LINEAR SYSTEMS ON METRIC GRAPHS AND SOME APPLICATIONS TO TROPICAL GEOMETRY AND NON-ARCHIMEDEAN GEOMETRY

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LINEAR SYSTEMS ON METRIC GRAPHS AND SOME APPLICATIONS TO TROPICAL GEOMETRY AND NON-ARCHIMEDEAN GEOMETRY

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To my daughter,

Sophia Zhouyao Luo,

with love.

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TABLE OF CONTENTS

LIST OF FIGURES

SUMMARY

The divisor theories on finite graphs and metric graphs were introduced systematically as an analogue to the divisor theory on algebraic curves, and these theories are deeply connected to each other via tropical geometry and non-archimedean geometry. In particular, rational functions, divisors and linear systems on algebraic curves can be specialized to those on finite graphs and metric graphs. Important results and interesting problems, including a graph-theoretic Riemann-Roch theorem, tropical proofs of conventional Brill-Noether theorem and Gieseker-Petri theorem, limit linear series on metrized complexes, and relations among moduli spaces of algebraic curves, nonarchimedean analytic curves, and metric graphs are discovered or under intense investigations. The content in this thesis is divided into three main subjects, all of which are based on my research and are essentially related to the divisor theory of linear systems on metric graphs and its application to tropical geometry and non-archimedean geometry. Chapter 1 gives an overview of the background and a general introduction of the main results. Chapter 2 is on the theory of rank-determining sets, which are subsets of a metric graph that can be used for the computation of the rank function. A general criterion is provided for rank-determining sets and certain specific examples of finite rank-determining sets are presented. Chapter 3 is on the subject of a tropical convexity theory on linear systems on metric graphs. In particular, the notion of general reduced divisors is introduced as the main tool used to study this tropical convexity theory. Chapter 4 is on the subject of smoothing of limit linear series of rank one on refined metrized complexes. A general criterion for smoothable limit g^1_d is presented and the relations between limit g^1_d and possible harmonic morphisms to genus 0 metrized complexes are studied.

CHAPTER I

INTRODUCTION

The divisor theory on finite graphs was introduced systematically as an analogue to the divisor theory on algebraic curves. Baker and Norine found a graph-theoretic version of the famous Riemann-Roch theorem on algebraic curves in their groundbreaking paper [13] as follows: if G is a graph of genus g and K the canonical divisor on G, then for all divisors $D \in \mathrm{Div}(G)$, $r(D) - r(K - D) = \deg(D) + 1 - g$ where $r(D)$ is the rank of D, which is conventionally the dimension of the linear system $|D|$ associated to D in the algebraic curve case. Such RR-type theorems have been extended to other combinatorial and geometric settings, such as weighted graphs [7], metric graphs/tropical curves [36, 54], lattices [8], and finite sets [45]. In addition, analogous notions of Jacobians, Abel-Jacobi maps and Picard groups have also been transplanted to combinatorial settings.

More than just a collateral theory, the divisor theory on graphs or metric graphs is actually deeply related to the divisor theory on curves, and these connections will benefit both sides [5, 6, 11, 15, 16, 20, 22–24, 28]: studying classical algebraic geometry problems using more combinatorial approaches and studying combinatorics using tools in algebraic geometry. For example, one connection is from the specialization map between rational functions (or equivalence classes) on curves and graphs. Let K be a non-Archimedean field with the valuation ring R and residue field k. For a complete nonsingular curve X/K together with a strongly semistable model $\mathfrak{X}$, consider the dual graph G of the special fiber of $\mathfrak{X}_s$ (when K is discretely valued) or a skeleton Γ of the Berkovich analytification X^{an} [15, 16, 21] (e.g. when K is algebraically closed). Then there exists a canonical specialization map τ_* from divisors on X to divisors on G (or divisors on Γ) which respects linear equivalence, and by Baker's specialization lemma [11, 15, 57], the rank of a divisor D (the dimension of $|D|$) on X is less than or equal to the rank of $\tau_*(D)$ on G (or on Γ). Based on such connections, recently people have obtained tropical proofs of the Brill-Noether theorem [28] and Gieseker-Petri theorem [20, 46]. In addition, the lifting problems on the other hand also attract lots of research efforts in recent years [5, 6, 17, 25], as well as the study of relations among moduli spaces of algebraic curves, nonarchimedean analytic curves, and metric graphs/tropical curves [1].

This thesis is mainly focused on three subjects I've worked on which are essentially related to linear systems on metric graphs and its application: (1) rank-determining sets of metric graphs [51], which provide algorithms to actually compute the rank function of arbitrary divisors on an arbitrary metric graph, (2) a tropical convexity theory for linear systems on metric graphs [52], and (3) smoothing of limit linear series of rank one on refined metrized complexes (an intermediate object between metric graphs and algebraic curves [4]), which is a lifting problem. In the following, each subject is expanded with main results listed.

1.1 Some basic notions

There are several very basic notions that will be mentioned throughout this thesis. Here we list some of them with conventional notations.

G**:** a connected, finite **graph** with vertex set $V(G)$ and edge set $E(G)$ (multiple edges allowed) with genus $g(G)$ (its first Betti number). A finite graph G also arises as the dual graph of a semistable algebraic curve (a reduced algebraic curve whose singular points are ordinary double points) in the following way: each irreducible component corresponds to a vertex of G, and each double point corresponds to an edge of G connecting vertices representing the two irreducible components containing the double point.

Γ**:** a (compact) **metric graph**, which can be considered as a geometric realization of a finite graph G by identifying each edge with a real interval. G is called a model of Γ. If we allow some edges incident with vertices of valence 1 (leaves) to have infinite length, then Γ can also be considered as an abstract **tropical curve** (a tropicalization of some algebraic curve over a field with non-archimedean valuation). Moreover, a skeleton of a **Berkovich analytic curve** is also a metric graph (refer to [15,16] for a framework relating non-archimedean analytification and tropicalization).

$\mathrm{Div}(G)$**:** the **divisor group** of G which is the free abelian group generated by $V(G)$. The elements D of $\mathrm{Div}(G)$ are called **divisors** on G. For $\sum_{v\in V(G)} D(v)\cdot(v)$, the degree $\deg(D)$ of D is $\sum_{v\in V(G)} D(v)$, and we say D is **effective** (written as $D \geqslant 0$) if $D(v) \geqslant 0$ for all $v \in V(G)$. We let $\mathrm{Div}_+(G) = \{D \in \mathrm{Div}(G) : D \geqslant 0\}$, $\mathrm{Div}^{\mathrm{d}}(G) = \{D \in \mathrm{Div}(G) : \deg(D) = d\}$, and $\mathrm{Div}_+^{\mathrm{d}}(G) = \{D \in \mathrm{Div}(G) : \deg(D) = d\}$. The canonical divisor is defined by $K = \sum_{v\in V(G)}(\deg(v) - 2)(v)$ which has degree $2g - 2$.

$\mathcal{M}(G)$**:** the space of all integer-valued functions (called **rational functions**) on $V(G)$. For $f \in \mathcal{M}(G)$, and define the Laplacian $\Delta : \mathcal{M}(G) \to \mathrm{Div}(G)$ by $\Delta f = \sum_{v\in V(G)} \mathrm{ord}_v(f)(v)$, where $\mathrm{ord}_v(f) = \sum_{\{v,w\}\in E(G)}(f(v) - f(w))(v)$. We call Δf the divisor associated to f (such divisors are also called **principal divisors**) and denote by $\mathrm{Prin}(G)$ the space of all principal divisors. We say two divisor D_1 and D_2 are linearly equivalent ($D_1 \sim D_2$) if they differ by a principal divisor.

$\mathrm{Div}(\Gamma)$**:** the **divisor group** of Γ which is the free abelian group generated by the points of Γ. We also have notions of degree of a divisor, effectiveness, rational functions, principal divisors, and linear equivalence defined in a similar way.

In particular, a rational function f on Γ is a continuous, piecewise linear real-valued function with integer slopes, and $\Delta f = \sum_{p\in\Gamma} \mathrm{ord}_p(f)(p)$ where $-\mathrm{ord}_p(f)$ is the sum of outgoing slopes of f at point p. Moreover, $\mathrm{supp}(D) = \{p \in \Gamma : D(v) \neq 0\}$.

$|D|$: A (complete) **linear system** (or **linear series**) on G or Γ which is defined as the set of all effective divisors linearly equivalent to D.

$r(D)$: the Baker-Norine [13] **rank** function of $D \in \mathrm{Div}(G)$ (or $D \in \mathrm{Div}(\Gamma)$ for metric-graph case), which is defined as the maximum value d such that for every $E \in \mathrm{Div}_+^{\mathrm{d}}(G)$ (respectively, $E \in \mathrm{Div}_+^{\mathrm{d}}(\Gamma)$), $|D - E| \neq \emptyset$.

1.2 Rank-determining sets of metric graphs

One may directly associate a metric graph Γ to a finite graph by requiring all edges to have the same length and a natural question is whether a divisor $D \in \mathrm{Div}(G)$ has the same rank when computed as a divisor on the associated metric graph Γ. More generally, the edge lengths of a metric graph Γ arising as a Berkovich skeleton can be arbitrary. Even though we define the rank functions on divisor groups $\mathrm{Div}(G)$ and $\mathrm{Div}(\Gamma)$ in the same way, there is a huge difference in practice that $\mathrm{Div}(G)$ is finitely generated while $\mathrm{Div}(\Gamma)$ is not, and one needs to verify infinitely many cases before being able to determine the rank of a divisor in $\mathrm{Div}(\Gamma)$. This problem is completely solved in Chapter 2 by introducing a notion of rank-determining sets which is a subset of Γ that can be finite [51]. Therefore, one only needs to verify finitely many cases to compute the rank of a divisor in $\mathrm{Div}(\Gamma)$. In particular, a full criterion for determining whether a subset of Γ is rank-determining is provided.

More precisely, **rank-determining sets** are defined in the following way: a subset A of Γ is rank-determining if for all $D \in \mathrm{Div}(\Gamma)$, $r(D)$ is equal to the maximum value d such that for every $E \in \mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ with $\mathrm{supp}(E) \subseteq A$, $|D - E| \neq \emptyset$. Therefore if there exists a finite rank-determining set, then we can use it to compute the rank of all divisors in $\mathrm{Div}(\Gamma)$.

Instead of stating the complete but technical criterion for rank-determining sets (stated in Theorem 2.3.17), here are some corollaries of the main criterion that are more useful in applications.

Theorem 1.2.1. *Any vertex set of a metric graph Γ is a rank-determining set of Γ.*

The following theorem was first proved in the preprint version of [44], but directly follows from the theory of rank-determining sets.

Theorem 1.2.2. *Let Γ be the metric graph corresponding to a graph G. Let D be a divisor on G. Let $r_G(D)$ be the rank of D on G, and $r_\Gamma(D)$ the rank of D on Γ. Then we have $r_G(D) = r_\Gamma(D)$.*

We may also define a similar notion of rank-determining sets on algebraic curves. In particular a subset of a curve of genus g is rank-determining if and only if it has cardinality at least $g+1$ (cf. Theorem 2.1.8). The following is a similar theorem for metric graphs.

Theorem 1.2.3. *For every metric graph Γ, there always exist finite rank-determining sets of with cardinality $g+1$.*

By definition, the rank function actually depends on the metric on Γ. However, a little surprisingly, rank-determining sets are not as stated in the following theorem.

Theorem 1.2.4. *Rank-determining sets are topological (preserved under homeomorphisms).*

The theory of rank-determining sets has been applied in the work of the tropical proof of a classical algebraic geometry problem, e.g., the Brill-Noether theorem [28]. They found a Brill-Noether general metric graph Γ (as a degeneration of a Brill-Noether general curve) by computing explicitly the rank function on all divisors.

Some other applications of rank-determining sets also directly use the fact that there exist finite ones [3, 4, 50]. In addition, the notion of rank-determining sets has been extended to metrized complexes [4] (also see Chapter 4).

1.3 Tropical convexity and general reduced divisors on linear systems

Recall that a complete linear system (or linear series) $|D|$ on an algebraic curve X is the space of global sections of a line bundle on X, and $|D|$ is just a projective space [43] with its linear subspaces called linear systems. On the other hand, a complete linear system on a finite graph G is a finite set. A complete linear system on a metric graph is actually a polyhedral complex without a pure dimension in general, which as a geometric object itself is the most complicated among the three.

There are lots of signs showing that specialization of a complete linear system on a curve X to a metric graph/tropical curve Γ does not result in a complete linear system on Γ. Natural questions are what should be the right notion of linear (sub)systems on Γ (subspaces of a complete linear system $|D|$), and what kind of rank function can be associated to those subsystems. More refined investigations of the linear systems on metric graphs are essential to answer what is actually happening during the specialization of linear systems from curves to metric graphs.

Tropical convexity is a convexity theory built on tropical semirings and semimodules [2, 29, 47, 48]. Conventionally, tropical convexity is studied on Euclidean spaces or tropical projective spaces (a Euclidean space modulo tropical translation). In particular, people also studied complete linear systems $|D|$ on metric graphs/tropical curves utilizing some tropical convexity techniques [39].

Indeed, linear systems on metric graphs have richer geometric properties and we have developed a geometric foundation for the notion of tropical convexity in the space of all divisors [52]. In particular, there is a canonical metric structure on the space of divisors with associated topological and geometric properties. The notion of tropical convexity is intrinsically built on this metric structure, and the linear systems $|D|$ are path-connected components which are finitely generated tropical convex hulls (also called tropical polytopes). Moreover, the sub-linear systems of $|D|$ are defined to be finitely generated tropical subconvex hulls of $|D|$.

The metric function on $|D|$ is defined in the following way. For D_1 and D_2 in $|D|$, since they are linearly equivalent, there exists a rational function f (unique up to translation by constants) such that $D_1 - D_2 = \Delta f$. The distance between D_1 and D_2 is defined to be $\max(f) - \min(f)$. This metric function is well-defined and induces a topology on $|D|$. Moreover, roughly speaking, the level sets of f also define a tropical segment connecting D_1 and D_2. Therefore, we can extend the notion of tropical segment to tropical convexity in a natural way as follows: a subset T of $|D|$ is **tropically convex** if the tropical segment connecting any two elements in T also lies in T.

The following theorems are some fundamental properties about tropical convexity.

Theorem 1.3.1. *Tropical convex sets are contractible.*

Theorem 1.3.2. *Every finitely generated tropical convex hull is compact.*

Simple as the above theorems appear, the proofs are actually technical and need to utilize a notion called "general reduced divisors" (or equivalently "canonical projections").

The notion of reduced divisors (under different names, e.g., critical configurations and G-parking functions) and a related notion of chip-firing games were originally introduced in a self-organized sandpile model on grids and then on arbitrary graphs [27,30,56], and have aroused interest in various fields of research (see the short survey article [49]) including combinatorics, theoretical physics, and arithmetic geometry. In particular, they are the fundamental tool employed by Baker and Norine [13] to prove the graph-theoretical Riemann-Roch theorem and they appear in lots of subsequent works by different authors including the tropical proof of Brill-Noether theorem [28].

In the context of metric graphs, reduced divisors arise in the following way: for any linear system $|D|$ (or more generally, any linear equivalence class of divisors) and any point p on the metric graph, there is a canonical divisor in $|D|$ which is "reduced" with respect to p.

There are several equivalent ways [54,56] to characterize reduced divisors. Recently, Baker and Shokrieh made a connection to potential theory on (metric) graphs [19]. The main tool in their theory is called the energy pairing, and for a fixed $q \in \Gamma$, it can be used to define two functions on the divisor group, the energy function $\mathcal{E}_q$ and the b-function b_q. Then the reduced divisor in $|D|$ with respect to q is the minimizer of either $\mathcal{E}_q$ or b_q. We generalize Baker-Shokrieh b-function to a function $\mathcal{B}(D_1, D_2)$ which can be consider as a pseudo-metric between two divisors D_1 and D_2 in $|D|$.

It is worth mentioning that the aforementioned conventional notion of reduced divisor is defined for a complete linear system $|D|$ and a specific point p on the metric graph which has to be specified first. In Chapter 3, the notion of reduced divisors is generalized in the following sense:

1. Reduced divisors exist not only just for complete linear systems $|D|$ but also for tropically convex sub-linear systems T of $|D|$.

2. Reduced divisors can be defined not only with respect to a point p on the metric graph but also any divisor E of the same degree as D. In particular, if $\deg(D) = d$, then the general reduced divisor in $|D|$ with respect to $d \cdot (p)$ is the conventional reduced divisor in $|D|$ with respect to p.

Based on this notion of general reduced divisors, several strong tools are developed to investigate tropical convexity theory and the geometric properties of linear systems on metric graphs. The existence and uniqueness of general reduced divisors can actually be used in a definition of **canonical projections** $\gamma : \mathrm{Div}_+^{\mathrm{d}}(\Gamma) \to T$ where $T \subseteq \mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ is tropically convex by sending a divisor $D \in \mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ to the reduced divisor in T with respect to D. In particular, the reduced-divisor maps introduced by Amini [3] (related to the induced map in [39] as an analogue of the map from a curve to its linear systems in algebraic geometry) are special cases of canonical projections restricted to the divisors of the form $d \cdot (p)$ where $p \in \Gamma$ and the codomain T is a complete linear system $|D|$. However, using general reduced divisors, the target linear system T does not need to be complete. Therefore, this notion of canonical projections is actually closer to the conventional map from an algebraic curve to its linear systems (not necessarily complete).

1.4 Smoothability of limit linear series of rank one

One of the most potent approaches to the study of smooth algebraic curves and their linear series is degeneration of smooth curves to singular curves and the respective degenerations of a family of linear series varying on the smooth fibers in the family. Fundamental results on algebraic curves such as the Brill-Noether theorem, Giesker-Petri Theorem, non-unirationality results of the moduli space of curves are established via degeneration to singular curves [31, 32, 34, 37, 38, 40, 41].

While degenerations to irreducible curves such as cuspidal curves and nodal curves were considered by Castelnuovo and several researchers subsequently [26], Eisenbud and Harris in a series of papers [32, 34] developed a theory of the degeneration of linear series, which is called limit linear series, to certain reducible curves. The theory of limit linear series had numerous applications, for instance proofs of non-unirationality of moduli spaces of curves M_g for $g \geqslant 23$ [34], a detailed study of the monodromy of Weierstrass points [33, 35], and an alternative simpler and complete proof of the Brill-Noether theorem [32]. This theory was further developed by several researchers. Osserman [55] generalized the theory of limit linear series to curves in positive characteristic and obtained a functorial description of limit linear series. However, the limit linear series was largely restricted to curves of compact-type i.e., reducible curves whose dual graphs are trees or equivalently those curves having compact Jacobians.

On the other hand, the tropical proof of the Brill-Noether theorem [28] made use of Baker's framework [11] of degenerating linear series on a regular semistable family of curves to a linear system on the dual graph of the special fiber which works best when the degeneration is maximal (the genus of the dual graph equals the genus of the curve as the generic fiber).

In a recent paper of Amini and Baker [4], a notion of metrized complexes of algebraic curves is introduced as a generalized framework of the above two orthogonal approaches. Roughly speaking, a metrized complex over an algebraically closed field k is a metric graph together with marked smooth algebraic curves C_v associated to all vertices v of Γ. They also generalized the notion of limit linear series, which only applies to curves of compact type in the Eisenbud-Harris theory, to all metrized complexes of curves. They show that their general notion of limit linear series coincides with that of Eisenbud and Harris on a metrized complex associated to a curve of compact-type.

Both the notions of limit linear series due to Eisenbud and Harris and due to Amini and Baker satisfy two key properties:

1. For any family of smooth curves degenerating to a curve of compact type (metrized complex respectively), any linear series on the smooth curves in the family degenerates to a limit linear series on the curve of compact type (metrized complex respectively).

2. The limit linear series is formulated in terms of linear series on each irreducible component and with relations between the linear series on each irreducible component that depends on the dual graph.

However, even in the case of curves of compact type the converse of Property 1 does not hold in general, in other words not every limit linear series arises as a "limit" of linear series. A limit g_d^r (a limit linear series of degree d and rank r) is said to be **smoothable** if it arises as a limit of linear series, i.e., it is specialized from a g_d^r (a linear series of degree d and rank r) on some smooth curve. Eisenbud and Harris [32] also considered a refinement of the notion of limit linear series called "refined" limit linear series, and showed that every refined limit g_d^1 is smoothable and for $r \geq 2$ a generic limit g_d^r is smoothable, which they call the regeneration theorem.

In this work, we consider a refinement of the notion of metrized complex called a refined metrized complex and undertake a detailed study of smoothability of a limit g_d^1 on refined metrized complexes of algebraic curves. Our following main result (also cf. Theorem 4.1.13 which actually states the theorem for pre-limit linear series) is an effective characterization of smoothable limit g_d^1.

Theorem 1.4.1. *A limit g_d^1 is smoothable if and only it is a diagrammatic limit g_d^1 that is solvable and satisfies the intrinsic global compatibility conditions.*

The terms "diagrammatic limit g_d^1" (see Definition 4.1.19), "solvability" (see Definition 4.1.20) and "intrinsic global compatibility" (see Definition 4.1.23) will be explained in details in Chapter 4. Here let us briefly explain the main technical ingredient of our effective characterization for smoothability of a limit g_d^1 on a refined metrized complex. Suppose that X is a smooth proper curve over $\mathbb{K}$ and let $\Sigma(X^{\mathrm{an}})$ be a skeleton of the Berkovich analytification $\Sigma(X^{\mathrm{an}})$ of X (see [15] for a precise definition). Let $\mathfrak{C}(\Sigma)$ be the refined metrized complex associated to $\Sigma(X^{\mathrm{an}})$. A base point free g_d^1 on X induces a morphism $f : X \to \mathbb{P}^1$ of degree d. By the functoriality of analytification, we have an induced map $f^{\mathrm{an}} : X^{\mathrm{an}} \to \mathbb{P}^1_{\mathrm{Berk}}$ where $\mathbb{P}^1_{\mathrm{Berk}}$ is the Berkovich projective line. The retraction from the X^{an} to the skeleton Σ induces a pseudo-harmonic morphism $\mathfrak{C}\phi$ from the refined metrized complex $\mathfrak{C}(\Sigma)$ to a refined metrized complex $\mathfrak{C}(T)$ of genus zero and $\mathfrak{C}(T)$ is a retract of $\mathbb{P}^1_{\mathrm{Berk}}$. The construction is summarized by the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{P}^1 \\ \downarrow & & \downarrow \\ X^{\mathrm{an}} & \xrightarrow{f^{\mathrm{an}}} & \mathbb{P}^1_{\mathrm{Berk}} \\ \downarrow & & \downarrow \\ \mathfrak{C}(\Sigma) & \xrightarrow{\mathfrak{C}\phi} & \mathfrak{C}(T) \end{array}$$

The main technical ingredient of our smoothing theorem is the characterization of all genus zero refined metrized complexes $\mathfrak{C}(T)$ that can arise in this commutative diagram in terms of the data of the limit g^1_d that is induced by f.

An alternative viewpoint on degeneration of g^1_d is via degeneration of maps from each smooth curve in the family to $\mathbb{P}^1$, giving rise to the notion of admissible covers between reducible curves developed by Mumford-Harris [42] and to the notion of harmonic morphism of refined metrized complexes developed by Amini et al. [5,6]. A harmonic morphism from a refined metrized complex to a refined metrized complex of genus zero is smoothable to a g^1_d on the smooth fiber. However, in several applications that involve the gonality stratification of $\bar{M}_g$, the harmonic morphism from a refined metrized complex to refined metrized complex of genus zero is not a part of the starting data. Instead, it is important to start with a limit g^1_d on a refined metrized complex and determine if it is smoothable or not.

1.5 Organization of the thesis

Chapter 2 is on the subject of rank-determining sets. In Section 2.2, we generalize Dhar's algorithm for determining whether a given effective divisor on a finite graph is v_0-reduced to the corresponding metric graph case, and then develop an algorithm for computing the v_0-reduced divisor linearly equivalent to a given effective divisor on Γ based on Dhar's algorithm. In Section 2.3, we introduce the notion of rank-determining sets and develop a criterion to characterize all rank-determining sets on a given metric graph (Theorem 2.3.17), from which more useful properties of rank-determining sets are developed. We also introduce the notion of minimal rank-determining sets. In Section 2.4, several applications of rank-determining sets are mentioned.

Chapter 3 is on the subject of a tropical convexity theory on linear systems on metric graphs. In Section 3.2, potential theory on metric graphs is briefly reviewed. In Section 3.3 and Section 3.4, we define a canonical metric structure on the set of real divisors $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ on Γ, and introduce the notion of tropical convexity with some basic properties stated. In Section 3.5, the notion of general reduced divisors is introduced. We provide several criteria for general reduced divisors and develop some useful tools with which to prove the theorems about the basic properties of tropical convex sets stated in Section 3.4. In Section 3.8, we discuss canonical projections which is derived from the notion of general reduced divisors.

Chapter 4 is on the subject of smoothing of limit linear series of rank one on refined metrized complexes, and is based on my collaboration with Madhusudan Manjunath. In Section 4.1, we state the main smoothing theorem for pre-limit g_d^1 with several related notions defined. In Section 4.2, we introduce two crucial technical concept-s: bifurcation trees and tropical dominant trees. In Section 4.3, 4.4, 4.5 and 4.6, we discuss the specialization map and the relation to our problem of smoothability. In Section 4.7, 4.8 and 4.9, we investigate the possible genus zero refined metrized complex and its underlying metric tree that can appear in the image of harmonic morphisms with a given pre-limit g_d^1, and then come to the proof of our main result.

CHAPTER II

RANK-DETERMINING SETS OF METRIC GRAPHS

2.1 Introduction

The work of Hladký, Král' and Norine [44] shows that the rank of a divisor D on a graph equals the rank of D on the corresponding metric graph Γ. However, their result requires that all the edges of Γ have length 1 and D is zero on the interiors of the edges. As an initial step, we assert that these restrictions are not necessary by proving that for an arbitrary metric graph Γ with a vertex set Ω and an arbitrary divisor D on Γ, the rank $r(D)$ of D equals the Ω-restricted rank $r_\Omega(D)$ of D. This result motivates us into further investigations on the subsets of Γ having such a property, to which we give the name *rank-determining sets.*

2.1.1 Preliminaries

Throughout this chapter, a *graph* G means a finite connected multigraph with no loop edges, and a *metric graph* Γ means a graph having each edge assigned a positive length. Roughly speaking, a *tropical curve* is a metric graph where we admit some edges incident with vertices of degree 1 having infinite length [53, 54]. We will expand our discussions within the framework of metric graphs, while the conclusions also apply for tropical curves. (We abuse notation throughout this chapter that the set of points of a metric graph Γ is also denoted by Γ.)

Denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The *genus* g of G is the first Betti number of G or the maximum number of independent cycles of G, which equals $\#E(G) - \#V(G) + 1$.

We can also define vertices and edges on a metric graph Γ. We call Ω a *vertex set* of Γ and the elements of Ω *vertices*, if Ω is a nonempty finite subset of Γ satisfying the following conditions:

(i) $\Gamma \setminus \Omega$ is a disjoint union of subspaces e_i^o isometric to open intervals.

(ii) Let e_i be the closure of e_i^o. For all i, $e_i \setminus e_i^o$ contains exactly two distinct points, which are both elements of Ω. We call e_i an *edge* of Γ, e_i^o the *interior* of e_i, and $v \in e_i^o$ an *internal point* of e_i. And we say that the two vertices in $e_i \setminus e_i^o$ are two *ends* (or *end-points*) of e_i or e_i^o, while e_i is an edge *connecting* these vertices.

Clearly, Γ is loopless with respect to a vertex set Ω. By our definition of a vertex set, there might be multiple edges between two vertices, which is not allowed in definitions of vertex sets by other authors (see, e.g., [12]).

By identifying each edge with a closed interval, the connected subsets or subintervals are called *segments* of Γ, which can be open, closed or half-open/half-closed. The boundary points of a segment are called the *ends* (or *end-points*) of that segment. For any point $v \in \Gamma$, we define the degree of v, denoted by $\deg(v)$, to be the maximum number of disjoint open segments with one end at v. Note that internal points always have degree 2, which means $\{v \in \Gamma : \deg(v) \neq 2\}$ is a finite subset of all vertex sets. We can refine any vertex set Ω by adding some internal points to Ω.

Throughout this chapter, whenever we mention a vertex or an edge of a metric graph Γ, we always assume a vertex set of Γ is predetermined, whether or not it is presented explicitly. Given a vertex set of Γ, the genus of Γ can be computed just like in the graph case (note that the genus is independent of how we choose vertex sets).

In addition, we transport the conventional notations for intervals onto metric graphs. For example, let w_1 and w_2 be two vertices that are neighbors, e be one of the edges connecting them, and v be an internal point e. Then (w_1, w_2) represents all the internal points of the edges connecting w_1 and w_2. And to avoid confusion in case of multiple edges, e can be represented by $[w_1, v, w_2]$. We use $\mathrm{dist}(x, y)$ to denote the distance between two points x and y measured on Γ, and define the distance between two subsets X and Y of Γ, denoted by $\mathrm{dist}(X, Y)$, to be $\inf\{\mathrm{dist}(x, y), x \in X, y \in Y\}$.

If e' is a segment, and $x, y \in e'$, then we use $\mathrm{dist}_{e'}(x, y)$ to denote the distance between x and y measured on e'.

For simplicity of notation, if v is a point of a metric graph, sometimes we refer to the singleton $\{v\}$ by just writing v.

Baker and Norine [13] systematically explored the analogies between finite graphs and Riemann surfaces in the context of linear equivalence of divisors. We give a series of definitions here following their work. A *divisor* D on G is an element of the free abelian group $\mathrm{Div}\, G$ on the vertex set of G. We can uniquely write a divisor $D \in \mathrm{Div}\, G$ as $D = \sum_{v \in V(G)} D(v)(v)$, where $D(v) \in \mathbb{Z}$ evaluates D at v. The *degree* of D is defined by the formula $\deg(D) = \sum_{v \in V(G)} D(v)$. A divisor D is called *effective* if $D(v) \geqslant 0$ for all $v \in V(G)$. We denote the set of all effective divisors on G by $\mathrm{Div}_+ G$, and the set of all effective divisors of degree s on G by $\mathrm{Div}_+^s G$. Provided a function $f : V(G) \to \mathbb{Z}$, the divisor associated to f is given by

$$\Delta f = \sum_{v \in V(G)} \sum_{e = wv \in E(G)} (f(v) - f(w))(v),$$

and called *principal*. It is easy to see that the principal divisors have degree 0. For two divisors D and D', we say that D is *linearly equivalent* to D' or $D \sim D'$ if $D - D'$ is principal. And we defined the *linear system associated to a divisor* D to be the set $|D|$ of all effective divisors linearly equivalent to D. The *rank* of a divisor D, denoted by $r_G(D)$, is an integer defined as, $r_G(D) = -1$ if $|D| = \emptyset$, and $r_G(D) \geqslant s \geqslant 0$ if and only if $|D - E| \neq \emptyset$ for all $E \in \mathrm{Div}_+^s G$. When it is clear that D is defined on G, we usually omit the subscript and write $r(D)$ instead of $r_G(D)$. Note that the rank of a divisor is invariant under linear equivalence.

Analogously, for a metric graph (or a tropical curve) Γ, elements of the free abelian group $\mathrm{Div}\, \Gamma$ on the set of points of Γ are called divisors on Γ. We can define the degree of a divisor and the notion of effective divisors in a similar way. A rational function f on Γ is a continuous, piecewise linear real function with integer slopes. The *order*

$\mathrm{ord}_v f$ of f at a point $v \in \Gamma$ is the negative of the sum of the outgoing slopes of all the segments emanating from v. Any rational function f has an associated divisor $\Delta f = \sum_{v\in\Gamma} \mathrm{ord}_v f \cdot (v)$ (we also write $(f) = \Delta f$ in this chapter for simplicity). We say (f) is *principal* for all rational functions f, and define linear equivalence relations and linear systems as on graphs. Also, we may define the *rank* $r_\Gamma(D)$ of a divisor D on Γ. Explicitly, $r_\Gamma(D) = -1$ if $|D| = \emptyset$, and $r_\Gamma(D) \geqslant s \geqslant 0$ if and only if $|D - E| \neq \emptyset$ for all $E \in \mathrm{Div}_+^s \Gamma$. We may omit the subscript and use $r(D)$ to represent the rank of a divisor D, when there is no confusion that D is defined on Γ.

Remark 2.1.1. In the classical Riemann surface case, the linear system $|D|$ associated to a divisor D is the $r(D)$-dimensional projective space of a $(r(D)+1)$-dimensional vector space. However, $|D|$ is a finite set in the finite graph case and a polyhedral complex in the metric graph case [36]. We give analogous definitions of rank $r(D)$ in these cases, even if $r(D)$ should no longer be interpreted as a dimension.

For a divisor D on Γ, let $\mathrm{supp}\, D = \{v \in \Gamma | D(v) \neq 0\}$ and $\mathrm{supp}\,|D| = \bigcup_{D'\in|D|} \mathrm{supp}\, D'$. We call $\mathrm{supp}\, D$ the *support of* D and call $\mathrm{supp}\,|D|$ the *support of* $|D|$. Note that even though $\mathrm{supp}\, D$ is always a finite subset of Γ, $\mathrm{supp}\,|D|$ is not in general.

2.1.2 Overview of related work

As an analogue of the classical Riemann-Roch theorem on Riemann surfaces, Baker and Norine formulated and proved the Riemann-Roch theorem for the rank of divisors on finite graphs [13]. We define the *canonical divisor* on a graph G to be the divisor K given by $K = \sum_{v\in V(G)} (\deg(v) - 2)(v)$.

Theorem 2.1.2 (Riemann-Roch theorem for graphs)**.** *Let G be a graph of genus g and K the canonical divisor on G. Then for all $D \in \mathrm{Div}\, G$, we have*

$$r_G(D) - r_G(K - D) = \deg(D) + 1 - g.$$

Not long after, such an analogy was extended to metric graphs and tropical curves by Gathmann and Kerber [36], by Hladký, Král' and Norine [44], and by Mikhalkin and Zharkov [54]. For a metric graph (or a tropical curve) Γ, we may also define the *canonical divisor* on Γ to be the divisor K given by $K = \sum_{v \in \Gamma}(\deg(v) - 2)(v)$. Here deg(v) is the number of outgoing segments at a point v.

Theorem 2.1.3 (Riemann-Roch thoerem for metric graphs and tropical curves)**.** *Let Γ be a metric graph (or a tropical curve) of genus g and K the canonical divisor on Γ. Then for all $D \in \operatorname{Div}\Gamma$, we have*

$$r_\Gamma(D) - r_\Gamma(K - D) = \deg(D) + 1 - g.$$

The following theorem, conjectured by Baker and proved by Hladký, Král' and Norine [44], states another important property about rank of divisors. For a graph G, by assigning all edges length 1, we obtain a metric graph *corresponding to G*.

Theorem 2.1.4 (Hladký, Král' and Norine)**.** *Let Γ be the metric graph corresponding to a graph G. Let D be a divisor on G. Let $r_G(D)$ be the rank of D on G, and $r_\Gamma(D)$ the rank of D on Γ. Then we have $r_G(D) = r_\Gamma(D)$.*

2.1.3 Main results in this chapter

We introduce a new notion of rank here.

Definition 2.1.5. Let Γ be a metric graph and A a nonempty subset of Γ. Let $\operatorname{Div}_+^s A$ be $\{E \in \operatorname{Div}_+^s \Gamma : \operatorname{supp} E \subseteq A\}$.

(i) Define the *A-restricted rank* $r_A(D)$ of a divisor $D \in \operatorname{Div}\Gamma$ by $r_A(D) = -1$ if $|D| = \emptyset$, and $r_A(D) \geqslant s \geqslant 0$ if and only if $|D - E| \neq \emptyset$ for all $E \in \operatorname{Div}_+^s A$.

(ii) A is said to be a *rank-determining set* of Γ, if it holds for every divisor $D \in \operatorname{Div}\Gamma$ that $r(D) = r_A(D)$.

One may also call $r_A(D)$ the rank of D restricted on A. Clearly, Γ itself is a rank-determining set of Γ and we say it is *trivial*. Following the definition, any superset of a rank-determining set is also rank-determining. It is natural to ask if all metric graphs have nontrivial rank-determining sets, or more ambitiously, finite ones? One of the main results of this chapter is the following theorem, which gives an affirmative answer.

Theorem 2.1.6. *Let Ω be a vertex set of a metric graph Γ. Then Ω is a rank-determining set of Γ.*

It is easy to see that Theorem 2.1.6 generalizes Theorem 2.1.4 to all metric graphs Γ and all divisors D on Γ. And since $\mathrm{Div}_+^s \Omega$ is always a finite set, this theorem also provides an algorithm for computing the rank of a divisor on Γ.

There exist finite rank-determining sets other than vertex sets. In particular, we will prove the following conjecture of Baker.

Theorem 2.1.7. *Let Γ be a metric graph of genus g. Then there exists a finite rank-determining set of cardinality $g+1$.*

Theorem 2.1.7 has a counterpart in the algebraic curve case, as stated in the following theorem. (See Remark 2.3.13 for a sketch of the proof.)

Theorem 2.1.8 (R. Varley)**.** *For a nonsingular projective algebraic curve C, any set of $g+1$ distinct points is a rank-determining set.*

The linear equivalence among divisors on Γ changes if we use a different metric. Actually, if $f : \Gamma \to \Gamma'$ is a homeomorphism between two metric graphs Γ and Γ', then by sending the supporting points of a divisor on Γ to points on Γ', f induces a push-forward map $f_* : \mathrm{Div}\,\Gamma \to \mathrm{Div}\,\Gamma'$ between divisors on Γ and Γ'. Consider two linear equivalent divisors D_1 and D_2 on Γ. Then $f_*(D_1)$ and $f_*(D_2)$ are not linearly equivalent in general. Example 2.1.9 shows a simple case that $r_\Gamma(D) \neq r_{\Gamma'}(f_*(D))$. However, we state in Theorem 2.1.10 that rank-determining sets will not be affected at all, even though their definition uses the notion of linear equivalence and linear systems.

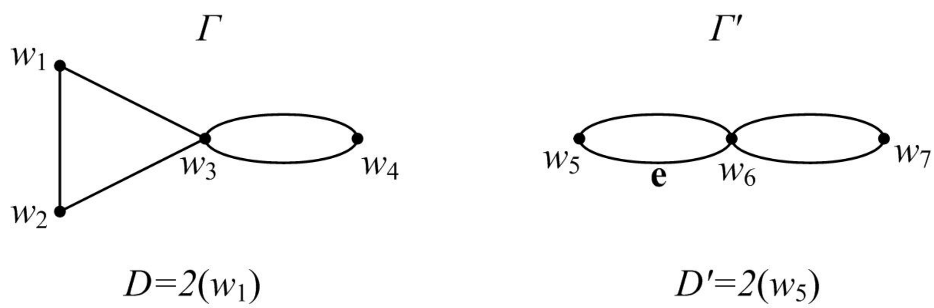

Figure 1: Two divisors, D and D', which are defined on two homeomorphic metric graphs Γ and Γ' respectively, and have different ranks.

Example 2.1.9. Let Γ and Γ' be two metric graphs with vertex sets $\{w_1, w_2, w_3, w_4\}$ and $\{w_5, w_6, w_7\}$ respectively (Figure 1). Assume all edges have length 1. By contracting $[w_1, w_2] \bigcup [w_2, w_3]$, the union of two edges of Γ, proportionally onto the edge $\mathbf{e}$ of Γ', we get a piecewise-linear homeomorphism $f : \Gamma \to \Gamma'$ between Γ and Γ' that is not an isometry. Let $D = 2(w_1)$ and $D' = 2(w_5)$. Then $D' = f_*(D)$ since $f(w_1) = w_5$. However, we observe that $r_\Gamma(D) = 0$, while $r_{\Gamma'}(D') = 1$. This is because the support of $|D|$ is $[w_1, w_2] \bigcup [w_1, w_3]$, which is a proper subset of Γ, and the support of $|D'|$ is the whole metric graph Γ'.

Theorem 2.1.10. *Rank-determining sets are preserved under homeomorphisms.*

In Section 2.2, we present an algorithm for computing the v_0-reduced divisor linearly equivalent to a given effective divisor on Γ. In Section 2.3, we investigate properties of rank-determining sets based on this algorithm, which are generalized into a criterion (Theorem 2.3.17) for rank-determining sets, from which Theorem 2.1.6, 2.1.7 and 2.1.10 easily follow. We also explore several concrete examples as applications of the criterion.

2.2 From effective divisors to reduced ones

2.2.1 Reduced divisors

The notion of *reduced divisors* was adopted in [13] as an important tool in the proof of the Riemann-Roch theorem for finite graphs. The definition of reduced divisors on finite graphs is based on the notion of *G-parking functions* [56].

Let G be a finite graph. For $A \subseteq V(G)$ and $v \in A$, the *out-degree* of v from A, denoted by $\mathrm{outdeg}_A(v)$, is defined as the number of edges of G with one end at v and the other end in $V(G)\setminus A$. Choose a vertex v_0. We say a function $f : V(G)\setminus\{v_0\} \to \mathbb{Z}$ is a *G-parking function* based at v_0 if

(i) $f(v) \geqslant 0$ for all $v \in V(G) \setminus \{v_0\}$, and

(ii) every nonempty subset A of $V(G) \setminus \{v_0\}$ contains a vertex v such that $f(v) < \mathrm{outdeg}_A(v)$.

A divisor $D \in \mathrm{Div}\, G$ is called *v_0-reduced* if the map $v \mapsto D(v)$ restricted to $V(G) \setminus \{v_0\}$ is a G-parking function based at v_0. An important property of reduced divisors is stated in the following proposition.

Proposition 2.2.1 (See Proposition 3.1 in [13])**.** *If we fix a base vertex $v_0 \in V(G)$, then for every $D \in \mathrm{Div}\, G$, there exists a unique v_0-reduced divisor $D' \in \mathrm{Div}\, G$ such that $D' \sim D$.*

Proposition 2.2.1 is quite useful when dealing with equivalence classes of divisors, since we can select a reduced divisor as a concrete representative for each equivalence class of divisors.

The notion of reduced divisors has been extended to metric graphs by several authors. We adopt the definition of reduced divisors on metric graphs as in [44], which follows closely the definition of reduced divisors on finite graphs as discussed above. Other authors suggest to define reduced divisors on metric graphs in more abstract ways [54], and it can be proved that these definitions are all equivalent.

Let Γ be a metric graph. If X is a subset of Γ with finitely many connected components, we use X^c to denote the complement of X on Γ, $\overline{X}$ the closure of X, X^o the interior of X, and ∂X the set of boundary points of X. Note that $\partial X = \partial(X^c)$. In addition, if X is closed, then for $v \in X$, we define the *out-degree* of v from X, denoted by $\mathrm{outdeg}_X(v)$, to be the number of segments leaving X at v, or more precisely, the maximum number of internally disjoint segments of X^c with an open end at v. Note that $\mathrm{outdeg}_X(v) = 0$ for all $v \in X \setminus \partial X$. For $D \in \mathrm{Div}\,\Gamma$, we call a boundary point v of X *saturated* with respect to X and D if $D(v) \geqslant \mathrm{outdeg}_X(v)$, and *non-saturated* otherwise.

Definition 2.2.2. Fix a base point $v_0 \in \Gamma$. We say that a divisor D is v_0-*reduced* if D is non-negative on $\Gamma \setminus v_0$, and every closed connected subset X of $\Gamma \setminus v_0$ contains a non-saturated point $v \in \partial X$.

As a counterpart of Proposition 2.2.1, the following theorem asserts the existence and uniqueness of a v_0-reduced divisor in any equivalence class of $\mathrm{Div}\,\Gamma$ [44] [54].

Theorem 2.2.3 (See Theorem 10 in [44])**.** *Let D be a divisor on a metric graph Γ. For any $v_0 \in \Gamma$, there exists a unique v_0-reduced divisor D_{v_0} that is linearly equivalent to D.*

For any finite subset S of Γ, we denote by $\mathcal{U}_{S,v_0}$ the connected component of S^c which contains v_0. In particular, if $v_0 \in S$, then $\mathcal{U}_{S,v_0} = \emptyset$. We emphasize here that $\mathcal{U}_{S,v_0}$ is connected and open, while $\mathcal{U}^c_{S,v_0}$ is closed and might have several connected components. We say that S is v_0-*minimal* if $\mathcal{U}^c_{S,v_0}$ is connected and S equals the set of boundary points of $\mathcal{U}^c_{S,v_0}$.

Assume now that D is effective. To verify if D is v_0-reduced, we do not need to go through all closed connected subsets of $\Gamma \setminus v_0$. The following lemma shows that we only need to consider finitely many of them.

Lemma 2.2.4. *Let v_0 be a point of Γ and D an effective divisor on Γ. Then D is v_0-reduced if and only if for any subset S of $\mathrm{supp}\, D \setminus v_0$, $\mathcal{U}^c_{S,v_0}$ contains a non-saturated boundary point with respect to D.*

Proof. First assume D is v_0-reduced and consider a subset S of $\mathrm{supp}\, D \setminus v_0$. Then $\mathcal{U}^c_{S,v_0}$ is a closed subset of Γ which has finitely many components. Apply the defining property of v_0-reduced divisors to any of these components, and we obtain non-saturated boundary points on each of them.

Conversely, assume that for any subset S of $\operatorname{supp} D \setminus v_0$, $\mathcal{U}^c_{S,v_0}$ contains a non-saturated point. If D is not v_0-reduced, then there exists a closed connected subset X of $\Gamma \setminus v_0$, such that every point of ∂X is saturated with respect to X and D. Since $\operatorname{outdeg}_X(v) > 0$ for all $v \in \partial X$, it follows that $\partial X \subseteq \operatorname{supp} D \setminus v_0$. And since $X \subseteq \mathcal{U}^c_{\partial X,v_0}$, the edges leaving $\mathcal{U}^c_{\partial X,v_0}$ must also be edges leaving X. Therefore, for every $v \in \partial \mathcal{U}_{\partial X,v_0}$, we have

$$D(v) \geqslant \operatorname{outdeg}_X(v) \geqslant \operatorname{outdeg}_{\mathcal{U}^c_{\partial X,v_0}}(v).$$

This is equivalent to saying that $\mathcal{U}^c_{\partial X,v_0}$ contains no non-saturated boundary points, which contradicts our assumption. $\square$

Lemma 2.2.4 tells us that to determine if an effective divisor D is v_0-reduced, it suffices to consider only the subsets of $\operatorname{supp} D \setminus v_0$. But the number of cases still grows exponentially with respect to $\#\{\operatorname{supp} D\}$. For finite graphs, there is an elegant algorithm for verifying if a given function is a G-parking function, which is adapted from an algorithm provided by Dhar [30] in the context of sandpile models (see [27]). Here we naturally extend Dhar's algorithm to metric graphs, as a consequence of which we just need to test the points in $\operatorname{supp} D \setminus v_0$ one by one in order to judge whether an effective divisor D is v_0-reduced.

Algorithm 2.2.5. (**Dhar's algorithm for metric graphs**)

Input: An effective divisor $D \in \operatorname{Div}_+ \Gamma$, and a point $v_0 \in \Gamma$.

Output: A subset S of $\operatorname{supp} D \setminus v_0$.

Initially, set $S_0 = \operatorname{supp} D \setminus v_0$, and $k = 0$.

(1) If $S_k = \emptyset$ or all the boundary points of $\mathcal{U}^c_{S_k,v_0}$ are saturated with respect to D, set $S = S_k$ and stop the procedure.

(2) Let N_k be the set of all non-saturated boundary points of $\mathcal{U}^c_{S_k,v_0}$. Set $S_{k+1} = S_k \setminus N_k$. Set $k \leftarrow k+1$ and go to step (1).

Lemma 2.2.6. *Run Dhar's algorithm for an effective divisor D and a point v_0. Then D is v_0-reduced if and only if the output S is empty.*

Proof. If S is nonempty, then all the boundary points of $\mathcal{U}^c_{S,v_0}$ are saturated. Thus D is not v_0-reduced by Lemma 2.2.4.

Otherwise, $S = \emptyset$. For a subset S' of $\operatorname{supp} D \setminus v_0$, let N_k be such that $N_k \bigcap S' \neq \emptyset$ and $N_{k'} \bigcap S' = \emptyset$ for $k' < k$. Note that $S' \subseteq S_k$. If $v \in N_k \bigcap S'$, then v must be a non-saturated boundary point of $\mathcal{U}^c_{S',v_0}$, since

$$D(v) < \operatorname{outdeg}_{\mathcal{U}^c_{S_k,v_0}}(v) \leqslant \operatorname{outdeg}_{\mathcal{U}^c_{S',v_0}}(v).$$

By Lemma 2.2.4, D is v_0-reduced. □

Remark 2.2.7. The out-degrees are topological invariants, which implies that whether or not a divisor is v_0-reduced is preserved under homeomorphisms. If we let $\operatorname{supp}|D|$ be a subset of the defined vertex set Ω, then Algorithm 2.2.5 reduces to a regular Dhar's algorithm on the underlying finite graph G of the metric graph Γ, where we require $V(G) = \Omega$ (edge lengths does not play a role here). This means Algorithm 2.2.5 has $O(\#\Omega)$ time complexity.

Remark 2.2.8. If an effective divisor D is not v_0-reduced, then running Algorithm 2.2.5 for D and v_0 can actually provide the unique "smallest" open neighborhood $\mathcal{U}_{S,v_0}$ of v_0 such that all its boundary points are saturated with respect to D and $\mathcal{U}^c_{S,v_0}$. Intuitively, "saturated" may be think of as "ready to move". When all the boundary points are saturated, we can launch a "move" of D towards the v_0-reduced divisor linearly equivalent to D. This motivates to develop an algorithm of computing reduced divisors (Algorithm 2.2.13), as will be discussed in the next subsection.

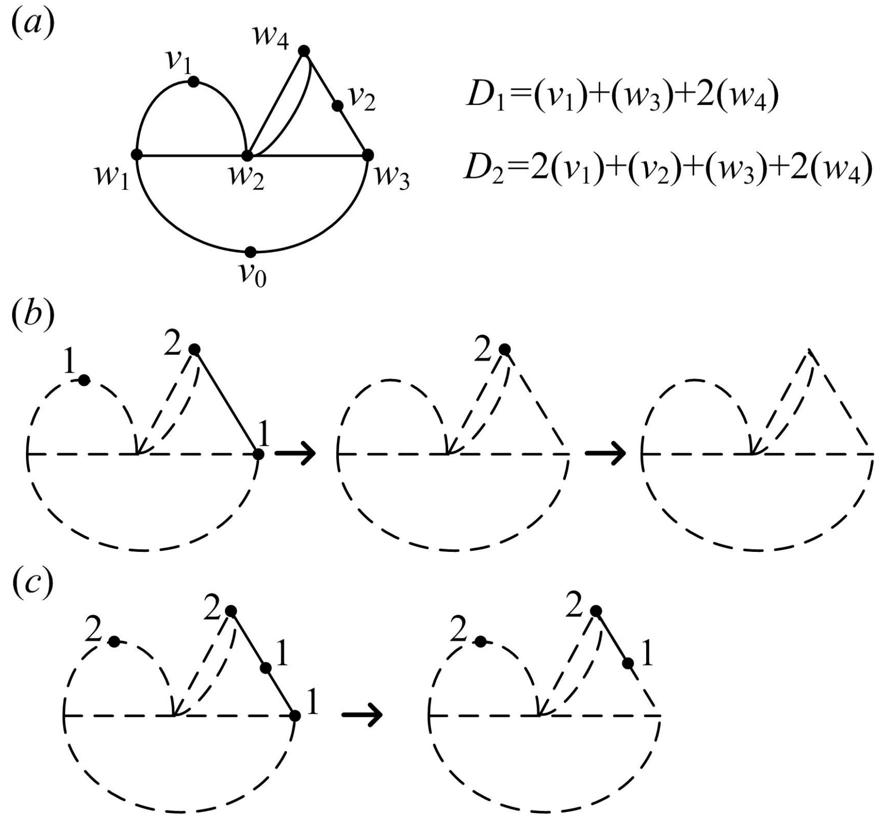

Figure 2: (a) A metric graph Γ and two effective divisors D_1 and D_2 on Γ. (b) Dhar's algorithm for D_1 and v_0. (c) Dhar's algorithm for D_2 and v_0.

Example 2.2.9. Let Γ be a metric graph as illustrated in Figure 2(a) with a vertex set $\{w_1, w_2, w_3, w_4\}$. Let $D_1 = (v_1)+(w_3)+2(w_4)$ and $D_2 = 2(v_1)+(v_2)+(w_3)+2(w_4)$. Run Dhar's algorithm for D_1 and v_0. The dashed areas in Figure 2(b) illustrate $\mathcal{U}_{S_k,v_0}$ step by step. Initially, we have $S_0 = \{v_1, w_3, w_4\}$ and $\mathcal{U}^c_{S_0,v_0} = \{v_1\} \bigcup [w_3, w_4]$. The set N_0 of all non-saturated boundary points of $\mathcal{U}^c_{S_0,v_0}$ is $\{v_1, w_3\}$. Then $S_1 = S_0 \setminus N_0 = \{w_4\}$ and $\mathcal{U}^c_{S_1,v_0} = \{w_4\}$. Since w_4 is a non-saturated point, we have $N_1 = \{w_4\}$ and $S_2 = \emptyset$. Now $\mathcal{U}^c_{S_2,v_0}$ is the whole graph and we get the output $S = \emptyset$. Therefore D_1 is v_0-reduced. We leave it to the readers to verify the output of Dhar's algorithm for D_2 and v_0 is $\{v_1, v_2, w_4\}$ and D_2 is not v_0-reduced (Figure 2(c)).

2.2.2 An algorithm for computing reduced divisors

Based on Dhar's algorithm and the criterion from Lemma 2.2.6, we formulate an algorithm to derive from an effective divisor D the unique v_0-reduced divisor linearly equivalent to D.

Recall from [44] the notion of *basic v_0-extremal functions* on Γ. We say a rational function f is a *basic v_0-extremal function* if there exist closed connected disjoint subsets $\Gamma_{\max}(f)$ and $\Gamma_{\min}(f)$ of Γ such that:

(i) $v_0 \in \Gamma_{\max}(f)$;

(ii) $\Gamma - \Gamma_{\min}(f) - \Gamma_{\min}(f)$ is the union of disjoint open segments of the same length;

(iii) f achieves its maximum on $\Gamma_{\max}(f)$ and its minimum on $\Gamma_{\min}(f)$;

Definition 2.2.10. Let D be an effective divisor on Γ and S a subset of $\operatorname{supp} D \setminus v_0$ such that all the boundary points of $\mathcal{U}^c_{S,v_0}$ are saturated with respect to D. Let Ω be a fixed vertex set of Γ. We call the following parameterizing process $\Delta_{D,S,v_0} : [0,1] \to \mathrm{Div}_+\Gamma$ the *v_0-move* of D with respect to S and Ω:

(i) $\Delta^{(0)}_{D,S,v_0} = D$.

(ii) Let J be the number of connected components of $\mathcal{U}^c_{S,v_0}$, and denote these components by X_1 through X_J.

For $j = 1, 2, \cdots, J$ and $t \in (0,1]$, let

$d^{(t)}_j = t \cdot \operatorname{dist}\big(X_j, \mathcal{U}_{S,v_0} \bigcap (\Omega \bigcup v_0)\big)$,

$P^{(t)}_j = \{p \in \mathcal{U}_{S,v_0} \mid \operatorname{dist}(X_j, p) = d^{(t)}_j\}$,

$Q^{(t)}_j = \{q \in \mathcal{U}_{S,v_0} \mid \operatorname{dist}(X_j, q) \leqslant d^{(t)}_j\}$, and

$f^{(t)}_j$ a basic v_0-extremal function such that

$\Gamma_{\min}(f^{(t)}_j) = X_j$, and $\partial\,\Gamma_{\max}(f^{(t)}_j) = P^{(t)}_j$.

(iii) $\Delta^{(t)}_{D,S,v_0} = D + \sum_{j=1}^{J}(f^{(t)}_j)$, for $t \in (0,1]$.

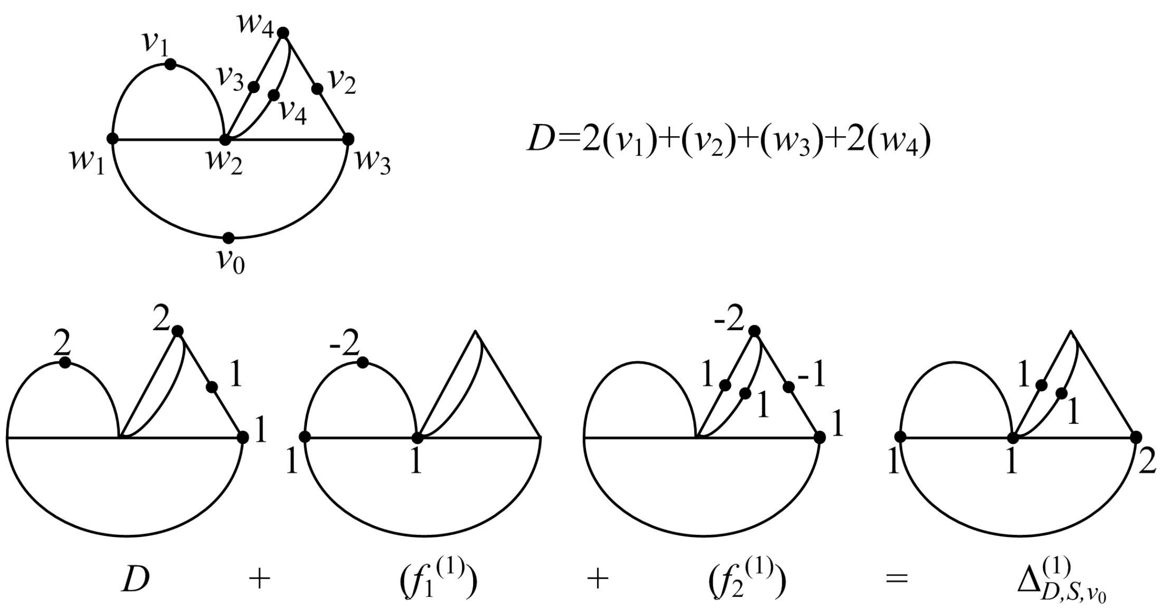

Figure 3: A v_0-move of D.

Example 2.2.11. Let Γ be the same metric graph as in Example 2.2.9 and $D = D_2$, as shown in Figure 3. In particular, we assign length 1 to all edges and let v_i be the middle point of the corresponding edge for $i = 0, 1, 2, 3, 4$. We know from Example 2.2.9 that the output S of Dhar's algorithm for D and v_0 is $\{v_1, v_2, w_4\}$. Let us consider a v_0-move Δ_{D,S,v_0}. Note that $\mathcal{U}^c_{S,v_0}$ has two connected components, v_1 and $[v_2, w_4]$, which we denote by X_1 and X_2 respectively. We observe that $d_1^{(t)} = d_2^{(t)} = 0.5t$ for $t \in (0, 1]$. And at the end of the move ($t = 1$), we get $P_1^{(1)} = \{w_1, w_2\}$, $Q_1^{(1)} = [w_1, v_1, w_2] \setminus v_1$, $P_2^{(1)} = \{v_3, v_4, w_3\}$, and $Q_2^{(1)} = (w_4, v_3] \bigcup (w_4, v_4] \bigcup (v_2, w_3]$. In addition, $(f_1^{(1)}) = (w_1) + (w_2) - 2(v_1)$ and $(f_2^{(1)}) = (v_3) + (v_4) + (w_3) - (v_2) - 2(w_4)$. Then we get $\Delta^{(1)}_{D,S,v_0} = D + (f_1^{(1)}) + (f_2^{(1)}) = (v_3) + (v_4) + (w_1) + (w_2) + 2(w_3)$.

The reader is suggested to go through the above example before reading the proofs of the following statements.

Lemma 2.2.12. *Let D be an effective divisor which is zero at v_0 and Δ_{D,S,v_0} a move of D. Denote* $\mathrm{supp}(\Delta^{(t)}_{D,S,v_0})$ *by $O^{(t)}$ for $t \in [0, 1]$. Then $\mathcal{U}_{O^{(t)},v_0}$ is non-expanding with respect to t. Moreover, $\mathcal{U}_{O^{(t)},v_0}$ evolves continuously unless possibly undergoing* *an abrupt shrink at $t = 1$.*

Proof. Let $Q_j^{(t)}$ be as defined in Definition 2.2.10 for $t \in (0,1]$. Let $Q^{(0)} = \partial \mathcal{U}_{S,v_0}$ and

$$Q^{(t)} = \bigcup_{j=1}^{J} Q_j^{(t)}, \text{ for } t \in (0,1].$$

Clearly, $Q^{(t)}$ continuously expands with respect to t. For $t \in [0,1)$, we have

$$\mathcal{U}_{O^{(t)},v_0} = \mathcal{U}_{O^{(0)},v_0} \setminus Q^{(t)},$$

which means $\mathcal{U}_{O^{(t)},v_0}$ is non-expanding as t increases and its evolution is continuous. The case $t = 1$ is somehow special, since the continuous expansion of $Q^{(t)}$ might result in a hit at certain vertices or v_0. But we still have

$$\mathcal{U}_{O^{(1)},v_0} \subseteq \mathcal{U}_{O^{(0)},v_0} \setminus Q^{(1)}.$$

This means that an abrupt shrink of $\mathcal{U}_{O^{(t)},v_0}$ might happen at $t = 1$. $\square$

Based on making v_0-moves iteratively, we propose the following algorithm to derive the v_0-reduced divisor linearly equivalent to an effective divisor D.

Algorithm 2.2.13. Input: An effective divisor $D \in \mathrm{Div}_+ \Gamma$, and a point $v_0 \in \Gamma$.
Output: The unique v_0-reduced divisor D_{v_0} linearly equivalent to D.
Initially, set $D^{(0)} = D$, and $i = 0$.

(1) Run Dhar's algorithm for $D^{(i)}$ and v_0 with the output denoted by $S^{(i)}$. If $S^{(i)} = \emptyset$, then set $D_{v_0} = D^{(i)}$ and stop the procedure. In addition, we say that the procedure *terminates* at i. And for convenience, we set $D^{(t)} = D^{(i)}$ for all real numbers $t > i$. Otherwise, go to step (2).
(2) Define $D^{(i+t)} = \Delta^{(t)}_{D^{(i)},S^{(i)},v_0}$ for $t \in (0,1]$. Set $i \leftarrow i+1$, and go to step (1).

If the procedure in Algorithm 2.2.13 terminates at I, then by Lemma 2.2.6, D_{v_0} is v_0-reduced as desired, and the evolution of D into D_{v_0} is parameterized by $D^{(t)}$, $t \in [0, I]$. The main goal of this section is to prove such a procedure always terminates

(Theorem 2.2.15), which means that we will always get to a reduced divisor using finitely many moves.

Lemma 2.2.14. *We have the following properties of the parameterizing procedure in Algorithm 2.2.13:*

(i) $D^{(t)}(v_0)$ *is integer-valued, bounded, and non-decreasing with respect to* t*, and it can jump only when* t *is an integer. In addition, there exists an integer* I_1 *such that* $D^{(t)}(v_0) = D^{(I_1)}(v_0)$ *for all* $t \geqslant I_1$*.*

(ii) For a non-negative integer i_0*, let* $d = D^{(i_0)}(v_0)$ *and* $D_0^{(t)} = D^{(t)} - d \cdot (v_0)$*. Then for all real numbers* $t \geqslant i_0$*,* $\mathcal{U}_{\operatorname{supp} D_0^{(t)}, v_0}$ *is non-expanding with respect to* t*. In particular,* $\mathcal{U}_{\operatorname{supp} D_0^{(t)}, v_0}$ *evolves continuously unless possibly undergoing an abrupt shrink when* t *is an integer.*

(iii) Denote $\mathcal{U}_{\operatorname{supp} D^{(t)} \setminus v_0, v_0}$ *by* $U(t)$*. For* $t \geqslant I_1$*, let* $K^{(t)} = \#\{\Omega \bigcap U(t)\}$*, which counts the number of vertices in* $U(t)$ *after* $D^{(t)}(v_0)$ *reaches its maximum. Then* $K^{(t)}$ *is integer-valued, bounded, and non-increasing with respect to* t*, and it can jump only when* t *is an integer. Furthermore, there exists an integer* $I_2 \geqslant I_1$ *such that* $K^{(t)} = K^{(I_2)}$ *for all* $t \geqslant I_2$*.*

Proof. Clearly $D^{(t)}(v_0)$ is integer-valued. Note that $v_0 \notin S^{(i)}$ for any i, which implies that $D^{(t)}(v_0)$ is non-decreasing and can only change its value when t is an integer. Moreover, $D^{(t)}(v_0)$ is bounded from below by $D(v_0)$ and from above by $\deg(D)$, which guarantees the existence of the finite integer I_1. Thus Property (i) holds.

$D_0^{(i_0)}$ has value 0 at v_0. Thus by Lemma 2.2.12, for $t \geqslant i_0$, $\mathcal{U}_{\operatorname{supp} D_0^{(t)}, v_0}$ is non-expanding, and evolves continuously unless possibly undergoing an abrupt shrink when t is an integer. In particular, whenever v_0 is hit by a move, $\mathcal{U}_{\operatorname{supp} D_0^{(t)}, v_0}$ will always be empty afterwards. And Property (ii) is proved.

After $D^{(t)}(v_0)$ reaches its maximum at $t = I_1$, v_0 will never be hit anymore. The above argument implies that for $t \geqslant I_1$, $U(t)$ is non-expanding, and continuously evolves unless possibly undergoing an abrupt shrink when t is an integer. It follows immediately that $K^{(t)}$ is integer-valued, and non-increasing with respect to t, while it only possibly changes when t is an integer. Clearly $K^{(t)}$ is lower-bounded by 0, which also implies the existence of I_2 and finishes the proof of Property (iii). $\square$

Theorem 2.2.15. *The procedure in Algorithm 2.2.13 always terminates.*

Proof. We proceed by induction on $\deg(D)$. Clearly Theorem 2.2.15 holds when $\deg D = 0$ since this implies that $D = 0$. Now suppose $\deg(D) > 0$.

By Lemma 2.2.14(i), if $D^{(I_1)}(v_0) > 0$, then $D^{(t)}(v_0) > 0$ for all $t \geqslant 0$ and the result follows by induction (applied to$D^{(I_1)} - (v_0)$). Now we assume $D^{(I_1)}(v_0) = 0$. By Lemma 2.2.14(iii), there exists an integer I_2, such that $K^{(t)} = K^{(I_2)}$ for all $t \geqslant I_2$. We let $t \geqslant I_2$ in the remaining parts of the proof. Note that $U(t)$ might keep shrinking. However, such a shrink can never hit a vertex anymore, which also means that $U(t)$ evolves continuously for $t \geqslant I_2$. Let X be a connected component of $U(I_2)^c$. Let U_0 be a subset of $U(I_2)$ derived by removing the open segments with one end a boundary point of $\partial U(I_2)$ and the other end a vertex or v_0. By definition U_0 is closed and connected, and $U(I_2) \setminus U_0$ is a union of some disjoint open segments. Denote by $\mathcal{E}_X$ the set of these segments. For $e \in \mathcal{E}_X$, we use w_e to denote the end of e on X. We say $e \in \mathcal{E}_X$ is *obstructed* at t if $\operatorname{supp} D^{(t)} \bigcap e \neq \emptyset$ or w_e is saturated with respect to $D^{(t)}$ and X. Note that if an edge is obstructed at t, then it is obstructed at all $t' \geqslant t$.

We claim that there exists $e \in \mathcal{E}_X$ that never becomes obstructed. Otherwise, there exists an integer I_3 such that for $t \geqslant I_3$, the component of $U(t)^c$ corresponding to X has all its boundary points saturated. Then one additional move from Algorithm 2.2.13 will result in a hit at a vertex, which contradicts the minimality of $K^{(I_2)}$. So let e be an element of $\mathcal{E}_X$ that never becomes obstructed. Then w_e does not belong to any output $S^{(i)}$ of Dhar's algorithm for $D^{(i)}$ when $i \geqslant I_2$. So Algorithm 2.2.13 for $D^{(I_2)}$ terminates if and only if the algorithm for $D^{(I_2)} - (w_e)$ terminates, and the induction applies. $\square$

Remark 2.2.16. What should X look like in the above proof? Since X must contain non-saturated boundary points with respect to $D^{(I_2)}$, there are only two possibilities. X can be a single non-vertex point with $D^{(I_2)}(X) = 1$, or else $X^{(I_2)}$ must contain a vertex on its boundary.

Remark 2.2.17. We know from the Riemann-Roch theorem that the rank of the divisor $n \cdot (v_0)$ as a function of n can be arbitrarily large. Hence given a divisor D (not necessarily effective) on Γ, there always exists a divisor D' which is non-negative on $\Gamma \setminus v_0$ and linearly equivalent to D. In particular, [44] presents an algorithm to construct such a divisor D' as the first step in the proof of the existence part of Theorem 2.2.3 (Theorem 10 in [44]). By running Algorithm 2.2.13 for $D' - D'(v_0) \cdot (v_0)$ and v_0, we can always obtain a v_0-reduced divisor D'' linearly equivalent to $D - D'(v_0) \cdot (v_0)$. Then $D'' + D'(v_0) \cdot (v_0)$ is a v_0-reduced divisor linearly equivalent to D. This provides an alternative proof of the existence part of Theorem 2.2.3.

Corollary 2.2.18. *Let D be a divisor on Γ and $|D|$ the linear system associated to D. For $v_0 \in \Gamma$, let D_{v_0} be the unique v_0-reduced divisor D_{v_0} in $|D|$.*

(i) If $v_0 \in \operatorname{supp}|D|$, then $D_{v_0}(v_0) > 0$.

(ii) If $|D| \neq \emptyset$ and $v_0 \notin \operatorname{supp}|D|$, then $\mathcal{U}_{\operatorname{supp}(D_{v_0}),v_0}$ is nonempty and for all $v \in \mathcal{U}_{\operatorname{supp}(D_{v_0}),v_0}$, we have $v \notin \operatorname{supp}|D|$ and D_{v_0} is also v-reduced.

Proof. If $v_0 \in \operatorname{supp}|D|$, let D' be an effective divisor such that $D' \in |D|$ and $D'(v_0) > 0$. Applying Algorithm 2.2.13 for D' and v_0, we can derive D_{v_0}. Note that $D_{v_0}(v_0) \geqslant D'(v_0)$. Thus $D_{v_0}(v_0) > 0$.

If $|D| \neq \emptyset$ and $v_0 \notin \operatorname{supp}|D|$, then $D_{v_0}(v_0) = 0$, which means $\mathcal{U}_{\operatorname{supp}(D_{v_0}),v_0}$ is nonempty. For all $v \in \mathcal{U}_{\operatorname{supp}(D_{v_0}),v_0}$, clearly $D_{v_0}(v) = 0$, and using Dhar's algorithm, it is easy to see that D_{v_0} is also v-reduced. Moreover, we have $v \notin \operatorname{supp}|D|$ by (i). $\square$

Remark 2.2.19. In the sense of Corollary 2.2.18(ii), if X is a subset of $\mathcal{U}_{\operatorname{supp} D_{v_0},v_0}$, then we may also say D_{v_0} is X-reduced.

Remark 2.2.20. Corollary 2.2.18 is what we are going to employ in the next section.

2.3 Rank-determining sets

We say a subset Γ' of a metric graph Γ is a *subgraph* of Γ if Γ' is connected and closed. Let Ω be a vertex set of Γ. Then $(\Omega \bigcap \Gamma') \bigcup \partial\Gamma'$ (considered in Γ) is automatically a vertex set of Γ', which we call the vertex set of Γ' induced by Γ. A *tree* on Γ is a subgraph of Γ with genus 0 (or equally a contractible subgraph), and a *spanning tree* of Γ is a tree on Γ that is minimal among those which contain all vertices of Γ. We call a point v a *cut point* in a metric graph if $\Gamma \setminus v$ is disconnected.

2.3.1 A is a rank-determining set if and only if $\mathcal{L}(A) = \Gamma$

Consider a point v in a metric tree T and an effective divisor D on T such that $v \in \operatorname{supp} D$. Then for all $v' \in T$, there exist an effective divisor D' such that $D' \sim D$ and $v' \in \operatorname{supp} D'$. Actually since all divisors on T of the same degree are linearly equivalent, we can let D' be any effective divisor which has the same degree as D and has v in its support. This means that for a linear system $|D|$, whenever we know $v \in \operatorname{supp} |D|$, we know $\operatorname{supp} |D| = T$. Now we want to generalize this observation from a metric tree T to an arbitrary metric graph and from a singleton $\{v\}$ to any subset of the metric graph.

For a nonempty subset A of a metric graph Γ, we use $\mathcal{L}(A)$ to denote the maximal subset of Γ such that $\mathcal{L}(A) \subseteq \operatorname{supp} |D|$ whenever $A \subseteq \operatorname{supp} |D|$. For simplicity of notation, we denote $\mathcal{L}(\bigcup_{i=1}^n A_i)$ by writing $\mathcal{L}(A_1, A_2, \cdots, A_n)$. Note that we can always find a linear system whose support contains A (for example, the support of the linear system associated to $\sum_{v\in\Omega}(v)$ is the whole graph Γ). Therefore we can write

$$\mathcal{L}(A) = \bigcap_{\operatorname{supp} |D| \supseteq A} \operatorname{supp} |D|.$$

Obviously, $A \subseteq \mathcal{L}(A)$, and if A' is a subset of $\mathcal{L}(A)$, then $\mathcal{L}(A, A') = \mathcal{L}(A)$. In case we want to emphasize that A and all the linear systems are defined on Γ, we may write $\mathcal{L}_\Gamma(A)$ in stead of $\mathcal{L}(A)$.

Proposition 2.3.1. *Let A be a nonempty subset of Γ. The following are equivalent.*

(i) $\mathcal{L}(A) = \Gamma$.

(ii) If $r_A(D) \geqslant 1$, then $r(D) \geqslant 1$.

(iii) A is a rank-determining set of Γ.

Proof. (i)$\Leftrightarrow$(ii). $\mathcal{L}(A) = \Gamma$, if and only if $A \subseteq \operatorname{supp}|D|$ implies $\operatorname{supp}|D| = \Gamma$, if and only if $|D - E_1'| \neq \emptyset$ for all $E_1' \in \operatorname{Div}_+^1 A$, implies $|D - E_1| \neq \emptyset$ for all $E_1 \in \operatorname{Div}_+^1 \Gamma$, if and only if $r_A(D) \geqslant 1$ implies $r(D) \geqslant 1$.

(iii)$\Rightarrow$(ii). This follows directly from the definition of rank-determining sets.

(ii)$\Rightarrow$(iii). If $|D| = \emptyset$, then $r_A(D) = r(D) = -1$. We will only consider the case $|D| \neq \emptyset$ in the following. Since A is a subset of Γ, it is easy to see that $r_A(D) \geqslant r(D)$ by definition. Therefore, to prove A is a rank-determining set, it suffices to show that $r_A(D) \geqslant s$ implies $r(D) \geqslant s$ for each integer $s \geqslant 0$. The case $s = 0$ is trivial, since $\operatorname{Div}_+^0 A = \operatorname{Div}_+^0 \Gamma = 0$. And the case $s = 1$ is stated in (ii).

Let $k \in \{0, 1, \cdots, s-1\}$. We claim that if $r_A(D - E_k) \geqslant s - k$ for all $E_k \in \operatorname{Div}_+^k \Gamma$, then $r_A(D - E_{k+1}) \geqslant s - k - 1$, for all $E_{k+1} \in \operatorname{Div}_+^{k+1} \Gamma$. This can be proved by the following deduction:

$$r_A(D - E_k) \geqslant s - k, \quad \forall E_k \in \operatorname{Div}_+^k \Gamma$$

$$\Longleftrightarrow$$

$$|D - E_k - E_{s-k}'| \neq \emptyset, \quad \forall E_k \in \operatorname{Div}_+^k \Gamma, \quad \forall E_{s-k}' \in \operatorname{Div}_+^{s-k} A$$

$$\Longleftrightarrow$$

$$|(D - E_k - E_{s-k-1}') - E_1'| \neq \emptyset, \qquad \forall E_k \in \operatorname{Div}_+^k \Gamma, \quad \forall E_{s-k-1}' \in \operatorname{Div}_+^{s-k-1} A, \quad \forall E_1' \in \operatorname{Div}_+^1 A$$

$$(\text{By (ii)}) \Longrightarrow$$

$$|(D-E_k-E'_{s-k-1})-E_1| \neq \emptyset, \qquad \forall E_k \in \operatorname{Div}_+^k \Gamma, \quad \forall E'_{s-k-1} \in \operatorname{Div}_+^{s-k-1} A, \quad \forall E_1 \in \operatorname{Div}_+^1 \Gamma$$

$$\Longleftrightarrow$$

$$|D - E_{k+1} - E'_{s-k-1}| \neq \emptyset, \quad \forall E_{k+1} \in \operatorname{Div}_+^{k+1} \Gamma, \quad \forall E'_{s-k-1} \in \operatorname{Div}_+^{s-k-1} A$$

$$\Longleftrightarrow$$

$$r_A(D - E_{k+1}) \geqslant s - k - 1, \quad \forall E_{k+1} \in \operatorname{Div}_+^{k+1} \Gamma.$$

Therefore, by applying the above deduction for k going from 0 through $s-1$, we have:

$$\begin{aligned}
r_A(D) \geqslant s \qquad & \Longrightarrow \\
r_A(D - E_1) \geqslant s - 1, \quad \forall E_1 \in \operatorname{Div}_+^1 \Gamma \qquad & \Longrightarrow \\
\cdots \qquad & \Longrightarrow \\
r_A(D - E_{s-1}) \geqslant 1, \quad \forall E_{s-1} \in \operatorname{Div}_+^{s-1} \Gamma \qquad & \Longrightarrow \\
r_A(D - E_s) \geqslant 0, \quad \forall E_s \in \operatorname{Div}_+^s \Gamma \qquad & \Longleftrightarrow \\
r(D) \geqslant s. \qquad &
\end{aligned}$$

Thus (ii) is sufficient to make A a rank-determining set of Γ. □

2.3.2 Special open sets and a criterion for $\mathcal{L}(A)$

By the definition of reduced divisors, we observe that by just knowing an effective divisor D is v_0-reduced, we can say something about $\mathcal{U}_{\operatorname{supp} D \setminus v_0, v_0}$. Actually it cannot be an arbitrary connected open set. We define "special open sets" to describe these sets.

Definition 2.3.2. A connected open subset U of Γ is called a *special open set* on Γ if either $U = \emptyset$ or Γ, or every connected component X of U^c contains a boundary point v such that $\operatorname{outdeg}_X(v) \geqslant 2$. In particular, we say Γ is *trivial* if $U = \emptyset$ or Γ. And we use $\mathcal{S}_\Gamma$ to denote the set of all special open sets on Γ.

Lemma 2.3.3 through 2.3.7 present some simple properties of special open sets.

Lemma 2.3.3. *Let U be a connected open set on Γ, and $D = \sum_{v\in\partial U}(v)$. Then U is a special open set if and only if D is U-reduced.*

Proof. We just need to consider U nontrivial. And it follows directly by running Dhar's algorithm for D and any point $v \in U$. $\square$

Lemma 2.3.4. *For $v_0 \in \Gamma$, if D is a v_0-reduced divisor, then $\mathcal{U}_{\operatorname{supp} D\setminus v_0, v_0}$ is a special open set.*

Proof. Let $D' = \sum_{v\in \operatorname{supp} D\setminus v_0}(v)$. Since D is a v_0-reduced divisor, D' must also be v_0-reduced. Thus $\mathcal{U}_{\operatorname{supp} D\setminus v_0, v_0}$ is a special open set by Lemma 2.3.3. $\square$

Lemma 2.3.5. *Let Γ be a metric graph of genus g. If U is a nontrivial special open set on Γ, then $\overline{U}$ has genus at least 1. In addition, every family of pairwise disjoint special open sets of Γ has at most g members.*

Proof. Suppose U is a nontrivial special open set such that $\overline{U}$ is a tree. Then for every $v \in \partial U$, $\operatorname{outdeg}_{U^c}(v) = 1$, which contradicts the definition of special open sets. And it follows immediately that Γ can sustain at most g disjoint nonempty special open set. $\square$

Lemma 2.3.6. *Let X be a nonempty connected subset of Γ, and $|D|$ a linear system such that $\operatorname{supp}|D| \bigcap X = \emptyset$. Then there exists a special open set U such that $X \subseteq U \subseteq (\operatorname{supp}|D|)^c$.*

Proof. Let $v \in X$ and D' be the v-reduced divisor in $|D|$. Then by Corollary 2.2.18 and Lemma 2.3.4, $\mathcal{U}_{\operatorname{supp} D', v}$ is a special open set with the desired properties. $\square$

Lemma 2.3.7. *Let D be a divisor on Γ and $|D|$ the corresponding linear system. Then $(\operatorname{supp}|D|)^c$ is a disjoint union of finitely many nonempty special open sets.*

Proof. Let v_1 and v_2 be two points in $(\operatorname{supp}|D|)^c$. Let D_1 and D_2 be elements of $|D|$ that are v_1-reduced and v_2-reduced, respectively. Let $U_1 = \mathcal{U}_{\operatorname{supp} D_1, v_1}$ and $U_2 = \mathcal{U}_{\operatorname{supp} D_2, v_2}$. Then by Lemma 2.3.4, U_1 and U_2 are special open sets. In addition, we have either $U_1 = U_2$ or $U_1 \bigcap U_2 = \emptyset$ by Corollary 2.2.18. Thus $(\operatorname{supp}|D|)^c$ must be a disjoint union of nonempty special open sets. And we know from Lemma 2.3.5 that there are only finitely many of them. $\square$

Based on the notion of special open sets, we formulate a sufficient condition for v to belong to $\mathcal{L}(A)$, as stated in the following theorem. (We will show in Theorem 2.3.17 that it is also a necessary condition.)

Theorem 2.3.8. *Let $v \in \Gamma$ and let A be a nonempty subset of Γ. Then $v \in \mathcal{L}(A)$ if for all special open sets U containing v, we have $A \bigcap U \neq \emptyset$. Moreover,*

$$\mathcal{L}(A) \supseteq \bigcap_{U \in \mathcal{S}_\Gamma, U \bigcap A = \emptyset} U^c.$$

In addition, A is a rank-determining set if all nonempty special open sets intersect A.

Proof. Suppose $|D|$ is a linear system such that $A \subseteq \operatorname{supp}|D|$. Then by Lemma 2.3.6, for every $v \notin \operatorname{supp}|D|$, there exists a neighborhood U of v which is a special open set disjoint from $\operatorname{supp}|D|$. Thus if all special open sets containing v intersect A, then $A \subseteq \operatorname{supp}|D|$ implies $v \in \operatorname{supp}|D|$, which means $v \in \mathcal{L}(A)$. It follows immediately that

$$\mathcal{L}(A) \supseteq \bigcap_{U \in \mathcal{S}_\Gamma, U \bigcap A = \emptyset} U^c.$$

If all nonempty special open sets intersect A, then $\mathcal{L}(A) = \Gamma$. Thus A is a rank-determining set by Proposition 2.3.1. $\square$

Proposition 2.3.9. *Let U be a nonempty connected open proper subset of Γ such that $\overline{U}$ is a tree. Then $\overline{U} \subseteq \mathcal{L}(\partial U)$.*

Proof. ∂U is nonempty since U is a proper subset of Γ. Then by Lemma 2.3.5, for every $v \in U$, if U' is a special open set containing v, then $\overline{U'}$ has genus at least 1 unless possibly U' is the whole graph. Thus U' must intersect ∂U, since any connected closed subset of $\overline{U}$ has genus 0. Therefore we have $v \in \mathcal{L}(\partial U)$ by Theorem 2.3.8. $\square$

Example 2.3.10. (a) Let Ω be an arbitrary vertex set of Γ. By Proposition 2.3.9, we immediately have $[w_i, w_j] \subseteq \mathcal{L}(w_i, w_j)$ for two adjacent vertices w_i and w_j (note that it doesn't matter whether there are multiple edges between w_i and w_j). Thus $\mathcal{L}(\Omega) = \Gamma$, which implies Ω is a rank-determining set of Γ, as claimed in **Theorem 2.1.6**.

(b) Let A be a finite set formed by choosing one internal point from each edge. Then it is also easy to show that A is a rank-determining set using Proposition 2.3.9.

Proposition 2.3.11. *Let U be a nonempty connected open proper subset of a metric graph Γ such that $\overline{U}$ has genus g'. Let T be a spanning tree of $\overline{U}$. Then $U \setminus T$ is a disjoint union of g' open segments. Choosing one point from each of these segments, we get a finite set B of cardinality g'. Then $\overline{U} \subseteq \mathcal{L}(\partial U, B)$*

Proof. If $g' = 0$, then $\overline{U} \subseteq \mathcal{L}(\partial U)$ by Proposition 2.3.9. Now we suppose $g' \geqslant 1$. Consider a point $v \in U$. If $v \notin \mathcal{L}(\partial U)$, then there exists a special open set U' such that $v \in U'$ and $U' \subseteq U$ by Theorem 2.3.8. We claim that $U' \bigcap B \neq \emptyset$, which implies $v \in \mathcal{L}(\partial U, B)$.

Denote the g' open segments of $U \setminus T$ by $e_1, e_2, \cdots, e_{g'}$. If $U' \bigcap T$ is not connected, then there must exist some $e_i \subseteq U' \setminus T$ to make U' connected. Thus $U' \bigcap B \neq \emptyset$. Now suppose $U' \bigcap T$ is connected. By definition of special open sets, every connected component of $(U')^c$ contains a boundary point with out-degree at least 2, which means that there exists some $e_i \subseteq U' \setminus T$ having one end in $\partial U'$ and the other in $U' \bigcap T$. Thus we also have $U' \bigcap B \neq \emptyset$. $\square$

Remark 2.3.12. **Theorem 2.1.7** can be deduced from Proposition 2.3.11 by the following argument. Let Γ be a metric graph of genus g and T a spanning tree of Γ.

Then $\Gamma \setminus T$ is a disjoint union of g open segments $e_1, e_2, \cdots, e_g$. Choose an arbitrary point v_0 from T, and an arbitrary point v_i from e_i for $i = 1, 2, \cdots, g$. Let $A = \{v_0, v_1, \cdots, v_g\}$. If v_0 is not a cut point, then we can directly apply Proposition 2.3.11 to $\Gamma \setminus v_0$ and conclude that $\mathcal{L}(A) = \Gamma$. Otherwise, applying Proposition 2.3.11 to each connected component X of $\Gamma \setminus v_0$ (note that the induced spanning tree of $\overline{X}$ is $T \bigcap \overline{X}$), we also get $\mathcal{L}(A) = \Gamma$. Therefore A is a rank-determining set of cardinality $g+1$ as desired.

Remark 2.3.13. For readers who know some algebraic geometry, we sketch Varley's proof of **Theorem 2.1.8** here (see chapter 4 of [43] for some terms used in this proof). Consider a nonsingular projective algebraic curve C. First note that the rank $r(D)$ of a divisor D on C has the same value as $\dim L(D) - 1$. Recall that we say a point $p \in C$ is a *base point* of a linear system $|D|$ if p belongs to the support of every element of $|D|$, i.e., $p \in \mathrm{BL}(|D|)$ where $\mathrm{BL}(|D|) = \bigcap_{D' \in |D|} \operatorname{supp} D'$ which is called the *base locus* of $|D|$. Varley's argument uses the fact that a point $p \in C$ is a base point of $|D|$ if and only if $r(D - (p)) = r(D)$. (Note that this is not true for metric graphs.) Take any set S of $g+1$ distinct points on C. To prove that S is a rank-determining set, it suffices to show that for a divisor D on C, if $r(D) \geqslant 0$, then there exists a point p in S such that $r(D - (p)) = r(D) - 1$. Let $B = \sum_{q \in \mathrm{BL}(|D|)} (q)$ which is the full base locus divisor of $|D|$. Note that $|B| = \{B\}$ since B cannot "move". If $\deg(B) \leqslant g$, then there is a point p of S not contained in $\mathrm{BL}(|D|)$, which means $r(D - (p)) = r(D) - 1$. If $\deg(B) \geqslant g+1$, then $r(B) \geqslant 1$ (by Riemann-Roch) which is impossible. The desired result follows by induction.

Example 2.3.14. Let Γ be a metric graph corresponding to K_4 with a vertex set Ω being $\{w_1, w_2, w_3, w_4\}$ as shown in Figure 4. Clearly Ω itself is a rank-determining set by Theorem 2.1.6. But a proper subset of Ω can also be a rank-determining set. Note that $[w_1, w_3] \bigcup [w_2, w_3] \bigcup [w_4, w_3]$ is a spanning tree of Γ, which implies $w_3 \in \mathcal{L}(w_1, w_2, w_4)$ by Proposition 2.3.9. Thus $\{w_1, w_2, w_4\}$ is a rank-determining

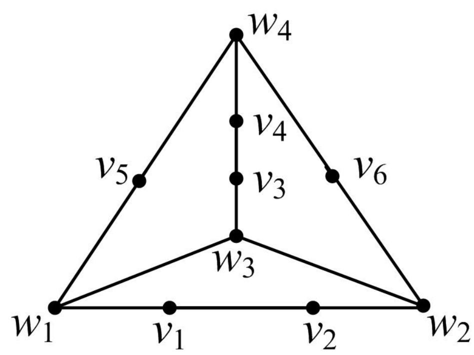

Figure 4: A metric graph corresponding to K_4.

set as desired. Let $v_1, v_2, \cdots, v_6$ be some internal points. It is also easy to see that $\{w_3, v_1, v_5, v_6\}$ and $\{v_1, v_3, v_5, v_6\}$ are rank-determining sets by Proposition 2.3.11. We recommend the reader to use Theorem 2.3.8 to verify that $\{v_1, v_2, v_3, v_4\}$ is another rank-determining set, which is not obvious at first sight.

Remark 2.3.15. We see from Example 2.3.14 that a proper subset of a vertex set can also be rank-determining. Recall that a vertex cover is a set of vertices such that each edge is incident to at least one vertex of the set. In fact, for every metric graph and a vertex set which does not allow multiple edges, all vertex covers are rank-determining sets, following from Proposition 2.3.9. We may even delete some points from a minimal vertex cover, while still keeping the set rank-determining. We will discuss such a problem in general using the notion *minimal rank-determining sets* in Section 2.3.4.

Proposition 2.3.16. *Let U be a special open set on Γ. Then there exists a divisor D such that* $\operatorname{supp}|D| = U^c$.

Proof. We only need to consider U nontrivial. Assume $(\partial U)^c$ has n connected components $X_1, X_2, \cdots, X_n$ other than U. Let T_i be a spanning tree of $\overline{X}_i$, $i = 1, 2, \cdots, n$. Then $X_i \setminus T_i$ is a disjoint union of g_i open segments. Choosing one point from each of these segments, we get a finite set B_i of cardinality g_i. Let $B = \bigcup_{i=1}^n B_i$ and $D = \sum_{v \in \partial U}(v) + \sum_{v \in B}(v)$. Then by Proposition 2.3.11, we have $U^c = \bigcup_{i=1}^n \overline{X}_i \subseteq$

$\mathcal{L}(\partial U, B) \subseteq \operatorname{supp}|D|$. Therefore, to prove $\operatorname{supp}|D| = U^c$, it suffices to show that D is U-reduced.

Let $D' = \sum_{v \in \partial U}(v)$. Then D' is U-reduced since U is a special open set. Thus by running Dhar's algorithm for D' and a point in U step by step and taking the set of non-saturated points in each step, we can get a partition of ∂U by $N'_0, N'_1, \cdots, N'_{K-1}$. Note that for every X_i, there exists some N'_k such that either ∂X_i is a subset of N'_k or X_i connects points in $\partial X_i \bigcap N'_k$ and $\partial X_i \bigcap N'_{k+1}$, i.e., $\partial X_i \bigcap N'_k$ and $\partial X_i \bigcap N'_{k+1}$ are nonempty and $\partial X_i \subseteq N'_k \bigcup N'_{k+1}$. Therefore we may define a function $\lambda : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, K-1\}$ by $\lambda(i) = k$ if $\partial X_i \bigcap N'_k \neq \emptyset$ and $\partial X_i \bigcap N'_{k-1} = \emptyset$. Let $N_k = (\bigcup_{\lambda(i)=k} B_i) \bigcup N'_k$ for $k = 0, 1, \cdots, K-1$. Obviously these N_k's form a partition of $\partial U \bigcup B$. Running Dhar's algorithm for D and a point in U step by step, we observe that the set of non-saturated points in each step is precisely $N_0, N_1, \cdots, N_{K-1}$ in sequence. Therefore the output is empty, which means D is U-reduced. $\square$

Now we come to the main conclusion of this subsection, which states that the condition in Theorem 2.3.8 is both necessary and sufficient.

Theorem 2.3.17 (Criterion for $\mathcal{L}(A)$ and rank-determining sets). *Let $v \in \Gamma$ and let A be a nonempty subset of Γ. Then $v \in \mathcal{L}(A)$ if and only if for all special open sets U containing v, we have $A \bigcap U \neq \emptyset$. Furthermore,*

$$\mathcal{L}(A) = \bigcap_{U \in \mathcal{S}_\Gamma, U \bigcap A = \emptyset} U^c.$$

In addition, A is a rank-determining set if and only if all nonempty special open sets intersect A.

Proof. We just need to prove that if $v \in \mathcal{L}(A)$, then all special open sets containing v must intersect A.

Suppose for the sake of contradiction that there exists $U \in \mathcal{S}_\Gamma$ such that $v \in U$ and $A \bigcap U = \emptyset$. Then by Proposition 2.3.16, there exists a divisor D such that supp $|D| = U^c$. Thus we have $A \subseteq$ supp $|D|$, which means that $\mathcal{L}(A) \subseteq$ supp $|D|$. But then $v \notin \mathcal{L}(A)$. $\square$

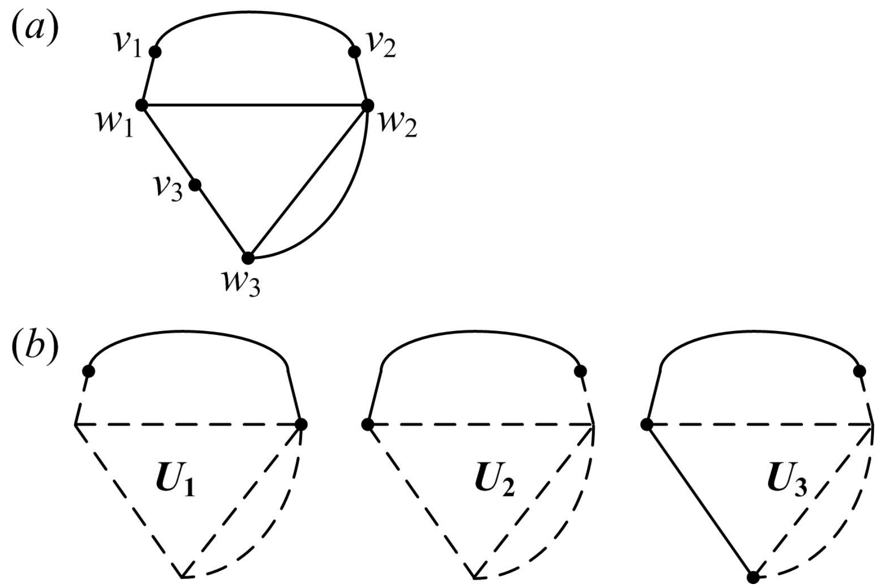

Figure 5: (a) A metric graph with a vertex set $\{w_1, w_2, w_3\}$. (b) Three examples of special open sets disjoint from $\{v_1, v_2\}$.

Example 2.3.18. Let Γ be a metric graph with a vertex set $\{w_1, w_2, w_3\}$ as shown in Figure 5(a), and let v_1, v_2, v_3 be some internal points. Clearly $[v_1, v_2] \subseteq \mathcal{L}(v_1, v_2)$. The dashed areas of Figure 5(b), U_1, U_2 and U_3, are three examples of special open sets disjoint from $\{v_1, v_2\}$. Hence we have $\mathcal{L}(v_1, v_2) = [v_1, v_2]$ by Theorem 2.3.17. Now let us consider $\mathcal{L}(v_1, v_2, v_3)$. We observe that any special open set disjoint from $\{v_1, v_2, v_3\}$ must be a subset of U_3, which implies $\mathcal{L}(v_1, v_2, v_3) = U_3^c$.

2.3.3 Consequences of the criterion

Corollary 2.3.19. *Let A be a nonempty subset of Γ. If A^c has n connected components $X_1, X_2, \cdots, X_n$, then A is a rank-determining set if and only if $X_i \subseteq \mathcal{L}(\partial X_i)$, for $i = 1, 2, \cdots, n$.*

Proof. For a point $v \in X_i$, if a special open set U containing v intersects A, then U must intersect ∂X_i. Thus by Theorem 2.3.17, A is a rank-determining set, if and only if all nonempty special open sets intersect A, if and only if for all $v \in \Gamma$, if $v \in X_i$, then all special open sets U containing v intersect ∂X_i, if and only if $X_i \subseteq \mathcal{L}(\partial X_i)$, for $i = 1, 2, \cdots, n$. □

Corollary 2.3.20. *Let Γ be a metric graph with a cut point v. Let Γ' be the closure of a connected component of $\Gamma \setminus v$. Then for every nonempty subset A of Γ', we have $\mathcal{L}_{\Gamma'}(A) \subseteq \mathcal{L}_{\Gamma}(A)$.*

Proof. For $v' \in \Gamma'$, if $v' \notin \mathcal{L}_{\Gamma}(A)$, then there exists $U \in \mathcal{S}_{\Gamma}$ such that $v' \in U$ and $U \bigcap A = \emptyset$ by Theorem 2.3.17. Then $U \bigcap \Gamma' \in \mathcal{S}'_{\Gamma}$, which means $v' \notin \mathcal{L}_{\Gamma'}(A)$. □

Proposition 2.3.21. *Let Γ be a metric graph with a vertex set Ω and A a finite rank-determining set of Γ. Suppose there exists a point v in A which has degree $m \geqslant 2$ and is not a cut point of Γ. Let U_v be an open neighborhood of v such that $(U_v \setminus v) \bigcap (\Omega \bigcup A) = \emptyset$. Denote $\Gamma - U_v$ by Γ'. Then Γ' is a subgraph of Γ and $A \setminus v$ is a rank-determining set of Γ'.*

Proof. Γ' is connected since v is not a cut point of Γ and $U_v \setminus v$ contains no vertices. Thus Γ' is a subgraph of Γ.

Clearly $U_v \setminus v$ is a disjoint union of m open segments. Denote these open segments by $e_1, e_2, \cdots, e_m$. Note that the total number of e_i's ends other than v may be strictly less than m because of the existence of multiple edges.

Suppose $A \setminus v$ is not a rank-determining set of Γ'. Then there exists $U' \in \mathcal{S}_{\Gamma'}$ disjoint from A by Theorem 2.3.17. Without loss of generality, we assume that m' is an integer such that e_i has an end in U' for $1 \leqslant i \leqslant m'$ and e_i has no end in U' for $m' < i \leqslant m$. Let $U = U' \bigcup (\bigcup_{i=1}^{m'} e_i)$. Obviously U is a connected open set on Γ disjoint from A. We claim $U \in \mathcal{S}_{\Gamma}$. This is because if $m' < m$, then $(\bigcup_{i=m'+1}^{m} e_i) \bigcup v$ may glue together some of the connected components of $\Gamma' - U'$ into one connected component of $\Gamma - U$ while the out-degrees of those boundary points are unchanged, and if $m' = m$, then v itself forms a connected component of $\Gamma - U$ and has out-degree at least 2. But this means A is not a rank-determining set of Γ by Theorem 2.3.17, a contradiction. $\square$

Remark 2.3.22. The converse proposition of Proposition 2.3.21 is not true. That is, A is not guaranteed to be a rank-determining set of Γ by $A \setminus v$ being a rank-determining set of Γ'. For example, let Γ be the metric graph corresponding to K_4 as shown in Figure 4. Let $\Gamma' = [w_1, w_2] \bigcup [w_2, w_4] \bigcup [w_4, w_1]$. Then $\{v_5, v_6\}$ is a rank-determining set of Γ'. However $\{v_5, v_6, w_3\}$ is not a rank-determining set of Γ.

It is clear that special open sets are preserved under homeomorphisms since out-degrees are topological invariants. Thus Theorem 2.3.17 tells us that rank-determining sets are also preserved under homeomorphisms (**Theorem 2.1.10**). The following theorem provides a more general description of this fact.

Theorem 2.3.23. *Let $f : \Gamma \to \Gamma'$ be a homeomorphism between two metric graphs Γ and Γ'. Let A be a nonempty subset of Γ. Then $\mathcal{L}_{\Gamma'}(f(A)) = f(\mathcal{L}_{\Gamma}(A))$. In particular, A is a rank-determining set of Γ if and only if $f(A)$ is a rank-determining set of Γ'.*

For a closed segment e on a metric graph Γ, we say $\phi_e : \Gamma \to \Gamma'$ is an *edge contraction* of Γ with respect to e if ϕ_e merges together all the points in e into a single point while keeping every point in $\Gamma \setminus e$ unchanged. Clearly an edge contraction ϕ_e may change the topology of Γ. We now give some some examples which show that rank-determining sets may not be preserved under edge contractions.

Example 2.3.24. (a) Consider a metric graph Γ corresponding to K_4 as in Example 2.3.14. An edge contraction with respect to $[w_2, w_3]$ results in a new graph Γ' (Figure 6(a)). Let v_1', v_2', v_3', v_4', w_1', w_4' and w' be the points in Γ' corresponding to v_1, v_2, v_3, v_4, w_1, w_4 and $[w_2, w_3]$, respectively. We know that $\{v_1, v_2, v_3, v_4\}$ is a rank-determining set of Γ. However, as shown in Figure 6(a), U is a special open set disjoint from $\{v_1', v_2', v_3', v_4'\}$. Thus $\{v_1', v_2', v_3', v_4'\}$ is not a rank-determining set of Γ'.

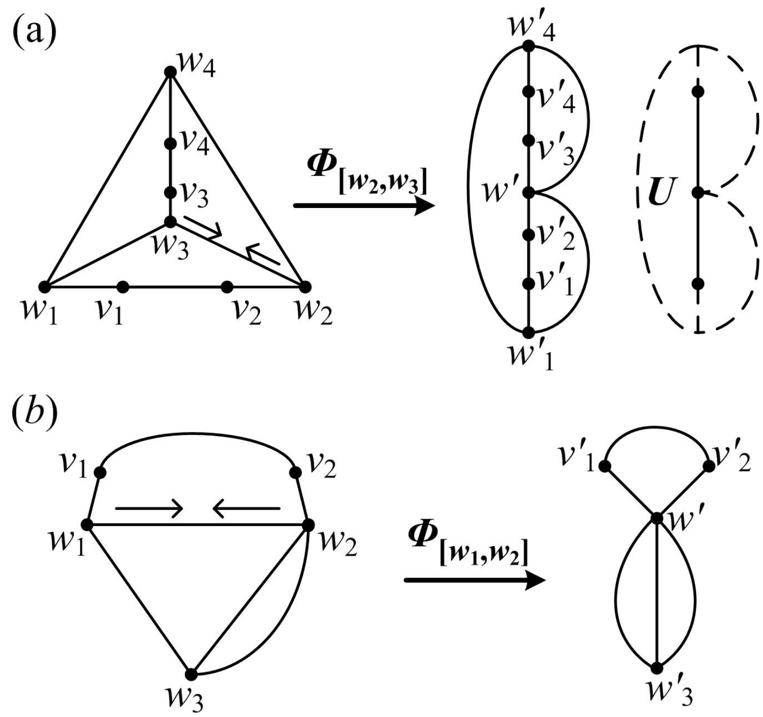

Figure 6: Two examples illustrating that edge contractions do not maintain rank-determining sets.

(b) Now let Γ be the metric graph as in Example 2.3.18. By contracting $[w_1, w_2]$, we get a new graph Γ' (Figure 6(b)). Let v_1', v_2', w_3' and w' be the points in Γ' corresponding to v_1, v_2, w_3 and $[w_1, w_2]$, respectively. Note that $w' \in \mathcal{L}_{\Gamma'}(v_1', v_2')$ by Corollary 2.3.20. Thus $\{v_1', v_2', w_3'\}$ is a rank-determining set of Γ'. However, $\{v_1, v_2, w_3\}$ is not a rank-determining set of Γ.

2.3.4 Minimal rank-determining sets

Definition 2.3.25. We say that a rank-determining set A of Γ is *minimal* if $A \setminus v$ is not a rank-determining set for every $v \in A$.

It is easy to see from Proposition 2.3.9 that minimal rank-determining sets must be finite. In particular, the intersection of a minimal rank-determining set and an edge contains at most 2 points. We have the following criterion for minimal rank-determining sets as an immediate corollary of Theorem 2.3.17.

Proposition 2.3.26. *Let A be a subset of a metric graph Γ. Then A is a minimal rank-determining set if and only if*

(i) all nonempty special open sets intersect A, and

(ii) for every point $v \in A$, there exists a special open set that intersects A only at v.

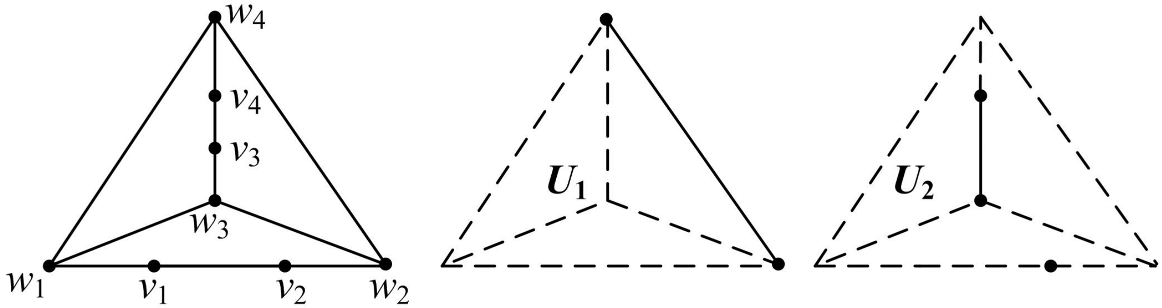

Figure 7: Two examples of special open sets on the metric graph corresponding to K_4.

Example 2.3.27. Let us reconsider a metric graph corresponding to K_4 as in Example 2.3.14. Let $U_1 = \Gamma \setminus [w_2, w_4]$ and $U_2 = \Gamma \setminus [w_3, w_4] \setminus \{v_2\}$, shown as the dashed areas of Figure 7. Then U_1 and U_2 are two special open sets. Let $A_1 = \{w_1, w_2, w_4\}$ and $A_2 = \{v_1, v_2, v_3, v_4\}$. By Example 2.3.14, A_1 and A_2 are both rank-determining sets. We claim that they are actually minimal rank-determining sets. Note that the points in $A_{1,2}$ are symmetrically distributed. Thus by Proposition 2.3.26, to show they are minimal, it only requires us to find some special open sets that intersect A_1 or A_2 at exactly one point. We observe that $U_1 \bigcap A_1 = \{w_1\}$ and $U_2 \bigcap A_2 = \{v_1\}$. Thus U_1 and U_2 are the desired special open sets.

We've given a proof of Theorem 2.1.7 by showing constructively that a family of finite subsets of Γ, all having cardinality $g + 1$, are rank-determining sets. Now we will prove that these rank-determining sets are minimal.

Proposition 2.3.28. *Let Γ be a metric graph of genus g and let T be a spanning tree of Γ. Denote the g disjoint open segments of $\Gamma \setminus T$ by $e_1, e_2, \cdots, e_g$. Choose arbitrarily* *a point v_0 from T and a point v_i from e_i for $i = 1, 2, \cdots, g$. Let $A = \{v_0, v_1, \cdots, v_g\}$. Then A is a minimal rank-determining set of Γ.*

Proof. It suffices to find $g+1$ special open sets $U_0, U_1, \cdots, U_g$ such that $U_i \bigcap A = \{v_i\}$ for $i = 0, 1, \cdots, g$ by Proposition 2.3.26.

Let $U_0 = \Gamma \backslash \{v_1, \cdots, v_g\}$. Clearly U_0 is connected and $U_0 \bigcap A = \{v_0\}$. It is easy to see that U_0 is a desired special open set. Now let us find the remaining g special open sets as required. Without loss of generality, we only need to find U_1 for v_1. Let u_a and u_b be the two ends of e_1. Note that if x and y are two points (not necessarily distinct) in T, then there exists a unique simple path (no repeated points) on T connecting x and y, which we denote $\Lambda_T^{[x,y]}$. We observe that $\Lambda_T^{[u_a,u_b]} \bigcap \Lambda_T^{[u_a,v_0]} \bigcap \Lambda_T^{[u_b,v_0]}$ contains exactly one point, which we denote u_c. Let $U_1 = \mathcal{U}_{\{u_c, v_2, \cdots, v_g\}, v_1}$. Then $U_1 \bigcap A = \{v_1\}$ and a connected component of U_1^c is either a single point in $\{v_2, \cdots, v_g\}$ or a closed subset X of Γ with u_c on its boundary such that $\mathrm{outdeg}_X(u_c) = 2$. Thus U_1 is a special open set intersecting A only at v_1. It follows that A is a minimal rank-determining set of Γ. □

Example 2.3.29. The cardinality of minimal rank-determining sets are not necessarily smaller than or equal to $g + 1$. Here is an example called "loops of loops" which is constructed by inserting a series of loops into one loop (this example first appears in [50]). Figure 8 shows such a metric graph of genus 4. Consider a vertex set $V = \{v_1, v_2, v_3, w_1, w_2, w_3\}$. Then V is a minimal rank-determining set since by removing one point from V one can always generate a special open set not intersecting the remaining points in V. However, the cardinality of V is 6, larger than $g+1$ where $g = 4$.

2.4 Further topics of rank-determining sets

Here we will mention several interesting topics related to the theory rank-determining sets.

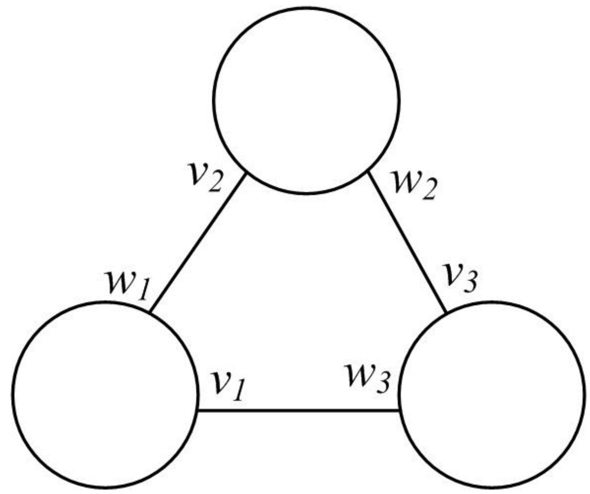

Figure 8: A "loops of loops" metric graph of genus 4.

Recently, Backman gave an elegant reproof to the main criterion for rank-determining sets (see Theorem 2.3.17) using a theory of generalized cycle, cocycle reversal systems he developed for investigating partial orientations on graphs in his thesis [9]. He showed that for each partial orientation $\mathcal{O}$ on a finite graph one may associate a divisor $D_{\mathcal{O}}$ with $D_{\mathcal{O}}(v) = \mathrm{indeg}(v) - 1$ where $\mathrm{indeg}(v)$ is the number of edges oriented towards v in $\mathcal{O}$, and a divisor associated to an acyclic orientation has negative rank. He proved that the Baker-Norine rank of a divisor $D_{\mathcal{O}}$ is one less than the minimum number of directed paths which need to be reversed in the generalized cocyle reverse system to produce an acyclic orientation. He also noticed that orientation problems on metric graph can be reduced to finite graph by "pushing" the change of direction to one of the two incident vertices. With a thorough investigation, he used his newly developed techniques to prove strengthened versions of Baker-Norine's criteria RR1 and RR2 for Riemann-Roch theorem [13], the criterion for rank-determining sets (Theorem 2.3.17) and some other interesting results.

In the tropical proof the Brill-Noether theorem in [28], a metric graph which is composed of a chain of loops with generic edge lengths (Figure 9) is proved to be Brill-Noether general metric graph, i.e., satisfying the following theorem. Then together with Baker's specialization lemma [11], the conventional Brill-Noether theorem for

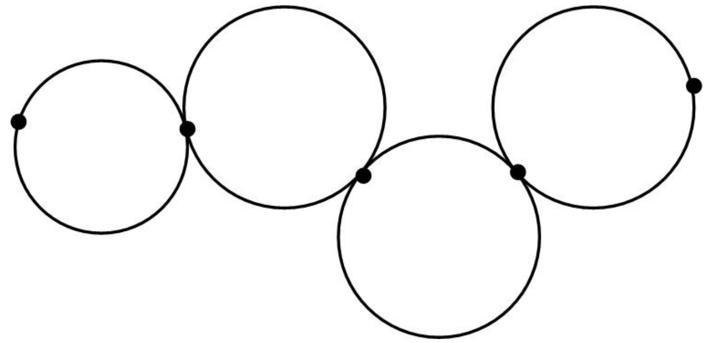

Figure 9: A "chain of loops" metric graph of genus 4.

algebraic curves is proved tropically.

Theorem 2.4.1. *Let Γ be a chain of g loops with generic edge lengths. Let $\rho = g-(r+1)(g-d+r)$. Let $W^r_d(\Gamma)$ be the subset of the real torus $Pic_d(\Gamma)$ parametrizing divisor classes of degree d and rank at least r.*

(i) If ρ is negative then Γ has no effective divisor of degree d and rank at least r.

(ii) If ρ is negative then the dimension of $W^r_d(\Gamma)$ is $\min(\rho, g)$.

The proof of the above theorem involves an explicit computation of rank of all divisors which use a vertex set of Γ as a rank-determining set.

Amini and Baker [4] introduced the notion of metrized complexes and developed a theory of limit linear series on metrized complexes (see also Chapter 4). Roughly speaking, a metrized complex is a metric graph with curves associated to its vertices. We have defined and investigated the notion of rank-determining sets for both metric graphs and algebraic curves in this chapter, and they extended this notion to metrized complexes which was applied in their proof of the equivalence of their notion of limit linear series when restricted to curves of compact type and the notion of limit linear series of Eisenbud-Harris theory [32].

CHAPTER III

TROPICAL CONVEXITY ON LINEAR SYSTEMS, GENERAL REDUCED DIVISORS AND CANONICAL PROJECTIONS

3.1 Introduction

3.1.1 Notations and terminologies

Throughout this chapter, we stick to the following very basic notations and terminologies. Let Γ be a compact metric graph with finite edge lengths. For simplicity, we also denote the set of points of Γ by Γ. Let $\mathrm{Div}(\Gamma)$ be the free abelian group on Γ. Let $\mathbb{R}\,\mathrm{Div}(\Gamma) = \mathrm{Div}(\Gamma)\otimes\mathbb{R}$. As in convention, we call the elements of $\mathrm{Div}(\Gamma)$ *divisors* (or $\mathbb{Z}$-*divisors* when we want to emphasize the integer coefficients), and elements of $\mathbb{R}\,\mathrm{Div}(\Gamma)$ $\mathbb{R}$-*divisors*. In cases of no confusion, we may also call $\mathbb{R}$-divisors just as divisors throughout this paper. Let $\mathrm{Div}_+(\Gamma)$ and $\mathbb{R}\,\mathrm{Div}_+(\Gamma)$ be the semigroups of effective $\mathbb{Z}$-divisors and effective $\mathbb{R}$-divisors respectively. If d is a nonnegative integer, denote the set of effective divisors of degree d by $\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$. If d is a nonnegative real, denote the set of effective $\mathbb{R}$-divisors of degree d by $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$.

For a continuous function f on Γ. Let $\mathcal{N}(f) = f - \min f$. Let $\Gamma_{\min}(f) := f^{-1}(\min f) = \{v \in \Gamma | f(v) = \min f\}$ and $\Gamma_{\max}(f) := f^{-1}(\max f) = \{v \in \Gamma | f(v) = \max f\}$. In other words, $\Gamma_{\min}(f)$ and $\Gamma_{\max}(f)$ are the minimizer and maximizer of f respectively.

3.1.2 Overview

There are several equivalent ways [54, 56] to characterize reduced divisors. Recently, Baker and Shokrieh made a connection to potential theory on (metric) graphs [19].

The main tool in their theory is the *energy pairing*, and for a fixed $q \in \Gamma$, it can be used to define two functions on the divisor group, the energy function $\mathcal{E}_q$ and the b-function b_q. Then the reduced divisor in $|D|$ with respect to q is the minimizer of either $\mathcal{E}_q$ or b_q. In this chapter, we are particularly interested in b-functions and have made an extension in our settings.

In [39], the authors studied the linear systems using the conventional theory of tropical convexity [29]. In this sense, complete linear systems are tropically convex. In this chapter, we have also generalized the notion of tropical convexity. More specifically, we have developed a geometric foundation for the notion of tropical convexity in the space of all $\mathbb{R}$-divisors. In particular, we have found a canonical metric structure on the space of divisors, which can be used to study the topology and geometry on it. The notion of tropical convexity is intrinsically built on this metric structure. In this sense, the linear systems $|D|$ are tropical-path-connected components of $\mathrm{Div}_+(\Gamma)$.

With our extended notions of b-functions and tropical convexity, we are able to generalize the notion of reduced divisors in the following sense:

1. Reduced divisors exist not only just for complete linear systems $|D|$ but also for any compact tropically convex subset of $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ with a given d.

2. Reduced divisors can be defined not only with respect to a point p on the metric graph but also any divisor $E \in \mathbb{R}\,\mathrm{Div}_+(\Gamma)$.

Using general reduced divisors, we further develop tools to investigate some basic properties of tropical convexity, e.g., the contractibility and compactness of tropical convex hulls. In addition, tropical projection maps are canonically derived from general reduced divisors.

The chapter is structured as follows. The potential theory on metric graphs is briefly reviewed in Section 3.2. We then define a metric structure on $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ and study the induced topology in Section 3.3. Our settings of tropical convexity are discussed in Section 3.4, where we also make statements of some basic properties of tropical convex sets. We introduce the notion of general reduced divisors and provide several criterions in Section 3.5. Then we investigate several particular cases about general reduced divisors on tropical segments and develop some useful tools in Section 3.6. As an application of these tools, the theorems about the basic properties of tropical convex sets (stated in Section 3.4) are proved in Section 3.7. Finally, we discuss canonical projections in Section 3.8.

3.2 Potential theory on metric graphs

We list here some standard terminologies and basic facts concerning potential theory on metric graphs. The reader may refer [10, 12] for details.

For a metric graph Γ, we let $\mathcal{C}(\Gamma)$ be the $\mathbb{R}$-algebra of continuous real-valued functions on Γ, and let $\mathrm{CPA}(\Gamma) \subset \mathcal{C}(\Gamma)$ be the vector space consisting of all continuous piecewise-affine (or piecewise-linear) functions on Γ. Note that $\mathrm{CPA}(\Gamma)$ is dense in $\mathcal{C}(\Gamma)$. Let $\mathrm{Meas}^0(\Gamma)$ be the vector space of finite signed Borel measures of total mass zero on Γ. Denote by $\mathbb{R} \in \mathcal{C}(\Gamma)$ the space of constant functions on Γ.

In terms of electric network theory, we may think of Γ as an electrical network with resistances given by the edge lengths. For $p, q, x \in \Gamma$, we define a j-function $j_q(x,p)$ as the potential at x when one unit of current enters the network at p and exits at q with q grounded (potential 0).

We have the following properties of the j-function.

1. $j_q(x,p)$ is jointly continuous in p, q and x.

2. $j_q(x,p) \in \mathrm{CPA}(\Gamma)$.

3. $j_q(q,p) = 0$.

4. $0 \leqslant j_q(x,p) \leqslant j_q(p,p)$.

5. $j_q(x,p) = j_q(p,x)$.

6. $j_q(x,p) + j_p(x,q)$ is constant for all $x \in \Gamma$. Denoted by $r(p,q)$, this constant is the effective resistance between p and q.

7. $r(p,q) = j_q(p,p) = j_p(q,q)$.

8. $r(p,q) \leqslant \mathrm{dist}_\Gamma(p,q)$ where $\mathrm{dist}_\Gamma(p,q)$ is the distance between p and q on Γ.

9. $\frac{r(p,q)}{\mathrm{dist}_\Gamma(p,q)} \to 1$ as $\mathrm{dist}_\Gamma(p,q) \to 0$.

Let $\mathrm{BDV}(\Gamma)$ be the vector space of functions of bounded differential variation [10]. Then we have $\mathrm{CPA}(\Gamma) \subset \mathrm{BDV}(\Gamma) \subset \mathcal{C}(\Gamma)$.

The Laplacian $\Delta : \mathrm{BDV}(\Gamma) \to \mathrm{Meas}^0(\Gamma)$ is defined as an operator in the following sense.

1. Δ induces an isomorphism between $\mathrm{BDV}(\Gamma) / \mathbb{R}$ and $\mathrm{Meas}^0(\Gamma)$ as vector spaces.

2. For $f \in \mathrm{CPA}(\Gamma)$, we have

$$\Delta f = \sum_{p \in \Gamma} \mathrm{ord}_p(f) \delta_p$$

where $-\mathrm{ord}_p(f)$ is the sum of the slopes of f in all tangent directions emanating from p and δ_p is the Dirac measure (unit point mass) at p. In particular, $\Delta j_q(x,p) = \delta_p(x) - \delta_q(x)$.

3. An inverse to Δ is given by

$$\nu \mapsto \int_\Gamma j_q(x,y) d\nu(y) \in \{f \in \mathrm{BDV}(\Gamma) : f(q) = 0\}.$$

3.3 A metric structure defined on $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$

If $D = \sum_{p \in \Gamma} m_p \cdot (p) \in \mathbb{R}\,\mathrm{Div}$, we let $\delta_D := \sum_{p \in \Gamma} m_p \cdot \delta_p$ with δ_p the Dirac measure at p. Let $D_1, D_2 \in \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$. Then based on the potential theory on Γ, there exists a piecewise-linear function $f_{D_2 - D_1} \in \mathrm{CPA}(\Gamma)$ on Γ such that $\Delta f_{D_2-D_1} = \delta_{D_2} - \delta_{D_1}$. Note that any two such associated functions differ in a constant. In this sense, we say

$\operatorname{div}(f) := D_2 - D_1$ is the *associated divisor* of $f_{D_2-D_1}$, and correspondingly $f_{D_2-D_1}$ is an *associated function* of $D_2 - D_1$. Then $\mathcal{N}(f_{D_2-D_1})$ has minimum 0 and is unique with D_1 and D_2 provided.

More precisely, if $D_1 = (q)$ and $D_2 = (p)$ for some $p, q \in \Gamma$, then $\mathcal{N}(f_{D_2-D_1})(x) = j_q(x,p)$. Now let $D_1 = \sum_{i=1}^{d_1} m_{1,i} \cdot (p_{1,i})$ and $D_2 = \sum_{i=1}^{d_2} m_{2,i} \cdot (p_{2,i})$ such that $D_1, D_2 \in \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$ (this means $d = \sum_{i=1}^{d_1} m_{1,i} = \sum_{i=1}^{d_2} m_{2,i}$). Then by the linearity of the Laplacian, for an arbitrary $q \in \Gamma$,

$$\sum_{i=1}^{d_1} m_{1,i} \cdot j_q(x, p_{1,i}) - \sum_{i=1}^{d_2} m_{2,i} \cdot j_q(x, p_{2,i})$$

is an associated function of $D_2 - D_1$.

Define the *distance function*

$$\rho(D_1, D_2) := \max(f_{D_2-D_1}) - \min(f_{D_2-D_1}) = \max(\mathcal{N}(f_{D_2-D_1})).$$

Immediately, we get $\rho(D_1, D_2) = 0$ if and only if $D_1 = D_2$. Furthermore, note that

$$\mathcal{N}(f_{D_3-D_1}) = \mathcal{N}(f_{D_2-D_1} + f_{D_3-D_2}).$$

By the linearity of the Laplacian, we get the triangle inequality

$$\rho(D_1, D_3) \leqslant \rho(D_1, D_2) + \rho(D_2, D_3)$$

since

$$\mathcal{N}(f_{D_2-D_1} + f_{D_3-D_2}) \leqslant \mathcal{N}(f_{D_2-D_1}) + \mathcal{N}(f_{D_3-D_2}),$$

while the equalities hold if and only if

$$\Gamma_{\min}(f_{D_2-D_1}) \bigcap \Gamma_{\min}(f_{D_3-D_2}) \neq \emptyset$$

and

$$\Gamma_{\max}(f_{D_2-D_1}) \bigcap \Gamma_{\max}(f_{D_3-D_2}) \neq \emptyset.$$

Thus ρ is well-defined as a metric on $\mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$.

Still, we let $D_1, D_2 \in \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$. Let $D_1 = D_{1,1} + D_{1,2}$ and $D_2 = D_{2,1} + D_{2,2}$. Here we suppose $D_{1,1}$ and $D_{2,1}$ are effective divisors of the same degree d_1, and $D_{1,2}$ and $D_{2,2}$ are effective divisors of the same degree d_2. By the linearity of the Laplacian, we get

$$\mathcal{N}(f_{D_2-D_1}) = \mathcal{N}(f_{D_{2,1}-D_{1,1}} + f_{D_{2,2}-D_{1,2}})$$

and

$$\rho(D_1, D_2) \leqslant \rho(D_{1,1}, D_{2,1}) + \rho(D_{1,2}, D_{2,2}),$$

since

$$D_2 - D_1 = (D_{2,1} - D_{1,1}) + (D_{2,2} - D_{1,2}).$$

The *tropical path* (or *t-path*) from D_1 to D_2 in $\mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$ is a map $P_{D_2-D_1} : [0,1] \to \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$ given by

$$P_{D_2-D_1}(t) = \Delta \min(t \cdot \rho(D_1, D_2), \mathcal{N}(f_{D_2-D_1})) + D_1.$$

In particular, $P_{D_2-D_1}(0) = D_1$ and $P_{D_2-D_1}(1) = D_2$.

Remark 3.3.1.

1. This map is well-defined since $P_{D_2-D_1}(t)$ lies in $\mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$. In other words, there exists a unique t-path from D_1 to D_2.

2. If we let $D(t) = P_{D_2-D_1}(t)$, then

$$\mathcal{N}(f_{D(t)-D_1}) = \min(t \cdot \rho(D_1, D_2), \mathcal{N}(f_{D_2-D_1})),$$

and

$$\mathcal{N}(f_{D_2-D(t)}) = \mathcal{N}(\max(t \cdot \rho(D_1, D_2), \mathcal{N}(f_{D_2-D_1}))).$$

3. $P_{D_1-D_2}$ is continuous.

We call $\operatorname{Im}(P_{D_2-D_1})$ the *tropical segment* (or *t-segment*) connecting D_1 and D_2. Note that $P_{D_2-D_1}(t) = P_{D_1-D_2}(1-t)$ and therefore $\operatorname{Im}(P_{D_2-D_1}) = \operatorname{Im}(P_{D_1-D_2})$. We say D_1 and D_2 are the *end points* of the t-segment $\operatorname{Im}(P_{D_2-D_1})$.

Given a function f with domain $[\kappa_1, \kappa_2]$ for some $\kappa_1 \leqslant \kappa_2$, we say the function $f \diamond s_\alpha$ is a *linear scaling* of f with $\alpha > 0$ the scaling factor such that $f \diamond s_\alpha(t) = f(t/\alpha)$, and the function $f \diamond \tau_\beta$ is a *linear translation* of f with β the translation factor such that $f \diamond \tau_\beta(t) = f(t - \beta)$. Then it is clear $f \diamond s_\alpha$ has domain $[\alpha\kappa_1, \alpha\kappa_2]$ and $f \diamond \tau_\beta$ has domain $[\kappa_1 + \beta, \kappa_2 + \beta]$.

$P_{D_2-D_1}$ is actually an isometry after a linear scaling. We give a basic characterization of $P_{D_2-D_1}$ in the following lemma.

Lemma 3.3.2. *For $D_1, D_2 \in \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$, we have the following fundamental properties of the t-path $P_{D_2-D_1}$.*

1. *For any $D_1', D_2' \in \operatorname{Im}(P_{D_2-D_1})$, the t-segment $\operatorname{Im}(P_{D_2'-D_1'})$ is a subset of the t-segment $\operatorname{Im}(P_{D_2-D_1})$.*

2. *Let $\hat{P}_{D_2-D_1} : [0, \rho(D_1, D_2)] \to \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$ be given by $\hat{P}_{D_2-D_1}(t) = P_{D_2-D_1} \diamond s_{\rho(D_1,D_2)}$ if $D_1 \neq D_2$ and $\hat{P}_{D_2-D_1}(0) = D_1$ if $D_1 = D_2$. Then $\hat{P}_{D_2-D_1}$ is an isometry from $[0, \rho(D_1, D_2)]$ to $\operatorname{Im}(P_{D_2-D_1})$.*

3. *The t-segment $\operatorname{Im}(P_{D_2-D_1})$ is compact and thus a closed subset of $\mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$.*

Proof. We may write uniquely $D_1' = P_{D_2-D_1}(t_1)$ and $D_2' = P_{D_2-D_1}(t_2)$ where $t_1, t_2 \in [0, 1]$. Switching the positions of D_1' and D_2' if necessary, we may assume $t_1 \leqslant t_2$. Then

$$\mathcal{N}(f_{D_2'-D_1'}) = \mathcal{N}(\max(t_1 \cdot \rho(D_1, D_2), \min(t_2 \cdot \rho(D_1, D_2), \mathcal{N}(f_{D_2-D_1})))).$$

Thus we have $\operatorname{Im}(P_{D_2'-D_1'}) = P_{D_2-D_1}([t_1, t_2]) \subseteq \operatorname{Im}(P_{D_2-D_1})$ (for statement (1)) and $\rho(D_1', D_2') = (t_2 - t_1) \cdot \rho(D_1, D_2)$ (for statement (2)).

The compactness of $\operatorname{Im}(P_{D_2-D_1})$ follows from the compactness of $[0, 1]$ and the continuity of $P_{D_2-D_1}$. □

Corollary 3.3.3. *The intersection of two t-segments in $\mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$ is again a t-segment in $\mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$.*

Proof. Let T_1 and T_2 be two t-segments in $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ with T being their intersection. Then by Lemma 3.3.2 (1), if T contains two divisors D_1 and D_2, then it must contain the whole t-segment connecting D_1 and D_2. This actually means that T is either a t-segment itself or a t-segment without one or both of the end points. But T must also be a compact closed subset of $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ by Lemma 3.3.2 (3). Thus T is a t-segment itself. $\square$

Remark 3.3.4. Suppose $D_1 \neq D_2$ and we have the t-path $P_{D_2-D_1}$ from D_1 to D_2 with an associated function $f_{D_2-D_1}$. In particular, we may assume $f_{D_2-D_1} = \mathcal{N}(f_{D_2-D_1})$. To simplify notation, we let $D(t) = P_{D_2-D_1}(t)$ and $l = \rho(D_1, D_2)$. Then it is easy to see that

1. $\Gamma_{\min}(f_{D(t)-D_1}) = \Gamma$ for $t = 0$, and $\Gamma_{\min}(f_{D(t)-D_1}) = \Gamma_{\min}(f_{D_2-D_1})$ for $t \in (0, 1]$;

2. $\Gamma_{\max}(f_{D(t)-D_1}) = f^{-1}_{D_2-D_1}([tl, l])$ for $t \in [0, 1]$, and $\Gamma_{\max}(f_{D(t)-D_1})$ shrinks as t increases; in addition, $\Gamma_{\max}(f_{D(t)-D_1})$ shrinks continuously as t increase in $(0, s)$ for some s small enough and $\lim_{t\searrow 0} \Gamma_{\max}(f_{D(t)-D_1}) = \overline{(\Gamma_{\min}(f_{D_2-D_1})^c}$.

3. $\Gamma_{\min}(f_{D_2-D(t)}) = f^{-1}_{D_2-D_1}([0, tl])$ for $t \in [0, 1]$, and $\Gamma_{\min}(f_{D_2-D(t)})$ expands as t increases; in addition, $\Gamma_{\min}(f_{D_2-D(t)})$ expands continuously as t increase in $(s', 1)$ for some s' big enough and $\lim_{t\nearrow 1} \Gamma_{\min}(f_{D_2-D(t)}) = \overline{(\Gamma_{\max}(f_{D_2-D_1})^c}$.

4. $\Gamma_{\max}(f_{D_2-D(t)}) = \Gamma$ for $t = 1$, and $\Gamma_{\max}(f_{D_2-D(t)}) = \Gamma_{\max}(f_{D_2-D_1})$ for $t \in [0, 1)$;

5. $\Gamma_{\min}(f_{D_2-D_1}) \bigcap \mathrm{supp}(D_1) \neq \emptyset$ and $\Gamma_{\max}(f_{D_2-D_1}) \bigcap \mathrm{supp}(D_2) \neq \emptyset$; and

6. Let $X = \Gamma_{\max}(f_{D(t)-D_1})$. Let X^o, X^c and ∂X be the interior, complement and boundary of X, respectively. Then $D(t)|_{X^o} = D_1|_{X^o}$, $D(t)|_{X^c} = D_2|_{X^c}$ and $D(t)|_{\partial X} \geqslant D_2|_{\partial X}$.

Lemma 3.3.5. *Let $D, D_1, D_2 \in \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$. Then the following properties are equivalent.*

1. $D \in \mathrm{Im}(P_{D_2-D_1})$.

2. $\mathrm{Im}(P_{D_2-D_1}) = \mathrm{Im}(P_{D_1-D}) \bigcup \mathrm{Im}(P_{D_2-D})$.

3. $\Gamma_{\min}(f_{D_1-D}) \bigcup \Gamma_{\min}(f_{D_2-D}) = \Gamma$.

Proof. The equivalence of (1) and (2) is straightforward from the definition of the tropical paths. The equivalence of (2) and (3) follows from the facts that $\Gamma_{\min}(f_{D_1-D}) = \Gamma_{\max}(f_{D-D_1})$ and $f_{D_2-D} + f_{D-D_1}$ is an associated function of $D_2 - D_1$. $\square$

Remark 3.3.6. One should be careful that $\rho(D_1, D_2) = \rho(D_1, D) + \rho(D_2, D)$ does not guarantee that D lies in the t-segment connecting D_1 and D_2.

Recall that Corollary 3.3.3 says we will get a t-segment by intersecting two t-segments. The following corollary tells us that if glued properly, the union of two t-segments will also be a t-segment.

Corollary 3.3.7. *For $0 \leqslant t_1 < t_2 \leqslant 1$, let $\Lambda : [0,1] \to \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ be a map such that $\Lambda|_{[0,t_2]} \diamond s_{\frac{1}{t_2}}$ is the t-path from $\Lambda(0)$ to $\Lambda(t_2)$ and $\Lambda|_{[t_1,1]} \diamond \tau_{-t_1} \diamond s_{\frac{1}{1-t_1}}$ is the t-path from $\Lambda(t_1)$ to $\Lambda(1)$. Then Λ is the t-path from $\Lambda(0)$ to $\Lambda(1)$.*

Proof. Under the assumptions, we have $\Lambda(t_1) \in \mathrm{Im}(P_{\Lambda(t_2)-\Lambda(0)}) = \Lambda([0,t_2])$ and $\Lambda(t_2) \in \mathrm{Im}(P_{\Lambda(1)-\Lambda(t_1)}) = \Lambda([t_1,1])$. Note that a special case is that $\Lambda(t_1) = \Lambda(t_2)$, which implies $\Lambda(0) = \Lambda(1) = \Lambda(t_1)$ since $t_2 > t_1$. Now we assume $\Lambda(t_1) \neq \Lambda(t_2)$. Applying Lemma 3.3.5, we get

$$\Gamma_{\min}(f_{\Lambda(0)-\Lambda(t_1)}) \bigcup \Gamma_{\min}(f_{\Lambda(t_2)-\Lambda(t_1)}) = \Gamma.$$

By Remark 3.3.4, we get

$$\Gamma_{\min}(f_{\Lambda(t_2)-\Lambda(t_1)}) = \Gamma_{\min}(f_{\Lambda(1)-\Lambda(t_1)}).$$

Therefore,

$$\Gamma_{\min}(f_{\Lambda(0)-\Lambda(t_1)}) \bigcup \Gamma_{\min}(f_{\Lambda(1)-\Lambda(t_1)}) = \Gamma,$$

and it again follows from Lemma 3.3.5 that $\Lambda(t_1) \in \mathrm{Im}(P_{\Lambda(1)-\Lambda(0)})$.

Using a similar argument, we get $\Lambda(t_2) \in \mathrm{Im}(P_{\Lambda(1)-\Lambda(0)})$. Thus

$$\mathrm{Im}(P_{\Lambda(1)-\Lambda(0)}) = \mathrm{Im}(P_{\Lambda(t_2)-\Lambda(0)}) \bigcup \mathrm{Im}(P_{\Lambda(1)-\Lambda(t_1)} = \mathrm{Im}(\Lambda).$$

Note that

$$\rho(\Lambda(t_1), \Lambda(t_2)) = \frac{t_2 - t_1}{t_2}\rho(\Lambda(0), \Lambda(t_2)) = \frac{t_2 - t_1}{1 - t_1}\rho(\Lambda(t_1), \Lambda(1)).$$

Therefore, we must have $\Lambda = P_{\Lambda(1)-\Lambda(0)}$ as claimed. □

If d is an integer and S_d is the symmetric group of degree d, then $\mathrm{Div}_+^{\mathrm{d}}(\Gamma) = \Gamma^d/S_d$ set-theoretically. Therefore, other than the metric topology, $\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ has a topology induced from Γ as a d-fold symmetric product. The following proposition says that these two topologies on $\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ are actually the same.

Proposition 3.3.8. *On* $\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$*, the metric topology is the same as the the induced topology as a d-fold symmetric product of* Γ*.*

Proof. Denote the first topology by $\mathcal{T}_1$ and the second by $\mathcal{T}_2$. To show $\mathcal{T}_1 = \mathcal{T}_2$, it suffices to show that for a divisor $D = \sum_{i=1}^{d}(q_i)$ with $q_i \in \Gamma$, a sequence $\{D^{(n)}\}_n$ converges to D in $\mathcal{T}_2$ if and only if $\rho(D^{(n)}, D) \to 0$. In addition, we note that to say $D^{(n)} \to D$ in $\mathcal{T}_2$ is equivalent to say that there exists d sequences of points on Γ, $\{p_i^{(n)}\}_n$ for $i = 1, \ldots, d$, such that $D^{(n)} = \sum_{i=1}^{d}(p_i^{(n)})$ and $p_i^{(n)} \to q_i$ on Γ.

Suppose $D^{(n)} \to D$ in $\mathcal{T}_2$. Since

$$\rho(D^{(n)}, D) \leqslant \sum_{i=1}^{d} \rho((p_i^{(n)}), (q_i)) = \sum_{i=1}^{d} r(p_i^{(n)}, q_i) \leqslant \sum_{i=1}^{d} \mathrm{dist}_\Gamma(p_i^{(n)}, q_i)$$

where $r(p_i^{(n)}, q_i)$ is the effective index between $p_i^{(n)}$ and q_i (see Section 3.2), we conclude that $D^{(n)} \to D$ in $\mathcal{T}_1$.

Now suppose $D^{(n)} \to D$ in $\mathcal{T}_1$ which means $\rho(D^{(n)}, D) = \max(\mathcal{N}(f_{D^{(n)}-D})) \to 0$. Considering the divisors D and $D^{(n)}$, for each point $q_i \in \mathrm{supp}\, D$, we will associate a point $p_i^{(n)}$ supp $D^{(n)}$ with an procedure as follows.

Let M be the maximum number of degrees among all the points in Γ. This means each point $p \in \Gamma$ has at most M adjacent edges. Denote the sum of slopes of $f_{D^{(n)}-D}$ for all outgoing directions from $p \in \Gamma$ by $\chi(p)$. Then $\chi(p) = -(\Delta f_{D^{(n)}-D})(p) = D(p) - D^{(n)}(p)$. Let $V(\Gamma)$ be a vertex set of Γ.

First, we will determine $p_1^{(n)}$ for q_1.

If $q_1 \in \operatorname{supp} D^{(n)}$, we let $p_1^{(n)} = q_1$.

Otherwise, we must have $\chi(q_1) \geqslant 1$ and there must be an outgoing direction $\vec{V}_{q_1}$ from q_1 with a slope at least $1/M$. Let $w(q_1) \in V(\Gamma)$ be the adjacent vertex of q_1 in direction $\vec{V}_{q_1}$. If there exists a point in $\operatorname{supp} D^{(n)}$ that lies in the half-open-half-closed segment $(q_1, w(q_1)]$, then we let $p_1^{(n)}$ be this point. Clearly, $f_{D^{(n)}-D}(q_1) < f_{D^{(n)}-D}(p_1^{(n)})$ and $\mathrm{dist}_\Gamma(p_1^{(n)}, q_1) \leqslant M \cdot \rho(D^{(n)}, D)$ in this case.

Otherwise, we must have $\chi(w(q_1)) \geqslant 0$. Since the slope of the outgoing direction from $w(q_1)$ to q_1 is at most $-1/M$, the sum of slopes in the remaining outgoing directions from $w(q_1)$ is at least $1/M$ and there must be an outgoing direction $\vec{V}_{w(q_1)}$ from $w(q_1)$ with a slope at least $1/(M(M-1))$. Let $w^2(q_1) \in V(\Gamma)$ be the adjacent vertex of $w(q_1)$ in direction $\vec{V}_{w(q_1)}$. Following the same procedure, we let $p_1^{(n)}$ be a point contained in both $\operatorname{supp} D^{(n)}$ and $(w(q_1), w^2(q_1)]$ if their intersection is nonempty, and otherwise keep seeking $p_1^{(n)}$ in the next outgoing direction from $w^2(q_1)$ with slope at least $1/(M(M-1)^2)$.

The procedure must terminate in finitely many steps since we only have finitely many elements in $V(\Gamma)$. Let $N = |V(\Gamma)|$. We conclude that we can find $p_1^{(n)}$ within N steps and $\mathrm{dist}_\Gamma(p_1^{(n)}, q_1) \leqslant C_1 \cdot \rho(D^{(n)}, D)$ where $C_1 = M(M-1)^N$.

Next we will determine $p_i^{(n)}$ one by one inductively. Suppose for $i = 2, \ldots, d'$ ($d' < d$), we have determined $p_i^{(n)}$ and known that $\mathrm{dist}_\Gamma(p_i^{(n)}, q_i) \leqslant C_i \cdot \rho(D^{(n)}, D)$ where C_i's are constants. We let $D_{d'}^{(n)} = D^{(n)} - \sum_{i=1}^{d'}(p_i^{(n)})$ and $D_{d'} = D - \sum_{i=1}^{d'}(q_i)$.

Then

$$
\begin{aligned}
\rho(D^{(n)}_{d'}, D_{d'}) &\leqslant \rho(D^{(n)}, D) + \sum_{i=1}^{d'} r(p^{(n)}_i, q_i) \\
&\leqslant \rho(D^{(n)}, D) + \sum_{i=1}^{d'} \mathrm{dist}_\Gamma(p^{(n)}_i, q_i) \\
&= (1 + \sum_{i=1}^{d'} C_i)\rho(D^{(n)}, D).
\end{aligned}
$$

Following exactly the same procedure we used to seek $p^{(n)}_1$, we can find $p^{(n)}_{d'+1} \in \operatorname{supp} D^{(n)}_{d'}$ such that

$$
\mathrm{dist}_\Gamma(p^{(n)}_{d'+1}, q_{d'+1}) \leqslant C_1 \cdot \rho(D^{(n)}_{d'}, D_{d'}) = C_{d'+1} \cdot \rho(D^{(n)}, D)
$$

where $C_{d'+1} = C_1(1 + \sum_{i=1}^{d'} C_i)$.

In this way, for each $D^{(n)}$, we can find $p^{(n)}_i$ such that $D^{(n)} = \sum_{i=1}^{d}(p^{(n)}_i)$ and $\mathrm{dist}_\Gamma(p^{(n)}_i, q_i)$ is bounded by $C_i \cdot \rho(D^{(n)}, D)$. This means $D^{(n)} \to D$ in $\mathcal{T}_1$ implies $D^{(n)} \to D$ in $\mathcal{T}_2$. □

Lemma 3.3.9. *The scaling map* $\phi : \mathbb{R}\,\mathrm{Div}^{d'}_+(\Gamma) \to \mathbb{R}\,\mathrm{Div}^{d}_+(\Gamma)$ *given by* $\phi(D) = \frac{d}{d'}D$ *is a homeomorphism. Moreover,*

$$
\rho(\phi(D_1), \phi(D_2)) = \frac{d}{d'}\rho(D_1, D_2)
$$

for $D_1, D_2 \in \mathbb{R}\,\mathrm{Div}^{d}_+(\Gamma)$.

Proof. It follows directly from the linearity of the Laplacian. □

3.4 Tropical convex sets: a generalization of complete linear systems

Definition 3.4.1. A set $T \subseteq \mathbb{R}\,\mathrm{Div}^{\mathrm{d}}_+(\Gamma)$ is *tropically convex* (*t-convex*) or equivalently *t-path-connected* of degree d if for every $D_1, D_2 \in T$, the whole t-segment $imag(P_{D_2 - D_1})$ connecting D_1 and D_2 is contained in T.

Note that the intersection of an arbitrary collection of tropically convex sets of the same degree is tropically convex. Thus we define the *tropical convex hull* generated by $S \subseteq \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$, denoted by $\mathrm{tconv}(S)$, as the intersection of all tropically convex sets in $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ containing S, and we say S is a *generating set* of $\mathrm{tconv}(S)$. If, in addition, $x \notin \mathrm{tconv}(S \setminus \{x\})$ for every $x \in S$, then we say S is *tropical convex (t-convex) independent*. We say a tropical convex hull is *finitely generated* if it can be generated by a finite set. In particular, we abuse notation here to write $\mathrm{tconv}(D_1, \ldots, D_n, S_1, \ldots, S_m)$ as a simplification of $\mathrm{tconv}(\{D_1, \ldots, D_n\} \bigcup S_1 \ldots \bigcup S_m)$ when it is clear that $D_1, \ldots, D_n$ are divisors in $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ and $S_1 \ldots \bigcup S_m$ are subsets of $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$. In particular, by Lemma 3.3.2 (1), it is easy to verify that $\mathrm{tconv}(D_1, D_2) = \mathrm{Im}(P_{D_2-D_1})$, and we use them both interchangeably to represent the t-segment connecting D_1 and D_2.

If d is an integer and $D_1, D_2 \in \mathrm{Div}_+^{\mathrm{d}}(\Gamma)$, we say D_1 is linearly equivalent to D_2 (denoted $D_1 \sim D_2$) if $f_{D_2-D_1}$ is rational, i.e., piecewise-linear with integral slopes. This is equivalent to say $\mathrm{tconv}(D_1, D_2) \subseteq \mathrm{Div}_+^{\mathrm{d}}(\Gamma)$. The complete linear system $|D|$ associated to $D \in \mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ is the set of effective divisors linearly equivalent to D.

We have the following facts:

1. All complete linear systems $|D|$ are t-path-connected.

2. $\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ is not t-path-connected in general, and the nonempty complete linear systems of degree d are the t-path-connected components in $\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$.

3. $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ is t-path-connected, but not finitely generated. When d is an integer, we have in general $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma) \supsetneq \mathrm{tconv}(\mathrm{Div}_+^{\mathrm{d}}(\Gamma))$.

Lemma 3.4.2. *Every complete linear system is finitely generated.*

Proof. A complete linear system $|D|$ can always be generated by the extremals (we only have finitely many of them) in $|D|$. $\square$

Remark 3.4.3. The extremals of complete linear systems are introduced in [39]. (They actually define extremals in $L(D)$ instead of $|D|$.) We will generalize this notion to all tropical convex sets in Section 3.7.

Now let us consider tropical convex sets in general. Theorem 3.4.4 and Theorem 3.4.5 state some fundamental properties of tropical convex sets. In particular, as it is well-known that conventional convex subsets of Euclidean spaces are contractible, Theorem 3.4.4 says this is also true for all tropical convex sets. Theorem 3.4.5 tells us how to generate a tropical convex set from its subsets and provides a compactness criterion. Then we may deduce an important conclusion immediately that finitely generated tropical convex hulls are always compact (Corollary 3.4.6). To prove these theorems, we need to employ a machinery based on *general reduced divisors* which will be introduced in the next section, and we will finish the proofs in Section 3.7.

Theorem 3.4.4. *Tropical convex sets are contractible.*

Theorem 3.4.5. *Let $T, T' \subseteq \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$ be tropically convex set. Then we have* $\operatorname{tconv}(T, T') = \bigcup_{D\in T, D'\in T'} \operatorname{tconv}(D, D')$. *If T and T' are compact in addition, then* $\operatorname{tconv}(T, T')$ *is compact.*

Corollary 3.4.6. *Every finitely generated tropical convex hull is compact.*

Proof. It follows immediately from Lemma 3.3.2 (3) and an induction on Theorem 3.4.5. □

Remark 3.4.7. The complete linear systems are finitely generated (Lemma 3.4.2) and thus compact in our metric topology (Corollary 3.4.6).

Example 3.4.8. As shown in Figure 10, we give an example of a tropical convex set and its tropical convex subsets. The graph Γ is a loop with vertices $\{v_1, w_{12}, v_2, w_{23}, v_3, w_{13}\}$. All edges have the same length. Then the following divisors are all linearly equivalent, $D_0 = (v_1) + (v_2) + (v_3)$, $D_1 = 3(v_1)$, $D_2 = 3(v_2)$, $D_3 = 3(v_3)$, $D_{12} = 2(w_{12}) + (v_3)$,

$D_{23} = 2(w_{23}) + (v_1)$, and $D_{13} = 2(w_{13}) + (v_2)$. Indeed, the complete linear system $|D_0|$ can be tropically generated by D_1, D_2 and D_3. We also show the tropical segments $\mathrm{tconv}(D_1, D_2)$, $\mathrm{tconv}(D_1, D_{23})$ and $\mathrm{tconv}(D_{12}, D_{23})$. In particular, D_0 lies on the segments $\mathrm{tconv}(D_1, D_{23})$ and $\mathrm{tconv}(D_{12}, D_{23})$. For tropical convex sets generated by three divisors, $\mathrm{tconv}(D_{12}, D_{23}, D_{13})$ is purely 1-dimensional, while the last three cases in Figure 10 does not have a pure dimension.

3.5 General reduced divisors

3.5.1 $\mathcal{B}$-functions

Let the $\mathcal{B}$-function $\mathcal{B} : \mathbb{R}\,\mathrm{Div}^0(\Gamma) \to \mathbb{R}_+$ be given by $\mathcal{B}(D_2 - D_1) = \int_\Gamma (f_{D_2-D_1} - \min(f_{D_2-D_1})) = \int_\Gamma \mathcal{N}(f_{D_2-D_1})$, where D_1 and D_2 are effective $\mathbb{R}$-divisors of the same degree. In addition, for $d \geqslant 0$, we define the $\mathcal{B}$-function restricted to degree d as $\mathcal{B}^d : \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma) \times \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma) \to \mathbb{R}_+$ given by $\mathcal{B}^d(D_1, D_2) = \mathcal{B}(D_2 - D_1)$. Unlike the distance function, we have $\mathcal{B}(D_2 - D_1) \neq \mathcal{B}(D_1 - D_2)$ in general. It is straightforward to verify that (1) $\mathcal{B}(D_1 - D_2) + \mathcal{B}(D_2 - D_1) = \rho(D_1, D_2) l_{\mathrm{tot}}$ where l_{tot} is the total length of Γ, and (2) $\mathcal{B}(D_1 - D_2) = 0$ if and only if $\rho(D_1, D_2) = 0$. Fixing D_1 or D_2, we get the functions $\mathcal{B}_{\star - D_1} : \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma) \to \mathbb{R}_+$ given by $\mathcal{B}_{\star - D_1}(D) = \mathcal{B}(D - D_1)$ and $\mathcal{B}_{D_2 - \star} : \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma) \to \mathbb{R}_+$ given by $\mathcal{B}_{D_2 - \star}(D) = \mathcal{B}(D_2 - D)$, respectively.

Remark 3.5.1. For $D \in \mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ and $q \in \Gamma$, the b-function $b_q(D)$ Baker and Shokrieh introduced in [19] is essentially a special case of the $\mathcal{B}$-function in the following sense:

$$b_q(D) = \mathcal{B}(D - d \cdot (q)).$$

Lemma 3.5.2. *1. For $D_1, D_2, D_3 \in \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$, we have the triangle inequality*

$$\mathcal{B}(D_3 - D_1) \leqslant \mathcal{B}(D_3 - D_2) + \mathcal{B}(D_2 - D_1).$$

The equality holds if and only if

$$\Gamma_{\min}(f_{D_3-D_2}) \bigcap \Gamma_{\min}(f_{D_2-D_1}) \neq \emptyset$$

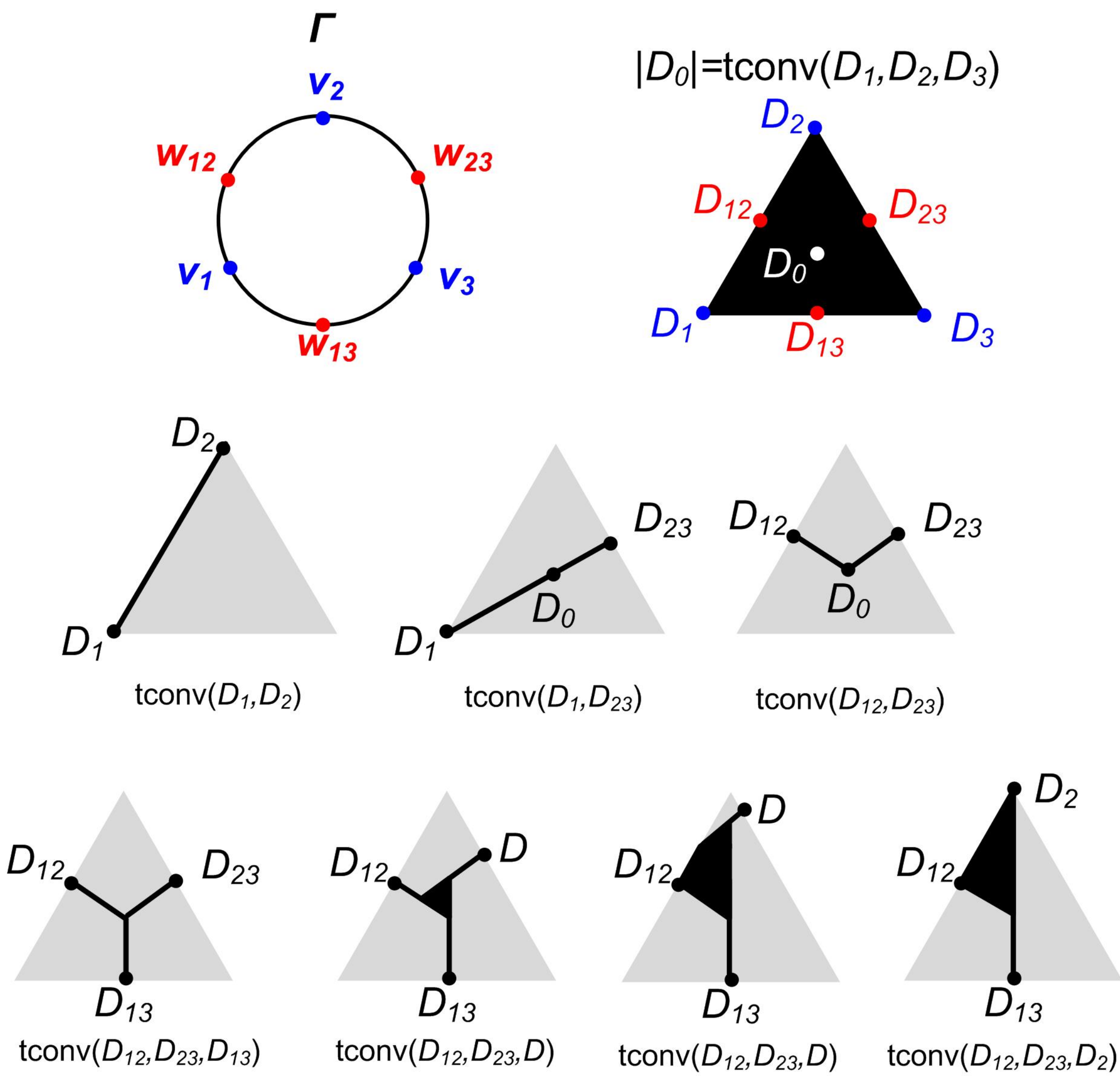

Figure 10: The loop Γ has a vertex set $\{v_1, w_{12}, v_2, w_{23}, v_3, w_{13}\}$, which are equally spaced for all adjacent vertices. The linear system $|D_0|$ is indeed a solid triangle. Here $D_0 = (v_1) + (v_2) + (v_3)$, $D_1 = 3(v_1)$, $D_2 = 3(v_2)$, $D_3 = 3(v_3)$, $D_{12} = 2(w_{12}) + (v_3)$, $D_{23} = 2(w_{23}) + (v_1)$, and $D_{13} = 2(w_{13}) + (v_2)$.

if and only if

$$\Gamma_{\min}(f_{D_3-D_1}) = \Gamma_{\min}(f_{D_3-D_2}) \bigcap \Gamma_{\min}(f_{D_2-D_1}).$$

2. *For* $D_1, D_2, D_3 \in \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$*,* $\rho(D_1, D_3) = \rho(D_1, D_2) + \rho(D_2, D_3)$ *if and only if*

$$\mathcal{B}(D_3 - D_1) = \mathcal{B}(D_3 - D_2) + \mathcal{B}(D_2 - D_1)$$

and

$$\mathcal{B}(D_1 - D_3) = \mathcal{B}(D_1 - D_2) + \mathcal{B}(D_2 - D_3).$$

3. *The functions* $\mathcal{B}^d$*,* $\mathcal{B}_{\star-D}$ *and* $\mathcal{B}_{D-\star}$ *are continuous.*

Proof. For the triangle inequality, we let $f_{D_3-D_2}$ and $f_{D_2-D_1}$ be associated to $D_3 - D_2$ and $D_2 - D_1$ respectively, and assume $f_{D_3-D_2} = \mathcal{N}(f_{D_3-D_2})$ and $f_{D_2-D_1} = \mathcal{N}(f_{D_2-D_1})$. Let $f_{D_3-D_1} = f_{D_3-D_2} + f_{D_2-D_1}$, which is associated to $D_3 - D_1$. Note that

$$\min(f_{D_3-D_1}) \geqslant \min(f_{D_3-D_2}) + \min(f_{D_2-D_1}) = 0,$$

while the equality holds if and only if

$$\Gamma_{\min}(f_{D_3-D_2}) \bigcap \Gamma_{\min}(f_{D_2-D_1}) \neq \emptyset$$

if and only if

$$\Gamma_{\min}(f_{D_3-D_1}) = \Gamma_{\min}(f_{D_3-D_2}) \bigcap \Gamma_{\min}(f_{D_2-D_1}).$$

Thus

$$\begin{aligned}
\mathcal{B}(D_3 - D_1) &= \int_\Gamma (f_{D_3-D_1} - \min(f_{D_3-D_1})) \\
&\leqslant \int_\Gamma f_{D_3-D_1} \\
&= \int_\Gamma f_{D_3-D_2} + \int_\Gamma f_{D_2-D_1} \\
&= \mathcal{B}(D_3 - D_2) + \mathcal{B}(D_2 - D_1),
\end{aligned}$$

with the equality holds under the same conditions.

For (2), $\rho(D_1, D_3) = \rho(D_1, D_2) + \rho(D_2, D_3)$ if and only if

$$\Gamma_{\min}(f_{D_2-D_1}) \bigcap \Gamma_{\min}(f_{D_3-D_2}) \neq \emptyset$$

and

$$\Gamma_{\max}(f_{D_2-D_1}) \bigcap \Gamma_{\max}(f_{D_3-D_2}) \neq \emptyset.$$

Note that $\Gamma_{\max}(f_{D_2-D_1}) = \Gamma_{\min}(f_{D_1-D_2})$ and $\Gamma_{\max}(f_{D_3-D_2}) = \Gamma_{\min}(f_{D_2-D_3})$, and hence (2) follows from (1).

For (3), it suffices to show $\mathcal{B}(D_2' - D_1') \to \mathcal{B}(D_2 - D_1)$ as $D_1' \to D_1$ and $D_2' \to D_2$. Actually, if l_{tot} is the total length of Γ, we have

$$\begin{aligned}
\mathcal{B}(D_2' - D_1') - \mathcal{B}(D_2 - D_1) &= \mathcal{B}((D_2' - D_2) + (D_2 - D_1) + (D_1 - D_1')) - \mathcal{B}(D_2 - D_1) \\
&\leqslant \mathcal{B}(D_2' - D_2) + \mathcal{B}(D_1 - D_1') \\
&\leqslant (\rho(D_2, D_2') + \rho(D_1, D_1'))l_{\text{tot}}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}(D_2 - D_1) - \mathcal{B}(D_2' - D_1') &= \mathcal{B}((D_2 - D_2') + (D_2' - D_1') + (D_1' - D_1)) - \mathcal{B}(D_2' - D_1') \\
&\leqslant \mathcal{B}(D_2 - D_2') + \mathcal{B}(D_1' - D_1) \\
&\leqslant (\rho(D_2, D_2') + \rho(D_1, D_1'))l_{\text{tot}}.
\end{aligned}$$

□

3.5.2 General reduced divisors

Theorem 3.5.3. *Let $T \subseteq \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$ be tropically convex and compact. For every $E \in \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$, there exists a unique $\mathbb{R}$-divisor $T_E \in T$, which minimizes $\mathcal{B}_{\star-E}|_T$.*

According to Lemma 3.5.2 (3), $\mathcal{B}_{\star-E}$ is a continuous function. Since T is compact, $\mathcal{B}_{\star-E}|_T$ can reach its minimal value. Hence, it only remains to show that the minimum can only be reached at a single divisor in T. We will finish our proof of Theorem 3.5.3

in Remark 3.5.9 after proving some useful facts in Proposition 3.5.7. Provided this theorem, we are now ready to bring up a central notion of this chapter.

Definition 3.5.4. Under the hypotheses of Theorem 3.5.3, we say the divisor T_E is the *(general) reduced divisor* in T with respect to E (or the E-reduced divisor in T).

Remark 3.5.5. For $D \in \mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ and $q \in \Gamma$, Baker and Shokrieh [19] showed that a conventional reduced divisor D_q is the unique divisor in the complete linear system $|D|$ such that the b-function $b_q(D)$ is minimized. Note that $|D|$ is compact (Remark 3.4.7) and we may express the b-function by an equivalent $\mathcal{B}$-function (Remark 3.5.1). Hence if we let $T = |D|$ and $E = d \cdot (q)$, the conventional reduced divisors fit well in our new setting by the identity $D_q = |D|_{d\cdot(q)}$.

Remark 3.5.6. Throughout this chapter, when we mention reduced divisors, we mean general reduced divisors unless otherwise stated.

Proposition 3.5.7. *Let $E, D_1, D_2 \in \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ and $D_1 \neq D_2$. Let $P_{D_2-D_1}$ be the t-path from D_1 to D_2. Let $D(t) = P_{D_2-D_1}(t)$ for $t \in [0,1]$. Consider the functions $g_\rho(t) = \rho(E, D(t))$ and $g_{\mathcal{B}}(t) = \mathcal{B}(D(t) - E)$ for $t \in [0,1]$. Then exactly one of the following two cases occur:*

1. $\Gamma_{\min}(f_{D_1-E}) \bigcap \Gamma_{\min}(f_{D_2-D_1}) \neq \emptyset$. *In this case, $g_\rho(t)$ is increasing and $g_{\mathcal{B}}(t)$ is strictly increasing for $t \in [0,1]$. And precisely, for $t \in (0,1]$, we have*

$$\begin{aligned}\Gamma_{\min}(f_{D(t)-E}) &= \Gamma_{\min}(f_{D(t)-D_1}) \bigcap \Gamma_{\min}(f_{D_1-E}) \\ &= \Gamma_{\min}(f_{D_2-E}) = \Gamma_{\min}(f_{D_2-D_1}) \bigcap \Gamma_{\min}(f_{D_1-E})\end{aligned}$$

   and $g_{\mathcal{B}}(t) = \mathcal{B}(D(t) - D_1) + \mathcal{B}(D_1 - E)$.

2. $\Gamma_{\min}(f_{D_1-E}) \bigcap \Gamma_{\min}(f_{D_2-D_1}) = \emptyset$. *In this case, at $t = 0$, $g_\rho(t)$ is decreasing and $g_{\mathcal{B}}(t)$ is strictly decreasing.*

Remark 3.5.8. We say a function $f(t)$ is increasing (resp. decreasing, strictly increasing, strictly decreasing, or locally constant) at t_0 if there exists $\delta > 0$ such that $g(t)$ is increasing (resp. decreasing, strictly increasing, strictly decreasing, or constant) on $[t_0, t_0 + \delta]$. Note that we adopt the usual definition of increasing (resp. decreasing) functions here, which actually means non-decreasing (resp. non-increasing).

Proof. Let $l = \rho(D_1, D_2)$. For simplicity of notations, we assume

$$\min(f_{D_1-E}) = \min(f_{D_2-D_1}) = \min(f_{D(t)-D_1}) = 0$$

from now on. It then follows $f_{D(t)-D_1} = \min(tl, f_{D_2-D_1})$. In addition, we let $f_{D_2-E} = f_{D_2-D_1} + f_{D_1-E}$, which is associated to $D_2 - E$, and $f_{D(t)-E} = f_{D(t)-D_1} + f_{D_1-E}$, which is associated to $D(t) - E$.

If

$$\Gamma_{\min}(f_{D_1-E}) \bigcap \Gamma_{\min}(f_{D_2-D_1}) \neq \emptyset,$$

then we have

$$\Gamma_{\min}(f_{D_2-E}) = \Gamma_{\min}(f_{D_2-D_1}) \bigcap \Gamma_{\min}(f_{D_1-E})$$

and

$$\min(f_{D_2-E}) = \min(f_{D_2-D_1} + f_{D_1-E}) = \min(f_{D_2-D_1}) + \min(f_{D_1-E}) = 0.$$

By Remark 3.3.4 (1), $\Gamma_{\min}(f_{D(t)-D_1}) = \Gamma_{\min}(f_{D_2-D_1})$ for $t \in (0, 1]$. Therefore,

$$\Gamma_{\min}(f_{D(t)-E}) = \Gamma_{\min}(f_{D(t)-D_1}) \bigcap \Gamma_{\min}(f_{D_1-E}) = \Gamma_{\min}(f_{D_2-D_1}) \bigcap \Gamma_{\min}(f_{D_1-E}) \neq \emptyset$$

and

$$\min(f_{D(t)-E}) = \min(f_{D(t)-D_1} + f_{D_1-E}) = \min(f_{D(t)-D_1}) + \min(f_{D_1-E}) = 0$$

for $t \in [0, 1]$. On the other hand, $\max(f_{D(t)-E})$ is an increasing function since $\max(f_{D(t)-E}) = \max(f_{D(t)-D_1} + f_{D_1-E})$ and the value of $f_{D(t)-D_1}(v)$ at any point $v \in \Gamma$ is an increasing function with respect to t. Therefore, $g_\rho(t)$ is also an increasing function since $g_\rho(t) = \max(f_{D(t)-E}) - \min(f_{D(t)-E}) = \max(f_{D(t)-E})$. Moreover, it follows from Lemma 3.5.2 that

$$\begin{aligned} g_{\mathcal{B}}(t) &= \mathcal{B}(D(t)-E) \\ &= \mathcal{B}(D(t)-D_1) + \mathcal{B}(D_1-E). \end{aligned}$$

Therefore $g_{\mathcal{B}}(t)$ is strictly increasing for $t \in [0,1]$ since $\mathcal{B}(D(t)-D_1)$ is strictly increasing.

Now consider the case $\Gamma_{\min}(f_{D_2-D_1}) \bigcap \Gamma_{\min}(f_{D_1-E}) = \emptyset$. Note that both $f^{-1}_{D_2-D_1}([0,\delta])$ and $f^{-1}_{D_1-E}([0,\delta])$ are closed subsets of Γ with finitely many connected components, and for a small enough positive δ_0, both $f^{-1}_{D_2-D_1}([0,\delta])$ and $f^{-1}_{D_1-E}([0,\delta])$ expand continuously as δ increases in $[0,\delta_0]$. In particular, we have

$$\lim_{\delta \searrow 0} f^{-1}_{D_2-D_1}([0,\delta]) = \Gamma_{\min}(f_{D_2-D_1})$$

and

$$\lim_{\delta \searrow 0} f^{-1}_{D_1-E}([0,\delta]) = \Gamma_{\min}(f_{D_1-E}).$$

Hence we may even choose δ_0 such that

$$f^{-1}_{D_2-D_1}([0,\delta]) \bigcap f^{-1}_{D_1-E}([0,\delta]) = \emptyset$$

for all $\delta \in [0,\delta_0]$. Then for $t \in [0,\delta_0/l]$, we have

- $f_{D(t)-D_1} = tl$, $f_{D_1-E}(v) = 0$ and $f_{D(t)-E}(v) = tl$ if $v \in \Gamma_{\min}(f_{D_1-E})$;
- $f_{D(t)-D_1} \geqslant 0$, $f_{D_1-E}(v) \geqslant tl$ and $f_{D(t)-E}(v) \geqslant tl$ if $v \in f^{-1}_{D_2-D_1}([0,tl])$; and
- $f_{D(t)-D_1} = tl$, $f_{D_1-E}(v) > 0$ and $f_{D(t)-E}(v) > tl$ if $v \in (f^{-1}_{D_2-D_1}([0,tl]) \bigcup \Gamma_{\min}(f_{D_1-E}))^c$.

Therefore, we conclude $\min(f_{D(t)-E}) = tl$ and $\Gamma_{\min}(f_{D_1-E}) \subseteq \Gamma_{\min}(f_{D(t)-E})$ for $t \in [0,\delta_0/l]$. Let $f_{D_1-D(t)} = \rho(D_1, D(t)) - f_{D(t)-D_1}$, and we have $\min(f_{D_1-D(t)}) = 0$ and the value of $f_{D_1-D(t)}(v)$ at any point $v \in \Gamma$ is an increasing function with respect to t. Then for $t \in [0, \delta_0/l]$,

$$\begin{aligned}
\mathcal{N}(f_{D(t)-E}) &= f_{D(t)-E} - tl \\
&= f_{D_1-E} + f_{D(t)-D_1} - \rho(D_1, D(t)) \\
&= f_{D_1-E} - f_{D_1-D(t)}.
\end{aligned}$$

Note that $g_\rho(t) = \max(\mathcal{N}(f_{D(t)-E}))$, which means $g_\rho(t)$ is decreasing for $t \in [0, \delta_0/l]$. Thus $g_\rho(t)$ is decreasing at $t = 0$. Moreover,

$$\begin{aligned}
g_{\mathcal{B}}(t) &= \mathcal{B}(D(t) - E) \\
&= \int_\Gamma \mathcal{N}(f_{D(t)-E}) \\
&= \int_\Gamma (f_{D_1-E} - f_{D_1-D(t)}) \\
&= \mathcal{B}(D_1 - E) - \mathcal{B}(D_1 - D(t)),
\end{aligned}$$

for $t \in [0, \delta_0/l]$. This means $g_{\mathcal{B}}(t)$ is strictly decreasing for $t \in [0, \delta_0/l]$ since $\mathcal{B}(D_1 - D(t))$ is strictly increasing. Thus $g_{\mathcal{B}}(t)$ is strictly decreasing at $t = 0$. $\square$

Remark 3.5.9. We observe some easy facts following from Proposition 3.5.7.

1. $g_\rho(t)$ can be locally constant, while $g_{\mathcal{B}}(t)$ cannot.

2. If $g_\rho(t)$ is *strictly increasing* at $t = 0$, then $g_\rho(t)$ is *increasing* on $[0, 1]$. If $g_{\mathcal{B}}(t)$ is *strictly increasing* at $t = 0$, then $g_{\mathcal{B}}(t)$ is *strictly increasing* on $[0, 1]$.

3. Recall that we've assumed $D_1 \neq D_2$. If $g_\rho(0) = g_\rho(1) = \kappa_\rho$, then $g_{\mathcal{B}}(t)$ is decreasing at $t = 0$ (locally constant is possible) and $g_\rho(t) \leqslant \kappa_\rho$ for $t \in (0, 1)$. If $g_{\mathcal{B}}(0) = g_{\mathcal{B}}(1) = \kappa_{\mathcal{B}}$, then $g_{\mathcal{B}}(t)$ is strictly decreasing at $t = 0$ and $g_{\mathcal{B}}(t) < \kappa_{\mathcal{B}}$ for $t \in (0, 1)$.

4. We can finish **the proof of Theorem 3.5.3** now. If there exist divisors D_1 and D_2 in T, both minimizing $\mathcal{B}_{\star-E}\,|_{D\in T}$, then we must have $D_1 = D_2$ by (3).

5. By applying Proposition 3.5.7 to the t-paths from D_1 to D_2 and from D_1 to D_2 respectively, we see that

$$\Gamma_{\min}(f_{D_2-D_1})\bigcap\Gamma_{\min}(f_{D_1-E})\neq\emptyset$$

implies

$$\Gamma_{\min}(f_{D_1-D_2})\bigcap\Gamma_{\min}(f_{D_2-E})=\emptyset$$

(still under the assumption $D_1\neq D_2$).

Proposition 3.5.7 can actually provide us with criterions of reduced divisors from different aspects, as summarized in the following corollary.

Corollary 3.5.10 (**Criterions for general reduced divisors**). *Let $T\subseteq\mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$ be tropically convex and compact. Let $E\in\mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$ and $D_0\in T$. The following properties are equivalent.*

1. *D_0 is the E-reduced divisor of T.*

2. *For every $D\in T$ and $t\in[0,1]$, the function $\mathcal{B}(P_{D-D_0}(t)-E)$ is strictly increasing.*

3. *For every $D\in T$ and $t\in[0,1]$, the function $\mathcal{B}(P_{D-D_0}(t)-E)$ is strictly increasing at $t=0$. (Equivalently, we say $\mathcal{B}_{\star-E}$ is strictly increasing at D_0 along all possible firing directions.)*

4. *For every $D\in T$,*

$$\Gamma_{\min}(f_{D-D_0})\bigcap\Gamma_{\min}(f_{D_0-E})\neq\emptyset.$$

5. *For every $D\in T$,*

$$\Gamma_{\min}(f_{D-E})=\Gamma_{\min}(f_{D-D_0})\bigcap\Gamma_{\min}(f_{D_0-E}).$$

6. *For every $D\in T$,*

$$\mathcal{B}(D-E)=\mathcal{B}(D-D_0)+\mathcal{B}(D_0-E).$$

7. For every $D \in T$ and $D \neq D_0$,

$$\Gamma_{\min}(f_{D_0-D}) \bigcap \Gamma_{\min}(f_{D-E}) = \emptyset.$$

Proof. All the criterions easily follows from Proposition 3.5.7. $\square$

3.5.3 Some properties of general reduced divisors

Unless otherwise stated, we let $T \subseteq \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$ be tropically convex and compact in the following discussions.

Lemma 3.5.11. *If $E \in T$, then $T_E = E$.*

Lemma 3.5.12. *Let T' be a compact tropical convex subset of T. For $E \in \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$, if $T_E \in T'$, then $T'_E = T_E$.*

The easy facts as stated in the above two lemmas can be verified using any criterion of reduced divisors in Corollary 3.5.10, and we skip the detailed proofs.

Lemma 3.5.13. *Let $E' \in \operatorname{tconv}(E, T_E)$. Then $T_{E'} = T_E$.*

Proof. By Corollary 3.5.10, we have

$$\Gamma_{\min}(f_{D-T_E}) \bigcap \Gamma_{\min}(f_{T_E-E}) \neq \emptyset$$

for every $D \in T$. Taking $D_1 = E$ and $D_2 = T_E$ in Remark 3.3.4 (3), we have $\Gamma_{\min}(f_{T_E-E'}) \supseteq \Gamma_{\min}(f_{T_E-E})$, which means

$$\Gamma_{\min}(f_{D-T_E}) \bigcap \Gamma_{\min}(f_{T_E-E'}) \neq \emptyset$$

for every $D \in T$. Using Corollary 3.5.10 again, we see that T_E is also E'-reduced in T. $\square$

Lemma 3.5.14. *For $D_0, E, E' \in \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$, suppose $D_0 \in T$ and $E' \in \operatorname{tconv}(E, D_0)$. Then $E' \in \operatorname{tconv}(E, T_{E'})$.*

Proof. By Corollary 3.5.10, $\Gamma_{\min}(f_{D_0-E'}) \subseteq \Gamma_{\min}(f_{T_{E'}-E'})$. By Lemma 3.3.5,

$$\Gamma_{\min}(f_{D_0-E'}) \bigcup \Gamma_{\min}(f_{E-E'}) = \Gamma.$$

Thus

$$\Gamma_{\min}(f_{T_{E'}-E'}) \bigcup \Gamma_{\min}(f_{E-E'}) = \Gamma.$$

Again, by Lemma 3.3.5, we have $E' \in \mathrm{tconv}(E, T_{E'})$. $\square$

Lemma 3.5.15. *Let $E_1, E_2 \in \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$. Then $\rho(T_{E_1}, T_{E_2}) \leqslant \rho(E_1, E_2)$. The equality holds if and only if*

$$\mathcal{B}(T_{E_2} - E_1) = \mathcal{B}(T_{E_2} - E_2) + \mathcal{B}(E_2 - E_1)$$

and

$$\mathcal{B}(T_{E_1} - E_2) = \mathcal{B}(T_{E_1} - E_1) + \mathcal{B}(E_1 - E_2).$$

Proof. Let l_{tot} be the total length of Γ. By Corollary 3.5.10, we have

$$\mathcal{B}(T_{E_2} - T_{E_1}) = \mathcal{B}(T_{E_2} - E_1) - \mathcal{B}(T_{E_1} - E_1)$$

and

$$\mathcal{B}(T_{E_1} - T_{E_2}) = \mathcal{B}(T_{E_1} - E_2) - \mathcal{B}(T_{E_2} - E_2).$$

By Lemma3.5.2, we have

$$\mathcal{B}(T_{E_2} - E_1) - \mathcal{B}(T_{E_2} - E_2) \leqslant \mathcal{B}(E_2 - E_1)$$

and

$$\mathcal{B}(T_{E_1} - E_2) - \mathcal{B}(T_{E_1} - E_1) \leqslant \mathcal{B}(E_1 - E_2).$$

Therefore,

$$
\begin{aligned}
&\rho(T_{E_1}, T_{E_2}) l_{\mathrm{tot}} \\
&= \mathcal{B}(T_{E_2} - T_{E_1}) + \mathcal{B}(T_{E_1} - T_{E_2}) \\
&= (\mathcal{B}(T_{E_2} - E_1) - \mathcal{B}(T_{E_1} - E_1)) + (\mathcal{B}(T_{E_1} - E_2) - \mathcal{B}(T_{E_2} - E_2)) \\
&= (\mathcal{B}(T_{E_2} - E_1) - \mathcal{B}(T_{E_2} - E_2)) + (\mathcal{B}(T_{E_1} - E_2) - \mathcal{B}(T_{E_1} - E_1)) \\
&\leqslant \mathcal{B}(E_2 - E_1) + \mathcal{B}(E_1 - E_2) \\
&= \rho(E_1, E_2) l_{\mathrm{tot}}.
\end{aligned}
$$

□

Corollary 3.5.16. *Let $E_1, E_2 \in \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$. If $\rho(E_1, E_2) = \rho(T_{E_1}, T_{E_2})$, then for each $E \in \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ such that $\rho(E_1, E) + \rho(E_2, E) = \rho(E_1, E_2)$, we have $\rho(E_1, E) = \rho(T_{E_1}, T_E)$ and $\rho(E_2, E) = \rho(T_{E_2}, T_E)$.*

Proof. By Lemma3.5.15, we get $\rho(T_{E_1}, T_E) \leqslant \rho(E_1, E)$ and $\rho(T_{E_2}, T_E) \leqslant \rho(E_2, E)$. Thus

$$\rho(E_1, E_2) = \rho(T_{E_1}, T_{E_2}) \leqslant \rho(T_{E_1}, T_E) + \rho(T_{E_2}, T_E) \leqslant \rho(E_1, E) + \rho(E_2, E) = \rho(E_1, E_2),$$

which implies $\rho(E_1, E) = \rho(T_{E_1}, T_E)$ and $\rho(E_2, E) = \rho(T_{E_2}, T_E)$. □

Remark 3.5.17. Each divisor $E \in \mathrm{tconv}(E_1, E_2)$ satisfies the condition $\rho(E_1, E) + \rho(E_2, E) = \rho(E_1, E_2)$ in Corollary 3.5.16. Therefore, we must have $\rho(E_1, E) = \rho(T_{E_1}, T_E)$ and $\rho(E_2, E) = \rho(T_{E_2}, T_E)$. However, we should note that the set $\{T_E : E \in \mathrm{tconv}(E_1, E_2)\}$ is not necessarily a tropical convex set.

Lemma 3.5.18. *Let $E \in \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ and T' be a compact tropical convex subset of T. Then $T'_E = T'_{T_E}$.*

Proof. To prove $T'_E = T'_{T_E}$, it suffices to show that

$$\mathcal{B}(D' - E) = \mathcal{B}(D' - T'_{T_E}) + \mathcal{B}(T'_{T_E} - E)$$

for every $D' \in T'$ by Corollary 3.5.10.

Actually, applying Corollary 3.5.10 to T with respect to E, we get

$$\mathcal{B}(D - E) = \mathcal{B}(D - T_E) + \mathcal{B}(T_E - E)$$

for every $D \in T$, and in particular

$$\mathcal{B}(T'_{T_E} - E) = \mathcal{B}(T'_{T_E} - T_E) + \mathcal{B}(D_1 - E).$$

Applying Corollary 3.5.10 to T' with respect to T_E, we get

$$\mathcal{B}(D' - T_E) = \mathcal{B}(D' - T'_{T_E}) + \mathcal{B}(T'_{T_E} - T_E)$$

for every $D' \in T'$.

Therefore,

$$\begin{aligned}
&\mathcal{B}(D' - E) = \mathcal{B}(D' - T_E) + \mathcal{B}(T_E - E) \\
&= \mathcal{B}(D' - T'_{T_E}) + \mathcal{B}(T'_{T_E} - T_E) + \mathcal{B}(T_E - E) \\
&= \mathcal{B}(D' - T'_{T_E}) + \mathcal{B}(T'_{T_E} - E)
\end{aligned}$$

for every $D' \in T'$, and T'_{T_E} is exactly the E-reduced divisor in T' as claimed.

$\square$

Let $E \in \mathbb{R}\,\mathrm{Div}^{\mathrm{d}}_{+}(\Gamma)$ and $r_{\min} = \inf_{D\in T} \rho(E, D)$ (knowing T is compact, actually we have $r_{\min} = \min_{D\in T} \rho(E, D)$). The following proposition shows that sublevel sets of the distance function $\rho_E := \rho(E, \star)$ and the $\mathcal{B}$-function $\mathcal{B}_{\star-E}$ on T are all tropically convex. For $r, s \in \mathbb{R}_+$, we let $L^T_{\leqslant r}(\rho_E) = \{D \in T | \rho(E, D) \leqslant r\}$, $L^T_{=r}(\rho_E) = \{D \in T | \rho(E, D) = r\}$, $L^T_{\leqslant s}(\mathcal{B}_{\star-E}) = \{D \in T | \,\mathcal{B}_{\star-E}(D) \leqslant s\}$, and $L^T_{=s}(\mathcal{B}_{\star-E}) = \{D \in T | \,\mathcal{B}_{\star-E}(D) = s\}$. In particular, we also denote the the level set $L^T_{=r_{\min}}(\rho_E)$ of ρ_E at the minimum distance by $L^T_{\min}(\rho_E)$.

Proposition 3.5.19. *Under the above hypotheses and notations, we have*

1. *The E-reduced divisor T_E lies in $L^T_{\min}(\rho_E)$.*

2. $L^T_{\min}(\rho_E)$, $L^T_{\leqslant r}(\rho_E)$, $L^T_{=r}(\rho_E)$, $L^T_{\leqslant s}(\mathcal{B}_{\star-E})$ *and* $L^T_{=s}(\mathcal{B}_{\star-E})$ *are all compact subsets of T.*

3. $L^T_{\min}(\rho_E)$, $L^T_{\leqslant r}(\rho_E)$ *and* $L^T_{\leqslant s}(\mathcal{B}_{\star-E})$ *are tropically convex with the compactness assumption of T removed.*

Proof. Let D be any divisor in T. By Corollary 3.5.10, we have

$$\Gamma_{\min}(f_{D-E}) = \Gamma_{\min}(f_{D-T_E}) \bigcap \Gamma_{\min}(f_{T_E-E}) \neq \emptyset.$$

Therefore,

$$\begin{aligned}\rho(E, D) = \max(\mathcal{N}(f_{D-E})) &= \max(\mathcal{N}(f_{D-T_E}) + \mathcal{N}(f_{T_E-E})) \\ &\geqslant \max(\mathcal{N}(f_{T_E-E})) = \rho(E, T_E),\end{aligned}$$

which implies $\rho(E, T_E) = r_{\min}$ and thus $T_E \in L^T_{\min}(\rho_E)$.

For (2), the compactness of $L^T_{\min}(\rho_E)$, $L^T_{\leqslant r}(\rho_E)$, $L^T_{=r}(\rho_E)$, $L^T_{\leqslant s}(\mathcal{B}_{\star-E})$ and $L^T_{=s}(\mathcal{B}_{\star-E})$ follows from the compactness of T and the continuity of the distance function and the $\mathcal{B}$-function.

Now let us show $L^T_{\leqslant r}(\rho_E)$ and $L^T_{\leqslant s}(\mathcal{B}_{\star-E})$ are tropically convex. In the following arguments, we do not require T to be compact. The tropical convexity of $L^T_{\min}(\rho_E)$ will follow from the tropical convexity of $L^T_{\leqslant r}(\rho_E)$ by setting $r = r_{\min}$. By Proposition 3.5.7 and Remark 3.5.9, if $D_1, D_2 \in L^T_{\leqslant r}(\rho_E)$, then

$$\rho(E, D) \leqslant \max\{\rho(E, D_1), \rho(E, D_2)\} \leqslant r$$

for all D in $\mathrm{tconv}(D_1, D_2)$ and thus $\mathrm{tconv}(D_1, D_2) \subseteq L^T_{\leqslant r}(\rho_E)$. Respectively, if $D_1, D_2 \in L^T_{\leqslant s}(\mathcal{B}_{\star-E})$, then

$$\mathcal{B}_{\star-E}(D) < \max\{\mathcal{B}_{\star-E}(D_1), \mathcal{B}_{\star-E}(D_2)\} \leqslant s$$

for all D in the interior of $\mathrm{tconv}(D_1, D_2)$ and thus $\mathrm{tconv}(D_1, D_2) \subseteq L^T_{\leqslant s}(\mathcal{B}_{\star-E})$. Therefore, both $L^T_{\leqslant r}(\rho_E)$ and $L^T_{\leqslant s}(\mathcal{B}_{\star-E})$ are tropically convex. $\square$

3.6 *Reduced divisors in tropical segments*

As t-segments are tropically convex and compact (Lemma 3.3.2), the reduced divisors are well-defined for t-segments. In this section, we study the properties of reduced divisors in t-segments, and the results will be employed intensively in the next section where we give proofs to some prestated theorems.

3.6.1 Basic properties

Lemma 3.6.1. *For $E, D_1, D_2 \in \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$, let D_0 be the E-reduced divisor in* $\operatorname{tconv}(D_1, D_2)$. *Then we have*

$$\Gamma_{\min}(f_{D_0-E}) = \Gamma_{\min}(f_{D_1-E}) \bigcup \Gamma_{\min}(f_{D_2-E}),$$

and for all $D \in \operatorname{tconv}(D_1, D_2)$,

$$\Gamma_{\min}(f_{D-E}) \subseteq \Gamma_{\min}(f_{D_1-E}) \bigcup \Gamma_{\min}(f_{D_2-E}).$$

Proof. Applying Corollary 3.5.10 to $\operatorname{tconv}(D_1, D_2)$ with respect to E and knowing that D_0 is the corresponding reduced divisor, we have

$$\Gamma_{\min}(f_{D_1-E}) = \Gamma_{\min}(f_{D_1-D_0}) \bigcap \Gamma_{\min}(f_{D_0-E}),$$

$$\Gamma_{\min}(f_{D_2-E}) = \Gamma_{\min}(f_{D_2-D_0}) \bigcap \Gamma_{\min}(f_{D_0-E}),$$

and

$$\Gamma_{\min}(f_{D-E}) = \Gamma_{\min}(f_{D-D_0}) \bigcap \Gamma_{\min}(f_{D_0-E}).$$

Moreover, we have

$$\Gamma_{\min}(f_{D_1-D_0}) \bigcap \Gamma_{\min}(f_{D_2-D_0}) = \Gamma$$

by Lemma 3.3.5. Therefore,

$$\Gamma_{\min}(f_{D-E}) \subseteq \Gamma_{\min}(f_{D_0-E}) = \Gamma_{\min}(f_{D_1-E}) \bigcup \Gamma_{\min}(f_{D_2-E}).$$

□

Lemma 3.6.2. *Let $E, D_1, D_2 \in \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$ and $D_1', D_2' \in \operatorname{tconv}(D_1, D_2)$. Suppose $D_1' \in \operatorname{tconv}(D_1, D_2')$. Let D_0 be the E-reduced divisor in $\operatorname{tconv}(D_1, D_2)$ and D_0' be the E-reduced divisor in $\operatorname{tconv}(D_1', D_2')$. Then*

1. *$D_0' = D_0$ if and only if $D_0 \in \operatorname{tconv}(D_1', D_2')$;*
2. *$D_0' = D_1'$ if and only if $D_0 \in \operatorname{tconv}(D_1, D_1')$;*
3. *$D_0' = D_2'$ if and only if $D_0 \in \operatorname{tconv}(D_2', D_2)$.*

Proof. This is an immediate consequence of the fact that the functions $\mathcal{B}(P_{D_1-D_0}(t) - E)$ and $\mathcal{B}(P_{D_2-D_0}(t) - E)$ are both strictly increasing (Corollary 3.5.10). □

Lemma 3.6.3. *For $D_1, D_2, D_3 \in \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$, we have*

1. *D_2 is the D_1-reduced divisor in $\operatorname{tconv}(D_2, D_3)$ if and only if*
$$\mathcal{B}(D_3 - D_1) = \mathcal{B}(D_3 - D_2) + \mathcal{B}(D_2 - D_1).$$
2. *D_2 is simultaneously the D_1-reduced divisor in $\operatorname{tconv}(D_2, D_3)$ and the D_3-reduced divisor in $\operatorname{tconv}(D_1, D_2)$ if and only if*
$$\rho(D_1, D_3) = \rho(D_1, D_2) + \rho(D_2, D_3).$$

Proof. (1) follows easily from Lemma 3.5.2 (1), Proposition 3.5.7 and the criterions for reduced divisors (Corollary 3.5.10).

Recall that by Lemma 3.5.2 (2), we have $\rho(D_1, D_3) = \rho(D_1, D_2) + \rho(D_2, D_3)$ if and only if

$$\mathcal{B}(D_3 - D_1) = \mathcal{B}(D_3 - D_2) + \mathcal{B}(D_2 - D_1)$$

and

$$\mathcal{B}(D_1 - D_3) = \mathcal{B}(D_1 - D_2) + \mathcal{B}(D_2 - D_3).$$

Then (2) follows from (1). □

Remark 3.6.4. By Lemma 3.6.3, the for the sufficient and necessary conditions for equality in Lemma 3.5.15 can be equivalently stated as E_2 is the E_1-reduced divisor in $\mathrm{tconv}(E_2, T_{E_2})$ and E_1 is the E_2-reduced divisor in $\mathrm{tconv}(E_1, T_{E_1})$.

3.6.2 Tropical triangles

Roughly, we may call the tropical convex hull generated by three divisors in $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ a tropical triangle. We will show that tropical triangles are made of tropical segments.

Proposition 3.6.5. *Let $D_0, D_1, D_2 \in \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ (see Figure 11), $D_3 \in \mathrm{tconv}(D_0, D_1)$ and $D_4 \in \mathrm{tconv}(D_0, D_2)$. Then we have we have the following properties.*

1. *For every $D_5 \in \mathrm{tconv}(D_3, D_4)$, there exists $D'_5 \in \mathrm{tconv}(D_1, D_2)$ such that $D_5 \in \mathrm{tconv}(D_0, D'_5)$. In particular, we can let D'_5 be the D_5-reduced divisor in $\mathrm{tconv}(D_1, D_2)$.*

2. *Conversely, for every $D'_5 \in \mathrm{tconv}(D_1, D_2)$, there exists $D_5 \in \mathrm{tconv}(D_3, D_4)$ such that $D_5 \in \mathrm{tconv}(D_0, D'_5)$. (In other words, $\mathrm{tconv}(D_3, D_4) \bigcap \mathrm{tconv}(D_0, D'_5) \neq \emptyset$.) More precisely, assuming D'_3 is the D_3-reduced divisor in $\mathrm{tconv}(D_1, D_2)$ and D'_4 is the D_4-reduced divisor in $\mathrm{tconv}(D_1, D_2)$, we have*

   - *if $D'_5 \in \mathrm{tconv}(D'_3, D'_4)$, then D_5 can be chosen such that D'_5 be the D_5-reduced divisor in $\mathrm{tconv}(D_1, D_2)$;*
   - *if $D'_5 \in \mathrm{tconv}(D_1, D'_3)$, then D_5 can be chosen to be D_3; and*
   - *if $D'_5 \in \mathrm{tconv}(D_2, D'_4)$, then D_5 can be chosen to be D_4.*

Proof. For (1), we suppose D'_5 is the D_5-reduced divisor in $\mathrm{tconv}(D_1, D_2)$, and claim that $D_5 \in \mathrm{tconv}(D_0, D'_5)$. By Lemma 3.6.1, we have

$$\Gamma_{\min}(f_{D'_5 - D_5}) = \Gamma_{\min}(f_{D_1 - D_5}) \bigcup \Gamma_{\min}(f_{D_2 - D_5}).$$

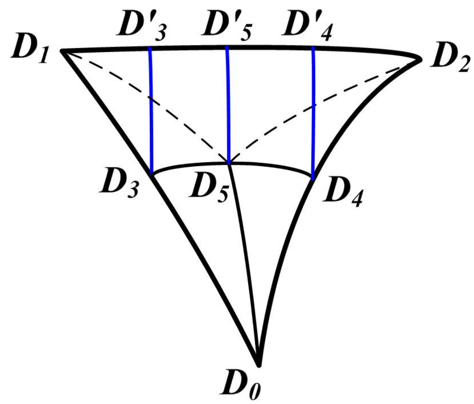

Figure 11: An illustration for Proposition 3.6.5.

Applying Lemma 3.6.1 again, we have

$$\Gamma_{\min}(f_{D_3-D_5}) \subseteq \Gamma_{\min}(f_{D_0-D_5}) \bigcup \Gamma_{\min}(f_{D_1-D_5})$$

and

$$\Gamma_{\min}(f_{D_4-D_5}) \subseteq \Gamma_{\min}(f_{D_0-D_5}) \bigcup \Gamma_{\min}(f_{D_2-D_5}).$$

Note that $\Gamma_{\min}(f_{D_3-D_5}) \bigcup \Gamma_{\min}(f_{D_4-D_5}) = \Gamma$ by Lemma 3.3.5. Therefore,

$$\begin{aligned}
&\Gamma_{\min}(f_{D_0-D_5}) \bigcup \Gamma_{\min}(f_{D'_5-D_5}) \\
&= \Gamma_{\min}(f_{D_0-D_5}) \bigcup (\Gamma_{\min}(f_{D_1-D_5}) \bigcup \Gamma_{\min}(f_{D_2-D_5})) \\
&= (\Gamma_{\min}(f_{D_0-D_5}) \bigcup \Gamma_{\min}(f_{D_1-D_5})) \bigcup (\Gamma_{\min}(f_{D_0-D_5}) \bigcup \Gamma_{\min}(f_{D_2-D_5})) \\
&\supseteq \Gamma_{\min}(f_{D_3-D_5}) \bigcup \Gamma_{\min}(f_{D_4-D_5}) = \Gamma,
\end{aligned}$$

which means $D_5 \in \mathrm{tconv}(D_0, D'_5)$ by Lemma 3.3.5.

For (2), we need to use a fact in Section 3.8 that reduced-divisor maps (Definition 3.8.1) are continuous (Lemma 3.8.2). Then it follows that if $D'_5 \in \mathrm{tconv}(D'_3, D'_4)$, then there exists $D_5 \in \mathrm{tconv}(D_3, D_4)$ such that D'_5 be the D_5-reduced divisor in $\mathrm{tconv}(D_1, D_2)$. By (1), this also means that $D_5 \in \mathrm{tconv}(D_0, D'_5)$ as expected.

If $D'_5 \in \mathrm{tconv}(D_1, D'_3)$, then by Proposition 3.5.7 and Corollary 3.5.10,

$$\Gamma_{\min}(f_{D'_5-D_3}) = \Gamma_{\min}(f_{D'_5-D'_3}) \bigcap \Gamma_{\min}(f_{D'_3-D_3})$$

and

$$\Gamma_{\min}(f_{D_1-D_3}) = \Gamma_{\min}(f_{D_1-D'_3}) \bigcap \Gamma_{\min}(f_{D'_3-D_3}),$$

which imply $\Gamma_{\min}(f_{D_1-D_3}) \subseteq \Gamma_{\min}(f_{D'_5-D_3})$. (Actually, if in addition $D'_5 \neq D'_3$, then $\Gamma_{\min}(f_{D_1-D_3}) = \Gamma_{\min}(f_{D'_5-D_3})$.) By Lemma 3.3.5, since $D_3 \in \mathrm{tconv}(D_0, D_1)$ which implies

$$\Gamma_{\min}(f_{D_0-D_3}) \bigcup \Gamma_{\min}(f_{D_1-D_3}) = \Gamma,$$

we have

$$\Gamma_{\min}(f_{D_0-D_3}) \bigcup \Gamma_{\min}(f_{D'_5-D_3}) = \Gamma$$

which implies $D_3 \in \mathrm{tconv}(D_0, D'_5)$.

If $D'_5 \in \mathrm{tconv}(D_2, D'_4)$, a similar argument can show that $D_4 \in \mathrm{tconv}(D_0, D'_5)$. □

Remark 3.6.6. In our proof of Proposition 3.6.5 (2), in the case that $D'_5 \in \mathrm{tconv}(D'_3, D'_4)$ and D'_5 is the D_5-reduced divisor in $\mathrm{tconv}(D_1, D_2)$, we do not need an additional assumption that D'_3, D'_5, D'_4 lie in $\mathrm{tconv}(D_1, D_2)$ in the same order as D_3, D_5, D_4 lie in $\mathrm{tconv}(D_3, D_4)$ as illustrated in Figure 11. But this is actually true, i.e., we must have $D'_3 \in \mathrm{tconv}(D_1, D'_4)$ (or equivalently $D'_4 \in \mathrm{tconv}(D_2, D'_3)$) and $D'_5 \in \mathrm{tconv}(D'_3, D'_4)$. Here is why. First we show that $D'_3 \in \mathrm{tconv}(D_1, D'_4)$. If $D'_3 \notin \mathrm{tconv}(D_1, D'_4)$, then $D_3 \neq D_4$. Referring to our proof of Proposition 3.6.5 (2), we see that $D_3, D_4 \in \mathrm{tconv}(D_0, D'_3)$ and $D_3, D_4 \in \mathrm{tconv}(D_0, D'_4)$. Let us draw contradictions from all possible cases. Recall that by Lemma 3.5.13, given a compact tropical convex set T, a divisor E of the same degree and T_E the corresponding E-reduced divisor in T, all the divisors on $\mathrm{tconv}(E, T_E)$ share the same reduced divisor in T.

- $D_4 \in \mathrm{tconv}(D_3, D'_3)$: It implies $D'_4 = D'_3$, a contradiction.
- $D_3 \in \mathrm{tconv}(D_4, D'_4)$: It implies $D'_3 = D'_4$, a contradiction.
- $D_4 \in \mathrm{tconv}(D_0, D_3)$: It goes back to the case $D_3 \in \mathrm{tconv}(D_4, D'_4)$. (To see

this, you may want to use Lemma 3.3.5 and refer to our proof of Proposition 3.6.5 (2).)

- $D_3 \in \text{tconv}(D_0, D_4)$: It goes back to the case $D_4 \in \text{tconv}(D_3, D'_3)$.

Thus we get $D'_3 \in \text{tconv}(D_1, D'_4)$ as claimed. Now suppose there exists $D_5 \in \text{tconv}(D_3, D_4)$ such that $D'_5 \notin \text{tconv}(D'_3, D'_4)$. Actually we may suppose $D'_5 \in \text{tconv}(D_1, D'_3) \setminus \{D'_3\}$ and $D'_3 \in \text{tconv}(D'_4, D'_5)$. Then by the continuity of reduced-divisor maps, there must exist $D_6 \in \text{tconv}(D_4, D_5)$ such that D'_3 is also the D_6-reduced divisor in $\text{tconv}(D_1, D_2)$. Then following from Proposition 3.6.5 (1), both D_3 and D_6 lie in $\text{tconv}(D_0, D'_3)$. Since $D_5 \in \text{tconv}(D_3, D_6)$, we get $D'_5 = D'_3$ no matter $D_6 \in \text{tconv}(D_3, D'_3)$ or $D_3 \in \text{tconv}(D_6, D'_3)$ by Lemma 3.5.13, which is a contradiction. □

Remark 3.6.7. There are several aspects of Proposition 3.6.5. First, as in (1), if we choose arbitrarily a divisor (e.g. D_3) in $\text{tconv}(D_0, D_1)$, a divisor (e.g. D_4) in $\text{tconv}(D_0, D_2)$, and then arbitrarily a divisor (e.g. D_5) in $\text{tconv}(D_3, D_4)$, we may add a t-segment $\text{tconv}(D_5, D'_5)$ with $D'_5 \in \text{tconv}(D_1, D_2)$ to the t-segment $\text{tconv}(D_0, D_5)$ while the result of such an extension is exactly $\text{tconv}(D_0, D'_5)$. With one step further, we can derive Corollary 3.6.8, which is a special case of Theorem 3.4.5. Second, the D_5-reduced divisor in $\text{tconv}(D_1, D_2)$ (as we've done throughout the proof) is a desired choice for D'_5. On the other hand, in some cases, we can choose D'_5 which is not necessarily D_5-reduced. Third, as in (2), it says that $\text{tconv}(D_3, D_4)$ and $\text{tconv}(D_0, D'_5)$ must intersect. But the intersection might not be just a single point.

Corollary 3.6.8. *For $D_0, D_1, D_2 \in \mathbb{R}\,\text{Div}_+^{\text{d}}(\Gamma)$, choose arbitrarily D'_1 in $\text{tconv}(D_0, D_1)$ and D'_2 in $\text{tconv}(D_0, D_2)$. Then we have*

$$\text{tconv}(D'_1, D'_2) \subseteq \bigcup_{D \in \text{tconv}(D_1, D_2)} \text{tconv}(D_0, D)$$

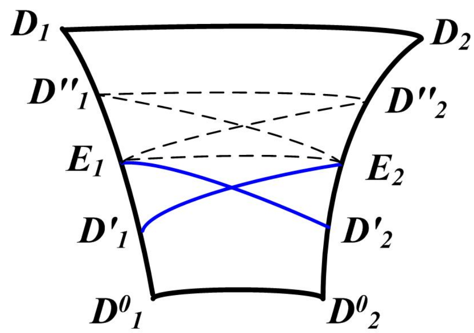

Figure 12: An illustration for Proposition 3.6.9.

and

$$\mathrm{tconv}(D_0, D_1, D_2) = \bigcup_{D \in \mathrm{tconv}(D_1, D_2)} \mathrm{tconv}(D_0, D).$$

Proof. By Proposition 3.6.5, we see immediately

$$\mathrm{tconv}(D'_1, D'_2) \subseteq \bigcup_{D \in \mathrm{tconv}(D_1, D_2)} \mathrm{tconv}(D_0, D).$$

Then $\bigcup_{D \in \mathrm{tconv}(D_1, D_2)} \mathrm{tconv}(D_0, D)$ is tropically convex by definition, and must be the minimal to contain D_0, D_1 and D_2. Thus

$$\mathrm{tconv}(D_0, D_1, D_2) = \bigcup_{D \in \mathrm{tconv}(D_1, D_2)} \mathrm{tconv}(D_0, D).$$

□

3.6.3 Useful length inequalities

Proposition 3.6.9. *For $D_1^0, D_2^0, D_1, D_2 \in \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}(\Gamma)$ (Figure 12), let $E_1 \in \mathrm{tconv}(D_1^0, D_1)$ and $E_2 \in \mathrm{tconv}(D_2^0, D_2)$. Let D'_1 be the E_2-reduced divisor in $\mathrm{tconv}(D_0, D_1)$ and D'_2 the E_1-reduced divisor in $\mathrm{tconv}(D_0, D_2)$. If $D'_1 \in \mathrm{tconv}(D_1^0, E_1)$ and $D'_2 \in \mathrm{tconv}(D_2^0, E_2)$, then $\rho(E_1, E_2) \leqslant \rho(D''_1, D''_2)$ for all $D''_1 \in \mathrm{tconv}(E_1, D_1)$ and $D''_2 \in \mathrm{tconv}(E_2, D_2)$.*

Proof. Let l_{tot} be the total length of Γ. Under the assumptions and applying Lemma 3.6.2, D'_1 must also be the E_2-reduced divisor in both $\mathrm{tconv}(D_1^0, E_1)$ and $\mathrm{tconv}(D_1^0, D''_1)$, and D_2' must also E_1-reduced divisor in both $\mathrm{tconv}(D_2^0, E_2)$ $\mathrm{tconv}(D_2^0, D_2'')$. Therefore, applying Corollary 3.5.10, we get the following equalities.

$$\begin{aligned}
&\mathcal{B}(D_1'' - E_2)\\
&= \mathcal{B}(D_1' - E_2) + \mathcal{B}(D_1'' - D_1')\\
&= \mathcal{B}(D_1' - E_2) + \mathcal{B}(D_1'' - E_1) + \mathcal{B}(E_1 - D_1')\\
&= \mathcal{B}(D_1'' - E_1) + \mathcal{B}(E_1 - E_2),
\end{aligned}$$

and analogously

$$\mathcal{B}(D_2'' - E_1) = \mathcal{B}(D_2'' - E_2) + \mathcal{B}(E_2 - E_1).$$

Therefore,

$$\begin{aligned}
\rho(E_1, E_2)l_{\mathrm{tot}} &= \mathcal{B}(E_1 - E_2) + \mathcal{B}(E_2 - E_1)\\
&= (\mathcal{B}(D_1'' - E_2) - \mathcal{B}(D_1'' - E_1)) + (\mathcal{B}(D_2'' - E_1) - \mathcal{B}(D_2'' - E_2))\\
&= (\mathcal{B}(D_1'' - E_2) - \mathcal{B}(D_2'' - E_2)) + (\mathcal{B}(D_2'' - E_1) - \mathcal{B}(D_1'' - E_1))\\
&\leqslant \mathcal{B}(D_1'' - D_2'') + \mathcal{B}(D_2'' - D_1'') = \rho(D_1'', D_2'')l_{\mathrm{tot}}.
\end{aligned}$$

The last inequality follows from the triangle inequality for $\mathcal{B}$-functions(Lemma 3.5.2).

□

The following corollaries of Proposition 3.6.9 are two special cases convenient for applications.

Corollary 3.6.10. *Let $D_1^0, D_2^0, D_1, D_2, E_1, E_2$ be under the same hypotheses as in Proposition 3.6.9. If E_1 is the E_2-reduced divisor in $\mathrm{tconv}(D_1^0, D_1)$, then*

$$\rho(E_1, E_2) \leqslant \max(\rho(D_1, D_2), \rho(D_1^0, D_2^0)).$$

In particular, if in addition $D_1^0 = D_2^0$, then $\rho(E_1, E_2) \leqslant \rho(D_1'', D_2'')$ for all $D_1'' \in \mathrm{tconv}(E_1, D_1)$ and $D_2'' \in \mathrm{tconv}(E_2, D_2)$.

Proof. Let D_1' be the E_2-reduced divisor in $\operatorname{tconv}(D_1^0, D_1)$ and D_2' the E_1-reduced divisor in $\operatorname{tconv}(D_2^0, D_2)$. Then D_1' is exactly E_1 which means $D_1' \in \operatorname{tconv}(D_0, E_1)$ automatically. Thus by Proposition 3.6.9, if $D_2' \in \operatorname{tconv}(D_2^0, E_2)$, then $\rho(E_1, E_2) \leqslant \rho(D_1, D_2)$, and if $D_2' \in \operatorname{tconv}(D_2, E_2)$, then $\rho(E_1, E_2) \leqslant \rho(D_1^0, D_2^0)$. In both cases, $\rho(E_1, E_2) \leqslant \max(\rho(D_1, D_2), \rho(D_1^0, D_2^0))$.

Recall that by Lemma3.5.15, the distance between reduced divisors is at most the distance between the original divisors. Thus if in addition $D_1^0 = D_2^0 = D_0$, then $\rho(D_0, D_2') \leqslant \rho(D_0, E_1) \leqslant \rho(D_0, E_2)$, which implies $D_2' \in \operatorname{tconv}(D_0, E_2)$. It follows from Proposition 3.6.9 that $\rho(E_1, E_2) \leqslant \rho(D_1'', D_2'')$. □

Corollary 3.6.11. *Let $D_1^0, D_2^0, D_1, D_2, E_1, E_2$ be under the same hypotheses as in Proposition 3.6.9 and suppose $D_1^0 = D_2^0 = D_0$. If $\rho(D_0, E_1) = \rho(D_0, E_2)$, then $\rho(E_1, E_2) \leqslant \rho(D_1'', D_2'')$ for all $D_1'' \in \operatorname{tconv}(E_1, D_1)$ and $D_2'' \in \operatorname{tconv}(E_2, D_2)$.*

Proof. By Lemma3.5.15 and get $\rho(D_0, D_1') \leqslant \rho(D_0, E_2)$ and $\rho(D_0, D_2') \leqslant \rho(D_0, E_1)$. Since $\rho(D_0, E_1) = \rho(D_0, E_2)$, we get $\rho(D_0, D_1') \leqslant \rho(D_0, E_1)$ and $\rho(D_0, D_2') \leqslant \rho(D_0, E_2)$. Thus we have $D_1' \in \operatorname{tconv}(D_0, E_1)$ and $D_2' \in \operatorname{tconv}(D_0, E_2)$, and it follows from Proposition 3.6.9 that $\rho(E_1, E_2) \leqslant \rho(D_1'', D_2'')$. □

3.7 A revisit of the general properties of tropical convex sets

3.7.1 Proofs of Theorem 3.4.4 and Theorem 3.4.5

Proof of Theorem 3.4.4. Let $T \subseteq \mathbb{R}\operatorname{Div}_+^{\mathrm{d}}(\Gamma)$ be tropically convex. To show T is contractible, it suffices to find a continuous function $h : [0,1] \times T \to T$ such that for some $D_0 \in T$ and all $D \in T$, $h(0, D) = D$ and $h(1, D) = D_0$. Indeed, we can define the contraction map h as follows. Choose D_0 arbitrarily from T and let $\kappa = \sup_{D' \in T} \rho(D_0, D')$. For any $D \in T$, we let $h(t, D) = D$ if $t \in [0, 1 - \frac{\rho(D_0,D)}{\kappa})$, and $h(t, D) = P_{D_0 - D}(\frac{\kappa}{\rho(D_0,D)}(t-1)+1)$ if $t \in [1 - \frac{\rho(D_0,D)}{\kappa}, 1]$. More explicitly, the contraction happens in the following way: for any $t \in [0, 1]$, if $\rho(D_0, D) < \kappa(1-t)$, then $h(t,D) = D$, and otherwise, $h(t,D)$ lies on the t-segment $\mathrm{tconv}(D_0, D)$ with distance $\kappa(1-t)$ to D_0. Then it is clear that $h(0,D) = D$ and $h(1,D) = D_0$. Therefore the only remaining fact to verify is the continuity of h. In other words, we need to show that $h(t_n, D_n) \to h(t,D)$ whenever $t_n \to t$ and $D_n \to D$ (we let $n > 0$ for D_n to avoid confusion with D_0). Note that

$$\rho(h(t_n, D_n), h(t,D)) \leqslant \rho(h(t_n, D_n), h(t, D_n)) + \rho(h(t, D_n), h(t,D)),$$

and

$$\rho(h(t_n, D_n), h(t, D_n)) \leqslant \rho(D_0, D_n)|t_n - t| \leqslant \kappa|t_n - t|.$$

Therefore, to show the continuity of h, it suffices to show $\rho(h(t, D_n), h(t,D)) \leqslant \rho(D_n, D)$.

Case (1): $\rho(D_0, D_n) < \kappa(1-t)$ and $\rho(D_0, D) < \kappa(1-t)$. In this case, $h(t, D_n) = D_n$ and $h(t,D) = D$.

Case (2): $\rho(D_0, D_n) < \kappa(1-t)$ and $\rho(D_0, D) \geqslant \kappa(1-t)$. In this case, $h(t, D_n) = D_n$ and $h(t,D) \in \mathrm{tconv}(D_0, D)$ with distance $\kappa(1-t)$ to D_0. Let $D' \in \mathrm{tconv}(D_0, D)$ be the D_n-reduced divisor in $\mathrm{tconv}(D_0, D)$. Then by Lemma 3.5.15,

$$\rho(D_0, D') \leqslant \rho(D_0, D_n) < \kappa(1-t) = \rho(D_0, h(t,D)).$$

This means $D' \in \mathrm{tconv}(D_0, h(t,D))$, and by Proposition 3.6.9,

$$\rho(h(t, D_n), h(t,D)) = \rho(D_n, h(t,D)) \leqslant \rho(D_n, D).$$

Case (3): $\rho(D_0, D_n) \geqslant \kappa(1-t)$ and $\rho(D_0, D) < \kappa(1-t)$. In this case, $h(t,D) = D$ and $h(t, D_n) \in \mathrm{tconv}(D_0, D_n)$ with distance $\kappa(1-t)$ to D_0. Let $D'_n \in \mathrm{tconv}(D_0, D_n)$ be the D-reduced divisor in $\mathrm{tconv}(D_0, D_n)$. Using an analogous argument as in case (2), we see that $\rho(h(t, D_n), h(t,D)) \leqslant \rho(D_n, D)$.

Case (4): $\rho(D_0, D_n) \geqslant \kappa(1-t)$ and $\rho(D_0, D) \geqslant \kappa(1-t)$. In this case, $h(t, D_n) \in \mathrm{tconv}(D_0, D_n)$ and $h(t,D) \in \mathrm{tconv}(D_0, D)$, both with distance $\kappa(1-t)$ to D_0. Therefore, by Corollary 3.6.11, we have $\rho(h(t, D_n), h(t,D)) \leqslant \rho(D_n, D)$. □

Remark 3.7.1. The contraction map h constructed in the above proof deforms the whole T to a point $D_0 \in T$. In particular, one can notice that at each $t \in [0,1]$, the set $h(t,T)$ is actually the sublevel set $L^T_{\leqslant r}(\rho_{D_0})$ of the distance function ρ_{D_0} to D_0 where $r = \kappa(1-t)$. Therefore, $h(t,T)$ is tropically convex by Proposition 3.5.19 (3).

Proof of Theorem 3.4.5. Denote $\bigcup_{D\in T, D'\in T'} \mathrm{tconv}(D,D')$ by $\tilde{T}$. Then clearly $\tilde{T} \subseteq \mathrm{tconv}(T,D)$. We claim that $\tilde{T}$ is tropically convex, which will imply $\tilde{T} = \mathrm{tconv}(T,D)$.

Choose arbitrarily E_1 and E_2 from $\tilde{T}$. Then there exist $D_1, D_2 \in T$ and $D'_1, D'_2 \in T'$ such that $E_1 \in \mathrm{tconv}(D_1, D'_1)$ and $E_2 \in \mathrm{tconv}(D_2, D'_2)$. Since T and T' are tropically convex, we have $\mathrm{tconv}(D_1, D_2) \subseteq T$ and $\mathrm{tconv}(D'_1, D'_2) \subseteq T'$. For every $E \in \mathrm{tconv}(E_1, E_2)$, let $D \in \mathrm{tconv}(D_1, D_2)$ be the E-reduced divisor in $\mathrm{tconv}(D_1, D_2)$ and $D' \in \mathrm{tconv}(D'_1, D'_2)$ be the E-reduced divisor in $\mathrm{tconv}(D'_1, D'_2)$. To show $\tilde{T}$ is tropically convex, it suffices to show that $E \in \mathrm{tconv}(D, D')$.

By Lemma 3.6.1, we have

$$\Gamma_{\min}(f_{D-E}) = \Gamma_{\min}(f_{D_1-E}) \bigcup \Gamma_{\min}(f_{D_2-E}),$$

$$\Gamma_{\min}(f_{D'-E}) = \Gamma_{\min}(f_{D'_1-E}) \bigcup \Gamma_{\min}(f_{D'_2-E}),$$

$$\Gamma_{\min}(f_{E_1-E}) \subseteq \Gamma_{\min}(f_{D_1-E}) \bigcup \Gamma_{\min}(f_{D'_1-E}),$$

and

$$\Gamma_{\min}(f_{E_2-E}) \subseteq \Gamma_{\min}(f_{D_2-E}) \bigcup \Gamma_{\min}(f_{D'_2-E}).$$

Note that since $E \in \mathrm{tconv}(E_1, E_2)$, we have $\Gamma_{\min}(f_{E_1-E}) \bigcup \Gamma_{\min}(f_{E_2-E}) = \Gamma$ by Lemma 3.3.5. Therefore,

$$\begin{aligned}
&\Gamma_{\min}(f_{D-E}) \bigcup \Gamma_{\min}(f_{D'-E}) \\
&= (\Gamma_{\min}(f_{D_1-E}) \bigcup \Gamma_{\min}(f_{D_2-E})) \bigcup (\Gamma_{\min}(f_{D'_1-E}) \bigcup \Gamma_{\min}(f_{D'_2-E})) \\
&= (\Gamma_{\min}(f_{D_1-E}) \bigcup \Gamma_{\min}(f_{D'_1-E})) \bigcup (\Gamma_{\min}(f_{D_2-E}) \bigcup \Gamma_{\min}(f_{D'_2-E})) \\
&\supseteq \Gamma_{\min}(f_{E_1-E}) \bigcup \Gamma_{\min}(f_{E_2-E}) = \Gamma,
\end{aligned}$$

which means $E \in \mathrm{tconv}(D, D')$ by Lemma 3.3.5.

Recall that a metric space is compact if and only if it is complete and totally bounded. Now let us show that if in addition T and T' are complete and totally bounded, then $\tilde{T}$ is also complete and totally bounded.

First, we show that $\tilde{T}$ is complete. Let $E_1, E_2, \ldots$ be a Cauchy sequence in $\tilde{T}$, i.e., $\rho(E_m, E_n) \to 0$ as $m, n \to \infty$. We claim that there exists $E_0 \in \tilde{T}$ such that $\rho(E_n, E_0) \to 0$ as $n \to \infty$, which implies the completeness of $\tilde{T}$. Since T is compact, there exist a unique E_i-reduced divisor D_i in T and a unique E_i-reduced divisor D'_i in T'. Then $D_1, D_2, \ldots$ is a Cauchy sequence in T and $D'_1, D'_2, \ldots$ is a Cauchy sequence in T', since $\rho(D_m, D_n) \leqslant \rho(E_m, E_n)$ and $\rho(D'_m, D'_n) \leqslant \rho(E_m, E_n)$ by Lemma3.5.15. Let $D_0 \in T$ be the limit of $D_1, D_2, \ldots$ and $D'_0 \in T$ be the limit of $D'_1, D'_2, \ldots$. Consider the t-segments $\mathrm{tconv}(D_0, D'_0)$. Then we get another Cauchy sequence $F_1, F_2, \ldots$ in $\mathrm{tconv}(D_0, D'_0)$, where F_i be the E_i-reduced divisor in $\mathrm{tconv}(D_0, D'_0)$. If $E_0 \in \mathrm{tconv}(D, D_0)$ is the limit of $F_1, F_2, \ldots$, then we have

$$\rho(E_n, E_0) \leqslant \rho(E_n, F_n) + \rho(F_n, E_0) \leqslant \max(\rho(D_n, D_0), \rho(D'_n, D'_0) + \rho(F_n, E_0),$$

where the second inequality follows from Corollary 3.6.10. Thus $\rho(E_n, E_0) \to 0$ as $n \to \infty$ as claimed.

Second, we show that $\tilde{T}$ is totally bounded, i.e., for every real $\epsilon > 0$, there exists a finite cover of $\tilde{T}$ by open balls of radius ϵ. We start with a finite cover of T by open balls $B^T(D_i, \epsilon/2) \subseteq T$ of radius $\epsilon/2$ with centers $D_i \in T$ for $i = 1, \ldots, n$, and a finite cover of T' by open balls $B^{T'}(D'_j, \epsilon/2) \subseteq T'$ of radius $\epsilon/2$ with centers $D'_j \in T'$ for $j = 1, \ldots, m$. Then for each $\mathrm{tconv}(D_i, D'_j)$, we have a finite cover by open balls $B^{(i,j)}(D^{(i,j)}_{k^{(i,j)}}, \epsilon/2) \subseteq \mathrm{tconv}(D_i, D'_j)$ of radius $\epsilon/2$ with the centers $D^{(i,j)}_{k^{(i,j)}} \in \mathrm{tconv}(D_i, D'_j)$ for $k^{(i,j)} = 1, \ldots, m^{(i,j)}$. We claim that there is a finite cover of $\tilde{T}$ by open balls $B^{\tilde{T}}(D^{(i,j)}_{k^{(i,j)}}, \epsilon) \subseteq \tilde{T}$ of radius ϵ with the centers $D^{(i,j)}_{k^{(i,j)}} \in \tilde{T}$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $k^{(i,j)} = 1, \ldots, m^{(i,j)}$. For any $E \in \tilde{T}$, there exist $D \in T$ and $D' \in T'$ such that $E \in \mathrm{tconv}(D, D')$. Suppose $D \in B^T(D_i, \epsilon/2)$ for some i and $D' \in$

$B^{T'}(D'_j, \epsilon/2)$ for some j. Furthermore, let F be the E-reduced divisor in $\operatorname{tconv}(D_i, D'_j)$ and suppose $F \in B^{(i,j)}(D^{(i,j)}_{k^{(i,j)}}, \epsilon/2)$ for some $D^{(i,j)}_{k^{(i,j)}}$. We have

$$\rho(E, D^{(i,j)}_{k^{(i,j)}}) \leqslant \rho(E,F)+\rho(F, D^{(i,j)}_{k^{(i,j)}}) \leqslant \max(\rho(D,D_i), \rho(D',D'_j))+\rho(D'', D^{(i,j)}_{k^{(i,j)}}) < \epsilon/2+\epsilon/2 = \epsilon,$$

where the second inequality follows from Corollary 3.6.10. Thus E lies in $B^{\tilde{T}}(D^{(i,j)}_{k^{(i,j)}}, \epsilon)$, which means $\tilde{T}$ is covered by this finite collection of open balls as claimed. □

3.7.2 Finitely generated tropical convex hulls

Recall that Lemma 3.3.5 provides a criterion for judging whether a divisor D lies in a tropical segment $\operatorname{tconv}(D_1, D_2)$, and Lemma 3.6.1 extends the criterion. The following theorem generalizes these results to all finitely generated tropical convex hulls, which are compact according to Corollary 3.4.6.

Theorem 3.7.2. *Let $T \subseteq \mathbb{R}\operatorname{Div}^{\mathrm{d}}_{+}$ be a tropical convex hull finitely generated by $D_1, \ldots, D_n$. Then for any $E \in \mathbb{R}\operatorname{Div}^{\mathrm{d}}_{+}$, we have $E \in T$ if and only if $\bigcup_{i=1}^{n} \Gamma_{\min}(f_{D_i-E}) = \Gamma$. Furthermore, if D_0 is the E-reduced divisor in T and D is an arbitrary divisor in T, then*

$$\Gamma_{\min}(f_{D-E}) \subseteq \Gamma_{\min}(f_{D_0-E}) = \bigcup_{i=1}^{n} \Gamma_{\min}(f_{D_i-E}).$$

Proof. We prove by induction on the number of generators. Suppose the statements are true for all tropical convex hulls generated by n divisors. Now consider a tropical convex hull T generated by $n+1$ divisors $D_1, \ldots, D_{n+1}$. Let $T' = \operatorname{tconv}(D_1, \ldots, D_n)$ be a t-convex subset of T. For $E \in \mathbb{R}\operatorname{Div}^{\mathrm{d}}_{+}$, let D_0 be the E-reduced divisor in T. By Theorem 3.4.5, there exists $D'_0 \in T'$ such that $D_0 \in \operatorname{tconv}(D'_0, D_n)$, which implies $\Gamma_{\min}(f_{D'_0-D_0}) \bigcup \Gamma_{\min}(f_{D_n-D_0}) = \Gamma$ by Lemma 3.3.5. By assumption, we have

$$\Gamma_{\min}(f_{D'_0-D_0}) \subseteq \bigcup_{i=1}^{n} \Gamma_{\min}(f_{D_i-D_0}).$$

Thus, $\bigcup_{i=1}^{n+1} \Gamma_{\min}(f_{D_i-D_0}) = \Gamma$.

In addition, $\Gamma_{\min}(f_{D_i-D_0}) = \Gamma_{\min}(f_{D_0-E}) \bigcap \Gamma_{\min}(f_{D_i-D_0})$. Therefore,

$$\begin{aligned}\Gamma_{\min}(f_{D_0-E}) &= \Gamma_{\min}(f_{D_0-E}) \bigcap (\bigcup_{i=1}^{n+1} \Gamma_{\min}(f_{D_i-D_0})) \\ &= \bigcup_{i=1}^{n+1} (\Gamma_{\min}(f_{D_0-E}) \bigcap \Gamma_{\min}(f_{D_i-D_0})) \\ &= \bigcup_{i=1}^{n+1} \Gamma_{\min}(f_{D_i-E}).\end{aligned}$$

And this also implies $E \in T$ if and only if $\bigcup_{i=1}^{n+1} \Gamma_{\min}(f_{D_i-E}) = \Gamma$. $\square$

Let T be a tropical convex set. For $D \in T$, if $D \notin \mathrm{tconv}(T \setminus \{D\})$, (note that equivalently this means $T \setminus \{D\}$ is also tropically convex) then we say D is an *extremal* of T. It is clear from definition that any generating set of T must contain all the extremals of T.

Theorem 3.7.3. *Every finitely generated tropical convex hull T contains finitely many extremals. The set S of all extremals of T generates T and is minimal among all generating sets of T.*

Proof. Let S' be a finite generating set of T, i.e., $\mathrm{tconv}(S') = T$. We may choose a subset S of S' such that $\mathrm{tconv}(S) = T$ and S is t-convex independent. (The uniqueness of the choice of S, which follows from the assertion in the theorem, is not required now.) We claim S is the set of all extremals of T, which also implies the minimality of S.

Let $S = \{D_0, D_1, D_2, \ldots, D_n\}$ and $T = \mathrm{tconv}(D_1, \ldots, D_n)$. Since S is t-convex independent, we must have $D_0 \notin T'$, which implies $\bigcup_{i=1}^{n} \Gamma_{\min}(f_{D_i-D_0}) \neq \Gamma$ by Theorem 3.7.2. It suffices to show that D_0 is an extremal of T, i.e., $T \setminus \{D_0\}$ is tropically convex. Choose arbitrarily E_1 and E_2 in $T \setminus \{D_0\}$. According to Theorem 3.4.5, there exist F_1 and F_2 in T' such that $E_1 \in \mathrm{tconv}(D_0, F_1)$ and $E_2 \in \mathrm{tconv}(D_0, F_2)$. Note that it follows $\Gamma_{\min}(f_{E_1-D_0}) = \Gamma_{\min}(f_{F_1-D_0})$ and $\Gamma_{\min}(f_{E_2-D_0}) = \Gamma_{\min}(f_{F_2-D_0})$. By

Theorem 3.7.2, we have

$$\Gamma_{\min}(f_{F_1-D_0}) \subseteq \bigcup_{i=1}^{n} \Gamma_{\min}(f_{D_i-D_0})$$

and

$$\Gamma_{\min}(f_{F_2-D_0}) \subseteq \bigcup_{i=1}^{n} \Gamma_{\min}(f_{D_i-D_0}).$$

Then,

$$\Gamma_{\min}(f_{E_1-D_0}) \bigcup \Gamma_{\min}(f_{E_2-D_0}) = \Gamma_{\min}(f_{F_1-D_0}) \bigcup \Gamma_{\min}(f_{F_2-D_0}) \subseteq \bigcup_{i=1}^{n} \Gamma_{\min}(f_{D_i-D_0}) \neq \Gamma,$$

which implies $D_0 \notin \mathrm{tconv}(E_1, E_2)$. Therefore, $T \backslash \{D_0\}$ is tropically convex as claimed.

□

3.8 *Canonical projections*

The existence and uniqueness of a reduced divisor in a compact tropical convex set T with respect to an effective $\mathbb{R}$-divisor of the same degree enable us to define a projection map to T.

Definition 3.8.1. For a compact tropical convex set T of degree d, the *canonical projection* to T, $\gamma^T : \mathbb{R}\,\mathrm{Div}_+ \to T$, is given by sending E to the E'-reduced divisor $T_{E'}$ in T where $E' = \frac{d}{\deg E} E$.

Lemma 3.8.2. *Restricted to degree d, a reduced-divisor map $\gamma^T|_{\mathbb{R}\,\mathrm{Div}_+^d}$ is continuous.*

Proof. This is an immediate corollary of Lemma 3.5.15 and Lemma 3.3.9. □

Remark 3.8.3. For a complete linear system $|D|$, Omini [3] defined the reduced-divisor map: $\mathrm{Red} : \Gamma \to |D|$ by sending a point $q \in \Gamma$ to the (conventional) reduced divisor $D_q \in |D|$. In our setting, the map Red is precisely $\gamma^{|D|}|_{\mathrm{Div}_+^1}$.

Let us recall some basic topological notions of retractions and retracts. If Y is a subspace of a topological space X, then a *retraction* of X onto Y is a continuous surjection $r : X \twoheadrightarrow Y$ such that $r|_Y = id_Y$. A *deformation retraction* of X onto Y is a homotopy between the identity map of X and a retraction of X onto Y, or more explicitly, a continuous map $h : [0,1] \times X \to X$ such that for all $x \in X$ and $y \in Y$, $h(0,x) = x$, $h(1,x) \in Y$, and $h(1,y) = y$. If in addition $h(t,y) = y$ for all $t \in [0,1]$ and $y \in Y$, then h is called a *strong deformation retraction.* With respect to the existence of a retraction, a deformation retraction or a strong deformation retraction of X onto Y, we say Y is a *retract*, a *deformation retract* or a *strong deformation retract* of X.

Now let $T \subseteq \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}$ be a compact tropical convex set. We know that the canonical projection $\gamma^T|_{\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}}$ is continuous (Lemma 3.8.2) and $\gamma^T|_T = id_T$ (Lemma 3.5.11). Therefore, T is a retract of $\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}$ with $\gamma^T|_{\mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}}$ the retraction. In addition, we can use the reduced-divisor map to construct a strong deformation retraction on T.

Definition 3.8.4. Let $X \subseteq \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}$ be tropically convex. Let $T \subseteq X$ be a compact tropical convex subset of X. Then we say a strong deformation retraction $h : [0,1] \times X \to X$ of X onto T is a *tropical retraction* if at each $t \in [0,1]$, the set $h(t,X)$ is tropically convex. In this sense, we say T is a *tropical retract* of X.

Theorem 3.8.5. *For each compact tropical convex subset T of a tropical convex set $X \subseteq \mathbb{R}\,\mathrm{Div}_+^{\mathrm{d}}$, there exists a tropical retraction of X onto T.*

Proof. Our proof will be very similar to the proof of Theorem 3.4.4. We will explicitly construct such a tropical retraction $h : [0,1]\times X \to X$. In particular, for each $D \in W$, we want $h(0,D) = D$ and $h(1,D) = \gamma^T(D)$.

Let $\rho_T(D) := \min_{D' \in T} \rho(D,D')$. Note that $\rho(D,\gamma^T(D)) = \rho_T(D)$ (Proposition 3.5.19 (1)). Let $\kappa = \sup_{D\in X} \rho_T(D)$. We define h in the following way. For any $D \in W$, we let $h(t,D) = D$ if $t \in [0, 1-\frac{\rho_T(D)}{\kappa})$, and $h(t,D) = P_{\gamma^T(D)-D}(\frac{\kappa}{\rho_T(D)}(t-1)+1)$ if $t \in [1-\frac{\rho_T(D)}{\kappa}, 1]$. In other words, if $\rho_T(D) < \kappa(1-t)$, then $h(t,D) = D$, and otherwise, $h(t,D)$ lies on the t-segment $\mathrm{tconv}(D,\gamma^T(D))$ with distance $\kappa(1-t)$ to $\gamma^T(D)$. It can be easily verified that $h(0,D) = D$ and $h(1,D) = \gamma^T(D)$. In addition, if

$D \in T$, then $h(t, D) = D = \gamma^T(D)$ for all $t \in [0, 1]$. Now, to show h is actually a tropical retraction of X onto T, it remains to show that h is continuous, and $h(t, X)$ is tropically convex for all $t \in [0, 1]$.

To say h is continuous is equivalent to say $h(t_n, D_n) \to h(t, D)$ whenever $t_n \to t$ and $D_n \to D$. We have

$$\rho(h(t_n, D_n), h(t, D)) \leqslant \rho(h(t_n, D_n), h(t, D_n)) + \rho(h(t, D_n), h(t, D)),$$

and

$$\rho(h(t_n, D_n), h(t, D_n)) \leqslant \rho(D_0, D_n)|t_n - t| \leqslant \kappa|t_n - t|.$$

In stead of proving $\rho(h(t, D_n), h(t, D)) \leqslant \rho(D_n, D)$ as in the proof of Theorem 3.4.4, here we claim that $\rho(h(t, D_n), h(t, D))$ is bounded by $2 \cdot \rho(D_n, D)$, which is still sufficient to guarantee the continuity of h.

Let $h(t, D_n) = D'_n$ and $h(t, D) = D'$. Note that $\gamma^T(D_n) = \gamma^T(D'_n)$ and $\gamma^T(D) = \gamma^T(D')$ (Lemma 3.5.13). Denote these reduced divisors by C_n and C respectively. Also, we note that $\rho(D_n, D) \geqslant \rho(C_n, C)$ (Lemma 3.5.15).

Case (1): $\rho_T(D_n) < \kappa(1 - t)$ and $\rho_T(D) < \kappa(1 - t)$. Then $D'_n = D_n$ and $D' = D$. We automatically have $\rho(D'_n, D') = \rho(D_n, D)$.

Case (2): $\rho_T(D_n) < \kappa(1 - t)$ and $\rho_T(D) \geqslant \kappa(1 - t)$. Then $D'_n = D_n$ and

$$\rho(D_n, C_n) = \rho_T(D_n) < \rho(D', C) = \rho_T(D') = \kappa(1 - t).$$

Let $D'' \in \mathrm{tconv}(C, D)$ be the D_n-reduced divisor in $\mathrm{tconv}(C, D)$. Depending on the relative positions of D' and D'' in $\mathrm{tconv}(C, D)$, there are two subcases.

Subcase (2a): $D'' \in \mathrm{tconv}(C, D')$. By Proposition 3.6.9, we have

$$\rho(D'_n, D') = \rho(D_n, D') \leqslant \rho(D_n, D).$$

Subcase (2b): $D' \in \mathrm{tconv}(C, D'')$. Now let C'' be the C_n-reduced divisor in tconv(C, D). Then we have $\rho(C'', C) \leqslant \rho(C_n, C)$ and $\rho(D'', C'') \leqslant \rho(D_n, C_n)$ (Lemma 3.5.15). Also, we note that $\rho(D_n, C_n) < \rho(D', C) = \kappa(1 - t)$. Therefore,

$$\begin{aligned}\rho(D'', D') = \rho(D'', C) - \rho(D', C) &\leqslant (\rho(D'', C'') + \rho(C'', C)) - \rho(D', C) \\ &\leqslant (\rho(D_n, C_n) + \rho(C_n, C)) - \rho(D', C) < \rho(C_n, C) \leqslant \rho(D_n, D).\end{aligned}$$

Moreover, by Corollary 3.6.10, we have

$$\rho(D_n, D'') \leqslant \max(\rho(C_n, C), \rho(D_n, D)) = \rho(D_n, D).$$

It follows

$$\rho(D'_n, D') = \rho(D_n, D') \leqslant \rho(D_n, D'') + \rho(D'', D') < 2 \cdot \rho(D_n, D).$$

Case (3): $\rho_T(D_n) \geqslant \kappa(1 - t)$ and $\rho_T(D) < \kappa(1 - t)$. Then $D' = D$ and $\rho_T(D) < \rho_T(D'_n) = \kappa(1 - t)$. Exchanging the roles of D_n and D, we may analyze this case in the same way as in Case (2), and conclude that $\rho(D'_n, D') < 2 \cdot \rho(D_n, D)$ in general.

Case (4): $\rho_T(D_n) \geqslant \kappa(1 - t)$ and $\rho_T(D) \geqslant \kappa(1 - t)$. In this case,

$$\rho(D'_n, C_n) = \rho_T(D_n) = \rho(D', C) = \rho_T(D') = \kappa(1 - t).$$

Let $D'' \in$ tconv(C, D) be the D'_n-reduced divisor in tconv(C, D) and $D''_n \in$ tconv(C_n, D_n) be the D'-reduced divisor in tconv(C_n, D_n). We need to consider the relative the positions of D' and D'' in tconv(C, D) and the relative positions of D'_n and D''_n in tconv(C_n, D_n).

Case (4a): $D'' \in$ tconv(C, D') and $D''_n \in$ tconv(C_n, D'_n). Then we can apply Proposition 3.6.9 and see that $\rho(D'_n, D') \leqslant \rho(D_n, D)$.

Case (4b): $D'' \in$ tconv(D', D) and $D''_n \in$ tconv(D'_n, D_n). Again we can apply Proposition 3.6.9 and get $\rho(D'_n, D') \leqslant \rho(C_n, C) \leqslant \rho(D_n, D)$.

Case (4c): $D'' \in$ tconv(D', D) and $D''_n \in$ tconv(C_n, D'_n). We can use an analogous analysis as in Case (2b) and get

$$\rho(D'_n, D') \leqslant \rho(D'_n, D'') + \rho(D'', D') \leqslant 2 \cdot \rho(D_n, D).$$

Note that we get " $\leqslant$" instead of " $<$" as in Case (2b) because we now have $\rho(D'_n, C_n) = \rho(D', C) = \kappa(1-t)$.

Case (4d): $D'' \in \mathrm{tconv}(C, D')$ and $D''_n \in \mathrm{tconv}(D'_n, D_n)$. Base on a similar analysis as in Case (4c), we get

$$\rho(D'_n, D') \leqslant \rho(D'_n, D''_n) + \rho(D''_n, D') \leqslant 2 \cdot \rho(D_n, D).$$

So far we've finished the proof of the continuity of h. To show $h(t, X)$ is tropically convex, we note that $h(t, T)$ is the sublevel set $L^T_{\leqslant r}(\rho_T)$ of the distance function ρ_T where $r = \kappa(1-t)$. Hence we only need to show that choosing arbitrarily D_1 and D_2 from X such that $\rho_T(D_1) \leqslant r$ and $\rho_T(D_2) \leqslant r$, we must have $\rho_T(D) \leqslant r$ for every $D \in \mathrm{tconv}(D_1, D_2)$. Let C_1 and C_2 the reduced divisors in T with respect to D_1, D_2 respectively. Let C be the D-reduced divisor in $\mathrm{tconv}(C_1, C_2)$. Then by Corollary 3.6.10, we must have

$$\rho_T(D) \leqslant \rho(D, C) \leqslant \max(\rho(D_1, C_1), \rho(D_2, C_2)) = \max(\rho_T(D_1), \rho_T(D_2)) \leqslant r.$$

$\square$

CHAPTER IV

SMOOTHING OF LIMIT LINEAR SERIES OF RANK ONE ON REFINED METRIZED COMPLEXES OF ALGEBRAIC CURVES

In this chapter, we study the smoothing problem of limit linear series of rank one on refined metrized complexes.

4.1 Statement of the main result

In the following section, we provide a self-contained explanation of the effective criterion for smoothing and the key steps in the proof.

4.1.1 Refined metrized complexes

Let $\mathbb{K}$ be an algebraically closed field complete with respect to a non-trivial non-archimedean absolute value. Let κ be the residue field of $\mathbb{K}$ and let R be the valuation ring of $\mathbb{K}$. Furthermore, we assume that κ has characteristic 0 the value group of $\mathbb{K}$ is $\mathbb{R}$.

Definition 4.1.1. A **refined metrized complex** $\mathfrak{C}$ over an algebraically closed field κ is the following data:

- A metric graph Γ.
- An algebraic curve C_p associated to each point $p \in \Gamma$ such that C_p is a projective line except for p in a finite subset of Γ.
- For every point $p \in \Gamma$, a bijection $\mathrm{red}_p : \mathrm{Tan}_\Gamma(p) \to A_p$ called the reduction map where $\mathrm{Tan}_\Gamma(p)$ is the set of outgoing tangent directions at p and

$A_p = \{x_t^p\}_{t \in \mathrm{Tan}_\Gamma(p)}$ is a finite subset of C_p where x_t^p is called the marked point associated to the tangent direction t.

Definition 4.1.2. The **genus** $g(\mathfrak{C})$ of a refined metrized complex $\mathfrak{C}$ is defined as $g(\Gamma) + \sum_{p \in \Gamma} g(C_p)$ where $g(\Gamma)$ is the genus of the metric graph Γ and $g(C_p)$ is the genus of the algebraic curve C_p. The genus of a refined metrized complex is finite since C_p has genus zero for all but a finite number of points p in Γ.

The notion of refined metrized complex comes from the notion of metrized complex introduced by Amini and Baker [4] where the difference is that they only associate algebraic curves to a vertex set of Γ.

4.1.2 Divisor theory on a refined metrized complex

The basic notions for the divisor theory on refined metrized complexes also come straightforwardly as correspondences to notions on metrized complexes developed by Baker and Amini [4].

Definition 4.1.3. Let $\mathfrak{C}$ be a refined metrized complex with underlying metric graph Γ and algebraic curve C_p at point p. Let D_Γ be a divisor on Γ and let D_p be a divisor on the curve C_p. A **pseudo-divisor** $\mathcal{D}$ on $\mathfrak{C}$ is the data $(D_\Gamma, \{D_p\}_{p \in \Gamma})$ satisfying the relation $D_\Gamma(p) = \deg(D_p)$ for every point $p \in \Gamma$. A **divisor** $\mathcal{D}$ on a refined metrized complex is a pseudo-divisor where $D_p = 0$ for all but finitely many points.

Note that for any pseudo-divisor $\mathcal{D}$ on a refined metrized complex, D_p will be a principal divisor for all but finitely many points $p \in \Gamma$. But, D_p can be non-zero for infinitely many points in Γ. This is unconventional from the viewpoint of divisor theory and the notion of divisor on a refined metrized complex rectifies this aspect.

Remark 4.1.4. Note that the space of pseudo-divisors on $\mathfrak{C}$ is a subgroup of $\mathrm{Div}(\Gamma) \bigoplus \prod_{p \in \Gamma} \mathrm{Div}(C_p)$. The space of divisors on $\mathfrak{C}$ is a subgroup of $\mathrm{Div}(\Gamma) \bigoplus (\bigoplus_{p \in \Gamma} \mathrm{Div}(C_p))$ that is itself isomorphic to $\bigoplus_{p \in \Gamma} \mathrm{Div}(C_p)$, which is the free abelian group on $\coprod_{p\in\Gamma} C_p$. Moreover, we say a pseudo-divisor $\mathcal{D} = (D_\Gamma, \{D_p\}_{p\in\Gamma})$ is effective if the divisor D_Γ is effective and D_p is effective for all points $p \in \Gamma$.

Definition 4.1.5. A **pseudo-rational function** $\mathfrak{f}$ on $\mathfrak{C}$ is the data $(f_\Gamma, \{f_p\}_{p\in\Gamma})$ where f_Γ is a rational function on Γ and f_p is a rational function on the algebraic curve C_p. Here we let the associated divisor of f_Γ be $\operatorname{div}(f_\Gamma) = \sum_{p\in\Gamma} \operatorname{ord}_p(f_\Gamma)(p)$ where $\operatorname{ord}_p(f_\Gamma)$ is the sum of slopes $\operatorname{sl}_t(f_\Gamma)$ of f_Γ along all outgoing tangent directions t at p. The **principal pseudo-divisor** $\operatorname{div}(\mathfrak{f})$ associated with the rational function is defined as $(\operatorname{div}(f_\Gamma), \{\operatorname{div}(f_p) + \operatorname{div}_p(f_\Gamma)\}_{p\in\Gamma})$ where $\operatorname{div}_p(f_\Gamma) = \sum_{t\in\operatorname{Tan}_\Gamma(p)} \operatorname{sl}_t(f_\Gamma)x_t^p$. A **rational function** is a pseudo rational function whose associated principal pseudo-divisor is a divisor. We also call a principal pseudo-divisor associated to a rational function a **principal divisor**. We say divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ are linearly equivalent if they differ by a principal divisor.

Remark 4.1.6. Note that just by definition, the associated divisor $\operatorname{div}(f_\Gamma)$ defined here is actually negative to the associated divisor Δf_Γ defined in the previous two chapters, since in this chapter we are following the conventional notions in paper [4].

As in the case of principal divisors on an algebraic curve or principal divisors on a metric graph, the principal divisors on a refined metrized complex form an Abelian group under addition.

4.1.3 Pre-limit linear series and limit linear series on a refined metrized Complex

We introduce notions of pre-limit linear series and limit linear series on a refined metrized complex.

Definition 4.1.7. A **pre-limit linear series** of rank r and degree d on a refined metrized complex $\mathfrak{C}$, also known as a pre-limit g_d^r, is the data $(\mathcal{D}, \mathcal{H})$ where $\mathcal{D}$ is an effective divisor of degree d on $\mathfrak{C}$ and $\mathcal{H} = \{H_p\}_{p\in\Gamma}$ where H_p is an $(r+1)$-dimensional subspace of the function field of C_p.

Definition 4.1.8. A **limit linear series** of rank r and degree d, also known as a limit g_d^r, on a refined metrized complex is a pre-limit g_d^r such that for every effective divisor $\mathcal{E}$ on $\mathfrak{C}$ of degree r, there exists a rational function $\mathfrak{f}$ such that $\mathcal{D} - \mathcal{E} + \mathrm{div}(\mathfrak{f}) \geq 0$.

Definition 4.1.9. A **refined limit** g_d^r (respectively, a **refined pre-limit** g_d^r) is a limit g_d^r (respectively, a pre-limit g_d^r) with the following additional properties:

1. The constant function is contained in H_p for every point $p \in \Gamma$.

2. For every point p, the support of D_p is disjoint from the set A_p of marked points of C_p.

Remark 4.1.10. More precisely, in all the above definitions of limit/pre-limit linear series, $(\mathcal{D}, \mathcal{H})$ is only a representative of the limit/pre-limit linear series which is actually an equivalence class defined in the following way: we say $(\mathcal{D}, \mathcal{H}) \sim (\mathcal{D}', \mathcal{H}')$ if (1) there exists a rational function $\mathfrak{f}$ such that $\mathcal{D}' = \mathcal{D} + \mathrm{div}(\mathfrak{f})$, (2) $f_p \in H_p$ for all $p \in \Gamma$, and (3) $H'_p = \{\frac{f}{f_p} | f \in H_p\}$. However, throughout this chapter, we directly call $(\mathcal{D}, \mathcal{H})$ limit/pre-limit linear series.

4.1.4 Statement of the smoothing theorem

The following notion of harmonic morphisms between refined metrized complexes is also a correspondence of harmonic morphisms on metrized complexes [5, 6]. (Here the notion harmonic morphism is actually an enrichment of the notion of admissible covers of Harris and Mumford [42]).

Definition 4.1.11. A **harmonic morphism** between refined metrized complexes $\mathfrak{C}'$ and $\mathfrak{C}$ is the data $\{\phi_{\Gamma'}, \{\phi_{p'}\}_{p' \in \Gamma'}\}$ where $\phi_{\Gamma'} : \Gamma' \to \Gamma$ is a harmonic morphism between Γ' and Γ (cf. [14] for harmonic morphisms between graphs) and $\phi_{p'} : C_{p'} \to C_{\phi_{\Gamma'}(p)}$ is a finite morphism of curves satisfying the following compatibility conditions:

1. Tangent directions $t'_1, t'_2 \in \mathrm{Tan}_{\Gamma'}(p')$ are mapped to the same tangent direction $t \in \mathrm{Tan}_{\Gamma}(\phi_{\Gamma'}(p'))$ by $\phi_{\Gamma'}$ if and only if the marked points corresponding to t'_1

and t'_2 are mapped to the marked point corresponding to the tangent direction t.

2. The expansion factor of $\phi_{\Gamma'}$ along a tangent direction $t' \in \mathrm{Tan}(v')$ coincides with the ramification index of $\phi_{v'}$ at the marked point corresponding to the tangent direction t'.

A characterization of smoothable limit g^1_d following the recent work in [5,6] is the starting point of our work.

Theorem 4.1.12. *A limit g^1_d $(\mathcal{D}, \{H_p\}_{p\in\Gamma})$ on $\mathfrak{C}$ is smoothable if and only if there exists a modification $\mathfrak{C}^{\mathrm{mod}}$ of $\mathfrak{C}$ and a harmonic morphism $(\phi_{\Gamma^{\mathrm{mod}}}, \{\phi_p\}_{p\in\Gamma^{\mathrm{mod}}})$ from $\mathfrak{C}^{\mathrm{mod}}$ to a genus zero refined metrized complex such that $\phi_p = f_p$ where f_p is a non-constant function in H_p.*

The characterization of a smoothable limit g^1_d in Theorem 4.1.12 is not effective since given a limit g^1_d the construction of the modification and the genus zero metrized complex is not evident. Furthermore, it is not evident from Theorem 4.1.12 if every limit g^1_d is smoothable. It turns out that not every limit g^1_d is smoothable. Our main result is an effective characterization of smoothable limit g^1_d, in particular we identify all the obstructions to smoothing a limit g^1_d. For the rest of this section, we focus on formulating a precise statement of the effective characterization of a smoothable limit g^1_d. In order to give the reader an impression of the statement, we start with the statement despite not having defined the obstructions to smoothing. We then explain the obstructions to smoothing.

Theorem 4.1.13. (**Smoothing Theorem**) *A pre-limit g^1_d is smoothable if and only it is a diagrammatic pre-limit g^1_d that is solvable and satisfies the intrinsic global compatibility conditions.*

In the following, we explain the terms "diagrammatic" pre-limit g^1_d, the notion of "solvability" and "intrinsic global compatibility" conditions. In particular, we explain that these conditions are effective.

4.1.5 Diagrammatic pre-limit g^1_d and solvability

Let $(\mathcal{D}, \mathcal{H})$ with $\mathcal{D} = (D_\Gamma, \{D_p\}_{p\in\Gamma})$ and $\mathcal{H} = \{H_p\}_{p\in\Gamma}$ be a refined limit g^r_d and let the subspace H_p have a basis of the form $\{1, f_p\}$ where f_p is a non-constant rational function on the algebraic curve C_p.

Definition 4.1.14. $R_{\mathcal{D},\mathcal{H}}$ is a subset of $|D_\Gamma|$ defined as follows: $D'_\Gamma \in R_{\mathcal{D},\mathcal{H}}$ if there exists an effective divisor $\mathcal{D}' = (D'_\Gamma, \{D'_p\}_{p\in\Gamma})$ and a rational function $\mathfrak{f} = (f_\Gamma, \{f_p\}_{p\in\Gamma})$ such that (1) $f_p \in H_p$ for all $p \in \Gamma$ and (2) $\mathcal{D}' - \mathcal{D} = \mathrm{div}(\mathfrak{f})$.

Suppose that $(\mathcal{D}, \mathcal{H})$ is smoothable and consider the metric tree T underlying a genus zero metrized complex satisfying Theorem 4.1.12. Then we can actually identify T to a subset of $R_{\mathcal{D},\mathcal{H}}$.

From the properties of the harmonic morphisms of tropical curves [14], we know that T is tropically convex and each point $p \in \Gamma$ is a supporting point of a divisor in T (actually we will call T satisfying these conditions tropical dominant trees in this chapter).

A key step in our smoothing theorem is a characterization of tropical dominant subtrees of $R_{\mathcal{D},\mathcal{H}}$. However not every such tropically convex dominant subtree of $R_{\mathcal{D},\mathcal{H}}$ can appear as the metric tree underlying the genus zero metrized complex in Theorem 4.1.12. We will characterize tropical dominant subtrees that can appear as the metric tree underlying the genus zero metrized complex.

From the point of view of investigating trees in $R_{\mathcal{D},\mathcal{H}}$, it is convenient to reorganize the information in the subspace H_p via local diagrams and global diagrams on Γ defined as follows:

Definition 4.1.15. A **local diagram** at a point p in a metric graph Γ is following data:

1. A nonzero integer $m(p,t)$ called the multiplicity associated to each tangent direction $t \in \mathrm{Tan}_\Gamma(p)$, where $\mathrm{Tan}_\Gamma(p)$ is the set of tangent directions emanating from p. We refer to those tangent directions with negative multiplicity as incoming tangent directions and denote it by $\mathrm{In}(p)$. Similarly, we refer to those tangent directions with positive multiplicity as outgoing tangent directions and denote it by $\mathrm{Out}(p)$.

2. The elements in $\mathrm{Tan}_\Gamma(p)$ are partitioned into equivalence classes and these equivalence classes satisfy the property that for every point p, the set $\mathrm{In}(p)$ is an equivalence class. We refer to this partition of $\mathrm{Tan}_\Gamma(p)$ as the local partition at p and refer to the tangent directions that belong to the same equivalence class as locally equivalent.

□

An open neighborhood of a point $p \in \Gamma$ is called a simple neighborhood if it is simply connected and every point in the neighborhood except possibly p has valence two.

Remark 4.1.16. For a simple neighborhood U of a point $p \in \Gamma$, a local diagram at a point p induces a local diagram on any point in U as follows: suppose the point $q \in U$ lies in the tangent direction $t \in \mathrm{Tan}_\Gamma(p)$. Note that q has valence two. Assign the integer $-m(p,t)$ to the tangent direction at q corresponding to the edge joining p and q and assign the integer $m(p,t)$ to the other tangent direction. Assign the two tangent directions at q to different equivalence classes.

Definition 4.1.17. A **global diagram** on a metric graph Γ is a collection of local diagrams at all the points in Γ such that the local diagrams satisfy the following continuity property: For every point p, there is a simple neighborhood U of p such that for every point $q \in U$ the local diagram induced by p at q coincides with the local diagram at q.

Remark 4.1.18. In the following, given a refined pre-limit g^1_d $(\mathcal{D}, \mathcal{H})$, we associate a local diagram to every point on the metric graph. From the definition of a refined pre-limit g^1_d, we know that the two dimensional subspace H_p of rational functions of C_p associated to a point p has a basis $\{1, f_p\}$ where f_p is a nonconstant rational function on C_p. We construct the local diagram at p as follows: for a tangent direction t of C_p, we assign the ramification index of f_p at the marked point corresponding to the tangent direction t as the multiplicity $m(p, t)$ with sign '$-$' if $\mathrm{red}_p(t)$ is a pole of f_p and sign '$+$' otherwise. The local partition at p is defined by declaring that two tangent directions are locally equivalent if and only if their marked points are in the same level set of f_p. Hence, $\mathrm{In}(p)$ is the set of tangent directions whose corresponding marked points are poles of f_p and the elements of $\mathrm{In}(p)$ are all locally equivalent. Note that this construction is independent of the choice of basis for H_p. In addition, we say a local diagram is **compatible** with $(\mathcal{D}, \mathcal{H})$ if $D_p - \Sigma_{t\in \mathrm{In}(p)} m(p, t)(x^p_t) \geqslant D^-_{f_p}$ where $D^-_{f_p}$ is $\mathrm{div}(f_p)$ restricted to the poles. For example, Figure 13 shows the local diagram generated by f_v of degree 3 which has the red points on v_v as poles, the black points as zeros and maps the blue points to value c. Two out of the three poles are marked points of C_v which correspond to the red incoming tangent directions at $v \in \Gamma$. Both zeros are marked points corresponding to the two black outgoing tangent directions, while one has multiplicity 1 and the other has multiplicity 2. Only one out of the three points mapped to c is a marked point, which correspond to the unique blue outgoing tangent direction.

Definition 4.1.19. We say a refined limit g^1_d $(\mathcal{D}, \mathcal{H})$ (respectively pre-limit g^1_d)on $\mathfrak{C} = (\Gamma, \{C_p\}_{p\in\Gamma})$ is a **diagrammatic limit** g^1_d (respectively **diagrammatic pre-limit** g^1_d) if the collection of associated local diagrams at all points of Γ is compatible with $(\mathcal{D}, \mathcal{H})$ and forms a global diagram.

As we shall see in Theorem 4.6.3, a smoothable limit g^1_d is a diagrammatic limit g^1_d.

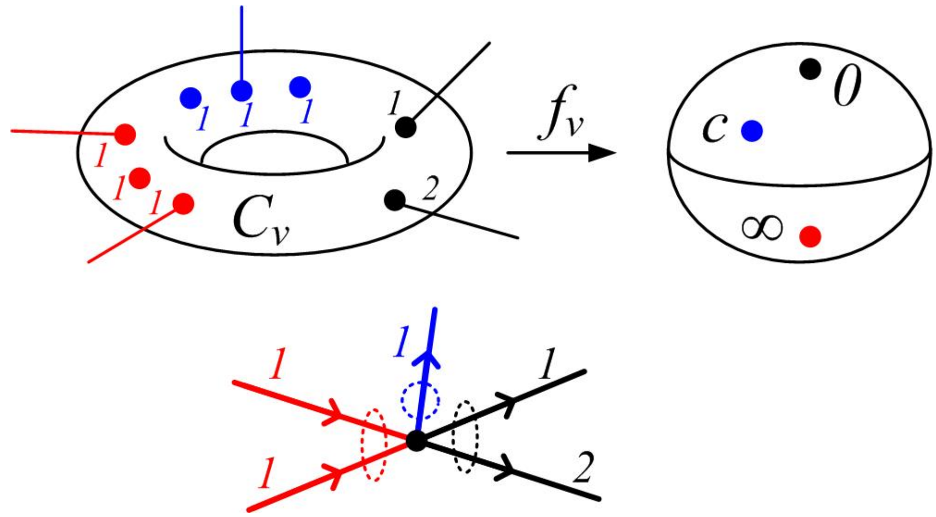

Figure 13: An illustration of a local diagram generated by H_v expanded by a basis $\{1, f_v\}$.

Definition 4.1.20. (**Characteristic Equation of a Global Diagram**) A global diagram on Γ is called **solvable** if there exists a tropical rational function ρ on Γ such that the outgoing slope of ρ at the tangent direction $t \in \mathrm{Tan}(p)$ for any point $p \in \Gamma$ coincides with the multiplicity. Formally, this means that the following differential equation is satisfied:

$$\mathrm{sl}_t(\rho) = m(p, t). \tag{1}$$

We refer to this equation as the **characteristic equation** of the global diagram. If the global diagram obtained from a diagrammatic limit/pre-limit g^1_d is solvable, then we may directly say that the diagrammatic limit/pre-limit g^1_d is solvable.

4.1.6 Intrinsic global compatibility conditions

In general, the characteristic equation of a global diagram associated to diagrammatic pre-limit g^1_d does not have a solution (see Example 4.7.7 and Section 4.7.2 for a detailed discussion). However, the global diagram associated to a smoothable limit g^1_d has a solution (cf. our main result in Theorem 4.1.13). When a global diagram is solvable, the solutions of the characteristic equation differ by a constant and have everywhere nonzero slopes. We refer to a solution to the characteristic equation as a **timing**

function if its minimum value is zero.

Solvability is only a necessary condition for a limit g^1_d being smoothable. In order to completely characterize a smoothable limit g^1_d we impose compatibility conditions between the bifurcation tree and the local partitions in the global diagram associated with the diagrammatic limit g^1_d. We refer to these compatibility conditions as intrinsic global compatibility conditions, which is stated in Definition 4.1.23.

For a point p in Γ, recall that $\mathrm{Tan}_\Gamma(p)$ is the set of tangent directions emanating from p. Let $\mathrm{Tan}_\Gamma^{\rho+}(p)$ denote the set of tangent directions in $\mathrm{Tan}_\Gamma(p)$ where ρ locally increases. Given a rational function ρ with everywhere nonzero slopes, we may canonically associate to ρ a pair $(\mathcal{B}, \pi)$ where $\mathcal{B}$ is rooted metric tree (having a specific point as the 'root') and $\pi_\mathcal{B} : \Gamma \to \mathcal{B}$ is a canonical projection from Γ onto $\mathcal{B}$ (see details in Section **??**). Moreover, $\pi_\mathcal{B}$ induces a pushforward map $\pi_{\mathcal{B}*}$ from the tangent directions on Γ to tangent directions on $\mathcal{B}$, and if we let $\mathrm{Tan}_\mathcal{B}^+(x)$ be the set of forward tangent directions (meaning the distance function from the root increases along these directions) at $x \in \mathcal{B}$, then for any $p \in \Gamma$ and $t \in \mathrm{Tan}_\Gamma^{\rho+}(p)$, we have $\pi_{\mathcal{B}*}(t) \in \mathrm{Tan}_\mathcal{B}^+(\pi(p))$.

Definition 4.1.21. A **bifurcation partition system** $\{\vec{P}_x\}_{x\in\mathcal{B}}$ on $\mathcal{B}$ is a collection of partitions $\vec{P}_x$ of tangent directions in $\mathrm{Tan}_\mathcal{B}^+(x)$ for all points $x \in \mathcal{B}$.

It is easy to see that there are only finitely many possible bifurcation partition systems on $\mathcal{B}$ since $\mathrm{Tan}_\mathcal{B}^+(x)$ is a singleton or for all but finitely many points $x \in \mathcal{B}$ whereas the exceptional partitions are also made over finite sets.

Definition 4.1.22. A point $p \in \Gamma$ is called an **ordinary point** of ρ if its valence is two and the slopes of ρ at p in the two tangent directions are opposite numbers. Denote the set of ordinary points by $\mathcal{O}_\rho$. The points in $\mathcal{E}_\rho := \Gamma \setminus \mathcal{O}_\rho$ and the values in the image of ρ restricted to $\mathcal{E}_\rho$ are called **exceptional points** and **exceptional values** of ρ, respectively.

Again, it is obvious that $\mathcal{E}_\rho$ is finite.

With the introduction of the above notions, we formulate the intrinsic global compatibility conditions as follows.

Definition 4.1.23. (Intrinsic global compatibility) A solvable diagrammatic pre-limit g_d^1 $(\mathcal{D}, \{H_p\}_{p\in\Gamma})$ with the normalized solution ρ and the corresponding projection π onto a bifurcation tree $\mathcal{B}$ is said to satisfy **intrinsic global compatibility** conditions if there exists a bifurcation partition system $\{\vec{P}_x\}_{x\in\mathcal{B}}$ on $\mathcal{B}$ and a collection $\{g_p\}_{p\in\mathcal{E}_\rho}$ of non-constant functions $g_p \in H_p$ over all exceptional points $p \in \mathcal{E}_\rho$, such that whenever $t_1 \in \mathrm{Tan}_\Gamma^{\rho+}(p_1)$ and $t_2 \in \mathrm{Tan}_\Gamma^{\rho+}(p_2)$ where $p_1, p_2 \in \mathcal{E}_\rho$ and $\pi_{\mathcal{B}}(p_1) = \pi_{\mathcal{B}}(p_2)$, we have $g_{p_1}(\mathrm{red}(t_1)) = g_{p_2}(\mathrm{red}(t_2))$ if and only if $\pi_{\mathcal{B}*}(t_1)$ and $\pi_{\mathcal{B}*}(t_2)$ are equivalent in $\{\vec{P}_x\}_{x\in\mathcal{B}}$.

Note that if we fix a basis $\{1, f_p\}$ of H_p, then any rational function $g_p \in H_p$ can be expanded linearly as $g_p = \alpha_{p,1} + \alpha_{p,2} f_p$. We emphasize here that whether a solvable diagrammatic pre-limit g_d^1 satisfies intrinsic global compatibility conditions is finitely verifiable, since (1) the number of possible bifurcation partition systems is finite, (2) $\mathcal{E}_\rho$ is finite, and (3) the intrinsic global compatibility actually provides finitely many linear equalities and inequalities on $\alpha_{p,1}$ and $\alpha_{p,2}$ for all $p \in \mathcal{E}_\rho$.

4.1.7 Key steps in the proof of Theorem 4.1.13

($\Rightarrow$) Given a smoothable limit g_d^1, by Theorem 4.1.12 it induces a harmonic morphism of refined metrized complexes. By Theorem 4.6.3, we know that this limit g_d^1 is a solvable diagrammatic limit g_d^1. Consider the genus zero metrized complex of the refined harmonic morphism. Using the compatibility properties of the harmonic morphisms we deduce that the diagrammatic limit g_d^1 is solvable and satisfies the intrinsic global compatibility condition.

($\Leftarrow$) We construct a refined harmonic morphism satisfying the conditions of Theorem 4.1.12. Intrinsic global compatibility conditions guarantee the existence of genus zero refined metrized complex in $R_{\mathcal{H},\mathcal{D}}$. We modify the metrized complex by adding marked points using H_p with additional care on the expansion factors. We construct $\{g_p\}_{p\in\Gamma}$ using the intrinsic global compatibility conditions and verify that this choice satisfies the compatibility properties of harmonic morphisms of refined metrized complexes. By Theorem 4.1.12, this implies that the limit g^1_d is smoothable.

4.1.8 Obstructions of pre-limit g^1_d's from being smoothable

By Theorem 4.1.13, solvability and intrinsic global compatibility are two levels of obstructions of a diagrammatic pre-limit g^1_d from being smoothable. While solvability is weaker and intrinsic global compatibility is stronger, we add two other intermediate levels of obstruction with natural characterizations.

For a diagrammatic pre-limit g^1_d which is solvable with a solution ρ and the corresponding bifurcation tree $\mathcal{B}$, we say a bifurcation partition system $\{\vec{P}_x\}_{x\in\mathcal{B}}$ is **admissible** (respectively, **strongly admissible)** if it satisfies the following property: for each exceptional point $p \in \mathcal{E}_\rho$, the tangent directions t_1 and t_2 in $\mathrm{Tan}^+_\Gamma(p)$ are equivalent in the local diagram at p (we may also say t_1 and t_2 are locally equivalent), if (respectively, if and only if) $\pi_{\mathcal{B}*}(t_1)$ and $\pi_{\mathcal{B}*}(t_2)$ are equivalent in $P_{\pi(p)}$.

Definition 4.1.24. A solvable diagrammatic pre-limit g^1_d is said to satisfy **local-bifurcation compatibility** (respectively, **strong local-bifurcation compatibility) conditions** if there exists a admissible (respectively, strongly admissible) bifurcation partition system compatible (respectively, strongly compatible).

Alternatively, we also call solvability (respectively, local-bifurcation compatibility, strong local-bifurcation compatibility and intrinsic global compatibility) Level I (respectively, II, III and IV) obstruction, and we have "Level IV $\Rightarrow$ Level III $\Rightarrow$ Level II $\Rightarrow$ Level I". For the above relations, we only need to verify Level IV impies Level III: by considering only the cases $p_1 = p_2$ in the definition of intrinsic global compatibility (Definition 4.1.23), it is easy to see that $\{P_x\}_{x\in\mathcal{B}}$ is strongly compatible to the local diagrams.).

For local-bifurcation compatibility (Level II obstruction), we have the following results.

Theorem 4.1.25. *Let $(\mathcal{D},\mathcal{H})$ be a solvable diagrammatic pre-limit g^1_d. Then $R_{\mathcal{D},\mathcal{H}}$ has a tropical dominant subtree which contains the tropical part D_Γ of $\mathcal{D}$ if and only if $(\mathcal{D},\mathcal{H})$ satisfies local-bifurcation compatibility conditions.*

Proposition 4.1.26. *A solvable diagrammatic pre-limit g^1_d satisfying local-bifurcation compatibility conditions is a limit g^1_d.*

The obstructions up to the strong local-bifurcation compatibility (Level III obstruction) are combinatoric, which means unlike H_p's are involved in determining whether Level IV is satisfied, we only need to refer to the local diagrams or the global diagram for levels $\leqslant 3$. Moreover, for certain metric graphs underlying a refined metrized complex, we only need to verify up to Level III to test smoothability.

4.2 Preliminaries

4.2.1 Bifurcation trees

We can associate a metric tree canonically to the timing function called called **bifurcation tree**. The bifurcation tree is "fundamental" to $R_{\mathcal{H},\mathcal{D}}$ in the sense that any tropical dominant subtree of $R_{\mathcal{H},\mathcal{D}}$ can be obtained by gluing branches of the bifurcation tree. Let ρ be a rational function on Γ with everywhere nonzero slopes, and let $\hat{\rho} := \rho - \min\rho$ be the normalized function of ρ with minimum value zero. For a real number c and $* \in \{\geq,\leq,<,>,=\}$, the set S^ρ_{*c} is defined as $\{p\in\Gamma|\ \rho(p) * c\}$. Denote the set of connected components of S^ρ_{*c} by $\mathrm{Comp}(S^\rho_{*c})$.

Definition 4.2.1. For each value $c \in \operatorname{Im}\rho$, the connected components of $S^\rho_{\geqslant c}$ are called **closed superlevel components** at c, and the connected components of $S^\rho_{>c}$ are called **open superlevel components** at c.

Remark 4.2.2. For $c \in \operatorname{Im}\rho$, for any open superlevel component $\beta \in S^{\rho}_{>c}$, there exists a unique closed superlevel component $\alpha \in S^{\rho}_{\geqslant c}$ such that $\alpha \supseteq \beta$. For each $c' \in \operatorname{Im}\rho$ such that $c' \leqslant c$, there exists a unique element $\beta' \in S^{\rho}_{>c'}$ such that $\beta' \supseteq \alpha$. There exists δ small enough such that for all $c'' \in (c, c+\delta)$, there exists a unique element $\alpha'' \in S^{\rho}_{\geqslant c''}$ such that $\beta \supseteq \alpha''$. Moreover, there exists exactly one element in $S^{\rho}_{\geqslant \min\rho}$ which is the whole metric graph Γ. These facts also imply that for $\alpha_1 \in S^{\rho}_{\geqslant c_1}$ and $\alpha_2 \in S^{\rho}_{\geqslant c_2}$, there exists a largest $c_3 \in \operatorname{Im}\rho$ such that there exists $\alpha_3 \in S^{\rho}_{\geqslant c_3}$ with $\alpha_3 \supseteq \alpha_1 \bigcup \alpha_2$. In particular, $c_3 \leqslant \min(c_1, c_2)$ and α_3 is the unique smallest closed superlevel component containing $\alpha_1 \bigcup \alpha_2$.

We now define the bifurcation tree associated to ρ as follows.

Definition 4.2.3. Consider a rational function ρ with everywhere nonzero slopes. The **bifurcation tree** $\mathcal{B}$ with respect to ρ is a rooted metric tree constructed in the following way:

1. By abuse of notation, we also use $\mathcal{B}$ to represent the set of points of $\mathcal{B}$. We identify the set of points of $\mathcal{B}$ with the set of all closed superlevel components of ρ by the bijection $\iota_{\mathcal{B}} : \mathcal{B} \to \coprod_{c \in \operatorname{Im}\rho} \operatorname{Comp}(S^{\rho}_{\geqslant c})$.

2. We assign a metric structure $d_{\mathcal{B}}$ to $\mathcal{B}$. For $x_1, x_2 \in \mathcal{B}$, denote $x_1 \vee x_2$ be the element in $\mathcal{B}$ such that $\iota_{\mathcal{B}}(x_1 \vee x_2)$ is the smallest closed superlevel component which contains $\iota_{\mathcal{B}}(x_1) \bigcup \iota_{\mathcal{B}}(x_2)$. Suppose $\iota_{\mathcal{B}}(x_1) \in \operatorname{Comp}(S^{\rho}_{\geqslant c_1})$, $\iota_{\mathcal{B}}(x_2) \in \operatorname{Comp}(S^{\rho}_{\geqslant c_2})$ and $\iota_{\mathcal{B}}(x_1 \vee x_2) \in \operatorname{Comp}(S^{\rho}_{\geqslant c_3})$. Then we let $d_{\mathcal{B}}(x_1, x_2) = c_1 + c_2 - 2c_3$.

3. The root $r(\mathcal{B})$ of $\mathcal{B}$ corresponds to the unique closed superlevel component at $\min(\rho)$.

If $\iota_{\mathcal{B}}(x) \in \operatorname{Comp}(S^{\rho}_{\geqslant c})$ where $c \in \operatorname{Im}\rho$, we let $d^{\rho}_{\mathcal{B}}(x) = c$ and $d^{0}_{\mathcal{B}}(x) = c - \min\rho$. We now show $\mathcal{B}$ is well-defined. In particular, a partial order is associated to $\mathcal{B}$.

Proposition 4.2.4. *$\mathcal{B}$ constructed in Definition 4.2.3 is a metric tree.*

Proof. We first note that a partial order can be associated to $\mathcal{B}$, i.e., for two points x_1 and x_2, we say $x \geqslant x'$ if $\iota_{\mathcal{B}}(x) \supseteq \iota_{\mathcal{B}}(x')$. By Remark 4.2.2, this partial order is well-defined and is actually a join-semilattice since any two elements x and x' in $\mathcal{B}$ always have a join $x \vee x'$ corresponding to the smallest closed superlevel component which contains $\iota_{\mathcal{B}}(x) \bigcup \iota_{\mathcal{B}}(x')$, and in addition, we have for each $x \in \mathcal{B}$, the set

$$\{x' \in \mathcal{B} | x' \geqslant x\} = \{x \vee x' | x' \in \mathcal{B}\}$$

is totally ordered.

Now let us show $d_{\mathcal{B}}$ is a metric on $\mathcal{B}$. For $x_1, x_2 \in \mathcal{B}$, suppose $d^{\rho}_{\mathcal{B}}(x_1) = c_1$ and $d^{\rho}_{\mathcal{B}}(x_2) = c_2$. If $x_1 \geqslant x_2$, then $c_1 \leqslant c_2$, $x_1 \vee x_2 = x_1$ and $d_{\mathcal{B}}(x_1, x_2) = c_2 - c_1$ by definition. In particular, since $r(\mathcal{B}) \geqslant x$ for all $x \in \mathcal{B}$, it follows $d^{0}_{\mathcal{B}}(x) = d_{\mathcal{B}}(r(\mathcal{B}), x)$. More generally, one may verify that $d_{\mathcal{B}}(x_1, x_2) = d_{\mathcal{B}}(x_1, x_1 \vee x_2) + d_{\mathcal{B}}(x_2, x_1 \vee x_2)$ from definition. For each $x_3 \in \mathcal{B}$, consider $x_1 \vee x_3$ and $x_2 \vee x_3$. Without loss of generality, we may assume: (1) $x_1 \vee x_3 > x_2 \vee x_3$ or (2) $x_1 \vee x_3 = x_2 \vee x_3$. For case (1), we have $x_1 \vee x_2 \vee x_3 = x_1 \vee x_2 = x_1 \vee x_3$, and thus

$$\begin{aligned}
d_{\mathcal{B}}(x_1, x_2) =& d_{\mathcal{B}}(x_1, x_1 \vee x_2 \vee x_3) + d_{\mathcal{B}}(x_1 \vee x_2 \vee x_3, x_2 \vee x_3) + d_{\mathcal{B}}(x_2 \vee x_3, x_2) \\
\leqslant& (d_{\mathcal{B}}(x_1, x_1 \vee x_2 \vee x_3) + d_{\mathcal{B}}(x_1 \vee x_2 \vee x_3, x_2 \vee x_3) + d_{\mathcal{B}}(x_2 \vee x_3, x_3)) \\
&+ (d_{\mathcal{B}}(x_2 \vee x_3, x_2) + d_{\mathcal{B}}(x_2 \vee x_3, x_3)) \\
=& d_{\mathcal{B}}(x_1, x_3) + d_{\mathcal{B}}(x_2, x_3)
\end{aligned}$$

where equality holds if and only if $x_3 \geqslant x_2$. For case (2), we have $x_1 \vee x_2 \vee x_3 = x_1 \vee x_3 = x_2 \vee x_3 \geqslant x_1 \vee x_2$, and thus

$$\begin{aligned}
d_{\mathcal{B}}(x_1, x_2) =& d_{\mathcal{B}}(x_1, x_1 \vee x_2) + d_{\mathcal{B}}(x_2, x_1 \vee x_2) \\
\leqslant& (d_{\mathcal{B}}(x_1, x_1 \vee x_2) + d_{\mathcal{B}}(x_1 \vee x_2, x_1 \vee x_2 \vee x_3) + d_{\mathcal{B}}(x_3, x_1 \vee x_2 \vee x_3)) \\
&+ (d_{\mathcal{B}}(x_2, x_1 \vee x_2) + d_{\mathcal{B}}(x_1 \vee x_2, x_1 \vee x_2 \vee x_3) + d_{\mathcal{B}}(x_3, x_1 \vee x_2 \vee x_3)) \\
=& d_{\mathcal{B}}(x_1, x_3) + d_{\mathcal{B}}(x_2, x_3)
\end{aligned}$$

where equality holds if and only if $x_3 = x_1 \vee x_2$. Therefore, the triangle equality is satisfied and $d_{\mathcal{B}}$ is a metric.

Then the construction of rooted metric tree can follow exactly from Appendix B5 of Baker-Rumley's book [18]. $\square$

Remark 4.2.5. The leafs of $\mathcal{B}$ other than $r(\mathcal{B})$ are in one-to-one correspondence with those closed superlevel sets which are singletons, and one-to-one correspondence with local maximum points of ρ (which we may also call sink points of ρ). Denote the set of leaf of $\mathcal{B}$ by $\mathrm{Leaf}(\mathcal{B})$. We call a point x of $\mathcal{B}$ with $|\operatorname{Tan}_{\mathcal{B}}^{+}(x)| \geqslant 2$ a **bifurcation point** of $\mathcal{B}$, and denote the set of bifurcation points by $\mathrm{Bif}(\mathcal{B})$. Then $\mathrm{Leaf}(\mathcal{B}) \bigcap \mathrm{Bif}(\mathcal{B}) = \emptyset$ and $\mathrm{Leaf}(\mathcal{B}) \bigcup \mathrm{Bif}(\mathcal{B})$ is the minimal vertex set of $\mathcal{B}$ (another translation of the minimal vertex set of $\mathcal{B}$ is the set of points of valence other than 2 in $\mathcal{B}$ together with $r(\mathcal{B})$). Note that we have either $r(\mathcal{B}) \in \mathrm{Leaf}(\mathcal{B})$ or $r(\mathcal{B}) \in \mathrm{Bif}(\mathcal{B})$. We call the image of $d_{\mathcal{B}}^{\rho}$ restricted to the minimal vertex set of $\mathcal{B}$ the set of **bifurcation values**, denoted by Bif_{ρ}. Then Bif_{ρ} is finite and we have $\mathrm{Bif}_{\rho} \subseteq \mathcal{E}_{\rho}$.

For a point p in Γ, recall that $\operatorname{Tan}_{\Gamma}^{\rho+}(p)$ is the set of tangent directions in $\operatorname{Tan}_{\Gamma}(p)$ emanating from p where ρ locally increases. Similarly, we let $\operatorname{Tan}_{\Gamma}^{\rho-}(p)$ be the set of tangent directions where ρ locally decreases. Then $\operatorname{Tan}_{\Gamma}(p) = \operatorname{Tan}_{\Gamma}^{\rho+}(p) \coprod \operatorname{Tan}_{\Gamma}^{\rho-}(p)$.

Let $\mathcal{T}$ be a metric tree rooted at $r(\mathcal{T})$. For a point x in $\mathcal{T}$, we say a tangent direction $t \in \operatorname{Tan}_{\mathcal{T}}(x)$ is a forward (respectively backward) tangent direction at x if the distance function from the root increases (respectively decreases) along t. Denote by $\operatorname{Tan}_{\mathcal{T}}^{+}(x)$ (respectively $\operatorname{Tan}_{\mathcal{T}}^{-}(x)$) the set of forward (respectively backward) tangent directions at x. Note that $\operatorname{Tan}_{\mathcal{T}}^{-}(x)$ is empty if x is the root of $\mathcal{T}$ and a singleton otherwise.

Based on the construction of the bifurcation tree $\mathcal{B}$ with respect to ρ, it follows the following simple lemmas. Lemma 4.2.6 shows ρ actually factors through $d_{\mathcal{B}}^{\rho}$ by a canonical projection $\pi_{\mathcal{B}} : \Gamma \to \mathcal{B}$. Lemma 4.2.7 shows the set of forward tangent directions on $\mathcal{B}$ can be identified with the set of all open superlevel components of ρ.

We state these lemmas without proofs.

Lemma 4.2.6. *For $p \in \Gamma$, there is a unique closed superlevel component α at $\rho(p)$ which contains p. By sending p to $\iota_{\mathcal{B}}^{-1}(\alpha)$, it induces a canonical projection $\pi_{\mathcal{B}}$: $\Gamma \to \mathcal{B}$. Moreover, the map $\pi_{\mathcal{B}}$ is continuous, piecewise-linear, surjective and satisfies $\rho = d^{\rho}_{\mathcal{B}} \circ \pi_{\mathcal{B}}$.*

Lemma 4.2.7. *There is a canonical bijection $\vec{\iota}_{\mathcal{B}} : \coprod_{x\in\mathcal{B}} \mathrm{Tan}^{+}_{\mathcal{B}}(x) \to \coprod_{c\in \mathrm{Im}\,\rho} \mathrm{Comp}(S^{\rho}_{>c})$. In particular, $\mathrm{Tan}^{+}_{\mathcal{B}}(x)$ is in bijection with $\{\beta \in \mathrm{Comp}(S^{\rho}_{>d^{\rho}_{\mathcal{B}}(x)}) | \beta \subseteq \iota_{\mathcal{B}}(x)\}$ by $\vec{\iota}_{\mathcal{B}}$.*

Remark 4.2.8. The projection $\pi_{\mathcal{B}}$ naturally induces a pushforward map $\pi_{\mathcal{B}*} : \coprod_{p\in\Gamma} \mathrm{Tan}_{\Gamma}(p) \to \coprod_{x\in\mathcal{B}} \mathrm{Tan}_{\mathcal{B}}(x)$. In particular, (1) if $t \in \mathrm{Tan}^{\rho-}_{\Gamma}(p)$, then $\pi_{\mathcal{B}*}(t)$ is the unique element in $\mathrm{Tan}^{-}_{\mathcal{B}}(\pi(p))$; (2) if $t \in \mathrm{Tan}^{\rho+}_{\Gamma}(p)$, then $\pi_{\mathcal{B}*}(t) \in \mathrm{Tan}^{+}_{\mathcal{B}}(\pi(p))$ and more precisely $\vec{\iota}_{\mathcal{B}}(\pi_{\mathcal{B}*}(t))$ is the unique open superlevel component of ρ with p on its boundary and t pointing inwards. It can be easily verified that $\pi_{\mathcal{B}*}$ is surjective.

Remark 4.2.9. Recall that $(d^{\rho}_{\mathcal{B}})^{-1}(c)$ is bijective to $\mathrm{Comp}(S^{\rho}_{\geqslant c})$ by $\iota_{\mathcal{B}}$. For $t \in \coprod_{x\in\mathcal{B}} \mathrm{Tan}^{+}_{\mathcal{B}}(x)$ with $\vec{\iota}_{\mathcal{B}}(t) \in \mathrm{Comp}(S^{\rho}_{>c})$ where $c \in \mathrm{Im}\,\rho$, we let $\vec{d}^{\rho}_{\mathcal{B}}(t) = c$ and $\vec{d}^{0}_{\mathcal{B}}(t) = c - \min\rho$. Then $(\vec{d}^{\rho}_{\mathcal{B}})^{-1}(c) = \coprod_{x\in(d^{\rho}_{\mathcal{B}})^{-1}(c)} \mathrm{Tan}^{+}_{\mathcal{B}}(x)$ and is bijective to $\mathrm{Comp}(S^{\rho}_{>c})$ by $\vec{\iota}_{\mathcal{B}}$. We may define natural maps for the fibers of $d^{\rho}_{\mathcal{B}}$ and $\vec{d}^{\rho}_{\mathcal{B}}$.

If $d \leqslant c$ and $c, d \in \mathrm{Im}\,\rho$, then there are natural maps

$$
\begin{aligned}
\omega^{c,d}_{\mathcal{B}} &: (d^{\rho}_{\mathcal{B}})^{-1}(c) \to (d^{\rho}_{\mathcal{B}})^{-1}(d) \text{ with } x_c \mapsto x_d,\\
\omega^{c*,d}_{\mathcal{B}} &: (\vec{d}^{\rho}_{\mathcal{B}})^{-1}(c) \to (d^{\rho}_{\mathcal{B}})^{-1}(d) \text{ with } t_c \mapsto x_d,\\
\omega^{c,d*}_{\mathcal{B}} &: (d^{\rho}_{\mathcal{B}})^{-1}(c) \to (\vec{d}^{\rho}_{\mathcal{B}})^{-1}(d) \text{ with } x_c \mapsto t_d \quad \text{(defined when } d < c),\\
\text{and } \omega^{c*,d*}_{\mathcal{B}} &: (\vec{d}^{\rho}_{\mathcal{B}})^{-1}(c) \to (\vec{d}^{\rho}_{\mathcal{B}})^{-1}(d) \text{ with } t_c \mapsto t_d.
\end{aligned}
$$

that are defined as follows: for each $t_c \in \mathrm{Tan}^{+}_{\mathcal{B}}(x_c)$, we let $x_d \in (\vec{d}^{\rho}_{\mathcal{B}})^{-1}(d)$ and $t_d \in \mathrm{Tan}^{+}_{\mathcal{B}}(x_d)$ be the unique elements in $(d^{\rho}_{\mathcal{B}})^{-1}(d)$ and $(\vec{d}^{\rho}_{\mathcal{B}})^{-1}(d)$ such that $\iota_{\mathcal{B}}(x_d) \supseteq$

$\vec{\iota}_{\mathcal{B}}(t_d) \supseteq \iota_{\mathcal{B}}(x_c) \supseteq \vec{\iota}_{\mathcal{B}}(t_c)$. For each $d' \in (d, c)$, the following diagram commutes.

$$\begin{tikzcd}
(d^{\rho}_{\mathcal{B}})^{-1}(c) \arrow[r, "\omega_{\mathcal{B}}^{c,d'}"] \arrow[rr, bend left, "\omega_{\mathcal{B}}^{c,d}"] \arrow[dr, "\omega_{\mathcal{B}}^{c,d'*}"] \arrow[drr, "\omega_{\mathcal{B}}^{c,d*}"'] & (d^{\rho}_{\mathcal{B}})^{-1}(d') \arrow[r, "\omega_{\mathcal{B}}^{d',d}"] \arrow[dr, "\omega_{\mathcal{B}}^{d',d*}"] & (d^{\rho}_{\mathcal{B}})^{-1}(d) \\
(\vec{d}^{\rho}_{\mathcal{B}})^{-1}(c) \arrow[u, "\omega_{\mathcal{B}}^{c*,c}"] \arrow[r, "\omega_{\mathcal{B}}^{c*,d'*}"'] \arrow[rr, bend right, "\omega_{\mathcal{B}}^{c*,d*}"'] \arrow[ur, "\omega_{\mathcal{B}}^{c*,d'}"] \arrow[urr, "\omega_{\mathcal{B}}^{c*,d}"] & (\vec{d}^{\rho}_{\mathcal{B}})^{-1}(d') \arrow[u, "\omega_{\mathcal{B}}^{d'*,d'}"] \arrow[r, "\omega_{\mathcal{B}}^{d'*,d*}"'] \arrow[ur, "\omega_{\mathcal{B}}^{d'*,d}"] & (\vec{d}^{\rho}_{\mathcal{B}})^{-1}(d) \arrow[u, "\omega_{\mathcal{B}}^{d*,d}"']
\end{tikzcd}$$

Let $\mathrm{Bif}_{\rho} = \{b_1, b_2, \cdots, b_m\}$ be the set of bifurcation values where $b_1 < b_2 < \cdots < b_m$. In particular, we note $b_1 = \min \rho$ and $b_m = \max \rho$. Then for each $i = 1, 2, \cdots, m-1$, we can define

$$\omega_{\mathcal{B}}^{c*,d} : (\vec{d}^{\rho}_{\mathcal{B}})^{-1}(c) \to (d^{\rho}_{\mathcal{B}})^{-1}(d) \text{ with } t_c \mapsto x_d \text{ for all } c \text{ and } d \text{ such that } b_i \leqslant c < d \leqslant b_{i+1}$$

$$\text{and } \omega_{\mathcal{B}}^{c*,d*} : (\vec{d}^{\rho}_{\mathcal{B}})^{-1}(c) \to (\vec{d}^{\rho}_{\mathcal{B}})^{-1}(d) \text{ with } t_c \mapsto t_d \text{ for all } c \text{ and } d \text{ such that } b_i \leqslant c \leqslant d < b_{i+1}$$

as follows: $x_d \in (d^{\rho}_{\mathcal{B}})^{-1}(d)$ (respectively $t_d \in (\vec{d}^{\rho}_{\mathcal{B}})^{-1}(d)$) is the unique element in $(d^{\rho}_{\mathcal{B}})^{-1}(d)$ (respectively $(\vec{d}^{\rho}_{\mathcal{B}})^{-1}(d)$) such that $\iota_{\mathcal{B}}(x_d) \subseteq \vec{\iota}_{\mathcal{B}}(t_c)$ (respectively $\vec{\iota}_{\mathcal{B}}(t_d) \subseteq \vec{\iota}_{\mathcal{B}}(t_c)$). It can be verified easily that $\omega_{\mathcal{B}}^{c*,d}$ (respectively $\omega_{\mathcal{B}}^{c*,d*}$) is always a bijection with inverse $\omega_{\mathcal{B}}^{d,c*}$ (respectively $\omega_{\mathcal{B}}^{d*,c*}$) in this case. $\square$

4.2.2 Tropical dominant trees

For a linear system $|D|$, we say a set $T \in |D|$ is a tropical dominant tree if T is tropically convex and 1-dimensional and for each point $p \in \Gamma$, there is a divisor

$D \in T$ such that $p \in \text{supp}(D)$.

4.3 Refined metrized complex associated to a Berkovich skeleton

We begin by briefly recalling the concept of Berkovich skeleton of X^{an}. A semistable vertex set V of X is a finite set of type-II of points of X^{an} such that the complement of V in X^{an} is a disjoint union of a finite number of open annuli and an infinite number of open balls. Let $\Sigma(X^{\text{an}}, V)$ be a skeleton of X^{an} with respect to the the semistable vertex set V.

In order to associate a refined metrized complex $\mathfrak{C}(V)$ to $\Sigma(X^{\text{an}}, V)$, we must associate the following data to it: a metric graph Γ, a smooth algebraic curve C_p for each point $p \in \Gamma$ and for each C_p, we must specify a set A_p of marked points that are in bijection with the set of tangent directions at p. The metric graph Γ underlying the refined metrized complex is defined as the metric graph associated to $\Sigma(X^{\text{an}}, V)$. We associate the algebraic curve C_p to each point $p \in \Gamma$ as follows: since the value group of $\mathbb{K}$ is $\mathbb{R}$, every point in $\Sigma(X^{\text{an}}, V)$ is a type (2) point [15]. Hence, the double residue field has transcendence one over κ and is isomorphic to the function field of a smooth curve over κ. This smooth curve is well defined up to isomorphism and we associate this curve C_p to the point $p \in \Gamma$. We defined marked points associated to the algebraic curve C_p using Berkovich theory as follows: let x be the type (2) point corresponding to p, the set of tangent directions at any type (2) point in X^{an} is in canonical bijection with the set of discrete valuations of the double residue field. The set of discrete valuations of the double residue field is in turn in bijection with the set of closed points of C_p. For each tangent direction $t \in \text{Tan}(p)$, we define its marked point as the point in C_p associated to the corresponding tangent direction in the skeleton $\Sigma(X^{\text{an}}, V)$. Note that the marked point associated to each tangent direction is distinct.

Lemma 4.3.1. *For any skeleton $\Sigma(X^{\mathrm{an}}, V)$ of X^{an}, the data $\mathfrak{C}(V)$ defines a refined metrized complex. In particular, for all but a finite number of points in Γ, the curve C_p is a projective line over κ.*

Proof. To show that $\mathfrak{C}(V)$ is a refined metrized complex, we must verify that the curve C_p has genus zero for all but finitely many points of Γ. Using Formula (5.45.1) of [], we have $g(X) = g(\Gamma) + \sum_{p\in\Gamma} g(C_p)$. Hence, $g(C_p) = 0$ for all but finitely many p. □

Remark 4.3.2. The semistable vertex sets of X are in one to one correspondence with the semistable models of X, we refer to [15] for a detailed treatment of the topic. Via this correspondence, we can associate a refined metrized complex to a semistable model of X. This refined metrized complex is the "limit" of the metrized complexes associated to semistable models obtained by successively blowing up the special fiber at its nodes.

4.4 Specialization and reduction map

We define a morphism from $\tau_* : \mathrm{Div}(X) \to \mathrm{Div}(\mathfrak{C}(V))$ called the **specialization map** and a map that takes a rational function on X to a rational function on $\mathfrak{C}(V)$ called the **reduction map**. We follow the analogous construction for metrized complexes by Amini and Baker [4].

4.4.1 Specialization map

Suppose that $r_V : X^{\mathrm{an}} \to \Sigma(X^{\mathrm{an}}, V)$ is the retraction map and let $\{r_{V,s}\}_{s\in[0,1]}$ be the continuous family of retraction maps associated with the deformation retraction from X^{an} to $\Sigma(X^{\mathrm{an}}, V)$. In particular, $r_{V,1} = r_V$. For a closed point $p \in X$, the point $r_V(p)$ has a unique tangent direction $t_V^{\mathrm{an}}(p)$ that lies in the image of the retraction map $r_{V,s}$ where s is in an open neighborhood of one. The map τ_* takes p to the divisor $(r_V(p), \mathrm{red}(t_V^{\mathrm{an}}(p)))$ on $\mathfrak{C}(V)$ where $\mathrm{red}(t_V^{\mathrm{an}}(p))$ is the marked point in C_p corresponding to the tangent direction $t_V^{\mathrm{an}}(p)$. We extend this map linearly to define a map from $\mathrm{Div}(X)$ to $\mathrm{Div}(\mathfrak{C}(V))$.

Lemma 4.4.1. *The specialization map τ_* is a homomorphism from $\mathrm{Div}(X) \to \mathrm{Div}(\mathfrak{C}(V))$ that takes effective divisors on $\mathrm{Div}(X)$ to effective divisors on $\mathrm{Div}(\mathfrak{C}(V))$. The image of τ_* is the set of all divisors $(D_\Gamma, \{D_p\}_{p\in\Gamma}) \in \mathrm{Div}(\mathfrak{C}(V))$ such that the support of D_p is contained in the set $C_p \setminus A_p$ for all $p \in \Gamma$.*

4.4.2 Reduction of rational functions

Consider a point $p \in \Gamma$ and let x be the corresponding type (2) point in $\Sigma(X^{\mathrm{an}}, V)$. By $f(x)$, we denote the multiplicative semi norm defined by x evaluated at f and let $|c| = |f(x)|$. Let $\tilde{H}(x)$ is the double residue field of x and note that the field $\tilde{H}(x)$ is isomorphic to the function field of C_p. Suppose that f maps to f_x in $\tilde{H}(x)$. The reduction map takes f to $c^{-1}f_x$, we denote $c^{-1}f_x$ by $\tilde{f}_x$ and the corresponding rational function in C_p by $\tilde{f}_p$. Note that f_x is only defined up to multiplication by κ^* and hence, its divisor is well-defined.

Lemma 4.4.2. *The dimension of any finite dimensional subspace of $\kappa(X)$ is preserved by reduction.*

Given a rational function f on X, we let f_Γ be a rational function on Γ given by the restriction to $\Sigma(X^{\mathrm{an}}, V)$ of the function $\log|f|$ on X^{an}. Hence, given a rational function f on X we associate a rational function $\mathfrak{f} = (f_\Gamma, \{\tilde{f}_p\}_{p\in\Gamma})$ on $\mathfrak{C}(V)$. The following version of the Poincare-Lelong Formula for refined metrized complexes establishes a compatibility between the specialization and the reduction maps.

Theorem 4.4.3. (**Poincaré-Lelong Formula**) *For any non-zero rational function f on X, suppose that $\mathfrak{f}$ is the reduction of f on $\mathfrak{C}(V)$, we have $\tau_*(\mathrm{div}(f)) = \mathrm{div}(\mathfrak{f})$. Hence, the map τ_* takes principal divisors in X to principal divisors in $\mathfrak{C}(V)$.*

Proof. For a point x in the the skeleton $\Sigma(X^{\mathrm{an}}, V)$ partition T_x^{an} the set of tangent directions at x into the tangent directions in $\Sigma(X^{\mathrm{an}}, V)$ and its complement and denote it by $T_{i,x}^{\mathrm{an}}$ and $T_{r,x}^{\mathrm{an}}$ respectively. By parts (2) and (5) of the slope formula [], we note that $\mathrm{ord}_t(\tilde{f}_x) = 0$ for all but points $x \in \Sigma(X^{\mathrm{an}}, V)$ and $t \in T_{r,x}^{\mathrm{an}}$ except those that lie in the image (under the retraction map) of the support of $\mathrm{div}(f)$. By part (2) of the slope formula, $\mathrm{sl}_t(f_\Gamma) = \mathrm{ord}_t(f_x)$. Hence, $\mathrm{div}(\mathfrak{f})$ has support at a finite number of points and its support coincides with the support of $\tau_*(\mathrm{div}(f))$. Hence, $\mathrm{div}(f)$ is a divisor (not just a pseudo-divisor). Let S be the the union of the support of $\mathrm{div}(\mathfrak{f})$ and the point of Γ of valence at least three. Thus, $\tau^*(\mathrm{div}(f))$ and $\mathrm{div}(\mathfrak{f})$ coincide on points in $\Gamma \setminus S$. Consider the metrized complex $\mathfrak{C}(V)|S$ obtained by restricting $\mathfrak{C}$ to S, by the Poincare-Lelong formula shown in Amini and Baker [4], we have $\tau^*(\mathrm{div}(f))$ and $\mathrm{div}(\mathfrak{f})$ coincide on $\mathfrak{C}(V)|S$. □

4.5 Smoothability

The following theorem is the analogue of Theorem 5.9 in [4] for refined metrized complexes.

Theorem 4.5.1. *For any g_d^r on X, given by the data (D, H) where H is an $(r+1)$-dimensional subspace of rational functions of X, the data $(\tau_*(D), \{H_p\}_{p\in\Gamma})$ where H_p is the image of H under the reduction map at p is a limit g_d^r on $\mathfrak{C}(V)$.*

Proof. By Lemma 4.4.2, the dimension of the space H is preserved by the specialization map. Using the notion rank-determining sets of algebraic curves, it suffices to exhibit a family $\{S_p\}_{p\in\Gamma}$ where $|S_p| = g(C_p)+1$ such that for every effective divisor $\mathcal{E} = (E_\Gamma, \{E_p\}_{p\in\Gamma})$ of degree r with the support of E_p in S_p there exists a rational function on $(f_\Gamma, \{f_p\}_{p\in\Gamma})$ with $f_p \in H_p$ such that $\tau_*(D) - \mathcal{E} + \mathrm{div}(\mathfrak{f}) \geq 0$. Taking cue from Lemma 4.4.1, we let S_p be any subset of $C_p \setminus A_p$ of size $g(C_p)+1$. From Lemma 4.4.1, we know that for any effective divisor $\mathcal{E} = (E_\Gamma, \{E_p\}_{p\in\Gamma})$ such that E_p has support in S_p for every $p \in \Gamma$, there exists an effective divisor E on X such that $\tau_*(E) = \mathcal{E}$. Since (D, H) is a g^r_d on X, there must a rational function $f \in H$ such that $D - E + \mathrm{div}(f) \geq 0$. We apply the specialization map to this inequality. Using the property that the specialization map is a homomorphism between divisor groups that preserves effective divisors, along with Theorem 4.4.3 , we conclude that $\rho(D) - \mathcal{E} + \mathrm{div}(\mathfrak{f}) \geq 0$. ☐

Definition 4.5.2. (Smoothable Limit Linear Series) A limit g^r_d (or a pre-limit g^r_d) given by the data $(\mathcal{D}, \mathcal{H})$ on a refined metrized complex $\mathfrak{C}$ is said to be **smoothable** if there exists a smooth proper curve X over $\mathbb{K}$ and a skeleton $\Sigma(X^{\mathrm{an}}, V)$ of the Berkovich analytification X^{an} of X with respect to a semistable vertex set V of X^{an} such that $\mathfrak{C}$ is isomorphic to the refined metrized complex associated to $\Sigma(X^{\mathrm{an}}, V)$ and there exists a g^r_d on X such that the associated limit g^r_d is $(\mathcal{D}, \mathcal{H})$.

4.6 Application of the specialization and reduction map

Let K be any algebraically closed field. Note that on a smooth proper curve over K, a two-dimensional subspace of rational functions containing the constant function defines a morphism from the curve to $\mathbb{P}^1_K$ and conversely, a morphism ϕ from the curve to $\mathbb{P}^1_K$ corresponds to a two dimensional space generated by a rational function that is equivalent to ϕ up to the action of $\mathrm{PGL}(2, K)$ on $\mathbb{P}^1_K$.

We restate Theorem 4.1.12 as follows.

Theorem 4.6.1. *A limit g^1_d given by the data $(\mathcal{D}, \{H_p\}_{p\in\Gamma})$ on $\mathfrak{C}$ is smoothable if and only if there exists a modification $\mathfrak{C}^{\mathrm{mod}}$ of $\mathfrak{C}$ and a harmonic morphism $\mathfrak{C}\phi^{\mathrm{mod}} = (\phi_{\Gamma^{\mathrm{mod}}}, \{\phi_p\}_{p\in\Gamma^{\mathrm{mod}}})$ of degree $\deg(\mathcal{D})$ from $\mathfrak{C}^{\mathrm{mod}}$ to a genus zero refined metrized complex such that $\mathcal{D}$ is the retract onto $\mathfrak{C}$ of a fiber of $\mathfrak{C}\phi^{\mathrm{mod}}$ and ϕ_p coincides with the map to $\mathbb{P}^1_{\mathbb{K}}$ defined by H_p for points $p \in \Gamma$.*

Proof. Suppose that $(\mathcal{D}, \{H_p\}_{p\in\Gamma})$ is smoothable, there exists a smooth proper curve X over $\mathbb{K}$ and a skeleton $\Sigma(X^{\mathrm{an}}, V)$ of X^{an} such that the refined metrized complex associated to X^{an} is isomorphic to $\mathfrak{C}$. Let (D, H) be a g^r_d on X corresponding to $(\mathcal{D}, \{H_p\}_{p\in\Gamma})$. Note that the divisor D is the fiber over the point ∞ of $\mathbb{P}^1_{\mathbb{K}}$. Consider the map $\phi : X \to \mathbb{P}^1_{\mathbb{K}}$ defined by H. The analytification functor induces a map $\phi^{\mathrm{an}} : X^{\mathrm{an}} \to \mathbb{P}^1{}_{\mathrm{Berk}}$, where $\mathbb{P}^1{}_{\mathrm{Berk}}$ is the Berkovich projective line over $\mathbb{K}$. We restrict the map ϕ^{an} to the skeleton $\Sigma(X^{\mathrm{an}}, V)$ and via the isometry between $\Sigma(X^{\mathrm{an}}, V)$ and $\mathfrak{C}$ to obtain a tropical rational function $\phi_\Gamma : \Gamma \to T$ where T is a retract of $\mathbb{P}^1{}_{\mathrm{Berk}}$. We apply the reduction map to H to obtain maps $\phi_p : C_p \to \mathbb{P}^1_\kappa$ for each $p \in \Gamma$. The data $(\phi_\Gamma, \{\phi_p\}_{p\in\Gamma})$ satisfies the compatibility properties (Definition 4.1.11) of a harmonic morphism and can be modified to a harmonic morphism of degree $\deg(\mathcal{D})$. Furthermore, from the slope formula, $\mathcal{D}$ is the fiber of $(\phi_\Gamma, \{\phi_p\}_{p\in\Gamma})$ over the image of the retraction map $\mathbb{P}^1_{\mathbb{K}} \to T$.

Conversely, suppose that there is a harmonic morphism $\mathfrak{C}\phi^{\mathrm{mod}} = (\phi_\Gamma, \{\phi_p\}_{p\in\Gamma^{\mathrm{mod}}})$ between refined metrized complexes $\mathfrak{C}^{\mathrm{mod}}$ and a genus zero metrized complex. Consider the retract onto $\mathfrak{C}$ of the fiber $\mathcal{D}$ over a point $u \in T$ and let H_p be the two-dimensional subspace of $\kappa(C_p)$ corresponding to ϕ_p. By the properties of a harmonic morphism $(\mathcal{D}, \{H_p\}_{p\in\Gamma})$ is a limit g^1_d. To show that $(\mathcal{D}, \{H_p\}_{p\in\Gamma})$ is smoothable, we use the lifting theorem of Amini et al. [5] that there exists a smooth proper curve X over $\mathbb{K}$ and a skeleton $\Sigma(X^{\mathrm{an}}, V)$ of X^{an} such that the refined metrized complex associated to X^{an} is isomorphic to $\mathfrak{C}$ and a morphism from $\phi : X \to \mathbb{P}^1_{\mathbb{K}}$ such that:

- ϕ_Γ is the restriction of the map $\phi^{\mathrm{an}} : X^{\mathrm{an}} \to \mathbb{P}^1_{\mathrm{Berk}}$ to $\Sigma(X^{\mathrm{an}}, V)$.
- ϕ_p coincides with the map to $\mathbb{P}^1_\kappa$ defined by the reduction, at the type-(2) point in $\Sigma(X^{\mathrm{an}}, V)$ corresponding to p, of the two-dimensional subspace of $\kappa(X)$ defined by ϕ.

Thus D is a fiber of ϕ over a point in the pre image of u under the retraction map. Let H be the subspace of $\kappa(X)$ corresponding to ϕ. Note that (D, H) is a g^1_d on X. In particular, $(\mathcal{D}, \{H_p\}_{p\in\Gamma})$ is smoothable to (D, H). $\square$

Lemma 4.6.2. *A smoothable limit g^r_d on a refined metrized complex is a refined limit g^r_d.*

Proof. We show that a smoothable limit g^r_d satisfies the two properties of refined limit g^r_d. For the first property of refined limit g^1_d: note that the constant function is contained in H where (D, H) is any smoothing of the limit g^r_d and the Poincaré-Lelong Formula (Theorem 4.4.3). Using the characterization of the image of the specialization map obtained in Lemma 4.4.1, we deduce that a smoothable limit g^1_d satisfies the second property of a refined limit g^r_d. □

Theorem 4.6.3. *A smoothable limit g^1_d on a refined metrized complex is a diagrammatic limit g^1_d.*

Proof. Let $(\mathcal{D}, \{H_p\}_{p\in\Gamma})$ be a smoothable limit g^1_d. Using Theorem 4.1.12, we know that there exists a harmonic morphism $\mathfrak{C}\phi = (\phi_{\Gamma^{\text{mod}}}, \{\phi_p\}_{p\in\Gamma^{\text{mod}}})$ of degree $\deg(D)$ from $\mathfrak{C}^{\text{mod}}$ to a genus zero refined metrized complex such that $\mathcal{D}$ is a fiber of $\mathfrak{C}\phi$ and ϕ_p coincides with the morphism to $\mathbb{P}^1_\kappa$ defined by H_p. By the compatibility property of the harmonic morphism we know that for any point $p \in \Gamma$ and for any tangent direction $t \in \text{Tan}(p)$, the ramification index of ϕ_p at the marked point $\text{red}(t)$ corresponding to t equals the slope of ϕ_Γ at t. Recall that the multiplicity $m(p, t)$ in the local diagram associated to $(\mathcal{D}, \{H_p\}_{p\in\Gamma})$ is the ramification index of ϕ_p at $\text{red}(t)$. Since ϕ_Γ is continuous, for any point $p \in \Gamma$, there exists a local neighborhood U of p such that the local diagram induced by p at any point $q \in U$ coincides with the local diagram at q. Hence, $(\mathcal{D}, \{H_p\}_{p\in\Gamma})$ is a diagrammatic limit g^1_d. □

4.7 Characterization of tropical dominant subtrees of $R_{\mathcal{D},\mathcal{H}}$

4.7.1 Slope-multiplicity principle for diagrammatic pre-limit g^1_d.

We formulate a property of a diagrammatic pre-limit g^1_d on a refined metrized complex that we call the **slope-multiplicity** principle. The slope-multiplicity principle states that the slope at certain tangent directions t of any tropical rational function g_Γ such that $D_\Gamma + \operatorname{div}(g_\Gamma)$ is in $R_{\mathcal{D},\mathcal{H}}$ (cf. Definition 4.1.14) coincides with the negative of the multiplicity of the global diagram at t. This will be very useful for the study of the space of tropical dominant subtrees of $R_{\mathcal{D},\mathcal{H}}$.

Definition 4.7.1. Let $(\mathcal{D},\mathcal{H})$ with $\mathcal{D} = (D_\Gamma, \{D_p\}_{p\in\Gamma})$ and $\mathcal{H} = \{H_p\}_{p\in\Gamma}$ be a diagrammatic pre-limit g^1_d on a refined metrized complex $\mathfrak{C}$ with underlying metric graph Γ and algebraic curves $\{C_p\}_{p\in\Gamma}$. An **effectivizing pair** at a point $u \in \Gamma$ is a pair $(\mathcal{E}, \mathfrak{g})$ with the effective divisor $\mathcal{E} = (E_\Gamma, \{E_p\}_{p\in\Gamma})$ of degree one and a rational function $\mathfrak{g} = (g_\Gamma, \{g_p\}_{p\in\Gamma})$ on $\mathfrak{C}$ satisfying the following properties:

1. For all $p \in \Gamma$, we have $g_p \in H_p$.
2. $E_\Gamma = (u)$ and the divisor E_u on the curve C_u is a point not contained in the support $\operatorname{supp}(D_u)$ of D_u and is not a marked point corresponding to any incoming tangent direction at u, i.e., $\operatorname{supp}(E_u) \nsubseteq \operatorname{red}_u(\operatorname{In}(u)) \bigcup \operatorname{supp}(D_u)$.
3. The divisor $\mathcal{D} - \mathcal{E} + \operatorname{div}(\mathfrak{g}) \geq 0$.

Remark 4.7.2. With the above notations, given an effectivizing pair $(\mathcal{E}, \mathfrak{g})$, we will have $D_\Gamma + \operatorname{div}(g_\Gamma) \in R_{\mathcal{D},\mathcal{H}}$.

Conversely, for any tropical rational function g_Γ such that $D_\Gamma + \operatorname{div}(g_\Gamma) \in R_{\mathcal{D},\mathcal{H}}$, by definition there exists a rational function $\mathfrak{g} = (g_\Gamma, \{g_p\}_{p\in\Gamma})$ such that $g_p \in H_p$ for all $p \in \Gamma$ and an effective divisor $\mathcal{D}' = (D'_\Gamma, \{D'_p\}_{p\in\Gamma})$ such that $\mathcal{D}' - \mathcal{D} = \operatorname{div}(\mathfrak{g})$. As long as $\mathcal{D}' \neq \mathcal{D}$, $(\mathcal{E}, \mathfrak{g})$ will be an effectivizing pair at $u \in \Gamma$ where $\mathcal{E} = ((u), \{E_p\}_{p\in\Gamma})$ is an effective divisor of degree one such that $\mathcal{E}$ is dominated by the effective part of $\operatorname{div}(\mathfrak{g})$. In particular, if in addition g_Γ itself is nonzero, then there always exists a rational function $\mathfrak{g}$ with g_Γ being the tropical part of $\mathfrak{g}$ such that for any u in the support of the effective part of $\operatorname{div}(g_\Gamma)$ we can always find an effectivizing pair $(\mathcal{E}, \mathfrak{g})$ at u. $\square$

Consider a closed directed path $\mathcal{P}$ on Γ. Then any point p in the interior of this path has two tangent directions on this path, we denote by $t^-(p, \mathcal{P})$ the tangent direction at p inverse to the orientation of $\mathcal{P}$ at p and by $t^+(p, \mathcal{P})$ the tangent direction at p oriented along $\mathcal{P}$. If p is the starting point $\mathcal{P}$, we denote the unique tangent direction on $\mathcal{P}$ by $t^+(p, \mathcal{P})$ and if p is the end point of $\mathcal{P}$, we denote the unique tangent direction on $\mathcal{P}$ by $t^-(p, \mathcal{P})$.

Lemma 4.7.3. *Let $(\mathcal{E}, \mathfrak{g})$ be an effectivizing pair with $\mathfrak{g} = (g_\Gamma, \{g_p\}_{p\in\Gamma})$. Then for any point $v \in \Gamma$ and any tangent direction $t \in \mathrm{Tan}_\Gamma(v)$, the outgoing slope $\mathrm{sl}_t(g_\Gamma)$ is either $-m(v, t)$ or 0.*

Proof. Let $\mathcal{P}$ be any directed path with the starting point being v and the end point being a valence-two point $v' \neq v$ along the tangent direction t. Let $n = \mathrm{sl}_t(g_\Gamma)$. Then from the piecewise-linear nature of g_Γ, the continuity property of the global diagram and the discreteness of $\mathrm{div}(\mathfrak{g})$, we have $\mathrm{sl}_{t^+(w,\mathcal{P})}(g_\Gamma) = n$, $\mathrm{sl}_{t^-(w,\mathcal{P}_1)}(g_\Gamma) = -n$, $m(w, t^+(w, \mathcal{P})) = m(v, t)$, $m(w, t^-(w, \mathcal{P})) = -m(v, t)$ and $\mathrm{div}(\mathfrak{g})(w) = 0$ for any point w in the interior of $\mathcal{P}$ as long as v' is sufficiently close to v. This implies that $\mathrm{div}(g_w) = -\mathrm{div}_w(g_\Gamma) = -n \cdot \mathrm{red}_w(t^+(w, \mathcal{P})) + n \cdot \mathrm{red}_w(t^-(w, \mathcal{P})) = -n \cdot x^w_{t^+(w,\mathcal{P})} + n \cdot x^w_{t^-(w,\mathcal{P})}$. Now since $g_w \in H_w$, we can only have $n = 0$ if g_w is a constant function or $n = m(w, t^-(w, \mathcal{P})) = -m(v, t)$ if g_w is a non-constant function in H_w. $\square$

Theorem 4.7.4. (**Slope-Multiplicity Principle**) *Let $(\mathcal{D}, \mathcal{H})$ with $\mathcal{D} = (D_\Gamma, \{D_p\}_{p\in\Gamma})$ and $\mathcal{H} = \{H_p\}_{p\in\Gamma}$ be a diagrammatic pre-limit g^1_d. Let $(\mathcal{E}, \mathfrak{g})$ be an effectivizing pair at $u \in \Gamma$ with $\mathfrak{g} = (g_\Gamma, \{g_p\}_{p\in\Gamma})$. Consider any closed directed path $\mathcal{P}$ with u as the end point which is compatible with the global diagram i.e., $\mathcal{P}$ is oriented along the global diagram. For every point r in the interior of this path $\mathcal{P}$ and for each of the two tangent directions at r on this path, the outgoing slope of g_Γ is equal to the negative of the multiplicity i.e., $\mathrm{sl}_{t^-(r,\mathcal{P})}(g_\Gamma) = -m(r, t^-(r, \mathcal{P}))$ and $\mathrm{sl}_{t^+(r,\mathcal{P})}(g_\Gamma) = -m(r, t^+(r, \mathcal{P}))$. Furthermore, the rational function g_r is non-constant.*

Proof. By assumption of $\mathfrak{g}$, $\mathcal{D} - \mathcal{E} + \mathrm{div}(\mathfrak{g}) \geq 0$, or equivalently, for every $w \in \Gamma$ and hence $D_w - E_w + \mathrm{div}(g_w) + \mathrm{div}_w(g_\Gamma) \geqslant 0$. By the assumption of E_u, we may let $E_u =$

(q_u) with $q_u \in C_u$ and $q_u \notin \mathrm{red}_u(\mathrm{In}(u)) \bigcup \mathrm{supp}(D_u)$. Since $D_u - E_u + \mathrm{div}(g_u) + \mathrm{div}_u(g_\Gamma)$ is effective, g_u must be non-constant with $\mathrm{div}(g_u)$ having poles in a nonempty subset of $\mathrm{red}_u(\mathrm{In}(u)) \bigcup \mathrm{supp}(D_u)$ and a zero at q_u, which also means $\mathrm{sl}_{t^-(u,\mathcal{P})}(g_\Gamma)$ cannot be 0. By Lemma 4.7.3, we know $\mathrm{sl}_{t^-(u,\mathcal{P})}(g_\Gamma)$ is exactly $-m(u, t^-(u,\mathcal{P}))$.

Now suppose for sake of contradiction that there exists a point r in the interior of the path $\mathcal{P}$ such that $r \neq u$ and $\mathrm{sl}_{t^+(r,\mathcal{P})}(g_\Gamma) \neq -m(r, t^+(u,\mathcal{P}))$. Then by Lemma 4.7.3, we get $\mathrm{sl}_{t^+(r,\mathcal{P})}(g_\Gamma) = 0$. Since g_Γ is piecewise-linear and continuous, there must exist a point v on the path $\mathcal{P}$ between r and u such that $\mathrm{sl}_{t^-(v,\mathcal{P})}(g_\Gamma) = 0$ and $\mathrm{sl}_{t^+(v,\mathcal{P})}(g_\Gamma)$ is nonzero. However, this is impossible since $D_v - E_v + \mathrm{div}(g_v) + \mathrm{div}_v(g_\Gamma) = D_v + \mathrm{div}(g_v) + \mathrm{div}_v(g_\Gamma) \geqslant 0$ won't be satisfied no matter g_v is a constant or non-constant function. Therefore, we conclude that $\mathrm{sl}_{t^+(r,\mathcal{P})}(g_\Gamma) = -m(r, t^+(u,\mathcal{P}))$ and furthermore, all g_r must be non-constant function with a zero at $x^r_{t^+(r,\mathcal{P})}$.

Finally, the continuity of the global diagram guarantees that $\mathrm{sl}_{t^-(r,\mathcal{P})}(g_\Gamma)$ is nonzero when r is a non-starting point on $\mathcal{P}$. Otherwise, you will be able to find a point r' on path $\mathcal{P}$ in the infinitesimal neighborhood of r with $\mathrm{sl}_{t^+(r',\mathcal{P})}(g_\Gamma) = 0$, contradicting our previous conclusion. Again, by Lemma 4.7.3, we get $\mathrm{sl}_{t^-(r,\mathcal{P})}(g_\Gamma) = -m(r, t^-(u,\mathcal{P}))$. $\square$

Corollary 4.7.5. *For a global diagram associated to an diagrammatic limit g^1_d, the orientation induced by the global diagram on the metric graph Γ is acyclic i.e., there are no directed cycles.*

Proof. Assume the contrary, suppose that there is a directed cycle $\mathcal{C}$ on the global diagram and consider any point w in $\mathcal{C}$. Consider an effectivizing pair $(\mathcal{E}, \mathfrak{g})$ at the point v. On the one hand, we know that the integral of the slopes of g_Γ over the directed cycle is zero. On the other hand, from the slope-multiplicity principle we know that the slope of g_Γ at any tangent to the directed cycle coincides with the negative of the multiplicity. Since, we are integrating along a directed cycle $\mathcal{C}$ in the global diagram, we know that for any point v on $\mathcal{C}$, the multiplicity $m(v, t^+(v,\mathcal{C})) =$

$m_{\mathcal{C}}(v)$ is a natural number. Hence, $|\int_{\mathcal{C}} m_{\mathcal{C}}|$ is at least the length of $\mathcal{C}$. This gives the desired contradiction. $\square$

4.7.2 Solvability of an diagrammatic pre-limit g_d^1

Consider any model G of Γ. We fix an arbitrary orientation on G and regard the multiplicity function m on Γ as an element in $C^1(G, \mathbb{Z})$ defined as $m(e)$ is the multiplicity of the edge e in the global diagram if the orientation on e in the global diagram is consistent with the orientation on e and the negative of the multiplicity of the edge e otherwise. Consider the integration pairing $C^1(G, \mathbb{Z}) \times C_1(G, \mathbb{Z}) \to \mathbb{Z}$ defined as $\int_\alpha f = \sum_e \alpha_e f(e)$ where $f \in C^1(G, \mathbb{Z})$ and $\alpha = \sum_e \alpha_e \cdot e \in C_1(G, \mathbb{Z})$. Recall that $H_1(\Gamma, \mathbb{Z})$ is the space of integer-valued flows on G. The following lemma provides alternate formulations for the existence of a solution to the characteristic equation of a global diagram.

Lemma 4.7.6. *The following statements are equivalent:*

1. *The characteristic equation of the global diagram has a solution.*

2. *The integral $\int_{\mathcal{C}_j} m = 0$ for a basis $\mathcal{C}_1, \dots, \mathcal{C}_g$ for $H_1(\Gamma, \mathbb{Z})$.*

3. *The integral $\int_{\mathcal{C}} m = 0$ for every element $\mathcal{C} \in H_1(\Gamma, \mathbb{Z})$.*

4. *For any pair of points u, v, the integral $\int_{\mathcal{P}} m$ of the multiplicity function along a closed directed path $\mathcal{P}$ between u and v does not depend on the path.* $\square$

Proof. $1 \Rightarrow 2$: If the characteristic equation of the global diagram has a solution then from the first fundamental theorem of calculus we know that every pair of points u, v of Γ, the integral of the multiplicity along any closed directed path between u and v is the same and is equal to $y(u) - y(v)$. If the closed directed path is a cycle then $u = v$ and this integral is zero.

$2 \Rightarrow 3$: Follows from linearity of the integration pairing with respect to the second argument.

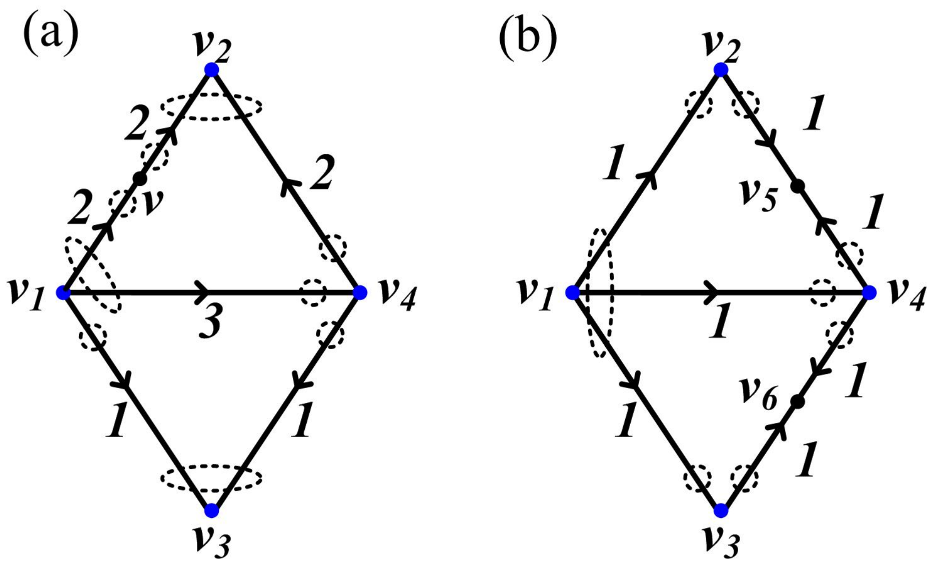

Figure 14: (a) An unsolvable global diagram (b) A solvable global diagram.

3 ⇒ 4: Consider any two closed directed paths $\mathcal{P}_1$, $\mathcal{P}_2$ between u and v, $\mathcal{P}_1 - \mathcal{P}_2$ is a directed cycle. Since the integral along this directed cycle is zero and since, the integration pairing is linear in the second argument, the integration of the multiplicity along $\mathcal{P}_1$ and $\mathcal{P}_2$ are equal.

4 ⇒ 1: If for any pair of points u, v of Γ, the integral of the multiplicity along any closed directed path between u and v is the same, we construct a solution y by choosing an arbitrary point w_0 and by setting $y(w_0) = 0$. For any other point w_1, consider any closed directed path $\mathcal{P}$ (not necessarily oriented according to the global diagram) from w_0 to w_1 in the metric graph and suppose $y(w_1, \mathcal{P})$ is the integral of the multiplicity function of the global diagram along this path. Since, $y(w_1, \mathcal{P})$ does not depend on $\mathcal{P}$, let $y(w_1, \mathcal{P}) = y(w_1)$. By construction, y is piecewise linear, continuous and has integral slopes. □

Note that the solutions to the characteristic equation differ by a constant. Hence, up to translation by a constant the characteristic equation can either have no solutions or have a unique solution. In general, the characteristic equation of a global diagram does not have a solution, as an example is shown in Figure 14(a) where a vertex set

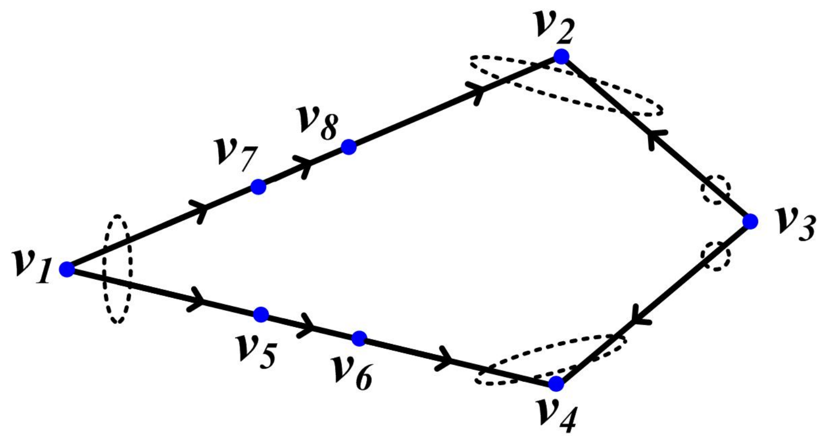

Figure 15: An example of a diagrammatic limit g^1_d such that the characteristic equation associated to the global diagram does not have a solution.

is $\{v_1, v_2, v_3, v_4\}$ and all edge lengths are the same. To see that the characteristic equation does not have a solution, suppose the contrary and assume without loss of generality that the solution ρ is zero at v_1. Integrating along the edge between v_1 and v_4 with multiplicity three, we find that $\rho(v_4) = 3$ and integrating along the path v_1 and v_4 that contains v_3, we find that $\rho(v_4) = 0$. On the contrary, we can easily verify that Figure 14(b) is a solvable global diagram. In fact, there exist diagrammatic limit g^1_d such that the characteristic equation associated to the global diagram does not have a solution as the following example shows:

Example 4.7.7. Consider the global diagram on a cycle shown in Figure 15 with the multiplicity on each edge one. The metric graph Γ has the lengths: $\ell_{v_2,v_3} = \ell_{v_2,v_8}$, $\ell_{v_4,v_3} = \ell_{v_4,v_5}$, $\ell_{v_1,v_5} = \ell_{v_1,v_7}$ and $\ell_{v_1,v_6} = \ell_{v_1,v_8}$. The algebraic curve C_p at every point $p \in \Gamma$ is a $\mathbb{P}^1$. The diagrammatic limit g^1_d is defined by the data: $D_\Gamma = 2(v_1) + (v_3)$ and D_{v_1} sum of two arbitrary non-marked points in C_{v_1} and D_{v_3} any non-marked point on C_{v_3}.

We claim that this is a limit g^1_d. To this end: we must show that for every effective divisor $\mathcal{E} = (u, q_u)$ where u is a point in Γ and $q_u \in C_u$ on the refined metrized complex of degree one, there exists a rational function $\mathfrak{g} = (g_\Gamma, \{g_p\}_{p\in\Gamma})$. We first specify g_Γ.

We describe g_Γ in terms of chip-firing moves: if u lies in $[v_1, v_8]$ and $[v_1, v_6]$, we can fire both chips from v_1 to u. If u lies in $[v_2, v_8]$ and $[v_2, v_3]$ then fire v_1 till a chip reaches v_8 and fire along v_3 and v_8 simultaneously. If u lies in $[v_4, v_5]$ and $[v_4, v_3]$ then fire v_1 till a chip reaches v_5 and then fire v_3 and v_5 simultaneously.

For a point $p \in \Gamma$, we construct g_p based on g_Γ in a neighborhood of p. Case i. Take g_p that satisfies $\mathrm{div}(g_p) = \mathrm{red}(t_i) - \mathrm{red}(t_o)$. Case ii. If $p = u$, then let g_p be the (unique) element in H_p such that $\mathrm{div}(g_p) \geq q_u$. Case iii. Take $g_p = 1$. Case iv. $-D_p + \mathrm{red}(t_1) + \mathrm{red}(t_2)$ where t_1, t_2 are the two outgoing directions. Case v. is not possible by the slope-multiplicity principle.

For this choice of $\mathfrak{g}$, we have: $\mathcal{D} - \mathcal{E} + \mathrm{div}(\mathfrak{g}) \geq 0$. On the other hand, the characteristic equation does not have a solution since the integral of the multiplicities along the cycle is nonzero and its absolute value is equal to ℓ_{v_5,v_6}. □

Nevertheless, the characteristic equation of a global diagram associated to a s-moothable limit g^1_d always has a solution.

Lemma 4.7.8. *Suppose ϕ is a harmonic morphism between tropical curves. The equation $\mathrm{sl}_t(g) = d_t(\phi)$ where $d_t(\phi)$ is the expansion factor of ϕ along the tangent direction t has a solution.* □

Theorem 4.7.9. *The characteristic equation of a global diagram associated to a s-moothable limit g^1_d has a solution and hence satisfies Level I.* □

Proof. Consider the harmonic morphism of refined metrized complexes $\mathfrak{C}\phi : \mathfrak{C}^{\mathrm{mod}} \to \mathfrak{C}\mathfrak{T}$ obtained from Theorem 4.1.12 where $\mathfrak{C}\phi = (\phi_\Gamma, \{\phi_p\}_{p\in\Gamma})$. By the first property of harmonic morphisms of refined metrized complexes, the expansion factor of ϕ_Γ along the tangent direction $t \in \mathrm{Tan}_\Gamma(p)$ of any point $p \in \Gamma$ coincides with the multiplicity $m(p,t)$. Lemma 4.7.8 completes the proof. □

Combining Example 4.7.7 and Theorem 4.7.9 leads us to the first obstruction towards smoothing a limit g^1_d:

Corollary 4.7.10. *There exists a non-smoothable diagrammatic limit g^1_d. In particular, there exists a diagrammatic limit g^1_d that does not satisfy Level I.*

4.7.3 Partition systems and partition trees

We characterize the set of tropical dominant subtrees of $R_{\mathcal{D},\mathcal{H}}$ in terms of certain partition systems that we call "admissible partition systems".

Definition 4.7.11. (Burning Function) Given a pair $(\mathcal{T}, \pi_{\mathcal{T}})$ where $\mathcal{T}$ is a metric tree rooted at a point $r(\mathcal{T}) \in \mathcal{T}$ and a continuous finite surjection $\pi_{\mathcal{T}}$ from Γ to $\mathcal{T}$ (finite means all fibers are finite), the function $b_{\mathcal{T}}$ that takes a point $p \in \Gamma$ to the distance between $r(\mathcal{T})$ and $\pi_{\mathcal{T}}(p)$ on $\mathcal{T}$ is called the **burning function** associated to $(\mathcal{T}, \pi_{\mathcal{T}})$.

Let $\rho : \Gamma \to \mathbb{R}$ be a rational function on Γ with everywhere nonzero slopes. By Λ_ρ, we denote the set of all pairs $(\mathcal{T}, \pi_{\mathcal{T}})$ where $\mathcal{T}$ is a rooted metric tree and a map π from Γ to $\mathcal{T}$ with burning function $b_{\mathcal{T}} = \hat{\rho}$ where $\hat{\rho} = \rho - \min \rho$.

Lemma 4.7.12. *Let $\mathcal{B}$ be the bifurcation tree with respect to ρ and π be the canonical projection from Γ onto $\mathcal{B}$. Then $(\mathcal{B}, \pi) \in \Lambda_\rho$.*

Proof. It follows from Lemma 4.2.6 directly. □

Definition 4.7.13. (**Partition Systems**) A **closed partition system** $\mathcal{P}_\rho$ with respect to ρ is a collection $\{P_c\}_{c \in \mathrm{Im}\,\rho}$ where P_c is a partition of $(d^\rho_{\mathcal{B}})^{-1}(c)$. An **open partition system** $\vec{\mathcal{P}}_\rho$ with respect to ρ is a collection $\{\vec{P}_c\}_{c \in \mathrm{Im}\,\rho}$ where $\vec{P}_c$ is a partition of $(\vec{d}^\rho_{\mathcal{B}})^{-1}(c)$. For $\mathbf{e} \in P_c$ (respectively, $\vec{\mathbf{e}} \in \vec{P}_c$), we say $\mathrm{supp}(\mathbf{e}) := \bigcup_{x \in \mathbf{e}} (\iota_{\mathcal{B}}(x))$ is the support of $\mathbf{e}$ (respectively, $\mathrm{supp}(\vec{\mathbf{e}}) := \bigcup_{t \in \vec{\mathbf{e}}} (\vec{\iota}_{\mathcal{B}}(t))$ is the support of $\vec{\mathbf{e}}$).

We denote the set of closed and open partition systems with respect to ρ by $\mathrm{CP}(\rho)$ and $\mathrm{OP}(\rho)$ respectively.

Example 4.7.14. One example of a closed (respectively, open) partition system is the finest closed (respectively, open) partition system with respect to ρ, i.e., for

$c \in \operatorname{Im}\rho$, each element in $(d_{\mathcal{B}}^{\rho})^{-1}(c)$ (respectively, $(\vec{d}_{\mathcal{B}}^{\rho})^{-1}(c)$) makes an equivalence class. Another example is the coarsest closed (respectively, open) partition systems with respect to ρ i.e., all elements in $(d_{\mathcal{B}}^{\rho})^{-1}(c)$ (respectively, $(\vec{d}_{\mathcal{B}}^{\rho})^{-1}(c)$) are equivalent.

Recall that in Property 4.2.9, we defined maps between partitions at different values of $\operatorname{Im}\rho$. Adding more restrictions based on these correlations of partitions, we have the following notion of properness for (closed and open) partition systems.

Definition 4.7.15. (**Proper Partition Systems**) A closed partition system $\{P_c\}_{c\in\operatorname{Im}\rho}$ with respect to ρ is called **proper** if for every pair of real numbers $d < c$ in $\operatorname{Im}\rho$, $x_1 \sim x_2$ in P_c implies $\omega_{\mathcal{B}}^{c,d}(x_1) \sim \omega_{\mathcal{B}}^{c,d}(x_2)$ in P_d. An open partition system $\{\vec{P}_c\}_{c\in\operatorname{Im}\rho}$ with respect to ρ is called **proper** if for each $c \in \operatorname{Im}\rho$ there exists a small enough neighborhood U of c such that (1) the map $\omega^{c*,d*}$ is well-defined for all $d \in U$, and (2) $t_1 \sim t_2$ in $\vec{P}_c$ implies $\omega_{\mathcal{B}}^{c*,d*}(t_1) \sim \omega_{\mathcal{B}}^{c*,d*}(t_2)$ in $\vec{P}_d$ for all $d \in U$.

We denote the set of proper closed and open partition systems with respect to ρ by $\mathrm{PCP}(\rho)$ and $\mathrm{POP}(\rho)$ respectively.

Example 4.7.16. The finest and coarsest partition system corresponding to ρ are both proper.

Lemma 4.7.17. *There is a one-to-one correspondence between* $\mathrm{PCP}(\rho)$ *and* $\mathrm{POP}(\rho)$.

Proof. For a proper closed partition system $\mathcal{P}_\rho$ and each $c \in [\min\rho, \max\rho)$, there exists δ small enough such that $\omega_{\mathcal{B}}^{c*,d}$ is a well-defined bijection with inverse $\omega_{\mathcal{B}}^{d,c*}$ where $d = c + \delta$. This induces an open partition system $\vec{\mathcal{P}}_\rho$ by letting $t_1 \sim t_2$ in $\vec{P}_c$ if and only if $\omega_{\mathcal{B}}^{c*,d}(t_1) \sim \omega_{\mathcal{B}}^{c*,d}(t_2)$ in P_d. By this construction, it is straightforwardly verifiable that $\vec{\mathcal{P}}_\rho$ is proper.

Conversely, given a proper open partition system $\vec{\mathcal{P}}_\rho$, for each $d \in (\min\rho, \max\rho]$, there exists δ small enough such that $\omega_{\mathcal{B}}^{c*,d}$ is a well-defined bijection with inverse $\omega_{\mathcal{B}}^{d,c*}$ where $c = d - \delta$. This induces a closed partition system $\mathcal{P}_\rho$ by letting $x_1 \sim x_2$

in P_d if and only if $\omega_{\mathcal{B}}^{d,c*}(x_1) \sim \omega_{\mathcal{B}}^{d,c*}(x_2)$ in $\vec{P}_c$. One may also verify straightforwardly that $\vec{\mathcal{P}}_\rho$ is proper and by all these constructions, the correspondence of proper closed partition systems and proper open partition systems is one-to-one. $\square$

Remark 4.7.18. By this lemma, when saying a proper partition system, we mean a pair $(\mathcal{P}_\rho, \vec{\mathcal{P}}_\rho)$ of a proper closed partition system $\mathcal{P}_\rho$ and its corresponding open partition system $\vec{\mathcal{P}}_\rho$, while only one may be referred to in practice. We also call $\mathcal{P}_\rho$ the closed section of $(\mathcal{P}_\rho, \vec{\mathcal{P}}_\rho)$ and $\vec{\mathcal{P}}_\rho$ the open section of $(\mathcal{P}_\rho, \vec{\mathcal{P}}_\rho)$.

Let $\mathcal{P}_\rho$ be a proper closed partition system. Guaranteed by the properness of the partition system, we have similar properties of the supports of equivalence classes in $\mathcal{P}_\rho$ as the properties stated in Remark 4.2.2. Therefore, we can construct a rooted metric tree $\mathcal{T}$ called the partition tree associated to this proper partition system essentially the same way as the construction of the bifurcation tree $\mathcal{B}$ with respect to ρ in Definition 4.2.3.

Definition 4.7.19. Let $\mathcal{B}$ be the bifurcation tree $\mathcal{B}$ with respect to ρ and $\mathcal{P}_\rho = \{P_c\}_{c\in \operatorname{Im}\rho}$ be a proper closed partition system. The **partition tree** $\mathcal{T}$ associated to $\mathcal{P}_\rho$ is a rooted metric tree constructed in the following way:

1. By abuse of notation, we also use $\mathcal{T}$ to represent the set of points of $\mathcal{T}$. We identify the set of points of $\mathcal{T}$ with the set of all equivalence classes in $\{P_c\}_{c\in \operatorname{Im}\rho}$ by the bijection $\iota_{\mathcal{T}} : \mathcal{T} \to \coprod_{c\in \operatorname{Im}\rho} P_c$.

2. We assign a metric structure $d_{\mathcal{T}}$ to $\mathcal{T}$. For $x_1, x_2 \in \mathcal{T}$, denote $x_1 \vee x_2$ be the element in $\mathcal{T}$ such that $\operatorname{supp}(\iota_{\mathcal{T}}(x_1 \vee x_2))$ is the smallest among all the supports of equivalence classes in $\{P_c\}_{c\in \operatorname{Im}\rho}$ which contains $\operatorname{supp}(\iota_{\mathcal{T}}(x_1)) \bigcup \operatorname{supp}(\iota_{\mathcal{T}}(x_2))$. Suppose $\iota_{\mathcal{T}}(x_1) \in P_{c_1}$, $\iota_{\mathcal{T}}(x_2) \in P_{c_2}$ and $\iota_{\mathcal{T}}(x_1 \vee x_2) \in P_{c_3}$. Then we let $d_{\mathcal{T}}(x_1, x_2) = c_1 + c_2 - 2c_3$.

3. The root $r(\mathcal{T})$ of $\mathcal{T}$ corresponds to the unique equivalence class $\{r(\mathcal{B})\}$ at $P_{\min\rho}$.

We note that the construction of the partition tree associated to the finest partition system is essentially the same as the construction of the bifurcation tree $\mathcal{B}$. In general, we can also assign a partial order to the points of $\mathcal{T}$ in the same way as to the points of $\mathcal{B}$, i.e., $x \geqslant x'$ if $\operatorname{supp}(\iota_{\mathcal{T}}(x)) \supseteq \operatorname{supp}(\iota_{\mathcal{T}}(x'))$. If $\iota_{\mathcal{T}}(x) \in P_c$ where $c \in \operatorname{Im}\rho$, we let $d^{\rho}_{\mathcal{T}}(x) = c$ and $d^0_{\mathcal{T}}(x) = c - \min \rho$. In particular, we also have $d^0_{\mathcal{T}}(x) = d_{\mathcal{T}}(r(\mathcal{T}), x)$.

Again, analogous to the canonical projection $\pi_{\mathcal{B}} : \Gamma \to \mathcal{B}$, we can construct an induced projection $\pi_{\mathcal{T}} : \Gamma \to \mathcal{T}$ (Lemma 4.7.20) such that $(\mathcal{T}, \pi_{\mathcal{T}}) \in \Lambda_\rho$. Let $\{\vec{P}_c\}_{c\in \operatorname{Im}\rho}$ be the proper open partition system corresponding to $\{P_c\}_{c\in\operatorname{Im}\rho}$. Lemma 4.7.21 shows the set of forward tangent directions on $\mathcal{T}$ can be identified with the set of all equivalence classes in $\{\vec{P}_c\}_{c\in\operatorname{Im}\rho}$. We state these lemmas without proofs since they are straightforward by the construction of partition trees.

Lemma 4.7.20. *For $p \in \Gamma$, there is a unique element x with $d^{\rho}_{\mathcal{T}}(x) = \rho(p)$ such that $p \in \operatorname{supp}(\iota_{\mathcal{T}}(x))$. By sending p to x, this induces a projection $\pi_{\mathcal{T}} : \Gamma \to \mathcal{T}$. Moreover, the map $\pi_{\mathcal{T}}$ is continuous, piecewise-linear, surjective and satisfies $\rho = d^{\rho}_{\mathcal{T}} \circ \pi_{\mathcal{T}}$.*

Lemma 4.7.21. *There is a canonical bijection $\vec{\iota}_{\mathcal{T}} : \coprod_{x\in\mathcal{T}} \operatorname{Tan}^+_{\mathcal{T}}(x) \to \coprod_{c\in\operatorname{Im}\rho} \vec{P}_c$. In particular, $\operatorname{Tan}^+_{\mathcal{B}}(x)$ is in bijection with $\{\vec{\mathbf{e}} \in \vec{P}_{d^{\rho}_{\mathcal{T}}(x)} | \operatorname{supp}(\vec{\mathbf{e}}) \subseteq \operatorname{supp}(\iota_{\mathcal{T}}(x))\}$ by $\vec{\iota}_{\mathcal{T}}$.*

Remark 4.7.22. Like the pushforward $\pi_{\mathcal{B}*}$ induced by the canonical projection $\pi_{\mathcal{B}}$, we also have the pushforward map $\pi_{\mathcal{T}*} : \coprod_{p\in\Gamma} \operatorname{Tan}_\Gamma(p) \to \coprod_{x\in\mathcal{T}} \operatorname{Tan}_{\mathcal{T}}(x)$ such that (1) if $t \in \operatorname{Tan}^{\rho-}_\Gamma(p)$, then $\pi_{\mathcal{T}*}(t)$ is the unique element in $\operatorname{Tan}^-_{\mathcal{T}}(\pi_{\mathcal{T}}(p))$; (2) if $t \in \operatorname{Tan}^{\rho+}_\Gamma(p)$, then $\pi_{\mathcal{T}*}(t) \in \operatorname{Tan}^+_{\mathcal{T}}(\pi(p))$.

The following theorem tells us that we may identify each element in Λ_ρ as a partition tree (with the induced projection).

Theorem 4.7.23. *There is a one to one correspondence between proper partition systems with respect to ρ and elements in Λ_ρ.*

Proof. Using the construction in Definition 4.7.19 and Lemma 4.7.20, we can associate a pair $(\mathcal{T}, \pi_{\mathcal{T}}) \in \Lambda_\rho$ to a proper partition of ρ.

Conversely, let $(\mathcal{T}, \pi_{\mathcal{T}})$ be an element in Λ_ρ. For $c \in \operatorname{Im}\rho$ and $y \in \mathcal{T}$, suppose $c = d_{\mathcal{T}}(r(\mathcal{T}), y) + \min\rho$. Note that for any $p \in \pi_{\mathcal{T}}^{-1}(y)$, $\rho(p) = c$ and hence $d_{\mathcal{B}}^{\rho}(\pi_{\mathcal{B}}(p)) = c$. Let $\mathbf{e}_y$ be $\pi_{\mathcal{B}}(\pi_{\mathcal{T}}^{-1}(y))$ which is a subset of $(d_{\mathcal{B}}^{\rho})^{-1}(c)$. By the continuity of $\pi_{\mathcal{T}}$, we must have $\pi_{\mathcal{T}}^{-1}(y) = \operatorname{supp}(\mathbf{e}_y) \bigcap \rho^{-1}(c)$, and hence $P_c := \{e_{y'} | d_{\mathcal{T}}(r(\mathcal{T}), y') + \min\rho = c\}$ is a partition of $(d_{\mathcal{B}}^{\rho})^{-1}(c)$. Therefore we derive a closed partition system $\{P_c\}_{c \in \operatorname{Im}\rho}$ from $(\mathcal{T}, \pi_{\mathcal{T}})$. Again the continuity of $\pi_{\mathcal{T}}$ implies $\{P_c\}_{c \in \operatorname{Im}\rho}$ is proper. Moreover, the partition tree associated to $\{P_c\}_{c \in \operatorname{Im}\rho}$ as constructed in Definition 4.7.19 is exactly $\mathcal{T}$ and the induced projection in Lemma 4.7.20 is exactly $\pi_{\mathcal{T}}$. Therefore, this correspondence between $\mathrm{PCP}(\rho)$ and Λ_ρ is one-to-one. □

Recall that the collection of all partitions of a finite set A is a lattice ordered by refinement. More precisely, for two partitions P and Q, we say $P_1 \leqslant P_2$ if P_1 is a refinement of P_2, i.e., $x \sim_{P_1} y$ implies $x \sim_{P_2} y$ for each two elements $x, y \in A$ where the equivalence relation associated to a partition P is denoted by $\sim_P$; the meet (greatest lower bound) $P_1 \bigwedge P_2$of P_1 and P_2 can be afforded in the way that $x \sim_{P_1 \bigwedge P_2} y$ if and only if both $x \sim_{P_1} y$ and $x \sim_{P_2} y$ for each two elements $x, y \in A$; the joint (least upper bound) $P_1 \bigvee P_2$ can be afforded in the way that $x \sim_{P_1 \bigvee P_2} y$ if and only if either $x \sim_{P_1} y$ or $x \sim_{P_2} y$ for each two elements $x, y \in A$.

Analogously such a partial order and lattice structure can be extended to $\mathrm{CP}(\rho)$ and $\mathrm{OP}(\rho)$. Let $\mathcal{P}_\rho = \{P_c\}_{c \in \operatorname{Im}\rho}$ and $\mathcal{Q}_\rho = \{\vec{Q}_c\}_{c \in \operatorname{Im}\rho}$ be closed partition systems. Then we say $\mathcal{P}_\rho \leqslant \mathcal{Q}_\rho$ if $P_c \leqslant Q_c$ for each $c \in \operatorname{Im}\rho$, and the meet $\bigwedge$ and join $\bigvee$ of partition systems as follows: $\mathcal{P}_\rho \bigwedge \mathcal{Q}_\rho = \{P_c \bigwedge Q_c\}_{c \in \operatorname{Im}\rho}$ and $\mathcal{P}_\rho \bigvee \mathcal{Q}_\rho = \{P_c \bigvee Q_c\}_{c \in \operatorname{Im}\rho}$ while it is straight forward that $\mathcal{P}_\rho \bigvee \mathcal{Q}_\rho$ and $\mathcal{P}_\rho \bigwedge \mathcal{Q}_\rho$ are both in $\mathrm{CP}(\rho)$. Similarly, for open partition systems $\vec{\mathcal{P}}_\rho = \{\vec{P}_c\}_{c \in \operatorname{Im}\rho}$ and $\vec{\mathcal{Q}}_\rho = \{\vec{Q}_c\}_{c \in \operatorname{Im}\rho}$, we have the partial order defined as $\vec{\mathcal{P}}_\rho \leqslant \vec{\mathcal{Q}}_\rho$ if $\vec{P}_c \leqslant \vec{Q}_c$ for each $c \in \operatorname{Im}\rho$, the meet defined as $\vec{\mathcal{P}}_\rho \bigwedge \vec{\mathcal{Q}}_\rho = \{\vec{P}_c \bigwedge \vec{Q}_c\}_{c \in \operatorname{Im}\rho}$ and the join defined as $\vec{\mathcal{P}}_\rho \bigvee \vec{\mathcal{Q}}_\rho = \{\vec{P}_c \bigvee \vec{Q}_c\}_{c \in \operatorname{Im}\rho}$. We then note that the maximum partition system is the coarsest partition system and the minimum partition system is the finest partition system.

The following simple lemma says the meet and join operations are closed when restricted to $\mathrm{PCP}(\rho)$ and $\mathrm{POP}(\rho)$.

Lemma 4.7.24. *Let $\mathcal{P}_\rho, \mathcal{Q}_\rho \in \mathrm{PCP}(\rho)$ and $\vec{\mathcal{P}}_\rho, \vec{\mathcal{Q}}_\rho \in \mathrm{POP}(\rho)$. Then $\mathcal{P}_\rho \bigwedge \mathcal{Q}_\rho \in \mathrm{PCP}(\rho)$, $\mathcal{P}_\rho \bigvee \mathcal{Q}_\rho \in \mathrm{PCP}(\rho)$, $\vec{\mathcal{P}}_\rho \bigwedge \vec{\mathcal{Q}}_\rho \in \mathrm{POP}(\rho)$, $\vec{\mathcal{P}}_\rho \bigvee \vec{\mathcal{Q}}_\rho \in \mathrm{POP}(\rho)$. In addition, if $\vec{\mathcal{P}}_\rho$ and $\vec{\mathcal{Q}}_\rho$ are the open partition systems corresponding to $\mathcal{P}_\rho$ and $\mathcal{Q}_\rho$ respectively, then $\vec{\mathcal{P}}_\rho \bigwedge \vec{\mathcal{Q}}_\rho$ and $\vec{\mathcal{P}}_\rho \bigvee \vec{\mathcal{Q}}_\rho$ are the open partition systems corresponding to $\mathcal{P}_\rho \bigwedge \mathcal{Q}_\rho$ and $\mathcal{P}_\rho \bigvee \mathcal{Q}_\rho$ respectively.*

Remark 4.7.25. Using the bijection provided by Theorem 4.7.23, we may associate Λ_ρ with the same partial order and lattice structure. In particular, let $\mathcal{T}_P$ and $\mathcal{T}_Q$ be the partition trees associated to $\mathcal{P}_\rho, \mathcal{Q}_\rho \in \mathrm{PCP}(\rho)$ with induced projections $\pi_{\mathcal{T}_P}$ and $\pi_{\mathcal{T}_Q}$ respectively. We say $(\mathcal{T}_P, \pi_{\mathcal{T}_P}) \leqslant (\mathcal{T}_Q, \pi_{\mathcal{T}_Q})$ or simply $\mathcal{T}_P \leqslant \mathcal{T}_Q$ if $\mathcal{P}_\rho \leqslant \mathcal{Q}_\rho$, i.e., $\mathcal{P}_\rho$ is finer than $\mathcal{Q}_\rho$. Moreover, if $\mathcal{T}_P \leqslant \mathcal{T}_Q$, there is a natural map: $\Theta^{\mathcal{T}_P}_{\mathcal{T}_Q} : \mathcal{T}_P \to \mathcal{T}_Q$ with $x \mapsto y$ such that $\iota_{\mathcal{T}_Q}(y)$ is the equivalence class dominating $\iota_{\mathcal{T}_P}(x)$. Clearly, $(\operatorname{Im}\rho, \rho)$ and $(\mathcal{B}, \pi_{\mathcal{B}})$ are the maximum and minimum of Λ_ρ respectively (as a rooted metric tree, $\operatorname{Im}\rho = [\min\rho, \max\rho]$ has its root at $\min\rho$), since they correspond to the coarsest and finest partition systems respectively.

In general, we write

$$(\mathcal{T}_P, \pi_{\mathcal{T}_P}) \bigwedge (\mathcal{T}_Q, \pi_{\mathcal{T}_Q}) = (\mathcal{T}_P \bigwedge \mathcal{T}_Q, \pi_{\mathcal{T}_P} \bigwedge \pi_{\mathcal{T}_Q})$$

and

$$(\mathcal{T}_P, \pi_{\mathcal{T}_P}) \bigvee (\mathcal{T}_Q, \pi_{\mathcal{T}_Q}) = (\mathcal{T}_P \bigvee \mathcal{T}_Q, \pi_{\mathcal{T}_P} \bigvee \pi_{\mathcal{T}_Q}),$$

where$\mathcal{T}_P \bigwedge \mathcal{T}_Q$ and $\mathcal{T}_P \bigvee \mathcal{T}_Q$ are the partition trees associated to $\mathcal{P}_\rho \bigwedge \mathcal{Q}_\rho$ and $\mathcal{P}_\rho \bigvee \mathcal{Q}_\rho$ with induced projections $\pi_{\mathcal{T}_P} \bigwedge \pi_{\mathcal{T}_Q} := \pi_{\mathcal{T}_P \bigwedge \mathcal{T}_Q}$ and $\pi_{\mathcal{T}_P} \bigwedge \pi_{\mathcal{T}_Q} := \pi_{\mathcal{T}_P \bigvee \mathcal{T}_Q}$ respectively.

$\square$

The following lemma follows straightforwardly from our definitions of partition trees and the maps between them.

Lemma 4.7.26. *If $\mathcal{T}_P \leqslant \mathcal{T}_Q \leqslant \mathcal{T}_R$ as partition trees, the following diagram commutes.*

$$\begin{tikzcd}
 & & \mathcal{T}_Q \arrow[dr, "\Theta^{\mathcal{T}_Q}_{\mathcal{T}_R}"] \arrow[drr, "d^{\rho}_{\mathcal{T}_Q}"] & & \\
\Gamma \arrow[r, "\pi_{\mathcal{T}_P}"] \arrow[urr, "\pi_{\mathcal{T}_Q}"] \arrow[rrr, bend right, "\pi_{\mathcal{T}_R}"'] \arrow[rrrr, bend right=60, "\rho"'] & \mathcal{T}_P \arrow[ur, "\Theta^{\mathcal{T}_P}_{\mathcal{T}_Q}"] \arrow[rr, "\Theta^{\mathcal{T}_P}_{\mathcal{T}_R}"] \arrow[rrr, bend right, "d^{\rho}_{\mathcal{T}_P}"'] & & \mathcal{T}_R \arrow[r, "d^{\rho}_{\mathcal{T}_R}"] & \operatorname{Im}\rho
\end{tikzcd}$$

Now we will associate another one-to-one correspondence to Λ_ρ, i.e., partition trees can be identified with tropical dominant trees.

Note that for every two points x_1 and x_2 in a partition tree $\mathcal{T}$, there is a unique segment $[x_1, x_2]_{\mathcal{T}}$ connecting x_1 and x_2. By sending a point $p \in \Gamma$ to $d^{\rho}_{\mathcal{T}}(x)$ where x is the retraction of $\pi_{\mathcal{T}}(p)$ onto $[x_1, x_2]_{\mathcal{T}}$, we can define a rational function on Γ, denoted by $f^{\rho}_{[x_1,x_2]_{\mathcal{T}}}$. With this definition, if $p \in (\pi_{\mathcal{T}})^{-1}([x_1, x_2]_{\mathcal{T}})$ then $f^{\rho}_{[x_1,x_2]_{\mathcal{T}}}(p) = \rho(p)$, otherwise for each connected components C of $\mathcal{T} \backslash [x_1, x_2]_{\mathcal{T}}$, $f^{\rho}_{[x_1,x_2]_{\mathcal{T}}}$ is a constant function restricted to $(\pi_{\mathcal{T}})^{-1}(C)$. Moreover, since $f^{\rho}_{[x_1,x_2]_{\mathcal{T}}}$ is composition of continuous functions, it is itself a continuous function and hence well-defined as a rational function on Γ. Clearly, with this construction, different segments on $\mathcal{T}$ induces rational functions.

Let $f^{\rho}_{\mathcal{T},x} := -f^{\rho}_{[r(\mathcal{T}),x]_{\mathcal{T}}}$ and D_ρ be the effective part of $\operatorname{div}(\rho)$. Then we observe that $f^{\rho}_{\mathcal{T},x_1} = f^{\rho}_{\mathcal{T},x_2}$ on $(\pi_{\mathcal{T}})^{-1}(x_1 \vee x_2)$.

Remark 4.7.27. Given a tropical dominant tree T, there is a natural map $\pi_T = \Gamma \to T$ such that π_T is continuous and for each $p \in \Gamma$, $p \in \operatorname{supp}(\pi_T(p))$. More specifically, using the language of chip firing, for a divisor $D \in T$, consider $p \in \operatorname{supp}(D)$, then $\pi_T(p) = D$ if and only if there exists at least one tangent direction t at D in T such that as D fires along t, a chip at p moves. Pick any point $r(T) \in T$ as the root of T. Let ρ_T be a function on Γ which sends $p \in \Gamma$ to the distance between $r(T)$ and $\pi_T(p)$. Then clearly ρ_T is a rational function on Γ with everywhere nonzero slopes and clearly $(T,\pi_T)\in\Lambda_{\rho_T}$. We call ρ_T the associated rational function with respect to T rooted at $r(T)$.

Theorem 4.7.28. *Elements in Λ_ρ is in one-to-one correspondence with tropical dominant subtrees of $|D_\rho|$ rooted D_ρ whose associated rational function is $\rho-\min\rho$. In addition, the correspondence on each element in Λ_ρ is an isometric embedding into Λ_ρ.*

Proof. For a partition tree $\mathcal{T}$ and every $x\in\mathcal{T}$ to $D_x:=D_\rho+\mathrm{div}(f^\rho_{\mathcal{T},x})$. Then $D_x\sim D_\rho$. By definition, $f^\rho_{\mathcal{T},x}=-\rho$ on $(\pi_{\mathcal{T}})^{-1}([r(\mathcal{T}),x]_{\mathcal{T}})$ and otherwise $f^\rho_{\mathcal{T},x}$ is componentwise constant. Therefore D_x is effective and we have $D_x\in|D_\rho|$. In particular, $D_{r(\mathcal{T})}=D_\rho$. Moreover, for any $p\in(\pi_{\mathcal{T}})^{-1}(x)$ where $x\neq r(\mathcal{T})$, $f^\rho_{\mathcal{T},x}(p)$ just reaches its minimum value since each $t\in\mathrm{Tan}^-_\Gamma(p)$ maps to the unique backward tangent direction at x by $\pi_{\mathcal{T}*}$, which means D_x must be effective at p. Therefore, by sending x to D_x, we can embed $\mathcal{T}$ into $|D_\rho|$. We will show that the image is tropically convex and the embedding is isometric. Let $T=\{D_x|x\in\mathcal{T}\}$. as a tropical dominant tree containing D_ρ.

Conversely, let T be a tropical dominant subtree of $|D_\rho|$ minimally generated by $\{D_\rho,D_1,\ldots,D_m\}$. Then $(T,\pi_T)\in\Lambda_{\rho_T}$.

□

4.7.4 Admissible and strongly admissible partition systems

Suppose $(\mathcal{D},\mathcal{H})$ is a solvable diagrammatic pre-limit g^1_d with a solution ρ and the corresponding bifurcation tree $\mathcal{B}$, where $\mathcal{D}=(D_\Gamma,\{D_p\}_{p\in\Gamma})$. Let $\Lambda^1_{\mathcal{D},\mathcal{H}}:=\Lambda_\rho$.

In general, we cannot tropically embed partition trees into $|D_\Gamma|$ as done for $|D_\rho|$ in Theorem 4.7.28, since $D_\Gamma+\mathrm{div}(f^\rho_{\mathcal{T},x})$ may not be effective. However, it is possible that some partition trees can be even embedded into $R_{\mathcal{D},\mathcal{H}}\in|D_\Gamma|$, which will be characterized in the this section.

Now we will consider the restriction of $\Lambda^1_{\mathcal{D},\mathcal{H}}$ to $R_{\mathcal{D},\mathcal{H}}$, i.e., the set of partition trees $(\mathcal{T},\pi_{\mathcal{T}}) \in \Lambda_\rho$ such that $\mathcal{T}$ tropically embeds into $R_{\mathcal{D},\mathcal{H}}$. We will show that these partition trees accounts for all possible tropical dominant subtrees of $R_{\mathcal{D},\mathcal{H}}$, and describe this set in terms of more restricted partition systems called admissible partition systems that take into account the local partitions at all points $p \in \Gamma$.

Definition 4.7.29. (**Admissible and strongly admissible partition systems**) A proper partition system with its open section $\vec{\mathcal{P}}_\rho$ is **admissible** (respectively, **strongly admissible**) if for every point $p \in \Gamma$ and each pair of tangent directions $t_1, t_2 \in \mathrm{Tan}^+_\Gamma(p)$, we have t_1 is locally equivalent to t_2 if (respectively, if and only if) $\pi_{\mathcal{B}*}(t_1)$ is equivalent to $\pi_{\mathcal{B}*}(t_2)$ in $\vec{\mathcal{P}}_\rho$.

In addition, we say a partition tree is admissible (respectively, strongly admissible) if it is associated to an admissible (respectively admissible) partition system. Denote by $\Lambda^2_{\mathcal{D},\mathcal{H}}$ the set of admissible partition trees, and by $\Lambda^3_{\mathcal{D},\mathcal{H}}$ the set of strongly admissible partition trees. Clearly $\Lambda^1_{\mathcal{D},\mathcal{H}} \supseteq \Lambda^2_{\mathcal{D},\mathcal{H}} \supseteq \Lambda^3_{\mathcal{D},\mathcal{H}}$.

Remark 4.7.30. Recall that a bifurcation partition system is determined by the partitions of forward tangent directions at finitely many points of $\mathcal{B}$, and thus there are only finitely many of them. Denote the set of bifurcation partition systems on $\mathcal{B}$ by $\mathrm{BP}(\rho)$. Then there is a natural map $\phi^{\mathrm{POP}} : \mathrm{POP}(\rho) \to \mathrm{BP}(\rho)$ such that for every $x \in \mathcal{B}$, every two forward tangent directions t_1 and t_2 in $\mathrm{Tan}^+_{\mathcal{B}}(x)$ are equivalent in $\phi^{\mathrm{POP}}(\vec{\mathcal{P}}_\rho)$ if and only if they are equivalent in $\vec{\mathcal{P}}_\rho$. In other words, the partition of $\mathrm{Tan}^+_{\mathcal{B}}(x)$ in $\phi^{\mathrm{POP}}(\vec{\mathcal{P}}_\rho)$ is exactly the partition at $d^\rho_{\mathcal{B}}(x)$ in $\vec{\mathcal{P}}_\rho$ restricted to $\mathrm{Tan}^+_{\mathcal{B}}(x)$.

We also define maps $\phi^{\mathrm{PCP}} : \mathrm{PCP}(\rho) \to \mathrm{BP}(\rho)$ and $\phi^{\Lambda} : \Lambda_\rho \to \mathrm{BP}(\rho)$ in the natural way based on the the correspondence between $\mathrm{POP}(\rho)$, $\mathrm{PCP}(\rho)$ and Λ_ρ. Then these maps are surjective for the following reasons. Consider a bifurcation partition system $\{\vec{P}_x\}_{x\in\mathcal{B}}$. For a small enough δ (precisely, we can let δ be less than the minimal distance between two bifurcation values) and each point $x \in \mathrm{Bif}(\mathcal{B})$, let $c = d^\rho_{\mathcal{B}}(x)$ and for all $d \in (c, c+\delta]$, we make $\omega^{c*,d}_{\mathcal{B}}(\mathrm{Tan}^+_{\mathcal{B}}(x))$ into an equivalence class. And by letting all the remaining equivalence classes being singletons, we derive a closed partition system $\mathcal{P}_\rho$ which is clearly proper. We call the partition tree $\mathcal{T}$ associated to $\mathcal{P}_\rho$ the δ-**glued partition tree** with respect to $\{\vec{P}_x\}_{x\in\mathcal{B}}$. Then it is easily verifiable that $\phi^{\mathrm{PCP}}(\mathcal{P}_\rho) = \phi^{\Lambda}(\mathcal{T}) = \{\vec{P}_x\}_{x\in\mathcal{B}}$.

$\square$

Denote the set of admissible bifurcation partition systems on $\mathcal{B}$ by $\mathrm{BP}^{\mathrm{a}}(\rho)$ and the set of strongly admissible bifurcation partition systems on $\mathcal{B}$ by $\mathrm{BP}^{\mathrm{sa}}(\rho)$. Then we have the following lemma which can be directly derived from the definitions.

Lemma 4.7.31. *The image of ϕ^{Λ} restricted to $\Lambda^2_{\mathcal{D},\mathcal{H}}$ is $\mathrm{BP}^{\mathrm{a}}(\rho)$ and the image of ϕ^{Λ} restricted to $\Lambda^3_{\mathcal{D},\mathcal{H}}$ is $\mathrm{BP}^{\mathrm{sa}}(\rho)$.*

Proof. Consider an admissible (respectively, strongly admissible) open partition system $\vec{\mathcal{P}}_\rho$. Then $\phi^{\mathrm{POP}}(\vec{\mathcal{P}}_\rho)$ is essentially a restriction of the partitions in $\vec{\mathcal{P}}_\rho$ to all $\mathrm{Tan}^+_{\mathcal{B}}(x)$ for all $x \in \Gamma$. Therefore, for all $p \in \Gamma$ and each pair $t_1, t_2 \in \mathrm{Tan}^+_{\Gamma}(p)$, we have $\pi_{\mathcal{B}*}(t_1), \pi_{\mathcal{B}*}(t_2) \in \mathrm{Tan}^+_{\mathcal{B}}(\pi_{\mathcal{B}}(p))$ and to say $\pi_{\mathcal{B}*}(t_1)$ and $\pi_{\mathcal{B}*}(t_2)$ are equivalent in $\vec{\mathcal{P}}_\rho$ is the same to say they are equivalent in $\phi^{\mathrm{POP}}(\vec{\mathcal{P}}_\rho)$. Therefore, by the definition of admissible (respectively, strongly admissible) partition systems, t_1 and t_2 are locally equivalent if (respectively, if and only if) $\pi_{\mathcal{B}*}(t_1)$ and $\pi_{\mathcal{B}*}(t_2)$ are equivalent in $\vec{\mathcal{P}}_\rho$ or equivalently in $\phi^{\mathrm{POP}}(\vec{\mathcal{P}}_\rho)$. Therefore, $\phi^{\mathrm{POP}}(\vec{\mathcal{P}}_\rho)$ is a admissible (respectively, strongly admissible) bifurcation partition systems on $\mathcal{B}$. $\square$

The following lemma says that $\Lambda^2_{\mathcal{D},\mathcal{H}}$ is lower closed.

Lemma 4.7.32. *If $\mathcal{T} \in \Lambda^2_{\mathcal{D},\mathcal{H}}$ is then any element $\mathcal{T}' \in \Lambda^1_{\mathcal{D},\mathcal{H}}$ with $\mathcal{T}' \leqslant \mathcal{T}$ is also in $\Lambda^2_{\mathcal{D},\mathcal{H}}$.*

Proof. Let $\vec{\mathcal{P}}_\rho$ and $\vec{\mathcal{P}'}_\rho$ be the admissible open partition systems corresponding to $\mathcal{T}$ and $\mathcal{T}'$ respectively. Then $\mathcal{T}' \leqslant \mathcal{T}$ means that $\vec{\mathcal{P}'}_\rho \leqslant \vec{\mathcal{P}}_\rho$, i.e., $\vec{\mathcal{P}'}_\rho$ is finer than $\vec{\mathcal{P}}_\rho$. Therefore, for all $p \in \Gamma$ and each pair $t_1, t_2 \in \mathrm{Tan}^+_{\Gamma}(p)$, $\pi_{\mathcal{B}*}(t_1)$ and $\pi_{\mathcal{B}*}(t_2)$ are equivalent in $\vec{\mathcal{P}'}_\rho$ implies they are equivalent in $\vec{\mathcal{P}}_\rho$ which further implies t_1 and t_2 are locally equivalent. Thus we conclude $\mathcal{T}' \in \Lambda^2_{\mathcal{D},\mathcal{H}}$. $\square$

Remark 4.7.33. If $\mathcal{T} \in \Lambda^2_{\mathcal{D},\mathcal{H}}$ is nonempty, then $\mathcal{B}$ is an element of $\Lambda^2_{\mathcal{D},\mathcal{H}}$ since $\mathcal{B}$ is the minimum of $\Lambda^1_{\mathcal{D},\mathcal{H}}$.

Lemma 4.7.34. *$\Lambda^2_{\mathcal{D},\mathcal{H}}$ and $\Lambda^3_{\mathcal{D},\mathcal{H}}$ are sublattices of $\Lambda^1_{\mathcal{D},\mathcal{H}}$.*

Proof. Just need to show $\Lambda^2_{\mathcal{D},\mathcal{H}}$ and $\Lambda^3_{\mathcal{D},\mathcal{H}}$ are closed under the meet $\bigwedge$ and join $\bigvee$ operations. We consider equivalent cases for open partition systems.

Let $\vec{\mathcal{P}}_\rho, \vec{\mathcal{Q}}_\rho \in \mathrm{POP}(\rho)$ be admissible. Then for all $p \in \Gamma$ and each pair $t_1, t_2 \in \mathrm{Tan}^+_\Gamma(p)$, we have t_1 and t_2 are locally equivalent if $\pi_{\mathcal{B}*}(t_1)$ and $\pi_{\mathcal{B}*}(t_2)$ are equivalent in $\vec{\mathcal{P}}_\rho$ or $\vec{\mathcal{Q}}_\rho$, which equally means they are equivalent in both $\vec{\mathcal{P}}_\rho \bigwedge \vec{\mathcal{Q}}_\rho$ and $\vec{\mathcal{P}}_\rho \bigvee \vec{\mathcal{Q}}_\rho$. Hence $\vec{\mathcal{P}}_\rho \bigwedge \vec{\mathcal{Q}}_\rho$ and $\vec{\mathcal{P}}_\rho \bigvee \vec{\mathcal{Q}}_\rho$ are both admissible.

Suppose now that $\vec{\mathcal{P}}_\rho$ and $\vec{\mathcal{Q}}_\rho \in \mathrm{POP}(\rho)$ are strongly admissible, then in addition we have t_1 and t_2 are locally equivalent implies $\pi_{\mathcal{B}*}(t_1)$ and $\pi_{\mathcal{B}*}(t_2)$ are equivalent in both $\vec{\mathcal{P}}_\rho$ and $\vec{\mathcal{Q}}_\rho$, while it follows they are equivalent in both $\vec{\mathcal{P}}_\rho \bigwedge \vec{\mathcal{Q}}_\rho$ and $\vec{\mathcal{P}}_\rho \bigvee \vec{\mathcal{Q}}_\rho$. Hence $\vec{\mathcal{P}}_\rho \bigwedge \vec{\mathcal{Q}}_\rho$ and $\vec{\mathcal{P}}_\rho \bigvee \vec{\mathcal{Q}}_\rho$ are both strongly admissible.

□

Theorem 4.7.35. *Elements in $\Lambda^2_{\mathcal{D},\mathcal{H}}$ is in one-to-one correspondence with tropical dominant subtrees of $R_{\mathcal{D},\mathcal{H}}$ containing D_Γ.*

Proof. First suppose $(\mathcal{D},\mathcal{H})$ is solvable with a solution ρ. For a partition tree $\mathcal{T}$ and every $x \in \mathcal{T}$ to $D_x := D_\rho + \mathrm{div}(f^\rho_{\mathcal{T},x})$. Then we show that $D_x \in R_{\mathcal{D},\mathcal{H}}$. Let $f_\Gamma = f^\rho_{\mathcal{T},x}$. For each $p \in \Gamma$, let $y = \pi_{\mathcal{T}}(p)$. Then if $y \in [r(\mathcal{T}), x)_{\mathcal{T}}$, we have

$$\mathrm{div}_p(f_\Gamma) = -\Sigma_{t \in \mathrm{Tan}^{\rho-}_\Gamma(p)} m(p,t)(\mathrm{red}_p(t)) - \Sigma_{t \in \mathrm{Tan}^{\rho+}_\Gamma(p) \bigcap (\pi_{\mathcal{T}*})^{-1}(t^+_y)} m(p,t)(\mathrm{red}_p(t))$$

where t^+_y is the forward tangent direction at y along the path from $r(\mathcal{T})$ to x. Since $\mathcal{T}$ is admissible, we see that the tangent directions in $\mathrm{Tan}^{\rho+}_\Gamma(p) \bigcap (\pi_{\mathcal{T}*})^{-1}(t^+_y)$ are locally equivalent. Therefore, there exist a rational function $f_p \in H_p$ such that $D_p + \mathrm{div}_p(f_\Gamma) + \mathrm{div}(f_p)$ is effective.

If $y = x$, we have $\mathrm{div}_p(f_\Gamma) = -\Sigma_{t \in \mathrm{Tan}^{\rho-}_\Gamma(p)} m(p,t)(\mathrm{red}_p(t))$. Let f_p be any nonconstant function in H_p. The remaining case is when $y \in \mathcal{T} \backslash [r(\mathcal{T}), x)_{\mathcal{T}}$ and $\mathrm{div}_p(f_\Gamma) = 0$.

Let f_p be a constant function. Then for all the cases, there will be a rational function $f_p \in H_p$ such that $D_p + \mathrm{div}_p(f_\Gamma) + \mathrm{div}(f_p)$ is effective. Let $\mathfrak{f}$ be $(f_\Gamma, \{f_p\}p \in \Gamma)$. Then $\mathcal{D} + \mathrm{div}((f))$ is effective, which proves $D_x \in R_{\mathcal{D},\mathcal{H}}$. Therefore, we can tropically and isometrically embed $\mathcal{T}$ into $R_{\mathcal{D},\mathcal{H}}$ exactly following the way we embed $\mathcal{T}$ into $|D_\rho|$ in the proof of Theorem 4.7.28.

Conversely, let $T \in R_{\mathcal{D},\mathcal{H}}$ be a tropical dominant tree rooted at D_Γ. Then by Remark 4.7.27, there is natural $\pi_T : \Gamma \to T$ and a function ρ_T such that $(T, \pi_T) \in \Lambda_{\pi_T}$. Therefore, T can be considered as a partition tree with respect to ρ_T. Then what we want to show is that ρ_T is a solution to the global diagram. But this follows from the slope-multiplicity principle. □

We now formulate a condition on the bifurcation map that characterizes the existence of a tropically convex burning subtree in $R_{\mathcal{D},\mathcal{H}}$.

Theorem 4.7.36. *A solvable diagrammatic pre-limit g^1_d $(\mathcal{D}, \mathcal{H})$ satisfies local-bifurcation condition (level II) if and only if $R_{\mathcal{D},\mathcal{H}}$ contain a tropical dominant subtree D_Γ.*

Proof. $(\mathcal{D}, \mathcal{H})$ satisfies local-bifurcation condition, if and only if $\mathrm{BP}^{\mathrm{a}}(\rho)$ is nonempty, if and only if $\Lambda^2_{\mathcal{D},\mathcal{H}}$ is nonempty, if and only if $R_{\mathcal{D},\mathcal{H}}$ contain a tropical dominant subtree D_Γ. □

4.8 Construction of a harmonic morphism from an diagrammatic limit g^1_d that satisfies the condition of Theorem 4.1.12

In this section, we explain the construction of a harmonic morphism of refined metrized complexes starting from an diagrammatic limit that is diagrammatic and satisfies the intrinsic global compatibility conditions. The diagrammatic limit g^1_d is diagrammatic the local diagram fits into a global diagram. Since, it is solvable the characteristic equation has a solution. Consider the bifurcation tree $\mathcal{B}$ associated to the timing function and let $\pi_{\mathcal{B}}$ be the canonical projection.

Modification: We perform two types of modification:

I. For every point u in Γ with only incoming tangent directions, by Property ii of a refined limit g^1_d (see Definition 4.1.9) we know that no point in the support of D_u is a marked point of C_u. We mark each point u in the support of the divisor D_u on C_u, glue a copy of the connected component of $\mathcal{B} \setminus \{\pi_{\mathcal{B}}(u)\}$ that contains the root of $\mathcal{B}$ and associate the multiplicity of D_u at the marked point as the expansion factor along this copy.

II. For every point $u \in \Gamma$ and for each class $\mathcal{C}$ of tangent directions consisting of all tangent directions that map to the same tangent direction via the push-forward map, since the limit g^1_d satisfies the local-bifurcation compatibility conditions, there exists a rational function $g_{\mathcal{C}} \in H_u$ that has zeroes at each tangent direction in $\mathcal{C}$. Mark the zeroes of $g_{\mathcal{C}}$ that are not already marked points of C_u. For each additionally marked zero, we glue in a copy of u of the connected component $\mathcal{B} \setminus \{\pi_{\mathcal{B}}(u)\}$ that does not contain the root. We associate multiplicity of $g_{\mathcal{C}}$ as the expansion factor along this branch.

Further branching: To extend the harmonic morphism of metrized complexes to metrized complexes to the modified metrized complex, it is important to further "split" the modification as follows: for each branch of $\mathcal{B}$ added in the modification, suppose that the multiplicity along $\mathcal{B}$ is d and q is the point of valence three or more that is closest to the root of the branch.

Note that the orientation of the bifurcation tree $\mathcal{B}$ induces an orientation on the newly added branches of Γ^{mod}. According to this orientation, there is one incoming tangent directions and d outgoing tangent directions at q. There are $(d-1)$ outgoing subranches rooted at q, replicate each of these branches by m-branches. We denote this metrized complex by $\mathfrak{X}^{\mathrm{mod}}$. Mark $m \cdot d$ generic points on C_q corresponding to the tangent directions.

Construction of the Harmonic Morphism of Tropical Curves: Extend $\pi_{\mathcal{B}}$

to a map ϕ such that a point in the newly added branches map to the corresponding point in $\mathcal{B}$ and with the prescribed expansion factor.

Promoting the Bifurcation tree to a Genus Zero Metrized Complex: For each $u \in \Gamma^{\text{mod}}$, we take C_u to be a rational curve for each $u \in \mathcal{B}$. We mark points on the curves we plugged in as follows: for each tangent direction t in Γ^{mod} mark the image of t under the pushforward $\pi^*_{\mathcal{B}}$. We denote the resulting genus zero metrized complex by $\mathfrak{T}$.

Construction of the Harmonic Morphism of Metrized Complexes:. We promote ϕ to a harmonic morphism of metrized complexes between $\mathfrak{X}^{\text{mod}}$ and $\mathfrak{T}$ as follows: we must additionally specify rational functions on the curves C_p for each point p of the metric graph. For every point $p \in \Gamma$, we take f_p as any non-constant element in H_p. For each branch B in $\Gamma^{\text{mod}} \setminus \Gamma$ with root $u \in \Gamma$ and suppose v is the point in $B \setminus \{q\}$ that is closest to the root u. For any point that is closer to the root u than v take f_v is a rational function of degree m with a pole of order m at the marked point corresponding to the incoming tangent direction and a zero of order m at the marked point corresponding to the outgoing tangent direction. Consider the rational function $f_v = P/Q$ where P is a generic polynomial of degree m and Q is a polynomial of degree m with a zero of order m at the marked point corresponding to incoming tangent direction on the rational curve C_q, where m is the multiplicity of the incoming tangent direction.

Lemma 4.8.1. *The map $\phi : \Gamma^{\text{mod}} \to \mathcal{B}$ is a harmonic morphism of tropical curves. The collection $(\phi, \{f_p\}_{p\in\Gamma})$ is a harmonic morphism of metrized complexes of degree $\deg(D_\Gamma)$.*

Proof. The map ϕ is a harmonic morphism between Γ^{mod} and the bifurcation tree since the first type of modification ensures that the map has degree $\deg(D_\Gamma)$ at the root of $\mathcal{B}$ and the second type modification ensures the harmonicity condition is satisfied at every point.

We show that the three compatibility conditions for a harmonic morphism of metrized complexes are satisfied:

Condition i: By the construction of the genus zero metrized complex $\mathfrak{T}$ enriching $\mathcal{B}$, we note that the marked points in the metrized complex $\mathfrak{X}^{\mathrm{mod}}$ are mapped to marked point in $\mathfrak{B}$.

Condition ii: By the construction of the genus zero metrized complex $\mathcal{B}$ enriching $\mathfrak{B}$, we know that for every point u, the function f_u takes a marked point to another marked point if and only if the corresponding tangent direction are mapped to a tangent direction by the harmonic morphism ϕ. Since the limit g^1_d satisfies the intrinsic global compatibility conditions, we know that for every pair of points $u,\ v \in \Gamma^{\mathrm{mod}}$ such that $\phi(u) = \phi(v)$, we know that the tangent directions $t_1,\ t_2 \in \mathrm{Tan}(p)$ are mapped to the same tangent direction by $\pi_{\mathcal{B}}$ if and only if $f_u(t_1) = f_v(t_2)$.

Condition iii: We must show that for every $p \in \Gamma_{\mathrm{mod}}$ and for every tangent direction $t \in \mathrm{Tan}(p)$ the expansion factor of ϕ is equal to ramification index of f_p at $\mathrm{Tan}(p)$. This is a consequence of the construction of the modification Γ^{mod}.

By construction, $(\phi, \{f_p\}_{p\in\Gamma})$ is a harmonic morphism of degree $\deg(D_\Gamma)$.

$\square$

4.9 Proof of the smoothing theorem.

This remaining section is sketchy. For more detailed explanations, readers should refer to the upcoming paper of me and Madhusudan Manjunath.

Suppose that the diagrammatic pre-limit g^1_d $(\mathcal{D}, \mathcal{H})$ is solvable and satisfies the intrinsic global compatibility conditions. Then we can construct a harmonic morphism from a modification of the metrized complex $\mathfrak{X}^{\mathrm{mod}}$ to a genus zero metrized complex with a δ-glued partition tree as the underlying metric tree, which implies $(\mathcal{D}, \mathcal{H})$ is smoothable by Theorem 4.1.12.

Conversely, suppose that the diagrammatic pre-limit g^1_d is smoothable and thus by Theorem 4.1.12 gives a harmonic morphism of metrized complexes between a modification $\mathfrak{X}^{\mathrm{mod}}$ and a genus zero metrized complex $\mathcal{T}$. Then we can show that the metric tree underlying this metrized complex is contained in $R_{\mathcal{D},\mathcal{H}}$. Using the classification of elements in $\Lambda^2_{\mathcal{D},\mathcal{H}}$, we conclude that this tree is dominated by the bifurcation tree and the local-bifurcation conditions (intrinsic global compatibility conditions resp.) with respect to the bifurcation tree $\mathcal{B}$ are satisfied.

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