ASYMPTOTIC PROPERTIES OF MÜNTZ ORTHOGONAL POLYNOMIALS

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Úlfar F. Stefánsson

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Approved by:

Professor Jeff Geronimo,
Committee Chair
School of Mathematics
Georgia Institute of Technology

Professor Doron S. Lubinsky, Advisor
School of Mathematics
Georgia Institute of Technology

Professor Plamen Iliev
School of Mathematics
Georgia Institute of Technology

Professor Christopher Heil
School of Mathematics
Georgia Institute of Technology

Professor Francisco Marcellán
Departamento de Matemáticas
Universidad Carlos III de Madrid

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INTRODUCTION

1.1 Orthogonal polynomials

The analysis of orthogonal polynomials associated with general weights is a major theme in classical analysis. The applications are rich, reaching different fields of mathematics, physics and engineering such as approximation theory, special functions, differential and integral equations, random matrix theory, number theory, quantum mechanics, statistics and image analysis.

The setting is the following: Given a weight function w on an interval (a, b), finite or infinite, such that the *moments*

$$c_n = \int_a^b x^n w(x)dx, \qquad n = 0, 1, 2, \ldots$$

exist and are finite, we say that the set of real polynomials $\{p_n\}_{n=0}^{\infty}$, with $p_n := p_n(w; \cdot)$ of strict degree n, is *orthogonal over* $[a, b]$ *with respect to the weight* w if

$$\int_a^b p_n(x)p_m(x)w(x)dx = 0, \qquad m \neq n.$$

The theory of orthogonal polynomials has its roots in Stieltjes' work on continued fractions. In his monumental final paper [53] he introduced the Stieltjes integral and used it to solve the classical *moment problem*, which asks whether a given weight w (or a more general measure $d\mu$) can be represented by its moments $\{c_n\}_{n=0}^{\infty}$ and vice versa. The moment problem is historically important because it gave rise to important tools in modern analysis. Among others, M. Riesz gave a proof and his techniques were later used to prove the well known Hahn-Banach theorem of functional analysis.

In the first half of the 20th century, the Hungarian mathematician Gábor Szegő had a profound influence on the theory of orthogonal polynomials and related fields.

His monograph [54] from 1939 still serves as the single most important reference in the field. He determined the asymptotic behavior of $p_n(w;\cdot)$ as $n \longrightarrow \infty$ for a large class of weights w on $[-1,1]$, characterized by the *Szegő condition*

$$\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}} dx > -\infty. \tag{1.1}$$

He was the first to consider orthogonal polynomials with respect to weights on arbitrary curves in the complex plane. An important case are orthogonal polynomials on the unit circle, which have applications in linear prediction and filtering theory, and are used extensively in spectral theory for certain linear operators, including discrete Schrödinger operators. Significant new developments due to B. Simon and his collaborators [45, 46] for orthogonal polynomials on the unit circle have major implications for weights on $[-1,1]$.

Motivated by applications from approximation theory, particularly Padé approximation and weighted polynomial approximation, problems concerning the asymptotics of orthogonal polynomials with respect to weights with support on the whole real line, have been in the spotlight since the school of G. Freud in the 1960's [15, 36]. Freud and P. Nevai considered weights of the form $w = W^2 = \exp(-2Q)$, where Q is even, convex and of smooth polynomial growth at infinity. Such weights are often called *Freud weights*. Particular examples are

$$w(x) = W_\alpha^2(x) = \exp(-|x|^\alpha), \qquad \alpha > 0.$$

E. Levin and D.S. Lubinsky (see the monograph [21] and the surveys [22, 25]) have done extensive work on orthogonal polynomials with respect to exponential weights.

In the 1980's, potential theory advanced as a vital tool in the development of orthogonal polynomials associated with weights on the real line. It is especially useful for investigating nth root asymptotics and zeros of orthogonal polynomials, see the monograph [44] of E.B. Saff and V. Totik. A breakthrough emerged in independent papers of E.A. Rakhmanov [41] and Mhaskar and Saff [29]: For Q even and convex, the authors defined what is now referred to as the *Mhaskar-Rakhmanov-Saff number* a_n, the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}} dt.$$

It turns out that $p_n(W^2, x)$ behaves on $[-a_n, a_n]$ much like an orthonormal polynomial for a Szegö weight on $[-1, 1]$, and this allowed the authors to establish nth root asymptotics for $p_n(W^2, \cdot)$.

Following the results on the nth root asymptotics, Lubinsky, Mhaskar and Saff [26] determined ratio asymptotics for orthogonal polynomials with weights on the real line, an in particular solved the Freud conjectures. The ratio asymptotics were quickly followed by strong (or Szegö) asymptotics, established independently by Lubinsky and Saff [27], and Rakhmanov [42]. In the various types of asymptotics, weighted polynomial approximation played an essential role. For more details, see Totik's excellent seminal lecture notes [55].

In the 1990's a new powerful approach was developed to analyse the asymptotics of orthogonal polynomials. First Fokas, Its and Kitaev [14] showed that you can look at polynomial orthogonality as a Riemann Hilbert problem. Through this approach Deift and Zhou [11] developed the powerful so-called non-linear steepest descent method. Via the Riemann-Hilbert techniques of Deift, Kriecherbauer, McLaughlin [9, 10], remarkably precise asymptotics have been obtained for orthogonal polynomials associated with weights on the real line.

Random matrix theory (see Mehta [28]) is a field that has gained a large following in recent years, and indeed is the main motivation for the Riemann-Hilbert approach for orthogonal polynomials. It has rich applications, such as nuclear physics, number theory, statistics, financial correlation, and the theory of integrable systems. Its usefulness becomes apparent when applied in modeling large samples with equally large population, take for example the world's wireless communication system. In many cases, important statistical properties of the modeled system can be deduced from exploring the behavior of the eigenvalues of the random matrix as the size of the matrix grows. Orthogonal polynomials come into play since some natural systems can be well modeled using random matrices on probability spaces constructed via so-called Gaussian unitary ensembles. In that case the growth of the eigenvalues can be determined from the asymptotic behavior of the associated orthogonal polynomials. A. Kuijlaars [19] and P. Miller [30] have both written excellent lecture notes on the subject. One of the most important statistics in random matrix theory relates to the associated correlation functions. As the size of the matrix grows, these essentialy behave identically for different weights and this phenomena has been refered to as the *universality law*. Using smoothing techniques, Lubinsky, in his landmark paper [23], has proved universality for very general weights on $[-1, 1]$.

1.2 Müntz polynomials

One of the fundamental theorems of functional analysis and approximation theory is Weierstrass's Theorem, proved in 1885 [57], which states that every continuous function on $[0, 1]$ (or any closed bounded interval) can be approximated arbitrarily closely by polynomials under the uniform norm. In other words, the space $\text{span}\{x^n : n = 0, 1, 2, \dots\}$ is dense in $C[0, 1]$. In 1912, S. Bernstein conjectured [3] that more generally the result holds if the exponents of x (the natural numbers) are replaced with an increasing sequence $0 \le \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ of real numbers that satisfy

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty.$$

In the case of the algebraic polynomials this is the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$, so Weierstrass's Theorem is a special case. Two years later the conjecture was proved (with $\lambda_0 = 0$) by the German mathematician Herman Müntz in his paper [34] (see a short biography in [40]). Since then this result has been generalized and numerous proofs have emerged. A recent research monograph [16] is devoted to ramifications of this theorem and further questions regarding density and different types of Müntz theorems are discussed in [1, 7]. The version that is most relevant to the topics of this text is the following:

Theorem 1.1 (Müntz's Theorem) *Let* $\{\lambda_k\}_{k=0}^{\infty}$ *be a sequence of real numbers such that* $\inf_k \lambda_k > -1/2$. *Then span*$\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ *is dense in* $L_2[0,1]$ *if and only if*

$$\sum_{k=0}^{\infty} \frac{1}{\lambda_k + \frac{1}{2}} = \infty. \tag{1.2}$$

This result is especially beautiful since it connects a topological property (the density of the space) to an arithmetic property (divergence of the series).

Following the considerations above, it is natural to ask further questions about these generalized polynomials, which have the form

$$\sum_{k=0}^{n} c_k x^{\lambda_k}.$$

We call them *Müntz polynomials*. It turns out that these functions share many of the basic properties of their algebraic polynomial cousins. They can be orthogonalized in a natural way on $L_2[0,1]$, and they form a Chebyshev system on $(0,\infty)$. These two properties give rise to applications in approximation theory. Using the substitution $x = e^{-t}$, we can alternatively look at the Müntz polynomials as exponential sums of the form

$$\sum_{k=0}^{n} c_k e^{-\lambda_k t},$$

which are important in non-linear approximation, particularly for decay processes.

A large part of the classic text of P. Borwein and T. Erdélyi [7] is devoted to Müntz spaces, and important questions involve Markov-Bernstein inequalities, Remez inequalities, the rate of approximation by Müntz polynomials and the distribution of their zeros. Müntz rational functions are also explored, and in [6], the authors investigate the corresponding Christoffel functions, which in a way provide a measure of the density of the space. Furthermore, in [7, Appendix 2], Müntz orthogonality is used to reproduce Apéry's proof of the irrationality of $\zeta(3)$. A Müntz-type of

Gauss-Jacobi quadrature has been developed by G.V. Milovanović and A.S. Cvetković [32, 33], especially with a view to numerical integration of functions with endpoint singularities. Furthermore, the zero distribution of the extremal Müntz polynomials has been investigated by Lubinsky and Saff [24].

1.3 Overview

The goal of this research is to determine asymptotic properties of the Müntz orthogonal polynomials with respect to Legendre and Jacobi weights in $L_2[0,1]$. An important special case is when the Müntz exponents $\{\lambda_k\}$ are asymptotic to an arithmetic progression, i.e.

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \rho$$

for some constant $\rho > 0$. Some of the classical orthogonal polynomials (e.g. Legendre, Jacobi with $\alpha = 0$, and Laguerre with $\alpha = 0$) can indeed be written in terms of the Müntz orthogonal polynomials, and our results yield new representations for these functions which in turn gives rise to new proofs for their asymptotic behavior. Furthermore, orthogonal exponential sums on $(0, \infty)$, as well as certain multiple orthogonal systems, can be represented by Müntz orthogonal polynomials, and our results apply to these as well.

In Chapter 3 we introduce a representation for the Müntz orthogonal polynomials as a real oscillatory integral, which holds for $x \in (0,1)$, i.e. on the interval of orthogonality. The formula holds for general real exponents $\{\lambda_k\}$ and it is quite special that we get the same critical path for the pure oscillatory behavior of these elements, independent of the λ_k's. This allows us to determine the asymptotics inside the interval under very mild conditions on the λ_k's using standard asymptotic analysis, and this is the topic of Chapter 4. We also consider endpoint asymptotics, as well as the behavior of the associated Christoffel functions for special cases. This is the first time that such asymptotics have been determined for general exponents λ_k.

Building on these results, we devote Chapter 5 to studies of the zeros of the Müntz orthogonal polynomials. We get a global bound for the smallest zero on the interval and asymptotically determine the position of the largest zeros. Moreover we determine the asymptotics of the spacing of the zeros in the bulk of the interval.

In Chapter 6, we turn our attention to asymptotics outside the interval of orthogonality. There, we don't have a nice formula as for $x \in (0,1)$, but using the method of steepest descent allows us to establish asymptotics for $x > 1$. In Chapter 7, we determine the asymptotic behavior of the Müntz-Christoffel functions at the endpoints $x = 0$ and 1. These functions can be written explicitly in terms of the Müntz orthogonal polynomials, which take on a simple form at the endpoints.

1.4 Notation

The notation for the asymptotic relations used throughout this dissertation are given in Table 1. Similar notation applies to sequences. We shall sometimes write "locally uniformly for x in U" when we mean "uniformly for x on compact subsets of U."

Table 1: Notation for asymptotic relations. Here, f and g are positive functions.

Notation	Relation
$f(x) = o(g(x))$	$\lim_{x\to\infty} f(x)/g(x) = 0$
$f(x) = \mathcal{O}(g(x))$	$f \le Ag$ for some constant A
$f(x) \sim g(x)$	$\lim_{x\to\infty} f(x)/g(x) = 1$
$f(x) \asymp g(x)$	$A_1 g \le f \le A_2 g$ for some constants A_1, A_2

We use the Kronecker-delta notation:

$$\delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \ne m. \end{cases}$$

The set of real polynomials of degree at most n is denoted by $\mathcal{P}_n$ and we let $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$ denote the set of all real polynomials. We define the space of continuous functions on $[a, b]$ by

$$C[a,b] := \{f : [a,b] \longrightarrow \mathbb{R} : f \text{ continuous}\}$$

and denote the supremum norm over $[a, b]$ by $\|f\|_{[a,b]} := \sup_{t\in[a,b]} |f(t)|$. We let

$$L_\infty[a,b] := \{f : [a,b] \longrightarrow \mathbb{R} : f \text{ measurable and } \|f\|_\infty < \infty\}.$$

and for each $1 \le p < \infty$,

$$L_p[a,b] := \left\{ f : [a,b] \longrightarrow \mathbb{R} : f \text{ measurable and } \|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p} < \infty \right\}.$$

We shall frequently use the notation

$$\lambda^* := \lambda + \frac{1}{2} \tag{1.3}$$

for each real number λ, and similarly for sums, we sometimes write

$$\sum_{k=0}^{n}{}^{*} a_k := \sum_{k=0}^{n-1} a_k + \frac{1}{2}a_n. \tag{1.4}$$

For a given sequence $\{\lambda_k\}_{k=0}^\infty$, let

$$\begin{aligned} S_n &:= \sum_{k=0}^{n}{}^{*} \lambda_k^* = \sum_{k=0}^{n-1} \lambda_k^* + \frac{1}{2}\lambda_n^*, \\ \sigma_n &:= \frac{S_n}{\lambda_n^{*2}}, \end{aligned}$$

and also let $\Sigma_n := 2S_n = \sum_{k=0}^{n-1}(2\lambda_k+1)+(2\lambda_n+1)/2$ for each n. Furthermore, define

$$T_n := \sum_{k=0}^{n}{}^{*} \frac{1}{\lambda_k^*} = \sum_{k=0}^{n-1} \frac{1}{\lambda_k^*} + \frac{1}{2\lambda_n^*}, \tag{1.5}$$

and note that the growth of T_n determines the denseness condition (1.2).

CHAPTER II

BACKGROUND

2.1 Orthogonal polynomials

Let $\mu(x)$ be a non-decreasing function with infinitely many points of increase on an interval (a,b) (finite of infinite) and assume that the moments are finite, that is

$$\int_a^b |x|^n d\mu(x) < \infty, \qquad n = 0,1,2,\ldots$$

Then a set of real polynomials $\{p_n\}_{n=0}^{\infty}$, with

$$p_n(x) := p_n(\mu;x) = \gamma_n x^n + \cdots \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}, \qquad n = 0,1,2,\ldots$$

is said to be *orthogonal over* (a,b) *with respect to the measure* μ if

$$\int_a^b p_n(x)p_m(x)d\mu(x) = 0, \qquad m \neq n. \tag{2.1}$$

The elements $\{p_n\}_{n=0}^{\infty}$ can be obtained by applying the Gram-Schmidt process to the monomials $1, x, x^2, x^3, \ldots$. They can be made uniquely determined by imposing some additional conditions on p_n, such as fixing the value of p_n at either one of the endpoints or taking the leading coefficients γ_n as positive and requiring the polynomials to be *orthonormal with respect to* μ, i.e.

$$\int_a^b p_n(x)p_m(x)d\mu(x) = \delta_{n,m}, \qquad m,n = 0,1,2,\ldots$$

If μ is absolutely continuous, then we can write $d\mu(x) = w(x)dx$, where the *weight function* $w(x)$ is non-negative and Lebesgue measurable with positive measure on (a,b). Given the condition (2.1), in this case we say that $\{p_n\}_{n=0}^{\infty}$ is *orthogonal with respect to the weight* $w(x)$.

In this setup, $\{p_n\}_{n=0}^{\infty}$ is linearly independent and every polynomial $Q_n \in \mathcal{P}_n$ can be uniquely written as a linear combination of $p_0, p_1, \ldots, p_n$. Then from (2.1) it follows that

$$\int_a^b p_n(x)x^k d\mu(x) = 0, \qquad k = 0, 1, \ldots, n-1.$$

The following property characterizes orthogonal polynomials and is the source of many important applications [7, 15, 45, 54]:

Theorem 2.1 *Each set of orthonormal polynomials (take $\gamma_n > 0$ for all n) satisfies a three-term recurrence relation of the form*

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1}p_{n-1}(x), \qquad n = 0, 1, 2, \ldots \tag{2.2}$$

where $p_{-1} := 0$, $a_{-1} = 0$, $a_n = \gamma_n/\gamma_{n+1} > 0$, and $b_n \in \mathbb{R}$.

A converse to this result is given by Favard's Theorem: given sequences $\{a_n\}$ and $\{b_n\}$ of real numbers with $a_n > 0$ for all n, if we define a set of polynomials recursively via (2.2), then there exists a measure μ with respect to which the polynomials form an orthogonal set.

Another fundamental result, with obvious connections to approximation theory, is the following extremal property [12, 15, 54]:

Theorem 2.2 *The polynomial*

$$Q_n(x) = \frac{1}{\gamma_n} p_n(\mu; x) = x^n + \cdots$$

is the unique monic polynomial of degree n of minimal $L_2(\mu)$-norm; that is, Q_n solves the extremal problem

$$\min_{x^n + \cdots \in \mathcal{P}_n} \int_a^b |x^n + \cdots|^2 d\mu(x).$$

The zeros of orthogonal polynomials play an important role in interpolation theory, Gauss-Jacobi quadrature, spectral theory and the design of digital filters [43, 44]. The following property is basic:

Theorem 2.3 *The zeros of each member of a set of orthogonal polynomials are real, simple, and lie in* (a, b).

Given a set of orthonormal polynomials $\{p_n\}_{n=0}^{\infty}$ associated with the measure μ on (a, b), define the associated *reproducing kernel* by

$$K_n(x, y) = \sum_{k=0}^{n} p_k(x)p_k(y)$$

The name given to $K_n(x, y)$ is drawn from its most fundamental property: for each polynomial $Q_n \in \mathcal{P}_n$,

$$Q_n(x) = \int_a^b K_n(t, x)Q_n(t)d\mu(t). \tag{2.3}$$

The classical *nth Christoffel function associated with* μ is defined by

$$\lambda_n(\mu; x) := \lambda(\mathcal{P}_n, \mu; x) := \inf_{\substack{Q_n \in \mathcal{P}_n \\ Q_n(x)=1}} \int_a^b |Q_n(t)|^2 d\mu(t). \tag{2.4}$$

It turns out that the reproducing kernel solves this minimizing problem, and consequently we can write the Christoffel function in terms of the orthogonal polynomials [15]:

Theorem 2.4 *Let* $\{p_n(x)\}_{n=0}^{\infty}$ *be the orthonormal polynomials with respect to the measure* μ *on* (a, b)*. Then*

$$\lambda_n(\mu; x)^{-1} = K_n(x, x) = \sum_{k=0}^{n} |p_k(x)|^2.$$

Using the three term recurrence relation (2.2), one can prove [54] the *Christoffel-Darboux formula*

$$K_n(x, y) = \frac{\gamma_n}{\gamma_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x - y}, \qquad x, y \in \mathbb{R}.$$

By taking the limit as $y \longrightarrow x$, this yields the identity

$$K_n(x, x) = \frac{\gamma_n}{\gamma_{n+1}} \left[p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)\right].$$

The Christoffel-Darboux formula can be used to prove the *Gauss-Jacobi quadrature formula* [15, 54]: for each polynomial Q_{2n-1} of degree at most $2n-1$,

$$\int_a^b Q_{2n-1}(x)d\mu(x) = \sum_{k=1}^{n} \lambda_{k,n} Q_{2n-1}(x_{k,n}),$$

where $x_{1,n} > x_{2,n} > \cdots > x_{n,n}$ are the (fixed) zeros of the orthogonal polynomial $p_n(\mu; x)$, and the *Cotes numbers* (or *Christoffel numbers*) $\lambda_{k,n} := \lambda_{k,n}(\mu)$ are given in terms of the Christoffel functions as

$$\lambda_{k,n} = \lambda_n(\mu; x_{k,n}), \qquad k = 1, 2 \ldots, n,$$

and they only depend on the measure μ.

Christoffel functions have proved to be important tools in the theory of orthogonal polynomials and approximation theory. They have been applied to problems involving quadrature formulas, interpolation theory, zeros of polynomials, polynomial inequalities and the moment problem [15, 36]. Furthermore, there is a close connection to the circular unitary ensemble of random unitary matrices, a field that has drawn a great deal of attention in recent years and has underlined the importance of orthogonal polynomials [19, 28, 30, 45].

2.2 Classical polynomials and their asymptotic properties

Here we introduce the classical polynomials (see Table 2.2), which are of importance in applied mathematics and numerical analysis. We shall look at some of their asymptotic properties, especially those that are important in the scope of this thesis. Indeed, some of the results of our research gives a new approach in establishing these classic asymptotics. Furthermore, we will look at the asymptotic behavior of the associated Christoffel functions.

The strong asymptotics are well known from Szegő's monograph [54]. For each case, there are two problems to consider. One is the behavior of the orthogonal polynomials $p_n(x)$ as $n \longrightarrow \infty$ for x outside the interval of orthogonality (real or

Table 2: Classical orthogonal polynomials, following Szegő [54]

Type	Weight	Leading coefficient	Normalization
Legendre poly. $P_n(x)$	$w(x) = 1$ on $[-1,1]$	$\frac{(2n)!}{2^n (n!)^2}$	$P_n(1) = 1$
Jacobi poly. $P_n^{(\alpha,\beta)}(x)$	$w(x) = (1-x)^\alpha (1+x)^\beta$ on $[-1,1]$ for $\alpha, \beta > -1$	$\frac{1}{2^n}\binom{2n+\alpha+\beta}{n}$	$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$
Laguerre poly. $\mathcal{L}_n^{(\alpha)}(x)$	$w(x) = e^{-x} x^\alpha$ on $[0,\infty)$ for $\alpha > -1$	$\frac{(-1)^n}{n!}$	$\mathcal{L}_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$
Hermite poly. $H_n(x)$	$w(x) = e^{-x^2}$ on $\mathbb{R}$	2^n	

complex) and the other for x on the interval of orthogonality. In general, the second problem is more difficult, since there the elements exhibit an oscillatory behavior. The classic approach is to apply the method of steepest descent (see [38, 54]) to a contour integral representation for the respective orthogonal polynomials. The results can also be obtained by examining the second order linear differential equations they satisfy, or via the associated generating function.

The following asymptotic formulas for the Legendre polynomials $P_n(x)$ are well known, see Szegő [54, p. 194]:

Theorem 2.5 (Formula of Laplace) *For each $\theta \in (0, \pi)$,*

$$P_n(\cos\theta) = \sqrt{\frac{2}{\pi n \sin\theta}} \cos\left(\left(n + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right) + \mathcal{O}\left(n^{-3/2}\right). \tag{2.5}$$

The bound for the error term holds uniformly for θ in compact subsets of $(0, \pi)$.

Theorem 2.6 (Formula of Laplace-Heine) *If $x \in \mathbb{C} \setminus [-1,1]$, then as $n \to \infty$,*

$$P_n(x) \sim \frac{1}{\sqrt{2\pi n}} \frac{\left[x + (x^2-1)^{1/2}\right]^{n+1/2}}{(x^2-1)^{1/4}}. \tag{2.6}$$

Here $(x^2-1)^{1/4}, (x^2-1)^{1/2}$ and $\left[x + (x^2-1)^{1/2}\right]^{n+1/2}$ are real and positive if x is real and greater than 1. This formula holds uniformly in the exterior of an arbitrary closed curve which encloses the segment $[-1,1]$.

G. Darboux, see [8] and [54, p. 196], was able to extend the results above to the Jacobi polynomials.

Theorem 2.7 *For each $\theta \in (0,\pi)$,*

$$P_n^{(\alpha,\beta)}(\cos\theta) = \frac{1}{\sqrt{\pi n}} \frac{\cos\left(\left[n + \frac{\alpha+\beta+1}{2}\right]\theta - \frac{(\alpha+1/2)\pi}{2}\right)}{[\sin(\theta/2)]^{\alpha+1/2}[\cos(\theta/2)]^{\beta+1/2}} + \mathcal{O}\left(n^{-3/2}\right). \tag{2.7}$$

The bound for the error term holds uniformly for θ in compact subsets of $(0,\pi)$

Theorem 2.8 *Let $\alpha,\beta > -1$. If $z \notin [-1,1]$, real or complex, then as $n \to \infty$,*

$$P_n^{(\alpha,\beta)}(z) \sim \frac{\left[(z+1)^{1/2} + (z-1)^{1/2}\right]^{\alpha+\beta}\left[z + (z^2-1)^{1/2}\right]^{n+1/2}}{(2\pi n)^{1/2}(z-1)^{\alpha/2}(z+1)^{\beta/2}(z^2-1)^{1/2}} \tag{2.8}$$

and this formula holds uniformly in the exterior af an arbitrary closed curve which encloses the segment $[-1,1]$. The determination of the multivalued functions is obvious.

An important consequence of this result is the nth root asymptotics for the Legendre and Jacobi polynomials: For $z \notin [-1,1]$,

$$\lim_{n\to\infty} |P_n^{(\alpha,\beta)}(z)|^{1/n} = |z + (z^2-1)^{1/2}|.$$

The function $\varphi(z) = z + (z^2-1)^{1/2}$ that appears on the right hand side is the well known conformal mapping that maps the cut plane $\mathbb{C} \setminus [-1,1]$ onto the exterior of the unit circle.

The endpoint limit asymptotics are also important, and the main result is the following [54, p. 192].

Theorem 2.9 (Formula of Mehler-Heine) *Let $\alpha,\beta > -1/2$. Then uniformly for z in $\mathbb{C}$,*

$$\lim_{n\longrightarrow\infty} P_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) = \lim_{n\longrightarrow\infty} P_n^{(\alpha,\beta)}\left(1 - \frac{z^2}{2n^2}\right) = \left(\frac{2}{z}\right)^{\alpha} J_\alpha(z), \tag{2.9}$$

where J_α is the Bessel function

$$J_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\alpha+2n}}{n!\Gamma(n+\alpha+1)}. \tag{2.10}$$

This gives the asymptotics as we approach the endpoint $x = 1$, and the asymptotics close to the left endpoint $x = -1$ follow directly via the identity [54, Section 4.1] $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$.

We can now proceed and obtain asymptotic formulas for the Christoffel functions. First we deal with the endpoints, and the result can be proved directly from Theorem 2.4 since the function values of $P_n^{(\alpha,\beta)}(-1)$ are well known [37, p. 85].

Theorem 2.10 *The Christoffel functions associated with the Jacobi weights $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ satisfy the following asymptotic relations at the endpoints,*

$$\begin{aligned}\lim_{n\to\infty} n^{2\alpha+2}\lambda_n(w^{(\alpha,\beta)}, 1) &= (\alpha+1)2^{\alpha+\beta+1}\Gamma(\alpha+1)^2,\\ \lim_{n\to\infty} n^{2\beta+2}\lambda_n(w^{(\alpha,\beta)}, -1) &= (\beta+1)2^{\alpha+\beta+1}\Gamma(\beta+1)^2.\end{aligned}$$

The following result gives the asymptotics on the interval of orthogonality [37, p. 85].

Theorem 2.11 *For each $x \in (-1, 1)$, we have*

$$\lim_{n\to\infty} n\lambda(w^{(\alpha,\beta)}, x) = \pi(1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}.$$

For x outside the interval we have the following result:

Theorem 2.12 *Let $\alpha, \beta > -1$. If $x \notin [-1, 1]$, real or complex, then*

$$\lambda(w^{(\alpha,\beta)}; x) \sim \frac{2^{\alpha+\beta+1}\pi|(x^2-1)(x-1)^\alpha(x+1)^\beta|}{|(x+1)^{1/2}+(x-1)^{1/2}|^{2(\alpha+\beta)}}\frac{1-|\phi(x)|^{-2}}{|\phi(x)|^{2n+1}}, \tag{2.11}$$

where $\phi(x) := x + (x^2-1)^{1/2}$. This formula holds uniformly in the exterior af an arbitrary closed curve which encloses the segment $[-1, 1]$.

Finally, we recall the asymptotics of the Laguerre polynomials on and outside the interval of orthogonality.

Theorem 2.13 (Formula of Fejér) *Let $\alpha \in \mathbb{R}$. For each $x > 0$, as $n \longrightarrow \infty$,*

$$\mathcal{L}_n^{(\alpha)}(x) = \frac{e^{x/2}}{\sqrt{\pi} x^{1/4+\alpha/2} n^{1/4-\alpha/2}} \cos\left(2\sqrt{nx} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}\left(n^{\alpha/2-3/4}\right), \tag{2.12}$$

and this holds uniformly for x in compact subsets of $(0, \infty)$.

Theorem 2.14 *Let $\alpha \in \mathbb{R}$. Then uniformly for bounded $z \in \mathbb{C}$,*

$$\lim_{n\to\infty} n^{-\alpha} \mathcal{L}^{(\alpha)}\left(\frac{y}{n}\right) = y^{-\alpha/2} J_\alpha(2y^{1/2}). \tag{2.13}$$

Theorem 2.15 (Formula of Perron) *Let $\alpha \in \mathbb{R}$. For each $x \in \mathbb{C} \setminus [0, \infty)$, as $n \longrightarrow \infty$,*

$$\mathcal{L}_n^{(\alpha)}(x) = \frac{e^{x/2}}{2\sqrt{\pi}(-x)^{1/4+\alpha/2} n^{1/4-\alpha/2}} e^{2(-nx)^{1/2}} \left[1 + \mathcal{O}\left(n^{-1/2}\right)\right], \tag{2.14}$$

and this holds locally uniformly for x in $\mathbb{C} \setminus [0, \infty)$. For $x < 0$, $(-x)^{1/4+\alpha/2}$ and $(-x)^{1/2}$ must be taken real and positive.

2.3 Müntz polynomials

In the introduction of this thesis, we saw how the density of the Müntz polynomials

$$\sum_{k=0}^{n} c_k x^{\lambda_k} \tag{2.15}$$

in $L_2[0, 1]$ is related to the growth of the exponents $\Lambda = \{\lambda_k\}$. The celebrated Theorem of Müntz [1, 7, 16] in one of its most general form asserts that if the λ_k's are distinct real numbers greater than $-1/p$, $p \in [1, \infty)$, then the functions (2.15) are dense in $L_p[0, 1]$ if and only if

$$\sum_{k=0}^{\infty} \frac{\lambda_k + \frac{1}{p}}{\left(\lambda_k + \frac{1}{p}\right)^2 + 1} = \infty. \tag{2.16}$$

If we assume that $\inf_{k\geq 0}\{\lambda_k\} > -1/p$, condition (2.16) is equivalent to

$$\sum_{k=0}^{\infty} \frac{1}{\lambda_k + \frac{1}{p}} = \infty. \tag{2.17}$$

Furthermore, assuming that the constant functions are included (i.e. $\lambda_0 = 0$), and $\inf_{k\geq 1} \lambda_k > 0$, (2.17) is also equivalent to the denseness of (2.15) under the supremum norm in $C[0,1]$.

A system of the form $(x^{\lambda_0}, x^{\lambda_1}, \dots)$ is called a *Müntz system*, and we denote the corresponding *Müntz space* by

$$M(\Lambda) := \bigcup_{n=0}^{\infty} M_n(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\},$$

where we let $M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ for each $n = 0, 1, 2, \dots$.

2.4 Müntz orthogonal polynomials

The nth *Müntz-Legendre polynomial* associated with Λ is defined by

$$L_n(x) := L_n(\Lambda; x) := \frac{1}{2\pi i} \int_\Gamma \prod_{k=0}^{n-1} \frac{t + \lambda_k + 1}{t - \lambda_k} \frac{x^t}{t - \lambda_n} dt, \tag{2.18}$$

where the simple contour Γ surrounds all the zeros of the denominator of the integrand. In the case when the Müntz sequence Λ satisfies the conditions

$$\lambda_n > -1/2, \quad n = 0, 1, \dots, \qquad \text{and} \qquad \lambda_k \neq \lambda_j, \quad j \neq k, \tag{2.19}$$

a straight-forward application of the Residue Theorem shows that the Müntz-Legendre polynomials are indeed elements of the corresponding Müntz space and for each $n = 0, 1, 2, \dots$,

$$L_n(\Lambda; x) = \sum_{k=0}^{n} c_{k,n} x^{\lambda_k} \tag{2.20}$$

with the coefficients

$$c_{k,n} = \frac{\prod_{j=0}^{n-1} (\lambda_k + \lambda_j + 1)}{\prod_{\substack{j=0 \\ j\neq k}}^{n} (\lambda_k - \lambda_j)}, \qquad k = 0, 1, \dots, n.$$

The Müntz-Legendre polynomials are orthogonal in $L_2[0,1]$ with respect to the Legendre weight. For the sake of completeness, we give a proof here [7]:

Theorem 2.16 *Let $\Lambda = \{\lambda_k\}$ be a sequence of real numbers greater than $-1/2$. For all $n, m = 0, 1, 2, \ldots,$*

$$\int_0^1 L_n(\Lambda; x) L_m(\Lambda; x) dx = \frac{\delta_{n,m}}{(2\lambda_n + 1)}. \tag{2.21}$$

Proof. It suffices to prove this for distinct λ_k's; if they are non-distinct we can use the fact that $L_n(\Lambda; x)$ is uniformly continuous with respect to individual λ_k's for x in compact subsets of $(0, 1)$ and therefore treat this case using a limit argument.

Without loss of generality, we can assume that $m \leq n$. Since $\lambda_k > -1/2$ for all k, we can choose the contour Γ in (2.18) so that $\mathrm{Re}(t) \geq -1/2$ for all t on Γ. Then for each t on Γ, $\mathrm{Re}(t + \lambda_m) > -1$ and $\int_0^1 x^{t+\lambda_m} dx = (t + \lambda_m + 1)^{-1}$. Applying Fubini's Theorem then yields

$$\int_0^1 L_n(\Lambda; x) x^{\lambda_m} dx = \frac{1}{2\pi i} \int_\Gamma \prod_{k=0}^{n-1} \frac{t + \lambda_k + 1}{t - \lambda_k} \frac{dt}{(t - \lambda_n)(t + \lambda_m + 1)}.$$

If $m < n$, then the new factor $t + \lambda_m + 1$ in the denominator can be cancelled and we have no new pole. Therefore we can change the contour Γ to $|t| = R > \max_{0 \leq j \leq n} \lambda_j$ and let $R \longrightarrow \infty$, and the integral then clearly vanishes. If however $m = n$, the factor $t + \lambda_m + 1$ gives a new pole at $t = -\lambda_n - 1 < -1/2$. Using the same treatment as above the only change is that we get a contribution when Γ passes through the new pole. The Residue theorem gives

$$\begin{aligned} \int_0^1 L_n(\Lambda; x) x^{\lambda_n} dx &= -\prod_{k=0}^{n-1} \frac{-\lambda_n + \lambda_k}{-\lambda_n - \lambda_k - 1} \frac{1}{-\lambda_n - \lambda_n - 1} \\ &= \frac{1}{(2\lambda_n + 1) c_{n,n}}, \end{aligned}$$

where $c_{n,n}$ is the leading coefficient from (2.20). It follows that

$$\int_0^1 L_n(x) L_m(x) dx = c_{m,m} \int_0^1 L_n(x) x^{\lambda_m} dx = \frac{\delta_{n,m}}{(2\lambda_n + 1)}.$$

$\square$

It immediately follows that the functions

$$L_n^* := (2\lambda_n + 1)^{1/2} L_n \tag{2.22}$$

are orthonormal in $L_2[0,1]$.

From the definition in (2.18) it is clear that the ordering of the first $n-1$ Müntz exponents does not make a difference, i.e.

$$L_n(\{\lambda_{\sigma(0)}, \lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n-1)}, \lambda_n\}; x) = L_n(\Lambda; x), \tag{2.23}$$

holds for all x and every permutation σ on $\{0, 1, 2, \ldots, n-1\}$.

The following identity is proved in [6] and will come of good use:

Theorem 2.17 *Let* $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ *a sequence of distinct real numbers greater than* $-1/2$. *Then the associated Müntz-Legendre polynomials* $L_n(x) = L_n(\Lambda;)$ *satisfy*

$$xL_n'(x) = \sum_{k=0}^{n-1} (2\lambda_k + 1) L_k(x) + \lambda_n L_n(x)$$

for every $x \in (0, 1]$ *and every* $n = 0, 1, 2, \ldots$

It turns out that the Müntz-Legendre polynomials are always 1 at $x = 1$ [6]:

Lemma 2.18 *For the Müntz-Legendre polynomials defined in* (2.18) *we have* $L_n(1) = 1$ *and* $L_n'(1) = \sum_{k=0}^{n-1}(2\lambda_k + 1) + \lambda_n$ *for all* $n = 0, 1, 2, \ldots$

We can extend the definition of the Christoffel functions (2.4) by taking more general function spaces. Here we are interested in examining the nth Christoffel function associated with the Müntz space $M_n(\Lambda)$ over the Legendre weight $w(x) = 1$ on $[0,1]$, namely

$$\lambda(M_n(\Lambda); x) := \inf_{\substack{Q \in M_n(\Lambda) \\ Q(x)=1}} \int_0^1 |Q^2(t)| dt. \tag{2.24}$$

In the same way as for the case of algebraic polynomials, one can show that

$$\lambda(M_n(\Lambda); x)^{-1} = \sum_{k=0}^{n} |L_k^*(x)|^2, \tag{2.25}$$

where the L_k^*'s are the orthonormal Müntz polynomials (2.22).

The Müntz-Christoffel functions have been used by Borwein and Erdélyi [6, 7] in establishing Markov-Bernstein inequalities, and they have been connected to the denseness of the corresponding Müntz-space:

Theorem 2.19 *Let $\Lambda = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ be a sequence of integers. Then the following statements are equivalent:*

(i) $M(\Lambda)$ is not dense in $C[0,1]$ in the uniform norm,

(ii) $\sum_{k=1}^{\infty} \lambda_k^{-1} < \infty$,

(iii) There is an $x \in [0,1)$, such that $\sum_{k=0}^{\infty} |L_k^*(x)|^2 < \infty$,

(iv) $\sum_{k=0}^{\infty} |L_k^*(x)|^2 < \infty$ converges uniformly on $[0, 1-\varepsilon]$ for every $0 < \varepsilon < 1$.

For parts (iii) and (iv), note that $\sum_{k=0}^{\infty} |L_k^*(x)|^2 = \lim_{n\to\infty} \lambda_n(M(\Lambda); x)^{-1}$.

We can also define the analogue of the Jacobi polynomials. For the weights $w^{(\beta)}(x) = x^\beta$, $\beta > -1$, the nth *Müntz-Jacobi polynomial* is defined by

$$L_n^{(\beta)}(x) := L_n^{(\beta)}(\Lambda; x) := \frac{x^{-\beta/2}}{2\pi i} \int_\Gamma \prod_{k=0}^{n-1} \frac{t + \lambda_k + \beta/2 + 1}{t - \lambda_k - \beta/2} \frac{x^t}{t - \lambda_n - \beta/2} dt, \tag{2.26}$$

where the contour Γ encloses the zeros of the denominator of the integrand. They satisfy the orthogonality condition

$$\int_0^1 L_n^{(\beta)}(x) L_m^{(\beta)}(x) x^\beta dx = \frac{\delta_{n,m}}{2\lambda_n + \beta + 1}.$$

Note that $w^{(\beta)}(x)$ corresponds to the classical Jacobi weight $w^{(0,\beta)}(x) = (1+x)^\beta$ on $[-1,1]$, and this can be seen by mapping $[0,1]$ to $[-1,1]$. It is easy to see that for each n,

$$L_n^{(\beta)}(\Lambda; x) = x^{-\beta/2} L_n(\Lambda + \beta/2; x), \tag{2.27}$$

so our result for the Müntz-Legendre polynomials also applies for this class of functions.

We pause here to emphasize an important aspect of the nature of Müntz-Legendre polynomials. The orthogonality in (2.21) is only with respect to the trivial Lebesgue measure $w(x) = 1$, which at first might appear highly restrictive. However, instead of looking at differents weights, here we have the freedom of manipulating the exponents $\{\lambda_k\}$. A simple demonstration is (2.27), where we have absorbed the Jacobi weight into the exponents of the Müntz-Legendre polynomial on the right hand side. We can also consider Müntz polynomials of the form

$$\sum_{k=0}^{n} c_k x^{\rho n},$$

for some constant ρ. Here $\lambda_n = \rho n$, and using the substitution $t = x^\rho$ in the orthogonality condition (2.21) yields

$$\frac{\delta_{n,m}}{(2\lambda_n + 1)} = \frac{1}{\rho} \int_0^1 L_n(\Lambda; t^{1/\rho}) L_m(\Lambda; t^{1/\rho}) t^{1/\rho - 1} dt.$$

Clearly $L_n(\Lambda; t^{1/\rho}) \in \mathcal{P}_n$ and it is easy to see that indeed, $L_n(\Lambda; t^{1/\rho}) = L_n^{(1/\rho-1)}(\{k\}; t)$.

In Section 2.4.2 we shall see that the Müntz-Legendre polynomials cover a large class of orthogonal systems, including orthogonal polynomials associated with different weights. Among these are many of the classical polynomials, orthogonal exponential sums and generalized Legendre polynomials. Our results on the asymptotic behavior of the Müntz-Legendre polynomials therefore also apply to these special cases.

2.4.1 Zeros of Müntz orthogonal polynomials

In their paper [6], Borwein, Erdélyi and Zhang study the zeros of the Müntz-Legendre polynomials. They discuss their interlacing and lexicographical properties and universally estimate the smallest and largest zeros through the zeros of Laguerre polynomials. Given (2.19), the Müntz polynomials form a Chebyshev system on $(0, \infty)$, so any nonzero element $\sum_{k=0}^{n} c_k x^{\lambda_k}$ has at most n zeros in $(0, 1]$. The following are basic properties.

Theorem 2.20 *Let* $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ *satisfy* (2.19). *Then for every* $n = 0, 1, 2, \ldots$

(i) $L_n(\Lambda; \cdot)$ *has exactly* n *zeros in* $(0, 1)$.

(ii) The zeros of $L_{n-1}(\Lambda; \cdot)$ *and* $L_n(\Lambda; \cdot)$ *strictly interlace.*

We denote these zeros by

$$0 < l_{n,n} < l_{n-1,n} < \cdots < l_{2,n} < l_{1,n} < 1. \tag{2.28}$$

In [7, pp. 136-137] a global estimate for the zeros is given: If we let $\lambda_{\min}^{(n)} := \min\{\lambda_0, \ldots, \lambda_n\}$ and $\lambda_{\max}^{(n)} := \max\{\lambda_0, \ldots, \lambda_n\}$ then

$$\exp\left(-2\frac{2n+1}{2\lambda_{\min}^{(n)}+1}\right) < l_{n,n} < \cdots < l_{1,n} < \exp\left(\frac{-j_1^2}{2(2n+1)(2\lambda_{\max}^{(n)}+1)}\right) \tag{2.29}$$

where j_1 is the smallest positive zero of the Bessel function J_0 defined in (2.10).

D.S. Lubinsky and E.B. Saff [24] have determined the zero distribution of the Müntz extremal polynomials $T_{n,p}(\Lambda)$ with respect to the L_p norm, which satisfy

$$\|T_{n,p}(\Lambda)\|_{L_p[0,1]} = \min_{c_0,\ldots,c_{n-1}} \left\| x^{\lambda_n} - \sum_{j=0}^{n-1} c_j x^{\lambda_j} \right\|_{L_p[0,1]}.$$

Indeed, if

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \alpha$$

for some $\alpha > 0$, then the normalized zero counting measure of $T_{n,p}(\Lambda)$ converges weakly to

$$\frac{\alpha}{\pi} \frac{t^{\alpha-1}}{\sqrt{t^{\alpha}(1-t^{\alpha})}} dt. \tag{2.30}$$

If $\alpha = 0$ or ∞, the limiting measure is a Dirac delta at 0 or 1 respectively. In the case when $p = 1$, we have the (monic) Müntz-Legendre polynomials.

2.4.2 Examples of Müntz orthogonal polynomials

Classical polynomials. If we let $\lambda_n = n$ for all n and map $[0, 1]$ to $[-1, 1]$ via $x \mapsto 2x - 1$, we can write the Legendre polynomials in terms of the Müntz-Legendre polynomials; indeed we have

$$P_n(x) = L_n\left(\mathbb{N}_0; \frac{x+1}{2}\right) \tag{2.31}$$

and if we take $x = \cos\theta \in [-1,1]$, $\theta \in [0,\pi]$ then $P_n(\cos\theta) = L_n\left(\cos^2(\theta/2)\right)$.

Similarly, if we let $\lambda_n = n + \beta/2$ for all n, then the Jacobi polynomials with respect to the weight $w^{(0,\beta)}(x) = (1+x)^\beta$ on $[-1,1]$ satisfy

$$\begin{aligned} P_n^{(0,\beta)}(x) &= L_n^{(\beta)}\left(\mathbb{N}_0; \frac{x+1}{2}\right) \\ &= \left(\frac{x+1}{2}\right)^{-\beta/2} L_n\left(\mathbb{N}_0 + \frac{\beta}{2}; \frac{x+1}{2}\right) \end{aligned} \tag{2.32}$$

where we use (2.27) in the last identity. It follows that

$$P_n^{(0,\beta)}(\cos\theta) = \frac{L_n\left(\mathbb{N}_0 + \frac{\beta}{2}; \cos^2(\theta/2)\right)}{\cos^\beta(\theta/2)}$$

for all $\theta \in [0,\pi]$.

In [6, pp. 525-526], an interesting identity is proved: If we let $\lambda_n = \lambda$, a constant, for all n, then we get the Laguerre polynomials via the formula

$$L_n(\Lambda; x) = x^\lambda \mathcal{L}_n(-(2\lambda+1)\log x).$$

In particular if we let $\lambda = 0$ and $y = -\log x$, we can write

$$\mathcal{L}_n(y) = L_n(\{0\}; e^{-y}). \tag{2.33}$$

Exponential sums. For a given sequence $\Pi = \{\mu_j\}$ of positive real numbers, consider the class of exponential sums of the form

$$\sum_{k=0}^{n} a_k e^{-\mu_k t}$$

which have applications in non-linear approximation, particularly for decay processes.

We define $E_n(\Pi; t) \in \operatorname{span}\{e^{-\mu_0 t}, e^{-\mu_1 t}, \ldots, e^{-\mu_n t}\}$ via the orthogonality condition

$$\int_0^\infty E_n(\Pi; t) E_m(\Pi; t)\, dt = \frac{\delta_{n,m}}{\sqrt{2\mu_n}}, \qquad n, m = 0, 1, 2, \ldots \tag{2.34}$$

Using the substitution $x = e^{-t}$ in the orthogonality condition (2.21) for the Müntz-Legendre polynomials $L_n(\Lambda; x)$ associated with the sequence $\Lambda = \{\lambda_k\}$, we can write

$$\begin{aligned}
\frac{\delta_{n,m}}{\sqrt{2\lambda_n + 1}} &= \int_0^\infty L_n(\Lambda; e^{-t}) L_m(\Lambda; e^{-t}) e^{-t} dt \\
&= \int_0^\infty \left[L_n(\Lambda; e^{-t}) e^{-t/2}\right] \left[L_m(\Lambda; e^{-t}) e^{-t/2}\right] dt.
\end{aligned}$$

Then since $L_n(e^{-t})e^{-t/2}$ is of the form $\sum_{k=0}^n a_k e^{-(\lambda_k + 1/2)t}$, we see that by letting $\mu_n = \lambda_n + 1/2$ for each n, we have

$$E_n\left(\Pi; t\right) = L_n(\Lambda; e^{-t}) e^{-t/2}. \tag{2.35}$$

In our results below, we introduce a formula (3.2) for the Müntz-Legendre polynomials $L_n(\Lambda; x)$. We shall see that the formula takes an especially nice form if written in terms of the orthogonal exponential sums $E_n(\Pi; t)$.

Generalized Legendre polynomials. Consider the Müntz system obtained by letting

$$\lambda_{2k} = \lambda_{2k+1} = k, \qquad k = 0, 1, 2, \ldots.$$

When orthogonalizing this system in the Müntz sense (2.21), one should interpret this by setting $\lambda_{2k} = k$, $\lambda_{2k+1} = k + \varepsilon$ and then letting $\varepsilon \longrightarrow 0$. The associated $(2n+1)$st Müntz-Legendre polynomial is

$$G_n(x) := L_{2n+1}(x) = \frac{1}{2\pi i} \int_\Gamma \frac{\prod_{k=0}^{n-1}(t + k + 1)}{\prod_{k=0}^{n}(t - k)^2} (t + n + 1) x^t dt,$$

where Γ encloses $t = 0, 1, \ldots, n$. Since each pole is double, and $\frac{d}{dt} x^t = (\log x) x^t$, it follows from the residue theorem that for each n, we can write

$$G_n(x) = p_n(x) \log x + q_n(x),$$

for some algebraic polynomials $p_n, q_n \in \mathcal{P}_n$ of degree n. It follows that [7, p. 373]

$$\int_0^1 G_n(x) x^k dx = 0, \qquad k = 0, 1, \ldots, n,$$

and

$$\int_0^1 G_n(x)(\log x)x^k dx = 0, \qquad k = 0, 1, \ldots, n-1.$$

Therefore, the functions $G_n(x)$ generalize the Legendre polynomials in the sense of multiple orthogonal polynomials of type II (see Aptekarev [2] for detailed definitions). In [7, Appendix 2], these elements are used to reproduce Apéry's proof of the irrationality of $\zeta(3)$.

Naturally, we can do this more generally: For a given natural number q, if we define the sequence $\Lambda = \{\lambda_k\}$ such that

$$\lambda_{qk+r} = k, \qquad r = 0, 1, \ldots, q-1, \quad k = 0, 1, \ldots,$$

then similarily we get orthogonal functions of the form

$$G_n(x) = \sum_{k=0}^{q-1} p_n^{(k)}(x)(\log x)^k,$$

where $p_n^{(k)}(x) \in \mathcal{P}_n$ and

$$\int_0^1 G_n(x)(\log x)^j x^k dx = 0, \qquad k = 0, 1, \ldots, n-j.$$

for all $j = 0, 1, \ldots, q-1$.

CHAPTER III

A NEW REPRESENTATION FOR MÜNTZ ORTHOGONAL POLYNOMIALS

In this chapter we present a formula which, for $x \in (0,1)$ on the interval of orthogonality, expresses the Müntz-Legendre polynomials as a real oscillatory integral on $[0,\infty)$. The formula involves a simple "measure" of the spacing of the exponents $\{\lambda_k\}$ in a form of a Riemann-like sum and is an actual mid-point Riemann sum in the case of the classical Legendre polynomials.

In particular, the formula gives a new expression for the classical Legendre, Jacobi and Laguerre polynomials (with $\alpha = 0$) on their respective intervals of orthogonality. Furthermore, since the representation is in the form of a Lebesgue-type oscillatory integral, this provides a direct way to obtain their asymptotics for x inside the interval of orthogonality. The result is published in the paper [48], "Asymptotic behavior of Müntz orthogonal polynomials."

3.1 Main Result

The formula is introduced here.

Theorem 3.1 *Let $\Lambda = \{\lambda_k\}$ be a sequence of real numbers. For all n and $x \in (0,1)$, we have the representation*

$$L_n(\Lambda; x) = \frac{1}{\pi\sqrt{x}} \int_0^\infty \frac{\sin[\Phi_n(s) - s\log x]}{\sqrt{{\lambda_n^*}^2 + s^2}} ds, \tag{3.1}$$

where

$$\Phi_n(s) = 2\sum_{k=0}^{n-1} \arctan\left(\frac{\lambda_k^*}{s}\right) + \arctan\left(\frac{\lambda_n^*}{s}\right)$$

and $\lambda_k^* = \lambda_k + 1/2$, for all k.

When determining the asymptotic behavior of $L_n(\Lambda; x)$ it will be useful to use a different scaling in the integral, and therefore we state an alternative representation.

Corollary 3.2 *Let $\Lambda = \{\lambda_k\}$ be a sequence of real numbers. For all n and $x \in (0,1)$, we have the representation*

$$L_n(\Lambda; x) = \frac{1}{\pi\sqrt{x}} \int_0^\infty \frac{\sin\left(2\lambda_n^*\left[R_n(t) - \frac{t}{2}\log x\right]\right)}{\sqrt{1+t^2}} dt, \tag{3.2}$$

where

$$R_n(t) = \frac{1}{\lambda_n^*}\left\{\sum_{j=0}^{n-1} \arctan \frac{\lambda_j^*}{\lambda_n^* t} + \frac{1}{2}\arctan\frac{1}{t}\right\},$$

and $\lambda_k^* = \lambda_k + 1/2$, for all k.

Remarks (1) Using the identity $\arctan(1/t) = \pi/2 - \arctan t$, it is easy to see that we can also write

$$L_n(\Lambda; x) = \frac{(-1)^n}{\pi\sqrt{x}} \int_0^\infty \frac{\cos\left(\lambda_n^*\left[2\widehat{R}_n(t) + t\log x\right]\right)}{\sqrt{1+t^2}} dt, \tag{3.3}$$

where

$$\widehat{R}_n(t) = \frac{1}{\lambda_n^*}\left\{\sum_{j=0}^{n-1} \arctan \frac{\lambda_n^*}{\lambda_j^*} t + \frac{1}{2}\arctan t\right\}.$$

(2) By introducing the probability counting measure

$$\nu_n(s) = \frac{1}{n^*}\left(\sum_{\lambda_j^*/\lambda_n^* < s} 1 + \frac{1}{2}\delta_{\{1\}}(s)\right),$$

it is easy to see that we can write

$$2\lambda_n^* R_n(t) = (2n+1)\int_0^1 \arctan\frac{s}{t} d\nu_n(s).$$

The proof of the formula above is similar to an approach of Milovanović's in his paper [31]. However our main observation is the simple fact that on the critical line $\mathrm{Re}(z) = -1/2$, the modulus of the product of the integrand in (2.18) is 1. This is true for all real sequences of exponents Λ and it is quite remarkable that we have the same critical path for this large class of functions. This allows us to write the Müntz orthogonal polynomials as a real integral over the positive real line.

3.2 Special cases

In the case of the classical Legendre polynomials P_n we have $\lambda_n = n$ for all n and we saw in (2.31) that

$$P_n(x) = L_n\left(\mathbb{N}_0; \frac{x+1}{2}\right).$$

What is interesting in this case is that the phase function R_n in (3.2) is precisely a mid-point Riemann sum with partition $\{0, 1/n^*, 2/n^*, \ldots, n/n^*, 1\}$, where each interval has length $1/n^*$ except the last one which has length $1/(2n^*)$. Then for each t, as $n \longrightarrow \infty$,

$$\begin{aligned} R_n(t) &= \int_0^1 \arctan\frac{u}{t}du + \mathcal{O}\left(\frac{1}{n^2}\right) \\ &= \arctan\frac{1}{t} - \frac{t}{2}\log\left(1+\frac{1}{t^2}\right) + \mathcal{O}\left(\frac{1}{n^2}\right). \end{aligned}$$

Therefore, and we shall do this rigorously in Chapter 4, as n grows we can replace $R_n(t)$ with the function $\arctan(t^{-1}) - (t/2)\log(1+t^{-2})$. Using the substitution $t = \cot u$ we see that $P_n(\cos 2\theta) = L_n(\mathbb{N}_0; \cos^2\theta)$, $\theta \in (0, \pi/2)$, will have the behavior of

$$\frac{1}{\pi\cos\theta}\int_0^{\pi/2}\frac{\sin\left((2n+1)\left[u+\cot u\log\frac{\cos u}{\cos\theta}\right]\right)}{\sin u}du.$$

The phase function $p(u) = u + \cot u \log(\cos u/\cos\theta)$ has a simple monotone derivative $p'(u) = -\csc^2 u \log(\cos u/\cos\theta)$, and hence the unique stationary point $u = \theta$. Thus a simple application of Kelvin's method of stationary phase (see Olver [38]) gives a new simple proof of the Formula of Laplace of Theorem 2.5.

Similarly, we get new proofs for the asymptotics of the Jacobi polynomials and Laguerre polynomials via (2.32) and (2.33) respectively. Indeed, for $y \in (0, \infty)$, via our formula (3.2), the Laguerre polynomials can be written in the simple form

$$\mathcal{L}_n(2y) = \frac{e^y}{\pi}\int_0^\infty \frac{\sin\left((2n+1)\arctan\frac{1}{t} + ty\right)}{\sqrt{1+t^2}}dt,$$

and standard asymptotic analysis yields Fejér's formula of Theorem 2.13.

Using the identity (2.35), it is easy to see that for the orthogonal exponential sums $E_n(\{\mu_k\};x)$ on the space span$\{e^{-\mu_0 t}, e^{-\mu_1 t}, \ldots,\}$ defined in (2.34), the formula yields

$$E_n(\{\mu_k\};y) = \frac{1}{\pi}\int_0^\infty \frac{\sin\left(\Psi_n(t)+ty\right)}{\sqrt{\mu_n^2+t^2}}dt,$$

with $\Psi_n(t) = 2\sum_{j=0}^{n-1}\arctan\left(\mu_j/t\right) + \arctan\left(\mu_n/t\right)$.

3.3 A proof of the formula

The main ingredient in our recipe for the formula for $L_n(\Lambda;x)$ is the simple fact that on the line $\mathrm{Re}(t) = -1/2$, the product of the integrand in the definition (2.18) has modulus 1; i.e. for all n and $t = -1/2 + is$, $s \in \mathbb{R}$,

$$\left|\prod_{k=0}^{n-1}\frac{t+\lambda_k+1}{t-\lambda_k}\right| = \prod_{k=0}^{n-1}\left|\frac{(\lambda_k+1/2)+is}{-(\lambda_k+1/2)+is}\right| = 1. \tag{3.4}$$

Hereafter, let Γ be the closed half-circle consisting of the line segment from $-1/2-iR$ to $-1/2+iR$, and the semi-circle $C_R = \{-1/2 + Re^{i\theta}; \theta \in [-\pi/2, \pi/2]\}$ with $R > \lambda_n + 1/2$ (see Figure 1). From now on we write $\lambda_k^* = \lambda_k + 1/2$ for all k.

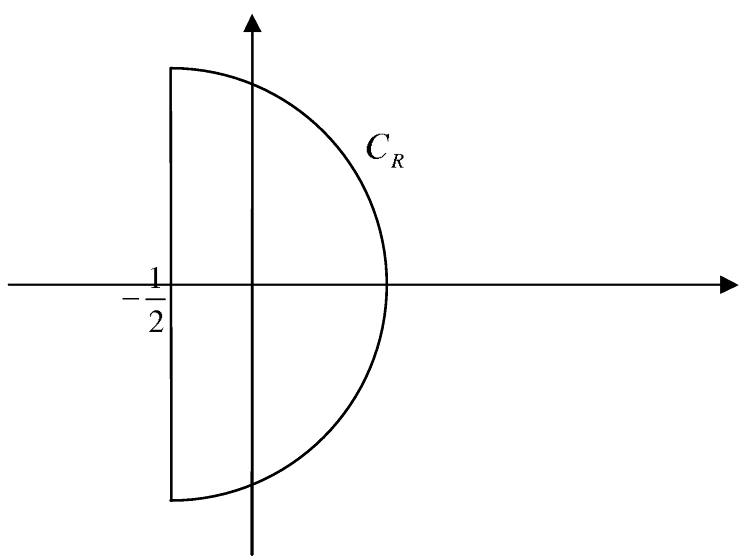

Figure 1: The contour Γ chosen in (2.18). We then let $R \longrightarrow \infty$.

First we prove a lemma (see a similar result in [31]):

Lemma 3.3 *We have*

$$\int_{C_R} \prod_{j=0}^{n-1} \frac{t+\lambda_j+1}{t-\lambda_j} \frac{x^t}{t-\lambda_n} dt \longrightarrow 0, \qquad R \longrightarrow \infty.$$

Proof. Denote the integral above by $I_n(x)$. Then we can write

$$I_n(x) = \frac{1}{\sqrt{x}} \int_{-\pi/2}^{\pi/2} \prod_{j=0}^{n-1} \frac{\lambda_j^* + Re^{i\theta}}{-\lambda_j^* + Re^{i\theta}} \frac{e^{Re^{i\theta}\log x}}{-\lambda_n^* + Re^{i\theta}} iRe^{i\theta} d\theta.$$

By writing

$$\left| \prod_{j=0}^{n-1} \frac{\lambda_j^* + Re^{i\theta}}{-\lambda_j^* + Re^{i\theta}} \frac{1}{-\lambda_n^* + Re^{i\theta}} \right| = \frac{1}{R} \prod_{j=0}^{n-1} \left| \frac{1 + \frac{\lambda_j^*}{R} e^{-i\theta}}{1 - \frac{\lambda_j^*}{R} e^{-i\theta}} \right| \frac{1}{\left| 1 - \frac{\lambda_n^*}{R} e^{-i\theta} \right|},$$

it is clear that for a sufficiently large R, there exists a constant $M > 1$ such that for all $\theta \in [-\pi/2, \pi/2]$,

$$\left| \prod_{j=0}^{n-1} \frac{\lambda_j^* + Re^{i\theta}}{-\lambda_j^* + Re^{i\theta}} \frac{1}{-\lambda_n^* + Re^{i\theta}} \right| \le \frac{M}{R}.$$

Hence if we let $c := -\log x > 0$, we obtain

$$|I_n(x)| \le \frac{2M}{\sqrt{x}} \int_0^{\pi/2} e^{-cR\cos\theta} d\theta,$$

and the result now follows from Lebesgue's Dominated Convergence Theorem. $\square$

Proof of Theorem 3.1. According to the lemma, the contour integral in (2.18) can be evaluated along the line $L : \mathrm{Re}(t) = -1/2$. We saw in (3.4) that for every $t = -1/2 + is$, we can write the product in the integrand as

$$w_n(s) := \prod_{j=0}^{n-1} \frac{\lambda_j^* + is}{-\lambda_j^* + is},$$

with $|w_n(s)| = 1$ for all n and $s \in \mathbb{R}$. We can write

$$w_n(s) = (-1)^n e^{i\theta_n(s)},$$

where $\theta_n(s) = 2\sum_{j=0}^{n-1} \arctan\left(s/\lambda_j^*\right) \in \mathbb{R}$. Furthermore, we have

$$\frac{w_n(s)}{-\lambda_n^* + is} = \frac{(-1)^{n+1} e^{i\Theta_n(s)}}{\sqrt{{\lambda_n^*}^2 + s^2}},$$

where

$$\Theta_n(s) = \theta_n(s) + \arctan\left(\frac{s}{\lambda_n^*}\right) = 2\sum_{j=0}^{n-1} \arctan\left(\frac{s}{\lambda_j^*}\right) + \arctan\left(\frac{s}{\lambda_n^*}\right).$$

Then if we let $c = -\log x > 0$ we can write (2.18) as (the negative sign comes from reversing the orientation)

$$L_n(\Lambda; x) = \frac{-1}{2\pi\sqrt{x}} \int_{-\infty}^{\infty} \frac{w_n(s)}{-\lambda_n^* + is} e^{-ics} ds = \frac{(-1)^n}{2\pi\sqrt{x}} \int_{-\infty}^{\infty} \frac{e^{i[\Theta_n(s) - cs]}}{\sqrt{{\lambda_n^*}^2 + s^2}} dt.$$

Since $s \mapsto \Theta_n(s) - cs$ is odd, it follows that

$$L_n(\Lambda; x) = \frac{(-1)^n}{\pi\sqrt{x}} \int_0^{\infty} \frac{\cos[\Theta_n(s) - cs]}{\sqrt{{\lambda_n^*}^2 + s^2}} ds. \tag{3.5}$$

Using the relation $\arctan x = \pi/2 - \arctan(1/x)$ we can write

$$\Theta_n(s) = (2n+1)\frac{\pi}{2} - \left(2\sum_{j=0}^{n-1} \arctan\frac{\lambda_j^*}{s} + \arctan\frac{\lambda_n^*}{s}\right) = (2n+1)\frac{\pi}{2} - \Phi_n(s).$$

The result now follows from the identity $\cos(\Theta_n(s) - cs) = \sin\left((2n+1)\frac{\pi}{2}\right)\sin(\Phi_n(s) + cs) = (-1)^n \sin(\Phi_n(s) - s\log x)$. $\square$

Proof of Corollary 3.2. The representation in (3.2) is arrived at by using the substitution $s = \lambda_n^* t$ in (3.1) and letting $R_n(t) = \Phi_n(\lambda_n^* t)/(2\lambda_n^*)$ for all n. $\square$

CHAPTER IV

ASYMPTOTICS ON THE INTERVAL OF ORTHOGONALITY

Most of the results of this chapter appear in "Asymptotic behavior of Müntz orthogonal polynomials" [48] and "Endpoint limit asymptotics of Müntz-Legendre polynomials" [49]. We shall use the representation (3.2) to determine the asymptotic behavior of the Müntz orthogonal polynomials for $x \in (0,1)$ on the interval of orthogonality.

First we present a result on the endpoint limit asymptotics close to the endpoint $x = 1$ under very weak conditions on the exponents. This generalizes the Formula of Mehler-Heine in Theorem 2.9 and the Formula of Fejér in Theorem 2.13 for the case $\alpha = 0$ and we believe the proof flows quite naturally from the perspective of Müntz orthogonality.

Next we prove strong asymptotics of $L_n(\Lambda; x)$ for x inside $(0,1)$. The formula (3.2) holds for all real exponents $\Lambda = \{\lambda_k\}$ and it turns out that the density condition

$$\sum_{k=0}^{\infty} \frac{1}{\lambda_k + \frac{1}{2}} = \infty \tag{4.1}$$

(this is condition (2.17) in Müntz's Theorem for $L_2[0,1]$) appears in a very natural way in the analysis of the phase function of (3.2), along with the sum of the exponents $\sum_{j=0}^{n}(2\lambda_j + 1)$. Essentially we only need to assume (4.1), apart from when we are very close to being non-dense, in which case we need to impose the very weak regularity condition

$$\lim_{n\to\infty} \frac{1}{2\lambda_n + 1} \sum_{k=0}^{n} (2\lambda_k + 1) = \infty. \tag{4.2}$$

We shall in particular look at the case when the Müntz exponents are asymptotic to an arithmetic progression, i.e. such that

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \rho, \tag{4.3}$$

for some constant $\rho > 0$. In particular, this provides a new proof for the asymptotic properties of the classical Legendre, Jacobi and Laguerre ($\alpha = 0$) polynomials inside their respective intervals of orthogonality.

Finally for special cases we get immediate corollaries for the asymptotic behavior of the associated Müntz-Christoffel functions on the interval.

4.1 Main results

We start with the endpoint limit asympotics as we approach $x = 1$. Here we don't need to assume the density condition (4.1).

Theorem 4.1 *Let $\Lambda : -1/2 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ be a sequence of real numbers and let $\Sigma_n = \sum_{k=0}^{n-1}(2\lambda_k + 1) + (2\lambda_n + 1)/2$. If Λ satisfies the regularity condition* (4.2) *(i.e. $\Sigma_n/(2\lambda_n + 1) \longrightarrow \infty$), then uniformly for bounded $y \ge 0$,*

$$\lim_{n\to\infty} L_n\left(e^{-y^2/4\Sigma_n}\right) = \lim_{n\to\infty} L_n\left(1 - \frac{y^2}{4\Sigma_n}\right) = J_0(y),$$

where J_0 is the Bessel function of order 0 as defined in (2.10)*. The error term is $\mathcal{O}\left(\sqrt{(2\lambda_n + 1)/\Sigma_n}\right)$ as $n \longrightarrow \infty$.*

Remarks Let us discuss the analogy to the classic results (2.9) and (2.13). The quality of the error term in Theorem 4.1 implies that the scaling with Σ_n is the "correct" one.

(1) It follows from (2.32) that we can write

$$P_n^{(0,\beta)}\left(1 - \frac{y^2}{2n^2}\right) = \left(1 - \frac{y^2}{4n^2}\right)^{-\beta/2} L_n\left(\left\{k + \frac{\beta}{2}\right\}_{k=0}^{\infty}; \left(1 - \frac{y^2}{4n^2}\right)\right).$$

where $P_n^{(0,\beta)}$ is the nth Jacobi polynomial with $\alpha = 0$. For $\lambda_k = k + \beta/2$ we have $\Sigma_n = n^2 + (\beta + 1)(n + 1/2)$ which of course grows like n^2. Thus (2.9) for $\alpha = 0$ is a special case of Theorem 4.1.

(2) We saw in (2.33) that if we let $\lambda_n = 0$ for all n we get the Laguerre polynomials $\mathcal{L} := \mathcal{L}^{(0)}$. Here $\Sigma_n = n + 1/2$ and we can write $\mathcal{L}(y/n^*) = L_n(\{0\}; e^{-y/n^*})$. Then it is easy to see that (2.13) (for $\alpha = 0$) follows from our result.

Before we give the strong asymptotics for fixed x, we give a name to the phase function that appears in the formula for $L_n(\Lambda; x)$ in (3.2), namely we let

$$h_n(t) := h_n(t, x) := R_n(t) - \frac{t}{2}\log x, \qquad t > 0, \quad x \in (0,1).$$

The following result is the most general one, where we only assume the denseness condition $T_n \longrightarrow \infty$, as $n \longrightarrow \infty$, and the regularity condition (4.2).

Theorem 4.2 *Let $\Lambda : -1/2 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ be a sequence of real numbers that satisfies the Müntz condition* (4.1) *and the regularity condition* (4.2). *Then for each $x \in (0,1)$, as $n \longrightarrow \infty$,*

$$L_n(\Lambda; x) = \frac{\cos\left(2\lambda_n^*\left[R_n(t_n) - \frac{t_n}{2}\log x\right] - \pi/4\right)}{\sqrt{\pi x \lambda_n^* R_n''(t_n)(1+t_n^2)}} + o\left(\frac{1}{\sqrt{\lambda_n^* R_n''(t_n)(1+t_n^2)}}\right),$$

where $t_n = t_n(x) \in (0,\infty)$ is the unique stationary point of $h_n(t) = h_n(t,x)$, i.e. such that $R_n'(t_n) = \frac{1}{2}\log x$. The result holds uniformly for x in compact subsets of $(0,1)$.

Remarks (1) The regularity condition (4.2) is weak: Even having $\lambda_n^* \le C\lambda_{n-m_n}^*$ for some increasing unbounded sequence $\{m_n\}$ and a positive constant C would be stronger.

(2) Note that using (2.23), we can relax the monotonicity condition of the exponents, and only require that the λ_k's are eventually non-decreasing.

In the case when $\lambda_n \sim n/\rho$, for some constant $\rho > 0$, and the rate of convergence is strong enough, the stationary point t_n converges, as well as $h_n(t_n) = R_n(t_n) - \frac{t_n}{2}\log x$ and $R_n''(t_n)$. However, in order to replace the phase $h_n(t_n)$ in Theorem 4.2, we need an accuracy of order $o(1/n)$. In Theorems 4.3 and 4.4, the asymptotics are explicit.

Theorem 4.3 *Let $\Lambda : -1/2 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ be a sequence of real numbers, and let $\rho > 0$ and $\theta \in (0, \pi/2)$.*

(i) If $\lambda_n = (n + \beta/2)/\rho + o(1)$, as $n \longrightarrow \infty$, for some constant $\beta > -1$, then as $n \longrightarrow \infty$,

$$L_n(\Lambda; \cos^{2\rho}\theta) = \frac{\cos([2n + \beta + \rho]\theta - \pi/4)}{\sqrt{\pi n \sin\theta \cos^{2\rho-1}\theta}} + o\left(\frac{1}{\sqrt{n}}\right).$$

(ii) If $\lambda_n = n/\rho + o(\sqrt{n})$, then, as $n \longrightarrow \infty$,

$$L_n(\Lambda; \cos^{2\rho}\theta) = \frac{\cos([2\lambda_n + 1]h_n(\cot\theta) - \pi/4)}{\sqrt{\pi n \sin\theta \cos^{2\rho-1}\theta}} + o\left(\frac{1}{\sqrt{n}}\right),$$

where $h_n(t) = R_n(t) - t\log(\cos\theta)$.

The results hold uniformly for θ in compact subsets of $(0, \pi/2)$.

The following theorem is a more general version of Theorem 4.3. Since t_n is not explicit, but h_n is, it is computationally practical to use successive approximations from the limit point $\cot\theta$ of t_n. We demonstrate this using a type of Newton iteration.

Theorem 4.4 *Let $\Lambda : -1/2 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ be a sequence of real numbers such that $\lambda_n = n/\rho + o\left(n^{1-\delta}\right)$ for some constant $\rho > 0$ and $0 < \delta < 1$. Then for each $\theta \in (0, \pi/2)$, as $n \longrightarrow \infty$,*

$$L_n(\Lambda; \cos^{2\rho}\theta) = \frac{\cos([2\lambda_n + 1]h_n(\gamma_{N,n}) - \pi/4)}{\sqrt{\pi n \sin\theta \cos^{2\rho-1}\theta}} + o\left(\frac{1}{\sqrt{n}}\right),$$

where $h_n(t) = R_n(t) - t\log(\cos\theta)$, and $\gamma_{N,n}$ is defined recursively by

$$\gamma_{0,n} = \gamma_0 = \cot\theta, \qquad \gamma_{k+1,n} = \gamma_{k,n} - \frac{h_n'(\gamma_k)}{h_n''(\gamma_k)}, \qquad k = 0, 1, 2, \ldots, N-1,$$

and N is the smallest integer such that $1/2^{N+1} \le \delta$. The result holds uniformly for θ in compact subsets of $(0, \pi/2)$.

Remarks (1) These results can easily be transferred to the Müntz-Jacobi polynomials via equation (2.27).

(2) Under the assumptions of Theorem 4.4, $\lim_{n\to\infty} t_n = \cot\theta$ and $\lim_{n\to\infty} h_n(t_n) = \rho\theta$.

(3) The classical results of Theorem 2.5 and Theorem 2.7 (with $\alpha = 0$) are special cases of Theorem 4.3, part (i).

The following result is a direct consequence of Theorem 4.2. Here we assume that $\lim_{n\to\infty} \sigma_n = \infty$, which covers all the cases $\lambda_n = o(n)$ as $n \longrightarrow \infty$.

Theorem 4.5 *Let $\Lambda : -1/2 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ be a sequence of real numbers that satisfies*

$$\lim_{n\to\infty} \frac{1}{(2\lambda_n+1)^2} \sum_{j=0}^{n} (2\lambda_j + 1) = \infty.$$

Then for each $x \in (0,1)$, as $n \longrightarrow \infty$,

$$L_n(\Lambda; x) = \frac{\cos\left(2\lambda_n^* h_n(t_n) - \pi/4\right)}{\sqrt{\pi x}|\log x|^{1/4}} \left(\sum_{j=0}^{n}(2\lambda_j+1)\right)^{-\frac{1}{4}} + o\left(\left(\sum_{j=0}^{n}(2\lambda_j+1)\right)^{-\frac{1}{4}}\right) \tag{4.4}$$

where $t_n = t_n(x) \in (0,\infty)$ is the unique stationary point of $h_n(t) = R_n(t) - \frac{t}{2}\log x$. The result holds uniformly for x in compact subsets of $(0,1)$.

In Theorem 4.5, $t_n = t_n(x)$ is not explicit. In the next result we make a stronger assumption for which $h_n(t_n)$ can be explicitly written in the phase. This includes all cases when $\lambda_n = o\left(n^{1/3}\right)$, and a special case are the asymptotics of the Laguerre polynomials (2.12) (recall the relation (2.33) we obtain by letting $\lambda_k = 0$ for all k).

Theorem 4.6 *Let $\Lambda : -1/2 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ be a sequence of real numbers that satisfies*

$$\lim_{n\to\infty} \frac{1}{(2\lambda_n+1)^4} \sum_{j=0}^{n} (2\lambda_j + 1) = \infty.$$

Then for each $x \in (0,1)$, as $n \longrightarrow \infty$,

$$L_n(\Lambda; x) = \frac{\cos\left(2\sqrt{\Sigma_n |\log x|} - \pi/4\right)}{\sqrt{\pi x}\,(\Sigma_n |\log x|)^{1/4}} + o\left(\frac{1}{\Sigma_n^{1/4}}\right)$$

where $\Sigma_n = \sum_{j=0}^{n}(2\lambda_j+1)+(2\lambda_n+1)/2$. The result holds uniformly for x in compact subsets of $(0,1)$.

In the following result, we state the asymptotic bounds obtained from Theorem 4.2 and summarize the bounds from the special cases above. Note that we don't have an explicit bound for the case $\sigma_n \longrightarrow 0$, $n \longrightarrow \infty$ (then we are close to being non-dense), since then it is more difficult to estimate the stationary point t_n.

Corollary 4.7 *Let $\Lambda : -1/2 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ be a sequence of real numbers that satisfies (4.1). Then uniformly for x in compact subsets of $(0,1)$,*

$$|L_n(\Lambda; x)|^2 = \mathcal{O}\left(\frac{1}{\lambda_n^* t_n}\right), \qquad n \longrightarrow \infty,$$

where $t_n = t_n(x) \in (0,\infty)$ is the unique stationary point of $h_n(t) = R_n(t) - \frac{t}{2}\log x$.

In particular, if $\sigma_n \longrightarrow \infty$, then

$$|L_n(\Lambda; x)|^2 = \mathcal{O}\left(\left(\sum_{j=0}^{n}(2\lambda_j+1)\right)^{-\frac{1}{2}}\right), \qquad n \longrightarrow \infty,$$

and if $\lambda_n \asymp n$ as $n \longrightarrow \infty$, then

$$|L_n(\Lambda; x)|^2 = \mathcal{O}\left(n^{-1}\right), \qquad n \longrightarrow \infty.$$

In some cases we can use the asymptotics of the Müntz orthogonal polynomials to determine the asymptotics of the associated Christoffel functions, which were presented in (2.24). The following result is a consequence of Theorem 4.3 (i).

Theorem 4.8 *Let $\Lambda : -1/2 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ be a sequence that satisfies $\lambda_n = (n+\beta/2)/\rho + o(1)$ as $n \longrightarrow \infty$. Then uniformly for x in compact subsets of $(0,1)$,*

$$\lim_{n\to\infty} n\lambda\left(M_n(\Lambda); x\right) = \rho\pi\sqrt{x^{2-1/\rho}(1-x^{1/\rho})}.$$

We also get uniform bounds:

Theorem 4.9 *Let $a \in (0,1]$ and let $\Lambda : -1/2 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ be a sequence that satisfies $\lambda_n \asymp n^a$ as $n \longrightarrow \infty$. Then uniformly for x in compact subets of $(0,1)$,*

$$\lambda\left(M_n(\Lambda); x\right)^{-1} = \mathcal{O}\left(n^{\frac{a+1}{2}}\right), \qquad n \longrightarrow \infty.$$

It is worth mentioning that we always have

$$\int_0^1 \lambda\left(M_n(\Lambda); x\right)^{-1} dx = \int_0^1 K_n(x,x)dx = n,$$

and therefore, in the case $a < 1$, Theorem 4.9 implies that the growth of the Christoffel functions is concentrated at the endpoints. The asymptotic behavior of the Müntz-Christoffel functions at the endpoints is addressed Chapter 7.

4.2 Proof for endpoint limit asymptotics

Here we present a proof of Theorem 4.1. For notational convenience, we use the sum $S_n = \Sigma_n/2$ defined in (1.5) instead of Σ_n. We define a sequence of real numbers $\{b_n\}$ by letting

$$b_n := \left(\frac{S_n}{\lambda_n^*}\right)^{1/2}, \qquad n = 0, 1, 2, \ldots \tag{4.5}$$

By hypothesis $S_n/\lambda_n^* \longrightarrow \infty$ as $n \longrightarrow \infty$, and therefore $b_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and

$$\lim_{n\to\infty} \frac{\lambda_n^* b_n}{S_n} = 0. \tag{4.6}$$

With $x = e^{-y^2/2S_n}$, (3.2) becomes

$$L_n\left(\Lambda; e^{-y^2/2S_n}\right) = \frac{e^{y^2/4S_n}}{\pi} \int_0^\infty \frac{\sin\left(\Theta_n(t) + \frac{\lambda_n^* y^2}{2S_n} t\right)}{\sqrt{1+t^2}} dt, \tag{4.7}$$

where $\Theta_n(t) = 2\lambda_n^* R_n(t) = 2\sum_{j=0}^{n-1} \arctan \frac{\lambda_j^*}{\lambda_n^* t} + \arctan \frac{1}{t}$.

We need to analyse the integral

$$I_y(n) = \int_0^\infty \frac{\exp i\left(\Theta_n(t) + \frac{\lambda_n^* y^2}{2S_n} t\right)}{\sqrt{1+t^2}} dt.$$

From Lemma 4.19 it follows (here $c = -\frac{1}{2}\log x = y^2/4S_n$) that $t_n \asymp S_n/\lambda_n^*$ as $n \longrightarrow \infty$ uniformly for y bounded. We will show that the main contribution will come from the part of the integral on $[b_n, \infty)$. We can write

$$I_y(n) = \int_{b_n}^{\infty} \frac{\exp i\left(\frac{2S_n}{\lambda_n^* t} + \frac{\lambda_n^* y^2}{2S_n}t\right)}{t} dt + \delta_0(n) + \delta_1(n) + \delta_2(n) \tag{4.8}$$

where

$$\begin{aligned}
\delta_0(n) &= \int_0^{b_n} \frac{\exp i\left(\Theta_n(t) + \frac{\lambda_n^* y^2}{2S_n}t\right)}{\sqrt{1+t^2}} dt \\
\delta_1(n) &= \int_{b_n}^{\infty} \frac{\exp i\left(\frac{2S_n}{\lambda_n^* t} + \frac{\lambda_n^* y^2}{2S_n}t\right)}{\sqrt{1+t^2}} \left[\exp i\left(\Theta_n(t) - \frac{2S_n}{\lambda_n^* t}\right) - 1\right] dt \\
\delta_2(n) &= \int_{b_n}^{\infty} \exp i\left(\frac{2S_n}{\lambda_n^* t} + \frac{\lambda_n^* y^2}{2S_n}t\right)\left[\frac{1}{\sqrt{1+t^2}} - \frac{1}{t}\right] dt.
\end{aligned}$$

We first estimate the error terms.

Lemma 4.10 *Uniformly for real bounded y, $\delta_0(n) = \mathcal{O}\left(\sqrt{\lambda_n^*/S_n}\right)$ as $n \longrightarrow \infty$.*

Proof. First we take out the linear part of the phase function that appears in $\delta_0(n)$ and write

$$\delta_0(n) = \int_0^{b_n} \frac{e^{i\Theta_n(t)}}{\sqrt{1+t^2}} dt + \varepsilon_0(n),$$

where

$$\varepsilon_0(n) = \int_0^{b_n} \frac{e^{i\Theta_n(t)}}{\sqrt{1+t^2}} \left[e^{i\frac{\lambda_n^* y^2}{2S_n}t} - 1\right] dt.$$

According to (4.6), we have $\lambda_n^* t/2S_n = o(1)$ for $t \le b_n$, so this yields

$$|\varepsilon_0(n)| \le \frac{\lambda_n^* y^2}{2S_n} \int_0^{b_n} \frac{t}{\sqrt{1+t^2}} dt \le y^2 \frac{\lambda_n^* b_n}{2S_n} = o(1).$$

The function $\Theta_n(t)$ is monotone on $[0, \infty)$ so we can make the substitution $u = \Theta_n(t)$. Then using integration by parts gives

$$\int_0^{b_n} \frac{e^{i\Theta_n(t)}}{\sqrt{1+t^2}} dt = i\left(e^{i\Theta_n(b_n)} p_n(b_n) - e^{i\Theta_n(0)} p_n(0) - \int_0^{b_n} p_n'(t) e^{i\Theta_n(t)} dt\right) \tag{4.9}$$

where we have defined $p_n(t) := -1/\Theta_n'(t)\sqrt{1+t^2}$, a non-negative function. The derivative of $p_n(t)$ is

$$p_n'(t) = \frac{(1+t^2)\Theta_n''(t) + t\Theta_n'(t)}{[\Theta_n'(t)]^2(1+t^2)^{3/2}}.$$

Since $-\Theta_n'(t) = 2\sum_{j=0}^{n-1} \frac{\lambda_j^*/\lambda_n^*}{(\lambda_j^*/\lambda_n^*)^2+t^2} + \frac{1}{1+t^2}$ and $\Theta_n''(t) = 2t\left(2\sum_{j=0}^{n-1} \frac{\lambda_j^*/\lambda_n^*}{\left[(\lambda_j^*/\lambda_n^*)^2+t^2\right]^2} + \frac{1}{[1+t^2]^2}\right)$, we have

$$\begin{aligned}
(1+t^2)\Theta_n(t) + t\Theta_n'(t) &= t\left(2\sum_{j=0}^{n-1} \frac{\frac{\lambda_j^*}{\lambda_n^*}}{\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 + t^2}\left[\frac{2(1+t^2)}{\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 + t^2} - 1\right] + \frac{1}{1+t^2}\right) \\
&= t\left(2\sum_{j=0}^{n-1} \frac{\frac{\lambda_j^*}{\lambda_n^*}}{\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 + t^2}\,\frac{2-\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 + t^2}{\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 + t^2} + \frac{1}{1+t^2}\right).
\end{aligned}$$

Since $\lambda_j^*/\lambda_n^* \le 1$ it follows that $p_n'(t) > 0$ for all $t > 0$. Hence we can estimate (4.9) by

$$\begin{aligned}
\left|\int_0^{b_n} \frac{e^{i\Theta_n(t)}}{\sqrt{1+t^2}}dt\right| &\le |p_n(b_n)| + |p_n(0)| + \int_0^{b_n} |p_n'(t)|dt \\
&\le p_n(b_n) + p_n(0) + p_n(b_n) - h_n(0) \\
&\le 3p_n(b_n).
\end{aligned}$$

Using the inequality $|\Theta_n'(t)| \ge (2S_n/\lambda_n^*)/(1+t^2)$ and the hypothesis $b_n \longrightarrow \infty$, this yields

$$\left|\int_0^{b_n} \frac{e^{i\Theta_n(t)}}{\sqrt{1+t^2}}dt\right| \le \frac{3}{|\Theta_n'(b_n)|\sqrt{1+b_n^2}} = \mathcal{O}\left(\frac{\lambda_n^* b_n}{S_n}\right)$$

and the result now follows since $b_n = (S_n/\lambda_n^*)^{1/2}$. □

Lemma 4.11 *Uniformly for real bounded y, $\delta_1(n) = \mathcal{O}\left(\sqrt{\lambda_n^*/S_n}\right)$ as $n \longrightarrow \infty$.*

Proof. Using the inequality $|\arctan x - x| \le |x|^3/3$ for $|x| < 1$ we see that for

$t \geq b_n \gg 1$,

$$\begin{aligned}
\left|\Theta_n(t) - \frac{2S_n}{\lambda_n^* t}\right| &\leq 2\sum_{j=0}^{n-1}\left|\arctan\frac{\lambda_j^*}{\lambda_n^* t} - \frac{\lambda_j^*}{\lambda_n^* t}\right| + \left|\arctan\frac{1}{t} - \frac{1}{t}\right| \\
&\leq \frac{1}{3t^3}\left(2\sum_{j=0}^{n-1}\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^3 + 1\right) \\
&\leq \frac{1}{3t^3}\frac{2S_n}{\lambda_n^*},
\end{aligned}$$

where in the last step we have used $\lambda_j^*/\lambda_n^* \leq 1$ for all $j = 0, 1, \ldots, n$. It follows that

$$|\delta_1(n)| \leq \frac{2S_n}{3\lambda_n^*}\int_{b_n}^{\infty}\frac{dt}{t^3\sqrt{1+t^2}} = \mathcal{O}\left(\frac{S_n}{\lambda_n^* b_n^3}\right),$$

and (4.5) then gives $|\delta_1(n)| = \mathcal{O}\left((\lambda_n^*/S_n)^{1/2}\right)$ as $n \longrightarrow \infty$. □

Lemma 4.12 *Uniformly for real bounded y, $\delta_2(n) = \mathcal{O}(\lambda_n^*/S_n)$ as $n \longrightarrow \infty$.*

Proof. We trivially have

$$|\delta_2(n)| \leq \int_{b_n}^{\infty}\frac{\sqrt{1+t^2}-t}{t\sqrt{1+t^2}}dt = \mathcal{O}\left(\frac{1}{b_n^2}\right),$$

and the result follows since $b_n = (S_n/\lambda_n^*)^{1/2}$. □

By using the substitution $v = \lambda_n^* ty/2S_n$ in (4.8) and applying Lemmas 1-3, we have established that

$$I_y(n) = \int_{\frac{\lambda_n^* b_n}{2S_n}y}^{\infty}\frac{\exp iy\left(\frac{1}{v}+v\right)}{v}dv + \mathcal{O}\left(\left(\frac{\lambda_n^*}{S_n}\right)^{1/2}\right) \tag{4.10}$$

as $n \longrightarrow \infty$. To complete the proof of Theorem 4.1 we need the following estimate:

Lemma 4.13 *Uniformly for real bounded y,*

$$\int_0^{\frac{\lambda_n^* b_n}{2S_n} y} \frac{\exp iy\left(\frac{1}{v}+v\right)}{v} dv = \mathcal{O}\left(\left(\frac{\lambda_n^*}{S_n}\right)^{1/2}\right), \qquad n \longrightarrow \infty.$$

Proof. Using the substitution $s = 1/v$ we can write

$$\begin{aligned}
\int_0^{\frac{\lambda_n^* b_n}{2S_n} y} \frac{\exp iy\left(\frac{1}{v}+v\right)}{v} dv &= \int_{\frac{2S_n}{\lambda_n^* b_n}\frac{1}{y}}^{\infty} \frac{\exp iy\left(s+\frac{1}{s}\right)}{s} ds \\
&= \int_{\frac{2S_n}{\lambda_n^* b_n}\frac{1}{y}}^{\infty} \frac{e^{iys}}{s} ds + \int_{\frac{2S_n}{\lambda_n^* b_n}\frac{1}{y}}^{\infty} \frac{e^{iys}}{s}\left[e^{iy\frac{1}{s}} - 1\right] ds \\
&= \int_{\frac{2S_n}{\lambda_n^* b_n}}^{\infty} \frac{e^{is}}{s} ds + \mathcal{O}\left(\int_{\frac{2S_n}{\lambda_n^* b_n}\frac{1}{y}}^{\infty} \frac{ds}{s^2}\right) \qquad (4.11) \\
&= \mathcal{O}\left(\frac{\lambda_n^* b_n}{S_n}\right),
\end{aligned}$$

where the estimate for the first integral in (4.11) follows by using integration by parts,

$$\int_{\frac{2S_n}{\lambda_n^* b_n}}^{\infty} \frac{e^{is}}{s} ds = -i\left.\frac{e^{is}}{s}\right|_{\frac{2S_n}{\lambda_n^* b_n}}^{\infty} - i\int_{\frac{2S_n}{\lambda_n^* b_n}}^{\infty} \frac{e^{is}}{s^2} ds.$$

$\square$

Proof of Theorem 4.1. Recall that

$$L_n(\Lambda; e^{-y^2/2S_n}) = \frac{e^{-y^2/4S_n}}{\pi} \operatorname{Im}[I_y(n)]$$

and trivially $e^{-y^2/4S_n} = 1 + \mathcal{O}(1/S_n) = 1 + \mathcal{O}\left(\sqrt{\lambda_n^*/S_n}\right)$ as $n \longrightarrow \infty$. Then (4.10) and Lemma 4.13 give

$$L_n(\Lambda; e^{-y^2/2S_n}) = \frac{1}{\pi}\int_0^{\infty} \frac{\sin\left(y\left[\frac{1}{v}+v\right]\right)}{v} dv + \mathcal{O}\left(\sqrt{\frac{\lambda_n^*}{S_n}}\right) \qquad (4.12)$$

as $n \longrightarrow \infty$ and this holds uniformly for bounded $y > 0$.

If we replace $x = e^{-y^2/2S_n}$ with $x = 1 - y^2/2S_n$ and repeat the treatment above, the estimates for the error terms are the same, but the main term in (4.10) becomes

$$\int_{\frac{\lambda_n^* b_n}{2S_n} y}^{\infty} \frac{\exp iy\left(\frac{1}{v} - v\frac{2S_n}{y^2}\log\left(1 - \frac{y^2}{2S_n}\right)\right)}{v} dv$$

We have $(2S_n/y^2)\log\left(1-y^2/2S_n\right) = -1+\mathcal{O}\left(S_n^{-2}\right)$ as $n \longrightarrow \infty$ and this holds uniformly for bounded y. For any $N_n \longrightarrow \infty$,

$$\begin{aligned}
&\left|\int_{\frac{\lambda_n^* b_n}{2S_n}y}^{N_n} \frac{\exp iy\left(\frac{1}{v}+v\right)}{v}dv - \int_{\frac{\lambda_n^* b_n}{2S_n}y}^{N_n} \frac{\exp iy\left(\frac{1}{v}-v\frac{2S_n}{y^2}\log\left(1-\frac{y^2}{2S_n}\right)\right)}{v}dv\right| \\
&= \quad \left|\int_{\frac{\lambda_n^* b_n}{2S_n}y}^{N_n} \frac{e^{iy\left(\frac{1}{v}+v\right)}}{v}\left[1-e^{-iyv\left(1+\frac{2S_n}{y^2}\log\left(1-\frac{y^2}{2S_n}\right)\right)}\right]dv\right| \\
&\leq \quad N_n y\left(1+\frac{2S_n}{y^2}\log\left(1-\frac{y^2}{2S_n}\right)\right) \\
&= \quad \mathcal{O}\left(\frac{N_n}{S_n^2}\right).
\end{aligned}$$

Just as in the proof of Lemma 4.13 we can show that

$$\int_{N_n}^{\infty} \frac{\exp iy\left(\frac{1}{v}-v\frac{2S_n}{y^2}\log\left(1-\frac{y^2}{2S_n}\right)\right)}{v}dv = \mathcal{O}\left(\frac{1}{N_n}\right).$$

By choosing $N_n = S_n$, we then have

$$\int_{\frac{\lambda_n^* b_n}{2S_n}y}^{\infty} \frac{\exp iy\left(\frac{1}{v}-v\frac{2S_n}{y^2}\log\left(1-\frac{y^2}{2S_n}\right)\right)}{v}dv = \int_{\frac{\lambda_n^* b_n}{2S_n}y}^{\infty} \frac{\exp iy\left(\frac{1}{v}+v\right)}{v}dv + \mathcal{O}\left(\frac{1}{S_n}\right)$$

as $n \longrightarrow \infty$, and it follows that $L_n(\Lambda; e^{-y^2/2S_n}) - L_n(\Lambda; 1-y^2/2S_n) = \mathcal{O}\left(\sqrt{\lambda_n^*/S_n}\right)$.

To complete the proof we make the substitution $v = e^s$ in the main term in (4.12) which gives

$$\frac{1}{\pi}\int_0^{\infty} \frac{\sin\left(y\left[\frac{1}{v}+v\right]\right)}{v}dv = \frac{2}{\pi}\int_0^{\infty} \frac{\sin\left(2y\cosh s\right)}{s}ds.$$

This is a well known representation for the Bessel function $J_0(2y)$ (see [39]). The case when $y = 0$ is trivial since $L_n(1) = 1 = J_0(0)$ (see [6] for the first identity). Finally, we note that $\Sigma_n = 2S_n$. □

4.3 Proofs for strong asymptotics; general results

4.3.1 Estimates for the phase function and its stationary point

In our formula (3.2) for the nth Müntz-Legendre polynomial, a key role is played by the function

$$h_n(t) = h_n(t,x) = R_n(t) + tc,$$

where we have fixed $c = -\frac{1}{2}\log x > 0$ for our analysis below, and

$$R_n(t) = \frac{1}{\lambda_n^*}\left(\sum_{j=0}^{n-1} \arctan\frac{\lambda_j^*}{\lambda_n^* t} + \frac{1}{2}\arctan\frac{1}{t}\right).$$

Note that $h_n(t) > 0$ for all $t \geq 0$. Also $h_n'(t) = R_n'(t) + c$ where

$$R_n'(t) = -\frac{1}{\lambda_n^*}\left(\sum_{j=0}^{n-1} \frac{\frac{\lambda_j^*}{\lambda_n^*}}{\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 + t^2} + \frac{1}{2}\frac{1}{1+t^2}\right) \tag{4.13}$$

and therefore

$$h_n'(0) = -\left(\sum_{j=0}^{n-1}\frac{1}{\lambda_j^*} + \frac{1}{2\lambda_n^*}\right) + c = -T_n + c$$

where T_n is the sum defined in (1.5). Hence if we assume the Müntz condition $T_n \longrightarrow \infty$, then $h_n'(0) < 0$ for n large enough. Furthermore

$$\lim_{t\longrightarrow\infty} h_n'(t) = c > 0,$$

and the second derivative

$$h_n''(t) = R_n''(t) = \frac{2t}{\lambda_n^*}\left(\sum_{j=0}^{n-1}\frac{\frac{\lambda_j^*}{\lambda_n^*}}{\left[\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 + t^2\right]^2} + \frac{1}{2}\frac{1}{\left[1+t^2\right]^2}\right) \tag{4.14}$$

is positive for all $t > 0$. Hence we have the following result:

Lemma 4.14 *If the Müntz condition $\sum_{k=0}^{\infty}\frac{1}{\lambda_k^*} = \infty$ is satisfied, then for all n large enough, the phase function $h_n(t)$ has a unique stationary point t_n in $(0,\infty)$.*

Recall the notation (1.5), (1.5), (1.5) for S_n, σ_n and T_n respectively. Hereafter we assume that the sequence $\Lambda = \{\lambda_k\}$ is non-decreasing.

Lemma 4.15 *For all n and $t > 0$ we have $-T_n < R_n'(t) < 0$ and*

$$\frac{1}{1+t^2} \le \frac{|R_n'(t)|}{\sigma_n} < \frac{1}{t^2}.$$

Proof. From (4.13) we see that $0 > R_n'(t) \ge R_n'(0) = -T_n$. Furthermore,

$$|R_n'(t)| < \frac{1}{\lambda_n^*}\left(\sum_{j=0}^{n-1} \frac{\lambda_j^*}{\lambda_n^*}\frac{1}{t^2} + \frac{1}{2}\frac{1}{t^2}\right) = \frac{\sigma_n}{t^2},$$

and since $\lambda_j^*/\lambda_n^* \le 1$ for all $j = 0, 1, \ldots, n$,

$$|R_n'(t)| \ge \frac{1}{\lambda_n^*}\left(\sum_{j=0}^{n-1} \frac{\lambda_j^*}{\lambda_n^*}\frac{1}{1+t^2} + \frac{1}{2}\frac{1}{1+t^2}\right) = \frac{\sigma_n}{1+t^2}.$$

□

Lemma 4.16 *For all n and $t > 0$,*

$$\frac{2t}{1+t^2} \le \frac{R_n''(t)}{|R_n'(t)|} < \frac{2}{t}.$$

Proof. From (4.14), we obtain

$$R_n''(t) < \frac{2t}{\lambda_n^*} \cdot \frac{1}{t^2}\left(\sum_{j=0}^{n-1} \frac{\frac{\lambda_j^*}{\lambda_n^*}}{\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 + t^2} + \frac{1}{2}\frac{1}{1+t^2}\right) = \frac{2}{t}|R_n'(t)|,$$

and using $\lambda_j^*/\lambda_n^* \le 1$ for all $j = 0, 1, \ldots, n$,

$$R_n''(t) \ge \frac{2t}{\lambda_n^*}\left(\sum_{j=0}^{n-1} \frac{\frac{\lambda_j^*}{\lambda_n^*}}{\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 + t^2} \cdot \frac{1}{1+t^2} + \frac{1}{2}\frac{1}{1+t^2} \cdot \frac{1}{1+t^2}\right) = \frac{2t}{1+t^2}|R_n'(t)|.$$

□

Combining the results of the previous two lemmas yields the following corollary:

Corollary 4.17 *For all n and $t > 0$,*

$$\frac{t}{[1+t^2]^2} \le \frac{R_n''(t)}{2\sigma_n} < \frac{1}{t^3}.$$

Lemma 4.18 *For all n and $t > 0$,*

$$|R_n^{(3)}(t)| \leq \frac{3}{t} R_n''(t).$$

Proof. First, we have

$$R_n^{(3)}(t) = \frac{2}{\lambda_n^*}\left(\sum_{j=0}^{n-1} \frac{\lambda_j^*}{\lambda_n^*} \frac{\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 - 3t^2}{\left[\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 + t^2\right]^3} + \frac{1}{2}\frac{1-3t^2}{[1+t^2]^3}\right).$$

Here, for all $r \in [0,1]$, $\left|\frac{r^2-3t^2}{r^2+t^2}\right| = \left|-3 + \frac{4r^2}{r^2+t^2}\right| \leq 3$, so

$$|R_n^{(3)}(t)| \leq 3\frac{2}{\lambda_n^*}\left(\sum_{j=0}^{n-1} \frac{\frac{\lambda_j^*}{\lambda_n^*}}{\left[\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 + t^2\right]^2} + \frac{1}{2}\frac{1}{[1+t^2]^2}\right) = \frac{3}{t} R_n''(t).$$

□

In the following results we give restrictions on the growth of the stationary point $t_n = t_n(x)$ as n grows:

Corollary 4.19 *Assume the Müntz condition* (4.1)*. Then for all n,*

$$t_n^2 < \frac{\sigma_n}{c} < t_n^2 + 1,$$

and

$$\frac{t_n^2}{t_n^2+1} < \frac{t_n R_n''(t_n)}{2c} < 1.$$

Proof. This follows directly from Lemmas 4.15 and 4.16, using $|R_n'(t_n)| = c$ for all n. □

Lemma 4.20 *Suppose the sequence $\Lambda = \{\lambda_j\}$ satisfies (4.1). Then, uniformly for x in compact subsets of $(0,1)$, $t_n\lambda_n^* \longrightarrow \infty$ as $n \longrightarrow \infty$. Furthermore, if the regularity condition*

$$\lim_{n\to\infty} \frac{1}{\lambda_n^*} \sum_{j=0}^{n-1} \lambda_j^* = \infty \tag{4.15}$$

is satisfied, then uniformly for x in compact subsets of $(0,1)$,

$$\lim_{n\longrightarrow\infty} \lambda_n^* t_n^2 R_n''(t_n) = \infty.$$

Proof. Assume $t_n < 1$. Then for some $l = l(n) \le n$, we have

$$\lambda_{l-1}^* < t_n\lambda_n^* \le \lambda_l^*.$$

First we have

$$\begin{aligned} 2c = -2R_n'(t_n) &= \frac{2}{\lambda_n^*}\left(\sum_{j=0}^{n-1} \frac{\frac{\lambda_j^*}{\lambda_n^*}}{\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 + t_n^2} + \frac{1}{2}\frac{1}{1+t_n^2}\right) \\ &> \frac{1}{(t_n\lambda_n^*)^2} \sum_{\frac{\lambda_j^*}{\lambda_n^*} < t_n} \lambda_j^* + \sum_{t_n \le \frac{\lambda_j^*}{\lambda_n^*} < 1} \frac{1}{\lambda_j^*} &(4.16) \\ &> \frac{1}{\lambda_l^{*2}} \sum_{j=0}^{l-1} \lambda_j^* + \sum_{j=l}^{n-1} \frac{1}{\lambda_j^*}. &(4.17) \end{aligned}$$

Then since $\sum_{j=l}^{n-1} \frac{1}{\lambda_j^*} \le 2c$ while $T_n = \sum_{j=0}^{n-1} \frac{1}{\lambda_j^*} + \frac{1}{2\lambda_n^*} \longrightarrow \infty$, it follows that

$$T_{l(n)} = \sum_{j=0}^{l-1} \frac{1}{\lambda_j^*} \sim T_n, \qquad n \longrightarrow \infty,$$

and this clearly also implies that $l(n) \longrightarrow \infty$ as $n \longrightarrow \infty$. This holds uniformly for x in compact subsets of $(0,1)$ since $c = -\frac{1}{2}\log x \asymp 1$ in such sets. Using the first sum in (4.16) gives

$$2ct_n^2\lambda_n^{*2} > \sum_{\frac{\lambda_j^*}{\lambda_n^*} < t_n} \lambda_j^* > \lambda_0^{*2} \sum_{\frac{\lambda_j^*}{\lambda_n^*} < t_n} \frac{1}{\lambda_j^*} \sim \lambda_0^{*2} T_n,$$

and the first result follows from our assumption $T_n \longrightarrow \infty$, $n \longrightarrow \infty$.

The same way as in (4.16), we obtain the inequality

$$\begin{aligned} 2t_n R_n''(t_n) &= \frac{4t_n^2}{\lambda_n^*}\left(\sum_{j=0}^{n-1} \frac{\frac{\lambda_j^*}{\lambda_n^*}}{\left[t_n^2 + \left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2\right]^2} + \frac{1}{2}\frac{1}{t_n^2+1}\right) \\ &> \frac{4t_n^2}{\lambda_n^*} \sum_{\frac{\lambda_j^*}{\lambda_n^*}<t_n} \frac{\frac{\lambda_j^*}{\lambda_n^*}}{[t_n^2+t_n^2]^2} \\ &= \frac{1}{t_n^2 {\lambda_n^*}^2} \sum_{\frac{\lambda_j^*}{\lambda_n^*}<t_n} \lambda_j^*. \end{aligned} \tag{4.18}$$

Therefore,

$$2\lambda_n^* t_n^2 R_n''(t_n) > \frac{1}{t_n \lambda_n^*} \sum_{\frac{\lambda_j^*}{\lambda_n^*}<t_n} \lambda_j^* \geq \frac{1}{\lambda_l^*}\sum_{j=0}^{l-1} \lambda_j^* = \frac{S_l}{\lambda_l^*},$$

and since $l = l(n) \longrightarrow \infty$, the result follows from our assumption (4.15).

If $t_n \geq 1$, then use $l = n$ above and the result is trivial. □

Corollary 4.21 *If $\lambda_n \sim n$, then $t_n \asymp 1$ as $n \longrightarrow \infty$, uniformly for x in compact subsets of* $(0,1)$.

Proof. Using the proof from Lemma 4.20, we see from (4.17), that in the case $\lambda_n \sim n$,

$$2c > \sum_{j=l}^{n-1} \frac{1}{\lambda_j^*} \sim \log n - \log l(n),$$

since $l(n) \longrightarrow \infty$. Note that $c = -\frac{1}{2}\log x \asymp 1$ for x in compact subsets of $(0,1)$ so $l(n) \longrightarrow \infty$ uniformly of x in such sets. This implies that $l(n) \sim n$, so the first sum in (4.17) gives

$$2ct_n^2 > \frac{1}{{\lambda_n^*}^2} \sum_{j=0}^{l-1} \lambda_j^* \sim \frac{l(n)^2}{{\lambda_n^*}^2} \asymp 1.$$

The upper bound for t_n^2 follows from Corollary 4.19 since $\sigma_n \asymp 1$ as $n \longrightarrow \infty$. □

4.3.2 Some technical lemmas

We will need the following lemmas:

Lemma 4.22 *If $\xi_n \sim t_n$, as $n \longrightarrow \infty$, then uniformly for x in compact subsets of $(0,1)$,*

$$R_n''(\xi_n) \sim R_n''(t_n), \qquad n \longrightarrow \infty.$$

Proof. First let $\xi_n = t_n + \eta_n$, with $\eta_n = o(t_n)$, and for each $j = 0, 1, \ldots, n$ let $r_j := r_{j,n} := \lambda_j^*/\lambda_n^*$. Recall that $R_n''(t)$ is given by (4.14). The factors of the terms of $R_n''(\xi_n) - R_n''(t_n)$ that depend on ξ_n and t_n are

$$\begin{aligned}
\frac{\xi_n}{[r_j^2+\xi_n^2]^2} - \frac{t_n}{[r_j^2+t_n^2]^2} &= t_n\left(\frac{1}{[r_j^2+\xi_n^2]^2} - \frac{1}{[r_j^2+t_n^2]^2}\right) + \frac{\eta_n}{[r_j^2+\xi_n^2]^2} \\
&= (t_n^2-\xi_n^2)\frac{t_n}{[r_j^2+t_n^2]^2}\frac{r_j^2+t_n^2+r_j^2+\xi_n^2}{[r_j^2+\xi_n^2]^2} + \frac{\eta_n}{[r_j^2+t_n^2]^2}\left(\frac{r_j^2+t_n^2}{r_j^2+\xi_n^2}\right)^2 \\
&= \frac{\eta_n}{t_n}\left[-\frac{t_n(t_n+\xi_n)}{r_j^2+\xi_n^2}\frac{r_j^2+t_n^2+r_j^2+\xi_n^2}{r_j^2+\xi_n^2} + \left(\frac{r_j^2+t_n^2}{r_j^2+\xi_n^2}\right)^2\right]\frac{t_n}{[r_j^2+t_n^2]^2}, \qquad (4.19)
\end{aligned}$$

where in the last step we use $t_n^2 - \xi_n^2 = -\eta_n(t_n+\xi_n)$.

If $\eta_n \geq 0$, i.e. $\xi_n \geq t_n$, then $\frac{t_n(t_n+\xi_n)}{r_j^2+\xi_n^2} \leq \frac{\xi_n(\xi_n+\xi_n)}{\xi_n^2} = 2$, $\frac{r_j^2+t_n^2}{r_j^2+\xi_n^2} \leq 1$, and $\frac{r_j^2+t_n^2+r_j^2+\xi_n^2}{r_j^2+\xi_n^2} \leq 2$ for all $j = 0, 1, \ldots, n$. If however $\eta_n < 0$, i.e. $\xi_n < t_n$, then $\frac{t_n(t_n+\xi_n)}{r_j^2+\xi_n^2} \leq 2\frac{t_n^2}{\xi_n^2}$, $\frac{r_j^2+t_n^2}{r_j^2+\xi_n^2} \leq \frac{t_n^2}{\xi_n^2}$, and $\frac{r_j^2+t_n^2+r_j^2+\xi_n^2}{r_j^2+\xi_n^2} = 2+\frac{t_n^2-\xi_n^2}{r_j^2+\xi_n^2} \leq 2+\frac{t_n^2-\xi_n^2}{\xi_n^2} = \frac{t_n^2+\xi_n^2}{\xi_n^2} \leq \frac{2t_n^2}{\xi_n^2}$ for all $j = 0, 1, \ldots, n$. Then from (4.19) and (4.14) it follows that

$$\begin{aligned}
|R_n''(\xi_n) - R_n''(t_n)| &= \frac{2}{\lambda_n^*}\left|\sum_{j=0}^{n-1} r_j\left(\frac{\xi_n}{[r_j^2+\xi_n^2]^2} - \frac{t_n}{[r_j^2+t_n^2]^2}\right) + \frac{1}{2}\left(\frac{\xi_n}{[1+\xi_n^2]^2} - \frac{t_n}{[1+t_n^2]^2}\right)\right| \\
&\leq \frac{\eta_n}{t_n}\cdot 5\cdot\max\left\{1, \frac{t_n^2}{\xi_n^2}\right\}^2 R_n''(t_n),
\end{aligned}$$

and since $\eta_n = o(t_n)$ and $\xi_n \sim t_n$ as $n \longrightarrow \infty$, we have

$$|R_n''(\xi_n) - R_n''(t_n)| = o\left(R_n''(t_n)\right).$$

The proof is completed by noting that the x dependence of $t_n = t_n(x)$ comes from the factor $c = -\frac{1}{2}\log x$ which is uniformly continuous in compact subsets of $(0,1)$. □

Lemma 4.23 *Let* $q(t) = (1+t^2)^{-1/2}$. *For all* $0 < t \neq t_n$,

$$\frac{d}{dt}\left(\frac{q(t)}{h_n'(t)}\right) < 0.$$

Proof. The derivative is

$$\frac{d}{dt}\left(\frac{q(t)}{h_n'(t)}\right) = \frac{q'(t)h_n'(t) - q(t)h_n''(t)}{h_n'(t)^2}. \tag{4.20}$$

We have $q(t)h_n''(t) > 0$ and $q'(t) = -t/(1+t^2)^{3/2} < 0$ for all $t > 0$ so the result is trivial for the case when $t > t_n$, since there $h_n'(t) > 0$ holds, ensuring a negative numerator above.

Now consider the case $0 < t < t_n$, for which $h_n'(t) < 0$. According to Lemma 4.16, we have $(1+t^2)h_n''(t) = (1+t^2)R_n''(t) > 2t|R_n'(t)| = -2tR_n'(t)$, and thus

$$\begin{aligned} th_n'(t) + (1+t^2)h_n''(t) &> th_n'(t) - 2tR_n'(t) \\ &= -th_n'(t) + 2tc \\ &> 0, \end{aligned}$$

where in the last step, we use $h_n'(t) < 0$ and $c = -\frac{1}{2}\log x > 0$. Since $q'(t) = -t(1+t^2)^{-3/2}$, it follows that the numerator in (4.20) satisfies

$$q'(t)h_n'(t) - q(t)h_n''(t) = -\frac{1}{(1+t^2)^{3/2}}\left[th_n'(t) + (1+t^2)h_n''(t)\right] < 0,$$

and we are done. □

The following is a standard integral (see [38, p. 97]).

Lemma 4.24

$$\int_0^\infty e^{is^2}\,ds = \frac{\sqrt{\pi}}{2}e^{i\frac{\pi}{4}}.$$

4.3.3 Estimation of the integral

The formula (3.2) can be written as

$$L_n(\Lambda; x) = \frac{1}{\pi\sqrt{x}} \operatorname{Im}\left[\int_0^\infty \frac{e^{i2\lambda_n^* h_n(t,x)}}{\sqrt{1+t^2}} dt\right], \tag{4.21}$$

where $h_n(t,x) = R_n(t) - \frac{t}{2}\log x$, so we need to estimate the integral

$$I_x(n) := \int_0^\infty \frac{e^{i2\lambda_n^* h_n(t,x)}}{\sqrt{1+t^2}} dt.$$

Here, we shall assume the Müntz condition $\sum_{j=0}^\infty \frac{1}{\lambda_j^*} = \infty$, in which case Lemma 4.14 states that $h_n(t,x)$ has a unique stationary point $t_n \in (0,\infty)$ for all n large enough. We split the integral $I_x(n)$ up in three parts; a central integral around t_n and the two integrals on each side of the point.

Recall from Lemma 4.20 that locally uniformly for x in $(0,1)$, $\lambda_n^* t_n^2 R_n''(t_n) \longrightarrow \infty$ as $n \longrightarrow \infty$ if (4.2) is satisfied. For a given Müntz space $M(\Lambda)$, define the sequence (well defined for all n large enough)

$$\eta_n = \left[\frac{\log\left(\lambda_n^* t_n^2 R_n''(t_n)\right)}{\lambda_n^* R_n''(t_n)}\right]^{1/2}. \tag{4.22}$$

Then we have

$$\frac{\eta_n^2}{t_n^2} = \frac{\log\left(\lambda_n^* t_n^2 R_n''(t_n)\right)}{\lambda_n^* t_n^2 R_n''(t_n)},$$

and hence

$$\eta_n = o(t_n), \qquad n \longrightarrow \infty. \tag{4.23}$$

Furthermore, we have

$$\eta_n \sqrt{\lambda_n^* R_n''(t_n)} = \sqrt{\log\left(\lambda_n^* t_n^2 R_n''(t_n)\right)} \longrightarrow \infty, \qquad n \longrightarrow \infty. \tag{4.24}$$

We note that the estimates below will hold uniformly for x in compact subsets of $(0,1)$. We mention this here, but neglect to repeat it in the proofs. This is trivial when we have explicit bounds, but otherwise it follows from the uniform continuity of $c = -\frac{1}{2}\log x$ on such sets.

First we consider the central integral:

Lemma 4.25 *Let $\Lambda = \{\lambda_j\}$ be a sequence of numbers such that the Müntz condition $\sum_{j=0}^{\infty} \frac{1}{\lambda_j^*} = \infty$ and (4.2) are satisfied. Then as $n \longrightarrow \infty$,*

$$\int_{t_n-\eta_n}^{t_n+\eta_n} \frac{e^{i2\lambda_n^* h_n(t)}}{\sqrt{1+t^2}} dt \sim \sqrt{\frac{\pi}{\lambda_n^* R_n''(t_n)(1+t_n^2)}} e^{i[2\lambda_n^* h_n(t_n)+\pi/4]},$$

where t_n is the unique stationary point of $h_n(t) = h_n(t;x)$ and η_n is defined in (4.22). This holds locally uniformly for x in $(0,1)$.

Proof. We expand the phase function $h_n(t)$ about t_n and write (recall $h_n^{(r)} = R_n^{(r)}$ for $r = 2, 3, \dots$)

$$h_n(t) = h_n(t_n) + \frac{R_n''(t_n)}{2}(t-t_n)^2 + \frac{R_n^{(3)}(\xi_n)}{3!}(t-t_n)^3, \tag{4.25}$$

with $\xi_n = \xi_n(t)$ between t_n and t. First we have

$$\int_{t_n}^{t_n+\eta_n} \frac{e^{i2\lambda_n^* h_n(t)}}{\sqrt{1+t^2}} dt = \frac{1}{\sqrt{1+t_n^2}} \int_{t_n}^{t_n+\eta_n} e^{i2\lambda_n^* h_n(t)} dt + \epsilon(n),$$

where

$$\epsilon(n) = \int_{t_n}^{t_n+\eta_n} \left[\frac{1}{\sqrt{1+t^2}} - \frac{1}{\sqrt{1+t_n^2}} \right] e^{i2\lambda_n^* h_n(t)} dt.$$

Then using the expansion (4.25), we can go on and write

$$\int_{t_n}^{t_n+\eta_n} \frac{e^{i2\lambda_n^* h_n(t)}}{\sqrt{1+t^2}} dt = \frac{e^{i2\lambda_n^* h_n(t_n)}}{\sqrt{1+t_n^2}} \int_{t_n}^{t_n+\eta_n} e^{i\lambda_n^* R_n''(t_n)(t-t_n)^2} dt + \delta(n) + \epsilon(n), \tag{4.26}$$

where

$$\delta(n) = \frac{1}{\sqrt{1+t_n^2}} \int_{t_n}^{t_n+\eta_n} e^{i2\lambda_n^* h_n(t)} \left[1 - e^{i\lambda_n^* \frac{R_n^{(3)}(\xi_n)}{3}(t-t_n)^3} \right] dt,$$

and $\xi_n = \xi_n(t)$ is between t_n and $t_n + \eta_n$ for all t. Using the substitution $s = \sqrt{\lambda_n^* R_n''(t_n)}(t-t_n)$, we can write the main contribution term as

$$\begin{aligned} \int_{t_n}^{t_n+\eta_n} e^{i\lambda_n^* R_n''(t_n)(t-t_n)^2} dt &= \frac{1}{\sqrt{\lambda_n^* R_n''(t_n)}} \int_0^{\eta_n \sqrt{\lambda_n^* R_n''(t_n)}} e^{is^2} ds \\ &= \frac{1}{\sqrt{\lambda_n^* R_n''(t_n)}} \left[\frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{4}} + o(1) \right], \end{aligned} \tag{4.27}$$

where in the last step, we have used Lemma 4.24 and (4.24). To summarize, combining (4.26) and (4.27) we have

$$\int_{t_n}^{t_n+\eta_n} \frac{e^{i2\lambda_n^* h_n(t)}}{\sqrt{1+t^2}} dt = \frac{\sqrt{\pi}}{2} \frac{e^{i[2\lambda_n^* h_n(t_n)+\pi/4]}}{\sqrt{\lambda_n^* R_n''(t_n)(1+t_n^2)}} [1+o(1)] + \delta(n) + \epsilon(n).$$

It remains to show that $\delta(n)$ and $\epsilon(n)$ are $o\left([\lambda_n^* R_n''(t_n)(1+t_n^2)]^{-1/2}\right)$ as $n \longrightarrow \infty$.

First we estimate $\epsilon(n)$:

$$|\epsilon(n)| \leq \int_{t_n}^{t_n+\eta_n} \left| \frac{1}{\sqrt{1+t^2}} - \frac{1}{\sqrt{1+t_n^2}} \right| dt \leq \eta_n \frac{\sqrt{1+(t_n+\eta_n)^2} - \sqrt{1+t_n^2}}{\sqrt{(1+(t_n+\eta_n)^2)(1+t_n^2)}}.$$

Then using $\eta_n = o(t_n)$ and the inequality $\sqrt{1+(a+\eta)^2} - \sqrt{1+a^2} \leq \eta$, we see that

$$|\epsilon(n)| = \mathcal{O}\left(\frac{\eta_n^2}{1+t_n^2}\right) = \mathcal{O}\left(\frac{\log(\lambda_n^* t_n^2 R_n''(t_n))}{\lambda_n^* R_n''(t_n)(1+t_n^2)}\right)$$

Then, using the inequalities $[R_n''(t_n)(1+t_n^2)]^{-1} \leq [2ct_n]^{-1}$ and $t_n R_n''(t_n) < 2c$ from Corollary 4.19, we obtain

$$\begin{aligned} \left[\lambda_n^* R_n''(t_n)(1+t_n^2)\right]^{1/2} |\epsilon(n)| &= \mathcal{O}\left(\frac{\log(\lambda_n^* t_n^2 R_n''(t_n))}{\sqrt{\lambda_n^* R_n''(t_n)(1+t_n^2)}}\right) \\ &= \mathcal{O}\left(\frac{\log(\lambda_n^* t_n)}{\sqrt{\lambda_n^* t_n}}\right) \\ &= o(1), \end{aligned}$$

as required. In the last step we have used the fact that $\lambda_n^* t_n \longrightarrow \infty$ as $n \longrightarrow \infty$, as shown in Lemma 4.20.

Now we estimate the error term $\delta(n)$. We have

$$\begin{aligned} |\delta(n)| &\leq \frac{1}{\sqrt{1+t_n^2}} \int_{t_n}^{t_n+\eta_n} \left| 1 - e^{i\lambda_n^* \frac{R_n^{(3)}(\xi_n)}{3}(t-t_n)^3} \right| dt \\ &\leq \frac{1}{\sqrt{1+t_n^2}} \frac{\lambda_n^*}{3} \int_{t_n}^{t_n+\eta_n} |R_n^{(3)}(\xi_n)(t-t_n)^3| dt \\ &= \frac{1}{\sqrt{1+t_n^2}} \lambda_n^* \eta_n^4 \frac{1}{12} \left| R_n^{(3)}(\nu_n) \right|, \end{aligned}$$

for some $\nu_n \in [t_n, t_n+\eta_n]$. Applying Lemmas 4.18 and 4.22 (note $\nu_n \sim t_n$ since

$\eta_n = o(t_n)$) gives $\left|R_n^{(3)}(\nu_n)/3\right| \le R_n''(\nu_n)/\nu_n \sim R_n''(t_n)/t_n$ as $n \longrightarrow \infty$, and hence

$$\begin{aligned} |\delta(n)| &= \mathcal{O}\left(\frac{1}{\sqrt{1+t_n^2}}\lambda_n^*\eta_n^4\frac{R_n''(t_n)}{t_n}\right) \\ &= \mathcal{O}\left(\frac{\lambda_n^*}{\sqrt{1+t_n^2}}\frac{R_n''(t_n)}{t_n}\left[\frac{\log(\lambda_n^* t_n^2 R_n''(t_n))}{\lambda_n^* R_n''(t_n)}\right]^2\right) \\ &= \mathcal{O}\left(\frac{1}{\sqrt{1+t_n^2}}\frac{[\log(\lambda_n^* t_n^2 R_n''(t_n))]^2}{t_n\lambda_n^* R_n''(t_n)}\right). \end{aligned}$$

Therefore,

$$\left[\lambda_n^* R_n''(t_n)(1+t_n^2)\right]^{1/2}|\delta(n)| = \mathcal{O}\left(\frac{[\log(\lambda_n^* t_n^2 R_n''(t_n))]^2}{\sqrt{\lambda_n^* t_n^2 R_n''(t_n)}}\right) = o(1),$$

and thus $\delta(n) = o\left([\lambda_n^* h_n''(t_n)(1+t_n^2)]^{-1/2}\right)$, as needed.

The proof is completed by noting that we can treat the left side of the central integral on $(t_n - \eta_n, t_n)$ the same way, and the same estimates hold. ☐

It remains to estimate the tail integrals, which are dealt with in the following two lemmas.

Lemma 4.26 *Let $\Lambda = \{\lambda_j\}$ be a sequence of numbers such that the Müntz condition* (4.1) *is satisfied. Then, as $n \longrightarrow \infty$,*

$$\int_{t_n+\eta_n}^{\infty} \frac{e^{i2\lambda_n^* h_n(t)}}{\sqrt{1+t^2}}dt = \mathcal{O}\left(\frac{1}{\lambda_n^* h_n'(t_n+\eta_n)\sqrt{1+t_n^2}}\right),$$

and

$$\int_0^{t_n-\eta_n} \frac{e^{i2\lambda_n^* h_n(t)}}{\sqrt{1+t^2}}dt = \mathcal{O}\left(\frac{1}{\lambda_n^* |h_n'(t_n-\eta_n)|\sqrt{1+t_n^2}}\right),$$

where t_n is the unique stationary point of $h_n(t)$ and η_n is defined in (4.22). *This holds locally uniformly for x in* $(0,1)$.

Proof. Let $A_n = t_n + \eta_n$ and $q(t) = (1+t^2)^{-1/2}$. Recall that $h_n'(t) > 0$ for $t > t_n$, and $h_n''(t) > 0$ for all t. Hence on (t_n, ∞), $h_n(t)$ has an inverse function and we can use the change of variables $s = h_n(t)$, $ds = h_n'(t)dt$. Then using integration by parts, we can write

$$\begin{aligned}
\int_{A_n}^{\infty} \frac{e^{i2\lambda_n^* h_n(t)}}{\sqrt{1+t^2}} dt &= \int_{A_n}^{\infty} e^{i2\lambda_n^* h_n(t)} q(t) dt \\
&= \left. \frac{e^{i2\lambda_n^* h_n(t)}}{i2\lambda_n^*} \frac{q(t)}{h_n'(t)} \right|_{A_n}^{\infty} - \frac{1}{i2\lambda_n^*} \int_{A_n}^{\infty} \frac{d}{dt} \left\{ \frac{q(t)}{h_n'(t)} \right\} e^{i2\lambda_n^* h_n(t)} dt \\
&= \frac{i e^{i2\lambda_n^* h_n(A_n)}}{2\lambda_n^*} \frac{q(A_n)}{h_n'(A_n)} + \frac{i}{2\lambda_n^*} \int_{A_n}^{\infty} \frac{d}{dt} \left\{ \frac{q(t)}{h_n'(t)} \right\} e^{i2\lambda_n^* h_n(t)} dt,
\end{aligned} \tag{4.28}$$

where we have used

$$\lim_{t\to\infty} \frac{q(t)}{h_n'(t)} = \frac{1}{c} \lim_{t\to\infty} \frac{1}{\sqrt{1+t^2}} = 0.$$

By Lemma 4.23, we have

$$\frac{d}{dt}\left(\frac{q(t)}{h_n'(t)} \right) < 0, \tag{4.29}$$

for all $t \neq t_n$, so

$$\int_{A_n}^{\infty} \left| \frac{d}{dt} \left\{ \frac{q(t)}{h_n'(t)} \right\} \right| dt = -\int_{A_n}^{\infty} \frac{d}{dt} \left\{ \frac{q(t)}{h_n'(t)} \right\} dt = \frac{q(A_n)}{h_n'(A_n)}. \tag{4.30}$$

Then, from (4.28), we deduce the estimate

$$\left| \int_{A_n}^{\infty} \frac{e^{i2\lambda_n^* h_n(t)}}{\sqrt{1+t^2}} dt \right| \le \frac{1}{2\lambda_n^*} \left(\frac{q(A_n)}{h_n'(A_n)} + \int_{A_n}^{\infty} \left| \frac{d}{dt} \left\{ \frac{q(t)}{h_n'(t)} \right\} \right| \right) = \frac{q(A_n)}{\lambda_n^* h_n'(A_n)},$$

and the first result follows.

Now consider the second tail integral, and let $B_n = t_n - \eta_n$. On the interval $(0, B_n)$, we have $h_n'(t) < 0$, so we can proceed as in the proof above and write

$$\begin{aligned}
\int_0^{B_n} \frac{e^{i2\lambda_n^* h_n(t)}}{\sqrt{1+t^2}} dt &= \int_0^{B_n} e^{i2\lambda_n^* h_n(t)} q(t) dt \\
&= \left. \frac{e^{i2\lambda_n^* h_n(t)}}{i2\lambda_n^*} \frac{q(t)}{h_n'(t)} \right|_0^{B_n} - \frac{1}{i2\lambda_n^*} \int_0^{B_n} \frac{d}{dt} \left\{ \frac{q(t)}{h_n'(t)} \right\} e^{i2\lambda_n^* h_n(t)} dt \\
&= \frac{1}{i2\lambda_n^*} \left[\frac{q(B_n) e^{i2\lambda_n^* h_n(B_n)}}{h_n'(B_n)} - \frac{e^{i2\lambda_n^* h_n(0)}}{h_n'(0)} - \int_0^{B_n} \frac{d}{dt} \left\{ \frac{q(t)}{h_n'(t)} \right\} e^{i2\lambda_n^* h_n(t)} dt \right].
\end{aligned}$$

The same way as in (4.30), we have

$$\int_0^{B_n} \left| \frac{d}{dt} \left\{ \frac{q(t)}{h_n'(t)} \right\} \right| dt = -\int_0^{B_n} \frac{d}{dt} \left\{ \frac{q(t)}{h_n'(t)} \right\} dt = \frac{1}{h_n'(0)} - \frac{q(B_n)}{h_n'(B_n)},$$

and this yields the estimate

$$
\begin{aligned}
\left|\int_0^{B_n} \frac{e^{i(2\lambda_n+1)h_n(t)}}{\sqrt{1+t^2}} dt\right| &\leq \frac{1}{2\lambda_n^*}\left[\frac{q(B_n)}{|h_n'(B_n)|} + \frac{1}{|h_n'(0)|} + \int_0^{B_n}\left|\frac{d}{dt}\left\{\frac{q(t)}{h_n'(t)}\right\}\right| dt\right] \\
&= \frac{1}{\lambda_n^*}\left[\frac{q(B_n)}{|h_n'(B_n)|} + \frac{1}{|h_n'(0)|}\right] \\
&= \mathcal{O}\left(\frac{1}{\lambda_n^*|h_n'(B_n)|\sqrt{1+t_n^2}}\right),
\end{aligned}
$$

where in the last step we use

$$
\frac{1}{|h_n'(0)|} = \frac{q(0)}{|h_n'(0)|} \leq \frac{q(B_n)}{|h_n'(B_n)|},
$$

which follows from (4.29). □

It turns out that the conditions of Lemma 4.25, for the central integral, are precisely the ones that we need in order for the tail integrals to be insignificant relative to the main term.

Lemma 4.27 *Let $\Lambda = \{\lambda_j\}$ be a sequence of numbers such that the Müntz condition $\sum_{j=0}^{\infty} \frac{1}{\lambda_j^*} = \infty$ and (4.2) are satisfied. Then as $n \longrightarrow \infty$,*

$$
\int_{t_n+\eta_n}^{\infty} \frac{e^{i(2\lambda_n+1)h_n(t)}}{\sqrt{1+t^2}} dt = o\left(\frac{1}{\sqrt{\lambda_n^* R_n''(t_n)(1+t_n^2)}}\right).
$$

and

$$
\int_0^{t_n-\eta_n} \frac{e^{i(2\lambda_n+1)h_n(t)}}{\sqrt{1+t^2}} dt = o\left(\frac{1}{\sqrt{\lambda_n^* R_n''(t_n)(1+t_n^2)}}\right),
$$

where t_n is the unique stationary point of $h_n(t)$ and η_n is defined in (4.22). This holds locally uniformly for x in $(0,1)$.

Proof. We can expand $h_n'(t)$ about the stationary point t_n and write

$$
h_n'(t_n+\eta_n) = h_n'(t_n) + h_n''(\xi_n)\eta_n = R_n''(\xi_n)\eta_n,
$$

with ξ_n between t_n and $t_n+\eta_n$. According to (4.24), we have $\eta_n^{-1} = o\left(\sqrt{\lambda_n R_n''(t_n)}\right)$, so using the result of the previous lemma yields

$$\begin{aligned}
\int_{t_n+\eta_n}^{\infty} \frac{e^{i(2\lambda_n+1)h_n(t)}}{\sqrt{1+t^2}}dt &= o\left(\frac{\sqrt{\lambda_n^* R_n''(t_n)}}{\lambda_n^* R_n''(\xi_n)\sqrt{1+t_n^2}}\right) \\
&= o\left(\frac{1}{\sqrt{\lambda_n^* R_n''(t_n)(1+t_n^2)}}\frac{R_n''(t_n)}{R_n''(\xi_n)}\right)
\end{aligned}$$

Since $\eta_n = o(t_n)$, Lemma 4.22 gives $R_n''(t_n)/R_n''(\xi_n) = \mathcal{O}(1)$ as $n \longrightarrow \infty$, and the first result follows. The estimate for the second tail integral is proved in the same way. □

4.3.4 General results on asymptotics

At this point, we can prove our most general result on the asymptotics of the Müntz orthogonal polynomials.

Proof of Theorem 4.2. The result follows directly from combining the results of Lemmas 4.25 and 4.27 and using the identity (4.21):

$$\begin{aligned}
L_n(\Lambda; x) &= \frac{1}{\pi\sqrt{x}}\mathrm{Im}\left[\sqrt{\frac{\pi}{\lambda_n^* R_n''(t_n)(1+t_n^2)}}e^{i[2\lambda_n^* h_n(t_n)+\pi/4]}\left(1+o(1)\right)\right] \\
&= \frac{\sin\left(2\lambda_n^* h_n(t_n)+\pi/4\right)}{\sqrt{\pi x \lambda_n^* R_n''(t_n)(1+t_n^2)}} + o\left(\frac{1}{\sqrt{\lambda_n^* R_n''(t_n)(1+t_n^2)}}\right) \\
&= \frac{\cos\left(2\lambda_n^* h_n(t_n)-\pi/4\right)}{\sqrt{\pi x \lambda_n^* R_n''(t_n)(1+t_n^2)}} + o\left(\frac{1}{\sqrt{\lambda_n^* R_n''(t_n)(1+t_n^2)}}\right).
\end{aligned}$$

□

Using our estimates for t_n and $R_n''(t_n)$ as $n \longrightarrow \infty$, we can now prove Theorems 4.5 and 4.7.

Proof of Theorem 4.5. First note that $\sigma_n \le T_n$ and $\sigma_n \le {\lambda_0^*}^{-1}(S_n/\lambda_n^*)$ for all n, so if $\sigma_n \longrightarrow \infty$, then both (4.1) and (4.2) are satisfied. Hence Theorem 4.2 applies. Also Corollary 4.19 gives $t_n^2 \sim 2\sigma_n/|\log x|$ and $t_n R_n''(t_n) \sim |\log x|$ as $n \longrightarrow \infty$, and therefore

$$R_n''(t_n)(1+t_n^2) \sim t_n R_n''(t_n) \cdot t_n \sim \sqrt{2|\log x|\sigma_n}.$$

Since $\lambda_n^* \sqrt{\sigma_n} = \sqrt{S_n}$, the result now follows from Theorem 4.2. □

Proof of Theorem 4.6. Here we assume that $S_n/{\lambda_n^*}^4 = \sigma_n/{\lambda_n^*}^2 \longrightarrow \infty$ so in particular $\lim_{n\to\infty} \sigma_n = \infty$. This is therefore a special case of Theorem 4.5, so (4.4) holds. Therefore, we only need to show that

$$2\lambda_n^* h_n(t_n) = 2\lambda_n^* R_n(t_n) + \lambda_n^* t_n |\log x| = \sqrt{2S_n |\log x|} + o(1) \tag{4.31}$$

as $n \longrightarrow \infty$ (recall $\Sigma_n = 2S_n$). We know from Corollary 4.19 that in this case, $t_n^2 \sim 2\sigma_n/|\log x|$. The same corollary gives the inequality

$$\frac{2\sigma_n}{t_n^2+1} \le |\log x| \le \frac{2\sigma_n}{t_n^2}.$$

Since $2\sigma_n/t_n^2 - 2\sigma_n/(t_n^2+1) = 2\sigma_n/t_n^2(1+t_n^2)$, this implies that

$$\left|\frac{2\sigma_n}{t_n^2} - |\log x|\right| \le \frac{2\sigma_n}{t_n^2(1+t_n^2)} = \mathcal{O}\left(\frac{1}{\sigma_n}\right).$$

Multiplying through by $(\lambda_n^* t_n)^2$ then yields

$$\left|2S_n - (\lambda_n^* t_n)^2 |\log x|\right| = \mathcal{O}\left(\frac{(\lambda_n^* t_n)^2}{\sigma_n}\right) = \mathcal{O}\left({\lambda_n^*}^2\right). \tag{4.32}$$

Therefore

$$\begin{aligned} \lambda_n^* t_n |\log x| - \sqrt{2S_n|\log x|} &= |\log x| \frac{(\lambda_n^* t_n)^2 |\log x| - 2S_n}{\lambda_n^* t_n |\log x| + \sqrt{2S_n |\log x|}} \\ &= \mathcal{O}\left(\frac{{\lambda_n^*}^2}{\sqrt{S_n}}\right). \end{aligned}$$

This takes care of the second term in (4.31).

Now we look at $2\lambda_n^* R_n(t_n)$. Since $t_n \longrightarrow \infty$ and $\lambda_j^*/\lambda_n^* \leq 1$ for each $j = 0, 1, \ldots, n$, we get the estimate

$$\begin{aligned}
\frac{2S_n}{\lambda_n^* t_n} - 2\lambda_n^* R_n(t_n) &= 2\sum_{j=0}^{n-1}\left(\frac{\lambda_j^*}{\lambda_n^* t_n} - \arctan\frac{\lambda_j^*}{\lambda_n^* t_n}\right) + \left(\frac{1}{t_n} - \arctan\frac{1}{t_n}\right) \\
&\leq \frac{2}{3}\sum_{j=0}^{n-1}\left(\frac{\lambda_j^*}{\lambda_n^* t_n}\right)^3 + \frac{1}{3}\left(\frac{1}{t_n}\right)^3 \\
&\leq \frac{1}{3t_n^3}\frac{S_n}{\lambda_n^*} \\
&= \mathcal{O}\left(\frac{{\lambda_n^*}^2}{\sqrt{S_n}}\right)
\end{aligned}$$

where in the last step we used $t_n^3 \asymp \left(S_n/{\lambda_n^*}^2\right)^{3/2}$. Also, using (4.32) gives

$$\begin{aligned}
\frac{2S_n}{\lambda_n^* t_n} - \lambda_n^* t_n |\log x| &= \frac{2S_n - (\lambda_n^* t_n)^2 |\log x|}{\lambda_n^* t_n} \\
&= \mathcal{O}\left(\frac{{\lambda_n^*}^2}{\sqrt{S_n}}\right).
\end{aligned}$$

Combining the estimates above and using the hypothesis ${\lambda_n^*}^4/S_n = o(1)$ confirms (4.31). $\square$

Proof of Corollary 4.7. By Corollary 4.19, $(1+t_n^2)R_n''(t_n) > 2ct_n$ for all n, so we have

$$\frac{1}{\lambda_n^* R_n''(t_n)(1+t_n^2)} = \mathcal{O}\left(\frac{1}{\lambda_n^* t_n}\right), \qquad n \longrightarrow \infty. \tag{4.33}$$

If $\sigma_n \longrightarrow \infty$, then $t_n^2 \asymp \sigma_n$ according to Corollary 4.19, and the second bound then follows from (4.33) using $\lambda_n^* \sqrt{\sigma_n} = \sqrt{S_n}$ for all n. If $\lambda_n \asymp n$ as $n \longrightarrow \infty$, then $t_n \asymp 1$ by Corollary 4.21 and (4.34) below, and thus $L_n(\Lambda; x) = \mathcal{O}\left({\lambda_n^*}^{-1/2}\right) = \mathcal{O}\left(n^{-1/2}\right)$. $\square$

4.4 *Proofs for the case* $\lambda_n \sim \rho n$, $\rho > 0$

Here we consider the special case when the sequence $\Lambda = \{\lambda_j\}$ asymptotically satisfies an arithmetic progression, i.e. such that

$$\lim_{n\to\infty}\frac{n}{\lambda_n} = \rho,$$

for some constant $\rho > 0$. First we will see how it will be sufficient to examine the case when $\rho = 1$. Let $\Pi = \{\mu_j\}$, with $\mu_j = \rho\lambda_j$ for all j, so that

$$\lim_{n\to\infty} \frac{n}{\mu_n} = 1.$$

Then, in (2.21), using the change of variables $t = u^\rho$ yields

$$\frac{1}{2\lambda_n + 1} = \int_0^1 L_n(\Lambda; t)^2 dt = \rho \int_0^1 L_n(\Lambda; u^\rho)^2 u^{\rho-1} du,$$

and thus

$$\frac{1}{2\mu_n + \rho} = \int_0^1 L_n(\Lambda; u^\rho)^2 u^{\rho-1} du.$$

Then, since $L_n(\Lambda; u^\rho) \in M_n(\Pi)$, it follows that

$$L_n(\Lambda; t) = L_n^{(\rho-1)}(\Pi; t^{1/\rho}), \qquad t \in (0, 1],$$

where $L_n^{(\rho-1)}(\Pi; x)$ is the nth Müntz-Jacobi orthogonal polynomial associated with Π and the weight $x^{\rho-1}$, as defined in (2.26). Then according to (2.27), for all $t \in (0, 1]$,

$$\begin{aligned} L_n(\Lambda; t) &= \left(t^{1/\rho}\right)^{-\frac{1}{2}(\rho-1)} L_n\left(\Pi + \frac{1}{2}(\rho - 1); t^{1/\rho}\right) \\ &= t^{\frac{1}{2}(1/\rho-1)} L_n\left(\Pi + \frac{1}{2}(\rho - 1); t^{1/\rho}\right), \end{aligned} \tag{4.34}$$

and $\mu_n + (\rho - 1)/2 \sim n$ as $n \longrightarrow \infty$. Hence it suffices to look at the case when $\rho = 1$.

When $\lambda_n \sim n$ as $n \longrightarrow \infty$, we expect that $R_n(t)$ should behave like

$$R(t) = \int_0^1 \arctan\frac{u}{t} du = \arctan\frac{1}{t} - \frac{t}{2}\log\left(1 + \frac{1}{t^2}\right), \tag{4.35}$$

for which the first two derivatives are

$$\begin{aligned} R'(t) &= -\int_0^1 \frac{u}{u^2 + t^2} du = -\frac{1}{2}\log\left(1 + \frac{1}{t^2}\right), \\ R''(t) &= 2t\int_0^1 \frac{u}{(u^2 + t^2)^2} du = \frac{1}{t(1 + t^2)}. \end{aligned}$$

Lemma 4.28 *Let $x = \cos^2\theta$, $\theta \in (0, \pi/2)$, and $\gamma_\theta = \cot\theta > 0$. Then*

$$\begin{aligned} h_n(\gamma_\theta) &= \theta + (R_n - R)(\gamma_\theta), \\ h_n'(\gamma_\theta) &= (R_n' - R')(\gamma_\theta), \\ h_n''(\gamma_\theta) &= \tan\theta\sin^2\theta + (R_n'' - R'')(\gamma_\theta). \end{aligned}$$

Proof. First, we have

$$\frac{1}{2}\log x + \frac{1}{2}\log\left(1+\frac{1}{\gamma_\theta^2}\right) = \frac{1}{2}\log\left[\cos^2\theta\frac{1}{\cos^2\theta}\right] = 0.$$

Hence, using the formulas above, we can write

$$\begin{aligned}
h_n(\gamma_\theta) &= R_n(\gamma_\theta) - \frac{\gamma_\theta}{2}\log x \\
&= R_n(\gamma_\theta) - R(\gamma_\theta) + \left(\arctan\frac{1}{\gamma_\theta} - \frac{\gamma_\theta}{2}\log\left(1+\frac{1}{\gamma_\theta^2}\right)\right) - \frac{\gamma_\theta}{2}\log x \\
&= \arctan(\tan\theta) + R_n(\gamma_\theta) - R(\gamma_\theta) \\
&= \theta + (R_n - R)(\gamma_\theta).
\end{aligned}$$

Moreover,

$$\begin{aligned}
h_n'(\gamma_\theta) &= R_n'(\gamma_\theta) - \frac{1}{2}\log x \\
&= R_n'(\gamma_\theta) - R'(\gamma_\theta) - \frac{1}{2}\log\left(1+\frac{1}{\gamma_\theta^2}\right) - \frac{1}{2}\log x \\
&= R_n'(\gamma_\theta) - R'(\gamma_\theta),
\end{aligned}$$

and finally

$$\begin{aligned}
h_n''(\gamma_\theta) &= R_n''(\gamma_\theta) \\
&= R_n''(\gamma_\theta) - R''(\gamma_\theta) + \frac{1}{\cot\theta(1+\cot^2\theta)} \\
&= \tan\theta\sin^2\theta + R_n''(\gamma_\theta) - R''(\gamma_\theta).
\end{aligned}$$

□

Lemma 4.29 *Let $\{\lambda_k\}$ be an increasing sequence of real numbers such that $\lim_{n\to\infty} n/\lambda_n = 1$, and let $f \in C^2[0,1]$ with $f(x) = \mathcal{O}(x)$ as $x \longrightarrow 0$. Then for each constant β, as $n \longrightarrow \infty$,*

$$\begin{aligned}
&\left|\int_0^1 f(x)dx - \frac{1}{\lambda_n^*}\left[\sum_{j=0}^{n-1} f\left(\frac{\lambda_j^*}{\lambda_n^*}\right) + \frac{1}{2}f(1)\right]\right| \\
&= \mathcal{O}\left(\frac{1}{n^2}\sum_{j=0}^{n-1}|j+\beta-\lambda_j| + \frac{|n+\beta-\lambda_n|}{n}\right) + \mathcal{O}\left(\frac{1}{n^2}\right).
\end{aligned}$$

Proof. Let $K_r := \sup_{s\in[0,1]} |f^{(r)}(s)|$ for $r = 1, 2$. First we look at the midpoint Riemann sum for the integral with partition $\{0, 1/n^*, 2/n^*, \ldots, n/n^*, 1\}$. For each $j = 0, 1, \ldots, n-1$, j^*/n^* is the midpoint of $[j/n^*, (j+1)/n^*]$, so

$$\begin{aligned}
\left|\int_{\frac{j}{n^*}}^{\frac{j+1}{n^*}} f(x)dx - \frac{1}{n^*} f\left(\frac{j^*}{n^*}\right)\right| &= \left|\int_{\frac{j}{n^*}}^{\frac{j+1}{n^*}} \left[f(x) - f\left(\frac{j^*}{n^*}\right) - f'\left(\frac{j^*}{n^*}\right)\left(x - \frac{j^*}{n^*}\right)\right] dx\right| \\
&\leq \frac{K_2}{2} \int_{\frac{j}{n^*}}^{\frac{j+1}{n^*}} \left(x - \frac{j^*}{n^*}\right)^2 dx \\
&= \frac{K_2}{24 n^{*3}}.
\end{aligned}$$

The interval $[n/n^*, 1]$ is of length $1/2n^*$, so

$$\left|\int_{\frac{n}{n^*}}^{1} f(x)dx - \frac{1}{2\lambda_n^*} f(1)\right| \leq \int_{\frac{n}{n^*}}^{1} |f(x) - f(1)|\, dx \leq \frac{K_1}{8 n^{*2}}.$$

It follows that

$$\left|\int_0^1 f(x)dx - \frac{1}{n^*}\left[\sum_{j=0}^{n-1} f\left(\frac{j^*}{n^*}\right) + \frac{1}{2} f(1)\right]\right| = \mathcal{O}\left(\frac{1}{n^2}\right), \qquad n \longrightarrow \infty. \tag{4.36}$$

For each $j = 0, 1, \ldots, n-1$, we have

$$\begin{aligned}
\frac{1}{n^*} f\left(\frac{j^*}{n^*}\right) - \frac{1}{\lambda_n^*} f\left(\frac{\lambda_j^*}{\lambda_n^*}\right) &= \frac{1}{n^*}\left[f\left(\frac{j^*}{n^*}\right) - f\left(\frac{\lambda_j^*}{n^*}\right)\right] \\
&\quad + \frac{1}{n^*}\left[f\left(\frac{\lambda_j^*}{n^*}\right) - f\left(\frac{\lambda_j^*}{\lambda_n^*}\right)\right] + f\left(\frac{\lambda_j^*}{\lambda_n^*}\right)\left[\frac{1}{n^*} - \frac{1}{\lambda_n^*}\right] \\
&= \frac{f'(\xi_{j,n})}{n^{*2}}(j - \lambda_j) + \frac{f'(\tau_{j,n})}{n^{*2}\lambda_n^*}(\lambda_n - n)\lambda_j^* + f\left(\frac{\lambda_j^*}{\lambda_n^*}\right)\frac{\lambda_n - n}{n^*\lambda_n^*},
\end{aligned}$$

for some $\xi_{j,n}, \tau_{j,n} \in (0,1)$. We also have $f(1)/2n^* - f(1)/2\lambda_n^* = f(1)(\lambda_n - n)/2n^*\lambda_n^*$, and it follows that

$$\begin{aligned}
&\left|\frac{1}{n^*}\left[\sum_{j=0}^{n-1} f\left(\frac{j^*}{n^*}\right) + \frac{1}{2} f(1)\right] - \frac{1}{\lambda_n^*}\left[\sum_{j=0}^{n-1} f\left(\frac{\lambda_j^*}{\lambda_n^*}\right) + \frac{1}{2} f(1)\right]\right| \\
&\leq \frac{K_1}{n^{*2}} \sum_{j=0}^{n-1} |j - \lambda_j| + \frac{K_1 |\lambda_n - n|}{\lambda_n^* n^{*2}} \sum_{j=0}^{n-1} \lambda_j^* + \frac{|\lambda_n - n|}{\lambda_n^*} \frac{1}{n^*}\left(\sum_{j=0}^{n-1} f\left(\frac{\lambda_j^*}{\lambda_n^*}\right) + \frac{1}{2} f(1)\right) \\
&= \mathcal{O}\left(\frac{1}{n^2}\sum_{j=0}^{n-1} |j - \lambda_j| + \frac{|\lambda_n - n|}{n}\right).
\end{aligned}$$

In the steps above one can easily replace j with $j+\beta$, for any constant β. Then the Riemann sum partition above becomes

$$\left\{\frac{\beta}{(n+\beta)^*}, \frac{1+\beta}{(n+\beta)^*}, \frac{2+\beta}{(n+\beta)^*}, \ldots, \frac{n+\beta}{(n+\beta)^*}, 1\right\}$$

so one needs to verify that on the interval $[0, \beta/(n+\beta)^*]$ in (4.36), we get

$$\int_0^{\frac{\beta}{(n+\beta)^*}} f(x)dx = \mathcal{O}\left(\frac{1}{n^2}\right),$$

which holds since $f(x) = \mathcal{O}(x)$ as $x \longrightarrow 0$ (similar if $\beta < 0$ since $f\left(\beta^*/(n+\beta)^*\right)/n^* = \mathcal{O}\left(1/n^2\right)$). $\square$

Combining the two lemmas above yields the following result:

Corollary 4.30 *Let $x = \cos^2\theta$, $\gamma_\theta = \cot\theta$, for $\theta \in (0, \pi/2)$, and suppose $\lambda_n \sim n$ as $n \longrightarrow \infty$. Then $h_n(\gamma_\theta) \longrightarrow \theta$, $h_n'(\gamma_\theta) \longrightarrow 0$ and $h_n''(\gamma_\theta) \longrightarrow \tan\theta\sin^2\theta$, as $n \longrightarrow \infty$, and for all the limits, the rate of convergence is*

$$\mathcal{O}\left(\frac{1}{n^2}\sum_{j=0}^{n-1}|j+\beta-\lambda_j| + \frac{|n+\beta-\lambda_n|}{n}\right) + \mathcal{O}\left(\frac{1}{n^2}\right)$$

for each constant $\beta > -1/2$.

Proof. For a fixed $t > 0$, the integrand in each of the integrals in $R(t)$, $R'(t)$ and $R''(t)$, namely $\arctan(u/t)$, $u/(u^2+t^2)$ and $u/(u^2+t^2)^2$, are $\mathcal{O}(u)$ as $u \longrightarrow 0$, so the result of the previous lemma holds in each case. Then the result follows directly from Lemma 4.28. $\square$

Lemma 4.31 *Let $x = \cos^2\theta$, $\theta \in (0, \pi/2)$, and suppose that $\lambda_n \sim n$ as $n \longrightarrow \infty$. Then $h_n''(t_n) = \tan\theta\sin^2\theta + o(1)$, and moreover, if $\lambda_n = n + \beta + o(n^{1-\frac{1}{2^{N+1}}})$ for some integer N and constant β, then $h_n(t_n) = h_n(\gamma_{n,N}) + o\left(1/n\right)$ as $n \longrightarrow \infty$, where $\gamma_n = \gamma_{n,N}$ is defined recursively by*

$$\gamma_0 = \cot\theta, \qquad \gamma_k = \gamma_{k-1} - \frac{h_n'(\gamma_k)}{h_n''(\gamma_k)}, \qquad k = 1, 2, \ldots, N.$$

If $\lambda_n = n + \beta + o(1)$, then $h_n(t_n) = \theta + o(1/n)$ as $n \longrightarrow \infty$.

Proof. For all n, we have

$$h_n'(\gamma_0) = h_n'(t_n) + h_n''(\eta_n)(\gamma_0 - t_n) = h_n''(\eta_n)(\gamma_0 - t_n),$$

where η_n is between γ_0 and t_n. By Corollary 4.21, the condition $\lambda_n \sim n$ ensures that t_n is bounded above and below. Then according to Corollary 4.17, $h_n''(\eta_n)$ is bounded above and below, and thus

$$|t_n - \gamma_0| \asymp |h_n'(\gamma_0)|, \qquad n \longrightarrow \infty. \tag{4.37}$$

Expanding h_n'' about t_n then yields

$$h_n''(\gamma_0) - h_n''(t_n) = h_n^{(3)}(\nu_n)(\gamma_0 - t_n) = \mathcal{O}\left(|h_n'(\gamma_0)|\right), \tag{4.38}$$

where we have used the fact that $h_n^{(3)}(t)$ is bounded above for t bounded above (Lemma 4.18). The first result now follows from $|h_n'(\gamma_0)| = o(1)$.

Using the expansion $h_n'(t) = h_n'(\gamma_0) + h_n''(\gamma_0)(t - \gamma_0) + h_n^{(3)}(\xi_n)(t - \gamma_0)^2/2$ yields

$$t_n - \gamma_0 = -\frac{h_n'(\gamma_0)}{h_n''(\gamma_0)} - \frac{h_n^{(3)}(\xi_n)}{2h_n''(\gamma_0)}(t_n - \gamma_0)^2 = -\frac{h_n'(\gamma_0)}{h_n''(\gamma_0)} + \mathcal{O}\left(|h_n'(\gamma_0)|^2\right),$$

where we have used (4.37) (and again $h_n''(t)$ is bounded above and below and by Corollary 4.17 for t bounded and thus also $h_n^{(3)}(t)$ is bounded above by Lemma 4.18). We can rewrite this as $t_n - \gamma_1 = \mathcal{O}\left(|h_n'(\gamma_0)|^2\right)$, where $\gamma_1 = \gamma_0 - h_n'(\gamma_0)/h_n''(\gamma_0)$. Now expanding the same way about γ_1 then yields

$$t_n - \gamma_1 = -\frac{h_n'(\gamma_1)}{h_n''(\gamma_1)} + \mathcal{O}\left(|h_n'(\gamma_0)|^4\right), \qquad n \longrightarrow \infty,$$

and continuing this way gives

$$t_n - \gamma_N = \mathcal{O}\left(|h_n'(\gamma_0)|^{2^N}\right), \qquad n \longrightarrow \infty,$$

for each $N \in \mathbb{N}$. Then we obtain the following estimate:

$$\begin{aligned} h_n(\gamma_N) - h_n(t_n) &= h_n'(t_n)(\gamma_N - t_n) + \frac{h_n''(\nu_n)}{2}(\gamma_N - t_n)^2 \\ &= \mathcal{O}\left(|h_n'(\gamma_0)|^{2^{N+1}}\right), \end{aligned}$$

and since $|h_n'(\gamma_0)| = \mathcal{O}\left(\sum_{j=0}^{n-1} |j+\beta-\lambda_j|/n^2 + |n+\beta-\lambda_n|/n\right) + \mathcal{O}\left(1/n^2\right)$, we have $h_n(\gamma_N) - h_n(t_n) = o\left(1/n\right)$ if $\lambda_n - n - \beta = o\left(n^{1-2^{-N-1}}\right)$.

If $\lambda_n - n - \beta = o(1)$, then we have $h_n(\gamma_0) - h_n(t_n) = \mathcal{O}(|h_n'(t_0)|^2) = o\left(1/n^2\right)$, and by Corollary (4.30), $h_n(\gamma_0) - \theta = \mathcal{O}\left(\sum_{j=0}^{n-1} |j+\beta-\lambda_j|/n^2 + |n+\beta-\lambda_n|/n\right) = o\left(1/n\right)$, and the last statement of the lemma holds. □

We are now ready to prove Theorems 4.3 and 4.4.

Proof of Theorem 4.4. We are assuming that $\lambda_n \sim n/\rho$, as $n \longrightarrow \infty$. Define the sequence $\Pi = \{\mu_n\}$ by letting

$$\mu_n = \rho\lambda_n + \frac{1}{2}(\rho - 1), \qquad n \in \mathbb{N}_0.$$

Then from (4.34), we have (note that this is not the same Π as in (4.34))

$$\begin{aligned} L_n(\Lambda; \cos^{2\rho}\theta) &= (\cos\theta)^{\rho(1/\rho-1)} L_n\left(\Pi; \cos^2\theta\right) \\ &= (\cos\theta)^{1-\rho} L_n\left(\Pi; \cos^2\theta\right), \end{aligned} \tag{4.39}$$

for each $\theta \in (0, \pi/2)$.

In the following, we put Λ and Π in the superscript, to indicate which sequence we are using. Since $\mu_n^* = \rho\lambda_n^*$ for all n, we have $h_n^{(\Pi)}(t) = h_n^{(\Lambda)}(t)/\rho$ for all t. By hypothesis, $\mu_n = n + o\left(n^{1-\delta}\right) = n + o\left(n^{1-2^{-N-1}}\right)$, and thus according to Theorem

4.2 and using Lemma 4.31,

$$\begin{aligned} L_n\left(\Pi; \cos^2\theta\right) &= \frac{\cos\left[2\mu_n^* h_n^{(\Pi)}(t_n) - \pi/4\right]}{\cos\theta\sqrt{\pi\mu_n^* h_n^{\prime\prime(\Pi)}(t_n)(1+t_n^2)}} + o\left(\frac{1}{\sqrt{\mu_n^* h_n''(t_n)(1+t_n^2)}}\right) \\ &= \frac{\cos\left[2\rho\lambda_n^* \frac{1}{\rho} h_n^{(\Lambda)}(\gamma_N) - \pi/4\right]}{\cos\theta\sqrt{\pi n\tan\theta\sin^2\theta(1+\cot^2\theta)}} + o\left(\frac{1}{\sqrt{n}}\right) \\ &= \frac{\cos\left[2\lambda_n^* h_n^{(\Lambda)}(\gamma_N) - \pi/4\right]}{\sqrt{\pi n\sin\theta\cos\theta}} + o\left(\frac{1}{\sqrt{n}}\right), \end{aligned}$$

as $n \longrightarrow \infty$. The result now follows from (4.39). □

Proof of Theorem 4.3. (i) Using the same notation as in the proof above, if $\lambda_n = (n+\beta/2)/\rho + o(1)$, then $\mu_n = n + (\beta+\rho-1)/2 + o(1)$ as $n \longrightarrow \infty$. Then, according to Lemma 4.31, $h_n^{(\Pi)}(t_n) = \theta + o\left(1/n\right)$ and we can follow the proof above to reach the conclusion.

(ii) This is a special case of Theorem 4.4, with $\delta = 1/2$ and $N = 0$. □

4.5 Proofs for the Müntz-Christoffel function

Proof of Theorem 4.8. We know from Theorem 4.3 (i), that locally uniformly for θ in $(0, \pi/2)$,

$$L_n(\Lambda; \cos^{2\rho}\theta) = \frac{\cos\left([2n+\beta+\rho]\theta - \pi/4\right)}{\sqrt{\pi\rho\lambda_n^* \sin\theta\cos^{2\rho-1}\theta}} + o\left(\frac{1}{\sqrt{n}}\right)$$

as $n \longrightarrow \infty$. Using the identities (2.25) and (2.22), we can write the nth Müntz-Christoffel function as

$$\lambda_n^{-1}(\Lambda; x) = \sum_{k=0}^{n} |L_k^*(\Lambda; x)|^2 = 2\sum_{k=0}^{n} \lambda_k^* |L_k(\Lambda; x)|^2.$$

Using the trigonometric identities $2\cos^2\phi = 1+\cos 2\phi$ and $\cos(\phi-\pi/2) = \sin\phi$, it follows that

$$
\begin{aligned}
\lambda(M_n(\Lambda);\cos^{2\rho}\theta)^{-1} &= 2\sum_{k=0}^{n}\left(\frac{\cos^2([2k+\beta+\rho]\theta-\pi/4)}{\pi\rho\sin\theta\cos^{2\rho-1}\theta}+o(1)\right)\\
&= \frac{1}{\pi\rho\sin\theta\cos^{2\rho-1}\theta}\sum_{k=0}^{n}(1+\cos(2[2k+\beta+\rho]\theta-\pi/2))+o(n)\\
&= \frac{1}{\pi\rho\sin\theta\cos^{2\rho-1}\theta}\left(n+1+\sum_{k=0}^{n}\sin([2k+\beta+\rho]2\theta)\right)+o(n). \qquad (4.40)
\end{aligned}
$$

We have

$$
\begin{aligned}
\sum_{k=0}^{n} e^{i2\theta(2k+\beta+\rho)} &= e^{i2\theta(\beta+\rho)}\sum_{k=0}^{n}\left(e^{i4\theta}\right)^k\\
&= e^{i2\theta(\beta+\rho)}\frac{e^{i4\theta(n+1)}-1}{e^{i4\theta}-1}\\
&= e^{i2\theta(\beta+\rho-1)}\frac{e^{i4\theta(n+1)}-1}{2i\sin 2\theta},
\end{aligned}
$$

and therefore $\sum_{k=0}^{n}\sin([2k+\beta+\rho]2\theta) = \mathcal{O}(1)$ uniformly for θ in compact subsets of $(0,\pi/2)$. Thus (4.40) yields

$$\lim_{n\to\infty} n\lambda(M_n(\Lambda);\cos^{2\rho}\theta) = \pi\rho\sin\theta\cos^{2\rho-1}\theta,$$

and the result follows by letting $x=\cos^{2\rho}\theta$. $\square$

Proof of Theorem 4.9. If $\lambda_n \asymp n^a$, then $S_n \asymp n^{a+1}$, and thus it follows from Corollary 4.7 that $\lambda_n^* L_n(\Lambda;x)^2 = \mathcal{O}\left(\lambda_n^* S_n^{-1/2}\right) = \mathcal{O}\left(n^{(a-1)/2}\right)$. It follows that

$$
\begin{aligned}
\lambda_n^{-1}(\Lambda;x) &= 2\sum_{k=0}^{n}\lambda_k^*|L_k(\Lambda;x)|^2\\
&= \mathcal{O}\left(\sum_{k=0}^{n}k^{(a-1)/2}\right)\\
&= \mathcal{O}\left(n^{(a+1)/2}\right).
\end{aligned}
$$

$\square$

CHAPTER V

ZERO SPACING ASYMPTOTICS AND ESTIMATES FOR THE SMALLEST AND LARGEST ZEROS

Using our formula (3.2), in this chapter we introduce new estimates for the largest and smallest zeros of the Müntz-Legendre polynomials. The endpoint limit asymptotics of Theorem 4.1 give us the exact asymptotics of the largest zeros. We also obtain a sharp lower bound for the smallest zero. Furthermore, we determine the asymptotic behavior of the spacing of consecutive zeros of the Müntz-Legendre polynomials $L_n(\Lambda; x)$ via the strong asymptotics obtained in Chapter 4. In Theorems 4.2, 4.4 and 4.5, the error limits hold uniformly for x in compact subsets of $(0,1)$ and this implies that the zeros of $L_n(\Lambda; x)$ can be approximated by the zeros of the main asymptotic term. The results are from the papers [51] "Zero spacing of Müntz orthogonal polynomials" and [50] "On the smallest and largest zeros of Müntz-Legendre polynomials."

5.1 Main results

Recall from Theorem 2.20 that the nth Müntz-Legendre polynomial $L_n(\Lambda; x)$ has precisely n zeros on $(0,1)$. Let $l_{n,n} < l_{n-1,n} < \cdots < l_{1,n}$ denote these zeros.

First we get a global bound for the smallest zero.

Theorem 5.1 *Let $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ be a sequence of real numbers greater than $-\frac{1}{2}$. Then*

$$\exp\left(-2\sum_{j=0}^{n-1}\frac{1}{\lambda_j+\frac{1}{2}} - \frac{1}{\lambda_n+\frac{1}{2}}\right) < l_{n,n},$$

Remark This considerably improves the lower bound in (2.29) as can be seen from the inequality

$$\sum_{j=0}^{n-1}\frac{1}{\lambda_j+\frac{1}{2}} + \frac{1}{2}\frac{1}{\lambda_n+\frac{1}{2}} \le \frac{n+\frac{1}{2}}{\lambda_{\min}^{(n)}+\frac{1}{2}}.$$

An important corollary is that for non-dense Müntz spaces, $L_n(\Lambda; x)$ has no zeros close to 0 (compare to [7, Section 6.2, E.2]).

Corollary 5.2 *Let* $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ *be a sequence of real numbers greater than* $-1/2$ *such that*

$$T := \sum_{k=0}^{\infty} \frac{1}{\lambda_k + 1/2} < \infty.$$

Then the smallest zero of $L_n(\Lambda; x)$ *for all* n *is greater than*

$$\exp(-2T) > 0.$$

Next we obtain the asymptotic behavior of the largest zeros.

Theorem 5.3 *Let* $\Lambda : -1/2 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ *be a sequence of real numbers and let*

$$\Sigma_n := \sum_{k=0}^{n-1} (2\lambda_k + 1) + \frac{2\lambda_n + 1}{2}. \tag{5.1}$$

If Λ *satisfies the regularity condition* $\lim_{n\to\infty} \Sigma_n/(2\lambda_n + 1) = \infty$ *then for fixed* $k \geq 1$,

$$\lim_{n\to\infty} \Sigma_n |\log l_{k,n}| = \left(\frac{j_k}{2}\right)^2$$

where j_k *denotes the* k*th positive zero of the Bessel function* J_0 *as defined in* (2.10). *The error term is* $\mathcal{O}\left(\sqrt{(2\lambda_n + 1)/\Sigma_n}\right)$ *as* $n \longrightarrow \infty$.

Remark Theorem 5.3 gives $l_{1,n} \sim \exp\left(-j_1^2/4\Sigma_n\right)$ which, in the asymptotic sense, improves the upper bound in (2.29). We trivially have

$$2\Sigma_n \leq (2n + 1)(2\lambda_{\max}^{(n)} + 1).$$

We now turn our attention to the bulk of the zeros and consider their spacings. For the general case when the Müntz space is dense, the zero spacing of the Müntz orthogonal polynomials $L_n(\Lambda; x)$ depends nicely on the stationary point $t_n(x)$ of the phase function $R_n(t) - \frac{t}{2}\log x$ of the representation (3.2).

Theorem 5.4 *Let Λ be a sequence as in Theorem 4.2, and let $t_n(x)$ be the unique number that satisfies $R_n'(t_n(x)) = \frac{1}{2}\log x$. If locally uniformly for x, y in $(0,1)$,*

$$t_n(x) \asymp t_n(y), \qquad n \longrightarrow \infty,$$

then locally uniformly for zeros $l_{k,n}, l_{k+1,n}$ of $L_n(\Lambda; x)$ in $(0,1)$,

$$\lim_{n\to\infty} \frac{\lambda_n^* t_n(l_{k,n})}{\pi l_{k,n}} (l_{k,n} - l_{k+1,n}) = 1.$$

If $\lambda_n \sim \alpha n$ as $n \longrightarrow \infty$, then $R_n(t)$ is asymptotic to a mid-point Riemann sum, and the stationary point $t_n(x)$ converges as $n \longrightarrow \infty$.

Theorem 5.5 *Let $\Lambda : -\frac{1}{2} < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ be a sequence of numbers such that*

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \alpha$$

for some $\alpha > 0$. Then locally uniformly for zeros $l_{k,n}, l_{k+1,n}$ of $L_n(\Lambda; x)$ in $(0,1)$,

$$\lim_{n\to\infty} \frac{\alpha}{\pi} \frac{n\,(l_{k,n} - l_{k+1,n})}{\sqrt{l_{k,n}^{2-\alpha}(1 - l_{k,n}^{\alpha})}} = 1.$$

It is interesting to compare this result to the zero distribution given in (2.30). The next result covers all cases when $\lambda_n = o(n)$ as $n \longrightarrow \infty$. Then we know the asymptotic growth of $t_n(x)$ as $n \longrightarrow \infty$.

Theorem 5.6 *Let $\Lambda : -\frac{1}{2} < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ be a sequence of numbers such that*

$$\lim_{n\to\infty} \frac{1}{(2\lambda_n + 1)^2} \sum_{k=0}^{n} (2\lambda_k + 1) = \infty.$$

Then locally uniformly for zeros $l_{k,n}, l_{k+1,n}$ of $L_n(\Lambda; x)$ in $(0,1)$,

$$\lim_{n\to\infty} \frac{1}{\pi} \left(\sum_{j=0}^{n} (2\lambda_j + 1) \right)^{1/2} \frac{l_{k,n} - l_{k+1,n}}{l_{k,n} |\log l_{k,n}|^{1/2}} = 1.$$

Note that the result of Theorem 5.6 is essentially the case when $\alpha \longrightarrow 0$ in Theorem 5.5. If $\lambda_n \sim \alpha n$ as $n \longrightarrow \infty$, then $\left(\sum_{j=0}^{n}(2\lambda_j+1)\right)^{1/2} \sim \sqrt{\alpha}n$ as $n \longrightarrow \infty$, and furthermore we have for $x \in (0,1)$,

$$\lim_{\alpha\to 0}\left(\frac{x^{2-\alpha}(1-x^{\alpha})}{\alpha}\right)^{1/2} = x|\log x|^{1/2}.$$

In Figure 2 we graph the limit functions of the zero spacing asymptotics for different cases. Recall from (2.30) that if $\alpha \longrightarrow 0$ or ∞, then the zero probability distribution is Dirac delta at $x=0$ and 1 respectively.

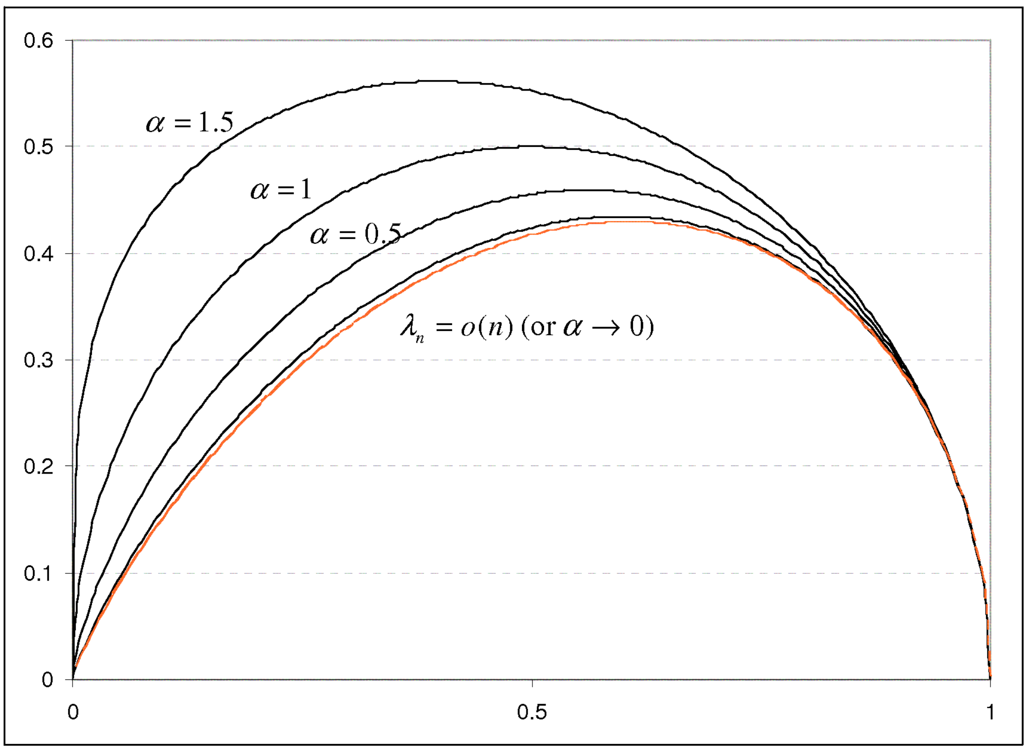

Figure 2: The graphs show the limit functions for the scaled spacing of consecutive zeros for different cases. This means that we show the graph of $f(x)$ for which $\lim_{n\to\infty}\frac{1}{\pi}\left(\sum_{j=0}^{n}(2\lambda_j+1)\right)^{1/2}(l_{k,n}-l_{k+1,n}) = f(l_{k,n})$. In the case of Theorem 5.5, $f(x) = \sqrt{x^{2-\alpha}(1-x^{\alpha})/\alpha}$ and if $\lambda_n = o(n)$ (or $\alpha \longrightarrow 0$) we have $f(x) = x\sqrt{|\log x|}$.

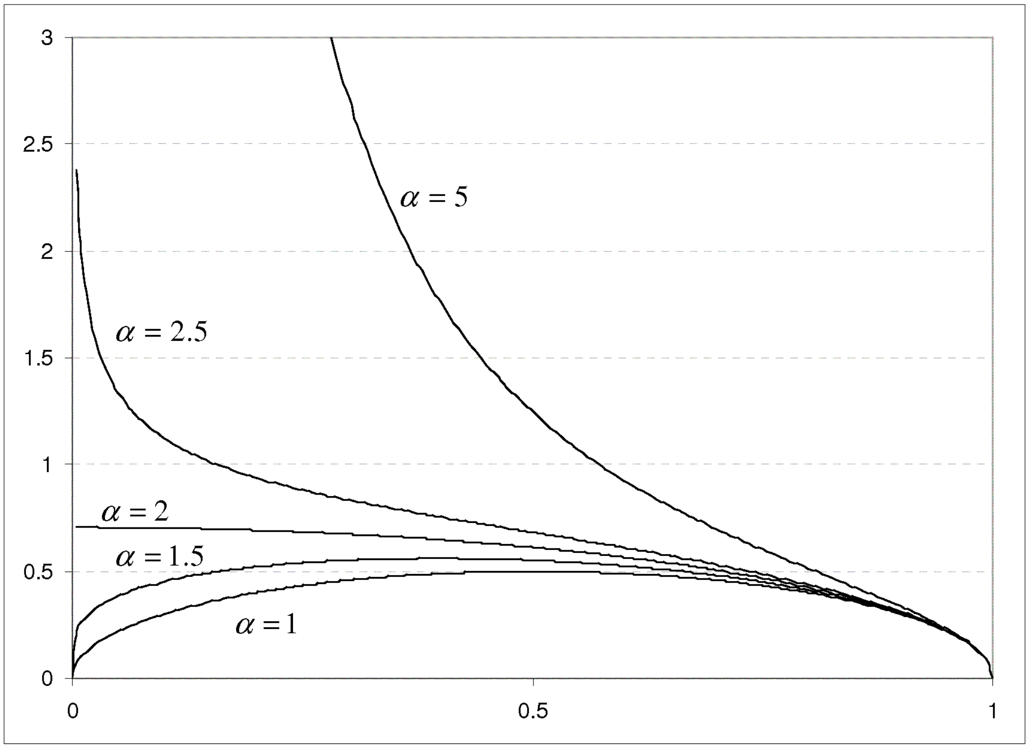

Figure 3: The same as for Figure 2 but for large values of α. As we know from Corollary 5.2 there are no zeros close to 0 when $\alpha \longrightarrow \infty$, hence the spacing diverges.

5.2 Proofs on smallest and largest zeros

Proof of Theorem 5.1. Recall that for each n we let $\lambda_n^* := \lambda_n + 1/2$ and $T_n := \sum_{k=0}^{n-1} \frac{1}{\lambda_k^*} + \frac{1}{2\lambda_n^*}$. We choose any $R_n \geq T_n$, and let $x_n = e^{-2R_n}$ so that $x_n \in (0, e^{-2T_n}]$. We need to show that $L_n(\Lambda; x_n) \neq 0$.

According to (3.5), we can write

$$L_n(\Lambda; x_n) = \frac{(-1)^n e^{R_n}}{\pi} \int_0^\infty \frac{\cos p_n(t)}{\left(\lambda_n^{*\,2} + t^2\right)^{1/2}} dt \tag{5.2}$$

where

$$p_n(t) = 2R_n t - \Phi_n(t).$$

The first two derivatives of p_n are $p_n'(t) = 2R_n - \Phi_n'(t)$ and $p_n''(t) = -\Phi_n''(t)$ where

$$\Phi_n'(t) = 2\sum_{k=0}^{n-1} \frac{\lambda_k^*}{\lambda_k^{*\,2} + t^2} + \frac{\lambda_n^*}{\lambda_n^{*\,2} + t^2}$$

and

$$\Phi_n''(t) = -2t\left(2\sum_{k=0}^{n-1}\frac{\lambda_k^*}{\left[\lambda_k^{*2}+t^2\right]^2}+\frac{\lambda_n^*}{\left[\lambda_n^{*2}+t^2\right]^2}\right).$$

Since $\Phi_n'(0) = 2T_n$, we therefore have $p_n'(0) = 2(R_n - T_n) \geq 0$ and $p_n''(t) > 0$ for $t > 0$. It follows that p_n is a strictly increasing function on $[0,\infty)$ which maps $[0,\infty)$ onto $[0,\infty)$ (note that $\Phi_n(t) \leq \pi n + \pi/2$)

We can therefore use the substitution $u = p_n(t)$ in integral of (5.2), and this gives

$$\int_0^\infty \frac{\cos p_n(t)}{\left(\lambda_n^{*2}+t^2\right)^{1/2}}dt = \int_0^\infty \frac{\cos u}{q_n(u)}du \tag{5.3}$$

where $q_n(u)$ is determined by

$$q_n(u) = \left(\lambda_n^{*2}+t^2\right)^{1/2} p_n'(t)$$

Then $q_n(0) = 2\lambda_n^*(R_n - T_n)$ and since $\lim_{t\to\infty} p_n'(t) = 2R_n$ we have

$$\lim_{u\to\infty} q_n(u) = \lim_{t\to\infty}\left(\lambda_n^{*2}+t^2\right)^{1/2} p_n'(t) = \infty.$$

We show that $q_n(u)$ is strictly increasing: The chain rule gives

$$\begin{aligned} p_n'(t)q_n'(u) &= \frac{d}{dt}\left(\left(\lambda_n^{*2}+t^2\right)^{1/2} p_n'(t)\right) \\ &= \frac{tp_n'(t)+\left(\lambda_n^{*2}+t^2\right)p_n''(t)}{\left(\lambda_n^{*2}+t^2\right)^{1/2}}, \end{aligned}$$

and since $p_n'(t), p_n''(t) > 0$ for $t > 0$ it follows that $q_n'(u) > 0$ for $u > 0$.

Using a standard argument we can write (5.3) as an alternating series $\sum_{k=0}^\infty (-1)^k a_k$ with $a_k > a_{k+1} > 0$ and $a_k \longrightarrow 0$, and the alternating series test shows that $\int_0^\infty \frac{\cos u}{q_n(u)}du \neq 0$. The result follows. □

Before we prove Theorem 5.3, we need two lemmas. First we define the function

$$f_n(y) := L_n\left(e^{-y^2/4\Sigma_n}\right), \qquad y \geq 0.$$

Then according to Theorem 4.1, uniformly for bounded $y \geq 0$,

$$f_n(y) - J_0(y) = \mathcal{O}\left(\sqrt{\frac{\lambda_n^*}{\Sigma_n}}\right) = o(1), \qquad n \longrightarrow \infty. \tag{5.4}$$

For each n and $k = 1, 2, \ldots, n$, we can write the zeros of $L_n(x)$ on the form

$$l_{k,n} = e^{-r_{k,n}^2/4\Sigma_n}$$

for some $0 < r_{1,n} < r_{2,n} < \cdots < r_{n,n}$. These are precisely the zeros of f_n, i.e.

$$f_n(r_{k,n}) = 0, \qquad k = 1, 2 \ldots, n. \tag{5.5}$$

Below, let $\|\cdot\|_{[0,y]}$ denote the supremum norm over $[0, y]$.

Lemma 5.7 *For each n and $y \geq 0$,*

$$\|f_n'\|_{[0,y]} \leq \frac{y}{2} \sup_k \|f_k\|_{[0,y]} < \infty.$$

Proof. We recall the identity from Theorem 2.17,

$$xL_n'(x) = \lambda_n L_n(x) + \sum_{k=0}^{n-1} (2\lambda_k + 1) L_k(x).$$

It follows that

$$\begin{aligned} f_n'(y) &= -\frac{y}{2\Sigma_n} e^{-y^2/4\Sigma_n} L_n'(e^{-y^2/4\Sigma_n}) \\ &= -\frac{y}{2\Sigma_n} \left[\lambda_n L_n(e^{-y^2/4\Sigma_n}) + \sum_{k=0}^{n-1} (2\lambda_k + 1) L_k(e^{-y^2/4\Sigma_n}) \right] \\ &= -\frac{y}{2\Sigma_n} \left[\lambda_n f_n(y) + \sum_{k=0}^{n-1} (2\lambda_k + 1) f_k\left(y\sqrt{\Sigma_k/\Sigma_n} \right) \right]. \end{aligned} \tag{5.6}$$

Therefore, since $0 \leq y\sqrt{\Sigma_k/\Sigma_n} \leq y$ for all $k = 0, 1, \ldots, n$,

$$\begin{aligned} |f_n'(y)| &\leq \frac{y}{2\Sigma_n} \left[\lambda_n + \sum_{k=0}^{n-1} (2\lambda_k + 1) \right] \max_{0 \leq k \leq n} \|f_k\|_{[0,y]} \\ &\leq \frac{y}{2} \sup_k \|f_k\|_{[0,y]}. \end{aligned}$$

Since f_k is continuous on $[0, y]$ for each k, and $f_n(t) \longrightarrow J_0(t)$ uniformly for t bounded, it follows from the inequality $\|f_k\|_{[0,y]} \leq \|J_0\|_{[0,y]} + \|f_k - J_0\|_{[0,y]} = 1 + \|f_k - J_0\|_{[0,y]}$ that

$$\sup_k \|f_k\|_{[0,y]} < \infty.$$

The result now follows from the trivial inequality $\frac{t}{2}\sup_k \|f_k\|_{[0,t]} \le \frac{y}{2}\sup_k \|f_k\|_{[0,y]}$ for each $t \le y$. □

Lemma 5.8 *For each n and $y \ge 0$, we have*

$$\|f_n''\|_{[0,y]} \le \frac{1}{2}\left(1+\frac{y^2}{2}\right)\sup_k \|f_k\|_{[0,y]} < \infty.$$

In particular, the family $\{f_n''\}$ is uniformly bounded on bounded sets $[0,y]$.

Proof. Using the identity (5.6) for $f_n'(y)$, we obtain

$$\begin{aligned} f_n''(y) &= -\frac{1}{2\Sigma_n}\left[\lambda_n f_n(y) + \sum_{k=0}^{n-1}(2\lambda_k+1) f_k\left(y\sqrt{\frac{\Sigma_k}{\Sigma_n}}\right)\right] \\ &\quad -\frac{y}{2\Sigma_n}\left[\lambda_n f_n'(y) + \sum_{k=0}^{n-1}(2\lambda_k+1)\sqrt{\frac{\Sigma_k}{\Sigma_n}} f_k'\left(y\sqrt{\frac{\Sigma_k}{\Sigma_n}}\right)\right] \\ &= \frac{f_n'(y)}{y} - \frac{y}{2\Sigma_n}\left[\lambda_n f_n'(y) + \sum_{k=0}^{n-1}(2\lambda_k+1)\sqrt{\frac{\Sigma_k}{\Sigma_n}} f_k'\left(y\sqrt{\frac{\Sigma_k}{\Sigma_n}}\right)\right] \end{aligned}$$

If we let $A := \frac{1}{2}\sup_k \|f_k\|_{[0,y]}$, then since $0 \le y\sqrt{\Sigma_k/\Sigma_n} \le y$ for all n and $k = 0, 1, \ldots, n$, the lemma above gives

$$\left| f_k'\left(y\sqrt{\frac{\Sigma_k}{\Sigma_n}}\right)\right| \le \frac{y}{2}\sqrt{\frac{\Sigma_k}{\Sigma_n}}\sup_k \|f_k\|_{[0,y\sqrt{\Sigma_k/\Sigma_n}]} \le Ay.$$

It follows that

$$\begin{aligned} |f_n''(y)| &\le A + \frac{y}{2}\cdot\frac{\lambda_n + \sum_{j=0}^{n-1}(2\lambda_j+1)}{\Sigma_n} Ay \\ &\le \left(1+\frac{y^2}{2}\right)A. \end{aligned}$$

The result now follows from the trivial inequality $\sup_k \|f_k\|_{[0,t]} \le \sup_k \|f_k\|_{[0,y]} = 2A$ for each $t \le y$. □

Proof of Theorem 5.3. Let $0 < j_1 < j_2 < \cdots$ denote the zeros of J_0 on the positive axis. According to the interlacing property of the zeros, for fixed k, $\{r_{k,n}\}_n$ is a decreasing sequence bounded below by 0, and thus has a limit. Then from (5.4) it is clear that for each k,

$$\lim_{n\to\infty} r_{k,n} = j_m$$

for some integer $m = m(k) \geq 1$. By the intermediate value theorem, for n large enough, f_n has a zero close to each j_k. Therefore, its smallest zero $r_{1,n}$ necessarily has j_1 as limit.

We need to show that $r_{2,n}$ does not approach j_1 as well. Suppose to the contrary that

$$\lim_{n\to\infty} r_{2,n} = j_1.$$

Then by the mean value theorem, there exists some $c_n \in (r_{1,n}, r_{2,n})$ such that

$$f_n'(c_n) = 0 \tag{5.7}$$

and of course by hypothesis $c_n \longrightarrow j_1$ as $n \longrightarrow \infty$.

Define a point

$$a_n = j_1 + \delta_n$$

where the error δ_n is chosen so that

$$\sqrt{\lambda_n^*/\Sigma_n} = o(\delta_n) = o(1)$$

(say $\delta_n = \log(\lambda_n^*/\Sigma_n)$). Then, since $f_n(y) \longrightarrow J_0(y)$ uniformly for bounded y with error $\mathcal{O}(\lambda_n^*/\Sigma_n)$, and $J_0(j_1) = 0$, we have for some ξ_n between j_1 and a_n,

$$\begin{aligned} f_n(a_n) &= J_0(a_n) + f_n(a_n) - J_0(a_n) \\ &= J_0'(\xi_n)(a_n - j_1) + \mathcal{O}\left(\sqrt{\frac{\lambda_n^*}{\Sigma_n}}\right) \\ &= J_0'(j_1)\delta_n[1 + o(1)] \end{aligned} \tag{5.8}$$

as $n \longrightarrow \infty$ (it is well known, see Olver [Section §7.6]), that the zeros the Bessel functions are simple, so $J_0'(\xi_n) \longrightarrow J_0'(j_1) \neq 0$). On the other hand, using (5.4) again with $J_0(j_1) = 0$ yields

$$\begin{aligned} f_n(a_n) &= f_n(j_1) + f_n'(\nu_n)(a_n - j_1) \\ &= \mathcal{O}\left(\sqrt{\frac{\lambda_n^*}{\Sigma_n}}\right) + f_n'(\nu_n)\delta_n. \end{aligned} \tag{5.9}$$

for some ν_n between j_1 and a_n. Expanding f' about the point c_n from (5.7) gives

$$f_n'(\nu_n) = f_n''(\eta_n)(\nu_n - c_n)$$

for some η_n between ν_n and c_n, and according to Lemma 5.8, since $c_n, \nu_n \longrightarrow j_1$ as $n \longrightarrow \infty$, we have $f_n'(\nu_n) = o(1)$ as $n \longrightarrow \infty$. Therefore, (5.9) gives $f_n(a_n) = o(\delta_n)$, which contradicts (5.8). Hence $\lim_{n\to\infty} r_{2,n} \neq j_1$.

Since f_n has a zero close to j_2 for n large enough, it follows that $r_{2,n} \longrightarrow j_2$. Now we can repeat the proof for $r_{3,n}$ and so on, and we have established that $\lim_{n\to\infty} r_{k,n} = j_k$ for each fixed k. The result now follows from $-4\Sigma_n \log l_{k,n} = r_{k,n}^2$.

As for the error, a linear approximation yields

$$J_0(r_{k,n}) = J_0(r_{k,n}) - J_0(j_k) = J_0'(\xi_n)(r_{k,n} - j_k),$$

for some $\xi_{k,n}$ between $r_{k,n}$ and j_k, and thus since the zeros of J_0 are simple, (5.4) and (5.5) yield

$$r_{k,n} - j_k = \mathcal{O}\left(J_0(r_{k,n})\right) = \mathcal{O}\left(\sqrt{\frac{\lambda_n *}{\Sigma_n}}\right), \qquad n \longrightarrow \infty.$$

□

5.3 Proofs on zero spacing asymptotics

In Theorem 4.2, we determined the asymptotic behavior of the Müntz orthogonal polynomials $L_n(\Lambda; x)$ for $x \in (0,1)$ when $n \longrightarrow \infty$ under some mild conditions on the sequence of exponents $\{\lambda_k\}$. Here we are interested in the zeros of $L_n(\Lambda; x)$ on the interval, so we look at the phase function that appears in the main asymptotic term as a function of x. Namely if we let

$$\varphi_n(x) := 2\lambda_n^* R_n(t_n(x)) - \lambda_n^* t_n(x) \log x, \tag{5.10}$$

then we have

$$L_n(\Lambda; x) = \frac{\cos\left(\varphi_n(x) - \frac{\pi}{4}\right)}{\sqrt{x\pi\lambda_n^* R_n''(t_n)(1+t_n^2)}} + o\left(\frac{1}{\sqrt{\lambda_n^* R_n''(t_n)(1+t_n^2)}}\right), \tag{5.11}$$

where

$$R_n(t) = \frac{1}{\lambda_n^*}\left[\sum_{j=0}^{n-1} \arctan\frac{\lambda_j^*}{\lambda_n^* t} + \frac{1}{2}\arctan\frac{1}{t}\right],$$

and $t_n(x) \in (0, \infty)$ is the unique point such that $R_n'(t_n(x)) = \frac{1}{2}\log x$, i.e. it is determined implicitly by

$$-\log x = |\log x| = \frac{2}{\lambda_n^*}\left[\sum_{j=0}^{n-1} \frac{\frac{\lambda_j^*}{\lambda_n^*}}{\left(\frac{\lambda_j^*}{\lambda_n^*}\right)^2 + t_n(x)^2} + \frac{1}{2}\frac{1}{1+t_n(x)^2}\right]. \tag{5.12}$$

We recall from (4.19) that for all n and $x \in (0,1)$,

$$(\lambda_n^* t_n)^2(x) < \frac{2S_n}{|\log x|} < (\lambda_n^*)^2(t_n(x)^2 + 1), \tag{5.13}$$

where $S_n = \sum_{j=0}^{n-1}\lambda_j^* + \frac{\lambda_n^*}{2}$.

The most important and beautiful aspect of the asymptotic analysis in Chapter 4 is that the slope of the phase function $R_n(t)$ at the left endpoint satisfies

$$-R_n'(0) = \sum_{k=0}^{n-1}\frac{1}{\lambda_k^*} + \frac{1}{2\lambda_n^*} =: T_n \tag{5.14}$$

which determines (2.17) and hence the denseness of the space (and the existence of the stationary point $t_n(x)$ for all $x \in (0,1)$). Also we saw in (4.14) that the second derivative $R_n''(t)$ is a strictly positive function.

Let us now look at the function $t_n(x)$. Taking the derivative of $R_n'(t_n(x)) = \frac{1}{2}\log x$ with respect to x gives

$$t_n'(x) R_n''(t_n(x)) = \frac{1}{2x}$$

and since $R_n''(t)$ is positive, this shows that $t_n(x)$ is an increasing function of $x \in (0,1)$. Also it is clear from (5.12) that

$$\lim_{x \to 1^-} t_n(x) = \infty, \tag{5.15}$$

and since $R_n'(t_n(e^{-2T_n})) = \frac{1}{2} \log e^{-2T_n} = -T_n$, it follows from (5.14) that

$$t_n\left(e^{-2T_n}\right) = 0.$$

It turns out that the first derivative of the phase function $\varphi_n(x)$ from (5.10) is surprisingly simple: Using $R_n'(t_n(x)) = \frac{1}{2} \log x$ we obtain

$$\begin{aligned} \varphi_n'(x) &= \lambda_n^* \left[2t_n'(x) R_n'(t_n(x)) - t_n'(x) \log x - \frac{t_n(x)}{x} \right] \\ &= -\lambda_n^* \frac{t_n(x)}{x}. \end{aligned} \tag{5.16}$$

Since $t_n(x) > 0$, this shows that $\varphi_n(x)$ is strictly decreasing on $(0,1)$. We've seen that $t_n(e^{-2T_n}) = 0$, so

$$\varphi_n(e^{-2T_n}) = 2\lambda_n^* R_n(0) = \pi n^* = \pi n + \frac{\pi}{2},$$

and (5.15) gives

$$\lim_{x \to 1^-} \varphi_n(x) = \lim_{x \to 1^-} \lambda_n^* t_n(x) |\log x| = \lim_{x \to 1^-} \frac{(\lambda_n^* t_n)^2(x) |\log x|}{\lambda_n^* t_n(x)} = 0,$$

where in the last step we have used that $(\lambda_n^* t_n)^2(x) |\log x|$ is bounded with respect to x, as seen in (5.13). This gives the following:

Lemma 5.9 *The phase function $\varphi_n(x)$ defined in (5.10) maps $[e^{-2T_n}, 1)$ bijectively onto $\left(0, \pi n + \frac{\pi}{2}\right]$.*

The zeros of the main term in (5.11) are the solutions of the equations

$$\varphi_n(x) = \pi k - \frac{\pi}{4},$$

for some integer k. Lemma 5.9 shows that there exists a unique solution $x_{k,n}$ for each $k = 1, 2, \ldots, n$. For two consecutive zeros $x_{k+1,n} < x_{k,n}$, using first order Taylor approximation, we have

$$\pi = \varphi_n(x_{k+1,n}) - \varphi_n(x_{k,n}) = -\varphi_n'(\xi_{k,n})(x_{k,n} - x_{k+1,n}),$$

for some $\xi_{k,n} \in (x_{k+1,n}, x_{k,n})$. Then using (5.16), we can write

$$x_{k,n} - x_{k+1,n} = \frac{\pi \xi_{k,n}}{\lambda_n^* t_n(\xi_{k,n})}. \tag{5.17}$$

Assuming the Müntz condition (2.17), Lemma 4.20 yields

$$0 < 1 - \frac{x_{k+1,n}}{x_{k,n}} < \frac{\pi}{(\lambda_n^* t_n)(x_{k+1,n})} = o(1), \tag{5.18}$$

for $x_{k+1,n}, x_{k,n}$ in compact subsets of $(0, 1)$. It follows that locally in $(0, 1)$,

$$x_{k+1,n} \sim x_{k,n} \qquad n \longrightarrow \infty.$$

Hence, asymptotically, we can replace $\xi_{k,n}$ with $x_{k,n}$ in the denominator of (5.17), but we also need to show that $t_n(\xi_{k,n}) \sim t_n(x_{k,n})$. For that purpose, we first need the following lemma:

Lemma 5.10 *For $0 < x < y < 1$, we have*

$$\frac{R_n''(t_n(y))}{t_n(y)} \leq \frac{\log y - \log x}{t_n^2(y) - t_n^2(x)} \leq \frac{R_n''(t_n(x))}{t_n(x)}$$

Proof. For each $j = 0, 1, \ldots, n$, let $r_j := r_{j,n} := \lambda_j^*/\lambda_n^*$. Then using (5.12) and the notation (1.4), we can write

$$\begin{aligned}
\log \frac{y}{x} &= -\log x + \log y \\
&= \frac{2}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{r_k^2 + t_n^2(x)} - \frac{2}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{r_k^2 + t_n^2(y)} \\
&= \frac{2}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} r_k \left(\frac{1}{r_k^2 + t_n^2(x)} - \frac{1}{r_k^2 + t_n^2(y)} \right) \\
&= \left[t_n^2(y) - t_n^2(x) \right] \frac{2}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{\left[r_k^2 + t_n^2(x)\right]\left[r_k^2 + t_n^2(y)\right]}.
\end{aligned}$$

Recalling that

$$R_n''(t) = \frac{2t}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{[r_k^2 + t^2]^2},$$

and using $t_n(x) < t_n(y)$, we get

$$\left[t_n^2(y) - t_n^2(x)\right] \frac{R_n''(t_n(y))}{t_n(y)} \leq \log \frac{y}{x} \leq \left[t_n^2(y) - t_n^2(x)\right] \frac{R_n''(t_n(x))}{t_n(x)}.$$

□

Lemma 5.11 *Assume the hypothesis of Lemma 4.20, and furthermore that locally uniformly for* x, y *in* $(0, 1)$,

$$t_n(x) \asymp t_n(y), \qquad n \longrightarrow \infty.$$

Then, if $x_n - y_n = \mathcal{O}\left(1/\lambda_n^* t_n(x_n)\right)$ *for* x_n, y_n *locally uniformly in* $(0, 1)$,

$$t_n(x_n) \sim t_n(y_n), \qquad n \longrightarrow \infty.$$

Proof. According to Lemma 5.10, we have

$$\left|t_n^2(x_n) - t_n^2(y_n)\right| \leq \max\left\{\frac{t_n(x_n)}{R_n''(t_n(x_n))}, \frac{t_n(y_n)}{R_n''(t_n(y_n))}\right\} \left|\log \frac{x_n}{y_n}\right|$$

By hypothesis we have $|1 - x_n/y_n| = \mathcal{O}\left(1/\lambda_n^* t_n(x_n)\right) = \mathcal{O}\left(1/\lambda_n^* t_n(y_n)\right)$, so

$$\left|1 - \frac{t_n^2(y_n)}{t_n^2(x_n)}\right| = \mathcal{O}\left(\max\left\{\frac{1}{\lambda_n^* t_n^2(x_n) R_n''(t_n(x_n))}, \frac{1}{\lambda_n^* t_n^2(y_n) R_n''(t_n(y_n))} \frac{t_n^2(y_n)}{t_n^2(x_n)}\right\}\right).$$

If $x_n, y_n \in [a, b] \subset (0, 1)$, then $t_n(y_n)/t_n(x_n) \leq t_n(b)/t_n(a) = \mathcal{O}(1)$, and since $\lambda_n^* t_n^2 R_n''(t_n) \longrightarrow \infty$ locally uniformly in $(0, 1)$ by Lemma 4.20, this gives $t_n(x_n) \sim t_n(y_n)$ as $n \longrightarrow \infty$. □

Given the assumptions of the lemma above, since $x_{k,n} - x_{k+1,n} = \mathcal{O}\left(1/\lambda_n^* t_n\right)$ by (5.17), we have

$$x_{k,n} - x_{k+1,n} \sim \frac{\pi x_{k,n}}{\lambda_n^* t_n(x_{k,n})}, \qquad n \longrightarrow \infty, \tag{5.19}$$

for $x_{k+1,n}, x_{k,n}$ on compact subsets of $(0,1)$. It remains to be shown that we can replace $x_{k,n}$ with $l_{k,n}$, where $l_{n,n} < l_{n-1,n} < \cdots < l_{2,n} < l_{1,n}$ are the actual zeros of $L_n(\Lambda; x)$ on $(0,1)$.

Lemma 5.12 *Assume the hypothesis of Lemma 4.20, and furthermore that locally uniformly for x, y in $(0,1)$,*

$$t_n(x) \asymp t_n(y), \qquad n \longrightarrow \infty.$$

Then locally uniformly for $x_{k,n}$ in $(0,1)$,

$$l_{k,n} - x_{k,n} = o\left(\frac{1}{\lambda_n^* t_n(l_{k,n})}\right), \qquad n \longrightarrow \infty,$$

and it follows that $l_{k,n} - l_{k+1,n} \sim x_{k,n} - x_{k+1,n}$ as $n \longrightarrow \infty$.

Proof. According to Lemma 4.20, locally uniformly for x in $(0,1)$, we have $\lambda_n^* t_n(x) \longrightarrow \infty$ as $n \longrightarrow \infty$. Let $\{\delta_n\}$ be a sequence of numbers such that

$$\delta_n = o\left(\frac{1}{\lambda_n^* t_n}\right) = o(1), \qquad n \longrightarrow \infty \tag{5.20}$$

(by hypothesis $t_n(x) \asymp t_n(y)$ for x, y on compact subsets of $(0,1)$ so we can simply write t_n instead of $t_n(x)$).

For each $k = 1, 2, \ldots, n$, $x_{k,n}$ is determined by $\varphi_n(x_{k,n}) = \pi k - \pi/4$, and therefore

$$\begin{aligned} \cos\left(\varphi_n(x_{k,n}) - \frac{\pi}{4}\right) &= 0, \\ \sin\left(\varphi_n(x_{k,n}) - \frac{\pi}{4}\right) &= (-1)^{k-1}. \end{aligned}$$

We can write

$$\varphi_n(x_{k,n} + \delta_n) = \varphi_n(x_{k,n}) + \varphi_n'(\nu_{k,n})\delta_n$$

for some $\nu_{k,n}$ between $x_{k,n}$ and $x_{k,n} + \delta_n$. Using the double angle formula $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, and the identity (5.16) for φ_n', we obtain

$$\begin{aligned} \cos\left(\varphi_n(x_{k,n} + \delta_n) - \frac{\pi}{4}\right) &= \cos\left(\varphi_n(x_{k,n}) - \frac{\pi}{4} + \varphi_n'(\nu_{k,n})\delta_n\right) \\ &= -\sin\left(\varphi_n(x_{k,n}) - \frac{\pi}{4}\right)\sin\left(\delta_n\varphi_n'(\nu_{k,n})\right) \\ &= (-1)^{k-1}\sin\left(\delta_n \frac{\lambda_n^* t_n(\nu_{k,n})}{\nu_{k,n}}\right). \end{aligned}$$

Given (5.20), we trivially have $x_{k,n} \sim x_{k,n} + \delta_n$ locally uniformly for $x_{k,n}$ in $(0,1)$ and by Lemma 5.11, $t_n(x_{k,n}+\delta_n) \sim t_n(x_{k,n})$ locally uniformly in $(0,1)$. According to Lemma 4.22, we also have $R_n''(t_n(x_{k,n}+\delta_n)) \sim R_n''(t_n(x_{k,n}))$ locally uniformly in $(0,1)$. Then, using the asymptotics formula of Theorem 4.2, we obtain locally uniformly for $x_{k,n} \pm \delta_n$ in $(0,1)$,

$$\begin{aligned}
L_n(x_{k,n} \pm \delta_n) &= \frac{\cos\left(\varphi_n(x_{k,n} \pm \delta_n) - \frac{\pi}{4}\right)}{\sqrt{\pi(x_{k,n} \pm \delta_n)\lambda_n^* R_n''(t_n(x_{k,n} \pm \delta_n))(1+t_n^2(x_{k,n} \pm \delta_n))}} \\
&\qquad + o\left(\frac{1}{\sqrt{\lambda_n^* R_n''(t_n(x_{k,n} \pm \delta_n))(1+t_n^2(x_{k,n} \pm \delta_n))}}\right) \\
&= \frac{\cos\left(\varphi_n(x_{k,n} \pm \delta_n) - \frac{\pi}{4}\right)}{\sqrt{\pi x_{k,n}\lambda_n^* R_n''(t_n(x_{k,n}))(1+t_n^2(x_{k,n}))}} \\
&\qquad + o\left(\frac{1}{\sqrt{\lambda_n^* R_n''(t_n(x_{k,n}))(1+t_n^2(x_{k,n}))}}\right) \\
&= \frac{(-1)^{k-1}}{\sqrt{\pi x_{k,n}\lambda_n^* R_n''(t_n)(1+t_n^2)}}\left[\pm \sin\left(\delta_n \frac{\lambda_n^* t_n(\nu_{k,n}^{\pm})}{\nu_{k,n}^{\pm}}\right) + \varepsilon_n(x)\right]
\end{aligned}$$

where the error term $\varepsilon_n(x)$ is $o(1)$ as $n \longrightarrow \infty$ locally uniformly for x in $(0,1)$. According to (5.20),

$$\sin\left(\delta_n \frac{\lambda_n^* t_n(\nu_{k,n}^{\pm})}{\nu_{k,n}^{\pm}}\right) \sim \frac{\delta_n \lambda_n^* t_n(x_{k,n})}{x_{k,n}} = o(1), \qquad n \longrightarrow \infty.$$

We still have the freedom of choosing the rate of convergence for δ_n in (5.20) and we can choose $\{\delta_n\}$ such that

$$\varepsilon_n(x) = o(\delta_n \lambda_n^* t_n(x)) = o\left(\sin\left(\delta_n \frac{\lambda_n^* t_n(\nu_{k,n}^{\pm})}{\nu_{k,n}^{\pm}}\right)\right)$$

as $n \longrightarrow \infty$, locally uniformly for x in $(0,1)$. Then by the intermediate value theorem, there is a zero $l_{k,n}$ of $L_n(\Lambda; x)$ that lies between $x_{k,n} - \delta_n$ and $x_{k,n} + \delta_n$. It follows that $l_{k,n} \sim x_{k,n}$ locally uniformly in $(0,1)$. The same holds for $x_{k+1,n}$ and this yields

$$\begin{aligned}
l_{k,n} - l_{k+1,n} &= x_{k,n} - x_{k+1,n} + (l_{k,n} - x_{k,n}) - (l_{k+1,n} - x_{k+1,n}) \\
&= x_{k,n} - x_{k+1,n} + \mathcal{O}\left(\delta_n\right).
\end{aligned}$$

According to (5.17), $x_{k,n} - x_{k+1,n} \asymp 1/\lambda_n^* t_n(x_{k,n})$, and it follows from (5.20) that $l_{k,n} - l_{k+1,n} \sim x_{k,n} - x_{k+1,n}$ as $n \longrightarrow \infty$. $\square$

Proof of Theorem 5.4. The result follows from replacing $x_{k,n}$ with $l_{k,n}$ in (5.19), which is justified by Lemmas 5.11 and 5.12. $\square$

Theorems 5.5 and 5.6 can now be proved with little effort by determining the behavior of the stationary point $t_n(x)$. In both cases we shall use the quantity

$$\sigma_n = \frac{1}{{\lambda_n^*}^2} \left(\sum_{k=0}^{n-1} \lambda_k^* + \frac{\lambda_n^*}{2} \right).$$

In the first result we assume that $\sigma_n \asymp 1$, and in the second $\sigma_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

Proof of Theorem 5.5. First assume that $\alpha = 1$, so $\lambda_n \sim n$ as $n \longrightarrow \infty$. Then $T_n \asymp \log n \longrightarrow \infty$ so (2.17) is satisfied. In Corollary 4.21 we showed that if $\lambda_n \sim n$, then $t_n(x) \asymp 1$ as $n \longrightarrow \infty$ locally uniformly for x in $(0, 1)$. Furthermore $\left(\sum_{j=0}^{n} \lambda_j^*\right)/\lambda_n^* \asymp n$ as $n \longrightarrow \infty$, so the conditions of Theorem 5.4 are satisfied. Define

$$R(t) := \int_0^1 \arctan \frac{s}{t} ds = \arctan \frac{1}{t} - \frac{t}{2} \log \left(1 + \frac{1}{t^2}\right).$$

Then $R'(t) = -\frac{1}{2} \log (1 + t^{-2})$, so if we let $\gamma_0 = (x/(1-x))^{1/2}$, then $R'(\gamma_0) = \frac{1}{2} \log x = R_n'(t_n)$. Using linear approximation then yields

$$(R_n - R)'(\gamma_0) = R_n'(\gamma_0) - R_n'(t_n) = R_n''(\xi_n)(\gamma_0 - t_n), \tag{5.21}$$

for some ξ_n between γ_0 and t_n. Then since $t_n \asymp \sigma_n \asymp 1$ and

$$\frac{t}{[1+t^2]^2} < \frac{R_n''(t)}{2\sigma_n} < \frac{1}{t^3},$$

(see Corollary 4.17), $R_n''(\xi_n)$ is bounded above and below. In Lemma 4.31 we showed that for $\lambda_n \sim n$, we have $(R_n - R)'(\gamma_0) = o(1)$ as $n \longrightarrow \infty$, and it now follows from

(5.21) that

$$t_n = \gamma_0 + o(1) = \left(\frac{x}{1-x}\right)^{1/2} + o(1), \qquad n \longrightarrow \infty.$$

Now consider the general case $\lambda_n \sim \alpha n$ with $\alpha > 0$. Then if we define the sequence $\Pi = \{\mu_n\}$ by letting $\mu_n = \lambda_n/\alpha$ for each n, then $\mu_n \sim n$, and the corresponding phase functions are related via

$$R_n^{(\Lambda)}(t) = \frac{1}{\alpha} R_n^{(\Pi)}(t).$$

Then $R_n^{(\Pi)}(\gamma_0) = \frac{1}{2}\log x$, where $\gamma_0 = (x/(1-x))^{1/2}$ was determined for the $\alpha = 1$ case above. Then

$$R_n^{(\Lambda)}(\gamma_0) = \frac{1}{\alpha}\frac{1}{2}\log x = \frac{1}{2}\log x^{1/\alpha},$$

so we can replace γ_0 with $\gamma_0^{(\Lambda)} = (x^\alpha/(1-x^\alpha))^{1/2}$. It follows from Theorem 5.4 that

$$\frac{\lambda_n^*}{\pi}\left(l_{k,n} - l_{k+1,n}\right) \sim \frac{l_{k,n}}{t_n(l_{k,n})} \sim l_{k,n}\left(\frac{1-l_{k,n}^\alpha}{l_{k,n}^\alpha}\right)^{1/2} = \sqrt{l_{k,n}^{2-\alpha}(1-l_{k,n}^\alpha)}.$$

$\square$

Proof of Theorem 5.6. Here the hypothesis $\sigma_n \longrightarrow \infty$ ensures that the conditions of Theorem 5.4 are satisfied, since $\sigma_n \le T_n$ and $\lambda_n^*\sigma_n \sim \left(\sum_{k=0}^{n-1}\lambda_k^*\right)/\lambda_n^*$, and furthermore Lemma 4.19 shows that $t_n^2 \asymp \sigma_n$ locally uniformly for x in $(0,1)$. More accurately, Lemma 4.19 gives $t_n^2(x) \sim 2\sigma_n/|\log x|$, which yields

$$(\lambda_n t_n)^2(x) \sim \frac{2}{|\log x|}\sum_{k=0}^{n}\lambda_k^*, \qquad n \longrightarrow \infty.$$

Therefore we get

$$\frac{l_{k,n} - l_{k+1,n}}{\pi l_{k,n}} \sim \frac{1}{\lambda_n^* t_n(l_{k,n})} \sim |\log l_{k,n}|^{1/2}\left(2\sum_{k=0}^{n}\lambda_k^*\right)^{-1/2}.$$

$\square$

CHAPTER VI

ASYMPTOTICS OUTSIDE THE INTERVAL OF ORTHOGONALITY

In this chapter we turn our attention to the asymptotic behavior of $L_n(\Lambda; x)$ as $n \longrightarrow \infty$, for $x > 1$ outside the interval of orthogonality. There, we don't have a nice formula as in the case for $x \in (0,1)$, so the approach here is not as clear-cut. However, given the standard form and revealing nature of the representation (3.2), we can emulate the manipulation of the contour integral definition (2.18) we performed in Chapter 3 to arrive at the "true" phase function for $x \notin (0,1)$. We then apply the method of steepest descent to determine the behavior of the Müntz orthogonal polynomials, and we present both strong asymptotics and root asymptotics. Again, we assume that the density condition

$$\sum_{k=0}^{\infty} \frac{1}{\lambda_k + 1/2} = \infty$$

is satisfied and we especially look at the cases when $\lambda_n = o(n)$ as $n \longrightarrow \infty$ and

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \alpha.$$

The results below appear in the manuscript [47], "Asymptotic behavior of Müntz-Legendre polynomials for $x > 1$."

6.1 Main results

Our most general result on the asymptotics of $L_n(\Lambda; x)$, stated in Theorem 6.10 below, requires a regularity condition on the Müntz exponents Λ and the asymptotic term depends on an implicit stationary point. For more special cases, which are presented here, the asymptotics are more explicit.

Theorem 6.1 *Let $\Lambda : -1/2 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots$ be a sequence of real numbers that satisfies*

$$\lambda_n = \frac{1}{\rho}\left(n + \frac{\beta}{2}\right) + o(1), \qquad n \longrightarrow \infty,$$

for some constants $\rho > 0$ and $\beta > -1$. Then locally uniformly for x in $(1, \infty)$, as $n \longrightarrow \infty$,

$$L_n(\Lambda; x^\rho) \sim \frac{1}{2\sqrt{\pi n}} \frac{\left[x^{1/2} + (x-1)^{1/2}\right]^{2n+\beta+\rho}}{x^{(2\rho-1)/4}(x-1)^{1/4}}.$$

Theorem 6.2 *Let $\Lambda : -1/2 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots$ be a sequence of real numbers that satisfies*

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \rho > 0.$$

Then locally uniformly for x in $(1, \infty)$,

$$\lim_{n\to\infty} L_n(\Lambda; x^\rho)^{1/n} = \left[x^{1/2} + (x-1)^{1/2}\right]^2.$$

Note that these results can easily be mapped to the Müntz-Jacobi polynomials $L_n^{(\beta)}(x)$ via (2.27).

The following result covers the cases when $\lambda_n = o(n)$ as $n \longrightarrow \infty$.

Theorem 6.3 *Let $\Lambda : -1/2 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots$ be a sequence of real numbers that satisfies*

$$\lim_{n\to\infty} \frac{1}{(2\lambda_n + 1)^2} \sum_{k=0}^{n} (2\lambda_k + 1) = \infty.$$

Then locally uniformly for x in $(1, \infty)$, as $n \longrightarrow \infty$,

$$L_n(\Lambda; x) \sim \frac{x^{\rho_n}}{2\sqrt{\pi}(\log x)^{1/4}} \left(\sum_{k=0}^{n}(2\lambda_k + 1)\right)^{-1/4} \prod_{k=0}^{n} \frac{\rho_n + \lambda_k + 1}{\rho_n - \lambda_k},$$

where $\rho_n = \rho_n(x) > \lambda_n$ is determined by the identity

$$\log x = 2\sum_{k=0}^{n-1} \frac{\lambda_k^*}{{\rho_n^*}^2 - {\lambda_k^*}^2} + \frac{\lambda_n^*}{{\rho_n^*}^2 - {\lambda_n^*}^2}.$$

and grows according to $\rho_n^2 \log x \sim \sum_{k=0}^{n}(2\lambda_k + 1)$.

Here the stationary point $\rho_n = \rho_n(x)$ is implicitly determined. In the following we reveal the root asymptotics in the same case, which does not depend on ρ_n.

Theorem 6.4 *Let $\Lambda : -1/2 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots$ be a sequence of real numbers that satisfies*

$$\lim_{n\to\infty} \frac{1}{(2\lambda_n+1)^2} \sum_{k=0}^{n} (2\lambda_k + 1) = \infty.$$

Then locally uniformly for x in $(1,\infty)$, as $n \longrightarrow \infty$,

$$\lim_{n\to\infty} L_n(\Lambda; x)^{1/\sqrt{\Sigma_n}} = e^{2(\log x)^{1/2}},$$

where $\Sigma_n = \sum_{k=0}^{n}(2\lambda_k+1) + (2\lambda_n+1)/2$ for each n.

6.2 Proofs

6.2.1 Setup and basic estimates

In the following we fix $x > 1$ and let

$$c := \log x > 0.$$

We start making the change of variables $t = \lambda_n^* u - \frac{1}{2}$ in (2.18) which yields

$$L_n(\Lambda; x) = \frac{1}{2\pi i x^{1/2}} \int_{\Gamma^*} \prod_{k=0}^{n-1} \frac{u + r_k}{u - r_k} \frac{x^{\lambda_n^* u}}{u-1} du, \tag{6.1}$$

where Γ^* encloses all the zeros of the denominator, and we let $r_k := r_{k,n} := \lambda_k^*/\lambda_n^*$ for all n and $k = 0, 1, \dots, n$.. Let $\log z$ and $z^{1/2}$ denote the principal brances of the logarithmic and square root functions, respectively. Note that since the λ_k's are real, u, $u + r_k$ and $u - r_k$ all lie in the same upper/lower half plane of $\mathbb{C}$, and if $\mathrm{Re}(u) > 0$, then $(u + r_k)(u - r_k)$ as well. If $\mathrm{Re}(u) > 0$, since $|\arg(u - r_k)| > |\arg(u + r_k)|$, it is easy to see that

$$(u^2 - r_k^2)^{1/2} = (u + r_k)^{1/2}(u - r_k)^{1/2} \qquad \text{and} \qquad \left(\frac{u + r_k}{u - r_k}\right)^{1/2} = \frac{(u + r_k)^{1/2}}{(u - r_k)^{1/2}}.$$

Then in the right half plane, and off the interval $(0, 1]$, we can write the integrand in (6.1) as

$$\begin{aligned} \prod_{k=0}^{n-1} \frac{u+r_k}{u-r_k} \frac{x^{\lambda_n^* u}}{u-1} &= \prod_{k=0}^{n-1} \frac{u+r_k}{u-r_k} \left(\frac{u+1}{u-1}\right)^{1/2} \frac{e^{\lambda_n^* uc}}{(u^2-1)^{1/2}} \\ &= \frac{e^{\lambda_n^* [\Theta_n(u)+uc]}}{(u^2-1)^{1/2}}, \end{aligned} \tag{6.2}$$

where $c = \log x$ and (see the definition for $\sum*$ in (1.4))

$$\Theta_n(u) = \frac{1}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \log \frac{u+r_k}{u-r_k}.$$

We then have

$$\Theta_n'(u) = -\frac{2}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{u^2 - r_k^2},$$

and since $\frac{d}{du}[\Theta_n(u) + uc] = \Theta_n'(u) + c$, the stationary points for the phase function $\Theta_n(u) + uc$ are the roots of the equation

$$\frac{2}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{u^2 - r_k^2} = c.$$

Remark Making the substitution $u = is$ in (6.2), we can write Θ_n as a sum of inverse tangent functions. In the case $x \in (0, 1)$, with some manipulations, this yields the representation in (3.2). Since $\log x < 0$ in that case, the stationary points lie on the imaginary axis, $u = \pm i\rho_n^*$.

Since

$$\lim_{u \to 1+} [-\Theta_n'(u)] = \infty, \qquad \lim_{u \to \infty} [-\Theta_n'(u)] = 0,$$

and

$$\Theta_n''(u) = \frac{4u}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{(u^2 - r_k^2)^2}$$

is strictly positive for $u > 1$, we see that there exists a unique stationary point $\tau_n = \tau_n(x)$ in $(1, \infty)$ of the phase function $\Theta_n(u) + uc$, i.e. τ_n satisfies

$$\frac{2}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{\tau_n^2 - r_k^2} = c = \log x. \tag{6.3}$$

For the following lemma, recall the definition for σ_n in (1.5).

Lemma 6.5 *(a) For all $x > 1$ and n, we have*

$$\tau_n{}^2 - 1 \leq \frac{2\sigma_n}{c} \leq \tau_n{}^2.$$

(b) We have $\lambda_n^* \tau_n \gtrsim S_n^{1/2} \longrightarrow \infty$ as $n \longrightarrow \infty$, locally uniformly for x in $(1, \infty)$. In particular, if $\sigma_n \longrightarrow \infty$, then $\tau_n{}^2 \sim 2\sigma_n/c$ as $n \longrightarrow \infty$.

Proof. Since $0 < r_k \leq 1$ for all $k = 0, 1, \ldots, n$, we have

$$c = \frac{2}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{\tau_n^2 - r_k^2} < \frac{2}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{\tau_n^2 - 1} = \frac{2\sigma_n}{\tau_n^2 - 1},$$

and similarly $c > 2\sigma_n/\tau_n^2$. Recalling that $S_n = {\lambda_n^*}^2 \sigma_n$, we can write the latter inequality as $c(\lambda_n^* \tau_n)^2(x) \geq 2S_n$. Since S_n is independent of x and $c = \log x$ is uniformly bounded above and below for x in compact subsets of $(1, \infty)$, the result follows. $\square$

Lemma 6.6 *(a) For all $x > 1$ and n, we have*

$$1 \leq \frac{\tau_n \Theta_n''(\tau_n)}{2c} \leq \frac{\tau_n^2}{\tau_n^2 - 1},$$

(b) We have $\lambda_n^* \tau_n^2 \Theta_n''(\tau_n) \gtrsim \lambda_n^* \tau_n \gtrsim S_n^{1/2} \longrightarrow \infty$ as $n \longrightarrow \infty$, locally uniformly for x in $(1, \infty)$. In particular, if $\sigma_n \longrightarrow \infty$, then $\tau_n \Theta_n''(\tau_n) \sim 2c$ as $n \longrightarrow \infty$.

Proof. By writing

$$\tau_n \Theta_n''(\tau_n) = \frac{4\tau_n^2}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{(\tau_n^2 - r_k^2)^2} = \frac{4}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{\tau_n^2 - r_k^2} \frac{\tau_n^2}{\tau_n^2 - r_k^2},$$

the first inequalities follow from the identity (6.3) and the trivial inequalities $\tau_n^2 - 1 \leq \tau_n^2 - r_k \leq \tau_n^2$. It follows from Lemma 6.5 that $\lambda_n^* \tau_n^2 \Theta_n''(\tau_n) > 2c\lambda_n^* \tau_n \gtrsim S_n^{1/2} \longrightarrow \infty$. $\square$

Applying Cauchy's Integral Theorem we can choose Γ^* to be the closed half circle $\{is : -\tau_n \leq s \leq \tau_n\} \cup C_n$, where C_n is the arc parametrized with $\gamma_n(s) = \tau_n e^{is}$, $s \in (-\pi/2, \pi/2)$. Then we can write (6.1) as

$$2\pi i x^{1/2} L_n(x) = i(-1)^n \int_{-\tau_n}^{\tau_n} \prod_{k=0}^{n-1} \frac{r_k - is}{r_k + is} \frac{e^{-i\lambda_n^* sc}}{1 + is} ds + \int_{C_n} \frac{e^{\lambda_n^*[\Theta_n(u) + uc]}}{(u^2 - 1)^{1/2}} du. \tag{6.4}$$

We estimate three parts of this integral. We split the integral over C_n up in two parts, one close to the stationary point $\tau_n + i0$, where we have the main contribution, and the other away from the point. Then we get a simple estimate for the part on the line segment $\{is : -\tau_n \leq s \leq \tau_n\}$.

Below, we shall assume that $\tau_n^2(x) - 1$ is bounded below away from 0, locally uniformly for x in $(1, \infty)$. This will ensure that the estimates and limits hold uniformly for x in compact subsets of $(1, \infty)$, but for the sake of brevity, we shall fail to mention this in every step. The key is that $c = \log x$ is uniformly bounded above and below in compact subsets of $(1, \infty)$.

6.2.2 Contribution near the stationary point

Here we look at the second integral on the right hand side of (6.4), over the arc C_n, and restrict to a part close to the stationary point $\tau_n + i0$. Define $\varepsilon_n > 0$ such that

$$\varepsilon_n^2 = \frac{\log(\lambda_n^* \tau_n)}{\lambda_n^* \tau_n}, \qquad n \geq 0. \tag{6.5}$$

In what follows, we assume that $\tau_n^2 - 1$ is bounded below away from 0, locally uniformly for x in $(1, \infty)$, so according to Lemma 6.5, $\varepsilon_n = o(1)$. We define the integral

$$I_x(\varepsilon_n) = \int_{C(\varepsilon_n)} \frac{e^{\lambda_n^* F_n(u)}}{(u^2 - 1)^{1/2}} du, \tag{6.6}$$

where $F_n(u) = F_n(u; x) = \Theta_n(u) + uc$ and $C(\varepsilon_n)$ is the arc parametrized by

$$\gamma_{\varepsilon_n}(s) = \tau_n e^{is} = \alpha_n(s) + i\beta_n(s), \qquad s \in (-\varepsilon_n, \varepsilon_n).$$

Here we let $\alpha_n(s) = \tau_n \cos s$ and $\beta_n(s) = \tau_n \sin s$ for simplification, and furthermore we write the real and imaginary parts of the phase function as

$$h_n(s) = \operatorname{Re}\left[F_n(\gamma_{\varepsilon_n}(s))\right] \qquad \text{and} \qquad k_n(s) = \operatorname{Im}\left[F_n(\gamma_{\varepsilon_n}(s))\right].$$

For $u = \gamma_{\varepsilon_n}(s)$, we have

$$\frac{u + r_k}{u - r_k} = \frac{(u + r_k)(\overline{u} - r_k)}{|u - r_k|^2} = \frac{({\tau_n}^2 - r_k^2) - i2r_k\beta_n(s)}{\tau_n^2 + r_k^2 - 2r_k\alpha_n(s)},$$

and thus

$$\left|\frac{u+r_k}{u-r_k}\right|^2 = \frac{({\tau_n}^2 - r_k^2)^2 + (2r_k\beta_n)^2}{[\tau_n^2 + r_k^2 - 2r_k\alpha_n]^2} = \frac{\tau_n^2 + r_k^2 + 2r_k\alpha_n}{\tau_n^2 + r_k^2 - 2r_k\alpha_n}$$

and

$$\arg\left(\frac{u+r_k}{u-r_k}\right) = -\arctan\frac{2r_k\beta_n(s)}{{\tau_n}^2 - r_k^2}.$$

It follows that (dropping the s-dependence of α_n and β_n from the notation)

$$h_n(s) = \alpha_n c + \frac{1}{2\lambda_n^*}\sum_{k=0}^{n}{}^{*} \log\frac{\tau_n^2 + r_k^2 + 2r_k\alpha_n}{\tau_n^2 + r_k^2 - 2r_k\alpha_n} \tag{6.7}$$

and

$$k_n(s) = \beta_n c - \frac{1}{\lambda_n^*}\sum_{k=0}^{n}{}^{*} \arctan\frac{2r_k\beta_n}{{\tau_n}^2 - r_k^2}. \tag{6.8}$$

First we show that the imaginary part $k_n(s)$ is insignificant for ε_n small enough.

Lemma 6.7 *If uniformly on $(1,\infty)$, $\tau_n^2(x) - 1$ is bounded below away from 0, then for $|s| \le \varepsilon_n$,*

$$k_n(s) = \mathcal{O}\left(\frac{\varepsilon_n^3}{\tau_n}\right), \qquad n \longrightarrow \infty.$$

and this holds locally uniformly for x in $(1,\infty)$.

Proof. Recall that for $|x| \le 1$, we have the alternating series expansion $\arctan x = x - x^3/3 + x^5/5 - \cdots$, and thus $|x - \arctan x| \le |x^3|/3$. By hypothesis,

$$\frac{\tau_n}{\tau_n^2 - 1} \asymp \frac{1}{\tau_n} \le 1,$$

and it follows that

$$\left|\frac{2r_k\beta_n(s)}{{\tau_n}^2 - r_k^2}\right| \le \frac{2\tau_n|\sin s|}{{\tau_n}^2 - 1} \le \frac{2\tau_n}{{\tau_n}^2 - 1}\varepsilon_n = \mathcal{O}\left(\varepsilon_n\right) = o(1)$$

as $n \longrightarrow \infty$, for all $k = 0, 1, \ldots, n$. Thus we get the following bound for (6.8),

$$\begin{aligned}
|k_n(s)| &\le |\beta_n(s)|\underbrace{\left[c - \frac{2}{\lambda_n^*}\sum_{k=0}^{n}{}^{*}\frac{r_k}{{\tau_n}^2 - r_k^2}\right]}_{=0} + \frac{(2|\beta_n(s)|)^3}{3\lambda_n^*}\sum_{k=0}^{n}{}^{*}\left(\frac{r_k}{{\tau_n}^2 - r_k^2}\right)^3 \\
&\le \frac{4\tau_n^3|s|^3}{3}\frac{1}{(\tau_n^2-1)^2}\underbrace{\frac{2}{\lambda_n^*}\sum_{k=0}^{n}{}^{*}\frac{r_k}{{\tau_n}^2 - r_k^2}}_{=c} \\
&= \mathcal{O}\left(\frac{\tau_n^3}{(\tau_n^2-1)^2}\varepsilon_n^3\right),
\end{aligned}$$

and the result follows since $\tau_n^4/(\tau_n^2-1)^2 \asymp 1$ as $n \longrightarrow \infty$, locally uniformly for x in $(1,\infty)$. $\square$

Using the parametrization γ_{ε_n} in (6.6), we can write

$$\begin{aligned} I_x(\varepsilon_n) &= i\tau_n \int_{-\varepsilon_n}^{\varepsilon_n} \frac{e^{\lambda_n^* h_n(s)} e^{i\lambda_n^* k_n(s)}}{(\tau_n^2 e^{2is}-1)^{1/2}} e^{is} ds \\ &= \frac{i\tau_n}{(\tau_n^2-1)^{1/2}} \int_{-\varepsilon_n}^{\varepsilon_n} e^{\lambda_n^* h_n(s)} ds + \delta_1(\varepsilon_n) + \delta_2(\varepsilon_n) + \delta_3(\varepsilon_n), \end{aligned}$$

where we have introduced the error terms

$$\begin{aligned} \delta_1(\varepsilon_n) &= i\tau_n \int_{-\varepsilon_n}^{\varepsilon_n} \frac{e^{\lambda_n^* h_n(s)} e^{i\lambda_n^* k_n(s)}}{(\tau_n^2 e^{2is}-1)^{1/2}} \left[e^{is}-1\right] ds \\ \delta_2(\varepsilon_n) &= i\tau_n \int_{-\varepsilon_n}^{\varepsilon_n} e^{\lambda_n^* h_n(s)} e^{i\lambda_n^* k_n(s)} \left[\frac{1}{(\tau_n^2 e^{2is}-1)^{1/2}} - \frac{1}{(\tau_n^2-1)^{1/2}}\right] ds \\ \delta_3(\varepsilon_n) &= \frac{i\tau_n}{(\tau_n^2-1)^{1/2}} \int_{-\varepsilon_n}^{\varepsilon_n} e^{\lambda_n^* h_n(s)} \left[e^{i\lambda_n^* k_n(s)}-1\right] ds \end{aligned}$$

Using the inequalities $|\tau_n^2 e^{2is}-1| \ge |\tau_n^2-1|$ and $|e^{is}-1| \le |s|$, for all s, we get

$$|\delta_1(\varepsilon_n)| \le \varepsilon_n \frac{\tau_n}{(\tau_n^2-1)^{1/2}} \int_{-\varepsilon_n}^{\varepsilon_n} e^{\lambda_n^* h_n(s)} ds.$$

By (6.5), we have

$$\left|\frac{\tau_n^2}{\tau_n^2-1}\left[e^{2i\varepsilon_n}-1\right]\right| \le \frac{2\tau_n^2}{\tau_n^2-1}\varepsilon_n = o(1), \qquad n \longrightarrow \infty,$$

and thus since $|(1+w)^{1/2}-1| = \mathcal{O}(|w|)$ if $|w| = o(1)$ we get the estimate

$$\begin{aligned} |\delta_2(\varepsilon_n)| &\le \max_{|s|\le\varepsilon_n} \left|1 - \frac{(\tau_n^2 e^{2is}-1)^{1/2}}{(\tau_n^2-1)^{1/2}}\right| \frac{\tau_n}{(\tau_n^2-1)^{1/2}} \int_{-\varepsilon_n}^{\varepsilon_n} e^{\lambda_n^* h_n(s)} ds \\ &= \mathcal{O}\left(\frac{\tau_n^2}{\tau_n^2-1}\varepsilon_n\right) \frac{\tau_n}{(\tau_n^2-1)^{1/2}} \int_{-\varepsilon_n}^{\varepsilon_n} e^{\lambda_n^* h_n(s)} ds. \end{aligned}$$

We are assuming that τ_n^2-1 is bounded below and since $\varepsilon_n = o(1)$, we have

$$\delta_1(\varepsilon_n), \delta_2(\varepsilon_n) = o\left(\frac{\tau_n}{(\tau_n^2-1)^{1/2}} \int_{-\varepsilon_n}^{\varepsilon_n} e^{\lambda_n^* h_n(s)} ds\right), \qquad n \longrightarrow \infty.$$

As for $\delta_3(\varepsilon_n)$, we apply Lemma 6.7, and get

$$|\delta_3(\varepsilon_n)| = \mathcal{O}\left(\frac{\lambda_n^* \tau_n^3}{(\tau_n^2-1)^2}\varepsilon_n^3\right) \frac{\tau_n}{(\tau_n^2-1)^{1/2}} \int_{-\varepsilon_n}^{\varepsilon_n} e^{\lambda_n^* h_n(s)} ds.$$

Since $\tau_n^4/(\tau_n^2-1)^2 \asymp 1$ as $n \longrightarrow \infty$, and recalling (6.5), we have

$$\begin{aligned}
\frac{\lambda_n^*\tau_n^3}{(\tau_n^2-1)^2}\varepsilon_n^3 &\asymp \frac{\lambda_n^*}{\tau_n}\varepsilon_n^3 = \frac{\lambda_n^*}{\tau_n}\frac{[\log(\lambda_n^*\tau_n)]^{3/2}}{(\lambda_n^*\tau_n)^{3/2}} \\
&= \frac{[\log(\lambda_n^*\tau_n)]^{3/2}}{(\lambda_n^*\tau_n)^{1/2}}\frac{1}{\tau_n^{3/2}} \\
&\leq \frac{[\log(\lambda_n^*\tau_n)]^{3/2}}{(\lambda_n^*\tau_n)^{1/2}} \\
&= o(1),
\end{aligned}$$

as $n \longrightarrow \infty$, and this holds locally uniformly for x in $(1,\infty)$. To summarize, we have shown that

$$I_x(\varepsilon_n) = \frac{i\tau_n}{(\tau_n^2-1)^{1/2}}\int_{-\varepsilon_n}^{\varepsilon_n} e^{\lambda_n^* h_n(s)}ds\,[1+o(1)]\,, \qquad n \longrightarrow \infty. \tag{6.9}$$

To deal with the main term in (6.9), we need to look at the function $h_n(s)$, defined in (6.7), in more detail. The first two derivatives are

$$h_n'(s) = -\beta_n\left[c + \frac{2}{\lambda_n^*}\sum_{k=0}^{n}{}^{*}\frac{r_k(\tau_n^2+r_k^2)}{(\tau_n^2-r_k^2)^2+(2r_k\beta_n)^2}\right] \tag{6.10}$$

and

$$h_n''(s) = -\alpha_n\left[c + \frac{2}{\lambda_n^*}\sum_{k=0}^{n}{}^{*}r_k(\tau_n^2+r_k^2)\frac{(\tau_n^2-r_k^2)^2-(2r_k\beta_n)^2}{[(\tau_n^2-r_k^2)^2+(2r_k\beta_n)^2]^2}\right]. \tag{6.11}$$

We note that (recall $\alpha_n(0)=\tau_n$ and $\beta_n(0)=0$) $h_n'(0)=0$,

$$\begin{aligned}
h_n(0) &= \tau_n c + \frac{1}{\lambda_n^*}\sum_{k=0}^{n}{}^{*}\log\frac{\tau_n+r_k}{\tau_n-r_k} \\
&= \tau_n c + \Theta_n(\tau_n) \\
&= F_n(\tau_n)
\end{aligned}$$

and

$$\begin{aligned}
h_n''(0) &= -\tau_n\left[c + \frac{2}{\lambda_n^*}\sum_{k=0}^{n}{}^{*}r_k\frac{\tau_n^2+r_k^2}{(\tau_n^2-r_k^2)^2}\right] \\
&= -\frac{4\tau_n^3}{\lambda_n^*}\sum_{k=0}^{n-1}\frac{r_k}{(\tau_n^2-r_k^2)^2} \\
&= -\tau_n^2\Theta_n''(\tau_n),
\end{aligned}$$

and here we have used the identity (6.3). Thus expanding $h_n(s)$ about $s=0$ yields

$$\begin{aligned} h_n(s) &= h_n(0) + h_n'(0)s + \frac{h_n''(0)}{2}s^2 + \cdots \\ &= F_n(\tau_n) - \frac{\tau_n^2 \Theta_n''(\tau_n)}{2}s^2 + \frac{h_n^{(3)}(\xi_{n,s})}{6}s^3, \end{aligned}$$

where $\xi_{n,s}$ is between 0 and s. Then we can write

$$\begin{aligned} \int_{-\varepsilon_n}^{\varepsilon_n} e^{\lambda_n^* h_n(s)} ds &= e^{\lambda_n^* F_n(\tau_n)} \int_{-\varepsilon_n}^{\varepsilon_n} e^{\lambda_n^*\left[-\frac{1}{2}\tau_n^2\Theta_n''(\tau_n)s^2 + \frac{1}{6}h_n^{(3)}(\xi_{n,s})s^3\right]} ds \\ &= e^{\lambda_n^* F_n(\tau_n)} \int_{-\varepsilon_n}^{\varepsilon_n} e^{-\frac{1}{2}\lambda_n^*\tau_n^2\Theta_n''(\tau_n)s^2} ds + \delta_4(\varepsilon_n), \end{aligned}$$

where

$$\delta_4(\varepsilon_n) = e^{\lambda_n^* F_n(\tau_n)} \int_{-\varepsilon_n}^{\varepsilon_n} e^{-\frac{1}{2}\lambda_n^*\tau_n^2\Theta_n''(\tau_n)s^2} \left[e^{\frac{\lambda_n^*}{6}h_n^{(3)}(\xi_{n,s})s^3} - 1\right] ds.$$

Recall from Lemma 6.6, that if $\tau_n^2 - 1$ is bounded below, then locally uniformly for x in $(1,\infty)$, $\tau_n\Theta_n''(\tau_n) \asymp 1$ as $n \longrightarrow \infty$. Therefore using (6.5), we get

$$\lambda_n^*\tau_n^2\Theta_n''(\tau_n)\varepsilon_n^2 \asymp \lambda_n^*\tau_n\varepsilon_n^2 = \log(\lambda_n^*\tau_n)$$

and by Lemma 6.5, $\lambda_n^*\tau_n \longrightarrow \infty$, locally uniformly for x in $(1,\infty)$. It follows that

$$\lim_{n\to\infty} \varepsilon_n\sqrt{\lambda_n^*\tau_n^2\Theta_n''(\tau_n)} = \infty. \tag{6.12}$$

Then using the substitution $v = \sqrt{\lambda_n^*\tau_n^2\Theta_n''(\tau_n)}s$ yields

$$\begin{aligned} \int_{-\varepsilon_n}^{\varepsilon_n} e^{-\frac{1}{2}\lambda_n^*\tau_n^2\Theta_n''(\tau_n)s^2} ds &= \frac{1}{\sqrt{\lambda_n^*\tau_n^2\Theta_n''(\tau_n)}} \int_{-\sqrt{\lambda_n^*\tau_n^2\Theta_n''(\tau_n)}\varepsilon_n}^{\sqrt{\lambda_n^*\tau_n^2\Theta_n''(\tau_n)}\varepsilon_n} e^{-\frac{1}{2}v^2} dv \\ &= \frac{1}{\sqrt{\lambda_n^*\tau_n^2\Theta_n''(\tau_n)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2} dv\,[1+o(1)] \\ &= \sqrt{\frac{2\pi}{\lambda_n^*\tau_n^2\Theta_n''(\tau_n)}}\,[1+o(1)], \end{aligned}$$

and the limits hold uniformly for x in compact subsets of $(1,\infty)$. It remains to estimate the error term $\delta_4(\varepsilon_n)$, and for that purpose we need an estimate for the third derivative $h_n^{(3)}(s)$.

Lemma 6.8 *If uniformly on $(1,\infty)$, $\tau_n^2(x)-1$ is bounded below away from 0, then for $|s|\le\varepsilon_n$,*

$$|h_n^{(3)}(s)| = \mathcal{O}\left(\tau_n\varepsilon_n\right), \qquad n\longrightarrow\infty,$$

and this holds locally uniformly for x in $(1,\infty)$.

Proof. From (6.11), and using $\alpha_n'(s)=-\beta_n(s)$, we get

$$h_n^{(3)}(s) = \beta_n c - \frac{2}{\lambda_n^*}\sideset{}{^*}\sum_{k=0}^{n} r_k(\tau_n^2+r_k^2)\frac{d}{ds}\left[\alpha_n\frac{(\tau_n^2-r_k^2)^2-(2r_k\beta_n)^2}{[(\tau_n^2-r_k^2)^2+(2r_k\beta_n)^2]^2}\right]. \qquad (6.13)$$

Letting $a_{k,n}=\tau_n^2-r_k^2$ for each n and $k=0,1,\dots,n$, and using $\frac{d}{ds}(2r_k\beta_n)^2 = 2(2r_k)^2\alpha_n\beta_n$, we compute

$$\begin{aligned}
\frac{d}{ds}\left[\alpha_n\frac{a_{k,n}^2-(2r_k\beta_n)^2}{[a_{k,n}^2+(2r_k\beta_n)^2]^2}\right] &= -\beta_n\frac{a_{k,n}^2-(2r_k\beta_n)^2}{[a_{k,n}^2+(2r_k\beta_n)^2]^2}\\
&\quad+\alpha_n\frac{-2(2r_k)^2\alpha_n\beta_n[a_{k,n}^2+(2r_k\beta_n)^2]-\left[a_{k,n}^2-(2r_k\beta_n)^2\right]2\cdot 2(2r_k)^2\alpha_n\beta_n}{[a_{k,n}^2+(2r_k\beta_n)^2]^3}\\
&= \frac{-\beta_n}{a_{k,n}^2+(2r_k\beta_n)^2}\left(\frac{a_{k,n}^2-(2r_k\beta_n)^2}{a_{k,n}^2+(2r_k\beta_n)^2}+\frac{2(2r_k\alpha_n)^2}{a_{k,n}^2+(2r_k\beta_n)^2}\frac{3a_{k,n}^3-(2r_k\beta_n)^2}{a_{k,n}^2+(2r_k\beta_n)^2}\right).
\end{aligned}$$

Since $\left|\frac{a_{k,n}^2-(2r_k\beta_n)^2}{a_{k,n}^2+(2r_k\beta_n)^2}\right|\le 1$, $\left|\frac{3a_{k,n}^2-(2r_k\beta_n)^2}{a_{k,n}^2+(2r_k\beta_n)^2}\right|\le 3$ and $a_{k,n}=\tau_n^2-r_k^2\ge\tau_n^2-1$ this yields

$$\begin{aligned}
\left|\frac{d}{ds}\left[\alpha_n\frac{a_{k,n}^2-(2r_k\beta_n)^2}{[a_{k,n}^2+(2r_k\beta_n)^2]^2}\right]\right| &\le \frac{|\beta_n|}{(\tau_n^2-r_k^2)^2}\left(1+\frac{2(2\tau_n)^2}{(\tau_n^2-1)^2}3\right)\\
&\le \frac{|\beta_n|}{(\tau_n^2-r_k^2)^2}\left(1+\frac{24\tau_n^2}{(\tau_n^2-1)^2}\right).
\end{aligned}$$

Using the identity (6.3), and $|\beta_n|=\tau_n|\sin s|\le\tau_n\varepsilon_n$, (6.13) yields

$$\begin{aligned}
|h_n^{(3)}(s)| &\le \tau_n\varepsilon_n\left[c+\frac{2}{\lambda_n^*}\sideset{}{^*}\sum_{k=0}^{n}\frac{r_k}{\tau_n^2-r_k^2}\frac{\tau_n^2+r_k^2}{\tau_n^2-r_k^2}\left(1+\frac{24\tau_n^2}{(\tau_n^2-1)^2}\right)\right]\\
&\le \tau_n\varepsilon_n c\left[1+\frac{\tau_n^2+1}{\tau_n^2-1}\left(1+\frac{24\tau_n^2}{(\tau_n^2-1)^2}\right)\right].
\end{aligned}$$

If τ_n^2-1 is bounded below, then

$$\frac{\tau_n^2+1}{\tau_n^2-1}\left(1+\frac{24\tau_n^2}{(\tau_n^2-1)^2}\right)=\mathcal{O}(1), \qquad n\longrightarrow\infty,$$

and the result follows. $\square$

The lemma above gives

$$\max_{0\le r,s\le\varepsilon_n}\left|e^{\frac{\lambda_n^*}{6}h_n^{(3)}(r)s^3}-1\right|=\mathcal{O}\left(\lambda_n^*\tau_n\varepsilon_n^4\right),$$

and using (6.5) and Lemma (6.5) yields $\lambda_n^*\tau_n\varepsilon_n^4=[\log(\lambda_n^*\tau_n)]^2/\lambda_n^*\tau_n=o(1)$. If follows that

$$\delta_4(\varepsilon)=o\left(e^{\lambda_n^*F_n(\tau_n)}\int_{-\varepsilon_n}^{\varepsilon_n}e^{-\frac{1}{2}\lambda_n^*\tau_n^2\Theta_n''(\tau_n)s^2}ds\right),$$

and this holds uniformly for x in compact subsets of $(1,\infty)$.

To conclude, combining this with (6.9), the main contribution is

$$\begin{aligned}\frac{1}{2\pi i}I_x(\varepsilon_n) &\sim \frac{1}{2\pi}\frac{\tau_n}{(\tau_n^2-1)^{1/2}}\int_{-\varepsilon_n}^{\varepsilon_n}e^{\lambda_n^*h_n(s)}ds\\ &\sim \frac{1}{2\pi}\frac{\tau_n}{(\tau_n^2-1)^{1/2}}e^{\lambda_n^*F_n(\tau_n)}\sqrt{\frac{2\pi}{\lambda_n^*\tau_n^2\Theta_n''(\tau_n)}}\\ &= \frac{e^{\lambda_n^*F_n(\tau_n)}}{\sqrt{2\pi\lambda_n^*(\tau_n^2-1)\Theta_n''(\tau_n)}}.\end{aligned}\tag{6.14}$$

6.2.3 Estimate of the integral on the arc away from the stationary point

Here using the same ε_n as in the section above, we estimate the integral

$$\begin{aligned}J_x(\varepsilon_n) &= \int_{C_n\setminus C(\varepsilon_n)}\frac{e^{\lambda_n^*F_n(u)}}{(u^2-1)^{1/2}}du\\ &= i\tau_n\int_{-\pi/2}^{-\varepsilon_n}\frac{e^{\lambda_n^*h_n(s)}e^{i\lambda_n^*k_n(s)}}{(\tau_n^2e^{2is}-1)^{1/2}}e^{is}ds+i\tau_n\int_{\varepsilon_n}^{\pi/2}\frac{e^{\lambda_n^*h_n(s)}e^{i\lambda_n^*k_n(s)}}{(\tau_n^2e^{2is}-1)^{1/2}}e^{is}ds\end{aligned}$$

Since $(\tau_n^2-1)^{1/2}\le|(\tau_n^2e^{2is}-1)^{1/2}|$ and $h_n(s)$ is even about $s=0$ (since $\alpha_n(s)=\tau_n\cos s$ is), we have

$$|J_x(\varepsilon_n)|\le\frac{2\tau_n}{(\tau_n^2-1)^{1/2}}\int_{\varepsilon_n}^{\pi/2}e^{\lambda_n^*h_n(s)}ds.\tag{6.15}$$

Lemma 6.9 *For $s\in[\varepsilon_n,\pi/2]$,*

$$\frac{1}{|h_n'(s)|}=\mathcal{O}\left(\frac{1}{\tau_n\varepsilon_n}\right),\qquad n\longrightarrow\infty,$$

and this holds locally uniformly for x in $(1,\infty)$.

Proof. From (6.10) we recall that

$$h_n'(s) = -\tau_n \sin s \left[c + \frac{2}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k(\tau_n^2 + r_k^2)}{(\tau_n^2 - r_k^2)^2 + (2r_k\tau_n \sin s)^2} \right].$$

Then using $\sin s \geq \sin \varepsilon_n \geq \frac{\varepsilon_n}{2}$ and the identity (6.3) we get

$$\begin{aligned}
|h_n'(s)| &\geq \frac{\tau_n \varepsilon_n}{2} \left[c + \frac{2}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k(\tau_n^2 + r_k^2)}{(\tau_n^2 - r_k^2)^2 + (2r_k\tau_n)^2} \right] \\
&= \frac{\tau_n \varepsilon_n}{2} \left[c + \frac{2}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{\tau_n^2 - r_k^2} \frac{\tau_n^2 - r_k^2}{\tau_n^2 + r_k^2} \right] \\
&\geq \frac{\tau_n \varepsilon_n}{2} \left[c + c\frac{\tau_n^2 - 1}{\tau_n^2 + 1} \right] \\
&= \tau_n \varepsilon_n c \frac{\tau_n^2}{\tau_n^2 + 1}.
\end{aligned}$$

The result follows from noting that $\tau_n^2/(\tau_n^2 + 1) \asymp 1$ as $n \longrightarrow \infty$. $\square$

Since $h_n'(s)$ is negative for $s > 0$, $h_n(s)$ is monotone on $[\varepsilon_n, \pi/2]$, and we can use the substitution $v = h_n(s)$, $dv = h_n'(s)ds$ in (6.15). Then using $h_n(\varepsilon_n) \leq h_n(0) = F_n(\tau_n)$ and the result of Lemma 6.9 yields

$$\begin{aligned}
|J_x(\varepsilon_n)| &\leq \frac{2\tau_n}{(\tau_n^2 - 1)^{1/2}} \frac{e^{\lambda_n^* h_n(\varepsilon_n)} - e^{\lambda_n^* h_n(\pi/2)}}{\lambda_n^* \min_{s\in[\varepsilon_n, \pi/2]} |h_n'(s)|} \\
&= \mathcal{O}\left(\frac{\tau_n}{(\tau_n^2 - 1)^{1/2}} \frac{e^{\lambda_n^* h_n(0)}}{\lambda_n^* \tau_n \varepsilon_n} \right) \\
&= \mathcal{O}\left(\frac{e^{\lambda_n^* F_n(\tau_n)}}{\sqrt{\lambda_n^*(\tau_n^2 - 1)\Theta_n''(\tau_n)}} \left(\frac{\tau_n \Theta_n''(\tau_n)}{\lambda_n^* \tau_n \varepsilon_n^2} \right)^{1/2} \right).
\end{aligned}$$

According to Lemma 6.6, if $\tau_n^2(x) - 1$ is unformly bounded away from 0 on $(1, \infty)$, then $\tau_n \Theta_n''(\theta_n) \asymp 1$, and using (6.5) and Lemma 6.5, we see that $\lambda_n^* \tau_n \varepsilon_n^2 = \log(\lambda_n^* \tau_n) \longrightarrow \infty$ as $n \longrightarrow \infty$. It follows that

$$J_x(\varepsilon_n) = o\left(\frac{e^{\lambda_n^* F_n(\tau_n)}}{\sqrt{\lambda_n^*(\tau_n^2 - 1)\Theta_n''(\tau_n)}} \right), \qquad n \longrightarrow \infty.$$

6.2.4 Estimate of the integral on the line segment

It remains to determine the contribution of the part of the integral in (6.4) that lies on the imaginary axis. We denote it by $l_x(n)$. Using $|r_k - is|/|r_k + is| = 1$ for all k

and $s \in \mathbb{R}$, and $c = \log x \in \mathbb{R}$ we get

$$\begin{aligned} l_x(n) &= \left| \int_{-\tau_n}^{\tau_n} \prod_{k=0}^{n-1} \frac{r_k - is}{r_k + is} \frac{e^{-i\lambda_n^* sc}}{1+is} ds \right| \\ &\leq \int_{-\tau_n}^{\tau_n} \frac{ds}{|1+is|} = 2\int_0^{\tau_n} \frac{ds}{\sqrt{1+s^2}} \\ &= 2\log\left(\tau_n + \sqrt{1+\tau_n^2}\right) \\ &= \mathcal{O}\left(\tau_n\right), \end{aligned}$$

where in the last step we have used our assumption that τ_n is bounded below away from 1. Recall that the main contribution term is $e^{\lambda_n^* F_n(\tau_n)}/\sqrt{2\pi\lambda_n^*(\tau_n^2-1)\Theta_n''(\tau_n)}$. First, since $\Theta_n(\tau_n) > 0$, we have the basic estimate

$$\lambda_n^* F_n(\tau_n) \geq \lambda_n^* \tau_n \log x.$$

According to Lemma 6.6, $(\tau_n^2 - 1)\Theta_n''(\tau_n) \asymp \tau_n$, and it follows that

$$\frac{\tau_n}{\frac{e^{\lambda_n^* F_n(\tau_n)}}{\sqrt{\lambda_n^*(\tau_n^2-1)\Theta_n''(\tau_n)}}} = \mathcal{O}\left(\frac{\tau_n\sqrt{\lambda_n^*\tau_n}}{e^{\lambda_n^*\tau_n \log x}}\right) = \mathcal{O}\left(\frac{(\lambda_n^*\tau_n)^{3/2}}{x^{\lambda_n^*\tau_n}}\frac{1}{\lambda_n^*}\right).$$

Then since $x > 1$ and $\lambda_n^*\tau_n(x) \longrightarrow \infty$ locally uniformly for x in $(1, \infty)$, it follows that $l_x(n) = o\left(I_x(\varepsilon_n)\right)$.

6.2.5 Proofs of main results

In (6.4), we wrote $2\pi i x^{1/2} L_n(\Lambda; x) = I_x(\varepsilon_n) + J_x(\varepsilon_n) + l_x(n)$. According to the treatment above, recalling the main contribution in (6.14); if uniformly on $(1, \infty)$, the stationary point $\tau_n = \tau_n(x) > 1$ of the phase function $F_n(t; x) = \Theta_n(t) + t\log x$ is bounded below from 1, then we have

$$L(\Lambda; x) \sim \frac{e^{\lambda_n^* F_n(\tau_n; x)}}{\sqrt{2\pi x \lambda_n^*(\tau_n^2-1)\Theta_n''(\tau_n)}}, \qquad n \longrightarrow \infty. \tag{6.16}$$

The following result will give these asymptotics, and we to find general conditions on the sequence Λ, under which the requirement that $\tau_n^2 - 1$ is bounded below holds.

Theorem 6.10 *Let $\Lambda : -1/2 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots$ be a sequence of real numbers such that*

$$(i) \qquad \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{\lambda_k^*}{{\lambda_n^*}^2 - {\lambda_k^*}^2} = \infty$$

$$(ii) \qquad \frac{1}{{\lambda_n^*}^2} \sum_{k=0}^{n} \lambda_k^* \text{ is bounded below away from zero as } n \longrightarrow \infty$$

$$(iii) \qquad \text{If } \varepsilon_n = o(1), \text{ then } \frac{1}{{\lambda_n^*}^2} \sum_{\lambda_k^* < \lambda_n^*(1-\varepsilon_n)} \lambda_k^* = o\left(\sum_{\lambda_k^* < \lambda_n^*(1-\varepsilon_n)} \frac{\lambda_k^*}{{\lambda_n^*}^2 - {\lambda_k^*}^2} \right).$$

Then locally uniformly for x in $(1,\infty)$, as $n \longrightarrow \infty$,

$$L(\Lambda; x) \sim \frac{1}{2} \left(2\pi\rho_n^* {\sum_{k=0}^{n}}^{*} \frac{\lambda_k^*}{{\rho_n^*}^2 - {\lambda_k^*}^2} \right)^{-1/2} \prod_{k=0}^{n-1} \frac{\rho_n + \lambda_k + 1}{\rho_n - \lambda_k} \frac{x^{\rho_n}}{\rho_n - \lambda_n},$$

where $\rho_n > \lambda_n$ is uniquely determined by

$$\log x = 2 \sum_{k=0}^{n-1} \frac{\lambda_k^*}{{\rho_n^*}^2 - {\lambda_k^*}^2} + \frac{\lambda_n^*}{{\rho_n^*}^2 - {\lambda_n^*}^2}.$$

Proof. We need to show that conditions (i), (ii) and (iii) ensure that $\tau_n^2(x) - 1$ is uniformly bounded below for x in compact subsets of $(1,\infty)$. If $\tau_n^2 - 1 \ge 1$, then there is nothing to prove, so we assume that $\tau_n^2 - 1 < 1$. Then we have

$$\tau_n^2 - r_k^2 = (1 - r_k^2) + (\tau_n^2 - 1) \le \begin{cases} 2(1 - r_k^2), & \text{if } \tau_n^2 - 1 \le 1 - r_k^2 \\ 2(\tau_n^2 - 1), & \text{if } \tau_n^2 - 1 \ge 1 - r_k^2. \end{cases}$$

and it is easy to see from (6.3) that,

$$\begin{aligned} \log x &= \frac{2}{\lambda_n^*} \sum_{1 - r_k^2 \ge \tau_n^2 - 1} \frac{r_k}{\tau_n^2 - r_k^2} + \frac{2}{\lambda_n^*} \sum_{0 < 1 - r_k^2 < \tau_n^2 - 1} \frac{r_k}{\tau_n^2 - r_k^2} + \frac{1}{\lambda_n^*} \frac{1}{\tau_n^2 - 1} \\ &\ge \frac{1}{\lambda_n^*} \sum_{1 - r_k^2 \ge \tau_n^2 - 1} \frac{r_k}{1 - r_k^2} + \frac{1}{\lambda_n^*} \sum_{0 < 1 - r_k^2 < \tau_n^2 - 1} \frac{r_k}{\tau_n^2 - 1} + \frac{1}{\lambda_n^*} \frac{1}{\tau_n^2 - 1} \\ &= \sum_{1 - r_k^2 \ge \tau_n^2 - 1} \frac{\lambda_k^*}{{\lambda_n^*}^2 - {\lambda_k^*}^2} + \frac{1}{{\lambda_n^*}^2 (\tau_n^2 - 1)} \sum_{0 \le 1 - r_k^2 < \tau_n^2 - 1} \lambda_n^*. \end{aligned} \qquad (6.17)$$

If we assume to the contrary that $\tau_n^2 - 1 = o(1)$ as $n \longrightarrow \infty$, then the bound

$$\log x \ge \frac{1}{{\lambda_n^*}^2 (\tau_n^2 - 1)} \sum_{0 \le 1 - r_k^2 < \tau_n^2 - 1} \lambda_n^*$$

yields

$$\frac{1}{\lambda_n^{*2}} \sum_{0 \le 1 - r_k^2 < \tau_n^2 - 1} \lambda_n^* = o(1),$$

and the condition (ii) then implies that

$$1 \lesssim \frac{1}{\lambda_n^{*2}} \sum_{k=0}^{n} \lambda_k^* \sim \frac{1}{\lambda_n^{*2}} \sum_{1 - r_k^2 \ge \tau_n^2 - 1} \lambda_n^*.$$

Letting $\varepsilon_n = \tau_n^2 - 1$, we can rewrite $1 - r_k^2 \ge \tau_n^2 - 1$ as $\lambda_k^* \le \lambda_n^* \sqrt{1 - \varepsilon_n}$, so we can apply (iii) and then the bound $\sum_{1 - r_k^2 \ge \tau_n^2 - 1} \lambda_k^* / \left(\lambda_n^{*2} - \lambda_k^{*2}\right) \le \log x$ from (6.17) then yields

$$1 = o\left(\sum_{1 - r_k^2 \ge \tau_n^2 - 1} \frac{\lambda_k^*}{\lambda_n^{*2} - \lambda_k^{*2}} \right) = o(1),$$

a contradiction. Hence $\tau_n^2 - 1$ is bounded below away from 0, and this holds uniformly for x in compact subsets of $(1, \infty)$ since $\log x$ is bounded above and below on such sets.

It follows that (6.16) holds. Letting $\rho_n^* = \lambda_n^* \tau_n^*$ for each n, we can write

$$\begin{aligned} \lambda_n^* (\tau_n^2 - 1) \Theta_n''(\tau_n) &= (\tau_n^2 - 1) 4 \tau_n \sideset{}{^*}\sum_{k=0}^{n} \frac{r_k}{(\tau_n^2 - r_k^2)^2} \\ &= 4(\rho_n^{*2} - \lambda_n^{*2}) \rho_n^* \sideset{}{^*}\sum_{k=0}^{n} \frac{\lambda_k^*}{(\rho_n^{*2} - \lambda_k^{*2})^2}. \end{aligned}$$

Then writing out the phase function

$$\lambda_n^* \Theta_n(\tau_n) = \sum_{k=0}^{n-1} \log \frac{\tau_n + r_k}{\tau_n - r_k} + \frac{1}{2} \log \frac{\tau_n + 1}{\tau_n - 1} = \sum_{k=0}^{n-1} \log \frac{\rho_n^* + \lambda_k^*}{\rho_n^* - \lambda_k^*} + \frac{1}{2} \log \frac{\rho_n^* + \lambda_n^*}{\rho_n^* - \lambda_n^*}$$

yields

$$\begin{aligned} L_n(\Lambda; x) &\sim \frac{x^{\rho_n^*}}{\sqrt{2\pi x \cdot 4(\rho_n^{*2} - \lambda_n^{*2}) \rho_n^* \sideset{}{^*}\sum_{k=0}^{n} \frac{\lambda_k^*}{(\rho_n^{*2} - \lambda_k^{*2})^2}}} \prod_{k=0}^{n-1} \frac{\rho_n^* + \lambda_k^*}{\rho_n^* - \lambda_k^*} \left(\frac{\rho_n^* + \lambda_n^*}{\rho_n^* - \lambda_n^*} \right)^{1/2} \\ &\sim \frac{1}{2} \left(2\pi \rho_n^* \sideset{}{^*}\sum_{k=0}^{n} \frac{\lambda_k^*}{\rho_n^{*2} - \lambda_k^{*2}} \right)^{-1/2} \prod_{k=0}^{n-1} \frac{\rho_n + \lambda_k + 1}{\rho_n - \lambda_k} \frac{x^{\rho_n}}{\rho_n - \lambda_n}. \end{aligned}$$

$\square$

We are now ready to explore the special cases.

Proof of Theorem 6.3. Here we assume that $\sigma_n \longrightarrow \infty$ as $n \longrightarrow \infty$, so according to Lemmas 6.5 and 6.6, we have $(\lambda_n^*\tau_n)^2 \sim 2S_n/\log x$ and $\tau_n\Theta_n(\tau_n) \sim 2\log x$. Therefore,

$$\lambda_n^*(\tau_n^2-1)\Theta_n''(\tau_n) \sim (\lambda_n^*\tau_n)(\tau_n\Theta_n(\tau_n)) \sim 2(2S_n\log x)^{1/2},$$

as $n \longrightarrow \infty$. Then writing out the phase function

$$\lambda_n^*F_n(\tau_n;x) = \lambda_n^*\Theta_n(t) + \lambda_n^*\tau_n\log x = \sideset{}{^*}\sum_{k=0}^{n} \log\frac{\tau_n+r_k}{\tau_n-r_k} + \lambda_n^*\tau_n\log x$$

in (6.16) and letting $\rho_n^* = \lambda_n^*\tau_n$ yields

$$\begin{aligned} L_n(\Lambda;x) &\sim \frac{x^{\lambda_n^*\tau_n}}{\sqrt{2\pi x\cdot 2(2S_n\log x)^{1/2}}}\prod_{k=0}^{n-1}\frac{\tau_n+r_k}{\tau_n-r_k}\left(\frac{\tau_n+1}{\tau_n-1}\right)^{1/2} \\ &\sim \frac{x^{\rho_n}}{2\sqrt{\pi}\,(2S_n\log x)^{1/4}}\prod_{k=0}^{n}\frac{\rho_n^*+\lambda_k^*}{\rho_n^*-\lambda_k^*}, \end{aligned}$$

and here we have used $(\tau_n+1)/(\tau_n-1) \sim 1$ as $n \longrightarrow \infty$ since $\tau_n \longrightarrow \infty$. The result now follows from $S_n = \sum_{k=0}^{n-1}\lambda_k^* + \lambda_n^*/2 \sim \sum_{k=0}^{n}\lambda_k^*$, which is guaranteed by our assumption ${\lambda_n^*}^2/\sum_{k=0}^{n}\lambda_k^* = o(1)$. □

Proof of Theorem 6.4. As in Theorem 6.3, we assume that $\sigma_n \longrightarrow \infty$, so $\tau_n \longrightarrow \infty$ by Lemma 6.5. Then we get

$$\begin{aligned} \Theta_n^*(\tau_n) &= \frac{1}{\lambda_n^*}\sideset{}{^*}\sum_{k=0}^{n}\log\left(1+\frac{2r_k}{\tau_n-r_k}\right) \\ &= \frac{1}{\lambda_n^*}\sideset{}{^*}\sum_{k=0}^{n}\frac{2r_k}{\tau_n-r_k}\,[1+o(1)] \\ &= \frac{2\tau_n}{\lambda_n^*}\sideset{}{^*}\sum_{k=0}^{n}\frac{r_k}{\tau_n^2-r_k^2}\,[1+o(1)] \\ &= \tau_n\log x\,[1+o(1)]\,, \end{aligned}$$

where in the last step we use the identity (6.3). Therefore we have $F_n(\tau_n;x) = \Theta_n(\tau_n) + \tau_n\log x = 2\tau_n\log x[1+o(1)]$. Using $(\lambda_n^*\tau_n)^2 \sim 2S_n/\log x$ then yields

$$\frac{\lambda_n^*F_n(\tau_n;x)}{(2S_n)^{1/2}} \sim \frac{2\lambda_n^*\tau_n\log x}{\lambda_n^*\tau_n(\log x)^{1/2}} = 2(\log x)^{1/2}.$$

We saw in the proof of Theorem 6.3 that the denomiator in (6.16) behaves like $2\sqrt{\pi}\,(2S_n \log x)^{1/4}$ and it follows that

$$L_n(\Lambda;x)^{(2S_n)^{-1/2}} \sim \left[e^{\lambda_n^* F_n(\tau_n;x)}\right]^{(2S_n)^{-1/2}} \sim e^{2(\log x)^{1/2}}.$$

The proof is completed by noting that $\Sigma_n = 2S_n$. $\square$

Now we turn to the case when $\lambda_n \sim \alpha n$ for some constant $\alpha > 0$. If $\alpha = 1$ we expect the phase function $F_n(t;x) = \Theta_n(t) + t\log x = \frac{1}{\lambda_n^*}\sum_{k=0}^{*n} \log\frac{t+r_k}{t-r_k} + t\log x$ to behave like the function

$$\begin{aligned} F(t) &= \int_0^1 \log\frac{t+u}{t-u}du + t\log x \\ &= \log\frac{t+1}{t-1} + t\left[\log\frac{t^2-1}{t^2} + \log x\right]. \end{aligned}$$

Then

$$F'(t) = \log\frac{t^2-1}{t^2} + \log x, \qquad F''(t) = \frac{2}{t(t^2-1)}.$$

If we let $\gamma_0 = (x/(x-1))^{1/2}$, i.e. $x = \gamma_0^2/(\gamma_0^2-1)$, then $F'(\gamma_0) = 0$ and

$$F(\gamma_0) = \log\frac{\gamma_0+1}{\gamma_0-1} = \log\frac{x^{1/2}+(x-1)^{1/2}}{x^{1/2}-(x-1)^{1/2}} = \log\left[x^{1/2}+(x-1)^{1/2}\right]^2,$$

which yields $e^{\lambda_n^* F(\gamma_0)} = \left[x^{1/2}+(x-1)^{1/2}\right]^{2\lambda_n+1}$. Furthermore

$$F''(\gamma_0) = \frac{2(x-1)^{3/2}}{x^{1/2}}.$$

If we let $\Theta(t) = \int_0^1 \log\frac{t+u}{t-u}du$, so that $F(t) = \Theta(t) + t\log x$, then we have

$$F_n(\gamma_0) = \log\left[x^{1/2}+(x-1)^{1/2}\right]^2 + (\Theta_n-\Theta)(\gamma_0) \tag{6.18}$$

$$F_n'(\gamma_0) = (\Theta_n-\Theta)'(\gamma_0) \tag{6.19}$$

$$F_n''(\gamma_0) = \frac{2(x-1)^{3/2}}{x^{1/2}} + (\Theta_n-\Theta)''(\gamma_0). \tag{6.20}$$

Now we recycle Lemma 4.29 from Chapter 4. This gives the following:

Lemma 6.11 *If $\lambda_n \sim n$ as $n \longrightarrow \infty$, then*

$$\begin{aligned}
F_n(\gamma_0) &= \log\left[x^{1/2} + (x-1)^{1/2}\right]^2 + o(1) \\
F_n'(\gamma_0) &= o(1) \\
F_n''(\gamma_0) &= \frac{2(x-1)^{3/2}}{x^{1/2}} + o(1),
\end{aligned}$$

where the rate of convergence for all the $o(1)$ errors is

$$\mathcal{O}\left(\frac{1}{n^2}\sum_{j=0}^{n-1}|j+\beta-\lambda_j| + \frac{|n+\beta-\lambda_n|}{n}\right) + \mathcal{O}\left(\frac{1}{n^2}\right)$$

for each constant β.

Proof. This follows directly from (6.18), (6.19) and (6.20) using Lemma 4.29 and the fact that the integrands in the integral representations of $\Theta(t)$, $\Theta'(t)$ and $\Theta''(t)$ are $\log((t-s)/(t+s))$, $s/(t^2-s^2)$ and $s/(t^2-s^2)^2$, respectively, and they are all $\mathcal{O}(s)$ as $s \longrightarrow \infty$. □

We also need the following results to ensure that in this case, (6.16) indeed holds, and that have $\tau_n \sim \gamma_0$ as $n \longrightarrow \infty$.

Lemma 6.12 *If $\lambda_n \sim n$ as $n \longrightarrow \infty$, then $\tau_n^2(x) - 1 \asymp 1$, and this holds locally uniformly for x in $(1, \infty)$.*

Proof. If $\lambda_n \sim n$, then $\sigma_n \asymp 1$ as $n \longrightarrow \infty$, so $\tau_n^2 - 1 = \mathcal{O}(1)$ by Lemma 6.5. If $\tau_n^2 - 1 \geq 1$, then we are done. Otherwise, since $\tau_n^2 - r_k^2 = (\tau_n^2 - 1) + (1 - r_k^2) \leq 2(1 - r_k^2)$ if $\tau_n^2 - 1 \leq 1 - r_k^2$, we have the bound

$$\log x = \frac{2}{\lambda_n^*}\sum_{k=0}^{n}{}^{*}\frac{r_k}{\tau_n^2 - r_k^2} \geq \frac{2}{\lambda_n^*}\sum_{1-r_k^2 \geq \tau_n^2 - 1}\frac{r_k}{1-r_k^2},$$

and since $\lambda_n \sim n$, this yields

$$\log x \gtrsim 2\int_0^{\sqrt{1-(\tau_n^2-1)}}\frac{s}{1-s^2}ds = -\int_1^{\tau_n^2-1}\frac{du}{u} = -\log(\tau_n^2 - 1).$$

Hence, $\tau_n^2 - 1$ is bounded below away from 0. Since $\log x$ is bounded above and below on compact subsets of $(1, \infty)$, the result follows. $\square$

Lemma 6.13 *If $\lambda_n \sim n$ as $n \longrightarrow \infty$, then*

$$\begin{aligned} \tau_n - \gamma_0 &= o(F_n'(\gamma_0)), \\ F_n(\tau_n) - F_n(\gamma_0) &= o(F_n'(\gamma_0)^2), \\ F_n''(\tau_n) - F_n''(\gamma_0) &= o(F_n'(\gamma_0)), \end{aligned}$$

Proof. Expanding $F_n'(t)$ about the stationary point $t = \tau_n$ yields

$$F_n'(\gamma_0) = F_n'(\tau_n) + F_n''(\xi_n)(\gamma_0 - \tau_n) = F_n''(\xi_n)(\gamma_0 - \tau_n), \tag{6.21}$$

where ξ_n is between γ_0 and τ_n. We have the basic estimate

$$F_n''(t) = \frac{4t}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{(t^2 - r_k^2)^2} \geq \frac{4t}{\lambda_n^*} \sum_{k=0}^{n}{}^{*} \frac{r_k}{t^4} = \frac{4\sigma_n}{t^3},$$

and since $\sigma_n \asymp 1$ as $n \longrightarrow \infty$, τ_n is bounded above by Lemma 6.5, and thus

$$F_n''(\xi_n) \geq \frac{4\sigma_n}{\min\{\gamma_0, \tau_n\}} \gtrsim 1, \qquad n \longrightarrow \infty.$$

It follows from (6.21) that $\gamma_0 - \tau_n = \mathcal{O}\left(F_n'(\gamma_0)\right)$.

Now we expand $F_n(t)$ about $t = \tau_n$ and get

$$F_n(\gamma_0) - F_n(\tau_n) = \frac{F_n(\nu_n)}{2}(\gamma_0 - \tau_n)^2,$$

for some ν_n between τ_n and γ_0. Note that for $t \geq 1$,

$$F_n''(t) \leq \frac{4t\sigma_n}{(t^2-1)^2},$$

so according to Lemma 6.12 we have $F_n''(\nu_n) = \mathcal{O}(1)$ as $n \longrightarrow \infty$. It follows that $|F_n(\gamma_0) - F_n(\tau_n)| = \mathcal{O}\left(|\gamma_0 - \tau_n|^2\right) = \mathcal{O}\left(F_n'(\gamma_0)^2\right)$.

The treatment above shows that $F_n''(\tau_n)$ and $F_n''(\gamma_0)$ are bounded above and below. Thus, the same way as above, $F_n''(\tau_n) - F_n''(\gamma_0) = \mathcal{O}\left(\gamma_0 - \tau_n\right)$ follows if we can show that $F_n^{(3)}(t)$ is bounded above when t is between τ_n and γ_0. This is clear since

$$F_n^{(3)}(t) = -\frac{4}{\lambda_n^*}\sum_{k=0}^{n}{}^{*} r_k \frac{3t^2 + r_k^2}{(t^2 - r_k^2)^3}$$

so for $t > 1$,

$$|F_n^{(3)}(t)| \leq 4\sigma_n \frac{3t^2+1}{(t^2-1)^3}.$$

□

Before we give the proofs for Theorems 6.1 and 6.2, we recall how the case $n/\lambda_n \sim \rho$ can be mapped to the case when $\rho = 1$. For a given sequence $\Lambda = \{\lambda_n\}$ define $\Pi = \{\mu_n\}$ by letting

$$\mu_n = \rho\lambda_n + \frac{1}{2}\left(\rho - 1\right),$$

or equivalently, $\mu_n^* = \rho\lambda_n^*$ for each n. Then as shown in (4.34)

$$L_n(\Lambda; u^\rho) = L_n(\Pi; u)u^{\frac{1}{2}(1-\rho)}. \tag{6.22}$$

Proof of Theorem 6.1. First assume that $\rho = 1$. Then Lemma 6.12 ensures that (6.16) holds in this case. If $\lambda_n = n + \beta/2 + o\left(1\right)$ as $n \longrightarrow \infty$, then

$$\frac{1}{n^2}\sum_{j=0}^{n-1}\left|j + \frac{\beta}{2} - \lambda_j\right| + \frac{|n+\beta/2-\lambda_n|}{n} = o\left(\frac{1}{n}\right),$$

so we have $F_n'(\gamma_0) = o\left(1/\lambda_n^*\right)$ as $n \longrightarrow \infty$ according to Lemma 6.11. Then Lemma 6.13 gives $F_n(\tau_n) - F_n(\gamma_0) = o\left(1/\lambda_n^*\right)$, $\tau_n \sim \gamma_0$ and $F_n''(\tau_n) \sim F_n''(\gamma_0)$ so (6.16) yields

$$L_n(\Lambda; x) \sim \frac{e^{\lambda_n^* F_n(\gamma_0)}}{\sqrt{2\pi x n(\gamma_0^2 - 1)F_n''(\gamma_0)}}.$$

Then Lemma 6.11 and $\gamma_0^2 - 1 = 1/(x-1)$ yield

$$L_n(\Lambda; x) \sim \frac{e^{\lambda_n^* \log\left[x^{1/2} + (x-1)^{1/2}\right]^2}}{\sqrt{2\pi x n \frac{1}{x-1}\frac{2(x-1)^{3/2}}{x^{1/2}}}} = \frac{\left[x^{1/2} + (x-1)^{1/2}\right]^{2\lambda_n^*}}{2\sqrt{\pi n}x^{1/4}(x-1)^{1/4}},$$

and by noting that $2\lambda_n^* = 2n + \beta + 1 + o(1)$, the proof is complete for this case.

Now assume that $\lambda_n = (n + \beta/2)/\rho + o(1)$ and define $\Pi = \{\mu_n\}$ by letting $\mu_n^* = \rho\lambda_n^*$ so that $\mu_n = n + (\beta + \rho - 1)/2 + o(1)$ as $n \longrightarrow \infty$. Then according to (6.22), and the proof for the special case $\rho = 1$ above,

$$\begin{aligned} L_n(\Lambda; x^\rho) &= L_n(\Pi; x) x^{\frac{1}{2}(1-\rho)} \\ &\sim \frac{\left[x^{1/2} + (x-1)^{1/2}\right]^{2\mu_n^*}}{2\sqrt{\pi n} x^{1/4}(x-1)^{1/4}} x^{\frac{1}{2}(1-\rho)} \\ &\sim \frac{\left[x^{1/2} + (x-1)^{1/2}\right]^{2n+\beta+\rho}}{2\sqrt{\pi n} x^{(2\rho-1)/4}(x-1)^{1/4}}. \end{aligned}$$

This completes the proof. □

Proof of Theorem 6.2. This follows from the proof of Theorem 6.1; the only difference is that since we are taking the nth root it suffices that $F_n(\tau_n) - F(\gamma_0) = o(1)$, and this holds if $\lambda_n \sim n/\rho$ according to Lemmas 6.11 and 6.13. □

CHAPTER VII

ASYMPTOTIC BEHAVIOR OF MÜNTZ-CHRISTOFFEL FUNCTIONS AT THE ENDPOINTS

This chapter is built on results from the paper [52]. At the endpoints $x = 0$ and 1, we compare the asymptotic behavior of Christoffel functions associated with different Müntz systems, for which the respective sequences of exponents are asymptotic. Along with the known asymptotics for the algebraic polynomials, we use this to establish the asymptotic behavior of the Christoffel functions in the case when the exponents are asymptotic to an arithmetic progression, i.e. such that

$$\lim_{k\to\infty} \frac{\lambda_k}{k} = \rho,$$

for some $\rho > 0$. We utilize the fact that the Christoffel functions can be written in terms of the Müntz-Legendre polynomials, which take on a simple form at the endpoints $x = 0$ and $x = 1$. Our main results are Theorems 4.1 and 4.2 below.

7.1 Lemmas

7.1.1 Asymptotics in polynomial spaces with Jacobi weights

Recall that on the interval $[-1, 1]$, the classical Jacobi weights are defined as

$$w^{(\alpha,\beta)}(x) := (1-x)^{\alpha}(1+x)^{\beta}, \qquad x \in [-1, 1],$$

for given real numbers $\alpha, \beta > -1$. In order to use these weights with the Müntz systems, we need to shift them to the interval $[0, 1]$, where the Müntz polynomials are well defined. Using the mapping $[-1, 1] \ni x \mapsto (x+1)/2 \in [0, 1]$ we obtain Jacobi weights on $[0, 1]$ as

$$u^{(\alpha,\beta)}(x) := (1-x)^{\alpha}x^{\beta}, \qquad x \in [0, 1]. \tag{7.1}$$

Note that $u^{(\alpha,\beta)}(x) = w^{(\alpha,\beta)}(2x-1)/2^{\alpha+\beta}$, for all $x \in [0,1]$.

Lemma 7.1 *Under the Jacobi weights $u^{(\alpha,\beta)}$ on $[0,1]$, the Christoffel functions over the algebraic polynomials satisfy the following asymptotic behavior at the endpoints,*

$$\lim_{n\to\infty} n^{2\alpha+2}\lambda_n(u^{(\alpha,\beta)},1) = (\alpha+1)\Gamma(\alpha+1)^2, \tag{7.2}$$

$$\lim_{n\to\infty} n^{2\beta+2}\lambda_n(u^{(\alpha,\beta)},0) = (\beta+1)\Gamma(\beta+1)^2. \tag{7.3}$$

Proof. We know from Theorem 2.10, that the Christoffel functions over the algebraic polynomials with respect to the Jacobi weights on $[-1,1]$ satisfy

$$\begin{aligned}\lim_{n\to\infty} n^{2\alpha+2}\lambda_n(w^{(\alpha,\beta)},1) &= (\alpha+1)2^{\alpha+\beta+1}\Gamma(\alpha+1)^2,\\ \lim_{n\to\infty} n^{2\beta+2}\lambda_n(w^{(\alpha,\beta)},-1) &= (\beta+1)2^{\alpha+\beta+1}\Gamma(\beta+1)^2.\end{aligned}$$

Then using the substitution $t = 2x-1$ and letting $Q_n(t) = P_n((t+1)/2)$, we obtain

$$\begin{aligned}\lambda_n(u^{(\alpha,\beta)},c) &= \inf_{P_n(c)=1}\int_0^1 P_n^2(x)u^{(\alpha,\beta)}(x)dx\\ &= \inf_{P_n(c)=1}\int_{-1}^1 P_n^2\left(\frac{t+1}{2}\right)\frac{w^{(\alpha,\beta)}(t)}{2^{\alpha+\beta}}\frac{dt}{2}\\ &= \frac{1}{2^{\alpha+\beta+1}}\inf_{Q_n(2c-1)=1}\int_{-1}^1 Q_n^2(t)w^{(\alpha,\beta)}(t)dt\\ &= \frac{1}{2^{\alpha+\beta+1}}\lambda_n(w^{(\alpha,\beta)},2c-1),\end{aligned}$$

and it follows that $\lambda_n(w,1) = 2^{\alpha+\beta+1}\lambda_n(u,1)$ and $\lambda_n(w,-1) = 2^{\alpha+\beta+1}\lambda_n(u,0)$. $\square$

Now let $\lambda_0, \lambda_1, \lambda_2, \dots$ and $\mu_0, \mu_1, \mu_2, \dots$ be sequences of real numbers with $\Lambda_n = \{\lambda_k\}_{k=0}^n$ and $\Pi_n = \{\mu_k\}_{k=0}^n$, and define the corresponding Müntz systems $M(\Lambda_n)$ and $M(\Pi_n)$ on $[0,1]$. Here, we wish to compare the Christoffel functions of the two spaces at the endpoints $x=0$ and $x=1$. First, for the right endpoint we have, according to equation (2.25) and Lemma 2.18,

$$\frac{1}{\lambda(M_n(\Lambda);1)} - \frac{1}{\lambda(M_n(\Pi);1)} = \sum_{k=0}^n[(2\lambda_k+1)-(2\mu_k+1)] = 2\sum_{k=0}^n(\lambda_k-\mu_k),$$

or equivalently,

$$\frac{\lambda(M_n(\Pi);1)}{\lambda(M_n(\Lambda);1)} = 1 + 2\lambda(M_n(\Pi);1)\sum_{k=0}^{n}(\lambda_k - \mu_k). \tag{7.4}$$

A simple consequence of this identity is the following lemma on monotonicity.

Lemma 7.2 *Let $\Lambda_n := \{\lambda_0, \lambda_1, \lambda_2 \ldots, \lambda_n\}$ and $\Pi_n := \{\mu_0, \mu_1, \mu_2 \ldots, \mu_n\}$ be sequences of positive real numbers such that $\lambda_k \le \mu_k$ for all $k = 0, 1, \ldots, n$, with strict inequality for at least one index. Then*

$$\lambda(M(\Lambda_n);1) > \lambda(M(\Pi_n);1).$$

As for the other endpoint, $x = 0$, we assume that $\lambda_0 = \mu_0 = 0$, so the Müntz-Legendre polynomials take the values $L_m(\Lambda;0) = c^{\Lambda}_{0,m}$ and $L_m(\Pi;0) = c^{\Pi}_{0,m}$, where $c^{\Lambda}_{0,m}$ and $c^{\Pi}_{0,m}$ are the coefficients from (2.20), for the respective systems. Then

$$\frac{|c^{\Pi}_{0,m}|}{|c^{\Lambda}_{0,m}|} = \frac{\frac{1}{\mu_m}\prod_{j=1}^{m-1}\frac{\mu_j+1}{\mu_j}}{\frac{1}{\lambda_m}\prod_{j=1}^{m-1}\frac{\lambda_j+1}{\lambda_j}} = \frac{\lambda_m}{\mu_m}\prod_{j=1}^{m-1}\frac{\left(1+\frac{1}{\mu_j}\right)}{\left(1+\frac{1}{\lambda_j}\right)}$$

and since

$$\frac{\lambda(M_n(\Lambda);0)}{\lambda(M_n(\Pi);0)} = \frac{\sum_{m=0}^{n}(2\mu_m+1)|c^{\Pi}_{0,m}|^2}{\sum_{m=0}^{n}(2\lambda_m+1)|c^{\Lambda}_{0,m}|^2}$$

the following monotonicity lemma follows directly:

Lemma 7.3 *Let $\Lambda_n := \{\lambda_0, \lambda_1, \lambda_2 \ldots, \lambda_n\}$ and $\Pi_n := \{\mu_0, \mu_1, \mu_2 \ldots, \mu_n\}$ be sequences of positive real numbers such that $\lambda_0 = \mu_0 = 0$, and $\lambda_k \le \mu_k$ for all $k = 0, 1, \ldots, n$, with strict inequality for at least one index. Then*

$$\lambda(M(\Lambda_n);0) < \lambda(M(\Pi_n);0).$$

7.2 A Comparison Theorem

Here, we wish to compare the Christoffel functions for two different Müntz systems, at the endpoints of $[0, 1]$.

Theorem 7.4 *Consider two sequences $\Lambda = \{\lambda_k\}$ and $\Pi = \{\mu_k\}$ of distinct non-negative real numbers such that $\lambda_k, \mu_k \longrightarrow \infty$ and*

$$\lim_{k\to\infty} \frac{\lambda_k}{\mu_k} = 1.$$

For each n, let $M_n(\Lambda) := span\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$ and $M_n(\Pi) := span\{x^{\mu_0}, x^{\mu_1}, \ldots, x^{\mu_n}\}$ be the corresponding Müntz spaces on $[0,1]$. Then,

(i)

$$\lim_{n\to\infty} \frac{\lambda(M_n(\Lambda);1)}{\lambda(M_n(\Pi);1)} = 1.$$

(ii) and if $\lambda_0 = \mu_0 = 0$, and the limit

$$\alpha := \lim_{n\to\infty} \prod_{k=1}^{n} \frac{\lambda_k}{\mu_k} \frac{\mu_k+1}{\lambda_k+1} \tag{7.5}$$

exists and is finite, then

$$\lim_{n\to\infty} \frac{\lambda(M_n(\Lambda);0)}{\lambda(M_n(\Pi);0)} = \alpha^2.$$

Proof. (i) At the endpoint $x = 1$, it follows from equation (7.4) that

$$\left| \frac{\lambda(M_n(\Pi);1)}{\lambda(M_n(\Lambda);1)} - 1 \right| \le 2\lambda(M_n(\Pi);1) \sum_{k=0}^{n} |\lambda_k - \mu_k|.$$

Since $\lambda_k/\mu_k \longrightarrow 1$ and $\mu_k \longrightarrow \infty$, we have

$$\sum_{k=0}^{n} |\lambda_k - \mu_k| = \sum_{k=0}^{n} \mu_k \left| \frac{\lambda_k}{\mu_k} - 1 \right| = o\left\{ \sum_{k=0}^{n} (2\mu_k + 1) \right\},$$

and the result follows from $\lambda(M_n(\Pi);1)^{-1} = \sum_{k=0}^{n} (2\mu_k+1)$.

(ii) Now, we turn our attention to the endpoint $x = 0$. For each $l \in \mathbb{N}$ define the number

$$\alpha_l = \prod_{j=1}^{l} \frac{\lambda_j(\mu_j+1)}{\mu_j(\lambda_j+1)}.$$

Then, since we assume here that $\lambda_0 = \mu_0 = 0$, we have

$$\frac{|c_{0,m}^{\Pi}|}{|c_{0,m}^{\Lambda}|} = \frac{\frac{1}{\mu_m} \prod_{j=1}^{m-1} \frac{\mu_j+1}{\mu_j}}{\frac{1}{\lambda_m} \prod_{j=1}^{m-1} \frac{\lambda_j+1}{\lambda_j}} = \frac{\lambda_m}{\mu_m} \alpha_{m-1}.$$

Therefore, we can write

$$\begin{aligned}
\frac{1}{\lambda(M_n(\Pi);0)} &= \sum_{m=0}^{n}(2\mu_m+1)|c_{0,m}^{\Pi}|^2 \\
&= \sum_{m=0}^{n}(2\mu_m+1)\frac{\lambda_m^2}{\mu_m^2}|c_{0,m}^{\Lambda}|^2\alpha_{m-1}^2 \\
&= \sum_{m=0}^{n}\frac{\lambda_m}{\mu_m}\left(2\lambda_m+\frac{\lambda_m}{\mu_m}\right)|c_{0,m}^{\Lambda}|^2\alpha_{m-1}^2.
\end{aligned}$$

Then, since $\lambda_k/\mu_k \longrightarrow 1$, $\alpha_k \longrightarrow \alpha$ and $\lambda_k \longrightarrow \infty$ as $k \longrightarrow \infty$, it follows that

$$\lambda(M_n(\Pi);0)^{-1} \sim \sum_{m=0}^{n}(2\lambda_m+1)|c_{0,m}^{\Lambda}|^2\alpha^2 = \alpha^2\lambda(M_n(\Lambda);0)^{-1},$$

as $n \longrightarrow \infty$, and we are done. □

Remark The condition in (7.5) can be replaced by requiring convergence of the series

$$\sum_{k=1}^{\infty}\left|\frac{1}{\lambda_k}-\frac{1}{\mu_k}\right|.$$

We see this by noticing that

$$\prod_{k=1}^{n}\frac{\lambda_k}{\mu_k}\frac{\mu_k+1}{\lambda_k+1} = \prod_{k=1}^{n}\left(1+\frac{\lambda_k-\mu_k}{\mu_k(\lambda_k+1)}\right) \le \prod_{k=1}^{n}\left(1+\left|\frac{\lambda_k-\mu_k}{\mu_k\lambda_k}\right|\right),$$

and recalling that $\prod(1+a_k)$ and $\sum a_k$ converge simultaneously for $a_k \ge 0$.

7.3 Asymptotic behavior of the Müntz-Christoffel function at the endpoints

7.3.1 Müntz systems with $\{\mu_k\} = \{k\rho\}$

Here we consider the simple case when the powers of the Müntz basis elements satisfy an arithmetic progression, that is we assume that there exists a positive number ρ such that

$$\mu_k = k\rho, \qquad \text{for all } k \ge 0.$$

Let $M_{n,\rho}$ be the span of $(1, x^\rho, \ldots, x^{n\rho})$, and note that for each element $S(x) = \sum_{k=0}^{n} c_k x^{k\rho}$ we can write

$$S(x) = P(x^\rho),$$

where $P(x) = \sum_{k=0}^{n} c_k x^k$ is a polynomial of degree n. Then, using the change of variables $u = t^\rho$, the Christoffel function over the Lebesque measure on $[0,1]$ can be written as

$$\begin{aligned}
\lambda(M_{n,\rho}; x) &= \inf_{S \in M_{n,\rho}} \frac{\int_0^1 S^2(t)dt}{|S(x)|^2} = \inf_{P \in \mathcal{P}_n} \frac{\int_0^1 P^2(t^\rho)dt}{|P(x^\rho)|^2} \\
&= \frac{1}{\rho} \inf_{P \in \mathcal{P}_n} \frac{\int_0^1 P^2(u) u^{1/\rho - 1} du}{|P(x^\rho)|^2} \\
&= \frac{1}{\rho} \cdot \lambda(\mathcal{P}_n, u^{(0,1/\rho-1)}; x^\rho),
\end{aligned}$$

where $u^{(0,1/\rho-1)}$ is the Jacobi weight $u^{(0,1/\rho-1)}(x) = u^{1/\rho-1}$ on $[0,1]$ with $\alpha = 0$ and $\beta = 1/\rho - 1$ in (7.1). Then, using (7.2), we obtain

$$\lim_{n\to\infty} n^2 \lambda(M_{n,\rho}; 1) = \frac{1}{\rho} \cdot \Gamma(1)^2 = \frac{1}{\rho}, \tag{7.6}$$

and since $\beta + 1 = 1/\rho$, equation (7.3) yields

$$\lim_{n\to\infty} n^{2/\rho} \lambda(M_{n,\rho}; 0) = \frac{1}{\rho} \cdot \frac{1}{\rho} \Gamma(1/\rho)^2 = \left(\frac{\Gamma(1/\rho)}{\rho}\right)^2. \tag{7.7}$$

7.3.2 Müntz systems with $\{\mu_k\}$ asymptotic to $\{k\rho\}$

From the comparison theorem in section 3, the following theorems are direct consequences of (7.6) and (7.7).

Theorem 7.5 *Let $\Pi = \{\mu_k\}$ be a sequence of real numbers such that*

$$\lim_{k\to\infty} \frac{\mu_k}{k} = \rho,$$

for some constant $\rho > 0$. If we let $M_n(\Pi) := span\{x^{\mu_0}, x^{\mu_1}, \ldots, x^{\mu_n}\}$ for each n, then

$$\lim_{n\to\infty} n^2 \lambda(M_n(\Pi); 1) = \frac{1}{\rho}.$$

Theorem 7.6 *Let $\Pi = \{\mu_k\}$ be a sequence of distinct real numbers such that*

$$\lim_{k\to\infty} \frac{\mu_k}{k} = \rho,$$

for some constant $\rho > 0$*, and such that the limit*

$$\alpha := \lim_{n\to\infty} \prod_{k=1}^{n} \frac{\mu_k}{k\rho} \frac{k\rho + 1}{\mu_k + 1}$$

exists and is finite. If we let $M_n(\Pi) := span\{x^{\mu_0}, x^{\mu_1}, \ldots, x^{\mu_n}\}$ *for each* n*, then*

$$\lim_{n\to\infty} n^{2/\rho} \lambda(M_n(\Pi); 0) = \frac{\alpha^2 \Gamma\left(1/\rho\right)^2}{\rho^2}.$$

CHAPTER VIII

FUTURE WORK

8.1 Random matrix theory: Müntz ensembles

8.1.1 Biorthogonal ensembles

In [35], K. A. Muttalib argued that for certain physical systems with two-body interactions, the standard random matrix model does not provide sufficient accuracy. He, and Borodin [5] did this more generally, proposed using ensembles built out of biorthogonal systems [17]. They turn out to have several of the same properties as their orthogonal ensemble cousins; most importantly the correlation functions also have determinantal form, so important statistics can be derived from the behavior of the biorthogonal functions themselves. Recently, a Riemann-Hilbert problem was formulated for biorthogonal polynomials by Kuijlaars et al. [20].

A *biorthogonal ensemble* is a probability density function of the form

$$\mathcal{P}(x_1, \ldots, x_n) = \frac{1}{Z_n} \det\left[f_i(x_j)\right]_{i,j=1}^n \det\left[g_i(x_j)\right]_{i,j=1}^n,$$

where $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are two given sequences of functions and the normalizing constant

$$Z_n = \int \cdots \int \det\left[f_i(x_j)\right]_{i,j=1}^n \det\left[g_i(x_j)\right]_{i,j=1}^n dx_1 \cdots dx_n$$

is defined such that $\mathcal{P}(x_1, \ldots, x_n)$ is indeed a probability density function.

Suppose that the systems $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ can be biorthogonalized with respect to each other. This means that we can find functions $\phi_1, \ldots, \phi_n$ in the span of $f_1, \ldots, f_n$ and $\psi_1, \ldots, \psi_n$ in the span of $g_1, \ldots, g_n$ such that

$$\int \phi_j(x)\psi_k(x)dx = \delta_{j,k}.$$

Then the biorthogonal ensemble is a *determinantal point process* with correlation kernel

$$K_n(x,y) = \sum_{j=0}^{n} \phi_j(x)\psi_k(x), \tag{8.1}$$

which means that we can write

$$\mathcal{P}(x_1,\ldots,x_n) = \frac{1}{n!}\det\left[K_n(x_i,x_j)\right]_{i,j=1}^{n}.$$

It follows that for every $k = 1,2,\ldots,n$, we can write the conditional probabilities (correlation functions) as

$$\int \cdots \int \mathcal{P}(x_1,\ldots,x_n)dx_{k+1}\cdots dx_n = \frac{(n-k)!}{n!}\det\left[K_n(x_i,x_j)\right]_{i,j=1}^{k}.$$

Therefore, just as in the case of orthogonal ensembles, the correlation kernel (8.1) is written in terms of the orthogonal functions and is a key element in examining the statistics of the biorthogonal ensembles. For the biorthogonal case however, we don't have an analogue of the Christoffel-Darboux formula, and this makes the treatment different from the standard case.

8.1.2 Müntz ensembles

The biorthogonal ensembles described above include a special case considered by Borodin [5]: here the biorthogonal functions are *Müntz biorthogonal polynomials*, i.e. the orthogonal elements obtained by orthogonalizing the systems $M(\Lambda) = \text{span}\{x^{\lambda_n}\}_{n=0}^{\infty}$ and $M(\Pi) = \text{span}\{x^{\mu_n}\}_{n=0}^{\infty}$ with respect to each other. Indeed, Müntz biorthogonal polynomials can be obtained by slightly altering the definition for the Müntz orthogonal polynomials (2.18): if we define

$$\varphi_n(\Lambda_\Pi;x) = \frac{1}{2\pi i}\int_\Gamma \prod_{k=0}^{n-1}\frac{t+\mu_k+1}{t-\lambda_k}\frac{x^t}{t-\lambda_n}dt, \tag{8.2}$$

and $\varphi_n(\Pi_\Lambda;x)$ in an analogous way, then (if the sequences consist of distinct entries) $\varphi_n(\Lambda_\Pi;\cdot) \in M_n(\Lambda)$, $\varphi_n(\Pi_\Lambda;\cdot) \in M_n(\Pi)$ and

$$\int_0^1 \varphi_n(\Lambda_\Pi;x)\varphi_m(\Pi_\Lambda;x)dx = \frac{\delta_{n,m}}{\lambda_n+\mu_m+1}.$$

Since Müntz biorthogonal polynomials have a contour integral representation which is similar to (2.18), one can hope that the approach used in this thesis can be applied to determine the asymptotic behavior of the elements (8.2). For example: is there an equation like (3.1) for Müntz biorthogonal polynomials?

Borodin [5] considered very special cases, namely $\lambda_k = k$ and $\mu_k = \theta k$ for all k, where θ is some constant, for the classical weights. This is a one parameter deformation of the standard random matrix model. His approach involved writing the correlation kernel (8.1) as

$$K_n(x, y) = \sum_{1 \le i,j \le n} c_{i,j}^{(n)} x^{\lambda_i} y^{\mu_j}$$

and estimating the coefficients $c_{i,j}^{(n)}$ as $n \longrightarrow \infty$.

It would be interesting to see whether we can apply the results in this thesis to shed light on the Müntz ensembles for more general λ_j and μ_j or obtain more accurate asymptotics for cases where λ_j and μ_j have regular growth (one important special case is discussed in the next section). Of course, when $\lambda_j = \mu_j$ for all j, then we have Müntz orthogonal polynomials. The following question is important: is there an analogue to the Christoffel-Darboux formula for Müntz biorthogonal (or just orthogonal) polynomials? We were able to obtain an asymptotic formula (Theorem 4.8) for $K_n(x, x)$ in the cases when $\lambda_n = (n + \beta/2)/\rho + o(1)$ as $n \longrightarrow \infty$. Otherwise, we only have the uniform bounds in Theorem 4.9.

8.1.3 Biorthogonal Müntz polynomials as multiple orthogonal polynomials of mixed type

Multiple orthogonal polynomials (see Aptekarev [2] for detailed definitions) have applications in number theory and approximation theory, and have recently been connected to certain models in random matrix theory [4, 18]. Asymptotic results for special cases [13] have been obtained through a Riemann Hilbert problem for multiple orthogonal polynomials, formulated by Van Assche, Geronimo and Kuijlaars [56].

Here we show that a certain class of multiple orthogonal polynomials can be written in terms of (non-multiple) Müntz orthogonal polynomials.

Let β be a real number greater than $-1/2$ and define the sequence $\Lambda = \{\lambda_k\}$ by letting

$$\lambda_{2k} = k \qquad \text{and} \qquad \lambda_{2k+1} = k + \beta$$

for all $k = 0, 1, 2 \ldots$. Then each associated Müntz polynomial can be written in the form

$$p(x) + q(x)x^{\beta},$$

where p, q are algebraic polynomials. Therefore it is easy to see that the corresponding Müntz-Legendre polynomials are precisely the multiple orthogonal polynomials of type I with respect to the Jacobi weights $w_1(x) = 1$ and $w_2(x) = x^{\beta}$ on $[0, 1]$. If we let $\beta \longrightarrow 0$, then (see [7, Appendix 2]) we get multiple orthogonal polynomials of type I with respect to the weights $w_1(x) = 1$ and $w_2(x) = \log x$.

More generally if we fix distinct numbers $\beta_0, \beta_1, \ldots, \beta_{r-1} > -1/2$, and for each $j = 0, 1, 2, \ldots, r-1$, let

$$\lambda_{rk+j} = k + \beta_j, \qquad k = 0, 1, 2, \ldots,$$

then each Müntz polynomial associated with $\lambda = \{\lambda_n\}$ is of the form

$$\sum_{k=0}^{r-1} p_k(x) w_k(x),$$

where $p_k(x)$ are algebraic polynomials and $w_k(x) = x^{\beta_k}$ for all $k = 0, 1, \ldots, r-1$. If we write the orthonormal nth Müntz-Legendre polynomial in this form with p_k of degree $n_k - 1$ and $n = n_0 + n_1 + \cdots + n_{r-1}$, then they satisfy the orthogonality condition

$$\int_0^1 \left(\sum_{k=0}^{r-1} p_k(x) w_k(x) \right) x^j dx = \begin{cases} 0 & \text{for } j = 0, 1, \ldots n-2 \\ 1 & \text{for } j = n-1 \end{cases}$$

and this is precisely the definition for the multiple orthogonal polynomials of type I with respect to the weights $w_k(x)$, $k = 0, 1, \ldots, r-1$ on $[0,1]$. By equating some of the β_k's we can more generally get weights of the form $x^{\beta}(\log x)^m$, $m \in \mathbb{N}_0$.

8.2 Müntz-Christoffel functions

In Section 2.4, we introduced the Müntz-Christoffel functions $\lambda(M_n(\Lambda); x)$ associated with $\Lambda = \{\lambda_k\}$ and just as in the algebraic polynomials case it can be written in terms of the reproducing kernel as

$$\lambda(M_n(\Lambda); x)^{-1} = K_n(x,x) = \sum_{k=0}^{n} |L_k^*(x)|^2 ,$$

where $L_k^*(x)$ is the kth orthonormal Müntz-Legendre polynomial (2.22). The Müntz-Christoffel functions have been used in relation to Müntz-type of Gauss-Jacobi quadrature [32], density questions [6] and, as described above, the reproducing kernels are the key functions for determinantal processes that arise in random matrix theory.

Using our formula (3.1), we can write the reproducing kernel as a double integral, namely

$$\begin{aligned} K_n(x,y) &= \sum_{k=0}^{n} 2\lambda_k^* L_k(x) L_k(y) \\ &= \frac{2}{\pi^2\sqrt{xy}} \sum_{k=0}^{n} \lambda_k^* \int_0^\infty \frac{\sin\left(\Psi_k(t) - \lambda_k^* t \log x\right)}{\sqrt{1+t^2}} dt \int_0^\infty \frac{\sin\left(\Psi_k(s) - \lambda_k^* s \log y\right)}{\sqrt{1+s^2}} ds \\ &= \frac{2}{\pi^2\sqrt{xy}} \int_0^\infty \int_0^\infty \frac{1}{\sqrt{1+t^2}\sqrt{1+s^2}} \\ &\quad \times \sum_{k=0}^{n} \lambda_k^* \sin\left(\Psi_k(t) - \lambda_k^* t \log x\right) \sin\left(\Psi_k(s) - \lambda_k^* s \log y\right) dt ds. \end{aligned}$$

where $\Psi_n(t) = 2\sum_{j=0}^{n-1} \arctan \frac{\lambda_j^*}{\lambda_n^* t} + \arctan \frac{1}{t}$. Letting $c = -\log x$ and using a trigonometric sum-to-product identity, we can write

$$\begin{aligned} \pi^2 x K_n(x,x) = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{1+t^2}\sqrt{1+s^2}} \sum_{k=0}^{n} \lambda_k^* \cos\left(\Psi_k(t) - \Psi_k(s) + \lambda_k^* c(t-s)\right) \\ - \int_0^\infty \int_0^\infty \frac{1}{\sqrt{1+t^2}\sqrt{1+s^2}} \sum_{k=0}^{n} \lambda_k^* \cos\left(\Psi_k(t) + \Psi_k(s) + \lambda_k^* c(t+s)\right) \end{aligned}$$

Can we look at this as a type of Christoffel-Darboux identity? For special cases, can we simplify this expression and obtain asymptotic formulas for $K_n(x,x)$ (or $K_n(x,y)$) via the asymptotics obtained in this thesis for the Müntz orthogonal polynomials?

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