Estimation Techniques for Nonlinear Functions of the Steady-State Mean in Computer Simulation

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Estimation Techniques for Nonlinear Functions of the Steady-State Mean in Computer Simulation

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To my mother Song-Ah Lim and my wife Geumsun Jo

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SUMMARY

A simulation study consists of several steps such as data collection, coding and model verification, model validation, experimental design, output data analysis, and implementation. Our research concentrates on output data analysis. In this field, many researchers have studied how to construct confidence intervals for the mean μ of a stationary stochastic process. However, the estimation of the value of a nonlinear function $f(\mu)$ has not received a lot of attention in the simulation literature. Towards this goal, a batch-means-based methodology was proposed by Muñoz and Glynn [51]. Their approach did not consider consistent estimators for the variance of the point estimator for $f(\mu)$. This thesis, however, will consider consistent variance estimation techniques to construct confidence intervals for $f(\mu)$. Specifically, we propose methods based on the combination of the delta method and nonoverlapping batch means (NBM), standardized time series (STS), or a combination of both. Our approaches are tested on moving average, autoregressive, and M/M/1 queueing processes. The results show that the resulting confidence intervals (CIs) perform often better than the CIs based on the method of Muñoz and Glynn in terms of coverage, the mean of their CI half-width, and the variance of their CI half-width.

CHAPTER I

INTRODUCTION

1.1 Simulation

Simulation is the imitation of the operation of a real-world process or system over time (Banks [5]). It concerns the study of the operating characteristics of real systems. A simulation project consists of several steps such as data collection, coding and model verification, model validation, experimental design, output data analysis, and implementation. In our thesis, we focus on statistical methods for computing confidence intervals for steady-state system performance measures.

To provide a framework for this thesis, it is useful to define the basic terms of simulation (Fishman [24] and Law and Kelton [39]).

- A *system* is defined to be a collection of entities, e.g., people or machines, that interact together toward the accomplishment of some logical end.

- The *state* of a system is that collection of variables necessary to describe the system at a particular time, relative to the objective of a study. In a study of a bank, examples of possible state variables are the number of busy tellers, the number of customers in the bank, and the arrival times of each customer in the bank.

- A *model* is a representation of the system that is used to study it as a surrogate for the actual system. In our thesis, we are interested in *mathematical models*, representing a system in terms of logical and quantitative relationships that are then manipulated and changed to see how the model reacts, and to see how the system would perform if the mathematical model were a valid one. We are considering highly complex mathematical models so that they must be studied by means of *simulation*, i.e., numerically exercising the model for the inputs in question to see how they affect the output measures of performance.

- *Deterministic* models do not contain any probabilistic (i.e., random) components. A complicated (and analytically intractable) system of differential equations describing a chemical reaction might constitute such a model. Many systems, however, must be modeled as having at least some random input components, and these give rise to *stochastic* simulation models. Most queueing and inventory systems have stochastic components. Stochastic simulation models produce output that is itself random, and can therefore only give estimates of the true characteristics of the model.

- If time is not a variable in a model, then the model is said to be *static* (e.g., a finite element model of a ship's hull). If the time horizon is a finite portion of time, say from the time that a particular bank opens to when it closes, then the model is said to be *transient*. On the other hand, if the time horizon is infinite, and interest lies in estimating characteristics of the model over the long run, then the model is said to be a *steady-state* model.

- *Discrete-event* simulation concerns the modeling of a system as it evolves over time by a representation in which the state variables change only at distinct points in time. These instances in times are the ones at which events occur, where an *event* is defined as an instantaneous occurrence that may change the state of the system. *Continuous simulation* concerns the modeling over time of a system by a representation in which the state variables change continuously with respect to time. Typically, continuous simulation models involve differential equations that give relationships for the rates of change of the state variables with time.

This research focuses on *discrete*, *stochastic*, *steady-state* simulation output data analysis.

1.2 Simulation Output Analysis

We start with some notation and abbreviations that will be used in the sequel.

Notation and Abbreviations

CI	confidence interval
CLT	central limit theorem
CvM	Cramér-von Mises
CMT	continuous mapping theorem
FCLT	functional central limit theorem
IID	independent and identically distributed
LHS	left-hand side
NBM	nonoverlapping batch means
RHS	right-hand side
RV	random variable
STS	standardized time series
$N(\mu, \sigma^2)$	RV following the normal distribution with mean μ and variance σ^2
t_n	t distribution with n degrees of freedom
$t_{n,1-\alpha}$	$1-\alpha$ quantile of the t distribution with n degrees of freedom
$z_{1-\alpha}$	$1-\alpha$ quantile of the $N(0,1)$ distribution
$\Rightarrow$	weak convergence
$\xrightarrow{\mathcal{P}}$	convergence in probability
$\xrightarrow{\mathcal{D}}$	convergence in distribution
$\xrightarrow{2}$	convergence in mean square

One objective of simulation output analysis is to estimate some unknown characteristic or parameter of the system being studied. The analyst often wants not only an estimate of this parameter value, but also some measure of the estimator's precision. Confidence intervals are widely used for this purpose.

Let $\{Y_i, i \geq 1\}$ be a stationary output stochastic process from a single simulation run. Assume that any transient portion of the simulation output has somehow been deleted. From this point, unless it is stated explicitly, we assume that the Y_j are univariate random variables with mean $\mu = \mathsf{E}[Y_1]$. For example, Y_j might be the throughput (production) in the jth hour for a stationary manufacturing system. Usually, the Y_j are not independent.

Let $y_{11}, y_{12}, \ldots, y_{1n}$ be a realization of the random variables $Y_1, Y_2, \ldots, Y_n$ resulting from a simulation run of length n observations (the jth random number used in the ith run is denoted y_{ij}). If we run the simulation with a different set of random numbers, then we will obtain a different realization $y_{21}, y_{22}, \ldots, y_{2n}$ of the random variables $Y_1, Y_2, \ldots, Y_n$. In general, suppose that we make l independent replications (runs) of the simulation (i.e., different random numbers are used for each replication) of length n, resulting in the observations:

$$
\begin{gathered}
y_{11}, \ldots, y_{1j}, \ldots, y_{1n} \\
y_{21}, \ldots, y_{2j}, \ldots, y_{2n} \\
\vdots \\
y_{l1}, \ldots, y_{lj}, \ldots, y_{ln}
\end{gathered}
$$

The observations from a particular replication (row) are clearly not independent. However, note that the data $y_{1j}, y_{2j}, \ldots, y_{lj}$ (from the jth column) are IID observations of the random variable Y_j, for $j = 1, 2, \ldots, n$. This independence across runs is the key to certain relatively simple output-data analysis methods. Then, from the observations y_{ij} ($i = 1, 2, \ldots, l;\ j = 1, 2, \ldots, n$), we can estimate some unknown parameter of the system and find a confidence interval.

Most research on steady-state simulation output analysis has focused on CIs for the mean μ. In our research, we are interested in estimating a nonlinear function $f(\mu)$ of the steady-state mean μ. Muñoz and Glynn [51] investigated the estimation of $f(\mu)$ in the steady-state simulation context. We would like to develop new asymptotically valid CIs for $f(\mu)$.

1.3 Confidence Interval Estimators for a Nonlinear Function of the Steady-State Mean

The vast majority of the existing articles on steady-state simulation output analysis try to find confidence intervals for the mean μ of a stationary, discrete-time stochastic process $\{Y_i, i \geq 1\}$. However, relatively few attempts have been made to investigate CIs for a nonlinear function $f(\mu)$. Muñoz and Glynn [51] proposed a batch-means-based methodology to estimate $f(\mu)$. Asymptotically valid confidence intervals for $f(\mu)$ were obtained by combining three different point estimators (classical, batch means and jackknife) with two different variability estimators (classical and jackknife). Muñoz and Glynn showed that if the run length is large enough, the jackknife point estimator usually has the smallest bias, with no significant increase in mean squared error. Our emphasis is on the development of the asymptotically valid confidence intervals for $f(\mu)$ using consistent variance parameter estimation methodology (Foley and Goldsman [25], Goldsman et al. [33], Goldsman and Schruben [31], Schruben [61]).

In the steady-state context, the sample average $\bar{Y}_n$ typically converges to the steady-state mean μ in terms of mean squared error (see Lehmann [40] and Serfling [63]), i.e., $\bar{Y}_n \xrightarrow{2} \mu$ as $n \to \infty$. By the continuous mapping theorem (CMT) (Billingsley [6]), if f is continuous, then $f(\bar{Y}_n)$ converges to $f(\mu)$ in mean square, i.e., $f(\bar{Y}_n)$ is a weakly consistent estimator for $f(\mu)$ (Lehmann and Casella [41]). In order to obtain a confidence interval for $f(\mu)$, we need a point estimator for $f(\mu)$ as well as the variance of the point estimator. As the point estimators for $f(\mu)$, we could use $f(\bar{Y}_n)$, the NBM, or the jackknife estimators, which will be discussed in Chapter III. That chapter also proposes estimators for the variances of the point estimators of $f(\mu)$ (hereafter referred to as variance estimators); these estimators are based on the delta method and consistent estimators of the variance parameter of the process $\{Y_i, i \geq 1\}$, defined in Section 2.2. The latter estimators are based on the methods of nonoverlapping batch means, standardized time series, or a combination of these methods, and are known to be consistent (Chien et al. [10] and Damerdji [12]).

The organization of the remainder of this thesis is as follows. Chapter II contains some background material on the nature of simulation output, several variance parameter estimation methods, and the jackknife and bootstrap resampling methods. Chapter III presents the proposed CIs for $f(\mu)$. Chapter IV contains several performance analyses for CIs based on a first-order moving average process. Chapter V presents experimental results based on an autoregressive process of order one. Chapter VI considers CIs in a stationary M/M/1 queueing system. Chapter VII summarizes our results and suggests topics for future research.

CHAPTER II

BACKGROUND

The purpose of this chapter is to introduce the nature of simulation output data, several variance parameter estimation methods, and the jackknife and bootstrap estimators of the variance of a certain estimator.

2.1 Simulation Output Data

Since most simulation models use random variables as input, the simulation output is itself random and care must be taken in drawing conclusions about a model's true characteristics. Consider an experiment in which we wish to estimate the mean μ of a stationary process, $\{Y_i, i \geq 1\}$, e.g., the mean waiting time in a stationary M/M/1 queueing system. If $Y_1, Y_2, \ldots, Y_n$ are IID RVs with finite population mean μ and finite population variance $\sigma_Y^2 \equiv \mathsf{Var}[Y_1]$, then the sample mean $\bar{Y}_n = \sum_{i=1}^n Y_i/n$ is an unbiased estimator of μ and the sample variance $S^2 = \sum_{i=1}^n (Y_i - \bar{Y}_n)^2/(n-1)$ is an unbiased estimator of σ_Y^2. The usual way to assess the precision of $\bar{Y}_n$ as an estimator of μ is to construct a confidence interval for μ. For IID samples, if n is sufficiently large, then an approximate $100(1-\alpha)\%$ confidence interval for μ is given by

$$\bar{Y}_n \pm z_{1-\alpha/2} \frac{S}{\sqrt{n}}. \tag{1}$$

Recall that if the Y_i are normal random variables, an exact $100(1-\alpha)\%$ confidence interval for μ is given by

$$\bar{Y}_n \pm t_{n-1,1-\alpha/2} \frac{S}{\sqrt{n}}. \tag{2}$$

If the distribution of the Y_i is not normal, the confidence interval given by Equation (2) will be approximate in terms of coverage. Since $t_{n-1,1-\alpha/2} > z_{1-\alpha/2}$, the confidence interval given by Equation (2) will usually be wider than the one given by Equation (1) and may have slightly higher coverage. Notice that $t_{n-1,1-\alpha/2} \to z_{1-\alpha/2}$, as $n \to \infty$.

Unfortunately, the independence assumption is rarely satisfied by simulation output data (e.g., consecutive waiting times) from a single run. In particular, assume that the RVs $Y_1, Y_2, \ldots, Y_n$ come from a stationary stochastic process. Then the sample mean $\bar{Y}_n$ is an unbiased estimator of μ; however, the sample variance S^2 is no longer an unbiased estimator of σ_Y^2, and the estimation of $\mathsf{Var}[\bar{Y}_n]$ by the "usual" estimator S^2/n induces serious errors. In fact, it can be shown that

$$\mathsf{Var}[\bar{Y}_n] = \frac{1}{n}\Big[R_0 + 2\sum_{i=1}^{n-1}(1-\frac{i}{n})R_i\Big] \tag{3}$$

and

$$\mathsf{E}[S^2] = R_0 - \frac{2}{n-1}\sum_{i=1}^{n-1}(1-\frac{i}{n})R_i, \tag{4}$$

where $R_k \equiv \mathsf{Cov}(Y_1, Y_{1+k})$, $k = 0, \pm 1, \pm 2, \ldots$, is the autocovariance function of the process $\{Y_i, i \geq 1\}$ (see Anderson [4], p. 448). Being very often the case in practice, if $R_i > 0$, then S^2/n will have negative bias as an estimator of $\mathsf{Var}[\bar{Y}_n]$: $\mathsf{E}[S^2/n] \ll \mathsf{Var}[\bar{Y}_n]$ (Law and Kelton [39] display a formula relating $\mathsf{E}[S^2/n]$ and $\mathsf{Var}[\bar{Y}_n]$). Therefore, we need other methods to estimate $\mathsf{Var}[\bar{Y}_n]$. Section 2.2 reviews such methods that will be used subsequently in our research.

2.2 Methods for Estimating $\mathsf{Var}[\bar{Y}_n]$

Recall that the usual estimator for μ is the sample mean $\bar{Y}_n$. A measure of the sample mean's variance from sample to sample is $\mathsf{Var}[\bar{Y}_n]$ (see Equation (3)), which is also unknown. To obtain confidence intervals for μ, it is common to provide an estimate of $\mathsf{Var}[\bar{Y}_n]$, or, almost equivalently, the *variance parameter* $\sigma^2 \equiv \lim_{n\to\infty} \sigma_n^2$, where $\sigma_n^2 \equiv n\,\mathsf{Var}[\bar{Y}_n]$ (provided that σ^2 exists and is positive and finite). We can show from Equation (3) that

$$\sigma_n^2 \;=\; R_0 + 2\sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)R_i \tag{5}$$

and, if $\sum_{j=1}^{\infty} |R_j| < \infty$,

$$\sigma^2 = \sum_{i=-\infty}^{\infty} R_i. \tag{6}$$

Generally, the observations will be divided into batches. We use the notations $\bar{Y}_{1,m}, \bar{Y}_{2,m}, \ldots$ and $\bar{Y}_n$ for the batched sample means and the grand sample mean from all of the batches, respectively (see Section 2.2.1 below). Also, assume that $\sum_{j=1}^{\infty} |jR_j| < \infty$ and let $\gamma \equiv -2\sum_{j=1}^{\infty} jR_j$ (Song and Schmeiser [67]).

Along the way, we will also assume that the process is ϕ-mixing (Billingsley [6]). As in Billingsley [6], we say that a σ*-field* is a class of subsets of a set that is closed under the operations of complements and countable unions. Let $\mathcal{M}_i^j$ $(i \le j)$ denote the σ-field generated by the RVs $Y_i, Y_{i+1}, \ldots, Y_j$. The sequence $\{Y_i, i \ge 1\}$ is ϕ*-mixing* if for all $i, j \ge 1$ and any events $E \in \mathcal{M}_1^i$ with $P(E) > 0$ and $F \in \mathcal{M}_{i+j}^{\infty}$, we have $|P(F|E) - P(F)| \le \phi_j$, where $\phi_j \to 0$ and, without loss of generality, the ϕ_j are nonincreasing in j. Intuitively, ϕ-mixing means that the distant future is essentially independent of the past or present. That is, the probability of F (a future event) conditioned on E (a past or present event) becomes very close to the unconditional probability of F as the time lag j increases.

We also use the "little-oh" notation $g(m) = o(h(m))$ to indicate that $g(m)/h(m) \to 0$ as $m \to \infty$. The "big-oh" notation $g(m) = O(h(m))$ is used when there is an integer $m_0 \ge 1$ such that $|g(m)/h(m)| \le C$ for some constant C and all $m \ge m_0$.

There are a number of estimators for σ^2, as described in standard references such as Law and Kelton [39]. One of the most popular techniques in practice is the NBM method; we will review this method in Section 2.2.1. Another class of estimators is based on Schruben's STS methodology (Schruben [61]). Two specific examples of STS estimators with good properties are the weighted area and weighted CvM estimators, described in Goldsman et al. [29] and [31], respectively. We will discuss weighted STS estimators in Section 2.2.2. Finally, we look at combined nonoverlapping batch means and STS (NBM+STS) estimators in Section 2.2.3.

2.2.1 The NBM Method for Estimating σ^2

The nonoverlapping batch means method is popular among experimenters because of its simplicity and effectiveness. This approach has been explored by many people, e.g., Conway [11], Fishman [22], Schmeiser [60], and so on. In the batch means approach, the sample $Y_1, Y_2, \ldots, Y_n$ is divided into sub-groups of samples, and each sub-group is reduced to a single average value — a batch mean. These batch means are then used to estimate σ^2.

Suppose that one forms b nonoverlapping batches, each of size m (assuming that $n = mb$):

$$\begin{array}{rl} \text{Batch 1:} & Y_1, Y_2, \ldots, Y_m \\ \text{Batch 2:} & Y_{m+1}, Y_{m+2}, \ldots, Y_{2m} \\ \vdots & \\ \text{Batch } b\text{:} & Y_{(b-1)m+1}, Y_{(b-1)m+2}, \ldots, Y_n. \end{array}$$

For $i = 1, 2, \ldots, b$ and $j = 1, 2, \ldots, m$, let

$$\bar{Y}_{i,j} \equiv \frac{1}{j}\sum_{k=1}^{j} Y_{(i-1)m+k}.$$

The NBMs are the averages $\bar{Y}_{i,m}$, $i = 1, 2, \ldots, b$, and form a stationary process themselves.

If we choose the batch size m large enough, it is reasonable to treat the $\bar{Y}_{i,m}$ as if they are IID normal random variables with mean μ (see details in Law and Carson [38], Alexopoulos and Seila [3], and Fishman [23]). Then, for sufficiently large m, the variance of the batch means can be estimated by their sample variance,

$$\begin{aligned} \widehat{\mathsf{Var}}[\bar{Y}_{1,m}] &= \frac{1}{b-1}\sum_{i=1}^{b}(\bar{Y}_{i,m} - \bar{Y}_n)^2 \\ &= \frac{1}{b-1}\left(\sum_{i=1}^{b}\bar{Y}_{i,m}^2 - b\bar{Y}_n^2\right), \end{aligned}$$

and the NBM estimator of σ_n^2 is given by

$$\hat{V}_B \equiv \widehat{\sigma_n^2} = m\widehat{\mathsf{Var}}[\bar{Y}_{1,m}] = \frac{m}{b-1}\sum_{i=1}^{b}(\bar{Y}_{i,m} - \bar{Y}_n)^2,$$

where we use the approximation $m\mathsf{Var}[\bar{Y}_m] \doteq n\mathsf{Var}[\bar{Y}_n] \doteq \sigma^2$ for sufficiently large m. That is, we estimate $\sigma^2 \doteq \sigma_m^2$ by m times the sample variance of the batch means.

Theorem 1 (Glynn and Whitt [27]) As the batch size $m \to \infty$,

$$\widehat{V}_B \xrightarrow{\mathcal{D}} \frac{\sigma^2\chi^2_{b-1}}{b-1}.$$

Further, if $\hat{V}_B^2$ is uniformly integrable (see Billingsley [6], p. 32), one has

$$\mathsf{E}[\widehat{V}_B] \to \sigma^2$$

and

$$\mathsf{Var}[\widehat{V}_B] \to \frac{2\sigma^4}{b-1}.$$

∎

Based on Theorem 1, we can form an asymptotic $100(1-\alpha)\%$ confidence interval for the mean μ similar to interval (2),

$$\mu \in \bar{Y}_n \pm t_{b-1,1-\alpha/2}\sqrt{\widehat{V}_B/n}. \tag{7}$$

The main problem with the application of the batch means method for fixed sample size is the choice of the batch size m. If m is too small, the batch means $\bar{Y}_{i,m}$ can be highly correlated and the resulting confidence interval may have coverage below the nominal value $1-\alpha$. Alternatively, a large batch size can result in very few batches and potential problems with the high variability of the CI half-width. For more detailed explanations, see Alexopoulos and Seila [3], Chien et al. [10], Fishman [24] and Steiger and Wilson [68].

Several variants of the batch-means estimation approach have been investigated. Meketon and Schmeiser [46] introduced the method of *overlapping batch means*, which has been explored further by Song and Schmeiser [66] and Pedrosa and Schmeiser [54]. The overlapping batch means estimator is generally offered as a variance-reducing modification to the NBM estimator. Also, Bischak [7] considered the formation of weighted nonoverlapping batch means.

2.2.2 STS Weighted Area Estimator for σ^2

The STS methodology, proposed by Schruben [61], uses a continuous-time random process to represent the original sequence of samples in a particularly useful form. Let

$$D_{i,n} \;=\; \bar{Y}_i - \bar{Y}_n, \;\; i = 1, 2, \ldots, n; \qquad D_{0,n} \equiv 0,$$

where $\bar{Y}_i \equiv \sum_{j=1}^{i} Y_j/i$ is the average of the first i observations in the sequence. Thus, $\mathsf{E}[D_{i,n}] = 0$, for $i = 0, 1, \ldots, n$. Then one scales the sequence by dividing by $\sqrt{n}\sigma/i$ and adjusts the time index of the sequence to the unit interval. The resulting process is

$$T_n(t) \;\equiv\; \frac{\lfloor nt \rfloor D_{\lfloor nt \rfloor, n}}{\sigma\sqrt{n}} \;=\; \frac{\lfloor nt \rfloor(\bar{Y}_{\lfloor nt \rfloor} - \bar{Y}_n)}{\sigma\sqrt{n}}, \qquad 0 \le t \le 1, \tag{8}$$

where $\lfloor \cdot \rfloor$ is the greatest integer function. Schruben pointed out that the original time series can be reconstructed from $T_n(t)$ and $\bar{Y}_n$; hence, no information is lost by the transformation. We assume that the variance parameter is positive and finite. We also assume that it is ϕ-mixing with a mixing sequence that goes to zero fast enough so that the series $\sum_{i=1}^{\infty} \sqrt{\phi_i}$ converges. Similar to Glynn and Iglehart [26], we need an additional reasonable assumption about $\{Y_i\}$ to derive our confidence intervals for μ. The following Assumption 1 is typically called the Funtional Central Limit Theorem (FCLT).

Assumption 1 (FCLT) (Billingsley [6]) There exist μ and $\sigma \in (0, \infty)$ such that as $n \to \infty$,

$$X_n \Rightarrow \sigma \mathcal{W},$$

where $\Rightarrow$ denotes weak convergence,

$$X_n(t) \equiv \frac{\lfloor nt \rfloor(\bar{Y}_{\lfloor nt \rfloor} - \mu)}{\sqrt{n}}, \quad \text{for } 0 \le t \le 1, \tag{9}$$

and $\mathcal{W}$ is a standard Brownian motion process. The sample paths of X_n lie in $\mathcal{D}[0,1]$, the space of functions on $[0,1]$ that are right-continuous and have left-hand limits, while the sample paths of $\mathcal{W}$ lie in $\mathcal{C}[0,1]$, the space of continuous functions on $[0,1]$. Glynn and Iglehart [27] studied conditions that imply the equality of μ and σ^2 with the steady-state mean and variance parameter, respectively.

Assumption FCLT leads to a result involving the standard *Brownian bridge* process, defined by $\mathcal{B}(t) = \mathcal{W}(t) - t\mathcal{W}(1)$. One has $\mathcal{B}(t) \sim N(0, t(1-t))$ and $\mathsf{Cov}[\mathcal{B}(s), \mathcal{B}(t)] = \min(s,t) - st,\ 0 \le s, t \le 1$. Further, $\mathcal{W}(1)$ and $\mathcal{B}(\cdot)$ are independent.

Theorem 2 Under Assumption FCLT,

$$(\sqrt{n}(\bar{Y}_n - \mu), \sigma T_n) \Rightarrow (\sigma\mathcal{W}(1), \sigma\mathcal{B}).$$

Proof: See Foley and Goldsman [25] or Glynn and Iglehart [26]. ∎

Remark 1 Theorem 2 implies three useful properties:

(1) $\sqrt{n}(\bar{Y}_n - \mu)$ is asymptotically $\sigma N(0,1)$.

(2) σT_n is asymptotically σ times a Brownian bridge.

(3) $\sqrt{n}(\bar{Y}_n - \mu)$ and σT_n are asymptotically independent; thus, all information gleaned from σT_n will be asymptotically independent of $\sqrt{n}(\bar{Y}_n - \mu)$.

The (weighted) area estimator for σ^2 is based on the statistic

$$S(w;n) \equiv \frac{1}{n}\sum_{k=1}^{n} w\left(\frac{k}{n}\right)\sigma T_n\left(\frac{k}{n}\right), \tag{10}$$

where $w(t)$ is a certain weight function that is continuous on [0,1] and chosen to satisfy $\mathsf{Var}[S(w)] = \sigma^2$, so that $S(w) \sim N(0, \sigma^2)$. This statistic is then used to form confidence intervals for μ.

The limiting functional of $S(w;n)$ is

$$S(w) \;\equiv\; \int_0^1 w(t)\sigma\mathcal{B}(t)dt.$$

In addition, let $A(w;n) \equiv S^2(w;n)$ and $A(w) \equiv S^2(w)$. Then under mild conditions, the CMT (see Billingsley [6], Theorem 5.1) implies $A(w;n) \xrightarrow{\mathcal{D}} A(w) \sim \sigma^2\chi_1^2$, and we call $A(w;n)$ the *weighted area* estimator for σ^2.

Example 1 (Goldsman and Schruben [33])

1. Schruben's [61] original area estimator has constant weight function $w_0(t) \equiv \sqrt{12}$, for all $t \in [0,1]$.

2. $w_1(t) \equiv \sqrt{45}t$ [or $w_1(t) \equiv \sqrt{45}(1-t)$] gives greater weight to "large" ["small"] values of t.

To this point, we have defined the STS of a sampled stochastic process for a single long run of data samples. Thus, variance parameter estimators utilizing the STS do not need to be based on batched data. However, batching the original sequence of data as described in the second paragraph in Section 2.2, generating a STS from each batch, and averaging the estimators from each batch can reduce the variance of the final variance parameter estimator.

Let us now consider the batched STS area estimator. This is the sample mean of the corresponding estimators from the individual batches, i.e.,

$$\widehat{V}_A \;\equiv\; \bar{A}(w;b,m) \;\equiv\; \frac{1}{b}\sum_{i=1}^{b} A_i(w;m),$$

where A_i denotes an estimator from the ith batch of size m ($n = bm$). Since the batched estimators are simply linear combinations of estimators from each contiguous batch of size m, we can produce the following result concerning $\mathsf{E}[\widehat{V}_A]$ and $\mathsf{Var}[\widehat{V}_A]$.

Theorem 3 (Goldsman et al. [28]) Suppose $\{Y_i\}$ is a stationary process for which Assumption FCLT holds and $\sum_{k=1}^{\infty} k^2|R_k| < \infty$. Then

$$\mathsf{E}[\widehat{V}_A] \;=\; \mathsf{E}[A_i(w;m)] \;=\; \sigma^2 + \frac{[(F-\bar{F})^2 + \bar{F}^2]\gamma}{2m} + o(1/m) \qquad (11)$$

and

$$\mathsf{Var}[\widehat{V}_A] \doteq \frac{\mathsf{Var}[A_i(w;m)]}{b} \doteq \frac{2\sigma^4}{b}, \tag{12}$$

where $F(t) \equiv \int_0^t w(s)\,ds$, $F \equiv F(1)$, $\bar{F}(t) \equiv \int_0^t F(s)\,ds$, and $\bar{F} \equiv \bar{F}(1)$. ∎

Remark 2 It is possible to choose weights $w(t)$ such that the first-order bias term in front of γ disappears. The antisymmetric function $w_2(t) \equiv \sqrt{840}(3t^2 - 3t + 1/2)$ (see Goldsman et al. [31]) has this property. Another class of weights yielding first-order unbiased estimators is given in Foley and Goldsman [25].

So we see that batching typically helps to decrease the variance of the STS estimator (by a factor of b), though this is achieved at the cost of a modest increase in bias (since m now appears instead of n in the expected value expressions). Recall that $A_i(w;m) \xrightarrow{\mathcal{D}} A_i(w) \sim \sigma^2\chi_1^2$. Further, under suitable moment and mixing conditions (see Glynn and Iglehart [26]), the RVs $A_i(w;m)$ are asymptotically independent; so the batched STS weighted area estimator for σ^2 converges to a χ^2 distribution with b degrees of freedom, that is,

$$\widehat{V}_A \xrightarrow{\mathcal{D}} \frac{\sigma^2\chi_b^2}{b}. \tag{13}$$

By Remark 1 and the definition of the t distribution, we have the following result:

$$\frac{(\bar{Y}_n - \mu)/(\sigma/\sqrt{n})}{(\widehat{V}_A/\sigma^2)^{1/2}} \xrightarrow{\mathcal{D}} \frac{N(0,1)}{(\chi_b^2/b)^{1/2}} \sim t_b, \qquad \text{as } m \to \infty. \tag{14}$$

This yields an asymptotic $100(1-\alpha)\%$ batched STS weighted area CI for μ:

$$\mu \in \bar{Y}_n \pm t_{b,1-\alpha/2}\sqrt{\widehat{V}_A/n}. \tag{15}$$

Another set of STS estimators with good properties are the CvM estimators in Goldsman et al. [29]. These estimators are based on weighted Cramér-von Mises statistics; certain weight functions yield estimators that are first-order unbiased for σ^2. Compared to the weighted area estimator, asymptotic variance reduction of up to 60% is achievable. However, in this research, we do not consider CvM estimators since they occasionally yield negative realizations (see Marshall et al. [45]).

2.2.3 NBM+STS estimators

We have discussed variance parameter estimators using the batch means and STS weighted area methods. Now, we review another variance parameter estimator that combines both.

Theorem 4 (Goldsman et al. [34]) Under the FCLT, as the batch size $m \to \infty$,

$$\widehat{V}_C \;\equiv\; \frac{(b-1)\widehat{V}_B + b\widehat{V}_A}{2b-1} \;\xrightarrow{\mathcal{D}}\; \frac{\sigma^2\chi^2_{2b-1}}{2b-1}, \tag{16}$$

so that

$$\mathsf{E}[\widehat{V}_C] \to \sigma^2$$

and

$$\mathsf{Var}[\widehat{V}_C] \to \frac{2\sigma^4}{2b-1}.$$

∎

Further, $\widehat{V}_C$ is asymptotically unbiased and has lower variance than $\widehat{V}_A$ or $\widehat{V}_B$. Similar to Equations (7) and (15), we have an approximate $100(1-\alpha)\%$ confidence interval for μ:

$$\mu \;\in\; \bar{Y}_n \pm t_{2b-1,1-\alpha/2}\sqrt{\widehat{V}_C/n}. \tag{17}$$

2.3 The Jackknife and Bootstrap Methods

According to Shao and Tu [64], the jackknife and bootstrap methods are the most popular data-resampling methods for estimating bias, variance and more general measures of error. Both methods replace theoretical derivations by repeatedly resampling the original data and making inferences from the resamples. Because of the availability of inexpensive and fast computing, these two computer-intensive methods have caught on very rapidly in recent years and are particularly appreciated by applied statisticians. The jackknife was introduced by Quenouille [56] and has been developed further by several authors including Tukey [70] and Miller [47, 48, 49]. The bootstrap was introduced by Efron [16], and has been developed further by others. In our thesis, we will consider the jackknife technique to reduce bias and estimate variance. In future research, we will investigate bootstrap techniques to construct confidence intervals for $f(\mu)$. In this section we briefly review these two techniques.

2.3.1 The Jackknife Method

We start with the jackknife method. For more-detailed explanations, see Miller [47, 48, 49] and Efron [17].

Quenouille [56] introduced a technique for reducing the bias of an estimator based on splitting the sample into two half-samples. The technique's properties were studied in some specific situations by Quenouille [57] and Durbin [14]. Tukey [70] proposed the general use of this technique in order to reduce the bias and obtain approximate confidence intervals in problems where standard statistical procedures may not exist or are difficult to apply. Suppose that $X_1, X_2, \ldots, X_n$ are IID random variables from a distribution with unknown parameter θ. Suppose that $\hat{\theta}$ is an estimator of the parameter θ based on the sample of size n. Further suppose that the data is divided into b groups of size m ($n = bm$), i.e.,

$$(X_1, \ldots, X_m), (X_{m+1}, \ldots, X_{2m}), \ldots, (X_{(b-1)m+1}, \ldots, X_{bm}).$$

Let $\hat{\theta}_{-i},\ i = 1, 2, \ldots, b$, denote the estimate of θ obtained by deleting the ith group and estimating θ from the remaining $(b-1)m$ observations. Define

$$\tilde{\theta}_i = b\hat{\theta} - (b-1)\hat{\theta}_{-i}, \quad i = 1, 2, \ldots, b, \tag{18}$$

which are called "pseudo-values" by Tukey. The jackknife estimator of θ is the average of the $\tilde{\theta}_i$; i.e.,

$$\tilde{\theta} \equiv \frac{1}{b}\sum_{i=1}^{b}\tilde{\theta}_i = b\hat{\theta} - \frac{b-1}{b}\sum_{i=1}^{b}\hat{\theta}_{-i}.$$

The jackknife typically eliminates bias of order n^{-1}. Namely, if

$$\mathsf{E}[\hat{\theta}] = \theta + \frac{a_1}{n} + O(1/n^2),$$

then

$$\mathsf{E}[\tilde{\theta}] = \theta + O(1/n^2).$$

Tukey [69] noted that in many instances the $\tilde{\theta}_i$ are approximately IID. Under this assumption,

$$\hat{V}_J \equiv \frac{\sum_{i=1}^b (\tilde{\theta}_i - \tilde{\theta})^2}{b(b-1)}$$

can be a estimator of $\mathsf{Var}[\tilde{\theta}]$, and

$$\frac{\sqrt{b}(\tilde{\theta} - \theta)}{\{\frac{1}{b-1}\sum_{i=1}^b (\tilde{\theta}_i - \tilde{\theta})^2\}^{1/2}} \approx t_{b-1}. \tag{19}$$

Property (19) could therefore be used to construct an approximate $100(1-\alpha)\%$ confidence interval for θ:

$$\theta \in \tilde{\theta} \pm t_{b-1,1-\alpha/2}\sqrt{\hat{V}_J/n}. \tag{20}$$

The jackknife method has become a more valuable tool since Tukey [69] found that it can also be used to construct variance estimators. However, it requires the computation of the pseudo-values — an expensive task in the early days (Shao and Tu [64]).

Since the original jackknife technique was developed for IID data, it may not be applicable to dependent data since it fails to capture the structure of dependencies. To overcome this problem, it needs nontrivial modifications. One modification takes repeated samples from appropriately defined residuals, whereas a second modification applies resampling to groups or blocks of the original data to maintain the dependence structure of the data. The interested reader can refer to Shao and Tu [64], Park and Willemain [52], Liu and Singh [43], Künsch [37], and references therein. In this thesis, we will consider the jackknife technique to estimate $f(\mu)$ and the variance of the estimators for $f(\mu)$.

2.3.2 The Bootstrap Method

In this section we will introduce the basic bootstrap technique. For details, we refer the reader to Efron [16, 17], Efron and Tibshirani [20], and Davison and Hinkley [13].

A data set of size n has $2^n - 1$ nonempty subsets; the jackknife method only utilizes n of them. The jackknife method may be improved by using statistics based on more than n (or even all $2^n - 1$) subsets. This idea was discussed by Hartigan [35], but it requires more computing power than the jackknife technique. Developments in computer technology over the last two decades have made this idea more attractive.

The bootstrap is a member of a larger class of methods that resample from the original data set, and thus are called resampling procedures. Since the publication of Efron [17], research activity on the bootstrap method has grown exponentially.

Suppose that our data consist of a random sample from an unknown probability distribution G,

$$U_1, U_2, \ldots, U_n \sim G. \tag{21}$$

Having observed $U_1 = u_1, U_2 = u_2, \ldots, U_n = u_n$, we compute the sample mean $\bar{U}_n = n^{-1} \sum_{i=1}^{n} U_i$. The issue now is: how accurate it is as an estimate of the true mean $\theta = \mathsf{E}[U]$.

The standard error $\sigma(G; n, \bar{U}_n)$, that is, the standard deviation of $\bar{U}_n$, is

$$\sigma(G) \;=\; [\mathsf{Var}(U_n)/n]^{1/2}. \tag{22}$$

In case we do not know $\mathsf{Var}(U_n)$, we can estimate the standard error by

$$\hat{\sigma}(G) \;=\; [\widehat{\mathsf{Var}}(U_n)/n]^{1/2}, \tag{23}$$

where $\widehat{\mathsf{Var}}(U_n) = (n-1)^{-1} \sum_{i=1}^{n} (U_i - \bar{U})^2$.

There is alternative way to estimate the quantity in Equation (22). Let $\hat{G}$ indicate the empirical probability mass function,

$$\hat{G} : \text{Probability mass } 1/n \text{ on } u_1, u_2, \ldots, u_n. \tag{24}$$

Then we can simply replace G by $\hat{G}$ in Equation (22), obtaining

$$\hat{\sigma}_{\text{boot}} \;\equiv\; \sigma(\hat{G}) \;=\; [\widetilde{\mathsf{Var}}(U_n)/n]^{1/2} \tag{25}$$

as the estimated standard error for $\bar{U}_n$, where $\widetilde{\mathsf{Var}}(U) = n^{-1}\sum_{i=1}^{n}(U_i - \bar{U}_n)^2$ is the true variance of $\hat{G}$. This is the *bootstrap estimate*.

As is easily seen, there is not much difference between $\hat{\sigma}(G)$ and $\hat{\sigma}_{\text{boot}}$ in this case. However, if the estimators are more complicated than $\bar{U}_n$ (for example, a median or a correlation or a slope coefficient from a robust regression), explicit formulas like (23) and (25) do not exist. This is where computing power becomes handy.

To illustrate, let $T_n = T_n(U_1, \ldots, U_n)$ be a given "complicated statistic" such as the median of the sample. Since the standard error $\sigma(G; n, T_n)$ does not have a simple form, we cannot easily evaluate $\sigma(G; n, T_n)$ exactly, even if G is known. Of course, we can use Monte Carlo sampling methods to approximate $\sigma(G; n, T_n)$ when G is known. That is, we repeatedly draw new data sets from G and then use the sample standard deviation of the values of T_n computed from the new data sets as an approximation to $\sigma(G; n, T_n)$. This idea can be used to approximate $\hat{\sigma}_{\text{boot}}$ since $\hat{G}$ is a known distribution. That is, we can draw $\{U_{1b}^*, \ldots, U_{nb}^*\}$, $b = 1, 2, \ldots, B$, independently from $\hat{G}$, conditionally on $U_1, U_2, \ldots, U_n$; compute $T_{n,b}^* = T_n(U_{1b}^*, U_{2b}^*, \ldots, U_{nb}^*)$; and approximate $\hat{\sigma}_{\text{boot}}$ as follows:

$$\hat{\sigma}_{\text{boot}}^B \equiv \sqrt{\frac{1}{B}\sum_{b=1}^{B}\left(T_{n,b}^* - \frac{1}{B}\sum_{l=1}^{B} T_{n,l}^*\right)^2}.$$

It is easy to see that as $B \to \infty$, $\hat{\sigma}_{\text{boot}}^B$ will approach $\hat{\sigma}_{\text{boot}} = \sigma(\hat{G})$, the bootstrap estimate of standard error. Both $\hat{\sigma}_{\text{boot}}^B$ and $\hat{\sigma}_{\text{boot}}$ can be called bootstrap estimators. In fact, $\hat{\sigma}_{\text{boot}}^B$ is more useful in practical applications, whereas in theoretical studies people usually focus on $\hat{\sigma}_{\text{boot}}$ (Efron [17], and Efron and Tibshirani [20]).

2.3.3 The Relationship Between the Jackknife and Bootstrap Methods

In the jackknife method, the given statistic is recalculated for b fixed data sets that are subsets of the original data set, whereas in the bootstrap method the recomputations are based on many *bootstrap data sets* $\{U_{ij}^*, i = 1, 2, \ldots, b\}$, $j = 1, 2, \ldots, B$, that are randomly generated from the original data set. This is why the jackknife and the bootstrap are called *resampling methods*. The bootstrap method often has a very close relationship with other data "reuse" methods such as the jackknife method (Efron [17], Rao and Wu [58], and Sitter [65]). The jackknife variance estimator $\hat{V}_J$ is an approximation to the bootstrap variance estimator $\hat{\sigma}^2_{\text{boot}}$ when the statistic T_n is sufficiently "smooth" (Shao and Tu [64]). But this does not imply that $\hat{\sigma}^2_{\text{boot}}$ or $(\hat{\sigma}^B_{\text{boot}})^2$ have smaller mean squared error than $\hat{V}_J$. We wish to remind the reader that the jackknife method often yields estimators with smaller mean squared error while the bootstrap method is more computer-intensive. In this thesis, we use only the jackknife technique to estimate $f(\mu)$ and the variance of the point estimator since it has been shown to construct asymptotically valid confidence intervals for $f(\mu)$.

CHAPTER III

ESTIMATION TECHNIQUES FOR A NONLINEAR FUNCTION OF THE STEADY-STATE MEAN

Since simulation output data are not IID, we cannot directly apply classical CIs to simulation data. A variety of methods exist for the computation of CIs for a steady-state mean μ. In this chapter we study new CIs for $f(\mu)$, where $f(\cdot)$ is a nonlinear that is continuously differentiable in a neighborhood of μ with $f'(\mu) \neq 0$. We also assume that the FCLT holds. As in Section 2.2.1, we collect n observations and form b nonoverlapping batches, each of size m.

Section 3.1 lists the point estimators for $f(\mu)$, Section 3.2 discusses variance estimators, and Section 3.3 describes asymptotically valid CIs for $f(\mu)$. In particular, we use the delta method to obtain consistent variance estimators and asymptotically valid CIs for $f(\mu)$.

3.1 Point Estimators for $f(\mu)$

We start with the three point estimators for $f(\mu)$ proposed by Muñoz and Glynn [51]:

(1) Classical estimator:

$$\hat{f}_C \equiv f(\bar{Y}_n). \tag{26}$$

(2) Batch means estimator:

$$\bar{f}_B \equiv \frac{1}{b}\sum_{i=1}^{b} f(\bar{Y}_{i,m}). \tag{27}$$

(3) Jackknife estimator:

$$\bar{f}_J \equiv \frac{1}{b}\sum_{i=1}^{b}\tilde{f}_i, \tag{28}$$

where

$$\tilde{f}_i \equiv bf(\bar{Y}_n) - (b-1)f(\bar{Y}_{-i,m})$$

and

$$\bar{Y}_{-i,m} \equiv \frac{1}{b-1}\sum_{j\neq i}\bar{Y}_{j,m}.$$

Since the process $\{Y_i\}$ is stationary, the sample average $\bar{Y}_n$ is a consistent estimator for the steady-state mean μ, so that the classical estimator $\hat{f}_C$ is also a consistent estimator for the continuous function $f(\mu)$. However, even for IID observations, the classical estimator $\hat{f}_C$ is often biased for finite n (Miller [49]). Due to the stationary assumption, the only significant bias effects on $\hat{f}_C$ must then ensue from the nonlinearity of f.

The batch means estimator $\bar{f}_B$ takes advantage of the methodology in Section 2.2.1, while the jackknife estimator $\bar{f}_J$ attempts to reduce the bias of the classical estimator $\hat{f}_C$.

The biases of the estimators $\bar{f}_B$, $\bar{f}_J$, and $\hat{f}_C$ can be compared under the assumptions of Theorem 3 of Muñoz and Glynn [51]. Namely, suppose that the FCLT holds, and that $f(\cdot)$ is bounded by a polynomial of degree $q \geq 0$ and is four times differentiable at μ. Further, let the number of batches $b \geq 1$ be fixed, and set $p = \max\{2, q\}$. If there exists an $n_0 > 0$ such that $\{n^{p/2}|\bar{Y}_n - \mu|^p : n \geq n_0\}$ is uniformly integrable, then for a sufficiently large n, we have

$$|\mathsf{Bias}[\bar{f}_J]| < |\mathsf{Bias}[\hat{f}_C]| < |\mathsf{Bias}[\bar{f}_B]|.$$

Theorem 3 of [51] also shows that all three point estimators exhibit similar performance with regard to their mean squared errors. Therefore, the jackknife point estimator appears to provide the smallest bias, with no significant increase in the mean squared error.

Theorems 1 and 2 in Miller [47] show that if $\{Y_i\}$ is an IID sequence with mean μ and standard deviation $\sigma_Y > 0$, and if f is a real function with bounded second derivative in a neighborhood of μ, then, as $n \to \infty$,

$$\sqrt{n}(\hat{f}_C - f(\mu)) \xrightarrow{\mathcal{D}} N[0, (f'(\mu))^2 \sigma_Y^2]$$

and

$$\sqrt{n}(\bar{f}_J - f(\mu)) \xrightarrow{\mathcal{D}} N[0, (f'(\mu))^2 \sigma_Y^2].$$

Since we are dealing with correlated processes, we will have to develop analogous results.

3.2 Point Estimators for $\mathsf{Var}[\hat{f}(\mu)]$

In this subsection, we study estimators for the variance of the point estimators for $f(\mu)$. The first two estimators, (a) and (b) below, are from Muñoz and Glynn [51]. Estimator (c) is based on the delta method (Lehmann et al. [40]), and the variance parameter estimators from Chapter II. We call this the *delta* variance estimator.

(a) The NBM variance estimator:

$$S_B^2 \equiv \frac{1}{b(b-1)} \sum_{i=1}^{b} [f(\bar{Y}_{i,m}) - \bar{f}_B]^2. \tag{29}$$

(b) The jackknife variance estimator:

$$S_J^2 \equiv \frac{1}{b(b-1)} \sum_{i=1}^{b} [\tilde{f}_i - \bar{f}_J]^2. \tag{30}$$

(c) The delta variance estimator:

$$S_*^2(\hat{f}'; \hat{V}) \equiv \frac{\hat{f}'(\mu)^2 \hat{V}}{n}, \tag{31}$$

where $\hat{f}'(\mu)$ is one of

$$\hat{f}'_C \equiv f'(\bar{Y}_n) = \frac{d}{d\mu} f(\mu)\bigg|_{\mu=\bar{Y}_n}, \tag{32}$$

$$\bar{f}'_B \equiv \frac{1}{b}\sum_{i=1}^{b} f'(\bar{Y}_{i,m}) = \frac{1}{b}\sum_{i=1}^{b} \frac{d}{d\mu}f(\mu)\bigg|_{\mu=\bar{Y}_{i,m}}, \tag{33}$$

and

$$\bar{f}'_J \equiv \frac{1}{b}\sum_{i=1}^{b} \tilde{f}'_i, \tag{34}$$

with $\tilde{f}'_i \equiv bf'(\bar{Y}_n) - (b-1)f'(\bar{Y}_{-i,m})$; and $\widehat{V}$ is any of $\widehat{V}_B$, $\widehat{V}_A$, or $\widehat{V}_C$.

Recall that the NBM variance estimator is the sample variance of the $f(\bar{Y}_{i,m})$ divided by b, and the jackknife variance estimator is the sample variance of the pseudo-values $\tilde{f}_i$ divided by b.

Theorem 5 (Muñoz and Glynn [51]) If the FCLT in Section 2.2.2 holds and f is differentiable in a neighborhood of μ, then the batch means and jackknife variance estimators obey

$$\frac{\hat{f}(\mu) - f(\mu)}{\sqrt{\widehat{\mathsf{Var}}[\hat{f}(\mu)]}} \xrightarrow{\mathcal{D}} t_{b-1}, \qquad \text{as } m \to \infty, \tag{35}$$

where $\hat{f}(\mu)$ can be $\hat{f}_C, \bar{f}_B$, or $\bar{f}_J$, and $\widehat{\mathsf{Var}}[\hat{f}(\mu)]$ can be S_B^2 or S_J^2. ∎

From Theorem 5, we can construct confidence intervals for $f(\mu)$. Clearly, any of the three point estimators for $f(\mu)$, in conjunction with either of the two variance estimators, (a) and (b), produces asymptotically valid confidence intervals for $f(\mu)$. One important property of the jackknife variance estimator that holds in the IID case is known as the Efron-Stein inequality (see Efron and Stein [19]); it states that the jackknife variance estimator overestimates the variance of a nonlinear function of the sample mean. This result implies that confidence intervals based on the jackknife variance estimator tend to have larger expected half-width than those based on the batch means variance estimator.

Section 3.3 proposes and studies several confidence intervals for $f(\mu)$ based on the point estimators (26)–(28) for $f(\mu)$ and the point estimators (29)–(31) for $\mathsf{Var}[\hat{f}(\mu)]$.

3.3 Confidence Intervals for $f(\mu)$

Since $\bar{Y}_n$, $\bar{Y}_{i,m}$, and $\bar{Y}_{-i,m}$ are consistent point estimators of μ, the three point estimators (26)–(28) for $f(\mu)$ (which is differentiable at μ) are also consistent. When f is a nonlinear function, the point estimator $f(\bar{Y}_n)$ is typically biased (even for IID observations); hence in general one is not able to produce unbiased estimators (Miller [49]). In this section, we will consider several CIs based on the point estimators for $f(\mu)$ and the variance estimators for $\hat{f}(\mu)$.

Based on Theorem 5, asymptotic $100(1-\alpha)\%$ confidence intervals for $f(\mu)$ are given by

$$f(\mu) \ \in \ \hat{f}(\mu) \pm t_{b-1,1-\alpha/2}\sqrt{\widehat{\mathsf{Var}}[\hat{f}(\mu)]}, \tag{36}$$

where $\hat{f}(\mu)$ can be $\hat{f}_C$, $\bar{f}_B$, or $\bar{f}_J$, and $\widehat{\mathsf{Var}}[\hat{f}(\mu)]$ can be S_B^2 or S_J^2.

To develop confidence interval estimators for $f(\mu)$ based on the delta variance estimator (31), we need to derive a CLT for the point estimators of $f(\mu)$ under consideration. We start with the first-order Taylor series expansion

$$f(x) = f(\mu) + f'(\mu)(x-\mu) + h(x-\mu), \quad x \in \mathbf{R}, \tag{37}$$

where the remainder $h : \mathbf{R} \to \mathbf{R}$ satisfies

$$\lim_{u \to 0} \frac{h(u)}{|u|} = 0. \tag{38}$$

From Equation (37), the expansion for $f(\bar{Y}_n)$ is given by

$$f(\bar{Y}_n) \ = \ f(\mu) + f'(\mu)(\bar{Y}_n - \mu) + h(\bar{Y}_n - \mu). \tag{39}$$

Lemma 1 Under the FCLT in Equation (9),

$$\sqrt{n} f'(\mu)(\bar{Y}_n - \mu) \ \xrightarrow{\mathcal{D}} \ |f'(\mu)|\sigma N(0,1), \quad \text{as } n \to \infty.$$

Proof The proof follows from $\sqrt{n}(\bar{Y}_n - \mu) \ \xrightarrow{\mathcal{D}} \ \sigma N(0,1)$. ∎

Now, if we scale both sides of Equation (39) by $\sqrt{n}$, we can show that the last term of Equation (39) converges to 0 in probability.

Lemma 2 Under the FCLT,

$$\sqrt{n}h(\bar{Y}_n - \mu) \xrightarrow{\mathcal{D}} 0, \quad \text{as } n \to \infty. \tag{40}$$

$$\frac{\sqrt{n}}{b}\sum_{i=1}^{b} h(\bar{Y}_{i,m} - \mu) \xrightarrow{\mathcal{D}} 0, \quad \text{as } m, b \to \infty, \tag{41}$$

and

$$\sqrt{n}\left\{bh(\bar{Y}_n - \mu) - \frac{b-1}{b}\sum_{i=1}^{b} h(\bar{Y}_{-i,m} - \mu)\right\} \xrightarrow{\mathcal{D}} 0, \quad \text{as } m, b \to \infty. \tag{42}$$

Proof We will prove Equation (40) only. The proofs of Equations (41) and (42) are similar. Our proof uses ideas from Section 3.1 of Muñoz and Glynn [51].

Define the function

$$h_1(x) = \begin{cases} h(x)/|x|, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

and write

$$\sqrt{n}h(\bar{Y}_n - \mu) = \sqrt{n}|\bar{Y}_n - \mu| h_1(\bar{Y}_n - \mu). \tag{43}$$

Equation (38) implies that h_1 is continuous at $x = 0$. Since $\bar{Y}_n \xrightarrow{\mathcal{P}} \mu$, the CMT implies

$$h_1(\bar{Y}_n - \mu) \xrightarrow{\mathcal{D}} 0, \quad \text{as } n \to \infty. \tag{44}$$

Recall that

$$\sqrt{n}(\bar{Y}_n - \mu) \xrightarrow{\mathcal{D}} \sigma N(0,1), \quad \text{as } n \to \infty. \tag{45}$$

Finally, Equations (44)–(45) and Slutsky's theorem imply Equation (40). The convergence in Equations (40)–(42) is also in probability because the limits are constant.

∎

The following result is an implication of Lemma 1–2 and Slutsky's theorem.

Theorem 6 Suppose that the FCLT holds and f is differentiable with a continuous first derivative in a neighborhood of μ. Then

$$\sqrt{n}(\hat{f}(\mu) - f(\mu)) \xrightarrow{\mathcal{D}} |f'(\mu)|\sigma N(0,1), \quad \text{as } m, b \to \infty,$$

where $\hat{f}(\mu)$ can be $\hat{f}_C$, $\bar{f}_B$, or $\bar{f}_J$.

Proof In Equation (39), we move $f(\mu)$ to the LHS and multiply both sides by $\sqrt{n}$. Then we get

$$\sqrt{n}(f(\bar{Y}_n) - f(\mu)) = \sqrt{n}f'(\mu)(\bar{Y}_n - \mu) + \sqrt{n}h(\bar{Y}_n - \mu). \tag{46}$$

Taking limits on both sides of Equation (46), the conclusion follows from Lemmas 1 and 2 and Slutsky's theorem. The proof for $\bar{f}_B$ and $\bar{f}_J$ is similar. ∎

Theorem 7 Suppose that $\hat{f}(\mu)$ is one of (26)–(28), $\hat{f}'(\mu)$ is one of (32)–(34), and $\widehat{V}$ is a consistent estimator of σ^2, that is, $\widehat{V} \xrightarrow{\mathcal{P}} \sigma^2$, as $m, b \to \infty$. Then

$$\frac{\sqrt{n}\big(\hat{f}(\mu) - f(\mu)\big)}{|\hat{f}'(\mu)|\sqrt{\widehat{V}}} \xrightarrow{\mathcal{D}} N(0,1), \quad \text{as } m, b \to \infty. \tag{47}$$

Proof Since f' is continuous in a neighborhood of μ, the estimators $\hat{f}'(\mu)$ in Equations (32)–(34) are consistent. First Theorem 6 implies

$$\frac{\sqrt{n}\big(\hat{f}(\mu) - f(\mu)\big)}{|f'(\mu)|\sigma} \xrightarrow{\mathcal{D}} N(0,1), \quad \text{as } m, b \to \infty. \tag{48}$$

The consistency of $\hat{V}$ and the CMT imply

$$\frac{\sigma}{\widehat{V}^{1/2}} \xrightarrow{\mathcal{P}} 1. \tag{49}$$

By Equations (48)–(49) and Slutsky's theorem, one has

$$\frac{\sqrt{n}\big(\hat{f}(\mu) - f(\mu)\big)}{|f'(\mu)|\sqrt{\widehat{V}}} \xrightarrow{\mathcal{D}} N(0,1), \quad \text{as } m, b \to \infty. \tag{50}$$

Since $\hat{f}'(\mu)$ is consistent for $f'(\mu)$, the CMT implies

$$\frac{f'(\mu)}{\hat{f}'(\mu)} \xrightarrow{\mathcal{P}} 1. \tag{51}$$

Equation (47) follows from Equations (50)–(51), the CMT, and Slutsky's theorem. ∎

Theorem yields the following asymptotically valid $100(1-\alpha)\%$ CI for $f(\mu)$:

$$f(\mu) \in \hat{f}(\mu) \pm z_{1-\alpha/2}|\hat{f}'(\mu)|\sqrt{\widehat{V}/n}. \tag{52}$$

Recall that the batch means estimator $\widehat{V}_B$, the area estimator $\widehat{V}_A$, and the combined estimator $\widehat{V}_C$ are consistent estimators of σ^2 (as both $m, b \to \infty$), yielding asymptotically valid CIs for $f(\mu)$.

Alternatively, we may use the limiting result,

$$\frac{\sqrt{n}\big(\hat{f}(\mu) - f(\mu)\big)}{|f'(\mu)|\sqrt{\sigma^2}} \Big/ \sqrt{\frac{\widehat{V}}{\sigma^2}} \;\; \xrightarrow{\mathcal{D}} \;\; t_{\mathrm{df}}, \tag{53}$$

where "df" refers to the appropriate degrees of freedom and $\widehat{V}$ can be $\widehat{V}_B$, $\widehat{V}_A$, or $\widehat{V}_C$. Property (53) yields the following asymptotically valid confidence interval for $f(\mu)$:

$$f(\mu) \;\in\; \hat{f}(\mu) \pm t_{\mathrm{df},1-\alpha/2} |\hat{f}'(\mu)| \sqrt{\widehat{V}/n}. \tag{54}$$

CHAPTER IV

ORDER-ONE MOVING AVERAGE PROCESS

This chapter evaluates the point and CI estimators for $f(\mu)$ using a first-order moving average [MA(1)] process. We present analytical and experimental results based on $f(\mu) = (\mu + 1)^2$ and $f(\mu) = 1/(1 + \mu)$. The point estimators are compared based on their mean and variance. The various CIs are compared based on the estimated coverage, the average half-width, and the variance of the half-width.

4.1 Preliminaries

Consider the stationary first-order moving average [MA(1)] process defined by $Y_i = \theta\epsilon_{i-1} + \epsilon_i$, $i \geq 1$, where the ϵ_is are IID $N(0, 1)$ RVs. Figure 1 shows a sample path of a Gaussian MA(1) process with $\mu = 0$ and $\theta = 0.9$ based on 100 observations. This process has autocovariance function $R_0 = 1 + \theta^2$, $R_{\pm 1} = \theta$, and $R_k = 0$, whence we have $\sigma^2 = \sum_{j=-\infty}^{\infty} R_j = (1 + \theta)^2$, $\gamma = -2\sum_{j=1}^{\infty} jR_j = -2\theta$, and

$$\mathsf{Cov}(\bar{Y}_j, \bar{Y}_k) = \begin{cases} \frac{\sigma^2}{k} + \frac{\gamma}{k^2}, & \text{for } j = k \\ \frac{\sigma^2}{k} + \frac{\gamma}{2jk}, & \text{for } j < k. \end{cases} \tag{55}$$

From (55), we have

$$\mathsf{Var}[\bar{Y}_{1,m}] = \frac{(1+\theta)^2}{m} - \frac{2\theta}{m^2}. \tag{56}$$

Furthermore,

$$\begin{aligned} \mathsf{Cov}[\bar{Y}_{1,m}, \bar{Y}_{2,m}] &= \mathsf{Cov}\left[\frac{1}{m}\sum_{i=1}^{m} Y_i, \frac{1}{m}\sum_{j=m+1}^{2m} Y_j\right] \\ &= \frac{1}{m^2}\mathsf{Cov}[Y_m, Y_{m+1}] \\ &= \frac{\theta}{m^2} \end{aligned} \tag{57}$$

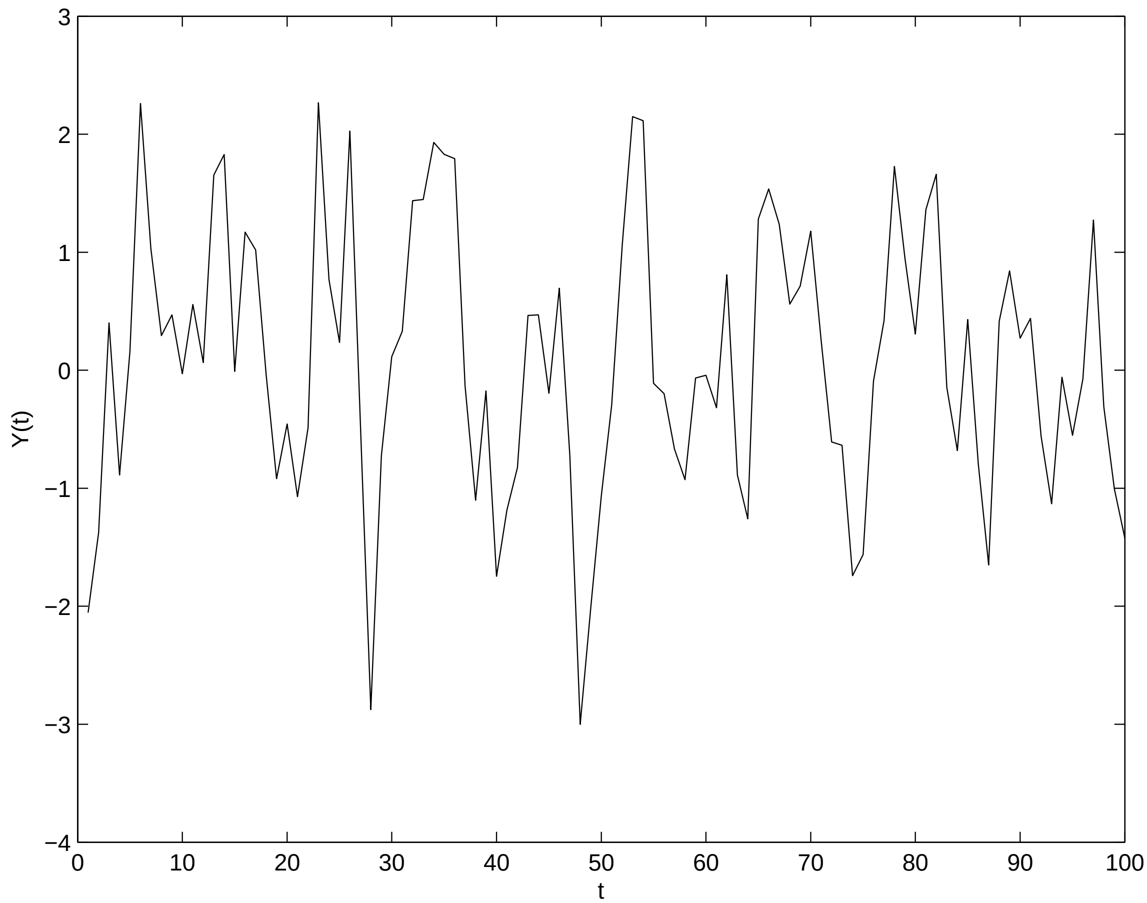

Figure 1: Sample Path of a Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$ based on 100 Observations

and

$$\mathsf{Cov}[\bar{Y}_{i,m}, \bar{Y}_{j,m}] = 0, \qquad \text{for } |j-i| \geq 2. \tag{58}$$

Later on, we will use the following lemma.

Lemma 3 If $X \sim N(0, \sigma^2)$ and r is a even number, then

$$\mathsf{Var}[X^r] = \left[\frac{(2r)!}{r!} - \left(\frac{r!}{(r/2)!} \right)^2 \right] \frac{\sigma^{2r}}{2^r}. \tag{59}$$

Proof Recall that if $X \sim N(0, \sigma^2)$, then

$$\mu_r = \begin{cases} 0, & \text{for } r \text{ odd} \\ \\ \frac{r!}{(r/2)!} \frac{\sigma^r}{2^{r/2}}, & \text{for } r \text{ even}, \end{cases} \tag{60}$$

where $\mu_r = \mathsf{E}[X^r]$ is the rth moment (cf. Mood et al. [50]). Then

$$\begin{aligned} \mathsf{Var}[X^r] &= \mathsf{E}[X^{2r}] - (\mathsf{E}[X^r])^2 \\ &= \frac{(2r)!}{r!} \frac{\sigma^{2r}}{2^r} - \left[\frac{r!}{(r/2)!} \frac{\sigma^r}{2^{r/2}} \right]^2 \\ &= \left[\frac{(2r)!}{r!} - \left(\frac{r!}{(r/2)!} \right)^2 \right] \frac{\sigma^{2r}}{2^r}. \blacksquare \end{aligned}$$

4.2 Performance Evaluation when $f(\mu) = (\mu + 1)^2$

In this subsection, we study the performance of the various estimators based on the quadratic function $f(\mu) = (\mu + 1)^2$. This function allows several analytical calculations.

4.2.1 Performance of Point Estimators

We first calculate the expectation and variance of point estimators for $f(\mu)$.

Case 1: $\hat{f}_C$ (classical estimator). By Equation (56),

$$\mathsf{E}[f(\bar{Y}_n)] = \mathsf{E}[(\bar{Y}_n + 1)^2] = \mathsf{E}[\bar{Y}_n^2] + 1 = \mathsf{Var}[\bar{Y}_n] + 1 = \frac{(1+\theta)^2}{n} - \frac{2\theta}{n^2} + 1. \tag{61}$$

In an attempt to shorten expression, we define

$$v_n^2 \equiv \mathsf{Var}[\bar{Y}_n] = \frac{(1+\theta)^2}{n} - \frac{2\theta}{n^2}, \quad n \geq 1.$$

One now has

$$\mathsf{Var}[f(\bar{Y}_n)] = \mathsf{Var}[(\bar{Y}_n+1)^2] = \mathsf{E}[(\bar{Y}_n+1)^4] - \left(\mathsf{E}[(\bar{Y}_n+1)^2]\right)^2. \tag{62}$$

Since $\bar{Y}_n$ has the normal distribution, we use Equation (60) to obtain

$$\begin{aligned} \mathsf{E}[(\bar{Y}_n+1)^4] &= \mathsf{E}[\bar{Y}_n^4] + 6\mathsf{E}[\bar{Y}_n^2] + 1 \\ &= 3v_n^4 + 6v_n^2 + 1. \end{aligned} \tag{63}$$

Putting Equation (63) into Equation (62),

$$\mathsf{Var}[f(\bar{Y}_n)] = 3v_n^4 + 6v_n^2 + 1 - (v_n^2+1)^2 = 2v_n^4 + 4v_n^2. \tag{64}$$

Case 2: $\bar{f}_B$ (batch means estimator).

$$\begin{aligned} \mathsf{E}[\bar{f}_B] &= \frac{1}{b}\sum_{i=1}^{b} \mathsf{E}[(\bar{Y}_{i,m}+1)^2] \\ &= \frac{1}{b}\left(\sum_{i=1}^{b} \mathsf{E}[\bar{Y}_{i,m}^2] + b\right) \\ &= \mathsf{Var}[\bar{Y}_{1,m}] + 1 \\ &= v_m^2 + 1. \end{aligned} \tag{65}$$

Also,

$$\begin{aligned} \mathsf{Var}[\bar{f}_B] &= \mathsf{Var}\left[\frac{1}{b}\sum_{j=1}^{b}(\bar{Y}_{j,m}+1)^2\right] \\ &= \frac{1}{b^2}\mathsf{Cov}\left[\sum_{j=1}^{b}\bar{Y}_{j,m}^2 + 2\sum_{j=1}^{b}\bar{Y}_{j,m} + b, \sum_{k=1}^{b}\bar{Y}_{k,m}^2 + 2\sum_{k=1}^{b}\bar{Y}_{k,m} + b\right] \\ &= \frac{1}{b^2}\left(\mathsf{Cov}\left[\sum_{j=1}^{b}\bar{Y}_{j,m}^2, \sum_{k=1}^{b}\bar{Y}_{k,m}^2\right] + 4\mathsf{Cov}\left[\sum_{j=1}^{b}\bar{Y}_{j,m}, \sum_{k=1}^{b}\bar{Y}_{k,m}\right]\right). \end{aligned} \tag{66}$$

Now,

$$\begin{aligned}
\mathsf{Cov}\left[\sum_{j=1}^{b}\bar{Y}_{j,m}^2, \sum_{k=1}^{b}\bar{Y}_{k,m}^2\right] &= 2\sum_{j=1}^{b}\sum_{k=1}^{b}\mathsf{Cov}^2\left[\,\bar{Y}_{j,m}, \bar{Y}_{k,m}\,\right] \\
&\quad \text{(by Patel and Read [53]) when expected value} = 0) \\
&= 2\left(b\mathsf{Var}^2\left[\bar{Y}_{1,m}\right] + 2(b-1)\mathsf{Cov}^2\left[\bar{Y}_{1,m}, \bar{Y}_{2,m}\right]\right) \\
&\quad \text{(since only adjacent batch means are correlated)} \\
&= 2[bv_m^4 + 2(b-1)\theta^2/m^4], \qquad (67)
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{Cov}\left[\sum_{j=1}^{b}\bar{Y}_{j,m}, \sum_{k=1}^{b}\bar{Y}_{k,m}\right] &= \sum_{j=1}^{b}\sum_{k=1}^{b}\mathsf{Cov}\left[\bar{Y}_{j,m}, \bar{Y}_{k,m}\right] \\
&= 2(b-1)\theta/m^2 + bv_m^2. \qquad (68)
\end{aligned}$$

Therefore, putting Equations (67) and (68) into Equation (66),

$$\mathsf{Var}[\bar{f}_B] = \frac{2}{b^2}\left[bv_m^4 + 2bv_m^2 + 2(b-1)(\theta/m^2)(\theta/m^2-2)\right]. \qquad (69)$$

Case 3: $\bar{f}_J$ (jackknife estimator). By Equation (18),

$$\bar{f}_J = \frac{1}{b}\sum_{i=1}^{b}\tilde{f}_i,$$

where

$$\tilde{f}_i = b(\bar{Y}_n+1)^2 - \frac{1}{b-1}\left(\sum_{j\neq i}\bar{Y}_{j,m}\right)^2 - 2\left(\sum_{j\neq i}\bar{Y}_{j,m}\right) - b + 1.$$

Therefore, after some algebra,

$$\mathsf{E}[\bar{f}_J] = b\mathsf{E}[\bar{Y}_n^2] - \frac{1}{b(b-1)}\mathsf{E}\left[\sum_{i=1}^{b}\left(\sum_{j\neq i}\bar{Y}_{j,m}\right)^2\right] + 1. \qquad (70)$$

In order to simplify Equation (70), notice that

$$\begin{aligned}
\mathsf{E}\left[\sum_{i=1}^{b}\left(\sum_{j\neq i}\bar{Y}_{j,m}\right)^2\right] &= \sum_{i=1}^{b}\mathsf{E}\left[\left(\sum_{j\neq i}\bar{Y}_{j,m}\right)^2\right] \\
&= \sum_{i=1}^{b}\left\{\mathsf{Var}\left[\sum_{j\neq i}\bar{Y}_{j,m}\right] + \left(\mathsf{E}\left[\sum_{j\neq i}\bar{Y}_{j,m}\right]\right)^2\right\} \\
&= \sum_{i=1}^{b}\mathsf{Var}\left[\sum_{j\neq i}\bar{Y}_{j,m}\right]. \qquad (71)
\end{aligned}$$

The RHS of Equation (71) can be expressed as

$$\begin{aligned}
\sum_{i=1}^{b} \mathsf{Var}\left[\sum_{j\neq i} \bar{Y}_{j,m}\right] &= \sum_{i=1}^{b}\sum_{j\neq i}\sum_{k\neq i} \mathsf{Cov}[\bar{Y}_{j,m}, \bar{Y}_{k,m}] \\
&= 2\left[(b-1)\mathsf{Var}[\bar{Y}_{1,m}] + 2(b-2)\mathsf{Cov}[\bar{Y}_{1,m}, \bar{Y}_{2,m}]\right] \\
&\quad +(b-2)\left[(b-1)\mathsf{Var}[\bar{Y}_{1,m}] + 2(b-3)\mathsf{Cov}[\bar{Y}_{1,m}, \bar{Y}_{2,m}]\right] \\
&\quad \text{(break into } i = 1, b \text{ and } i = 2, 3, \ldots, b-1 \text{ cases)} \\
&= b(b-1)\mathsf{Var}[\bar{Y}_{1,m}] + 2(b-1)(b-2)\mathsf{Cov}[\bar{Y}_{1,m}, \bar{Y}_{2,m}]. \qquad (72)
\end{aligned}$$

Plugging Equations (56) and (72) into Equation (70), we have

$$\begin{aligned}
\mathsf{E}[\bar{f}_J] &= bv_n^2 - \frac{1}{b(b-1)}\left[b(b-1)v_m^2 + 2(b-2)(b-1)\theta/m^2\right] + 1 \\
&= bv_n^2 - v_m^2 - 2\theta(b-2)/(bm^2) + 1 \\
&= 2\theta/(bm^2) + 1.
\end{aligned}$$

We skip the derivation for $\mathsf{Var}[\bar{f}_J]$ because it is very tedious when $b > 2$.

4.2.2 Performance of Variance Estimators

This section presents approximations for the means and variances of the various variance estimators. We start with the expectation and variance of point estimators for $\mathsf{Var}[\hat{f}(\mu)]$.

Case 1: S_*^2 (delta variance estimator). In this case we have

$$S_*^2(\hat{f}'; \widehat{V}) = \frac{\hat{f}'(\mu)^2\widehat{V}}{n} = \frac{4(\bar{Y}_n+1)^2\widehat{V}}{n},$$

where $\widehat{V}$ can be $\widehat{V}_B$, $\widehat{V}_A$, or $\widehat{V}_C$. Note that all three point estimators (26)–(28) for $f'(\mu) = 2(\mu+1)$ are equal to $2(\bar{Y}_n+1)$. Then

$$\mathsf{E}[S_*^2(2(\bar{Y}_n+1); \widehat{V})] = \frac{4}{n}\mathsf{E}[(\bar{Y}_n+1)^2\widehat{V}]$$

and

$$\begin{aligned}
\mathsf{Var}[S_*^2(2(\bar{Y}_n+1); \widehat{V})] &= \mathsf{Var}\left[\frac{f'(\mu)^2\widehat{V}}{n}\right] = \mathsf{Var}\left[\frac{4(\bar{Y}_n+1)^2\widehat{V}}{n}\right] \\
&= \frac{16}{n^2}\left(\mathsf{E}[(\bar{Y}_n+1)^4\widehat{V}^2] - \left\{\mathsf{E}[(\bar{Y}_n+1)^2\widehat{V}]\right\}^2\right). \qquad (73)
\end{aligned}$$

We consider the following three subcases, labeled 1(a)–1(c):

Case 1(a): $\widehat{V} = \widehat{V}_B$ (NBM estimator for σ^2).

$$
\begin{aligned}
\mathsf{E}[\widehat{V}_B] &= \mathsf{E}\left[\frac{m}{b-1}\sum_{i=1}^{b}(\bar{Y}_{i,m}-\bar{Y}_n)^2\right] \\
&\doteq \mathsf{E}\left[\,\sigma^2\chi^2_{b-1}/(b-1)\,\right] \quad \text{for large } m \\
&= \left(\frac{\sigma^2}{b-1}\right)\mathsf{E}[\chi^2_{b-1}] \\
&= \left(\frac{\sigma^2}{b-1}\right)(b-1) \\
&= \sigma^2. \qquad (74)
\end{aligned}
$$

Also, since $\bar{Y}_n$ and $\widehat{V}_B$ are asymptotically independent as $m \to \infty$,

$$
\begin{aligned}
\mathsf{E}[S^2_*(2(\bar{Y}_n+1);\widehat{V}_B)] &= \mathsf{E}\left[\frac{\hat{f}'(\mu)^2\widehat{V}_B}{n}\right] = \mathsf{E}\left[\frac{4(\bar{Y}_n+1)^2\widehat{V}_B}{n}\right] \\
&\doteq \frac{4}{n}\mathsf{E}[(\bar{Y}_n+1)^2]\mathsf{E}[\widehat{V}_B] \\
&\doteq \frac{4}{n}(v_n^2+1)\sigma^2 \qquad (75)
\end{aligned}
$$

and

$$
\mathsf{Var}[S^2_*(2(\bar{Y}_n+1);\widehat{V}_B)] \doteq \frac{16}{n^2}\left(\mathsf{E}[(\bar{Y}_n+1)^4]\mathsf{E}[\widehat{V}_B^2] - \left\{\mathsf{E}[(\bar{Y}_n+1)^2]\mathsf{E}[\widehat{V}_B]\right\}^2\right). \qquad (76)
$$

By Equation (63),

$$
\mathsf{E}[(\bar{Y}_n+1)^4] = 3v_n^4 + 6v_n^2 + 1 \qquad (77)
$$

and by Theorem 4,

$$
\begin{aligned}
\mathsf{E}[\hat{V}_B^2] &\doteq \mathsf{E}\left[\left(\sigma^2\chi^2_{b-1}/(b-1)\right)^2\right] \\
&= \left(\frac{\sigma^2}{b-1}\right)^2\mathsf{E}[(\chi^2_{b-1})^2] \\
&= \frac{\sigma^4}{(b-1)^2}(b-1)(b+1) \\
&= \frac{\sigma^4(b+1)}{b-1}, \qquad (78)
\end{aligned}
$$

where the third equality is due to

$$
\mathsf{E}\left[(\chi^2_d)^2\right] = \mathsf{Var}[\chi^2_d] + \left(\mathsf{E}[\chi^2_d]\right)^2 = 2d + d^2 = d(d+2). \qquad (79)
$$

Substitution of Equations (56), (74), (77), and (78) into the RHS of Equation (76) yields

$$\begin{aligned}\mathsf{Var}[S_*^2(2(\bar{Y}_n+1);\widehat{V}_B)] &\doteq \frac{16}{n^2}\left[(3v_n^4+6v_n^2+1)\frac{(b+1)}{b-1}\sigma^4-(v_n^2+1)^2\sigma^4\right] \\ &= \frac{32\sigma^4}{n^2(b-1)}\left[(b+2)v_n^4+2(b+2)v_n^2+1\right]. \end{aligned} \tag{80}$$

Remark 3 It has been shown that

$$\begin{aligned}\mathsf{E}[\widehat{V}_B] &= \sigma^2+\frac{\gamma(b+1)}{m}+o(1/m) \\ &= \sigma^2-\frac{2\theta(b+1)}{m}+o(1/m) \end{aligned} \tag{81}$$

(see Alexopoulos and Goldsman [1], Chien et al. [10], and references therein). Note that the RHS of Equation (81) converges to σ^2, as indicated by Equation (74).

Example 2 Consider the stationary Gaussian MA(1) process with mean 0 and $\theta = 0.9$. The variance parameter is $\sigma^2 = (1+\theta)^2 = 3.61$. Table 1 compares the analytical approximations obtained in this subsection to estimates obtained from 1000 independent experiments for various combinations of b and m.

Table 1: Performance Evaluation of $S_*^2(2(\bar{Y}_n+1);\widehat{V}_B)$ for an MA(1) Process with $\mu = 0$ and $\theta = 0.9$, and $f(\mu) = (\mu+1)^2$

	m	$\mathsf{E}[S_C^2]$	$\widehat{\mathsf{E}}[S_C^2]$	$\mathsf{Var}[S_C^2]$	$\widehat{\mathsf{Var}}[S_C^2]$
$b = 32$	128	3.525E-03	3.460E-03	8.499E-07	8.541E-07
	256	1.763E-03	1.757E-03	2.065E-07	2.077E-07
	512	8.814E-04	8.784E-04	5.087E-08	5.368E-08
$b = 128$	128	8.814E-04	8.814E-04	1.293E-08	1.265E-08
	256	4.407E-04	4.412E-04	3.146E-09	3.186E-09
	512	2.203E-04	2.200E-04	7.755E-10	7.544E-10

The results show that the analytical approximations are quite close to the simulation-based estimates. This is probably attributable to the consistency of the delta variance estimator $S_*^2(2(\bar{Y}_n+1);\widehat{V}_B)$.

Case 1(b): $\widehat{V} = \widehat{V}_A(w_i)$ (the batched STS area estimator with weight function $w(t) = w_i(t), i = 0,1,2$). We have

$$\mathsf{E}[\widehat{V}_A(w_i)] \doteq \mathsf{E}[\sigma^2\chi_b^2/b] = \left(\frac{\sigma^2}{b}\right) b = \sigma^2. \tag{82}$$

Since $\bar{Y}_n$ and $\widehat{V}_A$ are asymptotically independent,

$$\begin{aligned}
\mathsf{E}[S_*^2(2(\bar{Y}_n+1);\widehat{V}_A(w_i))] &= \mathsf{E}\left[\frac{\hat{f}'(\mu)^2\widehat{V}_A}{n}\right] \\
&= \mathsf{E}\left[\frac{(4\bar{Y}_n+1)^2\widehat{V}_A}{n}\right] \\
&\doteq \frac{4}{n}\mathsf{E}[(\bar{Y}_n+1)^2]\mathsf{E}[\widehat{V}_A(w_i)] \\
&\doteq \frac{4}{n}(v_n^2+1)\sigma^2. \qquad (83)
\end{aligned}$$

To obtain the variance of $S_*^2(2(\bar{Y}_n+1);\widehat{V}_A(w_i))$, we just need to know $\mathsf{E}[\widehat{V}_A(w_i)^2]$ in Equation (76) since the other terms are the same as in Equations (77), (82) and v_n^2. Using Equation (79) we have (cf. Goldsman and Schruben [32])

$$\begin{aligned}
\mathsf{E}[\hat{V}_A^2(w_i)] &\doteq \mathsf{E}\left[\left(\sigma^2\chi_b^2/(b)\right)^2\right] \\
&= \left(\frac{\sigma^2}{b}\right)^2 \mathsf{E}\left[\ (\chi_b^2)^2\ \right] \\
&= \left(\frac{b+2}{b}\right)\sigma^4.
\end{aligned}$$

Equation (73) can now have be written as

$$\begin{aligned}
&\mathsf{Var}[S_*^2(2(\bar{Y}_n+1);\widehat{V}_A(w_i))] \\
&\doteq \frac{16}{n^2}\left[(3v_n^4+6v_n^2+1)\frac{(b+2)}{b}\sigma^4-(v_n^2+1)^2\sigma^4\right] \\
&= \frac{32\sigma^4}{n^2 b}\left[(b+3)v_n^4+2(b+3)v_n^2+1\right].
\end{aligned}$$

Remark 4 From Equation (11) with $\gamma = -2\theta$, we have

$$\mathsf{E}[\widehat{V}_A] = \begin{cases} \sigma^2 - 6\theta/m + o(1/m) & \text{for } w_0(t) = \sqrt{12} \\ \sigma^2 - 6.25\theta/m + o(1/m) & \text{for } w_1(t) = \sqrt{45}t \\ \sigma^2 + o(1/m) & \text{for } w_2(t) = \sqrt{840}(3t^2 - 3t + 1/2). \end{cases} \tag{84}$$

(see Foley and Goldsman [25] and Goldsman et al. [31]). Note that the RHS of Equation (84) converges to σ^2, as indicated by Equation (82).

Example 3 Tables 2–4 compare the analytical approximations for the mean and variance of the estimators $S_*^2(2(\bar{Y}_n + 1); \widehat{V}_A(w_i))$ to Monte Carlo estimates obtained from 1000 independent experiments for various values of b and m. Again, the analytical approximations are quite close to the respective estimates. Also, the estimates of the mean and variance of $S_*^2(2(\bar{Y}_n + 1); \widehat{V}_A(w_i))$ are close to the respective estimates for $S_*^2(2(\bar{Y}_n + 1); \widehat{V}_B)$.

Table 2: Performance Evaluation of $S_*^2((2\bar{Y}_n + 1); \widehat{V}_A(w_0))$ for an MA(1) Process with $\mu = 0$ and $\theta = 0.9$, and $f(\mu) = (\mu + 1)^2$

	m	$\mathsf{E}[S_*^2]$	$\widehat{\mathsf{E}}[S_*^2]$	$\mathsf{Var}[S_*^2]$	$\widehat{\mathsf{Var}}[S_*^2]$
$b = 32$	128	3.525E-03	3.443E-03	8.247E-07	8.277E-07
	256	1.763E-03	1.744E-03	2.002E-07	2.036E-07
	512	8.814E-04	8.865E-04	4.930E-08	4.821E-08
$b = 128$	128	8.814E-04	8.724E-04	1.284E-08	1.356E-08
	256	4.407E-04	4.396E-04	3.122E-09	3.068E-09
	512	2.203E-04	2.201E-04	7.695E-10	7.807E-10

Table 3: Performance Evaluation of $S_*^2(2(\bar{Y}_n+1); \widehat{V}_A(w_1))$ for an MA(1) Process with $\mu = 0$ and $\theta = 0.9$, and $f(\mu) = (\mu+1)^2$

	m	$\mathsf{E}[S_*^2]$	$\widehat{\mathsf{E}}[S_*^2]$	$\mathsf{Var}[S_*^2]$	$\widehat{\mathsf{Var}}[S_*^2]$
$b = 32$	128	3.525E-03	3.438E-03	8.247E-07	8.279E-07
	256	1.763E-03	1.744E-03	2.002E-07	2.058E-07
	512	8.814E-04	8.873E-04	4.930E-08	4.821E-08
$b = 128$	128	8.814E-04	8.712E-04	1.284E-08	1.347E-08
	256	4.407E-04	4.394E-04	3.122E-09	3.130E-09
	512	2.203E-04	2.203E-04	7.695E-10	7.507E-10

Table 4: Performance Evaluation of $S_*^2(2(\bar{Y}_n+1); \widehat{V}_A(w_2))$ for an MA(1) Process with $\mu = 0$ and $\theta = 0.9$, and $f(\mu) = (\mu+1)^2$

	m	$\mathsf{E}[S_*^2]$	$\widehat{\mathsf{E}}[S_*^2]$	$\mathsf{Var}[S_*^2]$	$\widehat{\mathsf{Var}}[S_*^2]$
$b = 32$	128	3.525E-03	3.511E-03	8.247E-07	8.674E-07
	256	1.763E-03	1.751E-03	2.002E-07	2.195E-07
	512	8.814E-04	8.811E-04	4.930E-08	4.856E-08
$b = 128$	128	8.814E-04	8.822E-04	1.284E-08	1.316E-08
	256	4.407E-04	4.418E-04	3.122E-09	2.961E-09
	512	2.203E-04	2.200E-04	7.695E-10	7.816E-10

Case 1(c): $\widehat{V} = \widehat{V}_C(w_i)$ (combination of the NBM and batched STS area estimators).

From Theorem 4 we have

$$\mathsf{E}[\widehat{V}_C(w_i)] \doteq \sigma^2. \tag{85}$$

Again, since $\bar{Y}_n$ and $\widehat{V}_C$ are asymptotically independent,

$$\begin{aligned} \mathsf{E}[S_*^2(2(\bar{Y}_n+1); \widehat{V}_C(w_i))] &= \mathsf{E}\left[\frac{4(\bar{Y}_n+1)^2\widehat{V}_C(w_i)}{n}\right] \\ &\doteq \frac{4}{n}(v_n^2+1)\sigma^2. \end{aligned} \tag{86}$$

As with Equation (76), we just need to obtain $\mathsf{E}[\widehat{V}_C(w_i)^2]$ since the other terms come from Equations (56), (77), and (85). We have

$$
\begin{aligned}
\mathsf{E}\left[\widehat{V}_C(w_i)^2\right] &\doteq \mathsf{E}\left[\left(\sigma^2\chi^2_{2b-1}/(2b-1)\right)^2\right] \\
&= \left(\frac{\sigma^2}{2b-1}\right)^2 \mathsf{E}\left[(\chi^2_{2b-1})^2\right] \\
&= \left(\frac{\sigma^2}{2b-1}\right)^2 (2b-1)(2b+1) \\
&= \left(\frac{2b+1}{2b-1}\right)\sigma^4,
\end{aligned}
$$

where the third equality is due to Equation (79).

Finally,

$$
\begin{aligned}
&\mathsf{Var}[S^2_*(2(\bar{Y}_n+1);\widehat{V}_C(w_i))] \\
&\doteq \frac{16}{n^2}\left[(3v_n^4+6v_n^2+1)\frac{(2b+1)}{2b-1}\sigma^4-(v_n^2+1)^2\sigma^4\right] \\
&= \frac{32\sigma^4}{n^2(2b-1)}\left[2(b+1)v_n^4+4(b+1)v_n^2+1\right].
\end{aligned}
$$

Example 4 Tables 5–7 present experimental results related to the Gaussian MA(1) process with mean 0, $\theta = 0.9$, the function $f(\mu) = (\mu+1)^2$, and the estimators $S^2_*(2(\bar{Y}_n+1);\widehat{V}_C(w_i))$ based on the weight functions w_0, w_1, and w_2.

Table 5: Performance Evaluation of $S^2_*(2(\bar{Y}_n+1);\widehat{V}_C(w_0))$ for an MA(1) Process with $\mu = 0$ and $\theta = 0.9$, and $f(\mu) = (\mu+1)^2$

	m	$\mathsf{E}[S^2_*]$	$\widehat{\mathsf{E}}[S^2_*]$	$\mathsf{Var}[S^2_*]$	$\widehat{\mathsf{Var}}[S^2_*]$
$b = 32$	128	3.525E-03	3.451E-03	4.405E-07	4.301E-07
	256	1.763E-03	1.750E-03	1.044E-07	1.051E-07
	512	8.814E-04	8.825E-04	2.538E-08	2.535E-08
$b = 128$	128	8.814E-04	8.769E-04	6.785E-09	7.178E-09
	256	4.407E-04	4.404E-04	1.610E-09	1.617E-09
	512	2.203E-04	2.200E-04	3.916E-10	4.155E-10

Table 6: Performance Evaluation of $S_*^2(2(\bar{Y}_n+1);\widehat{V}_C(w_1))$ for an MA(1) Process with $\mu=0$ and $\theta=0.9$, and $f(\mu)=(\mu+1)^2$

	m	$\mathsf{E}[S_*^2]$	$\widehat{\mathsf{E}}[S_*^2]$	$\mathsf{Var}[S_*^2]$	$\widehat{\mathsf{Var}}[S_*^2]$
$b=32$	128	3.525E-03	3.449E-03	4.405E-07	4.317E-07
	256	1.763E-03	1.751E-03	1.044E-07	1.069E-07
	512	8.814E-04	8.829E-04	2.538E-08	2.528E-08
$b=128$	128	8.814E-04	8.763E-04	6.785E-09	7.169E-09
	256	4.407E-04	4.403E-04	1.610E-09	1.632E-09
	512	2.203E-04	2.201E-04	3.916E-10	4.024E-10

Table 7: Performance Evaluation of $S_*^2(2(\bar{Y}_n+1);\widehat{V}_C(w_2))$ for an MA(1) Process with $\mu=0$ and $\theta=0.9$, and $f(\mu)=(\mu+1)^2$

	m	$\mathsf{E}[S_*^2]$	$\widehat{\mathsf{E}}[S_*^2]$	$\mathsf{Var}[S_*^2]$	$\widehat{\mathsf{Var}}[S_*^2]$
$b=32$	128	3.525E-03	3.486E-03	4.405E-07	4.335E-07
	256	1.763E-03	1.754E-03	1.044E-07	1.084E-07
	512	8.814E-04	8.797E-04	2.538E-08	2.581E-08
$b=128$	128	8.814E-04	8.818E-04	6.785E-09	7.080E-09
	256	4.407E-04	4.415E-04	1.610E-09	1.644E-09
	512	2.203E-04	2.200E-04	3.916E-10	4.083E-10

The entries of columns 3 and 4 indicate that the estimates of $\mathsf{E}[S_*^2]$ are quite close to the respective analytical approximations. Also, the estimates of $\mathsf{Var}[S_*^2]$ are quite close to the respective analytical approximations. Finally, notice that the estimates of the variance of $S_*^2(2(\bar{Y}_n+1);\widehat{V}_C(w_i))$ are smaller than the respective estimates for $S_*^2(2(\bar{Y}_n+1);\widehat{V}_A(w_i))$ and $S_*^2(2(\bar{Y}_n+1);\widehat{V}_B)$.

Case 2: S_B^2 and S_J^2 (NBM and jackknife variance estimators). Since the analytic calculation of $\mathsf{E}[S_B^2]$, $\mathsf{E}[S_J^2]$, $\mathsf{Var}[S_B^2]$, and $\mathsf{Var}[S_J^2]$ is too tedious, we compute estimates based on 1000 independent experiments.

Example 5 This example estimates the mean and variance of S_B^2 and S_J^2 using a Monte Carlo experiment. We use a stationary MA(1) process with mean 0 and

$\theta = 0.9$. In Table 8, b and m denote the number of batches and batch size, respectively. One can see that S_J^2 has a slightly smaller sample mean and variance than the NBM variance estimator S_B^2.

Table 8: Performance Evaluation of S_B^2 and S_J^2 for an MA(1) Process with $\mu = 0$ and $\theta = 0.9$, and $f(\mu) = (\mu + 1)^2$

	m	$\widehat{\mathsf{E}}[S_B^2]$	$\widehat{\mathsf{E}}[S_J^2]$	$\widehat{\mathsf{Var}}[S_B^2]$	$\widehat{\mathsf{Var}}[S_J^2]$
$b = 32$	128	3.510E-03	3.460E-03	9.796E-07	8.535E-07
	256	1.766E-03	1.757E-03	2.196E-07	2.078E-07
	512	8.807E-04	8.784E-04	5.509E-08	5.368E-08
$b = 128$	128	8.939E-04	8.814E-04	1.424E-08	1.265E-08
	256	4.443E-04	4.412E-04	3.457E-09	3.186E-09
	512	2.207E-04	2.200E-04	7.884E-10	7.544E-10

4.2.3 Performance of Confidence Interval Estimators

Now we consider the performance of confidence intervals. First, we study the expected interval half-width $\mathsf{E}[H]$ and the variance of the confidence interval half-width, $\mathsf{Var}[H]$. Then, we deal with the coverage of the CIs. Unlike the case of the variance estimators for $\hat{f}(\mu)$, we start with the combined variance estimator. The CI in Equation (54) has half-width $H(\hat{f}'(\mu); \widehat{V}) = t_{\mathrm{df},1-\alpha/2}|\hat{f}'(\mu)|\sqrt{\widehat{V}/n}$. Therefore,

$$\mathsf{E}[H(\hat{f}'(\mu); \widehat{V})] = t_{\mathrm{df},1-\alpha/2}\frac{1}{\sqrt{n}}\mathsf{E}\left[|\hat{f}'(\mu)|\sqrt{\widehat{V}}\right]$$

and

$$\mathsf{Var}[H(\hat{f}'(\mu); \widehat{V})] = t^2_{\mathrm{df},1-\alpha/2}\frac{1}{n}\mathsf{Var}\left[|\hat{f}'(\mu)|\sqrt{\widehat{V}}\right].$$

Since $f(\mu) = (\mu + 1)^2$ and $\hat{f}'(\mu) = 2(\bar{Y}_n + 1)$, we have

$$H(\hat{f}'(\mu); \widehat{V}) = t_{\mathrm{df},1-\alpha/2}|2(\bar{Y}_n + 1)|\sqrt{\widehat{V}/n},$$

$$\mathsf{E}[H(\hat{f}'(\mu); \widehat{V})] \doteq t_{\mathrm{df},1-\alpha/2}\frac{2}{\sqrt{n}}\mathsf{E}[\widehat{V}^{1/2}], \quad \text{for large } m$$

and

$$\mathsf{Var}[H(\hat{f}'(\mu);\widehat{V})] = t^2_{\mathrm{df},1-\alpha/2}\frac{4}{n}\mathsf{Var}[|\bar{Y}_n+1|\widehat{V}^{1/2}].$$

Above, we have used the approximation $\mathsf{E}[|\bar{Y}_n+1|] \doteq \mathsf{E}[\bar{Y}_n+1] = 1$ because $\bar{Y}_n$ is normal with a variance that goes to 0 as $n\to\infty$.

Based on $\widehat{V}$, we have the following cases.

Case 1: $\widehat{V} = \widehat{V}_B$ (the batch means estimator for σ^2). We have

$$\begin{aligned}
\mathsf{E}\left[\widehat{V_B}^{1/2}\right] &= \mathsf{E}\left[\sqrt{\frac{m}{b-1}\sum_{i=1}^{b}(\bar{Y}_{i,m}-\bar{Y}_n)^2}\right]\\
&\doteq \mathsf{E}\left[\sqrt{\sigma^2\chi^2_{b-1}/(b-1)}\right]\\
&= \left(\frac{\sigma^2}{b-1}\right)^{1/2}\mathsf{E}\left[\sqrt{\chi^2_{b-1}}\right]\\
&\quad \text{(see Mood et al. [50], Appendix)}\\
&= \left(\frac{2\sigma^2}{b-1}\right)^{1/2}\frac{\Gamma(b/2)}{\Gamma((b-1)/2)},
\end{aligned}$$

where $\Gamma(\cdot)$ is defined by

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-1}dx \quad \mathrm{for} t>0.$$

Therefore,

$$\mathsf{E}[H(\hat{f}'(\mu);\widehat{V}_B)] \doteq t_{b-1,1-\alpha/2}\frac{2^{3/2}}{\sqrt{n}}\left(\frac{\sigma^2}{b-1}\right)^{1/2}\frac{\Gamma(b/2)}{\Gamma((b-1)/2)}. \tag{87}$$

Furthermore, since

$$\begin{aligned}
\mathsf{Var}\left[|\bar{Y}_n+1|\widehat{V}^{1/2}\right] &= \mathsf{E}[\widehat{V}_B(\bar{Y}_n+1)^2] - \left(\mathsf{E}\left[\widehat{V}_B^{1/2}|\bar{Y}_n+1|\right]\right)^2\\
&\doteq \mathsf{E}[\widehat{V}_B]\mathsf{E}[(\bar{Y}_n+1)^2] - \left(\mathsf{E}\left[\widehat{V}_B^{1/2}\right]\mathsf{E}[\bar{Y}_n+1]\right)^2 \quad \text{(for large } m)\\
&\doteq \sigma^2(v_n^2+1) - 2\left(\frac{\sigma^2}{b-1}\right)\left(\frac{\Gamma(b/2)}{\Gamma((b-1)/2)}\right)^2,
\end{aligned}$$

we have

$$\mathsf{Var}[H(\hat{f}'(\mu);\widehat{V}_B)] \doteq t^2_{b-1,1-\alpha/2}\frac{4}{n}\left[\sigma^2(v_n^2+1) - 2\left(\frac{\sigma^2}{b-1}\right)\left(\frac{\Gamma(b/2)}{\Gamma((b-1)/2)}\right)^2\right]. \tag{88}$$

Example 6 Table 9 contains analytical and experimental results related to the half-width of the 90% confidence interval resulting from the variance estimator $S_*^2(2(\bar{Y}_n+1);\widehat{V}_B)$. The entries in columns 3 and 5 are based on Equations (87) and (88), respectively, and the simulation results are based on 1000 independent experiments. The results show that the analytical approximations are close to the respective estimates.

Table 9: Performance Evaluation of the CI Half-width $H(2(\bar{Y}_n+1);\widehat{V}_B)$ for the Gaussian MA(1) Process with $\mu=0$ and $\theta=0.9$, and $f(\mu)=(\mu+1)^2$

	m	$\mathsf{E}[H]$	$\widehat{\mathsf{E}}[H]$	$\mathsf{Var}[H]$	$\widehat{\mathsf{Var}}[H]$
$b=32$	128	9.986E-02	9.885E-02	1.711E-04	1.744E-04
	256	7.061E-02	7.048E-02	8.330E-05	8.332E-05
	512	4.993E-02	4.982E-02	4.109E-05	4.334E-05
$b=128$	128	4.909E-02	4.909E-02	1.004E-05	9.806E-06
	256	3.471E-02	3.473E-02	4.887E-06	4.944E-06
	512	2.455E-02	2.453E-02	2.410E-06	2.345E-06

Case 2: $\widehat{V}=\widehat{V}_A(w_i)$ (the batched STS area estimators with $w_0(t)=\sqrt{12}, w_1(t)=\sqrt{45}t$, or $w_2(t)=\sqrt{840}(3t^2-3t+1/2)$). Since

$$\mathsf{E}\left[\widehat{V}_A^{1/2}\right]=\mathsf{E}\left[\sqrt{\sigma^2\chi_b^2/b}\right]=\left(\frac{2\sigma^2}{b}\right)^{1/2}\frac{\Gamma((b+1)/2)}{\Gamma(b/2)},$$

we have

$$\mathsf{E}[H]\doteq t_{b,1-\alpha/2}\frac{2^{3/2}}{\sqrt{n}}\left(\frac{\sigma^2}{b}\right)^{1/2}\frac{\Gamma((b+1)/2)}{\Gamma(b/2)}. \tag{89}$$

Also,

$$\begin{aligned}\mathsf{Var}\left[|\bar{Y}_n+1|\widehat{V}^{1/2}\right] &= \mathsf{E}[\widehat{V}_A(\bar{Y}_n+1)^2]-\left(\mathsf{E}\left[\widehat{V}^{1/2}|\bar{Y}_n+1|\right]\right)^2\\ &\doteq \sigma^2(v_n^2+1)-\frac{2\sigma^2}{b}\left(\frac{\Gamma((b+1)/2)}{\Gamma(b/2)}\right)^2.\end{aligned}$$

Therefore,

$$\mathsf{Var}[H]\doteq t_{b,1-\alpha/2}^2\frac{4}{n}\sigma^2\left[(v_n^2+1)-\frac{2}{b}\left(\frac{\Gamma((b+1)/2)}{\Gamma(b/2)}\right)^2\right]. \tag{90}$$

Example 7 Tables 10–12 contain analytical and experimental results relative to the half-width of the CI for $f(\mu) = (\mu+1)^2$ and the Gaussian MA(1) process with mean 0 and $\theta = 0.9$. The variance is estimated by $S_*^2(2(\bar{Y}_n+1); \widehat{V}_A(w_i))$ with weight functions w_0, w_1, and w_2. The entries of columns 3 and 5 are based on the approximations in Equations (89) and (90), respectively. Again, the estimates in columns 4 and 6 are quite close to the approximate values from Equations (89) and (90), respectively. Also, the mean and variance estimates of $H(2(\bar{Y}_n+1); \widehat{V}_A(w_i))$ are close to those of $H(2(\bar{Y}_n+1); \widehat{V}_B)$.

Table 10: Performance Evaluation of the CI Half-width $H(2(\bar{Y}_n+1); \widehat{V}_A(w_0))$ for the Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$, and $f(\mu) = (\mu+1)^2$

	m	$\mathsf{E}[H]$	$\widehat{\mathsf{E}}[H]$	$\mathsf{Var}[H]$	$\widehat{\mathsf{Var}}[H]$
$b = 32$	128	9.979E-02	9.853E-02	1.657E-04	1.692E-04
	256	7.056E-02	7.016E-02	8.063E-05	8.203E-05
	512	4.990E-02	5.005E-02	3.976E-05	3.854E-05
$b = 128$	128	4.909E-02	4.883E-02	9.965E-06	1.057E-05
	256	3.471E-02	3.467E-02	4.849E-06	4.771E-06
	512	2.455E-02	2.453E-02	2.391E-06	2.435E-06

Table 11: Performance Evaluation of the CI Half-width $H(2(\bar{Y}_n+1); \widehat{V}_A(w_1))$ for the Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$, and $f(\mu) = (\mu+1)^2$

	m	$\mathsf{E}[H]$	$\widehat{\mathsf{E}}[H]$	$\mathsf{Var}[H]$	$\widehat{\mathsf{Var}}[H]$
$b = 32$	128	9.979E-02	9.846E-02	1.657E-04	1.696E-04
	256	7.056E-02	7.016E-02	8.063E-05	8.279E-05
	512	4.990E-02	5.007E-02	3.976E-05	3.852E-05
$b = 128$	128	4.909E-02	4.880E-02	9.965E-06	1.051E-05
	256	3.471E-02	3.466E-02	4.849E-06	4.855E-06
	512	2.455E-02	2.454E-02	2.391E-06	2.338E-06

Table 12: Performance Evaluation of the CI Half-width $H(2(\bar{Y}_n + 1); \widehat{V}_A(w_2))$ for the Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$, and $f(\mu) = (\mu + 1)^2$

	m	$\mathsf{E}[H]$	$\widehat{\mathsf{E}}[H]$	$\mathsf{Var}[H]$	$\widehat{\mathsf{Var}}[H]$
$b = 32$	128	9.979E-02	9.950E-02	1.657E-04	1.734E-04
	256	7.056E-02	7.027E-02	8.063E-05	8.669E-05
	512	4.990E-02	4.989E-02	3.976E-05	3.909E-05
$b = 128$	128	4.909E-02	4.911E-02	9.965E-06	1.017E-05
	256	3.471E-02	3.476E-02	4.849E-06	4.580E-06
	512	2.455E-02	2.452E-02	2.391E-06	2.434E-06

Case 3: $\widehat{V} = \widehat{V}_C$ (the combination of NBM and batched STS area estimators). We have

$$\begin{aligned} \mathsf{E}\left[\widehat{V}_C^{1/2}\right] &\doteq \mathsf{E}\left[\sqrt{\sigma^2 \chi^2_{(2b-1)}/(2b-1)}\right] \\ &= \left(\frac{2\sigma^2}{2b-1}\right)^{1/2} \frac{\Gamma(b)}{\Gamma((2b-1)/2)} \end{aligned}$$

and

$$\mathsf{E}[H(\hat{f}'(\mu); \widehat{V}_C)] \doteq t_{2b-1,1-\alpha/2} \frac{2}{\sqrt{n}} \left(\frac{2\sigma^2}{2b-1}\right)^{1/2} \frac{\Gamma(b)}{\Gamma((2b-1)/2)}. \tag{91}$$

Then

$$\begin{aligned} \mathsf{Var}\left[|\bar{Y}_n + 1|\widehat{V}_C^{\frac{1}{2}}\right] &= \mathsf{E}[\widehat{V}_C(\bar{Y}_n + 1)^2] - \left(\mathsf{E}\left[\widehat{V}_C^{\frac{1}{2}}|\bar{Y}_n + 1|\right]\right)^2 \\ &\doteq \sigma^2(v_n^2 + 1) - \left[\frac{2\sigma^2}{2b-1}\left(\frac{\Gamma(b)}{\Gamma((2b-1)/2)}\right)^2\right] \end{aligned}$$

and

$$\mathsf{Var}[H(\hat{f}'(\mu); \widehat{V}_C)] \doteq t^2_{2b-1,1-\alpha/2} \frac{4}{n}\sigma^2 \left[(v_n^2 + 1) - \frac{2}{2b-1}\left(\frac{\Gamma(b)}{\Gamma((2b-1)/2)}\right)^2\right]. \tag{92}$$

Example 8 Tables 13–15 present analytical and experimental results relative to the half-width of the 90% CI for $f(\mu) = (\mu + 1)^2$ and an MA(1) process with mean 0 and $\theta = 0.9$. The variance is estimated by $S_C^2(2(\bar{Y}_n+1); \widehat{V}_C(w_i))$ based on $\widehat{V} = \widehat{V}_C$ and the weight functions w_0, w_1, and w_2. The entries of columns 3 and 5 are based on the approximations in Equations (91) and (92), respectively. The results show that the approximate values from Equations (91) and (92) are quite close to the estimates in columns 4 and 6, respectively. Also, the mean and variance estimates of $H(2(\bar{Y}_n+1); \widehat{V}_C(w_i))$ are less than those of $H(2(\bar{Y}_n+1); \widehat{V}_A(w_i))$ and $H(2(\bar{Y}_n+1); \widehat{V}_B)$.

Table 13: Performance Evaluation of the CI Half-width $H(2(\bar{Y}_n+1); \widehat{V}_C(w_0))$ for the Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$, and $f(\mu) = (\mu+1)^2$

	m	$\mathsf{E}[H]$	$\widehat{\mathsf{E}}[H]$	$\mathsf{Var}[H]$	$\widehat{\mathsf{Var}}[H]$
$b = 32$	128	9.873E-02	9.763E-02	8.632E-05	8.566E-05
	256	6.981E-02	6.955E-02	4.100E-05	4.108E-05
	512	4.936E-02	4.939E-02	1.996E-05	1.984E-05
$b = 128$	128	4.896E-02	4.883E-02	5.234E-06	5.521E-06
	256	3.462E-02	3.461E-02	2.485E-06	2.495E-06
	512	2.448E-02	2.446E-02	1.209E-06	1.283E-06

Table 14: Performance Evaluation of the CI Half-width $H(2(\bar{Y}_n+1); \widehat{V}_C(w_1))$ for the Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$, and $f(\mu) = (\mu+1)^2$

	m	$\mathsf{E}[H]$	$\widehat{\mathsf{E}}[H]$	$\mathsf{Var}[H]$	$\widehat{\mathsf{Var}}[H]$
$b = 32$	128	9.873E-02	9.760E-02	8.632E-05	8.640E-05
	256	6.981E-02	6.955E-02	4.100E-05	4.153E-05
	512	4.936E-02	4.940E-02	1.996E-05	1.985E-05
$b = 128$	128	4.896E-02	4.881E-02	5.234E-06	5.533E-06
	256	3.462E-02	3.460E-02	2.485E-06	2.522E-06
	512	2.448E-02	2.447E-02	1.209E-06	1.239E-06

Table 15: Performance Evaluation of the CI Half-width $H(2(\bar{Y}_n + 1); \widehat{V}_C(w_2))$ for the Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$, and $f(\mu) = (\mu + 1)^2$

	m	$\mathsf{E}[H]$	$\widehat{\mathsf{E}}[H]$	$\mathsf{Var}[H]$	$\widehat{\mathsf{Var}}[H]$
$b = 32$	128	9.873E-02	9.813E-02	8.632E-05	8.571E-05
	256	6.981E-02	6.961E-02	4.100E-05	4.202E-05
	512	4.936E-02	4.931E-02	1.996E-05	2.030E-05
$b = 128$	128	4.896E-02	4.897E-02	5.234E-06	5.402E-06
	256	3.462E-02	3.465E-02	2.485E-06	2.537E-06
	512	2.448E-02	2.446E-02	1.209E-06	1.257E-06

Let us now consider the expectation and variance of the half-width of CIs using S_B^2 (the NBM variance estimator) and S_J^2 (the jackknife variance estimator). The half-width of the CI resulting from Equation (35), is $H(S_l^2) = t_{\mathrm{df},1-\alpha/2}\sqrt{S_l^2/b}$, $l \in \{B, J\}$. Hence

$$\mathsf{E}[H(S_l^2)] = t_{b,1-\alpha/2}\mathsf{E}[S_l]/\sqrt{b}.$$

Since the derivation or approximation of $\mathsf{E}[S_l]$ is difficult, even for our Gaussian MA(1) process, we compute estimates of $\mathsf{E}[H]$ based on 1000 independent experiments with $\alpha = 0.10$. In Table 16:

- $\widehat{\mathsf{E}}[H(S_l^2)]$ is the sample mean of the half-widths of the approximate 90% CIs based on the variance estimator $S_l^2, l \in \{B, J\}$.
- $\widehat{\mathsf{Var}}[H(S_l^2)]$ is the sample variance of the half-widths of the approximate 90% CIs based on the variance estimator $S_l^2, l \in \{B, J\}$.

.

Table 16: Performance Evaluation of the CI Half-widths $H(S_B^2)$ and $H(S_J^2)$ for the Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$, and $f(\mu) = (\mu + 1)^2$

	m	$\widehat{\mathsf{E}}[H(S_B^2)]$	$\widehat{\mathsf{E}}[H(S_J^2)]$	$\widehat{\mathsf{Var}}[H(S_B^2)]$	$\widehat{\mathsf{Var}}[H(S_J^2)]$
$b = 32$	128	9.948E-02	9.885E-02	1.950E-04	1.743E-04
	256	7.064E-02	7.048E-02	8.630E-05	8.336E-05
	512	4.988E-02	4.982E-02	4.419E-05	4.334E-05
$b = 128$	128	4.943E-02	4.909E-02	1.088E-05	9.806E-06
	256	3.485E-02	3.473E-02	5.306E-06	4.944E-06
	512	2.457E-02	2.453E-02	2.434E-06	2.345E-06

The results indicate that the mean and variance of $H(S_J^2)$ seem to be higher than those of $H(2\bar{Y}_n; \widehat{V}_C(w_i))$. On the other hand, the mean and variance of $H(S_J^2)$ appear to be smaller than those of $H(S_B^2)$.

We now turn to the coverage of the CIs. Table 17 contains estimated coverage of 90% CIs for $f(\mu) = (\mu + 1)^2$. Since all estimates are based on 1000 independent runs, the standard errors are bounded by 1.5×10^{-3}. From now on, we will use the following abbreviations:

- BM: denotes $S_*^2(2(\bar{Y}_n + 1); \widehat{V}_B)$.
- STS(w_i): denotes $S_*^2(2(\bar{Y}_n + 1); \widehat{V}_A(w_0))$, $\quad i = 0, 1, 2$.
- BM+STS(w_i): denotes $S_*^2(2(\bar{Y}_n + 1); \widehat{V}_C(w_0))$, $\quad i = 0, 1, 2$.

Table 17: Coverage Estimates for 90% CIs for $f(\mu) = (\mu + 1)^2$ based on the Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$

		Variance Estimator								
Coverage	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$	128	0.913	0.913	0.912	0.919	0.907	0.908	0.920	0.912	0.913
Cl. for f	256	0.900	0.901	0.903	0.910	0.903	0.903	0.897	0.894	0.900
	512	0.911	0.908	0.903	0.907	0.911	0.907	0.909	0.908	0.911
$b = 128$	128	0.905	0.903	0.902	0.908	0.906	0.904	0.907	0.910	0.905
Cl. for f	256	0.901	0.898	0.900	0.901	0.899	0.900	0.902	0.898	0.901
	512	0.908	0.903	0.904	0.899	0.902	0.902	0.902	0.906	0.908
$b = 32$	128	0.883	0.882	0.883	0.891	0.888	0.887	0.887	0.885	0.884
BM for f	256	0.900	0.894	0.894	0.891	0.892	0.890	0.897	0.895	0.900
	512	0.900	0.905	0.903	0.901	0.904	0.904	0.896	0.900	0.900
$b = 128$	128	0.781	0.772	0.776	0.774	0.779	0.781	0.778	0.782	0.781
BM for f	256	0.818	0.812	0.814	0.818	0.820	0.822	0.816	0.818	0.818
	512	0.856	0.863	0.864	0.868	0.863	0.861	0.864	0.858	0.856
$b = 32$	128	0.912	0.913	0.910	0.920	0.907	0.908	0.917	0.912	0.911
Jk. for f	256	0.899	0.899	0.904	0.911	0.903	0.903	0.897	0.893	0.899
	512	0.910	0.908	0.902	0.907	0.909	0.905	0.906	0.907	0.910
$b = 128$	128	0.905	0.901	0.900	0.908	0.906	0.906	0.908	0.910	0.905
Jk. for f	256	0.901	0.897	0.898	0.901	0.899	0.900	0.902	0.899	0.901
	512	0.907	0.902	0.903	0.899	0.902	0.903	0.902	0.906	0.907

Below is a summary of observations based on Table 17:

- The classical and jackknife point estimation approaches induce good coverage for all variance estimators of $\hat{f}(\mu)$.

- The NBM point estimator induces lower coverage than the nominal 90% for all variance estimators of $\hat{f}(\mu)$ when $b = 128$. This can be attributed to the high bias of $\bar{f}_B$: Muñoz and Glynn [51] point out that the coefficient of n^{-1} in the asymptotic expansion for $\mathsf{Bias}[\bar{f}_B]$ is b times the corresponding coefficient in the asymptotic expansion for $\mathsf{Bias}[\hat{f}_C]$.

- In terms of coverage, CIs based on the jackknife point estimator and the delta variance estimator (in columns BM+STS(w_i)) for $\hat{f}(\mu)$ seem to perform best. Notice that the jackknife point estimator for $f(\mu)$ gives the lowest bias among others with no significant increase in variance (see Appendix 1).

4.2.4 Summary of Experimental Results

In this subsection, we summarize the results from Tables 1–17.

- In Tables 1–8, we considered the mean and variance of the variance estimators S_B^2, S_J^2, and S_*^2. The delta variance estimator (in columns BM+STS(w_i)) for $\hat{f}(\mu)$ is less variable than the other estimators.

- In Tables 9–16, we considered the mean and variance of the half-width of CIs for $f(\mu)$. The CIs based on the delta variance estimator (in columns BM+STS(w_i)) have smaller and less variable half-width than the alternative CIs.

- In Table 17, the CIs based on the jackknife point estimator and the delta variance estimator (in columns BM+STS(w_i)) for $\hat{f}(\mu)$ seem to perform best since the jackknife point estimator for $f(\mu)$ has the lowest bias with no significant increase in variance. The delta variance estimator (in columns BM+STS(w_i))

exhibits the solid statistical properties of the combined variance parameter estimator in Equations (16) and (31).

4.3 Performance Evaluation when $f(\mu) = 1/(\mu + 1)$

In this subsection, we evaluate our methods using the function $f(\mu) = 1/(\mu + 1)$. Since analytical derivations are very hard to obtain, we rely solely on Monte Carlo experiments. As performance measures for CIs, we will consider: $\mathsf{E}[H]$, the expected interval half-width; $\mathsf{Var}[H]$, the variance of the interval half-width; and C, the confidence interval achieved coverage. As performance measures for the estimators of $\mathsf{Var}[\hat{f}(\mu)]$ we consider their mean and variance. The Appendix contains the simulation results for the bias and variance of $\hat{f}(\mu)$.

As before, all estimates are based on 1000 independent experiments. Tables 18–24 contain the experimental results. We first look at the performance of the CIs; then we look at the variance estimates.

4.3.1 Discussion

• **Coverage**: We categorize our presentation based on the point estimator of $f'(\mu)$. Recall that the CIs based on the batch means and jackknife variance estimators for $\hat{f}(\mu)$ are not affected by the choice of point estimators for $|f'(\mu)|$—the entries of the last two columns of Tables 18–20 are identical.

▷ **Classical point estimator for $|f'(\mu)| = 1/(\mu + 1)^2$**: First, the batch means point estimator for $f(\mu)$ yields CIs with poor coverage. This is due to the relatively high bias and variance of the NBM point estimates for $f(\mu) = 1/(\mu+1)$ (see Appendix 1). Second, the classical point estimator for $f(\mu)$ yields CIs with good estimated coverage. This is probably due to the relatively lower bias and variance of the classical point estimator. Third, the jackknife point estimator for $f(\mu)$ yields CIs with coverage that is as good as that of the classical point estimator. Recall that the jackknife estimator for $f(\mu)$ has lower bias and a little lower variance than the classical estimator except when $b = 128$ and $m = 512$ (see Appendix 1). The CIs based on the jackknife point estimator for $f(\mu)$, the classical point estimator for $|f'(\mu)|$, and the delta variance estimator (in columns BM through BM+STS(w_2)) seem to perform as well, in terms of coverage, as the jackknife-based CIs from Muñoz and Glynn [51].

▷ **NBM point estimator for $|f'(\mu)| = 1/(\mu + 1)^2$**: All entries of Table 19 are a bit larger than the entries of Table 18 except for the entries in the last two columns. This is due to the relatively higher variance of the batch means point estimator for $|f'(\mu)|$. Except for the NBM point estimator for $f(\mu)$, all combinations have good coverage performance. As m increases for fixed b, the differences in the entries of Table 18 and 19 decrease. This is because the difference in the bias and variance between the classical point estimates and the NBM point estimates for $|f'(\mu)|$ decreases.

▷ **Jackknife point estimator for $|f'(\mu)| = 1/(\mu + 1)^2$**: Since the last two columns in Tables 18–20 are identical, we consider only columns BM through BM+STS(w_2). As with Tables 18 and 19, all combinations except for the batch means point estimator for $f(\mu)$ induce good CIs with coverage.

Based on the coverage estimates in Tables 18-20, we recommend the jackknife point estimators for $f(\mu)$ and $f'(\mu)$. With regard to the variance estimator for $\hat{f}(\mu)$, CIs using the delta variance estimator (in columns BM through BM+STS(w_2)) seem to perform as well as those based on the jackknife variance estimator.

• **Sample mean of half-width**: Among CIs with good coverage, the one with smaller sample mean half-width is preferable. Among the classical, batch means and jackknife point estimators for $|f'(\mu)| = 1/(\mu + 1)^2$, we recommend the jackknife point estimator. This is because for all entries corresponding to the delta variance estimators (in columns BM through BM+STS(w_2)) the jackknife point estimator for $|f'(\mu)|$ outperforms the other point estimators. Recall that the last two columns corresponding to the batch means and the jackknife variance estimators are identical. Therefore, if we consider both half-width and coverage as performance measures, we recommend the jackknife point estimators for $f(\mu)$ and $|f'(\mu)| = 1/(\mu+1)^2$. With regard to half-width, CIs based on the delta variance estimator (in columns BM through BM+STS(w_2)) seem to perform as well as those based on the jackknife variance estimator S_J^2.

• **Sample variance of half-width**: Among the classical, batch means and jackknife point estimators for $|f'(\mu)| = 1/(\mu+1)^2$, we recommend the jackknife point estimator. This is due to the lowest variance of the jackknife point estimator. Among the variance estimators, the delta variance estimators using the variance parameter estimator BM+STS(w_i) dominate all other estimators. This is probably due to their lower variance.

• **Best choice**: For this instance, we recommend the jackknife point estimators for $f(\mu)$ and $|f'(\mu)|$ and the delta variance estimator S_*^2 (preferably with weight function w_2). The resulting CIs exhibit good coverage, and have half-widths that are smaller and less variable than the half-widths of the CIs proposed by Muñoz and Glynn [51].

Table 18: Coverage Estimates of 90% CIs for $f(\mu) = 1/(\mu + 1)$ based on a Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$

		Variance Estimator								
Coverage	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$	128	0.913	0.920	0.918	0.919	0.917	0.918	0.922	0.936	0.913
Cl. for f	256	0.892	0.903	0.904	0.907	0.904	0.905	0.909	0.918	0.893
Cl. for f'	512	0.912	0.910	0.910	0.907	0.912	0.916	0.912	0.915	0.912
$b = 128$	128	0.905	0.903	0.898	0.907	0.903	0.903	0.907	0.932	0.905
Cl. for f	256	0.899	0.901	0.899	0.899	0.900	0.900	0.901	0.914	0.899
Cl. for f'	512	0.906	0.901	0.900	0.898	0.900	0.900	0.903	0.911	0.906
$b = 32$	128	0.728	0.713	0.719	0.719	0.706	0.714	0.714	0.801	0.726
BM for f	256	0.821	0.800	0.804	0.808	0.809	0.810	0.805	0.847	0.820
Cl. for f'	512	0.868	0.861	0.863	0.856	0.859	0.858	0.860	0.871	0.868
$b = 128$	128	0.345	0.347	0.343	0.345	0.341	0.343	0.343	0.424	0.345
BM for f	256	0.608	0.612	0.616	0.616	0.611	0.612	0.612	0.644	0.608
Cl. for f'	512	0.783	0.778	0.777	0.784	0.784	0.781	0.783	0.797	0.783
$b = 32$	128	0.916	0.919	0.921	0.919	0.918	0.920	0.921	0.937	0.916
Jk. for f	256	0.898	0.905	0.904	0.909	0.907	0.906	0.908	0.921	0.898
Cl. for f'	512	0.912	0.910	0.909	0.911	0.912	0.916	0.912	0.913	0.912
$b = 128$	128	0.905	0.905	0.903	0.907	0.903	0.903	0.906	0.931	0.905
Jk. for f	256	0.898	0.899	0.900	0.897	0.900	0.901	0.900	0.913	0.898
Cl. for f'	512	0.904	0.901	0.902	0.898	0.900	0.900	0.902	0.911	0.904

Table 19: Coverage Estimates of 90% CIs for $f(\mu) = 1/(\mu + 1)$ based on a Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$

		Variance Estimator								
Coverage	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$	128	0.937	0.937	0.935	0.943	0.940	0.937	0.941	0.936	0.913
Cl. for f	256	0.920	0.910	0.916	0.921	0.917	0.919	0.913	0.918	0.893
BM for f'	512	0.918	0.916	0.915	0.917	0.921	0.921	0.919	0.915	0.912
$b = 128$	128	0.928	0.935	0.935	0.933	0.932	0.933	0.932	0.932	0.905
Cl. for f	256	0.911	0.910	0.911	0.910	0.912	0.912	0.912	0.914	0.899
BM for f'	512	0.913	0.905	0.909	0.909	0.906	0.908	0.909	0.911	0.906
$b = 32$	128	0.786	0.767	0.768	0.774	0.770	0.772	0.779	0.801	0.726
BM for f	256	0.843	0.822	0.823	0.834	0.826	0.825	0.836	0.847	0.820
BM for f'	512	0.872	0.868	0.870	0.863	0.865	0.863	0.865	0.871	0.868
$b = 128$	128	0.401	0.403	0.404	0.410	0.397	0.400	0.402	0.424	0.345
BM for f	256	0.640	0.634	0.635	0.638	0.633	0.630	0.635	0.644	0.608
BM for f'	512	0.795	0.787	0.785	0.791	0.791	0.793	0.794	0.797	0.783
$b = 32$	128	0.935	0.935	0.936	0.944	0.938	0.940	0.940	0.937	0.916
Jk. for f	256	0.920	0.912	0.918	0.917	0.917	0.920	0.914	0.921	0.898
BM for f'	512	0.915	0.916	0.915	0.917	0.919	0.920	0.918	0.913	0.912
$b = 128$	128	0.928	0.934	0.935	0.932	0.930	0.934	0.931	0.931	0.905
Jk. for f	256	0.911	0.910	0.911	0.908	0.911	0.912	0.911	0.913	0.898
BM for f'	512	0.913	0.906	0.908	0.910	0.907	0.907	0.909	0.911	0.904

Table 20: Coverage Estimates of 90% CIs for $f(\mu) = 1/(\mu + 1)$ based on a Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$

		Variance Estimator								
Coverage	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$	128	0.913	0.918	0.917	0.918	0.917	0.918	0.922	0.936	0.913
Cl. for f	256	0.892	0.903	0.903	0.907	0.904	0.905	0.909	0.918	0.893
Jk. for f'	512	0.912	0.910	0.909	0.907	0.911	0.916	0.912	0.915	0.912
$b = 128$	128	0.905	0.903	0.898	0.907	0.903	0.903	0.907	0.932	0.905
Cl. for f	256	0.899	0.901	0.899	0.899	0.899	0.900	0.901	0.914	0.899
Jk. for f'	512	0.906	0.901	0.900	0.898	0.900	0.900	0.903	0.911	0.906
$b = 32$	128	0.726	0.711	0.717	0.716	0.705	0.709	0.713	0.801	0.726
BM for f	256	0.821	0.798	0.804	0.807	0.809	0.809	0.805	0.847	0.820
Jk. for f'	512	0.867	0.861	0.863	0.856	0.859	0.858	0.859	0.871	0.868
$b = 128$	128	0.345	0.347	0.343	0.345	0.341	0.342	0.343	0.424	0.345
BM for f	256	0.608	0.612	0.615	0.615	0.611	0.612	0.612	0.644	0.608
Jk. for f'	512	0.783	0.778	0.777	0.784	0.784	0.781	0.783	0.797	0.783
$b = 32$	128	0.915	0.918	0.920	0.918	0.918	0.919	0.919	0.937	0.916
Jk. for f	256	0.898	0.905	0.903	0.908	0.906	0.906	0.908	0.921	0.898
Jk. for f'	512	0.912	0.910	0.907	0.911	0.912	0.916	0.912	0.913	0.912
$b = 128$	128	0.905	0.903	0.901	0.906	0.903	0.902	0.906	0.931	0.905
Jk. for f	256	0.897	0.898	0.900	0.897	0.900	0.901	0.900	0.913	0.898
Jk. for f'	512	0.904	0.901	0.902	0.898	0.900	0.900	0.902	0.911	0.904

Table 21: Sample Mean of Half-widths for 90% CIs for $f(\mu) = 1/(\mu + 1)$ based on a Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$

		Variance Estimator								
$\widehat{\mathsf{E}}[H]$	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$ Cl. for f'	128	4.967E-02	4.950E-02	4.948E-02	5.000E-02	4.905E-02	4.904E-02	4.931E-02	5.567E-02	4.968E-02
	256	3.540E-02	3.524E-02	3.523E-02	3.529E-02	3.493E-02	3.493E-02	3.497E-02	3.745E-02	3.541E-02
	512	2.494E-02	2.505E-02	2.506E-02	2.497E-02	2.472E-02	2.473E-02	2.468E-02	2.561E-02	2.494E-02
$b = 128$ Cl. for f'	128	2.457E-02	2.443E-02	2.441E-02	2.457E-02	2.443E-02	2.443E-02	2.450E-02	2.792E-02	2.457E-02
	256	1.736E-02	1.732E-02	1.732E-02	1.737E-02	1.729E-02	1.729E-02	1.731E-02	1.840E-02	1.736E-02
	512	1.226E-02	1.226E-02	1.226E-02	1.226E-02	1.222E-02	1.223E-02	1.222E-02	1.261E-02	1.226E-02
$b = 32$ BM for f'	128	5.456E-02	5.419E-02	5.417E-02	5.474E-02	5.379E-02	5.378E-02	5.407E-02	5.567E-02	4.968E-02
	256	3.703E-02	3.680E-02	3.680E-02	3.686E-02	3.651E-02	3.651E-02	3.654E-02	3.745E-02	3.541E-02
	512	2.548E-02	2.559E-02	2.560E-02	2.550E-02	2.526E-02	2.526E-02	2.521E-02	2.561E-02	2.494E-02
$b = 128$ BM for f'	128	2.703E-02	2.685E-02	2.683E-02	2.701E-02	2.687E-02	2.686E-02	2.694E-02	2.792E-02	2.457E-02
	256	1.815E-02	1.811E-02	1.810E-02	1.815E-02	1.808E-02	1.808E-02	1.810E-02	1.840E-02	1.736E-02
	512	1.253E-02	1.252E-02	1.253E-02	1.252E-02	1.249E-02	1.249E-02	1.249E-02	1.261E-02	1.226E-02
$b = 32$ Jk. for f'	128	4.954E-02	4.937E-02	4.935E-02	4.987E-02	4.892E-02	4.891E-02	4.918E-02	5.567E-02	4.968E-02
	256	3.536E-02	3.519E-02	3.519E-02	3.525E-02	3.489E-02	3.488E-02	3.492E-02	3.745E-02	3.541E-02
	512	2.492E-02	2.504E-02	2.505E-02	2.496E-02	2.471E-02	2.471E-02	2.467E-02	2.561E-02	2.494E-02
$b = 128$ Jk. for f'	128	2.455E-02	2.441E-02	2.440E-02	2.456E-02	2.442E-02	2.441E-02	2.449E-02	2.792E-02	2.457E-02
	256	1.735E-02	1.732E-02	1.731E-02	1.736E-02	1.729E-02	1.729E-02	1.731E-02	1.840E-02	1.736E-02
	512	1.226E-02	1.226E-02	1.226E-02	1.225E-02	1.222E-02	1.223E-02	1.222E-02	1.261E-02	1.226E-02

Table 22: Sample Variance of Half-widths for 90% CIs for $f(\mu) = 1/(\mu + 1)$ based on a Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$

		Variance Estimator								
$\widehat{\mathsf{Var}}[H]$	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$	128	5.152E-05	4.933E-05	5.052E-05	5.243E-05	2.856E-05	2.925E-05	2.951E-05	1.260E-04	5.163E-05
Cl. for f'	256	2.274E-05	2.173E-05	2.183E-05	2.294E-05	1.175E-05	1.182E-05	1.203E-05	3.602E-05	2.273E-05
	512	1.153E-05	1.040E-05	1.032E-05	1.054E-05	5.680E-06	5.643E-06	5.790E-06	1.420E-05	1.153E-05
$b = 128$	128	2.976E-06	2.763E-06	2.757E-06	2.824E-06	1.694E-06	1.702E-06	1.745E-06	9.726E-06	2.976E-06
Cl. for f'	256	1.381E-06	1.218E-06	1.248E-06	1.194E-06	7.090E-07	7.204E-07	7.324E-07	2.168E-06	1.381E-06
	512	6.012E-07	6.076E-07	5.818E-07	6.157E-07	3.276E-07	3.161E-07	3.253E-07	7.579E-07	6.013E-07
$b = 32$	128	8.863E-05	6.419E-05	6.568E-05	6.810E-05	4.948E-05	5.034E-05	5.068E-05	1.260E-04	5.163E-05
BM for f'	256	2.945E-05	2.409E-05	2.426E-05	2.538E-05	1.522E-05	1.531E-05	1.550E-05	3.602E-05	2.273E-05
	512	1.308E-05	1.091E-05	1.082E-05	1.107E-05	6.457E-06	6.414E-06	6.575E-06	1.420E-05	1.153E-05
$b = 128$	128	5.292E-06	3.739E-06	3.719E-06	3.792E-06	3.083E-06	3.087E-06	3.135E-06	9.726E-06	2.976E-06
BM for f'	256	1.784E-06	1.358E-06	1.392E-06	1.345E-06	9.243E-07	9.372E-07	9.563E-07	2.168E-06	1.381E-06
	512	6.827E-07	6.388E-07	6.112E-07	6.467E-07	3.715E-07	3.591E-07	3.688E-07	7.579E-07	6.013E-07
$b = 32$	128	5.075E-05	4.904E-05	5.021E-05	5.212E-05	2.815E-05	2.883E-05	2.910E-05	1.260E-04	5.163E-05
Jk. for f'	256	2.256E-05	2.166E-05	2.176E-05	2.287E-05	1.166E-05	1.173E-05	1.194E-05	3.602E-05	2.273E-05
	512	1.148E-05	1.038E-05	1.031E-05	1.053E-05	5.657E-06	5.621E-06	5.767E-06	1.420E-05	1.153E-05
$b = 128$	128	2.965E-06	2.758E-06	2.752E-06	2.819E-06	1.688E-06	1.695E-06	1.739E-06	9.726E-06	2.976E-06
Jk. for f'	256	1.378E-06	1.217E-06	1.247E-06	1.193E-06	7.076E-07	7.190E-07	7.309E-07	2.168E-06	1.381E-06
	512	6.006E-07	6.073E-07	5.816E-07	6.155E-07	3.273E-07	3.158E-07	3.250E-07	7.579E-07	6.013E-07

Table 23: Sample Mean of Variance Estimators for $\hat{f}(\mu)$ when $f(\mu) = 1/(\mu + 1)$ based on a Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$

		Variance Estimator								
$\widehat{\mathsf{E}}[S^2]$	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$ Cl. for f'	128	8.761E-04	8.712E-04	8.707E-04	8.896E-04	8.736E-04	8.734E-04	8.830E-04	1.122E-03	8.764E-04
	256	4.439E-04	4.403E-04	4.402E-04	4.421E-04	4.421E-04	4.421E-04	4.430E-04	5.003E-04	4.439E-04
	512	2.203E-04	2.224E-04	2.225E-04	2.210E-04	2.214E-04	2.214E-04	2.207E-04	2.330E-04	2.203E-04
$b = 128$ Cl. for f'	128	2.210E-04	2.184E-04	2.181E-04	2.210E-04	2.197E-04	2.195E-04	2.210E-04	2.875E-04	2.210E-04
	256	1.102E-04	1.098E-04	1.097E-04	1.103E-04	1.100E-04	1.100E-04	1.103E-04	1.241E-04	1.102E-04
	512	5.494E-05	5.495E-05	5.500E-05	5.494E-05	5.495E-05	5.497E-05	5.494E-05	5.819E-05	5.494E-05
$b = 32$ BM for f'	128	1.066E-03	1.046E-03	1.045E-03	1.068E-03	1.056E-03	1.056E-03	1.067E-03	1.122E-03	8.764E-04
	256	4.872E-04	4.804E-04	4.803E-04	4.823E-04	4.837E-04	4.837E-04	4.847E-04	5.003E-04	4.439E-04
	512	2.305E-04	2.320E-04	2.321E-04	2.305E-04	2.312E-04	2.313E-04	2.305E-04	2.330E-04	2.203E-04
$b = 128$ BM for f'	128	2.680E-04	2.640E-04	2.637E-04	2.671E-04	2.660E-04	2.658E-04	2.675E-04	2.875E-04	2.210E-04
	256	1.206E-04	1.199E-04	1.199E-04	1.205E-04	1.203E-04	1.202E-04	1.206E-04	1.241E-04	1.102E-04
	512	5.739E-05	5.737E-05	5.742E-05	5.735E-05	5.738E-05	5.741E-05	5.737E-05	5.819E-05	5.494E-05
$b = 32$ Jk. for f'	128	8.712E-04	8.666E-04	8.661E-04	8.850E-04	8.689E-04	8.686E-04	8.782E-04	1.122E-03	8.764E-04
	256	4.427E-04	4.391E-04	4.391E-04	4.409E-04	4.409E-04	4.408E-04	4.418E-04	5.003E-04	4.439E-04
	512	2.200E-04	2.221E-04	2.223E-04	2.207E-04	2.211E-04	2.211E-04	2.204E-04	2.330E-04	2.203E-04
$b = 128$ Jk. for f'	128	2.207E-04	2.181E-04	2.178E-04	2.207E-04	2.194E-04	2.192E-04	2.207E-04	2.875E-04	2.210E-04
	256	1.102E-04	1.097E-04	1.096E-04	1.102E-04	1.099E-04	1.099E-04	1.102E-04	1.241E-04	1.102E-04
	512	5.492E-05	5.494E-05	5.499E-05	5.492E-05	5.493E-05	5.495E-05	5.492E-05	5.819E-05	5.494E-05

Table 24: Sample Variance of Variance Estimators for $\hat{f}(\mu)$ when $f(\mu) = 1/(\mu+1)$ based on a Gaussian MA(1) Process with $\mu = 0$ and $\theta = 0.9$

		Variance Estimator								
$\widehat{\mathsf{Var}}[S^2]$	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b=32$	128	6.381E-08	6.066E-08	6.249E-08	6.744E-08	3.613E-08	3.698E-08	3.800E-08	2.402E-07	6.398E-08
Cl. for f'	256	1.417E-08	1.340E-08	1.350E-08	1.440E-08	7.502E-09	7.597E-09	7.726E-09	2.625E-08	1.417E-08
	512	3.572E-09	3.256E-09	3.228E-09	3.306E-09	1.828E-09	1.807E-09	1.859E-09	4.731E-09	3.572E-09
$b=128$	128	9.714E-10	8.852E-10	8.840E-10	9.114E-10	5.491E-10	5.522E-10	5.706E-10	4.794E-09	9.715E-10
Cl. for f'	256	2.218E-10	1.955E-10	2.003E-10	1.916E-10	1.145E-10	1.160E-10	1.182E-10	3.967E-10	2.219E-10
	512	4.825E-11	4.862E-11	4.677E-11	4.931E-11	2.643E-11	2.564E-11	2.629E-11	6.466E-11	4.826E-11
$b=32$	128	1.401E-07	9.550E-08	9.789E-08	1.061E-07	7.839E-08	7.945E-08	8.148E-08	2.402E-07	6.398E-08
BM for f'	256	2.039E-08	1.623E-08	1.643E-08	1.742E-08	1.071E-08	1.085E-08	1.096E-08	2.625E-08	1.417E-08
	512	2.143E-09	1.451E-09	1.444E-09	1.485E-09	1.231E-09	1.233E-09	1.261E-09	4.731E-09	3.572E-09
$b=128$	128	2.143E-09	1.451E-09	1.444E-09	1.485E-09	1.231E-09	1.233E-09	1.261E-09	4.794E-09	9.715E-10
BM for f'	256	3.146E-10	2.382E-10	2.438E-10	2.358E-10	1.635E-10	1.652E-10	1.690E-10	3.967E-10	2.219E-10
	512	5.730E-11	5.336E-11	5.132E-11	5.407E-11	3.133E-11	3.044E-11	3.116E-11	6.466E-11	4.826E-11
$b=32$	128	6.244E-08	5.998E-08	6.179E-08	6.669E-08	3.540E-08	3.624E-08	3.726E-08	2.402E-07	6.398E-08
Jk. for f'	256	1.402E-08	1.332E-08	1.343E-08	1.432E-08	7.423E-09	7.516E-09	7.646E-09	2.625E-08	1.417E-08
	512	3.553E-09	3.247E-09	3.219E-09	3.297E-09	1.818E-09	1.797E-09	1.849E-09	4.731E-09	3.572E-09
$b=128$	128	9.664E-10	8.826E-10	8.814E-10	9.086E-10	5.463E-10	5.494E-10	5.677E-10	4.794E-09	9.715E-10
Jk. for f'	256	2.212E-10	1.952E-10	2.000E-10	1.914E-10	1.142E-10	1.157E-10	1.178E-10	3.967E-10	2.219E-10
	512	4.819E-11	4.858E-11	4.674E-11	4.927E-11	2.640E-11	2.560E-11	2.626E-11	6.466E-11	4.826E-11

CHAPTER V

ORDER-ONE AUTOREGRESSIVE PROCESS

This chapter considers a Gaussian stationary autoregressive process. We start out with some preliminaries and then we evaluate the performance of all estimators by Monte Carlo simulation.

5.1 Preliminaries

Consider the first-order Gaussian autoregressive [AR(1)] process defined by $Y_i = \phi Y_{i-1} + \varepsilon_{i-1}, i \geq 1$, where the ε_i are IID $N(0, 1-\phi^2)$ RVs, and Y_0 is a $N(0,1)$ RV initialized independently of the others. Figure 2 shows a sample path of a stationary Gaussian AR(1) process with $\mathsf{E}[Y_i] = 0$ and $\phi = 0.9$ based on 100 observations. This process has autocovariance function $R_k = \phi^{|k|}$, for all k, whence

$$\sigma^2 = \sum_{j=-\infty}^{\infty} R_j = \frac{1+\phi}{1-\phi}, \quad \text{and} \quad \gamma = -2\sum_{j=1}^{\infty} jR_j = \frac{-2\phi}{(1-\phi)^2}.$$

5.2 Performance Evaluation when $f(\mu) = (\mu+1)^2$

This subsection evaluates all methods using the function $f(\mu) = (\mu+1)^2$. Since analytical derivations are very hard, we rely on Monte Carlo simulations. Again, as performance measures for CIs, we will consider: $\mathsf{E}[H]$, the expected interval half-width; $\mathsf{Var}[H]$, the variance of the interval half-width; and C, the confidence interval achieved coverage. As performance measures for the estimators of $\mathsf{Var}[\hat{f}(\mu)]$ we consider the mean and variance. We will set coefficient $\phi = 0.9$ and variance parameter $\sigma^2 = (1+0.9)/(1-0.9) = 19$. All estimates are based on 1000 independent experiments.

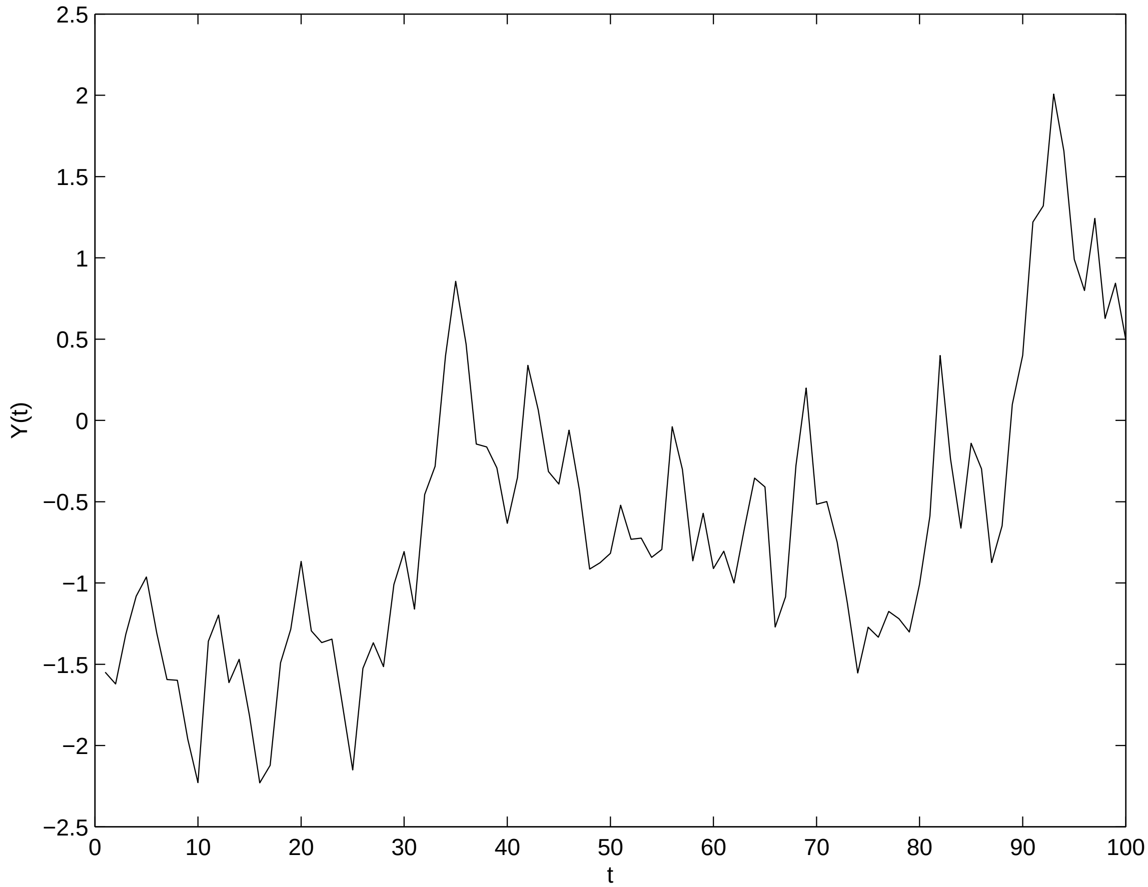

Figure 2: Sample Path of a Stationary Gaussian AR(1) Process with $\mathsf{E}[Y_i] = 0$ and $\phi = 0.9$ based on 100 Observations

5.2.1 Discussion

Now, let us discuss the simulation results of Tables 25–26. These results exhibit a similar pattern to the results in Tables 1–17 for the MA(1) process. The bias and variance of estimators for $f(\mu) = (\mu+1)^2$ and $|f'(\mu)| = 2|\mu+1|$ for this AR(1) process are studied in Appendix 2.

- In Table 25, we can see that the classical and jackknife estimators for $f(\mu) = (\mu + 1)^2$ give CIs with good coverage.
- In Table 25, the NBM point estimator for $f(\mu)$ yields CIs with lower coverage regardless of the variance estimator of $\hat{f}(\mu)$. The reason for this behavior is given in the discussion following Table 17.
- Considering Tables 25–26, the CIs based on the jackknife point estimator and the delta variance estimator (in columns BM+STS(w_i)) for $\hat{f}(\mu)$ seem to perform best. In particular, the delta variance estimator BM+STS(w_2) appears to have the same mean as the estimators S_B^2 and S_J^2 of Muñoz and Glynn [51], but is significantly less variable than either S_B^2 or S_J^2 (by roughly 1/2).

Table 25: Coverage Estimates of 90% CIs for $f(\mu) = (\mu + 1)^2$ based on a Gaussian AR(1) Process with $\mu = 0$ and $\phi = 0.9$

		Variance Estimator								
Coverage	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$ Cl. for f	128	0.898	0.870	0.858	0.890	0.879	0.879	0.890	0.906	0.900
	512	0.909	0.897	0.894	0.902	0.896	0.899	0.905	0.904	0.910
	2048	0.904	0.901	0.902	0.902	0.905	0.905	0.904	0.904	0.904
$b = 128$ Cl. for f	128	0.888	0.857	0.852	0.872	0.870	0.868	0.878	0.904	0.888
	512	0.905	0.893	0.891	0.896	0.895	0.895	0.900	0.903	0.905
	2048	0.905	0.902	0.899	0.901	0.909	0.908	0.907	0.907	0.905
$b = 32$ BM for f	128	0.759	0.706	0.701	0.726	0.725	0.728	0.741	0.777	0.760
	512	0.873	0.865	0.862	0.863	0.868	0.868	0.871	0.870	0.873
	2048	0.894	0.892	0.888	0.893	0.885	0.891	0.891	0.893	0.894
$b = 128$ BM for f	128	0.324	0.269	0.271	0.297	0.294	0.290	0.314	0.345	0.324
	512	0.729	0.707	0.706	0.721	0.713	0.712	0.722	0.732	0.729
	2048	0.855	0.850	0.847	0.852	0.852	0.854	0.855	0.856	0.855
$b = 32$ Jk. for f	128	0.898	0.868	0.859	0.889	0.878	0.879	0.885	0.902	0.898
	512	0.905	0.897	0.895	0.903	0.895	0.894	0.901	0.904	0.906
	2048	0.903	0.902	0.902	0.903	0.905	0.906	0.905	0.904	0.903
$b = 128$ Jk. for f	128	0.889	0.855	0.852	0.870	0.869	0.866	0.878	0.903	0.889
	512	0.906	0.893	0.890	0.895	0.894	0.893	0.897	0.903	0.906
	2048	0.905	0.900	0.899	0.901	0.907	0.908	0.908	0.908	0.905

Table 26: Performance Evaluation of 90% CIs for $f(\mu) = (\mu + 1)^2$ based on a Gaussian AR(1) Process with $\mu = 0$ and $\phi = 0.9$

		Variance Estimator								
Perf. Measure	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$	128	2.182E-01	2.012E-01	2.001E-01	2.104E-01	2.075E-01	2.069E-01	2.120E-01	2.245E-01	2.182E-01
$\widehat{\mathsf{E}}[H]$	512	1.133E-01	1.116E-01	1.115E-01	1.137E-01	1.112E-01	1.111E-01	1.122E-01	1.140E-01	1.133E-01
	2048	5.728E-02	5.706E-02	5.690E-02	5.744E-02	5.654E-02	5.646E-02	5.673E-02	5.741E-02	5.728E-02
$b = 128$	128	1.085E-01	9.958E-02	9.909E-02	1.038E-01	1.038E-01	1.036E-01	1.059E-01	1.121E-01	1.085E-01
$\widehat{\mathsf{E}}[H]$	512	5.578E-02	5.478E-02	5.475E-02	5.589E-02	5.513E-02	5.511E-02	5.569E-02	5.625E-02	5.578E-02
	2048	2.821E-02	2.806E-02	2.806E-02	2.818E-02	2.806E-02	2.806E-02	2.812E-02	2.828E-02	2.821E-02
$b = 32$	128	9.966E-04	8.514E-04	8.441E-04	9.079E-04	5.389E-04	5.366E-04	5.433E-04	1.392E-03	9.961E-04
$\widehat{\mathsf{Var}}[H]$	512	2.311E-04	2.029E-04	2.019E-04	2.135E-04	1.105E-04	1.108E-04	1.136E-04	2.540E-04	2.312E-04
	2048	5.701E-05	5.384E-05	5.276E-05	5.174E-05	2.769E-05	2.772E-05	2.751E-05	5.874E-05	5.702E-05
$b = 128$	128	5.765E-05	5.416E-05	5.419E-05	5.381E-05	3.399E-05	3.437E-05	3.380E-05	8.328E-05	5.766E-05
$\widehat{\mathsf{Var}}[H]$	512	1.286E-05	1.301E-05	1.224E-05	1.362E-05	7.251E-06	6.994E-06	7.424E-06	1.481E-05	1.286E-05
	2048	3.380E-06	3.209E-06	3.172E-06	3.062E-06	1.628E-06	1.613E-06	1.619E-06	3.504E-06	3.380E-06
$b = 32$	128	1.691E-02	1.441E-02	1.425E-02	1.574E-02	1.564E-02	1.556E-02	1.631E-02	1.801E-02	1.691E-02
$\widehat{\mathsf{E}}[S^2]$	512	4.542E-03	4.409E-03	4.400E-03	4.579E-03	4.474E-03	4.470E-03	4.561E-03	4.610E-03	4.543E-03
	2048	1.161E-03	1.153E-03	1.147E-03	1.168E-03	1.157E-03	1.154E-03	1.165E-03	1.167E-03	1.161E-03
$b = 128$	128	4.306E-03	3.632E-03	3.596E-03	3.944E-03	3.968E-03	3.950E-03	4.124E-03	4.605E-03	4.306E-03
$\widehat{\mathsf{E}}[S^2]$	512	1.138E-03	1.098E-03	1.096E-03	1.143E-03	1.118E-03	1.117E-03	1.141E-03	1.158E-03	1.138E-03
	2048	2.911E-04	2.880E-04	2.879E-04	2.904E-04	2.895E-04	2.895E-04	2.908E-04	2.925E-04	2.911E-04
$b = 32$	128	2.456E-05	1.753E-05	1.718E-05	2.043E-05	1.239E-05	1.226E-05	1.313E-05	3.752E-05	2.456E-05
$\widehat{\mathsf{Var}}[S^2]$	512	1.493E-06	1.272E-06	1.266E-06	1.379E-06	7.179E-07	7.183E-07	7.500E-07	1.693E-06	1.493E-06
	2048	9.337E-08	8.782E-08	8.674E-08	8.483E-08	4.625E-08	4.634E-08	4.611E-08	9.736E-08	9.337E-08
$b = 128$	128	3.640E-07	2.900E-07	2.878E-07	3.139E-07	2.002E-07	2.014E-07	2.079E-07	5.686E-07	3.640E-07
$\widehat{\mathsf{Var}}[S^2]$	512	2.140E-08	2.088E-08	1.960E-08	2.277E-08	1.195E-08	1.152E-08	1.250E-08	2.529E-08	2.139E-08
	2048	1.432E-09	1.348E-09	1.335E-09	1.293E-09	6.885E-10	6.829E-10	6.857E-10	1.495E-09	1.432E-09

5.3 Performance Evaluation when $f(\mu) = 1/(\mu + 1)$

In this subsection, we evaluate the methods using the function $f(\mu) = 1/(\mu+1)$. Since analytical comparisons remain intractable, we again rely on simulation experiments, each using 1000 independent replications. Since $f'(\mu)$ is a nonlinear function, the classical, jackknife and batch means point estimators for $|f'(\mu)|$ are not equal.

5.3.1 Discussion

The results in Tables 27–33 have similar patterns to the MA(1)–based simulation results in Tables 18–24. The bias and variance of the estimators of $f(\mu) = 1/(\mu + 1)$ and $|f'(\mu)| = 1/(\mu + 1)^2$ are studied in Appendix 2.

- In Tables 27–29, we can see that the classical and jackknife point estimators of $f(\mu) = 1/(\mu+1)$ yield confidence intervals with good coverage regardless of the variance estimators. This appears to be due to the lower bias and variance of $\hat{f}_C$ and $\bar{f}_J$ compared to $\bar{f}_B$.

- Tables 30–31 considers the sample mean and variance of CI half-widths. All variance estimators except for the NBM variance estimator S_B^2 yield similar sample averages for all combinations of b and m. In Table 31, the delta variance estimator based on $S_*^2(\bar{f}'_J; \widehat{V}_C(w_i))$ yields the least variable half-widths.

- Tables 32–33 list the sample means and sample variances of the variance estimators for $\hat{f}(\mu)$. These estimates are directly related to the sample means and sample variances of the half-widths. Therefore all variance estimators except for the NBM variance estimator S_B^2 yield similar sample means for all combinations of b and m. In Table 33, the delta variance estimators based on $S_*^2(\bar{f}'_J; \widehat{V}_C(w_i))$ exhibit the smallest sample variance.

- Overall, we recommend the jackknife point estimator for $f(\mu)$ and the delta variance estimator based on $S_*^2(\bar{f}'_J; \widehat{V}_C(w_i))$. These combinations yield CIs with good coverage and the best half-width in terms of mean and variance.

Table 27: Coverage Estimates for 90% CIs for $f(\mu) = 1/(\mu + 1)$ based on a Gaussian AR(1) Process with $\mu = 0$ and $\phi = 0.9$, and the classical point estimator for $f'(\mu)$

		Variance Estimator								
Coverage	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$	128	0.909	0.885	0.880	0.902	0.903	0.901	0.907	0.979	0.909
Cl. for f	512	0.904	0.903	0.905	0.912	0.905	0.908	0.908	0.942	0.904
Cl. for f'	2048	0.904	0.903	0.904	0.906	0.900	0.903	0.907	0.912	0.905
$b = 128$	128	0.897	0.862	0.859	0.879	0.881	0.882	0.888	0.997	0.897
Cl. for f	512	0.902	0.893	0.895	0.899	0.900	0.899	0.902	0.944	0.902
Cl. for f'	2048	0.903	0.906	0.902	0.904	0.901	0.902	0.905	0.920	0.903
$b = 32$	128	0.204	0.201	0.201	0.218	0.198	0.201	0.203	0.651	0.206
BM for f	512	0.682	0.682	0.683	0.686	0.682	0.677	0.681	0.785	0.684
Cl. for f'	2048	0.865	0.856	0.854	0.863	0.855	0.855	0.857	0.878	0.864
$b = 128$	128	0.051	0.047	0.047	0.049	0.050	0.050	0.050	0.363	0.051
BM. for f	512	0.232	0.234	0.237	0.236	0.232	0.232	0.231	0.322	0.232
Cl. for f'	2048	0.707	0.708	0.712	0.710	0.710	0.710	0.710	0.726	0.707
$b = 32$	128	0.910	0.884	0.888	0.896	0.899	0.896	0.898	0.976	0.909
Jk. for f	512	0.904	0.904	0.906	0.910	0.906	0.907	0.908	0.940	0.904
Cl. for f'	2048	0.905	0.903	0.905	0.907	0.902	0.903	0.907	0.911	0.905
$b = 128$	128	0.896	0.861	0.863	0.882	0.883	0.882	0.890	0.997	0.896
Jk. for f	512	0.904	0.891	0.897	0.899	0.898	0.897	0.903	0.946	0.904
Cl. for f'	2048	0.905	0.905	0.903	0.904	0.901	0.901	0.904	0.917	0.905

Table 28: Coverage Estimates for 90% CIs for $f(\mu) = 1/(\mu + 1)$ based on a Gaussian AR(1) Process with $\mu = 0$ and $\phi = 0.9$, and the batch means point estimator for $f'(\mu)$

		Variance Estimator								
Coverage	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$	128	0.979	0.976	0.976	0.981	0.980	0.978	0.982	0.979	0.909
Cl. for f	512	0.942	0.932	0.928	0.935	0.932	0.933	0.939	0.942	0.904
BM for f'	2048	0.915	0.915	0.911	0.913	0.912	0.914	0.914	0.912	0.905
$b = 128$	128	0.998	0.995	0.996	0.998	0.997	0.997	0.997	0.997	0.897
Cl. for f	512	0.940	0.936	0.935	0.943	0.937	0.939	0.940	0.944	0.902
BM for f'	2048	0.914	0.913	0.911	0.917	0.914	0.914	0.915	0.920	0.903
$b = 32$	128	0.722	0.636	0.633	0.682	0.680	0.666	0.698	0.651	0.206
BM for f	512	0.768	0.739	0.745	0.746	0.753	0.751	0.754	0.785	0.684
BM for f'	2048	0.876	0.866	0.869	0.873	0.870	0.871	0.876	0.878	0.864
$b = 128$	128	0.390	0.360	0.361	0.373	0.373	0.374	0.380	0.363	0.051
BM for f	512	0.295	0.290	0.291	0.311	0.294	0.292	0.302	0.322	0.232
BM for f'	2048	0.722	0.724	0.724	0.720	0.720	0.719	0.722	0.726	0.707
$b = 32$	128	0.976	0.975	0.974	0.980	0.976	0.975	0.978	0.976	0.909
Jk. for f	512	0.942	0.933	0.928	0.935	0.933	0.932	0.937	0.940	0.904
BM for f'	2048	0.915	0.914	0.914	0.912	0.912	0.912	0.914	0.911	0.905
$b = 128$	128	0.998	0.995	0.995	0.998	0.997	0.997	0.997	0.997	0.896
Jk. for f	512	0.940	0.938	0.935	0.943	0.938	0.939	0.939	0.946	0.904
BM for f'	2048	0.914	0.911	0.911	0.918	0.912	0.911	0.912	0.917	0.905

Table 29: Coverage Estimates for 90% CIs for $f(\mu) = 1/(\mu + 1)$ based on a Gaussian AR(1) Process with $\mu = 0$ and $\phi = 0.9$, and the jackknife point estimator for $f'(\mu)$

		Variance Estimator								
Coverage	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$	128	0.906	0.881	0.877	0.899	0.898	0.896	0.904	0.979	0.909
Cl. for f	512	0.903	0.900	0.905	0.910	0.903	0.908	0.907	0.942	0.904
Jk. for f'	2048	0.904	0.903	0.904	0.906	0.900	0.901	0.906	0.912	0.905
$b = 128$	128	0.897	0.862	0.857	0.878	0.881	0.882	0.888	0.997	0.897
Cl. for f	512	0.901	0.893	0.895	0.899	0.900	0.898	0.902	0.944	0.902
Jk. for f'	2048	0.903	0.905	0.901	0.904	0.901	0.902	0.905	0.920	0.903
$b = 32$	128	0.201	0.196	0.197	0.214	0.197	0.198	0.201	0.651	0.206
BM for f	512	0.682	0.679	0.681	0.684	0.680	0.676	0.678	0.785	0.684
Jk. for f'	2048	0.865	0.853	0.854	0.863	0.854	0.855	0.857	0.878	0.864
$b = 128$	128	0.051	0.047	0.047	0.049	0.050	0.049	0.050	0.363	0.051
BM for f	512	0.231	0.234	0.236	0.236	0.230	0.231	0.231	0.322	0.232
Jk. for f'	2048	0.707	0.708	0.712	0.710	0.710	0.710	0.710	0.726	0.707
$b = 32$	128	0.906	0.882	0.882	0.893	0.896	0.895	0.894	0.976	0.909
Jk. for f	512	0.904	0.904	0.905	0.909	0.905	0.906	0.906	0.940	0.904
Jk. for f'	2048	0.905	0.903	0.904	0.907	0.902	0.903	0.907	0.911	0.905
$b = 128$	128	0.896	0.861	0.859	0.881	0.883	0.881	0.890	0.997	0.896
Jk. for f	512	0.902	0.891	0.897	0.899	0.897	0.897	0.902	0.946	0.904
Jk. for f'	2048	0.905	0.905	0.902	0.904	0.901	0.901	0.904	0.917	0.905

Table 30: Sample Mean of Half-widths for 90% CIs for $f(\mu) = 1/(\mu + 1)$ based on a Gaussian AR(1) Process with $\mu = 0$ and $\phi = 0.9$

		Variance Estimator								
$\widehat{\mathsf{E}}[H]$	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$ Cl. for f'	128	1.112E-01	1.025E-01	1.019E-01	1.072E-01	1.057E-01	1.054E-01	1.080E-01	8.527E-01	1.113E-01
	512	5.686E-02	5.602E-02	5.597E-02	5.709E-02	5.582E-02	5.579E-02	5.636E-02	6.662E-02	5.687E-02
	2048	2.862E-02	2.851E-02	2.844E-02	2.871E-02	2.825E-02	2.821E-02	2.835E-02	2.961E-02	2.862E-02
$b = 128$ Cl. for f'	128	5.444E-02	4.996E-02	4.970E-02	5.208E-02	5.210E-02	5.198E-02	5.313E-02	2.067E+00	5.444E-02
	512	2.787E-02	2.737E-02	2.735E-02	2.793E-02	2.755E-02	2.754E-02	2.782E-02	3.321E-02	2.787E-02
	2048	1.410E-02	1.402E-02	1.402E-02	1.408E-02	1.402E-02	1.402E-02	1.405E-02	1.464E-02	1.410E-02
$b = 32$ BM for f'	128	3.934E+01	3.276E+01	3.262E+01	3.363E+01	3.567E+01	3.561E+01	3.607E+01	8.527E-01	1.113E-01
	512	6.473E-02	6.346E-02	6.339E-02	6.466E-02	6.338E-02	6.335E-02	6.399E-02	6.662E-02	5.687E-02
	2048	2.945E-02	2.932E-02	2.924E-02	2.952E-02	2.907E-02	2.902E-02	2.916E-02	2.961E-02	2.862E-02
$b = 128$ BM for f'	128	1.475E+03	1.223E+03	1.220E+03	1.304E+03	1.351E+03	1.349E+03	1.390E+03	2.067E+00	5.444E-02
	512	3.169E-02	3.108E-02	3.107E-02	3.172E-02	3.130E-02	3.129E-02	3.162E-02	3.321E-02	2.787E-02
	2048	1.451E-02	1.443E-02	1.443E-02	1.449E-02	1.443E-02	1.443E-02	1.446E-02	1.464E-02	1.410E-02
$b = 32$ Jk. for f'	128	1.097E-01	1.011E-01	1.006E-01	1.058E-01	1.043E-01	1.040E-01	1.066E-01	8.527E-01	1.113E-01
	512	5.666E-02	5.583E-02	5.578E-02	5.689E-02	5.563E-02	5.560E-02	5.616E-02	6.662E-02	5.687E-02
	2048	2.859E-02	2.849E-02	2.841E-02	2.868E-02	2.823E-02	2.819E-02	2.832E-02	2.961E-02	2.862E-02
$b = 128$ Jk. for f'	128	5.426E-02	4.979E-02	4.954E-02	5.192E-02	5.193E-02	5.181E-02	5.295E-02	2.067E+00	5.444E-02
	512	2.785E-02	2.735E-02	2.733E-02	2.790E-02	2.752E-02	2.751E-02	2.780E-02	3.321E-02	2.787E-02
	2048	1.409E-02	1.402E-02	1.402E-02	1.408E-02	1.402E-02	1.402E-02	1.405E-02	1.464E-02	1.410E-02

Table 31: Sample Variance of Half-widths for 90% CIs for $f(\mu) = 1/(\mu + 1)$ based on a Gaussian AR(1) Process with $\mu = 0$ and $\phi = 0.9$

		Variance Estimator								
$\widehat{\mathsf{Var}}[H]$	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$ Cl. for f'	128	4.440E-04	3.758E-04	3.753E-04	4.176E-04	3.059E-04	3.060E-04	3.197E-04	2.849E+01	4.452E-04
	512	7.161E-05	6.540E-05	6.471E-05	6.917E-05	4.141E-05	4.128E-05	4.271E-05	2.442E-04	7.160E-05
	2048	1.462E-05	1.427E-05	1.404E-05	1.381E-05	7.512E-06	7.542E-06	7.498E-06	1.946E-05	1.463E-05
$b = 128$ Cl. for f'	128	2.567E-05	1.994E-05	1.947E-05	2.209E-05	1.719E-05	1.705E-05	1.825E-05	5.271E+02	2.568E-05
	512	3.760E-06	3.636E-06	3.421E-06	3.844E-06	2.273E-06	2.198E-06	2.344E-06	2.115E-05	3.762E-06
	2048	8.874E-07	8.299E-07	8.255E-07	7.716E-07	4.424E-07	4.409E-07	4.285E-07	1.244E-06	8.874E-07
$b = 32$ BM for f'	128	5.467E+05	3.507E+05	3.536E+05	3.711E+05	4.338E+05	4.352E+05	4.439E+05	2.849E+01	4.452E-04
	512	1.540E-04	9.920E-05	9.789E-05	1.049E-04	9.013E-05	8.975E-05	9.222E-05	2.442E-04	7.160E-05
	2048	1.730E-05	1.517E-05	1.495E-05	1.472E-05	8.856E-06	8.899E-06	8.853E-06	1.946E-05	1.463E-05
$b = 128$ BM for f'	128	9.594E+08	6.789E+08	6.662E+08	7.910E+08	8.126E+08	8.063E+08	8.685E+08	5.271E+02	2.568E-05
	512	8.261E-06	5.581E-06	5.308E-06	5.997E-06	5.061E-06	4.966E-06	5.227E-06	2.115E-05	3.762E-06
	2048	1.052E-06	8.834E-07	8.789E-07	8.214E-07	5.262E-07	5.247E-07	5.114E-07	1.244E-06	8.874E-07
$b = 32$ Jk. for f'	128	4.145E-04	3.606E-04	3.601E-04	4.009E-04	2.867E-04	2.868E-04	3.000E-04	2.849E+01	4.452E-04
	512	7.022E-05	6.483E-05	6.415E-05	6.856E-05	4.064E-05	4.051E-05	4.192E-05	2.442E-04	7.160E-05
	2048	1.455E-05	1.424E-05	1.401E-05	1.379E-05	7.474E-06	7.504E-06	7.459E-06	1.946E-05	1.463E-05
$b = 128$ Jk. for f'	128	2.525E-05	1.974E-05	1.927E-05	2.187E-05	1.692E-05	1.678E-05	1.797E-05	5.271E+02	2.568E-05
	512	3.742E-06	3.628E-06	3.414E-06	3.836E-06	2.263E-06	2.188E-06	2.333E-06	2.115E-05	3.762E-06
	2048	8.862E-07	8.296E-07	8.251E-07	7.712E-07	4.418E-07	4.404E-07	4.279E-07	1.244E-06	8.874E-07

Table 32: Sample Mean of Variance Estimators for $\hat{f}(\mu)$ when $f(\mu) = 1/(\mu+1)$ based on a Gaussian AR(1) Process with $\mu = 0$ and $\phi = 0.9$

		Variance Estimator								
$\widehat{\mathsf{E}}[S^2]$	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$	128	4.456E-03	3.792E-03	3.752E-03	4.150E-03	4.118E-03	4.098E-03	4.300E-03	1.015E+01	4.461E-03
Cl. for f'	512	1.150E-03	1.117E-03	1.114E-03	1.160E-03	1.133E-03	1.132E-03	1.155E-03	1.629E-03	1.150E-03
	2048	2.900E-04	2.883E-04	2.867E-04	2.920E-04	2.891E-04	2.883E-04	2.910E-04	3.117E-04	2.900E-04
$b = 128$	128	1.089E-03	9.164E-04	9.070E-04	9.963E-04	1.002E-03	9.976E-04	1.042E-03	1.934E+02	1.089E-03
Cl. for f'	512	2.843E-04	2.742E-04	2.738E-04	2.855E-04	2.793E-04	2.791E-04	2.849E-04	4.095E-04	2.843E-04
	2048	7.271E-05	7.194E-05	7.191E-05	7.252E-05	7.232E-05	7.231E-05	7.262E-05	7.849E-05	7.271E-05
$b = 32$	128	1.905E+05	1.225E+05	1.235E+05	1.296E+05	1.560E+05	1.565E+05	1.596E+05	1.015E+01	4.461E-03
BM for f'	512	1.511E-03	1.438E-03	1.435E-03	1.494E-03	1.474E-03	1.472E-03	1.502E-03	1.629E-03	1.150E-03
	2048	3.078E-04	3.049E-04	3.032E-04	3.088E-04	3.063E-04	3.055E-04	3.083E-04	3.117E-04	2.900E-04
$b = 128$	128	3.499E+08	2.476E+08	2.430E+08	2.885E+08	2.985E+08	2.962E+08	3.191E+08	1.934E+02	1.089E-03
BM for f'	512	3.689E-04	3.540E-04	3.535E-04	3.687E-04	3.614E-04	3.612E-04	3.688E-04	4.095E-04	2.843E-04
	2048	7.708E-05	7.618E-05	7.615E-05	7.680E-05	7.663E-05	7.661E-05	7.694E-05	7.849E-05	7.271E-05
$b = 32$	128	4.329E-03	3.691E-03	3.652E-03	4.039E-03	4.005E-03	3.985E-03	4.182E-03	1.015E+01	4.461E-03
Jk. for f'	512	1.141E-03	1.109E-03	1.107E-03	1.152E-03	1.125E-03	1.124E-03	1.147E-03	1.629E-03	1.150E-03
	2048	2.894E-04	2.878E-04	2.862E-04	2.915E-04	2.886E-04	2.878E-04	2.905E-04	3.117E-04	2.900E-04
$b = 128$	128	1.082E-03	9.104E-04	9.011E-04	9.898E-04	9.957E-04	9.910E-04	1.036E-03	1.934E+02	1.089E-03
Jk. for f'	512	2.839E-04	2.737E-04	2.733E-04	2.850E-04	2.788E-04	2.786E-04	2.844E-04	4.095E-04	2.843E-04
	2048	7.268E-05	7.191E-05	7.188E-05	7.249E-05	7.229E-05	7.228E-05	7.258E-05	7.849E-05	7.271E-05

Table 33: Sample Variance of Variance Estimators for $\hat{f}(\mu)$ when $f(\mu) = 1/(\mu+1)$ based on a Gaussian AR(1) Process with $\mu = 0$ and $\phi = 0.9$

		Variance Estimator								
$\widehat{\mathsf{Var}}[S^2]$	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 32$	128	2.986E-06	2.181E-06	2.174E-06	2.681E-06	1.973E-06	1.974E-06	2.160E-06	3.470E+04	2.997E-06
Cl. for f'	512	1.170E-07	1.050E-07	1.035E-07	1.161E-07	6.909E-08	6.857E-08	7.295E-08	7.271E-07	1.169E-07
	2048	5.993E-09	5.855E-09	5.789E-09	5.742E-09	3.149E-09	3.161E-09	3.169E-09	8.730E-09	5.998E-09
$b = 128$	128	4.188E-08	2.716E-08	2.622E-08	3.289E-08	2.578E-08	2.545E-08	2.858E-08	1.622E+07	4.190E-08
Cl. for f'	512	1.565E-09	1.454E-09	1.372E-09	1.600E-09	9.329E-10	9.058E-10	9.796E-10	1.855E-08	1.566E-09
	2048	9.428E-11	8.717E-11	8.693E-11	8.168E-11	4.691E-11	4.686E-11	4.548E-11	1.446E-10	9.429E-11
$b = 32$	128	2.028E+13	7.421E+12	7.537E+12	8.447E+12	1.285E+13	1.290E+13	1.356E+13	3.470E+04	2.997E-06
BM for f'	512	3.563E-07	2.084E-07	2.042E-07	2.318E-07	2.060E-07	2.040E-07	2.156E-07	7.271E-07	1.169E-07
	2048	7.591E-09	6.576E-09	6.516E-09	6.472E-09	3.949E-09	3.966E-09	3.976E-09	8.730E-09	5.998E-09
$b = 128$	128	7.925E+19	4.408E+19	4.129E+19	6.217E+19	6.020E+19	5.860E+19	7.019E+19	1.622E+07	4.190E-08
BM for f'	512	4.588E-09	2.906E-09	2.787E-09	3.273E-09	2.757E-09	2.720E-09	2.916E-09	1.855E-08	1.566E-09
	2048	1.188E-10	9.820E-11	9.800E-11	9.198E-11	5.919E-11	5.915E-11	5.757E-11	1.446E-10	9.429E-11
$b = 32$	128	2.689E-06	2.030E-06	2.023E-06	2.498E-06	1.789E-06	1.790E-06	1.962E-06	3.470E+04	2.997E-06
Jk. for f'	512	1.137E-07	1.033E-07	1.019E-07	1.143E-07	6.728E-08	6.677E-08	7.106E-08	7.271E-07	1.169E-07
	2048	5.949E-09	5.834E-09	5.769E-09	5.721E-09	3.127E-09	3.139E-09	3.146E-09	8.730E-09	5.998E-09
$b = 128$	128	4.090E-08	2.671E-08	2.578E-08	3.234E-08	2.520E-08	2.487E-08	2.794E-08	1.622E+07	4.190E-08
Jk. for f'	512	1.555E-09	1.448E-09	1.367E-09	1.593E-09	9.270E-10	9.000E-10	9.735E-10	1.855E-08	1.566E-09
	2048	9.412E-11	8.709E-11	8.686E-11	8.161E-11	4.683E-11	4.678E-11	4.540E-11	1.446E-10	9.429E-11

CHAPTER VI

M/M/1 QUEUEING SYSTEM

This chapter presents experimental results related to a stationary M/M/1 system. Specifically, we consider the estimation of the variance of the steady-state distribution of the total time a customer spends in the system.

6.1 A Numerical Example

Consider a stationary M/M/1 queueing system (single-server queue with IID exponential interarrival times and IID exponential service time) with mean interarrival time α and mean service time β, where $\beta < \alpha$. The traffic intensity is $\rho = \beta/\alpha$. Let Z_i be the sojourn time of the ith customer (waiting time plus service time). It is known (cf. Kulkarni [36]) that the steady-state distribution of the sojourn time is exponential with rate $1/\beta - 1/\alpha$; hence its mean is $\mu = (\alpha\beta)/(\alpha - \beta)$. Figure 3 shows a plot of 100 steady-state sojourn times in an M/M/1 System with $\alpha = 10$ and $\beta = 8$. Our objective is to estimate the variance $f(\mu) = \mu^2$ of this exponential distribution. We use $\alpha = 10$ and $\beta = 8$. This implies $\rho = 0.8$, $\mu = 40$, and $f(\mu) = 1600$. To eliminate the effects of the initial transient, we sample the first customer's sojourn time from the limiting exponential distribution and generate the remaining sojourn times using Lindley's [42] recursion

$$Z_{i+1} = \max\{0, Z_i - A_{i+1}\} + S_{i+1}, \quad i \geq 1,$$

where A_i is the ith interarrival time and S_i is the ith service time. The variance parameter of the sojourn time process in the M/M/1 system is (see Blomqvist [8])

$$\sigma^2 = \frac{\rho\beta^2}{(1-\rho)^4}(\rho^3 - 4\rho^2 + 5\rho + 2) + \beta^2 = 126{,}528.$$

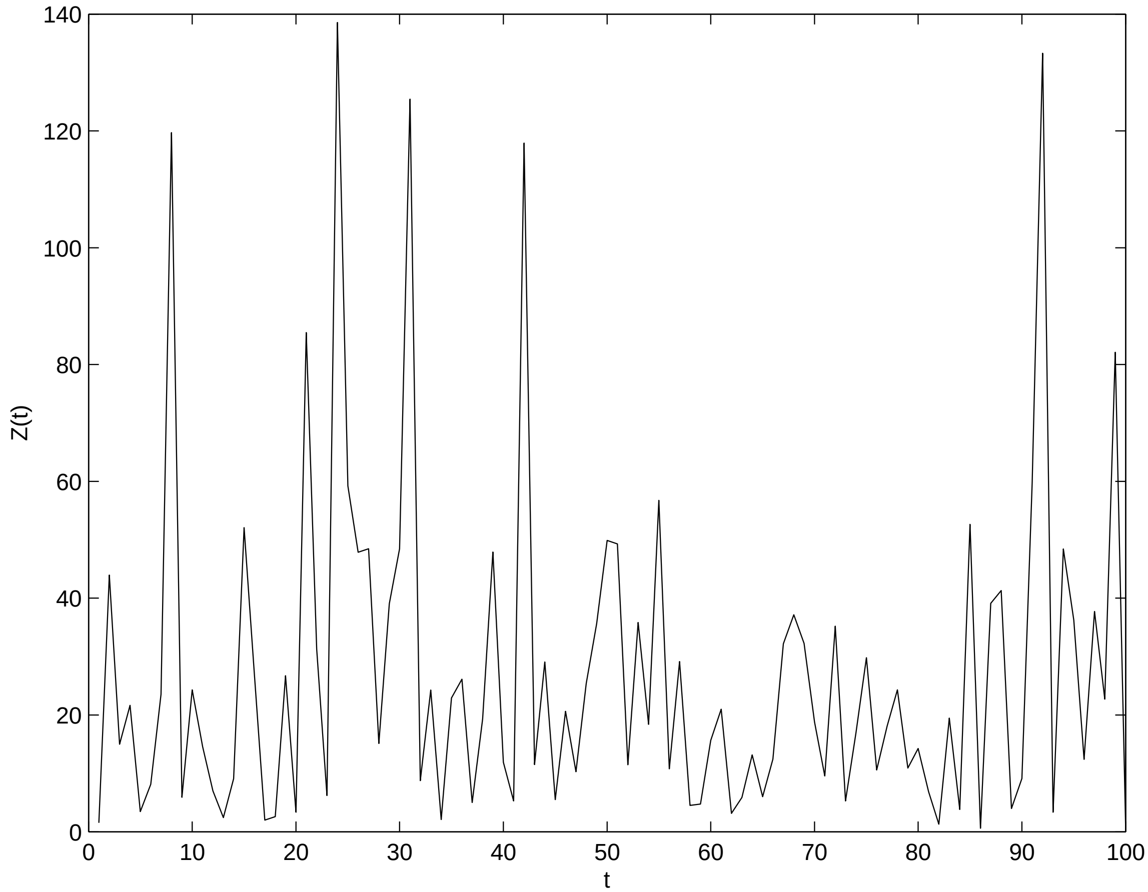

Figure 3: Plot of 100 Steady-state Sojourn Times in an M/M/1 System with $\alpha = 10$ and $\beta = 8$

The entries of Tables 35 and 36 are based on 1000 independent experiments. Again, as performance measures for the CIs we consider the expected interval half-width, the variance of the interval half-width, and the coverage of the confidence interval. Further, we judge the performance of the estimators of $\mathsf{Var}[\hat{f}(\mu)]$ based on their expectation and variance.

Table 34: Coverage Estimates for 90% CIs for the Variance of the Steady-State Customer Sojourn Time in an M/M/1 Queueing System with Traffic Intensity $\rho = 0.8$

		Variance Estimator								
Coverage	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 16$	2048	0.882	0.862	0.862	0.868	0.872	0.870	0.878	0.895	0.880
Cl. for f	4096	0.890	0.868	0.866	0.884	0.879	0.876	0.887	0.894	0.888
	8192	0.885	0.882	0.879	0.888	0.891	0.889	0.891	0.893	0.885
$b = 32$	2048	0.883	0.870	0.868	0.877	0.875	0.870	0.877	0.902	0.883
Cl. for f	4096	0.887	0.882	0.880	0.886	0.887	0.893	0.886	0.901	0.887
	8192	0.890	0.878	0.872	0.878	0.885	0.884	0.883	0.899	0.890
$b = 16$	2048	0.863	0.849	0.849	0.850	0.867	0.859	0.856	0.895	0.861
BM for f	4096	0.889	0.869	0.870	0.865	0.882	0.882	0.886	0.900	0.888
	8192	0.885	0.883	0.885	0.891	0.894	0.889	0.892	0.891	0.885
$b = 32$	2048	0.851	0.807	0.810	0.833	0.838	0.832	0.846	0.907	0.847
BM for f	4096	0.867	0.845	0.852	0.855	0.858	0.859	0.863	0.885	0.865
	8192	0.877	0.869	0.873	0.879	0.878	0.879	0.879	0.885	0.877
$b = 16$	2048	0.884	0.860	0.859	0.867	0.865	0.869	0.876	0.895	0.881
Jk. for f	4096	0.888	0.870	0.865	0.885	0.878	0.877	0.883	0.892	0.887
	8192	0.886	0.883	0.878	0.888	0.890	0.889	0.890	0.893	0.885
$b = 32$	2048	0.882	0.867	0.868	0.876	0.873	0.875	0.876	0.901	0.882
Jk. for f	4096	0.886	0.882	0.882	0.884	0.885	0.893	0.885	0.902	0.886
	8192	0.891	0.874	0.874	0.876	0.885	0.882	0.883	0.899	0.891

Table 35: Performance Evaluation for 90% CIs for the Variance of the Steady-State Customer Sojourn Time in an M/M/1 Queueing System with Traffic Intensity $\rho = 0.8$

		Variance Estimator								
Perf. Measure	m	BM	STS(w_0)	STS(w_1)	STS(w_2)	BM+STS(w_0)	BM+STS(w_1)	BM+STS(w_2)	S_B^2	S_J^2
$b = 16$	2048	2.630E+02	2.558E+02	2.556E+02	2.599E+02	2.537E+02	2.536E+02	2.554E+02	2.882E+02	2.614E+02
$\widehat{\mathsf{E}}[H]$	4096	1.917E+02	1.855E+02	1.858E+02	1.891E+02	1.845E+02	1.847E+02	1.862E+02	2.023E+02	1.910E+02
	8192	1.349E+02	1.350E+02	1.347E+02	1.363E+02	1.320E+02	1.318E+02	1.325E+02	1.388E+02	1.346E+02
$b = 32$	2048	1.854E+02	1.780E+02	1.778E+02	1.837E+02	1.800E+02	1.798E+02	1.827E+02	2.079E+02	1.848E+02
$\widehat{\mathsf{E}}[H]$	4096	1.315E+02	1.292E+02	1.288E+02	1.315E+02	1.290E+02	1.289E+02	1.301E+02	1.397E+02	1.312E+02
	8192	9.415E+01	9.356E+01	9.348E+01	9.431E+01	9.290E+01	9.287E+01	9.322E+01	9.732E+01	9.406E+01
$b = 16$	2048	6.684E+03	8.770E+03	8.845E+03	8.483E+03	6.059E+03	6.100E+03	6.092E+03	1.303E+04	6.380E+03
$\widehat{\mathsf{Var}}[H]$	4096	2.755E+03	2.888E+03	3.018E+03	2.775E+03	1.992E+03	2.036E+03	1.974E+03	4.307E+03	2.675E+03
	8192	9.185E+02	1.147E+03	1.209E+03	1.089E+03	6.590E+02	6.815E+02	6.558E+02	1.199E+03	9.032E+02
$b = 32$	2048	1.816E+03	2.120E+03	2.057E+03	2.349E+03	1.564E+03	1.545E+03	1.700E+03	4.106E+03	1.766E+03
$\widehat{\mathsf{Var}}[H]$	4096	6.226E+02	7.477E+02	7.661E+02	7.510E+02	4.946E+02	4.953E+02	5.172E+02	1.038E+03	6.127E+02
	8192	2.585E+02	2.652E+02	2.792E+02	2.614E+02	1.706E+02	1.754E+02	1.779E+02	3.594E+02	2.559E+02
$b = 16$	2048	2.467E+04	2.434E+04	2.433E+04	2.495E+04	2.450E+04	2.450E+04	2.481E+04	3.127E+04	2.431E+04
$\widehat{\mathsf{E}}[S^2]$	4096	1.285E+04	1.223E+04	1.231E+04	1.264E+04	1.253E+04	1.257E+04	1.274E+04	1.472E+04	1.274E+04
	8192	6.217E+03	6.352E+03	6.345E+03	6.453E+03	6.286E+03	6.283E+03	6.339E+03	6.662E+03	6.190E+03
$b = 32$	2048	1.259E+04	1.178E+04	1.174E+04	1.258E+04	1.218E+04	1.216E+04	1.259E+04	1.647E+04	1.249E+04
$\widehat{\mathsf{E}}[S^2]$	4096	6.227E+03	6.081E+03	6.053E+03	6.286E+03	6.153E+03	6.139E+03	6.257E+03	7.152E+03	6.201E+03
	8192	3.173E+03	3.143E+03	3.143E+03	3.191E+03	3.158E+03	3.158E+03	3.182E+03	3.419E+03	3.166E+03
$b = 16$	2048	3.168E+08	4.969E+08	4.902E+08	5.095E+08	3.420E+08	3.381E+08	3.593E+08	1.103E+09	2.913E+08
$\widehat{\mathsf{Var}}[S^2]$	4096	5.988E+07	5.874E+07	6.583E+07	5.593E+07	4.325E+07	4.467E+07	4.332E+07	1.415E+08	5.668E+07
	8192	8.246E+06	1.136E+07	1.276E+07	1.062E+07	6.519E+06	6.943E+06	6.350E+06	1.261E+07	8.032E+06
$b = 32$	2048	3.848E+07	4.445E+07	4.148E+07	5.323E+07	3.308E+07	3.192E+07	3.803E+07	1.525E+08	3.679E+07
$\widehat{\mathsf{Var}}[S^2]$	4096	5.877E+06	7.485E+06	7.872E+06	7.745E+06	4.947E+06	4.980E+06	5.306E+06	1.267E+07	5.742E+06
	8192	1.246E+06	1.277E+06	1.417E+06	1.249E+06	8.356E+05	8.757E+05	8.771E+05	2.067E+06	1.227E+06

6.2 Discussion

The following list contains a summary of the observations.

- Table 34 indicates that all point estimators $\hat{f}_C$, $\bar{f}_B$, and $\bar{f}_J$ of $f(\mu) = \mu^2$ result in CIs with coverage close to the nominal value regardless of the variance estimator.

- In Table 35, the sample mean and variance of the half-widths in columns BM+STS(w_i) are usually less than the respective quantities in columns BM, S_B^2 and S_J^2. In particular, the sample variance of half-widths in columns BM+STS(w_i) is up to 30% smaller than the sample variance in columns S_B^2 and S_J^2. We believe that this is due to the sound properties of the combined variance parameter estimator in Section 2.2.3.

- Figures 4 and 5 show the sample mean and sample variance of CI half-widths based on 32 batches. The CIs based on the delta variance estimator $S_*^2(\hat{f}'; \widehat{V}_C(w_2))$ have the lowest sample variance of half-width among others with competitive sample mean of half-width.

- Based on these experiments, we recommend the jackknife point estimator for $f(\mu)$ and the delta variance estimator based on $S_*^2(\hat{f}'; \widehat{V}_C(w_i))$.

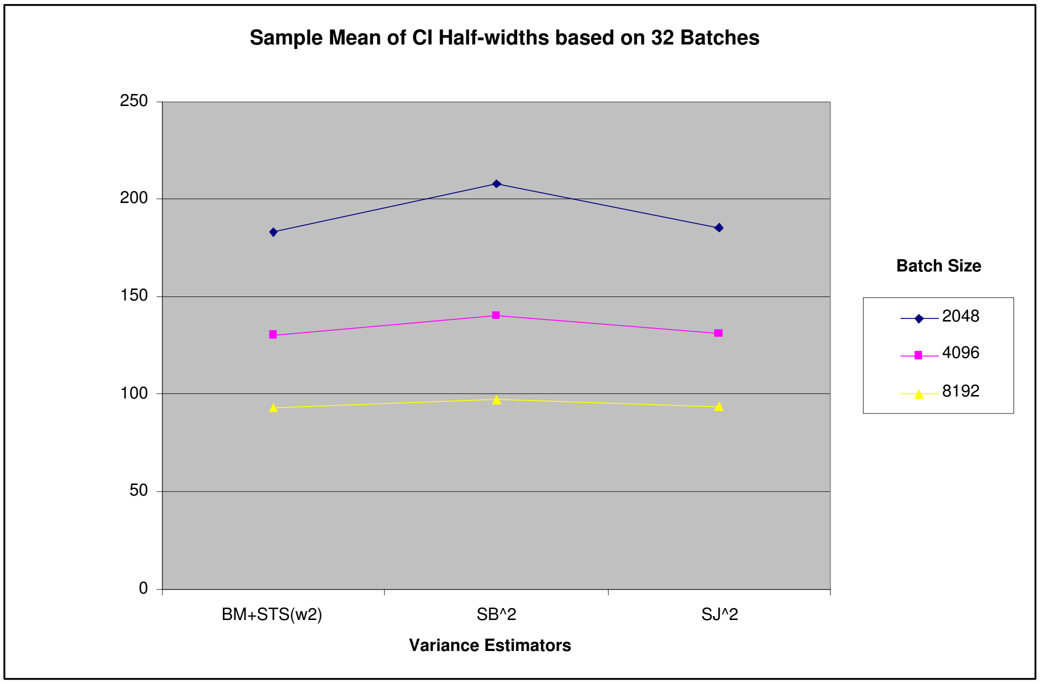

Figure 4: Sample Mean of the CI Half-widths for the Variance of the Customer Sojourn Time in a Stationary M/M/1 Queueing System with Traffic Intensity $\rho = 0.8$

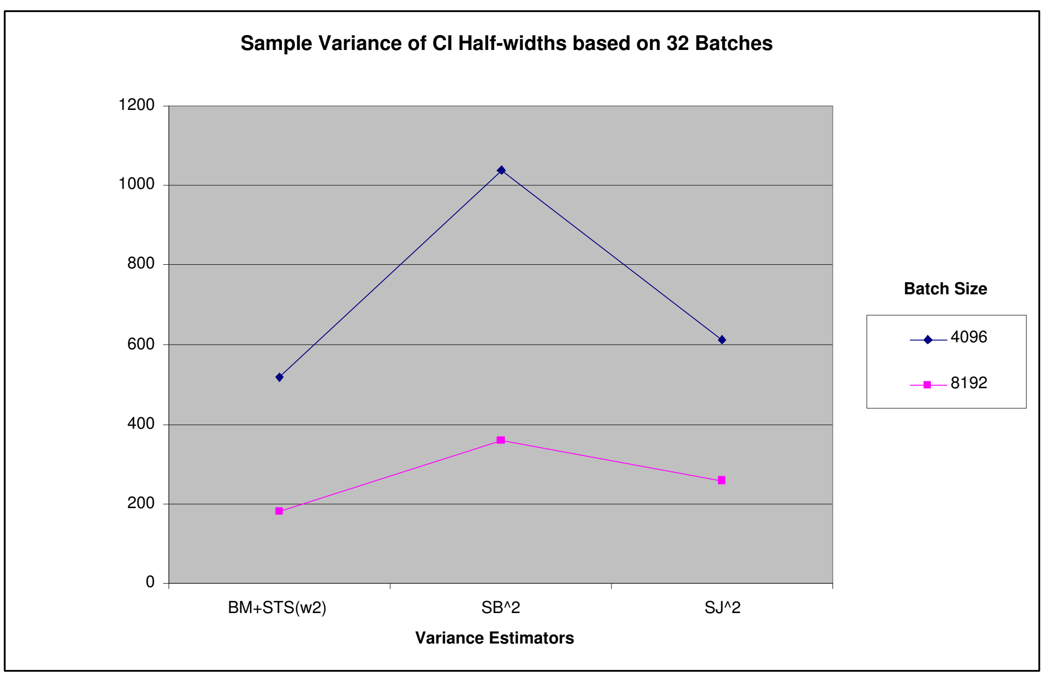

Figure 5: Sample Variance of the CI Half-widths for the Variance of the Customer Sojourn Time in a Stationary M/M/1 Queueing System with Traffic Intensity $\rho = 0.8$

CHAPTER VII

CONCLUSIONS AND FUTURE RESEARCH

This dissertation considered the derivation of point and confidence interval estimators for a function of the mean μ, say $f(\mu)$, of a stationary stochastic process $\{Y_i, i \geq 1\}$ with a finite variance parameter σ^2. The point estimators $\hat{f}(\mu)$ for $f(\mu)$ were proposed by Muñoz and Glynn [51]; the classical estimator is simply $f(\bar{Y}_n)$, while the batch means and jackknife estimators are based on nonoverlapping batches. The proposed estimators for the variance of $\hat{f}(\mu)$ are based on the delta method and consistent estimators of σ^2 from the literature.

The point estimators for $f(\mu)$ were evaluated by means of their mean and variance. The performance of a CI was evaluated by means of its coverage, mean half-width, and the variance of the half-width. The test beds were two Gaussian processes, an MA(1) process and an AR(1) process, and the process of customer sojourn times in a stationary M/M/1 system. We also used two functions, $f(\mu) = (\mu+1)^2$ and $f(\mu) = 1/(\mu+1)$ for the Gaussian processes and the function $f(\mu) = \mu^2$ for the M/M/1 system.

In the estimation problem of $f(\mu) = (\mu+1)^2$ for the MA(1) process, the analytical results show that the variance estimator $S_*^2(2(\bar{Y}_n+1); \widehat{V}_C)$ in Equations (16) and (31) has the smallest variance. This probably results from the fact that the variance parameter estimator $\hat{V}_C^2$ has smaller variance than its competitors. The results also show that the CIs based on the variance estimator $S_*^2(2(\bar{Y}_n+1); \widehat{V}_C)$ have half-widths with the smallest mean and variance. In the estimation problem of $f(\mu) = 1/(\mu+1)$ for the MA(1) process, the simulation results show that the variance estimators $S_*^2(\hat{f}'; \hat{V}_C(w_i))$ have the smallest variance among all counterparts. Further, these results show that the CIs based on the point estimators $\hat{f}_C$ and $\bar{f}_J$ and the variance estimator $S_*^2(\hat{f}'; \hat{V}_C(w_i))$ have the best performance in terms of coverage, mean and variance of half-width.

The experimental results from the estimation of $f(\mu) = (\mu + 1)^2$ and $f(\mu) = 1/(\mu + 1)$ under an AR(1) process are similar to the results for the MA(1) process. Since the AR(1) process has a more "prolonged" autocorrelation structure than an MA(1) process with equal correlation coefficient, the values of the mean and variance of the variance estimators for $\hat{f}(\mu)$ and half-widths are higher than their MA(1) counterparts.

In Chapter VI, we considered the steady-state sojourn time of customers in an the M/M/1 queueing system and presented experimental results for the variance $f(\mu) = \mu^2$ of the sojourn time. The experimental results show that the CIs based on the variance estimator $S_*^2(\hat{f}'; \hat{V}_C(w_i))$ have the best performance in terms of the mean and variance of their half-width while having proper coverage, regardless of the point estimator among $\hat{f}_C$, $\bar{f}_B$, or $\bar{f}_J$. This dominance appears to be due to the solid statistical properties of the combined variance parameter estimator in Equations (16) and (31).

Overall, the CIs based on the jackknife point estimator $\bar{f}_J$ and the variance estimator $S_*^2(\hat{f}'; \hat{V}_C(w_i))$ exhibit the best performance in terms of the mean and variance of their half-widths while achieving the nominal coverage.

With regard to future research, we are investigating CIs based on other variance parameter estimation methods, e.g., Cramér–von Mises (CvM) estimators and combinations of STS and CvM estimators (Goldsman et al. [28]). Another interesting problem arises when the first derivative $f'(\mu) = 0$. In this case, one could use a higher-order approximation; see Casella and Berger [9]. We also consider the application of the delta method to the estimation of functions of multivariate steady-state means.

APPENDIX A

ANCILLARY MATERIAL

A.1 Bias and Variance of Point Estimators for an MA(1) Process

This section presents experimental results for the bias and variance of the point estimators of $f(\mu)$ and $|f'(\mu)|$ for an MA(1) process with mean $\mu = 0$ and $\theta = 0.9$. The estimates are based on 1000 independent experiments. We define the bias as $\theta - \mathsf{E}[\hat{\theta}]$, where θ is a parameter and $\hat{\theta}$ is a estimator of θ.

Table 36: Bias and Variance Estimates of the Point Estimators for $f(\mu) = (\mu + 1)^2$ and $|f'(\mu)| = 2|\mu + 1|$ for an MA(1) Process with $\mu = 0$ and $\theta = 0.9$

BIAS	m	$\hat{f}_C$	$\bar{f}_B$	$\bar{f}_J$	$\lvert\hat{f}'_C\rvert$
$b = 32$	128	5.849E-04	-2.626E-02	1.451E-03	1.410E-03
	256	1.750E-03	-1.189E-02	2.190E-03	2.176E-03
	512	-5.854E-05	-6.868E-03	1.611E-04	1.466E-04
$b = 128$	128	-5.854E-05	-2.805E-02	1.618E-04	1.466E-04
	256	-7.436E-04	-1.474E-02	-6.334E-04	-6.336E-04
	512	-4.618E-04	-7.442E-03	-4.068E-04	-4.102E-04
VARIANCE	m	$\hat{f}_C$	$\bar{f}_B$	$\bar{f}_J$	$\lvert\hat{f}'_C\rvert$
$b = 32$	128	3.295E-03	3.322E-03	3.296E-03	3.302E-03
	256	1.693E-03	1.702E-03	1.693E-03	1.698E-03
	512	8.205E-04	8.192E-04	8.206E-04	8.213E-04
$b = 128$	128	8.205E-04	8.240E-04	8.205E-04	8.213E-04
	256	4.403E-04	4.404E-04	4.403E-04	4.400E-04
	512	2.066E-04	2.078E-04	2.066E-04	2.064E-04

Table 37: Bias and Variance Estimates of the Point Estimators for $f(\mu) = 1/(\mu+1)$ and $|f'(\mu)| = 1/(\mu+1)^2$ for an MA(1) Process with $\mu = 0$ and $\theta = 0.9$

BIAS	m	$\hat{f}_C$	$\bar{f}_B$	$\bar{f}_J$	$\lvert\hat{f}'_C\rvert$
$b = 32$	128	-1.535E-03	-3.110E-02	-6.626E-04	-3.908E-03
	256	-1.516E-03	-1.587E-02	-1.073E-03	-3.462E-03
	512	-2.788E-04	-7.248E-03	-5.872E-05	-7.636E-04
$b = 128$	128	-2.788E-04	-3.108E-02	-5.805E-05	-7.636E-04
	256	2.069E-04	-1.442E-02	3.171E-04	3.039E-04
	512	1.535E-04	-6.975E-03	2.085E-04	2.556E-04

VARIANCE	m	$\hat{f}_C$	$\bar{f}_B$	$\bar{f}_J$	$\lvert\hat{f}'_C\rvert$
$b = 32$	128	8.356E-04	1.094E-03	8.310E-04	3.375E-03
	256	4.289E-04	4.842E-04	4.277E-04	1.727E-03
	512	2.061E-04	2.208E-04	2.058E-04	8.268E-04
$b = 128$	128	2.061E-04	2.767E-04	2.058E-04	8.268E-04
	256	1.100E-04	1.256E-04	1.099E-04	4.400E-04
	512	5.153E-05	5.429E-05	5.152E-05	2.060E-04

A.2 Bias and Variance of Point Estimators for an AR(1) Process

This section presents experimental results for the bias and variance of the point estimators of $f(\mu)$ and $|f'(\mu)|$ for an AR(1) process with mean $\mu = 0$ and $\phi = 0.9$. The estimates are based on 1000 independent experiments.

Table 38: Bias and Variance Estimates of the Point Estimators for $f(\mu) = (\mu+1)^2$ and $|f'(\mu)| = 2|\mu+1|$ for an AR(1) Process with $\mu = 0$ and $\phi = 0.9$

BIAS	m	$\hat{f}_C$	$\bar{f}_B$	$\bar{f}_J$	$\lvert\hat{f}'_C\rvert$
$b = 32$	128	-1.015E-03	-1.320E-01	3.212E-03	3.323E-03
	512	-7.898E-04	-3.599E-02	3.455E-04	2.889E-04
	2048	-1.206E-03	-1.019E-02	-9.156E-04	-9.341E-04
$b = 128$	128	-7.898E-04	-1.374E-01	2.861E-04	2.889E-04
	512	-1.206E-03	-3.729E-02	-9.213E-04	-9.341E-04
	2048	-5.561E-04	-9.793E-03	-4.833E-04	-4.853E-04
VARIANCE	m	$\hat{f}_C$	$\bar{f}_B$	$\bar{f}_J$	$\lvert\hat{f}'_C\rvert$
$b = 32$	128	1.730E-02	1.824E-02	1.730E-02	1.736E-02
	512	4.309E-03	4.351E-03	4.311E-03	4.319E-03
	2048	1.088E-03	1.098E-03	1.088E-03	1.086E-03
$b = 128$	128	4.309E-03	4.528E-03	4.310E-03	4.319E-03
	512	1.088E-03	1.113E-03	1.088E-03	1.086E-03
	2048	2.834E-04	2.849E-04	2.834E-04	2.832E-04

Table 39: Bias and Variance Estimates of the Point Estimators for $f(\mu) = 1/(\mu+1)$ and $|f'(\mu)| = 1/(\mu+1)^2$ for an AR(1) Process with $\mu = 0$ and $\phi = 0.9$

BIAS	m	$\hat{f}_C$	$\bar{f}_B$	$\bar{f}_J$	$\lvert\hat{f}'_C\rvert$
$b = 32$	128	-6.089E-03	-1.062E-01	-1.720E-03	-1.676E-02
	512	-1.230E-03	-4.141E-02	-8.539E-05	-3.555E-03
	2048	1.961E-04	-9.033E-03	4.859E-04	1.216E-04
$b = 128$	128	-1.230E-03	2.563E-01	-1.458E-04	-3.555E-03
	512	1.961E-04	-4.077E-02	4.803E-04	1.216E-04
	2048	1.719E-04	-9.327E-03	2.446E-04	2.731E-04
VARIANCE	m	$\hat{f}_C$	$\bar{f}_B$	$\bar{f}_J$	$\lvert\hat{f}'_C\rvert$
$b = 32$	128	4.547E-03	1.003E+01	4.426E-03	1.901E-02
	512	1.095E-03	1.600E-03	1.087E-03	4.431E-03
	2048	2.709E-04	2.901E-04	2.705E-04	1.084E-03
$b = 128$	128	1.095E-03	1.933E+02	1.087E-03	4.431E-03
	512	2.709E-04	3.835E-04	2.704E-04	1.084E-03
	2048	7.074E-05	7.651E-05	7.071E-05	2.829E-04

A.3 Bias and Variance of Point Estimators for an M/M/1 Queueing System

This section presents experimental results for the bias and variance of the point estimators for $f(\mu) = \mu^2$ and $|f'(\mu)| = 2\mu$ in the case of the stationary M/M/1 system in Chapter VI. Recall that μ is the mean customer sojourn time. The estimates are based on 1000 independent experiments.

Table 40: Bias and Variance Estimates of the Point Estimators for $f(\mu) = \mu^2$ and $|f'(\mu)| = 2\mu$ for a Stationary M/M/1 System with Mean Interarrival Time $\alpha = 10$ and Mean Service Time $\beta = 8$

BIAS	m	$\hat{f}_C$	$\bar{f}_B$	$\bar{f}_J$	$\lvert\hat{f}'_C\rvert$
$b = 16$	2048	3.463E+00	-5.252E+01	7.195E+00	1.842E-01
	4096	1.924E+00	-2.767E+01	3.897E+00	9.576E-02
	8192	-4.478E-01	-1.491E+01	5.161E-01	1.399E-02
$b = 32$	2048	1.924E+00	-5.810E+01	3.860E+00	9.576E-02
	4096	-4.478E-01	-3.035E+01	5.168E-01	1.399E-02
	8192	9.120E-01	-1.440E+01	1.406E+00	3.555E-02

VARIANCE	m	$\hat{f}_C$	$\bar{f}_B$	$\bar{f}_J$	$\lvert\hat{f}'_C\rvert$
$b = 16$	2048	2.547E+04	3.292E+04	2.505E+04	1.560E+01
	4096	1.225E+04	1.429E+04	1.213E+04	7.626E+00
	8192	6.474E+03	6.864E+03	6.451E+03	4.033E+00
$b = 32$	2048	1.225E+04	1.627E+04	1.214E+04	7.626E+00
	4096	6.474E+03	7.401E+03	6.448E+03	4.033E+00
	8192	3.262E+03	3.515E+03	3.254E+03	2.041E+00

REFERENCES

[1] C. Alexopoulos and D. Goldsman (2004). "To Batch Or Not to Batch," *ACM Transactions on Modeling and Computer Simulation* **14(1)**, 76–114.

[2] C. Alexopoulos, D. Goldsman, and G. Tokol (2001). "Properties of Batched Quadratic-Form Variance Parameter Estimators for Simulations," *INFORMS Journal on Computing* **13**, 149–156.

[3] C. Alexopoulos and A. F. Seila (1998). *Output Data Analysis*, Chapter 7 in *Handbook of Simulation: Principles, Methodology, Advances, Applications, and Practice*, ed. J. Banks, 225–272, John Wiley & Sons, New York.

[4] T. W. Anderson (1971). *The Statistical Analysis of Time Series*, John Wiley & Sons, New York.

[5] J. Banks (1998). *Handbook of Simulation: Principles, Methodology, Advances, Applications, and Practice*, John Wiley & Sons, New York.

[6] P. Billingsley (1968). *Convergence of Probability Measures*, John Wiley & Sons, New York.

[7] D. P. Bischak (1988). *Weighted Batch Means for Improved Confidence Intervals for Steady-State Processes*, Ph.D. Thesis, Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, Michigan.

[8] N. Blomqvist (1967). "The covariance function of the M/G/1 queueing system," *Skandinavisk Aktuarietidskrift* **50**, 157–174.

[9] G. Casella and R. L. Berger (2001). *Statistical Inference*, Duxbury Press, California.

[10] C. Chien, D. Goldsman, and B. Melamed (1997). "Large–Sample Results for Batch Means," *Management Science* **43**, 1288–1295.

[11] R. W. Conway (1963). "Some Tactical Problems in Digital Simulation," *Management Science* **10**, 47–61.

[12] H. Damerdji (1995). "Mean–Square Consistency of the Variance Estimator in Steady-State Simulation Output Analysis," *Operations Research* **43**, 282–291.

[13] A. C. Davison and D. V. Hinkley (1997). *Bootstrap Methods and their Application*, Cambridge University Press, New York.

[14] J. Durbin (1959). "A Note on the Application of Quenouille's Method of Bias Reduction to the Estimation of Ratios," *Biometrika* **46**, 477–480.

[15] J. Durbin (1973). *Distribution Theory for Tests Based on the Sample Distribution Function*, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania.

[16] B. Efron (1979). "Bootstrap Methods: Another Look at The Jackknife," *Annals of Statistics* **7**, 1–26.

[17] B. Efron (1982). *The Jackknife, the Bootstrap, and Other Resampling Plans*, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania.

[18] B. Efron (1987). "Better Bootstrap Confidence Intervals (with discussion)," *Journal of American Statistical Association* **82**, 171–200.

[19] B. Efron and C. Stein (1981). "The Jackknife Estimate of Variance," *Annals of Statistics* **9**, 586–596.

[20] B. Efron and R.J. Tibshirani (1993). *An Introduction to the Bootstrap.* Chapman and Hall/CRC Press, New York.

[21] T. S. Ferguson (1996). *A Course in Large Sample Theory*. Chapman and Hall/CRC Press, New York.

[22] G. S. Fishman (1978). "Grouping Observations in Digital Simulation," *Management Science* **24**, 510–521.

[23] G. S. Fishman (1996). *Monte Carlo: Concepts, Algorithms and Applications*, Springer–Verlag, New York.

[24] G. S. Fishman (2001). *Discrete-Event Simulation: Modeling, Programming, and Analysis*, Springer–Verlag, New York.

[25] R. D. Foley and D. Goldsman (2000). "Confidence Intervals Using Orthonormally Weighted Standardized Time Series," *ACM Transactions on Modeling and Computer Simulation* **9**, 297–325.

[26] P. W. Glynn and D. L. Iglehart (1990). "Simulation Output Analysis Using Standardized Time Series," *Mathematics of Operations Research* **15**, 1–16.

[27] P. W. Glynn and W. Whitt (1991). "Estimating the Asymptotic Variance with Batch Means," *Operations Research Letters* **10**, 431–435.

[28] D. Goldsman, K. Kang, S. Kim, A. F. Seila, and G. Tokol (2003). "Combining Standardized Time Series Area and Cramér–von Mises Variance Estimators," Technical Report, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia.

[29] D. Goldsman, K. Kang, and A. F. Seila (1999). "Cramér–von Mises Variance Estimators for Simulations," *Operations Research* **47**, 299–309.

[30] D. Goldsman and M. S. Meketon (1986). "A Comparison of Several Variance Estimators," Technical Report, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia.

[31] D. Goldsman, M. S. Meketon, and L. W. Schruben (1990). "Properties of Standardized Time Series Weighted Area Variance Estimators," *Management Science* **36**, 602–612.

[32] D. Goldsman and L. W. Schruben (1984). "Asymptotic Properties of Some Confidence Interval Estimators for Simulation Output," *Management Science* **30**, 1217–1225.

[33] D. Goldsman and L. W. Schruben (1990). "New Confidence Interval Estimators Using Standardized Time Series," *Management Science* **36**, 393–397.

[34] D. Goldsman, L. W. Schruben, and J. J. Swain (1994). "Tests for Transient Means in Simulated Time Series," *Naval Research Logistics* **41**, 171–187.

[35] J. A. Hartigan (1969). "Using Subsamples Values as Typical Value," *Journal of American Statistical Association* **64**, 1303–1317.

[36] V. G. Kulkarni (1996). *Modeling and Analysis of Stochastic System*, Chapman and Hall/CRC Press, New York.

[37] H. R. Künsch (1989). "The Jackknife and the Bootstrap for General Stationary Observations," *Annals of Statistics* **17**, 1217–1241.

[38] A. M. Law and J. S. Carson (1979). "A Sequential Procedure for Determining the Length of a Steady-State Simulation," *Operations Research* **27**, 1011–1025.

[39] A. M. Law and W. D. Kelton (2000). *Simulation Modeling and Analysis*, 3rd Ed., McGraw-Hill, New York.

[40] E. L. Lehmann (1998). *Elements of Large–Sample Theory*, Springer-Verlag, New York.

[41] E. L. Lehmann, S. Fienberg, and G. Casella (1998). *Theory of Point Estimation*, 2nd Ed., Springer–Verlag, New York.

[42] D. V. Lindley (1952). "The Theory of Queues with a Single Server," *Proceedings of the Cambridge Philosophical Society* **48**, 277–289.

[43] R. Y. Liu and K. Singh (1992). "Moving Blocks Jackknife and Bootstrap Capture Weak Dependence," In *Exploring the Limit of Bootstrap*, eds. R. LePage and L. Billard, 168–177, John Wiley & Sons, New York.

[44] W. S. Marshall IV (1999). *Robust Variance Estimation for Ranking and Selection*, Ph.D. Thesis, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia.

[45] W. S. Marshall IV, S. J. Lee, D. Goldsman, and B. Y. Chang (2004). "Cramér–von Mises Estimators Don't Have to Be Negative," Working Paper, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia.

[46] M. S. Meketon and B. Schmeiser (1984). "Overlapping Batch Means: Something for Nothing?," In *Proceedings of the 1984 Winter Simulation Conference*, eds. S. Sheppard, U. W. Pooch, and C. D. Pegden, 227–230, Institute of Electrical and Electronics Engineers, Piscataway, New Jersey.

[47] R. G. Miller (1964). "A Trustworthy Jackknife," *Annals of Mathematical Statistics* **35**, 1594–1605.

[48] R. G. Miller (1968). "Jackknifing Variances," *Annals of Mathematical Statistics* **39**, 567–582.

[49] R. G. Miller (1974). "The Jackknife – A Review," *Biometrika* **61**, 1–15.

[50] A. M. Mood, F. A. Graybill, and D. C. Boes (1974). *Introduction to the Theory of Statistics*, McGraw–Hill, New York.

[51] D. F. Muñoz and P. W. Glynn (1997). "A Batch Means Methodology for Estimation of a Nonlinear Function of a Steady-State Mean," *Management Science* **43**, 1121–1135.

[52] D. Park and T. R. Willemain (1999). "The Threshold Bootstrap and Threshold Jackknife," *Computational Statistics and Data Analysis* **31**, 187–202.

[53] J. K. Patel and C. B. Read (1996). *Handbook of the Normal Distribution*, 2nd Ed., Marcel Dekker, New York.

[54] A. C. Pedrosa and B. W. Schmeiser (1994). "Estimating the variance of the sample mean: Optimal batch size estimation and 1–2–1 overlapping batch means," Technical Report SMS94–3, School of Industrial Engineering, Purdue University, West Lafayette, Indiana.

[55] D. N. Politis and J. P. Romano (1994). "The Stationary Bootstrap," *Journal of the American Statistical Association* **89**, 1303–1313.

[56] M. H. Quenouille (1949). "Approximate Tests of Correlation in Time-Series," *Journal of the Royal Statistical Society* B **11**, 68–84.

[57] M. H. Quenouille (1956). "Notes on Bias in Estimation," *Biometrika* **43**, 353–360.

[58] J. N. K. Rao and C. F. J. Wu (1988). "Resampling Inference with Complex Survey Data," *Journal of American Statistical Association* **83**, 231–241.

[59] R. G. Sargent, K. Kang, and D. Goldsman (1992). "An Investigation of Finite-Sample Behavior of Confidence Interval Estimators," *Operations Research* **40**, 898–913.

[60] B. W. Schmeiser (1982). "Batch Size Effects in the Analysis of Simulation Output," *Operations Research* **30**, 556–568.

[61] L. W. Schruben (1983). "Confidence Interval Estimation using Standardized Time Series," *Operations Research* **31**, 1090–1108.

[62] L. W. Schruben, H. Singh, and L. Tierney (1983). "Optimal Tests for Initialization Bias in Simulation Output," *Operations Research* **31**, 1167–1178.

[63] R. J. Serfling (1980). *Approximation Theorems of Mathematical Statistics*, John Wiley & Sons, New York.

[64] J. Shao and D. Tu (1996). *The Jackknife and Bootstrap*, Springer–Verlag, New York.

[65] R. R. Sitter (1992). "A Resampling Procedure for Complex Survey Data," *Journal of American Statistical Association* **87**, 755–765.

[66] W.-M. T. Song and B. W. Schmeiser (1993). "Variance of the Sample Mean: Properties and Graphs of Quadratic-form estimators," *Operations Research* **41**, 501–517.

[67] W.-M. T. Song and B. W. Schmeiser (1995). "Optimal Mean–Squared–Error Batch Sizes," *Management Science* **41**, 110–123.

[68] N. M. Steiger and J. R. Wilson (2002). "An Improved Batch Means Procedure for Simulation Output Analysis," *Management Science* **48**, 1569–1586.

[69] J. W. Tukey (1958). "Bias and Confidence in Not–Quite Large Samples (Abstract)," *Annals of Mathematical Statistics* **29**, 614.

[70] J. W. Tukey (1962). "Data Analysis and Behavioral Science," Unpublished manuscript.

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