JUDICIOUS PARTITIONS OF GRAPHS AND HYPERGRAPHS

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JUDICIOUS PARTITIONS OF GRAPHS AND HYPERGRAPHS

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To my parents and Gigi,

for their support and love

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SUMMARY

Classical partitioning problems, like the Max-Cut problem, ask for partitions that optimize one quantity, which are important to such fields as VLSI design, combinatorial optimization, and computer science. Judicious partitioning problems on graphs or hypergraphs ask for partitions that optimize several quantities simultaneously. In this dissertation, we work on judicious partitions of graphs and hypergraphs, and solve or asymptotically solve several open problems of Bollobás and Scott on judicious partitions, using the probabilistic method and extremal techniques.

We establish a conjecture of Bollobás and Scott in [12], by showing that: for any integer $k \geq 2$ and any hypergraph G with m_i edges of size i, $i = 1, 2$, there is a partition $V_1, \ldots, V_k$ of $V(G)$ such that for $i = 1, \ldots, k$, V_i contains at most $m_1/k + m_2/k^2 - o(m_2)$ edges. This is best possible since the expected bound in a random partition is $m_1/k + m_2/k^2$. We also prove that: for integer $k \geq 3$, any hypergraph with m_i edges of size i, $i = 1, 2$, has a partition $V_1, \ldots, V_k$ such that each V_i meets at least $m_1/k + m_2/(k-1) - o(m_2)$ edges. This result implies for large graphs the conjecture of Bollobás and Scott [9] that every graph with m edges admits a partition $V_1, \ldots, V_k$ such that each V_i meets at least $2m/(2k-1)$ edges. For $k = 2$, we prove that $V(G)$ admits a partition into two sets each meeting at least $m_1/2 + 3m_2/4 - o(m_2)$ edges, which solves a special case of a more general problem of Bollobás and Scott in [12].

Bollobás and Scott [12] asked for the smallest $f(k, m)$ such that for any integer $k \geq 2$ and any graph G with m edges, there is a partition $V(G) = \bigcup_{i=1}^{k} V_i$ such that for $1 \leq i \neq j \leq k$, $e(V_i \cup V_j) \leq f(k, m)$. They conjectured that $f(k, m) \leq \frac{12m}{(k+1)(k+2)} + O(n)$ for general graphs, and $f(k, m) \leq \frac{12m}{(k+1)(k+2)}$ for dense graphs. We obtain a general bound on $f(k, m)$, and prove conjecture for dense graphs and for $k = 3, 4, 5$ asymptotically.

We also work on a long standing conjecture of Bollobás and Thomason (see [7, 9, 11, 12]): for any integer $r \geq 3$, the vertex set of any r-uniform hypergraph with m edges admits a partition $V_1, ..., V_r$ such that for $i = 1, ..., r$, each V_i meets at least $\frac{r}{2r-1}m$ edges. We prove the bound $0.65m - o(m)$ for $r = 3$, which for large graph, is better than $0.6m$ suggested by this conjecture.

CHAPTER I

INTRODUCTION

We study judicious partitioning problems on graphs and hypergraphs. We solve or asymptotically solve several open problems of Bollobás and Scott on judicious partitions, using probabilistic method and extremal techniques. In this chapter we provide notation and terminology necessary for the subsequent chapters.

1.1 Notation

Let G be a graph or hypergraph, and let $S \subseteq V(G)$. We use $G[S]$ to denote the subgraph of G consisting of S and all edges of G with all incident vertices in S. Let A, B be subsets of $V(G)$ or subgraphs of G, we use (A, B) to denote the set of edges of G that have incident vertices in both A and B. For an edge (or hyperedge) e of G, we use $V(e)$ to denote the set of incident vertices of e. We write $e_G(S) := |\{e \in E(G) : V(e) \subseteq S\}|$, $e_G(S, T) := |\{e \in E(G) : V(e) \cap S \neq \emptyset \neq V(e) \cap T\}|$ for any $T \subseteq V(G)$, and $d_G(S) := |\{e \in E(G) : V(e) \cap S \neq \emptyset\}|$. When understood, the reference to G in the subscript may be dropped. Let $k \geq 2$ be an integer, a *k-partition* of $V(G)$ is a collection of subsets of $V(G)$, $V_1, V_2, ..., V_k$, such that $V_1 \cup V_2 \cup ... \cup V_k = V(G)$ and $V_i \cap V_j = \emptyset$ for any $1 \leq i < j \leq k$. We use $b(G)$ to denote the maximum number of edges in a bipartite subgraph of G.

We will also prove several results for weighted graphs. Let G be a graph and let $w : V(G) \cup E(G) \to \mathbf{R}^+$, where $\mathbf{R}^+$ represents the nonnegative reals. For $S \subseteq V(G)$ we write

$$w_G(S) = \sum_{u \in S} w(u) + \sum_{\{e \in E(G):\ V(e) \subseteq S\}} w(e)$$

and

$$\tau_G(S) = \sum_{u \in S} w(u) + \sum_{\{e \in E(G):\ V(e) \cap S \neq \emptyset\}} w(e).$$

If G is understood, we use $\tau(S), w(S)$ instead of $\tau_G(S), w_G(S)$, respectively. We point out that if H is an induced subgraph of G, then for any $S \subseteq V(H)$, we have $w_H(S) = w_G(S)$. Also, note that when $w(e) = 1$ for all $e \in E(G)$ and $w(v) = 0$ for all $v \in V(G)$, we have $w(S) = e(S)$ and $\tau(S) = d(S)$.

We will use the standard notation of probability theory. Given a sample space, let X be a random variable and A be an event. We use $\mathbb{P}(A)$ to denote the probability that A occurs, $\mathbb{E}(X)$ to denote the expectation of random variable X, and $\mathbb{E}(X|A)$ to denote the expectation of X conditional on A.

1.2 Background

Classical graph partitioning problems often ask for partitions of a graph that optimize a single quantity. For example, the well-known *Max-Cut Problem* asks for a partition V_1, V_2 of $V(G)$, where G is a weighted graph, that maximizes the total weight of edges with one end in each V_i. This problem is NP-hard, see [29]. It is shown [6] that it is also NP-hard to approximate the Max-Cut problem on cubic graphs beyond the ratio of 0.997. However, the Max-Cut problem is polynomial time solvable for planar graphs, see [25,36]. Goemans and Williamson [24] used semidefinite programming and hyperplane rounding to give a randomized algorithm with expected performance guarantee of 0.87856. Feige, Karpinski and Langberg [22] gave a similar randomized algorithm that improves this bound to 0.921 for subcubic graphs; a graph is called subcubic if it has maximum degree at most three.

The unweighted version of Max-cut problem is often called the *Maximum Bipartite Subgraph Problem*: Given a graph G, find a partition V_1, V_2 of $V(G)$ that maximizes $e(V_1, V_2)$, the number of edges with one end in each V_i. This is also NP-hard, see [21,23]. Moreover, Yannakakis [49] showed that the Maximum Bipartite Subgraph Problem is NP-hard even when restricted to triangle-free cubic graphs.

However, it is easy to prove that any graph with m edges has a partition V_1, V_2 with

$e(V_1, V_2) \geq m/2$: if one randomly picks a partition U_1, U_2, the probability of any edge belongs to (U_1, U_2) is exactly $1/2$, therefore $\mathbb{E}(e(U_1, U_2)) = m/2$ and the conclusion follows. Edwards [17, 18] improved the lower bound to $m/2 + \frac{1}{4}(\sqrt{2m + 1/4} - 1/2)$. This is best possible, as K_{2n+1} are extremal graphs. Alon [1] showed that for infinite many integers m, there exist graphs G_m such that $b(G_m) \geq m/2 + \frac{\sqrt{2m}}{4} + \Theta(m^{1/4})$, where $e(G_m) = m$, confirming a conjecture of Erdős in [20] that the gap between Edwards' bound and the truth can be arbitrary large. (Recall that $b(G)$ is the maximum number of edges in a bipartite subgraph of G.)

This lower bound may be improved by forbidding a fixed graph. For example, Erdős and Lovász (see [19]), Poljak and Tuza [37] and Shearer [43] made progress on improving the lower bound for triangle-free graphs. Alon [1] finally showed that $b(G) \geq m/2+\Theta(m^{4/5})$ for any triangle-free graph G with m edges, which is tight up to constant. For general H-free graphs, the Maximum Bipartite Subgraph Problem is studied in [4], i.e. H is an even cycle or a graph obtained by connecting a single vertex to all vertices of a fixed forest. But the main term of the best lower bound of $b(G)$ is still $m/2$, for H-free graph G with m edges, where H is triangle or one of the graphs studied in [4].

For some classes of graphs, the main term of the lower bound can exceed $|E(G)|/2$. Erdős [19] proved that if G is $2k$-colorable then $b(G) \geq \frac{k}{2k-1}|E(G)|$. As a consequence, if G is a graph with bounded maximum degree, then the lower bound can exceed $|E(G)|/2$. In particular, Erdős' result implies that $b(G) \geq \frac{2}{3}|E(G)|$ for cubic graph G. Locke [31] and Stanton [44] showed that $b(G) \geq \frac{7}{9}|E(G)|$ if G is cubic and G is not K_4. Hopkins and Stanton [28] showed that $b(G) \geq \frac{4}{5}|E(G)|$ if G is triangle-free cubic graph. More discussion on cubic (or subcubic) triangle-free graphs can be found in [12, 16, 35, 46, 48].

The Maximum Bipartite Subgraph Problem for integer weighted graphs also have been studied in [3] by N. Alon and E. Halperin. For other subsequent work of the Maximum Bipartition Subgraph Problem, we refer the reader to [30, 38, 45].

In practice one often needs to find a partition of a given graph or hypergraph to optimize several quantities simultaneously. Such problems are called *Judicious Partitioning Problems* by Bollobás and Scott [8]. One such example is the problem of finding a partition V_1, V_2 of the vertex set of a graph G that minimizes $\max\{e(V_1), e(V_2)\}$, or equivalently, maximizes $\min\{d(V_1), d(V_2)\}$ (since $d(V_i) = e(G) - e(V_{3-i})$ for $i = 1, 2$). This problem is also known as the *Bottleneck Bipartition Problem*, raised by Entringer (see, for example, [39, 40]). Shahrokhi and Székely [42] showed that this problem is also NP-hard. Porter [39] proved that any graph with m edges has a partition of its vertex set into V_1, V_2 with $e(V_i) \le m/4 + O(\sqrt{m})$ for $i = 1, 2$. Bollobás and Scott [10] improved this bound by proving

Theorem 1.2.1. *(Bollobás and Scott [10]) For any graph G with m edges, there exists a bipartition V_1, V_2 of $V(G)$ such that for $i = 1, 2$*

$$e(V_i) \le \frac{m}{4} + \frac{1}{8}(\sqrt{2m + 1/4} - 1/2).$$

They also showed that the complete graphs K_{2n+1} are the only extremal graphs (modulo isolated vertices).

Bollobás and Scott [10] further proved that for any integer $k \ge 1$ and any graph G with m edges, $V(G)$ has a k-partition $V_1, \dots, V_k$ such that

$$e(V_i) \le \frac{m}{k^2} + \frac{k-1}{2k^2}(\sqrt{2m + 1/4} - 1/2)$$

for $i \in \{1, 2, \dots, k\}$. The complete graphs of order $kn + 1$ are the only extremal graphs (modulo isolated vertices).

In fact, Bollobás and Scott [10] proved an even stronger result that any graph with m edges has a partition V_1, V_2 of its vertex set such that

$$e(V_1, V_2) \ge \frac{m}{2} + \frac{1}{4}(\sqrt{2m + 1/4} - 1/2)$$

and for $i = 1, 2$,

$$e(V_i) \le \frac{m}{4} + \frac{1}{8}(\sqrt{2m + 1/4} - 1/2).$$

Xu and Yu [47] recently generalized this result to k-partitions: any graph with m edges has a k-partition $V_1, ..., V_k$ of its vertex set such that the number of edges whose incident vertices are not in the same set

$$e(V_1, V_2, ..., V_k) \geq \frac{k-1}{k}m + \frac{1}{2k}(\sqrt{2m + 1/4} - 1/2)$$

and for $i \in \{1, 2, \ldots, k\}$,

$$e(V_i) \leq \frac{m}{k^2} + \frac{k-1}{2k^2}(\sqrt{2m + 1/4} - 1/2).$$

Alon *et al.* [2] showed that there is a connection between the Maximum Bipartite Subgraph Problem and the Bottleneck Bipartition Problem. More precisely, they proved the following: Let G be a graph with m edges and largest cut of size $m/2 + \delta$. If $\delta \leq m/30$ then $V(G)$ admits a partition V_1, V_2 such that for $i = 1, 2$,

$$e(V_i) \leq m/4 - \delta/2 + 10\delta^2/m + 3\sqrt{m};$$

and if $\delta \geq m/30$ then $V(G)$ admits a partition V_1, V_2 such that for $i = 1, 2$,

$$e(V_i) \leq m/4 - m/100.$$

Bollobás and Scott [15] recently extended this result to k-partitions: there is also a connection between the generalized "Maximum k-Partite Subgraph Problem" and the generalized "Bottleneck k-Partition Problem".

In their paper [7, 12, 13, 41], Bollobás and Scott studied k-partitions $V_1, ..., V_k$ in a graph or hypergraph that minimize $\max\{e(V_1), e(V_2), ..., e(V_k)\}$, or minimize $\max\{e(V_i \cup V_j) : 1 \leq i < j \leq k\}$, or maximize $\min\{d(V_1), d(V_2), ..., d(V_k)\}$. We have seen that when $k = 2$, minimizing $\max\{e(V_1), e(V_2)\}$ is equivalent to maximizing $\min\{d(V_1), d(V_2)\}$. However, when $k \geq 3$, minimizing $\max\{e(V_1), e(V_2), ..., e(V_k)\}$ is very different from maximizing $\min\{d(V_1), d(V_2), ..., d(V_k)\}$. These problems become more difficult if one imposes restrictions on the sizes of V_i, $1 \leq i \leq k$; for example, we have the *Balanced Bipartition Problem* when $k = 2$ and $||V_1| - |V_2|| \leq 1$. For more problems and references, we refer the reader to [12–14, 41].

1.3 Problems and results

We discuss several judicious partitioning problems which we are interested in and present our results to those problems in this section. In Section 1.3.1, we discuss several judicious partitioning problems about graphs with requirement on edges as well as on vertices. In Section 1.3.2, we consider judicious partitioning problems for bounding the size of all pairs in a k-partition of a graph. In Section 1.3.3, we focus a long standing conjecture of Bollobás and Thomason on 3-uniform hypergraphs. Our results on those problems can be found in [32–34].

1.3.1 Hypergraphs with edge size at most 2

We discuss several judicious partitioning problems about graphs with requirement on edges as well as on vertices, and such problems are called mixed partitioning problems. We follow Bollobás and Scott [12] to use the term hypergraphs with edge size at most 2.

Our first result is

Theorem 1.3.1. *If G is a hypergraph with m_i edges of size i, $i = 1, 2$, then $V(G)$ admits a partition V_1, V_2 such that for $i = 1, 2$*

$$d(V_i) \geq m_1/2 + 3m_2/4 + o(m_2).$$

Bollobás and Scott [12] suggested the lower bound $(m_1 - 1)/2 + 2m_2/3$ as a starting point for a more general problem, and Theorem 1.3.1 verifies this for large graphs. Note that if we take a partition V_1, V_2 randomly and uniformly, then $\mathbb{E}(d(V_i)) = m_1/2 + 3m_2/4$.

Next we attempt to generalize Theorem 1.3.1 to k-partitions. In particular, we prove

Theorem 1.3.2. *Let $k \geq 3$ be an integer and let G be a hypergraph with m_i edges of size i, $i = 1, 2$. Then there is a k-partition $V_1, \ldots, V_k$ of $V(G)$ such that for $i = 1, \ldots, k$,*

$$d(V_i) \geq \frac{m_1}{k} + \frac{m_2}{k-1} + o(m_2).$$

Note, if we take a k-partition $V_1, V_2, ..., V_k$ randomly and uniformly, then $\mathbb{E}(d(V_i)) = m_1/k + (2k-1)m_2/k^2$. Theorem 1.3.2 implies the following conjecture of Bollobás and Scott [11] for graphs with sufficiently many edges

Conjecture 1.3.3. *(Bollobás and Scott [11]) Every graph with m edges has a partition into k sets, each meeting at least* $2m/(2k-1)$ *edges.*

We also consider a generalization of the Bottleneck Bipartition Problem to hypergraphs. We have

Theorem 1.3.4. *Let G be a hypergraph with* m_i *edges of size i,* $i = 1, 2$. *Then for any integer* $k \geq 1$*, there is a k-partition* $V_1, \ldots, V_k$ *of* $V(G)$ *such that for* $i = 1, \ldots, k$,

$$e(V_i) \leq \frac{m_1}{k} + \frac{m_2}{k^2} + o(m_2).$$

Note that for a random k-partition $V_1, ..., V_k$ of $V(G)$, we have $\mathbb{E}(e(V_i)) = m_1/k + m_2/k^2$. In its special case, when $m_1 = o(m_2)$, Theorem 1.3.4 follows from Eq.2 in [12]. Theorem 1.3.4 establishes a conjecture of Bollobás and Scott [12] for large graphs that: any hypergraph with m_i edges of size i,$i = 1, 2$, admits a k-partition $V_1, ..., V_k$ such that for $i = 1, \ldots, k$,

$$e(V_i) \leq \frac{m_1}{k} + \frac{m_2}{\binom{k+1}{2}} + O(1).$$

In Chapter 2, we will prove weighted versions of Theorem 1.3.1,Theorem 1.3.2 and Theorem 1.3.4.

1.3.2 Bounds for pairs in partitions of graphs

The following judicious partitioning problem is proposed in [12]:

Problem 1.3.5. *(Bollobás and Scott [12]) What is the smallest* $f(k, m)$ *such that for any integer* $k \geq 2$*, every graph G with m edges has a k-partition* $V_1, ..., V_k$ *of* $V(G)$ *such that for* $1 \leq i < j \leq k$, $e(V_i \cup V_j) \leq f(k, m)$*?*

Note that the case $k = 2$ is trivial. For $k = 3$, we see that for each permutation ijk of $\{1, 2, 3\}$, $d(V_i) = m - e(V_j \cup V_k)$; so Problem 1.3.5 asks for a lower bound on $\min\{d(V_i) : i = 1, 2, 3\}$, and hence Theorem 1.3.2 provides an upper bound on $f(3, m)$. For $k \geq 4$, bounding $\max\{e(V_i \cup V_j) : 1 \leq i < j \leq k\}$ is much more difficult than bounding $\max\{e(V_i) : 1 \leq i \leq k\}$; in the former case one needs to bound $\binom{k}{2}$ quantities, while in the latter case one only needs to bound k quantities.

We prove the following general bound on $f(k, m)$:

Theorem 1.3.6. *For any integer $k \geq 3$, $f(k, m) < 1.6m/k + o(m)$, and $f(k, m) < 1.5m/k + o(m)$ for $k \geq 23$.*

We now show that $f(k, m) \geq m/(k - 1)$, which is close to $1.6m/k$ when k is small. For $k \geq 3$, take the graph $K_{1,n}$ with $n \geq k - 1$, and let x be the vertex of degree n. Let $V_1, ..., V_k$ be a k-partition of $V(G)$ with $x \in V_1$. Without loss of generality, we may assume that $|V_2| \geq (n + 1 - |V_1|)/(k - 1)$. Now $e(V_1 \cup V_2) \geq (n + 1 - |V_1|)/(k - 1) + (|V_1| - 1) = (n + (k - 2)(|V_1| - 1))/(k - 1) \geq n/(k - 1) = m/(k - 1)$, where $m = n$ is the number of edges in $K_{1,n}$.

The complete graph K_{k+2} has $m = \binom{k+2}{2}$ edges, and any k-partition $V_1, ..., V_k$ of $V(K_{k+2})$ has two sets, say V_1, V_2, such that $|V_1 \cup V_2| = 4$. So $e(V_1 \cup V_2) = 6 = \frac{12m}{(k+1)(k+2)}$. This shows that $f(k, m) \geq \frac{12m}{(k+1)(k+2)}$. For large n, a simple counting shows that for any k-partition $V_1, ..., V_k$ of $V(K_n)$, $k \geq 2$, there exist V_i, V_j such that $|V_i| + |V_j| \geq 2n/k$, and hence $e(V_i \cup V_j) \geq \binom{2n/k}{2}$. From this, we deduce that $f(k, m) \geq 4m/k^2 + O(n)$, and this bound is achieved by taking a balanced k-partition of $V(K_n)$ (i.e., any two partition sets differ in size by at most one).

The consideration of $K_{1,n}$ and K_{k+2} lead Bollobás and Scott [12] to the following conjecture. Note that $K_{1,n}$ is sparse, i.e. the number of edges is $O(n)$.

Conjecture 1.3.7. *(Bollobás and Scott [12]) For each integer $k \geq 2$, every graph G with m*

edges and n vertices has a k-partition $V_1, ..., V_k$ of $V(G)$ such that for $1 \le i < j \le k$,

$$e(V_i \cup V_j) \le \frac{12m}{(k+1)(k+2)} + O(n).$$

Conjecture 1.3.7 is trivial for $k = 2$, as the bound becomes $m + O(n)$. For $k = 3$, Conjecture 1.3.7 is equivalent to the following problem: Find a partition $V(G) = V_1 \cup V_2 \cup V_3$ so that $d(V_i) \ge 2m/5 + O(n)$. We point out that Theorem 1.3.2 implies $d(V_i) \ge m/2 + o(m)$; therefore Conjecture 1.3.7 holds for $k = 3$ and large m.

We show that Conjecture 1.3.7 holds for dense graphs as well:

Theorem 1.3.8. *Let $k \ge 2$ be an integer and let $\epsilon > 0$. If G is a graph with m edges and $\delta(G) \ge \epsilon n$, then there is a k-partition $V_1, \ldots, V_k$ of $V(G)$ such that for $1 \le i < j \le k$,*

$$e(V_i \cup V_j) \le \frac{4}{k^2}m + o_\epsilon(m).$$

Note that the main term $4m/k^2$ is tight because of the complete graphs K_n. Theorem 1.3.8 implies the following conjecture of Bollobás and Scott [12] for large graphs.

Conjecture 1.3.9. *(Bollobás and Scott [12]) For each $k \ge 2$ there is a constant $c_k > 0$ such that if G is a graph with m edges, n vertices, and minimum degree $\delta(G) \ge c_k n$, then there is a k-partition $V_1, ..., V_k$ of $V(G)$ such that for $1 \le i < j \le k$,*

$$e(V_i \cup V_j) \le \frac{12m}{(k+1)(k+2)}.$$

From Theorem 1.3.2, we have $f(3, m) \le m/2 + o(m)$, which is less than $\frac{12m}{(k+1)(k+2)} = \frac{3}{5}m$ for large m. We will show that $f(4, m) \le m/3 + o(m)$, which is less than $\frac{12m}{(k+1)(k+2)} = \frac{2}{5}m$ for large m. We will further show that $f(5, m) \le 4m/15 + o(m)$, which is less than $\frac{12m}{(k+1)(k+2)} = \frac{2}{7}m$ for large m. Therefore, Conjecture 1.3.7 holds for dense graph as well as for $k = 3, 4, 5$ and large m.

We also study the problem of finding a k-partitions $V_1, \ldots, V_k$ of $V(G)$ that satisfy bounds on both $\max\{e(V_i) : 1 \le i \le k\}$ and $\max\{e(V_i \cup V_j) : 1 \le i < j \le k\}$. It is proved in [10] that there exists a k-partition $V_1, ..., V_k$ of a graph with m edges such that

$e(V_i) \le \frac{m}{k^2} + \frac{k-1}{2k^2}(\sqrt{2m+1/4} - 1/2)$ for $1 \le i \le k$. Bollobás and Scott [12] asked whether it is possible to find a k-partition $V_1, \ldots, V_k$ such that $e(V_i) \le \frac{m}{k^2} + \frac{k-1}{2k^2}(\sqrt{2m+1/4} - 1/2)$ for $1 \le i \le k$, and $e(V_i \cup V_j) \le \frac{12m}{(k+1)(k+2)} + O(n)$ for $1 \le i < j \le k$. We will show that for $k = 3$ and $k = 4$ one can find a partition satisfying these bounds asymptotically.

1.3.3 3-Uniform hypergraphs

If V_1, V_2 is a bipartition of a graph G maximizing $e(V_1, V_2)$, then each $v \in V_i$ has at least as many neighbors in V_{3-i} as in V_i. Summing over all vertices in V_i, we get $e(V_1, V_2) \ge 2e(V_i)$ for $i = 1, 2$. Hence $e(V_i) \le m/3$, where m is the number of edges in G, so $d(V_i) \ge m - m/3 = 2m/3$ for $i = 1, 2$.

In an attempt to extend the above to hypergraphs, Bollobás and Thomason made the following conjecture (see [7,9,11,12]), one of the early problems about judicious partitions.

Conjecture 1.3.10. (*Bollobás and Thomason 1980s*) *For any integer $r \ge 3$, the vertex set of any r-uniform hypergraph with m edges admits a r-partition $V_1, \ldots, V_r$ such that for $i = 1, \ldots, r$,*

$$d(V_i) \ge \frac{r}{2r-1} m.$$

The conjectured bound is best possible; the complete r-uniform hypergraphs on $2r - 1$ vertices are such extremal hypergraphs. To see this, note that such a hypergraph has $m = \binom{2r-1}{r}$ edges, and any r-partition of such a hypergraph has a partition set with just one vertex, which meets $\binom{2r-2}{r-1}$ edges.

Bollobás, Reed and Thomason [7] proved that every 3-uniform hypergraph with m edges has a partition V_1, V_2, V_3 such that $d(V_i) \ge (1 - 1/e)m \approx 0.21m$ (here e is the base of the natural logarithm). In [11], this bound is improved to $(5/9)m$ by Bollobás and Scott using the following approach: find a reasonable partition, and remove vertices of one set and try to partition the remaining vertices into $r - 1$ parts in a better way. They [11] also proved a bound for general case: $d(V_i) \ge 0.27m$ for any integer $r \ge 3$ and $1 \le i \le r$. Note that the bound for $r = 3$ in Conjecture 1.3.10 is $0.6m$. Halesgrave [26] extended the idea of Bollobás and Scott in [11] and solved the case $r = 3$ completely. (Bollobás informed us that Halesgrave actually did it in 2006.) For large graphs, this bound may be improved. We prove the following result, which for large m gives an even better bound than what Conjecture 1.3.10 suggests for $r = 3$.

Theorem 1.3.11. *Every 3-uniform hypergraph with m edges has a 3-partition V_1, V_2, V_3 such that for $i = 1, 2, 3$,*

$$d(V_i) \geq 0.65m - o(m).$$

1.4 Azuma-Heoffding inequality

The approach we take is similar in spirit to that of Bollobás and Scott [9, 12]. First we partition a set of large degree vertices, then we establish a random process to partition the remaining vertices, and finally we apply a concentration inequality to bound the deviations. The key is to pick the probabilities appropriately so that the expectations of the process will be in a range that we want. This will be achieved by extremal techniques.

The concentration inequality we need is the Azuma-Heoffding inequality [5,27], which bounds deviations in a random process. We use the version given in [9].

Lemma 1.4.1. *(Azuma-Heoffding Inequality) Let $Z_1, \ldots, Z_n$ be independent random variables taking values in $\{1, \ldots, k\}$, let $Z := (Z_1, \ldots, Z_n)$, and let $f : \{1, \ldots, k\}^n \to \mathbf{N}$ such that $|f(Y) - f(Y')| \leq c_i$ for any $Y, Y' \in \{1, \ldots, k\}^n$ which differ only in the ith coordinate. Then for any $z > 0$,*

$$\mathbb{P}\left(f(Z) \geq \mathbb{E}(f(Z)) + z\right) \leq \exp\left(\frac{-2z^2}{\sum_{i=1}^n c_i^2}\right),$$

$$\mathbb{P}\left(f(Z) \leq \mathbb{E}(f(Z)) - z\right) \leq \exp\left(\frac{-2z^2}{\sum_{i=1}^n c_i^2}\right).$$

Before applying Lemma 1.4.1, we fix a k-partition $V_1, V_2, ..., V_k$ of the large degree vertices. In the application of 3-uniform hypergraphs, this k-partition will be chosen to satisfy certain requirements. We then order the remaining vertices as $v_1, v_2, ..., v_n$, and design a random process to assign every v_i to V_j with probability p_i^j independently, where p_i^j will be determined, $1 \leq i \leq n, 1 \leq j \leq k$. Then the choice of v_i goves a random variable Z_i. The quantities we are interested in, numbers of edges with certain requirements, are functions of $Z = (Z_1, ..., Z_n)$, which satisfy the condition in Lemma 1.4.1, namely, $|f(Z) - f(Z')| \leq c_i$ for any Z, Z' differing only in the ith coordinate Z_i, where c_i is the degree of vertex v_i in graph (or hypergrpah). This is because that if we change Z_i, i.e. the choice of vertex v_i, the edges affected are those incident with v_i; so the quantities change by at most the degree of v_i.

We have to make sure that those probabilities p_i^j can be chosen such that our random process gives us the desired expectations for the quantities we care. This turns out to be quite difficult when dealing with several quantities. We will also make sure that we can pick an appropriate set of large degree vertices, so that $\sum_{i=1}^n c_i^2$ is of order $o(m^2)$, where m is the number of edges. This will guarantee that after applying Lemma 1.4.1, z can be chosen to be of order $o(m)$.

We organize the rest of this dissertation as follows. In Chapter 2, we prove Theorems 1.3.1, 1.3.2 and 1.3.4. Chapter 3 concentrates on the bounds for pairs in k-partitions of graphs, where we will prove Theorems 1.3.6 and 1.3.8. In Chapter 4, we focus on 3-uniform hypergraphs and prove Theorem 1.3.11.

CHAPTER II

HYPERGRAPHS WITH EDGE SIZE AT MOST 2

There are three sections in this chapter. In Section 2.1, we prove Theorem 1.3.1. In Section 2.2, we prove Theorem 1.3.2. And in Section 2.3, we prove Theorem 1.3.4.

2.1 Bipartitions

In this section we consider the following problem of Bollobás and Scott [12]. Given a hypergraph G with m_i edges of size i, $1 \leq i \leq 2$, does there exist a partition of $V(G)$ into sets V_1 and V_2 such that $d(V_i) \geq \frac{m_1-1}{2} + \frac{2}{3}m_2$ for $i = 1, 2$. This problem was motivated by Conjecture 1.3.10, the Bollobás-Thomason conjecture on r-uniform hypergraphs. Bollobás and Scott [12] proved that if G is a hypergraph with m_i edges of size i, $i = 1, \ldots, k$, then $V(G)$ admits a partition V_1, V_2 such that for $i = 1, 2$,

$$d(V_i) \geq \frac{m_1 - 1}{3} + \frac{2m_2}{3} + \ldots + \frac{km_k}{k+1}.$$

They then used this to show that every 3-uniform hypergraph with m edges can be partitioned into three sets, each of which meets at least $\frac{5}{9}m$ edges.

In [11], Bollobás and Scott suggest that the following might hold. Given a hypergraph G with m_i edges of size i, $1 \leq i \leq k$, there exists a partition of $V(G)$ into sets V_1, V_2 such that for $i = 1, 2$,

$$d(V_i) \geq \frac{m_1 - 1}{2} + \frac{2m_2}{3} + \ldots + \frac{km_k}{k+1}.$$

In fact, they suggest in [12] that asymptotically the bound may be much larger, i.e. for $i = 1, 2$,

$$d(V_i) \geq \frac{1}{2}m_1 + \frac{3}{4}m_2 + \ldots + \left(1 - \frac{1}{2^k}\right)m_k + o(m_1 + \ldots + m_k).$$

In this section we confirm this for $k = 2$ by proving Theorem 2.1.3. Note that by taking a random bipartition V_1, V_2, we have $\mathbb{E}(d(V_i)) = \frac{m_1}{2} + \frac{3}{4}m_2 + ... + (1 - \frac{1}{2^k})m_k$.

We need a simple lemma to be used to pick probabilities in a random process.

Lemma 2.1.1. *Let $a, b, n \in \mathbf{R}^+$ with $a + b > 0$. Then there exists $p \in [0, 1]$ such that*

$$\min\{(n + b)p + a, (n + a)(1 - p) + b\} \geq \frac{n}{2} + \frac{3}{4}(a + b).$$

Proof. Setting $(n + b)p + a = (n + a)(1 - p) + b$, we obtain $p = \frac{n+b}{2n+a+b}$ and

$$(n + b)p + a = \frac{(n + b)^2}{2n + a + b} + a.$$

Clearly $p \in [0, 1]$. It is straightforward to show that

$$\frac{(n + b)^2}{2n + a + b} + a - \left(\frac{n}{2} + \frac{3}{4}(a + b)\right) = \frac{(a - b)^2}{4(2n + a + b)} \geq 0.$$

Hence, the assertion of the lemma holds. ∎

Remark. Note that $p = \dfrac{n + b}{2n + a + b}$ works for Lemma 2.1.1.

We now prove the main result in this section. Recall the notation $\tau(X)$.

Theorem 2.1.2. *Let G be a graph with n vertices and m edges and let $w : V(G) \cup E(G) \to \mathbf{R}^+$ such that $w(e) > 0$ for all $e \in E(G)$. Let $\lambda = \max\{w(x) : x \in V(G) \cup E(G)\}$, $w_1 = \sum_{v \in V(G)} w(v)$, and $w_2 = \sum_{e \in E(G)} w(e)$. Then there is a bipartition X, Y of $V(G)$ such that*

$$\min\{\tau(X), \tau(Y)\} \geq \frac{1}{2}w_1 + \frac{3}{4}w_2 - \lambda \cdot O(m^{4/5}).$$

Proof. We may assume that G is connected, since otherwise we simply consider the individual components. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n)$.

First, we need to deal with an appropriate number of vertices so that the remaining vertices have small degree (and hence will be useful when applying the Azuma-Hoeffding inequality in Lemma 1.4.1). Since G is connected, $n - 1 \leq m < \frac{1}{2}n^2$. Fix $0 < \alpha < \frac{1}{2}$ (to be optimized later), and let $V_1 = \{v_1, \ldots, v_t\}$ such that $t = \lfloor cm^\alpha \rfloor$, where c is some constant and $c < \sqrt{2}$. (Note that, since $\alpha < 1/2$, $c < \sqrt{2}$, and $m < \frac{1}{2}n^2$, we have $t < n$.) Then

$$e(V_1) \leq \binom{t}{2} < \frac{1}{2}t^2 \leq \frac{1}{2}c^2 m^{2\alpha}.$$

Since $\sum_{i=1}^{t+1} d(v_i) \le 2m$,

$$d(v_{t+1}) \le \frac{2m}{t+1} \le \frac{2}{c} m^{1-\alpha}.$$

Let $V_2 = V(G) \backslash V_1$, and rename the vertices in V_2 as $\{u_1, u_2, ..., u_{n-t}\}$ such that $e(\{u_i\}, V_1 \cup \{u_1, ..., u_{i-1}\}) > 0$ for $i = 1, \ldots, n-t$; which can be done since we assume that G is connected.

We now define a random process. First, fix an arbitrary partition $V_1 = X_0 \cup Y_0$, and assign color 1 to all vertices in X_0 and color 2 to all vertices in Y_0. The vertices $u_i \in V_2$ are independently colored 1 with probability p_i, and 2 with probability $1 - p_i$. (The p_i's are constants to be determined recursively.) Let Z_i denote the indicator random variable of the event of coloring u_i. Hence $Z_i = j$, $j \in \{1, 2\}$, iff u_i is assigned color j. When this process stops we obtain a bipartition of $V(G)$ into two sets X, Y, where X consists of all vertices with color 1 and Y consists of all vertices of color 2 (and hence $X_0 \subseteq X$ and $Y_0 \subseteq Y$).

We need additional notation to facilitate the choices of p_i ($1 \le i \le n-t$), the computations of expectations of $\tau(X)$ and $\tau(Y)$, and the estimations of concentration bounds. Let $G_i = G[V_1 \cup \{u_1, u_2, ..., u_i\}]$ for $i = 1, \ldots, n-t$, let $G_0 = G[V_1]$, and let the elements of $V(G_i) \cup E(G_i)$ inherit their weights from G. Let $x_0 = \tau(X_0)$ and $y_0 = \tau(Y_0)$, and define, for $i = 1, \ldots, n-t$,

$$X_i = \{\text{vertices of } G_i \text{ with color } 1\},$$

$$Y_i = \{\text{vertices of } G_i \text{ with color } 2\},$$

$$x_i = \tau_{G_i}(X_i),$$

$$y_i = \tau_{G_i}(Y_i),$$

$$\Delta x_i = x_i - x_{i-1},$$

$$\Delta y_i = y_i - y_{i-1},$$

$$a_i = \sum_{e \in (u_i, X_{i-1})} w(e),$$

$$b_i = \sum_{e \in (u_i, Y_{i-1})} w(e).$$

Note that x_i and y_i are random variables which depend on only $(Z_1, Z_2, \ldots, Z_i)$; and a_i and

b_i are random variables which depend on only $(Z_1, Z_2, \ldots, Z_{i-1})$. Thus,

$$\mathbb{E}(\Delta x_i |\, Z_1, \ldots, Z_{i-1}) = p_i(w(u_i) + b_i) + a_i,$$
$$\mathbb{E}(\Delta y_i |\, Z_1, \ldots, Z_{i-1}) = (1 - p_i)(w(u_i) + a_i) + b_i.$$

Hence,

$$\begin{aligned}\mathbb{E}(\Delta x_i) &= \mathbb{E}\left(\mathbb{E}(\Delta x_i | Z_1, \ldots, Z_{i-1})\right) \\ &= \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1})\left(p_i(w(u_i) + b_i) + a_i\right) \\ &= p_i\left(w(u_i) + \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) b_i\right) + \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) a_i.\end{aligned}$$

Similarly,

$$\mathbb{E}(\Delta y_i) = (1 - p_i)\left(w(u_i) + \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) a_i\right) + \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) b_i.$$

Let

$$\alpha_i = \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) a_i,$$
$$\beta_i = \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) b_i.$$

Then

$$\mathbb{E}(\Delta x_i) = p_i(w(u_i) + \beta_i) + \alpha_i,$$
$$\mathbb{E}(\Delta y_i) = (1 - p_i)(w(u_i) + \alpha_i) + \beta_i.$$

Note that α_i, β_i are determined by $p_1, \ldots, p_{i-1}$, since a_i and b_i are determined by $Z_1, \ldots, Z_{i-1}$. Also note that $e_i := a_i + b_i = \sum_{e \in (u_i,\, G_{i-1})} w(e)$ is the total weight of edges in $(u_i, V(G_{i-1}))$, which is independent of $Z_1, \ldots, Z_{i-1}$ and is the same in both G and G_i. Further, $e_i > 0$ by our choice of u_i and the assumption that $w(e) > 0$ for all $e \in E(G)$. Hence,

$$\begin{aligned}
\alpha_i + \beta_i &= \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1})(a_i + b_i) \\
&= \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1})e_i \\
&= e_i > 0.
\end{aligned}$$

Let $p_i = \dfrac{w(u_i) + \beta_i}{2w(u_i) + \alpha_i + \beta_i}$. Note that p_i is recursively defined, since α_i and β_i are determined by $p_1, \ldots, p_{i-1}$. It follows from Lemma 2.1.1 that $p_i \in [0, 1]$ and

$$\min\{\mathbb{E}(\Delta x_i), \mathbb{E}(\Delta y_i)\} \geq \frac{1}{2}w(u_i) + \frac{3}{4}(\alpha_i + \beta_i) = \frac{1}{2}w(u_i) + \frac{3}{4}e_i.$$

We can now compute the expectations of x_{n-t} and y_{n-t}:

$$\begin{aligned}
\mathbb{E}(x_{n-t}) = \mathbb{E}(x_0) + \sum_{i=1}^{n-t} \mathbb{E}(\Delta x_i) \geq \mathbb{E}(x_0) + \frac{1}{2}\sum_{i=1}^{n-t} w(u_i) + \frac{3}{4}\sum_{i=1}^{n-t} e_i, \\
\mathbb{E}(y_{n-t}) = \mathbb{E}(y_0) + \sum_{i=1}^{n-t} \mathbb{E}(\Delta y_i) \geq \mathbb{E}(y_0) + \frac{1}{2}\sum_{i=1}^{n-t} w(u_i) + \frac{3}{4}\sum_{i=1}^{n-t} e_i.
\end{aligned}$$

Let $X = X_{n-t}$, $Y = Y_{n-t}$. Then $X \cup Y = V(G)$ and $X \cap Y = \emptyset$. Note that $\tau(X) = x_{n-t}$, $\tau(Y) = y_{n-t}$, $\tau(X_0) = x_0$, $\tau(Y_0) = y_0$, $\mathbb{E}(x_0) = x_0$, and $\mathbb{E}(y_0) = y_0$. Also note that $w_2 = \sum_{V(e)\subseteq V_1} w(e) + \sum_{i=1}^{n-t} e_i$. Hence

$$\begin{aligned}
\mathbb{E}(\tau(X)) &\geq \frac{1}{2}\left(w_1 - \sum_{i=1}^{t} w(v_i)\right) + \frac{3}{4}\left(w_2 - \sum_{V(e)\subseteq V_1} w(e)\right) + \tau(X_0) \\
&\geq \frac{1}{2}w_1 + \frac{3}{4}w_2 - \left(\frac{1}{2}\sum_{i=1}^{t} w(v_i) + \frac{3}{4}\sum_{V(e)\subseteq V_1} w(e)\right) \\
&\geq \frac{1}{2}w_1 + \frac{3}{4}w_2 - \lambda\left(\frac{1}{2}t + \frac{3}{4}e(V_1)\right) \\
&\geq \frac{1}{2}w_1 + \frac{3}{4}w_2 - \lambda\left(\frac{1}{2}cm^\alpha + \frac{3}{8}c^2m^{2\alpha}\right).
\end{aligned}$$

Similarly,

$$\mathbb{E}(\tau(Y)) \geq \frac{1}{2}w_1 + \frac{3}{4}w_2 - \lambda\left(\frac{1}{2}cm^\alpha + \frac{3}{8}c^2m^{2\alpha}\right).$$

Next we show that $\tau(X)$ and $\tau(Y)$ are concentrated around their respective means. Note that changing the color of some u_i would affect $\tau(X)$ and $\tau(Y)$ by at most $d(u_i)\lambda + w(u_i) \le (d(u_i)+1)\lambda$. Hence by applying Lemma 1.4.1, we have

$$\begin{aligned}
\mathbb{P}\left(\tau(X) < \mathbb{E}(\tau(X)) - z\right) &\le \exp\left(-\frac{2z^2}{\lambda^2 \sum_{i=1}^{n-t}(d(u_i)+1)^2}\right) \\
&\le \exp\left(-\frac{2z^2}{\lambda^2 \sum_{i=1}^{n-t}(d(u_i)+1)\cdot(d(v_{t+1})+1)}\right) \\
&< \exp\left(-\frac{2z^2}{\lambda^2(1+\frac{2}{c}m^{1-\alpha})\cdot(2m+n-1)}\right) \\
&< \exp\left(-\frac{2z^2}{2\lambda^2\frac{2}{c}m^{1-\alpha}\cdot(2m+m)}\right) \\
&= \exp\left(-\frac{cz^2}{6\lambda^2 m^{2-\alpha}}\right).
\end{aligned}$$

Let $z = \lambda\left(\frac{6\ln 2}{c}\right)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}$. Then

$$\mathbb{P}\left(\tau(X) < \mathbb{E}(\tau(X)) - z\right) < \frac{1}{2}$$

and

$$\mathbb{P}\left(\tau(Y) < \mathbb{E}(\tau(Y)) - z\right) < \frac{1}{2}.$$

So there exists a partition $V(G) = X \cup Y$ such that

$$\tau(X) \ge \mathbb{E}(\tau(X)) - z \ge \frac{1}{2}w_1 + \frac{3}{4}w_2 + \lambda\cdot o(m)$$

and

$$\tau(Y) \ge \mathbb{E}(\tau(Y)) - z \ge \frac{1}{2}w_1 + \frac{3}{4}w_2 + \lambda\cdot o(m).$$

The $o(m)$ term in the above expressions is

$$-\left(\frac{1}{2}cm^{\alpha} + \frac{3}{8}c^2m^{2\alpha} + (\frac{6\ln 2}{c})^{\frac{1}{2}}m^{1-\frac{\alpha}{2}}\right).$$

So picking $\alpha = 2/5$ to minimize $\max\{2\alpha, 1-\frac{\alpha}{2}\}$, we have

$$\min\{\tau(X),\tau(Y)\} \ge \frac{1}{2}w_1 + \frac{3}{4}w_2 - \lambda\cdot O(m^{4/5}). \qquad \blacksquare$$

Note that a random bipartition V_1, V_2 shows that $\mathbb{E}(d(V_i)) = w_1/2 + 3w_2/4$. When G is a hypergraph whose edges are of size 1 or 2, we may view G as a weighted graph with weight function w such that $w(e) = 1$ for all $e \in E(G)$ with $|V(e)| = 2$, $w(v) = 1$ for all $v \in V(G)$ with $\{v\} \in E(G)$, and $w(v) = 0$ for all $v \in V(G)$ with $\{v\} \notin E(G)$. Theorem 2.1.2 then gives the following result which, in turn, implies Theorem 1.3.1.

Theorem 2.1.3. *Let G be a hypergraph with m_i edges of size i, $i = 1, 2$. Then there is a partition V_1, V_2 of $V(G)$ such that for $i = 1, 2$,*

$$d(V_i) \geq \frac{1}{2}m_1 + \frac{3}{4}m_2 - O(m_2^{4/5}).$$

The following is a consequence of Theorem 2.1.3.

Corollary 2.1.4. *Let $k \geq 2$ be an integer and G be a hypergraph with m_i edges of size i, $i = 1, 2, ..., k$. Then there is a partition V_1, V_2 of $V(G)$ such that for $i = 1, 2$, $d(V_i) \geq \frac{1}{2}m_1 + \frac{3}{4}(m_2 + m_3 + ... + m_k) + o(m_2 + m_3 + ... + m_k)$.*

Proof. For each $e \in E(G)$, if $|V(e)| \leq 2$ then let $e' := e$; otherwise, let e' be some 2-element subset of $V(e)$. Let G' denote the hypergraph with $V(G') = V(G)$ and $E(G') = \{e' : e \in E(G)\}$. Then G' has m_1 edges of size 1, and $m_2 + m_3 + ... + m_k$ edges of size 2.

By Theorem 2.1.3, $V(G')$ has a partition V_1, V_2 such that for $i = 1, 2$, $d(V_i) \geq \frac{m_1}{2} + \frac{3}{4}(m_2 + ... + m_k) + o(m_2 + ... + m_k)$ edges. By the construction of G', we see that V_1, V_2 is the desired partition of $V(G)$. ∎

2.2 *k-Partitions – bounding edges meeting each set*

In this section, we prove Conjecture 1.3.3 for graphs with large m. For $k = 2$, Conjecture 1.3.3 follows from the fact that every graph with m edges has a bipartition V_1, V_2 such that for $i \in \{1, 2\}$, each vertex in V_i has at least as many neighbors in V_{3-i} as in V_i.

We use the same approach as in the previous section, namely, first partition an appropriate set of vertices of larger degree, then establish a random process to compute expectations, and finally apply the Azuma-Hoeffding inequality to bound deviations. As before, we need to pick probabilities p_i in the process. To this end we need several lemmas. Our first lemma will be used to take care of critical points when applying Lagrange multipliers to optimize a function.

Lemma 2.2.1. *Let $a_i = a > 0$ for $i = 1, \ldots, l$, and let $a_j = 0$ for $j = l+1, \ldots, k$, where $k \geq l \geq 2$. Let $\delta \geq 0$ and $\alpha_i = \left(\sum_{j=1}^k a_j\right) + \delta - a_i$. Then*

$$1 + \sum_{i=1}^{k} \frac{a_i}{\alpha_i} \geq \left(\frac{\delta}{k} + \frac{2k-1}{k^2} \sum_{i=1}^{k} a_i\right) \sum_{i=1}^{k} \frac{1}{\alpha_i}.$$

Proof. By the assumption of the lemma, we have $\alpha_i = (l-1)a + \delta > 0$ for $1 \leq i \leq l$, and $\alpha_i = la + \delta > 0$ for $l+1 \leq i \leq k$. Let

$$f := 1 + \sum_{i=1}^{k} \frac{a_i}{\alpha_i} - \left(\frac{\delta}{k} + \frac{2k-1}{k^2} \sum_{i=1}^{k} a_i\right) \sum_{i=1}^{k} \frac{1}{\alpha_i}.$$

We need to prove $f \geq 0$. For convenience, let $\delta = a\varepsilon$. Then $\varepsilon \geq 0$ and

$$f = 1 + \frac{l}{l-1+\varepsilon} - \left(\frac{\varepsilon}{k} + \frac{2k-1}{k^2} l\right)\left(\frac{l}{l-1+\varepsilon} + \frac{k-l}{l+\varepsilon}\right).$$

A straightforward calculation shows that

$$(l-1+\varepsilon)(l+\varepsilon) f = \frac{l}{k^2}(k-1)(k-l) \geq 0.$$

Hence the assertion of the lemmas holds. ∎

Note that in the lemma below we are unable to require $p_i \geq 0$, and hence they cannot serve as probabilities in a random process. However, this lemma is needed to prove Lemma 2.2.3.

Lemma 2.2.2. *Let $\delta \geq 0$ and, for $i = 1, \ldots, k$, let $a_i \geq 0$ and $\alpha_i = \left(\sum_{j=1}^k a_j\right) + \delta - a_i$. Then there exist $p_i, i = 1, \ldots, k$, such that $\sum_{i=1}^{k} p_i = 1$ and, for $1 \leq i \leq k$,*

$$\alpha_i p_i + a_i \geq \frac{\delta}{k} + \frac{2k-1}{k^2} \sum_{i=1}^{k} a_i.$$

Proof. For convenience let $f_i(p_1, \ldots, p_k) := \alpha_i p_i + a_i$, $i = 1, \ldots, k$. If $a_i = 0$ for $i = 1, \ldots, k$, then the assertion of the lemma holds by letting $p_i = 1/k$ for $i = 1, \ldots, k$. So without loss of generality we may assume $a_1 > 0$.

Now assume $a_i = 0$ for $i = 2, \ldots, k$. Then $f_1 = \delta p_1 + a_1$ and $f_i = (a_1 + \delta)p_i$ for $2 \leq i \leq k$. Setting $f_i = f_1$ for $i = 2, \ldots, k$, we get $p_i = \dfrac{\delta p_1 + a_1}{a_1 + \delta}$. Requiring $\displaystyle\sum_{i=1}^{k} p_i = 1$, we obtain $p_1 = \dfrac{(2-k)a_1 + \delta}{a_1 + k\delta}$. Hence for $i = 1, \ldots, k$,

$$f_i = \delta p_1 + a_1 = \frac{(\delta + a_1)^2}{a_1 + k\delta},$$

and so,

$$f_i - \left(\frac{\delta}{k} + \frac{2k-1}{k^2}\sum_{i=1}^{k} a_i\right) = \frac{(k-1)^2 a_1^2}{(a_1 + k\delta)k^2} \geq 0.$$

Therefore, we may further assume that $a_2 > 0$. Hence $\alpha_i > 0$ for all $i = 1, \ldots, k$. Setting $f_i = f_1$ for $i = 2, \ldots, k$, we get $p_i = \dfrac{\alpha_1 p_1 + a_1 - a_i}{\alpha_i}$ for $i = 1, \ldots, k$. Requiring $\displaystyle\sum_{i=1}^{k} p_i = 1$ and noting that $a_i - a_1 = \alpha_1 - \alpha_i$ for $1 \leq i \leq k$, we have

$$p_1 = \frac{1 + \sum_{i=1}^{k} \frac{a_i - a_1}{\alpha_i}}{\alpha_1 \sum_{i=1}^{k} \frac{1}{\alpha_i}} = \frac{1 + \sum_{i=1}^{k} \frac{\alpha_1 - \alpha_i}{\alpha_i}}{\alpha_1 \sum_{i=1}^{k} \frac{1}{\alpha_i}} = 1 - \frac{k-1}{\alpha_1 \sum_{i=1}^{k} \frac{1}{\alpha_i}}.$$

Indeed, for $j = 1, \ldots, k$,

$$p_j = 1 - \frac{k-1}{\alpha_j \sum_{i=1}^{k} \frac{1}{\alpha_i}}.$$

Note that $\alpha_j + a_j = \alpha_i + a_i$ for any $1 \leq i, j \leq k$. Hence for $j = 1, 2, \ldots, k$, we have

$$f_j = \alpha_j p_j + a_j = \frac{\sum_{i=1}^{k} \frac{\alpha_j + a_j}{\alpha_i} - (k-1)}{\sum_{i=1}^{k} \frac{1}{\alpha_i}} = \frac{\sum_{i=1}^{k} \frac{\alpha_i + a_i}{\alpha_i} - (k-1)}{\sum_{i=1}^{k} \frac{1}{\alpha_i}} = \frac{1 + \sum_{i=1}^{k} \frac{a_i}{\alpha_i}}{\sum_{i=1}^{k} \frac{1}{\alpha_i}}.$$

Now define

$$f(a_1, a_2, \ldots, a_k) = 1 + \sum_{i=1}^{k} \frac{a_i}{\alpha_i} - \left(\frac{\delta}{k} + \frac{2k-1}{k^2}\sum_{i=1}^{k} a_i\right)\sum_{i=1}^{k} \frac{1}{\alpha_i}.$$

To complete the proof of this lemma, we need to show $f(a_1, \ldots, a_k) \geq 0$.

Case 1. $\delta = 0$.

Then $\alpha_i + a_i = \sum_{j=1}^k a_j$ for $j = 1, ..., k$. Set $\alpha = \sum_{j=1}^k a_j$; then $\sum_{i=1}^k \alpha_i = (k-1)\alpha$. Moreover,

$$\begin{aligned} f(a_1, ..., a_k) &= 1 + \sum_{i=1}^k \frac{a_i}{\alpha_i} - \frac{(2k-1)\alpha}{k^2} \sum_{i=1}^k \frac{1}{\alpha_i} \\ &= 1 + \sum_{i=1}^k \frac{\alpha - \alpha_i}{\alpha_i} - \frac{(2k-1)\alpha}{k^2} \sum_{i=1}^k \frac{1}{\alpha_i} \\ &= \frac{(k-1)^2\alpha}{k^2} \sum_{i=1}^k \frac{1}{\alpha_i} - (k-1) \\ &\geq \frac{(k-1)^2\alpha}{k^2} \frac{k^2}{\sum_{i=1}^k \alpha_i} - (k-1) \\ &= 0. \end{aligned}$$

Here the inequality follows from Cauchy-Schwarz, and the last equality follows from the face that $\sum_{i=1}^k \alpha_i = (k-1)\alpha$.

Case 2. $\delta > 0$.

Then $\alpha_i > 0$ for $i = 1, ..., k$. (So in this case we need not require $a_1 > 0$ and $a_2 > 0$.) Set $\alpha = \sum_{j=1}^k a_j$.

Let $g_l(a_1, \ldots, a_l) = f(a_1, \ldots, a_l, 0, \ldots, 0)$. It then suffices to show that $g_l(a_1, \ldots, a_l) \geq 0$ on the domain $D_l := [0, \alpha]^l$ for $l = 1, \ldots, k$.

First, we prove that for $l \in \{1, \ldots, k\}$, $g_l \geq 0$ at all possible critical points of g_l in D_l, subject to $\sum_{j=1}^k a_j - \alpha = 0$. For $j = 1, ..., l$,

$$\frac{\partial g_l}{\partial a_j} = -\sum_{i=1}^k \frac{a_i}{\alpha_i^2} + \frac{a_j}{\alpha_j^2} + \frac{1}{\alpha_j} + \frac{\delta}{k}\left(\sum_{i=1}^k \frac{1}{\alpha_i^2} - \frac{1}{\alpha_j^2}\right) - \frac{2k-1}{k^2}\left(\sum_{i=1}^k \frac{1}{\alpha_i} - \sum_{i=1}^k a_i \sum_{i=1}^k \frac{1}{\alpha_i^2} + \sum_{i=1}^k \frac{a_i}{\alpha_j^2}\right).$$

Using the method of Lagrange multipliers, we have $\frac{\partial g_l}{\partial a_j} = \lambda$ for all $j = 1, ..., l$. So $\frac{\partial g_l}{\partial a_j} = \frac{\partial g_l}{\partial a_1}$, which gives

$$\frac{a_j}{\alpha_j^2} + \frac{1}{\alpha_j} - \frac{\delta}{k}\frac{1}{\alpha_j^2} - \frac{2k-1}{k^2} \sum_{i=1}^k \frac{a_i}{\alpha_j^2} = \frac{a_1}{\alpha_1^2} + \frac{1}{\alpha_1} - \frac{\delta}{k}\frac{1}{\alpha_1^2} - \frac{2k-1}{k^2} \sum_{i=1}^k \frac{a_i}{\alpha_1^2}.$$

Since $\alpha_j + a_j = \alpha_1 + a_1 = \sum_{i=1}^k a_i + \delta$, we have

$$\frac{1}{\alpha_j^2}\left(\frac{(k-1)^2}{k^2} \sum_{i=1}^n a_i + \frac{k-1}{k}\delta\right) = \frac{1}{\alpha_1^2}\left(\frac{(k-1)^2}{k^2} \sum_{i=1}^n a_i + \frac{k-1}{k}\delta\right).$$

Hence $1/\alpha_j^2 = 1/\alpha_1^2$ for all $j = 1, \dots, l$. Therefore, $\alpha_j = \alpha_1$ for $j = 1, \dots, l$. This implies $a_j = a_1$ for $j = 1, \dots, l$. It now follows from Lemma 2.2.1 that $g_l \geq 0$ at all possible critical points of g_l in $[0, \alpha]^l$.

We now show that $g_l \geq 0$ on $[0, \alpha]^l$ by applying induction on l. Suppose $l = 1$. Then $\alpha = a_1$. So $\alpha_1 = \delta$, and $\alpha_i = a_1 + \delta$ for $i = 2, ..., k$. Hence,

$$g_1(a_1) = 1 + \frac{a_1}{\delta} - \left(\frac{\delta}{k} + \frac{(2k-1)a_1}{k^2}\right)\left(\frac{1}{\delta} + \frac{k-1}{a_1+\delta}\right) = \frac{(k-1)^2}{k^2}\left(\frac{a_1^2}{\delta(a_1+\delta)}\right) \geq 0.$$

So we may assume $l \geq 2$ and $g_i \geq 0$ for all $i = 1, \dots, l-1$. We now show $g_l \geq 0$ on the domain $[0, \alpha]^l$ by proving it for all points in the boundary of $[0, \alpha]^l$ (since $g_l \geq 0$ at all possible critical points of g_l). Let $(a_1, \dots, a_l)$ be in the boundary of $[0, \alpha]^l$. Then $a_j = 0$ or $a_j = \alpha$ for some $j \in \{1, \dots, l\}$. Note that g_l is a symmetric function. So we may assume without loss of generality that $a_l = 0$ or $a_1 = \alpha$. If $a_l = 0$, then $g_l(a_1, \dots, a_l) = g_{l-1}(a_1, \dots, a_{l-1}) \geq 0$ by induction hypothesis. If $a_1 = \alpha$ then $a_j = 0$ for $j = 2, ..., l$, and so, $g_l(a_1, ..., a_l) = g_1(a_1) \geq 0$. Again, we have $g_l(a_1, ..., a_l) \geq 0$. ∎

Note that in the proof of Lemma 2.2.2 when $\alpha_i > 0$, $1 \leq i \leq k$, we have

$$p_j = 1 - \frac{k-1}{\alpha_j \sum\limits_{i=1}^{k} \frac{1}{\alpha_i}}$$

for $j = 1, \dots, k$, which may be negative. We now apply Lemma 2.2.2 to prove the next result which gives the p_i's needed in a random process.

Lemma 2.2.3. *Let $\delta \geq 0$. For $i = 1, \dots, k$, where $k \geq 3$, let $a_i \geq 0$ and $\alpha_i = (\sum_{j=1}^{k} a_j) + \delta - a_i$. Then there exist $p_i \in [0, 1]$, $1 \leq i \leq k$, such that $\sum\limits_{i=1}^{k} p_i = 1$ and for $1 \leq i \leq k$,*

$$\alpha_i p_i + a_i \geq \frac{\delta}{k} + \frac{1}{k-1} \sum_{i=1}^{k} a_i.$$

Proof. If $a_i = 0$ for $1 \leq i \leq k$, then the assertion of the lemma holds by taking $p_i = 1/k$, $i = 1, \dots, k$. So we may assume without loss of generality that $a_1 > 0$. If $a_i = 0$ for $2 \leq i \leq k$ and $\delta = 0$, then $\alpha_1 = 0$ and $\alpha_i = a_1$ for $2 \leq i \leq k$; and the assertion of the lemma holds by setting $p_1 = 0$ and $p_i = \frac{1}{k-1}$ for $i = 2, \ldots, k$. Therefore, we may further assume that $a_2 > 0$ or $\delta > 0$. As a consequence, we have $\alpha_i > 0$ for $1 \le i \le k$.

We prove the assertion of this lemma by induction on k. For $1 \le i \le k$, let

$$f_i(p_1, \ldots, p_k) := \alpha_i p_i + a_i.$$

For $k = 3$, it follows from Lemma 2.2.2 (and the remark following its proof) that there exist p'_1, p'_2, p'_3 such that $p'_1 + p'_2 + p'_3 = 1$ and for $i = 1, 2, 3$,

$$p'_i = 1 - \frac{2}{\alpha_i \sum_{i=1}^{3} \frac{1}{\alpha_j}} \text{ and } f_i(p'_1, p'_2, p'_3) \ge \frac{\delta}{3} + \frac{5}{9} \sum_{i=1}^{3} a_i.$$

If $p'_i \ge 0$ for $i = 1, 2, 3$, then the assertion of the lemma holds by taking $p_i := p'_i$, $i = 1, 2, 3$. So we may assume that $p'_3 < 0$, which implies $a_3 > \alpha_3 p'_3 + a_3 = f_3(p'_1, p'_2, p'_3) \ge \frac{\delta}{3} + \frac{5}{9} \sum_{i=1}^{3} a_i$. By Lemma 2.1.1 (with $n := a_3 + \delta$), there exist $p_1, p_2 \in [0, 1]$ such that $p_1 + p_2 = 1$ and

$$f_1(p_1, p_2, 0) = (a_2 + a_3 + \delta)p_1 + a_1 \ge \frac{a_3 + \delta}{2} + \frac{3}{4}(a_1 + a_2),$$
$$f_2(p_1, p_2, 0) = (a_1 + a_3 + \delta)p_2 + a_2 \ge \frac{a_3 + \delta}{2} + \frac{3}{4}(a_1 + a_2).$$

Now, let $p_3 = 0$. Then $p_1 + p_2 + p_3 = 1$, $p_i \in [0, 1]$ for all $1 \le i \le 3$, and

$$f_1(p_1, p_2, p_3) = \alpha_1 p_1 + a_1 \ge \frac{\delta}{3} + \frac{1}{2}(a_1 + a_2 + a_3),$$
$$f_2(p_1, p_2, p_3) = \alpha_2 p_2 + a_2 \ge \frac{\delta}{3} + \frac{1}{2}(a_1 + a_2 + a_3),$$
$$f_3(p_1, p_2, p_3) = a_3 \ge \frac{\delta}{3} + \frac{1}{2}(a_1 + a_2 + a_3).$$

Hence Lemma 2.2.3 holds for $k = 3$.

Now let $n \ge 3$ be an integer, and assume that the assertion of the lemma holds when $k = n$. We prove the assertion of the lemma also holds when $k = n + 1$. By Lemma 2.2.2 (and the remark following its proof), there exist p'_i, $1 \le i \le n+1$, such that $\sum_{i=1}^{n+1} p'_i = 1$ and for $i = 1, \ldots, n + 1$,

$$f_i(p'_1, \ldots, p'_{n+1}) \ge \frac{\delta}{n+1} + \frac{2n+1}{(n+1)^2} \sum_{i=1}^{n+1} a_i,$$

and

$$p'_i = 1 - \frac{n}{\alpha_i \sum\limits_{j=1}^{n+1} \frac{1}{\alpha_j}} \leq 1.$$

If $p'_i \geq 0$ for $1 \leq i \leq n+1$, then let $p_i := p'_i$; and the lemma holds (since $\frac{2n+1}{(n+1)^2} > \frac{1}{n}$ when $n \geq 3$). So we may assume without loss of generality that $p'_{n+1} < 0$. Then

$$\begin{aligned}
a_{n+1} &> \alpha_{n+1} p'_{n+1} + a_{n+1} \\
&= f_{n+1}(p'_1, \ldots, p'_{n+1}) \\
&\geq \frac{\delta}{n+1} + \frac{2n+1}{(n+1)^2} \sum_{i=1}^{n+1} a_i \\
&\geq \frac{\delta}{n+1} + \frac{1}{n} \sum_{i=1}^{n+1} a_i
\end{aligned}$$

Let $\delta' = \delta + a_{n+1}$. Then for $1 \leq i \leq n$ we have $\alpha_i = (\sum_{j=1}^{n} a_j) + \delta' - a_i$. Hence by the induction hypothesis, there exist $p_i \in [0,1]$, $1 \leq i \leq n$, such that $\sum\limits_{i=1}^{n} p_i = 1$ and, for $i = 1, \ldots, n$,

$$\begin{aligned}
\alpha_i p_i + a_i &\geq \frac{\delta'}{n} + \frac{1}{n-1} \sum_{i=1}^{n} a_i \\
&= \frac{\delta}{n} + \frac{a_{n+1}}{n} + \frac{1}{n-1} \sum_{i=1}^{n} a_i.
\end{aligned}$$

Let $p_{n+1} = 0$. Then $\sum\limits_{i=1}^{n+1} p_i = 1$ and $p_i \in [0,1]$ for all $1 \leq i \leq n+1$. Also, for any $1 \leq i \leq n$,

$$\begin{aligned}
&f_i(p_1, \ldots, p_{n+1}) \geq \frac{\delta}{n} + \frac{a_{n+1}}{n} + \frac{1}{n-1} \sum_{i=1}^{n} a_i \geq \frac{\delta}{n+1} + \frac{1}{n} \sum_{i=1}^{n+1} a_i, \text{ and} \\
&f_{n+1}(p_1, \ldots, p_{n+1}) = a_{n+1} \geq \frac{\delta}{n+1} + \frac{1}{n} \sum_{i=1}^{n+1} a_i.
\end{aligned}$$

Hence, Lemma 2.2.3 holds for $k = n+1$, completing the proof of this lemma. ∎

We can now prove the following partition result on weighted graphs.

Theorem 2.2.4. *Let $k \geq 3$ be an integer, let G be a graph with m edges, and let $w : V(G) \cup E(G) \to \mathbf{R}^+$ such that $w(e) > 0$ for all $e \in E(G)$. Let $\lambda = \max\{w(x) : x \in V(G) \cup E(G)\}$,*

$w_1 = \sum_{v\in V(G)} w(v)$ and $w_2 = \sum_{e\in E(G)} w(e)$. Then there is a k-partition $V_1, \ldots, V_k$ of $V(G)$ such that for $1 \le i \le k$,

$$\tau(V_i) \ge \frac{1}{k} w_1 + \frac{1}{k-1} w_2 - \lambda \cdot O(m^{4/5}).$$

Proof. We may assume that G is connected. We use the same notation as in the proof of Theorem 2.1.2. Let $V(G) = \{v_1, \ldots, v_n\}$ such that $d(v_1) \ge d(v_2) \ge \ldots \ge d(v_n)$. Let $V_1 = \{v_1, \ldots, v_t\}$ with $t = \lfloor cm^\alpha \rfloor$, where $0 < \alpha < 1/2$ and $0 < c < \sqrt{2}$; and let $V_2 := V(G) \setminus V_1 = \{u_1, \ldots, u_{n-t}\}$ such that $e(u_i, V_1 \cup \{u_1, \ldots, u_{i-1}\}) > 0$ for $i = 1, \ldots, n-t$. Then

$$e(V_1) \le \frac{1}{2} c^2 m^{2\alpha} \quad \text{and} \quad d(v_{t+1}) \le \frac{2}{c} m^{1-\alpha}.$$

Fix an arbitrary partition $V_1 = Y_1 \cup Y_2 \cup \cdots \cup Y_k$ and, for each $i \in \{1, \ldots, k\}$, assign the color i to all vertices in Y_i. We extend this coloring to $V(G)$ such that each vertex $u_i \in V_2$ is independently assigned the color j with probability p^i_j, $\sum_{j=1}^k p^i_j = 1$. Let Z_i be the indicator random variable of the event of coloring u_i, i.e., $Z_i = j$ iff u_i is colored j. Let $G_i = G[V_1 \cup \{u_1, \cdots, u_i\}]$ for $i = 1, \ldots, n-t$, and let $G_0 = G[V_1]$. Let $X^0_j = Y_j$ and $x^0_j = \tau(X^0_j)$, and for $i = 1, \ldots, n-t$ and $j = 1, \ldots, k$, define

$$\begin{aligned}
X^i_j &= \{\text{vertices of } G_i \text{ with color } j\},\\
x^i_j &= \tau_{G_i}(X^i_j),\\
\Delta x^i_j &= x^i_j - x^{i-1}_j,\\
a^i_j &= \sum_{e\in(u_i, X^{i-1}_j)} w(e).
\end{aligned}$$

Note that a^i_j depends on only $(Z_1, \ldots, Z_{i-1})$. Hence, for $1 \le i \le n-t$ and $1 \le j \le k$,

$$\mathbb{E}(\Delta x^i_j | \, Z_1, \ldots, Z_{i-1}) = p^i_j \left(\sum_{l=1}^k a^i_l + w(u_i) - a^i_j \right) + a^i_j.$$

So

$$\mathbb{E}(\Delta x^i_j) = p^i_j \left(\sum_{l=1}^k b^i_l + w(u_i) - b^i_j \right) + b^i_j,$$

where for $1 \leq l \leq k$,

$$b_l^i = \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) a_l^i.$$

Since a_l^i is determined by $(Z_1, \ldots, Z_{i-1})$, b_l^i is determined by p_j^s, $1 \leq s \leq i-1$ and $1 \leq j \leq k$.

By Lemma 2.2.3 (with $\delta = w(u_i)$), there exist $p_j^i \in [0, 1]$, $1 \leq j \leq k$, such that $\sum_{j=1}^k p_j^i = 1$ and

$$\mathbb{E}(\Delta x_j^i) \geq \frac{w(u_i)}{k} + \frac{1}{k-1} \sum_{j=1}^k b_j^i.$$

Clearly, each p_j^i is dependent only on b_l^i, $1 \leq l \leq k$, and hence is determined (recursively) by p_l^s, $1 \leq l \leq k$ and $1 \leq s \leq i-1$. Note that $e_i := \sum\limits_{j=1}^k a_j^i = \sum_{e \in (u_i, G_{i-1})} w(e)$ is the total weight of the edges in (u_i, G_{i-1}), which is independent of $Z_1, \ldots, Z_{n-t}$. Thus,

$$\begin{aligned}
\mathbb{E}(\Delta x_j^i) &\geq \frac{w(u_i)}{k} + \frac{1}{k-1} \sum_{j=1}^k \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) a_j^i \\
&= \frac{w(u_i)}{k} + \frac{1}{k-1} \sum_{(Z_1,\ldots,Z_{i-1})} \left(\mathbb{P}(Z_1, \ldots, Z_{i-1}) \sum_{j=1}^k a_j^i \right) \\
&= \frac{w(u_i)}{k} + \frac{1}{k-1} \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) e_i \\
&= \frac{w(u_i)}{k} + \frac{1}{k-1} e_i.
\end{aligned}$$

Therefore, noting that $w_2 = \sum_{V(e) \subseteq V_1} w(e) + \sum_{i=1}^{n-t} e_i$, we have

$$\begin{aligned}
\mathbb{E}(x_j^{n-t}) &= \sum_{i=1}^{n-t} \mathbb{E}(\Delta x_j^i) + \mathbb{E}(x_j^0) \\
&\geq \frac{1}{k} \sum_{i=1}^{n-t} w(u_i) + \frac{1}{k-1} \sum_{i=1}^{n-t} e_i + x_j^0 \\
&\geq \frac{1}{k} \left(w_1 - \sum_{i=1}^t w(v_i) \right) + \frac{1}{k-1} \left(w_2 - \sum_{V(e) \subseteq V_1} w(e) \right) \\
&\geq \frac{1}{k} w_1 + \frac{1}{k-1} w_2 - \left(\frac{1}{k} \sum_{i=1}^t w(v_i) + \frac{1}{k-1} \sum_{V(e) \subseteq V_1} w(e) \right) \\
&\geq \frac{1}{k} w_1 + \frac{1}{k-1} w_2 - \lambda \left(\frac{1}{k} t + \frac{1}{k-1} e(V_1) \right).
\end{aligned}$$

Let $x_j := x_j^{n-t} = \tau_G(X_j^{n-t})$, $j = 1, \ldots, k$. Now changing the color of u_i only affects x_j by at most $d(u_i)\lambda + w(u_i) \le (d(u_i) + 1)\lambda$. Hence, by Lemma 1.4.1, we have, for $j = 1, \ldots, k$,

$$\begin{aligned}
\mathbb{P}\left(x_j < \mathbb{E}(x_j) - z\right) &\le \exp\left(-\frac{2z^2}{\lambda^2 \sum_{i=1}^{n-t}(d(u_i)+1)^2}\right) \\
&\le \exp\left(-\frac{2z^2}{\lambda^2 \sum_{i=1}^{n-t}(d(u_i)+1)\cdot(d(v_{t+1})+1)}\right) \\
&< \exp\left(-\frac{2z^2}{\lambda^2(2m+n-1)\cdot\frac{4}{c}m^{1-\alpha}}\right) \\
&\le \exp\left(-\frac{cz^2}{6\lambda^2 m^{2-\alpha}}\right).
\end{aligned}$$

Let $z = \lambda\left(\frac{6\ln k}{c}\right)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}$; then

$$\mathbb{P}\left(x_j < \mathbb{E}(x_j) - z\right) < \exp\left(-\ln k\right) = \frac{1}{k}.$$

So there exists a partition $V(G) = X_1 \cup X_2 \cup \cdots \cup X_k$ such that for $j = 1, \ldots, k$,

$$\begin{aligned}
x_j &\ge \mathbb{E}(x_j) - z \\
&\ge \frac{1}{k}w_1 + \frac{1}{k-1}w_2 - \lambda\left(\frac{1}{k}t + \frac{1}{k-1}e(V_1)\right) - z \\
&\ge \frac{1}{k}w_1 + \frac{1}{k-1}w_2 + \lambda\cdot o(m),
\end{aligned}$$

where the $o(m)$ term in the expression is

$$-\left(\frac{c}{k}m^{\alpha} + \frac{1}{2(k-1)}c^2m^{2\alpha} + \left(\frac{6\ln k}{c}\right)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}\right).$$

Picking $\alpha = \frac{2}{5}$ to minimize $\max\{2\alpha, 1-\alpha/2\}$, the $o(m)$ term becomes $O(m^{\frac{4}{5}})$. ∎

Suppose G is a hypergraph whose edges have size 1 or 2. We may view G as a weighted graph with weight function w such that $w(e) = 1$ for all $e \in E(G)$ with $|V(e)| = 2$, $w(v) = 1$ for all $v \in V(G)$ with $\{v\} \in E(G)$, and $w(v) = 0$ for all $v \in V(G)$ with $\{v\} \notin E(G)$. Theorem 2.2.4 then gives the following result, which implies Theorem 1.3.2.

Theorem 2.2.5. *Let $k \ge 3$ be an integer and let G be a hypergraph with m_i edges of size i, $i = 1, 2$. Then there is a partition $V_1, \ldots, V_k$ of $V(G)$ such that for $i = 1, \ldots, k$,*

$$d(V_i) \ge \frac{m_1}{k} + \frac{m_2}{k-1} - O(m_2^{4/5}).$$

We have the following corollary, which establishes Conjecture 1.3.3 for large graphs.

Corollary 2.2.6. *Let G be a graph with m edges and let $k \geq 3$ be an integer. Then there is an integer $f(k)$ such that if $m \geq f(k)$ then $V(G)$ has a partition $V_1, \ldots, V_k$ such that $d(V_i) \geq 2m/(2k-1)$ for $i = 1, \ldots, k$.*

Note that our proof gives $f(k) = O(k^{10}(\log k)^{5/2})$.

2.3 *k-Partitions – bounding edges inside each set*

Bollobás and Scott [8] proved that every graph with m edges can be partitioned into k sets each of which contains at most $m/\binom{k+1}{2}$ edges, with K_{k+1} as the unique extremal graph. For large graphs, they proved in [10] that this bound can be improved to $(1 + o(1))m/k^2$. They also [12] conjectured that:

Conjecture 2.3.1. *(Bollobás and Scott [12]) Any hypergraph with m_i edges of size $i, i = 1, 2$, admits a k-partition $V_1, ..., V_k$ such that for $i = 1, \ldots, k$,*

$$e(V_i) \leq \frac{m_1}{k} + \frac{m_2}{\binom{k+1}{2}} + O(1).$$

We now prove Conjecture 2.3.1. The following two lemmas will enable us to choose the probabilities in a random process.

Lemma 2.3.2. *Let $\delta \geq 0$ and, for integers $k \geq l \geq 1$, let $a_i = a > 0$ for $i = 1, \ldots, l$ and $a_j = 0$ for $j = l+1, \ldots, k$. Suppose $\delta + a_i > 0$ for all $1 \leq i \leq k$. Then*

$$\frac{1}{\sum_{i=1}^{k} \frac{1}{\delta + a_i}} \leq \frac{\delta}{k} + \frac{1}{k^2} \sum_{i=1}^{k} a_i.$$

Proof. If $l = k$ then the inequality holds with equality (both sides equal to $(\delta + a)/k$). So we may assume $k > l$. Then $\delta > 0$, since $\delta + a_k > 0$ by assumption. Thus $\sum_{i=1}^{k} \frac{1}{\delta + a_i} = \frac{l}{\delta + a} + \frac{k-l}{\delta}$ and $\sum_{i=1}^{k} a_i = la$. Hence

$$\frac{1}{\sum_{i=1}^{k} \frac{1}{\delta + a_i}} - \left(\frac{\delta}{k} + \frac{1}{k^2} \sum_{i=1}^{k} a_i \right) = \frac{-l(k-l)a^2}{k^2(k\delta + (k-l)a)} \leq 0.$$

Thus the assertion of the lemma holds. ∎

Lemma 2.3.3. *Let $\delta \geq 0$ and let $a_i \geq 0$ for $i = 1, \dots, k$. Then there exist $p_i \in [0, 1]$, $i = 1, \dots, k$, such that $\sum\limits_{i=1}^{k} p_i = 1$ and, for $1 \leq i \leq k$,*

$$(\delta + a_i)p_i \leq \frac{\delta}{k} + \frac{1}{k^2} \sum_{i=1}^{k} a_i.$$

Proof. If there exists some $1 \leq i \leq k$ such that $\delta + a_i = 0$, then $\delta = a_i = 0$. In this case let $p_i = 1$ and $p_j = 0$ for $j \neq i, 1 \leq j \leq k$. Then $(\delta + a_i)p_i = 0$ for $i = 1, \dots, k$; and clearly the assertion of the lemma holds.

Therefore, we may assume that $\delta + a_i > 0$, $1 \leq i \leq k$. Setting $(\delta + a_i)p_i = (\delta + a_1)p_1$ for $i = 2, \dots, k$, we have $p_i = \frac{\delta + a_1}{\delta + a_i} p_1$. Requiring $\sum\limits_{i=1}^{k} p_i = 1$ we have $(\delta + a_1)p_1 \sum\limits_{i=1}^{k} \frac{1}{\delta + a_i} = 1$. Hence for $i = 1, \dots, k$,

$$(\delta + a_i)p_i = \frac{1}{\sum\limits_{i=1}^{k} \frac{1}{\delta + a_i}}.$$

Let

$$f(a_1, a_2, \dots, a_k) := \frac{1}{\sum\limits_{i=1}^{k} \frac{1}{\delta + a_i}} - \left(\frac{\delta}{k} + \frac{1}{k^2} \sum_{i=1}^{k} a_i \right).$$

We need to show $f \leq 0$. This is clear if $a_i = 0$ for $i = 1, \dots, k$, since $f(0, \dots, 0) = 0$.

Let $g_l(a_1, \dots, a_l) := f(a_1, \dots, a_l, 0, \dots, 0)$ for $l = 1, \dots, k$. We now show that $g_l \leq 0$ on $D_l := [0, \alpha]^l$ for all $1 \leq l \leq k$; and hence $f = g_k \leq 0$. We apply induction on l.

Suppose $l = 1$. Clearly, $g_1(0) = f(0, 0, \dots, 0) = 0$, and if $a_1 = a > 0$ then by Lemma 2.3.2, $g_1(a_1) = f(a_1, 0, \dots, 0) \leq 0$.

Therefore, we may assume $l \geq 2$. It suffices to prove $g_l(a_1, \dots, a_l) \leq 0$ for all points $(a_1, \dots, a_l)$ that are on the boundary of D_l or critical points of g_l in D_l.

Let $(a_1, \dots, a_l)$ be a point on the boundary of D_l. Then there exists $j \in \{1, ..., l\}$ such that $a_j = 0$ or $a_j = \alpha$. Since g_l is a symmetric function, we may assume that $a_l = 0$ or $a_1 = \alpha$. If $a_l = 0$, then $g_l(a_1, \dots, a_{l-1}, 0) = g_{l-1}(a_1, \dots, a_{l-1}) \leq 0$, by induction hypothesis. If $a_1 = \alpha$, then $a_2 = ... = a_k = 0$, and so $g_l(a_1, ..., a_l) = g_1(a_1) \leq 0$ by induction basis.

Hence it remains to prove $g_l \leq 0$ at its critical points in D_l, subject to $\sum_{j=1}^{l} a_j - \alpha = 0$.

Note that for all $j = 1, \ldots, l$,

$$\frac{\partial f}{\partial a_j} = \frac{1}{\left(\sum_{i=1}^{k} \frac{1}{\delta + a_i}\right)^2} \cdot \frac{1}{(\delta + a_j)^2} - \frac{1}{k^2}.$$

Also note that $\frac{\partial g_l}{\partial a_j}$ is obtained from $\frac{\partial f}{\partial a_j}$ by setting $a_{l+1} = \ldots = a_k = 0$. Thus, letting $\dfrac{\partial g_l}{\partial a_j} = \lambda$ (the Lagrange multiplier) for $j = 1, \ldots, l$, we have for $1 \leq s \neq t \leq l$,

$$\frac{1}{\left(\sum_{i=1}^{k} \frac{1}{\delta + a_i}\right)^2} \cdot \frac{1}{(\delta + a_s)^2} - \frac{1}{k^2} = \frac{1}{\left(\sum_{i=1}^{k} \frac{1}{\delta + a_i}\right)^2} \cdot \frac{1}{(\delta + a_t)^2} - \frac{1}{k^2}.$$

As a consequence, $(\delta + a_s)^2 = (\delta + a_t)^2$ which implies $a_s = a_t$ for all $1 \leq s \neq t \leq l$. Thus, if $(a_1, a_2, \ldots, a_l)$ is a critical point of g_l in D_l, then there exists $a > 0$ such that $a_i = a > 0$ for $i = 1, \ldots, l$. Now it follows from Lemma 2.3.2 that $g_l \leq 0$. ∎

We now prove the following partition result for weighted graphs.

Theorem 2.3.4. *Let G be a graph with m edges, and let $w : V(G) \cup E(G) \to \mathbf{R}^+$ such that $w(e) > 0$ for all $e \in E(G)$. Let $\lambda := \max\{w(x) : x \in V(G) \cup E(G)\}$, $w_1 = \sum_{v \in V(G)} w(v)$ and $w_2 = \sum_{e \in E(G)} w(e)$. Then for any integer $k \geq 1$ there is a k-partition $V_1, \ldots, V_k$ of $V(G)$ such that for $i = 1, \ldots, k$,*

$$e(V_i) \leq \frac{1}{k} w_1 + \frac{1}{k^2} w_2 + \lambda \cdot O(m^{4/5}).$$

Proof. We may assume that G is connected. We use the same notation as in the proof of Theorem 2.1.2. Let $V(G) = \{v_1, \ldots, v_n\}$ such that $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$. Let $V_1 = \{v_1, \ldots, v_t\}$ with $t = \lfloor cm^\alpha \rfloor$, where $0 < \alpha < 1/2$ and $0 < c < \sqrt{2}$; and let $V_2 := V(G) \setminus V_1 = \{u_1, \ldots, u_{n-t}\}$ such that $e(u_i, V_1 \cup \{u_1, \ldots, u_{i-1}\}) > 0$ for $i = 1, \ldots, n - t$. Then $e(V_1) \leq \frac{1}{2} c^2 m^{2\alpha}$ and $d(v_{t+1}) \leq \frac{2}{c} m^{1-\alpha}$.

Fix an arbitrary k-partition $V_1 = Y_1 \cup Y_2 \cup \cdots \cup Y_k$, and assign each member of Y_i the color i, $1 \leq i \leq k$. Extend this coloring to $V(G)$, where each vertex $u_i \in V_2$ is independently assigned the color j with probability p^i_j and $\sum_{j=1}^{k} p^i_j = 1$. Let Z_i denote the indicator random variable of the event of coloring u_i. Hence $Z_i = j$ iff u_i is assigned the color j.

Let $G_i = G[V_1 \cup \{u_1, \cdots, u_i\}]$ for $i = 1, \ldots, n-t$, and let $G_0 = G[V_1]$. Let $X_j^0 = Y_j$ and $x_j^0 = w(X_j^0)$, and for $i = 1, \ldots, n-t$ define

$$\begin{aligned}
X_j^i &= \{\text{vertices of } G_i \text{ with color } j\},\\
x_j^i &= w(X_j^i),\\
\Delta x_j^i &= x_j^i - x_j^{i-1},\\
a_j^i &= \sum_{e \in (u_i, X_j^{i-1})} w(e).
\end{aligned}$$

Note that a_j^i depends on $(Z_1, \ldots, Z_{i-1})$ only. Hence for $1 \le i \le n-t$ and $1 \le j \le k$,

$$\mathbb{E}(\Delta x_j^i | Z_1, \ldots, Z_{i-1}) = (w(u_i) + a_j^i) p_j^i,$$

and so

$$\mathbb{E}(\Delta x_j^i) = (w(u_i) + b_j^i) p_j^i,$$

where here

$$b_j^i = \sum_{(Z_1, \ldots, Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) a_j^i.$$

Since a_j^i is determined by $(Z_1, \ldots, Z_{i-1})$, b_j^i is determined by p_j^s, $1 \le j \le k$ and $1 \le s \le i-1$. Note that $e_i := \sum\limits_{j=1}^{k} a_j^i = \sum_{e \in (u_i, G_{i-1})} w(e) > 0$, which is independent of $Z_1, \ldots, Z_{n-t}$. By Lemma 2.3.3, there exists $p_j^i \in [0,1]$, $1 \le j \le k$, such that $\sum_{j=1}^k p_j^i = 1$ and, for $1 \le i \le n-t$ and $j = 1, \ldots, k$,

$$\begin{aligned}
\mathbb{E}(\Delta x_j^i) &\le \frac{w(u_i)}{k} + \frac{1}{k^2} \sum_{j=1}^{k} b_j^i\\
&= \frac{w(u_i)}{k} + \frac{1}{k^2} \sum_{j=1}^{k} \sum_{(Z_1, \ldots, Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) a_j^i\\
&= \frac{w(u_i)}{k} + \frac{1}{k^2} \sum_{(Z_1, \ldots, Z_{i-1})} \left(\mathbb{P}(Z_1, \ldots, Z_{i-1}) \sum_{j=1}^{k} a_j^i \right)\\
&= \frac{w(u_i)}{k} + \frac{1}{k^2} \sum_{(Z_1, \ldots, Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) e_i\\
&= \frac{w(u_i)}{k} + \frac{1}{k^2} e_i.
\end{aligned}$$

Note that p^i_j is determined by b^i_l, $1 \le l \le k$; and hence p^i_j is recursively defined by p^s_l, $1 \le l \le k$ and $1 \le s \le i-1$. Also note that $w_2 = \sum_{e \in E(G_0)} w(e) + \sum_{i=1}^{n-t} e_i$. Now

$$\begin{aligned}
\mathbb{E}(x_j^{n-t}) &= \sum_{i=1}^{n-t} \mathbb{E}(\Delta x_j^i) + \mathbb{E}(x_j^0) \\
&\le \frac{1}{k} \sum_{i=1}^{n-t} w(u_i) + \frac{1}{k^2} \sum_{i=1}^{n-t} e_i + x_j^0 \\
&\le \frac{1}{k} w_1 + \frac{1}{k^2} w_2 + \sum_{i=1}^{t} w(v_i) + \sum_{V(e) \subseteq V_1} w(e) \\
&\le \frac{1}{k} w_1 + \frac{1}{k^2} w_2 + \lambda(t + e(V_1)).
\end{aligned}$$

Clearly, changing the color of u_i affects $x_j := x_j^{n-t}$ by at most $d(u_i)\lambda + w(u_i) \le (d(u_i)+1)\lambda$. So by Lemma 1.4.1,

$$\begin{aligned}
\mathbb{P}\left(x_j > \mathbb{E}(x_j) + z\right) &\le \exp\left(-\frac{2z^2}{\lambda^2 \sum_{i=1}^{n-t} (d(u_i)+1)^2}\right) \\
&\le \exp\left(-\frac{2z^2}{\lambda^2 \sum_{i=1}^{n-t} (d(u_i)+1)(d(v_{t+1})+1)}\right) \\
&< \exp\left(-\frac{2z^2}{\lambda^2 (2m+n-1)\frac{4}{c} m^{1-\alpha}}\right) \\
&\le \exp\left(-\frac{cz^2}{6\lambda^2 m^{2-\alpha}}\right).
\end{aligned}$$

Let $z = \lambda \left(\frac{6 \ln k}{c}\right)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}$. Then

$$\mathbb{P}\left(x_j > \mathbb{E}(x_j) + z\right) < \exp(-\ln k) = \frac{1}{k}.$$

So there exists a partition $V(G) = X_1 \cup X_2 \cup \cdots \cup X_k$, where $X_j := X_j^{n-t}$, such that for $1 \le j \le k$,

$$\begin{aligned}
x_j &\le \mathbb{E}(x_j) + z \\
&\le \frac{1}{k} w_1 + \frac{1}{k^2} w_2 + \lambda\,(t + e(V_1)) + z \\
&\le \frac{1}{k} w_1 + \frac{1}{k^2} w_2 + \lambda \cdot o(m).
\end{aligned}$$

The $o(m)$ term in the expression is

$$cm^\alpha + \frac{1}{2} c^2 m^{2\alpha} + \left(\frac{6 \ln k}{c}\right)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}.$$

Picking $\alpha = \frac{2}{5}$ to minimize $\max\{2\alpha, 1-\alpha/2\}$, the $o(m)$ term becomes $O(m^{\frac{4}{5}})$. ∎

For a hypergraph G whose edges are of size 1 or 2, we may view G as a weighted graph with weight function w such that $w(e) = 1$ for all $e \in E(G)$ with $|V(e)| = 2$, $w(v) = 1$ for all $v \in V(G)$ with $\{v\} \in E(G)$, and $w(v) = 0$ for $v \in V(G)$ with $\{v\} \notin E(G)$. Then Theorem 2.3.4 gives the following result, implying Theorem 1.3.4 and establishing Conjecture 2.3.1 raised by Bollobás and Scott [12].

Theorem 2.3.5. *Let G be a hypergraph with m_i edges of size i, $i = 1, 2$. Then for any integer $k \geq 1$, there is a k-partition $V_1, \ldots, V_k$ of $V(G)$ such that for $i = 1, \ldots, k$,*

$$e(V_i) \leq \frac{m_1}{k} + \frac{m_2}{k^2} + O(m_2^{4/5}).$$

Note that the term $m_1/k + m_2/k^2$ is the expected value of $e(V_i)$ if $V_1, ..., V_k$ is a random k-partition. Bollobás and Scott ask in [12] whether it is possible to replace $O(m_2^{4/5})$ in Theorem 2.3.5 with $O(\sqrt{m_1 + m_2})$. This is still open.

CHAPTER III

BOUNDS FOR PAIRS IN PARTITIONS OF GRAPHS

In this chapter we study Problem 1.3.5, Conjecture 1.3.7 and Conjecture 1.3.9. Recall $f(k,m)$ in Problem 1.3.5.

In Section 3.1, we show that $f(k,m) < 1.6m/k + o(m)$, and that $f(k,m) < 1.5m/k + o(m)$ for $k \geq 23$. In Section 3.2, we prove $f(k,m) \leq 4m/k^2 + o(m)$ for dense graphs, which confirms Conjecture 1.3.7 for such graphs, and we establish Conjecture 1.3.9 for graphs with $\Omega(k^{12}(\ln k)^3)$ edges.

In Section 3.3, we show $f(4,m) \leq m/3 + o(m)$ and $f(5,m) \leq 4m/15 + o(m)$, which imply Conjecture 1.3.7 for $k = 4$ and $k = 5$. In Section 3.4, we study the problem raised by Bollobás and Scott [12] that for any graph G with m edges, whether it is possible to find a k-partition $V_1, \ldots, V_k$ of $V(G)$ such that

$$e(V_i) \leq \frac{m}{k^2} + \frac{k-1}{2k^2}(\sqrt{2m + 1/4} - 1/2)$$

for $1 \leq i \leq k$, and

$$e(V_i \cup V_j) \leq \frac{12m}{(k+1)(k+2)} + O(n)$$

for $1 \leq i < j \leq k$. We show that for $k = 3$ and $k = 4$ one can find a partition satisfying these bounds asymptotically.

3.1 A general bound

In this section, we prove a bound on $f(k,m)$ in Problem 1.3.5. We need a simple lemma which will also be used in Section 3.3 for finding probabilities when dealing with 4-partitions.

Lemma 3.1.1. *Let $a_j \geq 0$ for $j \in \{1,2,3,4\}$ such that $\alpha := \sum_{j=1}^{4} a_j > 0$, and let $f_{ij}(x_i, x_j) = (a_i + a_j)(x_i + x_j)$ for $1 \leq i \neq j \leq 4$. Then there exist $p_i \in [0, 1/2]$, $1 \leq i \leq 4$, such that $\sum_{i=1}^{4} p_i = 1$ and, for $1 \leq i \neq j \leq 4$, $f_{ij}(p_i, p_j) \leq \alpha/3$.*

Proof. First, assume $a_i \leq \alpha/2$ for all $1 \leq i \leq 4$. Then $p_i := 1/2 - a_i/\alpha \in [0, \frac{1}{2}]$, and

$$f_{ij}(p_i, p_j) = (a_i + a_j)\left(1 - \frac{a_i + a_j}{\alpha}\right) = -\frac{1}{\alpha}\left(a_i + a_j - \frac{\alpha}{2}\right)^2 + \frac{\alpha}{4} \leq \frac{\alpha}{4}.$$

So we may assume without loss of generality that $a_4 > \alpha/2$. Then $a_i + a_j \leq \alpha/2$ for all $1 \leq i \neq j \leq 3$. Let $p_1 = p_2 = p_3 = 1/3$ and $p_4 = 0$. Then for $1 \leq i \leq 3$, $f_{i4} = (a_i + a_4)/3 \leq \alpha/3$; and for $1 \leq i \neq j \leq 3$, $f_{ij} = (a_i + a_j)(2/3) \leq (\alpha/2)(2/3) = \alpha/3$. ∎

Remark. From the above proof, we see that among the p_i satisfying the assertion of Lemma 3.1.1, we may choose $p_i = 0$ when $a_i > \alpha/2$, and $p_i \leq \max\{1/2 - a_i/\alpha, 1/3\}$ when $a_i \leq \alpha/2$.

We need another lemma.

Lemma 3.1.2. *Let $h_4 = 1/3$. There exist t_k, h_k for $k \geq 5$ such that*

$$h_k = \frac{2 - 2t_k}{k - 2t_k}, \text{ and}$$
$$\frac{2 - 2t_k}{k - 2t_k} = \frac{k-3}{k}h_{k-1} + \left(\frac{h_{k-1}}{k} + \frac{4}{k(k-1)}\right)2t_k.$$

Moreover, $h_k < 1.6/k$, and $h_k < 1.5/k$ for $k \geq 23$.

Proof. We first show that there exist $t_k \in (0, 1/2)$ and $h_k \in (1/(k-1), 2/k)$, $k \geq 5$, such that

$$h_k = \frac{2 - 2t_k}{k - 2t_k}, \text{ and}$$
$$\frac{2 - 2t_k}{k - 2t_k} = \frac{k-3}{k}h_{k-1} + \left(\frac{h_{k-1}}{k} + \frac{4}{k(k-1)}\right)2t_k.$$

Suppose $k \geq 5$. Let

$$f_k(t) = \frac{2 - 2t}{k - 2t}$$

and

$$g_k(t) = \frac{k-3}{k}h_{k-1} + \left(\frac{h_{k-1}}{k} + \frac{4}{k(k-1)}\right)2t.$$

It is easy to see that $f_k(t)$ is decreasing, and $g_k(t)$ is increasing. Now assume that $\frac{1}{k-2} \leq h_{k-1} < \frac{2}{k-1}$ for some $k \geq 5$. Note that

$$g_k(0) = \frac{k-3}{k}h_{k-1} < \frac{k-3}{k}\frac{2}{k-1} < \frac{2}{k} = f_k(0),$$

and

$$g_k(1/2) = \frac{k-2}{k}h_{k-1} + \frac{4}{k(k-1)} \geq \frac{1}{k} + \frac{4}{k(k-1)} > \frac{1}{k-1} = f_k(1/2).$$

Therefore, since $f_k(t)$ is decreasing and $g_k(t)$ is increasing and because both are continuous over $[0, 1/2]$, there exists $t_k \in (0, 1/2)$, for each $k \geq 5$, such that $f_k(t_k) = g_k(t_k)$. Let $h_k := f_k(t_k) = \frac{2-2t_k}{k-2t_k}$. Then since $t_k \in (0, 1/2)$, $1/(k-1) < h_k < 2/k$ for $k \geq 5$.

Next, we show that $h_k < 1.6/k$, and $h_k < 1.5/k$ for $k \geq 23$. Let $h_k = c_k/k$, and it suffices to show $c_k < 1.6$, and $c_k < 1.5 = 3/2$ for $k \geq 23$. Since $h_k \in (1/(k-1), 2/k)$, $c_k \in (1, 2)$. Note that

$$c_k = \frac{2-2t_k}{k-2t_k}k = (k-3)h_{k-1} + \left(h_{k-1} + \frac{4}{k-1}\right)2t_k = \frac{k-3}{k-1}c_{k-1} + \frac{4+c_{k-1}}{k-1}2t_k.$$

From $c_k = \frac{2-2t_k}{k-2t_k}k$ we deduce $t_k = \frac{2k-kc_k}{2k-2c_k}$; and so

$$c_k = \frac{k-3}{k-1}c_{k-1} + \frac{(4+c_{k-1})(2k-kc_k)}{(k-1)(k-c_k)}.$$

With $h_4 = 1/3$ (and hence $c_4 = 4/3$) and using *MATLAB*, we have $c_k < 1.6$ for $k = 5, \ldots, 22$, and $c_{23} \approx 1.4962 < 3/2$. Now assume $k \geq 24$ and $c_{k-1} < 3/2$. Then

$$c_k < \frac{k-3}{k-1} \times \frac{3}{2} + \frac{(4+3/2)(2k-kc_k)}{(k-1)(k-c_k)},$$

and so

$$2(k-1)c_k < 3(k-3) + 11(2-c_k) + 11(2-c_k)c_k/(k-c_k).$$

Hence, since $c_k \in (1, 2)$,

$$(2k+9)c_k < 3k + 13 + \frac{11(2-c_k)c_k}{k-c_k} = 3k + 13 + \frac{11(1-(1-c_k)^2)}{k-c_k} < 3k + 13 + 11/(k-2).$$

Therefore,

$$c_k < \frac{3k+13}{2k+9} + \frac{11}{(2k+9)(k-2)} \leq 3/2.$$

The last inequality holds since we assume $k \geq 24$. ∎

We can now prove the main lemma of this section for k-partitions.

Lemma 3.1.3. *Let $k \geq 4$ be an integer, let $a_j \geq 0$ for $j \in \{1, \dots, k\}$ such that $\alpha := \sum_{j=1}^{k} a_j > 0$, and let $f_{ij}(x_i, x_j) = (a_i + a_j)(x_i + x_j)$ for $1 \leq i \neq j \leq k$. Then there exist $p_i \in [0, 2/k]$, $1 \leq i \leq k$, such that $\sum_{i=1}^{k} p_i = 1$ and, for $1 \leq i \neq j \leq k$, $f_{ij}(p_i, p_j) \leq h_k\alpha$, where $h_k < 1.6/k$, and $h_k < 1.5/k$ for $k \geq 23$.*

Proof. We apply induction on k; the case $k = 4$ follows from Lemma 3.1.1 (as $h_4 = 1/3$). Suppose $k \geq 5$. By Lemma 3.1.2 and since $h_4 = 1/3$, there exist $t_k \in (0, 1/2), h_k \in (1/(k-1), 2/k)$ for $k \geq 5$ such that

$$h_k = \frac{2 - 2t_k}{k - 2t_k} = \frac{k-3}{k}h_{k-1} + \left(\frac{h_{k-1}}{k} + \frac{4}{k(k-1)}\right)2t_k,$$

$h_k < 1.6/k$, and $h_k < 1.5/k$ for $k \geq 23$.

First, assume that there exists some $l \in \{1, \dots, k\}$ such that $a_l \geq t_k\alpha$, say $l = k$. Let $p_i = x$ (x will be determined later) for $1 \leq i < k$, with $0 \leq x \leq \frac{1}{k-1}$, and let $p_k = 1 - (k-1)x$. Then $\sum_{i=1}^{k} p_i = 1$; for $1 \leq i \leq k - 1$,

$$f_{ik}(p_i, p_k) \leq (1 - (k-2)x)\alpha;$$

and for $1 \leq i \neq j \leq k - 1$,

$$f_{ij}(p_i, p_j) \leq 2x(a_i + a_j) \leq 2x(\alpha - a_k) \leq (1 - t_k)2x\alpha.$$

We wish to minimize $\max\{1 - (k-2)x, (1 - t_k)2x\}$. Setting $1 - (k-2)x = (1 - t_k)2x$, we have

$$x = \frac{1}{k - 2t_k}$$

and, for $1 \leq i \neq j \leq k$,

$$f_{ij}(p_i, p_j) \leq \frac{2 - 2t_k}{k - 2t_k}\alpha.$$

We point out that since $t_k \in (0, 1/2)$, indeed $x \in (0, 1/(k-1)]$ and so x is well-defined. Note that $p_i \in [0, 2/k]$ for $1 \leq i \leq k$.

Second, let us assume that $a_i \le t_k\alpha$ for all $1 \le i \le k$. By the induction hypothesis, for any $l \in \{1,\ldots,k\}$ there exist $p_i^l \in [0, 2/(k-1)]$, $i \in \{1,\ldots,k\}\setminus\{l\}$, such that $\sum_{i\in\{1,\ldots,k\}\setminus\{l\}} p_i^l = 1$ and for any $\{i, j\} \subseteq \{1,\ldots,k\} \setminus \{l\}$,

$$(a_i + a_j)(p_i^l + p_j^l) \le h_{k-1}(\alpha - a_l).$$

For $1 \le i \le k$, let

$$p_i = \frac{1}{k} \sum_{l\in\{1,\ldots,k\}\setminus\{i\}} p_i^l.$$

Since $p_i^l \le 2/(k-1)$ for $i \in \{1,\ldots,k\} \setminus \{l\}$, we have $p_i \in [0, 2/k]$ for $1 \le i \le k$. Also,

$$\sum_{i=1}^{k} p_i = \frac{1}{k}\sum_{i=1}^{k} \sum_{l\in\{1,\ldots,k\}\setminus\{i\}} p_i^l = \frac{1}{k}\sum_{l=1}^{k} \sum_{i\in\{1,\ldots,k\}\setminus\{l\}} p_i^l = \frac{1}{k}\sum_{l=1}^{k} 1 = 1.$$

Moreover, for $1 \le i \ne j \le k$,

$$\begin{aligned}
f_{ij}(p_i, p_j) &= (a_i + a_j)(p_i + p_j) \\
&= \frac{1}{k}(a_i + a_j)\left(\sum_{l\in\{1,\ldots,k\}\setminus\{i\}} p_i^l + \sum_{l\in\{1,\ldots,k\}\setminus\{j\}} p_j^l\right) \\
&= \frac{1}{k}\left(\sum_{l\in\{1,\ldots,k\}\setminus\{i,j\}} (a_i + a_j)(p_i^l + p_j^l)\right) + \frac{1}{k}(a_i + a_j)(p_i^j + p_j^i) \\
&\le \frac{h_{k-1}}{k} \sum_{l\in\{1,\ldots,k\}\setminus\{i,j\}} (\alpha - a_l) + \frac{1}{k}(a_i + a_j)(p_i^j + p_j^i) \\
&\le \frac{h_{k-1}}{k}\Big((k-3)\alpha + a_i + a_j\Big) + \frac{4}{k(k-1)}(a_i + a_j) \\
&\le \left(\frac{k-3}{k} h_{k-1} + \left(\frac{h_{k-1}}{k} + \frac{4}{k(k-1)}\right) 2t_k\right)\alpha.
\end{aligned}$$

Note that

$$h_k = \frac{2-2t_k}{k-2t_k} = \frac{k-3}{k} h_{k-1} + \left(\frac{h_{k-1}}{k} + \frac{4}{k(k-1)}\right) 2t_k,$$

$h_k < 1.6/k$, and $h_k < 1.5/k$ for $k \ge 23$. This completes the proof of the lemma. ∎

Theorem 3.1.4. *Let $k \ge 4$ be an integer. Then $f(k, m) \le h_k m + O(m^{4/5})$, where $h_k < 1.6/k$, and $h_k < 1.5/k$ for $k \ge 23$.*

Proof. Let G be a graph with m edges, and we may assume that G is connected (as otherwise we simply consider individual components). Let $V(G) = \{v_1, \ldots, v_n\}$ such that $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$. Let $V_1 = \{v_1, \ldots, v_t\}$ with $t = \lfloor m^\alpha \rfloor$, where $0 < \alpha < 1/2$ and will be optimized later. Then $t < n$ since $m < n^2/2$. Moreover,

$$e(V_1) < t^2/2 \leq \frac{1}{2}m^{2\alpha} \qquad \text{and} \qquad d(v_{t+1}) < 2m^{1-\alpha},$$

since $(t+1)d(v_{t+1}) \leq \sum_{i=1}^{t+1} d(v_i) \leq 2m$.

Label the vertices in $V_2 := V(G)\backslash V_1$ as $u_1, \ldots, u_{n-t}$ such that $e(u_i, V_1 \cup \{u_1, \ldots, u_{i-1}\}) > 0$ for $i = 1, \ldots, n-t$. Note that this can be done since G is connected.

Fix an arbitrary k-partition $V_1 = \bigcup_{i=1}^k Y_i$, and assign each member of Y_i the color i, $1 \leq i \leq k$. Extend this coloring to $V(G)$ such that each vertex $u_i \in V_2$ is independently assigned the color j with probability p_j^i, where $\sum_{j=1}^k p_j^i = 1$ and p_j^i will be determined later. Let Z_i denote the indicator random variable of the event of coloring u_i. Hence $Z_i = j$ iff u_i is assigned the color j.

Let $G_i = G[V_1 \cup \{u_1, \cdots, u_i\}]$ for $i = 1, \ldots, n-t$, and let $G_0 = G[V_1]$. Let $X_j^0 = Y_j$ for $1 \leq j \leq k$, and $x_{jl}^0 = e(X_j^0 \cup X_l^0)$ for $1 \leq j \neq l \leq k$. For $i = 1, \ldots, n-t$ and $1 \leq j, l \leq k$, define

$$\begin{aligned}
&X_j^i := \{\text{vertices of } G_i \text{ with color } j\},\\
&x_{jl}^i := e(X_j^i \cup X_l^i),\\
&\Delta x_{jl}^i := x_{jl}^i - x_{jl}^{i-1},\\
&b_j^i := e(u_i, X_j^{i-1}).
\end{aligned}$$

Note that b_j^i depends on $(Z_1, \ldots, Z_{i-1})$ only. Hence for $1 \leq i \leq n-t$ and $1 \leq j \neq l \leq k$,

$$\mathbb{E}(\Delta x_{jl}^i | \, Z_1, \ldots, Z_{i-1}) = (b_j^i + b_l^i)(p_j^i + p_l^i),$$

and so

$$\mathbb{E}(\Delta x_{jl}^i) = (a_j^i + a_l^i)(p_j^i + p_l^i),$$

where

$$a_j^i = \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1,\ldots,Z_{i-1}) b_j^i.$$

Since b_j^i is determined by $(Z_1,\ldots,Z_{i-1})$, a_j^i is determined by p_j^s, $1 \le j \le k$ and $1 \le s \le i-1$. Note that $\sum_{j=1}^{k} b_j^i = e(u_i, G_{i-1}) > 0$, and that $e(u_i, G_{i-1})$ is independent of $Z_1,\ldots,Z_{n-t}$. Moreover,

$$\begin{aligned}
\sum_{j=1}^{k} a_j^i &= \sum_{j=1}^{k} \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1,\ldots,Z_{i-1}) b_j^i \\
&= \sum_{(Z_1,\ldots,Z_{i-1})} \left(\mathbb{P}(Z_1,\ldots,Z_{i-1}) \sum_{j=1}^{k} b_j^i \right) \\
&= \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1,\ldots,Z_{i-1}) e(u_i, G_{i-1}) \\
&= e(u_i, G_{i-1}) \\
&> 0.
\end{aligned}$$

So by Lemma 3.1.3, there exist $p_j^i \in [0,1]$, $1 \le j \le k$, such that $\sum_{j=1}^{k} p_j^i = 1$ and, for $1 \le i \le n-t$ and $1 \le j \ne l \le k$,

$$\mathbb{E}(\Delta x_{jl}^i) \le h_k \sum_{j=1}^{k} a_j^i = h_k e(u_i, G_{i-1}),$$

where $h_k < 1.6/k$, and $h_k < 1.5/k$ for $k \ge 23$.

Note that p_j^i is determined by a_j^i, $1 \le i \le k$; and hence p_j^i is recursively determined by p_j^s, $1 \le j \le k$ and $1 \le s \le i-1$. Also note that $m = e(G_0) + \sum_{i=1}^{n-t} e(u_i, G_{i-1})$. Now

$$\begin{aligned}
\mathbb{E}(x_{jl}^{n-t}) &= \sum_{i=1}^{n-t} \mathbb{E}(\Delta x_{jl}^i) + \mathbb{E}(x_{jl}^0) \\
&\le h_k \sum_{i=1}^{n-t} e(u_i, G_{i-1}) + x_{jl}^0 \\
&\le h_k m + e(V_1) \\
&\le h_k m + \frac{1}{2} m^{2\alpha}.
\end{aligned}$$

Clearly, changing the color of u_i (i.e., changing Z_i) affects $x_{jl} := x_{jl}^{n-t}$ by at most $d(u_i)$. So by Lemma 1.4.1,

$$\begin{aligned}\mathbb{P}\left(x_{jl} > \mathbb{E}(x_{jl}) + z\right) &\leq \exp\left(-\frac{z^2}{2\sum_{i=1}^{n-t} d(u_i)^2}\right) \\ &\leq \exp\left(-\frac{z^2}{2\sum_{i=1}^{n-t} d(u_i)d(v_{t+1})}\right) \\ &< \exp\left(-\frac{z^2}{4m2m^{1-\alpha}}\right) \\ &\leq \exp\left(-\frac{z^2}{8m^{2-\alpha}}\right).\end{aligned}$$

Let $z = (8\ln(k(k-1)/2))^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}$. Then for $1 \leq j \neq l \leq k$,

$$\mathbb{P}\left(x_{jl} > \mathbb{E}(x_{jl}) + z\right) < \exp(-\ln(k(k-1)/2)) = \frac{2}{k(k-1)}.$$

So there exists a partition $V(G) = \bigcup_{i=1}^k X_i$ such that for $1 \leq j \neq l \leq k$,

$$e(X_j \cup X_l) \leq \mathbb{E}(x_{jl}) + z \leq h_k m + \frac{1}{2}m^{2\alpha} + z \leq h_k m + o(m),$$

where the $o(m)$ term in the expression is

$$\frac{1}{2}m^{2\alpha} + (8\ln(k(k-1)/2))^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}.$$

Choosing $\alpha = \frac{2}{5}$ to minimize $\max\{2\alpha, 1-\alpha/2\}$, the $o(m)$ term becomes $O(m^{\frac{4}{5}})$. ∎

3.2 Dense graphs

We now prove Conjecture 1.3.7 for graphs with large minimum degree. The approach is similar to that for proving Theorem 3.1.4, but simpler because the large minimum degree condition helps to bound $e(V_1, V_2)$. Note that the term $4m/k^2$ in the theorem below is best possible (by simply taking a random k-partition). The following result implies Theorem 1.3.8.

Theorem 3.2.1. *Let $k \geq 2$ be an integer and let $\epsilon > 0$. If G is a graph with m edges and $\delta(G) \geq \epsilon n$, then there is a k-partition $V_1, \ldots, V_k$ of $V(G)$ such that for $1 \leq i \neq j \leq k$,*

$$e(V_i \cup V_j) \leq \frac{4}{k^2}m + \left(\sqrt{2/\epsilon} + \sqrt{8\ln\frac{k(k-1)}{2}}\right)m^{5/6}.$$

Proof. We may assume that G is connected (otherwise it suffices to consider individual components). Let $V(G) = \{v_1, ..., v_n\}$ such that $d(v_1) \geq d(v_2) \geq ... \geq d(v_n)$. Let $V_1 = \{v_1, ..., v_t\}$ with $t = \lfloor m^\alpha \rfloor$, where $0 < \alpha < 1/2$. Then

$$t < n - 1, \quad e(V_1) < m^{2\alpha}/2, \quad \text{and} \quad d(v_{t+1}) \leq 2m^{1-\alpha}.$$

Let $V_2 = V(G) \backslash V_1 = \{u_1, ..., u_{n-t}\}$ such that $e(u_i, V_1 \cup \{u_1, ..., u_{i-1}\}) > 0$ for $i = 1, ..., n - t$.

Now assume $\delta(G) \geq \epsilon n$. Then $2m = \sum_{v \in V(G)} d(v) \geq \epsilon n^2$. So $n \leq \sqrt{2m/\epsilon}$. Thus,

$$e(V_1, V_2) + 2e(V_1) = \sum_{i=1}^{t} d(v_i) < tn \leq m^\alpha \sqrt{2m/\epsilon} = \sqrt{2/\epsilon}\, m^{1/2+\alpha}.$$

Fix an arbitrary partition $V_1 = Y_1 \cup Y_2 \cup ... \cup Y_k$ and, for each $i \in \{1, ..., k\}$, assign the color i to all vertices in Y_i. We extend this coloring to $V(G)$ by independently assigning the color j (for each $j \in \{1, ..., k\}$) to each vertex $u_i \in V_2$ with probability $1/k$. Let Z_i denote the indicator random variable of the event of coloring u_i.

Let X_i be the set of vertices of G with color i. Then $Y_i \subseteq X_i$ for $1 \leq i \leq k$; and for $1 \leq i \neq j \leq k$,

$$\begin{aligned}
\mathbb{E}(e(X_i \cup X_j)) &= \mathbb{E}(e((X_i \cup X_j) \cap V_2)) + \mathbb{E}(e((X_i \cup X_j) \cap V_2, Y_i \cup Y_j)) + e(Y_i \cup Y_j) \\
&\leq (2/k)^2 e(V_2) + e(V_1, V_2) + e(V_1) \\
&\leq \frac{4}{k^2} m + \sqrt{2/\epsilon}\, m^{1/2+\alpha}.
\end{aligned}$$

Clearly, changing the color of u_i (i.e., changing Z_i) affects $e(X_i \cup X_j)$ by at most $d(u_i)$. Then as in the proof of Theorem 3.1.4, we apply Lemma 1.4.1 to conclude that for any $1 \leq i \neq j \leq k$,

$$\mathbb{P}\left(e(X_i \cup X_j) > \mathbb{E}(e(X_i \cup X_j)) + z\right) \leq \exp\left(-\frac{z^2}{2\sum_{i=1}^{n-t} d(u_i)^2}\right) < \exp\left(-\frac{z^2}{8m^{2-\alpha}}\right).$$

Let $z = \sqrt{8\ln(k(k-1)/2)}\, m^{1-\alpha/2}$. Then for $1 \leq i \neq j \leq k$,

$$\mathbb{P}\left(e(X_i \cup X_j) > \mathbb{E}(e(X_i \cup X_j)) + z\right) < \exp\left(-\ln\frac{k(k-1)}{2}\right) = \frac{2}{k(k-1)}.$$

So there exists a partition $V(G) = X_1 \cup X_2 \cup ... \cup X_k$ such that, for $1 \le i \neq j \le k$,

$$\begin{aligned} e(X_i \cup X_j) &\le \frac{4}{k^2}m + \sqrt{2/\epsilon}m^{1/2+\alpha} + z \\ &\le \frac{4}{k^2}m + \sqrt{2/\epsilon}m^{1/2+\alpha} + \sqrt{8\ln(k(k-1)/2)}m^{1-\alpha/2} \end{aligned}$$

Picking $\alpha = 1/3$ to minimize $\max\{1/2 + \alpha, 1 - \alpha/2\}$, we have the desired bound. ∎

As a corollary, Conjecture 1.3.9 holds for graphs with $\Omega(k^{12}(\ln k)^3)$ edges. Hence Conjecture 1.3.7 holds for all graphs G with $\delta(G) \ge \epsilon n$, for any fixed $k \ge 2$ and $\epsilon > 0$.

3.3 Bounds for 4-partitions and 5-partitions

In this section, we prove Conjecture 1.3.7 for 4-partitions and 5-partitions. For 4-partitions, we use Lemma 3.1.1. For 5-partitions, we need the following lemma.

Lemma 3.3.1. *Let $a_j \ge 0$ for $j \in \{1, \dots, 5\}$ such that $\alpha := \sum_{j=1}^5 a_j > 0$, and let $f_{ij}(x_i, x_j) = (a_i + a_j)(x_i + x_j)$ for $1 \le i \neq j \le 5$. Then there exist $p_i \in [0, 2/5]$, $1 \le i \le 5$, such that $\sum_{i=1}^5 p_i = 1$ and, for $1 \le i \neq j \le 5$, $f_{ij}(p_i, p_j) \le 4\alpha/15$.*

Proof. If there exists some $l \in \{1, \dots, 5\}$ such that $a_l \ge 5\alpha/11$, then $a_i + a_j \le 6\alpha/11$ for $\{i, j\} \subseteq \{1, \dots, 5\} \setminus \{l\}$. Let $p_l = 1/45$ and let $p_i = 11/45$ for $i \in \{1, \dots, 5\} \setminus \{l\}$. Then for $i \in \{1, \dots, 5\} \setminus \{l\}$,

$$f_{il}(p_i, p_l) = (a_i + a_l)(p_i + p_l) \le \alpha\left(\frac{11}{45} + \frac{1}{45}\right) = \frac{4}{15}\alpha;$$

and for $\{i, j\} \subseteq \{1, \dots, 5\} \setminus \{l\}$,

$$f_{ij} = (a_i + a_j)(p_i + p_j) \le \frac{6\alpha}{11}\left(\frac{11}{45} + \frac{11}{45}\right) = \frac{4}{15}\alpha.$$

Therefore, we may assume that $a_i < 5\alpha/11$ for all $1 \le i \le 5$. By Lemma 3.1.1, for any $1 \le l \le 5$ there exist $p_i^l \in [0, 1/2]$, $i \in \{1, \dots, 5\} \setminus \{l\}$, such that $\sum_{i\in\{1,\dots,5\}\setminus\{l\}} p_i^l = 1$ and, for $\{i, j\} \subseteq \{1, \dots, 5\} \setminus \{l\}$,

$$(a_i + a_j)(p_i^l + p_j^l) \le \frac{1}{3}(\alpha - a_l).$$

Indeed, by the remark following Lemma 3.1.1, we may choose p_i^l, $i \in \{1, \dots, 5\} \setminus \{l\}$, such that $p_i^l = 0$ when $a_i > (\alpha - a_l)/2$, and $p_i^l \leq \max\{1/2 - a_i/(\alpha - a_l), 1/3\}$ when $a_i \leq (\alpha - a_l)/2$.

For $1 \leq i \leq 5$, let $p_i = \frac{1}{5} \sum_{l \in \{1,\dots,5\} \setminus \{i\}} p_i^l$. Then $p_i \in [0, 2/5]$, and

$$\sum_{i=1}^{5} p_i = \frac{1}{5} \sum_{i=1}^{5} \sum_{l \in \{1,\dots,5\} \setminus \{i\}} p_i^l = \frac{1}{5} \sum_{l=1}^{5} \sum_{i \in \{1,\dots,5\} \setminus \{l\}} p_i^l = \frac{1}{5} \sum_{l=1}^{5} 1 = 1.$$

So for $1 \leq i \neq j \leq 5$,

$$\begin{aligned}
f_{ij}(p_i, p_j) &= (a_i + a_j)(p_i + p_j) \\
&= \frac{1}{5}(a_i + a_j)\left(\sum_{l \in \{1,\dots,5\} \setminus \{i\}} p_i^l + \sum_{l \in \{1,\dots,5\} \setminus \{j\}} p_j^l \right) \\
&= \frac{1}{5}\left(\sum_{l \in \{1,\dots,5\} \setminus \{i,j\}} (a_i + a_j)(p_i^l + p_j^l) \right) + \frac{1}{5}(a_i + a_j)(p_i^j + p_j^i) \\
&\leq \frac{1}{15}\left(\sum_{l \in \{1,\dots,5\} \setminus \{i,j\}} (\alpha - a_l) \right) + \frac{1}{5}(a_i + a_j)(p_i^j + p_j^i) \\
&= \frac{1}{15}(2\alpha + a_i + a_j) + \frac{1}{5}(a_i + a_j)(p_i^j + p_j^i) \\
&= \frac{2}{15}\alpha + (a_i + a_j)\left(\frac{1}{15} + \frac{1}{5}(p_i^j + p_j^i) \right).
\end{aligned}$$

We need to show that $f_{ij}(p_i, p_j) \leq \frac{4}{15}\alpha$ for $1 \leq i \neq j \leq 5$.

If $a_i > (\alpha - a_j)/2$ and $a_j > (\alpha - a_i)/2$, then $p_i^j = p_j^i = 0$, and hence

$$f_{ij}(p_i, p_j) \leq \frac{3}{15}\alpha < \frac{4}{15}\alpha.$$

Now assume $a_i > (\alpha - a_j)/2$ and $a_j \leq (\alpha - a_i)/2$. Then $p_i^j = 0$ and $p_j^i \leq \max\{1/2 - a_j/(\alpha - a_i), 1/3\}$. Suppose $1/2 - a_j/(\alpha - a_i) > 1/3$. Then $a_j < (\alpha - a_i)/6$; and hence, since $a_i > (\alpha - a_j)/2$, we have $a_i > (\alpha - \alpha/6 + a_i/6)/2$. Solving this inequality for a_i, we have $a_i > 5\alpha/11$ which contradicts our assumption. Therefore, $1/2 - a_j/(\alpha - a_i) \leq 1/3$, and so $p_j^i \leq 1/3$. Hence

$$f_{ij}(p_i, p_j) \leq \frac{2}{15}\alpha + (a_i + a_j)\left(\frac{1}{15} + \frac{1}{5}\frac{1}{3} \right) \leq \frac{4}{15}\alpha.$$

By symmetry, if $a_j > (\alpha - a_i)/2$ and $a_i \leq (\alpha - a_j)/2$, then $f_{ij}(p_i, p_j) \leq \frac{4}{15}\alpha$.

So we are left with the case when $a_i \le (\alpha - a_j)/2$ and $a_j \le (\alpha - a_i)/2$. Then $a_i + a_j \le \alpha - (a_i + a_j)/2$, and so $a_i + a_j \le 2\alpha/3$. Moreover, $p_i^j \le \max\{1/2 - a_i/(\alpha - a_j), 1/3\}$ and $p_j^i \le \max\{1/2 - a_j/(\alpha - a_i), 1/3\}$.

If $1/2 - a_i/(\alpha - a_j) > 1/3$ and $1/2 - a_j/(\alpha - a_i) > 1/3$, then $6a_i + a_j < \alpha$ and $6a_j + a_i < \alpha$. Hence $a_i + a_j < 2\alpha/7$, and so (since $p_i^j \le 1/2$ and $p_j^i \le 1/2$),

$$f_{ij}(p_i, p_j) \le \frac{2}{15}\alpha + (a_i + a_j)\left(\frac{1}{15} + \frac{1}{5}\left(\frac{1}{2} + \frac{1}{2}\right)\right) < \frac{2}{15}\alpha + \frac{2}{7}\frac{4}{15}\alpha < \frac{4}{15}\alpha.$$

If $1/2 - a_i/(\alpha - a_j) > 1/3$ and $1/2 - a_j/(\alpha - a_i) \le 1/3$, then $6a_i + a_j \le \alpha$ and $p_j^i \le 1/3$. Since $a_j \le (\alpha - a_i)/2$, $a_i + 2a_j \le \alpha$. So $11(a_i + a_j) = 6a_i + a_j + 5(a_i + 2a_j) \le 6\alpha$, and hence $a_i + a_j \le 6\alpha/11$. Then

$$f_{ij}(p_i, p_j) \le \frac{2}{15}\alpha + (a_i + a_j)\left(\frac{1}{15} + \frac{1}{5}\left(\frac{1}{2} + \frac{1}{3}\right)\right) \le \frac{2}{15}\alpha + \frac{6}{11}\frac{7}{30}\alpha < \frac{4}{15}\alpha.$$

The case when $1/2 - a_i/(\alpha - a_j) \le 1/3$ and $1/2 - a_j/(\alpha - a_i) > 1/3$ is symmetric.

Therefore, we may assume that $1/2 - a_i/(\alpha - a_j) \le 1/3$ and $1/2 - a_j/(\alpha - a_i) \le 1/3$. Then $p_i^j \le 1/3$ and $p_j^i \le 1/3$. Recall that $a_i + a_j \le 2\alpha/3$. Hence

$$f_{ij}(p_i, p_j) \le \frac{2}{15}\alpha + (a_i + a_j)\left(\frac{1}{15} + \frac{1}{5}\left(\frac{1}{3} + \frac{1}{3}\right)\right) \le \frac{2}{15}\alpha + \frac{2}{3}\frac{1}{5}\alpha = \frac{4}{15}\alpha.$$

∎

Using the same proof of Theorem 3.1.4, with Lemma 3.1.1 and Lemma 3.3.1 in place of Lemma 3.1.3, we have the following results on 4-partitions and 5-partitions.

Theorem 3.3.2. $f(4, m) \le m/3 + O(m^{4/5})$.

Theorem 3.3.3. $f(5, m) \le 4m/15 + O(m^{4/5})$.

Recall that the graphs $K_{1,n}$ give $f(4, m) \ge m/3$ and $f(5, m) \ge m/4$. When $k = 4$, $12/((k + 2)(k + 1)) = 3/5 > 1/3$. So as a consequence of Theorem 3.3.2, Conjecture 1.3.7 holds for $k = 4$ asymptotically. When $k = 5$, $12m/((k + 2)(k + 1)) = 2/7 > 4/15$. Hence, Theorem 3.3.3 establishes Conjecture 1.3.7 for $k = 5$ asymptotically.

3.4 Simultaneous bounds for 3-partitions and 4-partitions

In this section, we study the following problem suggested by Bollobás and Scott [12].

Problem 3.4.1. *For any integer $k \geq 2$ and for any graph G with m edges and n vertices, is it possible to find a k-partition $V_1, \ldots, V_k$ of $V(G)$ such that for $1 \leq i \leq k$,*

$$e(V_i) \leq \frac{m}{k^2} + \frac{k-1}{2k^2}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right),$$

and for $1 \leq i < j \leq k$,

$$e(V_i \cup V_j) \leq \frac{12m}{(k+1)(k+2)} + O(n)?$$

Recall that Bollobás and Scott [10] showed the existence of a k-partition satisfying the above bound on $e(V_i)$, and K_{kn+1} are the only extremal graphs. Also recall that the bound on $e(V_i \cup V_j)$ is best possible for K_{k+2}.

We show that for $k = 3$ and $k = 4$, one can find partitions that satisfy these bounds asymptotically. For large k, a similar approach as in the proofs of Lemma 3.1.3 and Theorem 3.1.4 may be used to give some bounds.

Note that in the proofs to follow, we will use the fact that the maximum of $x(a - x)$, $a > 0$, is $a^2/4$.

Lemma 3.4.2. *Let $a_j \geq 0$ for $j = 1, 2, 3$ such that $\alpha := a_1 + a_2 + a_3 > 0$, let $f_{ij}(x_i, x_j) = (a_i + a_j)(x_i + x_j)$ for $1 \leq i \neq j \leq 3$, and let $g_i(x_i) = a_i x_i$ for $1 \leq i \leq 3$. Then there exist $p_i \in [0, 2/3]$, $1 \leq i \leq 3$, such that $\sum_{i=1}^{3} p_i = 1$, $f_{ij}(p_i, p_j) \leq 5\alpha/9$ for $1 \leq i \neq j \leq 3$, and $g_i(p_i) \leq \alpha/9$ for $1 \leq i \leq 3$.*

Proof. First, assume that $a_i < 2\alpha/3$ for all $i = 1, 2, 3$. Let $p_i = 2/3 - a_i/\alpha$. Then $p_i \in [0, 2/3]$, $i = 1, 2, 3$, and $p_1 + p_2 + p_3 = 1$. Moreover, for $1 \leq i \neq j \leq 3$,

$$f_{ij}(p_i, p_j) = \frac{a_i + a_j}{\alpha}\left(\frac{4}{3} - \frac{a_i + a_j}{\alpha}\right)\alpha \leq \frac{4}{9}\alpha < \frac{5}{9}\alpha$$

and, for $i = 1, 2, 3$,

$$g_i(p_i) = \frac{a_i}{\alpha}\left(\frac{2}{3} - \frac{a_i}{\alpha}\right)\alpha \leq \frac{1}{9}\alpha.$$

Next assume that some $a_i > 5\alpha/6$, say $a_3 > 5\alpha/6$. So $a_1 + a_2 \le \alpha/6$. We choose $p_1 = p_2 = 4/9$ and $p_3 = 1/9$. Then $f_{12}(p_1, p_2) < \alpha/6 < 5\alpha/9$; $f_{i3}(p_i, p_3) \le 5\alpha/9$ for $i = 1, 2$; $g_3(p_3) \le \alpha/9$; and $g_i(p_i) \le (\alpha/6)(4/9) = 2\alpha/27 < \alpha/9$ for $i = 1, 2$.

Therefore, we may assume that there exists some a_i, say a_3, such that $2\alpha/3 \le a_3 \le 5\alpha/6$. Then $\alpha/6 \le a_1 + a_2 \le \alpha/3$. Let $p_3 = 0$ and $p_i = 2/3 - a_i/(3(a_1 + a_2))$ for $i = 1, 2$. Then $p_i \in [0, 2/3]$ and $p_1 + p_2 + p_3 = 1$.

Clearly, $g_3(p_3) = 0$ and, for $i = 1, 2$,

$$g_i(p_i) = \frac{a_i}{3(a_1 + a_2)}\left(\frac{2}{3} - \frac{a_i}{3(a_1 + a_2)}\right)3(a_1 + a_2) \le \frac{3}{9}(a_1 + a_2) \le \frac{1}{9}\alpha.$$

Note that $f_{12}(p_1, p_2) = a_1 + a_2 \le \alpha/3 < 5\alpha/9$. So it remains to show that $f_{13}(p_1, p_3) \le 5\alpha/9$ and $f_{23}(p_2, p_3) \le 5\alpha/9$. By symmetry we only need to prove $f_{13}(p_1, p_3) \le 5\alpha/9$.

Note that $f_{13}(p_1, p_3) = (a_1 + a_3)(2/3 - a_1/(3(\alpha - a_3)))$, which may be viewed as a function of a_1, a_3 (while fixing α). We look for the maximal value of $h(a_1, a_3) := f_{13}(p_1, p_3)$ subject to $2\alpha/3 \le a_1 + a_3 \le \alpha$ and $2\alpha/3 \le a_3 \le 5\alpha/6$. Taking partial derivatives and setting them to 0, we have

$$\frac{\partial h}{\partial a_1} = \frac{2}{3} - \frac{a_1}{3(\alpha - a_3)} - \frac{a_1 + a_3}{3(\alpha - a_3)} = 0,$$

and

$$\frac{\partial h}{\partial a_3} = \frac{2}{3} - \frac{a_1}{3(\alpha - a_3)} - \frac{1}{3}a_1\frac{a_1 + a_3}{(\alpha - a_3)^2} = 0.$$

Then $a_1/(\alpha - a_3) = 1$ (from $\frac{\partial h}{\partial a_1} = \frac{\partial h}{\partial a_3}$), and hence $a_3 = 0$ (from $\frac{\partial h}{\partial a_1} = 0$), a contradiction. So the maximal value of h occurs on the boundary of the region defined by $2\alpha/3 \le a_1 + a_3 \le \alpha$ and $2\alpha/3 \le a_3 \le 5\alpha/6$.

When $a_1 + a_3 = 2\alpha/3$, then $a_1 = 0$ and $a_3 = 2\alpha/3$, and hence $h = 4\alpha/9$. When $a_1 + a_3 = \alpha$ then $h = \alpha/3$. When $a_3 = 2\alpha/3$ then $h = (a_1 + 2\alpha/3)(2/3 - a_1/\alpha) = (2/3 + a_1/\alpha)(2/3 - a_1/\alpha)\alpha \le 4\alpha/9$. When $a_3 = 5\alpha/6$, then $h \le (a_1 + 5\alpha/6)(2/3 - 2a_1/\alpha) = (5/6 + a_1/\alpha)(2/3 - 2a_1/\alpha)\alpha \le 5\alpha/9$. Hence $f_{13}(p_1, p_3) \le 5\alpha/9$. ∎

The next lemma is for 4-partitions.

Lemma 3.4.3. *Let $a_j \geq 0$ for $j = 1, 2, 3, 4$ such that $\alpha := a_1 + a_2 + a_3 + a_4 > 0$, let $f_{ij}(x_i, x_j) = (a_i + a_j)(x_i + x_j)$ for $1 \leq i \neq j \leq 4$, and let $g_i(x_i) = a_i x_i$ for $1 \leq i \leq 4$. Then there exist $p_i \in [0, 1/2]$, $1 \leq i \leq 4$, such that $\sum_{i=1}^{4} p_i = 1$, $f_{ij}(p_i, p_j) \leq 2\alpha/5$ for $1 \leq i \neq j \leq 4$, and $g_i(p_i) \leq \alpha/16$ for $1 \leq i \leq 4$.*

Proof. First, suppose $a_i < \alpha/2$ for all $1 \leq i \leq 4$. Let $p_i = 1/2 - a_i/\alpha$. Then $p_i \in [0, 1/2]$ for $1 \leq i \leq 4$, and $\sum_{i=1}^{4} p_i = 1$. Moreover, for $1 \leq i \neq j \leq 4$,

$$f_{ij}(p_i, p_j) = \frac{a_i + a_j}{\alpha}\left(1 - \frac{a_i + a_j}{\alpha}\right)\alpha \leq \frac{1}{4}\alpha < \frac{2}{5}\alpha,$$

and for $1 \leq i \leq 4$,

$$g_i(p_i) = \frac{a_i}{\alpha}\left(\frac{1}{2} - \frac{a_i}{\alpha}\right)\alpha \leq \frac{1}{16}\alpha.$$

Now assume that some $a_i > 4\alpha/5$, say $a_4 > 4\alpha/5$. Then $a_1 + a_2 + a_3 \leq \alpha/5$. Let $p_1 = p_2 = p_3 = 5/16$ and $p_4 = 1/16$. Then for $i = 1, 2, 3$,

$$f_{i4}(p_i, p_4) \leq 6\alpha/16 < 2\alpha/5;$$

for $1 \leq i \neq j \leq 3$,

$$f_{ij}(p_i, p_j) \leq \alpha/5 < 2\alpha/5;$$

$g_4(p_4) \leq \alpha/16$; and for $i = 1, 2, 3$, $g_i(p_i) \leq (\alpha/5)(5/16) = \alpha/16$.

So we may assume that there exists some a_i, say a_4, such that $\alpha/2 \leq a_4 \leq 4\alpha/5$. Then $\alpha/5 \leq a_1 + a_2 + a_3 \leq \alpha/2$. Let $p_4 = 0$ and $p_i = 1/2 - a_i/(2(\alpha - a_4))$ for $i = 1, 2, 3$. Then $p_i \in [0, 1/2]$ and $\sum_{i=1}^{4} p_i = 1$.

Clearly, $g_4(p_4) = 0$. Note that $\alpha - a_4 \leq \alpha/2$. So for $i = 1, 2, 3$

$$g_i(p_i) = \frac{a_i}{2(\alpha - a_4)}\left(\frac{1}{2} - \frac{a_i}{2(\alpha - a_4)}\right)2(\alpha - a_4) \leq \frac{1}{16}\alpha;$$

and for $1 \leq i \neq j \leq 3$,

$$f_{ij}(p_i, p_j) = \frac{a_i + a_j}{2(\alpha - a_4)}\left(1 - \frac{a_i + a_j}{2(\alpha - a_4)}\right)2(\alpha - a_4) \leq \frac{1}{4}\alpha < \frac{2}{5}\alpha.$$

Thus it remains to prove $f_{i4}(p_i, p_4) \le 2\alpha/5$ for $i = 1, 2, 3$. By symmetry, we only prove $f_{14}(p_1, p_4) \le 2\alpha/5$. Note that $h(a_1, a_4) := f_{14}(p_1, p_4) = (a_1 + a_4)(1/2 - a_1/(2(\alpha - a_4)))$ may be viewed as a function of a_1, a_4 (while fixing α), and we look for its maximal value subject to $\alpha/2 \le a_1 + a_4 \le \alpha$ and $\alpha/2 \le a_4 \le 4\alpha/5$.

Taking partial derivatives and setting them to 0, we have

$$\frac{\partial h}{\partial a_1} = \frac{1}{2} - \frac{a_1}{2(\alpha - a_4)} - \frac{1}{2}\frac{a_1 + a_4}{\alpha - a_4} = 0,$$

and

$$\frac{\partial h}{\partial a_4} = \frac{1}{2} - \frac{a_1}{2(\alpha - a_4)} - \frac{1}{2}a_1\frac{a_1 + a_4}{(\alpha - a_4)^2} = 0.$$

Then $a_1/(\alpha - a_4) = 1$ (from $\frac{\partial h}{\partial a_1} = \frac{\partial h}{\partial a_4}$), and so $a_4 < 0$ (from $\frac{\partial h}{\partial a_1} = 0$), a contradiction. Thus, the maximal value of h occurs when $a_1 + a_4 \in \{\alpha/2, \alpha\}$ or $a_4 \in \{\alpha/2, 4\alpha/5\}$.

When $a_1 + a_4 = \alpha/2$, we have $a_1 = 0$ and $a_4 = \alpha/2$, and hence $h = \alpha/4$. When $a_1 + a_4 = \alpha$, then $h = 0$. When $a_4 = \alpha/2$ then $h = \alpha(1/2 + a_1/\alpha)(1/2 - a_1/\alpha) \le \alpha/4$. When $a_4 = 4\alpha/5$, then $h = \alpha(4/5 + a_1/\alpha)(1/2 - 5a_1/(2\alpha)) \le 2\alpha/5$. Hence $f_{14}(a_1, a_4) \le 2\alpha/5$. ∎

Now we use Lemma 3.4.2 and (essentially) the same proof of Theorem 3.1.4 to prove

Theorem 3.4.4. *Let G be a graph with m edges. Then there is a partition V_1, V_2, V_3 of $V(G)$ such that for $1 \le i \le 3$,*

$$e(V_i) \le \frac{1}{9}m + O(m^{4/5}),$$

and for $1 \le i \ne j \le 3$,

$$e(V_i \cup V_j) \le \frac{5}{9}m + O(m^{4/5}).$$

Proof. We may assume that G is connected. Let $V(G) = \{v_1, \ldots, v_n\}$ such that $d(v_1) \ge d(v_2) \ge \ldots \ge d(v_n)$. Let $V_1 = \{v_1, \ldots, v_t\}$ with $t = \lfloor m^\alpha \rfloor$, where $0 < \alpha < 1/2$. Then $t < n - 1$, $e(V_1) < \frac{1}{2}m^{2\alpha}$, and $d(v_{t+1}) \le 2m^{1-\alpha}$. Let $V_2 := V(G) \setminus V_1 = \{u_1, \ldots, u_{n-t}\}$ such that $e(u_i, V_1 \cup \{u_1, \ldots, u_{i-1}\}) > 0$ for $i = 1, \ldots, n - t$.

Fix an arbitrary 3-partition $V_1 = Y_1 \cup Y_2 \cup Y_3$, and assign each member of Y_i the color i, $1 \le i \le 3$. Extend this coloring to $V(G)$ such that each vertex $u_i \in V_2$ is independently assigned the color j with probability p_j^i, where $\sum_{j=1}^{3} p_j^i = 1$ and p_j^i will be determined later. Let Z_i denote the indicator random variable of the event of coloring u_i.

Let $G_i = G[V_1 \cup \{u_1, \cdots, u_i\}]$ for $i = 1, \ldots, n-t$, and let $G_0 = G[V_1]$. Let $X_j^0 = Y_j$ and $x_{jl}^0 = e(X_j^0 \cup X_l^0)$ for $1 \leq j, l \leq 3$. For $i = 1, \ldots, n-t$ and $1 \leq j, l \leq 3$, define

$$X_j^i := \{\text{vertices of } G_i \text{ with color } j\},$$

$$x_{jl}^i := e(X_j^i \cup X_l^i),$$

$$\Delta x_{jl}^i := x_{jl}^i - x_{jl}^{i-1},$$

$$b_j^i := e(u_i, X_j^{i-1}).$$

When $j = l$, let $x_j^i := x_{jl}^i$ and $\Delta x_j^i = \Delta x_{jl}^i$. Note that b_j^i depends on $(Z_1, \ldots, Z_{i-1})$ only and $\sum_{j=1}^{3} b_j^i = e(u_i, G_{i-1})$ is independent of $(Z_1, \ldots, Z_{i-1})$.

Let $a_j^i = \sum_{(Z_1,\ldots,Z_{i-1})} P(Z_1, \ldots, Z_{i-1}) b_j^i$, which is determined by p_j^s, $1 \leq j \leq 3$ and $1 \leq s \leq i-1$. As in the proof of Theorem 3.1.4, for $1 \leq i \leq n-t$ and $1 \leq j \neq l \leq 3$ we have

$$\mathbb{E}(\Delta x_{jl}^i) = (a_j^i + a_l^i)(p_j^i + p_l^i),$$

and for $1 \leq i \leq n-t$ we have

$$\mathbb{E}(\Delta x_j^i) = a_j^i p_j^i.$$

By Lemma 3.4.2, there exist $p_j^i \in [0, 2/3]$, $1 \leq j \leq 3$, such that $\sum_{j=1}^{3} p_j^i = 1$, for $1 \leq i \leq n-t$ and $1 \leq j \neq l \leq 3$,

$$\mathbb{E}(\Delta x_{jl}^i) \leq \frac{5}{9} \sum_{j=1}^{3} a_j^i = \frac{5}{9} \sum_{j=1}^{3} b_j^i = \frac{5}{9} e(u_i, G_{i-1}),$$

and for $1 \leq i \leq n-t$,

$$\mathbb{E}(\Delta x_j^i) \leq \frac{1}{9} \sum_{j=1}^{3} a_j^i = \frac{1}{9} \sum_{j=1}^{3} b_j^i = \frac{1}{9} e(u_i, G_{i-1}).$$

Note that p_j^i is determined by a_j^i, $1 \leq j \leq 3$; and hence p_j^i is recursively defined by p_j^s, $1 \leq j \leq 3$ and $1 \leq s \leq i-1$. Now

$$\mathbb{E}(x_{jl}^{n-t}) = \frac{5}{9} \sum_{i=1}^{n-t} e(u_i, G_{i-1}) + x_{jl}^0 \leq \frac{5}{9} m + e(V_1),$$

and

$$\mathbb{E}(x_j^{n-t}) \leq \frac{1}{9}\sum_{i=1}^{n-t} e(u_i, G_{i-1}) + x_j^0 \leq \frac{1}{9}m + e(V_1).$$

Clearly, changing the color of u_i (i.e., changing Z_i) affects $x_{jl} := x_{jl}^{n-t}$ and $x_j := x_j^{n-t}$ by at most $d(u_i)$. So by Lemma 1.4.1,

$$\mathbb{P}\left(x_{jl} > \mathbb{E}(x_{jl}) + z\right) < \exp\left(-\frac{z^2}{8m^{2-\alpha}}\right),$$

and

$$\mathbb{P}\left(x_j > \mathbb{E}(x_j) + z\right) < \exp\left(-\frac{z^2}{8m^{2-\alpha}}\right).$$

Let $z = (8\ln 6)^{\frac{1}{2}}\, m^{1-\frac{\alpha}{2}}$. Then for $1 \leq j \neq l \leq 3$,

$$\mathbb{P}\left(x_{jl} > \mathbb{E}(x_{jl}) + z\right) < \frac{1}{6},$$

and for $1 \leq j \leq 3$,

$$\mathbb{P}\left(x_j > \mathbb{E}(x_j) + z\right) < \frac{1}{6}.$$

So there exists a partition $V(G) = X_1 \cup X_2 \cup X_3$ such that for $1 \leq j \neq l \leq 3$,

$$e(X_j \cup X_l) \leq \mathbb{E}(x_{jl}) + z \leq \frac{5}{9}m + o(m),$$

and for $1 \leq j \leq 3$,

$$e(X_j) \leq \mathbb{E}(x_j) + z \leq \frac{1}{9}m + o(m).$$

The $o(m)$ term in both expressions is

$$\frac{1}{2}m^{2\alpha} + (8\ln 6)^{\frac{1}{2}}\, m^{1-\frac{\alpha}{2}}.$$

Picking $\alpha = \frac{2}{5}$ to minimize $\max\{2\alpha, 1-\alpha/2\}$, the $o(m)$ term becomes $O(m^{\frac{4}{5}})$. ∎

By the same argument as in the proof of Theorem 3.4.4, using Lemma 3.4.3 instead of Lemma 3.4.2, we have the following result.

Theorem 3.4.5. *Let G be a graph with m edges. Then there is a partition V_1, V_2, V_3, V_4 of $V(G)$ such that for $1 \leq i \leq 4$,*

$$e(V_i) \leq \frac{1}{16}m + O(m^{4/5}),$$

and for $1 \leq i \neq j \leq 4$,

$$e(V_i \cup V_j) \leq \frac{2}{5}m + O(m^{4/5}).$$

CHAPTER IV

3-UNIFORM HYPERGRAPHS

4.1 The main result

Recall Conjecture 1.3.10 (Bollobás and Thomason, see [7, 9, 11, 12]) that any r-uniform hypergraph with m edges has a r-partition $V_1, ..., V_r$ such that $d(V_i) \geq \frac{r}{2r-1}m$. For large graphs, the bound $r/(2r-1)$ may be improved. In this section, we prove the following result, which implies Theorem 1.3.11; hence Conjecture 1.3.10 holds for $r = 3$ asymptotically.

Theorem 4.1.1. *Every 3-uniform hypergraph with m edges has a partition into sets* V_1, V_2, V_3 *such that for* $i = 1, 2, 3$*,*

$$d(V_i) \geq 0.65m - O(m^{6/7}).$$

Bollobás and Scott [11,12] made a more general conjecture. For integers $r, k \geq 2$, every r-uniform hypergraph with m edges has a vertex-partition into k sets, each of which meets at least $(1 + o(1))(1 - (1 - 1/k)^r)m$ edges. In particular, for $r = k = 3$, the bound in this conjecture is $19/27m + o(m)$, where $19/27 \approx 0.7037$. Although our method can be modified to make further improvement on the current bound of 0.65, it is unlikely to yield a bound close to 19/27.

We organize this chapter as follows. In Section 4.2, we first state two lemmas, Lemmas 4.2.1 and 4.2.2, which assert that certain inequalities hold. We then use these two lemmas to prove Lemma 4.2.3 which, in turn, is used to prove Theorem 4.1.1. In Lemma 4.2.3, we need to bound three quantities simultaneously. In Section 4.3, we prove two lemmas that can be used to bound two quantities simultaneously. These lemmas will then be used in Section 4.4 to prove Lemmas 4.2.1 and 4.2.2.

4.2 *Proof of Theorem 4.1.1*

We need two lemmas which provide inequalities needed for our proof of Theorem 4.1.1. The meaning of the parameters in these lemmas will be clear from the proof of Lemma 4.2.3; each is related to the number of edges of a certain type. The first lemma tries to bound three quantities $f_i(p_i)$, $i = 1, 2, 3$, which will be proved in Section 4.4. It says that, under certain conditions, there exist p_i such that either all three functions are bounded from above, or can be made equal. We use $\mathbf{R}^+$ to denote the set of nonnegative reals.

Lemma 4.2.1. *Let* $b_{ij}, x_i, a_i, c \in \mathbf{R}^+$, $1 \le i \ne j \le 3$, *such that* $b_{ij} = b_{ji}$, $b_{ij} \ge \max\{2x_i, 2x_j\}$, *and* $b_{12} + b_{23} + b_{31} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 + c = 1$. *For any permutation* ijk *of* $\{1, 2, 3\}$, *let*

$$f_i := (1 - p_i)(b_{jk} + x_j + x_k) + (1 - p_i)^2(a_j + a_k) + (1 - p_i)^3 c.$$

Then there exists $p_1, p_2, p_3 \in [0, 1]$ *with* $p_1 + p_2 + p_3 = 1$ *such that*

(*i*) $f_i \le 0.35$ *for* $i = 1, 2, 3$, *or*

(*ii*) $f_1 = f_2 = f_3$ *and* $p_i \in (0, 1)$ *for* $i = 1, 2, 3$.

The second lemma (when combined with Lemma 4.2.1) deals with the case $c = 0$ of Lemma 4.2.3, and will be proved in Section 4.4.

Lemma 4.2.2. *Let* $a_i, x_i, b_{ij} \in \mathbf{R}^+$, $1 \le i \ne j \le 3$, *such that* $b_{ij} = b_{ji}$, $b_{ij} \ge \max\{2x_i, 2x_j\}$ *and* $b_{12} + b_{23} + b_{31} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 = 1$. *For any permutation* ijk *of* $\{1, 2, 3\}$, *let*

$$f_k := (1 - p_k)(b_{ij} + x_i + x_j) + (1 - p_k)^2(a_i + a_j)$$

Suppose there exist $p_1, p_2, p_3 \in (0, 1)$ *such that* $p_1 + p_2 + p_3 = 1$ *and* $f_1 = f_2 = f_3$. *Then for such* p_1, p_2, p_3, *we have* $f_k \le 0.35$ *for* $k = 1, 2, 3$.

We can now prove the main lemma by using Lemma 4.2.1 and Lemma 4.2.2.

Lemma 4.2.3. *Let $b_{ij}, x_i, a_i, c \in \mathbf{R}^+$, $1 \le i \ne j \le 3$, such that $b_{ij} = b_{ji}$, $b_{ij} \ge \max\{2x_i, 2x_j\}$ and $b_{12} + b_{23} + b_{31} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 + c = 1$. Then there exist $p_1, p_2, p_3 \in [0, 1]$ with $p_1 + p_2 + p_3 = 1$ such that for any $\{i, j, k\} = \{1, 2, 3\}$,*

$$f_i := (1 - p_i)(b_{jk} + x_j + x_k) + (1 - p_i)^2(a_j + a_k) + (1 - p_i)^3 c \le 0.35.$$

Proof. By Lemma 4.2.1, we may assume that there exist $p_1, p_2, p_3 \in (0, 1)$ with $p_1 + p_2 + p_3 = 1$ such that $f_1 = f_2 = f_3$. Let $\mathscr{D}$ be the set of points

$$(a_1, a_2, a_3, x_1, x_2, x_3, b_{12}, b_{23}, b_{31}, c, p_1, p_2, p_3) \in [0, 1]^{13}$$

satisfying

$$\begin{aligned}
&b_{ij} \ge \max\{2x_i, 2x_j\},\\
&b_{12} + b_{23} + b_{31} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 + c = 1,\\
&p_1 + p_2 + p_3 = 1,\\
&p_i \in [0, 1] \text{ for } i = 1, 2, 3, \text{ and}\\
&f_1 = f_2 = f_3.
\end{aligned}$$

Note that $\mathscr{D} \ne \emptyset$ and $\mathscr{D}$ is a compact subset of $[0, 1]^{13}$. So $f_1(\mathbf{v})$ has an absolute maximum over $\mathscr{D}$. Let $\mathscr{M}$ denote all $\mathbf{v} \in \mathscr{D}$ for which $f_1(\mathbf{v})$ is the maximum of f_1 over $\mathscr{D}$. It suffices to show that there is some $\mathbf{v} \in \mathscr{M}$ such that $f_i(\mathbf{v}) \le 0.35$ for $i = 1, 2, 3$. Let

$$\mathbf{v} := (a_1, a_2, a_3, x_1, x_2, x_3, b_{12}, b_{23}, b_{31}, c, p_1, p_2, p_3) \in \mathscr{M}.$$

We claim that $\mathbf{v}$ may be chosen so that $c = 0$. For, suppose $c \ne 0$. Define

$$\mathbf{v}' := (a_1 + p_1 c, a_2 + p_2 c, a_3 + p_3 c, x_1, x_2, x_3, b_{12}, b_{23}, b_{31}, 0, p_1, p_2, p_3).$$

It is easy to check that $\mathbf{v}' \in \mathscr{D}$ and $f_i(\mathbf{v}') = f_i(\mathbf{v})$ for $i = 1, 2, 3$. Since $\mathbf{v} \in \mathscr{M}$, we have $\mathbf{v}' \in \mathscr{M}$. Now it follows from Lemma 4.2.2 that for any $i = 1, 2, 3$, $f_i(\mathbf{v}) = f_i(\mathbf{v}') \le 0.35$. ∎

We also need the following lemma, which is easy to prove. Let G be a graph (multiple edges allowed) and let $w : E(G) \to \mathbf{R}^+$. Recall that for any $S \subseteq V(G)$, we write $w(S) =$

$\sum_{V(e)\subseteq S} w(e)$; for any $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, we use (S, T) to denote the set of edges st with $s \in S$ and $t \in T$; and we write $w(S, T) = \sum_{e\in(S,T)} w(e)$.

Lemma 4.2.4. *Let G be a graph and let $w : E(G) \to \mathbf{R}^+$, and let $V(G) = V_1 \cup \ldots \cup V_k$ be a k-partition minimizing $\sum_{i=1}^k w(V_i)$. Then for any $1 \leq i \neq j \leq k$*

$$w(V_i, V_j) \geq \max\{2w(V_i), 2w(V_j)\}.$$

Proof. For any $v \in V_i$ and for any $j \in \{1, \ldots, k\} \setminus \{i\}$, we have

$$\sum_{\{uv\in E(G):u\in V_i - v\}} w(uv) \leq \sum_{\{uv\in E(G):u\in V_j\}} w(uv).$$

Summing over $v \in V_i$, we get $2w(V_i) \leq w(V_i, V_j)$. ∎

Proof of Theorem 4.1.1. We may assume that G is connected; as otherwise, we may simply consider the individual components. Hence every vertex of G has positive degree.

Let $V(G) = \{v_1, \ldots, v_n\}$ such that $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$. Let $U_1 := \{v_1, \ldots, v_t\}$ and $U_2 := V(G) \setminus U_1$, with $t = \lfloor m^\alpha \rfloor$ and $0 < \alpha < 1/3$. Since $m \leq \binom{n}{3}$ and $t < m^{1/3}$, we have $t \leq n - 2$ for $n \geq 3$ (by a simple calculation). Moreover,

$$m^\alpha d(v_{t+1}) \leq (1 + t)d(v_{t+1}) \leq \sum_{i=1}^{t+1} d(v) < \sum_{v\in V(G)} d(v) = 3m;$$

so $d(v_{t+1}) < 3m^{1-\alpha}$. Hence

$$\sum_{i=t+1}^{n} d(v_i)^2 < 3m^{1-\alpha} \sum_{i=1}^{n} d(v_i) = 9m^{2-\alpha}.$$

For any partition $U_1 = X_1 \cup X_2 \cup X_3$ and for $1 \leq i \neq j \leq 3$, define

$$x_i = |\{e \in E(G) : |V(e) \cap X_i| = 2, |V(e) \cap U_2| = 1\}|,$$

$$a_i = |\{e \in E(G) : |V(e) \cap X_i| = 1, |V(e) \cap U_2| = 2\}|,$$

$$b_{ij} = |\{e \in E(G) : |V(e) \cap X_i| = |V(e) \cap X_j| = |V(e) \cap U_2| = 1\}|,$$

$$c = |\{e \in E(G) : |V(e) \cap U_2| = 3\}|.$$

Then $m = e(U_1) + b_{12} + b_{23} + b_{13} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 + c$.

By Lemma 4.2.4, we may choose the partition $U_1 = X_1 \cup X_2 \cup X_3$ such that for $1 \le i \ne j \le 3$,

$$b_{ij} \ge \max\{2x_i, 2x_j\}.$$

For $1 \le i \le 3$, assign color i to the vertices in X_i. We extend the coloring to U_2 as follows: each vertex in U_2 is independently colored i with probability p_i for $1 \le i \le 3$, where $p_1 + p_2 + p_3 = 1$ and p_i will be determined by an application of Lemma 4.2.3.

For $i = 1, 2, 3$, let V_i be the vertices with color i, and let

$$y_i = |\{e \in E(G) : V(e) \subseteq U_1 \text{ and } V(e) \cap X_i \ne \emptyset\}.$$

Then, for any permutation ijk of $\{1, 2, 3\}$,

$$\mathbb{E}(d(V_i)) = b_{ij} + b_{ik} + x_i + a_i + p_i(b_{jk} + x_j + x_k) + (1-(1-p_i)^2)(a_j + a_k) + (1-(1-p_i)^3)c + y_i.$$

Thus

$$f_i := m - \mathbb{E}(d(V_i)) - e(U_1) + y_i = (1-p_i)(b_{jk} + x_j + x_k) + (1-p_i)^2(a_j + a_k) + (1-p_i)^3 c,$$

and

$$\alpha := m - e(U_1) = b_{12} + b_{23} + b_{31} + a_1 + a_2 + a_3 + x_1 + x_2 + x_3 + c.$$

By applying Lemma 4.2.3 (with $b_{ij}/\alpha, a_i/\alpha, x_i/\alpha, c/\alpha$ as b_{ij}, a_i, x_i, c, respectively), there exist $p_i \in [0, 1]$ with $p_1 + p_2 + p_3 = 1$ such that for $1 \le i \le 3$, $f_i/\alpha \le 0.35$. So

$$f_i \le 0.35(m - e(U_1)).$$

Hence

$$\mathbb{E}(d(V_i)) = m - f_i - e(U_1) + y_i \ge 0.65m - 0.65e(U_1) + y_i.$$

Changing the color of any v_j, $t + 1 \le j \le n$, affects $d(V_i)$ by at most $d(v_j)$. So by Lemma 1.4.1, we have for $i = 1, 2, 3$,

$$\mathbb{P}(d(V_i) < \mathbb{E}(d(V_i)) - z) \le \exp\left(\frac{-z^2}{2\sum_{j=t+1}^{n} d(v_j)^2}\right) < \exp\left(\frac{-z^2}{18m^{2-\alpha}}\right).$$

Taking $z = \sqrt{18 \ln 3} m^{1-\alpha/2}$, we have for $i = 1, 2, 3$,

$$\mathbb{P}(d(V_i) < \mathbb{E}(d(V_i)) - z) < 1/3.$$

Therefore, there exists a partition $V(G) = V_1 \cup V_2 \cup V_3$ such that for $i = 1, 2, 3$,

$$d(V_i) \geq \mathbb{E}(d(V_i)) - z \geq 0.65m - 0.65e(U_1) + y_i - z \geq 0.65m - 0.65e(U_1) - z.$$

Since $|U_1| = t \leq m^\alpha$, $e(U_1) = O(m^{3\alpha})$. So

$$0.65e(U_1) + z = O(m^{3\alpha}) + \sqrt{18 \ln 2} m^{1-\alpha/2}.$$

Choosing $\alpha = \frac{2}{7}$ to minimize $\max\{3\alpha, 1 - \alpha/2\}$, we have the desired bound. ∎

4.3 Bounding two quantities

In this section, we prove two lemmas to be used in our proofs of Lemmas 4.2.1 and 4.2.2. The first is a slight variation of the main lemma in [9]. The difference is that here we relax the constraint $z \geq \max\{2x, 2y\}$ in [9] to $z \geq x + y$; as a consequence we have a weaker bound. Our proof mimics that in [9], where a more general result is proved.

Lemma 4.3.1. *Let $a, b, x, y, z, e \in \mathbf{R}^+$ such that $z \geq x + y$ and $a + b + x + y + z + e = 1$. Then there exists $p \in (0, 1)$ such that*

$$p^2 a + px + p^3 e \leq 1/7, \text{ and } (1 - p)^2 b + (1 - p)y + (1 - p)^3 e \leq 1/7.$$

Proof. For convenience, let

$$f_1 := p^2 a + px + p^3 e, \text{ and } f_2 := (1 - p)^2 b + (1 - p)y + (1 - p)^3 e.$$

Note that f_1 and f_2 are continuous functions of p on $[0, 1]$. We may assume that

(1) $a + x + e > 0$ and $b + y + e > 0$.

Otherwise, by symmetry, we may assume $a + x + e = 0$. Then $a = x = e = 0$ and $f_1 = 0 < 1/7$. Since f_2 is a continuous function of p, there exist $0 < \epsilon < 1$ such that

$|f_2(\epsilon) - f_2(1)| < 1/7$. Thus, because $f_2(1) = 0$, we have $f_2(\epsilon) < 1/7$. So letting $p = \epsilon$, the assertion of the lemma holds. Thus we may assume (1).

By (1), $f_1(1) = a + x + e > 0$ and $f_2(0) = b + y + e > 0$. Therefore, since $f_1(0) = 0 = f_2(1)$ and because $f_1(p)$ (respectively, $f_2(p)$) is increasing (respectively, decreasing) and continuous on $[0, 1]$, we have

(2) for any a, b, x, y, z, e satisfying (1), there exists a unique $p \in (0, 1)$ such that $f_1 = f_2$.

We call $\mathbf{v} := (a, b, x, y, z, e, p) \in [0, 1]^7$ a *satisfying point* if $a, b, x, y, z, e, p \in \mathbf{R}^+$, $a+b+x+y+z+e = 1$, $z \geq x+y$, $p \in [0, 1]$, and $f_1 = f_2$. (In fact, $p \in (0, 1)$ by (2).) Let $\mathscr{D}$ denote the set of all satisfying points. Note $\mathscr{D}$ is a compact subset of $[0, 1]^7$. A point in $\mathscr{D}$ is said to be a *maximal point* if the value of f_1 at that point is the maximum of f_1 over $\mathscr{D}$. Let $\mathscr{M}$ be the set of maximal points, which is nonempty since $\mathscr{D} \neq \emptyset$ (by (1) and (2)) and $\mathscr{D}$ is compact.

It then suffices to show that $f_1(\mathbf{v}) \leq 1/7$ for any $\mathbf{v} \in \mathscr{M}$. We do so by looking for a special maximal point. First, we show that

(3) there exists $(a, b, x, y, z, e, p) \in \mathscr{M}$ such that $e = 0$, $z = x + y$, and $ab = 0$.

Let $\mathbf{v} := (a, b, x, y, z, e, p) \in \mathscr{M}$. If $e > 0$, then let $\mathbf{v}' := (a + pe, b + (1 - p)e, x, y, z, 0, p)$. It is easy to check that $\mathbf{v}' \in \mathscr{D}$ and $f_i(\mathbf{v}') = f_i(\mathbf{v})$ for $i = 1, 2$. Hence $\mathbf{v}' \in \mathscr{M}$, since $\mathbf{v} \in \mathscr{M}$ and $f_1(\mathbf{v}') = f_1(\mathbf{v})$. So we may assume $e = 0$.

We may assume $z = x + y$. For, otherwise, assume $z > x + y$. Let $\mathbf{v}' := (a + z - -x - y, b, x, y, x + y, 0, p')$ with $p' \in [0, 1]$, which satisfies (1). So by (2), we may choose $p' \in (0, 1)$ so that $f_1(\mathbf{v}') = f_2(\mathbf{v}')$; then $\mathbf{v}' \in \mathscr{D}$. If $p' < p$, then $f_2(\mathbf{v}') > f_2(\mathbf{v})$, contradicting the assumption that $\mathbf{v} \in \mathscr{M}$. So $p' \geq p$. Then

$$
\begin{aligned}
f_1(\mathbf{v}') - f_1(\mathbf{v}) &\geq p^2(z - x - y) > 0, \quad \text{and} \\
f_2(\mathbf{v}') - f_2(\mathbf{v}) &= b((1 - p')^2 - (1 - p)^2) + y((1 - p') - (1 - p)) \\
&= -(p' - p)((2 - p - p')b + y) \\
&\leq 0.
\end{aligned}
$$

Hence $f_1(\mathbf{v}') > f_1(\mathbf{v}) = f_2(\mathbf{v}) \geq f_2(\mathbf{v}')$, a contradiction.

Now suppose $a > 0$ and $b > 0$. Let $\varepsilon = \min\{pa, (1-p)b\}$, and let

$$\mathbf{v}' = (a', b', x', y', z', e', p') := (a - \frac{\varepsilon}{p}, b - \frac{\varepsilon}{1-p}, x + \varepsilon, y + \varepsilon, z + 2\varepsilon, 0, p).$$

It is easy to see that $e' = 0$, $z' = x' + y'$, $a'b' = 0$, and $f_i(\mathbf{v}') = f_i(\mathbf{v})$ for $i = 1, 2$ (and hence $f_1(\mathbf{v}') = f_2(\mathbf{v}')$). Since $a + b + x + y + z = 1$,

$$a' + b' + x' + y' + z' = 1 + 4\varepsilon - \left(\frac{\varepsilon}{p} + \frac{\varepsilon}{1-p}\right).$$

Since $p(1-p) \leq 1/4$ (with equality iff $p = 1/2$),

$$4\varepsilon \leq \frac{\varepsilon}{p} + \frac{\varepsilon}{1-p}$$

So we have $a' + b' + x' + y' + z' \leq 1$.

If $a' + b' + x' + y' + z' = 1$ then $p = 1/2$ and $\mathbf{v}' \in \mathscr{D}$. Since $f_i(\mathbf{v}') = f_i(\mathbf{v})$, we have $\mathbf{v}' \in \mathscr{M}$; and hence (3) holds with $\mathbf{v}'$. We may thus assume that $a' + b' + x' + y' + z' < 1$. Let

$$\alpha = \frac{\varepsilon}{p} + \frac{\varepsilon}{1-p} - 4\varepsilon,$$

and let

$$\mathbf{v}'' := (a'', b'', x'', y'', z'', e'', p'') = (a' + \alpha, b', x', y', z', 0, p'')$$

with $p'' \in [0, 1]$.

Note that $e'' = 0$, $z'' = x'' + y''$, $a'' + b'' + x'' + y'' + z'' = 1$, and $\mathbf{v}''$ satisfies (1). So by (2), we may choose $p'' \in (0, 1)$ such that $f_1(\mathbf{v}'') = f_2(\mathbf{v}'')$, and hence $\mathbf{v}'' \in \mathscr{D}$. If $p'' \geq p'$ then $f_1(\mathbf{v}'') > f_1(\mathbf{v}') = f_1(\mathbf{v})$ (since $a'' > a'$ and f_1 increases with p). If $p'' < p'$ then $f_2(\mathbf{v}'') > f_2(\mathbf{v}') = f_2(\mathbf{v})$ (since f_2 decreases with p). In either case, we obtain a contradiction to the assumption that $\mathbf{v} \in \mathscr{M}$. Thus, (3) holds.

Let $\mathscr{M}' = \{(a, b, x, y, z, e, p) \in \mathscr{M} : a = b = e = 0 \text{ and } z = x + y\}$. We may assume that

(4) $\mathscr{M}' = \emptyset$.

For otherwise, let $\mathbf{v} = (0, 0, x, y, x + y, 0, p) \in \mathscr{M}'$. Then $f_1(\mathbf{v}) = px$, $f_2(\mathbf{v}) = (1 - p)y$, and $x + y = 1/2$. Since $f_1(\mathbf{v}) = f_2(\mathbf{v})$, we have $px = (1 - p)(1/2 - x)$. Hence, $p = 1 - 2x$, and $f_1(\mathbf{v}) = x(1 - 2x) = 1/8 - 2(1/4 - x)^2 \leq 1/8 < 1/7$. So the assertion of the lemma holds; and thus we may assume (4).

By (3) and (4), we may assume without losing generality that there exists $\mathbf{v} = (0, b, x, y, x+ y, 0, p) \in \mathscr{M}$ such that $b \neq 0$. Then $b + 2(x + y) = 1$, and hence $x = (1 - b)/2 - y$. So

$$f_1(\mathbf{v}) = xp = (1 - b)p/2 - yp, \text{ and } f_2(\mathbf{v}) = y(1 - p) + b(1 - p)^2.$$

Since $\mathbf{v} \in \mathscr{M}$, $f_1(\mathbf{v})$ is the maximum value of f_1 over $\mathscr{D}$ subject to $g := f_1 - f_2 = 0$, where f_1, f_2, g are considered as functions of b, y, p.

Case 1. $y \neq 0$.

Then $y \in (0, 1)$ and $b \in (0, 1)$; so $\mathbf{v}$ is a critical point of f_1 (as a function of b, y). Hence $\mathbf{v}$ must satisfy $\partial f_1/\partial b = \lambda \partial g/\partial b$ and $\partial f_1/\partial y = \lambda \partial g/\partial y$, where λ is a Lagrange multiplier. Thus

$$p = \lambda\left(p + 2(1 - p)^2\right), \text{ and } p = \lambda\,(p + (1 - p)) = \lambda.$$

Since $p \in (0, 1)$, we have $\lambda \neq 0$. So from the above equations we deduce that $(1 - p) = 2(1 - p)^2$. Again since $p \neq 1$, we have $p = 1/2$. Let

$$\mathbf{v}' := (a', b', x', y', z', e', p') = (0, 0, x, y + b/2, z + b/2, 0, p).$$

Then $a' + b' + x' + y' + z' + e' = 1$, $z' = x' + y'$, and $f_1(\mathbf{v}') = f_1(\mathbf{v})$. Since $p = 1/2$,

$$f_2(\mathbf{v}') = (1 - p)(y + b/2) = (1 - p)y + (1 - p)b/2 = (1 - p)y + (1 - p)^2 b = f_2(\mathbf{v}).$$

This implies $\mathbf{v}' \in \mathscr{M}'$, contradicting (4).

Case 2. $y = 0$.

Then $f_1(\mathbf{v}) = (1 - b)p/2$ and $f_2(\mathbf{v}) = b(1 - p)^2$. By (1) and (2) and since $f_1(\mathbf{v}) = f_2(\mathbf{v})$, we have $b \in (0, 1)$ and $p \in (0, 1)$. Since $f_1(\mathbf{v})$ is the maximum of f_1 over $\mathscr{D}$ subject to

$g := f_1 - f_2 = 0$ (considered as functions of p and b), $\mathbf{v}$ satisfies $\partial f_1/\partial p = \lambda \partial g/\partial p$ and $\partial f_1/\partial b = \lambda \partial g/\partial b$ for some λ. Therefore,

$$(1-b)/2 = \lambda\left((1-b)/2 + 2b(1-p)\right), \text{ and } p/2 = \lambda\left(p/2 + (1-p)^2\right).$$

Since $p \in (0,1)$, we have $\lambda \neq 0$; so we derive from above that $b = (1-p)/(1+p)$. From $f_1(\mathbf{v}) = f_2(\mathbf{v})$, we deduce $b = \frac{p}{p+2(1-p)^2}$. Hence

$$\frac{p}{p+2(1-p)^2} = \frac{1-p}{1+p}.$$

Simplifying this we get $p^3 - 2p^2 + 3p - 1 = 0$. Since the function $p^3 - 2p^2 + 3p - 1$ is always increasing and takes value 0.036125 when $p = 9/20$, so $p < 9/20$.

We now claim that $f_1 \leq 1/7$. For otherwise, we have $f_1 > 1/7$, i.e.,

$$\frac{(1-b)p}{2} = \frac{p^2}{1+p} > 1/7.$$

But this gives $p > \frac{1+\sqrt{29}}{14} > 9/20$, a contradiction. This proves Lemma 4.3.1. ∎

In the next lemma we show that under certain conditions two functions can be made equal and bounded from above. The proof is similar to that of Lemma 4.3.1.

Lemma 4.3.2. *Let $\mathscr{D}$ denote the set of all points (a,b,x,y,e,p) such that $a,b,x,y,e \in \mathbf{R}^+$, $p \in [0.18, 1]$, $a + b + 2(x+y+e) = 1$, and $p^2a + px + p^3e = (1.18-p)^2b + (1.18-p)y + (1.18-p)^3e$. Suppose $\mathscr{D} \neq \emptyset$. Then for any $(a,b,x,y,e,p) \in \mathscr{D}$, $p^2a + px + p^3e \leq (1.18^2/8)(1-0.82e)$.*

Proof. For convenience, let

$$g_1(a,b,x,y,e,p) := p^2a + px + p^3e, \text{ and}$$
$$g_2(a,b,x,y,e,p) := (1.18-p)^2b + (1.18-p)y + (1.18-p)^3e.$$

A point $\mathbf{v} := (a,b,x,y,e,p) \in \mathscr{D}$ is said to be *maximal* if $g_1(\mathbf{v})$ is the maximum of g_1 over $\mathscr{D}$. Let $\mathscr{M}$ denote the set of all maximal points. Since $\mathscr{D}$ is compact and $\mathscr{D} \neq \emptyset$, $\mathscr{M} \neq \emptyset$. Let $M := g(\mathbf{v})$ for $\mathbf{v} \in \mathscr{M}$. We claim that

(1) for any $\mathbf{v} = (a, b, x, y, e, p) \in \mathscr{D}$, we have $e = 0$ and $g_1(\mathbf{v}) \leq M(1 - 0.82e)$.

It is clear that (1) holds when $e = 0$. So assume $e \neq 0$. Let

$$\mathbf{v}' := (a', b', x', y', e', p') = \left(\frac{a + pe}{1 - 0.82e}, \frac{b + (1.18 - p)e}{1 - 0.82e}, \frac{x}{1 - 0.82e}, \frac{y}{1 - 0.82e}, 0, p\right).$$

Then $a' + b' + 2(x' + y' + e') = 1$, and $g_1(\mathbf{v}') = g_1(\mathbf{v})/(1 - 0.82e) = g_2(\mathbf{v})/(1 - 0.82e) = g_2(\mathbf{v}')$; so $\mathbf{v}' \in \mathscr{D}$. Now $g_1(\mathbf{v}) = g_1(\mathbf{v}')(1 - 0.82e) \leq M(1 - 0.82e)$, proving (1).

Therefore, it suffices to prove that $M \leq 1.18^2/8$. Let $\mathscr{M}' = \{(a, b, x, y, e, p) \in \mathscr{M} : x = y = e = 0\}$. We may assume

(2) $\mathscr{M}' = \emptyset$.

For, suppose there exists some $\mathbf{v} = (a, b, x, y, e, p) \in \mathscr{M}'$. Then $a + b = 1$,

$$g_1(\mathbf{v}) = p^2 a, \text{ and } g_2(\mathbf{v}) = (1.18 - p)^2 b.$$

Since $g_1(\mathbf{v}) = g_2(\mathbf{v})$, we have

$$b = \frac{p^2}{p^2 + (1.18 - p)^2}.$$

Note that for any $s, t \in \mathbf{R}^+$, we have $2\sqrt{st} \leq s + t$ and $2st \leq s^2 + t^2$; so $8s^2t^2 \leq (s+t)^2(s^2+t^2)$, which implies

$$\frac{s^2 t^2}{s^2 + t^2} \leq \frac{1}{2}\left(\frac{s + t}{2}\right)^2.$$

Thus

$$M = g_2(\mathbf{v}) = \frac{p^2 (1.18 - p)^2}{p^2 + (1.18 - p)^2} \leq \frac{1}{2}\left(\frac{1.18}{2}\right)^2 = \frac{1.18^2}{8},$$

and the assertion of the lemma holds. So we may assume (2).

By (1) and (2), there exists $\mathbf{v} = (a, b, x, y, e, p) \in \mathscr{M}$ such that $e = 0$, and $x \neq 0$ or $y \neq 0$. We now show that $\mathbf{v}$ may be chosen so that

(3) $y = 0$.

For, suppose $y \neq 0$. Since $a + b + 2(x + y + e) = 1$ and $e = 0$, $x = (1 - a - b - 2y)/2$. So

$$g_1(\mathbf{v}) = p^2 a + p\frac{1 - a - b - 2y}{2}, \text{ and}$$
$$g_2(\mathbf{v}) = (1.18 - p)^2\, b + (1.18 - p)\, y.$$

Suppose $b \neq 0$. Then since we assume $y \neq 0$ and because $\mathbf{v} \in \mathscr{M}$, $\mathbf{v}$ is a critical point of g_1 subject to $g := g_1 - g_2 = 0$, where g_1, g_2, g are considered as functions of b and y. By applying the method of Lagrange multipliers, we have $\partial g_1/\partial b = \lambda \partial g/\partial b$ and $\partial g_1/\partial y = \lambda \partial g/\partial y$. Hence

$$-\frac{p}{2} = \lambda\left(-\frac{p}{2} - (1.18 - p)^2\right), \text{ and } -p = \lambda\left(-p - (1.18 - p)\right).$$

Since $p \in [0.18, 1]$, $\lambda \neq 0$. Hence from the above expressions we deduce that $(1.18 - p)^2 = (1.18 - p)/2$. So $p = 0.68$, since $p \in [0.18, 1]$. Let

$$\mathbf{v}' := (a', b', x', y', e', p') = (a, b + 2y, x, 0, 0, p).$$

Then

$$a' + b' + 2(x' + y' + e') = a + b + 2(x + y) = 1,$$
$$g_1(\mathbf{v}') = p^2 a + px = g_1(\mathbf{v}), \text{ and}$$
$$g_2(\mathbf{v}') = (1.18 - p)^2 b' = (1.18 - p)^2 b + 2(1.18 - p)^2 y = (1.18 - p)^2 b + (1.18 - p)y = g_2(\mathbf{v}).$$

The last equality holds because $p = 0.68$. So $g_1(\mathbf{v}') = g_2(\mathbf{v}') = g_1(\mathbf{v})$. This means that $\mathbf{v}' \in \mathscr{M}$, with $e' = 0$ and $y' = 0$; and (3) holds by replacing $\mathbf{v}$ with $\mathbf{v}'$.

Now suppose $a = 0$ and $b = 0$. Then $g_1(\mathbf{v}) = p(1 - 2y)/2$ and $g_2(\mathbf{v}) = (1.18 - p)y$. So $g_1(\mathbf{v}) = g_2(\mathbf{v})$ implies $y = p/2.36$. Hence,

$$M = g_1(\mathbf{v}) = \frac{p}{2} - \frac{p^2}{2.36} = \frac{1.18}{8} - \frac{1}{2 \times 1.18}\left(p - \frac{1.18}{2}\right)^2 \leq \frac{1.18}{8} < \frac{1.18^2}{8},$$

and the assertion of the lemma holds.

So we may assume $a \neq 0$ and $b = 0$. Then

$$g_1(\mathbf{v}) = p^2 a + p(1 - a - 2y)/2, \text{ and } g_2(\mathbf{v}) = (1.18 - p)y.$$

Now $\mathbf{v}$ must be a critical point of g_1 subject to $g := g_1 - g_2 = 0$, where g_1, g_2, g are considered as functions of a and y. So there exists λ (Lagrange multiplier) such that $\partial g_1/\partial a = \lambda \partial g/\partial a$ and $\partial g_1/\partial y = \lambda \partial g/\partial y$. This gives

$$p^2 - \frac{p}{2} = \lambda\left(p^2 - \frac{p}{2}\right), \text{ and } -p = \lambda\,(-p - (1.18 - p)) = -1.18\lambda.$$

Since $p \in [0.18, 1]$, $\lambda \neq 1$ (from the second equation) and $p = 1/2$ (from the first equation). Hence, $g_1(\mathbf{v}) = (1-2y)/4$ and $g_2(\mathbf{v}) = 0.68y$. Since $g_1(\mathbf{v}) = g_2(\mathbf{v})$, we have $(1-2y)/4 = 0.68y$, and so $y = 1/4.72$. Hence $M = g_2(\mathbf{v}) = 0.68/4.72 < 1.18^2/8$. This completes the proof of (3).

By (2) and (3), $x \neq 0$ and $\mathbf{v} = (a, b, x, 0, 0, p)$. Hence $x = (1-a-b)/2$,

$$g_1(\mathbf{v}) = p^2 a + p\frac{1-a-b}{2}, \text{ and } g_2(\mathbf{v}) = (1.18-p)^2 b.$$

Note that when $b = 0$, we have $M = g_2(\mathbf{v}) = 0 < 1.18^2/8$. Hence, we may assume

(4) $b \neq 0$.

We consider two cases: $a \neq 0$, and $a = 0$.

Case 1. $a \neq 0$.

Then $\mathbf{v}$ is a critical point of g_1 subject to $g := g_1 - g_2 = 0$, all considered as functions of a and b. So there exists λ such that $\partial g_1/\partial a = \lambda \partial g/\partial a$ and $\partial g_1/\partial b = \lambda \partial g/\partial b$, which give

$$p^2 - \frac{p}{2} = \lambda\left(p^2 - \frac{p}{2}\right), \text{ and } -\frac{p}{2} = \lambda\left(-\frac{p}{2} - (1.18-p)^2\right).$$

Since $p \in [0.18, 1]$, we have $\lambda \neq 1$ from the second equation; so $p^2 - p/2 = 0$ (from the first equation), which implies $p = 1/2$. Define

$$\mathbf{v}' := (a', b', x', y', e', p') = (a + 2x, b, 0, 0, 0, p).$$

Then $a' + b' + 2(x' + y' + e') = a + b + 2x = 1$ and $g_2(\mathbf{v}) = g_2(\mathbf{v}')$. Also, because $p = 1/2$, $g_1(\mathbf{v}') = p^2 a' = p^2 a + 2p^2 x = p^2 a + px = g_1(\mathbf{v})$. Therefore, $\mathbf{v}' \in \mathscr{M}'$, contradicting (2).

Case 2. $a = 0$.

Then $g_1(\mathbf{v}) = p(1-b)/2$ and $g_2(\mathbf{v}) = (1.18-p)^2 b$. Since $g_1(\mathbf{v}) = g_2(\mathbf{v})$, we have

$$b = \frac{p/2}{(1.18-p)^2 + p/2}.$$

If $p = 0.18$ then $b = 0.18/2.18$; so $M = g_2(\mathbf{v}) = b < 1.18^2/8$. If $p = 1$ then $b = 1/1.0648$; so $M = g_2(v) = 0.18^2 b < 1.18^2/8$. Hence we may assume $p \in (0.18, 1)$.

Since $b \neq 0$ (by (4)) and $p \in (0.18, 1)$, $\mathbf{v}$ is a critical point of g_1 subject to $g := g_1 - g_2 = 0$, all considered as functions of b and p. So there exists λ such that $\partial g_1/\partial b = \lambda \partial g/\partial b$ and $\partial g_1/\partial p = \lambda \partial g/\partial p$, which gives

$$-\frac{p}{2} = \lambda\left(-\frac{p}{2} - (1.18-p)^2\right) \text{ and } \frac{1-b}{2} = \lambda\left(\frac{1-b}{2} + 2b(1.18-p)\right).$$

Since $p \in (0.18, 1)$, we have $\lambda \neq 0$ (from the first equation). So

$$\frac{p}{2}\left(\frac{1-b}{2} + 2b(1.18-p)\right) = \frac{1-b}{2}\left(\frac{p}{2} + (1.18-p)^2\right).$$

By a simple calculation, we derive

$$b = \frac{1.18-p}{1.18+p}.$$

Therefore, we have $(1.18-p)^3 = p^2$.

Note that $h(p) := (1.18-p)^3 - p^2$ is a decreasing function over $(0.18, 1)$, and a simple calculation shows $h(0.53) = -0.006275 < 0$. So $p < 0.53$. Also note that $g_1(\mathbf{v}) = p^2/(1.18+p)$ is an increasing function over $(0.18, 1)$. So

$$g_1(\mathbf{v}) = \frac{p^2}{1.18+p} < \frac{(0.53)^2}{1.18+0.53} < 0.165 < \frac{1.18^2}{8}.$$

This completes the proof of Lemma 4.3.2. ∎

4.4 Proofs of Lemmas 4.2.1 and 4.2.2

Proof of Lemma 4.2.1. For any permutation ijk of $\{1, 2, 3\}$, let

$$\alpha_i := b_{jk} + x_j + x_k,\ \beta_i := a_j + a_k, \text{ and } \gamma_i := \alpha_i + \beta_i + c.$$

Then for $i = 1, 2, 3$,

$$f_i(p_i) = (1 - p_i)\alpha_i + (1 - p_i)^2\beta_i + (1 - p_i)^3 c.$$

By symmetry, we may assume that

$$\gamma_1 \leq \gamma_2 \leq \gamma_3.$$

We may further assume that

(1) $\gamma_1 \geq 0.35$.

For, suppose $\gamma_1 < 0.35$. Let $p_1 = 0$; then $f_1 = \gamma_1 < 0.35$. We wish to apply Lemma 4.3.1 to show that there exist $p_2, p_3 \in (0, 1)$ such that $p_2 + p_3 = 1$ and $f_2 = f_3 \leq 0.35$. Let

$$m = \alpha_2 + \alpha_3 + \beta_2 + \beta_3 + (\alpha_2 + \alpha_3) + c.$$

Let $x = \alpha_2/m$, $y = \alpha_3/m$, $a = \beta_2/m$, $b = \beta_3/m$, $z = (\alpha_2 + \alpha_3)/m$, and $e = c/m$. Then $a + b + x + y + z + e = 1$ and $z \geq x + y$. Thus by Lemma 4.3.1, there exist $p_2, p_3 \in (0, 1)$ such that $p_2 + p_3 = 1$ and $f_2/m = f_3/m \leq 1/7$.

Note that

$$m = 2(b_{13} + x_1 + x_3 + b_{12} + x_1 + x_2) + (a_1 + a_2 + a_1 + a_3) + c \leq 2 + 2x_1.$$

Since $b_{ij} \geq \max\{2x_i, 2x_j\}$ for $1 \leq i \neq j \leq 3$, we have $5x_1 \leq x_1 + b_{12} + b_{13} \leq 1$. Hence $x_1 \leq 1/5$, and so $m \leq 12/5$. Therefore, $f_2 = f_3 \leq (12/5)/7 < 0.35$; so (i) holds and we may assume (1).

We now write $f_i(p_i)$ for f_i, considering it as a function of p_i over $[0, 1]$ (while fixing the other parameters). Differentiating with respect to p_i, we have $f_i'(p_i) = -\alpha_i - 2(1 - p_i)\beta_i - 3(1 - p_i)^2 c \leq 0$ and $f_i''(p_i) = 2\beta_i + 6(1 - p_i)c \geq 0$. Note from (1) that $f'(p_i) < 0$ with the possible exception when $p_i = 1$. So

(2) each $f_i(p_i)$ is both decreasing and convex over $[0, 1]$.

Because of (2), we approximate $f_i(p_i)$ (for each i) with the line $h_i(p_i)$ through the the points $(0, f_i(0))$ and $(1, f_i(1))$ in the Euclidean plane. Hence $h_i(p_i) = (1 - p_i)\gamma_i$. It is also convenient to consider the reflection of $f_3(p_3)$ with respect to the line $p_3 = 1/2$, namely $f_4(p_3) = f_3(1-p_3) = p_3\alpha_3 + p_3^2\beta_3 + p_3^3 c$. Let $h_4(p_3) = \gamma_3 p_3$, which is the reflection of $h_3(p_3)$ with respect to the line $p_3 = 1/2$.

By (2) and by definition, we have

(3) $f_4(p_3)$ is convex and increasing over $[0, 1]$; and for $i = 1, 2, 3, 4$, $f_i(p_i) \leq h_i(p_i)$ when $p_i \in [0, 1]$.

For each $0 \leq \alpha \leq \gamma_1$ and for $i = 1, 2, 3, 4$, let $p_i(\alpha)$ denote the unique root of $f_i(p_i) = \alpha$ in $[0, 1]$, and $q_i(\alpha)$ the unique root of $h_i(q_i) = \alpha$ in $[0, 1]$. Note that from (2) and (3), we have

(4) for $\alpha \in [0, \gamma_1]$ and for $i = 1, 2, 3$, $p_i(\alpha) \leq q_i(\alpha)$, $p_i(\alpha)$ and $q_i(\alpha)$ decreases with α; and $p_4(\alpha)$ and $q_4(\alpha)$ increases with α.

Let (a, b) be the point where f_2 and f_4 intersect, that is, $f_2(a) = f_4(a) = b$; so $p_2(b) = p_4(b) = a$. Let (a', b') be the point where h_2 and h_4 intersect, i.e., $h_2(a') = h_4(a') = b'$. By (2) and (3), we have $b \leq b'$. By solving $h_2(a') = h_4(a') = b'$, we have

$$a' = \frac{\gamma_2}{\gamma_2 + \gamma_3}, \text{ and } b' = \frac{\gamma_2\gamma_3}{\gamma_2 + \gamma_3}.$$

Since $h_3(1 - a') = h_4(a') = b'$ and by definition, we have $q_3(b') = 1 - q_2(b')$; and so $q_2(b') + q_3(b') = 1$.

We may assume

(5) $b' = \frac{\gamma_2\gamma_3}{\gamma_2+\gamma_3} \geq \gamma_1$.

For, suppose $b' < \gamma_1$. Then $b < \gamma_1$; so $p_i(b)$ is defined for $i = 1, 2, 3, 4$. Since f_3 and f_4 are reflections through the line $p_3 = 1/2$, $p_3(b) + p_4(b) = 1$. Since $p_2(b) = p_4(b) = a$ and $p_1(b) > 0$, we have $p_1(b) + p_2(b) + p_3(b) = p_1(b) + 1 > 1$. Also, $p_1(\gamma_1) = 0$, and

$p_2(\gamma_1) + p_3(\gamma_1) \le q_2(\gamma_1) + q_3(\gamma_1) < q_2(b') + q_3(b') = 1$; so $p_1(\gamma_1) + p_2(\gamma_1) + p_3(\gamma_1) < 1$. Since $p_1(\alpha)+p_2(\alpha)+p_3(\alpha)$ is a decreasing function of α, there exists $\alpha \in (b, \gamma_1)$ (and hence by (4), $p_i(\alpha) \in (0, 1)$ for $i = 1, 2, 3$) such that $p_1(\alpha) + p_2(\alpha) + p_3(\alpha) = 1$; so (ii) holds with $f_i(p_i) = \alpha$ for $i = 1, 2, 3$.

We claim that

(6) $\gamma_1 \le 1/2$, $0.4 \le \gamma_2 \le 1$, $0.7 \le \gamma_3 \le 1$, $\gamma_2 + \gamma_3 \ge 1.4$, and $c - \sum_{1\le i<j\le 3} b_{ij} \ge -0.25$.

By (5), $\frac{\gamma_2\gamma_3}{\gamma_2+\gamma_3} \ge \gamma_1$. So by Cauchy-Schwarz,

$$\gamma_2 + \gamma_3 \ge \frac{4}{\frac{1}{\gamma_2} + \frac{1}{\gamma_3}} \ge 4\gamma_1.$$

Hence by (1), $\gamma_2 + \gamma_3 \ge 1.4$. Then $\gamma_2 \ge 0.4$ and, since $\gamma_3 \ge \gamma_2$, $\gamma_3 \ge (\gamma_2 + \gamma_3)/2 \ge 0.7$. Since

$$\gamma_1 + \gamma_2 + \gamma_3 = 2 + c - \sum_{1\le i<j\le 3} b_{ij},$$

we have $5\gamma_1 \le \gamma_1 + \gamma_2 + \gamma_3 = 2 + c - \sum_{i<j} b_{ij}$, and so $\gamma_1 \le 2/5 + (c - \sum_{i<j} b_{ij})/5$. Therefore, since $\gamma_2 + \gamma_3 \le 2$,

$$2 + c - \sum_{i<j} b_{ij} = \gamma_1 + \gamma_2 + \gamma_3 \le 2 + \frac{2}{5} + \frac{c - \sum_{i<j} b_{ij}}{5}.$$

So $c - \sum_{i<j} b_{ij} \le 1/2$, which in turn implies $5\gamma_1 \le 2 + c - \sum_{i<j} b_{ij} \le 5/2$. Thus, $\gamma_1 \le \frac{1}{2}$. By (1), $1.75 \le 5\gamma_1 \le 2 + c - \sum_{i<j} b_{ij}$, which implies $c - \sum_{i<j} b_{ij} \ge -0.25$.

We also claim that

(7) $x_i \le 1.25/9$, for $i = 1, 2, 3$.

Since $b_{ij} \ge 2x_i$ and $b_{ij} \ge 2x_j$, $c + 5x_i \le 1$. By (6), $c - \sum b_{ij} \ge -0.25$; so $c - 4x_i \ge -0.25$. Hence $1 - 5x_i \ge 4x_i - 0.25$, which gives (7).

We now prove that

(8) $f_1(0.18) \le 0.35$.

This is true if $\gamma_1 \le 0.35/0.82$ as $f_1(0.18) \le 0.82\gamma_1$. So we may assume that $\gamma_1 > 0.35/0.82$. From the proof of (6) we see that $c \ge \sum_{i<j} b_{ij} + 5\gamma_1 - 2$. Then, since $b_{12} \ge 2x_2, b_{13} \ge 2x_3$ and $\alpha_1 = b_{23}+x_2+x_3$, we have $c \ge \alpha_1+5\gamma_1-2$. Also, $\gamma_1 \ge \alpha_1+c$. So $\gamma_1-\alpha_1 \ge \alpha_1+5\gamma_1-2$. Therefore, $2\gamma_1 + \alpha_1 \le 1$. Hence, since $\gamma_1 > 0.35/0.82$, we have $\alpha_1 \le 1 - 0.7/0.82$ and $c \ge 5\gamma_1-2 \ge 5\times(0.35/0.82)-2 = 0.11/0.82$. This implies that $0.82\alpha_1+0.82^3c < 0.7(\alpha_1+c)$. Hence, since $0.82^2 < 0.7$, $f_1(0.18) < 0.7\gamma_1 \le 0.35$ (as $\gamma_1 \le 1/2$ by (6)). So we have (8).

Now let $p_1 = 0.18$; then by (8), $f_1(p_1) \le 0.35$. We wish to apply Lemma 4.3.2 to prove the existence of p_2 and p_3 such that $p_2 + p_3 = 1 - p_1 = 0.82$, $f_2(p_2) \le 0.35$ and $f_3(p_3) \le 0.35$. Let $1 - p_2 = p$ and $1 - p_3 = 1.18 - p$. Let

$$m = \beta_2 + \beta_3 + 2(\alpha_2 + \alpha_3 + c),$$

and let $a = \beta_2/m$, $b = \beta_3/m$, $x = \alpha_2/m$, $y = \alpha_3/m$, $e = c/m$, $g_1(p) = f_2(p)/m$, and $g_2(p) = f_3(p)/m$. Then $a + b + 2(x + y + e) = 1$,

$$g_1(p) = p^2a + px + p^3e, \text{ and } \quad g_2(p) = (1.18 - p)^2b + (1.18 - p)y + (1.18 - p)^3e.$$

Note that

$$m = 2a_1 + a_2 + a_3 + 2(b_{12} + b_{13} + 2x_1 + x_2 + x_3 + c) = 2 + 2x_1 - (a_2 + a_3 + 2b_{23}) \le 2 + 2x_1,$$

and

$$\begin{aligned} m &= 2 + 2x_1 - (a_2 + a_3 + 2b_{23}) \\ &= 2 + 2x_1 - \gamma_1 + x_2 + x_3 + c - b_{23} \\ &\le 2 + 2x_1 - \gamma_1 + c \quad (\text{since } b_{23} \ge \max\{2x_1, 2x_3\}) \\ &\le 2 + 2(1.25/9) - 0.35 + c \quad (\text{by (1) and (7)}). \end{aligned}$$

We claim that

(9) $\gamma_2/m > 0.18$ and $\gamma_3/m > 0.18$.

By (7), $m \leq 2 + 2(1.25/9)$; so by (6), $\gamma_3/m \geq 0.7/(2 + 2.5/9) > 0.18$. If $\gamma_2 \geq 0.5$, then $\gamma_2/m \geq 0.5/(2 + 2.5/9) > 0.18$. So we may assume that $\gamma_2 < 0.5$. Then by (6), $\gamma_3 > 0.9$. Hence, $2x_1 \leq b_{13} \leq b_{13} + b_{23} + x_3 + a_3 = 1 - \gamma_3 < 0.1$. So $m \leq 2 + 2x_1 < 2.1$ and, by (6), $\gamma_2/m \geq 0.4/2.1 > 0.18$. Thus, we have (9).

In order to apply Lemma 4.3.2, we need to show that there exists $p \in [0.18, 1]$ such that $g_1(p) = g_2(p)$. To see this, consider g_1, g_2 as functions of p. By (9), we note that

$$g_1(0.18) \leq 0.18(a + x + e) \leq 0.18, \text{ and}$$
$$g_2(0.18) = b + y + e = \gamma_3/m > 0.18.$$

So $g_1(0.18) < g_2(0.18)$. Similarly, we can show $g_1(1) > 0.18 \geq g_2(1)$. By (2), $g_1(p)$ is an increasing function, and $g_2(p)$ is a decreasing function. So there exists $p \in (0.18, 1)$ such that $g_1(p) = g_2(p)$.

We can now apply Lemma 4.3.2. As a consequence, $g_1(p) = g_2(p) \leq (1.18^2/8)(1 - 0.82e)$, so $f_2(p) = f_3(p) \leq (1.18^2/8)(m - 0.82c)$. If $c \leq 0.35$ then, since $m \leq 2 + 2(1.25/9) - 0.35 + c$,

$$f_2(p) = f_3(p) \leq \frac{1.18^2}{8}(2 + 2.5/9 - 0.35 + 0.18 \times 0.35) < 0.347 < 0.35.$$

So we may assume $c > 0.35$. Then, since $m \leq 2 + 2x_1 \leq 2 + 2.5/9$ by (7),

$$f_2(p) = f_3(p) \leq \frac{1.18^2}{8}(2 + 2.5/9 - 0.82 \times 0.35) < 0.35.$$

Note that $p_2 = 1 - p$ and $p_3 = p - 0.18$. Since $p \in (0.18, 1)$, we have $p_2, p_3 \in (0, 1)$. Clearly, $p_1 + p_2 + p_3 = 1$. So (i) holds, which completes the proof of Lemma4.2.1. ∎

In order to prove Lemma 4.2.2, we first deal with the special case when $b_{ij} = x_i + x_j$ for $1 \leq i < j \leq 3$.

Lemma 4.4.1. *Let $b_i, y_i \in \mathbf{R}^+$ for $i = 1, 2, 3$ such that $\sum_{i=1}^{3}(3y_i + b_i) = 2$. Suppose there exist $q_i \in (0, 1)$, $i = 1, 2, 3$, such that $q_1 + q_2 + q_3 = 2$ and $2y_1q_1 + b_1q_1^2 = 2y_2q_2 + b_2q_2^2 = 2y_3q_3 + b_3q_3^2$. Then for $i = 1, 2, 3$, $2y_iq_i + b_iq_i^2 \leq 0.35$.*

Proof. For convenience, let $f_i := 2y_iq_i + b_iq_i^2$, $i = 1,2,3$. Let $\mathscr{D}$ denote the set of all points $(b_1, b_2, b_3, y_1, y_2, y_3, q_1, q_2, q_3)$ such that $b_i, y_i \in \mathbf{R}^+$ and $q_i \in [0,1]$ for $i = 1,2,3$,

$$\sum_{i=1}^{3}(3y_i + b_i) = 2,$$

$$q_1 + q_2 + q_3 = 2, \text{ and}$$

$$f_1 = f_2 = f_3.$$

So $\mathscr{D}$ is a compact subset of $[0,2]^3 \times [0,2/3]^3 \times [0,1]^3$. Note that $\mathscr{D} \neq \emptyset$ by assumption of the lemma. Let

$$\mathbf{v} := (b_1, b_2, b_3, y_1, y_2, y_3, q_1, q_2, q_3) \in \mathscr{D}$$

such that $f_1(\mathbf{v})$ is the maximum of f_1 over $\mathscr{D}$. It suffices to show that $f_1(\mathbf{v}) \leq 0.35$.

We may assume that $q_i \neq 0$ for $i = 1,2,3$; as otherwise we have $f_i(\mathbf{v}) = 0 < 0.35$ for $i = 1,2,3$. Thus, since $f_1 = f_2 = f_3$, we see that if $f_i = 0$ for some $i \in \{1,2,3\}$ then $b_i = y_i = 0$ for $i = 1,2,3$, contradicting the condition that $\sum_{i=1}^{3}(3y_i + b_i) = 2$. Hence, we have

(1) for each $i \in \{1,2,3\}$, $q_i > 0$, and $b_i > 0$ or $y_i > 0$.

We may assume that

(2) there exists some $i \in \{1,2,3\}$ such that $b_i > 0$.

For, suppose $b_i = 0$ for $i = 1,2,3$. Then $f_i = 2y_iq_i$ and $y_i > 0$ (by (1)) for $i = 1,2,3$, and $y_1 + y_2 + y_3 = 2/3$. Hence, by Cauchy-Schwarz,

$$\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} \geq \frac{9}{y_1 + y_2 + y_3} = \frac{27}{2}.$$

Setting $f_1 = f_2 = f_3 = \alpha$, we have $q_i = \alpha/2y_i$ for $i = 1,2,3$. Therefore, since $q_1+q_2+q_3 = 2$,

$$\alpha = \frac{4}{\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}} \leq \frac{8}{27} < 0.35.$$

We may also assume that

(3) there exists some $j \in \{1,2,3\}$ such that $y_j > 0$.

For, otherwise, $y_1 = y_2 = y_3 = 0$. Then $f_i = b_i q_i^2$ and $b_i > 0$ (by (1)) for $i = 1,2,3$, and $b_1 + b_2 + b_3 = 2$. Setting $f_1 = f_2 = f_2 = \alpha$, we have $q_i = \sqrt{\alpha/b_i}$. Since $q_1 + q_2 + q_3 = 2$, we have (by Cauchy-Schwarz),

$$\alpha = \frac{4}{\left(\frac{1}{\sqrt{b_1}} + \frac{1}{\sqrt{b_2}} + \frac{1}{\sqrt{b_3}}\right)^2} \leq \frac{4}{81}\left(\sqrt{b_1} + \sqrt{b_2} + \sqrt{b_3}\right)^2 \leq \frac{4}{9}\frac{b_1 + b_2 + b_3}{3} = \frac{8}{27} < 0.35.$$

We may further assume that

(4) there exists some $i \in \{1,2,3\}$ such that $b_i y_i \neq 0$.

Otherwise, we have two cases (by symmetry): $y_1 = y_2 = b_3 = 0$, or $b_1 = b_2 = y_3 = 0$

First, assume $y_1 = y_2 = b_3 = 0$. Then, $b_1 > 0$, $b_2 > 0$, $y_3 > 0$, $b_1 + b_2 + 3y_3 = 2$,

$$f_1 = b_1 q_1^2, \; f_2 = b_2 q_2^2, \text{ and } f_3 = 2y_3 q_3.$$

Setting $\alpha = f_1 = f_2 = f_3$ and using $q_1 + q_2 + q_3 = 2$, we have

$$\frac{\sqrt{\alpha}}{\sqrt{b_1}} + \frac{\sqrt{\alpha}}{\sqrt{b_2}} + \frac{\alpha}{2y_3} = 2.$$

So

$$\sqrt{\alpha} = \frac{4}{\sqrt{(1/\sqrt{b_1} + 1/\sqrt{b_2})^2 + 4/y_3} + (1/\sqrt{b_1} + 1/\sqrt{b_2})}.$$

Note that

$$\left(\frac{1}{\sqrt{b_1}} + \frac{1}{\sqrt{b_2}}\right)^2 \geq \frac{4}{\sqrt{b_1 b_2}} \geq \frac{8}{b_1 + b_2} = \frac{8}{2 - 3y_3},$$

so

$$\sqrt{\alpha} \leq \frac{4}{\sqrt{\frac{8}{2-3y_3} + \frac{4}{y_3}} + \sqrt{\frac{8}{2-3y_3}}}.$$

Let $f(y_3) := \sqrt{8/(2-3y_3) + 4/y_3} + \sqrt{8/(2-3y_3)}$. Note that $y_3 \in (0, 2/3)$, and

$$f(y_3) \geq \begin{cases} \sqrt{4+20} + \sqrt{4}, & \text{if } y_3 \in (0, 1/5]; \\ \sqrt{8/(7/5) + 16} + \sqrt{8/(7/5)}, & \text{if } y_3 \in (1/5, 1/4]; \\ \sqrt{8/(5/4) + 12} + \sqrt{8/(5/4)}, & \text{if } y_3 \in (1/4, 1/3]; \\ \sqrt{8+8} + \sqrt{8}, & \text{if } y_3 \in (1/3, 1/2]; \\ \sqrt{16+6} + \sqrt{16}, & \text{if } y_3 \in (1/2, 2/3). \end{cases}$$

Therefore, $f(y_3) \geq 6.819$, and hence $\alpha \leq (4/6.819)^2 < 0.35$

Now assume $b_1 = b_2 = y_3 = 0$. Then $y_1 > 0$, $y_2 > 0$, $b_3 > 0$, $3(y_1 + y_2) + b_3 = 2$,

$$f_1 = 2y_1q_1,\ f_2 = 2y_2q_2,\ \text{and } f_3 = b_3q_3^2.$$

Again, setting $\alpha = f_1 = f_2 = f_3$ and using $q_1 + q_2 + q_3 = 2$, we have

$$\frac{\alpha}{2y_1} + \frac{\alpha}{2y_2} + \frac{\sqrt{\alpha}}{\sqrt{b_3}} = 2,$$

So

$$\sqrt{\alpha} = \frac{4}{\sqrt{1/b_3 + 4(1/y_1 + 1/y_2)} + 1/\sqrt{b_3}}.$$

Note that $1/y_1 + 1/y_2 \geq 4/(y_1 + y_2) = 12/(2 - b_3)$. Hence

$$\sqrt{\alpha} \leq \frac{4}{\sqrt{1/b_3 + 48/(2 - b_3)} + 1/\sqrt{b_3}}.$$

Let $g(b_3) := \sqrt{1/b_3 + 48/(2 - b_3)} + 1/\sqrt{b_3}$. Note that $b_3 \in (0, 2)$, and

$$g(b_3) \geq \begin{cases} \sqrt{3 + 48/(2-0)} + \sqrt{3}, & \text{if } b_3 \in (0, 1/3]; \\ \sqrt{2 + 48/(2-1/3)} + \sqrt{2}, & \text{if } b_3 \in (1/3, 1/2]; \\ \sqrt{3/2 + 48/(2-1/2)} + \sqrt{3/2}, & \text{if } b_3 \in (1/2, 2/3]; \\ \sqrt{2/3 + 48/(2-2/3)} + \sqrt{2/3}, & \text{if } b_3 \in (2/3, 3/2]; \\ \sqrt{1/2 + 48/(2-3/2)} + \sqrt{1/2}, & \text{if } b_3 \in (3/2, 2). \end{cases}$$

Therefore, $g(b_3) \geq 6.87$, and hence $\alpha \leq (4/6.87)^2 < 0.35$.

By (4) and by symmetry, we may assume that

(5) $b_3y_3 \neq 0$.

We may further assume that

(6) $b_1y_1 = 0$ and $b_2y_2 = 0$.

For, otherwise, by symmetry, assume $b_2y_2 > 0$. Then $\mathbf{v}$ is a solution to the following optimization problem:

Maximize f_1

subject to

$$h_1 := f_1 - f_2 = 0,$$

$$h_2 := f_1 - f_3 = 0,$$

$$h_3 := 3(y_1 + y_2 + y_3) + (b_1 + b_2 + b_3) - 2 = 0,$$

$$h_4 := q_1 + q_2 + q_3 - 2 = 0.$$

Applying the method of Lagrange multipliers, we have, for each $u \in \{y_i, b_i : i = 2, 3\}$,

$$\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u.$$

Thus,

$$\text{for } u = y_2, \text{ we have } 0 = \lambda_1(-2q_2) + 3\lambda_3,$$

$$\text{for } u = y_3, \text{ we have } 0 = \lambda_2(-2q_3) + 3\lambda_3,$$

$$\text{for } u = b_2, \text{ we have } 0 = \lambda_1(-q_2^2) + \lambda_3,$$

$$\text{for } u = b_3, \text{ we have } 0 = \lambda_2(-q_3^2) + \lambda_3.$$

Clearly, if $\lambda_i = 0$ for some $i \in \{1, 2, 3\}$ then $\lambda_i = 0$ for all $i = 1, 2, 3$ (since $q_i > 0$ by (1)). In fact, $\lambda_i \neq 0$ for all $i = 1, 2, 3$. To see this we notice that either $b_1 > 0$ or $y_1 > 0$, so $\mathbf{v}$ also satisfies $\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u$ for $u = b_1$ or $u = y_1$. For $u = b_1$, we have $q_1^2 = \lambda_1 q_1^2 + \lambda_2 q_1^2 + \lambda_3$, and for $u = y_1$ we have $2q_1 = \lambda_1 2q_1 + \lambda_2 2q_1 + 3\lambda_3$. In either case, we see that $\lambda_i \neq 0$ (since $q_1 > 0$).

Now using the partial derivatives with respect to b_2 and y_2, we get $q_2 = 2/3$; and using the partial derivatives with respect to b_3 and y_3 we obtain $q_3 = 2/3$. So $q_1 = 2/3$ since $q_1 + q_2 + q_3 = 2$. Then for $i = 1, 2, 3$,

$$f_i = \frac{4}{3}y_i + \frac{4}{9}b_i = \frac{4}{9}(3y_i + b_i).$$

Since $f_1 = f_2 = f_3$ and $\sum_{i=1}^{3}(3y_i + b_i) = 2$, we get $3y_i + b_i = 2/3$ for $i = 1, 2, 3$, and hence $f_i = 8/27 < 0.35$. This proves (6)

By (5) and (6), we have three cases to consider: $b_1 = b_2 = 0$; $y_1 = y_2 = 0$; $y_1 = b_2 = 0$ or $b_1 = y_2 = 0$. Let h_1, h_2, h_3, h_4 be defined as in the proof of (6).

Case 1. $b_1 = b_2 = 0$.

Then $y_1 > 0$, $y_2 > 0$, $f_1 = 2y_1q_1$, $f_2 = 2y_2q_2$, $f_3 = 2y_3q_3 + b_3q_3^2$. Moreover, $\mathbf{v}$ is a critical point of f_1 subject to $h_1 = h_2 = h_3 = h_4 = 0$, all considered as functions of y_1, y_2, y_3, b_3. Hence for $u \in \{y_1, y_2, y_3, b_3\}$, $\mathbf{v}$ satisfies

$$\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u.$$

So

$$\begin{aligned}
&\text{for } u = y_1, \text{ we have } 2q_1 = \lambda_1(2q_1) + \lambda_2(2q_1) + 3\lambda_3,\\
&\text{for } u = y_2, \text{ we have } 0 = \lambda_1(-2q_2) + 3\lambda_3,\\
&\text{for } u = y_3, \text{ we have } 0 = \lambda_2(-2q_3) + 3\lambda_3,\\
&\text{for } u = b_3, \text{ we have } 0 = \lambda_2(-q_3^2) + \lambda_3.
\end{aligned}$$

Clearly, $\lambda_i \neq 0$ for $i = 1, 2, 3$. So from the partial derivatives with respect to b_3 and y_3, we have $q_3 = 2/3$, and hence $q_1 + q_2 = 4/3$. Set $\alpha := 2y_1q_1 = 2y_2q_2 = 4(3y_3 + b_3)/9$. In particular, $\alpha = 4(3y_3 + b_3)/9 = 4(2 - 3(y_1 + y_2))/9$, and so $y_1 + y_2 = 2/3 - 3\alpha/4$. Using $q_1 + q_2 = 4/3$ and Cauchy-Schwarz, we get

$$\frac{4}{3} = \frac{\alpha}{2y_1} + \frac{\alpha}{2y_2} \geq \frac{2\alpha}{y_1 + y_2} = \frac{2\alpha}{2/3 - 3\alpha/4}.$$

This implies $\alpha \leq 8/27 < 0.35$.

Case 2. $y_1 = y_2 = 0$.

Then $b_1 > 0$, $b_2 > 0$, $f_1 = b_1q_1^2$, $f_2 = b_2q_2^2$ and $f_3 = 2y_3q_3 + b_3q_3^2$. Now $\mathbf{v}$ is a critical point of f_1 subject to $h_1 = h_2 = h_3 = h_4 = 0$, all considered as functions of b_1, b_2, b_3, y_3.

Hence for $u \in \{b_1, b_2, b_3, y_3\}$, $\mathbf{v}$ satisfies

$$\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u.$$

Thus,

$$\begin{aligned}
&\text{for } u = b_1, \text{ we have } q_1^2 = \lambda_1(q_1^2) + \lambda_2(q_1^2) + \lambda_3,\\
&\text{for } u = b_2, \text{ we have } 0 = \lambda_1(-q_2^2) + \lambda_3,\\
&\text{for } u = b_3, \text{ we have } 0 = \lambda_2(-q_3^2) + \lambda_3\\
&\text{for } u = y_3, \text{ we have } 0 = \lambda_2(-2q_3) + 3\lambda_3.
\end{aligned}$$

Clearly, $\lambda_i \neq 0$ for $i = 1, 2, 3$. So from the partial derivatives with respect to b_3 and y_3, we have $q_3 = 2/3$, and hence $q_1 + q_2 = 4/3$. Setting $\alpha := y_1 q_1^2 = y_2 q_2^2 = 4(3y_3 + b_3)/9$, we have $q_i = \sqrt{\alpha}/\sqrt{b_i}$ for $i = 1, 2$, $3y_3 + b_3 = 9\alpha/4$, and $b_1 + b_2 = 2 - 9\alpha/4$. So

$$\frac{4}{3} = \frac{\sqrt{\alpha}}{\sqrt{b_1}} + \frac{\sqrt{\alpha}}{\sqrt{b_2}} \geq \frac{2\sqrt{\alpha}}{\sqrt{\sqrt{b_1}\sqrt{b_2}}} \geq \frac{2\sqrt{2\alpha}}{\sqrt{b_1 + b_2}} = \frac{2\sqrt{2\alpha}}{\sqrt{2 - 9\alpha/4}}.$$

This gives $\alpha \leq 8/27 < 0.35$.

Case 3. $y_1 = b_2 = 0$, or $y_2 = b_1 = 0$.

By symmetry, we may assume that $y_1 = b_2 = 0$. Then $b_1 > 0$, $y_2 > 0$, $b_1 + 3y_2 + (3y_3 + b_3) = 2$, $f_1 = b_1 q_1^2$, $f_2 = 2y_2 q_2$, and $f_3 = 2y_3 q_3 + b_3 q_3^2$.

So $\mathbf{v}$ is a critical point of f_1 subject to $h_1 = h_2 = h_3 = h_4 = 0$, all considered as functions of b_1, y_2, b_3, y_3. Hence $\mathbf{v}$ satisfies $\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u$ for $u \in \{b_1, y_2, b_3, y_3\}$. Thus,

$$\begin{aligned}
&\text{for } u = b_1, \text{ we have } q_1^2 = \lambda_1(q_1^2) + \lambda_2(q_1^2) + \lambda_3,\\
&\text{for } u = y_2, \text{ we have } 0 = \lambda_1(-2q_2) + 3\lambda_3,\\
&\text{for } u = b_3, \text{ we have } 0 = \lambda_2(-q_3^2) + \lambda_3\\
&\text{for } u = y_3, \text{ we have } 0 = \lambda_2(-2q_3) + 3\lambda_3.
\end{aligned}$$

Clearly, $\lambda_i \neq 0$ for $i = 1, 2, 3$. So from the partial derivatives with respect to b_3 and y_3, we have $q_3 = 2/3$, and hence $q_1 + q_2 = 4/3$.

Set $\alpha = f_1(\mathbf{v}) = f_2(\mathbf{v}) = f_3(\mathbf{v})$. Then

$$2 = b_1 + 3y_2 + (3y_3 + b_3) = \left(\frac{1}{q_1^2} + \frac{3}{2q_2} + \frac{9}{4}\right)\alpha = \left(\frac{1}{q_1^2} + \frac{3}{2(4/3 - q_1)} + \frac{9}{4}\right)\alpha.$$

Let $h(q_1) := 1/q_1^2 + 3/(2(4/3 - q_1))$. Note that $q_1 \in (0, 4/3)$ and

$$h(q_1) \geq \begin{cases} 4 + 3/(2(4/3 - 0)), & \text{if } q_1 \in (0, 1/2]; \\ 9/4 + 3/(2(4/3 - 1/2)), & \text{if } q_1 \in (1/2, 2/3]; \\ 25/16 + 3/(2(4/3 - 2/3))), & \text{if } q_1 \in (2/3, 4/5]; \\ 1 + 3/(2(4/3 - 4/5)), & \text{if } q_1 \in (4/5, 1]; \\ 9/16 + 3/(2(4/3 - 1)), & \text{if } q_1 \in (1, 4/3). \end{cases}$$

So $h(q_1) \geq 3.8125$, and hence $\alpha = 2/(h(q_1) + 9/4) \leq 2/(3.8125 + 9/4) < 0.35$. ∎

Proof of Lemma 4.2.2. For any permutation ijk of $\{1, 2, 3\}$, and let $y_k = x_i + x_j$ and $b_k = a_i + a_j$. Then

$$f_k = (1 - p_k)(b_{ij} + y_k) + (1 - p_k)^2 b_k.$$

Set $\alpha = f_1(p_1) = f_2(p_2) = f_3(p_3)$. Note that we may assume $\alpha > 0$ (otherwise we are done); and hence $b_{ij} + y_k + b_k > 0$ for $k = 1, 2, 3$. Since $p_k \in (0, 1)$, $1 - p_k \in (0, 1)$; and hence by solving $f_k(p_k) = \alpha$ we get

$$1 - p_k = \frac{2\alpha}{\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha} + (b_{ij} + y_k)}.$$

We wish to show that $\alpha \leq 0.35$; so we consider the following optimization problem.

Maximize α

Subject to

$$g_1 := \sum_{k=1}^{3} \frac{2\alpha}{\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha} + (b_{ij} + y_k)} - 2 = 0,$$

$$g_2 := b_{12} + b_{13} + b_{23} + \frac{1}{2}(y_1 + y_2 + y_3 + b_1 + b_2 + b_3) - 1 = 0,$$

$$b_{ij} \geq y_k \geq 0, \text{ for } \{i, j, k\} = \{1, 2, 3\}.$$

Here, g_1, g_2 are considered as functions of α, b_{ij}, b_k, y_k. By the assumption of the lemma, the feasible region of this optimization problem is nonempty.

Claim 1. α is maximized only when $b_{ij} = y_k$ or $y_k = 0$, for all $\{i, j, k\} = \{1, 2, 3\}$.

For, suppose $b_{ij} > y_k > 0$ for some permutation ijk of $\{1, 2, 3\}$. By applying the method of Lagrange multipliers, we have $\partial\alpha/\partial u = \lambda_1 \partial g_1/\partial u + \lambda_2 \partial g_2/\partial u$, where $u \in \{\alpha, b_{ij}, y_k\}$. So

$$\begin{aligned}
&\text{for } u = b_{ij}, \quad 0 = \lambda_1 \frac{-2\alpha\left(b_{ij} + y_k + \sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha}\right)}{\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha}\left(\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha} + (b_{ij} + y_k)\right)^2} + \lambda_2, \\
&\text{for } u = y_k, \quad 0 = \lambda_1 \frac{-2\alpha\left(b_{ij} + y_k + \sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha}\right)}{\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha}\left(\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha} + (b_{ij} + y_k)\right)^2} + \frac{\lambda_2}{2}, \\
&\text{for } u = \alpha, \quad 1 = \lambda_1 \frac{\partial g_1}{\partial \alpha} + \lambda_2 \frac{\partial g_2}{\partial \alpha}.
\end{aligned}$$

The first two equations give $\lambda_1 = \lambda_2 = 0$, which contradicts the third equation.

Therefore, the maximum of α is achieved when $b_{ij} = y_k$ for some permutation ijk of $\{1, 2, 3\}$, or when $y_k = 0$ for some $k \in \{1, 2, 3\}$; so Claim 1 follows.

Claim 2. We may assume that α is maximized when $b_{ij} > y_k$ for some $\{i, j, k\} = \{1, 2, 3\}$.

For, otherwise, the maximum of α is achieved when $b_{ij} = y_k$ for all permutations ijk of $\{1, 2, 3\}$. Set $q_k = 1 - p_k$ for $k = 1, 2, 3$; and so $f_k = 2y_k q_k + b_k q_k^2$ and $3(y_1 + y_2 + y_3) + b_1 + b_2 + b_3 = 2$. We can now apply Lemma 4.4.1 and conclude that $f_k \leq 0.35$ for $k = 1, 2, 3$. So Claim 2 holds.

From Claim 1 and Claim 2, we deduce

Claim 3. α is maximized when there exists a permutation ijk of $\{1, 2, 3\}$ such that $b_{ij} > 0$ and $y_k = 0$ (so $x_i = x_j = 0$).

We consider three cases.

Case 1. α is maximized when $x_k = b_{ik} = b_{jk} = 0$ and $b_k = 0$.

Then $b_{ij} + a_k = 1$, $f_k = (1 - p_k)b_{ij}$, $f_i = (1 - p_i)^2 a_k$, and $f_j = (1 - p_j)^2 a_k$.

Since $f_i = f_j$, we have $p_i = p_j$. In particular, $p_i \in (0, 1/2)$ as $p_i + p_j + p_k = 1$. Since $b_{ij} = 1 - a_k$ and $f_k = f_i$, we have $2p_i(1 - a_k) = (1 - p_i)^2 a_k$. Therefore, $a_k = 2p_i/(1 + p_i^2)$, and so,

$$\alpha = \frac{2p_i(1-p_i)^2}{1+p_i^2} = \frac{4}{1+p_i^2} + 2p_i - 4.$$

Differentiating with respect to p_i, we have $\alpha'(p_i) = 2 - 8p_i/(1+p_i^2)^2$ and $\alpha''(p_i) < 0$. Thus $\alpha(p_i)$ has maximum when $\alpha'(p_i) = 0$, i.e., when $(1+p_i^2)^2 = 4p_i$. We now estimate $\alpha(p_i)$ subject to $(1+p_i^2)^2 = 4p_i$. Considering the function $g(x) := (1+x^2)^2 - 4x$ for $x \in (0, 1/2)$, we see that $g'(x) = 4(1+x^2)x - 4 < 0$, $g(0.3) < 0$, and $g(0.29) > 0$; so $g(x) = 0$ implies that $x \in (0.29, 0.3)$. Hence, $(1+p_i^2)^2 = 4p_i$ implies $p_i \in (0.29, 0.3)$. On the other hand, $(1+p_i^2)^2 = 4p_i$ implies $\alpha(p_i) = 2/\sqrt{p_i} + 2p_i - 4$. Since the function $h(t) := 2/\sqrt{t} + 2t - 4$ is decreasing over $[0.29, 0.3]$ (because $h' = 2 - t^{-3/2} < 0$ for $t \in [0.29, 0.3]$), we have $\alpha \le \alpha(p_i) = h(p_i) \le h(0.29) = 2/\sqrt{0.29} + 2(0.29) - 4 < 0.35$.

Case 2. α is maximized when $x_k = b_{ik} = b_{jk} = 0$ and $b_k > 0$.

Then $b_{ij} + (b_i + b_j + b_k)/2 = 1$, $f_i = (1-p_i)^2 b_i$, $f_j = (1-p_j)^2 b_j$, and $f_k = (1-p_k)b_{ij} + (1-p_k)^2 b_k$. From $\partial\alpha/\partial b_k = \lambda_1 \partial g_1/\partial b_k + \lambda_2 \partial g_2/\partial b_k$, we obtain

$$0 = \lambda_1 \frac{-4\alpha^2}{\sqrt{(b_{ij}+y_k)^2 + 4b_k\alpha}\left(\sqrt{(b_{ij}+y_k)^2 + 4b_k\alpha} + (b_{ij}+y_k)\right)^2} + \frac{\lambda_2}{2}.$$

Using this and the partial derivatives with respect to $u \in \{\alpha, b_{ij}\}$ (as in the proof of Claim 1), we deduce that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, and

$$4\alpha = b_{ij} + \sqrt{b_{ij}^2 + 4b_k\alpha}.$$

Therefore, α is maximized when $4\alpha = b_{ij} + \sqrt{b_{ij}^2 + 4b_k\alpha}$, that is $4\alpha = b_k + 2b_{ij}$ which implies $p_k = 1/2$ (since $f_k(p_k)$ is decreasing and $f_k(p_k) = \alpha$ has a unique solution).

Write $b_k' := b_k + 2b_{ij}$; then $f_k = (1-p_k)^2 b_k'$ (because $p_k = 1/2$). Note that $(b_k' + b_i + b_j)/2 = b_{ij} + (b_k + b_i + b_j)/2 = 1$. Since $\alpha = f_1 = f_2 = f_3$ and $(1-p_i) + (1-p_j) + (1-p_k) = 2$, we have

$$\frac{\sqrt{\alpha}}{\sqrt{b_k'}} + \frac{\sqrt{\alpha}}{\sqrt{b_i}} + \frac{\sqrt{\alpha}}{\sqrt{b_j}} = 2.$$

Applying Cauchy-Schwarz, we have

$$\alpha = \left(\frac{2}{\frac{1}{\sqrt{b'_k}} + \frac{1}{\sqrt{b_i}} + \frac{1}{\sqrt{b_j}}} \right)^2 \leq 4 \left(\frac{\sqrt{b'_k} + \sqrt{b_i} + \sqrt{b_j}}{9} \right)^2 \leq \frac{4}{9} \frac{b'_k + b_i + b_j}{3} = \frac{8}{27} < 0.35.$$

Case 3. α is maximized when (i) $x_k > 0$, or (ii) $x_k = 0$ and $b_{ik} > 0$ or $b_{jk} > 0$.

We claim that there exist $a'_m, x'_m, b'_{mn} \in \mathbf{R}^+$, for any $1 \leq m \neq n \leq 3$, such that $b'_{mn} = b'_{nm}$,

$$b'_{mn} \geq \max\{2x'_m, 2x'_n\},$$

$$b'_{12} + b'_{23} + b'_{31} + x'_1 + x'_2 + x'_3 + a'_1 + a'_2 + a'_3 = 1,$$

$$b'_{mn} + x'_m + x'_n \geq b_{mn} + x_m + x_n,$$

$$a'_m + a'_n = a_m + a_n, \text{ and}$$

$$b'_{st} + x'_s + x'_t > b_{st} + x_s + x_t \text{ for some } 1 \leq s \neq t \leq 3.$$

There are two cases to consider. First, suppose $x_k > 0$. Then there exists $\delta > 0$ such that $x'_k = x_k - \delta > 0$ and $b'_{ij} = b_{ij} - 2\delta \geq 2\delta$. Let $b'_{ik} = b_{ik} + \delta, b'_{jk} = b_{jk}$ and $x'_i = x'_j = \delta$. In particular, $x_k > \delta$; and so $b_{ik} \geq 2x_k \geq 2\delta$ and $b_{jk} \geq 2x_k \geq 2\delta$. It is easy to verify that the claim holds by setting $a'_i = a_i$, $a'_j = a_j$ and $a'_k = a_k$. Now assume that $x_k = 0$, and $b_{ik} > 0$ or $b_{jk} > 0$. We may assume $b_{ik} > 0$; the case $b_{jk} > 0$ is symmetric. Then there exists $\delta > 0$ such that $b'_{ik} = b_{ik} - \delta/2 \geq \delta$ and $b'_{ij} = b_{ij} - \delta/2 \geq \delta$. Let $b'_{jk} = b_{jk} + \delta/2$ and $x'_i = \delta/2$. It is easy to verify that the claim holds by setting $x'_j = x_j = 0$, $x'_k = x_k = 0$, $a'_i = a_i$, $a'_j = a_j$ and $a'_k = a_k$.

For every permutation mnl of $\{1, 2, 3\}$, let

$$f'_l := (1 - p_l)(b'_{mn} + x'_m + x'_n) + (1 - p_l)^2(a'_m + a'_n).$$

For convenience of comparison, recall that

$$\alpha := f_l = (1 - p_l)(b_{mn} + x_m + x_n) + (1 - p_l)^2(a_m + a_n).$$

By Lemma 4.2.1, there exist $p'_i \in [0, 1]$ with $p'_1 + p'_2 + p'_3 = 1$ such that $f'_l(p'_l) \leq 0.35$ for $l = 1, 2, 3$, or $f'_1(p'_1) = f'_2(p'_2) = f'_3(p'_3)$ and $p'_i \in (0, 1)$. Since $p_i \in [0, 1]$ and $p_1 + p_2 + p_3 = 1$, there exists some l such that $1 - p_l \leq 1 - p'_l$.

If $f_i'(p_i') \leq 0.35$ for $i = 1, 2, 3$ then, since $b'_{mn}+x'_m+x'_n \geq b_{mn}+x_m+x_n$ and $a'_m+a'_n = a_m+a_n$ for all $\{m, n, l\} = \{1, 2, 3\}$, we have $f_l(p_l) \leq f_l'(p_l') \leq 0.35$. Hence $\alpha \leq 0.35$.

We may thus assume $f_1'(p_1') = f_2'(p_2') = f_3'(p_3')$. Suppose $1 - p_l < 1 - p_l'$. Then, since $b'_{mn} + x'_m + x'_n \geq b_{mn} + x_m + x_n$ and $a'_m + a'_n = a_m + a_n$, and because $b_{mn} + x_m + x_n + a_m + a_n > 0$ (see the beginning of the proof), we have $f_l(p_l) < f_l'(p_l')$, contradicting the maximality of α. So $1 - p_l = 1 - p_l'$. Then $(1 - p_m') + (1 - p_n') = (1 - p_m) + (1 - p_n)$. So we may assume that $1 - p_n \leq 1 - p_n'$. By the same argument above for $1 - p_l' = 1 - p_l$, we derive the contradiction $f_n(p_n) < f_n'(p_n')$ if $1 - p_n < 1 - p_n'$; and so we must have $1 - p_n' = 1 - p_n$. Hence we have $p_i' = p_i$ for $i = 1, 2, 3$. Recall that there exist $1 \leq s \neq t \leq 3$ such that $b'_{st}+x'_s+x'_t > b_{st}+x_s+x_t$. Let $r \in \{1, 2, 3\} \setminus \{s, t\}$. Then $f_r(p_r) < f_r'(p_r')$, again a contradiction to the maximality of α. This proves Lemma 4.2.2. ∎

CHAPTER V

CONCLUDING REMARKS

Theorem 1.3.2 implies Conjecture 1.3.3 when the number of edges in the graph is sufficiently large; however to prove the entire conjecture is quite challenging. The error term in Theorem 2.3.5 is $O(m_2^{4/5})$, but Bollobás and Scott ask in [12] whether it is possible to replace the error term by $O(\sqrt{m_1 + m_2})$ or $O(\sqrt{m_2})$, which is still open.

For Problem 1.3.5, the general bound in Theorem 1.3.6 does not seem to be optimal. Also it is interesting to ask a general version of Problem 1.3.5: for any integer $r \in [3, k-1]$, find a k-partition $V_1, ..., V_k$ that minimizes $\max\{e(V_{i_1} \cup .... \cup V_{i_r}) : 1 \leq i_1 < i_2 < ... < i_r \leq k\}$. In Chapter 3, we further show that Conjecture 1.3.7 holds for dense graphs as well as asymptotically for $k = 3, 4, 5$; to the best of our knowledge, Conjecture 1.3.7 is still standing in general.

Conjecture 1.3.10 is open for $r \geq 4$. In fact, Bollobás and Scott made an asymptotic version of Conjecture 1.3.10: for integers $r, k \geq 2$, every r-uniform hypergraph with m edges has a vertex-partition into k sets, each of which meets at least $(1+o(1))(1-(1-1/k)^r)m$ edges. Note that, this bound is the expected number of edges meeting each set in a random k-partition. For $r = k = 3$, the bound becomes $19m/27 + o(m)$. One of the reasons why our proof does not give a closer bound to $19/27$ is that in Lemma 4.2.1, we can not get a smaller bound than 0.35 for (i) of Lemma 4.2.1.

REFERENCES

[1] N. Alon, Bipartite subgraphs, *Combinatorica* **16** (1996) 301–311.

[2] N. Alon, B. Bollobás, M. Krivelevich and B. Sudakov, Maximum cuts and judicious partitions in graphs without short cycles, *J. Combin. Theory Ser. B* **88** (2003) 329–346.

[3] N. Alon and E.Halperin, Bipartite subgraphs of integer weighted graphs, *Discrete Math.* **181** (1998) 19–29.

[4] N. Alon, M. Krivelevich and B. Sudakov, MaxCut in H-free graphs, *Combinatorics, Probability and Computing* **14** (2005) 629-647.

[5] K. Azuma, Weighted sums of certain dependent random variables, *Tokuku Math. J.* **19** (1967) 357–367.

[6] P. Berman and M. Karpinski, On some tighter inapproximability results, (extended abstract) Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, **1644** (1999) 200–209.

[7] B. Bollobás, B. Reed and A. Thomason, An extremal function for the achromatic number *in Graph Structure Theory* (Eds. N. Robertson and P. Seymour) pp. 161–165, American Mathematical Society, Providence, Rhode Island, 1993.

[8] B. Bollobás and A. D. Scott, On judicious partitions, *Period. Math. Hungar.* **26** (1993) 127–139.

[9] B. Bollobás and A.D. Scott, Judicious partitions of hypergraphs, *J. Combin. Theory Ser. A* **78** (1997) 15–31.

[10] B. Bollobás and A. D. Scott, Exact bounds for judicious partitions of graphs, *Combinatorica* **19** (1999) 473–486.

[11] B. Bollobás and A.D. Scott, Judicious partitions of 3-uniform hypergraphs, *Europ. J. Combinatorics* **21** (2000) 289-300.

[12] B. Bollobás and A. D. Scott, Problems and results on judicious partitions, *Random Structures and Algorithms* **21** (2002) 414-430.

[13] B. Bollobás and A. D. Scott, Better bounds for Max Cut, in Contemporary Comb, Bolyai Soc Math Stud 10, *János Bolyai Math Soc*, Budapest, (2002) 185–246.

[14] B. Bollobás and A. D. Scott, Judicious partitions of bounded-degree graphs, *J. Graph Theory* **46** (2004) 131–143.

[15] B. Bollobás and A. D. Scott, Max k-cut and judicious k-partitions, *Discrete Math.* **310** (2010) 2126–2139.

[16] J. A. Bondy and S. C. Locke, Largest bipartite subgraphs in triangle-free graphs with maximum degree three, *J. Graph Theory* **10** (1986) 477–504.

[17] C. S. Edwards, Some extremal properties of bipartite graphs, *Canadian J. math.* **25** (1973) 475–485.

[18] C. S. Edwards, An improved lower bound for the number of edges in a largest bipartite subgraph, in *Proc. 2nd Czechoslovak Symposium on Graph Theory*, Prague (1975) 167–181.

[19] P. Erdős, Problems and results in Graph Theory and Combinatorial Analysis, in: *Graph Theory and Related Topics*, J. A. Bondy and U. S. R. Murty (Eds.), Proc. Conf. Waterloo, 1977, Academic Press, New York, 1979, 153–163.

[20] P. Erdős, Some recent problems in Combinatorics and Graph Theory, Proc. 26 th Southeastern International Conference on Graph Theory, Combinatorics and Computing, Boca Raton, 1995, Congressus Numerantium.

[21] S. Even and Y. Shiloach, NP-completeness of several arrangement problems, Technical Report **43**, Department of Computer Science, Technion, Haifa, Israel (1975).

[22] U. Feige, M. Karpinski, and M. Langberg, Improved approximation of max-cut on graphs of bounded degree, *J. Algorithms* **43** (2002) 201–219.

[23] M. R. Garey, D. S. Johnson, and L. J. Stockmeyer, Some simplified NP-complete graph problems, *Theor. Comput. Sci* **1** (1976) 237–267.

[24] M. X. Goemans and D. P. Williamson, Improved approximation algorithms for maximum cut and satisfiability using semidefinite programming, *J. ACM* **42** (1995) 1115–1145.

[25] F. Hadlock, Finding a maximum cut of a planar graph in polynomial time, *SIAM J. Computing* **4** (1975) 221–225.

[26] J. Haslegrave, Judicious partitions of 3-uniform hypergraphs, arXiv:0911.0563v1.

[27] W. Hoeffding, Probability inequalities for sums of bounded random variables, *J. Amer. Statist. Assoc.* **58** (1963) 13–30.

[28] G. Hopkins and W. Staton, Extremal bipartite subgraphs of cubic triangle-free graphs, *J. Graph Theory* **6** (1982) 115–121.

[29] R. M. Karp, Reducibility among combinatorial problems, in *Complexity of Computer Computations*, (R. Miller and J. Thatcher, eds) Plenum Press, New York, (1972) 85–103.

[30] M. Laurent, Max-Cut problem, in Annotated Bibliographies in Combinatorial Optimization, M. DellAmico, F. Maffioli, and S. Martello, Editors, Wiley, New York, (1997) 241–259.

[31] S. C. Locke, Maximum k-colourable subgraphs, *J. Graph Theory* **6** (1982) 123–132.

[32] J. Ma, P.-L. Yen, and X. Yu, On several partition problems of Bollobás and Scott, *J. Combin. Theory Ser. B* **100** (6) (2010) 631–649.

[33] J. Ma and X. Yu, Bounds for pairs in partitions of graphs, *Discrete Math.* **310** (15-16) (2010) 2069–2081.

[34] J. Ma and X. Yu, Partitioning 3-uniform hypergraphs, Submitted.

[35] G. Malle, On Maximum bipartite subgraphs, *J. Graph Theory* **6** (1982) 105–113.

[36] G. I. Orlova and Y. G. Dorfman, Finding the maximal cut in a graph, *Eng. Cyber.* **10** (1972) 502–506.

[37] S. Poljak and Zs. Tuza, Bipartite subgraphs of triangle-free graphs, SIAM J. Discrete Math. **7** (1994) 307–313

[38] S. Poljak and Zs. Tuza, Maximum cuts and large bipartite subgraphs, In: W. Cook, L. Lovász, and P. Seymour, Editors, *Combinatorial Optimization*, DIMACS Series in Disc. Math and Theor. Comp. Sci. **20** (1995) 181–244.

[39] D. Porter, On a bottleneck bipartition conjecture of Erdös, *Combinatorica* **12** (1992) 317–321.

[40] D. Porter and B. Yang, Graph partitions II, *J. Combin. Math. Combin. Comput.* **37** (2001) 149–158.

[41] A. Scott, Judicious partitions and related problems, in *Surveys in Combinatorics* (2005) pp. 95–117, London Math. Soc. Lecture Note Ser., 327, Cambridge Univ. Press, Cambridge, 2005.

[42] F. Shahrokhi and L. A. Székely, The complexity of the bottleneck graph bipartition problem, *J. Combin. Math. Combin. Comp.* **15** (1994) 221–226.

[43] J. B. Shearer, A note on bipartite subgraphs of triangle-free graphs, *Random Structures and Algorithms* **3** (1992) 223–226.

[44] W. Stanton, Edge deletions and the chromatic number, *Ars Combinatoria* **10** (1980) 103–106.

[45] Zs. Tuza, Maximum cuts: Improvements and local algorithmic analogues of the Edwards-ErdŐs inequality, *Discrete Math.* **194** (1999) 39–58.

[46] B.Xu and X.Yu, Triangle-free subcubic graphs with the minimum bipartite density , *J. Combin. Theory Ser. B* **98** (2008) 516–537.

[47] B. Xu and X. Yu, Judicious *k*-partitions of graphs, *J. Combin. Theory Ser. B* **99** (2009) 324–337.

[48] B.Xu and X.Yu, On a bipartition problem of Bollobas and Scott, *Combinatorica* **29** (2009) 595–618.

[49] M. Yannakakis, Node- and Edge-Deletion NP-Complete Problems, *STOC* (1978) 253–264.