiliary graph constructed from P_{10} as follows: subdivide edges e_1, f_1, e_2 and f_2 once, identify the vertices obtain by subdividing e_i and f_i and denote the resulting vertex by w_i for $i = 1, 2$. Note that Γ can be considered as a drawing of G with no crossings if we map w_i to the point where the images of edges e_i and f_i cross in Γ. Note also that G

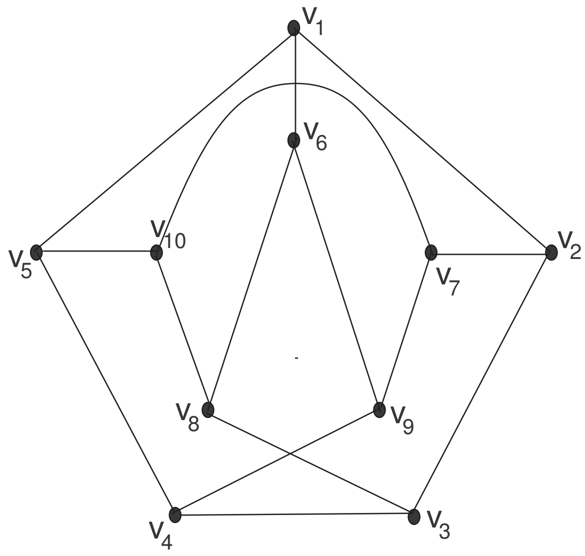


Figure 17: A drawing of the Petersen graph with two crossings

is 3-connected and therefore has a unique drawing in the sphere without crossings, up to s homeomorphism of the sphere, by a theorem of Whitney [58]. The theorem follows. A drawing of P_{10} with two crossings is shown on Figure 17. □

We say that a graph is *doublecross* if it can be drawn in the plane with two crossings in such a way that the two crossings belong to the same region. Doublecross graphs play important role in structural graph theory (see for example [43]). The following is an immediate corollary of Lemma 4.4.7.

Corollary 4.4.8. *The Petersen graph is not doublecross.*

CHAPTER V

DRAWING 4-PFAFFIAN GRAPHS ON THE TORUS

In this chapter we consider an extension of Theorem 4.3.1 to higher surfaces and k-Pfaffian graphs. We prove that 3-Pfaffian graphs are Pfaffian, 5-Pfaffian graphs are 4-Pfaffian and characterize 4-Pfaffian graphs in terms of their drawings on the torus. We prove partial results and state conjectures for higher surfaces and values of k.

The material presented in this chapter will also appear in [32].

5.1 Introduction

We now define drawings on surfaces. The definition is almost identical to the definition of drawings in the plane in Section 4.1. However, we find it convenient to allow self-intersections of edges. By a *drawing* Γ of a graph G on a surface S we mean an immersion of G in S such that edges are represented by locally homeomorphic images of $[0, 1]$, not containing vertices in their interiors. Edges are permitted to intersect, but there are only finitely many intersections and each intersection is a crossing. Let $cr_\Gamma(e, f)$ for distinct $e, f \in E(G)$ and $cr(M)$ for $M \in \mathcal{M}(G)$ be defined as in Section 4.1.

The main result of this chapter gives a characterization of 4-Pfaffian graphs, similar to the characterization of Pfaffian graphs given in Chapter 4.

Theorem 5.1.1. *A graph G is 4-Pfaffian if and only if there exists a drawing of G on the torus such that $cr(M)$ is even for every perfect matching M of G.*

In the next section we examine sequences of signs of perfect matchings in orientations of a k-Pfaffian graph. We prove that 3-Pfaffian graphs are Pfaffian and describe sequences of signs possible in 4-Pfaffian graphs. In Section 5.3 we prove that 5-Pfaffian graphs are 4-Pfaffian. Theorem 5.1.1 is proved in Section 5.4.

5.2 Admissible Sets of Sign Sequences

We say that a set $\mathcal{M}$ of $(1,-1)$-vectors of length k is *realizable* if there exists a graph G that is labeled k-Pfaffian, but not $(k-1)$-Pfaffian, orientations $D_1, D_2, \ldots, D_k$ of G and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$\mathcal{M} = \{(D_1(M), D_2(M), \ldots, D_k(M)) \mid M \text{ is a perfect matching of } G\}$$

and for every perfect matching M of G

$$\sum_{i=1}^{k} \alpha_i D_i(M) = 1.$$

We say that G *realizes* $\mathcal{M}$. Next we establish some conditions, which every realizable set of vectors has to satisfy.

Lemma 5.2.1. *Let G be a labeled graph, let k be an odd integer and let $S = (D_1, D_2, \ldots, D_k)$ be a sequence of orientations of G. Then there exists an orientation D_S of G such that*

$$D_S(M) = D_1(M)D_2(M)\ldots D_k(M).$$

Proof. Define D_S of G as follows. For every edge $uv \in E(G)$, let $uv \in E(D_S)$ if $|\{i \mid 1 \leq i \leq k, uv \in D_i\}|$ is odd and let $vu \in E(D_S)$ otherwise. Denote by S_i the set of edges on which D_S differs from D_i. We have

$$D_i(M) = (-1)^{|M \cap S_i|} D_S(M).$$

It follows that

$$D_1(M)D_2(M)\ldots D_k(M) = (-1)^{|M\cap S_1|+|M\cap S_2|+\ldots+|M\cap S_k|} D_S(M).$$

It remains to note that by definition of D_S

$$|E \cap S_1| + |E \cap S_2| + \ldots + |E \cap S_k|$$

is even for every $E \subseteq E(G)$. □

For a vector v of length k we denote its i-th coordinate by $v(i)$. We say that a set $\mathcal{V}$ of $(1,-1)$-vectors of length k is *admissible* if it satisfies the following properties

A1 for every odd $S \subseteq \{1, 2, \dots k\}$ there exist $v_1, v_2 \in \mathcal{V}$ such that for $i \in \{1, 2\}$ we have

$$\prod_{j \in S} v_i(j) = (-1)^i,$$

A2 for any set of real numbers $\{\beta_v\}_{v \in \mathcal{V}}$ such that $\sum_{v \in \mathcal{V}} \beta_v v$ is a zero vector, we have $\sum_{v \in \mathcal{V}} \beta_v = 0$.

Every realizable set $\mathcal{M}$ is admissible. Indeed, let G be a graph that realizes $\mathcal{M}$. Note that changing the orientation of all edges incident with any given vertex of G changes the sign of all perfect matchings of G. Therefore by Lemma 5.2.1 $\mathcal{M}$ has to satisfy **A1** as otherwise G is Pfaffian. The set $\mathcal{M}$ also satisfies condition **A2**, as it satisfies the following property

B2 there exist real numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ such that $\sum_{i=1}^{k} \alpha_i v(i) = 1$ for every $v \in \mathcal{V}$,

and by a standard linear algebra argument **A2** and **B2** are equivalent.

Note that **B2** also implies the following property, which we find useful to state separately.

A3 every two elements of $\mathcal{V}$ differ in at least two coordinates.

We say that sets $\mathcal{V}$ and $\mathcal{W}$ of $(1, -1)$-vectors of length k are *equivalent* if $\mathcal{W}$ can be obtained from $\mathcal{V}$ as follows: for some permutation π of the set $\{1, 2, \dots, k\}$ and some $S \subseteq \{1, 2, \dots, k\}$ apply π to the coordinates of all vectors in $\mathcal{V}$ and change the signs of all coordinates with indices in S for all vectors in $\mathcal{V}$. The above is clearly an equivalence relation. Trivially, if the sets $\mathcal{V}$ and $\mathcal{W}$ are equivalent then $\mathcal{V}$ is admissible (realizable) if and only if $\mathcal{W}$ is.

Lemma 5.2.2. *No set of* $(1, -1)$*-vectors of length two is admissible.*

Proof. Suppose $\mathcal{V}$ is an admissible set of $(1, -1)$-vectors of length two. Clearly $\mathcal{V}$ is equivalent to a set containing $(1, 1)$ and therefore without loss of generality we assume $(1, 1) \in \mathcal{V}$. By **A2** we know that $(-1, -1) \notin \mathcal{V}$ and therefore by **A1** applied to $S = \{1\}$ we have $(-1, 1) \in \mathcal{V}$ in contradiction with **A3**. $\square$

Lemma 5.2.3. *No set of* $(1, -1)$*-vectors of length three is admissible.*

Proof. Again without loss of generality we assume $(1,1,1) \in \mathcal{V}$. It implies by **A2** that $(-1,-1,-1) \notin \mathcal{V}$ and by **A1** applied to $S = \{1,2,3\}$ and equivalence we may assume $(1,1,-1) \in \mathcal{V}$ in contradiction with **A3**. □

The next theorem follows immediately from Lemmas 5.2.2 and 5.2.3 and the observations above.

Theorem 5.2.4. *Every* 3*-Pfaffian graph is Pfaffian.*

Next we examine admissible sets of sequences of length four. Denote $\{(-1,1,1,1)$, $(1,-1,1,1)$, $(1,1,-1,1)$,$(1,1,1,-1)\}$ by $\mathcal{S}$.

Lemma 5.2.5. *Every admissible set* $\mathcal{V}$ *of* $(1,-1)$*-vectors of length four is equivalent to* $\mathcal{S}$.

Proof. For a vector V of length four we denote $\sum_{i=1}^{4} v_i$ by $\sigma(V)$. Without loss of generality we assume $(1,1,1,1) \in \mathcal{V}$. By **A2** and **A3** we have $\sigma(V) \in \{-2,0,4\}$ for every $V \in \mathcal{V}$. Let n denote the number of elements $V \in \mathcal{V}$ with $\sigma(V) = -2$. By **A2** $n \leq 3$. We claim that $n = 0$.

Suppose not. If $n = 3$ without loss of generality we assume

$$(1,-1,-1,-1), (-1,1,-1,-1), (-1,-1,1,-1) \in \mathcal{V}.$$

By **A1** applied to $S = \{1,2,3\}$ we may assume $(-1,1,1,-1) \in \mathcal{V}$ in contradiction with **A3**. If $n = 2$ we assume $(1,-1,-1,-1)$, $(-1,1,-1,-1) \in \mathcal{V}$ and **A1** applied to $S = \{1,2,3\}$ and **A3** again lead to a contradiction. If $n = 1$ by equivalence, **A1** applied to $S = \{1,2,3\}$ and **A3** we may assume $(1,-1,-1,-1)$, $(-1,1,1,-1) \in \mathcal{V}$ and apply **A1** to $S = \{1,2,4\}$ for a contradiction.

Condition **A1** applied to all subsets of $\{1,2,3,4\}$ of size 3 implies $|\mathcal{V}| \geq 4$. By **A2** we know that for every $V_1, V_2 \in \mathcal{V}$ we have $V_1 + V_2 \neq 0$. Therefore up to equivalence $\mathcal{V} = \mathcal{V}_1$ or $\mathcal{V} = \mathcal{V}_2$, where

$$\mathcal{V}_1 = \{(1,1,1,1), (-1,1,1,-1), (1,-1,1,-1), (1,1,-1,-1)\},$$

and

$$\mathcal{V}_2 = \{(1,1,1,1), (1,1,-1,-1), (1,-1,1,-1), (1,-1,-1,1)\}.$$

Applying **A1** to $S = \{1\}$ we have $\mathcal{V} = \mathcal{V}_1$ and is equivalent to $\mathcal{S}$. $\square$

Before we can state the next lemma we need to strengthen our definition of admissibility. For a vector V of length k and $S \subseteq \{1, 2, \ldots k\}$ denote $\prod_{i \in S} V(i)$ by $V(S)$. We say that a set $\mathcal{V}$ of $(1, -1)$-vectors of length k is *strongly admissible* if it satisfies **B2** and

B1 For every odd $S_1, S_2, \ldots, S_{k-1} \subseteq \{1, 2, \ldots k\}$ and every real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ there exists $V \in \mathcal{V}$ such that

$$\sum_{i=1}^{k-1} \alpha_i V(S_i) \neq 1.$$

Every realizable set $\mathcal{M}$ is strongly admissible, as the negation of **B1** and Lemma 5.2.1 implies that every graph realizing $\mathcal{M}$ is $(k-1)$-Pfaffian. Note also that **B1** and implies **A1** and therefore every strongly admissible set of vectors is admissible.

Theorem 5.2.6. *No set of* $(1, -1)$*-vectors of length five is strongly admissible.*

Proof. The only argument we were able to find proceeds by exhaustive case analysis and is quite long. The proof will appear in a separate section. $\square$

The theorem below immediately follows from Theorem 5.2.6.

Theorem 5.2.7. *Every* 5*-Pfaffian graph is* 4*-Pfaffian.*

We now need to introduce some additional notation. Let V and W be $(1, -1)$-vectors of length m and n, respectively. We denote by $V \times W$ the vector of length mn defined by

$$(V \times W)((j-1)n + i) = V(i)W(j)$$

for all $1 \leq i \leq m, 1 \leq j \leq n$. For sets of $(1, -1)$-vectors $\mathcal{V}$ and $\mathcal{W}$ of length m and n correspondingly let $\mathcal{V} \otimes \mathcal{W} = \{V \times W \,|\, V \in \mathcal{V}, W \in \mathcal{W}\}$. We use the convention $\otimes^0 \mathcal{V} = \{(1)\}$ for any set $\mathcal{V}$ of $(1, -1)$-vectors.

Conjecture 5.2.8. *Let G be a labeled graph that is k-Pfaffian, but not $(k-1)$-Pfaffian, for some integer $k \geq 1$. Then $k = 4^g$ for some non-negative integer g and there exist orientations $D_1, D_2, \ldots, D_k$ of G such that for every perfect matching M of G*

$$(D_1(M), D_2(M), \ldots, D_k(M)) \in \otimes^g \mathcal{S}.$$

Note that the results of this section imply that Conjecture 5.2.8 holds for $k \leq 5$. Tardos [50] pointed out that there exists a strongly admissible set of $(1,-1)$-vectors of length six, namely the set of all vectors with exactly two negative coordinates. Therefore to prove Conjecture 5.2.8 one needs to use stronger properties of realizable sets then strong admissibility.

5.3 5-*Pfaffian Graphs are* 4-*Pfaffian*

In this section we give a proof of Theorem 5.2.6. Suppose $\mathcal{V}$ is a strongly admissible set of vectors of length five.

For a vector V of length five let $S(V) = \{i | V(i) = 1\}$ and let $\sigma(V) = |S(V)|$. We assume without loss of generality that $(1,1,1,1,1) \in \mathcal{V}$. By **A2** and **A3** we have $\sigma(V) \in \{1,2,3,5\}$ for every $V \in \mathcal{V}$. Let n_k denote the number of elements $V \in \mathcal{V}$ with $\sigma(V) = k$. We consider cases depending on n_1. Note that by **A2**, we have $n_1 \leq 4$.

Case 1: $\mathbf{n_1 = 4}$. We assume without loss of generality that

$$(1,-1,-1,-1,-1), (-1,1,-1,-1,-1), (-1,-1,1,-1,-1), (-1,-1,-1,1,-1) \in \mathcal{V}.$$

By **B2**, we have $V(1)+V(2)+V(3)+V(4)-3V(5) = 1$ for every $V \in \mathcal{V}$. Therefore $|\mathcal{V}| = 5$ and **A1** applied to $S = \{1,2,3,4,5\}$ yields a contradiction.

Case 2: $\mathbf{n_1 = 3}$. We assume that $(1,-1,-1,-1,-1)$, $(-1,1,-1,-1,-1)$, $(-1,-1,1,-1,-1) \in \mathcal{V}$. By **A1** applied to $S = \{1,2,3,4,5\}$ we have $n_2 > 0$ and therefore $(-1,-1,-1,1,1) \in \mathcal{V}$ by **A3**. We have

$$2(1,1,1,1,1) + (1,-1,-1,-1,-1) + (-1,-1,1,-1,-1) + (-1,-1,1,-1,-1) + \\ + (-1,-1,-1,1,1) = (0,0,0,0,0),$$

in contradiction with **A2**.

Case 3: $\mathbf{n_1 = 2}$. We assume that $(1,-1,-1,-1,-1)$, $(-1,1,-1,-1,-1) \in \mathcal{V}$. By **A1** applied to $S = \{1,2,3,4,5\}$ and **A3** without loss of generality we have $(-1,-1,1,1,-1) \in \mathcal{V}$.

By **A1** applied to $S = \{1,2,3\}$ there exists $W \in \mathcal{V}$ such that $S(W) \cap \{1,2,3\}$ is even. Suppose first $|S(W) \cap \{1,2,3\}| = 0$ then $W = \{-1,-1,-1,1,1\}$. It follows from **B2** that $V(1) + V(2) - 2V(3) + 3V(4) - 2V(5) = 1$ for every $V \in \mathcal{V}$. In particular $|S(V) \cap \{1,2,4\}|$ is odd for every $V \in \mathcal{V}$, in contradiction with **A1**.

Therefore $|S(W) \cap \{1,2,3\}| = 2$. It follows from **A3** that $\sigma(W) = 3$. We consider all possible choices for W up to the symmetry between the first and second coordinates.

W=(1,1,-1,1,-1) or W=(1,-1,1,-1,1): From **B2** we have $V(1) + V(2) + 2V(3) - V(4) - 2V(5) = 1$ for every $V \in \mathcal{V}$. Again it follows that $|S(V) \cap \{1,2,4\}|$ is odd for every $V \in \mathcal{V}$.

W=(1,1,-1,-1,1): (-1,-1,1,1,-1) and W contadict **A2**.

W=(1,-1,1,1,-1): (-1,-1,1,1,-1) and W contadict **A3**.

Case 4: $\mathbf{n_1 = 1}$. We assume that $(1,-1,-1,-1,-1) \in \mathcal{V}$. Note that by cases 1-3 we may assume that for every $V \in \mathcal{V}$ there exists at most one $W \in \mathcal{V}$ such that $|S(V) \triangle S(W)| = 4$. By **A1** applied to $S = \{1,2,3,4,5\}$ and **A3** we have $(-1,1,1,-1,-1) \in \mathcal{V}$ up to equivalence. We will proceed by considering subcases depending on n_3, but we would like to make a couple of observations first.

Note that if $W \in \mathcal{V}$, $\sigma(W) = 3$ then $1 \in S(W)$ by the observation above applied to $(1,-1,-1,-1,-1)$. Also $|S(W) \cap \{2,3\}| = 1$ by **A2** and **A3** applied to $(-1,1,1,-1,-1)$ and W. Moreover note that if $\mathcal{S}$ is a strongly admissible set and V is a $(1,-1)$-vector that lies in the affine space spanned by $\mathcal{S}$ then $\mathcal{S} \cup V$ is strongly admissible.

4.1: $\mathbf{n_3 \geq 3}$. We assume without loss of generality that $(1,1,-1,1,-1)$, $(1,1,-1,-1,1)$, $(1,-1,1,1,-1)$, $(1,-1,1,-1,1) \in \mathcal{V}$. Indeed, no other vector W with $\sigma(W) = 3$ can lie in $\mathcal{V}$ by an observation above, and these four vectors are affinely dependent: $(1,1,-1,1,-1) + (1,1,-1,-1,1) - (1,-1,1,1,-1) - (1,-1,1,-1,1) = (0,0,0,0,0)$. From **B2** we have $2V(1) + V(2) + V(3) - V(4) - V(5) = 2$ for every $V \in \mathcal{V}$. It follows that $|\mathcal{V}| = 7$. We have

$$\frac{1}{2}(V(1) + V(\{1,2,4\}) + V(\{1,2,5\}) - V(\{1,2,3\})) = 1$$

for every $V \in \mathcal{V}$ in contradiction with **B1**. Note that the set $\mathcal{V}$ is admissible.

4.2: $\mathbf{1 \le n_3 \le 2}$. We assume without loss of generality that $(1,-1,1,-1,1) \in \mathcal{V}$. If $(1,1,-1,-1,1) \in \mathcal{V}$ or $(1,-1,1,1,-1) \in \mathcal{V}$ then we again can conclude that $2V(1) + V(2) + V(3) - V(4) - V(5) = 2$ for every $V \in \mathcal{V}$ for a contradiction. By **A1** applied to $S = \{1,2,4\}$ there must exist $W \in \mathcal{V}$ with $\sigma(W) = 2$ such that $S(W) \cap \{1,2,4\}$ is even. If $S(W) \cap \{1,2,4\} = \emptyset$ then W and $(1,-1,1,-1,1)$ contradict **A2** and if $S(W) \subseteq \{1,2,4\}$ then W and $(1,-1,1,-1,1)$ contradict **A3**.

4.3: $\mathbf{n_3 = 0}$. Let $\mathcal{V}' = \{V \in \mathcal{V} | \sigma(V) = 2\}$. Note that $1 \notin S(W)$ for every $W \in \mathcal{V}'$ by **A3** applied to W and $(1,-1,-1,-1,-1)$. Also $S(W_1) \cap S(W_2) \neq \emptyset$ for every $W_1, W_2 \in V'$ by **A2** applied to $W_1, W_2, (1,-1,-1,-1,-1)$ and $(1,1,1,1,1)$. It follows that up to renumbering of the coordinates $\mathcal{V}$ is a subset of one of the following sets

$$\begin{aligned}\mathcal{V}_1 = \{&(1,1,1,1,1),(1,-1,-1,-1,-1),(-1,1,1,-1,-1),\\ &(-1,1,-1,1,-1),(-1,-1,1,1,-1)\}\end{aligned}$$

or

$$\begin{aligned}\mathcal{V}_2 = \{&(1,1,1,1,1),(1,-1,-1,-1,-1),(-1,1,1,-1,-1),\\ &(-1,1,-1,1,-1),(-1,1,-1,-1,1)\}.\end{aligned}$$

Moreover, $\mathcal{V}_1$ and $\mathcal{V}_2$ are equivalent. To verify that, consider changing signs of the last four coordinates of all vectors in $\mathcal{V}_1$. Therefore we assume $\mathcal{V} \subseteq \mathcal{V}_1$. Then

$$\frac{1}{2}(V(\{1,2,3\})) + V(\{1,2,4\} - V(\{1,2,5\}) - V(1)) = 1$$

for every $V \in \mathcal{V}$ in contradiction with **B1**.

Case 5: $\mathbf{n_1 = 0}$. Note that by the preceding cases we may assume that $|S(V) \triangle S(W)| \le 3$ for all $V, W \in \mathcal{V}$. By **A1** applied to $S = \{1,2,\ldots,5\}$ we assume without loss of generality that $(1,1,-1,-1,-1) \in \mathcal{V}$. Let $\mathcal{V}' = \{V \in \mathcal{V} \,|\, \sigma(V) = 2\}$ be defined as before. By the observation above either there exists $x \in \{1,2,\ldots,5\}$ such that $x \in W$ for every $W \in \mathcal{V}'$, or $n_2 = 3$ and there exists $S \subseteq \{1,2,\ldots,5\}$ such that $|S| = 3$ and $S(W) \subset S$ for every $W \in V'$. Suppose first that the second outcome holds. Without loss of generality $\mathcal{V}' = \{(1,1,-1,-1,-1),\ (1,-1,1,-1,-1),\ (-1,1,1,-1,-1)\}$. By **B1** there must exist

$U \in V$ such that $U(4) \neq U(5)$. It follows however that $\sigma(U) = 3$, $|S(U) \cap \{1,2,3\}| = 2$ and therefore there exists $W \in \mathcal{V}'$ such that $|S(W) \triangle S(U)| = 1$ in contradiction with **A3**.

We assume now without loss of generality that $1 \in W$ for every $W \in \mathcal{V}'$. By **A1** applied to $S = \{1\}$ there exists $U \in V$ such that $U(1) = -1$ and $\sigma(U) = 3$. Without loss of generality $U = \{-1, 1, 1, 1, -1\}$. By observations above $|S(W) \triangle S(U)| = 1$ for every $W \in V'$. By **A1** applied to $S = \{2,3,4\}$ there exists $T \in V$ such that $\sigma(T) = 3$ and $|S(T) \cap \{2,3,4\}| = 2$. Without loss of generality $T = (-1,1,1,-1,1)$, as $|S(T) \triangle S(U)| \leq 3$. It also follows that $n_3 = 2$. Indeed if $Z \in \mathcal{V}$, $Z \neq U, T$ and $\sigma(Z) = 3$ then $|S(Z) \cap \{2,3,4\}| = |S(T) \cap \{2,3,5\}| = 2$ and therefore $S(Z) = \{2,4,5\}$ or $S(Z) = \{1,2,3\}$ in contradiction with **A2** or **A3** respectively. By **A1** applied to $S = \{2\}$ we have $(1,-1,1,-1,-1) \in \mathcal{V}$. In fact it follows that

$$\mathcal{V} = \{(1,1,1,1,1), (1,1,-1,-1,-1), (1,-1,1,-1,-1), \\ (-1,1,1,1,-1), (-1,1,1,-1,1)\}.$$

But then $S(V) \cap \{1,4,5\}$ is odd for every $V \in \mathcal{V}$ in contradiction with **A1**. □

5.4 Drawing k-Pfaffian Graphs on Surfaces

The following theorem is the main result of this section.

Theorem 5.4.1. *For a labeled graph G and a non-negative integer g the following are equivalent*

1. *There exists a drawing of G on an orientable surface of genus g such that $cr(M)$ is even for every perfect matching M of G.*

2. *There exist orientations $D_0, D_1, \ldots, D_{4^g-1}$ of G such that for every perfect matching M of G*

$$(D_0(M), D_1(M), \ldots, D_{4^g-1}(M)) \in \otimes^g \mathcal{S}.$$

It is convenient to prove Theorem 5.4.1 in terms of special kinds of planar drawings. Let us from now on consider a plane with a fixed collection of g disjoint closed squares $S_1, S_2, \ldots, S_g$. We say that $S_1, S_2, \ldots, S_g$ are *singularities* and that a drawing of G in the

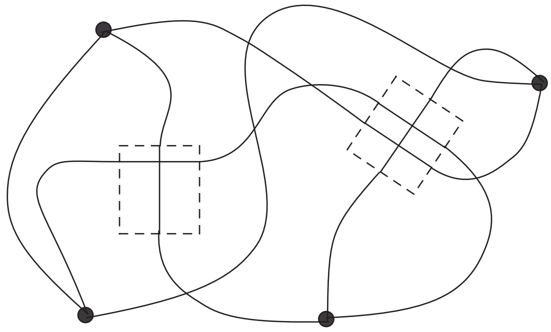


Figure 18: A 2-drawing of K_4

plane is a g-drawing if the images of all the vertices of G lie outside $S_1 \cup S_2 \cup \ldots \cup S_g$ and the images of the edges of G intersect each S_i in a finite number of straight line segments which are parallel to the sides of S_i. Figure 1 shows an example of a g-drawing.

For each singularity S_i fix one of its sides. For $e \in E(G)$ let e' be its image in Γ. Denote by $s_\Gamma(i,e)$ the number of segments in $e' \cap S_i$ parallel to the fixed side of S_i and by $s'_\Gamma(i,e)$ the number of segments in $e' \cap S_i$ perpendicular to this side. For $e, f \in E(G)$ let

$$cr'_\Gamma(e,f) = cr_\Gamma(e,f) - \sum_{i=1}^{g}(s_\Gamma(i,e)s'_\Gamma(i,f) + s'_\Gamma(i,e)s_\Gamma(i,f)).$$

In the notation introduced above we omit index Γ when the drawing is understood from context. Clearly for every drawing Γ of a graph G on an orientable surface of genus g there exists a g-drawing Γ' of a graph G in the plane such that $cr_\Gamma(e,f) = cr'_{\Gamma'}(e,f)$ for all $e, f \in E(G)$ and vice versa.

We say that $S \subseteq E(G)$ is a *marking* of a g-drawing Γ of G if $cr'_\Gamma(M)$ and $|M \cap S|$ have the same parity for every perfect matching M of G, where $cr'_\Gamma(M) = \sum_{\{e,f\} \subseteq M} cr'_\Gamma(e,f)$. Let L be a line in the plane and H one of the open half-planes determined by L such that all the singularities lie in H. We say that a g-drawing Γ of a labeled graph G is *standard* if the images of the vertices of G lie on L in order, and the images of the edges of G lie in $H \cup L$.

Lemma 5.4.2. *For a labeled graph G and a non-negative integer g the following are equivalent*

1. *There exists a standard g-drawing* Γ *of* G *and a marking* S *of* Γ.

2. *There exist orientations* $D_0, D_1, \dots, D_{4^g-1}$ *of* G *such that for every perfect matching* M *of* G
$$(D_0(M), D_1(M), \dots, D_{4^g-1}(M)) \in \otimes^g \mathcal{S}.$$

Proof. **(1)⇒(2).** Let $S' = \{e \in E(G) \mid \sum_{i=1}^{g} s(i,e)s'(i,e) \text{ is odd}\}$. For $i \in \{1,2,\dots,g\}$ let $E(i,0) = \emptyset$, $E(i,1) = \{e \in E(G) \mid s(i,e) \text{ is odd}\}$, $E(i,2) = \{e \in E(G) \mid s'(i,e) \text{ is odd}\}$ and let $E(i,3) = E(i,1) \triangle E(i,2)$. For an integer j let j_i denote the i-th digit from the right in a base two representation of j and let $j_* = \sum_{i=1}^{2g} j_{2i-1}j_{2i}$. For an orientation D let $\chi(D) = \{uv \in E(D) | u > v\}$. Note that χ is a bijection between orientations of G and subsets of $E(G)$. For $j \in \{0,1,\dots,4^g-1\}$ let
$$D'_j = \chi^{-1}(S \triangle S' \triangle E(1, j_1 + 2j_2) \triangle E(2, j_3 + 2j_4) \triangle \dots \triangle E(g, j_{2g-1} + 2j_{2g})).$$

We claim that $D_0, D_1, \dots, D_{4^g-1}$ satisfy (2). From the proof of Lemma 4.3.3 for an orientation D of G and a perfect matching M of G
$$(*) \qquad D(M) = (-1)^{cr(M) + |M \cap S(D)|}.$$

Let $s(i,M) = \sum_{e \in M} s(i,e)$ and $s'(i,M) = \sum_{e \in M} s'(i,e)$. For $j \in \{0,1,\dots,4^g-1\}$ the identities below hold modulo 2
$$\begin{aligned} cr(M) + |M \cap S(D'_j)| &= cr(M) + |M \cap S| + |M \cap S'| + \sum_{i=1}^{g} |M \cap E(i, j_{2i-1} + 2j_{2i})| = \\ &= (cr'(M) - cr(M)) + |M \cap S'| + \sum_{i:j_{2i-1}=1} s(i,M) + \sum_{i:j_{2i}=1} s'(i,M) = \\ &= \sum_{i=1}^{g} \sum_{\{e,f\} \subseteq M} (s(i,e)s'(i,f) + s'(i,e)s(i,f)) + \sum_{e \in M} s(i,e)s'(i,e) + \sum_{i:j_{2i-1}=1} s(i,M) + \\ &+ \sum_{i:j_{2i}=1} s'(i,M) = \sum_{i=1}^{g} (s(i,M))(s'(i,M)) + \sum_{i:j_{2i-1}=1} s(i,M) + \sum_{i:j_{2i}=1} s'(i,M) = \\ &= \sum_{i=1}^{g} (s(i,M) + j_{2i})(s'(i,M) + j_{2i-1}) + j_*. \end{aligned}$$

Therefore
$$D'_j(M) = (-1)^{j_*} \prod_{i=1}^{g} (-1)^{(s(i,M)+j_{2i})(s'(i,M)+j_{2i-1})}.$$

Let $D_j(M) = D'_j(M)$ if j_* is even and let $D_j(M)$ be obtained from $D'_j(M)$ by switching orientation of all the edges incident with vertex 1 if j_* is odd. Then

$$D_j(M) = \prod_{i=1}^{g}(-1)^{(s(i,M)+j_{2i})(s'(i,M)+j_{2i-1})}.$$

Let $v_0 = (1, 1, -1, 1)$, $v_1 = (1, -1, 1, 1)$, $v_2 = (-1, 1, 1, 1)$, and $v_3 = (1, 1, 1, -1)$. Note that for all $k \in \{0, 1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$ we have $v_k(j) = (-1)^{(j_1+k_1)(j_2+k_2)}$. Let $m(i) = r'_i + 2r_i$, where r_i and r'_i are the remainders modulo 2 of $s(i, M)$ and $s'(i, M)$ respectively. We claim that

$$(D_0(M), D_1(M), \ldots, D_{4^g-1}(M)) = v_{m(1)} \otimes v_{m(2)} \otimes \ldots \otimes v_{m(g)}.$$

Indeed

$$v_{m(1)} \otimes v_{m(2)} \otimes \ldots \otimes v_{m(g)}(j) = \prod_{i=1}^{g} v_{m(i)}(j_{2i-1} + 2j_{2i}) =$$
$$= \prod_{i=1}^{g}(-1)^{(r'_i+j_{2i-1})(r_i+j_{2i})} = \prod_{i=1}^{g}(-1)^{(s'(i,M)+j_{2i-1})(s(i,M)+j_{2i})} = D_j(M).$$

(2)⇒(1). Denote by A_j the set of edges of G in which D_j differs from D_0. Let Γ be a standard drawing of G such that for every $e \in E(G)$ such that $s(i, e)$ is odd if and only if $e \in A_{2^{2i-2}}$ and $s'(i, e)$ is odd if and only if $e \in A_{2^{2i-1}}$. Such a drawing is not difficult to construct. We use the notation introduced in the proof of (1)⇒(2) implication. Let $S = S(D_0)\triangle S'$. We claim that S is a marking of Γ, i.e. that $cr'(M) + |M \cap S|$ is even for every perfect matching M of G or by $(*)$ that $D_0(M) = (-1)^{cr'(M)-cr(M)+|M\cap S'|}$. Repeating part of the argument above we have modulo 2

$$cr'(M) - cr(M) + |M \cap S'| = \sum_{i=1}^{g}(s(i, M))(s'(i, M)) = \sum_{i=1}^{g}|M \cap A_{2^{2i-1}}||M \cap A_{2^{2i}}|.$$

Note that $|M \cap A_j|$ is even if and only if $D_0(M)D_j(M) = 1$. Let

$$(D_0(M), D_1(M), \ldots, D_{4^g-1}(M)) = w_1 \otimes w_2 \otimes \ldots \otimes w_g$$

for some $w_1, w_2, \ldots, w_g \in \mathcal{S}$. Then

$$D_0(M)D_{2^{2i-1}}(M) = w_i(1)w_i(2) \text{ and } D_0(M)D_{2^{2i}}(M) = w_i(1)w_i(3).$$

It follows that $|M \cap A_{2^{2i-1}}||M \cap A_{2^{2i}}|$ is odd if and only if $w_i(1) = -1$ as in every element of $\mathcal{S}$ at most one coordinate is negative. Therefore

$$(-1)^{cr'(M)-cr(M)+|M\cap S'|} = \prod_{i=1}^{g} w_i(1) = D_0(M). \quad \square$$

We say that g-drawings Γ_1 and Γ_2 are *similar* if every vertex of G has the same image in Γ_1 and Γ_2 and for every edge of G the symmetric difference of its images in Γ_1 and Γ_2 is a union of a family of closed simple curves in the plane none of which intersects a singularity or has a singularity in its interior. The proof of the following lemma is analogous to the proof of Lemma 4.3.4.

Lemma 5.4.3. *Let Γ_1 and Γ_2 be similar g-drawings of a labeled graph G. If there exists a marking of Γ_1 then there exists a marking of Γ_2.*

If there exists a marking of a g-drawing Γ of a labeled graph G then there exists a g-drawing Γ' of G similar to Γ such that $\emptyset$ is a marking of Γ'.

Let L be a line in the plane such that all the singularities lie in one of the open half planes determined by L. Clearly every g-drawing Γ of G can be transformed by some homeomorphism of the plane that is identical on the singularities to a g-drawing Γ', such that the images of the vertices of G in Γ' lie on L in order. Such Γ' is similar to some standard drawing. This observation and Lemma 5.4.2 imply Theorem 5.4.1. The next theorem extends Theorem 1.3.2.

Theorem 5.4.4. *Let G be a graph. If there exists a drawing of G on an orientable surface of genus g such that $cr(M)$ is even for every perfect matching M of G then G is 4^g-Pfaffian.*

Proof. By Theorem 5.4.1 there exist orientations $D_0, D_1, \ldots, D_{4^g-1}$ of G such that for every perfect matching M of G

$$(D_0(M), D_1(M), \ldots, D_{4^g-1}(M)) \in \otimes^g \mathcal{S}.$$

It is easy to verify that the sum of the coordinates of every element of $\otimes^g \mathcal{S}$ is 2^g. Therefore for every perfect matching M of G

$$\frac{1}{2^g} \sum_{i=0}^{4^g-1} D(i) = 1. \quad \square$$

Theorems 5.2.3, 5.4.1, 5.4.4 and Lemma 5.2.5 imply Theorem 5.1.1. By Theorems 5.4.1 and 5.4.4 Conjecture 5.2.8, implies the following conjecture.

Conjecture 5.4.5. *For a graph G and a non-negative integer g the following are equivalent*

1. *There exists a drawing of G on an orientable surface of genus g such that $cr(M)$ is even for every perfect matching M of G.*

2. *G is 4^g-Pfaffian.*

3. *G is $(4^{g+1} - 1)$-Pfaffian.*

CHAPTER VI

PFAFFIAN LABELINGS AND SIGNS OF EDGE COLORINGS

In this chapter we address a conjecture of Goddyn that every k-edge-colorable k-regular Pfaffian graph is k-list-edge-colorable. We prove Goddyn's conjecture for a slightly larger class of graphs that admit a Pfaffian labeling. Conversely, we prove that if a multigraph does not admit a Pfaffian labeling, then by adding parallel edges we can obtain from it a d-regular multigraph with two d-edge colorings of different signs.

We also give two descriptions of graphs that admit a Pfaffian labeling. The first one utilizes the theory developed in Chapter 2 and characterizes graphs with a Pfaffian labeling in terms of bricks and braces in their tight cut decomposition. The second one, in the spirit of Theorem 4.3.1, describes graphs with a Pfaffian labeling in terms of their drawings in the projective plane.

The material presented in this chapter will also appear in [38].

6.1 Introduction

In a k-regular graph G one can define an equivalence relation on k-edge colorings as follows. Let $c_1, c_2 : E(G) \to \{1, \dots, k\}$ be two k-edge colorings of G. For $v \in V(G)$ let $\pi_v : \{1, \dots, k\} \to \{1, \dots, k\}$ be a permutation such that $\pi_v(c_1(e)) = c_2(e)$ for every $e \in E(G)$ incident with v, and let $c_1 \sim c_2$ if $\prod_{v \in V(G)} sgn(\pi_v) = 1$. Obviously $\sim$ is an equivalence relation on the set of k-edge colorings of G and $\sim$ has at most two equivalence classes. We say that c_1 and c_2 *have the same sign* if $c_1 \sim c_2$ and we say that c_1 and c_2 *have opposite signs* otherwise.

A powerful algebraic technique developed by Alon and Tarsi [1] implies that if in a k-edge-colorable k-regular graph G all k-edge colorings have the same sign then G is k-list-edge-colorable. In [14] Ellingham and Goddyn prove the following theorem.

Theorem 6.1.1. *In a k-regular planar graph all k-edge colorings have the same sign. Therefore every k-edge-colorable k-regular planar graph is k-list-edge-colorable.*

Goddyn conjectured that Theorem 6.1.1 generalizes to Pfaffian graphs. The main goal of this chapter is to prove this conjecture and to describe k-edge-colorable graphs in which all k-edge colorings have the same sign.

6.2 Pfaffian Labelings and Signs of Edge Colorings

We generalize Pfaffian orientations to Pfaffian labelings and prove that Goddyn's conjecture holds for those graphs that admit a Pfaffian labeling. Let Γ be an Abelian multiplicative group, denote by 1 the identity of Γ and denote by -1 some element of order two in Γ. Let G be a graph with $V(G) = \{1, 2, \ldots, 2n\}$. (We are only interested in graphs that have a perfect matching, and hence an even number of vertices.) For a perfect matching $M = \{u_1v_1, u_2v_2, \ldots, u_nv_n\}$ of G, where $u_i < v_i$ for every $1 \le i \le n$, define

$$sgn(M) = sgn \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_n & v_n \end{pmatrix}.$$

We say that $l : E(G) \to \Gamma$ is a *Pfaffian labeling* of G if for every perfect matching M of G, $sgn(M) = \prod_{e \in M} l(e)$. We say that G admits a *Pfaffian Γ-labeling* if there exists a Pfaffian labeling $l : E(G) \to \Gamma$ of G. We say that G admits a *Pfaffian labeling* if G admits a Pfaffian Γ-labeling for some Γ. It is easy to see that a graph G admits a Pfaffian $\mathbb{Z}_2$-labeling if and only if G admits a Pfaffian orientation. Note also that the existence of Pfaffian labeling of a graph does not depend on the ordering of its vertices.

We need the following technical lemma.

Lemma 6.2.1. *Let X be a set and let $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m \subseteq X$, such that $|A_i \cap B_j| = 1$ for every $1 \le i \le n, 1 \le j \le m$, and every $x \in X$ belongs to exactly two of the sets $A_1, A_2, \ldots, A_n$ and exactly two of the sets $B_1, B_2, \ldots, B_m$. For every $1 \le i \le n$ let*

$$S_i = \{\{x, y\} \subseteq X |\, x, y \in A_i, x \in B_{i_1} \cap B_{i_3}, y \in B_{i_2} \cap B_{i_4} \;\; for\ some \\ i_1 < i_2 < i_3 < i_4\}.$$

Symmetrically for every $1 \le j \le m$ *let*

$$T_j = \{\{x, y\} \subseteq X |\ x, y \in B_j, x \in A_{j_1} \cap A_{j_3}, y \in A_{j_2} \cap A_{j_4} \ \textit{for some} \\ j_1 < j_2 < j_3 < j_4\}.$$

Then

$$\sum_{i=1}^{n} |S_i| = \sum_{j=1}^{m} |T_j|$$

modulo 2.

Proof. For $1 \le i \le n, 1 \le j \le m$ denote by x_{ij} the unique vertex of $A_i \cap B_j$. Let $Z = \{(a_1, b_1, a_2, b_2)|\ 1 \le a_1 < a_2 \le n, 1 \le b_2 < b_1 \le m, x_{a_1 b_1} \ne x_{a_2 b_2}\}$. Clearly $|Z| = n(n-1)m(m-1)/4 - |X|$ and $|X| = nm/4$. Moreover n and m are even, as $n = \sum_{i=1}^{n} |B_1 \cap A_i| = 2|B_1|$ and, similarly, $m = 2|A_1|$. Consequently $|Z|$ is even. For $\{u, v\} \subseteq X$ let $Z_{uv} = \{(a_1, b_1, a_2, b_2) \in Z|\ \{u, v\} = \{x_{a_1 b_1}, x_{a_2 b_2}\}\}$.

We claim that Z_{uv} is odd if and only if $\{u, v\}$ belongs to exactly one of $\triangle_{i=1}^{n} S_i$ and $\triangle_{j=1}^{m} T_j$. While simple case analysis can be used to verify this claim, we would like to demonstrate another proof. Draw a blue straight line between points $(0, i)$ and $(1, j)$ in $\mathbb{R}^2$ if $\{u\} = A_i \cap B_j$ and a red straight line if $\{v\} = A_i \cap B_j$. Then the resulting lines form blue and red closed curves, and as such they cross an even number of times. Note that $|Z_{uv}|$ is equal to the number of such crossings in $\mathbb{R}^2$ strictly between the lines $x = 0$ and $x = 1$; the number of times $\{u, v\}$ occurs in the sets $S_1, \ldots, S_n$ is equal the number of such crossings on the line $x = 0$ and the number of times $\{u, v\}$ occurs in the sets $T_1, \ldots, T_m$ is equal the number of crossings on the line $x = 1$. The claim follows.

From the claim, $\sum_{i=1}^{n} |S_i| + \sum_{j=1}^{m} |T_j| = \sum_{\{u,v\} \subseteq X} |Z_{uv}| = |Z| = 0$ modulo 2. □

Corollary 6.2.2. *Let* c_1 *and* c_2 *be two* k*-edge-colorings of a* k*-regular graph* G *and let* $V(G) = \{1, \ldots, 2n\}$. *Then* c_1 *and* c_2 *have the same sign if and only if* $\prod_{i=1}^{k} sgn(c_1^{-1}(i)) = \prod_{i=1}^{k} sgn(c_2^{-1}(i))$.

Proof. Define for $1 \le i \le 2k$, $A_i = c_1^{-1}(i)$ for $1 \le i \le k$ and $A_i = c_2^{-1}(i-k)$ for $k+1 \le i \le 2k$. Let B_j be the set of all edges incident with the vertex j for $1 \le j \le 2n$. Note that the sets $A_1, A_2, \ldots, A_{2k}, B_1, B_2, \ldots, B_{2n}$ satisfy the conditions of Lemma 6.2.1. Let S_i and T_j be defined as in Lemma 6.2.1. Note that $sgn(A_i)$ is equal to

$$(-1)^{|\{\{u,v\},\{u',v'\}\in A_i | u<u'<v<v'\}|} = (-1)^{|S_i|}.$$

On the other hand $sgn(\pi_j) = (-1)^{|T_j|}$, where π_j is as in the definition of sign of edge-colorings. The colorings c_1 and c_2 have the same sign if and only if $\prod_{j=1}^{2n} sgn(\pi_j) = 1$, but by Lemma 6.2.1

$$\prod_{j=1}^{2n} sgn(\pi_j) = \prod_{i=1}^{2k} sgnA_i = \prod_{i=1}^{k} sgn(c_1^{-1}(i)) \prod_{i=1}^{k} sgn(c_2^{-1}(i)). \quad \square$$

Theorem 6.2.3. *Let G be a k-regular graph, $V(G) = \{1, \ldots, 2n\}$. If G admits a Pfaffian labeling then all k-edge-colorings of G have the same sign.*

Proof. Let c_1 and c_2 be two k-edge-colorings of G. By Corollary 6.2.2 c_1 and c_2 have the same sign if and only if

$$\prod_{i=1}^{k} sgn(c_1^{-1}(i)) \prod_{i=1}^{k} sgn(c_2^{-1}(i)) = 1.$$

Let $l : E(G) \to \Gamma$ be a Pfaffian labeling of G for some Abelian group Γ. Then

$$\begin{aligned}\prod_{i=1}^{k} sgn(c_1^{-1}(i)) \prod_{i=1}^{k} sgn(c_2^{-1}(i)) &= \prod_{e\in E(G)} l(e) \times \prod_{e\in E(G)} l(e) = \\ &= (\prod_{i=1}^{k} sgn(c_1^{-1}(i))^2 = 1. \quad \square\end{aligned}$$

By Theorem 2.1 in [14], as well as Corollary 3.9 in [2], a k-regular graph is k-list-edge-colorable if the sum of signs of all of its k-edge colorings is non-zero. Therefore the following corollary of Theorem 6.2.3 holds.

Corollary 6.2.4. *Every k-edge-colorable k-regular graph that admits a Pfaffian labeling is k-list-edge-colorable.*

Next we will prove a partial converse of Theorem 6.2.3. We have to precede it by another technical lemma.

Lemma 6.2.5. *Let m and n be positive integers. Let A be an integer matrix with m rows and n columns and let $\mathbf{b}$ be a rational column vector of length m. Then either there exists a rational vector $\mathbf{x}$ of length n such that $A\mathbf{x}-\mathbf{b}$ is an integer vector, or there exists an integer vector $\mathbf{c}$, such that $\mathbf{c}A = \mathbf{0}$ and $\mathbf{c} \cdot \mathbf{b}$ is not an integer.*

Proof. There exists a unimodular integer $m \times m$ matrix $U = (u_{ij})$ such that $H = UA$ is in the Hermitian normal form (see for example [45]): if $H = (h_{ij})$ then there exist $1 \le k_1 < k_2 < \ldots < k_l \le n$, such that

1. $l \le m$,
2. $h_{ik_i} \neq 0$ for every $1 \le i \le l$,
3. $h_{ij} = 0$ for every $1 \le i \le l, 1 \le j < k_i$,
4. $h_{ij} = 0$ for every $l < i \le m, 1 \le j \le n$.

There exists $\mathbf{x} \in \mathbb{Q}^n$ such that first l coordinates of $H\mathbf{x} - U\mathbf{b}$ are zeros. Let $U\mathbf{b} = (d_j)_{1 \le j \le m}$. If $d_j \notin \mathbb{Z}$ for some $j > l$ then $\mathbf{c} = \{u_{j1}, u_{j2}, \ldots, u_{jm}\}$ is as required. If, on the other hand, $d_{l+1}, \ldots d_m \in \mathbb{Z}$ then $H\mathbf{x} - U\mathbf{b}$ is an integer vector and therefore so is $U^{-1}(H\mathbf{x} - U\mathbf{b}) = A\mathbf{x} - \mathbf{b}$. □

Theorem 6.2.6. *Let G be a graph with $V(G) = \{1, \ldots, 2l\}$. If G does not admit a Pfaffian labeling then there exist an integer k, a k-regular graph G' whose underlying simple graph is a subgraph of G and two k-edge colorings of G' of different signs.*

Proof. Let $\mathcal{M}$ denote the set of all perfect matchings of G and let Γ be the additive group $\mathbb{Q}/\mathbb{Z}$. The identity of Γ is 0 and the only other element of order two is 1/2. We will use the additive notation in this proof, instead of the multiplicative one we used before; in particular $sgn(M) \in \{0, 1/2\}$ for $M \in \mathcal{M}$. The graph G does not admit a Pfaffian Γ-labeling; i.e., there exists no function $l : E(G) \to \mathbb{Q}/\mathbb{Z}$ such that $\sum_{e \in M} l(e) = sgn(M)$ for every $M \in \mathcal{M}$. By Lemma 6.2.5 there exists a function $f : \mathcal{M} \to \mathbb{Z}$ such that $\sum_{M \ni e} f(M) = 0$ for every $e \in E(G)$ and $\sum_{M \in \mathcal{M}} f(M) sgn(M) = 1/2$. For every edge $e \in E(G)$ let $m(e) = 1/2 \cdot \sum_{M \ni e} |f(M)|$; then $m(e)$ is an integer. Let G' be the graph constructed from G by duplicating every edge $m(e) - 1$ times (if $m(e) = 0$ we delete e). Then G' is k-regular, where $k = 1/2 \cdot \sum_{M \in \mathcal{M}} |f(M)|$. Moreover, there exist a k-edge coloring c_1 of G' such that a perfect matching M appears as a color class of c_1 if and only if $f(M)$ is positive, in which case it appears $f(M)$ times. Similarly, there exist a k-edge coloring c_2 of G' such that a perfect matching M appears as a color class of c_2 if and only if $f(M)$ is negative, in which case it appears $|f(M)|$ times. Note that $\sum_{i=1}^{k} c_1^{-1}(i) + \sum_{i=1}^{k} c_2^{-1}(i) = \sum_{M \in \mathcal{M}} |f(M)| sgn(M) = \sum_{M \in \mathcal{M}} f(M) sgn(M) = 1/2$. Therefore c_1 and c_2 have different signs by Corollary 6.2.2. $\square$

6.3 Pfaffian Labelings and Tight Cut Decomposition

The previous section established a relation between graphs that admit a Pfaffian labeling and k-regular graphs in which all k-edge colorings have the same sign. This motivates the study of graphs that admit a Pfaffian labeling. In this section we use the matching decomposition procedure developed by Kotzig, and Lovász and Plummer [28], which we briefly review, for this purpose.

We say that a graph is *matching-covered* if every edge in it belongs to a perfect matching. Let G be a graph, and let $X \subseteq V(G)$. We use $\delta(X)$ to denote the set of edges with one end in X and the other in $V(G) - X$. A *cut* in G is any set of the form $\delta(X)$ for some $X \subseteq V(G)$. A cut C is *tight* if $|C \cap M| = 1$ for every perfect matching M in G. Every cut of the form $\delta(\{v\})$ is tight; those are called *trivial*, and all other tight cuts are called *nontrivial*. Let $\delta(X)$ be a nontrivial tight cut in a graph G, let G_1 be obtained from G by identifying all vertices in X into a single vertex and deleting all resulting parallel edges, and let G_2 be defined analogously by identifying all vertices in $V(G) - X$. We say that G *decomposes* along C into G_1 and G_2. By repeating this procedure any matching-covered graph can be decomposed into graphs with no non-trivial tight cuts. This motivates the study of the graphs that have no non-trivial tight cuts.

The graphs with no non-trivial tight cuts were characterized in [12, 29]. A *brick* is a 3-connected bicritical graph, where a graph G is *bicritical* if $G \backslash u \backslash v$ has a perfect matching for every two distinct vertices $u, v \in V(G)$. A *brace* is a connected bipartite graph such that every matching of size at most two is contained in a perfect matching.

Theorem 6.3.1. [12, 29] *A matching covered graph has no non-trivial tight cuts if and only if it is either a brick or a brace.*

Thus every matching covered graph G can be decomposed into a set $\mathcal{J}$ of bricks and braces. Lovász [29] proved that, up to isomorphism, the set $\mathcal{J}$ does not depend on the choice of tight cuts in the course of the decomposition. We say that the members of $\mathcal{J}$ are the *bricks* and *braces of* G.

The following lemma reduces the study of graphs with Pfaffian labelings to bricks and braces. Its analogue for Pfaffian orientations is due to Vazirani and Yannakakis [57].

Lemma 6.3.2. *Let* Γ *be a group. A matching-covered graph* G *admits a Pfaffian* Γ*-labeling if and only if each of its bricks and braces admits a Pfaffian* Γ*-labeling.*

Proof. Let $C = \delta(X)$ be a tight cut in G and let G_1 and G_2 be obtained from G by identifying vertices in X and $V(G) - X$ respectively. It suffices to prove that G admits a Pfaffian Γ-labeling if and only if both G_1 and G_2 admit a Pfaffian Γ-labeling. Without loss of generality, we assume that $V(G) = \{1, 2, \ldots, 2n\}, V(X) = \{1, 2, \ldots 2k+1\}$ and that G_1 and G_2 inherit the order on vertices from G; in particular, the vertex produced by identifying vertices of $V(G)-X$ has number $2k+2$ in G_1, the vertex produced by identifying vertices of X has number 1 in G_2. For every perfect matching M of G the sets of edges $M \cap E(G_1)$ and $M \cap E(G_2)$ are perfect matchings of G_1 and G_2 respectively. Moreover, $sgn(M) = sgn(M \cap E(G_1))sgn(M \cap E(G_2))$.

Suppose first that $l : E(G) \to \Gamma$ is a Pfaffian labeling of G. For every $e \in C$ fix a perfect matching $M_2(e)$ of G_2 containing e. Define $l_1(e) = sgn(M_2(e)) \prod_{f \in M_2(e)} l(f)$ for every $e \in C$ and define $l_1(e) = l(e)$ for every $e \in E(G_1) \setminus C$. For a perfect matching M of G_1 let $e \in C \cap M$. We have

$$\prod_{f \in M} l_1(f) = \prod_{f \in M \setminus \{e\}} l(f) \prod_{f \in M_2(e)} l(f)\, sgn(M_2(e)) =$$

$$= sgn(M \cup M_2(e))sgn(M_2(e)) = sgn(M).$$

Therefore $l_1 : E(G_1) \to \Gamma$ is a Pfaffian labeling of G_1.

Suppose now that $l_i : E(G_i) \to \Gamma$ is a Pfaffian labeling of G_i for $i \in \{1, 2\}$. Define $l(e) = l_i(e)$ for every $e \in E(G_i) \setminus C$ and define $l(e) = l_1(e)l_2(e)$ for every $e \in C$. It is easy to see that $l : E(G) \to \Gamma$ is a Pfaffian labeling of G. $\square$

For our analysis of Pfaffian labelings of bricks and braces we will need two theorems. The first of them is proved in [8] for bricks and in [28] for braces. It also follows from the results of Chapter 2.

Theorem 6.3.3. *Let G be a brick or brace different from K_2, C_4, K_4, the prism and the Petersen graph. Then there exists $e \in E(G)$ such that $G \setminus e$ is a matching covered graph with at most one brick in its brick decomposition and this brick is not the Petersen brick.*

For a graph G let the *matching lattice*, lat(G), be the set of all linear combinations with integer coefficients of the incidence vectors of perfect matchings of G. The next theorem of Lovász [29] gives a description of the matching lattice.

Theorem 6.3.4. *[29] If G has no brick isomorphic to the Petersen graph, then*

$$lat(G) = \{x \in \mathbb{Z}^{E(G)} \mid x(C) = x(D) \text{ for any two tight cuts } C \text{ and } D\}.$$

Lemma 6.3.5. *A brace or a brick not isomorphic to the Petersen graph admits a Pfaffian labeling if and only if it admits a Pfaffian orientation.*

Proof. By induction on $|E(G)|$. The base holds for K_2, C_4, K_4 and the prism as all those graphs admit a Pfaffian orientation.

For the induction step let $e \in E(G)$ be as in Theorem 6.3.3 and denote $G \setminus e$ by G'. The bricks and braces of G' satisfy the induction hypothesis and therefore by Lemma 6.3.2 either G' admits a Pfaffian orientation or G' does not admit a Pfaffian labeling. If G' does not admit a Pfaffian labeling then neither does G.

Therefore we can assume that G' admits a Pfaffian labeling $l : E(G') \to \mathbb{Z}_2$. It will be convenient to use additive notation for the group operation. Suppose l does not extend to a Pfaffian labeling of G. Then there exist perfect matchings M_1 and M_2 in G such that $e \in M_1 \cap M_2$ and $\sum_{f \in M_1 \setminus \{e\}} l(f) - \sum_{f \in M_2 \setminus \{e\}} l(f) \neq sgn(M_1) - sgn(M_2)$.

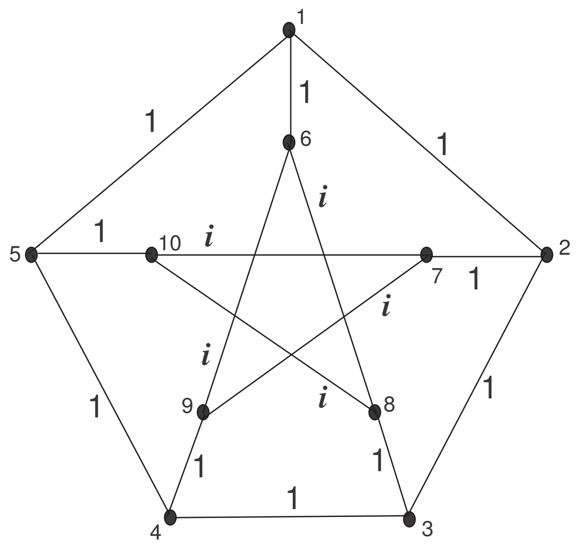


Figure 19: A μ_4-labeling of the Petersen graph

We claim that $|M_1 \cap C| = |M_2 \cap C|$ for any tight cut $C = \delta(X)$ in G'. Indeed, G' has at most one brick in its decomposition. Therefore we can assume that the graph G'' obtained from G' by identifying vertices in X is bipartite. It follows that $|M_1 \cap E(G'')| = |M_2 \cap E(G'')|$ and, consequently, that $|M_1 \cap C| = |M_2 \cap C|$.

By Theorem 6.3.4 we have $\chi_{M_1} - \chi_{M_2} = \sum_{M \in \mathcal{M}} c_M \chi_M$, where $\mathcal{M}$ denotes the set of perfect matchings of G' and c_M is an integer for every $M \in \mathcal{M}$. Therefore for every Pfaffian labeling $l' : E(G') \to \Gamma$ of G'

$$\sum_{M \in \mathcal{M}} c_M sgn(M) = \sum_{M \in \mathcal{M}} (c_M \sum_{f \in M} l'(f)) = \sum_{f \in M_1 \setminus \{e\}} l'(f) - \sum_{f \in M_2 \setminus \{e\}} l'(f).$$

But for $l' = l$ this expression is not congruent to $sgn(M_1) - sgn(M_2)$ modulo 2. It follows that $\sum_{f \in M_1 \setminus \{e\}} l'(f) - \sum_{f \in M_2 \setminus \{e\}} l'(f) \neq sgn(M_1) - sgn(M_2)$ for every Pfaffian labeling $l' : E(G') \to \Gamma$. Therefore no Pfaffian labeling of G' extends to a Pfaffian labeling of G, i.e. G does not admit a Pfaffian labeling. $\square$

Note that the Petersen graph admits a Pfaffian μ_4-labeling, where μ_n is the multiplicative group of nth roots of unity. Figure 1 shows an example of such labeling. Note that while the letter i was used for indexing above, from this point on it is used to denote a square root of -1.

The next theorem constitutes the main result of this section. It follows immediately from the observation above and Lemmas 6.3.2 and 6.3.5.

Theorem 6.3.6. *A graph G admits a Pfaffian labeling if and only if every brick and brace in its decomposition is either Pfaffian or isomorphic to the Petersen graph. If G admits a Pfaffian Γ-labeling for some Abelian group Γ then G admits a Pfaffian μ_4-labeling.*

6.4 Drawing Graphs with Pfaffian Labelings

By a *drawing* Φ of a graph G on a surface S we mean an immersion of G in S such that edges are represented by homeomorphic images of $[0, 1]$, not containing vertices in their interiors. Edges are permitted to intersect, but there are only finitely many intersections and each intersection is a crossing. For edges e, f of a graph G drawn on a surface S let $cr(e, f)$ denote the number of times the edges e and f cross. For a set $M \subseteq E(G)$ let $cr_\Phi(M)$, or $cr(M)$ if the drawing is understood from context, denote $\sum cr(e, f)$, where the sum is taken over all unordered pairs of distinct edges $e, f \in M$.

We use $sgn_D(M)$ to denote the sign of the perfect matching M in the directed labeled graph D. Note that it can differ from $sgn(M)$ defined in Section 6.2. The next lemma follows from the Theorem 4.3.2.

Lemma 6.4.1. *Let D be an orientation of a graph G and let $V(G) = \{1, 2, \ldots, 2n\}$. Then there exists a drawing Φ of G in the plane such that $sgn_D(M) = (-1)^{cr_\Phi(M)}$ for every perfect matching M of G. Moreover, for any $S \subseteq E(G)$ the drawing Φ can be chosen in such a way that there exists a point in the plane that belongs to the image of each edge in S and does not belong to the image of any other edge or vertex of G.*

Conversely, for any drawing Φ of G in the plane there exists an orientation D of G such that $sgn_D(M) = (-1)^{cr_\Phi(M)}$ for every perfect matching M of G.

For a point p and a drawing Φ of a graph G in the plane, such that Φ maps no vertex of G to p, let $cr_{p,\Phi}(e, f)$ denote the number of times the edges e and f cross at points other than p. For a perfect matching M of G let $cr_{p,\Phi}(M)$ denote $\sum cr_{p,\Phi}(e, f)$, where the sum is taken over all unordered pairs of distinct edges $e, f \in M$.

Lemma 6.4.2. *For a graph G the following are equivalent.*

1. *G admits a Pfaffian labeling,*

2. *There exists a point p and a drawing Φ of a graph G in the plane, such that Φ maps no vertex of G to p and $|M \cap S|$ and $cr_{p,\Phi}(M)$ are even for every perfect matching M of G, where $S \subseteq E(G)$ denotes the set of edges whose images contain p.*

Proof. We assume $V(G) = \{1, 2, \dots, 2n\}$.

(1) $\Rightarrow$ (2). By Theorem 6.3.6 there exists a Pfaffian μ_4-labeling $l : E(G) \to \{\pm 1, \pm i\}$ of G. Let D be the orientation of G such that $uv \in E(D)$ if and only if $u < v$ and $l(uv) \in \{1, i\}$, or $u > v$ and $l(uv) \in \{-1, -i\}$. Let $S = \{e \in E(G) \mid l(e) = \pm i\}$ and let $S' = \{e \in E(G) \mid l(e) \in \{-1, -i\}\}$. Note that $sgn_D(M) = (-1)^{|M \cap S'|} sgn(M)$ and $\prod_{e \in M} l(e) = (-1)^{|M \cap S'|} i^{|M \cap S|}$ for every perfect matching M of G.

By Lemma 6.4.1 there exist a point p and a drawing Φ of the graph G in the plane such that $sgn_D(M) = (-1)^{cr_\Phi(M)}$ for every perfect matching M of G, Φ maps no vertex of G to p, the images of the edges in S contain p and images of other edges do not contain p. Note that $\prod_{e \in M} l(e) \in \mathbb{R}$ for every perfect matching M and therefore $|M \cap S|$ is even. Denote $|M \cap S|/2$ by $z(M)$. We have $cr_{p,\Phi}(M) = cr_\Phi(M) + z(M)(2z(M) - 1)$. It follows that

$$\begin{aligned}(-1)^{cr_{p,\Phi}(M)} = sgn_D(M)(-1)^{z(M)} &= (-1)^{|M \cap S'|} i^{|M \cap S|} sgn(M) = \\ &= \prod_{e \in M} l(e) sgn(M) = 1.\end{aligned}$$

Therefore $cr_{p,\Phi}(M)$ is even.

(2) $\Rightarrow$ (1). By Lemma 6.4.1 there exists an orientation D of G such that $sgn_D(M) = (-1)^{cr_\Phi(M)}$ for every perfect matching M of G. For $uv \in E(G)$ with $u < v$ let $l_1(e) = 1$ if $uv \in E(D)$ and let $l_1(e) = -1$ otherwise; let $l_2(e) = i$ if $uv \in S$ and let $l_2(e) = 1$ otherwise. Finally, let $l(e) = l_1(e) l_2(e)$. One can verify that $l : E(G) \to \{\pm 1, \pm i\}$ is a Pfaffian labeling of G by reversing the argument used above. $\square$

We say that a region C of the projective plane is *a crosscap* if its boundary is a simple closed curve and its complement is a disc. We say that a drawing Φ of a graph G in the projective plane is *proper with respect to the crosscap C* if no vertex of G is mapped to C

and for every $e \in E(G)$ such that the image of e intersects C and every crosscap $C' \subseteq C$ the image of e intersects C'.

Now we can reformulate Lemma 6.4.2 in terms of drawings in the projective plane.

Theorem 6.4.3. *For a graph G the following are equivalent.*

1. *G admits a Pfaffian labeling,*

2. *There exists a crosscap C in the projective plane and a proper drawing Φ of G with respect to C, such that $|M \cap S|$ and $cr_\Phi(M)$ are even for every perfect matching M of G, where $S \subseteq E(G)$ denotes the set of edges whose images intersect C.*

CHAPTER VII

CONCLUDING REMARKS

In this chapter we discuss possible approaches to a structural characterization of Pfaffian graphs and a polynomial time recognition algorithm.

7.1 Even-faced embeddings in the Klein bottle

Let G be a graph embedded on a surface $\mathcal{S}$, which is obtained from a sphere by replacing k disjoint disks with Möbius strips. If $k = 1$ then $\mathcal{S}$ is the projective plane and if $k = 2$ then $\mathcal{S}$ is the Klein bottle. We say that a cycle C in G is *separating* if cutting $\mathcal{S}$ along C separates the surface, and we say that C is *non-separating* otherwise. Finally, we say that an embedding of G in $\mathcal{S}$ is *cross-cap-odd* if a non-separating cycle C in G is odd if and only if cutting $\mathcal{S}$ along C produces a surface with connected boundary.

Theorem 7.1.1. *Every graph that admits a cross-cap-odd embedding in the Klein bottle is Pfaffian.*

Proof. Let G be a graph and let Γ be a cross-cap-odd embedding of G in the Klein bottle. Without loss of generality, we assume that G is matching-covered and connected, and as such it is 2-connected. If G does not contain a non-separating cycle then G is planar, and hence Pfaffian by Theorem 1.2.1. Therefore we assume that G contains a non-separating cycle.

We claim that every separating cycle is even. We prove the claim by induction on $|E(G)|$. If G contains a vertex of degree two then the claim follows from induction hypothesis by considering the graph obtained from G by contracting one of the edges incident to such a vertex.

Therefore we assume that G has minimum degree three and fix a non-separating cycle C in G. By a standard "ear decomposition" argument (see for example [11, Proposition 3.1.1]), there exists $e = uv \in E(G) - E(C)$ such that $G \setminus e$ is 2-connected. We start by proving that there exists a non-separating cycle containing e in G. Let P_1 and P_2 be two vertex disjoint (possibly trivial) paths with ends u and u', and v and v' respectively, such that $u', v' \in V(C)$, and P_1 and P_2 are otherwise disjoint from C. The vertices u' and v' separate C into two paths Q_1 and Q_2. One of the cycles $P_1 \cup \{e\} \cup P_2 \cup Q_1$ and $P_1 \cup \{e\} \cup P_2 \cup Q_2$ is non-separating.

Suppose now that there exists an odd separating cycle in G. By induction hypothesis applied to $G \setminus e$ every such cycle contains e. We choose a separating cycle C' and a non-separating cycle C'', such that C' is odd, $e \in E(C') \cap E(C'')$ and subject to that $E(C') \cup E(C'')$ is minimal. If $C' \setminus C''$ is a path then the cycle D with edge set $E(C') \triangle E(C'')$ is non-separating and of the same homotopy type as C''. Therefore $|E(D)|$ and $|E(C'')|$ have the same parity, in contradiction with the parity of C'. If $C' \setminus C''$ is not a path then let P be a subpath of C'' with both ends in C' and otherwise disjoint from C'. Let P' be a subpath of C' with the same ends as P, such that $e \in P'$. By the choice of C and C' the cycle $D' = P \cup P'$ is non-separating and even. But then the cycle $D' \triangle C'$ is non-separating, odd and does not contain e, in contradiction with induction hypothesis. This finishes the proof of the claim.

Consider now the standard representation of the Klein bottle as a disk bounded by quadrilateral $ABCD$ with pairs of the quadrilateral's opposite sides identified as follows: AB with DC, and AD with CB. By bisubdividing edges of G if necessary we assume that every edge in $E(G)$ crosses the boundary of the quadrilateral at most once. Let $E_2, E_3 \subseteq E(G)$ be the sets of all edges of G that cross AB and AD, respectively, and let $E_1 = E(G) - E_2 - E_3$. Note that $(V(G), E_1 \cup E_2)$ is bipartite with bipartition (X, Y), and every edge of E_3 joins two vertices of X or two vertices of Y. We may extend Γ to a drawing Γ' of G in the plane such that for $e, f \in E(G)$ we have $cr(e, f) = 1$ if and only if $e \neq f$, $|\{e, f\} \cap E_3| \geq 1$ and $|\{e, f\} \cap E_1| = 0$, and we have $cr(e, f) = 0$, otherwise.

Let $k = (|X| - |Y|)/2$. Let E', E'' be the sets of all edges in E_3 joining two vertices of X and two vertices of Y, respectively. For a perfect matching M of G denote $|M \cap E'|$ by

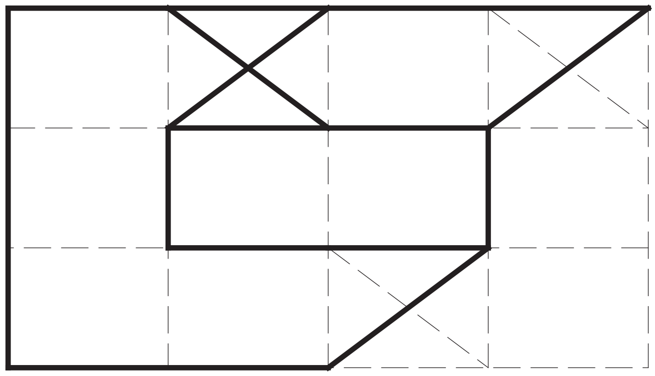


Figure 20: An cross-cap-odd embedding of $K_{3,3}$ on a surface with three "crosscaps".

n_M. We have $|M \cap E''| = n_M - k$ and $|M \cap E_3| = 2n_M - k$. Note that

$$cr_{\Gamma'}(M) = \frac{(2n_M + k)(2n_M + k - 1)}{2} + (2n_M + k)|M \cap E_2|.$$

We construct a Pfaffian marking S of Γ'. If k is even then $cr_{\Gamma'}(M) = n_M + k/2$ modulo 2 and therefore $S = E'$ is a Pfaffian marking of Γ' if k is divisible by four and $S = E' \triangle \delta(v)$ is a Pfaffian marking of Γ' for every $v \in V(G)$ otherwise. If k is odd then $cr_{\Gamma'}(M) = n_M + (k-1)/2 + |M \cap E_2|$ modulo 2 and $S = E' \cup E_2$ is a Pfaffian marking of Γ' if $k = 1$ modulo 4 and $S = (E' \cup E_2) \triangle \delta(v)$ is a Pfaffian marking of Γ' for every $v \in V(G)$ otherwise.

It follows from Theorem 4.3.2 that G is Pfaffian. $\square$

Note that Theorem 7.1.1 can not be extended to graphs that admit a cross-cap-odd embedding on surfaces of higher genus, as $K_{3,3}$ admits a cross-cap-odd embedding on a surface of Euler characteristic -1 (see Figure 20). Note also that non-bipartite graphs that admit an embedding in the projective plane with all faces even also admit a cross-cap-odd embedding in the Klein bottle and are therefore Pfaffian.

7.2 Matching width

For a cut C in a graph G let the *tightness* of C be defined as maximum of $|M \cap C|$ over all perfect matchings M of G. We say that a tree is *cubic* if the degree of every vertex in it is either one or three. A *matching tree decomposition* or an *MT-decomposition* of a graph G is a pair (T, W), where T is a cubic tree and $W = (W_t : t \in V(T))$, such that $\bigcup_{t \in V(T)} W_t = V(G)$, and $W_t \cap W_{t'} = \emptyset$ for every $t \neq t' \in V(T)$. For $e \in E(T)$ let the *order* of e be defined as the tightness of the cut $\delta(\bigcup_{t \in V(T_1)} W_t)$, where T_1 is a component of $T \setminus e$. Let the *adhesion* of an MT-decomposition be defined as the maximum order of an edge in it. For $t \in V(T)$ we say that the *bag* of (T, W) corresponding to t is the graph obtained from G by contracting $\bigcup_{t' \in T'} W_{t'}$ to a single vertex for every component T' of $T \setminus t$. Note that for every matching-covered graph G there exists an MT-decomposition (T, W) of G with adhesion one such that the bags of (T, W) are exactly the bricks and braces produced by the tight cut decomposition of G.

The *matching-width* of a graph G is the minimum integer k such that G admits an MT-decomposition (T, W) of adhesion at most k, in which $|W_t| = 1$ for every leaf $t \in V(T)$ and $|W_t| = 0$ for every non-leaf $t \in V(T)$.

Note that the family of dense Pfaffian bricks H_n defined in Section 1.3 has matching width 2. Let T be a cubic tree with $V(T) = \{t_1, t_2, \ldots, t_{n-2}, u_2, \ldots, u_{n-3}\}$ and edges $t_1 u_2, u_{n-3} t_{n-3}, u_{n-3} t_{n-2}$ and $u_i u_{i+1}, u_i t_i$ for every $2 \leq i \leq n-4$. Let $W_{t_i} = \{A_i, B_i\}$ for $1 \leq i \leq n-3$, let $W_{t_{n-2}} = \{A_{n-2}, A_{n-1}, A_n, B_{n-2}\}$ and let $W_{u_i} = \emptyset$ for $2 \leq i \leq n-3$.

We claim that the adhesion of (T, W) is two. It suffices to prove that the tightness of $\delta(X_i)$ is two for every $1 \leq i \leq n-3$, where $X_i = \{A_1, \ldots, A_i, B_1 \ldots B_i\}$. Clearly the tightness of $\delta(X_i)$ is at least two, as H_n is a brick and X_i is even. For every i all the edges crossing $\delta(X_i)$ either belong to the edge-set E_1 of the clique induced by $\{A_1, A_2, \ldots A_n\}$ or are incident with A_1. Every perfect matching contains at most one edge in E_1. It follows that the order of any edge in T is at most two. The claim follows.

A $k \times k$*-grid* is a planar graph with k^2 vertices, indexed with pairs of integers (i, j), such that $1 \leq i, j \leq k$ and a vertex (i_1, j_1) is joined to a vertex (i_2, j_2) if and only if $|i_1 - i_2| + |j_1 - j_2| = 1$.

Conjecture 7.2.1. *There exists a function $f : \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ such that every graph of matching-width at least $f(k)$ has $2k \times 2k$-grid as a matching minor.*

A similar result for the related concept of tree-width is known [40, 42] and a result for directed tree-width is conjectured in [18]. It is not hard to verify that the conjecture for directed tree-width in [18] would imply Conjecture 7.2.1 for bipartite graphs. We do not know whether the reverse implication holds.

Note that a standard uncrossing argument that is frequently used in similar problems can not be applied to prove Conjecture 7.2.1. It is essential in uncrossing arguments that the order $o(C)$ of a cut C be a submodular function, i.e.

$$o(\delta(X)) + o(\delta(Y)) \geq o(\delta(X \cap Y)) + o(\delta(X \cup Y)) \tag{9}$$

for every two subsets X and Y of the vertex set. Note that (9) does not necessarily hold if $o(C)$ denotes tightness of the cut C. An example follows.

Let G be a matching-covered graph and let $u, v \in V(G)$ be such that $G \setminus \{u, v\}$ consists of two even components G_1 and G_2. Let $X = V(G_1) \cup \{u\}$ and let $Y = V(G_1) \cup \{v\}$. Then $o(\delta(X)) = o(\delta(Y)) = 1$, $o(\delta(X \cap Y)) + o(\delta(X \cup Y)) = 2$ and (9) is violated. Similar examples can be constructed even if we further require that G is a brick.

It might be profitable to investigate Pfaffian bricks of "small" and "large" matching width separately. If Conjecture 7.2.1 holds then the techniques of Chapter 2 might help to obtain a structural characterization of Pfaffian bricks of "large" matching width. Known examples of Pfaffian bricks that do not adhere to surface-like behavior have "small" matching width, and so there is hope that graphs of "large" matching width have more structure.

In this section we present a polynomial time algorithm that produces a Pfaffian orientation or correctly identifies that a graph has no Pfaffian orientation for graphs of "small" matching width. In fact the algorithm that we present works for a larger class of graphs. The specifications follow.

Input: A graph G, an MT-decomposition (T, W) of G of adhesion at most k; for every $t \in V(T)$ a set $S_t \subseteq W_t$, $|S_t| \leq k$ such that every subgraph of $G[W_t] \setminus S_t$ is Pfaffian;

Output: A Pfaffian orientation of G, or a valid statement that G has no Pfaffian orientation;

Running time: Polynomial in $|V(G)|$ for fixed integer k.

Our method is influenced by the general purpose algorithm for graphs of bounded branch-width by Arnborg, Corneil and Proskurowski [3].

We repeatedly use the following result of Vazirani and Yannakakis [57, Theorem 3.1].

Theorem 7.2.2. *The problem of testing whether a graph has a Pfaffian orientation is polynomial-time equivalent to the problem of testing whether a given orientation of a graph is Pfaffian.*

In fact, the proof of Theorem 7.2.2 in [57] proves the following.

Theorem 7.2.3. *There exists a polynomial time algorithm that, given a Pfaffian graph G and an orientation D of G, tests whether D is Pfaffian.*

By Theorem 7.2.2 it suffices to test whether a given orientation D of G is Pfaffian. We assume that the vertices of G are ordered by a linear order $<$ and that G has a perfect matching. Let $E^- = \{uv \in E(D) \mid u > v\}$.

For two disjoint subsets V_1, V_2 of $V(G)$ let

$$I(V_1, V_2) = \{(v_1, v_2) | v_1 \in V_1, v_2 \in V_2, v_1 > v_2\}$$

and let $sgn(V_1, V_2) = (-1)^{|I(V_1,V_2)|}$. Recall that $V(e) = \{u, v\}$ for $e = uv \in E(G)$. For a matching M in G we define the *sign* of M, denoted by $sgn(M)$ as

$$(-1)^{|M \cap E^-|} \times \prod_{\{e_1,\, e_2\} \subseteq M,\, e_1 \neq e_2} sgn(V(e_1), V(e_2)).$$

Note that this definition is not limited to perfect matchings, and that for perfect matchings it is the same as the definition given in Section 1.3.

By possibly introducing new vertices to (T, W), we may assume that there exists a leaf $r \in V(T)$ with $W_r = \emptyset$. For every $t \in V(T) - \{r\}$ let $e_t \in E(T)$ be the edge incident with t in the unique path between t and r in T and let G_t denote $G[\bigcup_{t' \in T'} W_{t'}]$, where T' is the component of $T \setminus e$ containing t. For $t \in V(T) - \{r\}$ and $X \subseteq G_t$ let $f(X, t) = 1$ if $\mathcal{M}(G_t \setminus X) \neq \emptyset$ and $sgn(M) = 1$ for all $M \in \mathcal{M}(G_t \setminus X)$, let $f(X, t) = -1$ if $\mathcal{M}(G_t \setminus X) \neq \emptyset$ and $sgn(M) = -1$ for all $M \in \mathcal{M}(G_t \setminus X)$, and let $f(X, t) = 0$ otherwise.

We recursively compute

$$\mathcal{F}_t = \{(X, f(X,t)) \mid X \subseteq V(G_t), |X| \leq k\}$$

for $t \in V(T) - \{r\}$.

If $t \neq r$ is a leaf of T then we compute $\mathcal{F}_t$ as follows. Fix $X \subseteq V(G_t)$ such that $|X| \leq k$. It suffices to compute $f(X,t)$ in polynomial time. Let $G_{t,X} = G_t \setminus X$ and let $S_{t,X} = S_t \cap V(G_{t,X})$. Let $\mathcal{M}_{t,X}$ be the set of all matchings M in $G_{t,X}$, such that M covers all the vertices in $S_{t,X}$, no edge of M has both vertices in $V(G_{t,X}) - S_{t,X}$ and $V(G_{t,X}) - V(M)$ has a perfect matching. Note that $\mathcal{M}_{t,X}$ has polynomial size and can be computed in polynomial time.

For $M \in \mathcal{M}_{t,X}$ let $g(M,X,t) = 1$ if every perfect matching of $G_{t,X} \setminus V(M)$ has positive sign, let $g(M,X,t) = -1$ if every perfect matching of $G_{t,X} \setminus V(M)$ has negative sign and let $g(M,X,t) = 0$ otherwise. We can compute $g(M,X,t)$ for every $M \in \mathcal{M}_{t,X}$ in polynomial time by Theorem 7.2.3 as $G_{t,X} \setminus V(M)$ is Pfaffian. Note that $f(X,t) = c \neq 0$ if and only if $g(M,X,t)sgn(M)sgn(V(G_{t,X}) - V(M), V(M)) = c$ for every $M \in \mathcal{M}_{t,X}$. Indeed if a matching M is a disjoint union of matchings M_1 and M_2 then

$$sgn(M) = sgn(M_1)sgn(M_2)sgn(V(M_1), V(M_2)).$$

This finishes the computation of $f(X,t)$ in the base case.

Now we present the recursive step of our algorithm. Parts of it are similar to the base step described above. Let $t \in V(T)$ have degree three and let t_1 and t_2 be the two neighbors of t not incident with e_t. We compute $\mathcal{F}_t$ from $\mathcal{F}_{t_1}$ and $\mathcal{F}_{t_2}$. Again we fix $X \subseteq V(G_t)$ such that $|X| \leq k$ and compute $f(X,t)$. Let $G_{t,X}$ and $S_{t,X}$ be defined as above. Let $\mathcal{M}_{t,X}$ be the set of all matchings M in $G_{t,X}$ satisfying the following conditions:

1. M covers all the vertices in $S_{t,X}$,

2. every edge of M has at most one end in $V(G) - V(G_{t_1}) - V(G_{t_2})$ and at most one end in each of $V(G_{t_1})$ and $V(G_{t_2})$,

3. $|V(M) \cap V(G_{t_i})| \leq k$ for $i = 1, 2$,

4. $V(G_{t,X}) - V(M)$ has a perfect matching.

Again $\mathcal{M}_{t,X}$ has polynomial size and can be computed in polynomial time.

For $M \in \mathcal{M}_{t,X}$ and $i \in \{1,2\}$ denote $V(M) \cap V(G_{t_i})$ by V_i, let $V_3 = V(M)$ and let $V_4 = W_t - V(M)$. Let

$$s = sgn(M)f(V_1,t_1)f(V_2,t_2)\prod_{1\le i<j\le 4} sgn(V_i,V_j).$$

Finally, let $g(M,X,t) = c \neq 0$ if $sgn(M')s = c$ for every $M' \in \mathcal{M}(G[W_t - V(M)])$ and let $g(M,X,t) = 0$ otherwise. At this point we computed $g(M,X,t)$ for every $M \in \mathcal{M}_{t,X}$. Note that $g(M,X,t) = c \neq 0$ if and only if $sgn(M^*) = c$ for every $M^* \in \mathcal{M}(G_{t,X})$ such that $M \subseteq M^*$. Therefore $f(X,t) = c \neq 0$ if and only if $g(M,X,t) = 1$ for every $M \in \mathcal{M}_{t,X}$. This concludes the computation of $f(X,t)$.

To finish off the algorithm we compute $f(t_0,\emptyset)$, where $t_0 \in V(T)$ is the unique neighbor of r. As G has a perfect matching, $f(t_0,\emptyset) = 1$ if and only if D is Pfaffian.

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