EFFICIENT CONFIDENCE INTERVAL METHODOLOGIES FOR THE NONCENTRALITY PARAMETERS OF NONCENTRAL $T-$DISTRIBUTIONS

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EFFICIENT CONFIDENCE INTERVAL METHODOLOGIES FOR THE NONCENTRALITY PARAMETERS OF NONCENTRAL $T-$DISTRIBUTIONS

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To my wife, Jiyoung,

and my daughter, Suhyoon

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SUMMARY

The problem of constructing a confidence interval for the noncentrality parameter of a noncentral t-distribution based upon one observation from the distribution is an interesting problem with important applications. A general theoretical approach to the problem is provided by the specification and inversion of acceptance sets for each possible value of the noncentrality parameter. The standard method is based upon the arbitrary assignment of equal tail probabilities to the acceptance set, while the choices of the shortest possible acceptance sets and UMP unbiased acceptance sets provide even worse confidence intervals, which means that since the standard confidence intervals are uniformly shorter than those of UMPU method, the standard method are "biased". However, with the correct choice of acceptance sets it is possible to provide an improvement in terms of confidence interval length over the confidence intervals provided by the standard method for all values of observation.

The problem of testing the equality of the noncentrality parameters of two noncentral t-distributions is considered, which naturally arises from the comparison of two signal-to-noise ratios for simple linear regression models. A test procedure is derived that is guaranteed to maintain type I error while having only minimal amounts of conservativeness, and comparisons are made with several other approaches to this problem based on variance stabilizing transformations. In summary, these simulations confirm that the new procedure has type I error probabilities that are guaranteed not to exceed the nominal level, and they demonstrate that the new procedure has size and power levels that compare well with the procedures based on variance stabilizing transformations.

CHAPTER I

INTRODUCTION

1.1 Motivation and Literature review

The first topic of this thesis considers the problem of constructing two-sided efficient confidence interval methodologies for noncentrality parameter of noncentral $t-$distribution, which arises mainly in confidence interval of coefficient of variation, confidence interval of signal-to-noise ratio of simple regression and of process capability indices with only upper specification limit C_{pku} and with only lower specification limit C_{pkl}.

Suppose that the lower and upper tail critical points of a noncentral t-distribution with degrees of freedom ν and noncentrality parameter δ, $t_\nu(\delta)$, are denoted by $t^l_{\nu,\alpha}(\delta)$ and $t^u_{\nu,\alpha}(\delta)$ satisfying

$$\mathbb{P}(t_\nu(\delta) \leq t^l_{\nu,\alpha}(\delta)) = \mathbb{P}(t_\nu(\delta) \geq t^u_{\nu,\alpha}(\delta)) = \alpha$$

Then as explained on page 352 of Lehmann [10], for an observed value t one-sided confidence intervals for δ with a confidence level of exactly $1-\alpha$ can be obtained as

$$\{\, \delta \,:\, t^l_{\nu,\alpha}(\delta) \leq t \,\}$$

or

$$\{\, \delta \,:\, t \leq t^u_{\nu,\alpha}(\delta) \,\}$$

However, in chapter 3 the construction of efficient two-sided confidence interval is considered. The standard approach to this problem is described on page 510 of

Johnson *et. al.* [8], and it employs critical points for equal tail probabilities of $\alpha/2$. Specially, the standard confidence interval $C^s_{\nu,\alpha}(t)$, which is based on "equal tail probabilities", for δ is constructed as

$$C^s_{\nu,\alpha}(t) = \{\delta \,:\, t^l_{\nu,\alpha/2}(\delta) \le t \le t^u_{\nu,\alpha/2}(\delta)\} \tag{1}$$

which has a confidence level of exactly $1-\alpha$. Nevertheless, in spite of this method's widespread adoption, the use of critical points with equal tail probabilities of $\alpha/2$ is quite arbitrary, and the question arises as to whether improved confidence interval methodologies are possible. In fact, the duality property of confidence interval $C(t)$ and acceptance set $A(\delta)$, "$\delta \in C(t) \Leftrightarrow t \in A(\delta)$", leads us to the fact that the expected volume of confidence sets is equal to integrated sum of its false coverage probability (See Pratt [17] and Casella and Berger [3]). Thus the smaller expected length of confidence interval comes from better test for prefixed confidence level $1-\alpha$ and vice versa because the unknown parameter δ can be thought of as being uniformly distributed.

For example, the paramount work on the confidence intervals for the variance σ^2 of normally distributed data goes back as far as Tate and Klett's article [19]. When only S^2 is observed, they found the conditions for optimal confidence interval I_{op} for σ^2 satisfying

$$I_{op} = \left(\tfrac{1}{b_\nu}\textstyle\sum(X_i-\bar{X})^2, \quad \tfrac{1}{a_\nu}\sum(X_i-\bar{X})^2\right)$$
$$f_{\nu+4}(a_\nu) = f_{\nu+4}(b_\nu) \quad \text{and} \quad \textstyle\int_{a_\nu}^{b_\nu} f_\nu(x)dx = 1-\alpha$$

and conditions for uniformly most powerful unbiased confidence interval I_{umpu} for σ^2

satisfying

$$I_{umpu} = \left(\tfrac{1}{b_\nu}\sum(X_i - \bar{X})^2, \quad \tfrac{1}{a_\nu}\sum(X_i - \bar{X})^2\right)$$

$$f_{\nu+2}(a_\nu) = f_{\nu+2}(b_\nu), \quad \text{and} \quad \int_{a_\nu}^{b_\nu} f_\nu(x)dx = 1 - \alpha$$

where, $X_i,\ i = 1, \ldots, n$ are random sample from $N(\mu, \sigma^2)$, $\nu = n - 1$ and $f_\nu(\cdot)$ is pdf of chi-square distribution with degrees of freedom ν. As an example, the length of standard confidence interval (2.382, 23.031) from 6 observations is 20.649 while the length of I_{op}=(1.395, 16.546) is 15.151 and the length of I_{umpu} = (2.119, 19.466) is 17.347. (See Tate and Klett [19]).

Indeed, since the noncentral t-distribution is a skewed distribution, it may appear obvious that using equal tail probabilities cannot be optimal. In this thesis it is shown that there exist more efficient confidence interval methodologies that are uniformly better than this standard method in the sense that they provide confidence intervals with smaller lengths for all values of t. Computer programs are available to implement the new methodologies, and their development also provides some interesting theoretical results.

It should be noted that the advantages of the new methodologies over the standard method are paramount when the degrees of freedom ν are small, since this is when the skewness of the noncentral t-distribution is largest. When the degrees of freedom become large, the noncentral t-distribution is close to being symmetric, and in this case little improvement is possible over the standard method.

Most of the previous work on the noncentrality parameter of a noncentral t-distribution, going back as far as McKay [11] for example, is motivated by its connection with the coefficient of variation for normally distributed data. More recently, the

standard method (1) has been criticized as being too computationally cumbersome, and research has been directed towards constructing approximate confidence intervals for the coefficient of variation (see, for example, Vangel [20] and Wong & Wu [21]). However, the methodologies proposed in chapter 3 can provide efficient confidence intervals that are shorter than those provided by the standard method.

This thesis also considers the problem of testing equality of two noncentrality parameters of noncentral $t-$distribution in chapter 4, which arises in testing equality of two signal-to-noise ratios. For simple regression model, the natural estimator of Taguchi's signal-to-noise ratio (SN ratio) $\eta = \beta^2/\sigma^2$ is $\hat{\eta} = \hat{\beta}^2/\hat{\sigma}^2$, and the random variable $t = \sqrt{S_{xx}}\hat{\beta}/\hat{\sigma}$ is distributed as noncentral $t-$distributions with $n-2$ degrees of freedom and with noncentrality parameter $\sqrt{S_{xx}}\beta/\sigma$. Similarly, the variance ratio $F = MSR/MSE$ in the ANOVA table is distributed as noncentral $F-$distributions with 1 and $n-2$ degrees of freedom and with noncentrality parameter $S_{xx}\beta^2/\sigma^2$. Consequently, the problem of testing the equality of the two signal-to-noise ratios is equivalent to testing the equality of the two noncentrality parameters of either $t-$distributions or $F-$distributions with identical degrees of freedom.

In studying performance of measuring instruments, Taguchi's signal-to-noise ratio η is a useful criterion, and Taguchi [18] proposed the well-known two-step procedure for improving SN ratio. However, even when his procedure is successfully applied or when we have a problem of deciding which instrument is better, we couldn't have any methodologies for testing equality of two or more SN ratios until Miwa [12] and Nagata *et. al* [14]'s article were published. Let's consider the example in Nagata *et.*

Table 1: Percentage of cadmium permeability

		M_1	M_2	M_3	M_4	M_5
A_1	R_1	12.0	23.0	33.0	42.5	52.0
	R_2	11.0	22.5	34.0	44.0	54.0
A_2	R_1	3.5	7.0	10.5	14.5	18.0
	R_2	4.0	7.5	12.0	15.5	19.5

al [14]'s article, which will be considered in section 2.2 in detail. The data were measured in order to compare the SN ratios of two methods A_1 and A_2 for microanalysis of cadmium. Each value of the data set is percentage of cadmium permeability, and is tabulated in Table 1. M is a signal factor and M_k means the cadmium content of a sample in k ppm ($k = 1, \cdots, 5$). R_1 and R_2 which are considered as replications represent two analysts.

This testing problem was first studied by Miwa [12], and he derived a simple test statistic based on a variance stabilizing transformation of noncentral $F-$distributions and he proposed its improved test statistic using the Cornish-Fisher expansion. In his article, Monte Carlo simulation was employed to identify the sizes of two proposed test statistics, it is shown that the two test statistics except for small values of noncentrality parameter are a little liberal but that the statistics for small noncentrality parameters are very conservative due to unstability of the transformation. After Miwa's paper, Nagata *et. al* [14] proposed a simple test statistics based on variance stabilizing transformation of noncentral $t-$distributions by Laubscher [9] for two sample or more than two sample problems, and they showed that their test statistic was conservative for small noncentrality parameter. However, simulations of Appendix C show that their test statistic becomes more conservative as noncentrality parameter

increase.

The Miwa's and Nagata *et. al*'s test statistics were simple and convenient to be implemented for testing the equality of two or more SN ratios, but their statistics were more or less conservative or liberal due to instability of transformation. In chapter 4, a new test procedure is derived that is guaranteed to maintain type I error while having only minimal amounts of conservativeness, and comparisons are made with several other approaches to this problem based on variance stabilizing transformations. The new procedure derived in chapter 4 is shown to have good properties and will be useful for practitioners.

1.2 Thesis Outline

This thesis is organized as follows. In chapter 2, review on general methodology for confidence interval and the relationship between the length of confidence interval and power of test are presented in section 2.1.1 and 2.1.2 respectively. Section 2.2 considers literature review on methodologies for testing equality of two or more signal-to-noise ratios, and section 2.3 considers review and discussion on confidence interval methodologies for coefficient of variation. And some properties of noncentral $t-$distributions are discussed in section 2.4.

Chapter 3 considers three approaches for constructing efficient confidence intervals for noncentrality parameter δ of noncentral $t-$distributions. It is shown that the shortest possible acceptance sets methodology and UMP unbiased acceptance sets methodology are not efficient in section 3.2 and 3.3, while in section 3.4 new methodologies that are uniformly better than the standard method are described.

Chapter 4 considers the problem of testing the equality of the noncentrality parameters of two noncentral t-distributions is considered. Section 4.2 describes the new procedure, while section 4.3 presents simulations results comparing the new procedure with the other procedures in terms of type I error and power.

Finally, I summarize this work and review the implication of this work in chapter 5 and we will discuss the development of this work to other research areas such as uniformly shorter confidence interval for coefficient of variation of normal data and interval for noncentrality parameter of noncentral F distribution.

CHAPTER II

BACKGROUND INFORMATION

In section 2.1.1, we review how to construct the confidence set based on acceptance regions $A_{\nu,\alpha}(\delta)$, and section 2.1.2 shows the relationship between expected length of confidence interval and power of test, and considers the optimal, UMP unbiased and standard confidence interval of variance of normal distribution when only S^2 is observed. Section 2.2 introduces the problem of testing the equality of signal-to-noise ratios as the motivation of this thesis, and the section 2.3 considers some approximate confidence interval methodologies for coefficient of variation. Finally, we review the properties of noncentral $t-$distribution, and propose a new measure of skewness of noncentral $t-$distribution, which is fundamental to constructing efficient confidence interval of noncentral $t-$distribution.

2.1 General methodology for confidence intervals

Section 2.1.1 explains how confidence sets are in general obtained (see Lehamann [10]), and from the duality property between tests and confidence sets this concept is used to derive unbiased two-sided confidence sets with confidence level $1-\alpha$ if possible. Among a considerable number of papers dealing with these confidence intervals problems, section 2.1.2 reviews two interesting and fundamental articles related to the thesis. Pratt [17] obtained a different type of a confidence interval by minimizing the

expected length in the sense that this expected length is directly related to the type II error. As an example of the efficient confidence interval for skewed distributions, Tate & Klett [19] found the conditions for optimal confidence intervals for variance σ^2 of normally distributed data based only on observation of S^2, which showed that standard confidence intervals for skewed random variables is far from our satisfaction.

2.1.1 General Methodology for confidence intervals

The confidence interval for unknown parameter θ can be thought of as a family of tests of the hypothesis H_0 against H_A (see the Lehmann's book [10]). The one-sided confidence interval commonly used for exponential families, which corresponds to one-sided test for given size α, is unbeatable without any suspect. But for random variables of which skewness is not 0, the standard two-sided confidence interval, which is so called "equal tail probabilities", could be far from our satisfaction (for example, see the optimal confidence interval for variance σ^2 in Tate and Klett [19]). In this thesis, two-sided confidence intervals are considered and the general methodology for confidence intervals is as follows.

For each $\delta \in \Re$, let $A_{\nu,\alpha}(\delta)$ denote the acceptance set of confidence level $1-\alpha$ test of the hypothesis $H_0 : \theta = \delta$ versus $H_A : \theta \neq \delta$. And for each sample point t, let $C_{\nu,\alpha}(t)$ denote the set of parameter values

$$C_{\nu,\alpha}(t) \;=\; \{\, \delta : t \in A_{\nu,\alpha}(\delta), \quad \delta \in \Re \,\} = \{\, \delta : l_{\nu,\alpha}(\delta) \leq t \leq u_{\nu,\alpha}(\delta) \quad \delta \in \Re \,\} \qquad (2)$$

Then $C_{\nu,\alpha}(t)$ is a family of confidence sets for δ at confidence level $1-\alpha$ since we

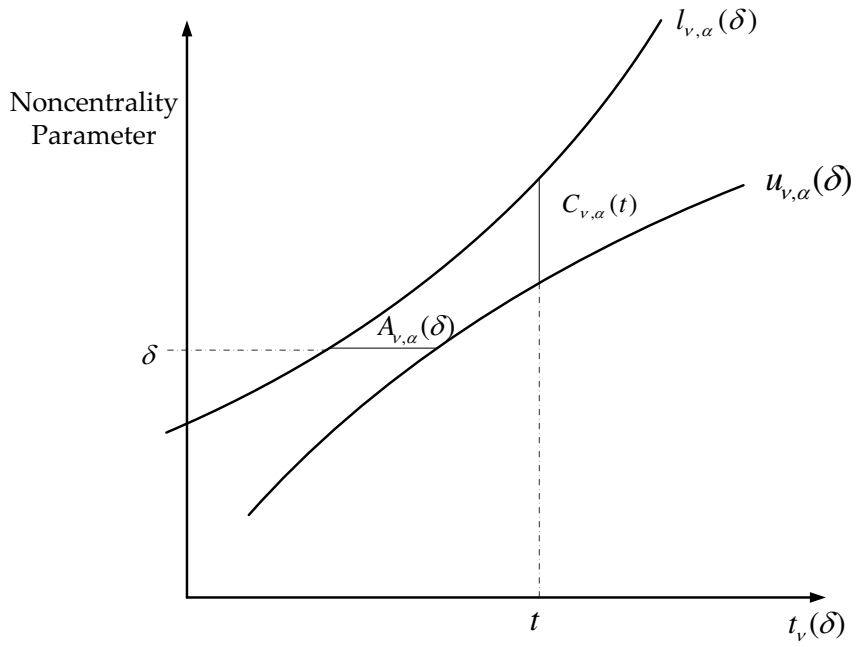

Figure 1: Graphical representation of general methodology for confidence set

have that

$$\delta \in C_{\nu,\alpha}(t) \quad \text{ if and only if } \quad t \in A_{\nu,\alpha}(\delta) \tag{3}$$

Hence we have that

$$\mathbb{P}_\delta\{\, \delta \in C_{\nu,\alpha}(t) \,\} \geq 1 - \alpha \quad \text{ for all } \delta \tag{4}$$

Thus any family of confidence level $1-\alpha$ acceptance sets, through the correspondence (3), leads to a family of confidence sets at confidence level $1 - \alpha$.

Conversely, given any class of confidence sets $C_{\nu,\alpha}(t)$ satisfying (4), let

$$A_{\nu,\alpha}(\delta) \,=\, \{\, t \,:\, \delta \in C_{\nu,\alpha}(t) \,\} \tag{5}$$

Then the sets $A_{\nu,\alpha}(\delta)$ are confidence level $1 - \alpha$ acceptance sets for testing the hypotheses $H_0 : \theta = \delta$ versus $H_A : \theta \neq \delta$, and the confidence sets $C_{\nu,\alpha}(t)$ show whether for the particular t observed the hypothesis $\theta = \delta$ is accepted or rejected for each δ

at confidence level $1 - \alpha$.

The graphical representation is made in Figure 1. From two boundaries $t = l_{\nu,\alpha}(\delta)$ and $t = u_{\nu,\alpha}(\delta)$ of the acceptance set $A_{\nu,\alpha}(\delta)$, one can obtain the confidence set $C_{\nu,\alpha}(t)$ as the interval of δ for which $l_{\nu,\alpha}(\delta) \leq t \leq u_{\nu,\alpha}(\delta)$ for each observation t.

2.1.2 Literature review on confidence intervals

2.1.2.1 The Length of confidence intervals and Power of tests

To measure the desirability of a confidence interval procedure, one would like to measure in some way the extent to which the interval includes false values. Since at most one point of the interval is the true value, one natural measure is the length of the interval(if it is fixed) or the expected length(if it is random; assume that both end-points are finite). That is, the expected length of the interval is a measure of the "average extent" of the false values included. There is another approach to the problem. A natural measure of the extent to which the confidence interval procedure includes a particular false value is the probability of including that particular value. To "average" this over all false value, one might simply integrate it over the false value. This gives an apparently different measure of the "average extent" of the false values included.

It turns out, however, that the two measures are equal. To be more specific, let $L \leq \theta \leq U$ be a confidence interval for a parameter θ. The expected length of the interval is $\mathbb{E}\{U - L\}$. The probability that the interval includes the false value θ is $\mathbb{P}\{L \leq \theta \leq U\}$. The fact stated above is that integrating this probability over θ

(omitting the true value θ_0) gives exactly the expected length

$$\mathbb{E}\{U - L\} = \int_{\theta \neq \theta_0} \mathbb{P}\{L \leq \theta \leq U\} d\theta \tag{6}$$

Let X be the observed random variable, and suppose $C(x)$ is a confidence set for a parameter θ. Suppose $\mu(S)$ is a measure of the size of the set C. Then we have that

$$\mu(C) \; = \; \int_{\theta \in C} d\mu(\theta) \tag{7}$$

And it is not difficult to show (see the Pratt [17]) that

$$\mathbb{E}\{\mu(C)\} \; = \; \int_{\theta \neq \theta_0} \mathbb{P}\{\theta \in C(X)\} d\mu(\theta) \tag{8}$$

$$= \int_{\theta \neq \theta_0} \mathbb{P}\{X \in A(\theta)\} d\mu(\theta) \tag{9}$$

If θ_0 is the true value of θ, this says the expected size of confidence set is the probability that $C(X)$ includes θ integrated over the false values of θ. So, if there is an "optimum" procedure as regards including false values, it is also "optimum" as regards expected length, and vice versa. And the probability that the confidence set $C(X)$ will include the false value θ equals the type II error probability of the associated test of null hypothesis that θ is the true value. Therefore, the most powerful test leads to the confidence set with minimum expected size because it minimizes the integrand of equation (8) at each θ. Moreover, if the uniformly most powerful unbiased (UMPU) test or uniformly most powerful invariant (UMPI) test are employed, the associated confidence sets will uniformly minimize the probability of including false values and the expected length of size of confidence sets among unbiased procedure or invariant procedure respectively.

The average expected size is minimized by the confidence procedure associated with a family of test $A(\theta)$ such that, for each θ, among tests of the null hypothesis that θ is true value, $A(\theta)$ is a most powerful test against the alternative; that is, the acceptance region $A(\theta)$ minimizes

$$\mathbb{P}\{X \in A(\theta)\} = \int \mathbb{P}_\theta\{X \in A(\theta)\} d\nu(\theta)$$

where $\nu(\cdot)$ is distribution of θ.

For example, Pratt considered the confidence interval for mean μ of a normal distribution with variance 1 . The standard $(1-\alpha)100\%$ confidence interval commonly used for the mean μ is

$$C^s(x) \;=\; x - z_{\alpha/2} \;\leq\; \mu \;\leq\; x + z_{\alpha/2} \tag{10}$$

where, $z_{\alpha/2}$ is the upper $\alpha/2$ critical point of a standard normal distribution. The length of this interval is constant as $2z_{\alpha/2}$ for all observation $x \in \mathbb{R}$.

The confidence interval with minimum expected length by Neyman-Pearson lemma for μ, when true μ is 0, is

$$\theta \in [\,\min\{0,\, X - z_\alpha\},\, \max\{0,\, X + z_\alpha\}\,] \tag{11}$$

because by Neyman-Pearson lemma, we have that most powerful test of null hypothesis θ against $\theta = 0$ accepts θ for

$$\begin{aligned} A(\theta) \;&= X \leq \theta + z_\alpha \quad \text{if } \theta < 0 \\ A(\theta) \;&= X \geq \theta - z_\alpha \quad \text{if } \theta > 0 \end{aligned} \tag{12}$$

Of course, if true μ is not 0, the interval (11) will not have minimum expected length any more. Even though the interval (11) of the observation x, which is far

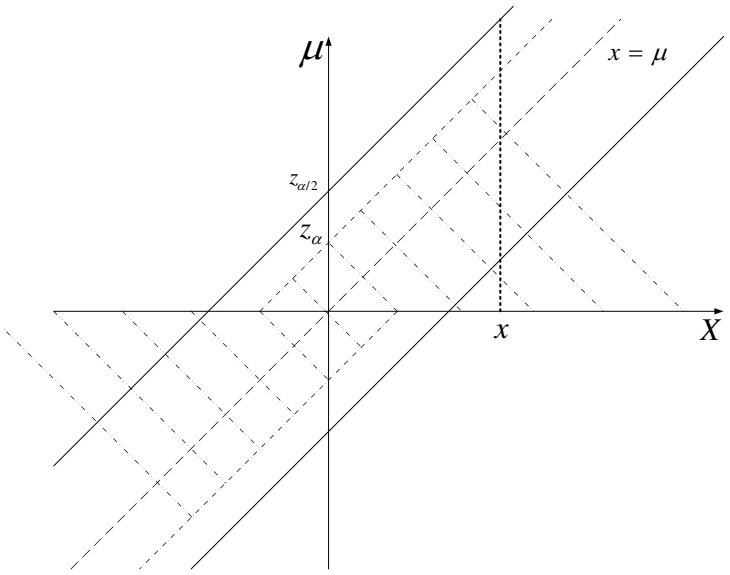

Figure 2: A comparison of between standard confidence interval and confidence interval for mean μ by UMP test under $H_0 : \mu = 0$

from 0 as in the Figure 2, has longer length of confidence interval than the standard interval (10), this expected length of confidence interval (11) is minimum only under assumption that the true μ is 0. However, since the true value μ is arbitrary, the expected length of interval (10) for arbitrary μ is unbeatable.

2.1.2.2 Optimal confidence intervals for variance σ^2

As an example in which the customary procedure for confidence interval for σ^2, so called "equal tails" intervals, is far from satisfactory, Tate and Klett [19] showed the conditions for optimal confidence intervals for the variance σ^2 of a normal distribution as follows. Suppose that chi-square density for ν degrees of freedom

$$f_\nu(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, \qquad x > 0$$

Assume at the outset a restriction to intervals of the form

$$I = \left(\frac{1}{b_\nu} \sum (X_i - \bar{X})^2,\ \frac{1}{a_\nu} \sum (X_i - \bar{X})^2 \right) \tag{13}$$

Then a confidence interval of type (13) which is to have minimum length and confidence level $1 - \alpha$ can be obtained by employing Lagrangian multiplier, and resulting conditions for a_ν and b_ν can be written as

$$\begin{aligned} f_{\nu+4}(a_\nu) &= f_{\nu+4}(b_\nu) \\ \int_{a_\nu}^{b_\nu} f_\nu(x)dx &= 1 - \alpha \end{aligned} \tag{14}$$

The conditions for shortest unbiased confidence intervals by Neyman and Pearson [15] are

$$\begin{aligned} f_{\nu+2}(a_\nu) &= f_{\nu+2}(b_\nu) \\ \int_{a_\nu}^{b_\nu} f_\nu(x)dx &= 1 - \alpha \end{aligned} \tag{15}$$

The standard confidence interval is given by

$$\begin{aligned} \int_{b_\nu}^{\infty} f_\nu(x)dx &= \alpha/2 \\ \int_{a_\nu}^{b_\nu} f_\nu(x)dx &= 1 - \alpha \end{aligned} \tag{16}$$

The values of a_ν and b_ν are tabulated in their article [19], and the comparisons of the intervals reviewed above at $\nu = 5$ and $\nu = 25$ are made in the Figure 3 in which the vertical lengths for given observation $S^2 = \sum (X_i - \bar{X})^2$ are the lengths of confidence intervals. As we know that the skewness of chi-square χ^2_ν distributions decreases in ν, the difference of lengths among the above intervals gets smaller as the degrees of freedom ν increases.

Under the assumption that the only S^2 is observable, the shortest confidence interval calculated by (14) is unbeatable, but Cohen [4] was able to construct improved

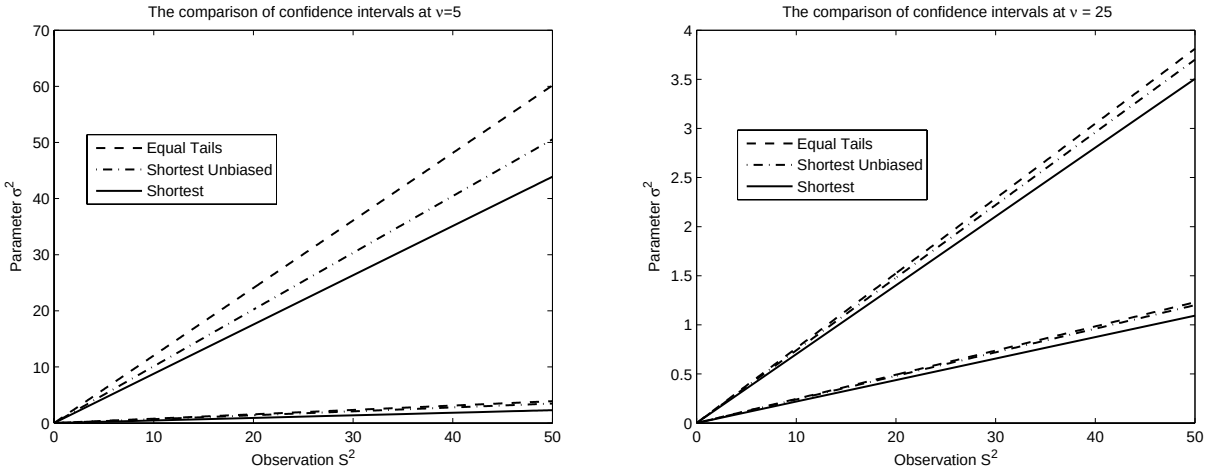

Figure 3: Comparisons of shortest confidence intervals, UMP unbiased confidence intervals and standard confidence intervals at $\nu = 5$ (left plot) and $\nu = 25$ (right plot), where $S^2 = \sum(X_i - \bar{X})^2$

confidence intervals adapting Brown's estimator [2] using the mean $\bar{X}$ and sample variance S^2 simultaneously.

2.2 Literature review on testing the equality of SN ratios

For an example of testing of the equality of signal-to-noise ratios, consider many measuring instruments utilizing a linear calibration procedure. With known input signals (standards) $x_1, x_2, \ldots, x_n$, the responses

$$y_i = \alpha + \beta x_i + \epsilon_i, \quad \text{where, } \epsilon_i \text{ i.i.d } N(0, \sigma^2) \quad i = 1, 2, \ldots, n \tag{17}$$

are observed.

In Taguchi's terminology, this ratio $\eta = \beta^2/\sigma^2$ is called the signal-to-noise ratio (SN ratio) for instrumentation. The data (17) follow the simple regression model and its ANOVA is given in Table 2. The variance ratio F is distributed as a noncentral F variables with $\nu_1 = 1$ and $\nu_2 = n - 2$ degrees of freedom and its noncentrality

Table 2: ANOVA table of a simple regression

Source	*df*	SS	MS	F	$\mathbb{E}[MS]$
Regression	1	SSR	MSR	$F = MSR/MSE$	$\sigma^2 + S_{xx}\beta^2$
Error	$n-2$	SSE	MSE		σ^2

parameter is

$$\lambda = \frac{\beta^2 S_{xx}}{\sigma^2} \tag{18}$$

Thus the noncentrality parameter λ is a multiple of the SN ratio $\eta = \beta^2/\sigma^2$, and the S_{xx} is common factor among the instruments.

And for an another example, let's consider problem for maximizing the *SN ratio* which has been developed in parameter design for simple response systems. Following two-step procedure for parameter design optimization is proposed by Taguchi.(see p.504 Wu and Hamada [22] and Taguchi [18])

Taguchi's Two-Step Procedure for Single-Response Systems

1. Select the levels of the significant control factors in the dynamical SN ratio model to maximize $\hat{\beta}^2/\hat{\sigma}^2$.

2. Select the level of an adjustment factor to bring the slope on target.

But the methodology for testing equality of two or more *SN ratios* is not given until Miwa's article [12] was published. In his article, the transformation of noncentral F variables to standard normal variable enables the statistician to test the equality of two or more SN ratios. After his article, Nagata *et al* used the variance stabilizing

transformation of noncentral t variables which converges to standard normal variables and their test statistic can be used for testing the equality of SN ratios.

2.2.1 Methodology based on transformation of noncentral F distributions

Let two measuring instruments be characterized by their SN ratios $\eta_1 = \beta_1^2/\sigma^2$ and $\eta_2 = \beta_2^2/\sigma^2$. We can compare them by carrying out a simple regression experiment for each of them with common standard inputs $x_1, x_2, \ldots, x_n$ and comparing the noncentrality parameters $\lambda_1 = \eta_1 S_{xx}$ and $\lambda_2 = \eta_2 S_{xx}$.

With known S_{xx}, statistical inference on the SN ratios reduces to that on noncentrality parameters. The statistical inference on noncentrality parameters is studied by means of a variance stabilizing transformation of a noncentral F statistic.

Noncentral F distributions with ν_1, ν_2 degrees of freedom and noncentrality parameter λ is defined as

$$F = \frac{\chi_1^2/\nu_1}{\chi_2^2/\nu_2}$$

where χ_1^2 is a noncentral χ^2 with ν_1 degree of freedom and noncentrality parameter λ and χ_2^2 is a central χ^2 with ν_2 degree of freedom. The mean and variance of F are

$$\begin{aligned}
\mathbb{E}[F] &= \frac{\nu_2}{\nu_2 - 2}\frac{\nu_1 + \lambda}{\nu_1} \\
\mathrm{Var}[F] &= \frac{2\nu_2^2\left[(\nu_1+\lambda)^2 + 2(\nu_2-2)(\nu_1+\lambda) - \nu_1(\nu_2-2)\right]}{\nu_1^2(\nu_2-2)^2(\nu_2-4)}
\end{aligned}$$

The variance depends on the noncentrality parameter λ to a large extent. Miwa [13] proposed a variance stabilizing transformation

$$g(F) = \ln\left(F + \frac{\nu_2}{\nu_1} + \sqrt{F^2 + 2\frac{\nu_2}{\nu_1}F}\right) \tag{19}$$

Expanding its mean and variance in terms of $1/\lambda$ and $1/\nu_2$, we obtain

$$\mathbb{E}[F] = g\left(1+\frac{\lambda}{\nu_1}\right) + \frac{3}{3\nu_2-1} + O\left(\frac{1}{\nu_2^3}\right) - \frac{1}{\lambda}\left[2+O\left(\frac{1}{\nu_2}\right)\right] + O\left(\frac{1}{\lambda^2}\right) \tag{20}$$

$$\mathrm{Var}[F] = \frac{2}{\nu_2-1} + O\left(\frac{1}{\nu_2^3}\right) + +O\left(\frac{1}{\lambda\nu_2^3}\right) + O\left(\frac{1}{\lambda^2}\right) \tag{21}$$

It eliminates the terms of order $1/\lambda, 1/\lambda\nu_2$ and $1/\lambda\nu_2^2$. So (20) and (21) can be approximately

$$\mathbb{E}[F] \simeq g\left(1+\frac{\lambda}{\nu_1}\right) + \frac{3}{3\nu_2-1}$$

$$\mathrm{Var}[F] \simeq \frac{2}{\nu_2-1}$$

as λ becomes large.

In practice, noncentrality parameter λ is usually large. As λ approaches to infinity, $\chi_1^2/(\nu_1+\lambda)$ converges to unity in probability. Then $g(F) - g(1+\lambda/\nu_1)$ converges to $Y = -\ln(\chi_2^2/\nu_2)$ in law. Its skewness and kurtosis are

$$\gamma_3 = \sqrt{\frac{2}{\nu_2}}\left[1+\frac{1}{2\nu_2}+O(1/\nu_2^2)\right]$$

$$\gamma_4 = \frac{4}{\nu_2}\left[1+\frac{1}{\nu_2}+O(1/\nu_2^2)\right]$$

And by the Monte Carlo calculations of mean and variance of $g(F)$, he showed that

$$g(F) \overset{approx.}{\sim} N\left(g(1+\lambda/\nu_1) + \frac{3}{3\nu_2-1}, \quad \frac{2}{\nu_2-1}\right) \tag{22}$$

Suppose that F_1, F_2 are noncentral F statistics with ν_1, ν_2 degrees of freedom and non-centrality parameters λ_1, λ_2 respectively and that they are independent. From (22) and their independence, we have that under $H_0 : \lambda_1 = \lambda_2$

$$g(F_1) - g(F_2) \quad \overset{approx.}{\sim} \quad N\left(0, \quad \frac{4}{\nu_2-1}\right) \tag{23}$$

So, for each hypothesis testing,

i. $H_0 : \lambda_1 = \lambda_2$ versus $H_A : \lambda_1 \neq \lambda_2$

ii. $H_0 : \lambda_1 = \lambda_2$ versus $H_A : \lambda_1 > \lambda_2$

iii. $H_0 : \lambda_1 = \lambda_2$ versus $H_A : \lambda_1 < \lambda_2$

we reject H_0 at level α if (i) $|\tau| > z_{\alpha/2}$ (ii) $\tau > z_\alpha$ (iii) $\tau < -z_\alpha$ at level α since we have that from (23)

$$\tau = \frac{g(F_1) - g(F_2)}{\sqrt{4/(\nu_2 - 1)}} \quad \overset{approx.}{\sim} \quad N(0,1)$$

Suppose that F_i, $i = 1, 2, \ldots, p$ are p independent noncentral F statistics with ν_1, ν_2 degrees of freedom and noncentrality parameters λ_i. As before, the transformation (19) gives

$$g(F_i) \quad \overset{approx.}{\sim} \quad N\left(g\left(1 + \frac{\lambda_i}{\nu_1}\right) + \frac{3}{3\nu_2 - 1}, \quad \frac{2}{\nu_2 - 1} \right)$$

Now we can test $H_0 : \lambda_1 = \lambda_2 = \cdots = \lambda_p$ vs H_A : not H_0. By (24) and their independence, we have that

$$\frac{1}{2/(\nu_2 - 1)} \sum_{i=1}^{p} (g(F_i) - \bar{g})^2 \quad \overset{approx.}{\sim} \quad \chi^2_{p-1}$$

Thus we can reject H_0 if

$$\sum_{i=1}^{p} (g(F_i) - \bar{g})^2 \quad > \quad \frac{2}{\nu_2 - 1} \chi^2_{p-1,\alpha}$$

where $\bar{g} = \frac{1}{p} \sum_{i=1}^{p} g(F_i)$ and $\chi^2_{p-1,\alpha}$ is upper $\alpha-$quantile of χ^2_{p-1}.

We can also reject H_0 if

$$\max_{i,j} |g(F_i) - g(F_j)| \quad > \quad \sqrt{2/(\nu_2 - 1)} Q(p, \infty; \alpha)$$

where, $Q(p, \infty; \alpha)$ is upper $\alpha-$quantile of the standard range distribution.

2.2.2 Methodology based on transformation of noncentral $t-$ distributions

In Nagata, *et. al*'s article [14], they use variance stabilizing transformation of noncentral $t-$distributions instead of noncentral $F-$distributions since $t_\nu(\delta)^2 = F_{1,\nu}(\delta^2)$.

Assume a simple linear regression model for the $k-$th signal-response system

$$y_{ki} = \beta_{k0} + \beta_{k1} x_i + \epsilon_{ki}, \quad \epsilon_{ki} \sim N(0, \sigma_k^2) \quad k = 1, 2, \ldots, p \quad i = 1, \ldots, n$$

where the signal factor values(standards) $x_1, x_2, \ldots, x_n$ are the same for all $p-$systems. A natural estimator of η_k is $\hat{\eta}_k = \hat{\beta}_{k1}^2 / \hat{\sigma}_k^2$, and we have that

$$t_k = \sqrt{S_{xx}} \frac{\hat{\beta}_{k1}}{\hat{\sigma}_k}$$

is distributed as noncentral $t-$distributions with degree of freedom $\nu = n - 2$ and the noncentrality parameter $\lambda_k = \frac{\beta_{k1}}{\sigma_k / S_{xx}} = \sqrt{S_{xx}\eta_k}$. Accordingly, the testing problem of the equality for the SN ratios is equivalent to the testing problem of the equality for the noncentrality parameter.

Suppose that t is distributed as the noncentral $t-$distribution with the degree of freedom ν and the noncentrality parameter λ. Note that

$$\mathbb{E}(t) = \frac{\sqrt{\nu/2}\Gamma((\nu-1)/2)}{\Gamma(\nu/2)}\lambda = c_{11}\lambda$$

$$\mathrm{Var}(t) = \left(\frac{\nu}{\nu-2} - c_{11}^2\right)\lambda^2 + \frac{\nu}{\nu-2} = c_{22}\lambda^2 + c_{20}$$

$$\text{where,} \quad c_{11} = \frac{\sqrt{\nu/2}\Gamma((\nu-1)/2)}{\Gamma(\nu/2)}, \quad c_{22} = \frac{\nu}{\nu-2} - c_{11}^2 \quad \text{and} \quad c_{20} = \frac{\nu}{\nu-2}$$

Put $\mu = c_{11}\lambda$ and $\sigma^2(\mu) = a^2 + b^2\mu^2$ where $a = \sqrt{c_{20}}$, $b = \sqrt{c_{22}}/c_{11}$. Consider a transformation $f(t)$ of t. Note that an approximation variance of $f(t)$ is $\{f'(\mu)\}^2\sigma^2(\nu)$

which is based on well known $\delta-$method. To obtain a variance stabilizing transformation, set $\{f'(\mu)\}^2\sigma^2(\nu) = 1$. Laubscher [9] derived

$$f(t) = \frac{1}{b}\ln(b\kappa + \sqrt{b^2t^2 + a^2}) \tag{24}$$

as the solution of $\{f'(\mu)\}^2\sigma^2(\nu) = 1$, and we have

$$f(t) \overset{approx.}{\sim} N\left(f(\mu),\, 1\right) = N\left(f(c_{11}\lambda),\, 1\right) \tag{25}$$

Furthermore, since $a \to 1$ and $b \to 0$ as $\nu \to \infty$, it can be shown that $f(t) \to t$ and $f(\mu) \to \lambda$. Thus based on this transformation (24), we can conduct the testing the following hypothesis

i. $H_0 : \beta_{11}^2/\sigma_1^2 = \beta_{21}^2/\sigma_2^2$ vs $H_A : \beta_{11}^2/\sigma_1^2 \neq \beta_{21}^2/\sigma_2^2$

ii. $H_0 : \beta_{11}^2/\sigma_1^2 = \beta_{21}^2/\sigma_2^2$ vs $H_A : \beta_{11}^2/\sigma_1^2 > \beta_{21}^2/\sigma_2^2$

iii. $H_0 : \beta_{11}^2/\sigma_1^2 = \beta_{21}^2/\sigma_2^2$ vs $H_A : \beta_{11}^2/\sigma_1^2 < \beta_{21}^2/\sigma_2^2$

And we can reject H_0 if (i) $|\tau| > z_{\alpha/2}$ (ii) $\tau > z_\alpha$ (iii) $\tau < -z_\alpha$ at level α since from (25) we have that

$$\tau \;=\; \frac{f(t_1) - f(t_2)}{\sqrt{2}} \;\overset{approx.}{\sim}\; N(0,1)$$

2.2.3 Discussion

From the simulation results in their articles, we can observe that the the sizes of two methodologies based on variance stabilizing transformation are likely to be stable near nominal level, and the their methodologies are convenient to implement the test procedure. However, their sizes are very conservative or liberal under some

conditions because of instability of the transformations they utilized. For example, at small degrees of freedom the sizes of Miwa's test statistics are too conservative and the sizes of Nagata *et al*'s test statistics get conservative as noncentrality parameter δ increases. Thus these methods cannot guarantee given size α test, which would be the most important motivations of this thesis.

2.3 Confidence interval for coefficient of variation

If X is a normal variable with mean μ and variance σ^2, then the parameter

$$\kappa \equiv \frac{\sigma}{\mu} \tag{26}$$

is called the population *coefficient of variance*, which is the degree to which a set of data point varies. The larger this number is, the greater the variability in the data. Let X_i for $i = 1, \ldots, n$ be an independent random sample, with $X_i \sim N(\mu, \sigma^2)$ for each i. In terms of usual sample estimates of normal parameters

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

a point estimator of (26) is

$$K \equiv \frac{S}{\bar{X}}$$

This statistic is widely calculated and interpreted, often for very small n, usually without an accompanying confidence interval.

2.3.1 Exact standard confidence interval for coefficient of variation

The confidence interval for $\theta = \mu/\sigma$ is obtained by Lehmann [10] as follows. Since $\bar{X} \sim N(\mu, \sigma^2/n)$, $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$ and $\bar{X}$ and S^2 are independent, we have

$$\frac{\sqrt{n}\bar{X}}{S} = \frac{\bar{X}}{\sigma/\sqrt{n}} \frac{1}{\sqrt{(n-1)S^2/\sigma^2(n-1)}} \sim t_{n-1}(\lambda) \tag{27}$$

where, $\lambda = \frac{\sqrt{n}\mu}{\sigma}$. Thus we can have $(1-\alpha)100\%$ standard confidence interval for $\lambda = \sqrt{n}\theta = \sqrt{n}/\kappa$ from (27)

$$\lambda = \frac{\sqrt{n}}{\kappa} \in \big(l(\sqrt{n}\theta_1), u(\sqrt{n}\theta_2)\big)$$

where $l(\sqrt{n}\theta_1)$ and $u(\sqrt{n}\theta_2)$ satisfy the following two conditions

$$\mathbb{P}_{\sqrt{n}\theta_1}\left(\frac{\sqrt{n}\bar{X}}{S} < l(\sqrt{n}\theta_1)\right) = \alpha/2$$
$$\mathbb{P}_{\sqrt{n}\theta_2}\left(\frac{\sqrt{n}\bar{X}}{S} > u(\sqrt{n}\theta_2)\right) = \alpha/2$$

Thus the exact $(1-\alpha)100\%$ standard confidence interval for κ can be obtained by

$$\kappa \in \left(\frac{\sqrt{n}}{u(\sqrt{n}\theta_2)}, \frac{\sqrt{n}}{l(\sqrt{n}\theta_1)}\right) \tag{28}$$

The calculation for obtaining (28) is so cumbersome, and Vangel [20] and Wong & Wu [21] proposed the approximated confidence intervals.

2.3.2 Approximate confidence intervals for coefficient of variation

This section considers the two articles by Vangel [20] and Wong & Wu [21], which considered approximate confidence interval methodologies for coefficient of variation.

2.3.2.1 Vangel's approximate confidence interval

In Vangel [20]'s article, he used an approximate pivotal quantity that can be used to easily calculate confidence intervals for κ that attain very nearly the nominal confidence for any sample size. These calculations require only a table of quantiles of the χ^2 distribution.

Let the random variable $W_\nu = \chi^2_\nu/\nu$ with $\nu = n-1$, let $\chi^2_{\nu,\alpha}$ denote the 100α percentile of the distribution of χ^2_ν for $\alpha \in (0,1)$, and let $t \equiv \chi^2_{\nu,\alpha}$ be corresponding quantile of W_ν. Define the random variable

$$Q \equiv \frac{K^2(1+\kappa^2)}{(1+\theta K^2)\kappa^2}$$

where $\theta = \theta(\nu, \alpha)$ is a known function and $K \equiv S/\bar{X}$. If we choose θ so that

$$\mathbb{P}(Q \leq t) \approx \mathbb{P}(W_\nu \leq t) \tag{29}$$

then, we can use Q as an approximate pivot for constructing hypothesis tests and confidence interval for κ because the distribution of W_ν is known and free of κ. Mckay [11] proposed that Q and W_ν are approximately equal in distribution when he choose $\theta = \nu/(\nu+1)$.

In Vangel [20]'s article, he investigate several θ for getting good *accuracy* of the approximation (29) to be $e(t) = \mathbb{P}(Q \leq t) - \alpha$.

An approximate $100(1-\alpha)\%$ confidence interval for $\kappa = \sigma/\mu$ based on this approach is

$$\kappa \in \left[\frac{K}{\sqrt{t_1(\theta_1 K^2 + 1) - K^2}}, \quad \frac{K}{\sqrt{t_2(\theta_2 K^2 + 1) - K^2}} \right] \tag{30}$$

where $t_1 = \chi^2_{\nu,1-\alpha/2}/\nu,\ t_2 = \chi^2_{\nu,\alpha/2}/\nu$. Vangel presented 4 kinds of function of θ such as

1. Mckay approximation $\theta = \nu/(\nu+1)$
2. David's approximation $\theta = 1$
3. Naive approximation $\theta_i = 1/t_i,\ i = 1, 2$
4. Modified Mckay approximation $\theta_i = \frac{2}{(\nu+1)t_i} + \frac{\nu}{\nu+1},\ i = 1, 2$

One minor problem of his confidence interval (30) is the possible existence of singularities. That is when

$$K^2 \geq \theta \frac{t}{1-t} \tag{31}$$

In order for (31) to hold for the choice of θ, either K^2 must be large or t must be small.

2.3.2.2 Wong & Wu's approximate confidence interval

Consider a model with log likelihood function $\mathcal{L}(\theta) = \mathcal{L}(\psi, \lambda)$ where ψ is a scalar parameter of interest and λ is the nuisance parameter. Inference based on the signed log likelihood ratio statistic

$$r(\psi) = sgn(\hat{\psi} - \psi) \left\{ 2 \left[\mathcal{L}(\hat{\theta}) - \mathcal{L}(\hat{\theta}_\psi) \right] \right\}^{1/2} \tag{32}$$

is a widely used method for inference concerning ψ. Note that $\hat{\theta} = (\hat{\psi}, \hat{\lambda})$ is overall maximum likelihood estimate of $\theta = (\psi, \lambda)$ and $\hat{\theta}_\psi = (\psi, \hat{\lambda}_\psi)$ is the constrained maximum likelihood estimate of θ for a given ψ. $r(\psi)$ is asymptotically distributed

as $N(0,1)$. Therefore a $100(1-\alpha)\%$ confidence interval for ψ based on $r(\psi)$ is

$$\left\{\psi \,:\, |r(\psi)| \leq z_{\alpha/2}\right\}$$

It is important to note that this asymptotic method has order of accuracy $O(n^{-1/2})$, and hence it can be quite inaccurate when the sample size is small (see Pierce and Peters [16]). Barndorff-Nielsen [1] defined the modified signed log likelihood ratio statistics as

$$r^*(\psi) = r(\psi) + r(\psi)^{-1} \ln\left\{\frac{u(\psi)}{r(\psi)}\right\} \tag{33}$$

where $r(\psi)$ is log likelihood ratio statistic (32) and $u(\psi)$ is a scalar quantity (see Wong & Wu [21]).

Barndorff-Nielsen showed that $r^*(\psi)$ is also asymptotically distributed as $N(0,1)$ with order of accuracy $O(n^{-3/2})$. Hence a $100(1-\alpha)$ confidence interval for ψ based on $r^*(\psi)$ is

$$\left\{\psi \,:\, |r^*(\psi)| \leq z_{\alpha/2}\right\} \tag{34}$$

For the normal coefficient of variation problem, the log likelihood function expressed in terms of the scalar parameter of interest $\kappa = \sigma/\mu$ and the nuisance parameter μ is

$$\mathcal{L}(\theta) = \mathcal{L}(\kappa, \mu) = -n \ln(\kappa\mu) - \frac{n}{2\kappa^2\mu^2}$$

where $\hat{\kappa} = S/\bar{X}$ and $S^2 = \sum(X_i - \bar{X})^2/n$ is maximum likelihood estimator of σ^2.

Then the signed log likelihood ratio statistic (32) can be simplified to

$$r(\kappa) = sgn(\hat{\kappa} - \kappa) \left\{ 2n \ln \left(\frac{\kappa \hat{\mu}_\kappa}{\hat{\kappa}\hat{\mu}} \right) + \frac{n}{\kappa^2 \hat{\mu}_\kappa^2} \left(\hat{\kappa}^2 \hat{\mu}^2 + (\hat{\mu} - \hat{\mu}_\kappa)^2 \right) - n \right\}^{1/2} \tag{35}$$

where, $\hat{\mu}_\kappa = \dfrac{-\bar{X} + \sqrt{\bar{X}^2 + 4\kappa^2(S^2 + \bar{X}^2)}}{2\kappa^2}$

Since the model is a full exponential model, Barndorff-Nielsen's modified signed log likelihood statistic can be employed and $u(\kappa)$ in (33) can be obtained

$$u(\kappa) \; = \; \sqrt{n} \left\{ \frac{1}{2} + \frac{1}{2} \left(\frac{\hat{\kappa}\hat{\mu}}{\kappa \hat{\mu}_\kappa} \right)^2 \right\} \left(\frac{\hat{\kappa}\hat{\mu}}{\kappa \hat{\mu}_\kappa} \right) \left(1 + \kappa^2 - \frac{\hat{\mu}}{2\hat{\mu}_\kappa} \right)^{-1/2} \tag{36}$$

Hence $r^*(\kappa)$ can be obtained from (33) with $r(\kappa)$ (35) and $u(\kappa)$ (36). Thus the $100(1-\alpha)\%$ confidence interval for κ by modified log likelihood ratio statistic can be obtained.

2.3.3 Discussion

We looked over three methodologies for $100(1-\alpha)\%$ confidence interval for κ, which are exact confidence interval, Vangel's approximate confidence interval, and Wong & Wu's approximate confidence interval. Wong & Wu's approximate confidence interval is based on log likelihood ratio statistic and modified log likelihood ratio statistics, and so they can be extended to non-normal models such as Gamma model and Weibull model. Thus, Wong & Wu's methodologies for constructing confidence interval can be thought of as more efficient methodologies.

In Wong & Wu's article [21], they made comparison to Vangel's computational results by utilizing same data in Vangel's article [20], but in both Vangel' and Wong

& Wu's articles, they did not make comparison to exact standard confidence interval for κ. Here, the comparisons to exact standard confidence interval will be added and summarized in Table 3.

Example (Vangel's example for normal model) This example was taken from both Wong & Wu's article [21] and Vangel's article [20]. The data is the tensile strength of five specimens of a composite material which is given as follows(in 1000 psi): 326, 302, 307, 299, 329. It is determined that the normal model is an approximate model.

1. Exact Confidence Interval : $\bar{X} = 312.6$ and $S = 13.94$ $K = S/\bar{X} = 0.045$. By numerical result, we have that $l(\sqrt{5}\theta_1) = 17.3789$, $u(\sqrt{5}\theta_2) = 83.7465$. Therefore the exact 95% confidence interval for κ is (0.0267, 0.1287).

2. Vangel's approximate Confidence Interval : Let $\theta_i = 1/t_i$ and then use (30) to find approximate confidence interval by Naive method. From $u_1 = \chi^2_{0.975,4} = 11.14$ and $u_2 = \chi^2_{0.025,4} = 0.4844$, find approximate confidence interval by modified Mckay method.

3. Wong & Wu's approximate Confidence Interval : Plug $\hat{\kappa} = \sqrt{(n-1)S^2/n}/\bar{X} = 0.03988$ and $\hat{\mu} = \bar{X} = 321.6$ in (35), (36), and then use (32) and (34) to construct approximate confidence intervals by signed log likelihood ratio statistic and modified signed log likelihood ratio statistic.

Since sample size is small, the approximate confidence interval by signed log likelihood ratio statistic of Wong & Wu is not accurate by comparison to exact confidence

Table 3: 95% confidence intervals for coefficient of variation κ

Source	Methodology	Interval	Length
Exact standard confidence interval		(0.0267, 0.1287)	0.1020
Vangel	Naive's method	(0.0267, 0.1281)	0.1014
	Modified Mckay	(0.0270, 0.1293)	0.1023
Wong & Wu	signed log likelihood ratio	(0.0238, 0.0872)	0.0634
	modified signed log likelihood ratio	(0.0265, 0.1247)	0.0982

interval as expected, while other approximate methodologies have relatively good accuracy. However, the calculation of exact confidence interval for coefficient of variation κ is actually not much difficult to obtain since cumulative distribution function of noncentral t variables can be provided in some softwares such as R, Matlab. As one of further research topics, I expect to obtain uniformly shorter confidence interval employing the methodologies in chapter 3.

2.4 Properties of noncentral $t-$distributions

2.4.1 The density and cumulative distribution function of noncentral $t-$distributions

Suppose that Z is normal distribution with mean δ and variance 1, χ^2_ν is chi-squared distribution with degree of freedom ν, and they are independent. Then the random variable $t_\nu(\delta)$ is defined as

$$t_\nu(\delta) = \frac{Z}{\sqrt{\chi^2_\nu/\nu}} \tag{37}$$

noncentral $t-$distribution with noncentrality parameter δ and degrees of freedom ν.

In order to find the density and cumulative distribution function of noncentral t variable with noncentrality parameter δ and degrees of freedom ν, we first consider

the density of $X = \chi_\nu$. Since $X^2 \sim \chi^2_\nu$, the cumulative distribution function of $X = \chi_\nu$ is

$$\begin{aligned}\mathbb{P}(X \le x) &= \mathbb{P}\left(\sqrt{\chi^2_\nu} \le x\right) = \mathbb{P}\left(\chi^2_\nu \le x^2\right) \\ &= \int_0^{x^2} \frac{1}{\Gamma(\nu/2)2^{\nu/2}} y^{\nu/2-1} e^{-y/2} dy \end{aligned} \tag{38}$$

Differentiating (38) gives us the density of X, $g_X(x)$,

$$g_X(x) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}} x^{\nu-2} e^{-x^2/2} 2x = \frac{1}{\Gamma(\nu/2)2^{\nu/2-1}} x^{\nu-1} e^{-x^2/2} \tag{39}$$

From the (37), (39) and conditioning on $X = x$, the cumulative distribution of $t_\nu(\delta)$ can be obtained by

$$\begin{aligned}\mathbb{P}(t_\nu(\delta) \le t) &= \int_0^\infty \mathbb{P}(Z \le X\nu^{-1/2}t - \delta | X = x) g_X(x) dx \\ &= \int_0^\infty \mathbb{P}(Z \le x\nu^{-1/2}t - \delta) \frac{1}{\Gamma(\nu/2)2^{\nu/2-1}} x^{\nu-1} e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{\pi}\Gamma(\nu/2)2^{(\nu-1)/2}} \int_0^\infty x^{\nu-1} e^{-x^2/2} \int_{-\infty}^{tx\nu^{-1/2}-\delta} e^{-z^2/2} dz dx \end{aligned} \tag{40}$$

Thus differentiating (40) gives us the density of $t_{\nu,\delta}$, $f_{\nu,\delta}(t)$,

$$\begin{aligned} f_{\nu,\delta}(t) &= \frac{\partial}{\partial t} \mathbb{P}(t_{\nu,\delta} \le t) = \frac{1}{\sqrt{\nu\pi}\Gamma(\nu/2)2^{(\nu-1)/2}} \int_0^\infty x^{\nu-1} e^{-x^2/2} e^{-(tx\nu^{-1/2}-\delta)^2/2} x\nu^{-1/2} dx \\ &= \frac{1}{\sqrt{\nu\pi}\Gamma(\nu/2)2^{(\nu-1)/2}} \int_0^\infty x^\nu e^{-(x^2+(tx\nu^{-1/2}-\delta)^2)/2} dx \end{aligned} \tag{41}$$

Figure 4 shows the sketches of the density functions $f_{\nu,\delta}(t)$ of noncentral $t-$distributions with $\nu = 5$ and $\nu = 15$. For fixed ν, the height of density $f_{\nu,\delta}(t)$ at mode decreases and the density $f_{\nu,\delta}(t)$ has heavier tail probability as the absolute value of δ increases. If we cut two density functions $f_{\nu,\delta}(t)$ along the plane $\delta = 15$ in Figure 4, Figure 5 can be obtained and this shows one of the properties of the density $f_{\nu,\delta}(t)$, which is that the skewness of noncentral $t-$distribution decreases in degrees of freedom ν.

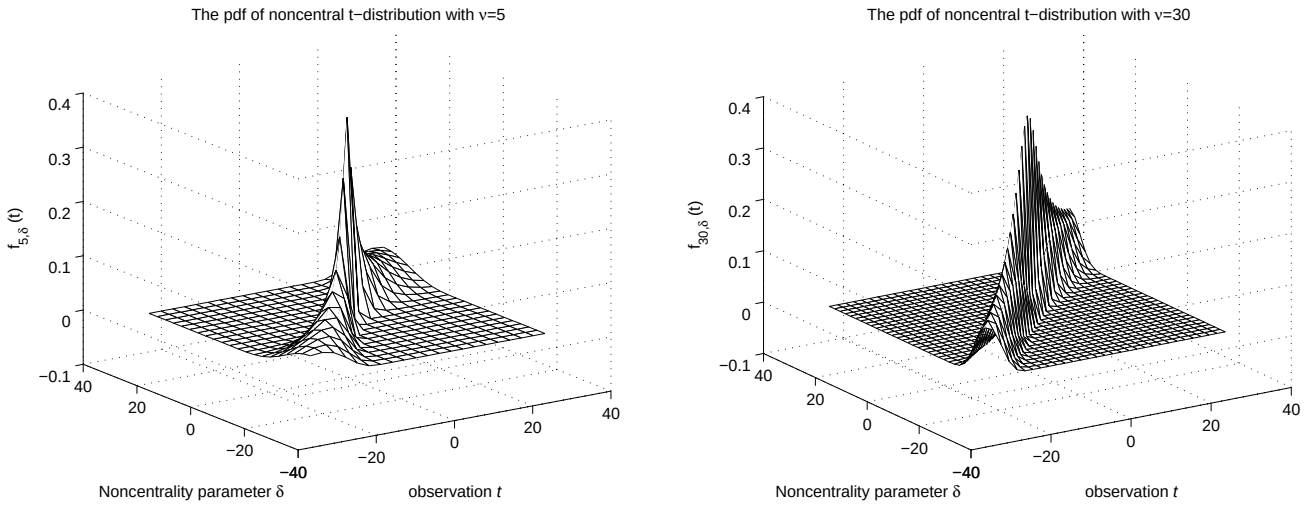

Figure 4: Graphical representation of the density functions $f_{\nu,\delta}(t)$ with $\nu = 5$ (left plot) and $\nu = 30$ (right plot)

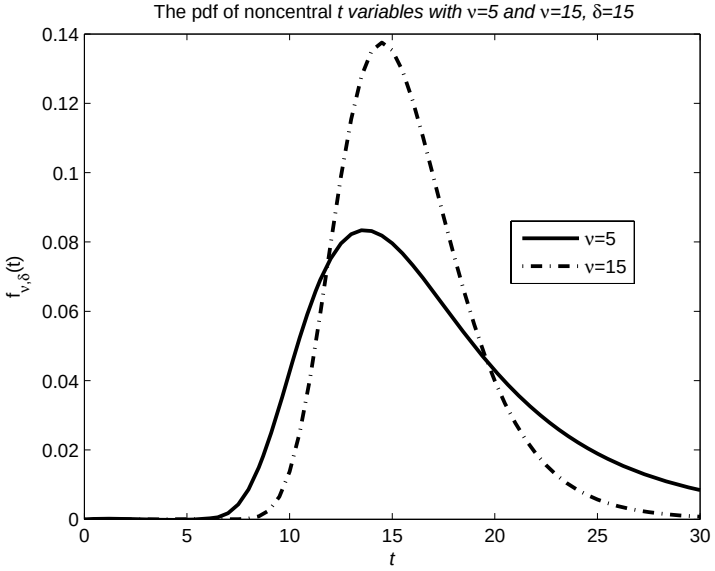

Figure 5: A comparison of two density functions $f_{\nu,\delta}(t)$ with $\nu = 5$ and $\nu = 15$ for given $\delta = 15$

2.4.2 Measures of skewness

The skewness η measures the degree of asymmetry of a distribution and can be usually defined as

$$\eta = \frac{\mathbb{E}(t_\nu(\delta) - \mu)^3}{\sigma^3} \tag{42}$$

The value of this measure may become arbitrary large, however, and thus difficult to interpret. For the Pareto distribution with density $f(x) = \theta/x^{\theta+1}$, for $\theta > 0$, $x \geq 1$ the noncentral moments are found to be $\mu_k = \theta/(\theta - k)$. Hence, for $\theta = 3 + \epsilon$ this skewness coefficient becomes arbitrarily large as $\epsilon \to 0^+$.

Other forms of skewness were also defined as follows. The *Pearson mode skewness* is

$$\eta = \frac{\text{mean - mode}}{\sigma}$$

and *Pearson's skewness coefficients* are

$$\eta = \frac{3(\text{mean - mode})}{\sigma} \quad \text{or} \quad \eta = \frac{3(\text{mean - median})}{\sigma}$$

The *Bowley skewness*, which is also known as quartile skewness coefficient, is defined as

$$\eta_1 = \frac{(Q_3 - Q_2) - (Q_2 - Q_1)}{Q_3 - Q_1} = \frac{Q_1 - 2Q_2 + Q_3}{Q_3 - Q_1} \tag{43}$$

where the Q_i denote the ith quartile of random variable $t_\nu(\delta)$. Only one of these coefficients will require the existence of the first moment of $t_\nu(\delta)$. Additionally, the values of the coefficients will have values on the interval (-1, 1) with 1 representing extreme right skewness and with -1 representing extreme left skewness. Further, these coefficients have quite natural interpretations as skewness measures. Groeneveld and

Meeden [5] investigated the properties of several extensions of the Bowley coefficient as measure of skewness. The Bowley coefficient (43) could be generalized in several ways. A general version of (43) was given by

$$\eta_2(\alpha) \;=\; \frac{F^{-1}_{\nu,\alpha}(1-\alpha) + F^{-1}_{\nu,\alpha}(\alpha) - 2\rho_\nu(\delta)}{F^{-1}_{\nu,\alpha}(1-\alpha) - F^{-1}_{\nu,\alpha}(\alpha)}, \qquad 0 < \alpha < 1/2 \tag{44}$$

where $\rho_\nu(\delta)$ is the median of $t_\nu(\delta)$. This measure was used by Hinkley [6] for estimation in the Box-Cox model. As $F^{-1}_{\nu,\delta}(\alpha)$ is αth quartile of $t_\nu(\delta)$, (44) reduces to the Bowley coefficient (43) for $\alpha = 1/4$. Another skewness coefficient $\eta_3(\alpha)$ could be obtained from (44) by integration of the numerator and denominator of (44) with respect to α on (0, 1/2), and we have

$$\eta_3(\alpha) \;=\; \frac{\int_0^{1/2}\{F^{-1}_{\nu,\alpha}(1-\alpha) + F^{-1}_{\nu,\alpha}(\alpha) - 2\rho_\nu(\delta)\}d\alpha}{\int_0^{1/2}\{F^{-1}_{\nu,\alpha}(1-\alpha) - F^{-1}_{\nu,\alpha}(\alpha)\}d\alpha} = \frac{\mathbb{E}\, t_\nu(\delta) - \rho_\nu(\delta)}{\mathbb{E}\,|t_\nu(\delta) - \rho_\nu(\delta)|} \tag{45}$$

A fourth skewness coefficient η_4 may be derived by letting $\alpha \to 0$ in (44). The limiting value is

$$\eta_4 \;=\; \lim_{\alpha\to 0}\eta_2(\alpha) = \frac{b + a - 2\rho_\nu(\delta)}{b - a} \tag{46}$$

where we assume the random variable $t_\nu(\delta)$ has a continuous distribution function with $F_{\nu,\alpha}(t)$ with differentiable density $f_{\nu,\alpha}(t)$ on the interval $I = (a,\, b)$. The interpretations of these coefficients, η_1, η_3, η_4, was given in Groeneveld and Meeden [5], suggesting how each reflects skewness.

Let's define the standard ratio of two distances $\xi^s_{\nu,\alpha}(\delta)$ of a noncentral $t-$distribution for given $\alpha \in (0, 1/2)$, ν and δ as

$$\xi^s_{\nu,\alpha}(\delta) = \frac{t^u_{\nu,\alpha/2}(\delta) - m_\nu(\delta)}{m_\nu(\delta) - t^l_{\nu,\alpha/2}(\delta)} \tag{47}$$

where, $m_\nu(\delta)$ is the mode of density $f_{\nu,\delta}(t)$, and $t^l_{\nu,\alpha/2}(\delta)$ and $t^u_{\nu,\alpha/2}(\delta)$ satisfy $\alpha/2 = \mathbb{P}(t_\nu(\delta) > t^u_{\nu,\alpha/2}(\delta)),\ \alpha/2 = \mathbb{P}(t_\nu(\delta) < t^l_{\nu,\alpha/2}(\delta))$.

Then, this ratio $\xi^s_{\nu,\alpha}(\delta)$ could be considered as a new measure of skewness of a continuous random variables similarly to the *Pearson's skewness coefficients* and *Bowley skewness*. If δ is equal to 0, then $\xi^s_{\nu,\alpha}(0) = 1$. For $\xi^s_{\nu,\alpha}(\delta) > 1$, this random variable can be thought of as right-skewed random variable while for $0 < \xi^s_{\nu,\alpha}(\delta) < 1$, this random variable can be thought of as left-skewed random variable.

In order to compare standard ratio $\xi^s_{\nu,\alpha}(\delta)$ with skewness η of (42), we could find $\eta(\nu,\delta)$ by the definition (37) of noncentral $t-$distributions as follows.

$$\mu = \mathbb{E}(t_\nu(\delta)) = \mathbb{E}(Z+\delta)\sqrt{\nu}\,\mathbb{E}(Y^{-1/2}) = \left(\frac{\nu}{2}\right)^{1/2}\frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)}\delta = c_1\delta \tag{48}$$

where, $Y \sim \chi^2_\nu$ and $c_1 = \left(\frac{\nu}{2}\right)^{1/2}\frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)}$. We have that

$$\begin{aligned}
\mathbb{E}(t^2_\nu(\delta)) &= \mathbb{E}(Z+\delta)^2\nu\,\mathbb{E}(Y^{-1}) = (\delta^2+1)\nu\frac{\Gamma(\nu/2-1)}{2\Gamma(\nu/2)} \\
&= \frac{\nu}{\nu-2}(\delta^2+1) = c_2(\delta^2+1) \quad \text{for } \nu > 2 \qquad (49)\\
\mathbb{E}(t^3_\nu(\delta)) &= \mathbb{E}(Z+\delta)^3\nu^{3/2}\,\mathbb{E}(Y^{-3/2}) \\
&= \mathbb{E}(\delta^3 + 3Z\delta + 3Z^2\delta + Z^3)(\nu/2)^{3/2}\frac{\Gamma((\nu-3)/2)}{\Gamma(\nu/2)} \\
&= (\nu/2)^{3/2}\frac{\Gamma((\nu-3)/2)}{\Gamma(\nu/2)}(\delta^3+3\delta) = c_3(\delta^3+3\delta) \quad \text{for } \nu > 3 \qquad (50)
\end{aligned}$$

where, $c_2 = \frac{\nu}{\nu-2}$ and $c_3 = (\nu/2)^{3/2}\frac{\Gamma((\nu-3)/2)}{\Gamma(\nu/2)}$. From (48) and (49), we have that

$$\sigma^2 = \mathbb{E}(t^2_\nu(\delta)) - \mu^2 = c_2\delta^2 + c_2 - c_1^2\delta^2 = c_{22}\delta^2 + c_{20} \tag{51}$$

where, $c_{22} = c_2 - c_1^2$ and $c_{20} = c_2 = \frac{\nu}{\nu-2}$.

Table 4: The limiting values of skewness $\eta(\nu, \delta)$

ν	5	10	20	30	50	100
limiting values	3.0981	1.4342	0.8837	0.6936	0.5215	0.3609

Thus, skewness $\eta(\nu, \delta)$ of noncentral $t-$distribution can be obtained as

$$\begin{aligned}
\eta(\nu, \delta) &= \mathbb{E}\left(\frac{(X-\mu)^3}{\sigma^3}\right) = \mathbb{E}\left(\frac{X^3 - 3X^2\mu + 3X\mu^2 + \mu^3}{\sigma^3}\right) \\
&= \frac{1}{(\sigma^2)^{3/2}}\left[\mathbb{E}(X^3) - 3\mu\,\mathbb{E}(X^2) + 2\mu^3\right] \\
&= \frac{c_3\delta^3 + 3c_3\delta - 3c_1\delta(c_2\delta^2 + c_2) + 2c_1^3\delta^3}{(c_{22}\delta^2 + c_{20})^{3/2}} \\
&= \frac{(c_3 - 3c_1c_2 + 2c_1^3)\delta^3 + (3c_3 - 3c_1c_2)\delta}{(c_{22}\delta^2 + c_{20})^{3/2}} = \frac{c_{33}\delta^3 + c_{31}\delta}{(c_{22}\delta^2 + c_{20})^{3/2}} \qquad (52)
\end{aligned}$$

where $c_{33} = c_3 - 3c_1c_2 + 2c_1^3$ and $c_{31} = 3c_3 - 3c_1c_2$.

Accordingly, we can observe an interesting property of noncentral $t-$distribution, which is that for each ν skewness $\eta(\nu, \delta)$ is converging to the limit $c_{33}(\nu)/c_{22}^{3/2}(\nu)$ as δ goes to infinity. And this property of noncentral $t-$distribution will play a key role in constructing efficient confidence intervals for noncentrality parameter δ of noncentral $t-$distribution in chapter 3. The limiting values of skewness $\eta(\nu, \delta)$ are tabulated in Table 4, which shows that the limiting values for ν greater than 30 become small. Hogben, *et al* [7] demonstrated that two coefficients, c_{33} and c_{31}, converge to 0, which implies that the limiting skewness $\eta(\nu, \delta)$ converges to 0 as ν goes to infinity.

Moreover inspection of the partial derivative of skewness $\eta(\nu, \delta)$ with respect to δ would show that the skewness $\eta(\nu, \delta)$ is monotonically increasing function at least for $\nu = 4, \ldots, 300$ as follows. The partial derivative of skewness $\eta(\nu, \delta)$ could be obtained

as

$$\begin{aligned}\frac{\partial}{\partial\delta}\eta &= \frac{(3c_{33}\delta^2+c_{31})(c_{22}\delta^2+c_{20})^{3/2}-3c_{22}\delta(c_{22}\delta^2+c_{20})^{1/2}(c_{33}\delta^3+c_{31}\delta)}{(c_{22}\delta^2+c_{20})^3}\\ &= \frac{(3c_{33}\delta^2+c_{31})(c_{22}\delta^2+c_{20})-3c_{22}\delta(c_{33}\delta^3+c_{31}\delta)}{(c_{22}\delta^2+c_{20})^{5/2}}\\ &= \frac{(3c_{20}c_{33}-2c_{22}c_{31})\delta^2+c_{20}c_{31}}{(c_{22}\delta^2+c_{20})^{5/2}}\\ &= \frac{h(\nu)\delta^2+c_{20}c_{31}}{(c_{22}\delta^2+c_{20})^{5/2}} = \frac{h(\nu)\delta^2+c_{20}c_{31}}{\sigma^5}\end{aligned}$$

It's easy to show that $c_{20}c_{31} > 0$ for $\nu > 3$ since we have that

$$c_3 = \left(\frac{\nu}{2}\right)^{3/2}\frac{\Gamma((\nu-3)/2)}{\Gamma(\nu/2)} = \frac{\nu}{\nu-3}\left(\frac{\nu}{2}\right)^{1/2}\frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)} > c_2c_1$$

and $c_{20}c_{31} = 3c_2(c_3 - c_1c_2) > 0$

And $h(\nu)$ could be rewritten as

$$\begin{aligned}h(\nu) &= 3c_{20}c_{33}-2c_{22}c_{31} = 3c_2(c_3-3c_1c_2+2c_1^3)-2(c_2-c_1^2)(3c_3-3c_1c_2)\\ &= 3(2c_1^2c_3-c_2c_3-c_1c_2^2),\end{aligned}$$

Figure 6 demonstrates that $h(\nu)$ is positive at least for $\nu = 4, \ldots, 300$. Hence, the skewness $\eta(\nu, \delta)$ is monotonically increasing and converging to limiting value at least for $\nu = 4, \ldots, 300$.

Hogben, *et al* [7] defined two coefficients c_{33} and c_{31} as

$$c_{33} = c_1\left(\frac{\nu(7-2\nu)}{(\nu-2)(\nu-3)}+2c_1^2\right) \quad \text{and} \quad c_{31} = \frac{3\nu}{(\nu-2)(\nu-3)}c_1 \tag{53}$$

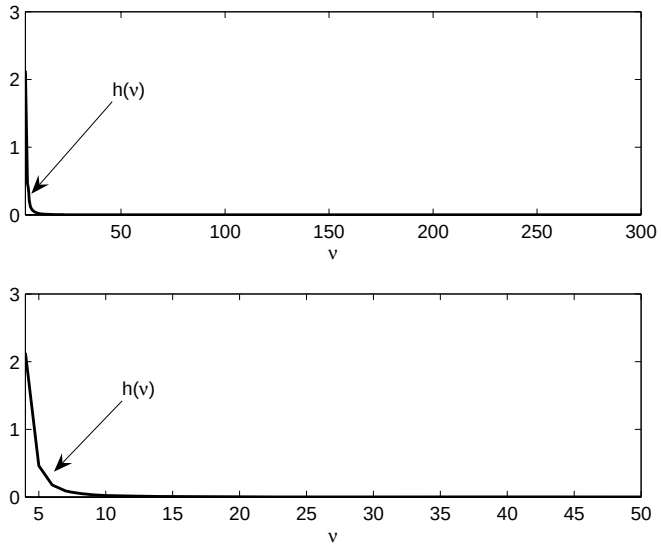

Figure 6: Computational results of $h(\nu)$ for $\nu = 4, \ldots, 300$

However, c_{33} and c_{31} of (52) are the same as (53) because we have that

$$\begin{aligned} c_{33} &= c_3 - 3c_1c_2 + 2c_1^3 = c_1\left(\frac{\nu}{\nu-3} - 3\frac{\nu}{\nu-2} + 2c_1^2\right) \\ &= c_1\left(\frac{\nu(7-2\nu)}{(\nu-2)(\nu-3)} + 2c_1^2\right) \\ c_{31} &= 3c_3 - 3c_1c_2 = 3c_1\left(\frac{\nu}{\nu-3} - \frac{\nu}{\nu-2}\right) = \frac{3\nu}{(\nu-2)(\nu-3)}c_1 \end{aligned}$$

Hogben, *et al* [7] tabulated the values of c_1, c_{20}, c_{22}, c_{31}, and c_{33} for each ν, and showed that c_1 and c_{20} converge to 1 and others converge to 0. This makes sense because skewness of χ^2_ν decreases and converges to 0 as ν goes to infinity.

The comparisons of skewness of noncentral t variables at $\nu = 10$, 20, 30, and 100 are made in Figure 7, which shows that the skewness $\eta(\nu, \delta)$ of noncentral $t-$distributions decreases in degrees of freedom ν, and monotonically increases and converges to their limits. Figure 8 shows a comparison of two measure of skewness, $\eta(\nu, \delta)$ and $\xi^s_{\nu,\alpha}(\delta)$, at $\nu = 10$ and $\alpha = 0.05$. The left y-axis reveals the values of

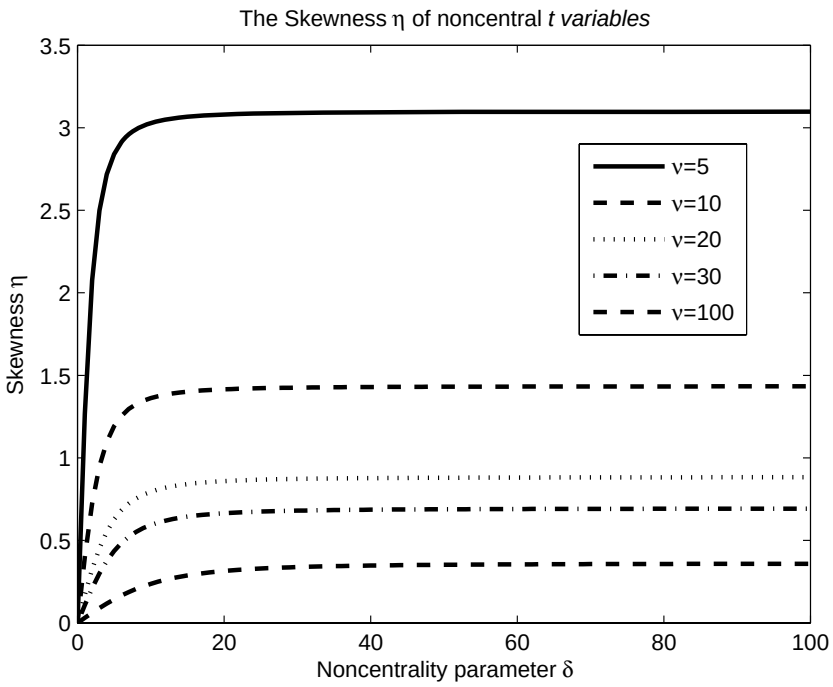

Figure 7: A comparison of skewness of noncentral $t-$distribution with $\nu = 5$, 10, 20, 30, 100

skewness ζ_s while right y-axis is the values of skewness $\xi^s_{\nu,\alpha}(\delta)$, and the x-axis is noncentrality parameter δ, which says that the two curves in the Figure 8 has exactly the same pattern and this means that the $\xi^s_{\nu,\alpha}(\delta)$ can be used for measure of skewness.

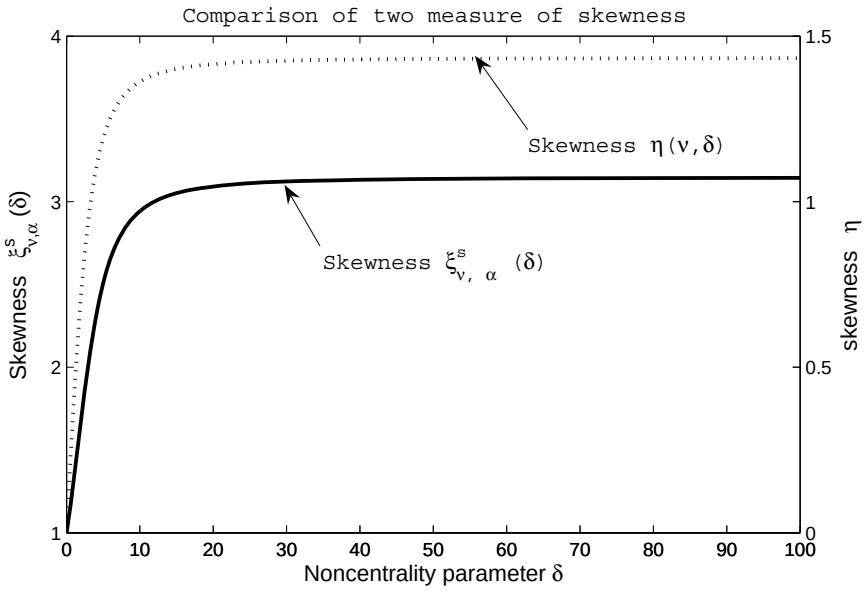

Figure 8: A comparison of skewness $\eta_1(\nu, \delta)$ for the ratio $\xi^s_{\nu,\alpha}(\delta)$ with $\nu = 10$ and $\alpha = 0.05$

CHAPTER III

EFFICIENT CONFIDENCE INTERVAL METHODOLOGIES FOR THE NONCENTRALITY PARAMETER OF A NONCENTRAL $T-$ DISTRIBUTION

This chapter considers the problem of constructing an efficient confidence interval for the noncentrality parameter of a noncentral $t-$distribution which is uniformly shorter than standard, so called "equal tail intervals", confidence interval. In section 3.2 and section 3.3, it is shown that the confidence intervals based on the shortest possible acceptance sets and intervals based on uniformly most powerful unbiased acceptance sets are not efficient, while in section 3.4 new methodologies that are uniformly better than the standard method are developed and described. A summary is provided in section 3.5.

3.1 Introduction

Suppose that an observation t is available from a noncentral t-distribution $t_\nu(\delta)$ defined as

$$\frac{N(\delta,\, 1)}{\sqrt{\chi^2_\nu/\nu}}$$

and that a confidence interval for the noncentrality parameter δ is required at a given confidence level. For example, this problem arises in simple linear regression where the ratio of the square of the slope parameter to the error variance $\eta = \beta_1^2/\sigma^2$ is

known as the signal to noise ratio (see Taguchi [18] and Miwa [12]), and the statistic $\sqrt{S_{XX}}\hat{\beta}_1/\hat{\sigma}$ has a noncentral t-distribution with noncentrality parameter $\sqrt{S_{XX}\eta}$ and degrees of freedom $\nu = n - 2$. Similarly, for normally distributed data, the statistic $\sqrt{n}\bar{X}/S$ has a noncentral t-distribution with noncentrality parameter $\sqrt{n}\mu/\sigma$, which is proportional to the reciprocal of the coefficient of variation, and degrees of freedom $\nu = n - 1$.

Suppose that the lower and upper tail critical points of a $t_\nu(\delta)$ distribution are denoted by $t^l_{\nu,\alpha}(\delta)$ and $t^u_{\nu,\alpha}(\delta)$ satisfying

$$P(t_\nu(\delta) \leq t^l_{\nu,\alpha}(\delta)) = P(t_\nu(\delta) \geq t^u_{\nu,\alpha}(\delta)) = \alpha.$$

Then as explained on page 352 of Lehmann [10], for an observed value t one-sided confidence intervals for δ with a confidence level of exactly $1 - \alpha$ can be obtained as

$$\{\delta \,:\, t^l_{\nu,\alpha}(\delta) \leq t\} \quad \text{or} \quad \{\delta \,:\, t \leq t^u_{\nu,\alpha}(\delta)\}.$$

However, in this chapter the construction of two-sided confidence intervals is considered. The standard approach to this problem is described on page 510 of Johnson *et al.* [8], and it employs critical points for equal tail probabilities of $\alpha/2$. Specifically, the standard confidence interval $C^s_{\nu,\alpha}(t)$ for δ is constructed as

$$C^s_{\nu,\alpha}(t) = \{\delta \,:\, t^l_{\nu,\alpha/2}(\delta) \leq t \leq t^u_{\nu,\alpha/2}(\delta)\} \tag{54}$$

which has a confidence level of exactly $1 - \alpha$. Nevertheless, in spite of this method's widespread adoption, the use of critical points with equal tail probabilities of $\alpha/2$ is quite arbitrary, and the question arises as to whether improved confidence interval methodologies are possible. Indeed, since the noncentral t-distribution is a skewed

distribution, it may appear obvious that using equal tail probabilities cannot be optimal. In this chapter it is shown that there exist more efficient confidence interval methodologies that are uniformly better than this standard method in the sense that they provide confidence intervals with smaller lengths for all values of t. The Matlab computer programs are available to implement the new methodologies, and their development also provides some interesting theoretical results.

It should be noted that the advantages of the new methodologies over the standard method are paramount when the degrees of freedom ν are small, since this is when the skewness of the noncentral t-distribution is largest. When the degrees of freedom become large, the noncentral t-distribution is close to being symmetric, and in this case little improvement is possible over the standard method.

Most of the previous work on the noncentrality parameter of a noncentral t-distribution, going back as far as McKay [11] for example, is motivated by its connection with the coefficient of variation for normally distributed data. More recently, the standard method (54) has been criticized as being too computationally cumbersome, and research has been directed towards constructing approximate confidence intervals for the coefficient of variation (see, for example, Vangel [20] and Wong & Wu [21]). However, the methodologies proposed in this chapter provide exact confidence intervals that are uniformly shorter than those provided by the standard method.

3.2 Inefficient confidence interval methodologies based on shortest acceptance sets

3.2.1 Shortest possible acceptance set $A^0_{\nu,\alpha}(\delta)$

Inspection of Figure 1 may suggest that the shortest confidence intervals $C_{\nu,\alpha}(t)$ ought to be obtained by taking the acceptance sets $A^0_{\nu,\alpha}(\delta) = (t^l_{\nu,\alpha/2}(\delta), t^u_{\nu,\alpha/2}(\delta))$ to be as short as possible. As Figure 9 shows, shorter acceptance sets than those employed by the standard procedure with equal tail probabilities are available. In fact, the shortest possible acceptance sets $A^0_{\nu,\alpha}(\delta)$ are obtained by following Lemma 3.2.1.

Lemma 3.2.1 *For a random variable X with unimodal continuous density $f(\cdot)$, the $(1-\alpha)100\%$ shortest possible acceptance interval $A^0 = (x^0_1, x^0_2)$ satisfies*

$$\begin{aligned} \mathbb{P}(x^0_1 \le X \le x^0_2) &= 1-\alpha \\ f(x^0_1) &= f(x^0_2) \end{aligned} \tag{55}$$

Proof : The problem of finding shortest acceptance interval is equivalent to solving the following optimization problem

$$\begin{aligned} &\text{Minimize} \quad x_2 - x_1 \\ &\text{Subject to } \mathbb{P}(x_1 \le X \le x_2) = 1-\alpha \end{aligned} \tag{56}$$

Introducing Lagrangian multiplier λ, define a function

$$h(x_1, x_2, \lambda, \alpha) = x_2 - x_1 + \lambda\{(1-\alpha) - \mathbb{P}(x_1 \le X \le x_2)\}$$

The problem (56) turns to be the problem of solving $\min_{x_1,x_2,\lambda} h(x_1, x_2, \lambda, \alpha)$. So by taking partial derivative $h(\cdot)$, we have that

$$\begin{aligned}
\frac{\partial}{\partial\lambda}h(x_1,x_2,\lambda,\alpha)\Big|_{x_1=x_1^0,x_2=x_2^0,\lambda=\lambda^0} = 0 \qquad &\Rightarrow \quad \mathbb{P}(x_1^0 \le X \le x_2^0) = 1-\alpha \\
\frac{\partial}{\partial t_1}h(x_1,x_2,\lambda,\alpha)\Big|_{x_1=x_1^0,x_2=x_2^0,\lambda=\lambda^0} = 0 \qquad &\Rightarrow \quad -1+\lambda^0 f(x_1^0) = 0 \\
\frac{\partial}{\partial t_2}h(x_1,x_2,\lambda,\alpha)\Big|_{x_1=x_1^0,x_2=x_2^0,\lambda=\lambda^0} = 0 \qquad &\Rightarrow \quad 1-\lambda^0 f(x_2^0) = 0
\end{aligned}$$

Hence it follows. ■

Corollary 3.2.1 *For the noncentral $t-$distribution with density $f_{\nu,\delta}(\cdot)$, the $(1-\alpha)100\%$ shortest possible acceptance interval $A^0_{\nu,\alpha}(\delta) = (l^0_{\nu,\alpha}(\delta),\, u^0_{\nu,\alpha}(\delta))$ satisfies*

$$\begin{aligned}
&\mathbb{P}(l^0_{\nu,\alpha}(\delta) \le t_\nu(\delta) \le u^0_{\nu,\alpha}(\delta)) = 1-\alpha \\
&\int_0^\infty x^\nu e^{-(x^2+(l^0_{\nu,\alpha}(\delta)x\nu^{-1/2}-\delta)^2)/2}dx = \int_0^\infty x^\nu e^{-(x^2+(u^0_{\nu,\alpha}(\delta)x\nu^{-1/2}-\delta)^2)/2}dx
\end{aligned} \tag{57}$$

Proof : The proof of this Corollary is straightforward from (41) since the density of $f_{\nu,\delta}(\cdot)$ is

$$f_{\nu,\delta}(t) = \frac{1}{\sqrt{\nu\pi}\Gamma(\nu/2)2^{(\nu-1)/2}}\int_0^\infty x^\nu e^{-(x^2+(tx\nu^{-1/2}-\delta)^2)/2}dx$$

Thus it follows. ■

From (57), we can observe that for $\delta = 0$, $u^0_{\nu,\alpha}(0) = -l^0_{\nu,\alpha}(0)$ is easily obtained, which could say that standard acceptance set of $t-$distribution is shortest as that of the normal is. However, the standard acceptance interval of the skewed random variables such as χ^2_ν, F_{ν_1,ν_2} and noncentral $t-$distribution is not shortest by Lemma 3.2.1 any more .

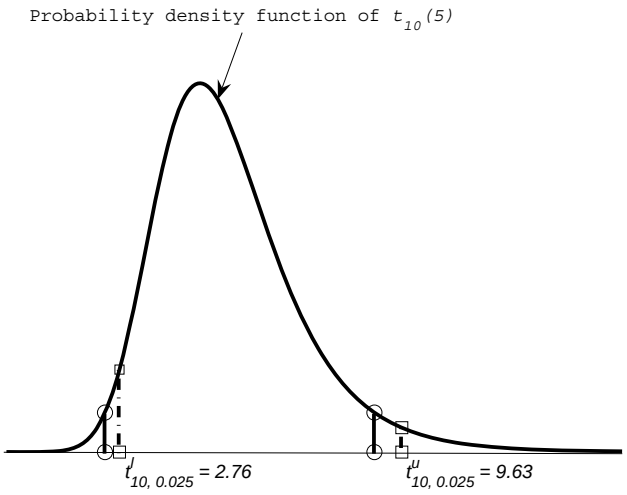

Figure 9: A comparison of the standard acceptance set and the shortest possible acceptance set for $\delta = 5$, $\nu = 10$, and $\alpha = 0.05$. The standard acceptance set is (2.76, 9.63) while the shortest possible acceptance set is (2.40, 8.97)

3.2.2 Computational results of shortest acceptance sets $A^0_{\nu,\alpha}(\delta)$

Define two functions $\lambda = f_{\nu,\delta}(t_1) = f_{\nu,\delta}(t_2)$, where $t_1 < m_\nu(\delta) < t_2$ and $G(\lambda) \to 1 - \alpha - \mathbb{P}(t_1 \le t_\nu(\delta) \le t_2)$ respectively. When $\lambda = \lambda_0$ in $\lambda \in [0, \lambda_0]$, where $\lambda_0 = f_{\nu,\delta}(m_\nu(\delta))$ and $m_\nu(\delta)$ is mode of density $f_{\nu,\delta}(\cdot)$ of noncentral $t-$distribution, then we have $G(\lambda) = 1 - \alpha$. And if $\lambda \to 0$, then $G(\lambda) = -\alpha$. Thus as λ decreases from λ_0 to 0, the unique solution of shortest acceptance set $A^0_{\nu,\alpha}(\delta)$ can be obtained, and the bisection algorithm may be employed to find the shortest possible acceptance interval $A^0_{\nu,\alpha}(\delta)$ by the Corollary 3.2.1. For given δ, ν, and α, the computer programs will stop if it satisfies

$$\left|\mathbb{P}(l^0_{\nu,\alpha}(\delta) \le t_\nu(\delta) \le u^0_{\nu,\alpha}(\delta)) - (1-\alpha)\right| < 10^{-10}$$

$$\left|\int_0^\infty x^\nu e^{-(x^2+(l^0_{\nu,\alpha}(\delta)x\nu^{-1/2}-\delta)^2)/2}dx - \int_0^\infty x^\nu e^{-(x^2+(u^0_{\nu,\alpha}(\delta)x\nu^{-1/2}-\delta)^2)/2}dx\right| < 10^{-10}$$

Figure 9 is an example of the comparisons between standard acceptance set $A^s_{\nu,\alpha}(\delta)$

$= (t^l_{\nu,\alpha/2}(\delta),\ t^u_{\nu,\alpha/2}(\delta))$, and shortest possible acceptance set $A^0_{\nu,\alpha}(\delta) = (l^0_{\nu,\alpha}(\delta),\ u^0_{\nu,\alpha}(\delta))$ with $\nu = 10$, $\delta = 5$ and $\alpha = 0.05$. The computational results are $A^s_{10,0.05}(5) = (2.7557,$ 9.6330) and $A^0_{10,0.05}(5) = (2.4036,\ 8.9667)$ and the ratio of two lengths $|A^0_{10,0.05}(5)|/$ $|A^s_{10,0.05}(5)| = 0.9543$. Thus, we could conjecture from properties of skewness of noncentral $t-$distribution described in section 2.4.2 that

- For $\delta > 0$, the two critical points $l^0_{\nu,\alpha}(\delta)$ and $u^0_{\nu,\alpha}(\delta)$ of shortest possible acceptance set are always smaller than the $t^l_{\nu,\alpha/2}(\delta)$ and $t^u_{\nu,\alpha/2}(\delta)$ of standard.

- For $\delta > 0$, the difference of $t^u_{\nu,\alpha/2}(\delta) - u^0_{\nu,\alpha}(\delta)$ is always larger than that of $t^l_{\nu,\alpha/2}(\delta) - l^0_{\nu,\alpha}(\delta)$, and we could conjecture that this gain increases as skewness increases.

- The ratio of two lengths, $|A^0_{\nu,\alpha}(\delta)|/|A^s_{\nu,\alpha}(\delta)|$, for given ν is monotonically decreasing in $|\delta|$ and looks converging to its limiting point as in the property of skewness.

- Accordingly, the lower tail probability of $\mathbb{P}(t_\nu(\delta) \leq l^0_{\nu,\alpha}(\delta))$ is also decreasing and looks converging to its limiting point as in the property of skewness.

Figure 10 and Table 5 are one of examples of computational results discussed above for $\nu = 10$ and $\alpha = 0.05$. As expected, Figure 10 and Table 5 show that the ratio of length of shortest possible set $|A^0_{\nu,\alpha}(\delta)|$ to the length of standard acceptance set $|A^s_{\nu,\alpha}(\delta)|$ for $\alpha = 0.05$ and $\nu = 10$ keeps decreasing in δ and converges to its limit ratio, and that the lower tail probability of $\mathbb{P}(t_\nu(\delta) \leq l^0_{\nu,\alpha}(\delta))$ is also decreasing in δ and looks converging to its limit probability as well in the property of skewness.

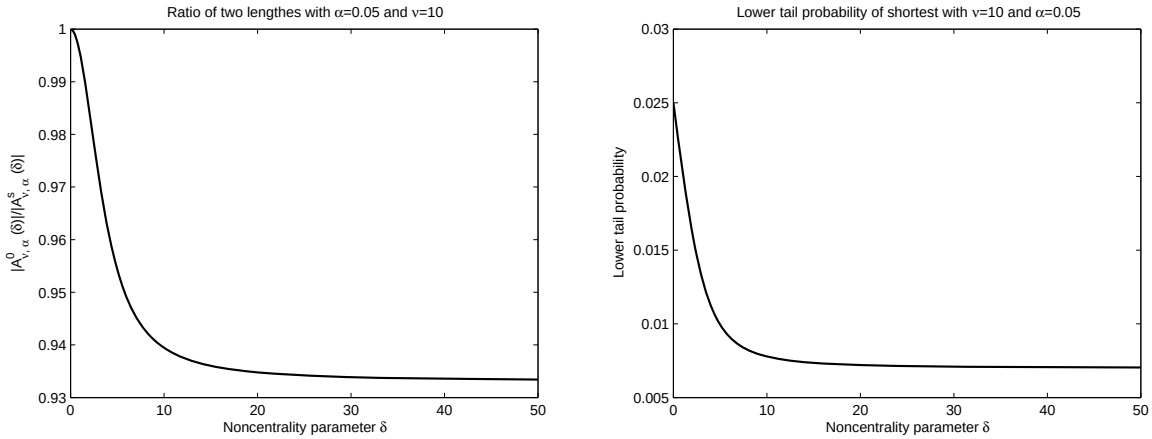

Figure 10: The left plot shows ratios of length of shortest possible set $|A^0_{\nu,\alpha}(\delta)|$ to the length of standard acceptance set $|A^s_{\nu,\alpha}(\delta)|$ for $\alpha = 0.05$ and $\nu = 10$, and the right plot is the lower tail probabilities of $\mathbb{P}(t_\nu(\delta) \leq l^0_{\nu,\alpha}(\delta))$ for $\alpha = 0.05$ and $\nu = 10$

Table 5: Comparisons of shortest possible acceptance set $A^0_{\nu,\alpha}(\delta)$ for standard $A^s_{\nu,\alpha}(\delta)$ for $\alpha = 0.05$ and $\nu = 10$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert / \lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.23, 2.23)	4.4563	(-2.23, 2.23)	4.4563	1.0000
5	(2.76, 9.63)	6.8774	(2.40, 8.97)	6.5631	0.9543
10	(6.59, 18.03)	11.4362	(5.99, 16.73)	10.7439	0.9395
20	(13.77, 35.35)	21.5731	(12.63, 32.79)	20.1662	0.9348
30	(20.83, 52.82)	31.9872	(19.12, 49.00)	29.8723	0.9339
50	(34.86, 87.85)	52.9917	(32.03, 81.49)	49.4634	0.9334
100	(69.83, 175.54)	105.7122	(64.20, 162.85)	98.6528	0.9332

The computational results of shortest possible acceptance sets $A^0_{\nu,\alpha}(\delta)$ and standard $A^s_{\nu,\alpha}(\delta)$ for $\alpha = 0.1, 0.05, 0.01$, $\nu = 5, 10, 20, 30$, and $\delta = 5, 10, 20, 30, 50, 100$ are summarized in Table 17 – Table 19 in Appendix A respectively. The inspection of these Tables may enable us to conjecture that

- For given ν and α, the ratio, $|A^0_{\nu,\alpha}(\delta)|/|A^s_{\nu,\alpha}(\delta)|$, is monotonically decreasing in δ.

- For given size α and δ, this ratio, $|A^0_{\nu,\alpha}(\delta)|/|A^s_{\nu,\alpha}(\delta)|$, is monotonically increases in ν.

3.2.3 Inefficient confidence intervals $C^0_{\nu,\alpha}(t)$ based on the shortest possible acceptance set

By contraries to our expectation, the inversion of the shortest acceptance sets $A^0_{\nu,\alpha}(\delta)$ is disappointing because the resulting confidence intervals $C^0_{\nu,\alpha}(t)$

$$C^0_{\nu,\alpha}(t) = \{\delta \,:\, l^0_{\nu,\alpha/2}(\delta) \leq t \leq u^0_{\nu,\alpha/2}(\delta)\}$$

are longer than those provided by the standard method. Figure 11 and Table 6 show how the resulting confidence interval lengths can be between 8.7% and 9.7% longer than the standard confidence interval lengths for $\nu = 10$ and $\alpha = 0.05$. The ratio of the length of confidence interval $C^0_{\nu,\alpha}(t)$ based on shortest possible acceptance sets to the length of standard confidence interval keeps increasing in $|t| > 0$. As Figure 12 shows, while $d_1 < d_2$ so that the new acceptance set is shorter, unfortunately $L_1 > L_2$ so that the resulting confidence interval is longer.

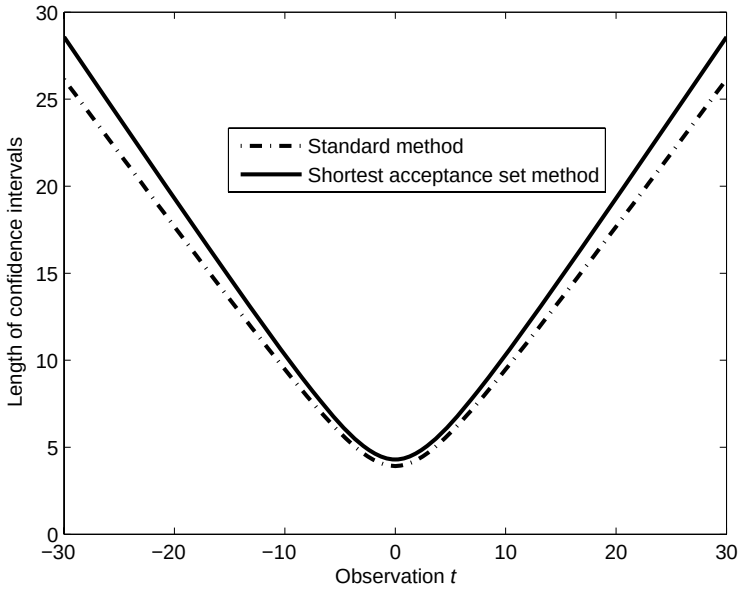

Figure 11: A comparison of the lengths of the confidence intervals provided by the standard method and the shortest acceptance set method for $\nu = 10$ and $\alpha = 0.05$.

Consequently, this analysis has demonstrated that confidence interval methodologies based upon the construction of the shortest possible acceptance sets are inefficient in comparison to the standard method. However, this failure can be used to guide the construction of methodologies that are uniformly better than the standard method, as shown in the section 3.4, which is that in order to obtain uniformly shorter confidence interval, the lower tail probability $\mathbb{P}(t_\nu(\delta) \leq l_{\nu,\alpha}(\delta))$ of efficient acceptance set $A_{\nu,\alpha}(\delta)$, which results in uniformly shorter confidence intervals than standard, would be larger than $\alpha/2$.

Computational results of confidence intervals $C^0_{\nu,\alpha}(t)$ for $\nu = 5, 10, 20, 30$, $\alpha = 0.1, 0.05, 0.01$, and $t = 0, 5, 10, 15, 20, 30$ are summarized in Table 20 – Table 22 in Appendix A.

Table 6: Comparisons of confidence interval $C^0_{\nu,\alpha}(t)$ based on shortest possible acceptance set for standard confidence interval $C^s_{\nu,\alpha}(t)$ for $\alpha = 0.05$ and $\nu = 10$

t	$C^s_{\nu,\alpha}(t)$	length	$C^0_{\nu,\alpha}(t)$	length	$\lvert C^0_{\nu,\alpha}(t)\rvert / \lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.96, 1.96)	3.9199	(-2.15, 2.15)	4.3021	1.0975
5	(2.03, 7.86)	5.8332	(2.23, 8.57)	6.3432	1.0874
10	(5.22, 14.70)	9.4740	(5.68, 16.00)	10.3134	1.0886
15	(8.22, 21.73)	13.5102	(8.91, 23.64)	14.7379	1.0909
20	(11.15, 28.82)	17.6743	(12.05, 31.35)	19.3016	1.0921
30	(16.93, 43.07)	26.1413	(18.27, 46.85)	28.5751	1.0931

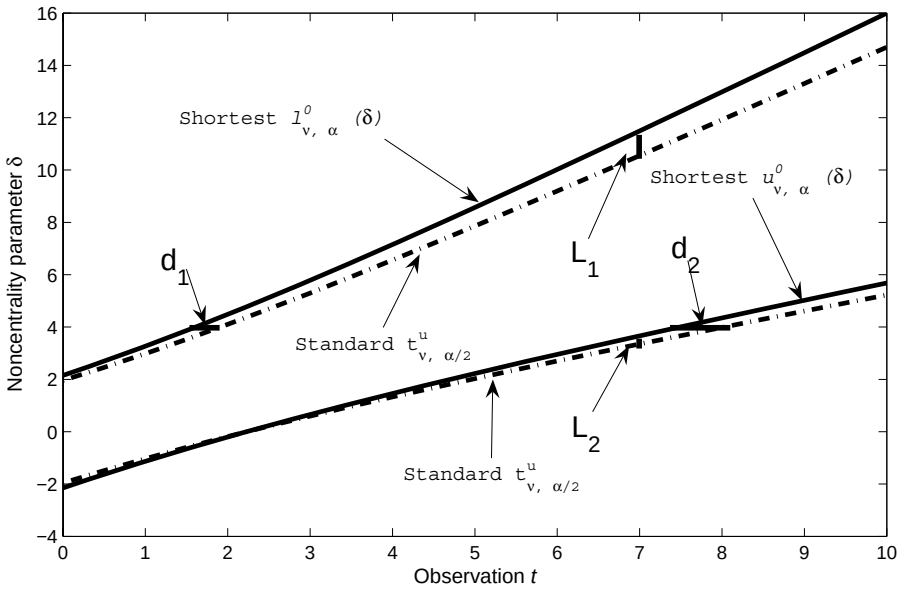

Figure 12: Since $L_2 < L_1$ the confidence interval obtained from the standard procedure is shorter than the confidence interval obtained from the shortest acceptance set procedure.

3.3 Inefficient confidence interval methodologies based on UMPU acceptance sets

3.3.1 UMPU acceptance sets $A^u_{\nu,\alpha}(\delta)$

As we reviewed in section 2.1.2 that the standard confidence interval of mean μ of normal distribution or $t-$distribution is unbeatable, the acceptance set $A^u_{\nu,\alpha}(\delta_0) = (l^u_{\nu,\alpha}(\delta_0),\, u^u_{\nu,\alpha}(\delta_0))$ of uniformly most powerful unbiased(UMPU) test for $H_0 : \delta = \delta_0 \neq 0$ against two-sided alternatives of noncentral $t-$distribution might result in uniformly shorter confidence interval than standard. The conditions for UMPU acceptance set $A^u_{\nu,\alpha}(\delta_0)$ of noncentral $t-$distribution on page 303 of Lehmann [10] should satisfy

$$\begin{aligned} \mathbb{P}(l^u_{\nu,\alpha}(\delta_0) \leq t_\nu(\delta_0) \leq u^u_{\nu,\alpha}(\delta_0)) &= 1-\alpha \\ \frac{\partial}{\partial\delta}\,\mathbb{P}_\delta(l^u_{\nu,\alpha} \leq t_\nu(\delta) \leq u^u_{\nu,\alpha})\Big|_{\delta=\delta_0} &= 0 \end{aligned} \tag{58}$$

The noncentral $t-$distribution can be shown to be STP_∞ and hence in particular STP_3. It follows that there exist a UMP unbiased one with acceptance region satisfying (58).

Corollary 3.3.1 *The conditions* (58) *can be equivalent to* (59)

$$\begin{aligned} &\mathbb{P}(l^u_{\nu,\alpha}(\delta_0) \leq t_\nu(\delta_0) \leq u^u_{\nu,\alpha}(\delta_0)) = 1-\alpha \\ &\int_0^\infty x^{\nu-1}e^{-\frac{1}{2}(x^2+(l^u_{\nu,\alpha}(\delta_0)x\nu^{-1/2}-\delta_0)^2)}dx = \int_0^\infty x^{\nu-1}e^{-\frac{1}{2}(x^2+(u^u_{\nu,\alpha}(\delta_0)x\nu^{-1/2}-\delta_0)^2)}dx \end{aligned} \tag{59}$$

Proof : It is sufficient to show second condition of (59) is equivalent to the second condition of (58). The partial derivative of $\mathbb{P}_\delta(t_\nu(\delta) \leq u^u_{\nu,\alpha}(\delta))$ with respect to δ gives

us

$$\begin{aligned}
\frac{\partial}{\partial\delta}\mathbb{P}_\delta(t_\nu(\delta)\le u^u_{\nu,\alpha}(\delta)) &= \frac{\partial}{\partial\delta}\frac{1}{\Gamma(\nu/2)2^{\nu/2-1}}\int_0^\infty x^{\nu-1}e^{-x^2/2}\int_{-\infty}^{u^u_{\nu,\alpha}(\delta)x\nu^{-1/2}-\delta}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dzdx\\
&= -\frac{1}{\Gamma(\nu/2)2^{\nu/2-1}}\int_0^\infty x^{\nu-1}e^{-x^2/2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u^u_{\nu,\alpha}(\delta)x\nu^{-1/2}-\delta)^2}dx\\
&= -\frac{1}{\sqrt{2\pi}\Gamma(\nu/2)2^{\nu/2-1}}\int_0^\infty x^{\nu-1}e^{-\frac{1}{2}[x^2+(u^u_{\nu,\alpha}(\delta)x\nu^{-1/2}-\delta)^2]}dx\\
\frac{\partial}{\partial\delta}\mathbb{P}_\delta(t_\nu(\delta)\le l^u_{\nu,\alpha}(\delta)) &= -\frac{1}{\sqrt{2\pi}\Gamma(\nu/2)2^{\nu/2-1}}\int_0^\infty x^{\nu-1}e^{-\frac{1}{2}[x^2+(l^u_{\nu,\alpha}(\delta)x\nu^{-1/2}-\delta)^2]}dx
\end{aligned}$$

Thus we have that

$$\begin{aligned}
&\frac{\partial}{\partial\delta}\mathbb{P}_\delta(l^u_{\nu,\alpha}\le t_\nu(\delta)\le u^u_{\nu,\alpha})\Big|_{\delta=\delta_0}=0 \quad\Leftrightarrow\\
&\int_0^\infty x^{\nu-1}e^{-\frac{1}{2}(x^2+(l^u_{\nu,\alpha}(\delta_0)x\nu^{-1/2}-\delta_0)^2)}dx=\int_0^\infty x^{\nu-1}e^{-\frac{1}{2}(x^2+(u^u_{\nu,\alpha}(\delta_0)x\nu^{-1/2}-\delta_0)^2)}dx
\end{aligned}$$

■

In particular, for $\delta_0 = 0$ this condition results in $u^u_{\nu,\alpha}(0) = -l^u_{\nu,\alpha}(0)$, which is exactly the same as the standard acceptance set of $t-$distribution.

3.3.2 Computational results of UMPU acceptance sets $A^u_{\nu,\alpha}(\delta)$

The conditions for UMPU acceptance set $A^u_{\nu,\alpha}(\delta)$ have similar form of those of shortest possible acceptance set $A^0_{\nu,\alpha}(\delta)$, and the bisection algorithm may be effective for obtaining numerical solutions of $A^u_{\nu,\alpha}(\delta)$. For given δ_0, ν, and α, the computer programs of Matlab will stop if it satisfies

$$\begin{aligned}
&\Big|\mathbb{P}(l^u_{\nu,\alpha}(\delta_0)\le t_\nu(\delta_0)\le u^u_{\nu,\alpha}(\delta_0))-(1-\alpha)\Big|<10^{-10}\\
&\left|\int_0^\infty x^{\nu-1}e^{-\frac{1}{2}(x^2+(l^u_{\nu,\alpha}(\delta_0)x\nu^{-1/2}-\delta_0)^2)}dx-\int_0^\infty x^{\nu-1}e^{-\frac{1}{2}(x^2+(u^u_{\nu,\alpha}(\delta_0)x\nu^{-1/2}-\delta_0)^2)}dx\right|<10^{-10}
\end{aligned}$$

The comparison of three acceptance sets for $\nu = 10$, $\delta = 5$ and $\alpha = 0.05$, which are that the standard acceptance set is (2.76, 9.63), and the shortest possible acceptance

set is (2.40, 8.97), and the UMPU acceptance set is (2.6078, 9.2538), are made in Figure 13. Figure 14 represents comparisons of standard, shortest, and UMPU acceptance sets for $\nu = 5$, $\alpha = 0.1$, $\delta = 0$ (left) and $\delta = 20$ (right) respectively. As expected, all three acceptance sets are the same for $\delta = 0$, while standard is (13.32 41.93), shortest is (11.27 36.22), and UMPU is (12.54, 38.55) for $\delta = 20$. Accordingly the length of UMPU acceptance set is longer than that of shortest but shorter than that of standard. Table 7 shows a comparison of standard $A^s_{\nu,\alpha}(\delta)$, shortest $A^0_{\nu,\alpha}(\delta)$ and UMPU acceptance set $A^u_{\nu,\alpha}(\delta)$ for $\alpha = 0.1$ and $\nu = 5$. As in Figure 14, the lengths of UMPU acceptance sets in the Table 7 are longer than those of shortest acceptance sets but shorter than those of standard.

The computational results of UMPU acceptance sets $A^u_{\nu,\alpha}(\delta)$ and standard $A^s_{\nu,\alpha}(\delta)$ for $\alpha = 0.1, 0.05, 0.01$, $\nu = 5, 10, 20, 30$, and $\delta = 5, 10, 20, 30, 50, 100$ are summarized in Table 23 – Table 25 in Appendix A respectively.

3.3.3 Inefficient UMPU confidence intervals $C^u_{\nu,\alpha}(t)$

Similarly to the confidence interval $C^0_{\nu,\alpha}(t)$ based on shortest acceptance set, inspection of Figure 14 may suggest that the confidence interval $C^u_{\nu,\alpha}(t)$

$$C^u_{\nu,\alpha}(t) = \left\{\delta : l^u_{\nu,\alpha}(\delta) \leq t \leq u^u_{\nu,\alpha}(\delta)\right\}$$

based on UMPU acceptance sets would not be uniformly shorter than that of standard $C^s_{\nu,\alpha}(t)$. Figure 15 shows comparisons of three confidence intervals, which are based on standard, shortest acceptance sets, and UMPU acceptance sets for $\nu = 5$, $\alpha = 0.1$, $t = 0$ (left plot) and $t = 20$ (right plot) respectively. This Figure 15 shows

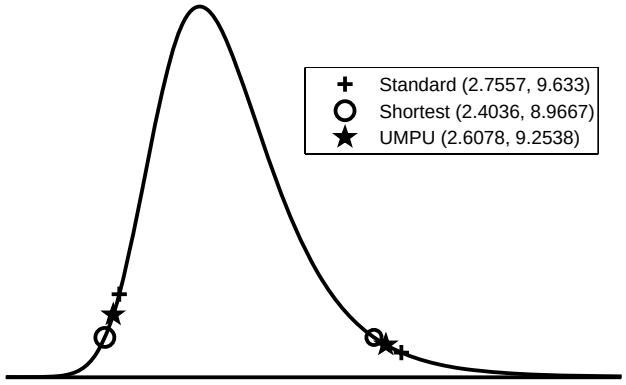

Figure 13: A comparison of the standard acceptance set,the shortest possible acceptance set, and UMPU acceptance set for $\delta = 5$, $\nu = 10$, and $\alpha = 0.05$. The standard acceptance set is (2.76, 9.63) and the shortest possible acceptance set is (2.40, 8.97) while the UMPU acceptance set is (2.61, 9.25)

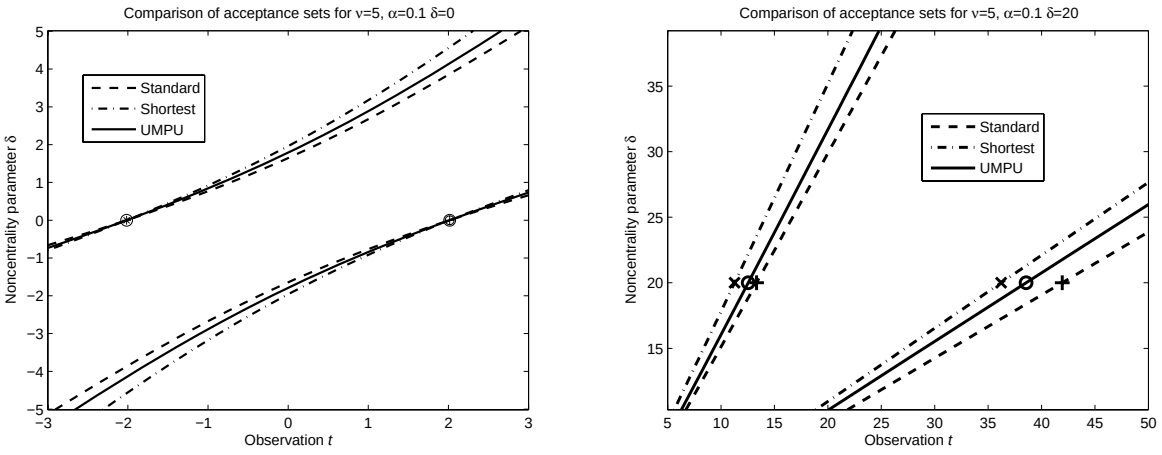

Figure 14: Comparisons of the standard acceptance set, the shortest possible acceptance set, and UMPU acceptance set for $\nu = 5$ and $\alpha = 0.1$. The left plot is for $\delta = 0$ and the right plot is for $\delta = 20$

Table 7: Comparisons of standard $A^s_{\nu,\alpha}(\delta)$, shortest $A^0_{\nu,\alpha}(\delta)$ and UMPU acceptance set $A^u_{\nu,\alpha}(\delta)$ for $\alpha = 0.1$ and $\nu = 5$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$
0	(-2.02, 2.02)	4.03	(-2.02, 2.02)	4.03
5	(2.90, 11.00)	8.10	(2.30, 9.53)	7.23
10	(6.48, 21.18)	14.70	(5.42, 18.30)	12.88
20	(13.32, 41.93)	28.61	(11.27, 36.22)	24.95
30	(20.08, 62.78)	42.69	(17.03, 54.23)	37.20
50	(33.55, 104.52)	70.97	(28.48, 90.29)	61.81
100	(67.18, 208.96)	141.77	(57.04, 180.49)	123.45

δ	UMPU	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$	$\lvert A^u_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.02, 2.02)	4.03	1.0000	1.0000
5	(2.65, 10.10)	7.45	0.8924	0.9199
10	(6.07, 19.46)	13.39	0.8763	0.9109
20	(12.54, 38.55)	26.01	0.8721	0.9089
30	(18.93, 57.72)	38.79	0.8713	0.9086
50	(31.64, 96.11)	64.47	0.8709	0.9084
100	(63.36, 192.14)	128.77	0.8707	0.9083

that the lengths of UMPU confidence interval are longer than those of standard, while they are shorter than those of intervals based on shortest acceptance sets. As an example of computational results, the comparisons of confidence intervals of standard method, UMPU method and shortest acceptance sets method for $\nu = 5$, $\alpha = 0.1$ and $t = 0, 5, 10, 15, 20, 30$, are made in following Table 8. The two ratios of lengths of confidence intervals, $|C^0_{\nu,\alpha}(t)|/|C^s_{\nu,\alpha}(t)|$ and $|C^u_{\nu,\alpha}(t)|/|C^s_{\nu,\alpha}(t)|$, increase in $|t|$ as expected, while we have $|C^0_{\nu,\alpha}(t)| < |C^u_{\nu,\alpha}(t)| < |C^s_{\nu,\alpha}(t)|$.

Consequently, this approach like shortest acceptance sets method has demonstrated that UMPU confidence interval $C^u_{\nu,\alpha}(t)$ method is inefficient as well. The computational results of $C^u_{\nu,\alpha}(t)$ for $\nu = 5, 10, 20, 30$, $\alpha = 0.1, 0.05, 0.01$ and $t = 0, 5, 10, 15, 20, 30$ are summarized in Table 26 – Table 28 in Appendix A respectively.

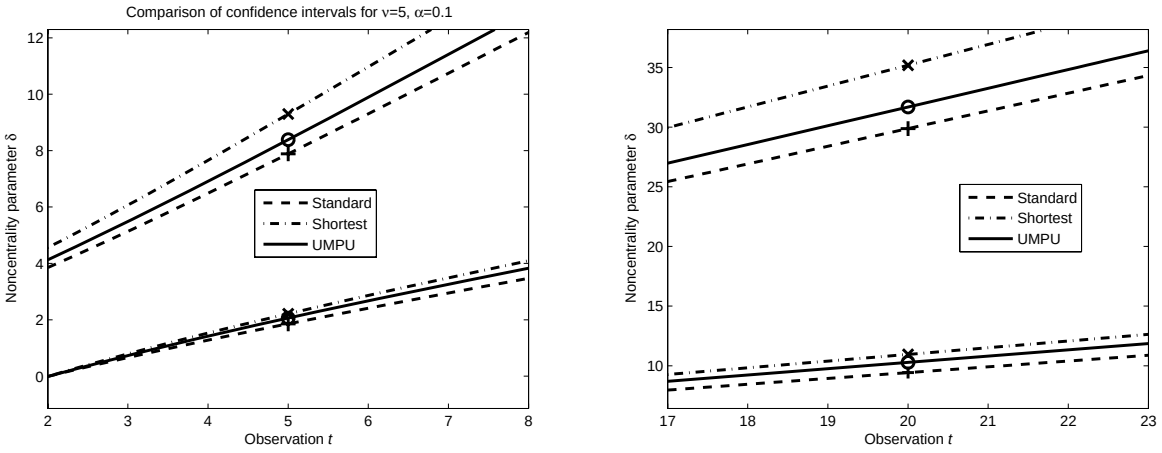

Figure 15: Comparisons of confidence intervals of the standard method, the shortest acceptance method, and UMPU method for $\nu = 5$ and $\alpha = 0.1$. The left plot is for observation $t = 5$ and the right plot is for $t = 20$

Table 8: Comparisons of confidence intervals of UMPU method $C^u_{\nu,\alpha}(t)$ for standard $C^s_{\nu,\alpha}(t)$, $C^0_{\nu,\alpha}(t)$ by shortest acceptance sets method for $\alpha = 0.1$ and $\nu = 5$

t	standard	$\lvert C^s_{\nu,\alpha}(t)\rvert$	shortest	$\lvert C^0_{\nu,\alpha}(t)\rvert$	UMPU	$\lvert C^u_{\nu,\alpha}(t)\rvert$
0	(-1.65, 1.65)	3.29	(-1.96, 1.96)	3.92	(-1.79, 1.79)	3.58
5	(1.86, 7.88)	6.03	(2.21, 9.30)	7.08	(2.06, 8.39)	6.32
10	(4.50, 15.11)	10.62	(5.27, 17.80)	12.53	(4.94, 16.04)	11.09
15	(6.98, 22.48)	15.49	(8.14, 26.47)	18.34	(7.64, 23.84)	16.20
20	(9.42, 29.88)	20.45	(10.95, 35.19)	24.23	(10.28, 31.68)	21.40
30	(14.26, 44.72)	30.46	(16.54, 52.66)	36.12	(15.53, 47.41)	31.88

t	$\lvert C^0_{\nu,\alpha}(t)\rvert / \lvert C^s_{\nu,\alpha}(t)\rvert$	$\lvert C^u_{\nu,\alpha}(t)\rvert / \lvert C^s_{\nu,\alpha}(t)\rvert$
0	1.1923	1.0876
5	1.1751	1.0493
10	1.1801	1.0450
15	1.1834	1.0458
20	1.1848	1.0463
30	1.1860	1.0468

3.4 Efficient confidence interval methodologies that are uniformly better than the standard method

The objective of this chapter is to find functions $l_{\nu,\alpha}(\delta)$ and $u_{\nu,\alpha}(\delta)$ satisfying

$$\mathbb{P}_\delta(l_{\nu,\alpha}(\delta) \leq t_\nu(\delta) \leq u_{\nu,\alpha}(\delta)) = 1 - \alpha \tag{60}$$

for all $\delta \in \Re$ such that the resulting confidence intervals $C_{\nu,\alpha}(t)$ have shorter lengths than those obtained from the standard method. While it is easy to achieve this goal at a given value of t, the results given in this section show that there are in fact functions $l_{\nu,\alpha}(\delta)$ and $u_{\nu,\alpha}(\delta)$ that achieve this goal for all observed values of $t \in \Re$.

Let $m_\nu(\delta)$ be the point at which the probability density function of a $t_\nu(\delta)$ distribution is maximized, and consider the function

$$\xi_{\nu,\alpha}(\delta) = \frac{u_{\nu,\alpha}(\delta) - m_\nu(\delta)}{m_\nu(\delta) - l_{\nu,\alpha}(\delta)}.$$

Notice that the function $\xi_{\nu,\alpha}(\delta)$ together with condition (60) specifies the functions $l_{\nu,\alpha}(\delta)$ and $u_{\nu,\alpha}(\delta)$. For example, as $\xi_{\nu,\alpha}(\delta)$ tends to infinity then $l_{\nu,\alpha}(\delta)$ tends to $t^l_{\nu,\alpha}(\delta)$ and $u_{\nu,\alpha}(\delta)$ tends to infinity, while if $\xi_{\nu,\alpha}(\delta)$ tends to zero then $l_{\nu,\alpha}(\delta)$ tends to negative infinity and $u_{\nu,\alpha}(\delta)$ tends to $t^u_{\nu,\alpha}(\delta)$. In general, it is sensible to have a procedure for which the confidence interval obtained at an observation $-t$ is the negative of the confidence interval obtained at t, and this property is satisfied if

$$\xi_{\nu,\alpha}(\delta) = \frac{1}{\xi_{\nu,\alpha}(-\delta)}. \tag{61}$$

Figure 16 shows the function $\xi_{10,0.05}(\delta)$, $\delta \geq 0$, for the standard method, for the method based on shortest acceptance sets described in section 3.2, and for the method based on UMPU acceptance sets in section 3.3. The functions for negative values of

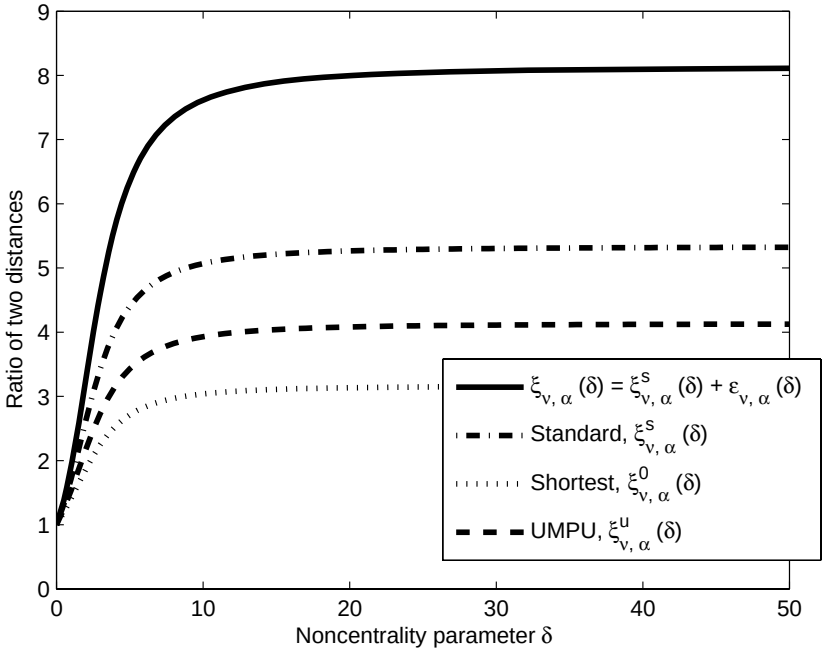

Figure 16: A comparison of the functions $\xi_{\nu,\alpha}(\delta)$ for the standard method, UMPU method, and the shortest acceptance set method

δ are obtained from the relationship (61). For $\delta \geq 0$ the function for the method based on shortest acceptance sets and the function for the method based on UMPU acceptance sets can be seen to be below the function for the standard method, and since the confidence intervals for these two methods are worse than the standard confidence intervals, this suggests that it would be better to have a function $\xi_{\nu,\alpha}(\delta)$, $\delta \geq 0$, that is larger than the corresponding function for the standard method as depicted in Figure 16.

Specifically, if $\xi^s_{\nu,\alpha}(\delta)$ is the function for the standard method, then attention has been directed towards functions

$$\xi_{\nu,\alpha}(\delta) = \xi^s_{\nu,\alpha}(\delta) + \epsilon_{\nu,\alpha}(\delta)$$

for $\delta \geq 0$, with $\epsilon_{\nu,\alpha}(\delta) \geq 0$ for $\delta \geq 0$. Notice that this is equivalent to choosing acceptance sets for $\delta > 0$ with upper tail probabilities strictly less than $\alpha/2$. Since $\xi^s_{\nu,\alpha}(\delta)$ could be thought of as new measure of skewness, the efficient confidence intervals could be obtained by considering $\epsilon_{\nu,\alpha}(\delta)$ as being function of $\xi^s_{\nu,\alpha}(\delta)$

$$\epsilon_{\nu,\alpha}(\delta) = h(\xi^s_{\nu,\alpha}(\delta) - 1)$$

A certain amount of trial and error is required in searching for the best choice of $\epsilon_{\nu,\alpha}(\delta) \geq 0$ (discussed in the Appendix B in detail), for example the four following functions of simple form are investigated.

$$\epsilon_{\nu,\alpha}(\delta) = \gamma(\xi^s_{\nu,\alpha}(\delta) - 1)$$

$$\epsilon_{\nu,\alpha}(\delta) = \gamma\sqrt{\xi^s_{\nu,\alpha}(\delta) - 1}$$

$$\epsilon_{\nu,\alpha}(\delta) = \gamma(\xi^s_{\nu,\alpha}(\delta) - 1)^2$$

$$\epsilon_{\nu,\alpha}(\delta) = \gamma(\xi^s_{\nu,\alpha}(\delta) - 1)^{1.5}$$

But good results have been found with functions of the form

$$\epsilon_{\nu,\alpha}(\delta) = \gamma(\xi^s_{\nu,\alpha}(\delta) - 1)^{1.5}$$

where the best choice of γ depends upon the particular values of ν and α. In fact, for given ν and α, there is $\gamma^{\max}_{\nu,\alpha}$ such that the new confidence intervals are uniformly shorter than standard confidence intervals for all γ with $0 < \gamma \leq \gamma^{\max}_{\nu,\alpha}$. The best improvement for t close to zero is obtained by taking $\gamma = \gamma^{\max}_{\nu,\alpha}$.

For example, with $\nu = 5$ and $\alpha = 0.05$ it is found that good results are obtained with $\gamma = 0.310$. This results in uniformly shorter confidence intervals than the

standard method for all values of t, and the improvement is between 6.13% and 0.40% for $|t| \leq 20$ as shown in Table 10, with the greatest improvement being for value of t close to zero. A comparison of acceptance sets by new method $A_{\nu,\alpha}(\delta)$ for standard $A^s_{\nu,\alpha}(\delta)$, shortest $A^0_{\nu,\alpha}(\delta)$ and UMPU acceptance set $A^u_{\nu,\alpha}(\delta)$ for $\alpha = 0.05$ and $\nu = 5$ is made in Table 9. As expected, Table 9 and Figure 17 represent that the lower tail probabilities of the acceptance sets of new method $A_{\nu,\alpha}(\delta)$ are strictly greater than $\alpha/2$ (see the left plot of Figure 17) and that the lengths of acceptance sets of new method $A_{\nu,\alpha}(\delta)$ in comparison with shortest acceptance sets are strictly greater than those of standard method (see the right plot of Figure 17). A comparison of the functions $l_{5,0.05}(\delta)$ and $u_{5,0.05}(\delta)$ for the standard method, shortest acceptance set method, UMPU method, and the new method with $\gamma = 0.310$ is made in Figure 18. Comparisons of confidence intervals of new method for standard method, shortest acceptance sets method, and UMPU method for $\nu = 5$, $\alpha = 0.05$, and $t = 0$ (left plot) and $t = 10$ (right plot) are made in Figure 19, which shows efficiently how new method is able to have paramount advantage over standard method. A comparison of lengths of confidence intervals of new method for those of standard, of shortest acceptance sets, and of UMPU acceptance sets with $\nu = 5$ and $\alpha = 0.05$ is made in Figure 20. The left plot of Figure 20 is for $-5 \leq t \leq 5$ and right plot is for $-10 \leq t \leq 10$.

With $\nu = 20$ and $\alpha = 0.05$ the value of $\gamma = 0.215$ is found to provide uniformly shorter confidence intervals for all value of t, and the improvement is between 1.24% and 0.13% for $|t| \leq 20$ as shown in Table 11. Comparisons of confidence intervals of new method for standard, shortest acceptance sets, and UMPU acceptance sets for

Table 9: Comparisons of acceptance set $A_{\nu,\alpha}(\delta)$ of new method for standard $A^s_{\nu,\alpha}(\delta)$, shortest $A^0_{\nu,\alpha}(\delta)$ and UMPU acceptance set $A^u_{\nu,\alpha}(\delta)$ with $\alpha = 0.05$ and $\nu = 5$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	UMPU
0	(-2.57, 2.57)	5.14	(-2.57, 2.57)	5.14	(-2.57, 2.57)
5	(2.57, 12.98)	10.41	(2.00, 11.28)	9.28	(2.34, 11.90)
10	(5.97, 24.91)	18.94	(4.98, 21.57)	16.59	(5.59, 22.82)
20	(12.35, 49.25)	36.90	(10.45, 42.63)	32.18	(11.65, 45.13)
30	(18.63, 73.71)	55.07	(15.82, 63.81)	47.99	(17.60, 67.56)
50	(31.16, 122.71)	91.55	(26.48, 106.23)	79.75	(29.43, 112.48)
100	(62.39, 245.30)	182.91	(53.05, 212.35)	159.30	(58.96, 224.85)

δ	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	new method	$\lvert A_{\nu,\alpha}(\delta)\rvert$	$\frac{\lvert A^0_{\nu,\alpha}(\delta)\rvert}{\lvert A^s_{\nu,\alpha}(\delta)\rvert}$	$\frac{\lvert A^u_{\nu,\alpha}(\delta)\rvert}{\lvert A^s_{\nu,\alpha}(\delta)\rvert}$	$\frac{\lvert A_{\nu,\alpha}(\delta)\rvert}{\lvert A^s_{\nu,\alpha}(\delta)\rvert}$
0	5.14	(-2.57, 2.57)	5.14	1.0000	1.0000	1.0000
5	9.56	(2.77, 15.50)	12.73	0.8912	0.9181	1.2234
10	17.22	(6.30, 30.33)	24.03	0.8760	0.9094	1.2690
20	33.49	(12.98, 60.27)	47.29	0.8721	0.9076	1.2817
30	49.96	(19.58, 90.29)	70.72	0.8714	0.9072	1.2840
50	83.04	(32.72, 150.39)	117.67	0.8711	0.9071	1.2853
100	165.90	(65.51, 300.69)	235.18	0.8709	0.9070	1.2858

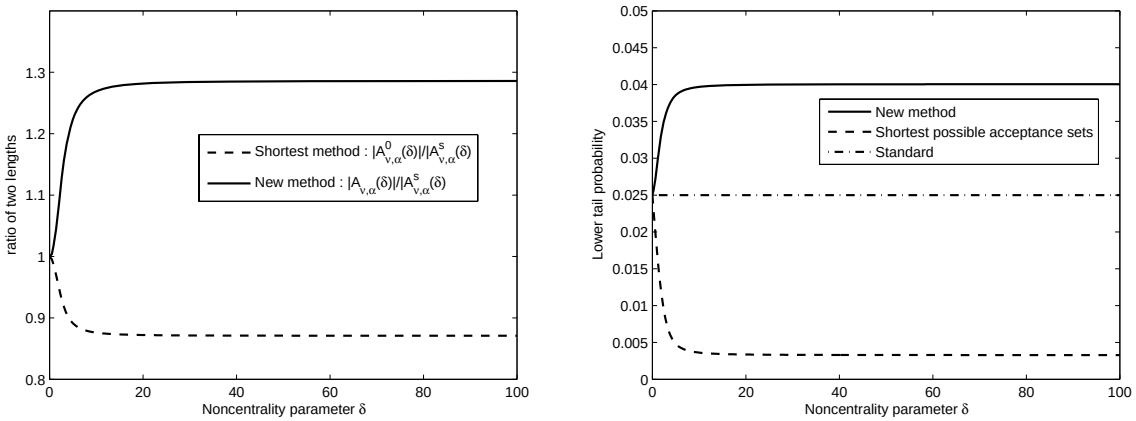

Figure 17: The left plot shows comparison of two ratios, which are ratio of length of shortest possible set $|A^0_{\nu,\alpha}(\delta)|$ to the length of standard acceptance set $|A^s_{\nu,\alpha}(\delta)|$ and ratio of length of new method $|A_{\nu,\alpha}(\delta)|$ to the length of standard for $\alpha = 0.05$ and $\nu = 10$, and the right plot reveals comparison of the two lower tail probabilities, which are $\mathbb{P}(t_\nu(\delta) \leq l^0_{\nu,\alpha}(\delta))$ of shortest method and $\mathbb{P}(t_\nu(\delta) \leq l_{\nu,\alpha}(\delta))$ of new method for $\alpha = 0.05$ and $\nu = 5$

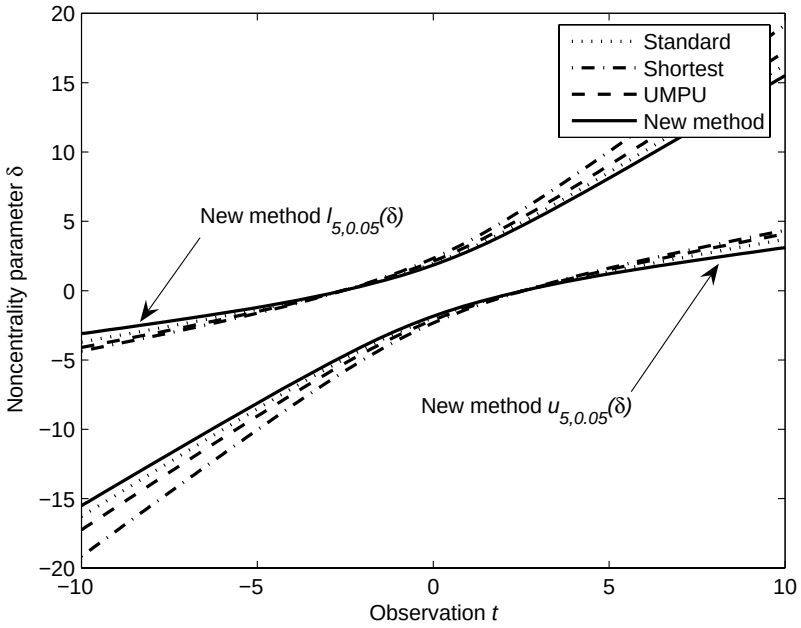

Figure 18: A comparison of the functions $l_{5,0.05}(\delta)$ and $u_{5,0.05}(\delta)$ of new method with $\gamma = 0.310$ for the standard method, shortest acceptance sets method and UMPU method

Table 10: A comparison of the lengths of the confidence intervals of new method for the standard method, shortest acceptance sets method, UMPU method with $\alpha = 0.05$ and $\nu = 5$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$
Standard method	3.9199	7.1686	12.5891	18.3519	24.2159
UMPU method	4.2478	7.5368	13.1568	19.1881	25.3313
Shortest method	4.6649	8.4137	14.8075	21.6418	28.5927
New method	3.6795	6.9016	12.4076	18.2285	24.1197
Ratio of New/Standard	93.87%	96.28%	98.56%	99.33%	99.60%

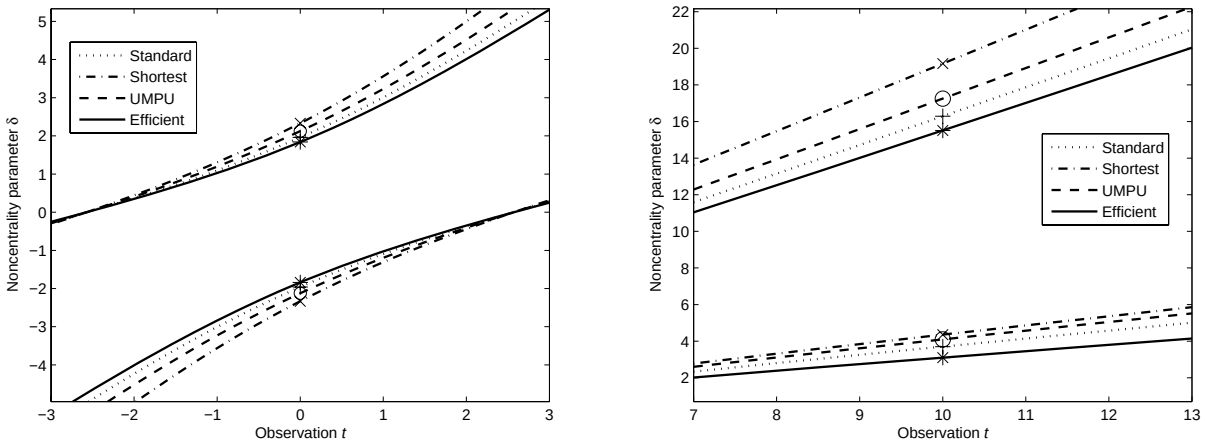

Figure 19: Comparisons of confidence intervals of new method for standard, shortest acceptance sets, and UMPU acceptance sets with $\nu = 5$ and $\alpha = 0.05$. The left plot is graphical representation of confidence intervals for $t = 0$ and right plot is for $t = 10$.

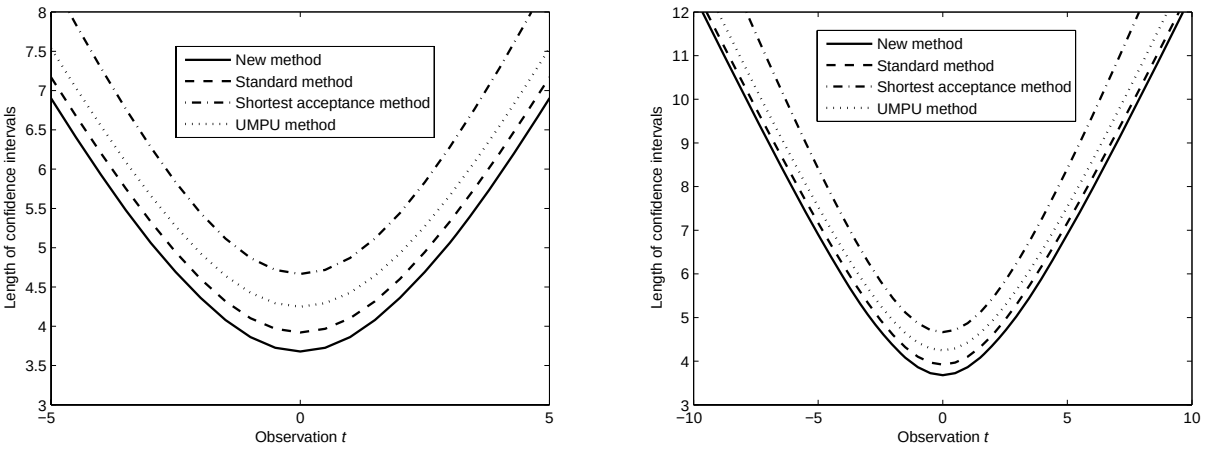

Figure 20: Comparisons of length of confidence intervals of new method for those of standard, of shortest acceptance sets, and of UMPU acceptance sets with $\nu = 5$ and $\alpha = 0.05$. The left plot is for $-5 \leq t \leq 5$ and right plot is for $-10 \leq t \leq 10$.

Table 11: A comparison of the lengths of the confidence interval of new method for the standard method, shortest acceptance sets method, UMPU method with $\alpha = 0.05$ and $\nu = 20$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$
Standard method	3.9199	4.9841	7.2948	10.0225	12.9060
UMPU method	4.0124	5.0600	7.3745	10.1283	13.0440
Shortest method	4.1135	5.2056	7.6148	10.4750	13.5002
New method	3.8715	4.9245	7.2591	10.0001	12.8895
Ratio of New/Standard	98.76%	98.80%	99.51%	99.77%	99.87%

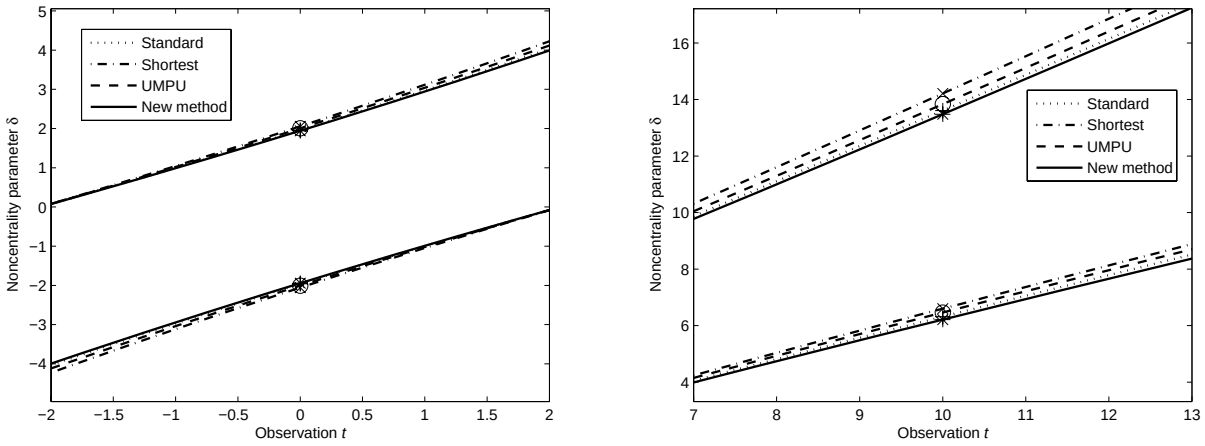

Figure 21: Comparisons of confidence intervals of new method for standard, shortest acceptance sets, and UMPU acceptance sets with $\nu = 20$ and $\alpha = 0.05$. The left plot is graphical representation of confidence intervals for $t = 0$ and right plot is for $t = 10$.

$\nu = 20$, $\alpha = 0.05$, and $t = 0$ (left plot) and $t = 10$ (right plot) are made in Figure 20. Again, this improvement of new method over standard is much less than that of $\nu = 5$ due to decrease of skewness.

As the computational results by a certain amount of trial and error, the recommended values of γ can be tabulated in Table 12. The resulting acceptance sets $A_{\nu,\alpha}(\delta)$ in comparisons with standard are made in Table 29 – Table 31 of Appendix A and confidence intervals $C_{\nu,\alpha}(t)$ are summarized for each α and ν in Table 32 – Table 34

Table 12: Recommended values of γ for $\nu = 5 \sim 30$ and $\alpha = 0.1,\ 0.05,\ 0.01$

ν	5	10	15	20	25	30
$\alpha = 0.10,\ \gamma$	0.350	0.290	0.260	0.250	0.245	0.240
$\alpha = 0.05,\ \gamma$	0.310	0.225	0.220	0.215	0.210	0.205
$\alpha = 0.01,\ \gamma$	0.290	0.175	0.150	0.140	0.135	0.130

of Appendix A.

Comparisons of lengths of confidence intervals of the sizes $\alpha = 0.01,\ 0.05,\ 0.1$ for given $\nu = 5$ and each observation t are made in Table 13 respectively, which demonstrates how well proposed new method for efficient confidence interval works over all values of α.

A very interesting property from Table 10 and Table 11 could be observed, which the length of standard confidence interval $C^s_{\nu,\alpha}(t) = (\delta^l,\ \delta^u)$ at $t = 0$ is unchanged in ν. This makes sense because the standard confidence interval at $t = 0$ depends only upon α. That is, the δ^l and δ^u of standard confidence interval $C^s_{\nu,\alpha}(t) = (\delta^l,\ \delta^u)$ at $t = 0$ satisfy

$$\mathbb{P}(t_\nu(\delta^u) \le t^l_{\nu,\alpha/2}(\delta^u) = 0) = \alpha/2 \quad \text{and} \quad \mathbb{P}(t_\nu(\delta^l) \ge t^u_{\nu,\alpha/2}(\delta^l) = 0) = \alpha/2$$

By the definition of noncentral $t-$distribution, we have that

$$\mathbb{P}(t_\nu(\delta^u) \le 0) = \mathbb{P}\left(\frac{X}{\sqrt{\chi^2_\nu/\nu}} \le 0\right) = \mathbb{P}(X \le 0) = \alpha/2$$

where $X \sim N(\delta^u, 1)$. Thus it follows.

Matlab programs are available to implement these new methodologies. For given values of ν and α the program selects an appropriate value of γ, and then constructs the confidence interval for the given value of t. In principle, the program operates in

Table 13: Comparisons of the lengths of the confidence intervals of new method for the standard method, shortest acceptance sets method, UMPU method $\nu = 5$.

$\alpha = 0.1$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$	$t = 30$
Standard method	3.2897	6.0280	10.6172	15.4938	20.4533	30.4585
UMPU method	3.5783	6.3250	11.0949	16.2034	21.4007	31.8832
Shortest method	3.9228	7.0837	12.5299	18.3353	24.2337	36.1224
New method	3.0871	5.8304	10.5001	15.4100	20.3803	30.3862
Ratio of New/Standard	93.84%	96.72%	98.90%	99.46%	99.64%	99.76%

$\alpha = 0.05$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$	$t = 30$
Standard method	3.9199	7.1686	12.5891	18.3519	24.2159	36.0498
UMPU method	4.2478	7.5368	13.1568	19.1881	25.3313	37.7273
Shortest method	4.6649	8.4137	14.8075	21.6418	28.5926	42.6080
New method	3.6795	6.9016	12.4076	18.2285	24.1197	35.9722
Ratio of New/Standard	93.87%	96.28%	98.56%	99.33%	99.60%	99.78%

$\alpha = 0.01$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$	$t = 30$
Standard method	5.1517	9.3807	16.3614	23.7849	31.3472	46.6222
UMPU method	5.5382	9.8888	17.1174	24.8574	32.7680	48.7566
Shortest method	6.0979	10.9867	19.1325	27.8564	36.7577	54.7307
New method	4.8411	8.9907	15.9937	23.5109	31.1522	46.5178
Ratio of New/Standard	93.97%	95.84%	97.75%	98.85%	99.38%	99.78%

the following manner.

1. Input t, ν, and α.

2. Select a value of γ depending on ν and α.

3. Construct the function $\xi_{\nu,\alpha}(\delta)$.

4. Construct the functions $l_{\nu,\alpha}(\delta)$ and $u_{\nu,\alpha}(\delta)$ using the function $\xi_{\nu,\alpha}(\delta)$ and condition (61).

5. Find δ^l satisfying $u_{\nu,\alpha}(\delta^l) = t$ and find δ^u satisfying $l_{\nu,\alpha}(\delta^u) = t$.

6. Output the confidence interval (δ^l, δ^u).

However, for standard values, UMPU values and values of shortest possible sets of ν and α the functions $l_{\nu,\alpha}(\delta)$ and $u_{\nu,\alpha}(\delta)$ are simply stored in form of data in the program.

3.5 Summary

The problem of constructing a two-sided confidence interval for the noncentrality parameter of a noncentral t-distribution based upon one observation from the distribution is an interesting problem with important applications. A general theoretical approach to the problem is provided by the specification and inversion of acceptance sets for each possible value of the noncentrality parameter. The standard method is based upon the arbitrary assignment of equal tail probabilities to the acceptance set, while the choice of the shortest possible acceptance sets and UMPU acceptance

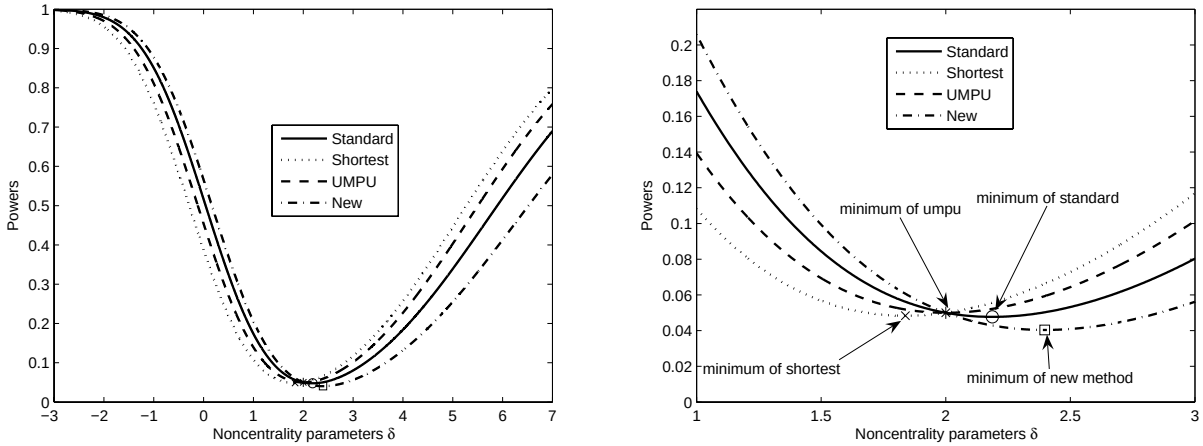

Figure 22: Power comparisons with $\nu = 5$, $\alpha = 0.05$ and true $\delta = 2$: Except UMPU method, other 3 methodologies are biased since power of standard is minimized at $\delta = 2.1870$ with minimal power 0.0477, power of shortest is minimized at $\delta = 1.8380$ with minimal power 0.0483 and new method is at 2.3970 with minimal power 0.0404

sets provides even worse confidence intervals, which means that since the standard confidence intervals are uniformly shorter than those of UMPU method, the standard method are "biased" as shown in Figure 22. However, with the correct choice of acceptance sets it is possible to provide an improvement in terms of confidence interval length over the confidence intervals provided by the standard method for all data values, and Matlab programs for this are available to implement these efficient new confidence interval methodologies.

CHAPTER IV

TESTING THE EQUALITY OF THE NONCENTRALITY PARAMETERS OF TWO NONCENTRAL T-DISTRIBUTIONS

The problem of testing the equality of the noncentrality parameters of two noncentral t-distributions is considered, which arises from the comparison of two signal-to-noise ratios for simple linear regression models. A test procedure is derived that is guaranteed to maintain type I error while having only minimal amounts of conservativeness, and comparisons are made with several other approaches to this problem based on variance stabilizing transformations. Section 4.1 contains some descriptions of other approaches to this problem based on variance stabilizing transformations of the random variables. Section 4.2 describes the new procedure, while section 4.3 presents some simulations results comparing the new procedure with the other procedures in terms of type I error and power.

4.1 Introduction

Suppose that X_1 and X_2 are independent random variables that have t-distributions with noncentrality parameters δ_1 and δ_2 respectively and identical degrees of freedom ν, designated $X_i \sim t_\nu(\delta_i)$ for $i = 1, 2$. The problem considered in this chapter is that of testing whether the two noncentrality parameters δ_1 and δ_2 are identical.

This problem naturally arises when comparisons are required of two signal-to-noise ratios in a simple linear regression model. For example, if two signal response systems are modeled as

$$y_{ij} = \beta_{i0} + \beta_{i1} x_j + \epsilon_{ij}, \qquad \epsilon_{ij} \sim N(0, \sigma_i^2), \quad i = 1, 2, \quad j = 1, \cdots, n$$

where the signal factor values x_j are the same for both systems, then the signal-to-noise ratios (see Taguchi [18] and Miwa [12]) are

$$\eta_i = \frac{\beta_{i1}^2}{\sigma_i^2} \quad i = 1, 2.$$

The natural estimators of these signal-to-noise ratios are

$$\hat{\eta}_i = \frac{\hat{\beta}_{i1}^2}{\hat{\sigma}_i^2} \quad i = 1, 2$$

and

$$t_i = \sqrt{S_{xx} \hat{\eta}_i} \sim t_\nu(\delta_i), \quad \nu = n - 2, \quad i = 1, 2$$

where $\delta_i = \sqrt{\eta_i S_{xx}}$. Consequently, The problem of testing the equality of the two signal-to-noise ratios is equivalent to testing the equality of the two noncentrality parameters of t-distributions with identical degrees of freedom.

The approach taken in this chapter is to reject $H_0 : \delta_1 = \delta_2$ if and only if

$$|X_1 - X_2| > K_{\alpha,\nu}(|\bar{X}|)$$

with $\bar{X} = (X_1 + X_2)/2$. For specific values of α and ν, functions $K_{\alpha,\nu}$ are derived which are guaranteed to maintain the type I error at no more than α (so that the rejection probability is no more than α for all $\delta \in \Re$ when $\delta_1 = \delta_2 = \delta$). Since the variance of a $t_\nu(\delta)$ random variable is increasing with δ (see, for example, Johnson

et al. [8]), $K_{\alpha,\nu}$ must necessarily be an increasing function. Suitable forms of these functions are found by direct computation of the type I error probabilities.

Previous approaches to this problem have been based on transformations of the noncentral t-distributions. Testing procedures are then based upon the approximate distributions of these transformed variables. (For detail explanations on these, see section 2.2).

Miwa procedure

If $t \sim t_\nu(\delta)$ then Miwa [12] considered the following variance stabilizing transformation of t^2 (which has a noncentral F-distribution)

$$g(t) = \ln\left(t^2 + \nu + \sqrt{t^4 + 2\nu t^2}\right).$$

This transformed variable is approximately distributed as a

$$N\left(g(\sqrt{1+\delta^2}) + 3/(3\nu - 1),\ 2/(\nu - 1)\right)$$

random variable. With

$$W = \frac{g(X_1) - g(X_2)}{\sqrt{4/(\nu - 1)}}$$

Miwa [12] used the rejection region $|W| > z_{\alpha/2}$, and using the Cornish-Fisher expansion Miwa [12] used the rejection region

$$\left| W - \frac{1}{12\nu}\left(W^3 - 3W\right) \right| > z_{\alpha/2}.$$

Nagata et al. procedure

Nagata et al. [14] used the variance stabilizing transformation of Laubsher [9]

$$f(t) = \frac{1}{b}\ln\left(bt + \sqrt{b^2t^2 + a^2}\right)$$

where $a = \sqrt{\frac{\nu}{\nu-2}}$ and $b = \sqrt{c_{22}}/c_{11}$ with $c_{11} = \frac{\sqrt{\nu/2}\,\Gamma((\nu-1)/2)}{\Gamma(\nu/2)}$ and $c_{22} = \sqrt{\frac{\nu}{\nu-2}} - c_{11}^2$. Laubsher showed that $f(t)$ is approximately normally distributed with a mean of $f(c_{11}\delta)$ and a variance of one. Consequently, Nagata *et al.* proposed the rejection region

$$\frac{|f(t_1) - f(t_2)|}{\sqrt{2}} > z_{\alpha/2}.$$

4.2 The new procedure for constructing acceptance set

4.2.1 Definition of acceptance set $K_{\alpha,\nu}$

The new procedure proposed in this chapter is based upon the construction of functions $K_{\alpha,\nu}$ with the property that

$$\mathbb{P}(|X_1 - X_2| > K_{\alpha,\nu}(|\bar{X}|)|\delta_1 = \delta_2 = \delta) \leq \alpha \quad \forall \delta \in \Re.$$

These functions $K_{\alpha,\nu}$ are constructed in the following manner. If $t_{\alpha,\nu}(\delta)$ denotes the upper α critical point of a $t_\nu(\delta)$ random variable and if $\alpha^* = 1 - \sqrt{1-\alpha}$, then

$$A_{\delta,\alpha,\nu} = \{(x_1, x_2) \,:\, x_i \in (t_{1-\alpha^*/2,\nu}(\delta),\ \ t_{\alpha^*/2,\nu}(\delta)),\ i = 1, 2\}$$

has the property that

$$\mathbb{P}((X_1, X_2) \in A_{\delta,\alpha,\nu}|\delta_1 = \delta_2 = \delta) = 1 - \alpha.$$

Now define the function $K^*_{\alpha,\nu}$ by

$$|X_1 - X_2| \leq K^*_{\alpha,\nu}(|\bar{X}|) \Leftrightarrow (X_1, X_2) \in \bigcup_{\delta \in \Re} A_{\delta,\alpha,\nu}.$$

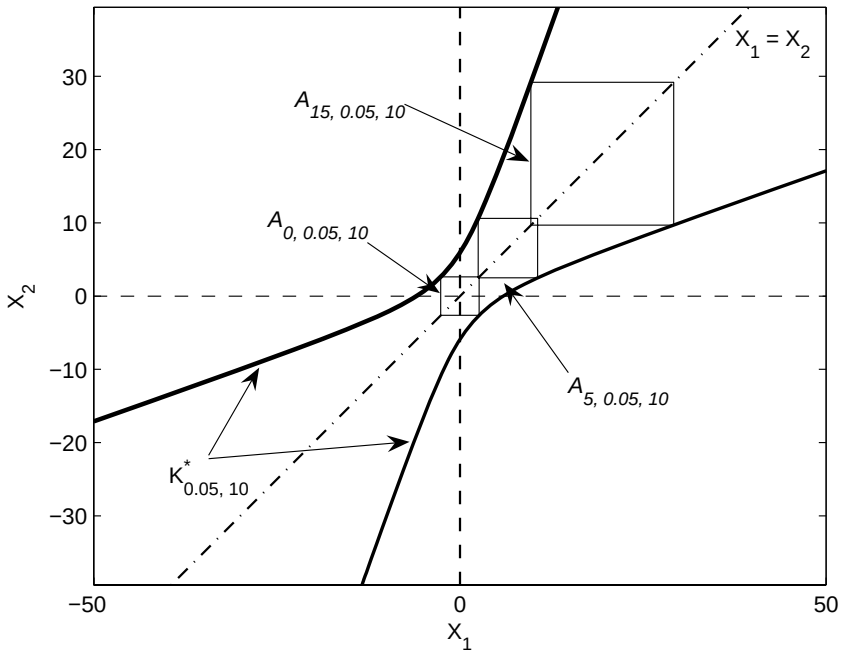

Figure 23: Graphical representation of the function $K^*_{0.05,10}$ and some of sets $A_{\delta,0.05,10}$

Figure 23 shows some of the sets $A_{\delta,\alpha,\nu}$ together with the region $|X_1 - X_2| \leq K^*_{\alpha,\nu}(|\bar{X}|)$ for $\nu = 10$ and $\alpha = 0.05$ that is formed by two bands symmetrically placed on either side of the line $X_1 = X_2$.

Notice that

$$\mathbb{P}(|X_1 - X_2| > K^*_{\alpha,\nu}(|\bar{X}|)|\delta_1 = \delta_2 = \delta) < \alpha \quad \forall \delta \in \Re$$

so that the function $K^*_{\alpha,\nu}$ does provide a size α hypothesis test for $H_0 : \delta_1 = \delta_2$. However, this hypothesis test is very conservative as shown in Figure 24 and improvements can be made by shrinking the bands in Figure 23 towards the line $X_1 = X_2$.

Specifically, functions of the form

$$K_{\alpha,\nu}(|\bar{X}|) = \epsilon_{\alpha,\nu}\left(1 + c_{\alpha,\nu}(1 - e^{-\gamma_{\alpha,\nu}\bar{X}^2})\right) K^*_{\alpha,\nu}(|\bar{X}|) \tag{62}$$

have been considered for positive values of $\epsilon_{\alpha,\nu}$, $c_{\alpha,\nu}$, and $\gamma_{\alpha,\nu}$ which shrink the bands

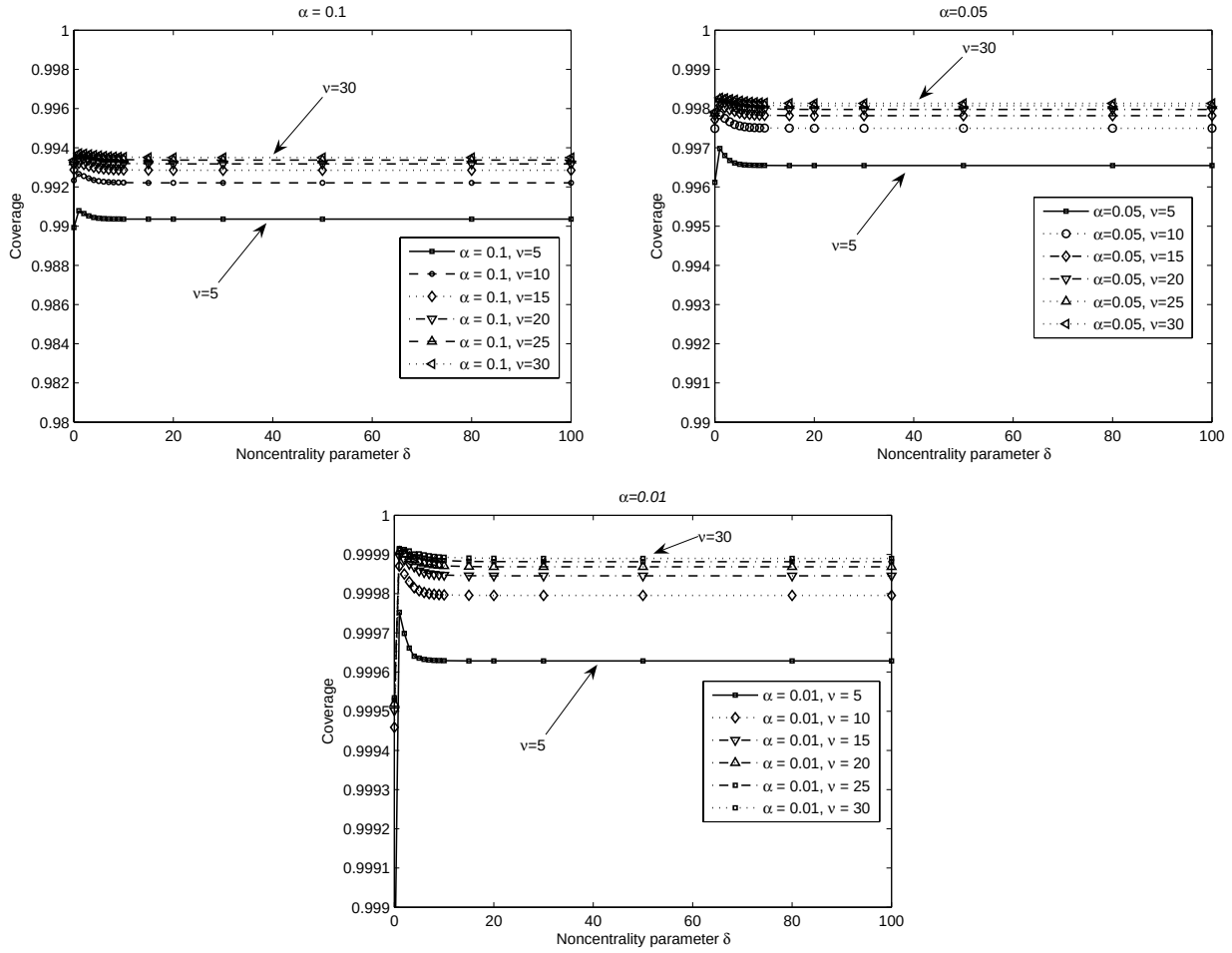

Figure 24: Comparisons of the probabilities of $\mathbb{P}((X_1, X_2) \in K^*_{\alpha,\nu})$ with $\alpha = 0.1, 0.05, 0.01$ and $\nu = 5 \sim 30$ respectively.

more for larger values of $|\bar{X}|$.

4.2.2 Selecting shrinkage coefficients and computational results

For each set of ν and α, the best values of $\epsilon_{\alpha,\nu}$, $c_{\alpha,\nu}$, and $\gamma_{\alpha,\nu}$ of (62) have been found by the direct evaluation of

$$\mathbb{P}(|X_1 - X_2| > K_{\alpha,\nu}(|\bar{X}|)|\delta_1 = \delta_2 = \delta) \tag{63}$$

for all values of δ. This probability is symmetric in δ, and the objective is to have

$$\max_{\delta \in \Re} P(|X_1 - X_2| > K_{\alpha,\nu}(\bar{X}) | \delta_1 = \delta_2 = \delta) = \alpha$$

while maximizing

$$\mathbb{P}_{min}(\alpha, \nu) = \min_{\delta \in \Re} P(|X_1 - X_2| > K_{\alpha,\nu}(|\bar{X}|) | \delta_1 = \delta_2 = \delta).$$

This ensures that the type I error of the hypothesis test is controlled at α while minimizing the amount of conservativeness. Inspection of (62) and Figure 25 may suggest that coefficient $\epsilon_{\alpha,\nu}$ of (62) affects size of acceptance set $K_{\alpha,\nu}(|\bar{X}|)$ for all δ, while coefficient $c_{\alpha,\nu}$ has influence on overall size of acceptance set except $|\delta|$ close to zero and coefficient $\gamma_{\alpha,\nu}$ has much effect on size for relatively small $|\delta|$. In fact, the size of acceptance set $K_{\alpha,\nu}(|\bar{X}|)$ for except relatively small $|\delta|$ would be approximately determined by the value $\epsilon_{\nu,\alpha}(1 + c_{\alpha,\nu})$. Figure 25 shows the procedure of selecting shrinkage coefficients of $\epsilon_{\alpha,\nu}$, $c_{\alpha,\nu}$ and $\gamma_{\alpha,\nu}$ with $\nu = 10$ and $\alpha = 0.1$, and Figure 26 shows how the resulting function $K_{\alpha,\nu}$ is a shrunken version of $K^*_{\alpha,\nu}$ for $\nu = 10$ and $\alpha = 0.1$.

For numerical computation of $\mathbb{P}(|X_1 - X_2| > K_{\alpha,\nu}(\bar{X}) | \delta_1 = \delta_2 = \delta)$, suppose that shrunken acceptance set $K_{\alpha,\nu}(\bar{X})$ is represented in $X_1 - X_2$ plane

$$K_{\alpha,\nu}(\bar{X}) = \{(x_1, x_2) \, : \, l_{\alpha,\nu}(x_1) \leq x_2 \leq u_{\alpha,\nu}(x_1), \; -\infty < x_1 < \infty\}$$

Since functions $u_{\alpha,\nu}(x_1)$ and $l_{\alpha,\nu}(x_1)$ is symmetric about $x_2 = x_1$, we have that

$$\mathbb{P}(|X_1 - X_2| > K_{\alpha,\nu}(\bar{X}) | \delta_1 = \delta_2 = \delta) = 1 - 2 \int_{-\infty}^{\infty} f_{\nu,\delta}(x_1)[F_{\nu,\delta}(u_{\alpha,\nu}(x_1)) - F_{\nu,\delta}(x_1)] dx_1$$

where $f_{\nu,\delta}(\cdot)$ and $F_{\nu,\delta}(\cdot)$ are pdf and cdf of noncentral $t-$distribution respectively.

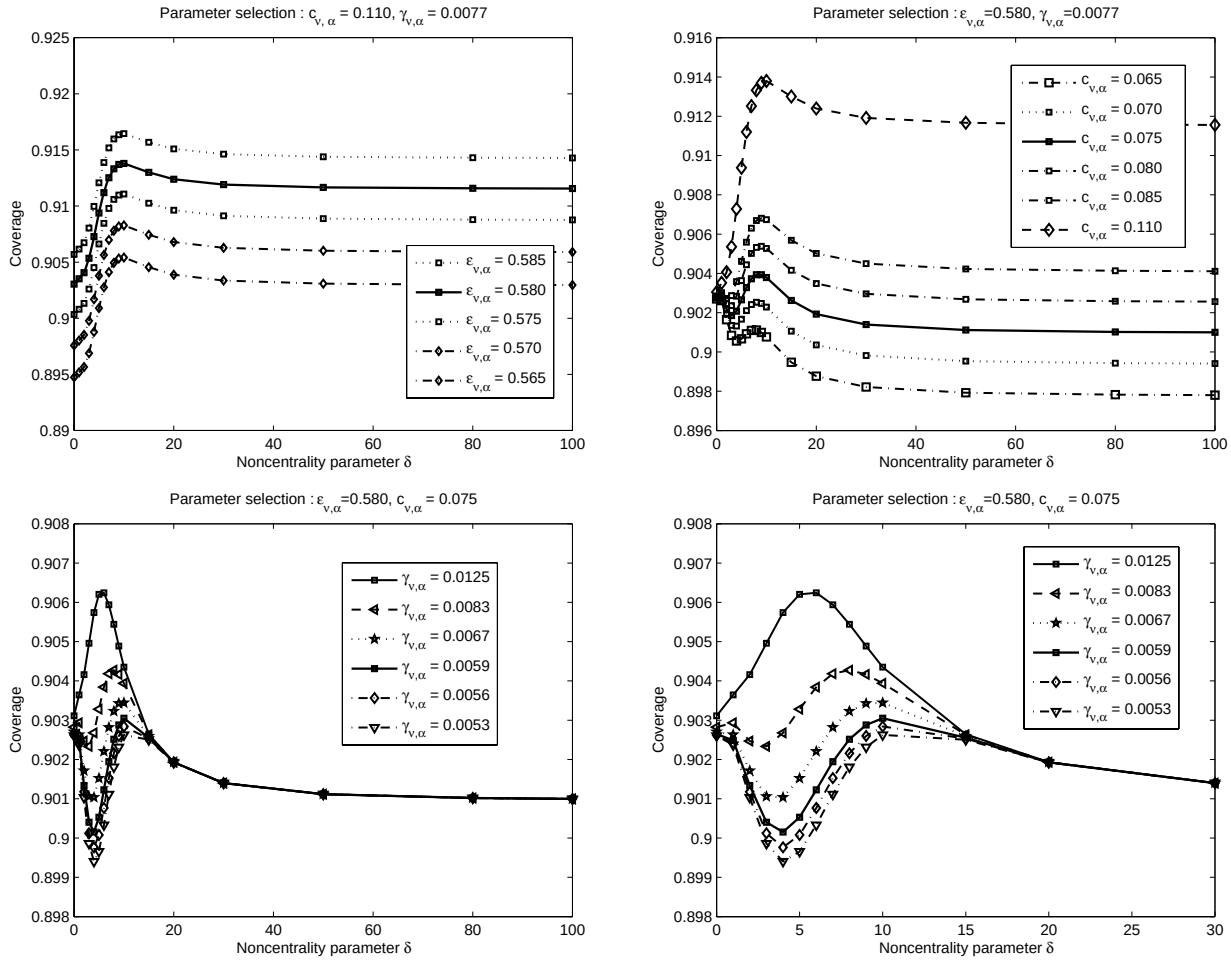

Figure 25: The procedure of selecting shrinkage coefficients of $\epsilon_{\alpha,\nu} = 0.580$, $c_{\alpha,\nu} = 0.075$ and $\gamma_{\alpha,\nu} = 0.0059$ with $\nu = 10$ and $\alpha = 0.1$. Upper left plot shows effect of coefficient $\epsilon_{\alpha,\nu}$ and upper right plot shows effect of coefficient $c_{\alpha,\nu}$. And lower plots reveal of effect of coefficient $\gamma_{\alpha,\nu}$.

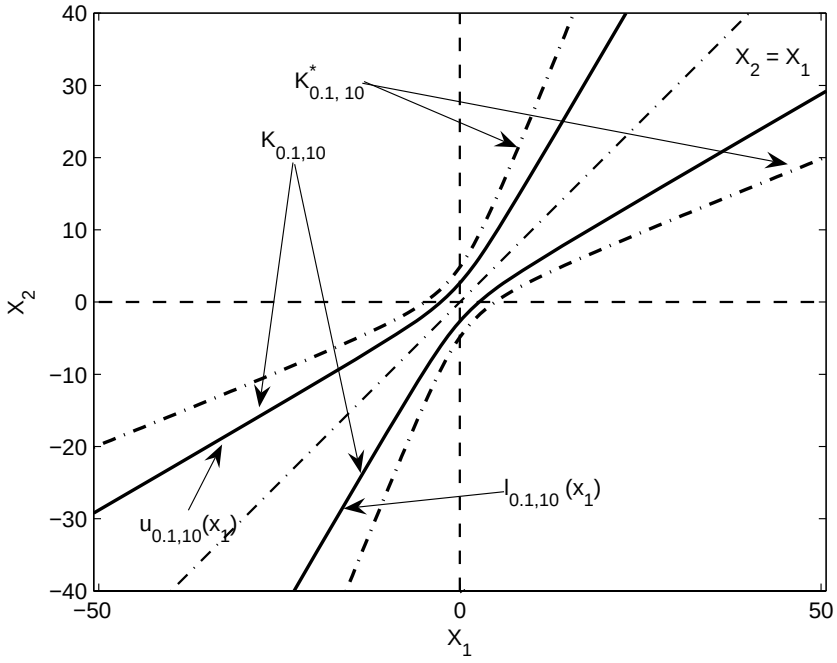

Figure 26: Graphical representation of the two functions $K_{0.1,10}$ and $K^*_{0.1,10}$

Table 14 shows the values of $\epsilon_{\alpha,\nu}$, $c_{\alpha,\nu}$, and $\gamma_{\alpha,\nu}$ which satisfy this objective, together with $P_{min}(\alpha,\nu)$. Also, Figure 27 shows how

$$\mathbb{P}(|X_1 - X_2| > K_{\alpha,\nu}(|\bar{X}|)|\delta_1 = \delta_2 = \delta)$$

and

$$\mathbb{P}(|X_1 - X_2| > K^*_{\alpha,\nu}(|\bar{X}|)|\delta_1 = \delta_2 = \delta)$$

vary with δ for $\nu = 5, 10, 15, 20, 25, 30$ and $\alpha = 0.1$. The computational results of $\alpha = 0.05,\ 0.01$ and $\nu = 5, 10, 15, 20, 25, 30$ are depicted in Appendix C.

Notice that as $\nu \to \infty$ then X_1 and X_2 tend to independent $N(\delta_1, 1)$ and $N(\delta_2, 1)$ random variables. In this limiting case $K^*_{\alpha,\nu} = 2z_{\alpha^*/2}$ and $K_{\alpha,\nu} = \sqrt{2}z_{\alpha/2}$, and these constant values provide straight line bands either side of the line $X_1 = X_2$. Consequently, $\epsilon_{\alpha,\nu}$ has a limiting value of $\frac{z_{\alpha/2}}{\sqrt{2}\ z_{\alpha^*/2}}$ while $c_{\alpha,\nu}$ and $\gamma_{\alpha,\nu}$ have limiting

Table 14: Selected shrinkage coefficients

ν	$\alpha = 0.1$					$\alpha = 0.05$				
	$\epsilon_{\alpha,\nu}$	$c_{\alpha,\nu}$	$\gamma_{\alpha,\nu}$	$P_{\min}$	$\frac{z_{\alpha/2}}{\sqrt{2}\ z_{\alpha^*/2}}$	$\epsilon_{\alpha,\nu}$	$c_{\alpha,\nu}$	$\gamma_{\alpha,\nu}$	$P_{\min}$	$\frac{z_{\alpha/2}}{\sqrt{2}\ z_{\alpha^*/2}}$
5	0.555	0.175	0.00100	0.0943	0.5968	0.560	0.235	0.00909	0.0462	0.6197
10	0.580	0.075	0.00590	0.0969	0.5968	0.590	0.110	0.00769	0.0471	0.6197
15	0.585	0.050	0.00480	0.0979	0.5968	0.600	0.072	0.00588	0.0478	0.6197
20	0.588	0.037	0.00380	0.0982	0.5968	0.606	0.052	0.00417	0.0485	0.6197
25	0.590	0.030	0.00290	0.0984	0.5968	0.608	0.042	0.00400	0.0487	0.6197
30	0.591	0.024	0.00240	0.0980	0.5968	0.612	0.032	0.00250	0.0490	0.6197
40	0.592	0.019	0.00227	0.0990	0.5968	0.613	0.025	0.00233	0.0492	0.6197
60	0.594	0.012	0.00135	0.0994	0.5968	0.616	0.016	0.00125	0.0494	0.6197
120	0.596	0.005	0.00048	0.0994	0.5968	0.618	0.008	0.00053	0.0496	0.6197

ν	$\alpha = 0.01$				
	$\epsilon_{\alpha,\nu}$	$c_{\alpha,\nu}$	$\gamma_{\alpha,\nu}$	$P_{\min}$	$\frac{z_{\alpha/2}}{\sqrt{2}\ z_{\alpha^*/2}}$
5	0.540	0.420	0.00588	0.0083	0.6491
10	0.600	0.180	0.00556	0.0088	0.6491
15	0.620	0.115	0.00417	0.0090	0.6491
20	0.630	0.080	0.00333	0.0090	0.6491
25	0.635	0.060	0.00250	0.0091	0.6491
30	0.637	0.049	0.00208	0.0092	0.6491
40	0.639	0.038	0.00200	0.0095	0.6491
60	0.643	0.024	0.00132	0.0096	0.6491
120	0.648	0.010	0.00019	0.0096	0.6491

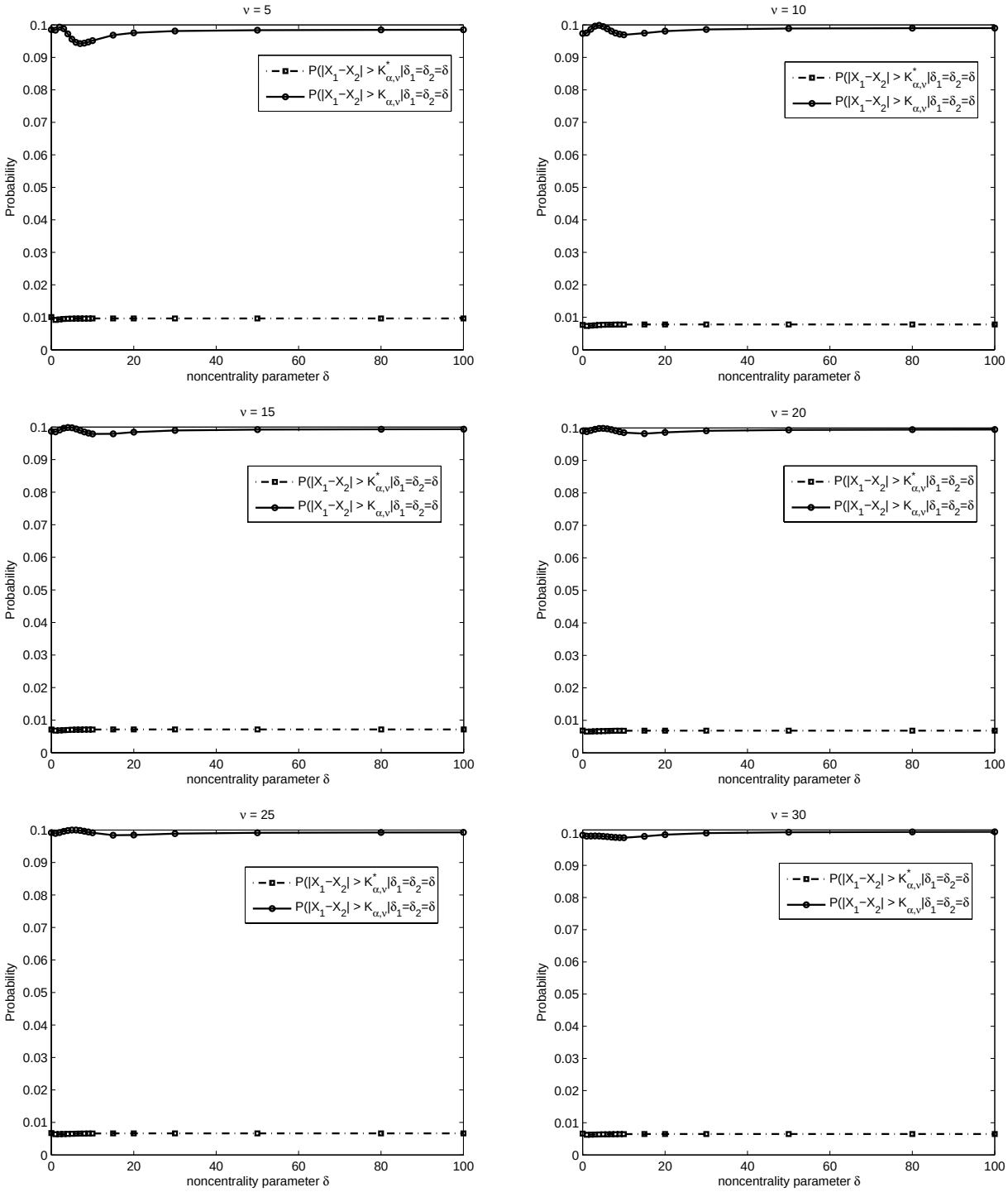

Figure 27: Comparisons of two probabilities $\mathbb{P}(|X_1 - X_2| > K_{\alpha,\nu}(|\bar{X}|)|\delta_1 = \delta_2 = \delta)$ and $\mathbb{P}(|X_1 - X_2| > K^*_{\alpha,\nu}(|\bar{X}|)|\delta_1 = \delta_2 = \delta)$ with $\alpha = 0.10$

values of zero, which can be seen from Table 14.

A Matlab program is available which constructs the function $K_{\alpha,\nu}$ for specified values of ν and α and which performs the hypothesis test of $H_0 : \delta_1 = \delta_2$ for specified values of X_1 and X_2.

4.3 Simulation results and Summary

Simulation results have been performed to compare the new procedure developed in this chapter with the approximate procedures described in section 2.2 and section 4.1 based on variance stabilizing transformations. The comparisons have been performed for both the size and the power of the hypothesis tests.

The sizes of the tests were investigated based on 100,000 simulated observations of X_1 and X_2. Values of the nominal size $\alpha = 0.10$, 0.05 and 0.01 were investigated together with a wide range of degrees of freedom ν. Table 15 contains the results for $\alpha = 0.05$, $\nu = 5 \sim 30$, and $\delta = 0 \sim 100$ which are similar to the other cases considered. Other simulation results for $\alpha = 0.1$, 0.01, $\nu = 5 \sim 30$, and $\delta = 0 \sim 100$ are summarized in Table 47 – Table 48 of Appendix C.

The following general observations can be made from the simulation results of the sizes of the tests. Firstly, the simulations confirm that the new procedure controls the error rate at the nominal level α for all values of δ, and the actual error rate is very close to the nominal level α (this is known to be true from the derivation of the new procedure). Secondly, the Miwa procedures may be very conservative for small values of δ, while for large values of δ the actual size is close to the nominal size, but may

Table 15: Comparisons of sizes of tests with $\nu = 5, 10$ and nominal size $\alpha = 0.05$ based on 100,000 simulations. The standard error of the table entries is about 0.0007.

	$\nu = 5$				$\nu = 10$			
δ	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.04762	0.03382	0.01145	0.01056	0.04810	0.04366	0.00623	0.00590
0.5	0.04773	0.03326	0.01578	0.01448	0.04928	0.04449	0.01070	0.01023
1.0	0.04949	0.03239	0.02830	0.02602	0.04896	0.04337	0.02198	0.02104
2.0	0.04871	0.02848	0.04795	0.04492	0.04936	0.04203	0.04617	0.04459
5.0	0.04672	0.02396	0.05161	0.04819	0.04921	0.03964	0.05280	0.05118
10.0	0.04752	0.02219	0.05102	0.04755	0.04764	0.03719	0.05142	0.04982
30.0	0.04960	0.02194	0.05076	0.04736	0.04881	0.03639	0.05064	0.04886
50.0	0.04865	0.02242	0.04949	0.04644	0.04923	0.03654	0.05067	0.04906
100.0	0.04897	0.02173	0.04977	0.04687	0.04989	0.03694	0.05120	0.04967

	$\nu = 15$				$\nu = 20$			
δ	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.04801	0.04519	0.00503	0.00483	0.04930	0.04770	0.00433	0.00420
0.5	0.04973	0.04660	0.00923	0.00900	0.04962	0.04791	0.00813	0.00795
1.0	0.04882	0.04573	0.01983	0.01928	0.04905	0.04732	0.01895	0.01852
2.0	0.04843	0.04424	0.04388	0.04277	0.04975	0.04712	0.04435	0.04364
5.0	0.04896	0.04205	0.05075	0.04988	0.04909	0.04458	0.05087	0.05010
10.0	0.04833	0.04202	0.05125	0.05017	0.04852	0.04411	0.05073	0.04992
30.0	0.04956	0.04139	0.05124	0.05009	0.04943	0.04313	0.05081	0.05000
50.0	0.04898	0.04109	0.05023	0.04916	0.04964	0.04330	0.05080	0.04990
100.0	0.04988	0.04120	0.05114	0.05006	0.04983	0.04346	0.05095	0.05009

	$\nu = 25$				$\nu = 30$			
δ	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.04973	0.04834	0.00410	0.00401	0.04975	0.04949	0.00368	0.00363
0.5	0.04994	0.04833	0.00782	0.00765	0.04924	0.04880	0.00725	0.00714
1.0	0.04826	0.04673	0.01685	0.01658	0.04872	0.04818	0.01741	0.01714
2.0	0.04943	0.04747	0.04310	0.04247	0.04796	0.04686	0.04138	0.04086
5.0	0.04928	0.04592	0.05122	0.05052	0.04948	0.04637	0.05093	0.05033
10.0	0.04894	0.04566	0.05079	0.05012	0.04937	0.04598	0.05060	0.05003
30.0	0.04857	0.04416	0.04967	0.04915	0.04846	0.04474	0.04949	0.04899
50.0	0.04977	0.04486	0.05076	0.04999	0.04962	0.04585	0.05055	0.05001
100.0	0.04959	0.04460	0.05026	0.04974	0.04996	0.04601	0.05070	0.05025

be slightly liberal. Thirdly, the Nagata procedure has a size that tends to decrease with δ, and while it is always less than the nominal size it may be substantially conservative.

Table 16 shows the simulated powers of the tests for $\alpha = 0.05$ and $\nu = 5, 10$ which are again similar to the other cases considered. The new procedure generally has substantially better power than the Nagata procedure. The Miwa procedures have power levels closer to those of the new procedure, and sometimes they may be slightly larger (although it should be remembered that the sizes of the Miwa procedures may be slightly larger than the nominal value). Other simulation results for $\alpha = 0.1, 0.05, 0.01$, $\nu = 5 \sim 30$, $\delta_1 = 1, 5$ and $\delta_2 = 0 \sim 100$ are summarized in Table 49 – Table 60 of Appendix C.

In summary, these simulations confirm that the new procedure has type I error probabilities that are guaranteed not to exceed the nominal level, and they demonstrate that the new procedure has size and power levels that compare well with the procedures based on variance stabilizing transformations.

Table 16: Comparisons of the powers of the tests with $\nu = 5, 10$ and $\alpha = 0.05$ based on 100,000 simulations.

	$\delta_1 = 1$							
	$\nu = 5$				$\nu = 10$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.08931	0.06673	0.02238	0.02088	0.10120	0.09309	0.01870	0.01792
1.0	0.04845	0.03212	0.02757	0.02553	0.04999	0.04439	0.02261	0.02162
2.0	0.08423	0.05655	0.06067	0.05666	0.09548	0.08544	0.06228	0.06033
3.0	0.18142	0.12725	0.14862	0.14076	0.22660	0.20587	0.18521	0.18088
4.0	0.32188	0.23585	0.29084	0.27809	0.41988	0.38981	0.38931	0.38287
5.0	0.47202	0.36436	0.45133	0.43726	0.61547	0.58225	0.60434	0.59833
6.0	0.60838	0.48942	0.60013	0.58589	0.76900	0.73983	0.76996	0.76512
7.0	0.71930	0.60674	0.71961	0.70735	0.87097	0.84875	0.87439	0.87140
8.0	0.80054	0.69948	0.80342	0.79300	0.93220	0.91789	0.93494	0.93298
9.0	0.86151	0.77706	0.86536	0.85767	0.96409	0.95553	0.96631	0.96497
10.0	0.90242	0.83399	0.90604	0.90004	0.98256	0.97761	0.98365	0.98309
15.0	0.97906	0.95999	0.98067	0.97680	0.99911	0.99886	0.99920	0.99919
20.0	0.99428	0.98852	0.99470	0.98498	0.99993	0.99992	0.99994	0.99983

	$\delta_1 = 5$							
	$\nu = 5$				$\nu = 10$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.72618	0.63449	0.55612	0.53946	0.83890	0.81791	0.71888	0.71285
1.0	0.47053	0.36152	0.45190	0.43667	0.61504	0.58183	0.60522	0.59871
2.0	0.25010	0.16530	0.26035	0.25008	0.35876	0.32321	0.37017	0.36474
3.0	0.11741	0.06761	0.12549	0.11883	0.16817	0.14490	0.17699	0.17285
4.0	0.06055	0.03170	0.06570	0.06182	0.07354	0.05974	0.07848	0.07618
5.0	0.04623	0.02353	0.05116	0.04752	0.04834	0.03798	0.05186	0.05024
6.0	0.05756	0.03060	0.06277	0.05918	0.06781	0.05490	0.07203	0.07029
7.0	0.08121	0.04383	0.08881	0.08364	0.12037	0.10084	0.12778	0.12477
8.0	0.12020	0.06885	0.13025	0.12356	0.19824	0.17015	0.20810	0.20381
9.0	0.16981	0.10218	0.18279	0.17388	0.30185	0.26646	0.31499	0.30981
10.0	0.22413	0.14050	0.24020	0.22960	0.40633	0.36628	0.42159	0.41537
15.0	0.52654	0.39004	0.54666	0.53340	0.82719	0.79848	0.83677	0.83315
20.0	0.74737	0.62656	0.76155	0.75111	0.96018	0.95075	0.96312	0.96180
30.0	0.92988	0.87503	0.93360	0.92826	0.99767	0.99678	0.99784	0.99775

CHAPTER V

CONCLUSIONS AND FUTURE RESEARCH

5.1 Conclusions

5.1.1 Efficient confidence interval methodology

Chapter 3 considers the problem of constructing two-sided efficient confidence interval for unknown parameter δ that is uniformly shorter than standard confidence interval $C^s_{\nu,\alpha}(t)$ based on acceptance sets $A^s_{\nu,\alpha}(\delta)$ of equal tail probabilities for all observed values $t \in \Re$. This problem is theoretically interesting because expected volume of confidence sets is equal to integrated sum of false coverage probability by Pratt [17] and because the standard method is based upon the arbitrary assignment of equal tail probabilities to the acceptance sets $A^s_{\nu,\alpha}(\delta)$.

The duality property of "$\delta \in C_{\nu,\alpha}(t) \Leftrightarrow t \in A_{\nu,\alpha}(\delta)$" could suggest that shortest acceptance sets $A^0_{\nu,\alpha}(\delta)$ or UMPU acceptance sets $A^u_{\nu,\alpha}(\delta)$ would result in the uniformly shorter confidence intervals. Surprisingly, these resulting confidence intervals are not efficient even though their lengths of acceptance sets are uniformly shorter than standard acceptance sets because of the property of skewness of non-central $t-$distribution. However, these failures can be used to guide the construction of methodologies that are uniformly shorter than the standard method, as shown in the section 3.4. That is, in order to obtain uniformly shorter confidence interval than standard, the lower tail probability $\mathbb{P}(t_\nu(\delta) \leq l_{\nu,\alpha}(\delta))$ of efficient acceptance set $A_{\nu,\alpha}(\delta)$ would be larger than $\alpha/2$.

The following function $\xi_{\nu,\alpha}(\delta)$, ratio of two lengths, is considered for constructing efficient acceptance set $A_{\nu,\alpha}(\delta) = (l_{\nu,\alpha}(\delta), u_{\nu,\alpha}(\delta))$

$$\xi_{\nu,\alpha}(\delta) = \frac{u_{\nu,\alpha}(\delta) - m_{\nu}(\delta)}{m_{\nu}(\delta) - l_{\nu,\alpha}(\delta)} = \xi^{s}_{\nu,\alpha}(\delta) + \epsilon_{\nu,\alpha}(\delta)$$

where, $m_{\nu}(\delta)$ is the point at which the probability density function of noncentral $t-$ distribution is maximized and $\xi^{s}_{\nu,\alpha}(\delta)$ is the function for the standard method. Deep inspection of the duality property may suggest that the appropriately selected function $\epsilon_{\nu,\alpha}(\delta)$ is able to result in efficient confidence intervals for all observed values $t \in \Re$. In fact, the function $\epsilon_{\nu,\alpha}(\delta) = \gamma(\xi^{s}_{\nu,\alpha}(\delta) - 1)^{1.5}$ by certain amount of trial and error is shown to be effective in section 3.4. Thus it is possible to provide an improvement in terms of confidence interval length over the confidence intervals provided by the standard method for all observed values, and a Matlab program for this is available to implement these efficient new confidence interval methodologies.

5.1.2 Testing equality of two noncentrality parameters

The problem of testing the equality of the noncentrality parameters of two noncentral t-distributions is considered in chapter 4, which naturally arises from the comparison of two signal-to-noise ratios of simple linear regression model. A test procedure is derived that is guaranteed to maintain type I error while having only minimal amounts of conservativeness, and comparisons are made with several other approaches to this problem based on variance stabilizing transformations. Section 4.1 and section 2.2 contain some descriptions of other approaches to this problem based on variance stabilizing transformations of the noncentral $t-$distribution and noncentral F distribution

by Miwa [12] and Nagata, et. al [14]. Section 4.2 describes the new procedure of acceptance set $K_{\nu,\alpha}$, which is shrunken set of $K^*_{\nu,\alpha}$ by the function

$$K_{\alpha,\nu}(|\bar{X}|) = \epsilon_{\alpha,\nu}\left(1 + c_{\alpha,\nu}(1 - e^{-\gamma_{\alpha,\nu}\bar{X}^2})\right) K^*_{\alpha,\nu}(|\bar{X}|)$$

Section 4.3 presents some simulations results comparing the new procedure with the other procedures in terms of type I error and power, and more simulation results are summarized in Appendix C. Simulation results say that our objective can successfully be satisfied by new procedure, while the sizes of Nagata, et. al's method get conservative for small degrees of freedom and as noncentrality parameter increases, and the sizes of Miwa's method are too conservative for small noncentrality parameter.

5.2 Future Research

For normally distributed data with mean μ and variance σ^2, the statistic $\sqrt{n}\bar{X}/S$ has a noncentral $t-$distribution with noncentrality parameter $\sqrt{n}\mu/\sigma$ and degrees of freedom $\nu = n - 1$. As discussed in section 2.3, the standard method for confidence interval for coefficient of variation $\kappa = \sigma/\mu$ has been criticized as being too computationally cumbersome, and current research has been directed towards constructing approximate confidence intervals for the coefficient of variation. Thus the methodologies considered in chapter 3 can be applied to this problem of finding the uniformly shorter confidence interval for κ than standard method because the estimator of coefficient of variation is nothing but the reciprocal of noncentral $t-$distribution.

Since noncentral F distributions are also right-skewed random variables, the new methodology discussed in section 3.4 could be applied for obtaining uniformly shorter

confidence interval for noncentrality parameter. In addition, we need to explore conditions for the optimal confidence interval of noncentral t and noncentral F distributions similarly to Tate and Klett's optimal confidence interval [19] for the variance σ^2 of normally distributed data.

APPENDIX A

COMPUTATIONAL RESULTS OF ACCEPTANCE SETS AND CONFIDENCE INTERVALS

The computational results of each acceptance set considered in the chapter 3 and resulting confidence intervals in comparison with standard are summarized in this Appendix.

Table 17 – Table 19 show shortest possible acceptance sets $A^0_{\nu,\alpha}(\delta)$ in comparison with standard acceptance sets $A^s_{\nu,\alpha}(\delta)$ with ν =5, 10, 20, 30, δ = 0, 5, 10, 20, 30, 50, 100 and α =0.1, 0.05, 0.01 respectively, and resulting confidence intervals $C^0_{\nu,\alpha}(t)$ in comparison with standard confidence interval $C_{\nu,\alpha}(t)$ are summarized with ν = 5, 10, 20, 30, t =0, 5, 10, 15, 20, 30 and α =0.1, 0.05, 0.01 in Table 20 – Table 22. Thus these Tables demonstrate that the resulting confidence intervals based on shortest acceptance sets are not efficient.

For UMP unbiased method, the computational results of UMPU acceptance sets $A^u_{\nu,\alpha}(\delta)$ in comparison with standard acceptance sets are tabulated in Table 23 – Table 25 with the same conditions, and resulting confidence intervals $C^u_{\nu,\alpha}(t)$ in comparison with standard are summarized in Table 26 – Table 28. Thus, this UMPU method is not efficient as well.

Finally, the computational results of acceptance sets $A_{\nu,\alpha}(\delta)$ of new method in

comparison with standard acceptance sets $A^s_{\nu,\alpha}(\delta)$ are tabulated in Table 29 – Table 31, and resulting confidence intervals $C_{\nu,\alpha}(t)$ in comparison with standard confidence intervals $C^s_{\nu,\alpha}(t)$ are summarized in Table 32 – Table 34. Thus these results demonstrate that this new method is efficient.

■ **Comparisons of acceptance sets of shortest acceptance sets method for standard acceptance sets**

Table 17: Comparisons of shortest possible acceptance set $A^0_{\nu,\alpha}(\delta)$ for standard $A^s_{\nu,\alpha}(\delta)$ with $\alpha = 0.1$

$\nu = 5$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.02, 2.02)	4.0301	(-2.02, 2.02)	4.0301	1.0000
5	(2.90, 11.00)	8.1032	(2.30, 9.53)	7.2311	0.8924
10	(6.48, 21.18)	14.7007	(5.42, 18.30)	12.8823	0.8763
20	(13.32, 41.93)	28.6111	(11.27, 36.22)	24.9514	0.8721
30	(20.08, 62.78)	42.6946	(17.03, 54.23)	37.2000	0.8713
50	(33.55, 104.52)	70.9675	(28.48, 90.29)	61.8059	0.8709
100	(67.18, 208.96)	141.7743	(57.04, 180.49)	123.4479	0.8707

$\nu = 10$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-1.81, 1.81)	3.6249	(-1.81, 1.81)	3.6249	1.0000
5	(3.07, 8.66)	5.5813	(2.71, 8.04)	5.3283	0.9547
10	(7.06, 16.32)	9.2599	(6.42, 15.12)	8.7008	0.9396
20	(14.61, 32.06)	17.4498	(13.40, 29.71)	16.3116	0.9348
30	(22.06, 47.93)	25.8677	(20.26, 44.41)	24.1560	0.9338
50	(36.89, 79.73)	42.8488	(33.90, 73.89)	39.9926	0.9333
100	(73.87, 159.35)	85.4740	(67.91, 147.67)	79.7588	0.9331

$\nu = 20$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-1.72, 1.72)	3.4494	(-1.72, 1.72)	3.4494	1.0000
5	(3.19, 7.63)	4.4344	(2.99, 7.35)	4.3601	0.9832
10	(7.52, 14.08)	6.5653	(7.15, 13.54)	6.3848	0.9725
20	(15.72, 27.42)	11.7067	(15.03, 26.36)	11.3293	0.9678
30	(23.77, 40.91)	17.1366	(22.77, 39.33)	16.5669	0.9668
50	(39.80, 67.99)	28.1931	(38.13, 65.37)	27.2409	0.9662
100	(79.75, 135.82)	56.0731	(76.43, 130.59)	54.1666	0.9660

$\nu = 30$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-1.70, 1.70)	3.3945	(-1.70, 1.70)	3.3945	1.0000
5	(3.24, 7.30)	4.0571	(3.10, 7.12)	4.0213	0.9912
10	(7.73, 13.32)	5.5975	(7.47, 12.97)	5.5043	0.9833
20	(16.26, 25.80)	9.5359	(15.78, 25.12)	9.3356	0.9790
30	(24.63, 38.43)	13.7951	(23.93, 37.42)	13.4915	0.9780
50	(41.27, 63.82)	22.5457	(40.11, 62.15)	22.0374	0.9775
100	(82.73, 127.43)	44.7099	(80.41, 124.10)	43.6914	0.9772

Table 18: Comparisons of shortest possible acceptance set $A^0_{\nu,\alpha}(\delta)$ for standard $A^s_{\nu,\alpha}(\delta)$ with $\alpha = 0.05$

$\nu = 5$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.57, 2.57)	5.1412	(-2.57, 2.57)	5.1412	1.0000
5	(2.57, 12.98)	10.4078	(2.00, 11.28)	9.2760	0.8912
10	(5.97, 24.91)	18.9400	(4.98, 21.57)	16.5916	0.8760
20	(12.35, 49.25)	36.8992	(10.45, 42.63)	32.1815	0.8721
30	(18.63, 73.71)	55.0733	(15.82, 63.81)	47.9928	0.8714
50	(31.16, 122.71)	91.5531	(26.48, 106.23)	79.7493	0.8711
100	(62.39, 245.30)	182.9070	(53.05, 212.35)	159.2972	0.8709

$\nu = 10$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.23, 2.23)	4.4563	(-2.23, 2.23)	4.4563	1.0000
5	(2.76, 9.63)	6.8774	(2.40, 8.97)	6.5631	0.9543
10	(6.59, 18.03)	11.4362	(5.99, 16.73)	10.7439	0.9395
20	(13.77, 35.35)	21.5731	(12.63, 32.79)	20.1662	0.9348
30	(20.83, 52.82)	31.9872	(19.12, 49.00)	29.8723	0.9339
50	(34.86, 87.85)	52.9917	(32.03, 81.49)	49.4634	0.9334
100	(69.83, 175.54)	105.7122	(64.20, 162.85)	98.6528	0.9332

$\nu = 20$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.09, 2.09)	4.1719	(-2.09, 2.09)	4.1719	1.0000
5	(2.88, 8.25)	5.3665	(2.68, 7.95)	5.2761	0.9832
10	(7.10, 15.06)	7.9550	(6.75, 14.49)	7.7356	0.9724
20	(15.02, 29.21)	14.1968	(14.36, 28.10)	13.7390	0.9678
30	(22.76, 43.55)	20.7862	(21.79, 41.89)	20.0954	0.9668
50	(38.14, 72.34)	34.2016	(36.54, 69.58)	33.0472	0.9662
100	(76.45, 144.47)	68.0269	(73.26, 138.98)	65.7158	0.9660

$\nu = 30$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.04, 2.04)	4.0845	(-2.04, 2.04)	4.0845	1.0000
5	(2.93, 7.81)	4.8830	(2.78, 7.62)	4.8397	0.9911
10	(7.34, 14.08)	6.7419	(7.09, 13.72)	6.6293	0.9833
20	(15.63, 27.13)	11.4937	(15.17, 26.42)	11.2521	0.9790
30	(23.74, 40.37)	16.6307	(23.05, 39.32)	16.2647	0.9780
50	(39.81, 67.00)	27.1833	(38.69, 65.26)	26.5706	0.9775
100	(79.84, 133.75)	53.9094	(77.60, 130.28)	52.6820	0.9772

Table 19: Comparisons of shortest possible acceptance set $A^0_{\nu,\alpha}(\delta)$ for standard $A^s_{\nu,\alpha}(\delta)$ with $\alpha = 0.01$

$\nu = 5$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-4.03, 4.03)	8.0643	(-4.03, 4.03)	8.0643	1.0000
5	(2.01, 18.57)	16.5620	(1.47, 16.19)	14.7173	0.8886
10	(5.11, 35.46)	30.3468	(4.24, 30.80)	26.5596	0.8752
20	(10.75, 70.01)	59.2532	(9.12, 60.79)	51.6741	0.8721
30	(16.27, 104.75)	88.4759	(13.85, 90.96)	77.1092	0.8715
50	(27.25, 174.36)	147.1142	(23.24, 151.41)	128.1723	0.8712
100	(54.60, 348.54)	293.9365	(46.60, 302.65)	256.0549	0.8711

$\nu = 10$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-3.17, 3.17)	6.3385	(-3.17, 3.17)	6.3385	1.0000
5	(2.18, 12.01)	9.8342	(1.83, 11.21)	9.3763	0.9534
10	(5.80, 22.24)	16.4420	(5.24, 20.68)	15.4397	0.9390
20	(12.35, 43.44)	31.0908	(11.31, 40.37)	29.0626	0.9348
30	(18.73, 64.86)	46.1231	(17.20, 60.28)	43.0772	0.9340
50	(31.40, 107.83)	76.4308	(28.87, 100.22)	71.3518	0.9335
100	(62.96, 215.45)	152.4882	(57.91, 200.24)	142.3283	0.9334

$\nu = 20$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.85, 2.85)	5.6907	(-2.85, 2.85)	5.6907	1.0000
5	(2.29, 9.62)	7.3304	(2.08, 9.29)	7.2056	0.9830
10	(6.36, 17.26)	10.8984	(6.03, 16.62)	10.5959	0.9722
20	(13.78, 33.27)	19.4903	(13.16, 32.03)	18.8612	0.9677
30	(20.97, 49.52)	28.5517	(20.07, 47.67)	27.6033	0.9668
50	(35.21, 82.20)	46.9925	(33.73, 79.13)	45.4086	0.9663
100	(70.64, 164.12)	93.4798	(67.69, 158.00)	90.3095	0.9661

$\nu = 30$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	shortest	$\lvert A^0_{\nu,\alpha}(\delta)\rvert$	$\lvert A^0_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.75, 2.75)	5.5000	(-2.75, 2.75)	5.5000	1.0000
5	(2.33, 8.91)	6.5789	(2.19, 8.71)	6.5203	0.9911
10	(6.63, 15.73)	9.0995	(6.39, 15.33)	8.9467	0.9832
20	(14.50, 30.04)	15.5398	(14.06, 29.27)	15.2128	0.9790
30	(22.12, 44.62)	22.4965	(21.48, 43.48)	22.0015	0.9780
50	(37.20, 73.98)	36.7816	(36.14, 72.09)	35.9534	0.9775
100	(74.67, 147.62)	72.9541	(72.57, 143.86)	71.2953	0.9773

■ **Comparisons of confidence intervals of shortest acceptance sets method for standard confidence intervals**

Table 20: Comparisons of confidence interval $C^0_{\nu,\alpha}(t)$ based on shortest possible acceptance set for standard confidence interval $C^s_{\nu,\alpha}(t)$ for $\alpha = 0.1$

$\nu = 5$

t	$C^s_{\nu,\alpha}(t)$	length	$C^0_{\nu,\alpha}(t)$	length	$\lvert C^0_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.64, 1.64)	3.2897	(-1.96, 1.96)	3.9228	1.1924
5	(1.86, 7.88)	6.0280	(2.21, 9.30)	7.0837	1.1751
10	(4.50, 15.11)	10.6172	(5.27, 17.80)	12.5299	1.1801
15	(6.98, 22.48)	15.4938	(8.14, 26.47)	18.3353	1.1834
20	(9.42, 29.88)	20.4533	(10.95, 35.19)	24.2337	1.1848
30	(14.26, 44.72)	30.4585	(16.54, 52.66)	36.1224	1.1860

$\nu = 10$

t	$C^s_{\nu,\alpha}(t)$	length	$C^0_{\nu,\alpha}(t)$	length	$\lvert C^0_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.64, 1.64)	3.2897	(-1.81, 1.81)	3.6122	1.0980
5	(2.47, 7.37)	4.8978	(2.71, 8.03)	5.3265	1.0875
10	(5.90, 13.86)	7.9655	(6.41, 15.09)	8.6795	1.0896
15	(9.15, 20.52)	11.3670	(9.92, 22.33)	12.4134	1.0921
20	(12.36, 27.23)	14.8752	(13.36, 29.63)	16.2626	1.0933
30	(18.70, 40.71)	22.0071	(20.20, 44.28)	24.0820	1.0943

$\nu = 20$

t	$C^s_{\nu,\alpha}(t)$	length	$C^0_{\nu,\alpha}(t)$	length	$\lvert C^0_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.64, 1.64)	3.2897	(-1.73, 1.73)	3.4526	1.0495
5	(2.86, 7.04)	4.1831	(3.00, 7.37)	4.3688	1.0444
10	(6.87, 12.99)	6.1254	(7.17, 13.57)	6.3955	1.0441
15	(10.70, 19.12)	8.4194	(11.15, 19.96)	8.8021	1.0455
20	(14.46, 25.31)	10.8441	(15.06, 26.41)	11.3469	1.0464
30	(21.92, 37.76)	15.8445	(22.81, 39.40)	16.5929	1.0472

$\nu = 30$

t	$C^s_{\nu,\alpha}(t)$	length	$C^0_{\nu,\alpha}(t)$	length	$\lvert C^0_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.64, 1.64)	3.2897	(-1.70, 1.70)	3.3986	1.0331
5	(3.01, 6.92)	3.9106	(3.11, 7.14)	4.0282	1.0301
10	(7.27, 12.63)	5.3575	(7.49, 13.00)	5.5136	1.0291
15	(11.36, 18.51)	7.1449	(11.69, 19.05)	7.3587	1.0299
20	(15.38, 24.45)	9.0738	(15.81, 25.17)	9.3516	1.0306
30	(23.33, 36.44)	13.1041	(23.98, 37.49)	13.5150	1.0314

Table 21: Comparisons of confidence interval $C^0_{\nu,\alpha}(t)$ based on shortest possible acceptance set for standard confidence interval $C^s_{\nu,\alpha}(t)$ for $\alpha = 0.05$

$\nu = 5$

t	$C^s_{\nu,\alpha}(t)$	length	$C^0_{\nu,\alpha}(t)$	length	$\lvert C^0_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.96, 1.96)	3.9199	(-2.33, 2.33)	4.6649	1.1901
5	(1.36, 8.53)	7.1686	(1.63, 10.04)	8.4137	1.1737
10	(3.70, 16.29)	12.5891	(4.36, 19.16)	14.8075	1.1762
15	(5.86, 24.21)	18.3519	(6.83, 28.48)	21.6418	1.1793
20	(7.96, 32.18)	24.2159	(9.25, 37.84)	28.5926	1.1807
30	(12.10, 48.15)	36.0498	(14.02, 56.62)	42.6080	1.1819

$\nu = 10$

t	$C^s_{\nu,\alpha}(t)$	length	$C^0_{\nu,\alpha}(t)$	length	$\lvert C^0_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.96, 1.96)	3.9199	(-2.15, 2.15)	4.3021	1.0975
5	(2.03, 7.86)	5.8332	(2.23, 8.57)	6.3432	1.0874
10	(5.22, 14.70)	9.4740	(5.68, 16.00)	10.3134	1.0886
15	(8.22, 21.73)	13.5102	(8.91, 23.64)	14.7379	1.0909
20	(11.15, 28.82)	17.6743	(12.05, 31.35)	19.3016	1.0921
30	(16.93, 43.07)	26.1413	(18.27, 46.85)	28.5751	1.0931

$\nu = 20$

t	$C^s_{\nu,\alpha}(t)$	length	$C^0_{\nu,\alpha}(t)$	length	$\lvert C^0_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.96, 1.96)	3.9199	(-2.06, 2.06)	4.1135	1.0494
5	(2.47, 7.45)	4.9841	(2.59, 7.80)	5.2056	1.0444
10	(6.32, 13.61)	7.2948	(6.60, 14.21)	7.6148	1.0439
15	(9.96, 19.98)	10.0225	(10.38, 20.86)	10.4750	1.0451
20	(13.52, 26.43)	12.9059	(14.08, 27.58)	13.5002	1.0460
30	(20.55, 39.40)	18.8531	(21.38, 41.12)	19.7374	1.0469

$\nu = 30$

t	$C^s_{\nu,\alpha}(t)$	length	$C^0_{\nu,\alpha}(t)$	length	$\lvert C^0_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.96, 1.96)	3.9199	(-2.02, 2.02)	4.0495	1.0330
5	(2.64, 7.30)	4.6597	(2.73, 7.53)	4.8001	1.0301
10	(6.78, 13.16)	6.3823	(6.98, 13.55)	6.5679	1.0291
15	(10.72, 19.23)	8.5093	(11.03, 19.79)	8.7628	1.0298
20	(14.57, 25.38)	10.8046	(14.98, 26.12)	11.1337	1.0305
30	(22.18, 37.78)	15.6008	(22.78, 38.87)	16.0876	1.0312

Table 22: Comparisons of confidence interval $C^0_{\nu,\alpha}(t)$ based on shortest possible acceptance set for standard confidence interval $C^s_{\nu,\alpha}(t)$ for $\alpha = 0.01$

$\nu = 5$

t	$C^s_{\nu,\alpha}(t)$	length	$C^0_{\nu,\alpha}(t)$	length	$\lvert C^0_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-2.58, 2.58)	5.1517	(-3.05, 3.05)	6.0979	1.1837
5	(0.43, 9.81)	9.3807	(0.51, 11.50)	10.9867	1.1712
10	(2.28, 18.65)	16.3614	(2.72, 21.85)	19.1325	1.1694
15	(3.90, 27.68)	23.7849	(4.58, 32.43)	27.8564	1.1712
20	(5.43, 36.78)	31.3472	(6.33, 43.09)	36.7577	1.1726
30	(8.40, 55.02)	46.6222	(9.73, 64.46)	54.7307	1.1739

$\nu = 10$

t	$C^s_{\nu,\alpha}(t)$	length	$C^0_{\nu,\alpha}(t)$	length	$\lvert C^0_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-2.58, 2.58)	5.1517	(-2.82, 2.82)	5.6452	1.0958
5	(1.18, 8.84)	7.6576	(1.30, 9.63)	8.3253	1.0872
10	(3.97, 16.36)	12.3969	(4.33, 17.80)	13.4692	1.0865
15	(6.49, 24.14)	17.6478	(7.04, 26.25)	19.2061	1.0883
20	(8.93, 31.99)	23.0680	(9.65, 34.78)	25.1318	1.0895
30	(13.69, 47.78)	34.0954	(14.76, 51.94)	37.1821	1.0905

$\nu = 20$

t	$C^s_{\nu,\alpha}(t)$	length	$C^0_{\nu,\alpha}(t)$	length	$\lvert C^0_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-2.58, 2.58)	5.1517	(-2.70, 2.70)	5.4034	1.0489
5	(1.71, 8.26)	6.5490	(1.79, 8.63)	6.8406	1.0445
10	(5.26, 14.83)	9.5739	(5.50, 15.49)	9.9896	1.0434
15	(8.55, 21.69)	13.1407	(8.92, 22.64)	13.7249	1.0445
20	(11.74, 28.65)	16.9114	(12.22, 29.90)	17.6776	1.0453
30	(17.98, 42.67)	24.6911	(18.70, 44.53)	25.8310	1.0462

$\nu = 30$

t	$C^s_{\nu,\alpha}(t)$	length	$C^0_{\nu,\alpha}(t)$	length	$\lvert C^0_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-2.58, 2.58)	5.1517	(-2.66, 2.66)	5.3208	1.0328
5	(1.92, 8.04)	6.1235	(1.98, 8.29)	6.3082	1.0302
10	(5.82, 14.21)	8.3826	(6.01, 14.63)	8.6252	1.0289
15	(9.48, 20.65)	11.1690	(9.76, 21.25)	11.4985	1.0295
20	(13.02, 27.20)	14.1755	(13.39, 27.99)	14.6026	1.0301
30	(19.97, 40.43)	20.4587	(20.51, 41.60)	21.0901	1.0309

■ **Comparisons of acceptance sets of UMPU method for standard acceptance sets**

Table 23: Comparisons of UMPU acceptance sets $A^u_{\nu,\alpha}(\delta)$ for standard $A^s_{\nu,\alpha}(\delta)$ with $\alpha = 0.1$

$\nu = 5$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	UMPU	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	$\lvert A^u_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.02, 2.02)	4.0301	(-2.02, 2.02)	4.0301	1.0000
5	(2.90, 11.00)	8.1032	(2.65, 10.10)	7.4538	0.9199
10	(6.48, 21.18)	14.7007	(6.07, 19.46)	13.3914	0.9109
20	(13.32, 41.93)	28.6111	(12.54, 38.55)	26.0054	0.9089
30	(20.08, 62.78)	42.6946	(18.93, 57.72)	38.7912	0.9086
50	(33.55, 104.52)	70.9675	(31.64, 96.11)	64.4665	0.9084
100	(67.18, 208.96)	141.7743	(63.36, 192.14)	128.7748	0.9083

$\nu = 10$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	UMPU	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	$\lvert A^u_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-1.81, 1.81)	3.6249	(-1.81, 1.81)	3.6249	1.0000
5	(3.07, 8.66)	5.5813	(2.92, 8.32)	5.3961	0.9668
10	(7.06, 16.32)	9.2599	(6.81, 15.68)	8.8697	0.9579
20	(14.61, 32.06)	17.4498	(14.16, 30.83)	16.6731	0.9555
30	(22.06, 47.93)	25.8677	(21.39, 46.09)	24.7055	0.9551
50	(36.89, 79.73)	42.8488	(35.78, 76.69)	40.9146	0.9549
100	(73.87, 159.35)	85.4740	(71.67, 153.28)	81.6076	0.9548

$\nu = 20$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	UMPU	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	$\lvert A^u_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-1.72, 1.72)	3.4494	(-1.72, 1.72)	3.4494	1.0000
5	(3.19, 7.63)	4.4344	(3.10, 7.48)	4.3799	0.9877
10	(7.52, 14.08)	6.5653	(7.37, 13.81)	6.4397	0.9809
20	(15.72, 27.42)	11.7067	(15.46, 26.91)	11.4535	0.9784
30	(23.77, 40.91)	17.1366	(23.40, 40.16)	16.7581	0.9779
50	(39.80, 67.99)	28.1931	(39.18, 66.74)	27.5641	0.9777
100	(79.75, 135.82)	56.0731	(78.52, 133.33)	54.8165	0.9776

$\nu = 30$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	UMPU	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	$\lvert A^u_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-1.70, 1.70)	3.3945	(-1.70, 1.70)	3.3945	1.0000
5	(3.24, 7.30)	4.0571	(3.18, 7.21)	4.0307	0.9935
10	(7.73, 13.32)	5.5975	(7.62, 13.15)	5.5322	0.9883
20	(16.26, 25.80)	9.5359	(16.07, 25.48)	9.4017	0.9859
30	(24.63, 38.43)	13.7951	(24.37, 37.96)	13.5944	0.9855
50	(41.27, 63.82)	22.5457	(40.84, 63.05)	22.2124	0.9852
100	(82.73, 127.43)	44.7099	(81.86, 125.91)	44.0445	0.9851

Table 24: Comparisons of UMPU acceptance sets $A^u_{\nu,\alpha}(\delta)$ for standard $A^s_{\nu,\alpha}(\delta)$ with $\alpha = 0.05$

$\nu = 5$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	UMPU	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	$\lvert A^u_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.57, 2.57)	5.1412	(-2.57, 2.57)	5.1412	1.0000
5	(2.57, 12.98)	10.4078	(2.34, 11.90)	9.5555	0.9181
10	(5.97, 24.91)	18.9400	(5.59, 22.82)	17.2247	0.9094
20	(12.35, 49.25)	36.8992	(11.65, 45.13)	33.4880	0.9076
30	(18.63, 73.71)	55.0733	(17.60, 67.56)	49.9640	0.9072
50	(31.16, 122.71)	91.5531	(29.43, 112.48)	83.0444	0.9071
100	(62.39, 245.30)	182.9070	(58.96, 224.85)	165.8955	0.9070

$\nu = 10$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	UMPU	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	$\lvert A^u_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.23, 2.23)	4.4563	(-2.23, 2.23)	4.4563	1.0000
5	(2.76, 9.63)	6.8774	(2.61, 9.25)	6.6461	0.9664
10	(6.59, 18.03)	11.4362	(6.36, 17.31)	10.9493	0.9574
20	(13.77, 35.35)	21.5731	(13.35, 33.96)	20.6044	0.9551
30	(20.83, 52.82)	31.9872	(20.21, 50.75)	30.5379	0.9547
50	(34.86, 87.85)	52.9917	(33.83, 84.41)	50.5800	0.9545
100	(69.83, 175.54)	105.7122	(67.80, 168.69)	100.8921	0.9544

$\nu = 20$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	UMPU	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	$\lvert A^u_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.09, 2.09)	4.1719	(-2.09, 2.09)	4.1719	1.0000
5	(2.88, 8.25)	5.3665	(2.79, 8.09)	5.3001	0.9876
10	(7.10, 15.06)	7.9550	(6.96, 14.77)	7.8018	0.9807
20	(15.02, 29.21)	14.1968	(14.77, 28.66)	13.8882	0.9783
30	(22.76, 43.55)	20.7862	(22.40, 42.73)	20.3250	0.9778
50	(38.14, 72.34)	34.2016	(37.55, 70.99)	33.4351	0.9776
100	(76.45, 144.47)	68.0269	(75.29, 141.78)	66.4960	0.9775

$\nu = 30$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	UMPU	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	$\lvert A^u_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.04, 2.04)	4.0845	(-2.04, 2.04)	4.0845	1.0000
5	(2.93, 7.81)	4.8830	(2.86, 7.71)	4.8511	0.9935
10	(7.34, 14.08)	6.7419	(7.23, 13.90)	6.6628	0.9883
20	(15.63, 27.13)	11.4937	(15.45, 26.79)	11.3313	0.9859
30	(23.74, 40.37)	16.6307	(23.48, 39.87)	16.3880	0.9854
50	(39.81, 67.00)	27.1833	(39.40, 66.18)	26.7802	0.9852
100	(79.84, 133.75)	53.9094	(79.02, 132.12)	53.1046	0.9851

Table 25: Comparisons of UMPU acceptance sets $A^u_{\nu,\alpha}(\delta)$ for standard $A^s_{\nu,\alpha}(\delta)$ with $\alpha = 0.01$

$\nu = 5$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	UMPU	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	$\lvert A^u_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-4.03, 4.03)	8.0643	(-4.03, 4.03)	8.0643	1.0000
5	(2.01, 18.57)	16.5620	(1.80, 16.94)	15.1394	0.9141
10	(5.11, 35.46)	30.3468	(4.80, 32.29)	27.4942	0.9060
20	(10.75, 70.01)	59.2532	(10.17, 63.76)	53.5888	0.9044
30	(16.27, 104.75)	88.4759	(15.42, 95.41)	79.9939	0.9041
50	(27.25, 174.36)	147.1142	(25.83, 158.82)	132.9902	0.9040
100	(54.60, 348.54)	293.9365	(51.78, 317.47)	265.6939	0.9039

$\nu = 10$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	UMPU	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	$\lvert A^u_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-3.17, 3.17)	6.3385	(-3.17, 3.17)	6.3385	1.0000
5	(2.18, 12.01)	9.8342	(2.04, 11.53)	9.4930	0.9653
10	(5.80, 22.24)	16.4420	(5.59, 21.31)	15.7243	0.9563
20	(12.35, 43.44)	31.0908	(11.98, 41.65)	29.6653	0.9542
30	(18.73, 64.86)	46.1231	(18.20, 62.19)	43.9912	0.9538
50	(31.40, 107.83)	76.4308	(30.52, 103.40)	72.8838	0.9536
100	(62.96, 215.45)	152.4882	(61.20, 206.60)	145.3981	0.9535

$\nu = 20$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	UMPU	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	$\lvert A^u_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.85, 2.85)	5.6907	(-2.85, 2.85)	5.6907	1.0000
5	(2.29, 9.62)	7.3304	(2.20, 9.44)	7.2382	0.9874
10	(6.36, 17.26)	10.8984	(6.23, 16.92)	10.6852	0.9804
20	(13.78, 33.27)	19.4903	(13.55, 32.62)	19.0616	0.9780
30	(20.97, 49.52)	28.5517	(20.65, 48.56)	27.9112	0.9776
50	(35.21, 82.20)	46.9925	(34.68, 80.61)	45.9281	0.9774
100	(70.64, 164.12)	93.4798	(69.60, 160.95)	91.3537	0.9773

$\nu = 30$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	UMPU	$\lvert A^u_{\nu,\alpha}(\delta)\rvert$	$\lvert A^u_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.75, 2.75)	5.5000	(-2.75, 2.75)	5.5000	1.0000
5	(2.33, 8.91)	6.5789	(2.26, 8.80)	6.5355	0.9934
10	(6.63, 15.73)	9.0995	(6.53, 15.52)	8.9916	0.9881
20	(14.50, 30.04)	15.5398	(14.34, 29.66)	15.3183	0.9858
30	(22.12, 44.62)	22.4965	(21.89, 44.06)	22.1656	0.9853
50	(37.20, 73.98)	36.7816	(36.82, 73.05)	36.2322	0.9851
100	(74.67, 147.62)	72.9541	(73.92, 145.78)	71.8573	0.9850

■ **Comparisons of confidence intervals of UMPU method for standard confidence intervals**

Table 26: Comparisons of UMPU confidence interval $C^u_{\nu,\alpha}(t)$ for standard confidence interval $C^s_{\nu,\alpha}(t)$ for $\alpha = 0.1$

$\nu = 5$

t	$C^s_{\nu,\alpha}(t)$	length	$C^u_{\nu,\alpha}(t)$	length	$\lvert C^u_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.64, 1.64)	3.2897	(-1.79, 1.79)	3.5783	1.0877
5	(1.86, 7.88)	6.0280	(2.06, 8.39)	6.3250	1.0493
10	(4.50, 15.11)	10.6172	(4.94, 16.04)	11.0949	1.0450
15	(6.98, 22.48)	15.4938	(7.64, 23.84)	16.2034	1.0458
20	(9.42, 29.88)	20.4533	(10.28, 31.68)	21.4007	1.0463
30	(14.26, 44.72)	30.4585	(15.53, 47.41)	31.8832	1.0468

$\nu = 10$

t	$C^s_{\nu,\alpha}(t)$	length	$C^u_{\nu,\alpha}(t)$	length	$\lvert C^u_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.64, 1.64)	3.2897	(-1.72, 1.72)	3.4426	1.0465
5	(2.47, 7.37)	4.8978	(2.60, 7.62)	5.0249	1.0260
10	(5.90, 13.86)	7.9655	(6.17, 14.30)	8.1370	1.0215
15	(9.15, 20.52)	11.3670	(9.55, 21.16)	11.6152	1.0218
20	(12.36, 27.23)	14.8752	(12.87, 28.08)	15.2054	1.0222
30	(18.70, 40.71)	22.0071	(19.46, 41.96)	22.5036	1.0226

$\nu = 20$

t	$C^s_{\nu,\alpha}(t)$	length	$C^u_{\nu,\alpha}(t)$	length	$\lvert C^u_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.64, 1.64)	3.2897	(-1.68, 1.68)	3.3688	1.0240
5	(2.86, 7.04)	4.1831	(2.93, 7.18)	4.2459	1.0150
10	(6.87, 12.99)	6.1254	(7.02, 13.21)	6.1918	1.0108
15	(10.70, 19.12)	8.4194	(10.92, 19.43)	8.5081	1.0105
20	(14.46, 25.31)	10.8441	(14.75, 25.71)	10.9602	1.0107
30	(21.92, 37.76)	15.8445	(22.34, 38.36)	16.0182	1.0110

$\nu = 30$

t	$C^s_{\nu,\alpha}(t)$	length	$C^u_{\nu,\alpha}(t)$	length	$\lvert C^u_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.64, 1.64)	3.2897	(-1.67, 1.67)	3.3431	1.0162
5	(3.01, 6.92)	3.9106	(3.06, 7.01)	3.9539	1.0111
10	(7.27, 12.63)	5.3575	(7.38, 12.78)	5.3980	1.0076
15	(11.36, 18.51)	7.1449	(11.52, 18.71)	7.1948	1.0070
20	(15.38, 24.45)	9.0738	(15.59, 24.72)	9.1374	1.0070
30	(23.33, 36.44)	13.1041	(23.63, 36.83)	13.1983	1.0072

Table 27: Comparisons of UMPU confidence interval $C^u_{\nu,\alpha}(t)$ for standard confidence interval $C^s_{\nu,\alpha}(t)$ for $\alpha = 0.05$

$\nu = 5$

t	$C^s_{\nu,\alpha}(t)$	length	$C^u_{\nu,\alpha}(t)$	length	$\lvert C^u_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.96, 1.96)	3.9199	(-2.12, 2.12)	4.2478	1.0836
5	(1.36, 8.53)	7.1686	(1.51, 9.05)	7.5368	1.0514
10	(3.70, 16.29)	12.5891	(4.10, 17.25)	13.1568	1.0451
15	(5.86, 24.21)	18.3519	(6.44, 25.63)	19.1881	1.0456
20	(7.96, 32.18)	24.2159	(8.73, 34.06)	25.3313	1.0461
30	(12.10, 48.15)	36.0498	(13.23, 50.96)	37.7273	1.0465

$\nu = 10$

t	$C^s_{\nu,\alpha}(t)$	length	$C^u_{\nu,\alpha}(t)$	length	$\lvert C^u_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.96, 1.96)	3.9199	(-2.05, 2.05)	4.0967	1.0451
5	(2.03, 7.86)	5.8332	(2.14, 8.13)	5.9892	1.0267
10	(5.22, 14.70)	9.4740	(5.48, 15.16)	9.6795	1.0217
15	(8.22, 21.73)	13.5102	(8.59, 22.39)	13.8050	1.0218
20	(11.15, 28.82)	17.6743	(11.63, 29.70)	18.0659	1.0222
30	(16.93, 43.07)	26.1413	(17.64, 44.37)	26.7301	1.0225

$\nu = 20$

t	$C^s_{\nu,\alpha}(t)$	length	$C^u_{\nu,\alpha}(t)$	length	$\lvert C^u_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.96, 1.96)	3.9199	(-2.01, 2.01)	4.0124	1.0236
5	(2.47, 7.45)	4.9841	(2.53, 7.59)	5.0600	1.0152
10	(6.32, 13.61)	7.2948	(6.46, 13.84)	7.3746	1.0109
15	(9.96, 19.98)	10.0225	(10.17, 20.30)	10.1283	1.0106
20	(13.52, 26.43)	12.9059	(13.80, 26.84)	13.0440	1.0107
30	(20.55, 39.40)	18.8531	(20.96, 40.02)	19.0596	1.0110

$\nu = 30$

t	$C^s_{\nu,\alpha}(t)$	length	$C^u_{\nu,\alpha}(t)$	length	$\lvert C^u_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.96, 1.96)	3.9199	(-1.99, 1.99)	3.9827	1.0160
5	(2.64, 7.30)	4.6597	(2.68, 7.40)	4.7117	1.0112
10	(6.78, 13.16)	6.3823	(6.88, 13.31)	6.4310	1.0076
15	(10.72, 19.23)	8.5093	(10.87, 19.44)	8.5689	1.0070
20	(14.57, 25.38)	10.8046	(14.77, 25.65)	10.8803	1.0070
30	(22.18, 37.78)	15.6008	(22.46, 38.18)	15.7129	1.0072

Table 28: Comparisons of UMPU confidence interval $C^u_{\nu,\alpha}(t)$ for standard confidence interval $C^s_{\nu,\alpha}(t)$ for $\alpha = 0.01$

$\nu = 5$

t	$C^s_{\nu,\alpha}(t)$	length	$C^u_{\nu,\alpha}(t)$	length	$\lvert C^u_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-2.58, 2.58)	5.1517	(-2.77, 2.77)	5.5382	1.0750
5	(0.43, 9.81)	9.3807	(0.47, 10.36)	9.8888	1.0542
10	(2.28, 18.65)	16.3614	(2.55, 19.67)	17.1174	1.0462
15	(3.90, 27.68)	23.7849	(4.34, 29.20)	24.8574	1.0451
20	(5.43, 36.78)	31.3472	(6.02, 38.79)	32.7680	1.0453
30	(8.40, 55.02)	46.6222	(9.26, 58.02)	48.7566	1.0458

$\nu = 10$

t	$C^s_{\nu,\alpha}(t)$	length	$C^u_{\nu,\alpha}(t)$	length	$\lvert C^u_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-2.58, 2.58)	5.1517	(-2.68, 2.68)	5.3679	1.0420
5	(1.18, 8.84)	7.6576	(1.25, 9.12)	7.8725	1.0281
10	(3.97, 16.36)	12.3969	(4.18, 16.85)	12.6719	1.0222
15	(6.49, 24.14)	17.6478	(6.81, 24.84)	18.0325	1.0218
20	(8.93, 31.99)	23.0680	(9.34, 32.92)	23.5766	1.0220
30	(13.69, 47.78)	34.0954	(14.30, 49.16)	34.8593	1.0224

$\nu = 20$

t	$C^s_{\nu,\alpha}(t)$	length	$C^u_{\nu,\alpha}(t)$	length	$\lvert C^u_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-2.58, 2.58)	5.1517	(-2.63, 2.63)	5.2681	1.0226
5	(1.71, 8.26)	6.5490	(1.75, 8.40)	6.6509	1.0156
10	(5.26, 14.83)	9.5739	(5.39, 15.07)	9.6811	1.0112
15	(8.55, 21.69)	13.1407	(8.75, 22.03)	13.2801	1.0106
20	(11.74, 28.65)	16.9114	(11.99, 29.08)	17.0923	1.0107
30	(17.98, 42.67)	24.6911	(18.35, 43.31)	24.9611	1.0109

$\nu = 30$

t	$C^s_{\nu,\alpha}(t)$	length	$C^u_{\nu,\alpha}(t)$	length	$\lvert C^u_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-2.58, 2.58)	5.1517	(-2.62, 2.62)	5.2316	1.0155
5	(1.92, 8.04)	6.1235	(1.95, 8.15)	6.1926	1.0113
10	(5.82, 14.21)	8.3826	(5.92, 14.37)	8.4479	1.0078
15	(9.48, 20.65)	11.1690	(9.62, 20.87)	11.2478	1.0071
20	(13.02, 27.20)	14.1755	(13.21, 27.48)	14.2750	1.0070
30	(19.97, 40.43)	20.4587	(20.24, 40.85)	20.6056	1.0072

■ **Comparisons of acceptance sets of new method for standard acceptance sets**

Table 29: Comparisons of acceptance set $A_{\nu,\alpha}(\delta)$ of new method for standard $A^s_{\nu,\alpha}(\delta)$ with $\alpha = 0.1$

$\nu = 5$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	new method	$\lvert A_{\nu,\alpha}(\delta)\rvert$	$\lvert A_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.02, 2.02)	4.0301	(-2.02, 2.02)	4.0301	1.0000
5	(2.90, 11.00)	8.1032	(3.11, 12.75)	9.6386	1.1895
10	(6.48, 21.18)	14.7007	(6.86, 24.89)	18.0353	1.2268
20	(13.32, 41.93)	28.6111	(14.04, 49.44)	35.3996	1.2373
30	(20.08, 62.78)	42.6946	(21.15, 74.06)	52.9085	1.2392
50	(33.55, 104.52)	70.9675	(35.33, 123.35)	88.0169	1.2402
100	(67.18, 208.96)	141.7743	(70.73, 246.63)	175.8952	1.2407

$\nu = 10$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	new method	$\lvert A_{\nu,\alpha}(\delta)\rvert$	$\lvert A_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-1.81, 1.81)	3.6249	(-1.81, 1.81)	3.6249	1.0000
5	(3.07, 8.66)	5.5813	(3.20, 9.09)	5.8926	1.0558
10	(7.06, 16.32)	9.2599	(7.27, 17.26)	9.9921	1.0791
20	(14.61, 32.06)	17.4498	(15.02, 33.99)	18.9688	1.0870
30	(22.06, 47.93)	25.8677	(22.67, 50.83)	28.1602	1.0886
50	(36.89, 79.73)	42.8488	(37.90, 84.58)	46.6813	1.0894
100	(73.87, 159.35)	85.4740	(75.90, 169.05)	93.1487	1.0898

$\nu = 20$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	new method	$\lvert A_{\nu,\alpha}(\delta)\rvert$	$\lvert A_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-1.72, 1.72)	3.4494	(-1.72, 1.72)	3.4494	1.0000
5	(3.19, 7.63)	4.4344	(3.26, 7.75)	4.4985	1.0144
10	(7.52, 14.08)	6.5653	(7.63, 14.37)	6.7383	1.0263
20	(15.72, 27.42)	11.7067	(15.93, 28.01)	12.0813	1.0320
30	(23.77, 40.91)	17.1366	(24.09, 41.79)	17.7060	1.0332
50	(39.80, 67.99)	28.1931	(40.31, 69.46)	29.1483	1.0339
100	(79.75, 135.82)	56.0731	(80.77, 138.76)	57.9884	1.0342

$\nu = 30$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	new method	$\lvert A_{\nu,\alpha}(\delta)\rvert$	$\lvert A_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-1.70, 1.70)	3.3945	(-1.70, 1.70)	3.3945	1.0000
5	(3.24, 7.30)	4.0571	(3.28, 7.36)	4.0828	1.0063
10	(7.73, 13.32)	5.5975	(7.80, 13.48)	5.6746	1.0138
20	(16.26, 25.80)	9.5359	(16.40, 26.11)	9.7103	1.0183
30	(24.63, 38.43)	13.7951	(24.84, 38.90)	14.0623	1.0194
50	(41.27, 63.82)	22.5457	(41.61, 64.61)	22.9955	1.0199
100	(82.73, 127.43)	44.7099	(83.41, 129.02)	45.6131	1.0202

Table 30: Comparisons of acceptance set $A_{\nu,\alpha}(\delta)$ of new method for standard $A^s_{\nu,\alpha}(\delta)$ with $\alpha = 0.05$

$\nu = 5$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	new method	$\lvert A_{\nu,\alpha}(\delta)\rvert$	$\lvert A_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.57, 2.57)	5.1412	(-2.57, 2.57)	5.1412	1.0000
5	(2.57, 12.98)	10.4078	(2.77, 15.50)	12.7332	1.2234
10	(5.97, 24.91)	18.9400	(6.30, 30.33)	24.0342	1.2690
20	(12.35, 49.25)	36.8992	(12.98, 60.27)	47.2919	1.2817
30	(18.63, 73.71)	55.0733	(19.58, 90.29)	70.7165	1.2840
50	(31.16, 122.71)	91.5531	(32.72, 150.39)	117.6708	1.2853
100	(62.39, 245.30)	182.9070	(65.51, 300.69)	235.1805	1.2858

$\nu = 10$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	new method	$\lvert A_{\nu,\alpha}(\delta)\rvert$	$\lvert A_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.23, 2.23)	4.4563	(-2.23, 2.23)	4.4563	1.0000
5	(2.76, 9.63)	6.8774	(2.87, 10.13)	7.2598	1.0556
10	(6.59, 18.03)	11.4362	(6.78, 19.13)	12.3436	1.0793
20	(13.77, 35.35)	21.5731	(14.13, 37.59)	23.4595	1.0874
30	(20.83, 52.82)	31.9872	(21.36, 56.19)	34.8353	1.0890
50	(34.86, 87.85)	52.9917	(35.73, 93.49)	57.7540	1.0899
100	(69.83, 175.54)	105.7122	(71.58, 186.83)	115.2495	1.0902

$\nu = 20$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	new method	$\lvert A_{\nu,\alpha}(\delta)\rvert$	$\lvert A_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.09, 2.09)	4.1719	(-2.09, 2.09)	4.1719	1.0000
5	(2.88, 8.25)	5.3665	(2.94, 8.39)	5.4522	1.0160
10	(7.10, 15.06)	7.9550	(7.21, 15.40)	8.1881	1.0293
20	(15.02, 29.21)	14.1968	(15.22, 29.92)	14.7026	1.0356
30	(22.76, 43.55)	20.7862	(23.06, 44.62)	21.5553	1.0370
50	(38.14, 72.34)	34.2016	(38.64, 74.13)	35.4919	1.0377
100	(76.45, 144.47)	68.0269	(77.45, 148.07)	70.6145	1.0380

$\nu = 30$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	new method	$\lvert A_{\nu,\alpha}(\delta)\rvert$	$\lvert A_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.04, 2.04)	4.0845	(-2.04, 2.04)	4.0845	1.0000
5	(2.93, 7.81)	4.8830	(2.97, 7.88)	4.9168	1.0069
10	(7.34, 14.08)	6.7419	(7.41, 14.26)	6.8439	1.0151
20	(15.63, 27.13)	11.4937	(15.77, 27.50)	11.7250	1.0201
30	(23.74, 40.37)	16.6307	(23.94, 40.93)	16.9851	1.0213
50	(39.81, 67.00)	27.1833	(40.15, 67.93)	27.7801	1.0220
100	(79.84, 133.75)	53.9094	(80.52, 135.63)	55.1080	1.0222

Table 31: Comparisons of acceptance set $A_{\nu,\alpha}(\delta)$ of new method for standard $A^s_{\nu,\alpha}(\delta)$ with $\alpha = 0.01$

$\nu = 5$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	new method	$\lvert A_{\nu,\alpha}(\delta)\rvert$	$\lvert A_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-4.03, 4.03)	8.0643	(-4.03, 4.03)	8.0643	1.0000
5	(2.01, 18.57)	16.5620	(2.18, 24.38)	22.1952	1.3401
10	(5.11, 35.46)	30.3468	(5.37, 48.12)	42.7453	1.4086
20	(10.75, 70.01)	59.2532	(11.24, 95.80)	84.5600	1.4271
30	(16.27, 104.75)	88.4759	(17.00, 143.57)	126.5705	1.4306
50	(27.25, 174.36)	147.1142	(28.45, 239.17)	210.7183	1.4323
100	(54.60, 348.54)	293.9365	(57.01, 478.25)	421.2395	1.4331

$\nu = 10$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	new method	$\lvert A_{\nu,\alpha}(\delta)\rvert$	$\lvert A_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-3.17, 3.17)	6.3385	(-3.17, 3.17)	6.3385	1.0000
5	(2.18, 12.01)	9.8342	(2.28, 12.77)	10.4912	1.0668
10	(5.80, 22.24)	16.4420	(5.96, 23.97)	18.0134	1.0956
20	(12.35, 43.44)	31.0908	(12.65, 47.01)	34.3604	1.1052
30	(18.73, 64.86)	46.1231	(19.18, 70.24)	51.0601	1.1070
50	(31.40, 107.83)	76.4308	(32.14, 116.82)	84.6861	1.1080
100	(62.96, 215.45)	152.4882	(64.42, 233.44)	169.0211	1.1084

$\nu = 20$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	new method	$\lvert A_{\nu,\alpha}(\delta)\rvert$	$\lvert A_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.85, 2.85)	5.6907	(-2.85, 2.85)	5.6907	1.0000
5	(2.29, 9.62)	7.3304	(2.34, 9.79)	7.4490	1.0162
10	(6.36, 17.26)	10.8984	(6.46, 17.68)	11.2239	1.0299
20	(13.78, 33.27)	19.4903	(13.95, 34.15)	20.1975	1.0363
30	(20.97, 49.52)	28.5517	(21.22, 50.85)	29.6271	1.0377
50	(35.21, 82.20)	46.9925	(35.63, 84.42)	48.7968	1.0384
100	(70.64, 164.12)	93.4798	(71.47, 168.57)	97.0984	1.0387

$\nu = 30$

δ	standard	$\lvert A^s_{\nu,\alpha}(\delta)\rvert$	new method	$\lvert A_{\nu,\alpha}(\delta)\rvert$	$\lvert A_{\nu,\alpha}(\delta)\rvert/\lvert A^s_{\nu,\alpha}(\delta)\rvert$
0	(-2.75, 2.75)	5.5000	(-2.75, 2.75)	5.5000	1.0000
5	(2.33, 8.91)	6.5789	(2.36, 8.99)	6.6237	1.0068
10	(6.63, 15.73)	9.0995	(6.69, 15.93)	9.2356	1.0150
20	(14.50, 30.04)	15.5398	(14.62, 30.47)	15.8489	1.0199
30	(22.12, 44.62)	22.4965	(22.30, 45.27)	22.9702	1.0211
50	(37.20, 73.98)	36.7816	(37.49, 75.07)	37.5794	1.0217
100	(74.67, 147.62)	72.9541	(75.24, 149.80)	74.5562	1.0220

■ **Comparisons of confidence intervals of new method for standard confidence intervals**

Table 32: Comparisons of confidence interval $C_{\nu,\alpha}(t)$ of new method for standard confidence interval $C^s_{\nu,\alpha}(t)$ for $\alpha = 0.1$

$\nu = 5$

t	$C^s_{\nu,\alpha}(t)$	length	$C_{\nu,\alpha}(t)$	length	$\lvert C_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.64, 1.64)	3.2897	(-1.54, 1.54)	3.0871	0.9384
5	(1.86, 7.88)	6.0280	(1.65, 7.48)	5.8304	0.9672
10	(4.50, 15.11)	10.6172	(3.85, 14.35)	10.5001	0.9890
15	(6.98, 22.48)	15.4938	(5.94, 21.35)	15.4100	0.9946
20	(9.42, 29.88)	20.4533	(8.00, 28.38)	20.3803	0.9964
30	(14.26, 44.72)	30.4585	(12.09, 42.47)	30.3862	0.9976

$\nu = 10$

t	$C^s_{\nu,\alpha}(t)$	length	$C_{\nu,\alpha}(t)$	length	$\lvert C_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.64, 1.64)	3.2897	(-1.60, 1.60)	3.1963	0.9716
5	(2.47, 7.37)	4.8978	(2.36, 7.16)	4.8049	0.9810
10	(5.90, 13.86)	7.9655	(5.57, 13.49)	7.9192	0.9942
15	(9.15, 20.52)	11.3670	(8.63, 19.97)	11.3399	0.9976
20	(12.36, 27.23)	14.8752	(11.65, 26.50)	14.8566	0.9987
30	(18.70, 40.71)	22.0071	(17.62, 39.62)	21.9958	0.9995

$\nu = 20$

t	$C^s_{\nu,\alpha}(t)$	length	$C_{\nu,\alpha}(t)$	length	$\lvert C_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.64, 1.64)	3.2897	(-1.63, 1.63)	3.2532	0.9889
5	(2.86, 7.04)	4.1831	(2.81, 6.95)	4.1365	0.9889
10	(6.87, 12.99)	6.1254	(6.72, 12.82)	6.0978	0.9955
15	(10.70, 19.12)	8.4194	(10.47, 18.87)	8.4006	0.9978
20	(14.46, 25.31)	10.8441	(14.15, 24.98)	10.8287	0.9986
30	(21.92, 37.76)	15.8445	(21.45, 37.28)	15.8305	0.9991

$\nu = 30$

t	$C^s_{\nu,\alpha}(t)$	length	$C_{\nu,\alpha}(t)$	length	$\lvert C_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.64, 1.64)	3.2897	(-1.63, 1.63)	3.2689	0.9937
5	(3.01, 6.92)	3.9106	(2.98, 6.86)	3.8797	0.9921
10	(7.27, 12.63)	5.3575	(7.18, 12.52)	5.3370	0.9962
15	(11.36, 18.51)	7.1449	(11.22, 18.35)	7.1306	0.9980
20	(15.38, 24.45)	9.0738	(15.19, 24.25)	9.0620	0.9987
30	(23.33, 36.44)	13.1041	(23.05, 36.14)	13.0933	0.9992

Table 33: Comparisons of confidence interval $C_{\nu,\alpha}(t)$ of new method for standard confidence interval $C^s_{\nu,\alpha}(t)$ for $\alpha = 0.05$

$\nu = 5$

t	$C^s_{\nu,\alpha}(t)$	length	$C_{\nu,\alpha}(t)$	length	$\lvert C_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.96, 1.96)	3.9199	(-1.84, 1.84)	3.6795	0.9387
5	(1.36, 8.53)	7.1686	(1.21, 8.11)	6.9016	0.9628
10	(3.70, 16.29)	12.5891	(3.10, 15.51)	12.4076	0.9856
15	(5.86, 24.21)	18.3519	(4.83, 23.06)	18.2285	0.9933
20	(7.96, 32.18)	24.2159	(6.53, 30.65)	24.1197	0.9960
30	(12.10, 48.15)	36.0498	(9.89, 45.86)	35.9722	0.9978

$\nu = 10$

t	$C^s_{\nu,\alpha}(t)$	length	$C_{\nu,\alpha}(t)$	length	$\lvert C_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.96, 1.96)	3.9199	(-1.91, 1.91)	3.8126	0.9726
5	(2.03, 7.86)	5.8332	(1.94, 7.66)	5.7187	0.9804
10	(5.22, 14.70)	9.4740	(4.93, 14.34)	9.4075	0.9930
15	(8.22, 21.73)	13.5102	(7.73, 21.20)	13.4665	0.9968
20	(11.15, 28.82)	17.6743	(10.48, 28.12)	17.6396	0.9980
30	(16.93, 43.07)	26.1413	(15.90, 42.02)	26.1116	0.9989

$\nu = 20$

t	$C^s_{\nu,\alpha}(t)$	length	$C_{\nu,\alpha}(t)$	length	$\lvert C_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.96, 1.96)	3.9199	(-1.94, 1.94)	3.8715	0.9876
5	(2.47, 7.45)	4.9841	(2.43, 7.35)	4.9246	0.9880
10	(6.32, 13.61)	7.2948	(6.17, 13.43)	7.2591	0.9951
15	(9.96, 19.98)	10.0225	(9.72, 19.72)	10.0001	0.9978
20	(13.52, 26.43)	12.9059	(13.19, 26.08)	12.8895	0.9987
30	(20.55, 39.40)	18.8531	(20.05, 38.89)	18.8412	0.9994

$\nu = 30$

t	$C^s_{\nu,\alpha}(t)$	length	$C_{\nu,\alpha}(t)$	length	$\lvert C_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-1.96, 1.96)	3.9199	(-1.95, 1.95)	3.8920	0.9929
5	(2.64, 7.30)	4.6597	(2.62, 7.24)	4.6205	0.9916
10	(6.78, 13.16)	6.3823	(6.69, 13.04)	6.3560	0.9959
15	(10.72, 19.23)	8.5093	(10.57, 19.06)	8.4917	0.9979
20	(14.57, 25.38)	10.8046	(14.37, 25.16)	10.7910	0.9987
30	(22.18, 37.78)	15.6008	(21.87, 37.46)	15.5898	0.9993

Table 34: Comparisons of confidence interval $C_{\nu,\alpha}(t)$ of new method for standard confidence interval $C^s_{\nu,\alpha}(t)$ for $\alpha = 0.01$

$\nu = 5$

t	$C^s_{\nu,\alpha}(t)$	length	$C_{\nu,\alpha}(t)$	length	$\lvert C_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-2.58, 2.58)	5.1517	(-2.42, 2.42)	4.8411	0.9397
5	(0.43, 9.81)	9.3807	(0.39, 9.38)	8.9907	0.9584
10	(2.28, 18.65)	16.3614	(1.86, 17.85)	15.9937	0.9775
15	(3.90, 27.68)	23.7849	(3.00, 26.51)	23.5109	0.9885
20	(5.43, 36.78)	31.3472	(4.07, 35.23)	31.1522	0.9938
30	(8.40, 55.02)	46.6222	(6.19, 52.70)	46.5178	0.9978

$\nu = 10$

t	$C^s_{\nu,\alpha}(t)$	length	$C_{\nu,\alpha}(t)$	length	$\lvert C_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-2.58, 2.58)	5.1517	(-2.50, 2.50)	5.0059	0.9717
5	(1.18, 8.84)	7.6576	(1.14, 8.63)	7.4920	0.9784
10	(3.97, 16.36)	12.3969	(3.71, 15.99)	12.2811	0.9907
15	(6.49, 24.14)	17.6478	(6.01, 23.59)	17.5744	0.9958
20	(8.93, 31.99)	23.0680	(8.25, 31.27)	23.0160	0.9977
30	(13.69, 47.78)	34.0954	(12.63, 46.70)	34.0607	0.9990

$\nu = 20$

t	$C^s_{\nu,\alpha}(t)$	length	$C_{\nu,\alpha}(t)$	length	$\lvert C_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-2.58, 2.58)	5.1517	(-2.54, 2.54)	5.0874	0.9875
5	(1.71, 8.26)	6.5490	(1.68, 8.16)	6.4721	0.9883
10	(5.26, 14.83)	9.5739	(5.14, 14.66)	9.5206	0.9944
15	(8.55, 21.69)	13.1407	(8.33, 21.44)	13.1049	0.9973
20	(11.74, 28.65)	16.9114	(11.43, 28.31)	16.8839	0.9984
30	(17.98, 42.67)	24.6911	(17.50, 42.17)	24.6688	0.9991

$\nu = 30$

t	$C^s_{\nu,\alpha}(t)$	length	$C_{\nu,\alpha}(t)$	length	$\lvert C_{\nu,\alpha}(t)\rvert/\lvert C^s_{\nu,\alpha}(t)\rvert$
0	(-2.58, 2.58)	5.1517	(-2.56, 2.56)	5.1139	0.9927
5	(1.92, 8.04)	6.1235	(1.90, 7.98)	6.0745	0.9920
10	(5.82, 14.21)	8.3826	(5.75, 14.10)	8.3462	0.9957
15	(9.48, 20.65)	11.1690	(9.35, 20.49)	11.1434	0.9977
20	(13.02, 27.20)	14.1755	(12.84, 26.99)	14.1549	0.9986
30	(19.97, 40.43)	20.4587	(19.68, 40.12)	20.4411	0.9991

APPENDIX B

APPROPRIATE CHOICE OF FUNCTION $\epsilon_{\nu,\alpha}(\delta)$ FOR EFFICIENT CONFIDENCE INTERVAL

As discussed in section 3.4, the appropriate choice of function $\epsilon_{\nu,\alpha}(\delta)$ plays key role of methodologies for uniformly shorter confidence intervals. Since the skewness of noncentral t-distribution converges to its limit, the function $\epsilon_{\nu,\alpha}(\delta)$ should converge to a limit at which the length of confidence interval can be shorter than standard as noncentrality parameter δ increases. And since $\xi^s_{\nu,\alpha}(\delta)$ could be considered as a measure of skewness, the function $\epsilon_{\nu,\alpha}(\delta)$ for efficient confidence interval would be in form of

$$\epsilon_{\nu,\alpha}(\delta) = h(\xi^s_{\nu,\alpha}(\delta) - 1) \tag{64}$$

where $\xi^s_{\nu,\alpha}(0) = 1$ for all ν and small α.

B.1 Computational results and selection of function

B.1.1 Function $\gamma(\xi^s_{\nu,\alpha}(\delta) - 1)$

The simplest form of function (64) can be considered in the form

$$\epsilon_{\nu,\alpha}(\delta) = \gamma(\xi^s_{\nu,\alpha}(\delta) - 1) \tag{65}$$

and the effect of γ is investigated to find best choices for each ν.

- $\nu = 5$: $\gamma = 0.60$ looks best and $C_{\nu,\alpha}(200)$ with $\gamma = 0.60$ is 238.3126 while standard $C^s_{\nu,\alpha}(200) = 238.8936$ and the ratio $C_{\nu,\alpha}(200)/C^s_{\nu,\alpha}(200) = 0.9975$. However, $C_{\nu,\alpha}(200)$ with $\gamma = 0.65$ is 238.7271 but the length for $t = 20$ in the Table 35 is not shorter than standard. The computational results are summarized in Table 35. The main drawback of this function is that the improvement after $t = 10$ is very small.

Table 35: Comparisons of lengths of new method $C_{\nu,\alpha}(t)$ of (65) for Standard $C^s_{\nu,\alpha}(t)$ with $\alpha = 0.05$, $\nu = 5$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$
Standard $\lvert C^s_{\nu,\alpha}(t)\rvert$	3.9199	7.1686	12.5891	18.3519	24.2159
$\gamma = 0.10$	3.8408	7.1124	12.5181	18.2509	24.0816
$\gamma = 0.30$	3.7174	7.0537	12.4769	18.1991	24.0127
$\gamma = 0.55$	3.6077	7.0369	12.5288	18.2895	24.1330
$\gamma = 0.60$	3.5900	7.0380	12.5469	18.3193	24.1727
$\gamma = 0.65$	3.5735	7.0402	12.5667	18.3516	24.2159
$\gamma = 0.70$	3.5581	7.0432	12.5879	18.3861	24.2619

- $\nu = 10$: $\gamma = 0.3$ looks best and $C_{\nu,\alpha}(200)$ with $\gamma = 0.3$ is 172.0729 while standard $C^s_{\nu,\alpha}(200) = 172.3198$ and the ratio $C^s_{\nu,\alpha}(200)/C^s_{\nu,\alpha}(200) = 0.9985$. However, $C_{\nu,\alpha}(200)$ with $\gamma = 0.35$ is 172.3436, which is longer than standard. The computational results are summarized in Table 36.

Table 36: Comparisons of lengths of new method $C_{\nu,\alpha}(t)$ of (65) for Standard $C^s_{\nu,\alpha}(t)$ with $\alpha = 0.05$, $\nu = 10$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$
Standard $\lvert C^s_{\nu,\alpha}(t)\rvert$	3.9199	5.8332	9.4740	13.5102	17.6743
$\gamma = 0.10$	3.8579	5.8010	9.4437	13.4701	17.6217
$\gamma = 0.20$	3.8032	5.7818	9.4384	13.4668	17.6181
$\gamma = 0.30$	3.7548	5.7726	9.4519	13.4915	17.6515
$\gamma = 0.40$	3.7116	5.7712	9.4799	13.5377	17.7134

- $\nu = 20$: $\gamma = 0.225$ looks best and $C_{\nu,\alpha}(200)$ with $\gamma = 0.225$ is 122.9293 while standard $C^s_{\nu,\alpha}(200) = 122.9832$, and the ratio $C\nu, \alpha(200)/C^s_{\nu,\alpha}(200) = 0.9995$. The computational results are summarized at Table 37.

Table 37: Comparisons of lengths of new method $C_{\nu,\alpha}(t)$ of (65) for Standard $C^s_{\nu,\alpha}(t)$ with $\alpha = 0.05$, $\nu = 20$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$
Standard $\lvert C^s_{\nu,\alpha}(t)\rvert$	3.9199	4.9841	7.2948	10.0220	12.9060
$\gamma = 0.10$	3.8790	4.9625	7.2801	10.0065	12.8864
$\gamma = 0.20$	3.8414	4.9479	7.2792	10.0110	12.8938
$\gamma = 0.225$	3.8324	4.9452	7.2809	10.0148	12.8993
$\gamma = 0.25$	3.8236	4.9429	7.2832	10.0196	12.9060
$\gamma = 0.275$	3.8150	4.9409	7.2861	10.0253	12.9140

B.1.2 Functions $\gamma(\xi^s_{\nu,\alpha}(\delta) - 1)^2$ and $\gamma\sqrt{\xi^s_{\nu,\alpha}(\delta) - 1}$

The function (65) could result in uniformly shorter confidence interval but the amount of improvement is not much. Thus, other types of function $\epsilon_{\nu,\alpha}(\delta)$ such as (66) and (67) are investigated as follows.

$$\epsilon_{\nu,\alpha}(\delta) = \gamma\sqrt{\xi^s_{\nu,\alpha}(\delta) - 1} \tag{66}$$

$$\epsilon_{\nu,\alpha}(\delta) = \gamma(\xi^s_{\nu,\alpha}(\delta) - 1)^2 \tag{67}$$

The comparison of (66) and (67) is made in first two rows of Table 38, and function (66) looks worse than (65) and even (65). Thus the effect of γ of only (67) will be investigated.

- $\nu = 5$: $\gamma = 0.15$ looks best and $C_{\nu,\alpha}(200)$ with $\gamma = 0.15$ is 238.7097 while standard $C^s_{\nu,\alpha}(200) = 238.8936$ and the ratio $C_{\nu,\alpha}(200)/C^s_{\nu,\alpha}(200) = 0.9992$.

However, $C_{\nu,\alpha}(200)$ with $\gamma = 0.175$ is 239.6969, which is longer than standard.

The computational results are summarized in Table 38.

Table 38: Comparisons of lengths of new method $C_{\nu,\alpha}(t)$ of (67) and (66) for Standard $C^s_{\nu,\alpha}(t)$ with $\alpha = 0.05$, $\nu = 5$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$
Standard $\lvert C^s_{\nu,\alpha}(t)\rvert$	3.9199	7.1686	12.5891	18.3519	24.2159
(66) $\gamma = 0.10$	3.8554	7.1640	12.5633	18.3036	24.1464
(67) $\gamma = 0.10$	3.8039	6.8998	12.3114	18.0878	23.9442
(67) $\gamma = 0.15$	3.7581	6.8375	12.2943	18.1295	24.0397
(67) $\gamma = 0.175$	3.7374	6.8161	12.2998	18.1684	24.1107
(67) $\gamma = 0.20$	3.7181	6.7995	12.3115	18.2146	24.1909

- $\nu = 10$: $\gamma = 0.155$ looks best and $C_{\nu,\alpha}(200)$ with $\gamma = 0.155$ is 172.2369 while standard $C^s_{\nu,\alpha}(200) = 172.3198$ and the ratio $C_{\nu,\alpha}(200)/C^s_{\nu,\alpha}(200) = 0.9995$. However, $C_{\nu,\alpha}(200)$ with $\gamma = 0.165$ is 172.3628, which is longer than standard.

  The computational results are summarized in Table 39.

Table 39: Comparisons of lengths of new method $C_{\nu,\alpha}(t)$ of (67) for Standard $C^s_{\nu,\alpha}(t)$ with $\alpha = 0.05$, $\nu = 10$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$
Standard $\lvert C^s_{\nu,\alpha}(t)\rvert$	3.9199	5.8332	9.4740	13.5102	17.6743
(67) $\gamma = 0.10$	3.8777	5.7320	9.3820	13.4260	17.5881
(67) $\gamma = 0.155$	3.8564	5.6934	9.3665	13.4341	17.6147
(67) $\gamma = 0.20$	3.8398	5.6685	9.3666	13.4602	17.6629

- $\nu = 20$: $\gamma = 0.20$ looks best and $C_{\nu,\alpha}(200)$ with $\gamma = 0.20$ is 122.9774 while standard $C^s_{\nu,\alpha}(200) = 122.9832$ and the ratio $C_{\nu,\alpha}(200)/C^s_{\nu,\alpha}(200) = 0.9999$. However, $C_{\nu,\alpha}(200)$ with $\gamma = 0.210$ is 123.0135, which is longer than standard.

  The computational results are summarized in Table 40.

Table 40: Comparisons of lengths of new method $C_{\nu,\alpha}(t)$ of (67) for Standard $C^s_{\nu,\alpha}(t)$ with $\alpha = 0.05$, $\nu = 20$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$
Standard $\lvert C^s_{\nu,\alpha}(t)\rvert$	3.9199	4.9841	7.2948	10.0220	12.9060
(67) $\gamma = 0.10$	3.9068	4.9471	7.2599	9.9909	12.8742
(67) $\gamma = 0.20$	3.8941	4.9165	7.2407	9.9852	12.8784

B.1.3 Function $\gamma(\xi^s_{\nu,\alpha}(\delta) - 1)^{1.5}$

The new function $\epsilon_{\nu,\alpha}(\delta)$ is defined in the form of (68)

$$\epsilon_{\nu,\alpha}(\delta) = \gamma(\xi^s_{\nu,\alpha}(\delta) - 1)^{1.5} \tag{68}$$

and the effect of γ of (68) is investigated.

- $\nu = 5 : \gamma = 0.310$ looks best and $C_{\nu,\alpha}(200)$ with $\gamma = 0.310$ is 238.6784 while standard $C^s_{\nu,\alpha}(200) = 238.8936$ and the ratio $C_{\nu,\alpha}(200)/C^s_{\nu,\alpha}(200) = 0.9983$. However, $C_{\nu,\alpha}(200)$ with $\gamma = 0.325$ is 238.9499, which is longer than standard. The computational results are summarized in Table 41.

Table 41: Comparisons of lengths of new method $C_{\nu,\alpha}(t)$ of (68) for Standard $C^s_{\nu,\alpha}(t)$ with $\alpha = 0.05$, $\nu = 5$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$
Standard $\lvert C^s_{\nu,\alpha}(t)\rvert$	3.9199	7.1686	12.5891	18.3519	24.2159
(68) $\gamma = 0.10$	3.8236	7.0259	12.4335	18.1682	23.9921
(68) $\gamma = 0.30$	3.6851	6.9045	12.4042	18.2190	24.1047
(68) $\gamma = 0.310$	3.6795	6.9016	12.4076	18.2285	24.1197
(68) $\gamma = 0.325$	3.6713	6.8977	12.4132	18.2435	24.1430

- $\nu = 10 : \gamma = 0.225$ looks best and $C_{\nu,\alpha}(200)$ with $\gamma = 0.225$ is 172.2228 while standard $C^s_{\nu,\alpha}(200) = 172.3198$ and the ratio $C_{\nu,\alpha}(200)/C^s_{\nu,\alpha}(200) = 0.9994$. However, $C_{\nu,\alpha}(200)$ with $\gamma = 0.25$ is 172.4420, which is longer than standard. The computational results are summarized in Table 42.

Table 42: Comparisons of lengths of new method $C_{\nu,\alpha}(t)$ of (68) for Standard $C^s_{\nu,\alpha}(t)$ with $\alpha = 0.05$, $\nu = 10$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$
Standard $\lvert C^s_{\nu,\alpha}(t)\rvert$	3.9199	5.8332	9.4740	13.5102	17.6743
(68) $\gamma = 0.10$	3.8688	5.7687	9.4176	13.4498	17.6038
(68) $\gamma = 0.20$	3.8232	5.7265	9.4054	13.4570	17.6240
(68) $\gamma = 0.225$	3.8126	5.7187	9.4075	13.4665	17.6396
(68) $\gamma = 0.25$	3.8023	5.7117	9.4112	13.4786	17.6585

- $\nu = 20$: $\gamma = 0.215$ looks best and $C_{\nu,\alpha}(200)$ with $\gamma = 0.20$ is 122.9605 while standard $C^s_{\nu,\alpha}(200) = 122.9832$ and the ratio $C_{\nu,\alpha}(200)/C^s_{\nu,\alpha}(200) = 0.9998$. However, $C_{\nu,\alpha}(200)$ with $\gamma = 0.225$ is 122.9916, which is longer than standard. The computational results are summarized in Table 43.

Table 43: Comparisons of lengths of new method $C_{\nu,\alpha}(t)$ of (68) for Standard $C^s_{\nu,\alpha}(t)$ with $\alpha = 0.05$, $\nu = 20$

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$
Standard $\lvert C^s_{\nu,\alpha}(t)\rvert$	3.9199	4.9841	7.2948	10.0220	12.9060
(68) $\gamma = 0.10$	3.8967	4.9526	7.2698	9.9990	12.8806
(68) $\gamma = 0.20$	3.8747	4.9278	7.2595	9.9985	12.8864
(68) $\gamma = 0.215$	3.8715	4.9245	7.2591	10.0001	12.8895
(68) $\gamma = 0.225$	3.8693	4.9225	7.2589	10.0013	12.8919
(68) $\gamma = 0.30$	3.8537	4.9085	7.2609	10.0162	12.9170

B.2 Discussions and selection

The final comparison of functions (65), (67) and (68) are summarized in Table 44 – Table 46 with degrees of freedom $\nu = 5$, 10, 20 respectively. The functions (67) and (68) are much better than function (65), but it is difficult to say which one is better between functions (67) and (68). However, the function (68) can be considered as better function since the function (67) has advantage over the function (68) only in

relatively small interval of t and since the unknown parameter δ is believed to be uniformly distributed over $\Re$.

Table 44: A Comparison of lengths of confidence intervals by new methodologies for correct choice of $\epsilon_{\nu,\alpha}(\delta)$ with $\alpha = 0.05$, $\nu = 5$

	$t=0$	$t=5$	$t=10$	$t=15$	$t=20$	$t=200$
Standard $\lvert C^s_{\nu,\alpha}(t)\rvert$	3.9199	7.1686	12.5891	18.3519	24.2159	238.8936
(65) $\gamma = 0.60$	3.5900	7.0380	12.5469	18.3193	24.1727	238.3126
(67) $\gamma = 0.15$	3.7581	6.8375	12.2943	18.1295	24.0397	238.7097
(68) $\gamma = 0.310$	3.6795	6.9016	12.4076	18.2285	24.1197	238.6784

Table 45: A Comparison of lengths of confidence intervals by new methodologies for correct choice of $\epsilon_{\nu,\alpha}(\delta)$ with $\alpha = 0.05$, $\nu = 10$

	$t=0$	$t=5$	$t=10$	$t=15$	$t=20$	$t=200$
Standard $\lvert C^s_{\nu,\alpha}(t)\rvert$	3.9199	5.8332	9.4740	13.5102	17.6743	172.3198
(65) $\gamma = 0.30$	3.7548	5.7726	9.4519	13.4915	17.6515	172.0729
(67) $\gamma = 0.155$	3.8564	5.6934	9.3665	13.4341	17.6147	172.2369
(68) $\gamma = 0.225$	3.8126	5.7187	9.4075	13.4665	17.6396	172.2228

Table 46: A Comparison of lengths of confidence intervals by new methodologies for correct choice of $\epsilon_{\nu,\alpha}(\delta)$ with $\alpha = 0.05$, $\nu = 20$

	$t=0$	$t=5$	$t=10$	$t=15$	$t=20$	$t=200$
Standard $\lvert C^s_{\nu,\alpha}(t)\rvert$	3.9199	4.9841	7.2948	10.0220	12.9060	122.9832
(65) $\gamma = 0.225$	3.8324	4.9452	7.2809	10.0148	12.8993	122.9293
(67) $\gamma = 0.20$	3.8941	4.9165	7.2407	9.9852	12.8784	122.9774
(68) $\gamma = 0.215$	3.8715	4.9245	7.2591	10.0001	12.8895	122.9605

APPENDIX C

COMPUTATIONAL RESULTS OF TWO PROBABILITIES IN CHAPTER IV AND SIMULATION RESULTS

Figure 28 and 29 show the comparisons of $\mathbb{P}(|X_1 - X_2| > K_{\alpha,\nu}(|\bar{X}|)|\delta_1 = \delta_2 = \delta)$ for $\mathbb{P}(|X_1 - X_2| > K^*_{\alpha,\nu}(|\bar{X}|)|\delta_1 = \delta_2 = \delta)$ with $\alpha = 0.05, 0.001$ and ν =5, 10, 15, 20, 25, 30. Thus these plots show how the conservativeness of the set $K^*_{\alpha,\nu}(|\bar{X}|)$ can be minimized by the shrunken function.

Table 47 – Table 48 show the sizes of new method $K_{\alpha,\nu}(|\bar{X}|)$ based on 100,000 simulation (The standard error of the table entries is about 0.0007) in comparison with those of Nagata, et al's method and Miwa's methods with $\alpha = 0.1, 0.01$.

Table 49 – Table 60 shows the comparison of powers of new procedure for the approximate methods by Miwa and Nagata, et al. with $\delta_1 = 1$ and $\delta_1 = 5$, $\nu = 5, 10, 15, 20, 25, 30$, $\alpha = 0.1, 0.05, 0.01$ and $\delta_2 = 0 \sim 30$.

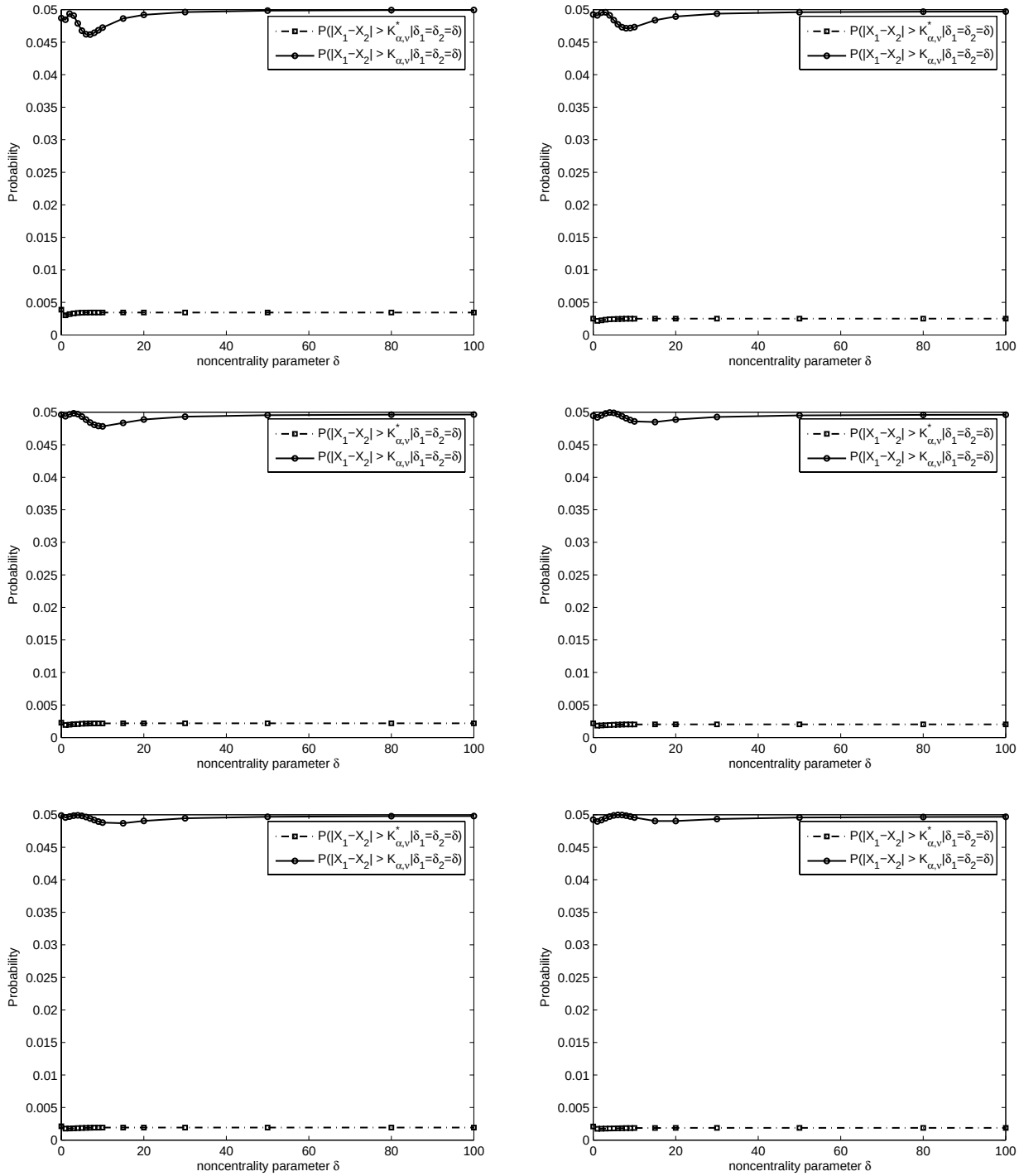

Figure 28: Comparisons of two probabilities, $\mathbb{P}(|X_1 - X_2| > K_{\alpha,\nu}(|\bar{X}|)|\delta_1 = \delta_2 = \delta)$ and $\mathbb{P}(|X_1 - X_2| > K^*_{\alpha,\nu}(|\bar{X}|)|\delta_1 = \delta_2 = \delta)$, with $\alpha = 0.05$

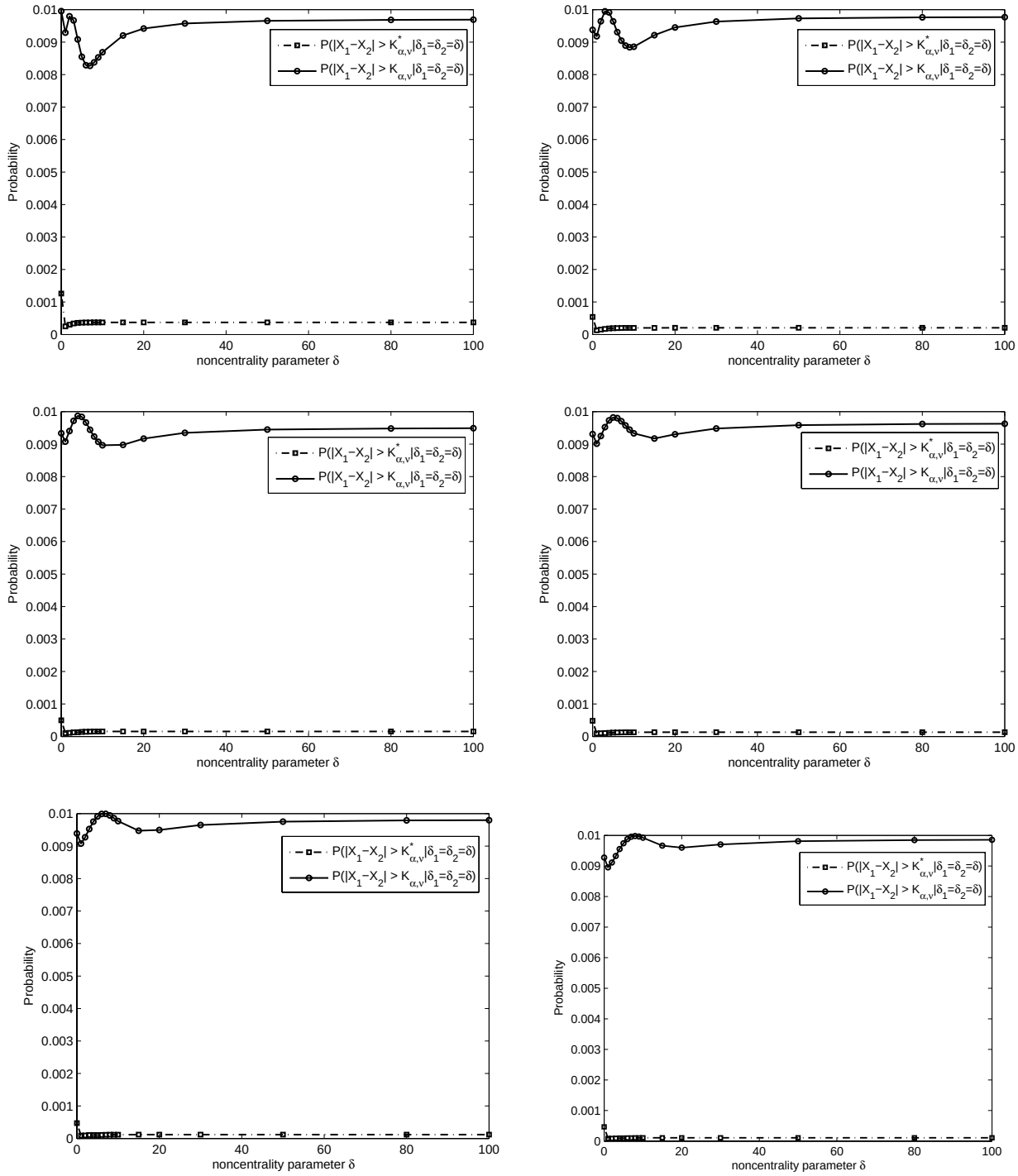

Figure 29: Comparisons of two probabilities, $\mathbb{P}(|X_1 - X_2| > K_{\alpha,\nu}(|\bar{X}|)|\delta_1 = \delta_2 = \delta)$ and $\mathbb{P}(|X_1 - X_2| > K^*_{\alpha,\nu}(|\bar{X}|)|\delta_1 = \delta_2 = \delta)$, with $\alpha = 0.01$

■ **Comparisons of size of new procedure for Miwa's method and Nagata's method**

Table 47: Comparisons of size of tests with $\nu = 5 \sim 30$ and nominal size $\alpha = 0.10$ based on 100,000 simulations.

	$\nu = 5$				$\nu = 10$			
δ	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.09752	0.06880	0.02446	0.02505	0.09663	0.08775	0.01723	0.01744
0.5	0.09877	0.06955	0.03446	0.03514	0.09660	0.08729	0.02720	0.02746
1.0	0.09841	0.06669	0.05664	0.05772	0.09755	0.08667	0.04928	0.04977
2.0	0.09997	0.06055	0.09413	0.09578	0.09889	0.08494	0.09349	0.09428
5.0	0.09682	0.05352	0.09778	0.09946	0.09843	0.07929	0.09901	0.09986
10.0	0.09371	0.04949	0.09393	0.09567	0.09441	0.07540	0.09602	0.09695
30.0	0.09895	0.05031	0.09591	0.09750	0.09810	0.07640	0.09794	0.09868
50.0	0.09992	0.05126	0.09689	0.09849	0.09887	0.07673	0.09845	0.09928
100.0	0.09816	0.04981	0.09469	0.09655	0.09820	0.07491	0.09768	0.09848

	$\nu = 15$				$\nu = 20$			
δ	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.09773	0.09189	0.01487	0.01497	0.09837	0.09446	0.01376	0.01384
0.5	0.09983	0.09367	0.02488	0.02507	0.09840	0.09477	0.02408	0.02423
1.0	0.09836	0.09210	0.04692	0.04733	0.09885	0.09444	0.04595	0.04623
2.0	0.09764	0.08953	0.09114	0.09176	0.09768	0.09195	0.09014	0.09059
5.0	0.09967	0.08733	0.10073	0.10127	0.09852	0.08993	0.09969	0.10018
10.0	0.09835	0.08581	0.09992	0.10038	0.09913	0.09000	0.10018	0.10052
30.0	0.09965	0.08516	0.09988	0.10043	0.09856	0.08819	0.09886	0.09931
50.0	0.09783	0.08347	0.09778	0.09827	0.09948	0.08862	0.09951	0.09983
100.0	0.09936	0.08470	0.09916	0.09964	0.09954	0.08864	0.09945	0.09982

	$\nu = 25$				$\nu = 30$			
δ	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.09916	0.09606	0.01242	0.01250	0.09975	0.09774	0.01249	0.01252
0.5	0.09859	0.09545	0.02210	0.02220	0.09920	0.09708	0.02105	0.02114
1.0	0.09872	0.09552	0.04476	0.04499	0.09940	0.09740	0.04516	0.04543
2.0	0.09883	0.09496	0.09088	0.09117	0.09778	0.09502	0.08935	0.08963
5.0	0.09954	0.09315	0.10085	0.10124	0.09879	0.09385	0.10018	0.10051
10.0	0.09935	0.09208	0.10015	0.10040	0.09968	0.09333	0.09997	0.10030
30.0	0.09842	0.09048	0.09918	0.09950	0.09913	0.09212	0.09934	0.09964
50.0	0.09719	0.08932	0.09757	0.09790	0.09925	0.09223	0.09917	0.09942
100.0	0.09725	0.08890	0.09745	0.09780	0.09983	0.09226	0.09966	0.09990

Table 48: Comparisons of size of tests with $\nu = 5 \sim 30$ and nominal size $\alpha = 0.01$ based on 100,000 simulations.

	$\nu = 5$				$\nu = 10$			
δ	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.00864	0.00753	0.00228	0.00132	0.00964	0.01056	0.00089	0.00061
0.5	0.00929	0.00775	0.00364	0.00200	0.00975	0.01050	0.00187	0.00137
1.0	0.00894	0.00681	0.00634	0.00360	0.00917	0.00968	0.00353	0.00269
2.0	0.00967	0.00576	0.01262	0.00710	0.00965	0.00908	0.01001	0.00779
5.0	0.00896	0.00479	0.01401	0.00852	0.00954	0.00776	0.01230	0.00974
10.0	0.00837	0.00410	0.01297	0.00769	0.00873	0.00716	0.01191	0.00950
30.0	0.00949	0.00390	0.01326	0.00764	0.00910	0.00692	0.01142	0.00900
50.0	0.00994	0.00449	0.01347	0.00820	0.00942	0.00731	0.01162	0.00923
100.0	0.00980	0.00410	0.01376	0.00799	0.00958	0.00726	0.01200	0.00944

	$\nu = 15$				$\nu = 20$			
δ	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.00869	0.00986	0.00062	0.00042	0.00934	0.01039	0.00042	0.00027
0.5	0.00862	0.00950	0.00095	0.00075	0.00882	0.00996	0.00076	0.00062
1.0	0.00946	0.01026	0.00272	0.00223	0.00916	0.01023	0.00238	0.00207
2.0	0.00929	0.00945	0.00864	0.00724	0.00912	0.00978	0.00810	0.00711
5.0	0.00944	0.00831	0.01139	0.00969	0.00981	0.00916	0.01151	0.01023
10.0	0.00858	0.00800	0.01138	0.00961	0.00938	0.00887	0.01098	0.00988
30.0	0.00911	0.00810	0.01127	0.00961	0.00925	0.00854	0.01056	0.00963
50.0	0.00916	0.00796	0.01094	0.00957	0.00890	0.00812	0.01024	0.00924
100.0	0.00975	0.00841	0.01183	0.01009	0.00969	0.00873	0.01108	0.00999

	$\nu = 25$				$\nu = 30$			
δ	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.00921	0.01020	0.00030	0.00026	0.00925	0.01031	0.00026	0.00023
0.5	0.00964	0.01067	0.00073	0.00061	0.00927	0.01045	0.00059	0.00052
1.0	0.00943	0.01042	0.00225	0.00198	0.00887	0.01014	0.00201	0.00180
2.0	0.00942	0.01006	0.00783	0.00700	0.00940	0.01030	0.00769	0.00710
5.0	0.00994	0.00941	0.01124	0.01030	0.00945	0.00932	0.01082	0.00998
10.0	0.00917	0.00867	0.01047	0.00943	0.00992	0.00931	0.01069	0.01001
30.0	0.00918	0.00854	0.01045	0.00954	0.00967	0.00907	0.01064	0.00998
50.0	0.00969	0.00889	0.01081	0.00980	0.00993	0.00929	0.01083	0.01004
100.0	0.00981	0.00900	0.01083	0.00994	0.00978	0.00897	0.01063	0.00987

■ **Comparisons of power of new procedure for Miwa's method and Nagata's method when** $\delta_1 = 1$

Table 49: Comparisons of powers of tests with $\nu = 5 \sim 20$ and $\alpha = 0.10$ based on 100,000 simulations.

	$\delta_1 = 1$							
	$\nu = 5$				$\nu = 10$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.16122	0.12137	0.04676	0.04765	0.17348	0.16021	0.04249	0.04291
1.0	0.09782	0.06635	0.05599	0.05724	0.09854	0.08857	0.05104	0.05155
2.0	0.15447	0.10714	0.11312	0.11509	0.16517	0.14956	0.11959	0.12057
3.0	0.29105	0.21507	0.24455	0.24762	0.33973	0.31401	0.29637	0.29791
4.0	0.46105	0.36154	0.42427	0.42822	0.55454	0.52258	0.52995	0.53182
5.0	0.62059	0.51057	0.59787	0.60220	0.73835	0.70893	0.72931	0.73104
6.0	0.74412	0.64172	0.73165	0.73496	0.86053	0.83845	0.85909	0.86005
7.0	0.83123	0.74523	0.82568	0.82789	0.93057	0.91635	0.93035	0.93086
8.0	0.88761	0.81925	0.88424	0.88618	0.96690	0.95811	0.96698	0.96724
9.0	0.92569	0.87477	0.92385	0.92490	0.98341	0.97908	0.98347	0.98359
10.0	0.95003	0.91171	0.94883	0.94960	0.99299	0.99065	0.99299	0.99312
15.0	0.99048	0.98136	0.99053	0.98915	0.99962	0.99956	0.99962	0.99962
20.0	0.99748	0.99477	0.99754	0.99221	0.99996	0.99996	0.99996	0.99990

	$\delta_1 = 1$							
	$\nu = 15$				$\nu = 20$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.17763	0.16961	0.03957	0.03980	0.17457	0.16897	0.03788	0.03813
1.0	0.09869	0.09254	0.04786	0.04829	0.09862	0.09422	0.04584	0.04615
2.0	0.17237	0.16254	0.12279	0.12339	0.17548	0.16886	0.12301	0.12343
3.0	0.36124	0.34631	0.31642	0.31749	0.37036	0.35989	0.32565	0.32643
4.0	0.59081	0.57366	0.56837	0.56966	0.61348	0.60161	0.59276	0.59365
5.0	0.78137	0.76607	0.77595	0.77673	0.80739	0.79851	0.80267	0.80318
6.0	0.90184	0.89141	0.90163	0.90210	0.91929	0.91384	0.91883	0.91916
7.0	0.95884	0.95300	0.95906	0.95928	0.97219	0.96943	0.97236	0.97245
8.0	0.98437	0.98188	0.98453	0.98465	0.99015	0.98897	0.99037	0.99038
9.0	0.99412	0.99282	0.99421	0.99426	0.99689	0.99659	0.99694	0.99700
10.0	0.99790	0.99745	0.99794	0.99798	0.99912	0.99900	0.99912	0.99914
15.0	0.99998	0.99998	0.99998	0.99998	1.00000	1.00000	1.00000	1.00000
20.0	0.99999	0.99999	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000

Table 50: Comparisons of powers of tests with $\nu = 25, 30$ and $\alpha = 0.10$ based on 100,000 simulations.

	$\delta_1 = 1$							
	$\nu = 25$				$\nu = 30$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.17722	0.17332	0.03685	0.03701	0.17903	0.17630	0.03726	0.03741
1.0	0.09811	0.09481	0.04414	0.04428	0.09918	0.09709	0.04481	0.04499
2.0	0.17430	0.16913	0.12212	0.12244	0.17932	0.17602	0.12606	0.12629
3.0	0.37870	0.37106	0.33363	0.33439	0.38333	0.37823	0.33830	0.33896
4.0	0.62391	0.61585	0.60329	0.60408	0.63269	0.62706	0.61362	0.61426
5.0	0.82279	0.81654	0.81834	0.81892	0.83153	0.82776	0.82804	0.82847
6.0	0.93141	0.92759	0.93121	0.93131	0.93854	0.93636	0.93870	0.93879
7.0	0.97724	0.97552	0.97750	0.97755	0.98175	0.98070	0.98183	0.98190
8.0	0.99416	0.99350	0.99426	0.99429	0.99567	0.99535	0.99574	0.99576
9.0	0.99838	0.99814	0.99841	0.99842	0.99884	0.99875	0.99886	0.99886
10.0	0.99963	0.99959	0.99963	0.99963	0.99976	0.99973	0.99977	0.99977
15.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
20.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

Table 51: Comparisons of powers of tests with $\nu = 5, 10$ and $\alpha = 0.05$ based on 100,000 simulations.

	$\delta_1 = 1$							
	$\nu = 5$				$\nu = 10$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.08931	0.06673	0.02238	0.02088	0.10120	0.09309	0.01870	0.01792
1.0	0.04845	0.03212	0.02757	0.02553	0.04999	0.04439	0.02261	0.02162
2.0	0.08423	0.05655	0.06067	0.05666	0.09548	0.08544	0.06228	0.06033
3.0	0.18142	0.12725	0.14862	0.14076	0.22660	0.20587	0.18521	0.18088
4.0	0.32188	0.23585	0.29084	0.27809	0.41988	0.38981	0.38931	0.38287
5.0	0.47202	0.36436	0.45133	0.43726	0.61547	0.58225	0.60434	0.59833
6.0	0.60838	0.48942	0.60013	0.58589	0.76900	0.73983	0.76996	0.76512
7.0	0.71930	0.60674	0.71961	0.70735	0.87097	0.84875	0.87439	0.87140
8.0	0.80054	0.69948	0.80342	0.79300	0.93220	0.91789	0.93494	0.93298
9.0	0.86151	0.77706	0.86536	0.85767	0.96409	0.95553	0.96631	0.96497
10.0	0.90242	0.83399	0.90604	0.90004	0.98256	0.97761	0.98365	0.98309
15.0	0.97906	0.95999	0.98067	0.97680	0.99911	0.99886	0.99920	0.99919
20.0	0.99428	0.98852	0.99470	0.98498	0.99993	0.99992	0.99994	0.99983

Table 52: Comparisons of powers of tests with $\nu = 15 \sim 30$ and $\alpha = 0.05$ based on 100,000 simulations.

	$\delta_1 = 1$							
	$\nu = 15$				$\nu = 20$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.10369	0.09886	0.01603	0.01573	0.10135	0.09892	0.01511	0.01485
1.0	0.04916	0.04588	0.01967	0.01908	0.04908	0.04721	0.01807	0.01770
2.0	0.09891	0.09302	0.06182	0.06069	0.10111	0.09763	0.06112	0.06026
3.0	0.24787	0.23611	0.20127	0.19826	0.25460	0.24759	0.20751	0.20512
4.0	0.45996	0.44275	0.42978	0.42540	0.48321	0.47223	0.45309	0.44980
5.0	0.67133	0.65328	0.66226	0.65842	0.70339	0.69292	0.69528	0.69283
6.0	0.83040	0.81581	0.83039	0.82762	0.85708	0.84989	0.85716	0.85558
7.0	0.92093	0.91197	0.92232	0.92097	0.94205	0.93784	0.94331	0.94262
8.0	0.96591	0.96126	0.96700	0.96622	0.97815	0.97590	0.97898	0.97860
9.0	0.98617	0.98385	0.98657	0.98624	0.99207	0.99121	0.99246	0.99228
10.0	0.99466	0.99354	0.99497	0.99480	0.99760	0.99733	0.99771	0.99763
15.0	0.99991	0.99988	0.99992	0.99992	1.00000	1.00000	1.00000	1.00000
20.0	0.99999	0.99999	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000

	$\delta_1 = 1$							
	$\nu = 25$				$\nu = 30$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.10541	0.10311	0.01459	0.01435	0.10572	0.10497	0.01457	0.01432
1.0	0.04944	0.04790	0.01730	0.01698	0.04970	0.04905	0.01720	0.01699
2.0	0.10239	0.09978	0.06102	0.06033	0.10415	0.10284	0.06366	0.06295
3.0	0.26212	0.25666	0.21300	0.21119	0.26757	0.26486	0.21912	0.21766
4.0	0.49752	0.48924	0.46684	0.46416	0.50550	0.50154	0.47789	0.47570
5.0	0.72345	0.71594	0.71480	0.71264	0.73606	0.73160	0.72953	0.72775
6.0	0.87762	0.87224	0.87731	0.87615	0.88748	0.88439	0.88798	0.88702
7.0	0.95263	0.95011	0.95357	0.95290	0.96075	0.95920	0.96198	0.96148
8.0	0.98527	0.98403	0.98577	0.98562	0.98897	0.98824	0.98940	0.98927
9.0	0.99578	0.99540	0.99596	0.99591	0.99696	0.99678	0.99712	0.99707
10.0	0.99893	0.99871	0.99901	0.99898	0.99922	0.99914	0.99923	0.99923
15.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
20.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

Table 53: Comparisons of powers of tests with $\nu = 5 \sim 20$ and $\alpha = 0.01$ based on 100,000 simulations.

	$\delta_1 = 1$							
	$\nu = 5$				$\nu = 10$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.02045	0.01808	0.00498	0.00299	0.02471	0.02636	0.00313	0.00251
1.0	0.00956	0.00718	0.00659	0.00380	0.00951	0.00977	0.00402	0.00299
2.0	0.01897	0.01354	0.01543	0.00929	0.02383	0.02404	0.01406	0.01096
3.0	0.05221	0.03763	0.04664	0.02935	0.07601	0.07395	0.05784	0.04750
4.0	0.11168	0.08146	0.10901	0.07097	0.18309	0.17573	0.16395	0.13998
5.0	0.19838	0.14512	0.20307	0.14072	0.33966	0.32472	0.33272	0.29637
6.0	0.30135	0.22515	0.32060	0.23414	0.51034	0.48950	0.52073	0.48026
7.0	0.41393	0.31778	0.44759	0.34557	0.66585	0.64238	0.68779	0.65012
8.0	0.51901	0.40992	0.56086	0.45656	0.78547	0.76402	0.80793	0.77871
9.0	0.61668	0.50271	0.66400	0.56228	0.86765	0.84956	0.88447	0.86456
10.0	0.69449	0.58260	0.74148	0.64960	0.92228	0.90958	0.93485	0.92117
15.0	0.90566	0.84618	0.93176	0.88323	0.99464	0.99326	0.99583	0.99480
20.0	0.96784	0.94340	0.97935	0.92982	0.99940	0.99923	0.99956	0.99927

	$\delta_1 = 1$							
	$\nu = 15$				$\nu = 20$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.02682	0.02923	0.00239	0.00198	0.02601	0.02861	0.00201	0.00178
1.0	0.00933	0.01005	0.00331	0.00262	0.00920	0.00991	0.00236	0.00203
2.0	0.02525	0.02650	0.01298	0.01077	0.02652	0.02840	0.01232	0.01086
3.0	0.08750	0.09014	0.06248	0.05526	0.09261	0.09696	0.06473	0.05889
4.0	0.21717	0.21912	0.19089	0.17355	0.23641	0.24229	0.20723	0.19399
5.0	0.40247	0.40070	0.39495	0.37024	0.43843	0.44323	0.42872	0.40995
6.0	0.60117	0.59640	0.60993	0.58532	0.65038	0.65223	0.65890	0.64172
7.0	0.76335	0.75649	0.77926	0.75994	0.81351	0.81315	0.82567	0.81359
8.0	0.87400	0.86828	0.88621	0.87408	0.91131	0.91014	0.92034	0.91318
9.0	0.93814	0.93367	0.94554	0.93894	0.96281	0.96168	0.96719	0.96392
10.0	0.96944	0.96705	0.97369	0.97000	0.98512	0.98438	0.98706	0.98556
15.0	0.99912	0.99906	0.99926	0.99915	0.99991	0.99990	0.99993	0.99991
20.0	0.99997	0.99997	0.99998	0.99998	1.00000	1.00000	1.00000	1.00000

Table 54: Comparisons of powers of tests with $\nu = 25, 30$ and $\alpha = 0.01$ based on 100,000 simulations.

	$\delta_1 = 1$							
	$\nu = 25$				$\nu = 30$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.02758	0.03018	0.00194	0.00169	0.02736	0.03039	0.00154	0.00143
1.0	0.00893	0.00986	0.00209	0.00183	0.00892	0.01003	0.00188	0.00170
2.0	0.02619	0.02818	0.01203	0.01094	0.02709	0.02945	0.01190	0.01098
3.0	0.09761	0.10173	0.06689	0.06243	0.09967	0.10567	0.06778	0.06407
4.0	0.24930	0.25622	0.21750	0.20643	0.25867	0.26847	0.22745	0.21868
5.0	0.46898	0.47532	0.45563	0.44168	0.48391	0.49321	0.47326	0.46149
6.0	0.68661	0.68978	0.69382	0.68038	0.70591	0.71242	0.71205	0.70252
7.0	0.84435	0.84521	0.85353	0.84551	0.86232	0.86556	0.87174	0.86581
8.0	0.93290	0.93243	0.93878	0.93453	0.94685	0.94778	0.95199	0.94898
9.0	0.97545	0.97495	0.97852	0.97655	0.98158	0.98162	0.98363	0.98238
10.0	0.99169	0.99148	0.99260	0.99190	0.99422	0.99423	0.99489	0.99447
15.0	0.99997	0.99996	0.99998	0.99997	1.00000	1.00000	1.00000	1.00000
20.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

■ **Comparisons of power of new procedure for Miwa's method and Nagata's method when $\delta_1 = 5$**

Table 55: Comparisons of powers of tests with $\nu = 5 \sim 20$ and $\alpha = 0.10$ based on 100,000 simulations.

	$\delta_1 = 5$							
	$\nu = 5$				$\nu = 10$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.84154	0.76794	0.70992	0.71345	0.91192	0.89757	0.83524	0.83634
1.0	0.61874	0.50936	0.59544	0.59965	0.73959	0.70907	0.73123	0.73271
2.0	0.37921	0.27373	0.37814	0.38161	0.48899	0.45125	0.49320	0.49506
3.0	0.20319	0.12956	0.20340	0.20610	0.26924	0.23549	0.27208	0.27333
4.0	0.11978	0.06910	0.12032	0.12193	0.13610	0.11162	0.13703	0.13781
5.0	0.09674	0.05264	0.09769	0.09934	0.09829	0.07882	0.09891	0.09981
6.0	0.11208	0.06446	0.11305	0.11472	0.12963	0.10595	0.13031	0.13132
7.0	0.15331	0.09214	0.15484	0.15713	0.20626	0.17386	0.20719	0.20849
8.0	0.20900	0.13069	0.21076	0.21348	0.31422	0.27441	0.31562	0.31720
9.0	0.27833	0.18439	0.28072	0.28367	0.43480	0.39004	0.43644	0.43826
10.0	0.35105	0.24134	0.35394	0.35753	0.55223	0.50622	0.55411	0.55595
15.0	0.67048	0.54666	0.67302	0.67657	0.90065	0.87958	0.90203	0.90259
20.0	0.84796	0.76212	0.84894	0.85096	0.98227	0.97708	0.98256	0.98276
30.0	0.96409	0.93418	0.96364	0.96370	0.99920	0.99895	0.99920	0.99920

	$\delta_1 = 5$							
	$\nu = 15$				$\nu = 20$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.93275	0.92666	0.87354	0.87415	0.94170	0.93790	0.88993	0.89043
1.0	0.78120	0.76624	0.77553	0.77637	0.80396	0.79505	0.79993	0.80054
2.0	0.53888	0.51753	0.54332	0.54445	0.57369	0.56004	0.57784	0.57860
3.0	0.30217	0.28179	0.30567	0.30675	0.31895	0.30435	0.32236	0.32302
4.0	0.14672	0.13160	0.14833	0.14906	0.15496	0.14466	0.15688	0.15738
5.0	0.09881	0.08687	0.09988	0.10041	0.09829	0.09018	0.09946	0.09982
6.0	0.13694	0.12174	0.13796	0.13853	0.14655	0.13539	0.14777	0.14829
7.0	0.24158	0.22128	0.24317	0.24410	0.26596	0.24982	0.26748	0.26820
8.0	0.37970	0.35273	0.38141	0.38250	0.42828	0.40971	0.43000	0.43082
9.0	0.52678	0.49961	0.52866	0.52985	0.59249	0.57372	0.59400	0.59481
10.0	0.66266	0.63749	0.66466	0.66568	0.73260	0.71655	0.73387	0.73454
15.0	0.96504	0.95913	0.96553	0.96572	0.98534	0.98339	0.98552	0.98563
20.0	0.99754	0.99703	0.99759	0.99760	0.99951	0.99945	0.99952	0.99952
30.0	0.99996	0.99996	0.99997	0.99997	1.00000	1.00000	1.00000	1.00000

Table 56: Comparisons of powers of tests with $\nu = 25, 30$ and $\alpha = 0.10$ based on 100,000 simulations.

	$\delta_1 = 5$							
	$\nu = 25$				$\nu = 30$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.94993	0.94738	0.90262	0.90299	0.95249	0.95087	0.91012	0.91034
1.0	0.82262	0.81614	0.81826	0.81859	0.82972	0.82534	0.82596	0.82635
2.0	0.58806	0.57719	0.59173	0.59253	0.59953	0.59233	0.60336	0.60394
3.0	0.33295	0.32291	0.33617	0.33677	0.34317	0.33599	0.34690	0.34747
4.0	0.15888	0.15136	0.16053	0.16098	0.15934	0.15357	0.16146	0.16174
5.0	0.09963	0.09293	0.10070	0.10101	0.09932	0.09449	0.10066	0.10097
6.0	0.15090	0.14196	0.15200	0.15239	0.15417	0.14782	0.15582	0.15626
7.0	0.28213	0.26977	0.28367	0.28422	0.29742	0.28783	0.29914	0.29971
8.0	0.46027	0.44569	0.46152	0.46234	0.48687	0.47550	0.48845	0.48890
9.0	0.63362	0.62016	0.63493	0.63561	0.67102	0.66028	0.67218	0.67273
10.0	0.77906	0.76734	0.77985	0.78035	0.81039	0.80208	0.81129	0.81165
15.0	0.99333	0.99271	0.99336	0.99339	0.99685	0.99656	0.99687	0.99688
20.0	0.99984	0.99982	0.99984	0.99984	0.99993	0.99992	0.99993	0.99993
30.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

Table 57: Comparisons of powers of tests with $\nu = 5, 10$ and $\alpha = 0.05$ based on 100,000 simulations.

	$\delta_1 = 5$							
	$\nu = 5$				$\nu = 10$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.72618	0.63449	0.55612	0.53946	0.83890	0.81791	0.71888	0.71285
1.0	0.47053	0.36152	0.45190	0.43667	0.61504	0.58183	0.60522	0.59871
2.0	0.25010	0.16530	0.26035	0.25008	0.35876	0.32321	0.37017	0.36474
3.0	0.11741	0.06761	0.12549	0.11883	0.16817	0.14490	0.17699	0.17285
4.0	0.06055	0.03170	0.06570	0.06182	0.07354	0.05974	0.07848	0.07618
5.0	0.04623	0.02353	0.05116	0.04752	0.04834	0.03798	0.05186	0.05024
6.0	0.05756	0.03060	0.06277	0.05918	0.06781	0.05490	0.07203	0.07029
7.0	0.08121	0.04383	0.08881	0.08364	0.12037	0.10084	0.12778	0.12477
8.0	0.12020	0.06885	0.13025	0.12356	0.19824	0.17015	0.20810	0.20381
9.0	0.16981	0.10218	0.18279	0.17388	0.30185	0.26646	0.31499	0.30981
10.0	0.22413	0.14050	0.24020	0.22960	0.40633	0.36628	0.42159	0.41537
15.0	0.52654	0.39004	0.54666	0.53340	0.82719	0.79848	0.83677	0.83315
20.0	0.74737	0.62656	0.76155	0.75111	0.96018	0.95075	0.96312	0.96180
30.0	0.92988	0.87503	0.93360	0.92826	0.99767	0.99678	0.99784	0.99775

Table 58: Comparisons of powers of tests with $\nu = 15 \sim 30$ and $\alpha = 0.05$ based on 100,000 simulations.

	$\delta_1 = 5$							
	$\nu = 15$				$\nu = 20$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.87856	0.86878	0.77840	0.77491	0.87856	0.86878	0.77840	0.77491
1.0	0.67210	0.65418	0.66123	0.65740	0.67210	0.65418	0.66123	0.65740
2.0	0.40998	0.38809	0.41742	0.41359	0.40998	0.38809	0.41742	0.41359
3.0	0.19622	0.18034	0.20161	0.19901	0.19622	0.18034	0.20161	0.19901
4.0	0.08262	0.07281	0.08559	0.08398	0.08262	0.07281	0.08559	0.08398
5.0	0.04998	0.04303	0.05198	0.05093	0.04950	0.04470	0.05138	0.05050
6.0	0.07569	0.06626	0.07827	0.07706	0.07569	0.06626	0.07827	0.07706
7.0	0.15010	0.13504	0.15446	0.15212	0.15010	0.13504	0.15446	0.15212
8.0	0.26065	0.23945	0.26663	0.26368	0.26065	0.23945	0.26663	0.26368
9.0	0.39431	0.36850	0.40225	0.39830	0.39431	0.36850	0.40225	0.39830
10.0	0.53600	0.50985	0.54434	0.54041	0.53600	0.50985	0.54434	0.54041
15.0	0.93038	0.92109	0.93370	0.93253	0.93038	0.92109	0.93370	0.93253
20.0	0.99303	0.99148	0.99340	0.99332	0.99303	0.99148	0.99340	0.99332
30.0	0.99986	0.99984	0.99987	0.99986	0.99986	0.99984	0.99987	0.99986

	$\delta_1 = 5$							
	$\nu = 25$				$\nu = 30$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.90528	0.90165	0.82153	0.81977	0.91087	0.90925	0.83487	0.83359
1.0	0.72286	0.71522	0.71392	0.71147	0.73476	0.73042	0.72751	0.72598
2.0	0.45995	0.45028	0.46636	0.46402	0.47366	0.46756	0.48064	0.47891
3.0	0.22539	0.21741	0.23079	0.22901	0.23467	0.22858	0.24022	0.23876
4.0	0.08875	0.08394	0.09133	0.09029	0.09168	0.08801	0.09422	0.09325
5.0	0.04962	0.04634	0.05133	0.05067	0.04927	0.04637	0.05101	0.05048
6.0	0.08268	0.07773	0.08496	0.08407	0.08562	0.08154	0.08764	0.08697
7.0	0.18174	0.17307	0.18553	0.18385	0.19448	0.18714	0.19797	0.19650
8.0	0.33391	0.32214	0.33906	0.33701	0.36210	0.35164	0.36595	0.36439
9.0	0.50496	0.49229	0.51123	0.50856	0.54338	0.53188	0.54711	0.54526
10.0	0.67233	0.66041	0.67802	0.67583	0.71392	0.70412	0.71713	0.71541
15.0	0.98475	0.98340	0.98529	0.98510	0.99172	0.99093	0.99197	0.99188
20.0	0.99978	0.99971	0.99980	0.99980	0.99988	0.99988	0.99988	0.99988
30.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

Table 59: Comparisons of powers of tests with $\nu = 5 \sim 20$ and $\alpha = 0.01$ based on 100,000 simulations.

	$\delta_1 = 5$							
	$\nu = 5$				$\nu = 10$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.40881	0.34726	0.26583	0.18557	0.60175	0.59370	0.42519	0.38206
1.0	0.19844	0.14444	0.20338	0.14117	0.34014	0.32486	0.33394	0.29811
2.0	0.07992	0.04949	0.10213	0.06849	0.14506	0.13232	0.16596	0.14370
3.0	0.02850	0.01603	0.03995	0.02556	0.05195	0.04479	0.06235	0.05261
4.0	0.01231	0.00645	0.01864	0.01129	0.01676	0.01395	0.02102	0.01702
5.0	0.00834	0.00467	0.01336	0.00810	0.00941	0.00759	0.01230	0.00967
6.0	0.01128	0.00607	0.01818	0.01085	0.01537	0.01278	0.01981	0.01579
7.0	0.01698	0.00935	0.02613	0.01635	0.03296	0.02735	0.04110	0.03359
8.0	0.02800	0.01568	0.04261	0.02695	0.06455	0.05551	0.07822	0.06622
9.0	0.04388	0.02528	0.06513	0.04249	0.11366	0.09994	0.13550	0.11666
10.0	0.06328	0.03762	0.09246	0.06144	0.17575	0.15763	0.20581	0.18068
15.0	0.22235	0.14944	0.29535	0.21897	0.58931	0.56287	0.63656	0.60100
20.0	0.43559	0.32609	0.52541	0.43000	0.85761	0.84196	0.88276	0.86458
30.0	0.75729	0.65311	0.81720	0.74383	0.98747	0.98502	0.99013	0.98808

	$\delta_1 = 5$							
	$\nu = 15$				$\nu = 20$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.67428	0.67762	0.50044	0.47283	0.70777	0.71558	0.54541	0.52479
1.0	0.40275	0.40131	0.39391	0.36854	0.44120	0.44631	0.43180	0.41312
2.0	0.18416	0.17975	0.20236	0.18604	0.20631	0.20704	0.22446	0.21146
3.0	0.06516	0.06162	0.07509	0.06749	0.07449	0.07370	0.08435	0.07797
4.0	0.01968	0.01806	0.02340	0.02039	0.02162	0.02086	0.02525	0.02265
5.0	0.00987	0.00883	0.01160	0.01006	0.00975	0.00916	0.01141	0.01015
6.0	0.01811	0.01641	0.02130	0.01860	0.01943	0.01841	0.02224	0.02007
7.0	0.04483	0.04144	0.05161	0.04585	0.05305	0.05067	0.05928	0.05467
8.0	0.09442	0.08807	0.10691	0.09648	0.12054	0.11533	0.13184	0.12323
9.0	0.17313	0.16299	0.19185	0.17662	0.22515	0.21757	0.24166	0.22897
10.0	0.27691	0.26370	0.30206	0.28231	0.35780	0.34776	0.37825	0.36271
15.0	0.78829	0.77895	0.81178	0.79603	0.88745	0.88304	0.89820	0.89087
20.0	0.96497	0.96315	0.97095	0.96746	0.99075	0.99036	0.99248	0.99149
30.0	0.99926	0.99914	0.99939	0.99932	0.99997	0.99997	0.99997	0.99997

Table 60: Comparisons of powers of tests with $\nu = 25, 30$ and $\alpha = 0.01$ based on 100,000 simulations.

	$\delta_1 = 5$							
	$\nu = 25$				$\nu = 30$			
δ_2	New	Nagata	Miwa 1	Miwa 2	New	Nagata	Miwa 1	Miwa 2
0.0	0.73439	0.74202	0.57028	0.55424	0.74651	0.75653	0.59168	0.57868
1.0	0.46731	0.47379	0.45582	0.44099	0.48448	0.49359	0.47269	0.46017
2.0	0.22313	0.22550	0.23817	0.22855	0.23449	0.24024	0.25157	0.24241
3.0	0.08140	0.08151	0.09016	0.08520	0.08365	0.08530	0.09283	0.08821
4.0	0.02331	0.02285	0.02582	0.02414	0.02393	0.02411	0.02699	0.02538
5.0	0.00962	0.00923	0.01077	0.00995	0.00949	0.00941	0.01072	0.00995
6.0	0.02138	0.02053	0.02370	0.02187	0.02155	0.02140	0.02366	0.02237
7.0	0.06116	0.05869	0.06610	0.06221	0.06739	0.06608	0.07244	0.06886
8.0	0.14181	0.13693	0.15109	0.14309	0.15975	0.15681	0.16880	0.16254
9.0	0.26581	0.25814	0.27864	0.26773	0.29635	0.29135	0.30883	0.29916
10.0	0.42218	0.41251	0.43707	0.42403	0.46661	0.45914	0.48148	0.46954
15.0	0.93738	0.93416	0.94165	0.93792	0.96218	0.96014	0.96438	0.96245
20.0	0.99762	0.99744	0.99795	0.99770	0.99919	0.99913	0.99928	0.99923
30.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

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