LARGE SCALE GROUP NETWORK OPTIMIZATION

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PREFACE

Most large scale optimization problems involve either column generation or constraint generation. An early example of column generation was the cutting stock problem by Gilmore and Gomory [10, 11] in which each column is generated by solving a knapsack problem. They observed a cyclic repetition of the values of nonbasic variables for the knapsack problem. The periodicity initiated the cyclic group problem and led Gomory [12] to the group problem for integer programming.

Every knapsack problem may be relaxed to a cyclic group problem. The cyclic group relaxation is much smaller than the original knapsack problem, and is polynomial if the coefficients remain fixed while the right-hand side increases. But, it is not polynomial if the coefficients are allowed to increase. The cyclic group subproblem is equivalent to a shortest path problem in the corresponding circulant digraph. If the circulant digraph has a large number of nodes and a small diameter in its degree, the dynamic programming algorithm given by Gomory [12] will perform poorly as we will see on Wong-Coppersmith digraphs in this thesis. We will develop a cutting plane algorithm based on shooting and empirically show that the cutting plane algorithm solves the shortest path problem on Wong-Coppersmith digraphs with only small number of cutting planes generated by shooting. Incorporating shooting into the subproblem of such a cutting plane algorithm improves the speed of the cyclic group relaxation.

Group network optimization in the title represents the shortest path problem in the Cayley digraph. It has been my topic since the research [29] in Slovakia with Jozef Širáň on interconnection network in Cayley graphs. We hope to show that the proposed methodology, that exploits the symmetric property of groups, will exhibit greater performance relative to Dijkstra's algorithm with respect to Cayley diagraphs.

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SUMMARY

Every knapsack problem may be relaxed to a cyclic group problem. Gomory [12] found the subadditive characterization of facets of the master cyclic group problem. We simplify the subadditive relations by the substitution of complementarities and discover a minimal representation of the subadditive polytope for the master cyclic group problem. By using the minimal representation, we characterize the vertices of cardinality length 3 and implement the shooting experiment from the natural interior point.

The shooting from the natural interior point is a shooting from the inside of the plus level set of the subadditive polytope. It induces the shooting for the knapsack problem. From the shooting experiment for the knapsack problem we conclude that the most hit facet is the knapsack mixed integer cut which is the 2-fold lifting of a mixed integer cut.

We develop a cutting plane algorithm augmenting cutting planes generated by shooting, and implement it on Wong-Coppersmith digraphs observing that only small number of cutting planes are enough to produce the optimal solution. We discuss a relaxation of shooting as a clue to quick shooting. A max flow model on covering space is shown to be equivalent to the dual of shooting linear programming problem.

CHAPTER I

INTRODUCTION

This thesis is organized into six chapters. This chapter presents a summary, literature and preliminaries related to the cyclic group problem. In the end of Section 1.1, we discuss computational difficulty of a procedure of solving the cyclic group problem using the example of Wong-Coppersmith digraphs [30]. Then, we see in Section 1.2 how we may relax the knapsack problem to the cyclic group problem, and conclude this chapter with a detailed summary of the thesis.

Chapter 2 focuses on the master knapsack problem $K(n)$ and its relation to the master cyclic group problem (C_{n+1}, n). Aráoz [1] defined the subadditive cone for the master knapsack problem by the complementarities and the knapsack subadditivities and showed that its minimal set of defining rays gives the nontrivial facets of the knapsack polytope. By substitution of the complementarities, we derive a minimal representation of the subadditive cone in Section 2.1. In Section 2.2, we explore the relation between the master knapsack problem $K(n)$ and the master cyclic group problem (C_{n+1}, n), and review the geometry of these two problems in Aráoz et. al [2].

Gomory [12] defined the *subadditive polytope* $\Pi(C_n, b)$ in which the extreme points are the nontrivial facets of the master cyclic group polyhedron $P(C_n, b)$. Substituting out a special choice of a variable from the two variables in each complementarity leads to a minimal representation of $\Pi(C_n, b)$ as shown in Chapter 3. We also derive a minimal representation of the subadditive polytope for the master binary group problem by the substitution of the complementarities in Section 3.3.3. In Chapter 4, the minimal representation of the subadditive polytope $\Pi(C_n, b)$ given in Chapter 3 is applied to find the vertices of cardinality length 3 of $P(C_n, b)$. We develop shooting from the natural interior point in Section 4.2 and implement the shooting experiment for the knapsack problem in Section 4.3 observing that the most hit facet is the knapsack mixed integer cut.

In Chapter 5, we introduce primal and fractional cutting plane algorithms for the cyclic group problem, which implement a scheme that generates a sequence of constraints by shooting. The development of cutting plane algorithms is motivated by Wong-Coppersmith digraphs on which Dijkstra's algorithm performs poorly. We implement the fractional cutting plane algorithm on Wong-Coppersmith digraphs and see that only small number of cutting planes are enough to produce the optimal solution. But, shooting takes much time generating a facet of the master cyclic group problem. In Chapter 6, the complementary relaxation of shooting is shown to be polynomial and the dual of shooting is shown to be transformed to a max flow problem on covering space.

1.1 The cyclic group and the circulant digraph

If the difference between two numbers a and b is an integer multiple of n, they are called *congruent modulo* n and denoted by

$$a \equiv b \mod n.$$

The congruence modulo n partitions the integers $\mathbb{Z}$ into the n classes each of which contains the integers with the same remainder when they are divided by n. The set of n classes is denoted by $C_n = \{[0], [1], ..., [n-1]\}$, where the remainders $0, ..., n-1$ are the natural representatives of the classes. We will just denote $C_n = \{0, 1, ..., n-1\}$. The usual addition on the integers is well-defined on C_n. With the addition, C_n is called the *cyclic group* of *order* n.

Cyclic groups show cyclic repetition like a clock. For example, the cyclic group of order 3 is the set $C_3 = \{0, 1, 2\}$ with addition modulo 3 such as

$$1 + 1 \equiv 2, 1 + 2 \equiv 3 \equiv 0, 2 + 2 \equiv 4 \equiv 1 \mod 3.$$

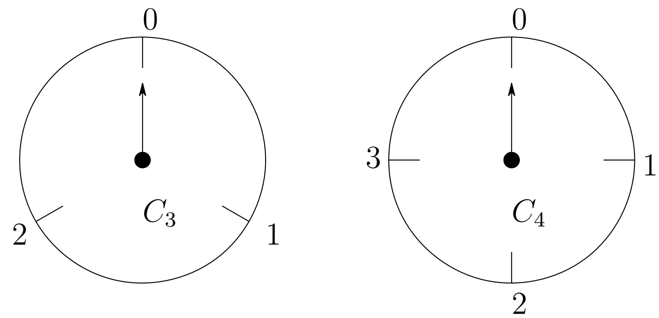

Figure 1: C_3 and C_4

In C_4, $3 + 3 \equiv 6 \equiv 2 \mod 4$. (See Figure 1)

For a cyclic group C_n and a subset $M \subseteq C_n - \{0\}$, we denote $Cay(C_n, M)$ as the *circulant digraph*[1] of order n generated by M to be the digraph with node set C_n and the arcs $(i, i + k)$ for all $i \in C_n$ and $k \in M$. The arc $(i, i + k)$ is said to be *labeled by* k (the group element k will be called the *label* of the arc.) If $M = C_n - \{0\}$, we call $Cay(C_n, M)$ a *complete* circulant digraph and denote it by $Cay(C_n)$ leaving off M.

The *cyclic group problem* (C_n, M, b) of C_n generated by $M \subseteq C_n - \{0\}$ with nonzero right-hand side b and nonnegative objective function c_g for $g \in M$ is

$$\begin{aligned} \min \quad & \sum_{g \in M} c_g t_g \\ st \quad & \sum_{g \in M} t_g g \equiv b \mod n \\ & t = (t_g : g \in M) \geq 0 \text{ integer.} \end{aligned}$$

The constraint means t_g arcs labeled by each $g \in M$ are used to get the right-hand side b. The used arcs form a path from node 0 to b in the circulant digraph $Cay(C_n, M)$. The cyclic group problem (C_n, M, b) is the shortest path problem from 0 to b in $Cay(C_n, M)$ with (arc) length c (*i.e.*, the arcs labeled by g are of length c_g for $g \in M$.) The convex hull of solutions to (C_n, M, b) is a polyhedron that is called the *cyclic group polyhedron* $P(C_n, M, b)$. It is known [12] to be a full dimensional

[1]For general groups, they are called *Cayley digraphs*. Arthur Cayley (1821-1895) introduced the drawing in groups in 1878. (For more history, see [15].)

polyhedron containing all nonnegativities $t_g \geq 0$ for $g \in M$ as the *trivial* facets; i.e., the facets with 0 right-hand side.

See the example of $Cay(C_{12}, M = \{1, 2, 4\})$ with length $c = \mathbf{1}$ in Figure 2. From each node, three arcs labeled by $M = \{1, 2, 4\}$ are going out. The cyclic group problem $(C_{12}, M = \{1, 2, 4\}, b = 11)$ with objective function $c = \mathbf{1}$ can be solved by the thick-lined shortest path $1+2+4+4 \equiv 11 \mod 12$ in Figure 2. The corresponding optimal solution is $t = (t_1 = t_2 = 1, t_4 = 2)$.

Graph algorithms such as Dijkstra's may perform poorly on the cyclic group problem (C_n, M, b) when n is much larger than $|M|$ and the *cardinality length* $\sum_{i \in M} t_i$ of a solution t that is the cardinality length of a path corresponding to t in $Cay(C_n, M)$. For example, $Cay(C_n, M)$ with $n = 2^m$ and $M = \{2^i : i = 0, 1, ..., m-1\}$ introduced by Wong and Coppersmith [30] has diameter m and degree m. To get the shortest path to node $b = 2^m - 1$, Dijkstra's algorithm (breadth-first search here) enumerates all the exponentially many nodes. See the expression

$$2^m - 1 = \sum_{i=0}^{m-1} 2^i.$$

This is the cyclic group subproblem $(C_n, M, b = n-1)$ with cost $\mathbf{1}$,

$$\begin{array}{ll} \min & \sum_{i=0}^{m-1} x_i \\ st & \sum_{i=0}^{m-1} 2^i x_i \equiv b \mod n \\ & x \geq 0 \text{ integer.} \end{array}$$

The group problem (G, M, b) is on a finite abelian group G. The non-abelian group problem is introduced in Aráoz and Johnson [3]. The non-abelian group problem with objective function $\mathbf{1}$ is called a *minimal word problem* in groups and is itself important in group theory. (See [15].) We will discuss more about non-abelian group problems in Section 3.3.2.

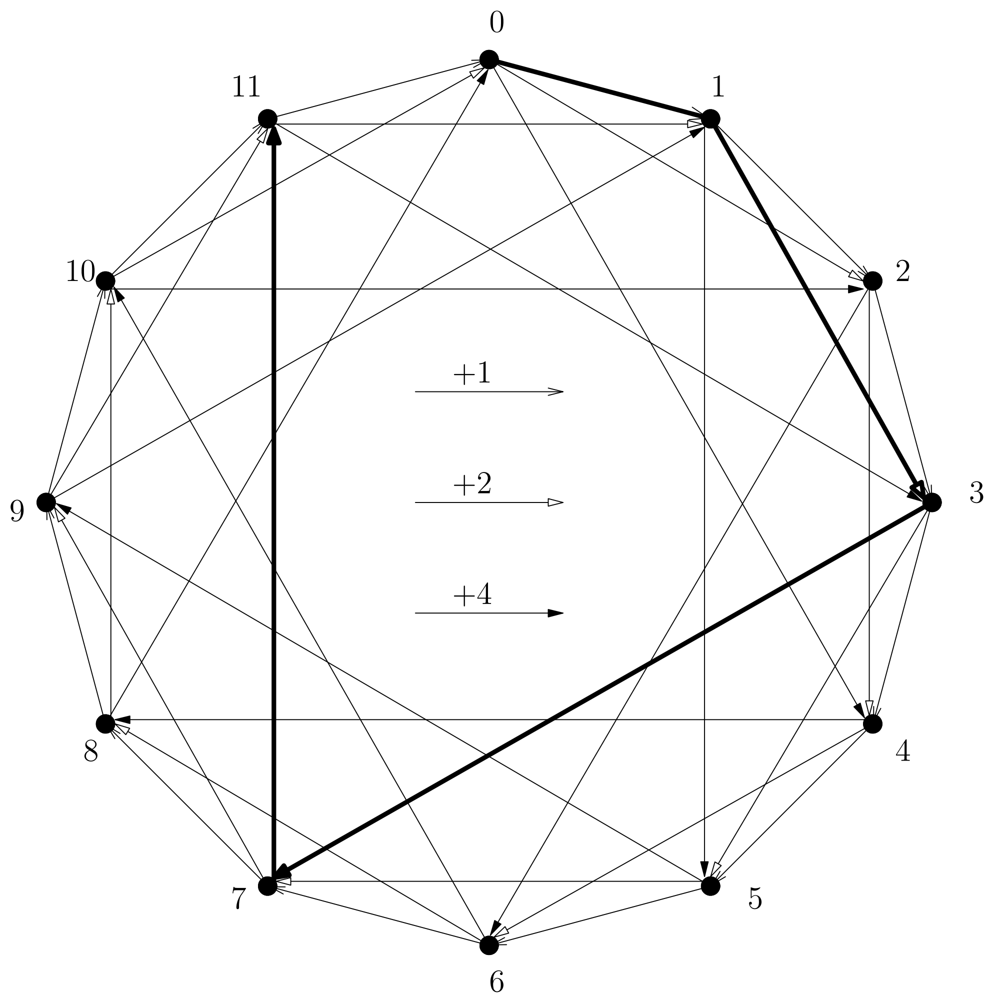

Figure 2: $Cay(C_{12}, M = \{1, 2, 4\})$

1.2 The knapsack problem

A *(packing) knapsack problem* is an integer programming with only one constraint in the form

$$\max\left\{\sum_{i=1}^{m} r_i x_i : \sum_{i=1}^{m} l_i x_i \leq l_0, x \geq 0 \text{ integer}\right\}.$$

We assume that l_0 and all r_i and l_i for $i = 1, ..., m$ are positive integers and $l_0 \geq l_i$ for all $i = 1, ..., m$. Adding a slack variable, we can also assume that the knapsack problems are *equality* knapsack problems

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} r_i x_i, \\ st \quad & \sum_{i=1}^{m} l_i x_i + s = l_0, \\ & x, s \geq 0 \text{ integer.} \end{aligned} \tag{1}$$

Knapsack problems may be solved by dynamic programming or recursion on the integer l_0. The next section introduces an enumeration algorithm for knapsack problems which also helps us understand the cyclic group problems relaxed for the knapsack problems.

Assuming

$$\frac{r_1}{l_1} \geq \frac{r_i}{l_i}, \text{ for all } i = 2, ..., m,$$

the linear programming relaxation yields the following optimal table

$$\begin{aligned} \min \quad & \sum_{i=2}^{m} c_i x_i + c_s s = z^{LP} - z(x_2, ..., x_m, s) = d(x_2, ..., x_m, s), \\ st \quad & x_1 = x_1(x_2, ..., x_m, s) = \frac{l_0}{l_1} - \sum_{i=2}^{m} \frac{l_i}{l_1} x_i - \frac{1}{l_1} s, \\ & x_1 \geq 0 \text{ integer}, \\ & x_2, ..., x_m, s \geq 0 \text{ integer}, \end{aligned} \tag{2}$$

where the objective function coefficients are the reduced costs, $c_i = r_1 l_i / l_1 - r_i \geq 0, i = 2, ..., m$, and $c_s = r_1/l_1 \geq 0$. The original knapsack problem is equivalent to minimization of the gap $d(x_2, ..., x_m, s)$ between the LP optimal value z^{LP} and the objective value $z(x_2, ..., x_m, s)$ subject to the basic variable x_1 being a nonnegative integer. Note that the basic (dependent) variable x_1 and the objective value z are functions of the nonnegative integral nonbasic (independent) variables $x_2, ..., x_m, s$.

Ironically, a knapsack problem becomes much easier due to a cyclic repetition whenever its right-hand side l_0 is large enough.

Theorem 1.1 (Gilmore-Gomory [11]) *If l_0 is large enough ($l_0 \geq l_1 \times \max\{l_2, ..., l_m, 1\}$ will suffice), then the optimum solution to the knapsack problem for $l_0' = l_0 + l_1$ is obtained from that for l_0 by changing only x_1 to $x_1 + 1$, and the objective value z increases by l_1. Thus, the solutions repeat with period l_1 and there are only l_1 different values of $x_2, ..., x_m, s$ needed in any optimum solution, once l_0 is large enough.*

Assuming l_0 is large enough, we drop the nonnegativity of the optimal basis x_1 in (2), or get the difference x_1 of the left-hand and the right-hand sides of the following relation integral,

$$\sum_{i=2}^{m} \frac{l_i}{l_1} x_i + \frac{s}{l_1} \equiv \frac{l_0}{l_1} \mod 1.$$

Multiply through by the common divisor l_1 and induce the following problem equivalent to (2) whenever l_0 is large enough,

$$\begin{array}{ll} \min & \sum_{i=2}^{m} c_i x_i + c_s s, \\ st & \sum_{i=2}^{m} l_i x_i + s \equiv l_0 \mod l_1, \\ & x_2, ..., x_m, s \geq 0 \text{ integers.} \end{array} \tag{3}$$

Set $n = l_1$, $b \equiv l_0 \mod n$, and $M = \{l_2', ..., l_m', 1\} \subseteq C_n$, where $l_i' \equiv l_i \mod n$ for $i = 2, ..., m$. We solve the cyclic group problem (C_n, M, b) or the shortest path problem from node 0 to node b in $Cay(C_n, M)$ with length c, and obtain an optimal solution to the knapsack problem by augmenting the basic variable x_1 that is decided by the nonbasic variables. We may assume that every pair of elements in $M = \{l_2', ..., l_m', 1\}$

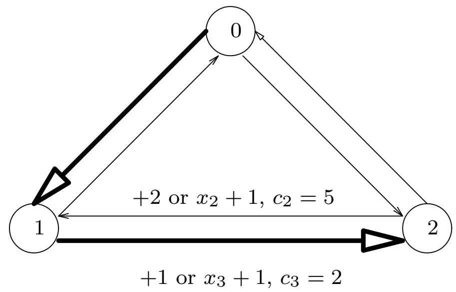

Figure 3: A shortest path from 0 to $b = 2$ on $Cay(C_3, M = \{1, 2\})$.

are different in C_n, since we do not need longer parallel arcs for a shortest path and so we can eliminate them.

Example 1.2 Consider the knapsack problem with a large right-hand side $l_0 = 101$.

$$\begin{aligned} \max \quad & 9x_1 + x_2 + 19x_3 \\ st \quad & 3x_1 + 2x_2 + 7x_3 + s = 101 \\ & x, s \geq 0 \text{ integer.} \end{aligned}$$

With the optimal basis x_1, we have the following equivalent problem

$$\begin{aligned} \min \quad & 5x_2 + 2x_3 + 3s = 24 - z(x_2, x_3, s) = d(x_2, x_3, s), \\ st \quad & x_1 = \frac{101}{3} - \left(\frac{2}{3}x_2 + \frac{7}{3}x_3 + \frac{1}{3}s\right) \geq 0 \text{ integer,} \\ & (x_2, x_3, s) \geq 0 \text{ integer.} \end{aligned}$$

It leads to the cyclic group problem

$$\begin{aligned} \min \quad & 5x_2 + 2x_3 + 3s \\ st \quad & 2x_2 + x_3 + s \equiv 2x_2 + 7x_3 + s \equiv 101 \equiv 2 \mod 3, \\ & x_2, x_3, s \geq 0 \text{ integers.} \end{aligned}$$

The coefficients of x_3, s are the same in C_3. Since the length $c_3 = 2$ of x_3 is smaller than the length $c_2 = 3$ of s, no shortest path contains an arc labeled by s. In Figure 3,

l_0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
z	0	1	9	9	10	18	19	19	27	28	28	36	37	38	45	46	47	54	55	56
x_1	0	0	1	0	1	2	0	0	3	1	1	4	2	0	5	3	1	6	4	2
x_2	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_3	0	0	0	0	0	0	1	1	0	1	1	0	1	2	0	1	2	0	1	2
s	1	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0

Table 1: Solutions to Example 1.2 with right-hand side l_0.

the outer arcs and the inner arcs denote adding 1 and 2 modulo 3 respectively and the thick lined shortest path from 0 to $b = 2$,

$$x_3.1 = 2.1 = 1 + 1 \equiv 2 = b \mod 3,$$

gives an optimal solution $(x_1 = 29, x_2 = 0, x_3 = 2, s = 0)$ to the knapsack problem with gap

$$d = 4 = z^{LP} - z = 303 - 299$$

which is the length of the shortest path. Note that the small right-hand side $l_0 = 8$ induces the same cyclic group problem because of the same right-hand side

$$b = 2 \equiv 8 \equiv 101 \mod 3,$$

which implies the infeasible solution $(x_1 = -2, x_2 = 0, x_3 = 2, s = 0)$ pruned in Example 1.3 of Section 1.3. See Table 1 and note the triples $(x_2, x_3, s)^T = (0,0,0)^T, (0,1,0)^T, (0,2,0)^T$ repeating from $l_0 = 12$ which is better than a sufficient condition given in Theorem 1.1.

1.3 Enumeration algorithm

If the knapsack problem (1) has a small right-hand side l_0, we may enumerate the solutions to the linear programming relaxation of the knapsack problem. Let $N = \{2, ..., m, s\}$ be the set of nonbasic variables and let e_j be the vector of all components 0 except the j-component equal to 1. To solve the minimization problem (2), we can think of an enumeration algorithm as follows:

0. Initialize $V = W = \emptyset \subseteq \mathbb{R}^N_+$ and set $d(\mathbf{0}) = 0, d(y) = +\infty$ for all $y \in \mathbb{R}^N_+ \setminus \{\mathbf{0}\}$.

1. Choose $x = (x_2, ..., x_m, s) \in \mathbb{R}^N_+ - V - W$ such that

$$d(x) = \min\{d(y) : y \in \mathbb{R}^N_+ - V - W\}.$$

2. If $x_1(x)$ is a nonnegative integer, return the optimal solution x and terminate.
3. If x_1 is strictly negative, then $W \leftarrow W \cup \{x\}$, which is called *pruning* x.
4. Otherwise, extend $V \leftarrow V \cup \{x\}$, update the neighbors as

$$d(x+e_j) = \min\{d(x+e_j), d(x)+c_j\} \text{ for all } x+e_j \in \mathbb{R}^N_+ - V - W, j = 2, ..., m, s$$

and go to Step 1.

The enumeration algorithm is a variant of Dijkstra algorithm and works for general IP without Step 3 (pruning condition) because of possible negative coefficients in the constraints. We observe that a smaller right-hand side l_0 enhances tractability of the algorithm by the function defining the basic variable x_1 in the problem (2). However, Gilmore and Gomory [11] showed that a knapsack problem becomes much easier whenever its right-hand side l_0 is large enough as shown in Theorem 1.1. The enumeration algorithm solves a shortest path problem to a node having a nonnegative integral x_1 on the Cayley digraph $Cay(\mathbb{Z}^N, \{e_j : j \in N\})$, which is a generalization of the cyclic group problem (C_n, M, b).

Example 1.3 Consider the knapsack problem with a small right-hand side $l_0 = 8$.

$$\begin{aligned} \max \quad & 9x_1 + x_2 + 19x_3 \\ st \quad & 3x_1 + 2x_2 + 7x_3 + s = 8 \\ & x, s \geq 0 \text{ integer.} \end{aligned}$$

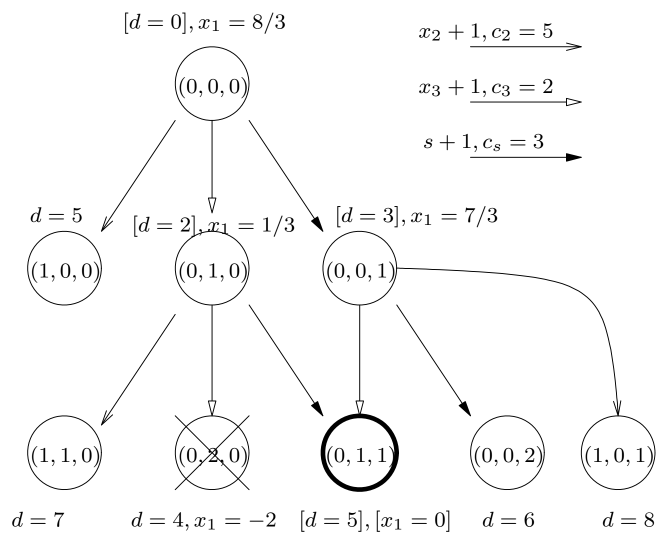

Figure 4: Enumeration of solutions

With the optimal basis x_1, we have the following equivalent problem

$$\begin{aligned}
\min \quad & 5x_2 + 2x_3 + 3s = 24 - z(x_2, x_3, s) = d(x_2, x_3, s), \\
st \quad & x_1 = \frac{8}{3} - \left(\frac{2}{3}x_2 + \frac{7}{3}x_3 + \frac{1}{3}s\right) \geq 0 \text{ integer}, \\
& (x_2, x_3, s) \geq 0 \text{ integer}.
\end{aligned}$$

See Figure 4. From each node marked by a triple (x_2, x_3, s), three arcs are going out when it enters S. One is corresponding to the increase of x_2 by 1, denoted by x_2++ with length $c_2 = 5$. The others are of lengths $c_3 = 2, c_s = 3$ and corresponding to the increases of x_3 and s by 1, respectively. The inspected nodes are entered into $S = \{(0,0,0), (0,1,0), (0,0,1), (0,1,1)\}$ in the order and have the real distance d from the root $(0,0,0)$. The pruned node $(0,2,0) \in W$ is marked by a cross. The thick circled node $(x_2 = 0, x_3 = 1, s = 1)$ gives an optimal solution $(x_1 = 0, x_2 = 0, x_3 = 1, s = 1)$ to the knapsack problem with the smallest gap $d(0,1,1) = 5$ from the LP optimal.

1.4 Overview

The thesis consists of one main theorem about a minimal representation of the subadditive polytope of the master cyclic group problem and three applications, the vertices of cardinality length 3, shooting experiment for the knapsack problem and integer primal simplex method for cyclic group problems. The integer simplex method is tested on Wong-Coppersmith digraphs compared with the dynamic programming (Dijkstra's algorithm) suggested by Gomory [12].

The *master cyclic group polyhedron* $P(C_n, b)$ is the convex hull of solutions to the *master cyclic group problem* (C_n, b),

$$\begin{array}{rl} \min & c_i t_i \\ st & \displaystyle\sum_{i \in C_n - \{0\}} t_i i \equiv b \neq 0 \mod n \\ & t \geq 0 \text{ integer}, \end{array}$$

where the objective c is assumed to be nonnegative. Gomory [12] showed that the nontrivial facets of $P(C_n, b)$ are the lower bounded inequalities

$$\pi t \geq \pi_b = 1$$

corresponding to the extreme points π of the *subadditive polytope* $\Pi(C_n, b)$ defined by the *complementarities* and the *subadditivities* given below,

$$\begin{aligned} \Pi(C_n, b) \quad = \quad & \Big\{ \pi \in \mathbb{R}_+^{C_n - \{0\}} : \\ & \pi_i + \pi_j = \pi_b = 1 \text{ if } i + j \equiv b \mod n, \\ & \pi_i + \pi_j \geq \pi_k \text{ if } i + j \equiv k \mod n \Big\}, \end{aligned}$$

where none of i, j, k is b. The nonnegativities of $\Pi(C_n, b)$ are all redundant for $n \geq 5$ except $(C_6, 2)$ and $(C_6, 4)$.

The main result of thesis is a minimal representation of the subadditive polytope $\Pi(C_n, b)$ for every master cyclic group problem (C_n, b). Firstly, we partition the nonzero cyclic group elements into two sets

$$O = \{i : \pi_i^{mic} < 1/2\} \text{ and } X = \{j : \pi_j^{mic} > 1/2\}$$

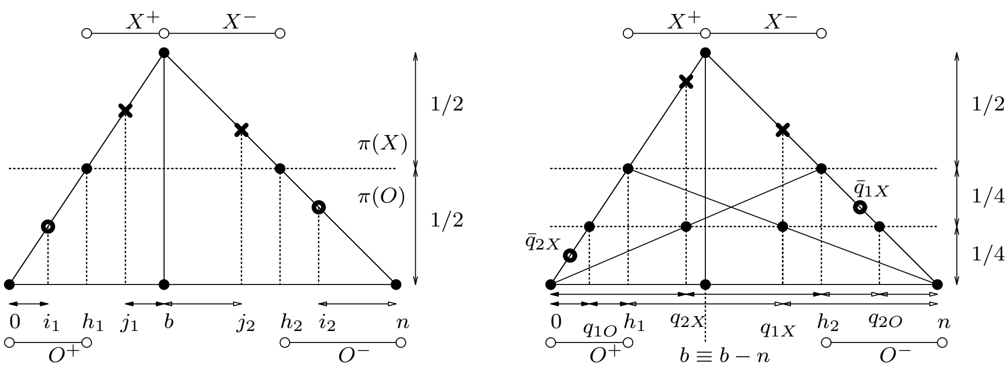

Figure 5: A partition of O and X given by π^{mic}.

except the *halves* h with $2h \equiv b \mod n$ by using the *mixed integer cut*

$$\pi^{mic} = \left(\pi_i^{mic} = i/b \text{ for } i \leq b; \pi_i^{mic} = (n-i)/(n-b) \text{ for } i \geq b\right).$$

The elements in O are chosen far from the right-hand side b so that b cannot be reached by the sum of any pair of elements in O (see Figure 5.) Each complementarity contains one index in O and the other in X. Then, we substitute out the variables π_j for $j \in X$ and have the projected image $\Pi(C_n, b)_O$ of the subadditive polytope $\Pi(C_n, b)$ onto $\mathbb{R}^O$. The *projected* subadditive polytope $\Pi(C_n, b)_O$ is equivalent to the subadditive polytope $\Pi(C_n, b)$ with one-to-one correspondence by the restriction π_O to O of $\pi \in \Pi(C_n, b)$, and full dimensional in $\mathbb{R}^O$ verified by the restriction $\dot{\pi}_O$ of the *natural interior point* $\dot{\pi}$ of $\Pi(C_n, b)$,

$$\dot{\pi} = (\dot{\pi}_i = 1/2 \text{ for } i \neq b; \dot{\pi}_b = 1),$$

satisfying all the subadditivities as strict inequalities.

By the substitution of complementarities, the subadditivities containing no variable π_h with $2h \equiv b \mod n$ become the four types of inequalities,

$$\begin{aligned}
\pi_i + \pi_j &\geq \pi_k && \text{whenever } i + j \equiv k \mod n,\\
\pi_i + \pi_j + \pi_k &\geq 1 && \text{whenever } i + j + k \equiv b \mod n,\\
\pi_i + \pi_j &\leq \pi_k + 1 && \text{whenever } i + j \equiv k + b \mod n,\\
\pi_i + \pi_j + \pi_k &\leq 2 && \text{whenever } i + j + k \equiv 2b \mod n,
\end{aligned}$$

where all i, j, k belong to O. The first and the second types are lower bound types. The third and the fourth types are upper bound types adding $\pi_b = 1$ to the right-hand sides of the first and the second types. The *quarters* are defined (see Fifure 5) as

$$q_{1O} = \frac{b}{4}, q_{1X} = \frac{b+2n}{4}, q_{2O} = \frac{b+3n}{4} \text{ and } q_{2X} = \frac{b+n}{4}.$$

By using the quarters, the *additional* inequalities are defined as

$$\pi_{q_{jO}} \geq 1/4 \text{ whenever } q_{jO} \text{ is an integer,}$$
$$\pi_{\bar{q}_{jX}} \leq 3/4 \text{ whenever } q_{jX} \text{ is an integer,}$$

where $j = 1, 2$. The subadditivities containing a variable π_h for a half h become equivalent to the additional inequalities.

The first type of inequality above may contain the *exceptional* inequality

$$\pi_i + \pi_j \geq \pi_k \text{ with } i = b/3, j \equiv (b-n)/3 \text{ and } k \equiv i + j \mod n,$$

where i, j, k are all in O. Fortunately, that is the only exception. The main theorem of thesis states that the four types of inequalities and the additional inequalities are the facets of $\Pi(C_n, b)_O$ except the exceptional inequality. A *certificate* $\breve{\pi}$ for a constraint of $\Pi(C_n, b)_O$ to be irredundant is a solution which satisfies the constraint as equality and the other constraints as strict inequalities. It verifies that the corresponding constraint is a facet. For convenience, we take an infeasible solution $\hat{\pi}$ that is infeasible only for the constraint. The intersection of the line segment between $\hat{\pi}$ and the natural interior point $\dot{\pi}$ with the hyperplane of the constraint is a certificate $\breve{\pi}$ for the constraint.

The first application is characterizing the vertices t of the master cyclic group polyhedron of *cardinality length* 3, $\sum_{i \in C_n - \{0\}} t_i = 3$. The subadditive polytope $\Pi(C_n, b)$ is the convex hull of the vertices of the blocking polyhedron $\mathfrak{B}(P(C_n, b))$ in Fulkerson's framework of blocking pairs of polyhedra. That is,

$$\mathfrak{B}(P(C_n, b)) = \Pi(C_n, b) + \mathbb{R}^{C_n - \{0\}}.$$

The facets of $\Pi(C_n, b)$ from our minimal representation can be tilted into facets of $\mathfrak{B}(P(C_n, b))$. *Tilting* an inequality is adding a linear combination of equality constraints or complementarities. In fact, we add (or substitute) only one complementarity to tilt a subadditivity $\pi_i + \pi_j \geq \pi_k$ into

$$\pi_i + \pi_j + \pi_{\bar{k}} \geq 1 = \pi_b,$$

where $\bar{k}$ is the *complementary* of k; i.e., $k + \bar{k} \equiv b \mod n$. By using that tilting into a path of triple to b, $i + j + \bar{k} \equiv b \mod n$, Ellis L. Johnson made short the theorem in Gomory [12] that the extreme points of the subadditive polytope are the facets of the master group polyhedron. The triples have one-to-one correspondence with the four types of inequalities:

$$\begin{aligned}
\pi_i + \pi_j + \pi_{\bar{k}} \geq \pi_b = 1 \quad &\Leftrightarrow \quad \pi_i + \pi_j \geq \pi_k \\
\pi_i + \pi_j + \pi_k \geq \pi_b = 1 \quad &\Leftrightarrow \quad \pi_i + \pi_j + \pi_k \geq \pi_b = 1 \\
\pi_{\bar{i}} + \pi_{\bar{j}} + \pi_k \geq \pi_b = 1 \quad &\Leftrightarrow \quad \pi_i + \pi_j \leq \pi_k + \pi_b = \pi_k + 1 \\
\pi_{\bar{i}} + \pi_{\bar{j}} + \pi_{\bar{k}} \geq \pi_b = 1 \quad &\Leftrightarrow \quad \pi_i + \pi_j + \pi_k \leq \pi_b + \pi_b = 2,
\end{aligned}$$

where all i, j, k are in O. The coefficients of the triples above are shown to be the *irreducible* solutions t of $P(C_n, b)$ with cardinality length 3, i.e., every integral t', $0 \leq t'_g \leq t_g, g \in C_n - \{0\}$, gives a different element $\sum_{g \in C_n - \{0\}} t'_g g$ in C_n. Moreover, except the triples from the exceptional inequality, they are the vertices of $P(C_n, b)$ with cardinality length 3. The solution $(t_b = 1)$ is the vertex of cardinality length 1. The coefficients of the complementarities are the vertices with cardinality length 2. The additional inequalities for $j = 1, 2$ give the vertices of cardinality length 4 with only one nonzero component,

$$(t_{q_{jO}} = 4) \text{ whenever } q_{jO} \text{ is an integer; } (t_{q_{jX}} = 4) \text{ whenever } q_{jX} \text{ is an integer.}$$

Since the vertices are known [12] to be irreducible, we characterized all the vertices and the irreducible solutions of cardinality length ≤ 3 by using our main theorem.

The second application is shooting for the knapsack problem based on the concept of shooting from the natural interior point. Shooting from the natural interior point shoots from the inside of the polyhedron, which would look more natural. Conceptually, *shooting in* $v \in \mathbb{R}_+^{C_n - \{0\}}$ is shooting an arrow from the origin $\mathbf{0}$ in the direction v and seeing which facet it hits. We mathematically define *shooting in* v as solving the linear programming problem

$$\begin{aligned} \min \quad & v\pi \\ st \quad & \pi \in \Pi(C_n, b). \end{aligned} \tag{4}$$

Equivalently, we can minimize $v\pi$ over the blocking polyhedron $\mathfrak{B}(P(C_n, b))$. Decreasing dimension at least by half, the linear programming problem (4) for shooting in v can be reduced by projection onto O to have the equivalent linear programming problem

$$\begin{aligned} \min \quad & \sum_{i \in O} (v_i - v_{\bar{i}})\pi_i \\ st \quad & \pi_O \in \Pi(C_n, b)_O. \end{aligned}$$

Its optimal basic solution π_O is the restriction to O of the facet π hit by shooting in v. *Shooting from the natural interior point* in $w \in \mathbb{R}^O$ is defined to "maximize" the objective function $w\varphi$ over the translate $\varphi \in \Pi(C_n, b)_O - \dot{\pi}_O$ of the projected subadditive polytope $\Pi(C_n, b)_O$ by $-\dot{\pi}_O$ so that the natural interior point $\dot{\pi}_O$ of $\Pi(C_n, b)_O$ is translated into the origin $\mathbf{0}$. Its optimal solution φ is the facet $\varphi x \leq 1$ of the *plus level set*

$$(\Pi(C_n, b)_O - \dot{\pi}_O)^+ = \{x : \mu(x) \leq 1\},$$

where $\mu(x)$ is the *support function* given by

$$\mu(x) = \sup\{x\varphi : \varphi \in \Pi(C_n, b)_O - \dot{\pi}_O\}.$$

The corresponding facet $\pi t \geq 1$ of $P(C_n, b)$ is derived by translating the resulting optimal basic solution φ back to $\pi_O = \varphi + \dot{\pi}_O$. We implement *shooting experiment* *from the natural interior point*; i.e., shooting from the natural interior point in a random vector w uniformly distributed on the unit sphere $||w|| = 1$. Note that shooting in $v \in \mathbb{R}_+^{C_n - \{0\}}$ with $||v|| = 1$ is equivalent to shooting in w from the natural interior point with $w = \big(w_i = (v_{\bar{i}} - v_i)/||(v_{\bar{i}} - v_i)|| \text{ for } i \in O\big)$. [2] The vector $v' = (v'_i = v_i - v_{\bar{i}} \text{ for } i \in O)$ is a point in the closed ball $||v'|| \leq 1$ for v with $||v|| = 1$ and $v \geq 0$.

We define *shooting for the knapsack problem* $K(n)$ in $v \geq 0$ as solving the linear programming problem $\min\{v\pi : \pi \in \Pi(K(n))\}$ over the *knapsack subadditive polytope*

$$
\begin{aligned}
\Pi(K(n)) = \quad & \{\pi \in \mathbb{R}^{\{1,\ldots,n\}} : \\
& \pi_i + \pi_{n-i} = \pi_n = 1 \text{ for } i = 1, ..., \left\lfloor \frac{n}{2} \right\rfloor, \\
& \pi_i + \pi_j \geq \pi_{i+j} \text{ whenever } i + j < n, \\
& \pi_1 = 1/2\},
\end{aligned}
$$

whose extreme points are the nontrivial facets of the convex hull $P(K(n))$ of the nonnegative integer solutions t to the *master knapsack problem* $K(n)$; i.e.,

$$\sum_{i=1}^{n} t_i i = n.$$

We implement shooting experiment for $K(n)$ and observe the amazing aspect that the most hit facet is always the *knapsack mixed integer cut* defined as

$$
\begin{aligned}
\pi_i^{mick} &= \frac{i}{2} \text{ for } i < n/2 \\
&= \frac{1}{2} \text{ for } i = n/2 \\
&= \frac{i - n + 1}{2} \text{ for } i > n/2,
\end{aligned}
$$

with empirical probability $\geq 1/5$ of hit.

The third application is integer primal simplex method for cyclic group problem. We are more interested in cyclic group subproblems, which is motivated by poor

[2] One important question here is whether shooting experiment is equivalent to shooting experiment from the natural interior point. Experimental results boost the equivalence.

performance of Dijkstra's algorithm to $b = 2^m - 1$ on Wong-Coppersmith digraphs $Cay(C_{2^m}, \{2^0, 2^1, ..., 2^{m-1}\})$. It suggests a primal cutting plane algorithm and a fractional cutting plane algorithm where cutting planes are generated by shooting. For convenience, we implement a fractional cutting plane algorithm generating cutting planes by shooting. A primal dual simplex method is introduced for shooting. We also observe complementary relaxation of shooting LP which sheds light on how to take advantage of sparsity of the objectives v.

The other parts of the thesis are instructive. We review the relation between the master knapsack problem $K(n)$ and the master cyclic group problem (C_{n+1}, n) that is explored by Aráoz et al [2]. We also derive a minimal representation of the subadditive cone for the master knapsack problem by the substitution of complementarities. The thesis introduces the lifting of facets by Gomory [12] and the group problems for general groups based on Aráoz and Johnson [3].

CHAPTER II

THE MASTER KNAPSACK PROBLEM

2.1 The master knapsack problem

We are interested in the important case of knapsack problems in which all possible coefficients occur in the constraint. Define the *master knapsack problem* $K(n)$ to be

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} r_i t_i, \\ st \quad & \sum_{i=1}^{n} i t_i = n, \\ & t \geq 0 \text{ and integer,} \end{aligned}$$

where we typically ignore the objective function. The *knapsack polytope* $P(K(n))$ is the convex hull of feasible solutions. (It is clear to see that $P(K(n))$ is a bounded polyhedron, i.e., a polytope.) Since the solutions $(t_1 = n)$ and $(t_1 = n - i, t_i = 1)$ for $i = 2, ..., n$ are linearly independent, $P(K(n))$ has dimension $n - 1$. For each $i = 2, ..., n$, dropping the solution $(t_1 = n - i, t_i = 1)$ from these n solutions results in the remaining $n - 1$ linearly independent solutions which verify that the nonnegativity $t_i \geq 0$ is a facet, called a *trivial* facet. The nonnegativity of the first component $t_1 \geq 0$ is redundant, since $t_1 = 0$ implies $t_{n-1} = 0$. General knapsack problems are missing some coefficients in the constraint and are equivalent to the integer program over

$$P(K(n)) \cap \{t : t_i = 0 \text{ whenever } i \text{ is not a coefficient in the constraint}\}.$$

Aráoz [1] defined the *knapsack subadditive cone* $S(K(n)) \subseteq \mathbb{R}^{\{1,\ldots,n\}}$ for $K(n)$ by the *complementarities* (5) and the *knapsack subadditivities* (6) given below,

$$\pi_i + \pi_{n-i} = \pi_n \text{ for } i = 1, ..., \left\lfloor \frac{n}{2} \right\rfloor, \tag{5}$$

$$\pi_i + \pi_j \geq \pi_{i+j} \text{ whenever } i + j < n, \tag{6}$$

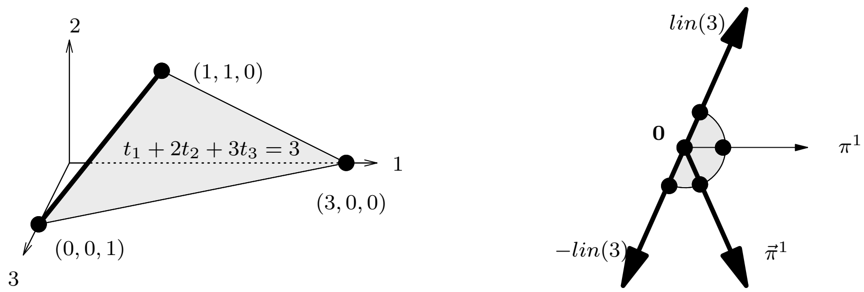

Figure 6: $P(K(3))$ and a defining set of rays of $S(K(3))$

and showed that its defining set of rays π are the nontrivial facets of $P(K(n))$

$$\sum_{i=1}^{n} \pi_i t_i \geq \pi_n.$$

The *lineality* L of $S(K(n))$ is the linear space generated by $lin(n) = (1/n, 2/n, ..., 1)$ and defines by $\pi t \geq \pi_n$ for $\pi \in L$ the hyperplane $\sum_{i=1}^{n} it_i = n$ containing $P(K(n))$. The extreme rays π of the cone $S(K(n)) \cap \{\pi_n = 0\}$ are the nontrivial facets $\pi t \geq \pi_n = 0$ of $P(K(n))$.

Example 2.1 See the knapsack polytope $P(K(3))$ and the knapsack subadditive cone $S(K(3))$ in Figure 6. The 2-dimensional half-plane $S(K(3)) = \{\pi : \pi_1 + \pi_2 = \pi_3$ and $2\pi_1 \geq \pi_2\}$ is defined by the lineality L and the ray $\vec{\pi}^1 = (2/3, -2/3, 0)$ that are thick-lined on the right side of the figure. The ray $\vec{\pi}^1$ is the nontrivial facet $\vec{\pi}^1 t \geq \vec{\pi}^1_3 = 0$ of $P(K(3))$ that is thick-lined on the left side of the figure. Two extreme rays $\pm lin(3) = \pm(1/3, 2/3, 1)$ define the lineality L and determine the hyperplane $t_1 + 2t_2 + 3t_3 = 3$ containing $P(K(3))$ by $\pm lin(3).t \geq \pm lin(3)_3 = \pm 1$. The cone $S(K(3)) \cap \{\pi_3 = 0\}$ is the ray $\vec{\pi}^1$ itself. Alternatively, $\pi^1 = (1, 0, 1)$ can be the nontrivial facet $\pi^1 t \geq \pi^1_3 = 1$ of $P(K(3))$. Multiple choices of ray π in $S(K(3))$ account for multiple choices of hyperplanes deciding the nontrivial facet of $P(K(3))$.

□

In fact, many knapsack subadditivities (6) are redundant. To extract a minimal representation of $S(K(n))$ assuming $n \geq 3$, we define the index sets for the left-hand sides of complementarities (5) with indices $< n/2$ by O and with indices $> n/2$ by X; i.e.,

$$O = \{i : i < n/2\},$$
$$X = \{n - i : i < n/2\}.$$

We project out the X-variables by the substitution of $\pi_{n-i} = \pi_n - \pi_i$ into the knapsack subadditivities (6) for $i < n/2$. If n is even, we distinguish the *half* $h = n/2$ and also project out π_h by the substitution of the complementarity

$$2\pi_h = \pi_n \tag{7}$$

to have the cone

$$S(K(n))_{O\cup\{n\}} \subseteq \mathbb{R}^{O\cup\{n\}}$$

which is equivalent to $S(K(n))$ as its projected image onto $\mathbb{R}^{O\cup\{n\}}$. If n is odd, each knapsack subadditivity (6) becomes one of the two types

$$\pi_i + \pi_j \geq \pi_{i+j}, \qquad \text{where } i, j, i + j \in O, \tag{8}$$

$$\pi_i + \pi_j + \pi_{n-i-j} \geq \pi_n, \qquad \text{where } i, j, n - i - j \in O. \tag{9}$$

If n is even, the knapsack subadditivities (6) without any term of π_h are of the two types above and those containing π_h become

$$\pi_i + \pi_{h-i} \geq \pi_h = \pi_n/2,$$

which is redundant in case $i > h-i$, as the average of $2\pi_{h-i} \geq \pi_{n-2i}$ and $2\pi_i + \pi_{n-2i} \geq \pi_n$. So, we have at most one *additional* inequality

$$\pi_{n/4} \geq \pi_n/4 \text{ when } n \text{ is divisible by 4.} \tag{10}$$

The two types of inequalities (8),(9) and the additional inequality (10) are enough to define $S(K(n))_{O\cup\{n\}}$.

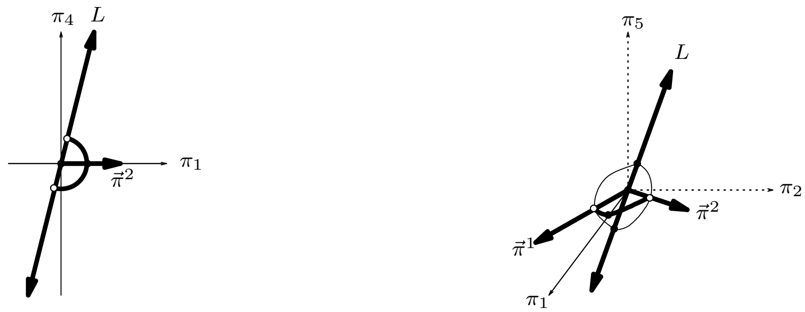

Figure 7: $S(K(4))_{\{1,4\}}$ and $S(K(5))_{\{1,2,5\}}$.

Example 2.2 See the 2-dimensional cone $S(K(4))_{\{1,4\}}$ on the left side in Figure 7. The thick curve is a set of interior points of $S(K(4))_{\{1,4\}}$. The additional inequality $\pi_1 \geq \pi_4/4$ in (10) determines $S(K(4))_{\{1,4\}}$ which can be defined by the lineality L and the ray $\vec{\pi}^2 = (5/12, 0, -5/12, 0)$. The nontrivial facet of $P(K(4))$ is $\vec{\pi}^2 t \geq \vec{\pi}^2_4 = 0$. Strictly speaking, the point $\vec{\pi}^2$ in the figure is its restriction $\vec{\pi}^2_{\{1,4\}}$ to $\{1,4\}$.

The 3-dimensional cone $S(K(5))_{\{1,2,5\}}$ on the right side is determined by $2\pi_1 \geq \pi_2$ in (8) and $\pi_1 + 2\pi_2 \geq \pi_5$ in (9) which are respectively binding the thin curves. The extreme rays $\vec{\pi}^1 = (4/5, -2/5, 2/5, -4/5, 0)$ and $\vec{\pi}^2 = (3/10, 3/5, -3/5, -3/10, 0)$ of the cone $S(K(5)) \cap \{\pi : \pi_5 = 0\}$ are the nontrivial facets $\vec{\pi}^1 t \geq \vec{\pi}^1_5 = 0$ and $\vec{\pi}^2 t \geq \vec{\pi}^2_5 = 0$ of $P(K(5))$.

The *natural interior point* is

$$\dot{\pi} = (\dot{\pi}_i = 1/2 \text{ for } i \neq n; \dot{\pi}_n = 1).$$

It is a relative interior of $S(K(n))$ satisfying the knapsack subadditivities (6) as strict inequalities. The projected cone $S(K(n))_{O\cup\{n\}}$ is full dimensional with the restriction of $\dot{\pi}$ to $O \cup \{n\}$ as an interior point,

$$\dot{\pi}_{O\cup\{n\}} = (\dot{\pi}_i = 1/2 \text{ for } i < n/2; \dot{\pi}_n = 1),$$

and therefore the dimension of $S(K(n))$ is $\lceil n/2 \rceil$. We show that the two types of inequalities (8),(9) and the additional inequality (10) are irredundant defining

$S(K(n))_{O\cup\{n\}}$. A *certificate* $\breve{\pi}$ for a constraint of $S(K(n))_{O\cup\{n\}}$ to be irredundant is a solution which satisfies the constraint as equality and the other constraints as strict inequalities. It verifies that the corresponding constraint is a facet. For convenience, we will take a point $\hat{\pi}$ that is infeasible only for the constraint. The intersection of the line segment between $\hat{\pi}$ and $\dot{\pi}$ with the hyperplane of the constraint is a certificate $\breve{\pi}$ for the constraint.

Theorem 2.3 *The two types of inequalities (8), (9) and the additional inequality (10) in case of n divisible by 4 form a minimal representation of $S(K(n))$ with the complementarities (5).*

Proof. For $\pi_{i_0} + \pi_{j_0} \geq \pi_{i_0+j_0}$ with $i_0, j_0, i_0 + j_0 \in O$, we change at most three components of $\dot{\pi}$ to get an infeasible solution

$$\hat{\pi} = (\hat{\pi}_{i_0} = \hat{\pi}_{j_0} = 1/3, \hat{\pi}_{i_0+j_0} = 3/4),$$

where the undefined variables remain the same as the corresponding components of $\dot{\pi}$. The infeasible solution $\hat{\pi}$ is shown to be feasible for all the other inequalities in (8):

$$\begin{aligned} &\hat{\pi}_i + \hat{\pi}_j = \frac{1}{3} + \frac{1}{3} \geq \frac{1}{2} \geq \hat{\pi}_{i+j} \qquad && \text{if } i_0 \neq j_0, i = j = i_0, \\ &\hat{\pi}_i + \hat{\pi}_j \geq \frac{1}{3} + \frac{1}{2} \geq \frac{3}{4} \geq \hat{\pi}_{i+j} && \text{otherwise.} \end{aligned}$$

It is also feasible for all the inequalities in (9):

$$\hat{\pi}_i + \hat{\pi}_j + \hat{\pi}_{n-i-j} \geq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 = \hat{\pi}_n.$$

The additional inequality (10) is trivially feasible.

For $\pi_{i_0} + \pi_{j_0} + \pi_{k_0} \geq \pi_n$ with $i_0 \leq j_0 \leq k_0$ and $i_0 + j_0 + k_0 = n$, we set $\hat{\pi}_{j_0} = 1/4$

in case $i_0 = j_0 = k_0 = n/3$. Otherwise, we set

$$\begin{aligned}\hat{\pi}_{j_0} &= \frac{1}{3};\\ \hat{\pi}_{i_0} &= \frac{1}{4} \text{ if } i_0 \neq j_0;\\ \hat{\pi}_{k_0} &= \frac{1}{4} \text{ if } k_0 \neq j_0,\end{aligned}$$

where the undefined variables are the same as the corresponding components of $\dot{\pi}$. It is infeasible for the inequality and feasible for all the inequalities in (8). Since all the other inequalities in (9) contain at least one term equal to $1/2$, the point $\hat{\pi}$ is feasible for all of them. The inequality (10) is trivially feasible here again.

In case n is a multiple of 4, the additional inequality (10) has $\hat{\pi}_{n/4} = 0$ in which the undefined variables are the same as the corresponding components of $\dot{\pi}$. It is trivially infeasible for (10) and feasible for all the inequalities in (8). Since $\pi_{n/4}$ comes up at most once in the left-hand side of each inequality in (9), $\hat{\pi}$ is feasible for all the inequalities, completing the proof of Theorem 2.3. □

We can translate the cone $S(K(n)) \cap \{\pi : \pi_n = 0\}$ by $lin(n) = (1/n, 2/n, ..., 1)$ into

$$S(K(n)) \cap \{\pi : \pi_n = 1\} = lin(n) + (S(K(n)) \cap \{\pi : \pi_n = 0\})$$

so that the origin $\mathbf{0}$ is translated into $lin(n)$. If $\vec{\pi} = \pi - lin(n)$ is an extreme ray of the cone $S(K(n)) \cap \{\pi : \pi_n = 0\}$, then π is a nontrivial facet $\pi t \geq \pi_n = 1$ of $P(K(n))$ which is equivalent to the facet $\vec{\pi} t \geq \vec{\pi}_n = 0$, and therefore $S(K(n)) \cap \{\pi : \pi_n = 1\}$ gives another set of the nontrivial facets π of $P(K(n))$. Fixing $\pi_1 = 1/2$, the *knapsack subadditive polytope* is defined to be

$$\begin{aligned}\Pi(K(n)) = \quad & \{\pi \in \mathbb{R}^{\{1,\dots,n\}} :\\ & \pi_i + \pi_{n-i} = \pi_n = 1 \text{ for } i = 1, ..., \left\lfloor \frac{n}{2} \right\rfloor,\\ & \pi_i + \pi_j \geq \pi_{i+j} \text{ whenever } i + j < n,\\ & \pi_1 = 1/2\}\end{aligned}$$

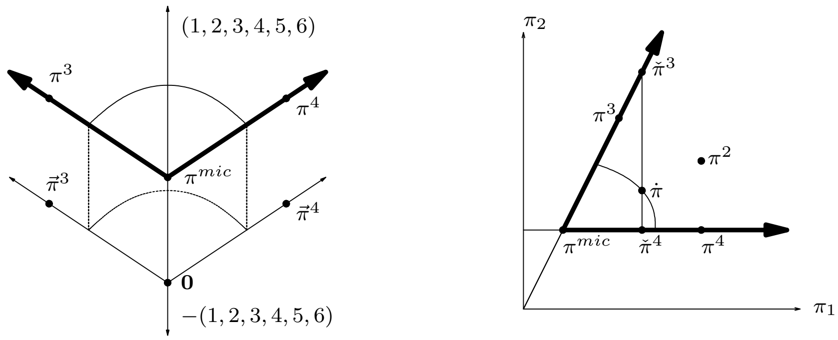

Figure 8: $S(K(6))$ and $S(K(6)) \cap \{\pi : \pi_6 = 1\}$.

in $S(K(n)) \cap \{\pi : \pi_n = 1\}$. Its extreme points are the nontrivial facets of $P(K(n))$ and its dimension is $|O| - 1 = \lfloor \frac{n-1}{2} \rfloor - 1$ with the natural interior point $\dot{\pi}$ as a relative interior point.

In Figure 8, the cone $S(K(6))_{\{1,2,6\}}$ equivalent to $S(K(6))$ has two facets, $2\pi_1 \geq \pi_2$ and $3\pi_2 \geq \pi_6$. Note that $\pi^2 = (6, 5, 4, 3, 2, 8)/8$ is not a facet of $P(K(6))$ but $\pi^3 = (4, 8, 5, 2, 6, 10)/10$ and $\pi^4 = (9, 4, 6, 8, 3, 12)/12$ are the nontrivial facets: Binding no inequality is

$$\vec{\pi}^2 = \pi^2 - lin(6) = (\frac{6}{8} - \frac{1}{6}, \frac{5}{8} - \frac{2}{6}, \frac{4}{8} - \frac{3}{6}, \frac{3}{8} - \frac{4}{6}, \frac{2}{8} - \frac{5}{6}, \frac{8}{8} - \frac{6}{6}),$$

but, $\vec{\pi}^3$ and $\vec{\pi}^4$ are binding $2\pi_1 \geq \pi_2$ and $3\pi_2 \geq \pi_6$, respectively, and the extreme rays of the cone $S(K(6)) \cap \{\pi : \pi_6 = 0\}$. The knapsack subadditive polytope $\Pi(K(6))$ is the line segment $[\breve{\pi}^3, \breve{\pi}^4]$ between $\breve{\pi}^3 = (1/2, 1, 1/2, 0, 1/2, 1)$ and $\breve{\pi}^4 = (1/2, 1/3, 1/2, 2/3, 1/2, 1)$. The extreme points $\breve{\pi}^3$ and $\breve{\pi}^4$ of $\Pi(K(6))$ are the nontrivial facets $\breve{\pi}^3 t \geq \breve{\pi}^3_6 = 1$ and $\breve{\pi}^4 t \geq \breve{\pi}^4_6 = 1$ of $P(K(6))$.

2.2 *The master cyclic group problem* (C_{n+1}, n)

The trivial facets of the master cyclic group polyhedron $P(C_{n+1}, n)$ are known [12] to be the nonnegativities $t_i \geq 0$ for $i = 1, ..., n$. Gomory [12] showed that the nontrivial facets $\pi t \geq \pi_n = 1$ of $P(C_{n+1}, n)$ are the extreme points π of the *subadditive polytope*

$$\begin{aligned}\Pi(C_{n+1}, n) \quad = \quad & \{\pi \in \mathbb{R}_+^{C_{n+1}-\{0\}} : \\ & \pi_i + \pi_j = \pi_n = 1 \text{ if } i + j \equiv n \mod n+1, \qquad (11) \\ & \pi_i + \pi_j \geq \pi_k \text{ if } i + j \equiv k \mod n\}, \qquad (12)\end{aligned}$$

where none of i, j, k is n. In fact, the nonnegativities of $\Pi(C_{n+1}, n)$ for $n \geq 4$ are all redundant and can be left out from the representation of $\Pi(C_{n+1}, n)$ above.

Aráoz et. al [2] showed the knapsack polytope $P(K(n))$ is a facet of the cyclic group polyhedron $P(C_{n+1}, n)$ and researched their relation. Fixing $\pi_n = 1$, the complementarities (11) are exactly (5) and the subadditivities (12) include the knapsack subadditivities (6) of $S(K(n))$. The subadditivities (12) which are not in (6) can be written as

$$\pi_i + \pi_j \geq \pi_{i+j-n-1} \text{ whenever } i + j > n + 1. \qquad (13)$$

Hence, the subadditive polytope $\Pi(C_{n+1}, n)$ is embedded into the translate $S(K(n)) \cap \{\pi : \pi_n = 1\}$ of the cone $S(K(n)) \cap \{\pi : \pi_n = 0\}$ by the *mixed integer cut* $lin(n) = (1/n, 2/n, ...1)$ of (C_{n+1}, n) which is known to be a facet of $P(C_{n+1}, n)$. Since the restriction $\dot{\pi}_O$ of $\dot{\pi}$ to $O = \{i : i < n/2\}$ is an interior point of $\Pi(C_{n+1}, n)_O$, $\Pi(C_{n+1}, n)$ is full dimensional in $S(K(n)) \cap \{\pi : \pi_n = 1\}$.

Assuming n is odd, the polytope $\Pi(C_{n+1}, n)$ has two more types of analogous but upper bound type inequalities with one more term $\pi_n = 1$ in the right-hand sides

$$\pi_i + \pi_j \leq \pi_{i+j+1} + \pi_n, \qquad \text{where } i, j, i + j + 1 \in O, \qquad (14)$$

$$\pi_i + \pi_j + \pi_{n-1-i-j} \leq \pi_n + \pi_n, \qquad \text{where } i, j, n - 1 - i - j \in O. \qquad (15)$$

If n is even, $\Pi(C_{n+1}, n)$ has the four types of inequalities in (8), (9), (14) and (15), and exactly one additional inequality in either (10) or

$$\pi_{\frac{n-2}{4}} \leq \frac{3\pi_n}{4} \text{ when } n \equiv 2 \mod 4. \qquad (16)$$

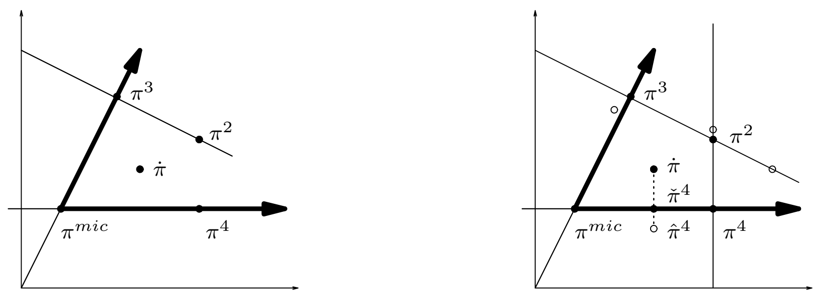

Figure 9: Augmentation of inequalities to have $\Pi(C_7, 6)$.

In other words, we augment (14), (15) and (16) to $S(K(n)) \cap \{\pi : \pi_n = 1\}$ defining $\Pi(C_{n+1}, n)$. Moreover, the four types of inequalities and the additional inequality are all irredundant.

Theorem 2.4 *The four types of inequalities in (8), (9), (14) and (15), and the additional inequality in either (10) or (16) in case of* n *even, form a minimal representation of* $\Pi(C_{n+1}, n)_O$ *fixing* $\pi_n = 1$.

In the next chapter, we prove Theorem 3.5 which implies Theorem 2.4.

Let's consider the previous example of $S(K(6))$. We augment the inequalities in (14), (15) and (16) to $S(K(6)) \cap \{\pi : \pi_6 = 1\}$, and have $\Pi(C_7, 6)$. See Figure 9 that follows Figure 8. There is no inequality in (14). The augmented inequalities in (15) and (16) are $\pi_1 + 2\pi_2 \leq 2$ and $\pi_1 \leq 3/4$, respectively. In the figure, the hollow dots denote the infeasible solutions $\hat{\pi}$ given in Chapter 3. Note that the line segment between $\dot{\pi}_O$ and the infeasible solution $\hat{\pi}^4 = (1/2, 1/4)$ intersects with $3\pi_2 = \pi_6 = 1$ at a certificate $\check{\pi}^4 = (1/2, 1/3)$ for $3\pi_2 \geq \pi_6 = 1$ to be irredundant.

Aráoz et. al [2] noted that each nontrivial facet π of $P(C_{n+1}, n)$ adjacent to the mixed integer cut $lin(n)$ of $P(C_{n+1}, n)$ is a nontrivial facet of the knapsack polytope $P(K(n))$, and that they are related through a process called *tilting*. Two facets are *facet-adjacent* if their intersection has one-less dimension than a facet in the polyhedron. Two extreme points are *vertex-adjacent* if they are linked by an edge of the polyhedron. Due to the special property of the mixed integer cut $lin(n)$ of $P(C_{n+1}, n)$ that it satisfies all the knapsack subadditivities (6) of $S(K(n)) \cap \{\pi : \pi_n = 1\}$ as equalities, vertex-adjacency to $lin(n)$ in $\Pi(C_{n+1}, n)$ is equivalent to facet-adjacency to $lin(n)$ in $P(C_{n+1}, n)$.

Theorem 2.5 (Aráoz et al. [2]) *A nontrivial facet π of $P(C_{n+1}, n)$ is vertex-adjacent to the mixed integer cut $lin(n)$ of $P(C_{n+1}, n)$ in $\Pi(C_{n+1}, n)$, if and only if it is facet-adjacent to $lin(n)$ in $P(C_{n+1}, n)$.*

Proof. Assume that π^k is vertex-adjacent to $lin(n)$. The difference $\vec{\pi}^k = \pi^k - lin(n)$ is an extreme ray of the cone $S(K(n)) \cap \{\pi : \pi_n = 0\}$ and has $n-2$ linearly independent constraints among the complementarities (5) fixing $\vec{\pi}_n = 0$ and the knapsack subadditivities (6) which are binding at π^k. We transform the binding knapsack subadditivities (6) by substitution of (5) into

$$\pi_i + \pi_j + \pi_{n-i-j} \geq 1. \tag{17}$$

The coefficients $t_1, ..., t_{n-1}$ of the binding constraints give $n-2$ linearly independent solutions followed by $t_n = 0$ to both $\pi^k t = 1$ and $lin(n).t = 1$, and form $n-1$ linearly independent solutions to $P(C_{n+1}, n) \cap \{t : lin(n).t = 1\} \cap \{t : \pi^k t = 1\}$ with $t = (t_i = 0, i = 1, ..., n-1; t_n = 1)$.

Conversely, if π^k is facet-adjacent to $lin(n)$, then $P(C_{n+1}, n) \cap \{t : lin(n).t = 1\} \cap \{t : \pi^k t = 1\}$ contains $n-2$ linearly independent solutions t with $t_n = 0$ to the master cyclic group problem. The solutions are also solutions t to the master knapsack problem $K(n)$, since $lin(n).t = 1$ holds. For each t, we can rewrite $\pi t = 1$ as

$$\sum_{j=1}^{J} \pi_{i_j} = 1, \text{ where } \sum_{j=1}^{J} i_j = n,$$

and, if $J \geq 3$, decompose it into the telescoping sum of a complementarity (5) and knapsack subadditivities (6) which are binding at π^k

$$\begin{aligned}
\pi_{i_1} + \pi_{i_2} &\geq \pi(i_1 + i_2) \\
\pi(i_1 + i_2) + \pi_{i_3} &\geq \pi(i_1 + i_2 + i_3) \\
&\vdots \\
\pi\left(\sum_{j<J-1} i_j\right) + \pi_{i_{J-1}} &\geq \pi\left(\sum_{j<J} i_j\right) \\
\pi\left(\sum_{j<J} i_j\right) + \pi_{i_J} &= 1.
\end{aligned}$$

All the decomposed contain $n-2$ linearly independent binding constraints at π^k fixing $\pi_n = 1$ among the complementarities (5) and the knapsack subadditivities (6), and verify that π^k is vertex-adjacent to $lin(n)$ in $\Pi(C_{n+1}, n)$. □

The proof above gives us a hint on how to decompose a cyclic group solution into small solutions. Suppose that our solution t corresponds to a path $(i_1, i_2, ..., i_J)$ with $J > 3$ in $Cay(C_{n+1}, n)$,

$$n \equiv \sum_{j=1}^{J} i_j \mod n+1.$$

For any $\pi \in \Pi(C_{n+1}, n)$, it holds that

$$\pi t = \sum_{j=1}^{J} \pi_{i_j} \geq \pi_n = 1,$$

which can be decomposed into paths of cardinality length 3 and 2 alternatively,

$$\begin{aligned}
+(\pi_{i_1} + \pi_{i_2} + \pi_{b-(i_1+i_2)} &\geq 1) \\
-(\pi_{b-(i_1+i_2)} + \pi_{i_1+i_2} &= 1) \\
+(\pi_{i_1+i_2} + \pi_{i_3} + \pi_{b-(i_1+i_2+i_3)} &\geq 1) \\
&\vdots \\
-(\pi_{b-\sum_{j=1}^{J-2} i_j} + \pi_{\sum_{j=1}^{J-2} i_j} &= 1) \\
+(\pi_{\sum_{j=1}^{J-2} i_j} + \pi_{i_{J-1}} + \pi_{i_J} &\geq 1).
\end{aligned}$$

In other words, t can be written as

$$\begin{aligned} t &= t^{\gamma}(i_1, i_2, n-(i_1+i_2)) - t^{\beta}(n-(i_1+i_2), i_1+i_2) \\ &\vdots \\ &+t^{\gamma}\left(\sum_{j=1}^{J-3} i_j, i_{J-2}, n-\sum_{j=1}^{J-2} i_j\right) - t^{\beta}\left(n-\sum_{j=1}^{J-2} i_j, \sum_{j=1}^{J-2} i_j\right) \\ &+t^{\gamma}\left(\sum_{j=1}^{J-2} i_j, i_{J-1}, i_J\right), \end{aligned}$$

where $t^{\gamma}(i,j,k)$ denote $\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_k$ for triples $i+j+k \equiv n \mod n+1$ and $t^{\beta}(i,n-i)$ denote $\mathbf{e}_i+\mathbf{e}_{n-i}$ for complementary pairs $(i,n-i)$. Note that we have one more t^{γ}'s than t^{β}'s. We will use the decomposition later.

2.3 The knapsack mixed integer cut

Assume that n is an odd number, and that $d=(n+1)/2$ and $r=d-1$. By Theorem 19 of Gomory [12], we can construct the 2-fold lifting of the mixed integer cut $lin(r)=(1/r, 2/r, ..., 1)$ of (C_d, r),

$$\rho = (lin(r), 0, lin(r)),$$

which is a facet $\rho t \geq \rho_n = 1$ of (C_{n+1}, n). The valid inequality of $P(K(n))$,

$$\rho t \geq \rho_n = 1,$$

is a facet of $P(K(n))$ given by Theorem 6.9 of Aráoz et al. [2] and will be called the *knapsack mixed integer cut* of $P(K(n))$. We can induce an equivalent facet of $P(K(n))$ by tilting,

$$\pi^{mick} t \geq \pi^{mick}_n = 1,$$

corresponding to the intersection point $\pi^{mick}=\bar{\rho}$ of the knapsack subadditive polytope $\Pi(K(n))$ with the straight line going through both ρ and the mixed integer cut $lin(n)=(1/n, 2/n, ..., 1)$ of (C_{n+1}, n).

If n is even, we set $d = (n+2)/2 = \lceil (n+1)/2 \rceil$ and $r = d - 2$. The 2-fold lifting of the *mixed integer cut* $(1/r, 2/r, ..., 1, 1/2)$ *of* (C_d, r),

$$(\rho, 1/2) = (1/r, 2/r, ..., 1, 1/2, 0, 1/r, 2/r, ..., 1, 1/2),$$

corresponds to the facet of (C_{n+2}, n),

$$\rho t + \frac{1}{2} t_{n+1} \geq \rho_n = 1,$$

where (C_{n+2}, n) is the master cyclic group problem

$$\sum_{i \in C_{n+2} - \{0\}} t_i i \equiv n \mod n+2, \; t \geq 0 \text{ integer}.$$

As above, $\rho t \geq \rho_n = 1$ will be called the knapsack mixed integer cut of $P(K(n))$. The intersection point $\pi^{mick} = \bar{\rho}$ given by tilting $\rho t \geq \rho_n = 1$ is shown to be a facet of $P(K(n))$ as an extreme point of the knapsack subadditive polytope $\Pi(K(n))$ by $|O| - 1$ linearly independent binding constraints

$$\pi_1 + \pi_i \geq \pi_{i+1} \text{ for } i = 1, ..., |O| - 1.$$

The facet $\pi^{mick} t \geq \pi_n^{mick} = 1$ of $P(K(n))$ is written as

$$\begin{aligned} \pi_i^{mick} &= \frac{i}{2} \text{ for } i < n/2 \\ &= \frac{1}{2} \text{ for } i = n/2 \\ &= \frac{i-n+2}{2} \text{ for } i > n/2. \end{aligned}$$

Example 2.6 Let $d = 5$ and let $r = 4$. The mixed integer cut of the master cyclic group polyhedron $P(C_5, 4)$ is

$$lin(4)t = 0.25t_1 + 0.5t_2 + 0.75t_3 + t_4 \geq 1.$$

Lift $lin(4)$ above up and obtain the 2-fold lifting

$$\rho t = 0.25t_1 + 0.5t_2 + 0.75t_3 + t_4 + 0.25t_6 + 0.5t_7 + 0.75t_8 + t_9 \geq 1$$

which is the third facet $\pi^3 t = \rho t \geq 1$ of the master cyclic group polyhedron $P(C_{10}, 9)$ in Gomory's cyclic group facet table [12].

Let's compute the knapsack mixed integer cut π^{mick} of the knapsack problem $K(9)$ which is the extreme point of the knapsack subadditive polytope $\Pi(K(9))$ on the ray going out from the mixed integer cut $lin(9) = (1/9, 2/9, ..., 1)$ of $P(C_{10}, 9)$ through the 2-fold lifting ρ; that is, there are scalars x and $y > 0$ satisfying

$$\pi^{mick} = x.lin(9) + y.\rho.$$

The scalars $x = -1.8$ and $y = 2.8$ are determined by the system of equations

$$\begin{aligned} \frac{x}{9} + \frac{y}{4} &= \frac{1}{2} \text{ and} \\ x + y &= 1 \end{aligned}$$

which are respectively from the first component $\pi_1^{mick} = 1/2$ and the last component $\pi_9^{mick} = lin(9)_9 = \rho_9 = 1$. Now, we have the knapsack mixed integer cut of $P(K(9))$,

$$\pi^{mick} = -1.8lin(9) + 2.8\rho = (0.5, 1, 1.5, 2, -1, -0.5, 0, 0.5, 1).$$

□

CHAPTER III

THE MASTER CYCLIC GROUP PROBLEM

3.1 General partition of indices

We denote (C_n, b) as the *master* cyclic group problem to be the cyclic group problem (C_n, M, b) with $M = C_n - \{0\}$,

$$\begin{aligned} \min \quad & \sum_{i \in C_n - \{0\}} c_i t_i, \\ st \quad & \sum_{i \in C_n - \{0\}} i t_i \equiv b \mod n, \\ & t \geq 0 \text{ integer,} \end{aligned} \tag{18}$$

emphasizing that the congruent relation (18) contains all possible nonzero group elements. The master cyclic group polyhedron $P(C_n, b)$ is the convex hull of solutions to the master cyclic group problem (C_n, b). In Gomory [12], the nontrivial facets $\pi t \geq \pi_b = 1$ of $P(C_n, b)$ are known to be the extreme points π of the *subadditive polytope* $\Pi(C_n, b)$ defined by the *complementarities* (19) and the *subadditivities* (20) given below,

$$\begin{aligned} \Pi(C_n, b) \quad = \quad & \{\pi \in \mathbb{R}_+^{C_n - \{0\}} : \\ & \pi_i + \pi_j = \pi_b = 1 \text{ if } i + j \equiv b \mod n, & (19) \\ & \pi_i + \pi_j \geq \pi_k \text{ if } i + j \equiv k \mod n\}, & (20) \end{aligned}$$

where none of i, j, k is b. The nonnegativities of $\Pi(C_n, b)$ are all redundant.[1]

Let a *half* h satisfy $2h \equiv b \mod n$. The variable π_h is a constant $1/2$ by the complementarity $2\pi_h = \pi_b = 1$. Assume O and X respectively contain one term and

[1]It is true for $n \geq 5$ except $(C_6, 2)$ and $(C_6, 4)$.

the other of every complementarity (19) without any π_h. In other words,

$$i \in O, j \in X \text{ whenever } i \not\equiv j \equiv b - i \mod n. \tag{21}$$

The subadditivities (20) containing no π_h with $2h \equiv b \mod n$ are classified into the following 6 cases

$$\begin{aligned}
&\pi_i + \pi_j \geq \pi_k \qquad \text{where } k \equiv i + j \mod n \text{ and } i \in O, j \in O, k \in O,\\
&\pi_i + \pi_j \geq \pi_k \qquad \text{where } k \equiv i + j \mod n \text{ and } i \in O, j \in O, k \in X,\\
&\pi_i + \pi_j \geq \pi_k \qquad \text{where } k \equiv i + j \mod n \text{ and } i \in X, j \in X, k \in X,\\
&\pi_i + \pi_j \geq \pi_k \qquad \text{where } k \equiv i + j \mod n \text{ and } i \in X, j \in X, k \in O,\\
&\pi_i + \pi_j \geq \pi_k \qquad \text{where } k \equiv i + j \mod n \text{ and } i \in O, j \in X, k \in O,\\
&\pi_i + \pi_j \geq \pi_k \qquad \text{where } k \equiv i + j \mod n \text{ and } i \in O, j \in X, k \in X.
\end{aligned}$$

Note that the fifth and the sixth cases are equivalent to the third and the first, respectively, which reduces the number of such inequalities by half.

The inequalities (20) containing more than one π_h for (same or different) halves h with $2h \equiv b \mod n$ are trivially redundant. Each inequality (20) containing exactly one π_h is assumed to have the π_h in its right-hand side:

$$\begin{aligned}
\pi_i + \pi_j \geq \pi_h \qquad & \text{with } i + j \equiv h \mod n \text{ and } i, j \in O\\
& \Leftrightarrow \pi_h + \pi_j \geq \pi_k \text{ with } k \equiv h + j \mod n \text{ and } j \in O, k \in X\\
\pi_i + \pi_j \geq \pi_h \qquad & \text{with } i + j \equiv h \mod n \text{ and } i \in O, j \in X\\
& \Leftrightarrow \pi_h + \pi_j \geq \pi_k \text{ with } k \equiv h + j \mod n \text{ and } j, k \in O\\
& \Leftrightarrow \pi_h + \pi_j \geq \pi_k \text{ with } k \equiv h + j \mod n \text{ and } j, k \in X\\
\pi_i + \pi_j \geq \pi_h \qquad & \text{with } i + j \equiv h \mod n \text{ and } i, j \in X\\
& \Leftrightarrow \pi_i + \pi_j \leq \pi_b + \pi_h \text{ with } i + j \equiv b + h \mod n \text{ and } i, j \in O\\
& \Leftrightarrow \pi_h + \pi_j \geq \pi_k \text{ with } k \equiv h + j \mod n \text{ and } j \in X, k \in O.
\end{aligned}$$

Moreover, if $i \not\equiv j \mod n$ above, $\pi_i + \pi_j \geq \pi_h = 1/2, i + j \equiv h \mod n$ is verified to be redundant by the average of two inequalities $2\pi_i \geq \pi_{2i}$ and $2\pi_j \geq \pi_{2j}$. Therefore, we can simplify the left-hand sides of (20).

Proposition 3.1 *The following inequalities are enough to define* $\Pi(C_n, b)_O$*:*

$$\pi_i + \pi_j \geq \pi_k \qquad \text{whenever } k \equiv i + j \mod n \text{ and } i, j \in O,$$

$$\pi_i + \pi_j \geq \pi_k \qquad \text{whenever } k \equiv i + j \mod n \text{ and } i, j \in X.$$

By substitution of the complementarities (19), we simplify the first four cases out of the six above without $2h \equiv b \mod n$ to have the following four types of inequalities:

Proposition 3.2 *Every inequality in (20) containing no* π_h *with* $2h \equiv b \mod n$ *is equivalent to one of the following four types of inequalities:*

$$\pi_i + \pi_j \geq \pi_k \qquad \text{whenever } i + j \equiv k \mod n, \tag{22}$$

$$\pi_i + \pi_j + \pi_k \geq 1 \qquad \text{whenever } i + j + k \equiv b \mod n, \tag{23}$$

$$\pi_i + \pi_j \leq \pi_k + 1 \qquad \text{whenever } i + j \equiv k + b \mod n, \tag{24}$$

$$\pi_i + \pi_j + \pi_k \leq 2 \qquad \text{whenever } i + j + k \equiv 2b \mod n, \tag{25}$$

where all i, j, k *belong to* O*.*

If there is no half h such that $2h \equiv b \mod n$, that is, n is even and b is odd, the four types in Proposition 3.2 are enough to define $\Pi(C_n, b)_O$. They gave the facets of $\Pi(C_{n+1}, n)_O$ in the special case of n odd and $O = \{i : i < n/2\}$ in Theorem 2.4.

Proposition 3.3 *The following additional inequalities are enough to define* $\Pi(C_n, b)_O$ *with the four types of inequalities (22)-(25) above:*

$$2\pi_i \geq \pi_h = 1/2 \text{ whenever } 2i \equiv h \mod n \text{ and } i \in O \cup X.$$

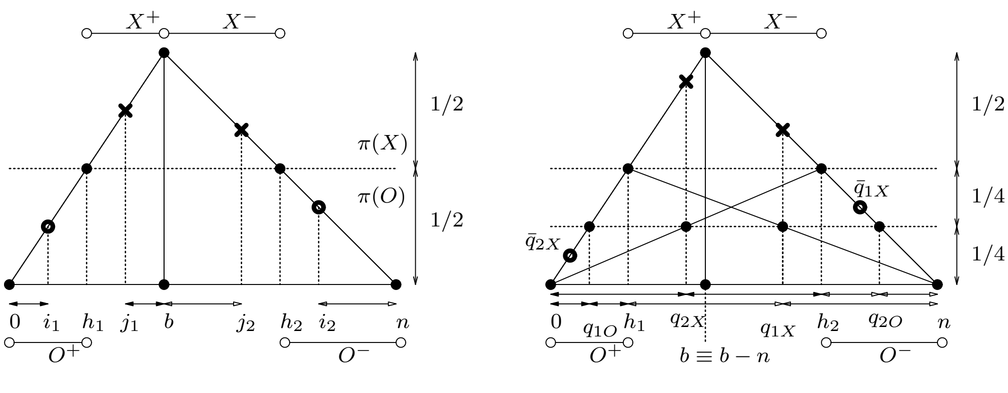

Figure 10: A partition of O and X given by π^{mic}.

The *complementary* $\bar{i}$ of i is defined to satisfy $i + \bar{i} \equiv b \mod n$. The additional inequalities in Proposition 3.3 can be written as

$$\begin{aligned} &\pi_i \geq 1/4 \text{ whenever } 2i \equiv h \mod n \text{ and } i \in O, \\ &\pi_{\bar{i}} \leq 3/4 \text{ whenever } 2i \equiv h \mod n \text{ and } i \in X. \end{aligned} \tag{26}$$

We generalize the special case of $\Pi(C_{n+1}, n)_O$ with n even to (C_n, b) with n odd as follows.

Corollary 3.4 *The four types of inequalities (22)-(25) and the following additional inequality (27) are enough to define* $\Pi(C_n, b)_O$ *with* n *odd:*

$$\begin{aligned} \textit{either} \qquad &\pi_i \geq \frac{1}{4} \textit{ if } 2i \equiv h \mod n \textit{ and } i \in O, \\ \textit{or} \qquad &\pi_i \leq \frac{3}{4} \textit{ if } 2(b-i) \equiv h \mod n \textit{ and } i \in O. \end{aligned} \tag{27}$$

In fact, Proposition 3.1, 3.2 and 3.3 are true for general group problems.

3.2 The partition given by the mixed integer cut

The mixed integer cut $lin(n)$ of $P(C_{n+1}, n)$ is generalized to the *mixed integer cut* π^{mic} of every master cyclic group polyhedron $P(C_n, b)$, which is defined by

$$\pi^{mic} = (\pi_i^{mic} = i/b \text{ for } i \leq b; \pi_i^{mic} = (n-i)/(n-b) \text{ for } i \geq b).$$

In Figure 10, the mixed integer cut gives a partition

$$O = \{i : \pi_i^{mic} < 1/2\}, X = \{j : \pi_j^{mic} > 1/2\}.$$

Let $h_1 = b/2$ and let $h_2 = (b+n)/2$. They may be fractional. They bisect the partition given by π^{mic} into the intervals

$$\begin{aligned} O^+ &= \left(0, \frac{b}{2}\right) \cap \mathbb{Z}, \\ X^+ &= \left(\frac{b}{2}, b\right) \cap \mathbb{Z}, \\ X^- &= \left(b, \frac{b+n}{2}\right) \cap \mathbb{Z}, \\ O^- &= \left(\frac{b+n}{2}, n\right) \cap \mathbb{Z}. \end{aligned}$$

Each complementary pair $i_1 \in O$ and $j_1 \in X$ with $i_1 + j_1 = b$ are symmetric in h_1. The other kinds of pairs $i_2 \in O$ and $j_2 \in X$ with $i_2 + j_2 = b + n \equiv b \mod n$ are symmetric in h_2. The two kinds of symmetries are true for any $\pi \in \Pi(C_n, b)$ because of the complementarities. The *quarters* are defined as

$$\begin{aligned} q_{1O} &= \frac{h_1}{2} = \frac{b}{4} \in O^+ \\ q_{1X} &= \frac{h_1 - n}{2} + n = \frac{b+2n}{4} \in X^- \\ q_{2O} &= \frac{h_2 - n}{2} + n = \frac{b+3n}{4} \in O^- \\ q_{2X} &= \frac{h_2}{2} = \frac{b+n}{4} \in X^+. \end{aligned}$$

We rewrite the *additional* inequalities (26) as

$$\begin{aligned} &\pi_{q_{jO}} \geq 1/4 \text{ whenever } q_{jO} \text{ is an integer,} \\ &\pi_{\bar{q}_{jX}} \leq 3/4 \text{ whenever } q_{jX} \text{ is an integer,} \end{aligned} \tag{28}$$

where $j = 1, 2$.

The partition of O and X given by π^{mic} yields a redundant constraint among the four types (22)-(25) and the additional (26) in case of $b \equiv 0 \mod 3$ and $n \equiv 0 \mod 3$.

Consider the cyclic group problem $(C_9, 6)$. The partition by π^{mic} is $O = \{1, 2, 8\}$. We see that the three inequalities, $3\pi_2 \geq 1 = \pi_6$, $3\pi_8 \geq 1$ and $3\pi_1 \leq 2 = 2\pi_6$, imply $\pi_2 + \pi_8 \geq 2/3 \geq \pi_1$. Assuming $b \equiv n \equiv 0 \mod 3$, the example can be generalized to the *exceptional* inequality in (22)

$$\pi_i + \pi_j \geq \pi_k \text{ with } i = b/3, j \equiv (b-n)/3 \text{ and } k \equiv i + j \mod n, \tag{29}$$

where i, j, k are all in O. The exceptional inequality is induced by

$$3\pi_i \geq 1, 3\pi_j \geq 1, 3\pi_k \leq 2.$$

Note that (C_{n+1}, n) does not satisfy $b \equiv n \equiv 0 \mod 3$. In general, we have no such exceptional inequality unless $b \equiv n \equiv 0 \mod 3$. Fortunately, the other inequalities are irredundant even though $b \equiv n \equiv 0 \mod 3$ holds:

Theorem 3.5 *Let O and X be given by the mixed integer cut π^{mic}. The four types of inequalities (22)-(25) and the additional inequalities (28) without the exceptional inequality (29) in (22) are irredundant defining $\Pi(C_n, b)_O$.*

In order to show the theorem, we shift the intervals O^- and X^- by modulo n onto

$$O^- = \left(\frac{b-n}{2}, 0\right) \cap \mathbb{Z} \text{ and } X^- = \left(b-n, \frac{b-n}{2}\right) \cap \mathbb{Z}$$

in the rest of this chapter.

Remark. The interval $O \cup \{0\} = ((b-n)/2, b/2) \cap \mathbb{Z}$ implies the following:

(a) If $i, j \in O$, then $i + j \not\equiv b$.

(b) If $i + j + k \equiv b \mod n$ with $i, j, k \in O$, then $i + j + k = b$ or $b - n$.

(c) If $i + j + k \equiv b \mod n$ with $i, j, k \in O$, then all $i, j, k > 0$ or all $i, j, k < 0$.

(d) If $i + j \equiv k + b \mod n$ with $i, j, k \in O$, then $i + i + j \equiv i + k + b \not\equiv 2b \mod n$.

(e) If $i + j + k \equiv 2b \mod n$ with $i, j, k \in O$, then $i + j + k = 2b - n$.

We list the infeasible solutions $\hat{\pi}$ before the proof: All i_0, j_0, k_0 below belong to O. The undefined variables are assumed to be $\hat{\pi}_b = 1$ and $\hat{\pi}_i = 1/2$ for $i \neq b$. For $\pi_{i_0} + \pi_{j_0} \geq \pi_{k_0}$ with $i_0 + j_0 \equiv k_0 \mod n$, $i_0 \neq b/3$ and $i_0 \neq (b-n)/3$,

$$\begin{aligned}\hat{\pi} &= \left(\hat{\pi}_{i_0} = \hat{\pi}_{j_0} = \frac{1}{3}; \hat{\pi}_{k_0} = \frac{3}{4}\right) \text{ when } n \not\equiv 2b \mod 3, \\ &= \left(\hat{\pi}_{i_0} = \frac{1}{4}; \hat{\pi}_{j_0} = \frac{1}{3} \text{ if } j_0 \neq i_0; \hat{\pi}_{k_0} = \frac{2}{3}\right) \text{ when } n \equiv 2b \mod 3.\end{aligned} \tag{30}$$

For $\pi_{i_0} + \pi_{j_0} + \pi_{k_0} \geq 1$ with $|i_0| \leq |j_0| \leq |k_0|$ and $i_0 + j_0 + k_0 \equiv b \mod n$,

$$\begin{aligned}\hat{\pi} &= \left(\hat{\pi}_{j_0} = \frac{1}{4}\right) \text{ when } i_0 = j_0 = k_0 = \frac{b}{3} \text{ or } \frac{b-n}{3}; \\ &= \left(\hat{\pi}_{j_0} = \frac{1}{3}; \hat{\pi}_{i_0} = \frac{1}{4} \text{ if } i_0 \neq j_0; \hat{\pi}_{k_0} = \frac{1}{4} \text{ if } k_0 \neq j_0\right) \text{ otherwise.}\end{aligned} \tag{31}$$

For $\pi_{i_0} + \pi_{j_0} \leq \pi_{k_0} + 1$ with $i_0 + j_0 \equiv k_0 + b \mod n$ and $3i_0 \neq 2b - n$,

$$\begin{aligned}\hat{\pi} &= \left(\hat{\pi}_{k_0} = \frac{1}{4}; \hat{\pi}_{i_0} = \frac{2}{3}; \hat{\pi}_{j_0} = \frac{5}{8} \text{ if } j_0 \neq i_0\right) \text{ when } b \not\equiv 0 \text{ and } b \not\equiv n \mod 3 \\ &= \left(\hat{\pi}_{k_0} = \frac{1}{3}; \hat{\pi}_{i_0} = \frac{3}{4}; \hat{\pi}_{j_0} = \frac{5}{8} \text{ if } j_0 \neq i_0\right) \text{ otherwise.}\end{aligned} \tag{32}$$

For $\pi_{i_0} + \pi_{j_0} + \pi_{k_0} \leq 2$ with $i_0 \leq j_0 \leq k_0$ and $i_0 + j_0 + k_0 = 2b - n \equiv 2b$,

$$\begin{aligned}\hat{\pi} &= \left(\hat{\pi}_{j_0} = \frac{3}{4}\right) \text{ when } i_0 = j_0 = k_0 = \frac{2b-n}{3}; \\ &= \left(\hat{\pi}_{j_0} = \frac{2}{3}; \hat{\pi}_{i_0} = \frac{3}{4} \text{ if } i_0 \neq j_0; \hat{\pi}_{k_0} = \frac{3}{4} \text{ if } k_0 \neq j_0\right) \text{ otherwise.}\end{aligned} \tag{33}$$

For the additional constraints $\pi_{q_{jO}} \geq 1/4$ with q_{jO} integer, $j = 1, 2$,

$$\hat{\pi} = \left(\hat{\pi}_{q_{jO}} = 0\right). \tag{34}$$

For $\pi_{\bar{q}_{jX}} \leq 3/4$ with q_{jX} integer, $j = 1, 2$,

$$\hat{\pi} = \left(\hat{\pi}_{\bar{q}_{jX}} = 1\right). \tag{35}$$

Proof. For the infeasible solution (30) of (22), we may assume that $i_0 \neq b/3$ and $i_0 \neq (b-n)/3$, since

$$\{2b/3, 2(b-n)/3\} \subset (b/2, b) \cup (b-n, (b-n)/2) = X.$$

The infeasible solution (30) is feasible for all the other inequalities in (22), since

$$\hat{\pi}_i + \hat{\pi}_j = \frac{1}{2} + \frac{1}{2} \geq \hat{\pi}_k = \hat{\pi}_{k_0}, \text{ if } k \equiv i + j \equiv i_0 + j_0 \mod n,$$
$$\hat{\pi}_i + \hat{\pi}_j \geq \frac{1}{4} + \frac{1}{4} \geq \hat{\pi}_k \neq \hat{\pi}_{k_0}, \text{ if } k \equiv i + j \not\equiv i_0 + j_0 \mod n.$$

Its feasibility for (23) can be shown excluding all possibly infeasible cases,

$$\begin{aligned} (i, j, k) &= (i_0, i_0, i_0), \\ &= (i_0, i_0, j_0), \\ &= (i_0, j_0, j_0), \end{aligned} \tag{36}$$

by noting that

$$3i_0 \not\equiv b \mod n \text{ assuming } i_0 \neq b/3 \text{ and } i_0 \neq (b-n)/3,$$
$$\text{and } (i_0 + j_0) + k \equiv k_0 + k \not\equiv b \mod n \text{ for any } k \in O \text{ (Remark (a))}.$$

It is verified to be feasible for (24) by

$$\hat{\pi}_i + \hat{\pi}_j \leq \frac{3}{4} + \frac{3}{4} = \frac{1}{2} + 1 \leq \hat{\pi}_k + 1, \text{ if } i = j = k_0;$$
$$\hat{\pi}_i + \hat{\pi}_j \leq \frac{3}{4} + \frac{1}{2} = \frac{1}{4} + 1 \leq \hat{\pi}_k + 1, \text{ otherwise};$$

since $i = j = k_0 \mod n$ implies $k \not\equiv i_0 \mod n$ and $k \not\equiv j_0 \mod n$ by the last two impossible cases in (36). The only possibly infeasible inequality in (25) with $3k_0 = 2b - n \equiv 2b \mod n$ contradicts the integrality of k_0 when $n \not\equiv 2b \mod 3$. Winding up the irredundancy of (22) without $\{i, j\} = \{b/3, (b-n)/3\}$, its feasibility for the additional inequalities (28) is trivially verified by

$$\frac{1}{4} \leq \hat{\pi}_i \leq \frac{3}{4} \text{ for all } i.$$

The infeasible solution (31) of (23) is feasible for (22), since

$$\hat{\pi}_i + \hat{\pi}_j \geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \geq \hat{\pi}_k \text{ with } k \equiv i + j \mod n.$$

It can be shown to be feasible for all the other inequalities in (23), say

$$\hat{\pi}_i + \hat{\pi}_j + \hat{\pi}_k \geq 1 \text{ with } |i| \leq |j| \leq |k| \text{ with } i + j + k \equiv b \mod n.$$

If $i_0 = j_0 = k_0 = b/3$ or $(b-n)/3$, then one of $\hat{\pi}_i, \hat{\pi}_j$ and $\hat{\pi}_k$ is $1/2$ implying feasibility. Otherwise, we consider two cases, $j = j_0$ and $j \neq j_0$. If $j = j_0$, then $i \neq i_0$ and $k \neq k_0$, and so $\hat{\pi}_i \geq 1/3$ and $\hat{\pi}_k \geq 1/3$ implying feasibility. Assume that $0 < i \leq j \leq k$ and $0 < i_0 \leq j_0 \leq k_0$. If $j < j_0$, then $k > k_0$ implies $\hat{\pi}_k = 1/2$, or $i_0 < i \leq j < j_0$ implies $\hat{\pi}_i = \hat{\pi}_j = 1/2$, both of which imply feasibility. If $j > j_0$, *vice versa*. The other cases are analogous or trivial. (See Remark (c) above.) It is also feasible for (24), since

$$\hat{\pi}_i + \hat{\pi}_j \leq \frac{1}{2} + \frac{1}{2} \leq \frac{1}{4} + 1 \leq \hat{\pi}_k + 1.$$

And, the feasibilities of $\hat{\pi}$ to (25) and (28) are trivial.

For the infeasible solution (32) of (24), we may assume that $3i_0 \neq 2b - n$, since $2i_0 \equiv k_0 + b \mod n$ and $3i_0 = 2b - n$ would contradict Remark (a). The infeasible solution (32) is feasible for (22), since

$$\begin{aligned} &\hat{\pi}_{k_0} + \hat{\pi}_{k_0} \geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \geq \hat{\pi}_k, \text{ if } k \equiv 2k_0 \in O \mod n, \text{ and} \\ &\hat{\pi}_i + \hat{\pi}_j \geq \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \geq \hat{\pi}_k \text{ with } i + j \equiv k \mod n, \text{ otherwise,} \end{aligned}$$

where the inequality $1/2 \geq \hat{\pi}_k$ in the first line is by Remark (a). The only possibly infeasible inequality in (23)

$$\hat{\pi}_{k_0} + \hat{\pi}_{k_0} + \hat{\pi}_{k_0} \geq 1 \text{ with } 3k_0 \equiv b \mod n$$

contradicts the assumption, $b \not\equiv 0$ and $\not\equiv n \mod 3$. It is feasible for all the other inequalities in (24),

$$\begin{aligned} \hat{\pi}_i + \hat{\pi}_j &= \frac{1}{2} + \frac{1}{2} \leq \hat{\pi}_{k_0} + 1 \text{ if } i + j \equiv k + b \equiv k_0 + b \mod n, \\ \hat{\pi}_i + \hat{\pi}_j &\leq \frac{3}{4} + \frac{3}{4} = \frac{1}{2} + 1 \leq \hat{\pi}_k + 1 \text{ otherwise.} \end{aligned}$$

The possibly infeasible cases for the inequalities in (25) are

$$\begin{aligned}(i,j,k) &= (i_0,i_0,i_0)\\ &= (i_0,i_0,j_0)\end{aligned}$$

with $i+j+k = 2b-n$ when $b \equiv 0 \mod 3$ or $b \equiv n \mod 3$. The first case is impossible by the assumption $3i_0 \neq 2b-n$. The second case contradicts Remark (e). And, it is trivially feasible for (28).

The relation (25) can be verified to be irredundant by the infeasible solution (33) in a very similar way to (23) above. The infeasible solutions (34) and (35) trivially prove the irredundancies of (28), respectively. □

The only remaining case that we have not considered before Theorem 3.5 is (C_n, b) with n even and b even, that is, having two halves $h_1 = \frac{b}{2}$ and $h_2 \equiv \frac{b-n}{2} \mod n$.

Corollary 3.6 *Let n and b be both even. The irredundant subadditivities (20) having a term $\pi_{h_j}, j = 1,2,$ are*

$$\begin{aligned}&\pi_{q_{j1}} \geq \frac{1}{4} \text{ with } 2q_{j1} \equiv h_j \mod n \text{ and } q_{j1} \in O, \text{ and}\\ &\pi_{\bar{q}_{j2}} \leq \frac{3}{4} \text{ with } 2q_{j2} \equiv h_j,\ \bar{q}_{j2} + q_{j2} \equiv h_j \mod n, \text{ and } q_{j2} \in X,\end{aligned}$$

whenever h_j is even.

3.3 Digression: General group problem

3.3.1 Nonabelian group problem

Aráoz and Johnson [3] generalized the group problems defined by Gomory [12] which are abelian group problems in their generalized viewpoint. A *group* is a set G with an addition $+$ such that for every i and j in G, $i+j$ is also in G and such that the following properties hold:

1. $i + (j+k) = (i+j)+k$ for all i,j,k in G (associativity).

2. $i + 0 = 0 + i$ for all i in G (zero element).

3. $i + (-i) = (-i) + i = 0$ for all $i \in G$ and some $-i \in G$ (negation).

The zero element 0 of G is easily seen to be unique, as is the negative $-i$ for a given $i \in G$. An *abelian* group satisfies, in addition:

$i + j = j + i$ for all i, j in G (commutativity).

All of our groups will be finite.

The group problem (G, M, b) is determined by a group G, a generating set $M \subseteq G - \{0\}$, a right-hand side vector $b \in G - \{0\}$, and an objective function $c(j) \geq 0, j \in M$. A *solution expression* to the group problem is an expression whose sum is b:

$$i + j + \cdots + k = b. \tag{37}$$

An *optimum solution expression* is a solution expression $i+j+\cdots+k$ which minimizes $c(i)+c(j)+\cdots+c(k)$ over all solution expressions. The *incidence vector* of the solution expression is $(t(j) : j \in M)$ where $t(j)$ is the number of times the group element j appears in the solution expression. The *group polyhedron* $P(G, M, b)$ is the convex hull of the incidence vectors t of solution expressions. The *group problem* (G, M, b) with objective c is to find the incidence vector of an optimum solution expression. In group theory, a *minimal word problem* is to find the optimal value of a group problem (G, M, b) with all-1 objective $\mathbf{1}$. The minimal word problem in groups is known to be NP-hard [7] if the set M of generators is not fixed in advance. Let $Cay(G, M)$ denote the *Cayley digraph* of the group G and the subset M. The vertices of $Cay(G, M)$ are the elements of G and the arcs are all ordered pairs $(i, i + j)$ for $i \in G, j \in M$. The arc $(i, i + j)$ is said to be *labeled by* j. Let $c : M \to \mathbb{R}_+$ be the length function; i.e., $c(j) \geq 0$ is the length of each arc labeled by $j \in M$. The group problem (G, M, b) is to find an optimum solution expression corresponding to a shortest path from node

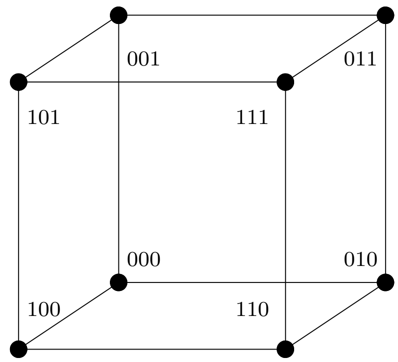

Figure 11: Diagram of Rubik's cube

0 to node b in $Cay(G, M)$ with length function c. Note that a path from 0 to b corresponds to an incidence vector t of a solution expression.

For an example of general group problem, let's consider the Rubik's cube. Define one move by rotating one side by $\pi/2$ clockwise (+) or counterclockwise (−) while fixing the sides. We want to know the minimum number of moves to solve it. Rubik's cube have 6 sides F, B, U, D, L, R (front, back, up, down, left, right sides in Figure 11), 8 vertices 0=000, 1=001, 2=010, 3=011, 4=100, 5=101, 6=110, 7=111, and 12 edges

$$LB, BD, LD, BU, LU, RB, RD, LF, FD, RU, FU, FR,$$

where $LB = BL$ is the intersection of L and B. For one move, we have 12 generators

$$M = \{L+, L-, R+, R-, U+, U-, D+, D-, F+, F-, B+, B-\},$$

where we denote the counterclockwise rotation of L by $L-$. We can express $L-$ by the permutation $(5, 1, 0, 4)(LU, LB, LD, LF)$ meaning the vertex at position 5 goes to the position 1, 1 goes to 0, and so on.

Let G be the group of permutations of Rubik's cube. Then, the minimum number of moves to solve a Rubik's cube is the length between 0 and the corresponding permutation on $Cay(G, M)$. If we use Dijkstra, we may enumerate all the nodes on the Cayley digraph. If the number of nodes is much larger than the diameter, Dijkstra doesn't look good for this shortest path problem. We can write the minimum number of moves as

$$\min\{\mathbf{1}.t : t \in P(G, M, b)\},$$

where $-b$ is the permutation of our given Rubik's cube.

3.3.2 Lifting

Gomory [12] showed a facet of a group polyhedron with a 0 coefficient can be constructed by repeating a facet of a smaller group polyhedron. His lifting theorem enables us to assemble small facets as building blocks for big facets. Aráoz and Johnson [3] generalized Gomory's lifting theorem to nonabelian group problems.

A *subgroup* K of a group G is a subset of the group which is a group with the inherited (same as G's) addition. The subgroup K partitions G into the left *cosets*

$$G/K = \{g + K : g \in G\}.$$

The set G/K of left cosets forms a group, called a *factor (or a quotient) group* of G modulo K, with the natural addition from G

$$(g_1 + K) + (g_2 + K) = (g_1 + g_2) + K \text{ for all } g_1, g_2 \in G,$$

if each left coset is the same as the corresponding right coset; i.e.,

$$g + K = K + g \text{ for all } g \in G,$$

in which case we call K a *normal subgroup* of G. Note that a subgroup of an abelian group is always a normal subgroup.

A *homomorphism* ϕ of a group G into a group H is a map $\phi : G \to H$ which preserves the addition; i.e.,

$$\phi(g_1 + g_2) = \phi(g_1) + \phi(g_2) \text{ for all } g_1, g_2 \in G.$$

The *kernel* $Ker(\phi)$ of the map $\phi : G \to H$ is defined to be the set of elements g in G which are mapped to $\phi(g) = 0$. Since ϕ is a homomorphism, the kernel $Ker(\phi)$ is a normal subgroup of G. We can immediately and partially generalize Gomory's lifting theorem to nonabelian group problems, which is a corollary of Theorem 7.1 in Aráoz and Johnson [3]:

Theorem 3.7 (Thm 4.1 in [3]) *If K is a normal subgroup of G and if $\hat{\pi}$ is a facet for the factor group G/K with right-hand side $\hat{b} \neq \hat{0}$, then a facet for G with right-hand side b, where $\phi(b) = \hat{b}$, is given by*

$$\pi(g) = \hat{\pi}(\phi(g)),$$

where ϕ is the homomorphism from G to G/K.

We can prove the theorem above by using the strong product of graphs. The *strong product* $\Gamma \boxtimes \Omega$ of two graphs Γ and Ω is defined on the Cartesian product of the vertex sets of the factors, two distinct vertices (u, v) and (x, y) of $\Gamma \boxtimes \Omega$ of $\Gamma \boxtimes \Omega$ being adjacent with respect to the strong product if $u = x$ and $vy \in E(\Omega)$, or $ux \in E(\Gamma)$ and $v = y$, or $ux \in E(\Gamma)$ and $vy \in E(\Omega)$. The basic example is $K_4 = K_2 \boxtimes K_2$. It is best described by the symbol $\boxtimes$. More generally, note that $K_{mn} = K_m \boxtimes K_n$.

Let's denote $Cay(G, G_+)$ simply by $Cay(G)$. The complete Cayley digraph $Cay(G)$ can be regarded as the complete graph in which each edge have two opposite directions of arcs in the digraph. Let

$$R = \{b_0 = 0, b_1, b_2, ..., b_{\frac{|G|}{|K|}-1} = b\}$$

be coset representatives. Then, we can write

$$Cay(G) = Cay(K) \boxtimes Cay(G/K) = Cay(K) \boxtimes K_R,$$

where K_R is the complete graph with the nodes R. For each $k \in K$, k-row of $Cay(K) \boxtimes K_R$ is another representatives $k + R = \{k, k + b_1, k + b_2, ..., k + b\}$, and, for each $j = 0, 1, 2, ..., |G/K| - 1$, the j-th column is the coset $K + b_j$.

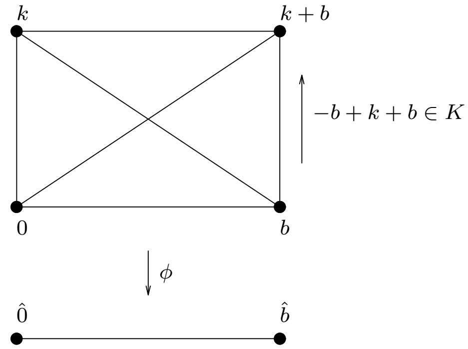

Figure 12: $\phi : Cay(G) = Cay(K) \boxtimes K_R \rightarrow Cay(G/K)$

We think of ϕ as the projection to $Cay(G/K)$: (See Fifure 12.) Each edge $u \sim (u+g)$ is projected to $\phi(u) \sim (\phi(u+g) = \phi(u) + \phi(g))$ (for $g \in K$, we may consider a loop.) Lift-up $\pi(g) = \hat{\pi}(\phi(g))$ is giving the edges of the strong product the values of the projected images.

To show Theorem 3.7, validity of π is easy to get because the projected image of a walk W from 0 to b gives a walk from $\hat{0}$ to $\hat{b}$. We take $|G/K|-1$ affinely independent paths $\hat{P}_1, ..., \hat{P}_{|G/K|-1}$ satisfying $\hat{\pi}\hat{t}[\hat{P}_i] = \hat{\pi}_{\phi(b)}$. Firstly, we lift up the paths $|K|$ times, and then add $|K|-1$ more independent paths in the strong product.

For $\hat{P}_1$, let $\hat{P}_1 = \hat{b}_{j_1} + \hat{b}_{j_2} + ... + \hat{b}_{j_{l_1}}$. By using each row $k+R$ of the strong product, we have $|K|$ lift-ups of $\hat{P}_1$ as

$$(k + b_{j_1}) + ... + (k + b_{j_{l_1}}) + k' = b, \text{ for some } k' \in K.$$

Then we have $|K|(|G|-1)$ independent paths lifted up.

To get $|K|-1$ more, fix one path $b_{j_1} + ... + b_{j_{l_1}} + k' = b$. Before start, run a self-loop of $\hat{0}$:

$$p(k) \cdot k + b_{j_1} + ... + b_{j_{l_1}} + k' = b, \text{ for each } k \in K_+,$$

where $p(k) \geq 2$ is the order of $k \in K$.

3.3.3 Binary group problem

The cyclic group problems are one of the two easiest group problems in having only one row. The other easiest group problems are the binary group problems in the smallest possible order 2. The *binary group* C_2^n is the direct product of n cyclic groups C_2 of order 2 with componentwise addition modulo 2. In the rest of this paper, we deal with only *master* binary group problems. We only need to consider the binary group problem $(C_2^n, \mathbf{1})$ with all-1 vector $\mathbf{1}$ as the right-hand side, for every nonzero b is an automorphic image of $\mathbf{1}$. Let's define O to be the set of the nonzero elements with 0 in the first component. The other elements belong to X. Note that we have no half h for any binary group problem with nonzero right-hand side. Gomory [12] implies the convex hull of nontrivial facets $\pi t \geq 1$ of $P(C_2^n, \mathbf{1})$

$$\begin{aligned}\Pi(C_2^n, \mathbf{1}) \quad = \quad & \{\pi \in \mathbb{R}^{C_n - \{0\}} : \\ & \pi_i + \pi_j = \pi_{\mathbf{1}} = 1 \text{ if } i + j = \mathbf{1} \text{ in } C_2^n, && (38) \\ & \pi_i + \pi_j \geq \pi_k \text{ if } i + j = k \text{ in } C_2^n\}, && (39)\end{aligned}$$

where none of i, j, k is $\mathbf{1}$.

The subadditive relations don't have any analogue of (24) and (23) because of the first component. They have only the other two analogues of (22) and (25)

$$\pi_i + \pi_j \geq \pi_{i+j} \text{ for the unordered pairs } \{i, j\} \subseteq O \text{ and } i + j \in O, \tag{40}$$

$$\pi_i + \pi_j + \pi_{i+j} \leq 2 \text{ for the unordered triples } \{i, j, i + j\} \subseteq O. \tag{41}$$

They define the projected image $\Pi(C_2^n, \mathbf{1})|_O$. We show that they are irredundant or the facets of $\Pi(C_2^n, \mathbf{1})|_O$. Note that $i, j, i + j$ are different from each other and the sum of each pair is the other one.

For $\pi_{i_0} + \pi_{j_0} \geq \pi_{i_0+j_0}$, a certificate is

$$\hat{\pi} = \left(\hat{\pi}_{i_0} = \hat{\pi}_{j_0} = \frac{1}{3}, \hat{\pi}_{i_0+j_0} = \frac{3}{4}\right), \tag{42}$$

and, for $\pi_{i_0} + \pi_{j_0} + \pi_{i_0+j_0} \leq 2$, a certificate is

$$\hat{\pi} = \left(\hat{\pi}_{i_0} = \hat{\pi}_{j_0} = \hat{\pi}_{i_0+j_0} = \frac{3}{4}\right). \tag{43}$$

Observe that they are infeasible for the corresponding inequalities. We need to show each certificate is feasible for the other constraints. For (42), the other inequalities in (40) are feasible

$$\begin{aligned} \pi_i + \pi_j &\geq \frac{1}{2} + \frac{1}{2} \geq \frac{3}{4} = \pi_{i+j}, \text{ if } i + j = i_0 + j_0, \\ \pi_i + \pi_j &\geq \frac{1}{3} + \frac{1}{3} \geq \frac{1}{2} \geq \pi_{i+j}, \text{ otherwise.} \end{aligned}$$

The inequalities in (41) are feasible

$$\pi_i + \pi_j + \pi_{i+j} \leq \frac{3}{4} + \frac{1}{2} + \frac{1}{2} \leq 2,$$

since $i + j \neq i$.

For (43), the inequalities in (40) are feasible

$$\pi_i + \pi_j \geq \frac{1}{2} + \frac{1}{2} \geq \frac{3}{4} \geq \pi_{i+j}.$$

The other inequalities (41) cannot have all terms π_i, π_j, π_{i+j} equal to $3/4$ and are feasible

$$\pi_i + \pi_j + \pi_{i+j} \leq 2 = \frac{3}{4} + \frac{3}{4} + \frac{1}{2}.$$

CHAPTER IV

BLOCKING POLYHEDRA AND SHOOTING

We now consider Fulkerson's [9] framework of blocking pairs of polyhedra. The *blocking polyhedron* $\mathfrak{B}(P(C_n, b))$ of the master cyclic group polyhedron $P(C_n, b)$ is

$$\mathfrak{B}(P(C_n, b)) = \{\pi \in \mathbb{R}_+^{C_n - \{0\}} : \pi t \geq 1 \text{ for all } t \in P(C_n, b)\}.$$

Since $P(C_n, b)$ satisfies

$$P(C_n, b) = P(C_n, b) + \mathbb{R}_+^{\{1,\ldots,n-1\}},$$

it is easy to see that $\mathfrak{B}(\mathfrak{B}(P(C_n, b))) = P(C_n, b)$. Let the rows of the matrix Π be the nontrivial facets π of $P(C_n, b)$ and let those of T be the vertices. The cyclic group polyhedron $P(C_n, b)$ can be written as

$$\begin{aligned} P(C_n, b) &= \{t : \Pi t \geq 1, t \geq 0\} \\ &= CONV(T) + \mathbb{R}_+^{C_n - \{0\}}, \end{aligned}$$

where $CONV(T)$ denotes the convex hull of rows of T. It can be easily shown that

$$\mathfrak{B}(P(C_n, b)) = \{\pi : T\pi \geq 1, \pi \geq 0\}.$$

That is, the vertices of $P(C_n, b)$ are the facets of $\mathfrak{B}(P(C_n, b))$. The blocking polyhedron has a similar representation:

$$\begin{aligned} \mathfrak{B}(P(C_n, b)) &= CONV(\Pi) + \mathbb{R}_+^{C_n - \{0\}} \\ &= \Pi(C_n, b) + \mathbb{R}_+^{C_n - \{0\}}. \end{aligned}$$

4.1 The vertices of cardinality length 3

The vertex of cardinality length 1 of $P(C_n, b)$ is the trivial solution ($t_b = 1$) corresponding to the facet $\pi_b \geq 1$ of $\mathfrak{B}(P(C_n, b))$. The vertices of cardinality length 2 are the coefficient vectors of the complementarities,

$$\pi_i + \pi_{\bar{i}} = \pi_b = 1 \text{ for all } i \in C_n - \{0, b\}.$$

In Chapter 3, we discovered the facets of the subadditive polytope $CONV(\Pi) = \Pi(C_n, b)$. They can be tilted to be facets of $\mathfrak{B}(P(C_n, b))$ which are vertices of $P(C_n, b)$. They don't provide all the facets. For example, the 27th vertex $t^{27} = (t_4 = 3, t_9 = 1)$ of $P(C_{11}, 10)$ in the Gomory's table [12] is the facet $3\pi_4 + \pi_9 \geq 1$ of $\mathfrak{B}(P(C_{11}, 10))$ that is equivalent to $3\pi_4 \geq \pi_1$ and not a facet of $\Pi(C_{11}, 10)$. In this section, we characterize the vertices of cardinality length 3 with the facets of $\Pi(C_n, b)$.

A solution t to the master cyclic group problem (C_n, b) is called *irreducible* if every integral t', $0 \leq t'_g \leq t_g, g \in C_n - \{0\}$, gives a different element $\sum_{g \in C_n - \{0\}} t'_g g$ in C_n. Gomory [12] showed that the vertices of $P(C_n, b)$ are irreducible. The four types of inequalities for the subadditive polytope give irreducible solutions of cardinality length 3:

$$\pi_i + \pi_j \geq \pi_k \quad \Leftrightarrow \quad \pi_i + \pi_j + \pi_{\bar{k}} \geq \pi_b = 1 \tag{44}$$

$$\pi_i + \pi_j + \pi_k \geq \pi_b = 1 \quad \Leftrightarrow \quad \pi_i + \pi_j + \pi_k \geq \pi_b = 1 \tag{45}$$

$$\pi_i + \pi_j \leq \pi_k + \pi_b = \pi_k + 1 \quad \Leftrightarrow \quad \pi_{\bar{i}} + \pi_{\bar{j}} + \pi_k \geq \pi_b = 1 \tag{46}$$

$$\pi_i + \pi_j + \pi_k \leq \pi_b + \pi_b = 2 \quad \Leftrightarrow \quad \pi_{\bar{i}} + \pi_{\bar{j}} + \pi_{\bar{k}} \geq \pi_b = 1, \tag{47}$$

where all i, j, k are in $O = ((b-n)/2, b/2) \cap \mathbb{Z} - \{0\} \subseteq C_n$. We see that there is a one-to-one correspondence between the four types of inequalities and the solutions of cardinality length 3 without any half. Note that the solutions of cardinality length 3 with a half are reducible.

Lemma 4.1 *A solution of cardinality length 3 is irreducible if and only if it has no half.*

Proof. If a solution of cardinality length 3 contains a half h, then it is trivially reducible, since the sum of the other two elements is the half h. Conversely, consider a solution t of cardinality length 3 without any half. If it has only one nonzero component, it is trivially irreducible. If it has exactly two nonzero components, say $(t_i = 2, t_j = 1)$, then each pair of elements in $\{i, 2i, j, i+j, 2i+j \equiv b \mod n\}$ are different from each other. If it has exactly three nonzero components, say $(t_i = t_j = t_k = 1)$, then each pair of elements in $\{i, j, k, i+j, j+k, k+i, i+j+k\}$ are different from each other, completing the proof. □

Corollary 4.2 *Each inequality in the four types (22)-(25) corresponds to each irreducible solution of cardinality length 3.*

The additional inequalities (28) for $j = 1, 2$ give the vertices of cardinality length 4 with only one nonzero component,

$$(t_{q_{jO}} = 4) \text{ whenever } q_{jO} \text{ is an integer,} \tag{48}$$

$$(t_{q_{jX}} = 4) \text{ whenever } q_{jX} \text{ is an integer.} \tag{49}$$

Now, consider the exceptional inequality (29) in the first type (22),

$$\pi_i + \pi_j \geq \pi_k, \{i, j\} = \{b/3, (b+2n)/3\} \text{ whenever } b \equiv n \equiv 0 \mod 3.$$

Its corresponding path of cardinality length 3 is the coefficients of

$$\pi_{b/3} + \pi_{(b+2n)/3} + \pi_{(b+n)/3} \geq 1. \tag{50}$$

We know that the exceptional inequality is not a facet of the subadditive polytope. The solution[1] $(t_{b/3} = t_{(b+2n)/3} = t_{(b+n)/3} = 1)$ is the average of the three solutions $(t_{b/3} = 3), (t_{(b+2n)/3} = 3), (t_{(b+n)/3} = 3)$, and so, is not a vertex of $P(C_n, b)$. Therefore, the facets of the subadditive polytope give vertices of the cyclic group problem.

Theorem 4.3 *Each inequality in the four types (22)-(25) without the exceptional inequality (29) corresponds to each vertex of cardinality length 3.*

[1]From the solution, Steve Tyber constructed infinitely many non-extreme irreducible solutions to the master knapsack problem.

Proof. Consider an irreducible solution of cardinality length 3

$$i + j + k \equiv b \mod n,$$

where repetition is allowed. If the solution is not extreme, there is no solution of cardinality length ≤ 2 containing only group elements in $\{i, j, k\}$ and so the solution is a convex combination of solutions of cardinality length 3. In the two cases of $i = j = k$ and $i = j \neq k$, the solution is easily shown to be extreme. We may assume i, j, k are all different from each other. Only possible combination for the solution can be shown to be of

$$3i \equiv 3j \equiv 3k \equiv b \mod n$$

which corresponds to the exceptional inequality. □

4.2 Shooting from the natural interior point

Based on the nonnegativities of $\pi \in \Pi(C_n, b)$, Gomory, Johnson and Evans [13] explained geometry of *shooting in* $v > 0$ as follows: Increase the vector v by multiplying by a scalar λ, test the vector $v\lambda$ against the various facets $\pi^i t \geq 1$ to see which side of each facet it is on, or, equivalently, which facet defining inequalities are satisfied by $v\lambda$. Keep increasing λ until $v\lambda$ gets to be on the far side, the master cyclic group polyhedron side, of all those facets. When $v\lambda$ gets beyond all the facets, it is in the polyhedron, so the last facet it gets beyond is the sought-after facet containing the point where $v\lambda$ hits the polyhedron. Put another way, if $v\lambda$ lies beyond all the facets but one, and lies on that one, that facet is the one hit by the vector v.

This conceptual process requires having the facets to test against the direction $v\lambda$. Except for the smallest master cyclic group polyhedra, we do not have those facets. Based on a lecture by Ralph Gomory [13] at Georgia Tech, we have a way of doing shooting experiments without knowledge of the facets. We can define *shooting in* v

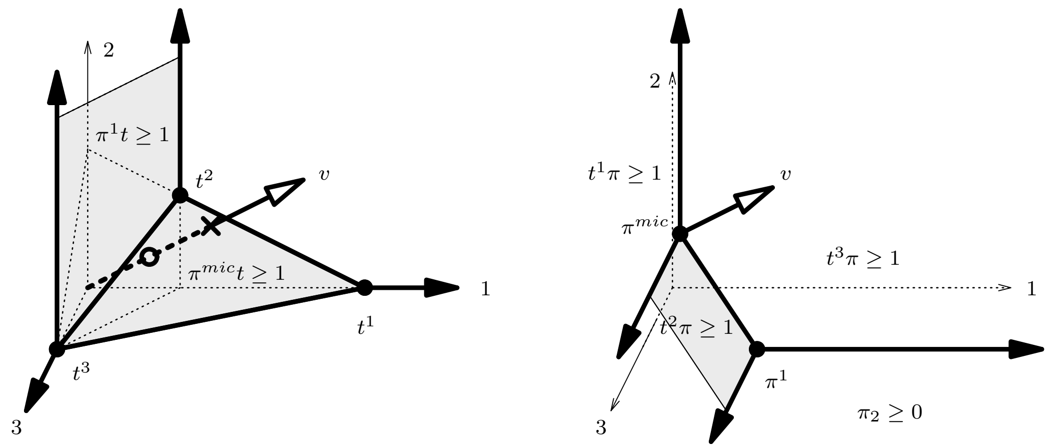

Figure 13: Shooting in v over $P(C_4, 3)$ and $\mathfrak{B}(P(C_4, 3))$

as solving the linear programming problem,

$$\begin{aligned} \min \quad & v\pi \\ st \quad & \pi \in \Pi(C_n, b), \end{aligned} \tag{51}$$

which is shown in [13] to be equivalent to the conceptual process of shooting as mentioned above. Note that we can replace (52) by $\pi \in \mathfrak{B}(P(C_n, b))$ to have another equivalent linear programming problem,

$$\begin{aligned} \min \quad & v\pi \\ st \quad & \pi \in \mathfrak{B}(P(C_n, b)). \end{aligned} \tag{52}$$

The objective vector v is contained in the *reverse polar cone* of its optimal solution π; i.e., the cone of points x satisfying $xy \geq 0$ for all y in the translate $\mathfrak{B}(P(C_n, b)) - \pi$ of the blocking polyhedron $\mathfrak{B}(P(C_n, b))$ by $-\pi$ so that π is translated into the origin $\mathbf{0}$.

Consider the example of shooting in $v = (3, 2, 1)$ over the master cyclic group problem $(C_4, 3)$ in Figure 13. The thick-lined polyhedron $P(C_4, 3)$ on the left side is defined by the nonnegativities and two nontrivial facets $\pi^1 t = (1, 0, 1).t \geq 1$ and $\pi^{mic} t = (1/3, 2/3, 1).t \geq 1$. The arrow v goes through the hollow dot $(3/4, 1/2, 1/4)$

on $\pi^1 t = 1$ and hits the facet $\pi^{mic} t \geq 1$ at the cross $(9/10, 6/10, 3/10)$ that is a boundary point of $P(C_4, 3)$, which is corresponding to the solution π^{mic} on the right side to the linear programming problem over the thick-lined polyhedron $\mathfrak{B}(P(C_4, 3))$,

$$\begin{aligned} \min \quad & v_1\pi_1 + v_2\pi_2 + v_3\pi_3 && (53) \\ st \quad & t^1\pi = 3\pi_1 \geq 1 && (54) \\ & t^2\pi = \pi_1 + \pi_2 \geq 1 && (55) \\ & t^3\pi = \pi_3 \geq 1 && (56) \\ & \pi \geq 0, && (57) \end{aligned}$$

which is equivalent to the linear programming problem over the line segment $\Pi(C_4, 3) = [\pi^1, \pi^{mic}]$. Note that the objective vector $v = (3, 2, 1)$ is contained in the reverse polar cone of the optimal solution π^{mic}. If $v = (0, 1, 0)$, the arrow goes through $(0, 3/2, 0)$ on $\pi^{mic} t = 1$ and goes on forever below $\pi^1 t \geq 1$. We regard $v = (0, 1, 0)$ as hitting $\pi^1 t \geq 1$ at ∞. Note that shooting in $(t_b = 1)$ hits all the nontrivial facets of $P(C_n, b)$ in general.

We can reduce shooting in v by projecting (51) onto $O = ((b-n)/2, b/2) \cap \mathbb{Z} - \{0\}$ to have the equivalent linear programming problem

$$\begin{aligned} \min \quad & \sum_{i \in O} (v_i - v_{\bar{i}})\pi_i \\ st \quad & \pi_O \in \Pi(C_n, b)_O. \end{aligned}$$

Its optimal basic solution π_O is the restriction to O of the optimal basic solution π for the shooting linear programming (51).

The *natural interior point* of the subadditive polytope $\Pi(C_n, b)$ is

$$\dot{\pi} = (\dot{\pi}_i = 1/2 \text{ for all } i \neq b; \dot{\pi}_b = 1).$$

Every subadditivity has the same amount of slack that is $1/2$. Let's translate $\Pi(C_n, b)_O$ into $\Pi(C_n, b)_O - \dot{\pi}_O$ so that $\dot{\pi}_O$ is translated into the origin $\mathbf{0}$ of $\mathbb{R}^O$. That is, we translate the facets of $\Pi(C_n, b)_O$ into the following upper bound type inequalities with right-hand side 1 to have $\Pi(C_n, b)_O - \dot{\pi}_O$:

$$-2\pi_i - 2\pi_j + 2\pi_k \leq 1 \qquad \text{whenever } i + j \equiv k, i \not\equiv \frac{b}{3}, i \not\equiv \frac{b-n}{3} \mod n, \tag{58}$$

$$-2\pi_i - 2\pi_j - 2\pi_k \leq 1 \qquad \text{whenever } i + j + k \equiv b \mod n, \tag{59}$$

$$2\pi_i + 2\pi_j - 2\pi_k \leq 1 \qquad \text{whenever } i + j \equiv k + b \mod n, \tag{60}$$

$$2\pi_i + 2\pi_j + 2\pi_k \leq 1 \qquad \text{whenever } i + j + k \equiv 2b \mod n, \tag{61}$$

$$-4\pi_{q_{jO}} \leq 1 \qquad \text{whenever } q_{jO} \text{ is an integer}, \tag{62}$$

$$4\pi_{\bar{q}_{jO}} \leq 1 \qquad \text{whenever } q_{jX} \text{ is an integer}, \tag{63}$$

where all i, j, k are in O. For example, (58) is the translate of (22) by $-\dot{\pi}$; i.e.,

$$-2\pi_i - 2\pi_j + 2\pi_k \leq 1 \quad \Leftarrow \quad \left(\pi_i + \frac{1}{2}\right) + \left(\pi_j + \frac{1}{2}\right) \geq \left(\pi_k + \frac{1}{2}\right).$$

Note that the constraints (58)-(61) correspond to the paths (44)-(47) with cardinality length 3 containing no half which we may enumerate to construct $\Pi(C_n, b)_O$ for shooting from the natural interior point.

Define the *support function* $\mu(x)$ by

$$\mu(x) = \sup\{x\pi : \pi \in \Pi(C_n, b)_O - \dot{\pi}_O\}$$

for all $x \in \mathbb{R}^O$. The *plus level set* of $\Pi(C_n, b)_O - \dot{\pi}_O$ is

$$(\Pi(C_n, b)_O - \dot{\pi}_O)^+ = \{x : \mu(x) \leq 1\}.$$

Each facet $\pi t \geq 1$ of $P(C_n, b)$ corresponds to each facet $\varphi x = (\pi_O - \dot{\pi}_O)x \leq 1$ of $(\Pi(C_n, b)_O - \dot{\pi}_O)^+$. Note that $(\Pi(C_n, b)_O - \dot{\pi}_O)^+ = \{x : \mu(x) \leq 1\}$ is the convex hull of the vertices that are the coefficients of the facets (58)-(63) of $\Pi(C_n, b)_O - \dot{\pi}_O$.

Shooting from the natural interior point in $w \in \mathbb{R}^O$ is solving the linear programming problem

$$\begin{array}{ll} \max & w\pi_O \\ st & \pi_O \in \Pi(C_n, b)_O, \end{array}$$

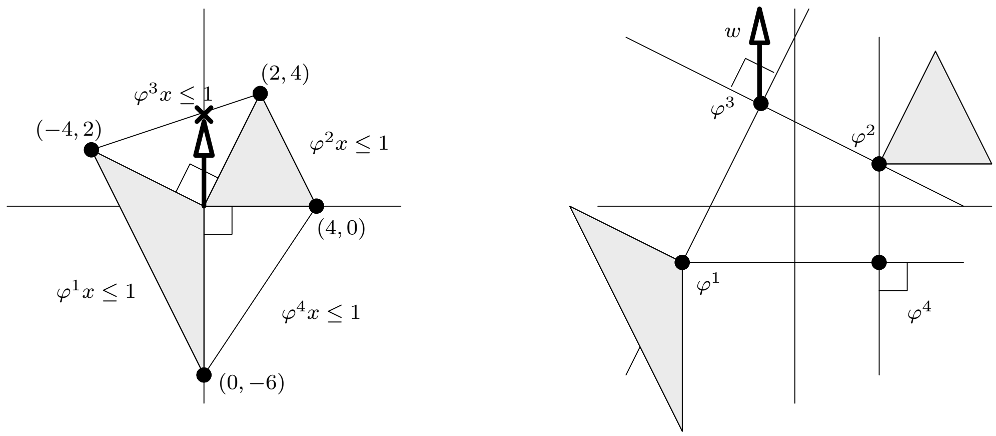

Figure 14: $(\Pi(C_7, 6)_O - \dot{\pi}_O)^+$ and $\Pi(C_7, 6)_O - \dot{\pi}_O$

which is equivalent by translation $\pi_O = \varphi + \dot{\pi}_O$ to

$$\begin{aligned} \max \quad & w\varphi \\ st \quad & \varphi \in \Pi(C_n, b)_O - \dot{\pi}_O. \end{aligned} \tag{64}$$

The objective vector w is contained in the *polar cone* of its optimal solution φ; i.e., the cone of points x satisfying $xy \leq 0$ for all y in the translate $\Pi(C_n, b)_O - \dot{\pi}_O - \varphi$ of $\Pi(C_n, b)_O - \dot{\pi}_O$ by $-\varphi$ so that φ is translated into the origin $\mathbf{0}$. Geometrically, the vertex φ is the facet $\varphi x = (\pi_O - \dot{\pi}_O)x \leq 1$ of $(\Pi(C_n, b)_O - \dot{\pi}_O)^+$ hit by the arrow shot in w from the origin. We think of the facet $\pi t \geq 1$ of $P(C_n, b)$ with $\pi_O = \varphi + \dot{\pi}$ as hit by that shooting from the natural interior point in w. Note that shooting from the natural interior point in w is *equivalent* (yielding the corresponding optimal solution) to shooting in $v \geq 0$ with the relation

$$w = -\lambda v' = -\lambda(v_i' = v_i - v_{\bar{i}} : i \in O) \text{ for some positive scalar } \lambda > 0. \tag{65}$$

Example 4.4 Let's consider $(\Pi(C_7, 6)_O - \dot{\pi}_O)^+$ and $\Pi(C_7, 6)_O - \dot{\pi}_O$ in Figure 14. We set $O = \{1, 2\}$ and translate $\Pi(C_7, 6)_O$ into $\Pi(C_7, 6)_O - \dot{\pi}_O$ as follows:

$$\begin{aligned} 2\pi_1 \geq \pi_2 &\;\;\Rightarrow\;\; -4\varphi_1 + 2\varphi_2 \leq 1 \\ 3\pi_2 \geq 1 &\;\;\Rightarrow\;\; -6\varphi_2 \leq 1 \\ \pi_1 + 2\pi_2 \leq 2 &\;\;\Rightarrow\;\; 2\varphi_1 + 4\varphi_2 \leq 1 \\ \pi_1 \leq 3/4 &\;\;\Rightarrow\;\; 4\varphi_1 \leq 1. \end{aligned}$$

The objective $w = (w_1 = 0, w_2 = 1)$ of the linear programming problem (64) belongs to the polar cone of the vertex φ^3 of $\Pi(C_7, 6)_O$ on the right side of the figure and the corresponding arrow hits the facet $\varphi^3 x \leq 1$ of $(\Pi(C_7, 6)_O - \dot{\pi}_O)^+$ on the left side, where $\varphi^3 = (-1/10, 3/10) = (\pi_1^3, \pi_2^3) - (1/2, 1/2)$. The shooting in v satisfying the relation (65) hits the facet π^3 of $P(C_7, 6)$ in Figure 9,

$$\pi^3 t = \frac{4}{10}t_1 + \frac{8}{10}t_2 + \frac{5}{10}t_3 + \frac{2}{10}t_4 + \frac{6}{10}t_5 + \frac{10}{10}t_6 \geq 1.$$

Gomory, Johnson and Evans [13] implemented *shooting experiment*; i.e., shooting in a random vector v uniformly distributed on the positive part of sphere, $||v|| = 1$ and $v > 0$. We can generate the random vector v by changing the signs of negative components of a random vector $u/||u||$ uniformly distributed on the whole unit sphere $||v|| = 1$, which is derived by Knuth [22] from a random vector u having distribution $N(0, I)$; i.e., the components of u are independent and have normal distribution $N(0, 1)$ each. In fact, we don't have to scale u into the unit vector $u/||u||$ in shooting experiment.

Shooting experiment from the natural interior point is shooting from the natural interior point in a random vector w uniformly distributed on the sphere $||w|| = 1$, which is similar to traditional shootings for other problems such as the traveling salesman problem. The probability of a facet φ of $(\Pi(C_7, 6)_O - \dot{\pi}_O)^+$ hit in the shooting experiment from the natural interior point is proportional to the size of the intersection of the sphere $||x|| = 1$ with the cone generated by the facet $\{x : \varphi x =$

$1\}\cap(\Pi(C_7,6)_O-\dot{\pi}_O)^+$. In Example 4.4, the angle for each facet φ of $(\Pi(C_7,6)_O-\dot{\pi}_O)^+$ on the left side in Figure 14 is the same as the angle of the polar cone of φ on the right side and is proportional to the probability of the facet φ hit in the shooting experiment from $\dot{\pi}_O$. The ratios of angles for φ^1, φ^2, φ^3 and φ^4 are 0.3238, 0.1762, 0.2500 and 0.2500, respectively.

The following tables are results of shooting experiment from the natural interior point. The results are almost the same as those of usual shooting experiment. Gomory, Johnson and Evans [13] observed the concentration of hits; i.e., only a small percentage of facets absorb 50% of the hits. Each superscript in the most left columns is the number of the facet in the tables given by Gomory [12].

Table 2: The facets of $P(C_8,7)$ and the numbers of hits by 10,000 shots

	π_1	π_2	π_3	# hits/10000
π^1	1	0	1	3081
π^2	0.33333333	0.66666667	1	1915
π^5	0.14285714	0.28571429	0.42857143	1682
π^7	0.6	0.66666667	0.2	1283
π^4	0.6	0.4	0.2	815
π^3	0.33333333	0.66666667	0.33333333	633
π^6	0.77777778	0.66666667	0.55555556	591

Table 3: The facets of $P(C_9,8)$ and the numbers of hits by 10,000 shots

	π_1	π_2	π_3	# hits
π^1	0.5	1	0	2915
π^3	0.125	0.25	0.375	1860
π^5	0.28571429	0.57142857	0.85714286	1675
π^7	0.8	0.25	0.6	1256
π^4	0.8	0.7	0.6	1008
π^6	0.6875	0.25	0.375	664
π^2	0.5	0.25	0.75	622

Table 4: The facets of $P(C_{10}, 9)$ and the numbers of hits by 10,000 shots

	π_1	π_2	π_3	π_4	# hits
π^1	1	0	1	0	2757
π^3	0.25	0.5	0.75	1	1572
π^9	0.11111111	0.22222222	0.33333333	0.44444444	1322
π^4	0.66666667	0.5	0.33333333	1	1312
π^{12}	0.42857143	0.85714286	0.33333333	0.28571429	837
π^{11}	0.42857143	0.85714286	0.57142857	0.28571429	526
π^{10}	0.81818182	0.72727273	0.63636364	0.54545455	411
π^5	0.66666667	0.5	0.33333333	0.16666667	358
π^6	0.25	0.5	0.75	0.375	273
π^7	0.66666667	0.77777778	0.33333333	0.44444444	256
π^8	0.66666667	0.22222222	0.33333333	0.44444444	217
π^2	0.33333333	0.66666667	0.33333333	0.66666667	159

Table 5: The facets of $P(C_{11}, 10)$ and the numbers of hits by 10,000 shots

	π_1	π_2	π_3	π_4	# hits
π^1	0.1	0.2	0.3	0.4	1354
π^7	0.22222222	0.44444444	0.66666667	0.88888889	1291
π^{13}	0.375	0.75	0.66666667	0.125	935
π^2	0.83333333	0.75	0.66666667	0.58333333	872
π^{16}	0.57142857	0.75	0.14285714	0.71428571	816
π^{17}	0.83333333	0.2	0.66666667	0.4	813
π^5	0.375	0.75	0.4375	0.125	646
π^3	0.57142857	0.35714286	0.14285714	0.71428571	579
π^6	0.83333333	0.44444444	0.66666667	0.27777778	404
π^{11}	0.83333333	0.29166667	0.66666667	0.58333333	403
π^9	0.65	0.2	0.3	0.4	402
π^4	0.375	0.75	0.4375	0.8125	297
π^8	0.22222222	0.44444444	0.66666667	0.27777778	276
π^{14}	0.375	0.75	0.20833333	0.58333333	250
π^{18}	0.46666667	0.2	0.66666667	0.4	187
π^{15}	0.69230769	0.53846154	0.38461538	0.23076923	173
π^{10}	0.65	0.75	0.3	0.4	162
π^{12}	0.375	0.75	0.66666667	0.58333333	140

Table 6: 22 facets of $P(C_{12}, 11)$ hit by 10,000 shots

π_1	π_2	π_3	π_4	π_5	# hits
1	0	1	0	1	2226
0.5	1	0	0.5	1	1809
0.33333333	0.66666667	1	0	0.33333333	1288
0.090909091	0.18181818	0.27272727	0.36363636	0.45454545	881
0.2	0.4	0.6	0.8	1	849
0.71428571	0.4	0.42857143	0.8	0.14285714	419
0.71428571	0.57142857	0.42857143	0.28571429	0.14285714	387
0.84615385	0.76923077	0.69230769	0.61538462	0.53846154	325
0.2	0.4	0.6	0.8	0.4	282
0.6	0.4	0.2	0.8	0.6	223
0.71428571	0.28571429	0.42857143	0.57142857	0.14285714	187
0.71428571	0.57142857	0.42857143	0.71428571	0.14285714	181
0.8	0.4	0.6	0.8	0.4	157
0.63636364	0.18181818	0.27272727	0.36363636	0.45454545	143
0.6	0.4	0.6	0.8	0.2	119
0.5	0.25	0.75	0.5	0.25	116
0.33333333	0.66666667	0.33333333	0.66666667	0.33333333	94
0.33333333	0.66666667	0.33333333	0.33333333	0.33333333	79
0.44	0.4	0.36	0.8	0.28	66
0.56521739	0.60869565	0.65217391	0.69565217	0.2173913	66
0.33333333	0.66666667	0.66666667	0.66666667	0.33333333	55
0.38461538	0.76923077	0.69230769	0.61538462	0.53846154	48

Table 7: 47 facets of $P(C_{13}, 12)$ hit by 10,000 shots

π_1	π_2	π_3	π_4	π_5	# hits
0.083333333	0.16666667	0.25	0.33333333	0.41666667	1052
0.18181818	0.36363636	0.54545455	0.72727273	0.90909091	827
0.85714286	0.28571429	0.71428571	0.57142857	0.57142857	684
0.4	0.4	0.4	0.8	0.8	543
0.85714286	0.16666667	0.71428571	0.33333333	0.57142857	522
0.85714286	0.28571429	0.71428571	0.33333333	0.57142857	463
0.6	0.2	0.8	0.33333333	0.4	406
0.66666667	0.33333333	0.33333333	0.66666667	0.66666667	350
0.625	0.16666667	0.25	0.33333333	0.41666667	323
0.6	0.2	0.8	0.4	0.4	314
0.22222222	0.44444444	0.44444444	0.66666667	0.88888889	309
0.44444444	0.16666667	0.61111111	0.33333333	0.77777778	297
0.66666667	0.33333333	0.33333333	0.33333333	0.66666667	234
0.44444444	0.38888889	0.38888889	0.33333333	0.77777778	227
0.1	0.2	0.3	0.4	0.4	221
0.625	0.16666667	0.79166667	0.33333333	0.41666667	214
0.18181818	0.36363636	0.54545455	0.72727273	0.72727273	190
0.22222222	0.44444444	0.55555556	0.66666667	0.88888889	189
0.1	0.2	0.25	0.35	0.4	183
0.44444444	0.38888889	0.61111111	0.33333333	0.77777778	181
0.80555556	0.16666667	0.61111111	0.33333333	0.41666667	165
0.625	0.2	0.25	0.4	0.4	153
0.8	0.2	0.6	0.33333333	0.4	152
0.33333333	0.33333333	0.66666667	0.33333333	0.66666667	139
0.1	0.2	0.3	0.33333333	0.4	134
0.8	0.2	0.6	0.4	0.4	121
0.73333333	0.16666667	0.46666667	0.33333333	0.63333333	110
0.15	0.2	0.25	0.4	0.4	91
0.2	0.4	0.6	0.7	0.8	87
0.625	0.25	0.25	0.5	0.5	87
0.625	0.25	0.25	0.33333333	0.5	86
0.66666667	0.33333333	0.66666667	0.66666667	0.66666667	81
0.63333333	0.2	0.26666667	0.33333333	0.4	79
0.11111111	0.22222222	0.25	0.33333333	0.44444444	75
0.21428571	0.42857143	0.57142857	0.71428571	0.85714286	74
0.625	0.2	0.25	0.35	0.4	67
0.1	0.2	0.26666667	0.33333333	0.4	66
0.125	0.25	0.25	0.375	0.5	66
0.33333333	0.33333333	0.66666667	0.66666667	0.66666667	57
0.10416667	0.20833333	0.25	0.33333333	0.41666667	55
0.28571429	0.42857143	0.57142857	0.71428571	0.85714286	55
0.16666667	0.25	0.25	0.33333333	0.5	53
0.2	0.4	0.6	0.6	0.8	53
0.625	0.20833333	0.25	0.33333333	0.41666667	49
0.21052632	0.42105263	0.52631579	0.73684211	0.84210526	47
0.25	0.25	0.25	0.5	0.5	40
0.11111111	0.22222222	0.33333333	0.33333333	0.44444444	29

Table 8: 65 facets of $P(C_{14}, 13)$ hit by 10,000 shots

π_1	π_2	π_3	π_4	π_5	π_6	# hits
1	0	1	0	1	0	2231
0.16666667	0.33333333	0.5	0.66666667	0.83333333	1	863
0.076923077	0.15384615	0.23076923	0.30769231	0.38461538	0.46153846	649
0.4	0.8	0.5	0.2	0.6	1	601
0.75	0.33333333	0.5	0.66666667	0.25	1	572
0.75	0.625	0.5	0.375	0.25	1	397
0.27272727	0.54545455	0.81818182	0.66666667	0.090909091	0.36363636	387
0.86666667	0.8	0.73333333	0.66666667	0.6	0.53333333	357
0.55555556	0.8	0.11111111	0.66666667	0.6	0.22222222	324
0.27272727	0.54545455	0.81818182	0.45454545	0.090909091	0.36363636	278
0.55555556	0.33333333	0.11111111	0.66666667	0.44444444	0.22222222	274
0.75	0.625	0.5	0.375	0.25	0.125	261
0.16666667	0.33333333	0.5	0.66666667	0.83333333	0.41666667	190
0.55555556	0.72222222	0.11111111	0.66666667	0.44444444	0.22222222	161
0.55555556	0.59259259	0.11111111	0.66666667	0.7037037	0.22222222	150
0.61538462	0.15384615	0.23076923	0.30769231	0.38461538	0.46153846	133
0.64705882	0.47058824	0.29411765	0.11764706	0.76470588	0.58823529	133
0.4	0.8	0.5	0.2	0.6	0.3	120
0.47368421	0.21052632	0.68421053	0.42105263	0.15789474	0.63157895	111
0.4	0.8	0.26666667	0.2	0.6	0.53333333	103
0.55555556	0.33333333	0.88888889	0.66666667	0.44444444	0.22222222	98
0.4	0.8	0.73333333	0.2	0.6	0.53333333	96
0.16666667	0.33333333	0.5	0.66666667	0.25	0.41666667	91
0.4	0.8	0.73333333	0.66666667	0.6	0.53333333	85
0.68	0.8	0.36	0.48	0.6	0.16	83
0.75	0.1875	0.5	0.375	0.25	0.5625	65
0.55555556	0.59259259	0.62962963	0.66666667	0.18518519	0.22222222	62
0.27272727	0.54545455	0.18181818	0.45454545	0.72727273	0.36363636	61
0.4	0.8	0.15	0.55	0.6	0.3	59
0.86666667	0.33333333	0.73333333	0.66666667	0.6	0.53333333	58
0.33333333	0.66666667	0.33333333	0.66666667	0.33333333	0.66666667	56
0.4	0.33333333	0.73333333	0.66666667	0.13333333	0.53333333	54
0.27272727	0.54545455	0.81818182	0.24242424	0.51515152	0.36363636	52
0.27272727	0.54545455	0.81818182	0.45454545	0.72727273	0.36363636	51
0.75	0.33333333	0.5	0.66666667	0.25	0.41666667	45
0.55555556	0.33333333	0.5	0.66666667	0.83333333	0.22222222	41
0.75	0.33333333	0.5	0.66666667	0.83333333	0.41666667	41
0.4	0.8	0.26666667	0.66666667	0.6	0.53333333	39
0.34615385	0.69230769	0.23076923	0.30769231	0.38461538	0.46153846	36
0.4	0.8	0.26666667	0.66666667	0.6	0.3	36
0.27272727	0.54545455	0.81818182	0.66666667	0.51515152	0.36363636	32
0.68	0.8	0.36	0.48	0.6	0.72	32
0.55555556	0.59259259	0.62962963	0.66666667	0.18518519	0.74074074	29
0.63333333	0.8	0.26666667	0.66666667	0.6	0.53333333	27
0.35483871	0.70967742	0.16129032	0.51612903	0.41935484	0.32258065	26
0.75	0.625	0.5	0.66666667	0.25	0.70833333	26
0.2173913	0.43478261	0.65217391	0.26086957	0.47826087	0.69565217	25
0.55555556	0.8	0.42222222	0.66666667	0.6	0.22222222	24
0.61538462	0.69230769	0.23076923	0.30769231	0.38461538	0.46153846	24
0.75	0.625	0.5	0.66666667	0.25	0.41666667	23
0.33333333	0.66666667	0.66666667	0.66666667	0.33333333	0.66666667	22
0.4	0.8	0.5	0.66666667	0.6	0.3	21
0.64705882	0.47058824	0.29411765	0.66666667	0.76470588	0.58823529	21
0.66666667	0.66666667	0.33333333	0.66666667	0.66666667	0.66666667	20
0.33333333	0.66666667	0.33333333	0.66666667	0.33333333	0.33333333	17
0.6	0.8	0.2	0.4	0.6	0.4	16
0.33333333	0.33333333	0.33333333	0.66666667	0.33333333	0.66666667	14
0.33333333	0.66666667	0.33333333	0.33333333	0.33333333	0.66666667	14
0.33333333	0.66666667	0.66666667	0.66666667	0.66666667	0.33333333	14
0.55555556	0.59259259	0.62962963	0.66666667	0.7037037	0.22222222	14
0.68	0.24	0.36	0.48	0.6	0.72	13
0.33333333	0.66666667	0.33333333	0.33333333	0.33333333	0.33333333	12
0.66666667	0.66666667	0.33333333	0.66666667	0.33333333	0.66666667	12
0.61538462	0.69230769	0.76923077	0.30769231	0.38461538	0.46153846	11
0.66666667	0.33333333	0.33333333	0.66666667	0.66666667	0.66666667	7

Table 9: 68 facets of $P(C_{15}, 14)$ hit by 10,000 shots

π_1	π_2	π_3	π_4	π_5	π_6	# hits
0.5	1	0	0.5	1	0	1852
0.25	0.5	0.75	1	0	0.25	1134
0.66666667	0.5	0.33333333	1	0	0.66666667	904
0.071428571	0.14285714	0.21428571	0.28571429	0.35714286	0.42857143	698
0.15384615	0.30769231	0.46153846	0.61538462	0.76923077	0.92307692	638
0.36363636	0.72727273	0.75	0.090909091	0.45454545	0.81818182	449
0.875	0.8125	0.75	0.6875	0.625	0.5625	423
0.875	0.14285714	0.75	0.28571429	0.625	0.42857143	300
0.36363636	0.72727273	0.40909091	0.090909091	0.45454545	0.81818182	298
0.60714286	0.14285714	0.21428571	0.28571429	0.35714286	0.42857143	222
0.66666667	0.5	0.33333333	0.16666667	0.83333333	0.66666667	194
0.875	0.34375	0.75	0.6875	0.625	0.5625	155
0.76470588	0.64705882	0.52941176	0.41176471	0.29411765	0.17647059	150
0.36363636	0.72727273	0.40909091	0.77272727	0.45454545	0.81818182	124
0.15384615	0.30769231	0.46153846	0.61538462	0.76923077	0.34615385	123
0.5	0.25	0.75	0.5	0.25	0.75	123
0.875	0.5	0.75	0.375	0.625	0.25	123
0.875	0.34375	0.75	0.21875	0.625	0.5625	110
0.25	0.5	0.75	0.375	0.625	0.875	108
0.60714286	0.14285714	0.75	0.28571429	0.35714286	0.42857143	102
0.78571429	0.14285714	0.57142857	0.28571429	0.71428571	0.42857143	92
0.78571429	0.14285714	0.57142857	0.28571429	0.35714286	0.42857143	89
0.57894737	0.36842105	0.15789474	0.73684211	0.52631579	0.31578947	88
0.40625	0.8125	0.75	0.21875	0.625	0.5625	84
0.875	0.1875	0.75	0.375	0.625	0.5625	81
0.40625	0.8125	0.75	0.6875	0.625	0.5625	78
0.25	0.5	0.75	0.16666667	0.41666667	0.66666667	75
0.25	0.5	0.75	0.375	0.625	0.25	75
0.25	0.5	0.75	0.375	0.3125	0.25	66
0.42857143	0.14285714	0.57142857	0.28571429	0.71428571	0.42857143	59
0.25	0.5	0.75	0.6875	0.625	0.25	56
0.5	0.625	0.75	0.125	0.625	0.75	56
0.875	0.25	0.75	0.5	0.625	0.375	48
0.73076923	0.30769231	0.46153846	0.61538462	0.76923077	0.34615385	47
0.60714286	0.67857143	0.21428571	0.28571429	0.35714286	0.42857143	46
0.33928571	0.67857143	0.21428571	0.28571429	0.35714286	0.42857143	40
0.36363636	0.72727273	0.75	0.43181818	0.45454545	0.81818182	37
0.73076923	0.30769231	0.46153846	0.61538462	0.19230769	0.34615385	37
0.60714286	0.14285714	0.75	0.28571429	0.625	0.42857143	35
0.66666667	0.5	0.33333333	0.16666667	0.41666667	0.66666667	34
0.32352941	0.64705882	0.52941176	0.41176471	0.29411765	0.17647059	32
0.8	0.4	0.6	0.2	0.4	0.6	32
0.66666667	0.22222222	0.33333333	0.44444444	0.27777778	0.66666667	31
0.5	0.625	0.75	0.5	0.25	0.75	30
0.66666667	0.77777778	0.33333333	0.72222222	0.55555556	0.66666667	30
0.76470588	0.20588235	0.52941176	0.41176471	0.29411765	0.61764706	28
0.5	0.25	0.75	0.5	0.625	0.75	27
0.73076923	0.30769231	0.46153846	0.61538462	0.76923077	0.63461538	27
0.76470588	0.20588235	0.52941176	0.41176471	0.73529412	0.61764706	27
0.66666667	0.5	0.33333333	0.58333333	0.83333333	0.66666667	26
0.5	0.25	0.375	0.5	0.25	0.75	25
0.30434783	0.60869565	0.26086957	0.56521739	0.2173913	0.52173913	22
0.5625	0.5	0.75	0.375	0.3125	0.25	21
0.66666667	0.5	0.75	0.16666667	0.41666667	0.66666667	19
0.60714286	0.67857143	0.75	0.28571429	0.35714286	0.42857143	18
0.5625	0.5	0.75	0.6875	0.625	0.25	16
0.66666667	0.66666667	0.33333333	0.33333333	0.33333333	0.66666667	16
0.66666667	0.77777778	0.33333333	0.44444444	0.55555556	0.66666667	15
0.66666667	0.33333333	0.33333333	0.66666667	0.66666667	0.66666667	14
0.60714286	0.67857143	0.75	0.28571429	0.35714286	0.69642857	13
0.640625	0.34375	0.75	0.6875	0.625	0.328125	13
0.40625	0.34375	0.75	0.6875	0.625	0.328125	12
0.76470588	0.64705882	0.52941176	0.41176471	0.29411765	0.61764706	12
0.33928571	0.67857143	0.75	0.28571429	0.35714286	0.42857143	11
0.66666667	0.22222222	0.33333333	0.44444444	0.55555556	0.66666667	11
0.25	0.5	0.75	0.6875	0.625	0.5625	9
0.40625	0.34375	0.75	0.21875	0.625	0.5625	9
0.40625	0.34375	0.75	0.6875	0.625	0.5625	1

Table 10: 173 facets of $P(C_{16}, 15)$ hit by 10,000 shots

π_1	π_2	π_3	π_4	π_5	π_6	π_7	# hits
1	0	1	0	1	0	1	1923
0.33333333	0.66666667	1	0	0.33333333	0.66666667	1	922
0.066666667	0.13333333	0.2	0.26666667	0.33333333	0.4	0.46666667	574
0.14285714	0.28571429	0.42857143	0.57142857	0.71428571	0.85714286	1	497
0.23076923	0.46153846	0.69230769	0.92307692	0.33333333	0.15384615	0.38461538	342
0.6	0.66666667	0.2	0.8	0.33333333	0.4	1	342
0.88235294	0.82352941	0.76470588	0.70588235	0.64705882	0.58823529	0.52941176	272
0.77777778	0.66666667	0.55555556	0.44444444	0.33333333	0.22222222	0.11111111	268
0.45454545	0.90909091	0.2	0.36363636	0.81818182	0.4	0.27272727	230
0.6	0.4	0.2	0.8	0.6	0.4	1	214
0.23076923	0.46153846	0.69230769	0.92307692	0.53846154	0.15384615	0.38461538	200
0.77777778	0.28571429	0.55555556	0.57142857	0.33333333	0.85714286	0.11111111	175
0.77777778	0.66666667	0.55555556	0.44444444	0.33333333	0.22222222	1	175
0.14285714	0.28571429	0.42857143	0.57142857	0.71428571	0.85714286	0.42857143	171
0.45454545	0.90909091	0.63636364	0.36363636	0.81818182	0.54545455	0.27272727	164
0.33333333	0.66666667	0.33333333	0.66666667	0.33333333	0.66666667	1	157
0.6	0.4	0.2	0.8	0.6	0.4	0.2	157
0.6	0.13333333	0.2	0.26666667	0.33333333	0.4	0.46666667	127
0.45454545	0.90909091	0.39393939	0.36363636	0.81818182	0.3030303	0.27272727	115
0.68421053	0.52631579	0.36842105	0.21052632	0.89473684	0.73684211	0.57894737	114
0.77777778	0.37037037	0.55555556	0.74074074	0.33333333	0.51851852	0.11111111	112
0.52380952	0.28571429	0.80952381	0.57142857	0.33333333	0.85714286	0.61904762	104
0.45454545	0.90909091	0.27272727	0.36363636	0.81818182	0.54545455	0.27272727	86
0.77777778	0.66666667	0.55555556	0.44444444	0.33333333	0.66666667	0.11111111	76
0.77777778	0.22222222	0.55555556	0.44444444	0.33333333	0.66666667	0.11111111	75
0.39130435	0.7826087	0.47826087	0.86956522	0.56521739	0.26086957	0.65217391	67
0.6	0.8	0.2	0.8	0.6	0.4	0.6	65
0.45454545	0.18181818	0.63636364	0.36363636	0.81818182	0.54545455	0.27272727	59
0.77777778	0.37037037	0.55555556	0.44444444	0.33333333	0.81481481	0.11111111	55
0.52380952	0.28571429	0.80952381	0.57142857	0.33333333	0.095238095	0.61904762	51
0.14285714	0.28571429	0.42857143	0.57142857	0.71428571	0.28571429	0.42857143	49
0.45454545	0.42424242	0.39393939	0.36363636	0.81818182	0.78787879	0.27272727	48
0.6	0.66666667	0.2	0.26666667	0.86666667	0.4	0.46666667	48
0.71428571	0.28571429	0.42857143	0.57142857	0.71428571	0.85714286	0.42857143	48
0.52380952	0.28571429	0.42857143	0.57142857	0.71428571	0.85714286	0.23809524	44
0.88235294	0.35294118	0.76470588	0.70588235	0.64705882	0.58823529	0.52941176	42
0.41176471	0.82352941	0.76470588	0.70588235	0.64705882	0.58823529	0.52941176	41
0.71428571	0.28571429	0.42857143	0.57142857	0.42857143	0.85714286	0.14285714	40
0.82857143	0.28571429	0.65714286	0.57142857	0.48571429	0.85714286	0.31428571	40
0.6	0.66666667	0.2	0.26666667	0.33333333	0.4	0.46666667	39
0.33333333	0.66666667	0.2	0.26666667	0.33333333	0.4	0.46666667	38
0.6	0.66666667	0.2	0.8	0.33333333	0.4	0.46666667	38
0.68421053	0.52631579	0.36842105	0.21052632	0.33333333	0.73684211	0.57894737	38
0.23076923	0.46153846	0.69230769	0.51282051	0.74358974	0.15384615	0.38461538	37
0.39130435	0.7826087	0.47826087	0.17391304	0.56521739	0.26086957	0.65217391	37
0.33333333	0.66666667	0.55555556	0.88888889	0.33333333	0.22222222	0.55555556	36
0.6	0.8	0.2	0.4	0.6	0.4	0.2	36
0.77777778	0.48888889	0.55555556	0.62222222	0.33333333	0.75555556	0.11111111	35
0.23076923	0.46153846	0.69230769	0.30769231	0.53846154	0.15384615	0.38461538	34
0.42222222	0.84444444	0.2	0.26666667	0.68888889	0.4	0.46666667	34
0.28	0.56	0.2	0.48	0.76	0.4	0.68	31
0.33333333	0.66666667	0.33333333	0.66666667	0.33333333	0.66666667	0.33333333	29
0.65517241	0.20689655	0.31034483	0.4137931	0.51724138	0.62068966	0.17241379	29
0.45454545	0.54545455	0.63636364	0.36363636	0.81818182	0.18181818	0.27272727	27
0.6	0.4	0.2	0.8	0.33333333	0.4	0.46666667	27
0.45454545	0.42424242	0.39393939	0.84848485	0.33333333	0.3030303	0.27272727	26
0.6	0.13333333	0.73333333	0.26666667	0.33333333	0.4	0.46666667	26
0.52380952	0.28571429	0.42857143	0.57142857	0.33333333	0.85714286	0.23809524	25
0.65517241	0.75862069	0.31034483	0.4137931	0.51724138	0.62068966	0.17241379	25
0.18518519	0.37037037	0.55555556	0.74074074	0.33333333	0.51851852	0.7037037	24
0.23076923	0.46153846	0.69230769	0.30769231	0.53846154	0.76923077	0.38461538	24
0.23076923	0.46153846	0.69230769	0.51282051	0.33333333	0.15384615	0.38461538	24
0.28	0.56	0.84	0.48	0.76	0.4	0.68	24
0.33333333	0.66666667	0.33333333	0.33333333	0.33333333	0.66666667	0.33333333	23
0.77777778	0.13333333	0.55555556	0.26666667	0.33333333	0.4	0.46666667	23
0.77777778	0.37037037	0.55555556	0.14814815	0.33333333	0.51851852	0.7037037	23
0.71428571	0.85714286	0.42857143	0.57142857	0.71428571	0.28571429	0.42857143	22
0.77777778	0.37037037	0.55555556	0.74074074	0.33333333	0.51851852	0.7037037	22
0.5483871	0.58064516	0.61290323	0.64516129	0.67741935	0.19354839	0.22580645	21
0.6	0.4	0.2	0.8	0.33333333	0.4	0.73333333	21
0.6	0.88	0.2	0.48	0.76	0.4	0.36	21
0.84615385	0.46153846	0.69230769	0.30769231	0.53846154	0.76923077	0.38461538	21
0.28	0.56	0.2	0.48	0.76	0.4	0.36	18
0.37142857	0.28571429	0.65714286	0.57142857	0.48571429	0.85714286	0.31428571	18
0.52380952	0.28571429	0.61904762	0.57142857	0.33333333	0.85714286	0.23809524	17
0.71428571	0.28571429	0.42857143	0.57142857	0.33333333	0.85714286	0.23809524	17
0.33333333	0.66666667	0.33333333	0.22222222	0.33333333	0.66666667	0.55555556	16
0.33333333	0.66666667	0.66666667	0.66666667	0.33333333	0.66666667	0.33333333	15
0.45454545	0.61818182	0.2	0.36363636	0.81818182	0.4	0.27272727	15
0.6	0.4	0.6	0.8	0.6	0.4	0.2	15
0.63265306	0.28571429	0.59183673	0.57142857	0.55102041	0.85714286	0.18367347	15
0.77777778	0.66666667	0.55555556	0.74074074	0.33333333	0.51851852	0.7037037	15
0.33333333	0.66666667	0.66666667	0.33333333	0.33333333	0.33333333	0.33333333	14
0.33333333	0.66666667	0.66666667	0.33333333	0.33333333	0.66666667	0.33333333	14
0.33333333	0.66666667	0.77777778	0.66666667	0.33333333	0.66666667	0.55555556	14
0.52380952	0.28571429	0.42857143	0.57142857	0.33333333	0.85714286	0.61904762	14
0.64102564	0.87179487	0.28205128	0.51282051	0.74358974	0.56410256	0.38461538	14

To be continued on the next page...

Table 10 is continued ...

0.64705882	0.82352941	0.29411765	0.70588235	0.64705882	0.58823529	0.52941176	14
0.6	0.4	0.2	0.4	0.6	0.4	0.2	13
0.71428571	0.28571429	0.42857143	0.57142857	0.33333333	0.85714286	0.42857143	13
0.77777778	0.48888889	0.55555556	0.26666667	0.33333333	0.75555556	0.46666667	13
0.33333333	0.66666667	0.2	0.53333333	0.33333333	0.4	0.73333333	12
0.70212766	0.38297872	0.40425532	0.42553191	0.44680851	0.80851064	0.14893617	12
0.77777778	0.22222222	0.55555556	0.44444444	0.33333333	0.22222222	0.55555556	12
0.23076923	0.46153846	0.69230769	0.61538462	0.53846154	0.76923077	0.38461538	11
0.37142857	0.74285714	0.2	0.57142857	0.48571429	0.4	0.31428571	11
0.23076923	0.46153846	0.38461538	0.30769231	0.53846154	0.76923077	0.38461538	10
0.33333333	0.33333333	0.33333333	0.66666667	0.33333333	0.66666667	0.33333333	10
0.33333333	0.66666667	0.33333333	0.66666667	0.66666667	0.66666667	0.33333333	10
0.41176471	0.82352941	0.29411765	0.23529412	0.64705882	0.58823529	0.52941176	10
0.42222222	0.84444444	0.2	0.62222222	0.68888889	0.4	0.46666667	10
0.45454545	0.42424242	0.39393939	0.36363636	0.81818182	0.3030303	0.27272727	10
0.45454545	0.42424242	0.39393939	0.84848485	0.33333333	0.3030303	0.75757576	10
0.5483871	0.32258065	0.61290323	0.64516129	0.67741935	0.70967742	0.22580645	10
0.6	0.66666667	0.73333333	0.8	0.33333333	0.4	0.46666667	10
0.18518519	0.37037037	0.55555556	0.74074074	0.33333333	0.51851852	0.40740741	9
0.33333333	0.66666667	0.73333333	0.8	0.33333333	0.4	0.46666667	9
0.37142857	0.74285714	0.2	0.57142857	0.71428571	0.4	0.31428571	9
0.45454545	0.42424242	0.39393939	0.36363636	0.33333333	0.78787879	0.27272727	9
0.66666667	0.33333333	0.33333333	0.33333333	0.33333333	0.66666667	0.66666667	9
0.77777778	0.28571429	0.55555556	0.57142857	0.33333333	0.85714286	0.36507937	9
0.25490196	0.50980392	0.76470588	0.70588235	0.64705882	0.58823529	0.52941176	8
0.33333333	0.66666667	0.33333333	0.33333333	0.33333333	0.33333333	0.33333333	8
0.41176471	0.82352941	0.29411765	0.70588235	0.64705882	0.58823529	0.52941176	8
0.65517241	0.75862069	0.31034483	0.4137931	0.51724138	0.62068966	0.72413793	8
0.66666667	0.66666667	0.33333333	0.33333333	0.33333333	0.66666667	0.33333333	8
0.66666667	0.66666667	0.33333333	0.66666667	0.33333333	0.66666667	0.33333333	8
0.23076923	0.46153846	0.69230769	0.67692308	0.66153846	0.64615385	0.38461538	7
0.26315789	0.52631579	0.36842105	0.21052632	0.47368421	0.73684211	0.57894737	7
0.33333333	0.66666667	0.33333333	0.66666667	0.33333333	0.33333333	0.33333333	7
0.40350877	0.24561404	0.64912281	0.49122807	0.33333333	0.73684211	0.29824561	7
0.45454545	0.18181818	0.63636364	0.36363636	0.33333333	0.54545455	0.27272727	7
0.65517241	0.20689655	0.31034483	0.4137931	0.51724138	0.62068966	0.72413793	7
0.65517241	0.75862069	0.31034483	0.68965517	0.51724138	0.62068966	0.72413793	7
0.23076923	0.46153846	0.69230769	0.30769231	0.33333333	0.35897436	0.38461538	6
0.26315789	0.52631579	0.78947368	0.63157895	0.47368421	0.73684211	0.57894737	6
0.33333333	0.33333333	0.66666667	0.66666667	0.33333333	0.66666667	0.33333333	6
0.33333333	0.66666667	0.66666667	0.66666667	0.66666667	0.66666667	0.33333333	6
0.37142857	0.74285714	0.2	0.57142857	0.48571429	0.4	0.77142857	6
0.45454545	0.42424242	0.39393939	0.36363636	0.33333333	0.78787879	0.75757576	6
0.5483871	0.58064516	0.61290323	0.64516129	0.67741935	0.70967742	0.22580645	6
0.66666667	0.66666667	0.33333333	0.66666667	0.66666667	0.66666667	0.66666667	6
0.68421053	0.52631579	0.36842105	0.63157895	0.47368421	0.73684211	0.15789474	6
0.77777778	0.66666667	0.55555556	0.66666667	0.33333333	0.66666667	0.55555556	6
0.23076923	0.46153846	0.69230769	0.71794872	0.33333333	0.56410256	0.38461538	5
0.33333333	0.33333333	0.33333333	0.66666667	0.33333333	0.66666667	0.66666667	5
0.33333333	0.66666667	0.33333333	0.33333333	0.33333333	0.33333333	0.66666667	5
0.45454545	0.42424242	0.63636364	0.36363636	0.33333333	0.78787879	0.27272727	5
0.55555556	0.66666667	0.33333333	0.22222222	0.33333333	0.66666667	0.55555556	5
0.6	0.4	0.6	0.4	0.6	0.8	0.2	5
0.66666667	0.66666667	0.33333333	0.66666667	0.66666667	0.66666667	0.33333333	5
0.77777778	0.66666667	0.55555556	0.74074074	0.33333333	0.51851852	0.40740741	5
0.23076923	0.46153846	0.69230769	0.30769231	0.33333333	0.56410256	0.38461538	4
0.23076923	0.46153846	0.69230769	0.51282051	0.74358974	0.56410256	0.38461538	4
0.33333333	0.33333333	0.66666667	0.33333333	0.33333333	0.33333333	0.33333333	4
0.33333333	0.66666667	0.55555556	0.44444444	0.77777778	0.22222222	0.55555556	4
0.33333333	0.66666667	0.55555556	0.66666667	0.33333333	0.22222222	0.33333333	4
0.3559322	0.71186441	0.52542373	0.33898305	0.42372881	0.23728814	0.3220339	4
0.55555556	0.66666667	0.77777778	0.66666667	0.33333333	0.66666667	0.55555556	4
0.66666667	0.66666667	0.33333333	0.33333333	0.33333333	0.66666667	0.66666667	4
0.66666667	0.66666667	0.33333333	0.66666667	0.33333333	0.66666667	0.66666667	4
0.77777778	0.22222222	0.55555556	0.44444444	0.33333333	0.66666667	0.55555556	4
0.25490196	0.50980392	0.76470588	0.70588235	0.33333333	0.58823529	0.52941176	3
0.33333333	0.33333333	0.66666667	0.66666667	0.66666667	0.66666667	0.33333333	3
0.6	0.4	0.73333333	0.8	0.33333333	0.4	0.46666667	3
0.6	0.8	0.2	0.4	0.6	0.4	0.6	3
0.66666667	0.66666667	0.66666667	0.66666667	0.33333333	0.66666667	0.66666667	3
0.68421053	0.24561404	0.36842105	0.49122807	0.33333333	0.73684211	0.57894737	3
0.6969697	0.42424242	0.39393939	0.36363636	0.33333333	0.78787879	0.27272727	3
0.77777778	0.66666667	0.55555556	0.44444444	0.33333333	0.66666667	0.55555556	3
0.77777778	0.66666667	0.55555556	0.66666667	0.33333333	0.66666667	0.33333333	3
0.33333333	0.33333333	0.66666667	0.33333333	0.33333333	0.66666667	0.33333333	2
0.33333333	0.66666667	0.2	0.53333333	0.33333333	0.4	0.46666667	2
0.33333333	0.66666667	0.55555556	0.44444444	0.33333333	0.22222222	0.33333333	2
0.33333333	0.66666667	0.66666667	0.66666667	0.33333333	0.33333333	0.33333333	2
0.37142857	0.74285714	0.2	0.34285714	0.48571429	0.4	0.31428571	2
0.41176471	0.35294118	0.76470588	0.70588235	0.64705882	0.58823529	0.52941176	2
0.66666667	0.33333333	0.33333333	0.66666667	0.33333333	0.33333333	0.33333333	2
0.66666667	0.33333333	0.33333333	0.66666667	0.33333333	0.66666667	0.66666667	2
0.66666667	0.66666667	0.66666667	0.66666667	0.33333333	0.66666667	0.33333333	2
0.33333333	0.66666667	0.55555556	0.44444444	0.33333333	0.22222222	0.55555556	1
0.40350877	0.52631579	0.36842105	0.21052632	0.33333333	0.73684211	0.57894737	1
0.5483871	0.58064516	0.35483871	0.64516129	0.67741935	0.70967742	0.22580645	1

Table 11: The 46 most hit facets totalling 50% of hits out of 390 facets of $P(C_{17}, 16)$ hit by 10,000 shots.

π_1	π_2	π_3	π_4	π_5	π_6	π_7	# hits
0.0625	0.125	0.1875	0.25	0.3125	0.375	0.4375	582
0.30769231	0.61538462	0.92307692	0.25	0.23076923	0.53846154	0.84615385	359
0.7	0.55	0.4	0.25	0.1	0.8	0.65	305
0.21428571	0.42857143	0.64285714	0.85714286	0.33333333	0.33333333	0.28571429	252
0.30769231	0.61538462	0.92307692	0.57692308	0.23076923	0.53846154	0.84615385	224
0.7	0.42857143	0.4	0.85714286	0.1	0.8	0.28571429	193
0.59375	0.125	0.1875	0.25	0.3125	0.375	0.4375	153
0.44444444	0.66666667	0.16666667	0.61111111	0.33333333	0.33333333	0.77777778	136
0.77777778	0.66666667	0.55555556	0.5	0.33333333	0.33333333	0.16666667	124
0.26666667	0.53333333	0.8	0.25	0.33333333	0.6	0.86666667	113
0.30769231	0.61538462	0.26923077	0.57692308	0.23076923	0.53846154	0.84615385	108
0.26666667	0.53333333	0.3	0.56666667	0.33333333	0.6	0.86666667	105
0.44444444	0.44444444	0.88888889	0.61111111	0.33333333	0.33333333	0.77777778	102
0.7	0.26666667	0.4	0.53333333	0.1	0.8	0.36666667	99
0.77777778	0.42857143	0.55555556	0.85714286	0.33333333	0.72222222	0.28571429	97
0.7	0.66666667	0.4	0.5	0.33333333	0.8	0.16666667	95
0.7	0.55	0.4	0.675	0.1	0.8	0.65	92
0.21428571	0.42857143	0.64285714	0.85714286	0.33333333	0.54761905	0.28571429	90
0.7	0.42857143	0.4	0.85714286	0.33333333	0.8	0.28571429	90
0.77777778	0.66666667	0.55555556	0.5	0.33333333	0.72222222	0.16666667	89
0.21428571	0.42857143	0.64285714	0.85714286	0.26190476	0.47619048	0.28571429	87
0.066666667	0.13333333	0.2	0.26666667	0.33333333	0.4	0.46666667	83
0.7	0.55	0.4	0.675	0.1	0.8	0.225	81
0.58333333	0.66666667	0.16666667	0.75	0.33333333	0.33333333	0.5	78
0.77777778	0.125	0.55555556	0.25	0.33333333	0.375	0.45833333	75
0.77777778	0.125	0.55555556	0.25	0.33333333	0.375	0.4375	73
0.33333333	0.66666667	0.58333333	0.5	0.33333333	0.33333333	0.16666667	69
0.30769231	0.61538462	0.26923077	0.57692308	0.23076923	0.53846154	0.19230769	65
0.61904762	0.42857143	0.23809524	0.85714286	0.33333333	0.47619048	0.28571429	63
0.75	0.66666667	0.58333333	0.5	0.33333333	0.33333333	0.16666667	63
0.44444444	0.66666667	0.88888889	0.25	0.33333333	0.55555556	0.77777778	61
0.59375	0.125	0.71875	0.25	0.3125	0.375	0.4375	61
0.58333333	0.66666667	0.16666667	0.64583333	0.33333333	0.33333333	0.70833333	60
0.61904762	0.42857143	0.23809524	0.85714286	0.26190476	0.47619048	0.28571429	60
0.066666667	0.13333333	0.2	0.26666667	0.33333333	0.4	0.43333333	59
0.60416667	0.125	0.72916667	0.25	0.33333333	0.375	0.45833333	59
0.33333333	0.66666667	0.25	0.58333333	0.33333333	0.5	0.83333333	58
0.77083333	0.125	0.54166667	0.25	0.3125	0.375	0.4375	58
0.26666667	0.53333333	0.8	0.56666667	0.33333333	0.6	0.86666667	55
0.7	0.55	0.4	0.25	0.33333333	0.8	0.65	55
0.51785714	0.42857143	0.64285714	0.85714286	0.16071429	0.67857143	0.28571429	54
0.46666667	0.66666667	0.4	0.5	0.33333333	0.8	0.16666667	53
0.58333333	0.16666667	0.16666667	0.33333333	0.33333333	0.33333333	0.5	53
0.58333333	0.33333333	0.16666667	0.66666667	0.33333333	0.33333333	0.66666667	53
0.44444444	0.44444444	0.16666667	0.61111111	0.33333333	0.33333333	0.77777778	52
0.46666667	0.42857143	0.4	0.85714286	0.33333333	0.8	0.28571429	52

Table 12: The 7 most hit facets totalling 50% of hits out of 326 facets of $P(C_{18}, 17)$ hit by 10,000 shots.

π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	# hits
1	0	1	0	1	0	1	0	1811
0.5	1	0	0.5	1	0	0.5	1	1186
0.125	0.25	0.375	0.5	0.625	0.75	0.875	1	533
0.2	0.4	0.6	0.8	1	0	0.2	0.4	511
0.058823529	0.11764706	0.17647059	0.23529412	0.29411765	0.35294118	0.41176471	0.47058824	413
0.28571429	0.57142857	0.85714286	0.5	0.14285714	0.42857143	0.71428571	1	380
0.8	0.7	0.6	0.5	0.4	0.3	0.2	1	235

Table 13: The 31 most hit facets totalling 50% of hits out of 608 facets of $P(C_{19}, 18)$ hit by 10,000 shots.

π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	# hits
0.055555556	0.11111111	0.16666667	0.22222222	0.27777778	0.33333333	0.38888889	0.44444444	476
0.11764706	0.23529412	0.35294118	0.47058824	0.58823529	0.70588235	0.82352941	0.94117647	464
0.9	0.85	0.8	0.75	0.7	0.65	0.6	0.55	297
0.26666667	0.53333333	0.8	0.75	0.066666667	0.33333333	0.6	0.86666667	276
0.1875	0.375	0.5625	0.75	0.9375	0.33333333	0.125	0.3125	228
0.46153846	0.92307692	0.16666667	0.38461538	0.84615385	0.33333333	0.30769231	0.76923077	206
0.72727273	0.59090909	0.45454545	0.31818182	0.18181818	0.90909091	0.77272727	0.63636364	205
0.26666667	0.53333333	0.8	0.43333333	0.066666667	0.33333333	0.6	0.86666667	201
0.1875	0.375	0.5625	0.75	0.9375	0.53125	0.125	0.3125	186
0.58333333	0.71428571	0.16666667	0.75	0.42857143	0.33333333	0.91666667	0.14285714	174
0.9	0.11111111	0.8	0.22222222	0.7	0.33333333	0.6	0.44444444	169
0.58333333	0.375	0.16666667	0.75	0.54166667	0.33333333	0.91666667	0.70833333	168
0.35714286	0.71428571	0.8	0.071428571	0.42857143	0.78571429	0.6	0.14285714	154
0.72727273	0.375	0.45454545	0.75	0.18181818	0.90909091	0.125	0.63636364	151
0.80952381	0.71428571	0.61904762	0.52380952	0.42857143	0.33333333	0.23809524	0.14285714	147
0.58333333	0.11111111	0.16666667	0.22222222	0.27777778	0.33333333	0.38888889	0.44444444	142
0.35714286	0.71428571	0.39285714	0.071428571	0.42857143	0.78571429	0.46428571	0.14285714	134
0.46153846	0.92307692	0.65384615	0.38461538	0.84615385	0.57692308	0.30769231	0.76923077	134
0.58333333	0.375	0.16666667	0.75	0.54166667	0.33333333	0.125	0.70833333	120
0.35714286	0.71428571	0.73214286	0.071428571	0.42857143	0.78571429	0.46428571	0.14285714	100
0.35714286	0.71428571	0.61904762	0.071428571	0.42857143	0.78571429	0.69047619	0.14285714	98
0.46153846	0.92307692	0.28846154	0.38461538	0.84615385	0.57692308	0.30769231	0.76923077	90
0.9	0.375	0.8	0.75	0.7	0.65	0.6	0.55	87
0.11764706	0.23529412	0.35294118	0.47058824	0.58823529	0.70588235	0.82352941	0.38235294	86
0.58333333	0.375	0.16666667	0.75	0.54166667	0.33333333	0.91666667	0.3125	84
0.58333333	0.63888889	0.16666667	0.75	0.27777778	0.33333333	0.91666667	0.44444444	84
0.46153846	0.19230769	0.65384615	0.38461538	0.84615385	0.57692308	0.30769231	0.76923077	80
0.26666667	0.53333333	0.8	0.43333333	0.7	0.33333333	0.6	0.86666667	79
0.58333333	0.11111111	0.69444444	0.22222222	0.80555556	0.33333333	0.38888889	0.44444444	74
0.46153846	0.19230769	0.65384615	0.38461538	0.11538462	0.57692308	0.30769231	0.76923077	73
0.72727273	0.59090909	0.45454545	0.31818182	0.18181818	0.90909091	0.34090909	0.63636364	71

Table 14: The 11 most hit facets totalling 50% of hits out of 783 facets of $P(C_{20}, 19)$ hit by 10,000 shots.

π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	# hits
1	0	1	0	1	0	1	0	1	1601
0.33333333	0.66666667	1	0	0.33333333	0.66666667	1	0	0.33333333	761
0.25	0.5	0.75	1	0	0.25	0.5	0.75	1	632
0.66666667	0.5	0.33333333	1	0	0.66666667	0.5	0.33333333	1	509
0.11111111	0.22222222	0.33333333	0.44444444	0.55555556	0.66666667	0.77777778	0.88888889	1	368
0.052631579	0.10526316	0.15789474	0.21052632	0.26315789	0.31578947	0.36842105	0.42105263	0.47368421	348
0.17647059	0.35294118	0.52941176	0.70588235	0.88235294	0.66666667	0.058823529	0.23529412	0.41176471	211
0.42857143	0.85714286	0.33333333	0.28571429	0.71428571	0.66666667	0.14285714	0.57142857	1	201
0.9047619	0.85714286	0.80952381	0.76190476	0.71428571	0.66666667	0.61904762	0.57142857	0.52380952	191
0.81818182	0.72727273	0.63636364	0.54545455	0.45454545	0.36363636	0.27272727	0.18181818	0.090909091	164
0.17647059	0.35294118	0.52941176	0.70588235	0.88235294	0.47058824	0.058823529	0.23529412	0.41176471	126

4.3 Shooting for the master knapsack problem

We define *shooting in* $w \in \mathbb{R}^O$ *for the knapsack problem* $K(n)$ to be solving the linear programming problem over the *projected* knapsack subadditive polytope $\Pi(K(n))_O$,

$$\begin{aligned} \max \quad & w\pi_O \\ st \quad & \pi_O \in \Pi(K(n))_O, \end{aligned}$$

where $O = \{i : i < n/2\}$. Since the first component π_1 is constant $1/2$ for every point $\pi_O \in \Pi(K(n))_O$, we may assume that the first component w_1 is 0. The facet $\pi t \geq \pi_n = 1$ of the knapsack polytope $P(K(n))$ is obtained from the optimal solution π_O to the problem above and is thought of as being hit by the shooting in w for $K(n)$. The optimal solution $\pi_{O-\{1\}}$ is the point translated by $\dot{\pi}_{O-\{1\}}$ from the optimal solution φ for

$$\begin{aligned} \max \quad & (w_2, ..., w_n).\varphi \\ st \quad & \varphi \in \Pi(K(n))_{O-\{1\}} - \dot{\pi}_{O-\{1\}}, \end{aligned}$$

where $\Pi(K(n))_{O-\{1\}} - \dot{\pi}_{O-\{1\}}$ is the translate of $\Pi(K(n))_{O-\{1\}}$ by $-\dot{\pi}_{O-\{1\}}$ so that the natural interior point $\dot{\pi}_{O-\{1\}} = \mathbf{1}/2$ is translated into the origin $\mathbf{0}$. Shooting in $(w_2, .., w_n)$ for $K(n)$ hits the facet $\varphi x' \leq 1$ of the plus level set of $\Pi(K(n))_{O-\{1\}} - \dot{\pi}_{O-\{1\}}$,

$$(\Pi(K(n))_{O-\{1\}} - \dot{\pi}_{O-\{1\}})^+ = \{x' : \mu(x') \leq 1\},$$

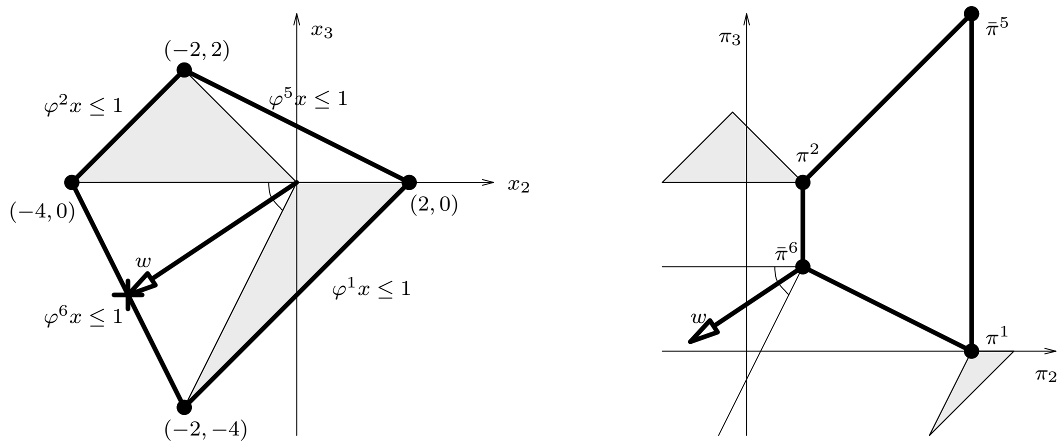

Figure 15: Shooting for $K(8)$ over $(\Pi(K(8))_{O-\{1\}} - \dot\pi_{O-\{1\}})^+$ and $\Pi(K(8))_{O-\{1\}}$

where the support function $\mu(x')$ is

$$\mu(x') = \sup\{x'\pi : \pi \in \Pi(K(n))_{O-\{1\}} - \dot\pi_{O-\{1\}}\}.$$

Example 4.5 Consider shooting in w for $K(8)$ in Figure 15. The knapsack subadditive polytope $\Pi(K(8))$ on the right side is the convex hull of 4 extreme points

$$\begin{aligned}
\pi^1_{O-\{1\}} &= (1,0)\\
\pi^2_{O-\{1\}} &= (1/4,3/4)\\
\bar\pi^5_{O-\{1\}} &= (1,3/2)\\
\bar\pi^6_{O-\{1\}} &= (1/4,3/8).
\end{aligned}$$

The first and the second extreme points, π^1 and π^2, are the first and the second facets of $P(C_9,8)$ in Gomory's table [12]. The third and the fourth extreme points, $\bar\pi^5$ and $\bar\pi^6$, are equivalent to the fifth and the sixth facets of $P(C_9,8)$ in the table,

$$\begin{aligned}
\pi^5 t &= \frac{4}{14}t_1 + \frac{8}{14}t_2 + \frac{12}{14}t_3 + \frac{7}{14}t_4 + \frac{2}{14}t_5 + \frac{6}{14}t_6 + \frac{10}{14}t_7 + \frac{14}{14}t_8 \ge \pi^5_8 = 1\\
\pi^6 t &= \frac{11}{16}t_1 + \frac{4}{16}t_2 + \frac{6}{16}t_3 + \frac{8}{16}t_4 + \frac{10}{16}t_5 + \frac{12}{16}t_6 + \frac{5}{16}t_7 + \frac{16}{16}t_8 \ge \pi^5_8 = 1.
\end{aligned}$$

The polar cone of $\bar\pi^6$ contains the arrow w. On the left side of the figure, the arrow

w hits the facet of the plus level set $\Pi(K(n))_{O-\{1\}} - \dot{\pi}_{O-\{1\}}$,

$$\varphi^6 x = (\bar{\pi}_{\{2,3\}} - \dot{\pi}_{\{2,3\}})x = -\frac{1}{4}x_2 - \frac{1}{8}x_3 \leq 1.$$

The ratios of angles for the facets φ^5, φ^1, φ^6 and φ^2 are 0.3750, 0.3238, 0.1762 and 0.1250, respectively. Compare them with the experimental result in Table 16. □

Shooting experiment for the knapsack problem $K(n)$ is shooting for $K(n)$ in a random vector w uniformly distributed on the unit sphere, $\{w \in \mathbb{R}^O : ||w|| = 1\}$. We observe that the most hit knapsack facet is always the mixed integer cut π^{mick} of the knapsack polytope $P(K(n))$. The results of shooting experiments for $K(8)$ through $K(19)$ are as follows:[2] Each superscript in the most left columns in the tables is the number of the corresponding facet of the master cyclic group problem in the tables given by Gomory [12]. The knapsack facets denoted by $\bar{\pi}$ are equivalent to the corresponding facets π of the master cyclic group problem.

Table 15: 4 facets of $P(K(7))$ hit by 10,000 shots

	π_2	π_3	# hits/10000
$\bar{\pi}^2$	1	1.5	3708
$\bar{\pi}^3$	1	0.25	2468
$\bar{\pi}^1$	0.16666667	0.66666667	2009
$\bar{\pi}^4$	0.375	0.25	1815

Table 16: 4 facets of $P(K(8))$ hit by 10,000 shots

	π_2	π_3	# hits
$\bar{\pi}^5$	1	1.5	3708
π^1	1	0	3213
$\bar{\pi}^6$	0.25	0.375	1804
π^2	0.25	0.75	1275

[2]Based on the Gurobi-code written by Helder Inacio.

Table 17: 8 facets of $P(K(9))$ hit by 10,000 shots

	π_2	π_3	π_4	# hits
$\bar{\pi}^3$	1	1.5	2	3397
$\bar{\pi}^6$	1	1.5	0.25	1827
$\bar{\pi}^1$	0.125	0.625	0.25	1337
$\bar{\pi}^{12}$	1	0.33333333	0.25	1287
$\bar{\pi}^8$	0.22222222	0.33333333	0.44444444	659
$\bar{\pi}^2$	1	0.33333333	0.83333333	574
$\bar{\pi}^4$	0.41666667	0.33333333	0.83333333	515
$\bar{\pi}^5$	0.41666667	0.33333333	0.25	404

Table 18: 7 facets of $P(K(10))$ hit by 10,000 shots

	π_2	π_3	π_4	# hits
$\bar{\pi}^7$	1	1.5	2	3291
$\bar{\pi}^8$	1	1.5	0	2105
$\bar{\pi}^5$	1	0.5	0	1143
$\bar{\pi}^9$	0.2	0.3	0.4	1089
$\bar{\pi}^{14}$	1	0.16666667	0.66666667	968
$\bar{\pi}^3$	0.33333333	0.16666667	0.66666667	839
$\bar{\pi}^{18}$	0.2	0.7	0.4	565

Table 19: 13 facets of $P(K(11))$ hit by 10,000 shots

π_2	π_3	π_4	π_5	# hits
1	1.5	2	2.5	2914
1	1.5	-0.25	0.25	1961
1	1.5	2	0.25	1649
1	0	0.5	1	1008
0.1	0.6	0.2	0.7	634
0.18181818	0.27272727	0.36363636	0.45454545	492
1	0.375	0.3125	0.25	320
1	0.375	0.875	0.25	226
0.25	0.375	0.5	0.25	188
0.4375	0.375	0.875	0.25	181
0.35714286	0.21428571	0.71428571	0.57142857	167
0.25	0.75	0.5	0.25	142
0.4375	0.375	0.3125	0.25	118

Table 20: 11 facets of $P(K(12))$ hit by 10,000 shots

π_2	π_3	π_4	π_5	# hits
1	1.5	2	2.5	2914
1	1.5	2	0	1721
1	-0.16666667	0.33333333	0.83333333	1363
1	1.5	0.33333333	0	1083
0.16666667	0.25	0.33333333	0.41666667	582
0.28571429	0.071428571	0.57142857	0.35714286	531
1	0.25	0.75	0	484
1	1.5	0.33333333	0.83333333	435
0.16666667	0.66666667	0.33333333	0.83333333	373
1	0.66666667	0.33333333	0	262
0.16666667	0.66666667	0.33333333	0.41666667	252

Table 21: 24 facets of $K(13)$ hit by 10,000 shots

π_2	π_3	π_4	π_5	π_6	# hits
1	1.5	2	2.5	3	2796
1	1.5	2	2.5	0.25	1462
1	1.5	2	-0.25	0.25	1432
1	1.5	0.625	-0.25	0.25	662
0.083333333	0.58333333	0.16666667	0.66666667	0.25	553
1	1.5	0.16666667	0.66666667	0.25	367
1	1.5	0.16666667	0.66666667	1.1666667	367
1	0.125	0.625	1.125	0.25	360
0.15384615	0.23076923	0.30769231	0.38461538	0.46153846	349
0.3125	0.125	0.625	0.4375	0.25	325
1	0.125	0.625	0.4375	0.25	203
0.21428571	0.71428571	0.42857143	0.14285714	0.64285714	193
1	0.58333333	0.16666667	0.66666667	1.1666667	193
1	0.27777778	0.16666667	0.66666667	0.55555556	106
1	0.4	0.35	0.3	0.25	97
1	0.23076923	0.30769231	0.38461538	0.46153846	86
0.38888889	0.27777778	0.16666667	0.66666667	0.55555556	84
1	0.58333333	0.16666667	0.66666667	0.25	76
1	0.4	0.35	0.3	0.8	69
0.26666667	0.4	0.53333333	0.3	0.8	66
0.45	0.4	0.9	0.3	0.8	53
1	0.4	0.9	0.3	0.8	44
0.45	0.4	0.35	0.3	0.8	36
0.45	0.4	0.35	0.3	0.25	21

Table 22: 19 facets of $K(14)$ hit by 10,000 shots

π_2	π_3	π_4	π_5	π_6	# hits
1	1.5	2	2.5	3	2796
1	1.5	2	-0.5	0	1806
1	1.5	2	2.5	0	1490
1	0	0.5	1	0	750
1	1.5	0	0.5	1	633
0.14285714	0.21428571	0.28571429	0.35714286	0.42857143	599
1	0.5	0	0.5	1	375
1	1.5	0.5	0.25	0	294
1	1.5	0.5	1	0	186
0.14285714	0.64285714	0.28571429	0.78571429	0.42857143	167
0.4	0.3	0.8	0.1	0.6	152
1	0.21428571	0.28571429	0.35714286	0.42857143	148
0.14285714	0.64285714	0.28571429	0.35714286	0.42857143	142
1	0.3	0.8	0.1	0.6	130
0.25	0.375	0.5	0.25	0.75	88
0.33333333	0.16666667	0.66666667	0.5	0.33333333	85
0.25	0.75	0.5	0.25	0.75	64
1	0.75	0.5	0.25	0	56
0.4	0.3	0.2	0.7	0.6	39

Table 23: The most hit 20 facets out of 65 facets of $P(K(15))$ hit by 10,000 shots

π_2	π_3	π_4	π_5	π_6	π_7	# hits
1	1.5	2	2.5	3	3.5	2597
1	1.5	2	2.5	3	0.25	1348
1	1.5	2	2.5	-0.25	0.25	1202
1	1.5	2	0.33333333	-0.25	0.25	731
1	1.5	-0.16666667	0.33333333	0.83333333	1.3333333	676
0.13333333	0.2	0.26666667	0.33333333	0.4	0.46666667	378
1	1.5	0.375	0.875	-0.25	0.25	338
0.071428571	0.57142857	0.14285714	0.64285714	0.21428571	0.71428571	316
1	1.5	2	0.33333333	0.83333333	1.3333333	214
1	1.5	2	0.33333333	0.83333333	0.25	213
1	0.2	0.26666667	0.33333333	0.4	0.46666667	144
1	1.5	0.91666667	0.33333333	-0.25	0.25	132
1	1.5	0.375	0.875	1.375	0.25	99
1	0.2	0.7	1.2	0.4	0.25	97
1	0.2	0.7	0.33333333	0.4	0.9	82
1	0.41666667	0.91666667	0.33333333	0.83333333	1.3333333	78
1	0.2	0.7	1.2	0.4	0.9	70
1	1.5	0.375	0.33333333	0.83333333	0.25	63
1	1.5	0.375	0.33333333	0.29166667	0.25	60
1	0.2	0.375	0.875	0.4	0.25	58

Table 24: The most hit 20 facets out of 53 facets of $P(K(16))$ hit by 10,000 shots

π_2	π_3	π_4	π_5	π_6	π_7	# hits
1	1.5	2	2.5	3	3.5	2597
1	1.5	2	2.5	3	0	1365
1	1.5	2	2.5	-0.5	0	1320
1	1.5	2	0.75	-0.5	0	606
0.125	0.1875	0.25	0.3125	0.375	0.4375	456
1	1.5	2	0.16666667	0.66666667	0	352
1	1.5	2	0.16666667	0.66666667	1.1666667	280
1	1.5	0.25	0.16666667	0.66666667	1.1666667	207
1	1.5	0.25	0.75	1.25	1.75	199
0.3	0.1	0.6	0.4	0.2	0.7	184
1	1.5	0.25	0.75	1.25	0	181
1	0.33333333	0.83333333	0.16666667	0.66666667	0	175
1	1.5	0.25	0.75	0.375	0	145
1	0.1	0.6	1.1	0.2	0.7	136
1	0.1875	0.25	0.3125	0.375	0.4375	132
0.125	0.625	0.25	0.75	0.375	0.875	129
1	0.33333333	0.83333333	0.16666667	0.66666667	1.1666667	126
1	0.1	0.6	0.4	0.2	0.7	116
0.125	0.625	0.25	0.3125	0.375	0.4375	98
1	0.625	0.25	0.75	1.25	0	98

Table 25: The most hit 20 facets out of 93 facets of $P(K(17))$ hit by 10,000 shots.

π_2	π_3	π_4	π_5	π_6	π_7	π_8	# hits
1	1.5	2	2.5	3	3.5	4	2458
1	1.5	2	2.5	-0.75	-0.25	0.25	1337
1	1.5	2	2.5	3	3.5	0.25	1261
1	1.5	2	2.5	3	-0.25	0.25	1117
1	1.5	0.125	0.625	1.125	-0.25	0.25	380
1	1.5	2	0	0.5	1	1.5	298
1	0	0.5	1	0	0.5	1	288
0.11764706	0.17647059	0.23529412	0.29411765	0.35294118	0.41176471	0.47058824	286
1	1.5	2	0	0.5	1	0.25	259
0.0625	0.5625	0.125	0.625	0.1875	0.6875	0.25	246
1	0.25	0.75	0	0.5	1	0.25	199
1	1.5	2	0.625	0.1875	-0.25	0.25	190
1	1.5	0.125	0.625	1.125	1.625	0.25	187
1	1.5	0.75	0	0.5	1	1.5	137
1	1.5	2	0.625	1.125	-0.25	0.25	129
1	0.17647059	0.23529412	0.29411765	0.35294118	0.41176471	0.47058824	93
1	1.5	0.125	0.625	0.1875	0.6875	0.25	83
1	1.5	0.33333333	0	0.5	1	0.66666667	83
1	0.25	0.125	0.625	0.5	0.375	0.25	56
1	1.5	0.75	0	0.5	1	0.25	53

Table 26: The most hit 20 facets out of 104 facets of $P(K(18))$ hit by 10,000 shots.

π_2	π_3	π_4	π_5	π_6	π_7	π_8	# hits
1	1.5	2	2.5	3	3.5	4	2458
1	1.5	2	2.5	3	3.5	0	1269
1	1.5	2	2.5	3	-0.5	0	1181
1	1.5	2	2.5	0.33333333	-0.5	0	612
1	1.5	2	-0.16666667	0.33333333	0.83333333	1.3333333	467
0.11111111	0.16666667	0.22222222	0.27777778	0.33333333	0.38888889	0.44444444	383
1	1.5	0	0.5	1	1.5	0	310
1	1.5	2	0.5	1	-0.5	0	301
1	1.5	0.66666667	-0.16666667	0.33333333	0.83333333	1.3333333	242
1	1.5	2	2.5	0.33333333	0.83333333	0	213
1	1.5	2	2.5	0.33333333	0.83333333	1.3333333	169
1	1.5	0	0.5	1	0.5	0	164
1	0.5	0	0.5	1	0.5	0	147
1	0.16666667	0.22222222	0.27777778	0.33333333	0.38888889	0.44444444	146
1	1.5	2	1.1666667	0.33333333	-0.5	0	141
1	0.16666667	0.66666667	1.1666667	0.33333333	0.83333333	1.3333333	130
1	0.16666667	0.66666667	1.1666667	0.33333333	0.83333333	0	124
0.2	0.7	0.4	0.1	0.6	0.3	0.8	91
1	0.16666667	0.66666667	0.5	0.33333333	0.83333333	0	89
1	1.5	2	0.5	1	1.5	0	88

Table 27: The most hit 20 facets out of 170 facets of $P(K(19))$ hit by 10,000 shots.

π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	# hits
1	1.5	2	2.5	3	3.5	4	4.5	2334
1	1.5	2	2.5	3	3.5	4	0.25	1249
1	1.5	2	2.5	3	-0.75	-0.25	0.25	1041
1	1.5	2	2.5	3	3.5	-0.25	0.25	1000
1	1.5	2	-0.33333333	0.16666667	0.66666667	1.1666667	1.6666667	510
1	1.5	2	2.5	0.875	-0.75	-0.25	0.25	477
1	1.5	-0.125	0.375	0.875	1.375	-0.25	0.25	401
1	1.5	2	2.5	0.16666667	0.66666667	-0.25	0.25	236
0.10526316	0.15789474	0.21052632	0.26315789	0.31578947	0.36842105	0.42105263	0.47368421	222
0.055555556	0.55555556	0.11111111	0.61111111	0.16666667	0.66666667	0.22222222	0.72222222	154
1	1.5	2	2.5	0.16666667	0.66666667	1.1666667	1.6666667	154
1	1.5	2	2.5	0.16666667	0.66666667	1.1666667	0.25	126
0.29166667	0.083333333	0.58333333	0.375	0.16666667	0.66666667	0.45833333	0.25	107
1	1.5	2	0.375	0.875	1.375	-0.25	0.25	106
1	0.083333333	0.58333333	1.0833333	0.16666667	0.66666667	1.1666667	0.25	103
1	1.5	2	0.375	0.875	0.3125	-0.25	0.25	88
1	0.15789474	0.21052632	0.26315789	0.31578947	0.36842105	0.42105263	0.47368421	69
0.15	0.65	0.3	0.8	0.45	0.1	0.6	0.25	67
1	0.083333333	0.58333333	0.375	0.16666667	0.66666667	0.45833333	0.25	55
1	1.5	2	0.8	0.45	0.1	-0.25	0.25	55

CHAPTER V

SUBPROBLEMS

If $M \neq C_n - \{0\}$, we call the cyclic group problem (C_n, M, b) a cyclic group *subproblem*. We can solve the cyclic group subproblem (C_n, M, b) with objective $c \in \mathbb{R}_+^M$ by solving the linear programming problem over $P(C_n, M, b)$. Especially, we are interested in the case of $|M| \ll n$, and assume that 1 and b belong to M by setting c_1 and c_b large numbers otherwise.

5.1 Shooting for subproblems

Gomory [12] obtained $P(C_n, M, b)$ from the master cyclic group polyhedron $P(C_n, b)$ by intersecting $P(C_n, b)$ with the subspace $E(M) = \{t \in \mathbb{R}^{C_n - \{0\}} : t_i = 0 \text{ for } i \notin M\}$ as follows:

Theorem 5.1 (Gomory [12])

$$P(C_n, M, b) = P(C_n, b) \cap E(M).$$

The geometry of shooting over the cyclic group subproblem is the same as that over the master cyclic group problem. Since $t_i = 0$ for $i \notin M$ and $t \in P(C_n, M, b) = P(C_n, b) \cap E(M)$, a nontrivial facet $\pi t \geq 1$ of $P(C_n, b)$ is a valid inequality $\pi_M . t_M \geq 1$ for $P(C_n, M, b)$ in $\mathbb{R}^M$. The facets of $P(C_n, M, b)$ are among the valid inequalities π_M given by deleting π_i for $i \notin M$ from the facets π of the master cyclic group polyhedron $P(C_n, b)$, and so the vertices of $\mathfrak{B}(P(C_n, M, b))$ are among the projections π_M of the vertices π of $\mathfrak{B}(P(C_n, b))$. It is easy to see that

$$\mathfrak{B}(P(C_n, M, b)) = \mathfrak{B}(P(C_n, b))_M,$$

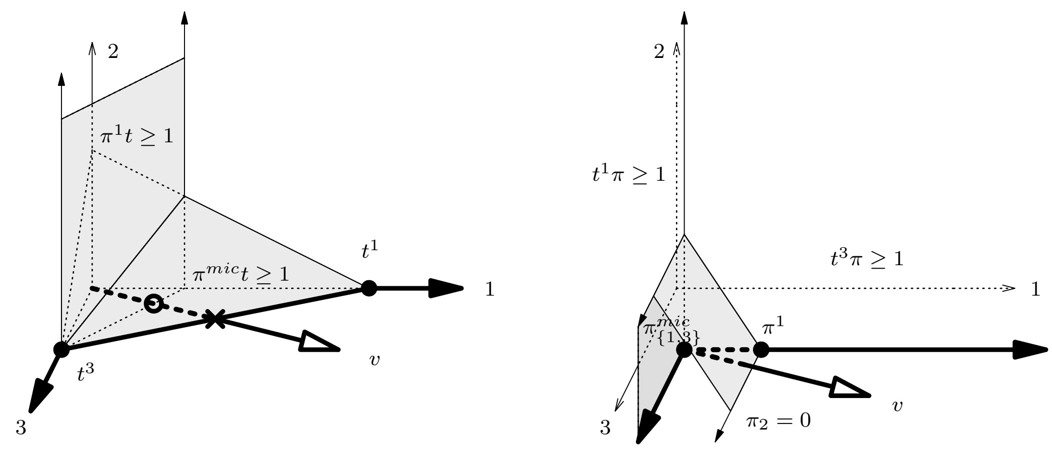

Figure 16: Shooting over $P(C_4, M = \{1, 3\}, 3)$ and $\mathfrak{B}(P(C_4, M, 3))$

where $\mathfrak{B}(P(C_n, b))_M$ denotes the projected image of $\mathfrak{B}(P(C_n, b))$ onto $\{\pi : \pi_i = 0 \text{ for } i \notin M\}$.

We define *shooting in* $v \in \mathbb{R}^M_+$ over the cyclic group subproblem (C_n, M, b) as solving the linear programming problem

$$\begin{array}{ll} \min & v\pi \\ st & \pi \in \mathfrak{B}(P(C_n, M, b)), \end{array}$$

which is equivalent to shooting in $(v, 0) \in \mathbb{R}^M_+ \times \mathbb{R}^{C_n - \{0\} - M}_+$ over the master cyclic group problem (C_n, b). Since $v_i = 0$ for $i \notin M$, any inequality $t\pi \geq 1$ for $\mathfrak{B}(P(C_n, b))$ in (52) holds with a large value of π_i and so can be eliminated for any vertex t of $P(C_n, b)$ with $t_i > 0$ or with $t \notin P(C_n, M, b)$.

Consider shooting in $v = (v_1, v_3) \geq 0$ over the cyclic group subproblem $(C_4, M = \{1, 3\}, 3)$ in Figure 16. On the left side in the figure, the thick-lined polyhedron $P(C_4, M, 3) = P(C_4, 3) \cap E(M)$ is defined by the nontrivial facet $\pi^{mic} t \geq 1$ and the nonnegativities $t_1 \geq 0$ and $t_3 \geq 0$ having two vertices $t^1 = (t^1_1 = 3)$ and $t^3 = (t^3_3 = 1)$ given by intersection of the vertices of $P(C_4, 3)$ with $E(M)$ on the left side in Figure 13. Since $t_2 = 0$ for $t \in P(C_4, M, 3) \subseteq E(M)$, the nontrivial facet $\pi^{mic} t \geq 1$ is written in

$E(M) = \mathbb{R}^{\{1,3\}}$ deleting π_2^{mic} as

$$\pi_{\{1,3\}}^{mic}.(t_1, t_3) = \pi_1^{mic} t_1 + \pi_3^{mic} t_3 = \frac{1}{3} t_1 + t_3 \geq 1,$$

where $\pi_{\{1,3\}}^{mic}$ is the restriction to $\{1, 3\}$ or the projection onto $\{\pi : \pi_2 = 0\}$ of π^{mic}. The 2-dimensional thick-lined polyhedron $\mathfrak{B}(P(C_4, M, 3))$ on the right side is defined by two facets (54) and (56) having the vertex $\pi_{\{1,3\}}^{mic}$ given by projection from the vertex π^{mic} of $\mathfrak{B}(P(C_4, 3))$. The vertex π^1 of $\mathfrak{B}(P(C_4, 3))$ on the right side in Figure 13 is not a vertex of $\mathfrak{B}(P(C_4, M, 3)) = \mathfrak{B}(P(C_4, 3))_{\{1,3\}}$ any more. The constraint (55) among (54)-(56) with a nonzero coefficient of π_2 holds for any big value of π_2 due to $v_2 = 0$, and so can be eliminated resulting in the linear programming problem

$$\begin{array}{rl} \min & v_1\pi_1 + v_3\pi_3 \\ st & t^1\pi = 3\pi_1 \geq 1 \\ & t^3\pi = \pi_3 \geq 1 \\ & \pi_1, \pi_3 \geq 0. \end{array}$$

5.2 Integer primal simplex method

The master cyclic group problem (C_n, b) without ignoring the objective $c \geq 0$ can be solved by a graph algorithm such as Dijkstra's algorithm to find a shortest path from 0 to b on the complete circulant digraph $Cay(C_n)$ with arc length c. On the other hand, we develop another method to solve the shortest path problem that generates facets by shooting, which originated in constraint generation [6] and shooting [23] for the traveling salesman problem. An *integer primal simplex method* based on shooting is as follows:

1. Initial LP relaxation is $P^1 = \{t \geq 0 : \pi^{mic} t \geq 1\}$. Let $j = 1$ and let t^1 be an initial basic feasible solution to P^1, preferably a good one, which is also an extreme point to the cyclic group polyhedron.

2. Iterate primal simplex algorithm over P^j with initial basic feasible solution t^j until encounter the first infeasible solution $\hat{t}^j$ for the master cyclic group problem. Otherwise, return the LP optimal solution that is an optimal solution for the master cyclic group problem and terminate.

3. Generate a facet π by shooting in $\hat{t}^j$ and augment $\pi t \geq 1$ to have $P^{j+1} = P^j \cap \{t : \pi t \geq 1\}$. Set t^{j+1} to be the last feasible solution in the above iteration of simplex algorithm over P^j. Increase $j \leftarrow j+1$ and go to Step 2.

A good initial basic feasible solution in Step 2 can be $(t_b^1 = 1)$, $(t_1^1 = b)$ or any other extreme point of both P^1 and $P(C_n, b)$. As $(t_b^1 = 1)$ is binding at every facet, the cyclic group polyhedra are suspected of high degeneracy. To avoid degeneracy, we may replace primal simplex above by double description method introduced in Motzkin et al [25].

To solve a cyclic group subproblem (C_n, M, b) by using the integer primal simplex method above, we generate cutting planes by shooting ideally over the convex hull of π's corresponding to the nontrivial facets $\pi t \geq \pi_b = 1$ of $P(C_n, M, b)$. For generalization of the subadditive polytope $\Pi(C_n, b)$, refer to Chopra and Johnson [4] and Johnson [21]. For now, we will shoot over the projected image $\Pi(C_n, b)_M$ of the subadditive polytope $\Pi(C_n, b)$ onto M.

Example 5.2 Consider the cyclic group subproblem for the shortest path problem in Figure 2,

$$\begin{aligned} \min \quad & t_1 + t_2 + t_4 + 100t_{11} \\ s.t. \quad & t \in P(C_{12}, M = \{1, 2, 4, 11\}, 11), \end{aligned}$$

where the objective 100 of t_{11} is assigned as an arbitrary large cost to make $t_{11} = 0$ in the optimal solution t. While we implement the integer primal simplex method above, we will shoot over the projected image[1] of the subadditive polytope $\Pi(C_{12}, 11)$ onto $\{1, 2, 4, 11\}$,

$$\begin{aligned}\Pi(C_{12}, 11)_{\{1,2,4,11\}} = \Big\{(\pi_1, \pi_2, \pi_4, \pi_{11}) :& \\ \pi_{11} &= 1 \\ 2\pi_1 - \pi_2 &\geq 0 \\ 2\pi_2 - \pi_4 &\geq 0 \\ \pi_1 + \pi_2 + 2\pi_4 &\geq 1 \\ -\pi_2 - 2\pi_4 &\geq -2 \\ -4\pi_1 - \pi_4 &\geq -4 \\ -\pi_1 - \pi_2 + \pi_4 &\geq -1\Big\}.\end{aligned}$$

We solve the first linear programming problem over P^1,

$$\begin{aligned} \min \quad & t_1 + t_2 + t_4 + 100t_{11} \\ s.t. \quad & t \geq 0 \\ & t_1 + 2t_2 + 4t_4 + 11t_{11} \geq 11, \end{aligned}$$

with the initial basis $t^1_{11} = 1$ until we meet the first infeasible basis $\hat{t}^1_4 = 11/4$.

By shooting in the infeasible solution ($\hat{t}^1_4 = 11/4$) as

$$\begin{aligned} \min \quad & \frac{11}{4}\pi_4 \\ s.t. \quad & \pi \in \Pi(C_{12}, 11)_{\{1,2,4,11\}}, \end{aligned}$$

we generate and augment a second valid inequality

$$\pi_1 = \pi_{11} = 1, \pi_2 = \pi_4 = 0$$

[1] The projection is done by Fourier-Motzkin elimination.

cutting off the infeasible solution. We solve the augmented LP over P^2,

$$\begin{array}{ll} \min & t_1 + t_2 + t_4 + 100t_{11} \\ s.t. & t \geq 0 \\ & t_1 + 2t_2 + 4t_4 + 11t_{11} \geq 11 \\ & t_1 + t_{11} \geq 1, \end{array}$$

with the initial basis $t^2_{11} = 1, t^2_1 = 0$, and meet the first infeasible basis $\hat{t}^2_4 = 5/2, \hat{t}^2_1 = 1$.

By shooting in the infeasible solution $(\hat{t}^2_4 = 5/2, \hat{t}^2_1 = 1)$, we generate and augment a third valid inequality

$$\pi_1 = \frac{1}{3}, \pi_2 = \frac{2}{3}, \pi_{11} = 1$$

cutting off the infeasible solution $\hat{t}^2$, and solve the LP over P^3,

$$\begin{array}{ll} \min & t_1 + t_2 + t_4 + 100t_{11} \\ s.t. & t \geq 0 \\ & t_1 + 2t_2 + 4t_4 + 11t_{11} \geq 11, \\ & t_1 + t_{11} \geq 1, \\ & t_1 + 2t_2 + 3t_{11} \geq 3. \end{array}$$

with the initial basis $t^3_{11} = 1, t^3_1 = 0, t^3_4 = 0$. The LP ends up with an optimal basis $t_2 = 1, t_1 = 1, t_4 = 2$ which is feasible for the cyclic group subproblem $(C_{12}, \{1, 2, 4, 11\}, 11)$. See the thick lined path in Figure 2. □

The integer primal simplex method based on shooting is a primal cutting plane algorithm. A primal cutting plane algorithm is known to be successful especially for the traveling salesman problem developed by Padberg and Hong [27]. The strongest point of a primal cutting plane algorithm is to keep feasibility of intermediate solutions showing up during the algorithm. We may have a fairly good improved feasible solution even though we stop before termination. However, intermediate solutions can hardly escape from the initial basic feasible solution t^1 before getting the final optimal solution if the primal cutting plane algorithm begins with the most degenerate solution $t^1 = (t^1_b = 1)$, as we see $t^1 = t^2 = t^3 = (t_{11} = 1)$ in Example 5.2, which makes us pay attention to the claim by Nemhauser and Wolsey [26] that a primal cutting plane algorithm tends to be inferior to a fractional cutting plane algorithm. The *fractional cutting plane algorithm* is as follows:

1. Initial LP relaxation is $P^1 = \{t \geq 0 : \pi^{mic} t \geq 1\}$. Let $j = 1$ and let $\hat{t}^0 = \mathbf{0}$ with the surplus variable as dual-feasible basis.

2. Iterate dual simplex algorithm over P^j with initial dual-feasible solution $\hat{t}^{j-1}$ to get an optimal solution $\hat{t}^j$. If $\hat{t}^j$ is a solution to the master cyclic group problem, return it as an optimal solution for the master cyclic group problem and terminate.

3. Generate a facet π by shooting in $\hat{t}^j$ and augment $\pi t \geq 1$ to have $P^{j+1} = P^j \cap \{t : \pi t \geq 1\}$. Increase $j \leftarrow j + 1$ and go to Step 2.

For example, let's solve the cyclic group subproblem in Example 5.2 by using the fractional cutting plane algorithm. The sequence of polyhedra P^1, P^2 and P^3 are the same as those in the primal cutting plane algorithm of Example 5.2, and the sequence of intermediate solutions $\hat{t}^1 = (\hat{t}^1_4 = 11/4)$ and $\hat{t}^2 = (\hat{t}^2_4 = 5/2, \hat{t}^2_1 = 1)$ are exactly those in Example 5.2. The small example does not show any difference between the two cutting plane algorithms in performance. But, the master cyclic group problem with a random cost $c \geq 0$ will be a good testing material for comparison of the two cutting plane algorithms.

5.3 *Primal-dual simplex method for shooting*

We implement shooting in v as a subroutine of the cutting plane algorithms introduced in the previous section. For shooting in v, we will solve the following equivalent problem,

$$\begin{aligned} \max \quad & w\varphi \\ st \quad & \varphi \in \Pi(C_n, b)_O - \dot{\pi}_O, \end{aligned} \tag{66}$$

where w is given by

$$w_i \quad = \quad v_{\bar{i}} - v_i \text{ for all } i \in O.$$

In this section, we assume that our master cyclic group problem has no half h with $2h \equiv b \mod n$. We only need to augment at most a couple constraints including quarters if it has a half.

We can write the above problem explicitly as follows:

$$\begin{aligned} \max \quad & w\varphi \\ st \quad & -\varphi_i - \varphi_j + \varphi_k \leq \tfrac{1}{2} \quad \text{if } i + j + \bar{k} \equiv b \mod n \\ & -\varphi_i - \varphi_j - \varphi_k \leq \tfrac{1}{2} \quad \text{if } i + j + k \equiv b \mod n \\ & \varphi_i + \varphi_j - \varphi_k \leq \tfrac{1}{2} \quad \text{if } \bar{i} + \bar{j} + k \equiv b \mod n \\ & \varphi_i + \varphi_j + \varphi_k \leq \tfrac{1}{2} \quad \text{if } \bar{i} + \bar{j} + \bar{k} \equiv b \mod n \end{aligned}$$

where all i, j, k are in O given by the mixed integer cut π^{mic}. We can rewrite the four types of inequalities above briefly as one line,

$$\operatorname{sgn}(i)\varphi_{p(i)} + \operatorname{sgn}(j)\varphi_{p(j)} + \operatorname{sgn}(k)\varphi_{p(k)} \leq \frac{1}{2} \text{ whenever } i + j + k \equiv b \mod n, \tag{67}$$

where $\mathrm{sgn}(g)$ and $p(g)$ are respectively defined by

$$\begin{aligned} \mathrm{sgn}(g) &= -1 && \text{if } g \in O, \\ &= +1 && \text{if } g \in X, \\ p(g) &\equiv g \mod n && \text{if } g \in O \\ &\equiv \bar{g} \equiv b - g \mod n && \text{if } g \in X. \end{aligned}$$

For convenience, let's define the left-hand side of (67) as a function LHS_φ on the triples (i, j, k) with $i + j + k \equiv b \mod n$ evaluated by φ as follows:

$$LHS_\varphi(i, j, k) = \mathrm{sgn}(i)\varphi_{p(i)} + \mathrm{sgn}(j)\varphi_{p(j)} + \mathrm{sgn}(k)\varphi_{p(k)} \text{ for } i + j + k \equiv b \mod n.$$

Let $\varphi^0 = \mathbf{0}$ be the initial feasible solution and let $\rho^1 = w$ be the initial improving direction. We decide the step size θ_τ, $\tau \geq 1$, so that $\varphi^\tau = \varphi^{\tau-1} + \theta_\tau \rho^\tau$ are inductively constructed to be feasible for (67) and are binding at some constraints. Compute the maximum value of left-hand sides of the constraints above when evaluated by $\rho^1 = w$. Since the subadditive polytope $\Pi(C_n, b)$ is bounded, the maximum value is strictly positive. Then, the first step size θ_1 can be given by the equation,

$$\theta_1 \times \max\{LHS_{\rho^1}(i, j, k) = LHS_w(i, j, k) : i + j + k \equiv b \mod n\} = \frac{1}{2}.$$

Let E^1 consist of the rows which are the coefficient vectors of the constraints yielding the maximum value of the left-hand sides $LHS_{\rho^1}(i, j, k) = LHS_w(i, j, k)$ evaluated by $\rho^1 = w$. The feasible solution $\varphi^1 = \varphi^0 + \theta_1\rho^1 = \theta_1 w$ is binding at the constraints corresponding to the rows of E^1. The number of constraints to enumerate for the maximum value is n multiplied by the number of nonzero components of $\rho^1 = w$ that is at most $n|M|$, since we don't need to enumerate the constraints with all 0 terms in their left-hand sides evaluated by $\rho^1 = w$.

Let $\tau \geq 2$ and consider the restricted problem to $E^{\tau-1}$ of the dual problem of the problem (66). The phase 1 of the restricted problem has the following dual problem with a feasible solution $\rho = 0$:

$$\begin{aligned} \max \quad & w\rho \\ st \quad & E^{\tau-1}\rho \leq \mathbf{0} \\ & \rho_i \leq 1 \text{ if } w_i > 0 \\ & \rho_i \geq -1 \text{ if } w_i < 0. \end{aligned} \tag{68}$$

The objective w is sparse for subproblems (C_n, M, b) with $|M| \ll n$ and there are only small number of upper bounded variables $\rho_i \leq 1$ and lower bounded variables $\rho_i \geq -1$. Almost all variables are unrestricted in sign. If the optimal value of (68) is 0, the current solution $\varphi^{\tau-1}$ is optimal. Otherwise, its optimal solution ρ^τ is a nontrivial improving direction for the next feasible solution $\varphi^\tau = \varphi^{\tau-1} + \theta_\tau \rho^\tau$. If $LHS_{\rho^\tau}(i,j,k)$ is strictly positive, the step size θ_τ should satisfy

$$LHS_{\varphi^\tau}(i,j,k) = LHS_{\varphi^{\tau-1}}(i,j,k) + \theta_\tau LHS_{\rho^\tau}(i,j,k) \leq \frac{1}{2}.$$

Therefore, the step size θ_τ, $\tau \geq 1$, is given by

$$\theta_\tau = \min\left\{ \frac{\frac{1}{2} - LHS_{\varphi^{\tau-1}}(i,j,k)}{LHS_{\rho^\tau}(i,j,k)} : i+j+k \equiv b \mod n \text{ and } LHS_{\rho^\tau}(i,j,k) > 0 \right\}.$$

The binding constraints of φ^τ consists of the constraints yielding the minimum value θ_τ and the constraints from $E^{\tau-1}$ which correspond to $i+j+k \equiv b \mod n$ with $LHS_{\rho^\tau}(i,j,k) = 0$. The inductive construction of φ^τ terminates as an optimal solution $\varphi = \varphi^\tau$ for (66) when 0 is the optimal value $w\rho^{\tau+1}$ of the restricted Phase-1's dual problem,

$$\begin{aligned} \max \quad & w\rho \\ st \quad & E^{\tau}\rho \leq \mathbf{0} \\ & \rho_i \leq 1 \text{ if } w_i > 0 \\ & \rho_i \geq -1 \text{ if } w_i < 0. \end{aligned}$$

As an example, let's consider the cyclic group subproblem $(C_8, M = \{1, 2, 4\}, b = 7)$. Our initial linear programming relaxation is

$$\begin{aligned}
\min \quad & t_1 + t_2 + t_4 \\
st \quad & \frac{1}{7}t_1 + \frac{2}{7}t_2 + \frac{4}{7}t_4 \geq 1 \\
& t \geq 0.
\end{aligned}$$

containing the mixed integer cut restricted to M with an initial basic solution $(t_1 = 7)$. The basic feasible solution $\hat{t}^1 = (\hat{t}^1_4 = \frac{7}{4})$ is its first solution infeasible for $(C_8, M = \{1, 2, 4\}, b = 7)$ in simplex method iterations. We generate a cutting plane by shooting in the infeasible solution $\hat{t}^1 = (\hat{t}^1_4 = \frac{7}{4})$ to cut off. Let's solve the following problem equivalent to the shooting:

$$\begin{aligned}
\max \quad & \frac{7}{4}\varphi_3 \\
st \quad & \varphi \in \Pi(C_8, 7)_O - \dot{\pi}_O.
\end{aligned} \tag{69}$$

To solve it by using primal-dual simplex method, set the initial improving direction $\rho^1 = w = (\rho^1_3 = 7/4)$. The step size $\theta^1 = 2/7$ is decided at two constraints

$$\begin{aligned}
-\varphi_1 - \varphi_2 + \varphi_3 &\leq \frac{1}{2} \\
+\varphi_1 + \varphi_2 + \varphi_3 &\leq \frac{1}{2}.
\end{aligned}$$

That is, $\varphi^1 = \theta_1\rho^1 = (\varphi^1_3 = 1/2)$ is binding the two constraints above. Let E^1 be the coefficient matrix corresponding to the two constraints.

The dual of Phase 1 restricted to E^1 is

$$\begin{aligned}
\max \quad & \frac{7}{4}\rho_3 \\
st \quad & -\rho_1 - \rho_2 + \rho_3 \leq 0 \\
& \rho_1 + \rho_2 + \rho_3 \leq 0 \\
& \rho_3 \leq 1.
\end{aligned}$$

Since 0 is an optimal solution to the problem, φ^1 is optimal for (69). Translation of $\varphi^1 = (\varphi^1_3 = 1/2)$ by the natural interior point $\dot{\pi}$ and restriction to M induce the cutting plane

$$\pi_M = (\pi_1 = \varphi^1_1 + \dot{\pi}_1 = 1/2, \pi_2 = \varphi^1_2 + \dot{\pi}_2 = 1/2, \pi_4 = 1 - (\varphi^1_3 + \dot{\pi}_3) = 0).$$

We go on the primal cutting plane algorithm with the stronger relaxation augmented by π_M above,

$$\begin{array}{ll} \min & t_1 + t_2 + t_4 \\ st & \frac{1}{7}t_1 + \frac{2}{7}t_2 + \frac{4}{7}t_4 \geq 1 \\ & \frac{1}{2}t_1 + \frac{1}{2}t_2 \geq 1 \\ & t \geq 0. \end{array}$$

An initial basis may be $t_1 = 7$ and the surplus variable of the second constraint equal to 5/2. The first infeasible solution is $\hat{t}^2 = (\hat{t}^2_1 = 2, \hat{t}^2_4 = 5/4)$ in which we shoot to generate the next cutting plane. Our second shooting is

$$\begin{array}{ll} \max & -2\varphi_1 + \frac{5}{4}\varphi_3 \\ st & \varphi \in \Pi(C_8, 7)_O - \dot{\pi}_O. \end{array} \tag{70}$$

The feasible solution in primal-dual simplex method is

$$\varphi^1 = \theta^1 \rho^1 = \frac{1}{8}\left(\rho^1_1 = -2, \rho^1_3 = \frac{5}{4}\right) = \left(\varphi^1_1 = -\frac{1}{4}, \varphi^1_3 = \frac{5}{32}\right),$$

where θ^1 is given at the binding constraint

$$-2\varphi^1_1 + \varphi^1_2 \leq \frac{1}{2}.$$

The dual of restricted Phase-1 is

$$\begin{array}{ll} \max & -2\rho^2_1 + \frac{5}{4}\rho^2_3 \\ st & -2\rho^2_1 + \rho^2_2 \leq 0 \\ & \rho^2_1 \geq -1 \\ & \rho^2_3 \leq 1. \end{array} \tag{71}$$

An optimal solution $\rho^2 = (\rho_1^2 = -1, \rho_2^2 = -2, \rho_3^2 = 1)$ is binding at the constraint (71). The step size $\theta^2 = 3/128$ is decided at

$$-\varphi_1^2 - \varphi_2^2 + \varphi_3^2 \leq \frac{1}{2}. \tag{72}$$

The next feasible solution is

$$\varphi^2 = \varphi^1 + \theta^2\rho^2 = \left(\varphi_1^2 = -\frac{35}{128}, \varphi_2^2 = -\frac{3}{64}, \varphi_3^2 = \frac{23}{128}\right).$$

We solve the dual of restricted Phase-1 corresponding to (71) and (72),

$$\begin{aligned} \max \quad & -2\rho_1^3 + \frac{5}{4}\rho_3^3 \\ st \quad & -2\rho_1^3 + \rho_2^3 \leq 0 && (73) \\ & -\rho_1^3 - \rho_2^3 + \rho_3^3 \leq 0 && (74) \\ & \rho_1^3 \geq -1 \\ & \rho_3^3 \leq 1. \end{aligned}$$

Its optimal solution is

$$\rho^3 = \left(\rho_1^3 = \frac{1}{3}, \rho_2^3 = \frac{2}{3}, \rho_3^3 = 1\right)$$

binding at (73) and (74). The step size $\theta^3 = \frac{41}{128}$ is decided at the two constraints

$$\varphi_1^3 + \varphi_2^3 + \varphi_3^3 \leq \frac{1}{2} \tag{75}$$

$$3\varphi_2^3 \leq \frac{1}{2}. \tag{76}$$

The next feasible solution is

$$\varphi^3 = \varphi^2 + \theta^3\rho^3 = \left(\varphi_1^3 = -\frac{1}{6}, \varphi_2^3 = \frac{1}{6}, \varphi_3^3 = \frac{1}{2}\right).$$

We solve the dual of restricted Phase-1 corresponding to (73), (74), (75) and (76);

i.e.,

$$\begin{aligned} \max \quad & -2\rho_1^4 + \frac{5}{4}\rho_3^4 \\ st \quad & -2\rho_1^4 + \rho_2^4 \leq 0 \\ & -\rho_1^4 - \rho_2^4 + \rho_3^4 \\ & \rho_1^4 + \rho_2^4 + \rho_3^4 \leq 0 \\ & 3\rho_2^4 \leq 0 \\ & \rho_1^4 \geq -1 \\ & \rho_3^4 \leq 1. \end{aligned}$$

It has $\rho^4 = 0$ as an optimal solution. Therefore, the feasible solution φ^3 is an optimal solution for (70) generating the cutting plane

$$\pi_M = (\pi_1 = \frac{1}{6} + \frac{1}{2} = \frac{1}{3}, \pi_2 = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}, \pi_4 = 1 - (\frac{1}{2} + \frac{1}{2})) = 0.$$

We augment it to have the stronger relaxation,

$$\begin{aligned} \min \quad & t_1 + t_2 + t_4 \\ st \quad & \frac{1}{7}t_1 + \frac{2}{7}t_2 + \frac{4}{7}t_4 \geq 1 \\ & \frac{1}{2}t_1 + \frac{1}{2}t_2 \geq 1 \\ & \frac{1}{3}t_1 + \frac{2}{3}t_2 \geq 1 \\ & t \geq 0, \end{aligned}$$

and end up with optimal solution $(t_1 = t_2 = t_4 = 1)$ for the cyclic group subproblem $(C_8, \{1, 2, 4\}, 7)$.

5.4 Wong-Coppersmith digraphs

We implement fractional cutting plane algorithm introduced in Section 5.2 to solve the cyclic group subproblem $(C_{2^m}, M = \{2^0, 2^1, ..., 2^{m-1}\}, 2^m - 1)$ with cost $\mathbf{1}$. The cyclic group subproblem is the shortest path problem from 0 to $b = 2^m - 1$ in the circulant digraph $Cay(C_n, M)$ with cardinality length $\mathbf{1}$ where $n = 2^m$. The shortest path is

$$2^m - 1 = 2^0 + 2^1 + ... + 2^{m-1}.$$

That is, the optimal solution is $t = (t_{2^0} = t_{2^1} = ... = t_{2^{m-1}} = 1) = \mathbf{1}$.

For $m \leq 10$, the fractional cutting plane algorithm augments the cutting planes in Table 28 through Table 35 and yields the optimal solution $t = \mathbf{1}$. The cutting planes are generated in the order of the table by shooting over the subadditive polytope.

From $11 \leq m \leq 13$, the fractional cutting plane algorithm takes too many hours because of the big size of each shooting linear programming problem. For each shooting, primal-dual simplex method is implemented as introduced in Section 5.3. Instead of the trivial initial feasible solution $\varphi^0 = 0$, we adopt the mixed integer cut $\varphi^0 = \pi_O^{mic} - \dot{\pi}_O$ as our initial feasible solution to speed up the primal dual simplex method. See Table 36 through Table 38.[2]

We see the number of cutting planes is reasonably small in m. So, the running time of our cutting plane algorithm depends on shooting linear programming problems.

[2]The C++ code of our cutting plane algorithm is written in Appendix A calling CPLEX for the dual of each restricted Phase 1 in primal dual simplex method.

Table 28: The facets $\pi t \geq 1$ augmented for $m = 3$

order	π_{2^0}	π_{2^1}	π_{2^2}
1:	0.142857142857143	0.285714285714286	0.571428571428571
2:	0.333333333333333	0.666666666666667	0
3:	1	0	0

Table 29: The facets $\pi t \geq 1$ augmented for $m = 4$

order	π_{2^0}	π_{2^1}	π_{2^2}	π_{2^3}
1:	0.0666666666666667	0.133333333333333	0.266666666666667	0.533333333333333
2:	0.777777777777778	0.666666666666667	0.444444444444444	0
3:	0.142857142857143	0.285714285714286	0.571428571428571	0
4:	0.333333333333333	0.666666666666667	0	0
5:	1	0	0	0

Table 30: The facets $\pi t \geq 1$ augmented for $m = 5$

order	π_{2^0}	π_{2^1}	π_{2^2}	π_{2^3}	π_{2^4}
1:	0.032258064516129	0.0645161290322581	0.129032258064516	0.258064516129032	0.516129032258065
2:	0.6	0.4	0.8	0.266666666666667	0
3:	1	0	0	0	0
4:	0.0666666666666667	0.133333333333333	0.266666666666667	0.533333333333333	0
5:	0.142857142857143	0.285714285714286	0.571428571428571	0	0
6:	0.333333333333333	0.666666666666667	0	0	0

Table 31: The facets $\pi t \geq 1$ augmented for $m = 6$

order	π_{2^0}	π_{2^1}	π_{2^2}
1:	0.0158730158730159	0.0317460317460317	0.0634920634920635
2:	0.706666666666667	0.133333333333333	0.266666666666667
3:	1	0	0
4:	0.0666666666666665	0.133333333333333	0.266666666666666
5:	0.142857142857143	0.285714285714286	0.571428571428571
6:	0.333333333333333	0.666666666666667	0
7:	0.032258064516129	0.064516129032258	0.129032258064516

order	π_{2^3}	π_{2^4}	π_{2^5}
1:	0.126984126984127	0.253968253968254	0.507936507936508
2:	0.533333333333333	0.853333333333333	0
3:	0	0	0
4:	0.533333333333334	- 3.33066907387547e-16	1.11022302462516e-16
5:	0	0	0
6:	0	0	1.11022302462516e-16
7:	0.258064516129032	0.516129032258065	0

Table 32: The facets $\pi t \geq 1$ augmented for $m = 7$

order	π_{2^0}	π_{2^1}	π_{2^2}	π_{2^3}
1:	0.0078740157480315	0.015748031496063	0.031496062992126	0.062992125984252
2:	0.693121693121693	0.821869488536155	0.514991181657848	0.578483245149912
3:	0.454545454545455	0.909090909090909	0.363636363636364	0.727272727272727
4:	1	5.55111512312578e-17	0	0
5:	0.015873015873016	0.0317460317460317	0.0634920634920633	0.126984126984126
6:	0.0322580645161291	0.064516129032258	0.129032258064516	0.258064516129032
7:	0.0666666666666667	0.133333333333333	0.266666666666667	0.533333333333333
8:	0.142857142857143	0.285714285714286	0.571428571428571	- 1.11022302462516e-16
9:	0.333333333333333	0.666666666666667	- 1.11022302462516e-16	1.66533453693773e-16

order	π_{2^4}	π_{2^5}	π_{2^6}
1:	0.125984251968504	0.251968503937008	0.503937007874016
2:	0.253968253968254	0.507936507936508	0
3:	0.484848484848485	5.55111512312578e-17	0
4:	1.11022302462516e-16	0	0
5:	0.253968253968254	0.507936507936508	- 1.11022302462516e-16
6:	0.516129032258064	5.55111512312578e-17	0
7:	0	0	0
8:	0	0	2.22044604925031e-16
9:	0	2.77555756156289e-16	- 2.22044604925031e-16

Table 33: The facets $\pi t \geq 1$ augmented for $m = 8$

order	π_{2^0}	π_{2^1}	π_{2^2}	π_{2^3}
1:	0.00392156862745098	0.00784313725490196	0.0156862745098039	0.0313725490196078
2:	0.333333333333333	0.333333333333333	0.333333333333333	0.333333333333333
3:	0.333333333333333	0.333333333333333	0.333333333333333	0.666666666666667
4:	0.777777777777778	0.666666666666667	0.444444444444445	0
5:	0.142857142857143	0.285714285714286	0.571428571428572	1.11022302462516e-16
6:	0.333333333333333	0.666666666666667	- 1.11022302462516e-16	0
7:	1	0	1.11022302462516e-16	- 1.11022302462516e-16
8:	0.00787401574803143	0.0157480314960631	0.0314960629921262	0.0629921259842515
9:	0.0158730158730159	0.0317460317460316	0.0634920634920634	0.126984126984127
10:	0.0322580645161292	0.0645161290322579	0.129032258064516	0.258064516129032
11:	0.0666666666666668	0.133333333333333	0.266666666666666	0.533333333333333

order	π_{2^4}	π_{2^5}	π_{2^6}	π_{2^7}
1:	0.0627450980392157	0.125490196078431	0.250980392156863	0.501960784313725
2:	0.333333333333333	0.666666666666667	0.666666666666667	0
3:	0.333333333333333	0.333333333333333	0	0
4:	0	0	0	0
5:	1.11022302462516e-16	2.22044604925031e-16	2.22044604925031e-16	- 1.11022302462516e-16
6:	- 1.55431223447522e-15	0	0	1.11022302462516e-16
7:	- 2.22044604925031e-16	- 4.44089209850063e-16	1.66533453693773e-16	0
8:	0.125984251968504	0.251968503937008	0.503937007874016	- 1.11022302462516e-16
9:	0.253968253968254	0.507936507936508	0	1.11022302462516e-16
10:	0.516129032258065	0	0	0
11:	- 1.11022302462516e-16	0	2.22044604925031e-16	- 1.11022302462516e-16

Table 34: The facets $\pi t \geq 1$ augmented for $m = 9$

order	π_{20}	π_{21}	π_{22}	π_{23}	π_{24}
1:	0.00195694716242661	0.00391389432485323	0.00782778864970646	0.0156555772994129	0.0313111545988258
2:	0.666666666666667	0.333333333333334	0.666666666666667	0.333333333333333	0.666666666666667
3:	0.666666666666667	0.666666666666667	0.333333333333333	0.666666666666667	0.666666666666667
4:	0.333333333333333	0.666666666666667	0.333333333333333	0.666666666666667	1.11022302462516e-16
5:	0.333333333333333	0.666666666666667	3.33066907387547e-16	0	3.88578058618805e-16
6:	1	2.22044604925031e-16	4.44089209850063e-16	1.18793863634892e-14	- 1.32116539930394e-14
7:	0.00392156862745108	0.00784313725490193	0.015686274509804	0.0313725490196077	0.0627450980392147
8:	0.00787401574803137	0.015748031496063	0.031496062992126	0.0629921259842527	0.125984251968505
9:	0.0158730158730157	0.0317460317460316	0.0634920634920644	0.126984126984127	0.253968253968254
10:	0.0322580645161288	0.0645161290322582	0.129032258064517	0.258064516129032	0.516129032258065
11:	0.0666666666666666	0.133333333333334	0.266666666666667	0.533333333333333	0
12:	0.142857142857143	0.285714285714286	0.571428571428571	1.11022302462516e-16	0

order	π_{25}	π_{26}	π_{27}	π_{28}
1:	0.0626223091976517	0.125244618395303	0.250489236790607	0.500978473581213
2:	0.666666666666666	0.333333333333333	0.666666666666666	0
3:	0.333333333333333	0.333333333333333	0	0
4:	- 1.11022302462516e-16	- 2.22044604925031e-16	4.9960036108132e-16	1.66533453693773e-16
5:	0	2.77555756156289e-16	0	- 1.11022302462516e-16
6:	- 2.94209101525666e-14	- 5.6621374255883e-15	0	- 2.22044604925031e-16
7:	0.125490196078431	0.250980392156863	0.501960784313725	0
8:	0.251968503937007	0.503937007874016	0	0
9:	0.507936507936508	0	1.11022302462516e-16	5.55111512312578e-17
10:	- 1.11022302462516e-16	- 1.11022302462516e-16	1.11022302462516e-16	0
11:	- 2.22044604925031e-16	- 1.11022302462516e-16	5.55111512312578e-17	2.22044604925031e-16
12:	2.22044604925031e-16	2.22044604925031e-16	0	- 2.22044604925031e-16

Table 35: The facets $\pi t \geq 1$ augmented for $m = 10$

order	π_{20}	π_{21}	π_{22}	π_{23}	π_{24}
1:	0.000977517106549365	0.00195503421309873	0.00391006842619746	0.00782013685239492	0.0156402737047898
2:	0.333333333333334	0.666666666666667	0.666666666666667	0.666666666666667	0.333333333333334
3:	0.333333333333328	0.666666666666666	0.666666666666669	0	0
4:	0.333333333333347	0.666666666666693	5.55111512312578e-17	1.11022302462516e-16	2.22044604925031e-16
5:	1	0	0	0	0
6:	0.0019569471624265	0.00391389432485334	0.00782778864970635	0.0156555772994129	0.0313111545988258
7:	0.00392156862745069	0.00784313725490215	0.0156862745097943	0.0313725490196082	0.062745098039215
8:	0.00787401574803143	0.0157480314960631	0.031496062992126	0.0629921259842525	0.125984251968504
9:	0.0158730158730159	0.0317460317460316	0.0634920634920637	0.126984126984127	0.253968253968254
10:	0.0322580645161287	0.0645161290322582	0.129032258064516	0.258064516129033	0.516129032258066
11:	0.0666666666666668	0.133333333333333	0.266666666666667	0.533333333333333	2.22044604925031e-16
12:	0.142857142857143	0.285714285714286	0.571428571428573	5.55111512312578e-17	1.11022302462516e-16

order	π_{25}	π_{26}	π_{27}	π_{28}	π_{29}
1:	0.0312805474095797	0.0625610948191593	0.125122189638319	0.250244379276637	0.500488758553275
2:	0.333333333333334	0.333333333333334	0.333333333333333	0.333333333333333	0
3:	- 2.66453525910038e-15	- 8.88178419700125e-16	- 1.77635683940025e-15	0	0
4:	4.44089209850063e-16	- 9.2148511043888e-15	- 1.84297022087776e-14	0	0
5:	- 1.11022302462516e-16	- 2.22044604925031e-16	3.33066907387547e-16	4.38538094726937e-15	0
6:	0.0626223091976517	0.125244618395303	0.250489236790606	0.500978473581213	0
7:	0.125490196078432	0.250980392156863	0.501960784313725	1.11022302462516e-16	1.11022302462516e-16
8:	0.251968503937007	0.503937007874016	1.11022302462516e-16	0	- 1.11022302462516e-16
9:	0.507936507936509	1.11022302462516e-16		- 1.11022302462516e-16	1.11022302462516e-16
10:	1.11022302462516e-16	2.22044604925031e-16	- 2.22044604925031e-16	1.11022302462516e-16	- 2.22044604925031e-16
11:	5.55111512312578e-17	- 3.33066907387547e-16	1.66533453693773e-16	5.55111512312578e-17	0
12:	0	- 1.11022302462516e-16	1.11022302462516e-16	0	2.22044604925031e-16

Table 36: The facets $\pi t \geq 1$ augmented for $m = 11$

order	$\pi_2 0$	$\pi_2 1$	$\pi_2 2$	$\pi_2 3$	$\pi_2 4$	$\pi_2 5$
1:	0.000488519785051295	0.000977039570102589	0.00195407914020518	0.00390815828041036	0.00781631656082071	0.0156326331216414
2:	0.5	0.5	0.5	0.5	0.5	0.5
3:	0.875	0.875	0.833333333333333	0.75	0.666666666666667	0.666666666666667
4:	1	0	0	5.55111512312578e-17	0	5.55111512312578e-17
5:	0.00392156862745086	0.00784313725490166	0.0156862745098035	0.0313725490196069	0.0627450980392158	0.125490196078429
6:	0.00787401574803126	0.0157480314960626	0.0314960629921252	0.0629921259842499	0.125984251968501	0.251968503937005
7:	0.0158730158730153	0.0317460317460306	0.0634920634920612	0.126984126984123	0.253968253968253	0.507936507936496
8:	0.0322580645161281	0.0645161290322562	0.129032258064513	0.258064516129027	0.516129032258054	4.44089209850063e-16
9:	0.0666666666666647	0.133333333333329	0.266666666666659	0.533333333333313	3.33066907387547e-16	7.21644966006352e-16
10:	0.142857142857138	0.285714285714276	0.571428571428552	1.11022302462516e-16	3.88578058618805e-16	5.55111512312578e-16
11:	0.000977517106549364	0.00195503421309873	0.00391006842619745	0.00782013685239485	0.0156402737047898	0.0312805474095795
12:	0.333333333333305	0.66666666666661	2.22044604925031e-16	3.33066907387547e-16	6.66133814775094e-16	1.38777878078145e-15
13:	0.00195694716242684	0.00391389432485323	0.00782778864970618	0.0156555772994124	0.0313111545988251	0.0626223091976505

order	$\pi_2 6$	$\pi_2 7$	$\pi_2 8$	$\pi_2 9$	$\pi_2 10$
1:	0.0312652662432829	0.0625305324865657	0.125061064973131	0.250122129946263	0.500244259892526
2:	0.5	0.5	0.5	0.5	0
3:	0.666666666666667	0.666666666666667	0.458333333333333	1.11022302462516e-16	5.55111512312578e-17
4:	2.22044604925031e-16	7.21644966006352e-16	3.33066907387547e-16	6.66133814775094e-16	1.33226762955019e-15
5:	0.250980392156854	0.501960784313712	5.21804821573824e-15	1.04360964314765e-14	1.50990331349021e-14
6:	0.503937007874006	1.72084568816899e-15	3.44169137633799e-15	6.88338275267597e-15	1.26565424807268e-14
7:	1.27675647831893e-15	2.4980018054066e-15	5.32907051820075e-15	1.06581410364015e-14	1.77635683940025e-14
8:	8.88178419700125e-16	1.77635683940025e-15	3.49720252756924e-15	5.99520433297585e-15	1.28785870856518e-14
9:	1.4432899320127e-15	2.99760216648792e-15	7.32747196252603e-15	1.16573417585641e-14	6.88338275267597e-15
10:	1.38777878078145e-15	2.55351295663786e-15	4.88498130835069e-15	1.14352971536391e-14	1.4210854715202e-14
11:	0.0625610948191591	0.125122189638318	0.250244379276635	0.500488758553272	5.10702591327572e-15
12:	2.77555756156289e-15	5.93969318174459e-15	1.18793863634892e-14	2.37587727269783e-14	3.77475828372553e-14
13:	0.1252446183953	0.2504892367906	0.500978473581199	1.35447209004269e-14	1.22124532708767e-14

Table 37: The facets $\pi t \geq 1$ augmented for $m = 12$

order	π_{20}	π_{21}	π_{22}	π_{23}	π_{24}	π_{25}
1:	0.000244200244200244	0.000488400488400488	0.000976800976800977	0.00195360195360195	0.00390720390720391	0.00781440781440781
2:	0.5	0.5	0.5	0.5	0.5	0.5
3:	0.875	0.875	0.833333333333333	0.75	0.666666666666667	0.666666666666667
4:	1	0	5.55111512312578e-17	1.11022302462516e-16	2.22044604925031e-16	3.88578058618805e-16
5:	0.00195694716242628	0.00391389432485262	0.00782778864970524	0.0156555772994105	0.0313111545988211	0.0626223091976397
6:	0.00392156862745091	0.00784313725490171	0.0156862745098035	0.0313725490196073	0.062745098039214	0.125490196078428
7:	0.00787401574803043	0.015748031496061	0.0314960629921218	0.0629921259842439	0.125984251968486	0.251968503936971
8:	0.0158730158730149	0.0317460317460299	0.0634920634920597	0.126984126984124	0.253968253968233	0.507936507936482
9:	0.0322580645161231	0.0645161290322462	0.129032258064492	0.258064516128961	0.516129032257972	1.49880108324396e-15
10:	0.0666666666666624	0.133333333333325	0.26666666666665	0.533333333333301	2.77555756156289e-16	4.44089209850063e-16
11:	0.142857142857128	0.285714285714255	0.571428571428511	2.22044604925031e-16	4.9960036108132e-16	7.21644966006352e-16
12:	0.000488519785051211	0.000977039570102478	0.00195407914020523	0.00390815828041058	0.00781631656082049	0.015632633121641
13:	0.333333333333248	0.666666666666496	3.33066907387547e-16	5.55111512312578e-16	9.43689570931383e-16	2.16493489801906e-15
14:	0.000977517106549142	0.00195503421309851	0.00391006842619712	0.0078201368523943	0.0156402737047884	0.0312805474095769

order	π_{26}	π_{27}	π_{28}	π_{29}	π_{210}	π_{211}
1:	0.0156288156288156	0.0312576312576313	0.0625152625152625	0.125030525030525	0.25006105006105	0.5001221001221
2:	0.5	0.5	0.5	0.5	0.5	0
3:	0.666666666666667	0.666666666666667	0.666666666666667	0.458333333333333	1.11022302462516e-16	5.55111512312578e-17
4:	8.88178419700125e-16	1.55431223447522e-15	2.38697950294409e-15	6.88338275267597e-15	1.37667655053519e-14	2.75335310107039e-14
5:	0.125244618395279	0.250489236790566	0.500978473581126	2.95319324550292e-14	5.90638649100583e-14	6.66133814775094e-14
6:	0.250980392156859	0.501960784313712	2.05391259555654e-15	4.05231403988182e-15	8.10462807976364e-15	1.29896093881143e-14
7:	0.503937007873943	4.6074255521944e-15	9.32587340685131e-15	3.56381590904675e-14	4.45199432874688e-14	6.19504447740837e-14
8:	1.0547118733939e-15	2.1094237467878e-15	4.21884749357559e-15	8.43769498715119e-15	1.68753899743024e-14	2.8199664825479e-14
9:	2.99760216648792e-15	6.10622663543836e-15	1.22124532708767e-14	2.38142838782096e-14	4.86277684785819e-14	8.78186412478499e-14
10:	1.0547118733939e-15	2.05391259555654e-15	4.10782519111308e-15	7.99360577730113e-15	1.60982338570648e-14	3.07531777821168e-14
11:	1.77635683940025e-15	3.5527136788005e-15	7.105427357601e-15	1.41553435639707e-14	3.04201108747293e-14	4.9515946898282e-14
12:	0.0312652662432824	0.0625305324865645	0.125061064973128	0.250122129946257	0.500244259892521	2.06501482580279e-14
13:	4.27435864480685e-15	8.54871728961371e-15	1.73194791841524e-14	3.46389583683049e-14	6.90558721316847e-14	1.19015908239817e-13
14:	0.062561094819154	0.125122189638308	0.250244379276615	0.500488758553246	3.66373598126302e-14	5.21804821573824e-14

Table 38: The facets $\pi t \geq 1$ augmented for $m = 13$

order	$\pi_{2}0$	$\pi_{2}1$	$\pi_{2}2$	$\pi_{2}3$	$\pi_{2}4$	$\pi_{2}5$	$\pi_{2}6$
1:	0.000122085215480405	0.000244170430960811	0.000488340861921621	0.000976681723843243	0.00195336344768649	0.00390672689537297	0.00781345379074594
2:	0.5	0.5	0.5	0.5	0.5	0.5	0.5
3:	0.875	0.875	0.833333333333333	0.75	0.666666666666667	0.666666666666667	0.666666666666667
4:	0.999999999998871	0	0	5.55111512312578e-17	3.27515792264421e-15	- 1.33226762955019e-15	- 7.7715611723761e-16
5:	0.000977517106549142	0.00195503421309834	0.00391006842619657	0.00782013685238436	0.0156402737047715	0.0312805474095695	0.0625610948191453
6:	0.0019569471624265	0.00391389432485317	0.00782778864970635	0.015655577299413	0.0313111545988249	0.0626223091976497	0.125244618395303
7:	0.00392156862744958	0.00784313725489927	0.0156862745097985	0.0313725490196129	0.0627450980391959	0.125490196078423	0.250980392156775
8:	0.00787401574803126	0.0157480314960624	0.0314960629921249	0.06299212598425	0.1259842519685	0.251968503936999	0.503937007873997
9:	0.0158730158730152	0.0317460317460304	0.063492063492061	0.12698412698412	0.253968253968249	0.507936507936487	4.44089209850063e-16
10:	0.0322580645161218	0.0645161290322436	0.129032258064487	0.258064516128971	0.51612903225795	9.43689570931383e-16	1.88737914186277e-15
11:	0.0666666666666611	0.133333333333322	0.266666666666644	0.533333333333293	2.22044604925031e-16	2.77555756156289e-16	7.7715611723761e-16
12:	0.142857142857138	0.285714285714275	0.57142857142855	5.55111512312578e-17	5.55111512312578e-17	2.22044604925031e-16	3.88578058618805e-16
13:	0.00024420024420041	0.000488400488400875	0.000976800976800973	0.00195360195360184	0.00390720390720395	0.00781440781440756	0.0156288156288152
14:	0.333333333333078	0.666666666666156	6.10622663543836e-16	5.55111512312578e-16	1.55431223447522e-15	3.05311331771918e-15	6.0507154842071e-15
15:	0.000488519785051211	0.000977039570102367	0.00195407914020657	0.00390815828041008	0.00781631656081977	0.0156326331216398	0.0312652662432797

order	$\pi_{2}7$	$\pi_{2}8$	$\pi_{2}9$	$\pi_{2}10$	$\pi_{2}11$	$\pi_{2}12$
1:	0.0156269075814919	0.0312538151629838	0.0625076303259675	0.125015260651935	0.25003052130387	0.50006104260774
2:	0.5	0.5	0.5	0.5	0.5	0
3:	0.666666666666667	0.666666666666667	0.666666666666667	0.458333333333333	1.11022302462516e-16	5.55111512312578e-17
4:	- 6.66133814775094e-16	1.23789867245705e-14	- 1.29896093881143e-14	3.5527136788005e-15	4.44089209850063e-15	8.43769498715119e-15
5:	0.125122189638274	0.250244379276569	0.500488758553163	4.2854608750531e-14	8.57092175010621e-14	9.41469124882133e-14
6:	0.2504892367906	0.500978473581205	1.4432899320127e-15	2.88657986402541e-15	5.77315972805081e-15	7.88258347483861e-15
7:	0.501960784313587	1.14352971536391e-14	2.28705943072782e-14	4.57411886145564e-14	9.14823772291129e-14	1.738609256563e-13
8:	5.55111512312578e-16	1.16573417585641e-15	2.38697950294409e-15	1.60982338570648e-15	8.99280649946377e-15	1.62092561595273e-14
9:	8.88178419700125e-16	1.72084568816899e-15	3.44169137633799e-15	7.71605002114484e-15	1.37667655053519e-14	1.29896093881143e-14
10:	3.71924713249427e-15	7.43849426498855e-15	1.49324996812084e-14	2.03170813506404e-14	5.99520433297585e-14	1.05138120432002e-13
11:	1.49880108324396e-15	3.10862446895044e-15	6.27276008913213e-15	1.06581410364015e-14	2.4757973449141e-14	3.50830475781549e-14
12:	7.7715611723761e-16	1.55431223447522e-15	3.10862446895044e-15	6.21724893790088e-15	1.26565424807268e-14	1.22124532708767e-14
13:	0.0312576312576304	0.062515262515261	0.125030525030522	0.250061050061044	0.500122100122088	2.35367281220533e-14
14:	1.25455201782643e-14	2.50910403565285e-14	4.96269692007445e-14	1.15019105351166e-13	1.98063787593128e-13	3.69704267200177e-13
15:	0.0625305324865594	0.125061064973119	0.250122129946238	0.500244259892468	6.12843109593086e-14	3.86357612569554e-14

CHAPTER VI

DUAL OF SHOOTING

6.1 Complementary relaxation of shooting

Assuming $|M| \ll n$, the objective v of the linear programming problem for shooting in v is sparse with almost all components $v_i = 0$ for $i \notin M$. We want to take advantage of 0-valued components to reduce the size of the problem. Let's attempt to find a clue in the complementary relaxation of shooting.

Consider the triple system of the subadditive inequalities suggested by Ellis Johnson for the proof of Theorem 18 in Gomory [12]:

$$\begin{aligned}
\Pi(C_n, b) \quad = \quad & \{\pi \in \mathbb{R}_+^{C_n - \{0\}} : \\
& \pi_i + \pi_j = \pi_b = 1 \text{ if } i + j \equiv b \mod n, \\
& \pi_i + \pi_j + \pi_k \geq \pi_b = 1 \text{ if } i + j + k \equiv b \mod n\},
\end{aligned}$$

where none of i, j, k is b. The shooting linear programming problem can be written as

$$\begin{aligned}
\min \quad & v\pi \\
st \quad & \pi \geq 0, \\
& \pi_b \geq 1 \\
& \pi_i + \pi_j \geq 1 \text{ if } i + j \equiv b \mod n, \\
& \pi_i + \pi_j \leq 1 \text{ if } i + j \equiv b \mod n, \\
& \pi_i + \pi_j + \pi_k \geq 1 \text{ if } i + j + k \equiv b \mod n. \qquad (77)
\end{aligned}$$

Its dual is

$$\begin{array}{ll} \max & \mathbf{1}.(\alpha, \beta^+, -\beta^-, \gamma) \\ st & f = \alpha \mathbf{e}_b + \beta^+ T^2 - \beta^- T^2 + \gamma T^3 \le v \\ & (\alpha, \beta^+, \beta^-, \gamma) \ge 0, \end{array} \tag{78}$$

where the rows of T^2 and T^3 are the indicating vectors of solutions with cardinality lengths 2 and 3, respectively. Some rows t of T^2 and T^3 may contain a nonzero component $t_h > 0$ at a half h with $2h \equiv b \mod n$. That is, we consider all possible complementary pairs and triples including the halves h with $2h \equiv b \mod n$. Since the nonnegativity $t \ge 0$ is redundant in the master cyclic group polyhedron $P(C_n, b)$, there exists an optimal solution $(\tilde{\alpha}, \tilde{\beta}^+, \tilde{\beta}^-, \tilde{\gamma})$ such that the equality $f = v$ in (78); i.e.,

$$\begin{aligned} f &= \tilde{\alpha}\mathbf{e}_b \\ &\quad + \sum_{i+j \equiv b \mod n} \Big(\tilde{\beta}^+(i+j \equiv b \mod n) - \tilde{\beta}^-(i+j \equiv b \mod n) \Big).(\mathbf{e}_i + \mathbf{e}_j) \\ &\quad + \sum_{i+j+k \equiv b \mod n} \tilde{\gamma}(i+j+k \equiv b \mod n).(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k) \\ &= v \ge 0. \end{aligned}$$

Let's define the *complementary relaxation* of the shooting linear programming problem to be

$$\begin{array}{ll} \min & v\pi \\ st & \pi \ge 0 \\ & \pi_b \ge 1 \\ & \pi_i + \pi_j \ge 1 \text{ if } i+j \equiv b \mod n, \\ & \pi_i + \pi_j + \pi_k \ge 1 \text{ if } i+j+k \equiv b \mod n. \end{array}$$

The dual of the relaxation is written as

$$\begin{aligned} \max \quad & (\alpha, \beta^+, \gamma).\mathbf{1} \\ st \quad & f = \alpha \mathbf{e}_b + \beta^+ T^2 + \gamma T^3 \leq v \\ & (\alpha, \beta^+, \gamma) \geq 0. \end{aligned}$$

We think of the nonnegative dual variables $(\alpha, \beta^+, \gamma)$ as the flows going through the paths from 0 to b in the circulant digraph $Cay(C_n, b)$ corresponding to the solutions in $\mathbf{e}_b, T^2, T^3$, respectively. At an optimal solution $(\tilde{\alpha}, \tilde{\beta}^+, \tilde{\gamma})$, the flow $\tilde{\alpha}$ over the arc $(0, b)$ is trivially equal to the capacity v_b. For $v_i = 0$ with $i \not\equiv b \mod n$, all 0 are the nonnegative dual variables corresponding to the pairs and the triples summing up to b containing i. That is,

$$\begin{aligned} \tilde{\beta}^+(i + j \equiv b \mod n) = 0 \qquad & \text{if any of } v_i, v_j \text{ is } 0 \\ \tilde{\gamma}(i + j + k \equiv b \mod n) = 0 \qquad & \text{if any of } v_i, v_j, v_k \text{ is } 0. \end{aligned}$$

Therefore, the dual problem (78) of shooting contains only the constraints with strictly positive right-hand side $v_i > 0$ and only the variables, $\beta^+(i + j \equiv b \mod n)$ with both $v_i, v_j > 0$, $\gamma(i+j+k \equiv b \mod n)$ with all $v_i, v_j, v_k > 0$. The dual problem of complementary relaxation has only polynomially many variables and constraints.

Let's go back to the shooting linear programming problem (77) keeping in mind the polynomial size of dual problem of complementary relaxation. The dual problem of shooting relaxes the nonnegativity of β^+ in (78); i.e., the dual problem of the shooting linear programming problem is

$$\begin{aligned} \max \quad & (\alpha, \beta, \gamma).\mathbf{1} \\ st \quad & \alpha \mathbf{e}_b + \beta T^2 + \gamma T^3 \leq v \\ & (\alpha, \gamma) \geq 0. \end{aligned} \tag{79}$$

Consider the sum of all components in (79). If we increase a variable $\beta(i + \bar{i} \equiv b \mod n)$ by 1, then the objective value will increase by 1 and the left-hand side of the sum of the components in (79) will increase by 2. That is, a β-variable contributes to the objective value's increase by 1/2 per increase of the left-hand side of the sum of the components of (79). A γ-variable does by 1/3. Therefore, the maximizing problem (79) tends to increase β-variables first. That is, negative β-variables tend to get some penalty and to be held up.

6.2 Network flow on covering space

The directed graph B_m of only one node and m loops is called the *bouquet of m circles* (or *m-bouquet*). Consider the circulant digraph $Cay(C_n, M)$ with $|M| = m$, and assign each group element in M to a loop of B_m. We say the group element j on a loop to be the *voltage* of the loop. The *covering projection* ϱ of $Cay(C_n, M)$ onto B_m is the onto function which maps each arc of $Cay(C_n, M)$ labeled by j to the loop of B_m with voltage j for all j in M. Note that the same number of arcs (n arcs) in $Cay(C_n, M)$ are mapped to a loop of B_m. At each node of $Cay(C_n, M)$, an arc labeled by j is going out and is coming in, as an arc (a loop) with voltage j is going out and coming in at the node of B_m. In topological sense, the covering space $Cay(C_n, M)$ is locally isomorphic everywhere to the image B_m of the covering projection. See Figure 17.

Reversely, $Cay(C_n, M)$ can be derived from B_m by the voltages. In general, the *graph $(\bar{V}, \bar{E})$ derived by the voltages j in a group G on the arcs (u, v) of a base graph* (V, E) is defined by $\bar{V} = V \times G$ and the arcs $((u, i), (v, i + j))$ for all $i \in G$ and all the voltages j. The derived graphs are completely equivalent to the regular covering graphs of their natural covering projections onto the base graphs. For covering space in topology, refer to Kwak and Shim [24], and Gross and Tucker [14]. For now, we only need to see the graph derived from B_m with the set M of voltages in C_n is exactly the circulant digraph $Cay(C_n, M)$.

A *flow* F is a real valued function on the paths from 0 to b in $Cay(C_n)$. As usual,

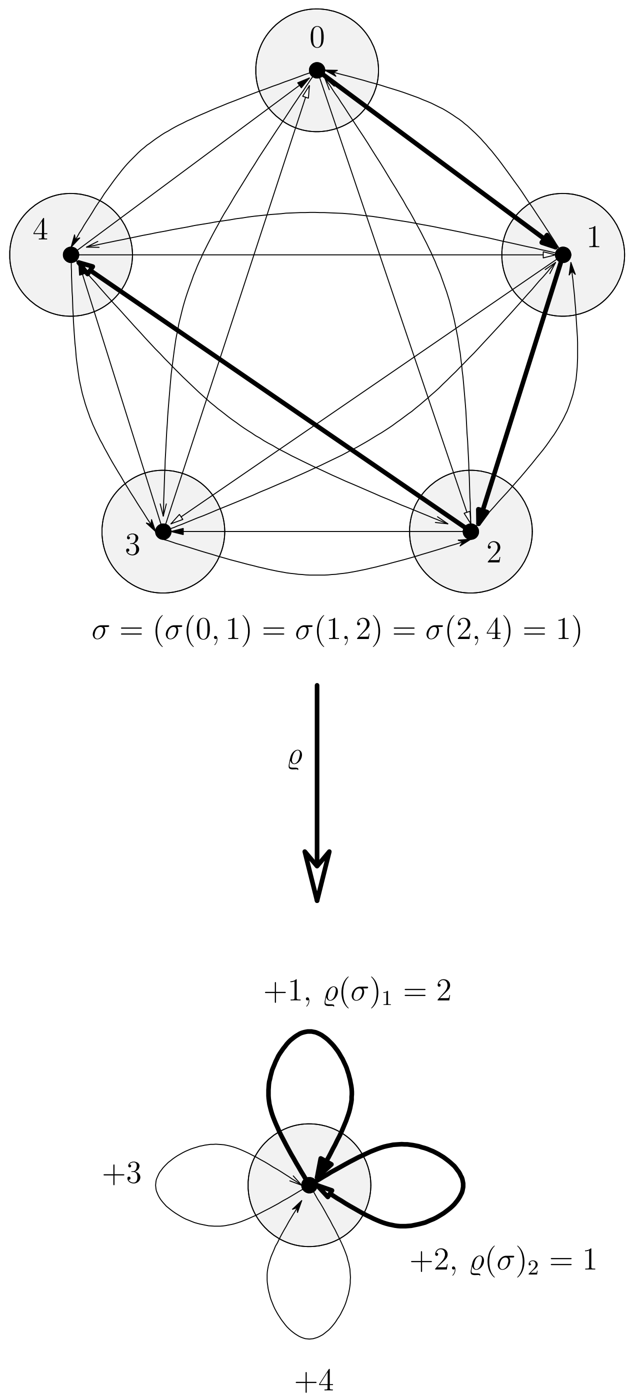

Figure 17: Covering projection $\varrho : Cay(C_5) \to B_4$.

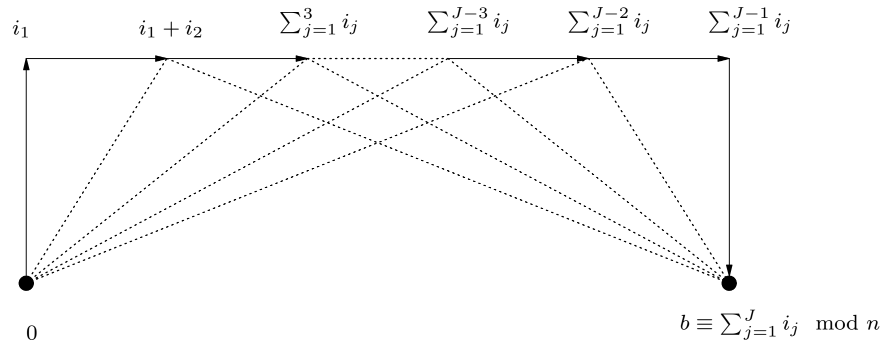

Figure 18: Decomposition of a path with cardinality length ≥ 4.

the *flow* F_p *at path* p is nonnegative for all the paths p. The *flow* σ^F in arc form with respect to a flow F is the real valued function whose value on each arc $(i, i+j)$ is the sum of flows at the paths containing the arc $(i, i+j)$; i.e.

$$\sigma^F(i, i+j) = \sum_{p \ni (i,i+j)} F_p.$$

If a function σ on the arcs is σ^F for some flow F, it will be called a flow in arc form.

What is the projected image $\varrho(\sigma)$ of a flow σ in arc form? The *base-flow* $\varrho(\sigma)_j$ *on the loop with voltage* j of a flow σ in arc form is defined to be the sum of flows over the arcs labeled by j; i.e.,

$$\varrho(\sigma)_j = \sum_{i \in C_n} \sigma(i, i+j).$$

It is given by projecting down and accumulating the flows over the arcs labeled by j in $Cay(C_n)$ onto the loop of B_m with voltage j. The base-flow has the capacity v; i.e.,

$$\varrho(\sigma)_j \leq v_j \text{ for all } j \in M.$$

For example, see Figure 17 again. The flow $\sigma = (\sigma(0,1) = \sigma(1,2) = \sigma(2,4) = 1)$ in arc form is projected onto the base-flow $\varrho(\sigma) = (\varrho(\sigma)_1 = 2, \varrho(\sigma)_2 = 1)$.

Let $\mathbf{e}_p$ or $\mathbf{e}(p)$ denote the flow with all 0 components but 1 at the path p. The flow F_p at a path p of cardinality length ≥ 4 with the ordered list of nodes $p = (0, i_1, i_1 + i_2, ..., \sum_{j=1}^{J} i_j \equiv b \mod n)$, can be decomposed into the flows at paths of cardinality lengths 2 and 3 alternatively,

$$
\begin{aligned}
\sigma^{F_p.\mathbf{e}_p} &= F_p.\sigma^{\mathbf{e}_p} = \\
&+F_p.\sigma^{\mathbf{e}(0,i_1,i_1+i_2,b)} \\
&-F_p.\sigma^{\mathbf{e}(0,i_1+i_2,b)} \\
&+F_p.\sigma^{\mathbf{e}(0,i_1+i_2,i_1+i_2+i_3,b)} \\
&\vdots \\
&+F_p.\sigma^{\mathbf{e}\left(0,\sum_{j=1}^{J-3} i_j,\sum_{j=1}^{J-2},b\right)} \\
&-F_p.\sigma^{\mathbf{e}\left(0,\sum_{j=1}^{J-2} i_j,b\right)} \\
&+F_p.\sigma^{\mathbf{e}\left(0,\sum_{j=1}^{J-2} i_j,\sum_{j=1}^{J-1} i_j,b\right)}.
\end{aligned}
$$

See Figure 18. The corresponding base-flow is

$$
\begin{aligned}
\varrho(\sigma^{F_p.\mathbf{e}_p}) &= F_p.\varrho(\sigma^{\mathbf{e}_p}) \\
&= F_p.\varrho\left(\sigma^{\mathbf{e}(0,i_1,i_1+i_2,b)}\right) \\
&-F_p.\varrho\left(\sigma^{\mathbf{e}(0,i_1+i_2,b)}\right) \\
&+F_p.\varrho\left(\sigma^{\mathbf{e}(0,i_1+i_2,i_1+i_2+i_3,b)}\right) \\
&\vdots \\
&+F_p.\varrho\left(\sigma^{\mathbf{e}\left(0,\sum_{j=1}^{J-3} i_j,\sum_{j=1}^{J-2},b\right)}\right) \\
&-F_p.\varrho\left(\sigma^{\mathbf{e}\left(0,\sum_{j=1}^{J-2} i_j,b\right)}\right) \\
&+F_p.\varrho\left(\sigma^{\mathbf{e}\left(0,\sum_{j=1}^{J-2} i_j,\sum_{j=1}^{J-1} i_j,b\right)}\right)
\end{aligned}
$$

That is,

$$\begin{aligned}
\varrho(\sigma^{F_p.\mathbf{e}_p}) &= F_p.\varrho(\sigma^{\mathbf{e}_p}) \\
&= F_p.T^3(i_1, i_2, b-(i_1+i_2)) \\
&-F_p.T^2(i_1+i_2, b-(i_1+i_2)) \\
&+F_p.T^3(i_1+i_2, i_3, b-(i_1+i_2+i_3)) \\
&\vdots \\
&+F_p.T^3\left(\sum_{j=1}^{J-3} i_j, i_{J-2}, b-\sum_{j=1}^{J-2} i_j\right) \\
&-F_p.T^2\left(\sum_{j=1}^{J-2} i_j, b-\sum_{j=1}^{J-2} i_j\right) \\
&+F_p.T^3\left(\sum_{j=1}^{J-2} i_j, i_{J-1}, b-\sum_{j=1}^{J-1} i_j \equiv i_J \mod n\right),
\end{aligned}$$

where $T^2(i, b-i) = \mathbf{e}_i + \mathbf{e}_{b-i}$ and $T^3(i,j,k) = \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k$ are respectively the rows of T^2 and T^3 in (78) corresponding to the complementary pairs and the triples $i+j+k \equiv b \mod n$. The nonnegative value F_p contributes to the dual variables γ corresponding to the triples for $1 < k < J$,

$$\left(\sum_{j=1}^{k-1} i_j\right) + i_k + \left(b - \sum_{j=1}^{k} i_j\right) \equiv b \mod n.$$

The nonpositive value $-F_p$ is added to the dual variables β corresponding to the complementarities for $1 < k < J-1$,

$$\left(\sum_{j=1}^{k} i_j\right) + \left(b - \sum_{j=1}^{k} i_j\right) \equiv b \mod n.$$

Therefore, we can transform a flow F to the dual variables $(\alpha, \beta = \beta^+ - \beta^-, \gamma)$ in the dual of shooting linear programming problem (78).

Reversely, we can compose the optimal solution $(\alpha, \beta = \beta^+ - \beta^-, \gamma)$ in the dual of shooting linear programming problem (78) to a flow F exhausting back flows β^-.

Assume that $\beta(i+\bar{i} \equiv b \mod n) < 0$. Since all $v_i, v_{\bar{i}}$ are nonnegative, there should be positive γ-variables,

$$\gamma(i+j+(b-i-j) \equiv b \mod n) \text{ and } \gamma(\bar{i}+k+(b-\bar{i}-k) \equiv b \mod n),$$

where the two γ-variables are assumed to correspond to a pair of different triples. If there are no such pair of different triples, then split the positive γ-variable into half and half, and substitute them for the two γ-variables above. They are composed into a path $\bar{i}+k+(b-\bar{i}-k) \equiv b \mod n$ of cardinality length 4:

$$\begin{aligned}
& \beta(i,\bar{i}).(\mathbf{e}_i+\mathbf{e}_{\bar{i}}) \\
& +\gamma(i,j,b-i-j).(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_{b-i-j}) \\
& +\gamma(\bar{i},k,b-\bar{i}-k).(\mathbf{e}_{\bar{i}}+\mathbf{e}_k+\mathbf{e}_{b-\bar{i}-k}) \\
& -\gamma(j,b-i-j,k,b-\bar{i}-k).(\mathbf{e}_j+\mathbf{e}_{b-i-j}+\mathbf{e}_k+\mathbf{e}_{b-\bar{i}-k}) \\
& +\gamma(j,b-i-j,k,b-\bar{i}-k).(\mathbf{e}_j+\mathbf{e}_{b-i-j}+\mathbf{e}_k+\mathbf{e}_{b-\bar{i}-k}) \\
= & \left\{\beta(i,\bar{i})+\gamma(j,b-i-j,k,b-\bar{i}-k)\right\}.(\mathbf{e}_i+\mathbf{e}_{\bar{i}}) && (80) \\
+ & \left\{\gamma(i,j,b-i-j)-\gamma(j,b-i-j,k,b-\bar{i}-k)\right\}.(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_{b-i-j}) && (81) \\
+ & \left\{\gamma(\bar{i},k,b-\bar{i}-k)-\gamma(j,b-i-j,k,b-\bar{i}-k)\right\}.(\mathbf{e}_{\bar{i}}+\mathbf{e}_k+\mathbf{e}_{b-\bar{i}-k}) && (82) \\
+ & \gamma(j,b-i-j,k,b-\bar{i}-k).(\mathbf{e}_j+\mathbf{e}_{b-i-j}+\mathbf{e}_k+\mathbf{e}_{b-\bar{i}-k}). && (83)
\end{aligned}$$

The scalars of (80), (81), (82), (83) are new dual variables $\beta(i,\bar{i})$, $\gamma(i,j,b-i-j)$, $\gamma(\bar{i},k,b-\bar{i}-k)$, $\gamma(j,b-i-j,k,b-\bar{i}-k)$ of the shooting LP

$$\begin{aligned}
\min \quad & v\pi \\
st \quad & \pi_b = 1 \\
& \pi_i+\pi_j = 1 \text{ for } i+j \equiv b \mod n \\
& \pi_i+\pi_j+\pi_k \equiv b \mod n,
\end{aligned}$$

augmented by the redundant inequality

$$\pi_j+\pi_{b-i-j}+\pi_k+\pi_{b-\bar{i}-k} \geq 1.$$

We increase the new dual variable $\gamma(j, b-i-j, k, b-\bar{i}-k)$ until one of (81) and (82) gets 0. The nonnegativity of v enables us to repeat the process until all dual variables are nonnegative. Note that the process does not change the objective value that is the sum of all dual variables.

Therefore, the dual of shooting linear programming problem is equivalent to the max flow problem subject to base-flow with capacity v,

$$\begin{array}{rl} \max & \sum_p F_p \\ st & \varrho(\sigma^F) \leq v \\ & F \text{ is a flow on } Cay(C_n) \text{ with source 0 and sink } b. \end{array}$$

APPENDIX A

C++ CODE FOR WONG-COPPERSMITH

```
//Shooting with Primal-Dual

//Wenwei Cao and Sangho Shim

//Last update: 10/1/2009

#include <ilcplex/ilocplex.h>

#include <vector>

#include <list>

#include <math.h>

#include <time.h>


ILOSTLBEGIN


int exp2Func(int exp);


const int PARAM = 14;// m

const int ORDER = exp2Func(PARAM); //n

const int RHS = ORDER - 1; //b

const double EPSILON = 1.0e-9;


class Triple

{

public:

Triple(int ii,int jj, int kk)
```

```
{
i = ii;
j = jj;
k = kk;
}
int i,j,k;//set public for convenience
protected:
private:
};


class ObjCoeffTuple
{
public:
ObjCoeffTuple(int ind, IloNum coe)
{
index = ind;
coeff = coe;
}
int index;//set public for convenience
IloNum coeff;
protected:
private:
};


int exp2Func(int exp)
{
int n = 1;
```

```
for (int i=0;i<exp;i++)
{
n = n * 2;
}
return n;
}

static void
shooting (IloModel model, IloNumVarArray var, IloRangeArray con, int mm,
  vector<ObjCoeffTuple>& objCoeff);

int sgn(int i, int b)
{
if (i >= (b+1)/2 )
{
return 1;
}
else
{
return -1;
}
}

int parityFunc(int i, int b)
{
if (i >= (b+1)/2 )
{
```

```
return b-i;
}
else
{
return i;
}
}

bool isSolRight(IloNumArray& val)
{
int i;
for (i = 0; i < val.getSize(); i++)
{
if (fabs(val[i] - floor(val[i]+0.5)) > EPSILON )
{
return false;
}
}
int sumOfVal = 0;
for (i = 0; i < val.getSize(); i++)
{
sumOfVal += (exp2Func(i) * int(floor(val[i] + 0.5)));
}
return (sumOfVal % ORDER == RHS);
}

IloNum lhsVal(IloNumArray& solVals, int i, int j, int k) /*size of solVals
```

```
 must be (RHS-1) / 2, back to original*/
{
return ( sgn(i, RHS) * solVals[parityFunc(i, RHS) - 1]
+ sgn(j, RHS) * solVals[parityFunc(j, RHS) - 1]
+ sgn(k, RHS) * solVals[parityFunc(k, RHS) - 1] );
}

void shootingPrimalDual(IloEnv env, IloNumArray& tVals,
IloNumArray& shootingSolVals, int iterOfShooting)
{
IloNum theta = IloInfinity;
char strBuf[40];

//initialize shootingSolVals (phi)
for (int i=0; i<shootingSolVals.getSize() - 1; i++)
{
shootingSolVals[i] = (i + 1) / ORDER - 0.5;
}
shootingSolVals[shootingSolVals.getSize() - 1] = (RHS - 1) / 2 / ORDER
- 0.5;

IloNumArray phiBar(env, shootingSolVals.getSize());

for (int iterNum = 0; ;iterNum++ )
{
//rebuild model for efficient cleaning
IloEnv subEnv;
```

```
IloModel subPrimalDual (subEnv);
IloObjective subObj = IloMaximize(subEnv);
IloNumVarArray subVar(subEnv);//rho
IloRangeArray subConRho(subEnv);/*constraints associated with rho,
second kind of constraints*/

//add rho's
for (int i = 1; i <= (RHS-1) / 2; i++)
{
subVar.add(IloNumVar(subEnv, -IloInfinity, IloInfinity));
sprintf(strBuf, "rho_%d", i);
subVar[i-1].setName(strBuf);
subPrimalDual.add(subVar[i-1]);
}

//set obj coeff for rho's (w) and subConRho
int conRhoIndex = 0;
for (int i=0; i < tVals.getSize() - 1; i++)
{
subObj.setLinearCoef(subVar[exp2Func(i) - 1], -tVals[i]);
if (tVals[i] > 0) //w_exp(i) < 0
{
subConRho.add(IloRange(subEnv, -IloInfinity, 1));
subConRho[conRhoIndex].setLinearCoef(subVar[exp2Func(i) - 1],
-1);

conRhoIndex++;
```

```
}
}

subObj.setLinearCoef(subVar[RHS - exp2Func(PARAM - 1) - 1],
tVals[PARAM - 1]);

if (tVals[PARAM - 1] > 0) //w_exp(i) > 0
{
subConRho.add(IloRange(subEnv, -IloInfinity, 1));
subConRho[conRhoIndex].setLinearCoef(subVar[RHS -
exp2Func(PARAM - 1) - 1], 1);

conRhoIndex++;
}

subPrimalDual.add(subObj);
subPrimalDual.add(subConRho);
IloCplex subSolver(subPrimalDual);

int sizeE = 0;
IloRangeArray subConEscratch(subEnv);
cout << "Enumerating triples to get E from scratch..." <<endl;

for (int i=1;i<RHS;i++)
{
for(int j=i;j<RHS;j++)
{
```

```
if ( ((i + j) % ORDER != 0) && ((i + j) %ORDER != RHS))
{
int k = (RHS - i - j + ORDER) % ORDER;
if (k >= j)
{
//use this triple to get E
if ( fabs(lhsVal(shootingSolVals, i, j, k)  - 0.5) < EPSILON)
{
//add ijk to E
subConEscratch.add(IloRange(subEnv, -IloInfinity, 0));
int coeff1 = 0;
int coeff2 = 0;
int coeff3 = 0;
int p_i = parityFunc(i,RHS);
int p_j = parityFunc(j,RHS);
int p_k = parityFunc(k,RHS);
int flag1 = p_i;
int flag2 = 0;
int flag3 = 0;
coeff1 = sgn(i,RHS);


//p_j
if (p_j == flag1)
{
coeff1 += sgn(j,RHS);
}
else
```

```
{
coeff2 = sgn(j,RHS);
flag2 = p_j;

}

//p_k
if (p_k == flag1)
{
coeff1 += sgn(k,RHS);
}
else
{
if (0 == flag2)
{
coeff2 += sgn(k,RHS);
flag2 = p_k;
}
else
{
if (p_k == flag2)
{
coeff2 += sgn(k,RHS);
}
else
{
coeff3 += sgn(k,RHS);
```

```
flag3 = p_k;
}


}
}
subConEscratch[sizeE].setLinearCoef(subVar[flag1-1],
coeff1);
if (flag2 > 0)
{
subConEscratch[sizeE].setLinearCoef(subVar[flag2-1],
coeff2);
if (flag3 > 0)
{
subConEscratch[sizeE].setLinearCoef(subVar[flag3-1],
coeff3);
}
}


sizeE++;/*because we added one constraint,
we need to update the last index*/
}
}
}
}
}


if (sizeE > 0) subPrimalDual.add(subConEscratch);//------------------
```

```
cout << endl << "Subproblem Iteration: " << iterNum + 1 <<
"  (Shooting Iteration: " << iterOfShooting +1 << ")" <<endl;
subSolver.solve();
// get sub solution (rho)
IloNumArray subSolVals(subEnv); //rho
subEnv.out() << "Sub Obj Value  = " << subSolver.getObjValue() << endl;
subSolver.getValues(subSolVals, subVar); //rho

if (0 == subSolver.getObjValue())//optimal
{
subEnv.end();
break; //all opt obj == 0, optimal
}

/*else, continue to calc theta and do NOT keep thetaList,
b/c we don't need it in ScratchE*/
theta = IloInfinity;
cout << "Enumerating triples to get theta..." <<endl;
for (int i=1;i<RHS;i++)
{
for(int j=i;j<RHS;j++)
{
if ( ((i + j) % ORDER != 0) && ((i + j) %ORDER != RHS))
{
int k = (RHS - i - j + ORDER) % ORDER;
if (k >= j)
```

```
{
//update theta
IloNum lhsRho = lhsVal(subSolVals, i, j, k);
if ( lhsRho > EPSILON)
{

IloNum thetaCurrent = ( 0.5 - lhsVal(shootingSolVals, i, j,
k)) / lhsRho;

if ( thetaCurrent < theta )
{
theta = thetaCurrent;
}
}
}
}
}
}
cout << "theta: " <<theta <<endl;

if (theta == IloInfinity) { cout << "ERROR in Primal Dual: Unbounded!"
<<endl; break; }

for (int i=0; i<shootingSolVals.getSize(); i++)
{
shootingSolVals[i] += theta * subSolVals[i];
}
```

```
subEnv.end();
}
}

int
main(int argc, char **argv)
{
//initialize
IloEnv env;
int i;
char strBuf[40];
clock_t start,finish;
double totaltime;

try {
start=clock();

cout << "Building Initial Relaxation of Cyclic Group Problem." <<endl;
IloModel masterProblem (env);
IloObjective   masterObjective = IloAdd(masterProblem,
IloMinimize(env));
IloNumVarArray masterVar(env);//t
IloRangeArray masterCon(env);

//Build Relaxation Model---------------------------
//feed initial constraints
```

```
masterCon.add(IloRange(env, 1, IloInfinity));

for (i = 0; i < PARAM; i++)
{
masterVar.add(IloNumVar(env, 0, IloInfinity));
sprintf(strBuf, "t_%d", i+1);
masterVar[i].setName(strBuf);
}

for (i = 0; i < PARAM; i++)
{
masterObjective.setLinearCoef(masterVar[i], 1);
}

for (i = 0; i < PARAM; i++)
{
masterCon[0].setLinearCoef(masterVar[i], double(exp2Func(i))/RHS);
}

masterProblem.add(masterCon);

IloCplex masterSolver(masterProblem);
masterSolver.exportModel("InitialRelaxation.lp");

//Build shooting problem model----------------------------------
int i;
IloNumArray shootingSolVals(env, (RHS-1) / 2); /* store shooting
```

```
   problem solutions*/
for (int iterNum = 0; ;iterNum++ )
{
//Solve current Relaxation
masterSolver.solve();
IloNumArray solVals(env);
env.out() << "Relaxation Obj Value  = " << masterSolver.getObjValue()
<< endl;
masterSolver.getValues(solVals, masterVar);
env.out() << "Relaxation Solution = " << solVals << endl;

//check if current solution is integral
if (isSolRight(solVals))
{
cout << "Total Number of Cuts Added: " << iterNum << endl;
break; //optimal, TERMINATE.
}

cout <<endl <<"----Shooting Iteration: " << iterNum + 1 << " ----" <<endl;

shootingPrimalDual(env, /*vecTriple,*/  solVals,  shootingSolVals,
iterNum);



cout <<"Adding a cut to Relaxation." <<endl;
IloRange   aCut = IloAdd(masterProblem, IloRange(env, 1,
IloInfinity));
```

```
for (i = 0; i < PARAM - 1; i++)
{
aCut.setLinearCoef(masterVar[i], shootingSolVals[exp2Func(i) - 1]
+ 0.5);
}
aCut.setLinearCoef(masterVar[PARAM - 1], - shootingSolVals[RHS -
exp2Func(PARAM - 1) - 1] + 0.5);
}


//get Final Relaxation solution
IloNumArray masterSolVals(env);


env.out() << "Final Relaxation Obj Value  = "
<< masterSolver.getObjValue() << endl;
masterSolver.getValues(masterSolVals, masterVar);
env.out() << "Final Relaxation Solution = " << masterSolVals << endl;
masterSolver.exportModel("FinalRelaxation.lp");
finish=clock();
totaltime=(double)(finish-start); //in millisecond
int minutes = int (totaltime / (60*CLOCKS_PER_SEC));
totaltime -= minutes * (60*CLOCKS_PER_SEC);
int seconds = int (totaltime /CLOCKS_PER_SEC);
cout <<"\nTotal Time: "<<minutes<<" min "<< seconds <<" sec."<<endl;//----
}
catch (IloException& ex) {
cerr << "Error: " << ex << endl;
}
```

```
catch (...) {
cerr << "Error" << endl;
}


env.end();


return 0;
}
```

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