ASYMPTOTIC THEORY FOR DECENTRALIZED SEQUENTIAL HYPOTHESIS TESTING PROBLEMS AND SEQUENTIAL MINIMUM ENERGY DESIGN ALGORITHM

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ASYMPTOTIC THEORY FOR DECENTRALIZED SEQUENTIAL HYPOTHESIS TESTING PROBLEMS AND SEQUENTIAL MINIMUM ENERGY DESIGN ALGORITHM

Approved by:

Professor Yajun Mei, Advisor,
Committee Chair
School of Industrial and System
Engineering
Georgia Institute of Technology

Professor C. F. Jeff Wu, Advisor
School of Industrial and System
Engineering
Georgia Institute of Technology

Professor Alexander Shapiro
School of Industrial and System
Engineering
Georgia Institute of Technology

Professor Roshan Joseph Vengazhiyil
School of Industrial and System
Engineering
Georgia Institute of Technology

Professor Chuanyi Ji
School of Electrical and Computer
Engineering
Georgia Institute of Technology

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SUMMARY

The dissertation investigates asymptotic theory of decentralized sequential hypothesis testing problems as well as asymptotic behaviors of the Sequential Minimum Energy Design (SMED). The main results are summarized as follows.

1. We develop the first-order asymptotic optimality theory for decentralized sequential multi-hypothesis testing under a Bayes framework. Asymptotically optimal tests are obtained from the class of "two-stage" procedures and the optimal local quantizers are shown to be the "maximin" quantizers that are characterized as a randomization of at most $M-1$ Unambiguous Likelihood Quantizers (ULQ) when testing $M \geq 2$ hypotheses.

2. We generalize the classical Kullback-Leibler inequality to investigate the quantization effects on the second-order and other general-order moments of log-likelihood ratios. It is shown that a quantization may increase these quantities, but such an increase is bounded by a universal constant that depends on the order of the moment. This result provides a simpler sufficient condition for asymptotic theory of decentralized sequential detection.

3. We propose a class of multi-stage tests for decentralized sequential multi-hypothesis testing problems, and show that with suitably chosen thresholds at different stages, it can hold the second-order asymptotic optimality properties when the hypotheses testing problem is "asymmetric."

4. We characterize the asymptotic behaviors of SMED algorithm, particularly the denseness and distributions of the design points. In addition, we propose a simplified version of SMED that is computationally more efficient.

CHAPTER I

INTRODUCTION

This dissertation investigates two distinct topics: the decentralized sequential multi-hypothesis testing problem and the sequential black-box design.

Sequential hypothesis testing has many important real-world applications such as target detection in multiple-resolution radar (Marcus and Swerling [23]), serial acquisition of direct-sequence spread spectrum signals (Simon et al. [35]) and statistical pattern recognition (Fu [6]). The centralized version, in which all observations are available at a single central location, has been well studied in the statistics literature. For example, when testing $M = 2$ hypotheses, a well-known optimal centralized test is the sequential probability ratio test (SPRT) developed by Wald [44], also see Wald and Wolfowitz [45]. When testing $M \geq 3$ hypotheses, i.e., in the sequential multihypothesis testing problem, there is no tractable closed-form expression for the optimal centralized sequential tests, although various asymptotically optimal sequential tests have been proposed and investigated in the literature, see, for example, Kiefer and Sacks [12], Lorden [21], Draglin, Tartakovsky and Veeravalli [3, 4].

In recent years, the decentralized version of sequential hypothesis testing problems has gained a great amount of attention and has been applied into a wide range of applications such as military surveillance (Tenney and Sandell [38]), target tracking and classification (Li et al. [16]), and data filtering (Ye et al. [47]). Under a widely used decentralized setting, raw data are observed at a set of geographically deployed sensors, whereas the final decision is made at a central location, often called the fusion center. The key feature here is that raw observations at the local sensors are generally not directly accessible by the fusion center, and the local sensors need to send quantized summary messages (generally belonging to a finite alphabet set) to the fusion center. This is due to limited communication bandwidth and requirements of high communication robustness.

Decentralized sequential hypothesis testing problems are very challenging, and to the best of our knowledge, existing research is restricted to testing two simple hypotheses on the distributions of raw data, for example, see Veeravalli [41], Veeravalli, Basar and Poor [43], Nguyen, Wainwright and Jordan [27], and Mei [25]. It has been an open problem to find any sort of asymptotically optimal solutions for the decentralized sequential testing problem when testing $M \geq 3$ hypotheses. This is not surprising, because even in the centralized version, it requires sophisticated mathematical and statistical techniques and only asymptotic optimality results are available.

Chapter 2 of the dissertation offers the first family of decentralized sequential procedures that is asymptotically optimal up to first-order when testing $M \geq 3$ simple hypotheses. A major challenge we face is to find the "optimal quantizers" that can best send quantized summary sensor messages from the local sensors to the fusion center so as to lose as little information as possible. Intuitively, such a quantizer should depend on the true distribution of the raw data, which is unknown, and thus stationary quantizers are generally not optimal. In addition, since a quantizer can be any measurable function as long as its range is in the given finite alphabet set, it resides in an infinite dimensional functional space. Hence it is essential to investigate the form of the "optimal quantizers" so that one can reduce the infinite dimensional functional space to a finite-dimensional parameter space for the purpose of theoretical analysis and numerical computation. Note that when testing $M = 2$ hypotheses, Tsitsiklis [40] and Veeravalli et al. [43] showed that the optimal quantizers can be found from the family of monotone likelihood ratio quantizers (MLRQ), whose form is defined up to a finite number of parameters. Unfortunately, such a result does not apply to the case of testing $M \geq 3$ hypotheses. To find the form of the optimal quantizers for multi-hypotheses, we propose to combine three existing methodologies together: two-stage tests in Stein [36] and Kiefer and Sacks [12] (or equivalently, tandem quantizers in Mei [25]), unambiguous likelihood quantizers (ULQ) in Tsitsiklis [40], and randomized quantizers (see Chernoff [2] for a closely related topic on randomized experiments).

Chapter 3 aims for providing a simpler sufficient condition for asymptotic optimality theory in decentralized sequential detection problems. In the statistics literature, a standard

regularity condition often assumed is the finiteness of the second (or other higher) moments of the log-likelihood ratios, and for decentralized sequential hypothesis testing, one often need to verify that the second (or other higher) moments of the log-likelihood ratios of quantized sensor messages are uniformly bounded for a class of quantization functions. See, for example, the condition (5.9) of Kiefer and Sacks [12]. Unfortunately, it can be analytically challenging or intractable to check this directly, even if the distributions of the unobservable raw data are known to belong to some simple families of distributions, since one may only have very limited knowledge about the quantization functions. To overcome such a difficulty, it is natural to investigate whether the quantized sensor messages will satisfy the regularity conditions regardless of the quantization functions as long as the raw observations have certain properties. For that purpose, we investigate the quantization effects on the second or other higher-order moments of the log-likelihood ratios, which is of interest on its own.

Chapter 4 extends the first-order optimality theory in Chapter 2 to the second-order optimality theory for certain scenarios of decentralized sequential hypothesis testing problems. The two-stage tests can hold the first-order asymptotic optimality properties, but are not second-order asymptotically optimal, since they spend too many steps on their first stage to get a preliminary guess. In order to improve the efficiency, we propose recursively applying the two-stage procedure itself to the first stage. This motivates us to define the multi-stage tests where the fusion center has more opportunities to correct inaccurate guesses. It is shown that if the thresholds at each stage satisfy an easy-to-check condition, then the multi-stage tests can achieve the second order asymptotic optimality properties when the hypothesis testing problem is "asymmetric."

In Chapter 5, we deal with the second topic of this dissertation: the sequential black-box design. In a typical computer or laboratory experimental design problem, researchers often want to find the global optimum of a black-box response function $p(x)$ over a given design region $\mathcal{X}$, but one often has very little prior knowledge of the function $p(x)$ which can be thought of as the experimental yields. To find the global optimum the researchers need to select a set of design points and evaluate the response function $p(x)$ over them.

A good design should be able to approximate the global optimum accurately with as few design points as possible, and one attractive design is the sequential minimum energy design (SMED) proposed in Joseph, Dasgupta and Wu [11] that selects the design points sequentially by minimizing the potential energy. This research focuses on the asymptotic theoretical properties of the SMED, particularly the denseness and the distribution laws of the selected design points. Numerical simulation results are also reported to verify the theoretical properties.

CHAPTER II

SEQUENTIAL DECENTRALIZED MULTIHYPOTHESIS TESTING AND THE TWO-STAGE PROCEDURE

2.1 Sequential Mutihypothesis Testing with Sensor Network

By decentralized detection we mean to perform (statistical) detections by sensor network systems. As illustrated in Fig. 1, in a widely used configuration, a sensor network consists of K local sensors labeled by $S^1, \ldots, S^K$ and a fusion center which makes a final decision when stopping taking observations. At each time step $n = 1, 2, \ldots$, each local sensor S^k observes raw data $\{X_n^k\}$ and sends quantized summary messages $\{U_n^k\}$ to the fusion center. Here the quantized messages $\{U_n^k\}$ are required to belong to a finite alphabet, say, $\{0, 1, \ldots, l^k - 1\}$, due to limited communication bandwidth or requirements of high communication robustness. In other words, the fusion center does not have direct access to the raw data $\{X_n^k\}$, and has to utilize the quantized sensor messages $\{U_n^k\}$ to make a final decision. If necessary, the fusion center can send feedback $\{V_n^k\}$ to the local sensors so as to improve the system efficiency.

To be more rigorous, we need to further specify the form of the sensor message functions. In this chapter, we focus on *systems with full feedback, but local memories restricted to past decisions*, e.g., *Case E* of Veeravalli et al. [43]. Mathematically, at time n, for each $k = 1, 2, \ldots, K$, the quantized sensor message at the kth local sensor is assumed to be of

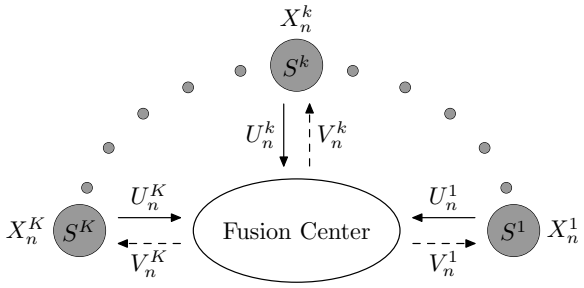


Figure 1: A widely used configuration of sensor network

the form

$$U_n^k = \phi_n^k(X_n^k; V_{n-1}^k) \ \in \{0, 1, \ldots, l^k - 1\} \tag{1}$$

where the feedback V_{n-1}^k is defined by

$$V_{n-1}^k = \psi_n^k(U_{[1,n-1]}^1, \ldots, U_{[1,n-1]}^K) \tag{2}$$

and $U_{[1,n-1]}^k = (U_1^k, \ldots, U_{n-1}^k)$ denotes all past local sensor messages. That is, the quantizer ϕ_n^k is a function used by sensor S^k to map the local raw data X_n^k into $\{0, 1, \ldots, l^k - 1\}$, and the choice of ϕ_n^k can depend on the feedback V_{n-1}^k and can be a randomized function (to be discussed later).

In decentralized sequential multihypothesis testing problems, there are M hypotheses regarding the distribution P of the raw data $\{X_n^k\}$:

$$H_m : \quad P = P_m, \quad m = 0, 1, \ldots, M - 1. \tag{3}$$

Under each P_m, the raw data X_n^k at local sensor S^k are i.i.d. with density $f_m^k(\cdot)$ with respect to a common underlying measure, and the raw data $\{X_n^k\}$ are assumed to be independent across different sensors. Hence the distributions of the raw data under P_m are completely determined by the K densities: $f_m^1, \ldots, f_m^K$. Below we simply state that the true state of nature is m or P_m if the hypothesis H_m is true.

A decentralized sequential test δ consists of a rule to determine the sensor messages, a stopping time N used by the fusion center and a final decision rule $D \in \{0, 1, \ldots, M - 1\}$ that chooses one of the M probability measures P_m's based on the information up to time N at the fusion center. As in Wald [44], Veeravalli et al. [43], and Veeravalli [41], let $c > 0$ be the cost per time step until stopping, and let $W(m, m')$ be the loss of making decision $D = m'$ when the true state is P_m. It is standard to assume that $W(m, m) = 0$ but $W(m, m') > 0$ for any $m \neq m'$, i.e., no loss occurs when a correct decision is made. Then when the true state of nature is P_m, the expected cost of a decentralized test δ conditioned on the m-th hypothesis is

$$\mathcal{R}_c(\delta; m) = cE_m\{N\} + \sum_{m'} W(m, m') P_m\{D = m'\}$$

where E_m is the expectation operator under P_m. In a Bayesian formulation, we assign prior probabilities $\pi = (\pi_0, \ldots, \pi_{M-1})$ to the M hypotheses $H_0, \cdots, H_{M-1}$. Hence, the (total) Bayes risk of the decentralized test δ is

$$\mathcal{R}_c(\delta) = \sum_{m=0}^{M-1} \pi_m \mathcal{R}_c(\delta; m). \tag{4}$$

The Bayes formulation of the decentralized sequential multihypothesis testing problem can then be stated as follows.

Problem (P1): Minimize the Bayes risk $\mathcal{R}_c(\delta)$ in (4) among all possible decentralized sequential multihypothesis test procedures δ.

Denote by $\delta_B^*(c)$ a Bayes solution to (*P1*). In Veeravalli et al. [43], $\delta_B^*(c)$ is obtained through dynamic programming for the simplest case of testing binary hypotheses, i.e., $M = 2$. Unfortunately, in a general multihypothesis setting when $M \geq 3$, it is impractical to find $\delta_B^*(c)$ via dynamical programming, since the problem is intractable even for the centralized version, see, for example, Dragalin, Tartakovsky and Veeravalli [3]. This prompts us to adopt the following asymptotic optimization approach in which the cost c per time step goes to 0.

Problem (P2): Find a family of decentralized sequential multihypothesis testing procedures $\{\delta_A(c)\}$ that is *asymptotically optimal* of the first order in the sense that

$$\lim_{c \to 0} \mathcal{R}_c(\delta_B^*(c)) / \mathcal{R}_c(\delta_A(c)) = 1. \tag{5}$$

Any procedure $\delta_A(c)$ which is a solution to Problem (P1) is refered to as having asymptotic optimality up to the first order.

In this chapter, we introduce a class of "two-stage" decentralized sequential tests in which each local sensor uses two stationary (possibly randomized) local quantizers with at most one switch between these two quantizers. This type of tests are useful because they allows the fusion center to first make a preliminary guess about the true state of nature and then optimize the procedure accordingly. However, before we get into the details of the two-stage procedure, we need to first define the concepts of quantizers and their Kullback-Leibler divergences.

2.2 *Quantizers and Kullback-Leibler Divergence in Decentralized Sequential Detection*

Now let us discuss the concepts of quantizers and their Kullback-Leibler (K-L) divergences, both of which will be essential in our asymptotic optimality theory. A quantizer is either a deterministic measurable function or a randomization of some (possibly infinitely many) deterministic measurable functions that maps the raw data into a finite alphabet set, e.g., the function ϕ_n^k in (1) is a quantizer. The quantizer is called a deterministic quantizer if the corresponding measurable function is deterministic. At a given local sensor S (here and below we miss the superscript k for simplicity), denote by Φ the set of all possible local deterministic quantizers ϕ's and let $f_m(\cdot;\phi)$ be the induced probability mass function of quantized message $U_n = \phi(X_n)$ when the raw observation X_n is distributed according to $f_m(\cdot)$ under P_m, i.e.,

$$f_m(u;\phi) = P_m(\phi(X_n) = u), \quad \text{for } u = 0, 1, \ldots, l-1. \tag{6}$$

For the deterministic quantizer ϕ, it is easy to see that its K-L divergences are defined by

$$I(m, m';\phi) = \sum_{u=0}^{l-1} f_m(u;\phi) \log \frac{f_m(u;\phi)}{f_{m'}(u;\phi)} \tag{7}$$

for all $m \neq m'$. However, we need to be very careful when defining the K-L divergences of a randomized quantizer $\bar{\phi} = \sum p^j \phi^j$ that assigns probability masses $\{p^j\}$ onto some countable subset of deterministic quantizers $\{\phi^j\} \subset \Phi$. On the one hand, one can directly substitute the ϕ in (7) by $\bar{\phi}$, i.e.,

$$\tilde{I}(m, m';\bar{\phi}) = \sum_{u=0}^{l-1} f_m(u;\bar{\phi}) \log \frac{f_m(u;\bar{\phi})}{f_{m'}(u;\bar{\phi})} \tag{8}$$

where

$$f_m(u;\bar{\phi}) = P_m(\bar{\phi}(X) = u), \quad u = 0, 1, \ldots, l-1.$$

This type of the K-L divergence has been defined for randomized quantizers in the engineering literature, e.g., Tsitsiklis [40]. On the other hand, one can also define the K-L divergence of the randomized quantizer $\bar{\phi}$ by the weighted average of those of the deterministic quantizers it randomizes:

$$I(m, m';\bar{\phi}) = \sum p^j I(m, m';\phi^j), \quad 0 \leq m \neq m' \leq M-1. \tag{9}$$

By Jensen's inequality, we have $\tilde{I}(m, m'; \bar{\phi}) \leq I(m, m'; \bar{\phi})$, i.e., the K-L divergence defined in (8) is dominated by that in (9), also see Section 2.7 for more discussion.

It turns out that the K-L divergence in (9) instead of that in (8) will play a central role in our asymptotic theory, and to the best of our knowledge, the definition in (9) has not been well studied in the literature (an exception is the paper by Nguyen, Wainwright and Jordan [27]). The reason why our asymptotic theory involves the K-L divergence in (9) instead of that in (8) is due to our novel way of implementing randomized quantizers to minimize loss of information. Roughly speaking, when implementing randomized quantizers, it is essential for the fusion center to know which specific deterministic quantizer is going to be used at the local sensor at each time step, since otherwise randomization will only make decision-making more challenging and less efficient. This issue will be discussed further in Subsection 2.3.2. Also note that a deterministic quantizer can also be thought as a randomized quantizer that assigns probability one to itself. Denote by $\bar{\Phi}$ the set of all possible quantizers at the local sensor S, deterministic or randomized.

Throughout this chapter we make the following standard assumption to ensure the finiteness of the expectation of the raw data's log-likelihood ratios.

Assumption 1. *For any two different states* $0 \leq m \neq m' \leq M-1$ *and local sensor* S^k*,*

$$0 < E_m \left\{ \log \frac{f_m^k(X_n^k)}{f_{m'}^k(X_n^k)} \right\} < \infty.$$

In the literature, researchers often assume a uniform bound on the second moments of $\log \frac{f_m^k(u^k;\phi)}{f_{m'}^k(u^k;\phi)}$, the log-likelihood ratio for quantized sensor messages, under P_m. See, for example, Kiefer and Sacks [12] and Mei [25]. Here our assumption is much weaker, and it turns out that it will be sufficient for the first-order asymptotic optimality under our setting.

2.3 Two-Stage Test Procedures

In this section, we introduce a class of "two-stage" decentralized sequential tests in which each local sensor uses two stationary (possibly randomized) local quantizers with at most one switch between these two quantizers. This type of tests are useful because they allow

the fusion center to first make a preliminary guess about the true state of nature and then optimize the procedure accordingly.

To highlight our main ideas, in the present and next sections we assume that the sensor network system consists of a single local sensor, i.e., $K = 1$ and all quantized messages are binary, i.e., $U_n \in \{0, 1\}$. Extensions to general cases are presented in Section 2.5. To simplify notation, we drop all the superscripts denoting the sensors. That is, in this and next sections we assume that one observes raw data $X_1, X_2, \cdots$, which are i.i.d. with density $f_m(x)$ under the hypothesis H_m. The final decision is based on quantized messages $U_n = \phi_n(X_n; V_{n-1}) \in \{0, 1\}$ with the feedback $V_{n-1} = \psi_{n-1}(U_1, \cdots, U_{n-1})$. For a given (randomized) quantizer ϕ, the K-L divergence of $P_{m'}$ from P_m is $I(m, m'; \phi)$ defined in (9).

2.3.1 Our Proposed Test

Our proposed two-stage test $\delta(c)$ can be defined as follows. In the *first stage* of $\delta(c)$, the local sensor can use any "reasonable" stationary deterministic quantizer and the fusion center needs to make a preliminary guess about the true state of nature. The only requirement is that as the cost $c \to 0$, the probabilities of making incorrect preliminary guess go to zero but the time steps taken at this first stage become negligible as compared to those of the overall procedure (or the second stage).

To be more concrete, let $u(c) \in (0, 1/2)$ be a function of c such that $u(c) \to 0$ and $\log u(c)/\log c \to 0$ when $c \to 0$, e.g., $u(c) = 1/|\log c|$. Choose a deterministic quantizer ϕ^0 such that $I(m, m'; \phi^0) > 0$ for any two states $0 \le m \ne m' \le M - 1$, and let the local sensor use the stationary quantizer ϕ^0 to send i.i.d. sensor messages $U_n = \phi^0(X_n)$ to the fusion center. Then the fusion center faces a classical sequential detection problem with the i.i.d. sensor messages U_n's as inputs, and thus it is intuitively appealing to make a preliminary decision based on posterior distributions. Specifically, at each time step $n = 1, 2, 3, \cdots$, the fusion center updates recursively the posterior distribution $(\pi_{0,n}, \pi_{1,n}, \ldots, \pi_{M-1,n})$ as follows:

$$\pi_{m,n} = \frac{\pi_{m,n-1} f_m(U_n; \phi^0)}{\sum_{m'=0}^{M-1} \pi_{m',n-1} f_{m'}(U_n; \phi^0)} \tag{10}$$

with the initial value $\pi_{m,0} = \pi_m$, the prior probability of the m-th hypothsis. Then the

fusion center will stop the first stage at time step

$$N_0 = \min\{n \geq 0 : \max_{0 \leq m \leq M-1} \{\pi_{m,n}\} \geq 1 - u(c)\}$$

and when stopped, the fusion center makes a preliminary decision

$$D_0 = \arg\max_{0 \leq m \leq M-1} \pi_{m,N_0}.$$

Note that the preliminary decision D_0 is well-defined because the maximum value of π_{m,N_0} is attained at only one index m due to the definition of N_0 and the fact that $u(c) < 1/2$. For the purpose of practical implementation, the preliminary decision D_0 can be transmitted to the local sensor through a feedback of $\log_2 M$ bits and thus a one-shot $\log_2 M$-bits feedback will be sufficient.

In the *second stage* of our proposed test $\delta(c)$, the local sensor will switch to another stationary (likely randomized) quantizer that may depend on the preliminary decision D_0. Without loss of generality, we assume that the local sensor uses the stationary quantizer $\bar{\phi}_m$ when the preliminary decision at the first stage is $D_0 = m$ for $m = 0, 1, \ldots, M-1$. Here we put a bar over $\bar{\phi}_m$ to emphasize that it is likely a randomized quantizer when optimized, and we will postpone the detailed discussion about how to implement randomized quantizers to the next subsection.

Now at the second stage, the fusion center shall ignore the preliminary decision D_0 and *continue to update* the posterior distribution $(\pi_{0,n}, \ldots, \pi_{M-1,n})$ with the sensor messages generated from the new quantizer $\bar{\phi}_m$ when $D_0 = m$ (how to update will be discussed in the next subsection). Then the fusion center will stop the second stage (hence the whole procedure) at time step

$$N = \min\{n \geq N_0 : \max_{0 \leq m \leq M-1} \{\pi_{m,n}\} \geq 1 - c\} \tag{11}$$

and when stopped, the fusion center makes a final decision

$$D = \arg\max_{0 \leq m \leq M-1} \pi_{m,N}.$$

From the asymptotic point of view, many other decision rules can also be used at the fusion center. For instance, let $r_{m,n} = \sum_{m' \neq m} \pi_{m',n} W(m', m)$ be the average posterior cost

when making a decision m at time n, and then the fusion center can stop the second stage at time

$$N' = \min\{n \geq N_0 : \min_{0 \leq m \leq M-1} r_{m,n} \leq c\}. \tag{12}$$

When the costs $W(m', m) = I(m' = m)$ are a simple $0-1$ cost function, the stopping time N' in (12) becomes N in (11), and our experience suggests that N' in (12) is slightly better than N in (11) for other cost functions in finite-sample numerical simulations. In addition, when updating the posterior distribution at the second stage, we can also adopt a standard approach by starting afresh as in those two-stage tests in the literature, see Section V of Kiefer and Sacks [12] or Section IV of Mei [25]. In this chapter, we propose a new approach by continuing to update the posterior distribution from the first stage so as to further utilize information gathered from the first stage. This allows us to improve the efficiency in finite-sample simulations, although it also means extra treatments in asymptotic arguments.

2.3.2 Implementing Randomized Quantizers and Updating Posterior Distribution

When testing $M \geq 3$ hypotheses, randomized quantizers are likely needed in the second stage in order to develop the optimal two-stage tests, and thus it is necessary to determine the appropriate approach to implement them as well as how to update posterior distributions at the fusion center, especially during the second stage. Assume a randomized quantizer is given by $\bar{\phi} = \sum p^j \phi^j$. The key requirements for randomization in our two-stage test is that the fusion center must know which deterministic quantizer is picked to quantize the raw observation, since otherwise the randomization can cause confusion when making decision at the fusion center. The most straightforward (though practically infeasible) implementation is to let the fusion center do the randomization directly. Specifically, at time step n the fusion center will choose the deterministic quantizer ϕ^j with probability p^j, say choosing the deterministic quantizer $\phi^{j(n)}$. Through a feedback from the fusion center, the local sensor will then use the chosen deterministic quantizer $\phi^{j(n)}$ at time step n to quantize the raw observation. After receiving the quantized sensor message U_n at time step n, the fusion

center then update the posterior distribution as follows:

$$\pi_{m,n} = \frac{\pi_{m,n-1} f_m(U_n; \phi^{j(n)})}{\sum_{m'=0}^{M-1} \pi_{m',n-1} f_{m'}(U_n; \phi^{j(n)})} \tag{13}$$

because the fusion center knows that U_n comes from the deterministic quantizer $\phi^{j(n)}$ at time step n.

In practice a more feasible implementation is to adopt "pseudo-randomization" to reduce the communication from the fusion center to the local sensors. One approach is to let the fusion center and the local sensor(s) use the same random-number generation mechanism that can be initialized with the same seed. Another simpler approach is to adopt a "periodic block design" at the local level (see Section V of Kiefer and Sacks [12]). To be specific, suppose $\bar{\phi}$ randomizes a finite number (say i) of deterministic quantizers, and all p^j's are (or can be approximated by) rational numbers with b a common denominator. Then we divide the time steps into blocks of size b, and within each block, the raw data are quantized with deterministic quantizers $\{\phi^1, \ldots, \phi^i\}$ following a fixed order such that each ϕ^j is used for exactly $p^j b$ times. Under these pseudo-randomization implementations, the fusion center again knows which deterministic quantizer is used at each time step, and thus can update the posterior distribution as in (13).

We would like to point out that our implementation of randomized quantizers is very different from those existing implementations in the literature (see Tsitsiklis [40]). In the latter the randomization is done at the local level in the sense that the local sensor randomly picks one of the deterministic quantizer ϕ^j's, and the fusion center will only get the quantized message U_n without knowing exactly which deterministic quantizer is used to generate U_n. In this case, to update the posterior distribution, the fusion center has to plug in $\bar{\phi}$ (instead of $\phi^{j(n)}$) into (13), i.e.,

$$\pi_{m,n} = \frac{\pi_{m,n-1} f_m(U_n; \bar{\phi})}{\sum_{m'=0}^{M-1} \pi_{m',n-1} f_{m'}(U_n; \bar{\phi})}.$$

Since our proposed implementation and the local randomization implementation lead to different likelihood ratios, it is not surprising that there are two different K-L divergences for the same randomized quantizer in Section 2.2: one defined in (8) and the other in (9).

2.4 Main Results

In the present section, we show that a two-stage test can be an asymptotic optimal solution to problem (P2) by carefully choosing the quantizers used in the second stage. We also give characterizations of these optimal quantizers as well as the corresponding numerical computation.

2.4.1 Maximin Quantizers and Asymptotic Theory

Let us begin with the definition of some useful information numbers. For a given (deterministic or randomized) quantizer $\bar{\phi} \in \bar{\Phi}$, define

$$I(m; \bar{\phi}) = \min_{m' \neq m} I(m, m'; \bar{\phi}). \tag{14}$$

for each state $m = 0, 1, \ldots, M-1$. That is, $I(m; \bar{\phi})$ characterizes the least divergence from the state m to other states.

The following theorem, whose proof is presented in Section 2.8, establishes the asymptotic properties of a two-stage test $\delta(c)$ as the cost c goes to 0.

Theorem 2.4.1. *Let $\delta(c)$ be a two-stage test with $\{\bar{\phi}_0, \ldots, \bar{\phi}_{M-1}\}$ being the set of (possibly randomized) quantizers used in its second stage. Assume each $\bar{\phi}_m$ randomizes a finite number of deterministic quantizers, and suppose that the prior probabilities $\pi_m > 0$ and $I(m; \bar{\phi}_{m'}) > 0$ for all states $m = 0, 1, \ldots, M-1$ and $m' = 0, 1, \ldots, M-1$. Then as $c \to 0$, the time step N taken by the two-stage test $\delta(c)$ satisfies*

$$E_m\{N\} = (1 + o(1))|\log c|/I(m; \bar{\phi}_m), \quad m = 0, 1, \ldots, M-1, \tag{15}$$

and the final decision D of the two-stage test $\delta(c)$ satisfies

$$P_m\{D \neq m\} = O(c), \quad m = 0, 1, \ldots, M-1. \tag{16}$$

Thus, the Bayes risk of the two-stage test $\delta(c)$ is

$$\mathcal{R}_c(\delta) = c|\log c|(1 + o(1)) \sum_{m=0}^{M-1} \pi_m / I(m; \bar{\phi}_m). \tag{17}$$

In light of Theorem 2.4.1, for the proposed test $\delta(c)$, the costs of making wrong decisions become negligible as compared to the cost of the time step. Moreover, to asymptotically minimize the Bayes risk within the class of two-stage tests, it is clear that one should maximize the information numbers $I(m;\bar{\phi}_m)$ for $m = 0, 1, \ldots, M-1$. This leads to a natural definition of the optimal quantizers that we should use in the second stage:

Definition 2.4.1. *For $m = 0, 1, \ldots, M-1$, define the maximin quantizer with respect to P_m as*

$$\bar{\phi}_m^{max} = \arg\sup_{\bar{\phi}\in\bar{\Phi}} I(m;\bar{\phi}) = \arg\sup_{\bar{\phi}\in\bar{\Phi}} \min_{m'\neq m} I(m,m';\bar{\phi})$$

and define the corresponding maximin information number by $I(m) = \sup_{\bar{\phi}\in\bar{\Phi}} I(m;\bar{\phi})$.

As shown later in Theorems 2.4.3 and 2.5.1, the supremum of $I(m,\bar{\phi})$ is attainable, and the maximin quantizers not only exist, but also can be realized as randomization of a finite number of deterministic quantizers. Now we are ready to investigate the asymptotic optimality properties of the two-stage test when the maximin quantizers are used in the second stage. Denote by $\delta_A(c)$ such a two-stage test. Then by Theorems 2.4.1, we have

$$\mathcal{R}_c(\delta_A(c)) = (1+o(1))c|\log c| \sum_{m=0}^{M-1} \pi_m/I(m). \tag{18}$$

as $c \to 0$. What is surprising is that $\delta_A(c)$ is not only the best within the class of two-stage tests, but also asymptotically optimal among all possible decentralized tests. A key step in the proof is the following important theorem which establishes asymptotic lower bounds on the expected time steps of any decentralized tests with "suitably small" probabilities of making incorrect decisions.

Theorem 2.4.2. *Assume that $\delta(c)$ is a decentralized (not necessarily a two-stage) test that makes a final decision D and*

$$P_m\{D \neq m\} = O(c|\log c|), \quad m = 0, 1, \ldots, M-1,$$

as $c \to 0$. Then the time step N taken by $\delta(c)$ satisfies

$$E_m\{N\} \quad \geq \quad (|\log c| - \log|\log c| + O(1))/I(m)$$

$$= \quad (1 + o(1))|\log c|/I(m) \tag{19}$$

for all $m = 0, 1, \ldots, M-1$.

The proof of Theorem 2.4.2 is presented in Section 2.9. The first-order asymptotic lower bound will be sufficient to prove the first-order asymptotic optimality of $\delta_A(c)$, and the reason why we present a higher order lower bounds is due to its potential usefulness in higher-order analysis in further research. By relation (18) and Theorem 2.4.2, we have

Corollary 2.4.1. *The procedure* $\delta_A(c)$ *is first-order asymptotically Bayes.*

Proof. Let $\delta_B^*(c)$ be the Bayes procedure. By definition, $\mathcal{R}_c(\delta_B^*(c)) \leq \mathcal{R}_c(\delta_A(c))$. Using the relation (18) and the definition of Bayes risk $\mathcal{R}_c(\delta_B^*(c))$, the probabilities for the Bayes procedure $\delta_B^*(c)$ to make incorrect decisions are at most $O(c \log c)$. By Theorem 2.4.2, the stopping time N_c^* of the Bayes procedure $\delta_B^*(c)$ satisfies (19). Now using the definition of Bayes risk again, for any test, the cost of time steps taken to make the final decision is only portion of the Bayes risk. In particular,

$$\mathcal{R}_c(\delta_B^*(c)) \geq c \sum_{m=0}^{M-1} \pi_m E_m \{N_c^*\} \geq (1 + o(1))c|\log c| \sum_{m=0}^{M-1} \pi_m / I(m).$$

Combining all arguments yields that $\mathcal{R}_c(\delta_B^*(c))/\mathcal{R}_c(\delta_A(c)) \to 1$ as $c \to 0$, completing the proof of the corollary. □

It is useful to point out that the test $\delta_A(c)$ is asymptotic Bayes mainly because the local sensor uses the maximin quantizers $\bar{\phi}_m^{\max}$'s in the second stage. Since the maximin quantizers do not depend on the prior distribution $\{\pi_m\}$'s, it is easy to see from (15) and (16) that the asymptotic optimality properties of $\delta_A(c)$ are actually robust with respect to $\{\pi_m\}$ as long as all prior probabilities are positive. Likewise, the asymptotic optimality properties still hold if the stopping times of $\delta_A(c)$ at the fusion center are replaced by other efficient multi-hypotheses tests, e.g., those in Dragalin, Tartakovsky and Veeravalli [3, 4].

2.4.2 Characterizing the Maximin Quantizers.

In this subsection, we provide a deeper understanding of the maximin quantizers $\{\bar{\phi}_m^{\max} : m = 0, 1, \ldots, M-1\}$ and also illustrate how to compute them explicitly when the sensor messages are binary.

Let us first introduce the Unambiguous Likelihood Quantizer (ULQ) which was first proposed in Tsitsiklis [40] as a generalization of the Monotone Likelihood Ratio Quantizer (MLRQ). For notational convenience, here we give the definition of ULQ only for the case of binary sensor messages, and the general definition will be provided in Definition 2.5.1 in Subsection 2.5.1.

Definition 2.4.2. *A deterministic quantizer $\phi \in \Phi$ is said to be an Unambiguous Likelihood Quantizer (ULQ) if there exist real numbers $\{a_m : m = 0, \ldots, M-1\}$ such that*

$$\phi(X) = \begin{cases} 1, & \text{if } \sum_{m=0}^{M-1} a_m f_m(X) > 0 \\ 0, & \text{if } \sum_{m=0}^{M-1} a_m f_m(X) \leq 0 \end{cases} \tag{20}$$

and for any $0 \leq m' \leq M-1$, the set $\{a_m\}$ satisfies the following condition

$$P_{m'} \left\{ \sum_{m=0}^{M-1} a_m f_m(X) = 0 \right\} = 0. \tag{21}$$

When relation (21) holds for any set of $\{a_m\}$ that are not simultaneous zero, the set of pdf's $\{f_m\}$ are said to be *linearly independent*. With the definition of ULQs, the following theorem characterizes the form of the maximin quantizers $\bar{\phi}_m^{\max}$. The proof is very technical and is deferred to Section 2.7.

Theorem 2.4.3. *For each $m = 0, 1, \ldots, M-1$, the maximin quantizer $\bar{\phi}_m^{max}$ exists and can be chosen as a randomization of at most $M-1$ deterministic quantizers. Moreover, if the pdf's $\{f_m\}$ are linearly independent, then it can actually be chosen as a randomization of at most $M-1$ deterministic ULQ quantizers.*

Clearly, when testing $M = 2$ simple hypotheses, the ULQs become MLRQs, and thus the maximin quantizer in the second stage is just the deterministic MLRQ, which is consistent with the corresponding results in Mei [25].

Note that Theorem 2.4.3 reduces the search of the maximin quantizers from an infinite dimensional function space to a parameter space of dimension $O(M^2)$. To see this, fix a state m and define $M^2 - 1$ parameters as probability masses $\{p_m^j : 1 \leq j \leq M-1, p_m^j \geq 0, \sum_{j=1}^{M-1} p_m^j = 1\}$, and ULQ coefficients $\{a_{m,m'}^j : 1 \leq j \leq M-1, 0 \leq m' \leq$

$M-1, \sum_{m'=0}^{M-1}(a_{m,m'}^j)^2 = 1\}$. Based on every combination of these parameters, define by $\bar{\phi}$ the quantizer randomizing $M-1$ ULQs: $\bar{\phi} = \sum_{j=1}^{M-1} p_m^j \phi_m^j$, where

$$\phi_m^j(X) = I(\sum_{m'=0}^{M-1} a_{m,m'}^j f_{m'}(X) > 0).$$

The maximin quantizer $\bar{\phi}_m^{\max}$ can then be found as $\bar{\phi}$ that maximizes

$$\min_{l \neq m} I(m, l; \bar{\phi}) \tag{22}$$

among all possible combinations of $\{p_m^j; a_{m,m'}^j\}_{1 \leq j \leq M-1, 0 \leq m' \leq M-1}$.

To numerically find the maximin quantizer $\bar{\phi}_m^{\max}$ for a given state m, one can discretize all parameters and then solve the optimization problem (22) in a brute force approach. Of course, this still requires considerable computation power. To further reduce computational complexity, we can apply the following lemma which provides a sufficient condition that a deterministic MLRQ quantizer is the maximin quantizer.

Lemma 2.4.1. *Fix a state m, for another state $m' \neq m$, let $\phi_{m,m'}$ be the deterministic MLRQ quantizer that maximizes the K-L divergence of m' from m, i.e.,*

$$\phi_{m,m'} = \arg\sup_{\phi \in \Phi} I(m, m'; \phi) = \arg\sup_{\bar{\phi} \in \bar{\Phi}} I(m, m'; \bar{\phi}).$$

If for any other state $m'' \neq m$:

$$I(m, m''; \phi_{m,m'}) \geq I(m, m'; \phi_{m,m'})$$

then $\phi_{m,m'}$ is also the maximin quantizer for state m.

Proof. By our assumption,

$$I(m; \phi_{m,m'}) = \min_{m'' \neq m} I(m, m''; \phi_{m,m'}) = I(m, m'; \phi_{m,m'}).$$

Take any $\bar{\phi} \in \bar{\Phi}$,

$$I(m; \bar{\phi}) \leq I(m, m'; \bar{\phi}) \leq I(m, m'; \phi_{m,m'}) = I(m; \phi_{m,m'})$$

and thus $\phi_{m,m'}$ is the maximin quantizer for state m. □

To better understand this lemma, it is useful to imagine an extreme case that there is only one state m' that is very "close" to the given state m and all other states m'' are very far away from the state m. Then when the state m is the true state, the difficulty of testing multiple hypotheses is mainly due to testing state m versus state m', no matter whether the decision making is based on raw observations and quantized sensor messages. Hence the information number $I(m;\bar{\phi})$ in (14) becomes $I(m,m';\bar{\phi})$, and finding a maximin quantizer is equivalent to finding a quantizer that maximizes $I(m,m';\bar{\phi})$, which is a deterministic MLRQ as shown in Tsitsiklis [40].

2.5 Extensions

Section 2.4 deals with the simplest case when the network only has a single sensor with binary sensor messages. In this section, we extend our results to three more general scenarios: 1) the sensor messages belong to a finite alphabet (not necessarily binary); 2) there is more than one sensor in the network (though observations are independent between different sensors); and 3) the hypotheses are composite.

2.5.1 Sensor Messages Belonging to a Finite Alphabet

Suppose the network still consists of only one sensor, but now the sensor messages belong to a finite alphabet, say, $\{0,1,\ldots,l-1\}$ with $l \geq 2$. In this scenario, the definitions of two-stage tests (Subsection 2.3.1) and maximin quantizers (Subsection 2.4.1) are still applicable, and Theorem 2.4.1 and Theorem 2.4.2 also hold. The only change is Theorem 2.4.3, as we need to consider the following general definition of ULQ, which was originally proposed in Tsitsiklis [40] and includes Definition 2.4.2 as a special case.

Definition 2.5.1. *When the sensor messages belong to a finite alphabet $\{0,1,\ldots,l-1\}$, a deterministic quantizer $\phi \in \Phi$ is said to be an unambiguous likelihood quantizer (ULQ) if and only if there exist real numbers $\{a_{i,m} : 0 \leq i \leq l-1, 0 \leq m \leq M-1\}$ such that*

$$\phi(X) = \arg\min_{0 \leq i \leq l-1} \sum_{m=0}^{M-1} a_{i,m} f_m(X) \tag{23}$$

and the probability of a tie is zero under every P_m for $m = 0,1,\ldots,M-1$.

With this definition, Theorem 2.4.3 can be generalized as follows.

Theorem 2.5.1. *Suppose the sensor messages belong to a finite alphabet* $\{0, 1, \ldots, l-1\}$ *with* $l \geq 2$. *Then for* $m = 0, 1, \ldots, M-1$, *the maximin quantizer* $\bar{\phi}_m^{max}$ *can be realized as a randomization of at most* $M-1$ *deterministic quantizers. Moreover, for every* m, *there exists quantizer* $\{\bar{\phi}_m\}$ *randomizing at most* $M-1$ *ULQs that can approximate* $\bar{\phi}_m^{max}$ *arbitrarily close.*

The proof of Theorem 2.5.1 is presented in Section 2.7. Note that there is a significant difference between Theorem 2.4.3 and Theorem 2.5.1. When the sensor messages are binary (i.e., $l = 2$), we are sure that the maximin quantizers can be attained by randomizing $M-1$ ULQs if the pdfs f_0,..., f_{M-1} are linearly independent. However, this is no longer true for $l \geq 3$, also see Section 2.7 for more explanations. Fortunately, since the maximin quantizers can always be approximated as described in Theorem 2.5.1, the issue is not essential from the viewpoint of numerical computation, as we can compute the maximin quantizers (or their approximations) in the same way as in Subsection 2.4.2 except that each ULQ is now associated with an l by M matrix $A = \{a_{i,m}\}$.

Another benefit of Theorem 2.5.1 is that it can deal with the case of binary sensor messages but with the pdf's that are not linearly independent. Such a case was not addressed by Theorem 2.4.3, and Theorem 2.5.1 shows that although the maximin quantizer $\bar{\phi}_m^{\max}$ may no longer be a randomization of at most $M-1$ ULQs, it can still be approximated arbitrarily close by a quatizer $\bar{\phi}_m$, randomizing at most $M-1$ ULQs.

2.5.2 Multiple Sensors

We now assume that there are $K \geq 2$ sensors in the system in which all raw observations are independent from sensor to sensor conditioned on each P_m, $m = 0, 1, \ldots, M-1$. In the following notation, we use the superscripts to denote different sensors as in Section 2.2. For simplicity, we assume the sensor messages are binary, since the extension to the scenario with a finite alphabet sensor messages can be easily done as in Subsection 2.5.1. The key to extend our results is to treat the quantizers in Sections 2.3 and 2.4 as vectors. Specifically, a (deterministic) vector quantizer is $\vec{\phi} = (\phi^1, \ldots, \phi^K)$, where each local sensor S^k uses the

deterministic scalar quantizer ϕ^k to quantize the raw data. Denote by $\Phi^{(K)}$ the set of all (deterministic) vector quantizers, and define a randomized vector quantizer

$$\vec{\varphi} = \sum_j p^j \vec{\phi}^j$$

where $\vec{\phi}^j = (\phi^{1,j}, \ldots, \phi^{K,j}) \in \Phi^{(K)}$, and $\{p^j\}$ are the probability masses assigned to the set of deterministic quantizer vectors $\{\vec{\phi}^j\} \subset \Phi^{(K)}$. Let the set of all (deterministic or randomized) vector quantizers be $\overline{\Phi^{(K)}}$. The implementation of a randomized vector quantizer $\vec{\varphi} = \sum p^j \vec{\phi}^j$ is the same as that in Subsection 2.3.2, i.e., the fusion center knows about which deterministic vector quantizer is picked, either letting the fusion center conduct the randomization directly or using the pseudo-randomization block design at the local sensor level. Likewise, for a deterministic vector quantizer $\vec{\phi} = (\phi^1, \ldots, \phi^K)$, the K-L divergence of state m' from state m is defined as

$$I(m, m'; \vec{\phi}) = \sum_{k=1}^{K} I(m, m'; \phi^k) \tag{24}$$

and for a randomized vector quantizer $\vec{\varphi} = \sum_j p^j \vec{\phi}^j$, the K-L divergence is a weighted average as in Section 2.2:

$$I(m, m'; \vec{\varphi}) = \sum_j p^j I(m, m'; \vec{\phi}^j) = \sum_j \sum_{k=1}^{K} p^j I(m, m'; \phi^{k,j}). \tag{25}$$

Now the maximin vector quantizers $\{\vec{\varphi}_m^{\max}\}$ and maximin information numbers $\{I(m)\}$ for vector quantizers can be defined in exactly the same way as in Subsection 2.4.1, and the theory developed for single-sensor networks, i.e., Theorems 2.4.1-2.4.3, also holds for the multiple sensor cases except replacing the scalar quantizers by vector quantizers.

A special case is when the sensors are homogeneous, i.e., when the observations are independent and identically distributed across different sensors. In this case, the maximin vector quantizer contains simple replicate of the maximin quantizers in the corresponding single-sensor case. This result is summarized in the following proposition.

Proposition 2.5.1. *Assume that $f_m^1 = \cdots = f_m^K = f_m$ for $m = 0, 1, \ldots, M-1$. Fix a state m, let $\vec{\phi}_m^{0,max} = \sum_j p_m^j \phi_m^{0,j}$ be the maximin quantizer in the corresponding single sensor case where the system has only one sensor and the raw data are distributed according to*

$\{f_m\}$. *Define randomized vector quantizer* $\vec{\varphi}_m^* = \sum_j p_m^j \vec{\phi}_m^j$ *with each* $\vec{\phi}_m^j$ *being a* K*-time replication of* $\phi_m^{0,j}$*, i.e.,* $\vec{\phi}_m^j = (\phi_m^{0,j}, \ldots, \phi_m^{0,j})$*. Then* $\bar{\phi}_m^*$ *is a maximin quantizer vector for the state* m*.*

Proof. The proof follows from (24) and (25). □

2.5.3 Composite Multihypothesis Testing

Our theory can also be extended to the scenario of composite hypothesis with finitely many points. Suppose that there are B composite hypotheses, $H_0, \ldots, H_{B-1}$, where

$$H_b = \{P_{i_b}, P_{i_b+1}, \ldots, P_{i_{b+1}-1}\}$$

include $i_{b+1} - i_b$ points for $b = 0, 1, \ldots, B-1$, and $i_0 = 0$. Without loss of generality, let us assume $M = i_B$. Then there are a total of $i_B = M$ simple hypotheses, and the decision maker is required to pick one of the B hypotheses that most likely includes the true state of nature P_m. We further assume that the prior for each simple hypothesis P_{i_b} is known to be π_{i_b} (we will not consider another scenario when the prior for H_b is given but the priors for each simple hypothesis are unknown, since it is against the Bayesian spirit and belongs to a semi-Bayesian approach). Hence, the problem formulation is the same as that in Section II, except that the cost function $W(m, m')$ needs to be re-defined to reflect composite hypotheses in the multihypothesis testing problem. To simplify our notation, for $m = 0, 1, \ldots, M-1$, denote by $[m]$ the hypothesis that contains P_m, i.e., $[m] = H_b$ if and only if $P_m \in H_b$. In composite multihypothesis testing problem, the loss function W has the form $\{W(m, [m'])\}$, where $W(m, [m'])$ indicates the loss caused by making a decision $D = [m']$ when the state of nature is P_m. We assume $W(m, [m']) \geq 0$ and $W(m, [m']) = 0$ if and only if $m \notin [m']$, i.e., no loss in making a correct decision.

As in Section 2.2, the total expected cost or risk of a test δ when the true state of nature is m is:

$$\mathcal{R}_c(\delta; m) = cE_m\{N\} + \sum_{[m']} W(m, [m'])P_m\{D = [m']\}$$

and the Bayes risk of δ is

$$\mathcal{R}_c(\delta) = \sum_m \pi_m \mathcal{R}_c(\delta; m) \tag{26}$$

where the prior probability of the hypothesis H_b is $\pi_{i_b} + \ldots + \pi_{i_{b+1}-1}$.

In the scenario of composite hypotheses, the definition of the two-stage tests is similar except a slight modification of the stopping time N and the final decision D of the fusion center in the second stage. For simplicity, let us consider the simplest case of the single-sensor and binary sensor messages. At time step n in the second stage, the fusion center computes

$$r_{[m],n} = \sum_{m' \notin [m]} \pi_{m',n} W(m', [m])$$

which is the average loss if one makes a final decision $D = [m]$. Then the fusion center stops at time $N = \min\{N_{[m]}\}$, where

$$N_{[m]} = \{n \geq N_0 : r_{[m],n} \leq c\}$$

and N_0 is the stopping time for the first stage. When stopped, the fusion center makes a final decision $D = [m]$ if $N = N_{[m]}$.

Note that we do not change the fusion center policy in the first stage, i.e., the preliminary decision D_0 at the fusion center still picks the most promising state among the M states instead of picking one of the B hypotheses. When the preliminary decision at the first stage is $D_0 = m \in \{0, 1, \ldots, M-1\}$, then the optimal quantizer at the second stage should be chosen to maximize the information number

$$I(m; \bar{\phi}) = \min_{m' \notin [m]} I(m, m'; \bar{\phi}).$$

In other words, to find the asymptotically optimal tests among the two-stage tests, we need to modify the definition of $I(m; \bar{\phi})$ by ignoring those states grouped into the same hypothesis with m. With these new definitions, Theorems 2.4.1 and 2.4.2 remain valid, and we can still use Theorem 2.4.3 to numerically compute each maximin quantizer $\bar{\phi}_m^{\max}$ by pretending $[m] = \{P_m\}$, i.e., by temporarily discarding other states in $[m]$.

2.6 Examples

In this section we illustrate our theory via numerical simulations. Suppose we are interested in testing the mean of a normal distribution with unit variance in a network with a single

sensor and binary sensor messages. That is, the raw data observed at the local sensor follow a normal distribution $P \sim N(\theta, 1)$. In the hypothesis testing problem, we consider three hypotheses regarding θ, say, $H_0 : \theta = \theta_0$, $H_1 : \theta = \theta_1$ and $H_1 : \theta = \theta_2$, and assign the prior probability of $1/3$ to each of these three hypotheses. In addition, as in Dragalin et al. [4], we assume 0-1 loss for decision-making, i.e., $W(m, m') = 1$ if $m \neq m'$ and $= 0$ if $m = m'$. Two different scenarios will be considered:

1) **Asymmetric (HT1):** $(\theta_0, \theta_1, \theta_2) = (-0.5, 0, 1)$.

2) **Symmetric (HT2):** $(\theta_0, \theta_1, \theta_2) = (-0.5, 0, 0.5)$.

For our proposed asymptotic optimal decentralized test δ_A in these scenarios, it suffices to determine the local quantizers. The stationary quantizer in the first stage of δ_A is easy, as we can simply use $\phi^0(X) = I(X \geq 0)$, which satisfies the conditions in Subsection 2.3.1. It is a little more challenging to characterize the maximin quantizers used in the second stage of δ_A. For the asymmetric case (HT1), it is straightforward to show from Lemma 2.4.1 that the three maximin quantizers are all deterministic MLRQs. Numerical computations illustrate that the three maximin quantizers are $\phi_0 = I(X \geq -0.3963), \phi_1 = I(X \geq -0.1037), \phi_2 = I(X \geq 0.7941)$ and the corresponding maximin information numbers are $I_0 = 0.0796, I_1 = 0.0796, I_2 = 0.3186$, respectively.

The maximin quantizers of the symmetric case (HT2) are a little tricky. It is easy to check that Lemma 2.4.1 can be applied to state $m = 0$ and $m = 2$, yielding two maximin quantizers $\phi_0 = I(X \geq -0.1037)$ and $\phi_2 = I(X \geq 0.3963)$ with maximin information numbers $I_0 = I_2 = 0.07959$. However, we need to pay special attention to the maximin quantizer for the state $m = 1$ since the other two states $m = 0$ and $m = 2$ are symmetric with respect to $m = 1$. Since the three pdfs are obviously linearly independent as defined in Subsection 2.4.2, by Theorem 2.4.3, the maximin quantizer for state $m = 1$ can be realized as a randomization of at most two ULQs. The following lemma, whose proof is straightforward and thus is omitted, gives more convenient descriptions of the ULQs in (HT2) when the observations are normally distributed.

Lemma 2.6.1. *For the symmetric case (HT2), up to a permutation of the values it takes, a ULQ always takes one of the following two forms:* $I(X \geq \lambda)$ *or* $I(\lambda_1 \leq X \leq \lambda_2)$*, where* λ *and* $\lambda_1 \leq \lambda_2$ *are real numbers.*

This allows us to do numerical computation of the maximin quantizer for state $m = 1$ as in Subsection 2.4.2. Numerical computations show that the maximin quantizer for state $m = 1$ is also the deterministic quantizer defined by $\phi_1 = I(X > 0)$ up to the precision of 5 decimal digits, and $I_1 = 0.07928$.

For each of the two scenarios, (HT1) and (HT2), we will consider two versions of our proposed tests: one is $\delta_A(c)$ for the system with a single sensor, and the other is $\delta'_A(c)$ for the system with two independent and identical sensors. As a comparison of our proposed tests, we also consider an asymptotically optimal *centralized* test δ_a proposed in Dragalin et al. [3, 4] for the system with a single sensor (we omitted another family of asymptotically optimal *centralized* test δ_b proposed in Dragalin et al. [3, 4], since its performance is similar to that of δ_a). For δ_a, the fusion center updates the posterior distribution $\{\pi_{m,n}\}$ based on the raw data $\{X_n\}$ and its stopping time is defined as $N(a) = \min_{1 \leq m \leq M} N_m(a)$, where $N_m(a) = \inf\{n \geq 1 : \pi_{m,n} \geq A_m\}$. In other words, δ_a stops as soon as one of the posterior probability $\pi_{m,n}$ passes the threshold A_m, which can take different values for different m. In the numerical simulation given in [4], the values of these thresholds are as follows. For the asymmetric case (HT1), $A_0 = A_1 = 1 - 3.99 \times 10^{-3}$, $A_2 = 1 - 5.33 \times 10^{-3}$, while for the symmetric case (HT2), $A_0 = A_1 = A_2 = 1 - 3.99 \times 10^{-3}$. These particular values for the thresholds tune the overall probabilities of making incorrect decisions with test δ_a to $1.0 \pm 0.1 \times 10^{-3}$.

In our simulations, the cost $c = 3.6 \times 10^{-3}$, and the threshold $u(c)$ at the first stage of our proposed tests $\delta_A(c)$ and $\delta'_A(c)$ is set as 0.1. Because of the selection of the parameters, δ_A, δ'_A, and δ_a have similar probabilities of making incorrect decisions, i.e., $1.0 \pm 0.1 \times 10^{-3}$. Thus it suffices to report the simulated expected time steps $E_m\{N\}$ under each of the three hypotheses H_m for $m = 0, 1, 2$, as smaller values of $E_m\{N\}$ imply better performance of the test (in the sense of smaller Bayes risks). These results are reported in Table 1.

Table 1: Expected values of time steps taken for each of the three tests.

	$E_m\{N\}$	δ_a	$\delta_A(c)$	$\delta'_A(c)$
Asymmetric (HT1)	$m=0$	46.48	73.5±0.9	36.8±0.7
	$m=1$	48.39	77.7±0.9	38.9±0.7
	$m=2$	11.90	19.8±0.2	9.9±0.1
Symmetric (HT2)	$m=0$	46.59	73.4±0.9	37.8±0.6
	$m=1$	69.43	110.2±0.9	55.2±0.7
	$m=2$	46.60	73.4±0.9	37.8±0.6

The numerical results illustrate that the centralized test, δ_a, indeed performs better than the decentralized test $\delta_A(c)$ that makes a final decision based on binary sensor messages instead of raw normal observations. However, Table 1 demonstrates that for (HT1) and (HT2), the loss of information is not so significant. As in Mei [24, 25], if we define the efficiency of a decentralized test δ as compare to the centralized test δ_a by $e(\delta) = \min_{1\le m\le M} E_m\{N_a\}/E_m\{N\}$, where N and N_a are the respective stopping times of δ and δ_a that satisfy a given probability of making incorrect decisions. In our simulation, the efficiency of $\delta_A(c)$ is 60.1% in (HT1) and 63.0% in (HT2), respectively. This implies that when more than one sensors provide information to the fusion center, we will need $1/e(\delta) \approx 1.6$ as many sensors in order for the decentralized test to beat the centralized test. This is consistent with the existing decentralized detection literature, also see Tartakovsky et al. [37]. In particular, our simulation result of $\delta'_A(c)$ illustrate that if we have two identical sensors the expected sample size $E_m\{N\}$ will be cut roughly into half, and a decentralized test with two sensors can outperform a centralized test with a single sensor. All these are consistent with our asymptotic analysis.

2.7 Proofs of Theorems 2.4.3 and 2.5.1

Since quantizers, especially randomized quantizers, play an important role in our theorems, we will gather some useful results for quantizers in this section, including the proofs of Theorems 2.4.3 and 2.5.1. Without loss of generality, we focus on the system with a single sensor, and assume that the quantized messages belong to a finite alphabet, say, $\{0, 1, \ldots, l-1\}$. For a (deterministic or randomized) quantizer $\bar{\phi} \in \bar{\Phi}$, define its distribution vector as a

vector of length Ml:

$$q(\bar{\phi}) = (q(i; m, \bar{\phi}))_{\ 0 \le i \le l-1;\ 0 \le m \le M-1}$$

where $q(i; m, \bar{\phi}) = P_m\left\{\bar{\phi}(X) = i\right\}$. As in Tsitsiklis [40], the key observation is that while a (randomized) quantizer $\bar{\phi}$ belongs to infinite dimensional functional space, its distribution quantizer vector $q(\bar{\phi})$ not only belongs to a finite-dimensional space of dimension $= Ml$, but also captures all (statistical) information of $\bar{\phi}$. Thus a crucial idea of deriving optimal quantizers is to reduce the space dimension from infinite to finite by investigating the properties of $q(\bar{\phi})$.

First, let us consider four subspaces induced by the distribution vectors $q(\bar{\phi})$:

- Let Q be the set formed by the distribution vectors of all *deterministic* quantizers, i.e., $Q = \{q(\phi) : \phi \in \Phi\}$;
- Let $\bar{Q} = \{q(\bar{\phi}) : \bar{\phi} \in \bar{\Phi}\}$ be the set formed by the distribution vectors of all quantizers, deterministic or random;
- Denote by $Q_U \subset Q$ the set of distribution vectors of all ULQs (see Definition 2.5.1);
- Denote by Q_α the set of extreme points (or corners) of (the compact convex set) $\bar{Q}$.

By Tsitsiklis [40], Q is compact and $\bar{Q}$ is the compact convex hull of Q. By the Krein-Milman theorem, the compact convex set $\bar{Q}$ is also the convex hull of its extreme points Q_α. Thus it is useful to characterize Q_α. Tsitsiklis [40] showed that $Q_U \subset Q_\alpha \subset Q$, and Q_U is a dense subset of Q_α. Moreover, it also studied in detail the case of testing $M = 2$ hypotheses. The case of $M \ge 3$ hypotheses is clearly more challenging. Fortunately, below we are able to show that $Q_\alpha = Q_U$ for $M \ge 3$ hypotheses under some reasonable additional assumptions.

Lemma 2.7.1. *If the sensor messages are binary (i.e., $l = 2$) and the pdf's $\{f_0, \ldots, f_{M-1}\}$ are linearly independent (as defined in Subsection 2.4.2), then $Q_\alpha = Q_U$.*

Proof. Since Q_U is a subset of Q_α, it is sufficient to show that if the $2M$-dimensional vector $q^0 = (q^0_{0m}, q^0_{1m})_{0 \le m \le M-1} \in Q_\alpha$ (with $q^0_{0m} + q^0_{1m} = 1$), then $q^0 \in Q_U$. Since Q_U is

dense in Q_α, there is a sequence of ULQs ϕ^j, say, $\phi^j(X) = I(\sum_{m=0}^{M-1} a_m^j f_m(X) > 0)$ with $\sum_{m=0}^{M-1}(a_m^j)^2 = 1$, such that $q(\phi^j) \to q^0$, or equivalently, $\lim_{j\to\infty} P_m\left\{\phi^j(X) = 1\right\} = q_{1m}^0$ for each m.

By the Bolzano-Weierstrass theorem, each bounded sequence has a convergent subsequence. By passing to subsequences, we can simply assume that a_m^j converges to a_m^0 for each state m. Then $\sum_{m=0}^{M-1}(a_m^0)^2 = 1$, and we can define two quantizers:

$$\phi^0(X) = I(\sum_{m=0}^{M-1} a_m^0 f_m(X) > 0)$$

and

$$\varphi^0(X) = I(\sum_{m=0}^{M-1} a_m^0 f_m(X) \geq 0).$$

By the dominated convergence theorem, for each m,

$$E_m\left\{\phi^0(X)\right\} \leq \liminf_{j\to\infty} E_m\left\{\phi^j(X)\right\} \leq \limsup_{j\to\infty} E_m\left\{\phi^j(X)\right\} \leq E_m\left\{\varphi^0(X)\right\}.$$

Since the pdf's are assumed to be linearly independent, $\phi^0(X)$ is a ULQ, and $E_m\left\{\phi^0(X)\right\} = E_m\left\{\varphi^0(X)\right\}$. Thus, $\lim E_m\left\{\phi^j(x)\right\} = E_m\left\{\phi^0(X)\right\}$ and $q_{1m}^0 = \lim P_m\left\{\phi^j(x) = 1\right\} = P_m\left\{\phi^0(X) = 1\right\}$ for each m, since the sensor messages are binary. This implies that $q^0 = q(\phi^0) \in Q_U$, and the lemma is proved. □

Second, we need to pass the definition of the K-L divergences from quantizer $\bar{\phi}$ to the distribution vectors $q(\bar{\phi})$. Given $q = q(\bar{\phi}) \in \bar{Q}$, denote $q_{i,m} = q(i; m, \bar{\phi}) = P_m\left\{\bar{\phi}(X) = i\right\}$, where $i = 0, \ldots, l-1$ and $m = 0, \ldots, M-1$. For $0 \leq m \neq m' \leq M-1$, define the K-L divergence of the distribution vector q of state m' from state m by

$$J(m, m'; q) = \sum_{i=0}^{l-1} q_{i,m} \log \frac{q_{i,m}}{q_{i,m'}} \tag{27}$$

where as conventional $0 \log \frac{0}{0} = 0$.

Note that for a randomized quantizer $\bar{\phi}$, the definition of $J(m, m'; q(\bar{\phi}))$ is equivalent to the K-L divergence defined in (8), and Tsitsiklis [40] investigated the corresponding optimal quantization problems. However, in our context, the K-L divergence of a randomized quantizer $\bar{\phi}$ is defined in (9), and thus the results of Tsitsiklis [40] are not directly applicable.

Fortunately, the idea can be salvaged by introducing a new definition of K-L divergence. To do so, let $\bar{\mathcal{M}}$ be the set of Borel probability measures on $\bar{Q}$, for each $\mu \in \bar{\mathcal{M}}$ and two states $0 \leq m \neq m' \leq M-1$ define

$$J^*(m, m'; \mu) = \int_{\bar{Q}} J(m, m'; q) d\mu(q) \tag{28}$$

and

$$J^*(m; \mu) = \min_{m' \neq m} J^*(m, m'; \mu). \tag{29}$$

Then for a randomized quantizer $\bar{\phi} \in \bar{\Phi}$, the K-L divergence defined in (9) is equivalent to $J^*(m, m'; \mu)$ for some suitably chosen μ. To see this, note that $\bar{\phi}$ assigns probability masses to a finite or countable subset of Φ, and thus induces a probability measure $\mu(\bar{\phi})$ on Q. Hence, $I(m, m'; \bar{\phi}) = J^*(m, m'; \mu(\bar{\phi}))$ and

$$I(m; \bar{\phi}) = J^*(m; \mu(\bar{\phi})). \tag{30}$$

Next, we need to investigate how to use the $J^*(m; \mu)$'s in (29) to characterize the maximin information number $I(m)$ in Definition 2.4.1 in Subsection 2.4.1. Our next result provides an alternative representation of $I(m)$.

Lemma 2.7.2. *The maximin information number*

$$I(m) = \sup_{\mu \in \mathcal{M}} J^*(m; \mu) = \sup_{\mu \in \bar{\mathcal{M}}} J^*(m; \mu),$$

where $\mathcal{M} \subset \bar{\mathcal{M}}$ is the set of probability measures supported on Q.

Proof. Denote by $\mathcal{M}^0$ and $\bar{\mathcal{M}}^0$ the set of probability measures on Q and $\bar{Q}$ that have at most countable supports, respectively. By (30), $\sup_{\mu \in \mathcal{M}^0} J^*(m; \mu) = I(m)$, and thus

$$I(m) \leq \sup_{\mu \in \mathcal{M}} J^*(m; \mu) \leq \sup_{\mu \in \bar{\mathcal{M}}} J^*(m; \mu).$$

By Tsitsiklis [40], $J(m, m'; q)$ is bounded and continuous as a function of $q \in \bar{Q}$. Hence $J^*(m, m'; \mu)$ and $J^*(m; \mu)$ are also continuous viewed as functions of $\mu \in \bar{\mathcal{M}}$ (under weak-convergence). Thus the lemma follows at once from the denseness of $\mathcal{M}^0$ (or $\bar{\mathcal{M}}^0$) in $\mathcal{M}$ (or $\bar{\mathcal{M}}$), provided that $I(m) \geq \sup_{\mu \in \bar{\mathcal{M}}^0} J^*(m; \mu)$. Hence, it suffices to show that for

each $\mu \in \bar{\mathcal{M}}^0$, there exists a $\mu' \in \mathcal{M}^0$ such that $J^*(m, m'; \mu) \leq J^*(m, m'; \mu')$ for each $m' \neq m$. By linearity, we only need to prove it under the further assumption that $\mu \in \bar{\mathcal{M}}^0$ is supported on a single point $q = q(\bar{\phi})$ for a randomized quantizer $\bar{\phi} \in \bar{\Phi}$. In this case $J^*(m, m'; \mu) = J(m, m'; q) \leq I(m, m'; \bar{\phi})$. By our previous argument, $\bar{\phi}$ can be identified to a probability measure $\mu' = \mu(\bar{\phi}) \in \mathcal{M}^0$ with the property $I(m, m'; \bar{\phi}) = J^*(m, m'; \mu')$. Therefore $J^*(m, m'; \mu) \leq J^*(m, m'; \mu')$, completing the proof of the lemma. □

Finally, we are in a position to prove Theorems 2.4.3 and 2.5.1.

Proofs of Theorem 2.4.3 and Theorem 2.5.1. Note that Theorem 2.4.3 is a special case of Theorem 2.5.1, and follows at once from Theorem 2.5.1 and Lemma 2.7.1 under the assumption of binary sensor messages and linearly independent pdf's in which $Q_U = Q_\alpha$. By symmetry and the fact that Q_U is a dense subset in Q_α, it is sufficient to show that under the assumption of Theorem 2.5.1, for the state $m = 0$, there exists a maximin quantizer which is a randomization of at most $M-1$ quantizers with their distribution vectors in Q_α.

Define two sets in the $M-1$ dimensional space, $\mathscr{I} = \{(J(0,1;q), \ldots, J(0,M-1;q)) : q \in Q\}$, and $\mathscr{I}_\alpha = \{(J(0,1;q), \ldots, J(0,M-1;q)) : q \in Q_\alpha\}$. Define the same for $\mathscr{I}^*$ and $\mathscr{I}^*_\alpha$ when $J(0,m;q)$ is replaced by $J^*(0,m;\mu)$ with $\mu \in \mathcal{M}$ and $\mu \in \mathcal{M}_\alpha$, respectively, where $\mathcal{M}_\alpha$ is the set of probability measures supported in Q_α. As we have mentioned earlier, $J(0,m;q)$ is continuous if viewed as a function of $q \in Q$, so both $\mathscr{I}$ and $\mathscr{I}_\alpha$ are compact. Obviously, $\mathscr{I}^*$ and $\mathscr{I}^*_\alpha$ are convex hulls of $\mathscr{I}$ and $\mathscr{I}_\alpha$, so they are compact as well. The main idea of the proof is to relate the maximin information number $I(0)$ with the set $\mathscr{I}^*_\alpha$.

First, we claim that $I(0) = \sup_{J \in \mathscr{I}^*_\alpha} h(J)$, where $h(\cdot)$ is a function on the $M-1$ dimensional space defined by $h(x_1, \ldots, x_{M-1}) = \min\{x_1, \ldots, x_{M-1}\}$. By Lemma 2.7.2, we have $I(0) = \sup_{J \in \mathscr{I}^*} h(J)$. Since $\mathscr{I}^*_\alpha \subset \mathscr{I}^*$, to prove the claim, we only need to show, for any $J \in \mathscr{I}^*$, there exists $J' \in \mathscr{I}^*_\alpha$, such that each component of J is less than or equal to the corresponding component of J'. By linearity, it is sufficient to prove for $J \in \mathscr{I}$, say, $J = (J(0,1;q), \ldots, J(0,M-1;q))$ for some $q \in Q$. Decompose q as a convex combination of points in Q_α: $q = \sum p^j q^j$, then

$$J(m, m'; q) \leq \sum p^j J(m, m'; q^j), \quad 0 \leq m \neq m' \leq M-1.$$

Let $J' = (J^*(0,1;\mu), \ldots, J^*(0,M-1;\mu))$ with μ assigns probability mass p^j to q^j for each j, and our claim is justified.

Second, we will show that

$$\sup_{J \in \mathscr{I}_\alpha^*} h(J) = \min_{1 \le m \le M-1} J^*(0,m;\mu_0) \tag{31}$$

for a probability $\mu_0 \in \mathcal{M}_\alpha$ whose support includes at most $M-1$ points. To see this, note that $\mathscr{I}_\alpha^*$ is a compact convex subset in $M-1$ dimensional space. Thus $h(\cdot)$ attains its maximum at a point $\tilde{J}$ on the surface of $\mathscr{I}_\alpha^*$ and $\tilde{J}$ can be realized as a convex combination of at most $M-1$ points in $\mathscr{I}_\alpha^*$, see, for example, Hormander [9]. Suppose that $\tilde{J} = \sum_{j=1}^{M-1} p_0^j J^j$, where $\sum p_0^j = 1$ and $J^j \in \mathscr{I}_\alpha^*$. For each j, let $J^j = (J(0,1;q_0^j), \ldots, J(0,M-1;q_0^j))$, with $q_0^j \in Q_\alpha$. Define $\mu_0 \in \mathcal{M}_\alpha$ be a probability measure such that $\mu_0(q_0^j) = p_0^j$, for $j = 1, \ldots, M-1$, then (31) holds.

Finally, define the randomized quantizer $\bar{\phi}_0$ as the one induced by the measure μ_0 in (31). Then $I(0) = \min_{m \neq 0}\{I(0,m;\bar{\phi}_0)\}$ and $\bar{\phi}_0$ can be rewritten as $\sum_{j=1}^{M-1} p_0^j \phi_0^j$ where ϕ_0^j has q_0^j as its distribution vector. Equivalently, $\bar{\phi}_0$ is just the maximin quantizer $\bar{\phi}_0^{\max}$, and it can be taken as a randomization of at most $M-1$ quantizers with their distribution vectors in Q_α. This completes our proof. □

As a final remark about maximum quantizers, the main difference between Theorem 2.4.3 and Theorem 2.5.1 is due to Lemma 2.7.1 that shows $Q_U = Q_\alpha$ under the assumption of binary sensor messages and linearly independent pdf's. It is natural to ask whether we can extend Lemma 2.7.1 beyond binary sensor messages? In other words, when the sensor messages belong to a finite alphabet $\{0, 1, \cdots, l-1\}$ with $l \ge 3$, will we still have $Q_U = Q_\alpha$ for linearly independent pdf's? Unfortunately, the answer is "No!" The following illustrates why the proof of Lemma A.1 cannot go through when $l = 3$.

For $l = 3$, note that a ULQ of form $\phi(X) = \arg\min_{0 \le i \le 2} \sum_{m=0}^{M-1} a_{i,m}^* f_m(X)$ can be rewritten as

$$\phi(X) = \begin{cases} 0, & \text{if } \sum_{m=0}^{M-1} a_{1,m} f_m(X) < 0 \text{ and } \sum_{m=0}^{M-1} a_{2,m} f_m(X) < 0 \\ 1, & \text{if } \sum_{m=0}^{M-1} a_{1,m} f_m(X) > \max(0, \sum_{m=0}^{M-1} a_{2,m} f_m(X)) \\ 2, & \text{if } \sum_{m=0}^{M-1} a_{2,m} f_m(X) > \max(0, \sum_{m=0}^{M-1} a_{1,m} f_m(X)) \end{cases},$$

where $a_{i,m} = a^*_{0,m} - a^*_{i,m}$. That is, without loss of generality, we can set $\sum_{m=0}^{M-1} a^*_{i,m} f_m(X)$ as the baseline and focus on the difference between these functions, which also leads to our definition of ULQ in Definition 2.4.2 for binary sensor messages. For $l \geq 3$, the normalization constraint imposed the coefficients $a_{1,m}$'s and $a_{2,m}$'s is

$$\sum_{m=0}^{M-1} (a_{1,m})^2 + \sum_{m=0}^{M-1} (a_{2,m})^2 = 1,$$

since we cannot normalize $a_{1,m}$'s and $a_{2,m}$'s individually. Now let us proceed as in Lemma A.1 for $l \geq 3$. Given an extreme point of quantizer $q^0 \in Q_\alpha$ and since Q_U is dense in Q_α, we have a sequences of ULQs ϕ^j, with coefficients $a^j_{1,m}$'s and $a^j_{2,m}$'s, such that $q(\phi^j) \to q^0$. Again, by the Bolzano-Weierstrass theorem, without loss of generality, we can assume that for each m, the sequences $a^j_{1,m}$ and $a^j_{2,m}$ converge to $a^0_{1,m}$ and $a^0_{2,m}$, respectively, as j goes to ∞. Denote by $\phi^0(X)$ the "ULQ" defined by the limits $a^0_{1,m}$'s and $a^0_{2,m}$'s. In Lemma A.1 we show that $q^0 = q(\phi^0)$ for binary sensor messages. When $l \geq 3$, it can be shown that this is still true except the following three "degenerate" cases: (i) $a^0_{1,m}$'s are zero for all $m = 0, 1, \ldots, M-1$; (ii) $a^0_{2,m}$'s are zero for all m; and (iii) $a^0_{1,m} = a^0_{2,m}$ for all m. For example, in the first degenerate case when $a^0_{1,m}$'s are zero for all m, in general $q^0 \neq q(\phi^0)$, since $\mathbf{P}_m(\phi^0(X) = 1) = 0$ as $\phi^0(X) = 2 \times 1\{\sum_{m=0}^{M-1} a^0_{2,m} f_m(X) > 0\}$ is a ULQ for binary messages of "0" and "2." It is interesting to see that in this case, the extreme point q^0 is not a ULQ, but it can be written as a combination of two ULQ's for binary sensor messages:

$$q^0 = \begin{cases} 0, & \text{if } \phi^0(X) = 0 \text{ and } \psi^*(X) = 0 \\ 1, & \text{if } \phi^0(X) = 0 \text{ and } \psi^*(X) = 1 \\ 2, & \text{if } \phi^0(X) = 2 \end{cases},$$

where $\phi^0(X) = 2 \times 1\{\sum_{m=0}^{M-1} a^0_{2,m} f_m(X) > 0\}$ and $\psi^*(X) = I\{\sum_{m=0}^{M-1} a^{*0}_{1,m} f_m(X) > 0\}$. Here we normalize the sequences $a^j_{1,m}$ by

$$a^{*j}_{1,m} = \frac{a^j_{1,m}}{\sum_{m=0}^{M-1} a^j_{1,m}}$$

and assume that for each m, (some subsequence of) $a^{*j}_{1,m}$ converges to $a^{*0}_{1,m}$ as j goes to ∞. The result can be extended to characterize the extreme quantizers that correspond to extreme points of quantizers in the general case of $l \geq 3$ in a recursive approach: all extreme

quantizers for l sensor messages are either ULQs or a combination of extreme quantizers for $i < l$ sensor messages.

2.8 Proof of Theorem 2.4.1

The main idea in the proof of Theorem 2.4.1 is to condition on the preliminary decision D_0 of the two-stage test $\delta(c)$. In the following we will focus on the proof of (15) to highlight the associated technical mathematical problems that need special attention. Denote by N_0 and N_1 the total time steps of the first and second stages of the two-stage test $\delta(c)$, respectively, then the total time step N taken by $\delta(c)$ satisfies

$$\begin{aligned} E_m\{N\} &= E_m\{N_0\} + E_m\{N_1\} \\ &= E_m\{N_0\} + E_m\{N_1|D_0 = m\}\, P_m\{D_0 = m\} + E_m\{N_1 1\{D_0 \neq m\}\}. \end{aligned}$$

Note that at the first stage of our proposed two-stage test $\delta(c)$, since the local sensor uses stationary (deterministic) quantizers, the sensor messages U_n's are i.i.d. and the fusion center essentially faces the classical centralized sequential multi-hypothesis testing problems. Thus by standard arguments, the stopping boundary of $1-u(c)$ at the first stage guarantees that $P_m\{D_0 = m\} = 1 - O(u(c))$ and $E_m\{N_0\} = O(|\log u(c)|)$. Since $u(c) \to 0$ satisfies $|\log u(c)|/|\log c| \to 0$, e.g., $u(c) = 1/|\log c|$, we have $P_m\{D_0 = m\} = 1-o(1)$ and $E_m\{N_0\} = o(|\log c|)$. Hence, equation (15) holds if we can further show that

$$E_m\{N_1|D_0 = m\} = (1 + o(1))|\log c|/I(m; \bar{\phi}_m) \tag{32}$$

$$E_m\{N_1 1\{D_0 \neq m\}\} = o(|\log c|). \tag{33}$$

To prove (32) and (33), note that at time n of the second stage of our proposed two-stage test $\delta(c)$, the log-likelihood ratio statistic of the latest sensor message at the fusion center is

$$\Delta Z_n(m, m'; \phi^{j(n)}) = \log \frac{f_m(U_n; \phi^{j(n)})}{f_{m'}(U_n; \phi^{j(n)})},$$

where $\phi^{j(n)}$ is the deterministic quantizer selected through the randomization at time step n and $U_n = \phi^{j(n)}(X_n)$ is the quantized sensor message. Hence, for our proposed two-stage

test, the log-likelihood ratio statistic of sensor messages from the beginning of the second stage to the time n of the second stage is

$$Z_n(m, m'; \bar{\phi}) = \sum_{i=1}^{n} \Delta Z_i(m, m'; \phi^{j(i)}). \tag{34}$$

Furthermore, since $\bar{\phi}$ is assumed to be a randomization of a finite number of deterministic quantizers, our implementation of randomized quantizers implies that $\{\Delta Z_n(m, m'; \phi^{j(i)}), n = 1, 2, \dots\}$ is a sequence of i.i.d. random variables with mean $I(m, m'; \bar{\phi})$ in (9) and finite variance (If we adopt periodic block design, then the arguments need some modifications by letting U_n represent each block).

To prove (32), let us focus on the " $\leq$ " part, as the " $\geq$ " part can be proved similarly (or through the general result of Theorem 4.2 on lower bound). It is sufficient to prove the following stronger result:

$$\sup_{\vec{\pi}_{N_0}} E_m \{N_1 | D_0 = m, \vec{\pi}_{N_0}\} \leq (1 + o(1))|\log c|/I(m; \bar{\phi}_m)$$

where $\vec{\pi}_{N_0} = (\pi_{0,N_0}, \dots, \pi_{M-1,N_0})$ denotes the posterior distribution at time N_0. For a given $\vec{\pi}_{N_0}$, this relation follows at once from the fact that $Z_n(m, m'; \bar{\phi})\}$ is the sum of i.i.d. random variables with mean $I(m, m'; \bar{\phi})$ in (9) and finite variance, but we need some extra work to prove that it also holds uniformly regardless of the values of $\vec{\pi}_{N_0}$.

For that purpose, let $B_c = |\log \frac{c}{1-c}| + |\log(1-u(c))|$ and consider the following stopping time:

$$T(B_c; \bar{\phi}_m) = \inf\{n \geq 1 : \min_{m':m' \neq m} Z_n(m, m'; \bar{\phi}_m) \geq B_c\} \tag{35}$$

where $Z_n(m, m'; \bar{\phi}_m)$ is the log-likelihood ratio in (34) except that the quantizer $\bar{\phi}$ is now replaced by $\bar{\phi}_m$, i.e., $Z_n(m, m'; \bar{\phi}_m)$ is the log-likelihood ratio of sensor messages during the time step $N_0 + 1$ and $N_0 + n$ when conditioning on $\{D_0 = m\}$. By (13), given the state m, at time n of the second stage (i.e., at the time step $N_0 + n$), the posterior probability

$$\pi_{m,n} = \frac{\pi_{m,N_0}}{\pi_{m,N_0} + \sum_{m' \neq m} \pi_{m',N_0} \exp(-Z_n(m, m', \bar{\phi}))}. \tag{36}$$

Hence, at the time $n^* = T(B_c; \bar{\phi}_m)$, we have

$$\pi_{m,n^*} \geq \frac{\pi_{m,N_0}}{\pi_{m,N_0} + (1 - \pi_{m,N_0}) \exp(-B_c)},$$

where we use the fact that $\sum_{m' \neq m} \pi_{m',N_0} = 1 - \pi_{m,N_0}$. Now conditioning on $P_m\{\cdot | D_0 = m, \vec{\pi}_{N_0}\}$, we have $\pi_{m,N_0} \geq 1 - u(c)$. By our choice of B_c, we have $\exp(-B_c) = \frac{c}{1-c}(1-u(c)) \leq \frac{c}{1-c}\pi_{m,N_0}$. Hence,

$$\pi_{m,n^*} \geq \frac{1}{1 + (1 - \pi_{m,N_0})c/(1-c)} \geq \frac{1}{1 + c/(1-c)} = 1 - c,$$

and thus $N_1 \leq T(B_c; \bar{\phi}_m)$. In other words, conditioning on $P_m\{\cdot | D_0 = m, \vec{\pi}_{N_0}\}$, the stopping time N_1 of our proposed two-stage test is dominated by $T(B_c; \bar{\phi}_m)$, which does not depend on $\vec{\pi}_{N_0}$. By the law of large numbers, we have $E_m\{T(B_c; \bar{\phi}_m)\}/B_c \to 1/I(m; \bar{\phi}_m)$, also see Theorem 5.1 of Baum and Veeravalli [1]. Since $\log u(c) = o(|\log c|)$ and B_c is the same order as $|\log c|$, the $\leq$ part of relation (32) is proved. The $\geq$ part of the relation can be proved similarly and thus relation (32) holds.

The proof of (33) is more technically involved. It suffices to show that $E_m\{N_1 1\{D_0 = m'\}\} = o(|\log c|)$ for each $m' \neq m$. Now when $\{D_0 = m'\}$, our proposed two-stage procedure $\delta(c)$ uses the stationary (likely randomized) quantizer $\bar{\phi}_{m'}$ at the second stage. Intuitively, due to a wrong preliminary decision, a suboptimal quantizer $\bar{\phi}_{m'}$ is used at the second stage when the true state of nature is m. Recall that a key assumption of our theorem is that $I(m, \bar{\phi}_{m'}) > 0$, and thus when the true state of nature is m, this suboptimal quantizer still brings in positive information in favor of making a correct final decision $\{D = m\}$. On average the proposed two-stage test takes roughly $N_2 \approx |\log c|/I(m, \bar{\phi}_{m'}) = O(|\log c|)$ steps at the second stage to increase the posterior probability $\pi_{m,n}$ from π_{m,N_0} to $\pi_{m,N}(\geq 1 - c)$. The difficulty arises from the fact that π_{m,N_0} can be truly small if the preliminary decision $\{D_0 = m'\}$ is wrong, as $\pi_{m',N_0} \geq 1 - u(c)$ by definition, and we need to show that such event is negligible (exponentially bounded).

To be more rigorous, as in (35), define a new stopping time (on the second stage)

$$T^* = \inf\{n \geq 1 : \min_{m'':m'' \neq m} Z_n(m, m''; \bar{\phi}_{m'}) \geq B^*\},$$

where $Z_n(m, m''; \bar{\phi}_{m'})$ is defined as in (34) except that the stationary quantizer is given by $\bar{\phi}_{m'}$, and

$$B^* = |\log \frac{c}{1-c}| + |\log \pi_{m,N_0}|.$$

As compared to $T(B_c;\bar{\phi}_m)$ in (35), T^* has a different quantizer $\bar{\phi}_{m'}$ and a different threshold B^*. By assumption, $I(m;\bar{\phi}_{m'}) = \min_{m''\neq m} I(m,m'';\bar{\phi}_{m'}) > 0$ and thus $E_m\{T^*\} = (1+o(1))B^*/I(m;\bar{\phi}_m)$. Furthermore, it follows from (36) that conditioning on $P_m\{\cdot|D_0 = m',\vec{\pi}_{N_0}\}$, the stopping time N_1 of our proposed two-stage test is dominated by T^*. Then

$$\begin{aligned}
E_m\left\{N_1 1\{D_0=m'\}\right\} &\leq E_m\left\{T^* 1\{D_0=m'\}\right\} \\
&= E_m\left\{(1+o(1))B^* 1\{D_0=m'\}/I(m;\bar{\phi}_{m'})\right\} \\
&= (1+o(1))/I(m;\bar{\phi}_{m'})E_m\left\{B^* 1\{D_0=m'\}\right\} \\
&\leq O(1)E_m\left\{(|\log\frac{c}{1-c}| + |\log\pi_{m,N_0}|)1\{D_0=m'\}\right\} \\
&= O(|\log\frac{c}{1-c}|)P_m\left\{D_0=m'\right\} + O(1)E_m\left\{|\log\pi_{m,N_0}|1\{D_0=m'\}\right\} \\
&= o(|\log c|) + O(1)E_m\left\{|\log\pi_{m,N_0}||1\{D_0=m'\}\right\},
\end{aligned}$$

where we use the fact that $P_m\{D_0=m'\} = o(1)$. Thus, to prove (33), it remains to show that $E_m\{|\log\pi_{m,N_0}|1\{D_0=m'\}\} = o(|\log c|)$. Below we will prove a stronger statement that

$$E_m\{|\log\pi_{m,N_0}|1\{D_0\neq m\}\} = o(1). \tag{37}$$

To prove (37), it is sufficient to focus on the first stage of our proposed two-stage test. Note that by our construction, when $N_0 = n$ (a finite time step), we have

$$\frac{1-\pi_{m,n}}{\pi_{m,n}} = \sum_{m'\neq m}\frac{\pi_{m'}}{\pi_m}\exp(-Z_n(m,m',\phi^0)).$$

By definition, at time N_0, if $D_0 = m'$ then $\pi_{m',N_0} \geq 1-u(c) > 1/2$. So $\pi_{m,N_0} < u(c) < 1/2$ and for all $L > 0$,

$$\begin{aligned}
& P_m\{|\log\pi_{m,N_0}| > L, D_0\neq m\} \\
\leq\ & P_m\left\{\frac{1-\pi_{m,N_0}}{\pi_{m,N_0}} > e^L/2, D_0\neq m\right\} \\
\leq\ & P_m\left\{\sup_{n\geq 1}\sum_{m':m'\neq m}\frac{\pi_{m'}}{\pi_m}\exp\{-Z_n(m,m';\phi^0)\} > e^L/2\right\} \\
\leq\ & P_m\left\{\sup_{n\geq 1}\Big[\frac{1-\pi_m}{\pi_m}\max_{m':m'\neq m}\exp\{-Z_n(m,m';\phi^0)\}\Big] > e^L/2\right\} \\
=\ & P_m\left\{\min_{m':m'\neq m}\inf_{n\geq 1} Z_n(m,m';\phi^0) < -L + \log\frac{2(1-\pi_m)}{\pi_m}\right\},
\end{aligned}$$

where in the second to last relation, we replace each term by the maximum term and then use the fact that $\sum_{m' \neq m} \pi_{m'} = 1 - \pi_m$. Assume for a moment that the minimum $Z^* = \min_{m':m' \neq m} \inf_{n \geq 0} Z_n(m, m'; \phi^0)$ is exponentially bounded in the sense that there exists a constant $C_1 > 0$ and $0 < \rho < 1$ such that for any $L > 0$,

$$P_m \{Z^* \leq -L\} \leq C_1 \rho^L. \tag{38}$$

Then we have

$$P_m \{|\log \pi_{m,N_0}| > L, D_0 \neq m\} \leq C_2 \rho^L$$

with the constant $C_2 = C_1 \exp(-\log \rho \log \frac{2(1-\pi_m)}{\pi_m})$. Consequently,

$$\begin{aligned}
& E_m \{|\log \pi_{m,N_0}| 1\{D_0 \neq m\}\} \\
= \; & E_m \{|\log \pi_{m,N_0}| 1\{D_0 \neq m, |\log \pi_{m,N_0}| \geq |\log u(c)|\}\} \\
\leq \; & C_2 \int_{|\log u(c)|}^{\infty} \rho^L dL \\
= \; & \frac{C_2}{|\log \rho|} \rho^{|\log u(c)|}
\end{aligned}$$

which goes to 0 as $c \to 0$. Thus (37) is proved and the theorem holds.

It remains to prove (38). Note that such an exponential boundness is not surprising, and it is instructive to look at a well-known result for Brownian motion. Let $B(t)$ denote standard Brownian motion with mean zero and variance parameter 1. Then for all positive L, μ and σ,

$$\mathbf{P}(\inf_{t \geq 0} \{\sigma B(t) + \mu t\} \leq -L) = \exp(-2\mu\sigma^{-2} L).$$

To prove (38), since the log-likelihood ratio statistic $Z_n(m, m'; \bar{\phi})$ in (34) is the sum of i.i.d. random variables with positive mean and finite variance under $\mathbf{P}_m$, the minimum

$$Z^*_{m'} = \inf_{n \geq 0} Z_n(m, m'; \bar{\phi})$$

is a well-defined (non-positive valued) random variable under $\mathbf{P}_m$. Moreover,

$$P_m \{Z^* \leq -L\} \quad \leq \quad \sum_{m':m' \neq m} P_m \{Z^*_{m'} \leq -L\}.$$

Thus, to prove (38), it suffices to show that $Z^*_{m'}$ is exponentially bounded for each m'. Define a stopping time $\tau_- = \inf\{n : Z_n(m, m'; \bar{\phi}) < 0\}$ and let $Y_1, Y_2, \ldots$ be i.i.d. random variables, where $Y_1 = Z_{\tau_-}(m, m'; \bar{\phi})$ conditional on the event $\tau_- < \infty$. Then it is well-known that $Z^*_{m'}$ has the same distribution as $\sum_{i=1}^{\tilde{N}} Y_i$, where $\tilde{N}$ is a geometric random variable independent of Y_i's such that $P(\tilde{N} = n) = p(1-p)^n$ with $p = P_m\{Z^*_{m'} = 0\} > 0$, see Klass [13], or Lemma 11.3 and Remark 11.3 of Gut [7]. Now in our case, since U_n is discrete and $\bar{\phi}$ is randomization of a finite number of deterministic quantizer, $\Delta Z_n(m, m'; \bar{\phi})$ has a lower bound, say $-C$ for some $C > 0$. Thus $Y_1 = Z_{\tau_-}(m, m'; \bar{\phi})$ also has a lower bound $-C$. So

$$\begin{aligned} P_m\{Z^*_{m'} \le -L\} &= P(\sum_{i=1}^{\tilde{N}} Y_i \le -L) \\ &= P(\tilde{N} \ge L/C) \\ &= (1-p)^{[L/C]} \end{aligned}$$

where the last relation uses the fact that $\tilde{N}$ is geometrically distributed. Hence $Z^*_{m'}$ is exponentially bounded and this completes the proof of the theorem.

2.9 Proof of Theorem 2.4.2

To prove Theorem 2.4.2, the main idea is to construct a martingale based on log-likelihood ratios and then apply the optional stopping theorem and Wald's inequalities. Denote by $\tilde{\phi}_n$ the quantizer used at the fusion center for decision making at time step n. Note that since Theorem 2.4.2 deals with general decentralized sequential tests, randomized quantizers may or may not be implemented as we proposed for the two-stage tests. For example, when a randomized quantizer $\bar{\phi} = \sum p^j \phi^j$ is implemented and the fusion center knows that the deterministic quantizer $\phi^{j(n)}$ is picked at time step n, then $\tilde{\phi}_n = \phi^{j(n)}$. Meanwhile, if the randomization is done at the local sensor and the fusion center has no access about which deterministic quantizer is picked, then $\tilde{\phi}_n = \bar{\phi}$.

Let U_n be the sensor message at time step n and let $q(\tilde{\phi}_n)$ be the distribution vector of $\tilde{\phi}_n$. For $n = 1, 2, \ldots$, define $\mathcal{F}_{n-1}$ as the σ-algebra generated by $U_1, \ldots, U_{n-1}$ and $q(\tilde{\phi}_1), \ldots, q(\tilde{\phi}_n)$. In other words, $\mathcal{F}_{n-1}$ is all the past information available to the fusion center before the nth time step. Then at time step n, the log-likelihood ratio of state m

with respect to state m' is $Z_n = \sum_{i=1}^{n} \Delta Z_i$, where

$$\Delta Z_i = \log \frac{f_m(U_i|\mathcal{F}_{i-1})}{f_{m'}(U_i|\mathcal{F}_{i-1})}$$

and $f_m(\cdot|\mathcal{F}_{i-1})$ is the conditional probability mass function induced on U_i under P_m. Since U_i depends on $\mathcal{F}_{i-1}$ only through $\tilde{\phi}_i$, $f_m(\cdot|\mathcal{F}_{i-1})$ is simply $f_m(\cdot;\tilde{\phi})$ in (8), and thus $E_m\{\Delta Z_i|\mathcal{F}_{i-1}\} = J(m,m';q(\tilde{\phi}_i))$ in (27). Therefore,

$$M_n = \sum_{i=1}^{n} \Big[\Delta Z_i - J(m,m';q(\tilde{\phi}_i))\Big] = Z_n - \sum_{i=1}^{n} J(m,m';q(\tilde{\phi}_i))$$

forms a martingale under P_m with respect to $\{\mathcal{F}_n\}$. Applying the optional stopping theorem to the martingale $\{M_n;\mathcal{F}_n\}$, for the stopping time N of a decentralized test $\delta(c)$, we have $\mathbf{E}_m(M_N) = 0$, or equivalently,

$$E_m\{Z_N\} = E_m\left\{\sum_{i=1}^{N} J(m,m';q(\tilde{\phi}_i))\right\}. \tag{39}$$

Now let us go back to the proof of Theorem 2.4.2. Obviously, for a decentralized test $\delta(c)$ satisfying the error probability assumption in Theorem 2.4.2, if the sample size N satisfies $E_m\{N\} = \infty$, then Theorem 2.4.2 holds. Thus we only need to consider the case when $E_m\{N\} < \infty$. To derive the asymptotic lower bound on $E_m\{N\}$, we construct a new test $\delta'(c)$ that accepts H_m if the final decision of $\delta(c)$ is $D = m$ but accepts $H_{m'}$ (for a given $m' \neq m$) if $D \neq m$. Then this new test $\delta'(c)$ is a well-defined sequential test in the problem of testing a simple hypothesis H_m against a simple alternative $H_{m'}$. Moreover, the assumption of Theorem 2.4.2 guarantees that both type I and type II errors of $\delta'(c)$ are less than $\alpha_c = Ac|\log c|$, where $A > 0$ is a constant. Hence, Z_N represents the log-likelihood ratio of the test $\delta'(c)$ when stopped and by Wald's inequalities (also see Theorem 2.39 of Siegmund [34]),

$$\begin{aligned} E_m\{Z_N\} &\geq (1-\alpha_c)\log(\frac{1-\alpha_c}{\alpha_c}) + \alpha_c \log(\frac{\alpha_c}{1-\alpha_c}) \\ &\geq (1-\alpha_c)|\log \alpha_c| - \log 2 \\ &= |\log c| - \log|\log c| + O(1) \end{aligned}$$

as $c \to 0$, where the $O(1)$ term depends only on A. Here the second inequality follows from the facts that $\alpha \log(1-\alpha)^{-1}$ is nonnegative and that $(1-\alpha)\log(1-\alpha) + \alpha \log \alpha$ attains

minimum value $-\log 2$ when $\alpha = \frac{1}{2}$. By (39), we have

$$E_m \left\{ \sum_{i=1}^{N} J(m, m'; q(\tilde{\phi}_i)) \right\} \geq |\log c| - \log |\log c| + O(1). \tag{40}$$

Now we claim that the left-hand side of (40) can be rewritten as $J^*(m, m'; \mu_m) E_m \{N\}$ for a suitably chosen probability measure μ_m on $\bar{Q}$, where $J^*(m, m'; \mu_m)$ is defined as in (28). Then the theorem follows at once from this claim, relation (40), and Lemma 2.7.2. It remains to prove this claim. To do so, define μ_m as a convex combination of a sequence of probability measures $\{\mu_{m,n,i} : i \leq n\}$ as follows.

$$\mu_m = \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{P_m\{N = n\}}{E_m\{N\}} \mu_{m,n,i}.$$

Then let $\mu_{m,n,i}$ be the distribution of $q(\tilde{\phi}_i)$ under P_m and conditioned on the event $N = n$. In other words, for any Borel set $A \subset \bar{Q}$, $\mu_{m,n,i}(A) = P_m\{q(\tilde{\phi}_i) \in A | N = n\}$. We have

$$\begin{aligned}
& E_m \{N\} J^*(m, m'; \mu_m) \\
= \; & E_m \{N\} \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{P_m \{N = n\}}{E_m \{N\}} J^*(m, m'; \mu_{m,n,i}) \\
= \; & \sum_{n=1}^{\infty} P_m \{N = n\} \sum_{i=1}^{n} E_m \left\{ J(m, m'; q(\tilde{\phi}_i)) \Big| N = n \right\} \\
= \; & \sum_{n=1}^{\infty} \sum_{i=1}^{n} E_m \left\{ J(m, m'; q(\tilde{\phi}_i)) 1\{N = n\} \right\} \\
= \; & E_m \left\{ \sum_{i=1}^{N} J(m, m'; q(\tilde{\phi}_i)) \right\}.
\end{aligned}$$

CHAPTER III

A GENERALIZATION OF KULLBACK-LEIBLER INEQUALITY

In this chapter we extend the Kullback-Leibler inequality to investigate the quantization effects on the second-order (or other general order) moments of likelihood ratios, thereby simplifying our asymptotic arguments in decentralized sequential detection problems. We devote this chapter to discuss this mathematical result, since it stands out by its own importance.

3.1 Introduction

In Subsection 2.2, we already defined the Kullback-Leibler divergence for the distributions of quantizers, which are discrete. In information theory and statistics, the K-L divergence can be defined for two general distributions P_0 and P_1 of any random variable X and is a fundamental quantity that characterizes the difference between them. Denote by $f_1(x)$ and $f_0(x)$ the densities of P_1 and P_0 with respect to some common underlying probability measure μ, then the Kullback-Leibler divergence from P_0 to P_1 is

$$I(f_0, f_1) = \int f_0(x) \log \frac{f_0(x)}{f_1(x)} d\mu = E_0\{Z\}$$

where Z is the log-likelihood ratio

$$Z = \log \frac{f_0(X)}{f_1(X)}$$

and E_0 means taking the expectation over the distribution P_0.

In some applications, the random variable X itself may be unobservable and what is actually observed is another variable Y that is a quantization of X, or more generally, a function of X, say $Y = \phi(X)$. Denote by P_i^{ϕ} and $f_i(y;\phi)$ the probability distribution and probability mass (or density) function of Y when X has a distribution P_i. Then the Kullback-Leibler divergence of the Y is

$$I(f_0, f_1; \phi) = E_0\left\{Z_\phi\right\}$$

where the log-likelihood ratio of $Y = \phi(X)$

$$Z_\phi = \log \frac{dP_0^\phi}{dP_1^\phi}(Y) = \log \frac{f_0(Y;\phi)}{f_1(Y;\phi)}.$$

Obviously, the theories of Subsection 2.2 just correspond to the case where ϕ is a quantizer, i.e., a function taking value in a finite set.

An important property is that quantization cannot increase the Kullback-Leibler divergence, that is,

$$I_\phi(f_1, f_0) \le I(f_1, f_0) \tag{41}$$

with equality if and only if $Y = \phi(X)$ is a sufficient statistics of X, see Theorem 4.1 of Kullback and Leibler [14]. This is consistent with our intuition that $Y = \phi(X)$ is generally less informative than the X itself. Note that the inequality (41), which will be referred as *Kullback-Leibler's inequality* below, deals with the expected value or first moment of the log-likelihood ratio.

In this section, we extend the Kullback-Leibler inequality (41) to investigate the quantization effects on the second or other higher moments of the log-likelihood ratio. Such a result will be closely related to the decentralized sequential detection problem introduced in Chapter 2. Specifically, it yields a uniform bound on the second moments of the log-likelihood ratios over all allowable quantization function ϕ's. With such a bound, many results can be greatly improved and proofs can be simplified.

3.2 Second-order Moments

For the X and $Y = \phi(X)$, define their respective second moments of log-likelihood ratios as

$$V(f_0, f_1) = E_0\left\{Z^2\right\} = E_0\left\{\left(\log \frac{f_0(X)}{f_1(X)}\right)^2\right\}$$

and

$$V_\phi(f_0, f_1) = E_0\left\{Z_\phi^2\right\} = E_0\left\{\left(\log \frac{f_0(Y;\phi)}{f_1(Y;\phi)}\right)^2\right\},$$

where Z and Z_ϕ are the log-likelihood ratios of X and Y.

Our main result is as follows.

Theorem 3.2.1. *For any measurable function ϕ, we have*

$$V_\phi(f_0, f_1) \le V(f_0, f_1) + \frac{2}{e}. \tag{42}$$

Proof. Let $L = e^Z = f_0(X)/f_1(X)$ and $L_\phi = e^{Z_\phi} = f_0(Y;\phi)/f_1(Y;\phi)$ be the likelihood ratios. To simplify notation, let $E_0\{\cdot|Y\}$ denote the conditional expectation with respect to a given value of the observed data $Y = \phi(X)$, then

$$E_0\left\{ L^{-1} \middle| Y \right\} = E_0\left\{ \frac{f_1(X)}{f_0(X)} \middle| Y \right\} = \frac{f_1(Y;\phi)}{f_0(Y;\phi)} = L_\phi^{-1}.$$

Recall that in the proof of the Kullback-Leibler's inequality (41), beside the above equality, another key step is to observe that the function $H(t) = -\log t$ is convex when $t > 0$. Then, by Jensen's inequality, $Z_\phi = \log L_\phi = H(L_\phi^{-1}) = H(E_0(L^{-1}|Y)) \le E_0(H(L^{-1})|Y)$, and relation (41) is proved by taking expectations on both sides. Unfortunately, this approach fails for the second moment case since the function $H_2(t) = (-\log t)^2 = (\log t)^2$ is no longer convex (nor is it concave). Fortunately, we can salvage this approach by finding a convex function that is larger, but not too much larger, than $H_2(t)$. Specifically, for the function $H_2(t) = (\log t)^2$, taking derivatives leads to $H_2'(t) = 2\log t/t$ and $H_2''(t) = 2(1 - \log t)/t^2$. Thus $H_2(t) = (\log t)^2$ is convex on $t \le e$ but is concave on $t \ge e$. Hence, if we consider the following new function

$$\tilde{H}_2(t) = \begin{cases} H_2(t) = (\log t)^2 & \text{if } 0 < t \le e \\ H_2(e) + H_2'(e)(t-e) = \frac{2}{e}t - 1 & \text{if } t > e \end{cases} \tag{43}$$

then $\tilde{H}_2(t)$ is a continuous convex function of t when $t \ge 0$. Moreover, the concavity of $H_2(t)$ on $t \ge e$ implies that $\tilde{H}_2(t)$ dominates $H_2(t)$, see Fig. 2.

To prove our theorem, by the definitions of $H_2(t), \tilde{H}_2(t)$ and $V_\phi(f_0, f_1)$, we have

$$\begin{aligned} V_\phi(f_0, f_1) &= E_1\left\{ H_2(L_\phi^{-1}) \right\} \\ &\le E_0\left\{ \tilde{H}_2(L_\phi^{-1}) \right\} \\ &= E_0\left\{ \tilde{H}_2\left(E_0\left\{ L^{-1} \middle| Y \right\} \right) \right\} \\ &\le E_0\left\{ E_0\left\{ \tilde{H}_2\left(L^{-1} \right) \middle| Y \right\} \right\} \\ &= E_0\left\{ \tilde{H}_2\left(L^{-1} \right) \right\} \end{aligned}$$

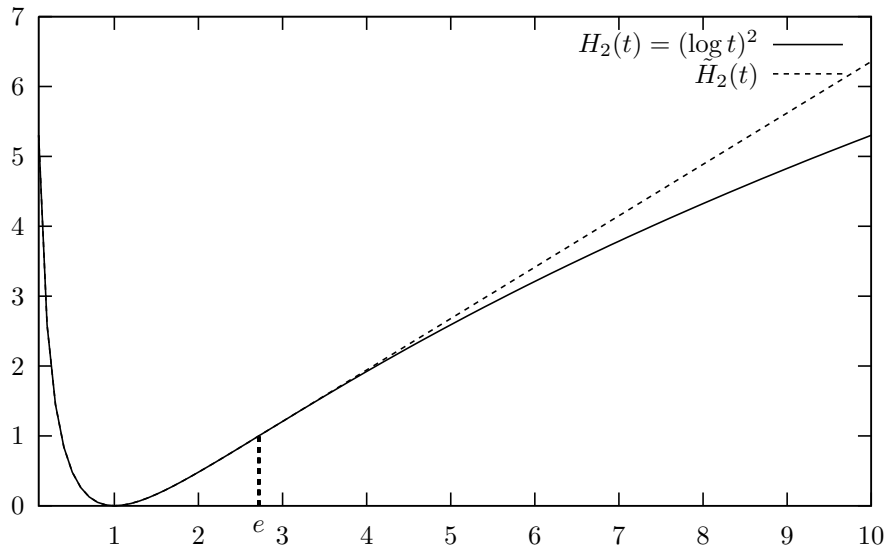


Figure 2: Dominating Function $\tilde{H}_2(t)$

where the first inequality follows from $H_2(t) \leq \tilde{H}_2(t)$, and the second inequality is to apply Jensen's inequality to the convex function $\tilde{H}_2(t)$.

Meanwhile, the difference between $E_0\{\tilde{H}_2\left(L^{-1}\right)\}$ and $V(f_0, f_1) = E_0\{H_2\left(L^{-1}\right)\}$ turns out to be insignificant. By the definition of $\tilde{H}_2(t)$ on (43), we have

$$\begin{aligned} E_0\left\{\tilde{H}_2\left(L^{-1}\right)\right\} &= E_0\left\{\tilde{H}_2\left(L^{-1}\right) I\left\{L^{-1} \leq e\right\}\right\} + E_0\left\{\tilde{H}_2\left(L\right) I\left\{L^{-1} > e\right\}\right\} \\ &= E_0\left\{H_2\left(L^{-1}\right) I\left\{L^{-1} \leq e\right\}\right\} + E_0\left\{\left(\frac{2}{e}L^{-1} - 1\right) I\left\{L^{-1} > e\right\}\right\} \\ &\leq E_0\left\{H_2\left(L^{-1}\right)\right\} + \frac{2}{e}E_0\left\{L^{-1}\right\} - P_0\left\{L^{-1} > e\right\} \\ &\leq V(f_0, f_1) + \frac{2}{e} \end{aligned}$$

where we use the fact that

$$E_0\{L^{-1}\} = \int (f_1(x)/f_0(x)) f_0(x) d\mu = \int f_1(x) d\mu = 1.$$

Combining the above inequalities yields (42), completing the proof of the theorem. □

It is useful to provide some comments to better understand our theorem. First, the discrete version of the Kullback-Leibler's inequality (41) is the well-known log-sum inequality: for non-negative numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, denote the sum of all a_i's by a and the

sum of all b_i's by b, and then we have

$$a \log \frac{a}{b} \leq \sum_{i=1}^{n} a_i \log \frac{a_i}{b_i}$$

with equality if and only if a_i/b_i are constant. Meanwhile, the discrete version of our main result (42) becomes that

$$a\big(\log \frac{a}{b}\big)^2 \leq \sum_{i=1}^{n} a_i \big(\log \frac{a_i}{b_i}\big)^2 + \frac{2}{e} b,$$

where the extra term on the right side is $2b/e$ instead of $2/e$ since we do not put any normalization conditions on a or b.

Second, a comparison of (41) and (42) shows that we have an extra constant term $2/e$ for the second moment case, and thus it is natural to ask whether or not the term can be eliminated, i.e., whether it is always true that $V_\phi(f_0, f_1) \leq V(f_0, f_1)$. The following counterexample provides a negative answer. Suppose that the X takes three distinct values 0, 1, 2 with probabilities 29/36, 1/9, 1/12 under P_0 and equal probabilities 1/3 under P_1. Let ϕ be a function with a binary range $\{0, 1\}$ such that $\phi(0) = 0$, $\phi(1) = \phi(2) = 1$. Then it is easy to verify that $V(f_0, f_1) = 0.9215 \leq V_\phi(f_0, f_1) = 0.9224$. More generally, other counterexamples can be easily found by choosing two distributions P_0 and P_1 of X, both of which are supported on $n + 1$ ($n \geq 2$) points $x_0, \ldots, x_n$ such that the likelihood ratio $L_0 = f_1(x_0)/f_0(x_0) < e$ and $L_i = f_1(x_i)/f_0(x_i) > e$ for $i = 1, \ldots, n$ with $L_1, \ldots, L_n$ being n distinct values. Then if we consider a quantization function ϕ that maps all $x_1, \ldots, x_n$ to a single point y_1 but maps x_0 to another point y_0, then $V(f_0, f_1) \leq V_\phi(f_0, f_1)$. To see this, note that $H_2(t) = (\log t)^2$ is strictly concave on $t \geq e$, so

$$\begin{aligned}
& \sum_{i=1}^{n} f_0(x_i) H_2\left(\frac{f_1(x_i)}{f_0(x_i)}\right) \\
< \;& (1 - f_0(x_0)) H_2\left(\sum_{i=1}^{n} \frac{f_0(x_i)}{1 - f_0(x_0)} \frac{f_1(x_i)}{f_0(x_i)}\right) \\
= \;& f_0(y_1; \phi) H_2\left(\frac{f_1(y_1; \phi)}{f_0(y_1; \phi)}\right)
\end{aligned}$$

and

$$\begin{aligned} V(f_0, f_1) &= \sum_{i=0}^{n} f_0(x_i) H_2\left(\frac{f_1(x_i)}{f_0(x_i)}\right) \\ &< f_0(x_0) H_2\left(\frac{f_1(x_0)}{f_0(x_0)}\right) + f_0(y_1;\phi) H_2\left(\frac{f_1(y_1;\phi)}{f_0(y_1;\phi)}\right) \\ &= V_\phi(f_0, f_1). \end{aligned}$$

In other words, unlike the case of Kullback-Leibler's inequality (41), a quantization indeed can increase the second moment of the log-likelihood ratio. Fortunately, our theorem shows that such an increase is at most $2/e$.

3.3 General Higher-order Moments

The technique we developed in proving Theorem 3.2.1 can be useful to deal with higher-order moments of the log-likelihood ratios. To be specific, for a positive integer $j = 1, 2, \ldots,$ define

$$W_j(f_0, f_1) = E_0\left\{(Z)^j\right\} = E_0\left\{\left(\log\frac{f_0(X)}{f_1(X)}\right)^j\right\} \tag{44}$$

and

$$W_{\phi,j}(f_0, f_1) = E_0\left\{(Z_\phi)^j\right\} = E_0\left\{\left(\log\frac{f_0(Y;\phi)}{f_1(Y;\phi)}\right)^j\right\}. \tag{45}$$

It turns out that we need to consider two different cases, depending on whether j is even or odd. For the purpose of our theorem, let us define two sequence of constants. For any integer $j \geq 1$, define

$$C_j = \frac{j(j-1)^{j-1}}{e^{j-1}}$$

and when j is odd, further define C_j^* to be the only real number $x \geq 0$ that satisfies the equation

$$x = (j-1)^{j-1} - C_j \exp(-x^{1/j}). \tag{46}$$

By convention we set $0^0 = 1$, and thus $C_1 = 1$ and $C_1^* = 0$.

The following theorem involves higher-order moments of the log-likelihood ratios, and includes the Kullback-Leibler's inequality (41) and relation (42) for second-order moment as special cases.

Theorem 3.3.1. *For any measurable function ϕ and any integer $j \geq 1$, we have*

$$W_{\phi,j}(f_0, f_1) \leq W_j(f_0, f_1) + B, \tag{47}$$

where the constant $B = C_j$ if j is even and $B = C_j^*$ if j is odd. Moreover, $W_{\phi,j}(f_0, f_1)$ and $W_j(f_0, f_1)$ have a lower bound 0 when j is even, and have a lower bound $-j(j-1)^{j-1}/e^{j-1} - (j-1)^j$ when j is odd.

We will prove Theorem 3.3.1 in two separate cases, depending on whether j is even or odd. Let us begin with the case when j is even, and we will prove a more general result on the α-moments of the absolute values of the log-likelihood ratios Z and Z_ϕ for any real number $\alpha \geq 1$. Specifically, define

$$\tilde{W}_\alpha(f_0, f_1) = E_1\left\{|Z|^\alpha\right\} = E_0\left\{\left|\log\frac{f_0(X)}{f_1(X)}\right|^\alpha\right\}$$

and

$$\tilde{W}_{\phi,\alpha}(f_0, f_1) = E_1\left\{|Z_\phi|^\alpha\right\} = E_0\left\{\left|\log\frac{dP_0^\phi}{dP_1^\phi}(Y)\right|^\alpha\right\} = E_0\left\{\left|\log\frac{f_0(Y;\phi)}{f_1(Y;\phi)}\right|^\alpha\right\}.$$

Lemma 3.3.1. *For any $\alpha \geq 1$,*

$$\tilde{W}_{\phi,\alpha}(f_0, f_1) \leq \tilde{W}_\alpha(f_0, f_1) + C_\alpha, \tag{48}$$

where the constant $C_\alpha = \frac{\alpha(\alpha-1)^{\alpha-1}}{e^{\alpha-1}} > 0$ and $C_1 = 1$ by convention that $0^0 = 1$.

Proof. When $\alpha \geq 1$, the function $H_\alpha(t) = |\log t|^\alpha$ is convex on $0 < t \leq t_\alpha$ but is concave on $t > t_\alpha$, where $t_\alpha = e^{\alpha-1} \geq 1$. Hence, the function $H_\alpha(t) = |\log t|^\alpha$ is dominated by the following convex function

$$\tilde{H}_\alpha(t) = \begin{cases} H_\alpha(t) & \text{if } 0 < t \leq t_\alpha \\ C_\alpha t - d_\alpha & \text{if } t > t_\alpha (\geq 1) \end{cases},$$

where $C_\alpha = \frac{\alpha(\alpha-1)^{\alpha-1}}{e^{\alpha-1}} > 0$ and $d_\alpha = (\alpha-1)^{\alpha-1} \geq 0$. The remaining proof is identical to those of Theorem 3.2.1 and thus omitted. □

As in Theorem 3.2.1, it is generally not true that $\tilde{W}_{\phi,\alpha}(f_0, f_1) \leq \tilde{W}_\alpha(f_0, f_1)$, and counterexamples can be easily found by exploring the fact that for any $\alpha \geq 1$, the function

$H_\alpha(t)$ is always strictly concave when $t \geq t_\alpha$. In other words, the counterexamples can be constructed by picking $n+1$ ($n \geq 2$) points $x_0, \ldots, x_n$ and two distributions P_0 and P_1 such that $L_0 = f_1(x_0)/f_0(x_0) < t_\alpha$ while $L_i = f_1(x_i)/f_0(x_i) > t_\alpha$ are n distinct values for $i = 1, \ldots, n$, and then proceeding as in the case of $\alpha = 2$.

It is also interesting to compare the Kullback-Leibler's inequality (41) with the case $\alpha = 1$ in Lemma 3.3.1: we have $E_0\{Z_\phi\} \leq E_0\{Z\}$ and $E_0|Z_\phi| \leq E_0|Z| + 1$. In other words, while the first moment of the log-likelihood ratio always decrease after a mapping, the first moment of its absolute value can indeed increase although such an increase is at most 1. This is because the function $-\log t$ is convex on $t > 0$ but the function $|\log t|$ is not convex.

Now let us prove Theorem 3.3.1 when $j \geq 1$ is odd. Fix the odd integer $j \geq 1$, and the key is to find a convex function that dominates $H(t) = (-\log t)^j$. By taking derivatives, it is easy to see that $H(t) = (-\log t)^j$ is convex on $0 < t \leq 1$ or $t \geq e^{j-1}$ but is concave when $1 \leq t \leq e^{j-1}$. Thus, if we let $t_0 = e^{j-1}$, then $H(t) \leq H(t_0) + H'(t_0)(t - t_0) = -C_j t + d_j$ when $1 \leq t \leq t_0$, where $C_j = \frac{j(j-1)^{j-1}}{e^{j-1}} > 0$ and $d_j = (j-1)^{j-1} \geq 0$. A simple calculation shows that the line $y = -C_j t + d_j$ intersects the curve $y = H(t)$ at two points: one of them is $t = t_0 = e^{j-1} \geq 1$ and the other one is in the interval $(0, 1]$ and denoted by $t^* \leq 1$. Therefore, by our construction, the following function $\tilde{H}(t)$ is convex on $t > 0$ and dominates $H(t) = (-\log t)^j$:

$$\tilde{H}(t) = \begin{cases} H(t) = (-\log t)^j & \text{if } 0 < t \leq t^* (\leq 1) \\ -C_j t + d_j & \text{if } t^* \leq t < t_0 = e^{j-1} \\ H(t) = (-\log t)^j & \text{if } t \geq t_0 (\geq 1) \end{cases}.$$

Next, we claim that $0 \leq \tilde{H}(t) - H(t) \leq C_j^*$ for all $t > 0$, where C_j^* is defined in (46). To prove this claim, first note that $C_j^* = H(t^*) \geq 0$ and it suffices to prove the claim when $t^* \leq t < t_0$, i.e., when $\tilde{H}(t)$ is decreasing as it is a linear function with negative slope. The proof needs to consider two scenarios, depending on whether $t \leq 1$ or ≥ 1. If $t^* \leq t \leq 1$, then the claim clearly holds since $\tilde{H}(t) \leq \tilde{H}(t^*) = C_j^*$ and $H(t) \geq 0$. Meanwhile, if $1 \leq t < t_0$, then by taking derivatives, $\tilde{H}(t) - H(t)$ is a decreasing function and thus

$$\tilde{H}(t) - H(t) \leq \tilde{H}(1) - H(1) = \tilde{H}(1) \leq \tilde{H}(t^*) = C_j^*.$$

Therefore, for all $t > 0$ we have $0 \leq \tilde{H}(t) - H(t) \leq C_j^*$, and our claim is proved.

For an odd integer $j \geq 1$, relation (47) of Theorem 3.3.1 can then be easily proved along the same line as in Theorem 3.2.1, and it remains to show that $W_j(f_1, f_0)$ in (44) and $W_{\phi,j}(f_1, f_0)$ in (45) are bounded below, since the random variables Z^j or Z_ϕ^j may take both positive and negative values. For any random variable X, let $X_+ = \max\{X, 0\}$ be the positive part of X and let $X_- = -\min\{X, 0\}$ be the negative part of X. Then $X = X_+ - X_-$, and it is evident that $X \geq -X_-$. The following lemma completes the proof of Theorem 3.3.1.

Lemma 3.3.2. *When $j \geq 1$ is an odd integer,*

$$E_0\left\{\left(Z^j\right)_-\right\} \leq j(j-1)^{j-1}/e^{j-1} + (j-1)^j$$

where $0^0 = 1$ by convention.

Proof. Fix the odd integer $j \geq 1$, consider the function

$$\psi(t) = -\min\{0, (-\log t)^j\} = \max\{0, (\log t)^j\}. \tag{49}$$

By taking derivatives, it is easy to see that as a non-decreasing function, $\psi(t)$ is concave on $t \geq t_0$, where $t_0 = e^{j-1}$. Thus

$$\psi(t) \leq \begin{cases} \psi(t_0) & \text{if } t \leq t_0 \\ \psi(t_0) + \psi'(t_0)(t - t_0) & \text{if } t \geq t_0 \end{cases}$$

or equivalently,

$$\psi(t) \leq (j-1)^j I\{t \leq t_0\} + (C_j t - d_j) I\{t > t_0\}$$

where $C_j = \frac{j(j-1)^{j-1}}{e^{j-1}} > 0$ and $d_j = (j-1)^{j-1} \geq 0$. Recall that $L = e^Z = f_0(X)/f_1(X)$ is the likelihood ratio, and thus

$$\begin{aligned} E_0\left\{\left(Z^j\right)_-\right\} &= E_0\left\{\psi\left(L^{-1}\right)\right\} \\ &\leq (j-1)^j P_0\left\{L^{-1} \leq t_0\right\} + C_j E_0\left\{L^{-1} I\{L > t_0\}\right\} - d_j P_0\left\{L^{-1} > t_0\right\} \\ &\leq (j-1)^j + C_j E_0\left\{L^{-1}\right\} \\ &= (j-1)^j + C_j, \end{aligned}$$

completing the proof of the lemma. □

3.4 Application into Decentralized Sequential Detection of Change Points

In this section we discuss the relation between the generalized K-L inequality as in Theorem 3.2.1 and decentralized sequential detection.

In Chapter 2 we proved the first order optimality of the two-stage procedure based on the Assumption 1, which assumes that for any $0 \leq m \neq m' \leq M-1$ and local sensor S^k, the divergence

$$E_m \left\{ \log \frac{f_m^k(X_n^k)}{f_{m'}^k(X_n^k)} \right\}$$

is positive and finite. For simplicity of notations let's focus on the case of single quantizer and drop the superscript k. Now define I_{tot} as follows:

$$I_{tot} = \sup_{\bar{\phi} \in \bar{\Phi}} \max_{m,m'} I(m, m'; \bar{\phi})$$

then by the classic Kullback-Leibler inequality (41),

$$I_{tot} < \infty.$$

In other words, the Kullback-Leibler divergence $I(m, m'; \bar{\phi})$ is uniformly bounded w.r.t. the set of quantizers $\bar{\phi}$. In the proof of Theorem 2.4.2, this fact is essential to the existence of a common lower bound for expected time steps of *any* test procedures with accepted error rates.

However, in many similar problems regarding decentralized detection, a stronger assumption, namely a uniform upper bound for the *second* moments of the log-likelihood ratios of the quantized data is necessary for any meaningful conclusions. As an example, let's consider the simplest version of decentralized quickest change detection problems with the same configuration of sensor networks we introduced in Chapter 2. In particular, the *limited local memory and full feedback* feature is assumed to hold. Now instead of doing the hypothesis testing problem, we assume that initially the underlying probability distribution of the raw data X_n^k is P_0 (with density f_0^k). Then an event occurs to the network system at some unknown time ν, and changes the measure to another given probability measure P_1 (with density f_1^k). Furthermore, we assume that the observations are independent over

time and from sensor to sensor. The objective is to jointly optimize the policies at the local sensors and fusion center levels so as to detect the change as soon as possible subject to a constraint on the false alarm rate.

A crucial challenge in decentralized quickest change detection is which kind of local quantizers should be used at each local sensor. On the one hand, this is easy if one further assumes that each local sensor uses a stationary local quantizer, as the corresponding problem reduces to the classical centralized case and various well-developed optimal or asymptotic optimal theories are applicable, see for example Lorden [20], Moustakides [26], Page [29], Pollak [30], Shiryayev [32] and [33], etc. In fact, it is not difficult to see that the optimal stationary quantizer ϕ^* for any local sensor S^k is the one that maximizes the local Kullback-Leibler divergence $I(f_1^k, f_0^k; \phi)$, and such an optimal quantizer ϕ^* is a Monotone Likelihood Ratio Quantizer (MLRQ) by our theories in Subsection 2.4.2.

On the other hand, the scenario becomes more complicated if the local quantizers are allowed to be non-stationary. By comparing with Bayes procedures, Veeravalli [42] conjectures that the schemes based on the optimal stationary MLRQ ϕ^* are asymptotically optimal regardless whether the quantizers are stationary or not. While this conjecture sounds reasonable as maximizing the Kullback-Leibler divergence seems to be natural to construct optimal local quantizers, it is very challenging to prove or disprove it, partly because of the regularity conditions of the quantized observations. For example, a sequence of non-stationary quantizers may outperform that of stationary quantizers when the second order moments of the log-likelihood ratios of non-stationary quantizers can go to infinity.

Some sufficient conditions under which this conjecture holds are available in the literature. By Lai [15], this conjecture is true under the following sufficient condition:

$$\lim_{n\to\infty} \sup_{\nu\geq 1} \operatorname{ess\,sup} P^{(\nu)} \left\{ \max_{t\leq n} \sum_{i=\nu}^{\nu+t} \sum_{k=1}^{K} Z_{i,\phi}^k \geq I_{tot}(1+\delta)n \middle| U_1, \ldots, U_{\nu-1} \right\} = 0 \qquad (50)$$

where $P^{(\nu)}$ is the probability measure when the change occurs at time ν, $Z_{i,\phi}^k$ is the likelihood ratio for the quantized data U_i^k, i.e.,

$$Z_{i,\phi}^k = \log \frac{f_1^k(U_i^k; \phi_i^k)}{f_0^k(U_i^k; \phi_i^k)}$$

and $I_{tot} = \sum_{k=1}^{K} I_{\max}^k$ with $I_{\max}^k = \sup_\phi I(f_1^k, f_0^k; \phi)$. Here $f_m^k(u; \phi_i^k)$ is probability mass function, i.e.,

$$f_m^k(u; \phi_i^k) = P_m^k \left\{ \phi_i^k(X_i^k) = u \right\}, \quad m = 0, 1.$$

Unfortunately, condition (50) involves all possible non-stationary quantizers, and it is impossible to verify it directly. By using Kolmogorov's inequality for martingales, Mei [24] provides a stronger sufficient condition, and shows that the conjecture holds if there is a uniform bound on the second moments of the log-likelihood ratios of quantized observations. Specifically, Mei [24] showed that condition (50) holds if for all $k = 1, \ldots, K$,

$$\sup_\phi V(f_1^k, f_0^k; \phi) < \infty. \tag{51}$$

Moreover, condition (51) holds when the quantized messages belong to binary sensor messages with $l = 2$ and when f_0 and f_1 belong to the same one-parameter exponential family satisfying certain restrictions, see Theorem 2 of [24]. However, it is still an open problem whether condition (51) holds in general or not, as the quantizers can have arbitrary forms and belong to the infinite dimensional functional space.

Our main theorem allows us to tackle more general scenarios. Specifically, by Theorem 3.2.1, if for all $k = 1, \ldots, K$,

$$V(f_1^k, f_0^k) = \int \left(\log \frac{f_1^k(x)}{f_0^k(x)} \right)^2 f_1^k(x) dx < \infty, \tag{52}$$

then condition (51) holds and so does (50). Note that condition (52) only deals with the densities f_i^k of raw observations and does not involve the stationary or non-stationary quantizers. Moreover, it is a standard assumption in the statistical literature as a regularity condition for the raw density functions. Therefore, condition (52) provides a simple and reasonable sufficient condition under which the long-standing conjecture of asymptotic optimality of the schemes with the optimal stationary MLRQ ϕ^* is true regardless whether the quantizers are stationary or not.

3.5 Generalization to Ali-Silvey Distance Measures

The K-L divergence is a special case of the general Ali-Silvey distance measure, which can be defined as follows. Let H^* be a general convex, continuous function with domain $(0, \infty)$

and

$$I^*(f_1, f_0) = E_1 \left\{ H^* \left(L^{-1} \right) \right\}$$

as well as

$$I^*_\phi(f_1, f_0) = E_1 \left\{ H^* \left(L_\phi^{-1} \right) \right\}.$$

By Jensen inequality and the convexity of H^*, it is easy to show that for any quantization ϕ, $I^*_\phi(f_1, f_0) \leq I^*(f_1, f_0)$.

Parallel to the case of K-L divergence, for any integer $j \geq 1$, define

$$W^*_j(f_1, f_0) = E_1 \left\{ \left(H^* \left(L^{-1} \right) \right)^j \right\}, \quad W^*_{\phi,j}(f_1, f_0) = E_1 \left\{ \left(H^* \left(L_\phi^{-1} \right) \right)^j \right\}$$

and for any real number $\alpha \geq 1$, define

$$\tilde{W}^*_\alpha(f_1, f_0) = E_1 \left\{ \left| H^* \left(L^{-1} \right) \right|^\alpha \right\}, \quad \tilde{W}^*_{\phi,\alpha}(f_1, f_0) = E_1 \left\{ \left| H^* \left(L_\phi^{-1} \right) \right|^\alpha \right\}.$$

Then for the relation between $W^*_j(f_1, f_0)$ and $W^*_{\phi,j}(f_1, f_0)$, we have

Theorem 3.5.1. *Suppose exists $A^* > 0$ such that we always have one of the following two cases:*

1. $\frac{d^2}{dx^2}(H^*(x))^j \geq 0$ *for any* $x \geq A^*$.
2. $\frac{d^2}{dx^2}(H^*(x))^j \leq 0$ *and* $H^*(x) > 0$ *for any* $x \geq A^*$.

Then exists constants $B < \infty$, $B' > -\infty$ such that

$$W^*_{\phi,j}(f_1, f_0) \leq W^*_j(f_1, f_0) + B$$

and

$$W^*_j(f_1, f_0) \geq B'$$

and the constants B and B' only depend on the form of $H^*(x)$ and the number j, i.e., they are uniform w.r.t. ϕ.

For the relation between $\tilde{W}^*_\alpha$ and $\tilde{W}^*_{\phi,\alpha}$,

Lemma 3.5.1. *Suppose exists $A^* > 0$ such that when $x \geq A^*$, we always have either $\frac{d^2}{dx^2}(H^*(x))^j \geq 0$ or $\frac{d^2}{dx^2}(H^*(x))^j \leq 0$, then exists a constant $\tilde{B} < \infty$ such that*

$$\tilde{W}^*_{\phi,\alpha}(f_1, f_0) \leq \tilde{W}^*_{\alpha}(f_1, f_0) + \tilde{B}$$

and the $\tilde{B}$ here only depends on the form of $H^*(x)$ and α and is uniform w.r.t. ϕ.

The proof of Theorem 3.5.1 and Lemma 3.5.1 utilizes the convex domination method we developed in Section 3.2 and Section 3.3. For convenience, we first summarize the essence of such method in the following proposition in a more general form. The proof is omitted because it can be done in the same way as Theorem 3.2.1.

Proposition 3.5.1. *Let L or L_ϕ be defined as in Section 3.2, and G is any function defined on $(0, \infty)$ such that $E_1\left\{\left|G(L^{-1})\right|\right\} < \infty$. Suppose that exists convex function $\tilde{G}(x)$ defined on $(0, \infty)$, and constants $0 < C < \infty$, $0 < A \leq \infty$ (note that $A = \infty$ is allowed), such that*

$$|G(x) - \tilde{G}(x)| \leq C \quad \text{when } x \leq A$$
$$\tilde{G}(x) = B_1 x + B_2 \geq G(x) - C \text{ and } G(x) \geq 0 \quad \text{when } x > A.$$

Then

$$E_1\left\{G(L_\phi^{-1})\right\} \leq E_1\left\{G(L^{-1})\right\} + D$$

with $D < \infty$ being a constant only depending on C, B_1, B_2.

Now we first provide the proof of Lemma 3.5.1 and then that of Theorem 3.5.1, because when j is even, Theorem 3.5.1 is only a special case of Lemma 3.5.1.

Proof of Lemma 3.5.1. Let $G(x) = |H^*(x)|^\alpha$. We claim that exists $\delta > 0$ such that on $(0, \delta)$, $G(x)$ is either bounded or positive convex with $G(x) \to +\infty$ when $x \to 0$. Indeed, because $H^*(x)$ is convex, then either $H^*(x)$ is bounded on an interval $(0, \delta)$ or $H^*(x) \to \infty$ on $(0, \delta)$. If the latter is the case, we can also assume that $H^*(x) \geq 0$ on $(0, \delta)$ by choosing δ small enough. Then if $H^*(x)$ is bounded on $(0, \delta)$, so is $G(x)$, if $H^*(x) > 0$ on $(0, \delta)$ and $H^*(x) \to \infty$ when $x \to 0$, by taking second derivative of $G(x)$, it is easy to see that $G(x)$ is convex on $(0, \delta)$ as well. Now we divide the remainder of the proof into four parts to deal with the corresponding four cases of the behavior of $G(x)$ around 0 and ∞.

1. $G(x)$ is convex on both $x \geq A^*$ and $x \leq \delta$, and $G(x) \to \infty$ when $x \to 0$. In this case, pick $x_1 = A^* + 1$, without loss of generality, suppose $\delta < x_1$. We claim that exists $x_0 < \delta$ such that $G'(x_0) \leq \frac{G(x_1)-G(x_0)}{x_1-x_0} \leq G'(x_1)$. If the claim is true, we can construct a convex $\tilde{G}(x)$ as follows:

$$\tilde{G}(x) = \begin{cases} G(x), & \text{if } x \leq x_0 \text{ or } x \geq x_1 \\ G(x_0) + \frac{G(x_1)-G(x_0)}{x_1-x_0}(x - x_0) & \text{otherwise.} \end{cases}$$

Such a $\tilde{G}(x)$ certainly satisfies the condition of Proposition 3.5.1 because $G(x) - \tilde{G}(x)$ is bounded on $(0, \infty)$. So it remains to prove the claim. First we can pick a $x^* < \delta$ such that

$$\frac{G(x_1) - G(x^*)}{x_1 - x^*} \leq G'(x_1)$$

if $G'(x^*) \leq \frac{G(x_1)-G(x^*)}{x_1-x^*}$, just pick $x_0 = x^*$, if not, we can pick x_0 as the unique point in $(0, x^*)$ such that

$$\frac{G(x_1) - G(x_0)}{x_1 - x_0} = \frac{G(x_1) - G(x^*)}{x_1 - x^*}$$

i.e., $(x_0, G(x_0))$ stay on the line connecting $(x^*, G(x^*))$ and $(x_1, G(x_1))$.

2. $G(x)$ is bounded on $(0, \delta)$ but convex on (A^*, ∞). Let x_1 be defined as in first case, and

$$\tilde{G}(x) = \begin{cases} G(x), & \text{if } x \geq x_1 \\ G(x_1) + G'(x_1)(x - x_1) & \text{if } x < x_1. \end{cases}$$

3. $G(x) \to \infty$ when $x \to 0$ but concave on $x \geq A^*$. Let $x_1 = A^* + 1$, because $G(x) \geq 0$, $G'(x_1) \geq 0$. As in first case, we can find $x_0 \leq \delta$ such that $G'(x_0) \leq \frac{G(x_1)-G(x_0)}{x_1-x_0} \leq G'(x_1)$, then define

$$\tilde{G}(x) = \begin{cases} G(x), & \text{if } x \leq x_0 \\ G(x_0) + \frac{G(x_1)-G(x_0)}{x_1-x_0}(x - x_0) & \text{if } x_0 < x \leq x_1 \\ G(x_1) + G'(x_1)(x - x_1) & \text{if } x > x_1. \end{cases}$$

4. $G(x)$ is bounded when $x \leq \delta$ and concave on $x \geq A^*$. In this case, let

$$\tilde{G}(x) = G(x_1) + G'(x_1)(x - x_1)$$

i.e., $\tilde{G}(x)$ is linear.

□

Finally we are ready to prove Theorem 3.5.1.

Proof of Theorem 3.5.1. We only need to prove the case that j is odd. It is easy to show that when $x \to 0$, $H^*(x)$ is either bounded or tends to ∞, and so is $G(x) = (H^*(x))^j$. Then the dominant function $\tilde{G}(x)$ can be constructed just as in the proof of Lemma 3.5.1, and we only needs to show the lower bound $B' > -\infty$ exists. For that, if exists A^* such that $H^*(x) \geq 0$ when $x \geq A^*$ then $H^*(x)$ or $G(x)$ itself is lower bounded, and the claim must be true. Otherwise, by the condition stated, we must have that exists A^* s.t. $G(x)$ is convex when $x \geq x_1 = A^* + 1$, which means that

$$G(x) \geq Gl(x) = G(x_1) + G'(x_1)(x - x_1) \quad x \geq x_1$$

here $Gl(x)$ is a linear function. Because $G(x)$ is also lower bounded on $(0, \delta)$, there must exists C such that

$$G(x) \geq Gl(x) - C$$

for any $x > 0$. Hence

$$E_1\left\{G(L^{-1})\right\} \geq E_1\left\{Gl(L^{-1})\right\} = G(x_1) - G'(x_1)x_1 - C > -\infty.$$

□

CHAPTER IV

MULTI-STAGE PROCEDURES FOR ASYMMETRIC MULTIHYPOTHESIS TESTING

In this chapter, we extend our results in the previous chapters to develop decentralized sequential tests that hold second-order asymptotic optimality properties under certain scenarios. Since it is very challenging to develop second-order optimality theory, here we will focus on the case of a single sensor in the network system when testing simple hypotheses. Extensions to multisensor network and composite hypothesis testing can be done in exactly the same way as Section 2.5 and will be omitted here. Consequently, we will drop the superscriptions denoting sensors as in Section 2.3.

4.1 Problem Statement and Background

Assume that there is $K = 1$ sensor in a system. At time n, the local sensor observes a raw observation X_n and sends a quantized message $U_n = \phi_n(X_n, \mathcal{F}_{n-1})$ to the fusion center, where $\mathcal{F}_{n-1} = (U_1, \cdots, U_{n-1})$ denotes the past messages. There are M hypotheses about the distribution of X_n's and the X_n's are i.i.d. with density f_m under hypothesis H_m, for $m = 0, 1, \ldots, M-1$. We consider a Bayes formulation: assign a prior density π_m to each hypothesis and assume the cost of making incorrect decision is 1 and the cost of each time step is c. Then the Bayes risk of a decentralized test δ is

$$\mathcal{R}_c(\delta) = \sum_{m=0}^{M} \pi_m [E_m(N) + P_m(D \neq m)],$$

where N is the stopping time of δ and D is the final decision of δ.

The present chapter will address the following second order asymptotic optimization problem.

Problem (P3): Find a family of decentralized sequential multihypothesis testing procedures $\{\delta_A(c)\}$ that is *asymptotically optimal* up to second-order in the sense that

$$\mathcal{R}_c(\delta_A(c)) = \mathcal{R}_c(\delta_B^*(c)) + O(c) \tag{53}$$

as $c \to 0$, where $\delta_B^*(c)$ is the corresponding Bayes solution.

Our main results are that while a two-stage procedure can not achieve second order optimality on its own, it can be modified in a "recursive" manner to become a multi-stage procedure that leads to the solutions to Problem (P3) when the hypothesis testing problem is of an "asymmetric" type (to be defined later).

To achieve the second or higher order asymptotic optimality, we require an assumption that is stronger than Assumption 1. Throughout the present chapter, we make the following assumption that is essential to our results regarding second order asymptotically optimal procedures for testing $M \geq 3$ hypotheses.

Assumption 2. *There exists $\lambda > 1$ and $\rho(\lambda) < \infty$ such that for $m = 0, \ldots, M-1$ and $m' \neq m$,*

$$E_m \left\{ \left(\frac{f_{m'}(X_n)}{f_m(X_n)} \right)^{\lambda} \right\} \leq \rho(\lambda)$$

or equivalently

$$E_m \left\{ \exp \left\{ \lambda \log \left(\frac{f_m(X_n)}{f_{m'}(X_n)} \right) \right\} \right\} \leq \rho(\lambda).$$

When testing $M = 2$ hypotheses, Assumption 2 can be weakened to the following Assumption 3.

Assumption 3. *There exists $V^* < \infty$ such that for $m = 0, 1$, and $m' \neq m$*

$$0 \leq E_m \left\{ \left(\log \frac{f_m(X_n)}{f_{m'}(X_n)} \right)^2 \right\} \leq V^*.$$

Note that Assumptions 2 and 3 deal with the raw observation X_n's. The following lemma shows that similar results also hold for quantized messages U_n's.

Lemma 4.1.1. *Under Assumption 2, let ϕ be any quantizer or general measurable function, and denote by $f_m(\cdot; \phi)$ the probability mass function of $U_n = \phi(X_n)$ when X_n has density f_m. Then*

$$E_m \left\{ \exp \left\{ \lambda \log \left(\frac{f_m(U_n; \phi)}{f_{m'}(U_n; \phi)} \right) \right\} \right\} \leq \rho(\lambda) < \infty. \tag{54}$$

If Assumption 3 holds, then

$$\max_{m, m'} E_m \left\{ \left(\log \frac{f_m(U_n; \phi)}{f_{m'}(U_n; \phi)} \right)^2 \right\} \leq V^* + \frac{2}{e}. \tag{55}$$

Proof. By Jensen inequality, we have

$$E_m\left\{\exp\left\{\lambda\log\left(\frac{f_m(U_n;\phi)}{f_{m'}(U_n;\phi)}\right)\right\}\right\}\leq E_m\left\{\exp\left\{\lambda\log\left(\frac{f_m(X_n)}{f_{m'}(X_n)}\right)\right\}\right\},$$

and then (54) follows directly from Assumption 2. Meanwhile, relation (55) follows at once from our generalized Kullback-Leibler inequality, i.e., Theorem 3.2.1. □

In the following let us first provide a background on two-stage procedure and illustrate their shortcomings in the context of the second or higher-order optimality problem. Assume that we are testing $M=2$ hypothesis H_0 and H_1. Recall that a two-stage procedure $\delta_A(c)$ that is asymptotically optimal up to first-order can be defined as follows. In the first stage, $\delta_A(c)$ uses a preliminary quantizer ϕ^0 and makes a preliminary decision $D_0=m\in\{0,1\}$. In the second stage, based on the preliminary decision, it switches to one of the maximin quantizers ϕ_0 and ϕ_1 to do further refined test (these two quantizers will be deterministic MLRQs when $M=2$, and they are generally different due to asymmetry of K-L divergence).

To be more concrete and to facilitate our statements, let us define the stopping times of $\delta_A(c)$ in terms of the log-likelihood ratios. For the simple hypothesis testing problem, let

$$Z_n(0,1)=\sum_{k=1}^{n}\Delta Z_n(0,1)$$

where

$$\Delta Z_n(0,1)=\log\frac{f_0(U_n;\phi^{(n)})}{f_1(U_n;\phi^{(n)})}$$

and $\phi^{(n)}$ is the quantizer applied by the local sensor at time step n. It is easy to see that the stopping policies of the two-stage procedures defined in Chapter 2 can be equivalently stated in terms of $Z_n(0,1)$ as follows:

Define by $A_0=|\log u(c)|$ and $A_1=|\log c|$ the thresholds for the first and second stages, respectively. Recall that the only requirement of A_0 is that $A_0\to\infty$ while $A_0/A_1\to 0$ when $c\to 0$. Then the first stage will be stopped whenever

$$|Z_n(0,1)|>A_0$$

and the preliminary decision $D_0=0$ if $Z_n(0,1)>A_0$ and $D_0=1$ if $Z_n(0,1)<-A_0$. Meanwhile, the second stage stops (so does the whole test procedure) whenever

$$|Z_n(0,1)|>A_1$$

and the final decision $D = 0$ or 1, depending on whether $Z_n(0,1) > A_1$ or $< -A_1$ when stopped.

Assume for a moment that $m = 0$ is the true state of nature. In the first stage, $\Delta Z_n(0,1)$ becomes an i.i.d. with a positive mean $I(0,1;\phi^0)$, and thus it is most likely that the realized log likelihood ratio $Z_n(0,1)$ will hit A_0 earlier than $-A_0$, and on average the time step of the first stage of the two-stage test $\delta_A(c)$ will be roughly $A_0/I(0,1;\phi^0)$. This implies that $\delta_A(c)$ most likely makes the preliminary decision $D_0 = 0$ at the first stage, and if this is the case then in the second stage, the fusion center will switch to the maximin quantizer ϕ_0. This makes $\Delta Z_n(0,1)$ again an i.i.d. sequence with mean $I(0) > 0$ in the second stage, and in order for the realized log likelihood ratio $Z_n(0,1)$ increase from A_0 to $A_1 = |\log c| > A_0$, on average the time step of the second stage is roughly $(|\log c| - A_0)/I(0)$. Hence, when $m = 0$ is the true state, the expected sample size of the two-stage test $\delta_A(c)$ is roughly

$$E_0\{N\} \sim \frac{A_0}{I(0,1;\phi^0)} + \frac{|\log c| - A_0}{I(0)} = \frac{|\log c|}{I(0)} + A_0\Big(\frac{1}{I(0,1;\phi^0)} - \frac{1}{I(0)}\Big).$$

Likewise, when $m = 1$ is the true state,

$$E_1\{N\} \sim \frac{A_0}{I(1,0;\phi^0)} + \frac{|\log c| - A_0}{I(1)} = \frac{|\log c|}{I(1)} + A_0\Big(\frac{1}{I(1,0;\phi^0)} - \frac{1}{I(1)}\Big)$$

Let $\epsilon_0 = 1/I(0,1;\phi^0) - 1/I(0) \geq 0$ and $\epsilon_1 = 1/I(1,0;\phi^0) - 1/I(1) \geq 0$. Using the fact that $P_m\{D \neq m\} = O(c)$, the Bayes risk of the two-stage test $\delta_A(c)$ is roughly

$$\mathcal{R}_c(\delta_A(c)) \sim \Big(\frac{\pi_0}{I(0)} + \frac{\pi_1}{I(1)}\Big)c|\log c| + cA_0(\pi_0\epsilon_0 + \pi_1\epsilon_1). \tag{56}$$

As we have stated before, we cannot have $\epsilon_0 = 0$ and $\epsilon_1 = 0$ simultaneously, and thus $\pi_0\epsilon_0 + \pi_1\epsilon_1 > 0$ unless in some trivial cases when π_0 or π_1 is 0. Now $A_0 \to \infty$ as $c \to 0$, the term cA_0 will be much larger than $O(c)$, and thus $\delta_A(c)$ will not be second-order asymptotically optimal by the asymptotic lower bound in Theorem 2.4.2. In other words, the two-stage procedure $\delta_A(c)$ takes too many steps in its first stage when the suboptimal quantizer ϕ^0 is used, and that causes extra large term in the risk function.

A natural reaction is to let $A_0 = O(1)$ when $c \to 0$, i.e., we can choose A_0 as a large constant which does not change as $c \to 0$. Then on average the time steps in the first stage is reduced to the order of $O(1)$. Unfortunately, such an approach does not work, since it can

be shown that $P_m\{D_0 \neq m\} \sim e^{-A_0}$ and thus the probabilities of an incorrect preliminary decision do not go to zero as $c \to 0$. With a wrong preliminary guess, the fusion center picks up a sub-optimal quantizer in the second (and major) stage, which causes the time steps used to reach the final decision increase by an amount of order as $|\log c|$. That is, if $A_0 = O(1)$, then the two-stage test $\delta_A(c)$ does not even possess the first-order asymptotic optimality. In the literature, a popular choice of A_0 is $A_0 = \log|\log c|$, which is a sufficient condition for the first-order asymptotic optimality of the two-stage test $\delta_A(c)$.

It is clear that it is necessary to modify the two-stage test in order to develop second-order asymptotically optimal tests, and some major modification to the first stage is needed to accelerate the pace of the two-stage test $\delta_A(c)$. A key observation is that the first stage of $\delta_A(c)$ is itself a decision making procedure just like the whole test itself. As we stated before, as a "preliminary" stage, the first stage is of low efficiency because it uses a suboptimal stationary quantizer for too long. Therefore, a natural idea is to apply the two-stage test to the first-stage, thereby leading to the three-stage tests.

Specifically, to define a three-stage test $\delta_{A,3}(c)$, we can choose three thresholds $0 < A_0 < A_1 < A_2 = |\log c|$ satisfying that $A_0 \to \infty$, $A_0 = o(A_1)$, $A_1 = o(A_2)$ as $c \to 0$, e.g., $A_0 = \log\log|\log c|$ and $A_1 = \log|\log c|$. Then the three stages of $\delta_{A,3}(c)$ can be defined as the first times when $|Z_n(0,1)|$ passes A_0, A_1 and $A_2 = A$, respectively. Denote by N_0, N_1 and N_2 the time steps of these three stages. Compared with the two-stage test $\delta_A(c)$ with $A_0 = \log|\log c|$, this three-stage test allows the quantizers to be updated for an extra stage, and it can be shown that the extra term cA_0 in the risk is decreased from $c\log|\log c|$ to $c\log\log|\log c|$.

While the three stage tests still cannot be second-order asymptotically optimal, we can just keep on exploiting the idea of dividing the first stage via another two-stage test, and this leads to a multi-stage test $\delta_{MS}(c)$ as follows. Given a sequence of thresholds $0 < A_0 < A_1 < \cdots < A_J = |\log c|$ such that $A_0 = O(1)$, $A_{j-1} = o(A_j)$ for $j = 1, \cdots, J$, where the total number J of thresholds may depend on c. For each $j = 0, 1, \ldots, J$, we stop the $(j+1)$-th stage at the first time n whenever $|Z_n(0,1)|$ cross the the threshold A_j and when stopped, we will make a "preliminary" decision $D = 0$ or 1 depending on whether

$Z_n(0,1) > A_j$ or $Z_n(0,1) < -A_j$, and will optimize the local quantizers accordingly.

In the following we will show that by tuning these thresholds in the multi-stage test, we can finally develop a test that reaches the second order asymptotic optimality, thereby offering a solution to (P3). A feature of such a multi-stage procedure is that the total number of stages J increases to ∞ as c decreases to 0. In addition, it turns out that the thresholds can be quite flexible to ensure second order optimality, and a simple sufficient condition is of the following form

$$\sum_{j=1}^{J} A_j e^{-A_{j-1}} = O(1)$$

as $c \to 0$.

It is much more complicated to find a solution to (P3) when testing $M \geq 3$ hypotheses. The fundamental reason is that our asymptotic analysis utilizes the lower bound in Theorem 2.4.2, which may not be sharp for multihypothesis testing.

When testing $M \geq 3$ hypotheses, in this thesis we will need to distinguish two different cases: *asymmetric* and *symmetric*. For instance, when testing three hypotheses on a normal mean, say, $\mu = 0, \mu_1, \mu_2$, it is symmetric when $\mu_1 = -\mu_2$ but is asymmetric otherwise. A more general definition of asymmetric and symmetric cases is based on Kullback-Leibler information divergence as follows.

Suppose $\bar{\phi}$ be a possibly randomized quantizer, we say it is asymmetric w.r.t. the state m if there is a unique state $m' \neq m$ that minimizes $I(m, m'; \bar{\phi})$, i.e., there is a single $m' \neq m$ satisfying

$$I(m, m'; \bar{\phi}) = \min_{m'' \neq m} I(m, m''; \bar{\phi}) (= I(m; \bar{\phi}));$$

otherwise we say that $\bar{\phi}$ is symmetric w.r.t. the state m. Alternatively, let $I_{gap}(m; \bar{\phi})$ be the difference between the smallest two values of $\{I(m, m'; \bar{\phi}) : m' \neq m\}$, i.e.,

$$I_{gap}(m; \bar{\phi}) = \min_{m'' \neq m, m'} I(m, m''; \bar{\phi}) - I(m, m'; \bar{\phi}) \geq 0,$$

where $I(m, m'; \bar{\phi}) = I(m; \bar{\phi})$. Then $\bar{\phi}$ is asymmetric w.r.t. state m if and only if $I_{gap}(m; \bar{\phi}) > 0$. Note that I_{gap} exists only when testing $M \geq 3$ hypotheses.

Let $\bar{\phi}_m^{\max}$ be the maximin quantizer w.r.t. a given state m (see Definition 2.4.1). Then we say the multi-hypothesis testing problem is asymmetric w.r.t. state m if $\bar{\phi}_m^{\max}$ is asymmetric

w.r.t. state m. Otherwise the multi-hypothesis testing problem is said to be symmetric w.r.t. state m. If the multi-hypothesis testing problem is asymmetric w.r.t. **all states**, then we say the problem itself is asymmetric. Otherwise the multi-hypothesis testing problem is symmetric if it is symmetric w.r.t at least one state m.

It turns out that we will be able to show that the multi-stage tests can still hold second-order optimality properties for the asymmetric case in the multihypotheses testing problems, although it is still an open problem to find a test that is second-order asymptotically optimal for the symmetric case.

4.2 Definition of Multi-Stage Procedures

The purpose of this section is to define the multi-stage procedures for testing $M \geq 2$ hypotheses more rigorously.

Denote by $\delta_{MS}(c)$ a multi-stage test when the cost of taking observations per time step is c. To define it, we first need to specify the thresholds to stop stages: $0 = A_{-1} < A_0 < A_1 < \cdots < A_J = |\log c|$. Note that both $J+1$, the number of the stages, and the threshold values $A_0, \ldots, A_J$ may depend on c. Next, we need to specify (at most) $M+1$ different (possibly randomized) quantizers that will be used by $\delta_{MS}(c)$ when testing $M \geq 2$ hypotheses, say, $\bar{\phi}^0, \bar{\phi}_0, \ldots, \bar{\phi}_{M-1}$. For our multi-stage test, quantizers are stationary within each stage, with $\bar{\phi}^0$ being used in the first or preliminary stage and one of the remaining M quantizers used in each of the following J stages. Ideally, at the beginning of each stage, the local quantizers will be optimized according to the decision at the previous stage. In other words, $\bar{\phi}_0, \ldots, \bar{\phi}_{M-1}$ are the M "maximin" quantizers and at the jth stage the local quantizers will be switched to the maximin quantizer $\bar{\phi}_m$ if the decision at the $(j-1)$-th stage is $\{D = m\}$.

Now let us define the stopping rules of $\delta_{MS}(c)$ at different stages. At each time step n, the fusion center updates the log-likelihood vector

$$\tilde{Z}_n = (Z_n(0), \ldots, Z_n(M-1)),$$

where for each $m = 0, \ldots, M-1$,

$$Z_n(m) = \sum_{i=0}^{n} \Delta Z_i(m) = Z_{n-1}(m) + \Delta Z_n(m)$$

and

$$\Delta Z_i(m) = \log f_m(U_i; \phi^{(i)}).$$

Here $\phi^{(i)}$ is the *deterministic* quantizer applied by the local sensor to generate the sensor message U's at time step i. For instance, in the first or preliminary stage of $\delta_{MS}(c)$, the possibly randomized quantizer $\bar{\phi}^0$ is used at the fusion center, then $\phi^{(i)}$ is obtained from randomization schemes described as in Subsection 2.3.2.

Then the first stage of $\delta_{MS}(c)$ is stopped at time

$$N_0 = \inf\{n : \max_{1 \le m \le M} \min_{m' \ne m} (Z_n(m) - Z_n(m')) \ge A_0\}.$$

That is, the first stage stops if for some m, the difference between the log-likelihood $Z_n(m)$ and the log-likelihood of the next most likely hypothesis crosses the threshold A_0. To simplify our notation below, for each $j = 0, \ldots, J$, define stopping regions $\mathcal{O}_j = \bigcup_{m=0}^{M-1} \mathcal{O}_j^m \subset \mathbb{R}^M$, where the sub-stopping region $\mathcal{O}_j^m$ is defined by

$$\mathcal{O}_j^m = \{z = \left(z^0, \ldots, z^{M-1}\right) \in \mathbb{R}^M : \min_{m' \ne m} (z^m - z^{m'}) \ge A_j\}.$$

Under this new notation, the first or preliminary stage of $\delta_{MS}(c)$ stops at time

$$N_0 = \min\{n : \tilde{Z}_n \in \mathcal{O}_0\}$$

and makes a preliminary decision $D_0 = m$ if $\tilde{Z}_{N_0} \in \mathcal{O}_0^m$ for some $m \in \{0, \ldots, M-1\}$.

The other stages of $\delta_{MS}(c)$ can also be defined similarly except a change in local quantizors. Specifically, for $j = 1, \ldots, J$, if the intermediate decision from the j-th stage is $D_{j-1} = m$, then in the $j+1$-th stage, the fusion center switches to a (likely randomized) quantizer $\bar{\phi}_m$ and continue to update the vector $\tilde{Z}_n$. The $j+1$-th stage of $\delta_{MS}(c)$ is stopped at time

$$N_j = \min\{n \ge N_{j-1} : \tilde{Z}_n \in \mathcal{O}_j\}$$

and makes a intermediate decision $D_j = m$ if $\tilde{Z}_{N_j} \in \mathcal{O}_j^m$. The final stopping time for the test is $N = N_J$ with a final decision $D = D_J$ at the fusion center.

It is easy to see that the two-stage test defined in Chapter 2 corresponds to a special case of multi-stage procedure with $J+1=2$ stages (or $J=1$), and the only difference is that the stopping strategies of the two-stage test are expressed in terms of the posterior probabilities in Chapter 2 instead of the log-likelihoods.

4.3 Asymptotic Properties When Testing Asymmetric Multihypotheses

Let us begin with the Bayes risk of the proposed multi-stage test $\delta_{MS}(c)$. This includes two estimations: the probability of making incorrect decisions and the expected sample sizes under each hypothesis $\mathbf{H}_m$. The former is relatively easy and is summarized in the following lemma.

Lemma 4.3.1. *Given a multi-stage test $\delta_{MS}(c)$, for any state $m=0,\ldots,M-1$, we have*

$$P_m\{D\neq m\} < Mc.$$

For the expected sample size of $\delta_{MS}(c)$, it is standard in the literature to derive the asymptotic expression when the cost per time step c goes to 0. To do so, we need to make additional assumptions on the thresholds A_j's, which may depend on the cost c: we assume that the thresholds $A_0,\ldots,A_J$ satisfy the constraint

$$\limsup_{c\to 0}\sum_{j=0}^{J} A_j e^{-A_{j-1}} \leq A^* \tag{57}$$

for some finite constant A^* and, as $c\to 0$,

$$\lim_{c\to 0}\max_{1\leq j\leq J} A_{j-1}/A_j = 0. \tag{58}$$

Relation (58) means that A_{j-1}/A_j goes to zero uniformly as $c\to 0$.

Now we can summarize the asymptotic expression of the expected sample size of $\delta_{MS}(c)$ in the following theorem when it is equipped with asymmetric quantizers.

Theorem 4.3.1. *Given a multi-stage test $\delta_{MS}(c)$ equipped with possibly randomized quantizers $\{\bar{\phi}_m\}$ and a preliminary quantizer $\bar{\phi}^0$, and assume that the quantizer $\bar{\phi}_m$ is asymmetric w.r.t. the state m for a given $0\leq m\leq M-1$, i.e., $I_{gap}(\bar{\phi}_m;m)>0$. Furthermore, we*

assume that the K-L divergences of the quantized data have a common positive lower bound $I_* > 0$*, i.e.,*

$$\min_{0\le m\neq m'\le M-1} I(m,m';\bar{\phi}) \ge I_*, \quad \textit{for any } \bar{\phi} \in \left\{\bar{\phi}^0, \bar{\phi}_0, \ldots, \bar{\phi}_{M-1}\right\}.$$

If the thresholds A_j*'s satisfy (57) and (58), then the expected sample size* N *of* $\delta_{MS}(c)$ *satisfies*

$$E_m\{N\} = |\log c|/I(m;\bar{\phi}_m) + O(1) \tag{59}$$

as $c \to 0$.

It is informative to comment on the uniformity of the $O(1)$ term in (59) in the sense that it holds regardless of specific choices of quantizers as along as the quantizers share a common $I_* > 0$ and a common lower bound $I_{gap} > 0$ for the quantities $I_{gap}(m;\bar{\phi}_m)$. For that purpose, let A^*_{ratio} be any common upper bound of the ratios $\{A_{j-1}/A_j\}$ such that

$$\max_{1\le j\le J} A_{j-1}/A_j \le A^*_{ratio} \quad \text{and} \quad \lim_{c\to 0} A^*_{ratio} = 0. \tag{60}$$

Then it can be shown in Section 4.5 that the $O(1)$ term in (59) is related to the set of quantizers, $\left\{\bar{\phi}^0, \bar{\phi}_0, \ldots, \bar{\phi}_{M-1}\right\}$, only through the lower bound of the K-L divergence I_* and the gap $I_{gap}(m;\bar{\phi}_m)$. In addition, other factors involved in the $O(1)$ term in (59) include: A^*_{ratio} in (60), A^* in (57), I^* in Assumption 1, V^* in Assumption 3, λ and $\rho(\lambda)$ in Assumption 2.

In addition, it is worth emphasizing that the asymmetric assumption of $\bar{\phi}_m$ w.r.t. the state m is a crucial condition for Theorem 4.3.1. If this condition is not satisfied, i.e., if $\bar{\phi}_m$ is symmetric w.r.t. the state m, then we can only drive a weaker result, which is summarized in the next theorem.

Theorem 4.3.2. *Let* $\delta_{MS}(c)$ *be defined as in Theorem 4.3.1, however* $\bar{\phi}_m$ *is not necessarily asymmetric w.r.t. state* m*, then when the thresholds* $A_0, \ldots, A_J$ *satisfy*

$$A_{j-1}/\sqrt{A_j} \le B_1 < \infty, \quad A_j e^{-A_{j-1}} \le B_2 < \infty$$

then as $c \to 0$

$$E_m\{N\} = |\log c|/I(m;\bar{\phi}_m) + O(\sqrt{|\log c|})$$

where the $O(\sqrt{|\log c|})$ *can be made uniform for problems sharing common* I_*, V^*, B_1 *and* B_2.

Now we are ready to derive the Bayes risk of the proposed multi-stage test $\delta_{MS}(c)$. Combining Lemma 4.3.1 and Theorem 4.3.1 yields that if every quantizer $\bar{\phi}_m$ is asymmetric w.r.t. the corresponding state m and relations (57) and (58) hold, then

$$\mathcal{R}_c(\delta_{MS}(c)) = c|\log c| \sum_{m=0}^{M-1} \pi_m / I(m; \bar{\phi}_m) + O(c)$$

where the $O(c)$ term can be made uniform w.r.t. groups of quantizers with common I_* and positive lower bound for $\{I_{gap}(\bar{\phi}_m; m)\}$. In particular, denote by $\delta^*_{MS}(c)$ the multi-stage test that is equipped with the maximin quantizers $\{\bar{\phi}_m^{\max}\}$ and thresholds $\{A_j\}$ satisfying (57) and (58). Then if all states m's are asymmetric, that is, each maximin quantizer $\bar{\phi}_m^{\max}$ is asymmetric w.r.t. the corresponding state m, then

$$\mathcal{R}_c(\delta^*_{MS}(c)) = c|\log c| \sum_{m=0}^{M-1} \pi_m / I(m) + O(c). \tag{61}$$

Next, the following theorem provides a lower bound for the Bayes risk of the Bayes procedure.

Theorem 4.3.3. *Let* $\delta^*_B(c)$ *be the Bayes test procedure, with parameter* c *being the cost per time step. Then as* $c \to 0$,

$$\mathcal{R}_c(\delta^*_B(c)) \geq c|\log c| \sum_{m=0}^{M-1} \pi_m / I(m) + O(c).$$

Finally, by (61) and Theorem 4.3.3, it is clear that $\delta^*_{MS}(c)$ is a solution to (P3), as it achieves the asymptotic lower bound in Theorem 4.3.3.

Theorem 4.3.4. *If every state* $m = 0, \ldots, M-1$ *is asymmetric, then the multi-stage test* $\delta^*_{MS}(c)$ *is asymptotically optimal to the second order.*

It is useful to add several remarks. First, condition (57) on the thresholds A_j is crucial for the second order asymptotic optimality of $\delta^*_{MS}(c)$. In order to satisfy this condition, the thresholds A_j's should not increase too fast. There are many possible choices to satisfy (57). As an illustration, let $\gamma > 1$ be a constant that do not depend on c, and a valid choice of A_j's

can be constructed backward by setting $J = [\log\log|\log c|/\gamma] + 1$ and defining $A_J = |\log c|$ and $A_{j-1} = A_j^{\frac{1}{\gamma}}$ for $j = J, J-1, \ldots, 1$.

A notable choice that violates (57) is as follows. Let $C > 0$ be a constant and define $A_J = |\log c|$ and $A_{j-1} = \log A_j$ recursively until $0 < A_0 < C$. In this case, $J = \min\{j : \log^{(j)}|\log c| < C\}$. In this case, $A_j = e^{A_{j-1}}$ increases too fast, and $\sum_{j=0}^{J} A_j e^{-A_{j-1}} = J + A_0$, which goes to ∞ as $c \to 0$ (since $J \to \infty$ although extremely slowly). Thus condition (57) does not hold in this case.

Second, in real-world application, a multi-stage test with a smaller number of stages may be preferred. By Corollary 4.4.1, to reach the second-order asymptotic optimality properties, the total number J of stages in a multi-stage test always tend to infinite when $c \to 0$. Since we can always choose $\{A_j\}$ in a way that J increases very slowly, say, with the recursive rule $A_{j-1} = A_j^{\frac{1}{2}}$, we can have J increase to infinite by the extremely slow rate of $\log\log|\log c|$. Therefore, if the cost c per time step is only moderately small, then the two-stage test, i.e., $J = 1$, is usually sufficient in many applications for the practical purposes.

Third, the expansion (61) depends heavily on the asymmetric assumption of each maximin quantizer $\bar{\phi}_m^{\max}$. When one or more $\bar{\phi}_m^{\max}$ is symmetric w.r.t. the corresponding state m, (61) does not hold and the asymmetric lower bound in Theorem 4.3.3 is no longer sharp. In this case, $\delta_{MS}^*(c)$ only has the first-order asymptotic optimality properties, as the difference between its Bayes risk and the asymptotic lower bound on the Bayes risk increases from $O(c)$ to the order $O(c\sqrt{|\log c|})$. It is useful to point out that a two-stage test can be designed to enjoy similar properties as illustrated by the following theorem.

Theorem 4.3.5. *Let $\delta_{II}^*(c)$ be a two-stage test equipped with maximin quantizers $\{\bar{\phi}_m^{max}\}$. If $A_0 = O(\sqrt{A_1})$ as $c \to 0$, then the expected sample size N of $\delta_{II}^*(c)$ satisfy*

$$E_m\{N\} = |\log c|/I(m) + O(\sqrt{|\log c|})$$

and its Bayes risk is given by

$$\mathcal{R}_c(\delta_{II}^*(c)) = c|\log c| \sum_{m=0}^{M-1} \pi_m/I(m) + O(c\sqrt{|\log c|}).$$

Hence $\delta_{II}^*(c)$ is asymptotically optimal up to first-order.

4.4 *Testing $M = 2$ Simple Hypotheses as a Special Case*

In this section let us focus on the problem of testing $M = 2$ simple hypotheses. This is a special case of multihypotheses testing and thus the theorems in Section 4.3 are applicable. Because now there are only one null and one alternative hypotheses, the asymmetric assumption of Theorem 4.3.1 is satisfied. Hence, the test $\delta^*_{MS}(c)$ equipped with the maximin quantizers is second-order asymptotically optimal.

Compared with the multihypotheses testing problem, a much stronger result can be obtained for testing $M = 2$ hypothesis. First, the condition of Theorem 4.3.1 and Theorem 4.3.4 can be weakened.

Proposition 4.4.1. *When testing $M = 2$ hypotheses, the conclusions of Theorem 4.3.1 and Theorem 4.3.4 remain valid with only Assumption 3 and without condition (58).*

The proof is omitted because it can be done by following the same steps as in the proofs of the original theorems regarding general multihypotheses testing.

By Theorem 4.3.1 or Proposition 4.4.1, in order for a multi-stage test to reach second order asymptotic optimality, the threshold condition (57) is crucial. Indeed, in the case of testing $M = 2$ hypotheses, (57) is not only sufficient as shown by Proposition 4.4.1, by also necessary provided that the tests implement maximin quantizers.

Theorem 4.4.1. *Assume $\phi_0^{max} \neq \phi_1^{max}$, i.e., assume that there is no one single quantizer $\bar{\phi}^{max}$ that maximizes $I(0, 1; \bar{\phi})$ and $I(1, 0; \bar{\phi})$ simultaneously. Consider a multi-stage procedure using the maximin quantizers, i.e., $\bar{\phi}_m = \phi_m^{max}$ for $m = 0, 1$. Then in order for this test to have second order asymptotic optimality, the thresholds $\{A_j\}$ must satisfy condition (57).*

The proof of this theorem is presented in the next section. One may wonder whether the second-order asymptotic optimality can also be obtained without the maximin quantizers. In the case of testing $M = 2$ hypotheses, a negative answer can be obtained with a mild regularity condition on $\{A_j\}$.

Proposition 4.4.2. *For testing $M = 2$ hypotheses, define a generalized multi-stage test as follows. For each stage $j = 1, \ldots, J$, a pair of quantizers $\bar{\phi}_0^{(j)}$, $\bar{\phi}_1^{(j)}$ is prepared such that after the log-likelihood ratio Z_n crosses the boundary A_j or $-A_j$, quantizer $\bar{\phi}_0^{(j)}$ or $\bar{\phi}_1^{(j)}$ will be used based on the intermediate decision $D_j = 0$ or $D_j = 1$. Suppose for each $j = 1, 2, \ldots$, $A_j - A_{j-1}$ goes to infinite as $c \to 0$, then for the generalized multi-stage test to have second order asymptotic optimality, condition (57) for the thresholds has to be satisfied and $\bar{\phi}_m^{(j)} = \phi_m^{max}$ for any j and $m = 0, 1$.*

The proof of Proposition 4.4.2 is similar to that of Theorem 4.4.1 and thus omitted.

Another consequence of Theorem 4.4.1 is that a two-stage test can never reach second-order asymptotic optimality no matter how well the thresholds are set.

Corollary 4.4.1. *When testing $M = 2$ hypotheses, if a multi-stage procedure $\delta_{MS}(c)$ has second order asymptotic optimality, the total number of its thresholds, i.e., $J + 1$, must go to infinite as $c \to 0$.*

4.5 Proofs

4.5.1 Proofs for Section 4.3

Proof of Theorem 4.3.1. Without loss of generality, suppose the quantizer $\bar{\phi}_0$ is asymmetric w.r.t. state 0, and

$$I(0; \bar{\phi}_0) = I(0, 1; \bar{\phi}_0) < I(0, m; \bar{\phi}_0), \quad \text{for any } m \notin \{0, 1\}.$$

Now we reformulate the stopping criteria of the test in terms of the quantities $Z_n(0, m)$, where $m = 1, \ldots, M-1$. To do so, let $x = (x^1, \ldots, x^{M-1})$ represent point in $\mathbb{R}^{M-1}$ and for $j = 0, \ldots, J$, define the region $\tilde{\mathcal{O}}_j = \bigcup_{m=0}^{M-1} \tilde{\mathcal{O}}_j^m \in \mathbb{R}^{M-1}$ with

$$\tilde{\mathcal{O}}_j^0 = \bigcap_{m'=0}^{M-1} \{x^{m'} \geq A_j\}$$

and

$$\tilde{\mathcal{O}}_j^m = \bigcap_{m'=0}^{M-1} \left\{x^{m'} - x^m \geq A_j\right\}$$

for $m \neq 0$, where we set $x^0 = 0$. Define the $M-1$ dimensional random vector $\tilde{Z}_n = (Z_n(0,1), \ldots, Z_n(0,M-1))$, then the stopping time N_j can be represented as

$$N_j = \min\left\{n : \tilde{Z}_n \in \tilde{\mathcal{O}}_j\right\}$$

and the decision $D_j = m$ if $\tilde{Z}_{N_j} = \tilde{\mathcal{O}}_j^m$.

When $m = 0$ is true, let's define another stopping time N' as follows. $N' = N$ if the final decision $D = 0$. When $D = m \neq 0$, i.e., $\tilde{Z}_N$ stopps in the region $\tilde{\mathcal{O}}_J^m$, then let the local sensor continue to sample the data and the fusion center continue to use the same quantizer as it did at step N for quantization, and let the continued procedure stop at time $N' = \min\left\{n \geq N : \tilde{Z}_n \in \tilde{\mathcal{O}}_J^0\right\}$. Obviously $N' \geq N$, and to prove Theorem 4.3.1, we only needs to show as $c \to 0$

$$E_0\left\{N'\right\} = |\log c|/I(0,1;\bar{\phi}_0) + O(1) \tag{62}$$

and

$$E_0\left\{N' - N\right\} = O(1) \tag{63}$$

with both $O(1)$ terms only depending on $I_*, I^*, V^*, \lambda, \rho(\lambda), A^*$ and A_{J-1}/A_J.

It is relatively easy to show (63). To do so, pick $m \neq 0$ and $L \geq A_J$, by a change of measure,

$$P_0\left\{Z_N(0,m) \leq -L, D = m\right\} = E_m\left\{e^{Z_N(0,m)}, D = m, Z_N(0,m) \leq -L\right\} \leq e^{-L}. \tag{64}$$

When $\tilde{Z}_N \in \tilde{\mathcal{O}}_J^m$, let $a = -Z_N(0,m) \geq A_J$, and define another stopping time $N^{''}$ as

$$N^{''} = \min\left\{n \geq N : \min_{1 \leq m' \leq M-1}\left(Z_n(0,m') - Z_N(0,m')\right) \geq A_J + a\right\}.$$

Because for any $m' \neq m$, $Z(0,m') > Z(0,m)$ on $\tilde{\mathcal{O}}_J^m$, so when $D_J = m$, $N^{''} \geq N'$. By Lemma 4.5.3, when $D_J = m$,

$$E_0\left\{N^{''} - N \middle| \mathcal{F}_N\right\} \leq C_1(A_J + a + 1).$$

Together with (64),

$$E_0\left\{N' - N, D_J = m\right\} \leq C_2 A_J e^{-A_J} \leq C_2 \tag{65}$$

with C_2 a constant only depending on I_*, I^*, V^*.

Now it remains to prove (62). For $n = 1, 2, \ldots$, denote the possibly randomized quantizer chosen by the fusion center at time step n be $\tilde{\phi}_n$, where $\tilde{\phi}_n \in \{\bar{\phi}^0, \bar{\phi}_0, \ldots, \bar{\phi}_{M-1}\}$. By definition, $E_0\{\Delta Z_n(0,1)\} = I(0,1;\tilde{\phi}_n)$. Optional stopping theorem for margingales implies that

$$E_0\left\{Z_{N'}(0,1)\right\} = E_0\left\{\sum_{n=1}^{N'} \Delta Z_n(0,1)\right\} = E_0\left\{\sum_{n=1}^{N'} I(0,1;\tilde{\phi}_n)\right\}.$$

By the definition of the multi-stage test $\delta_{MS}(c)$, we have

$$\begin{aligned}
E_0\left\{Z_{N'}(0,1)\right\} &= I(0,1;\bar{\phi}^0)E_0(N_0) + I(0,1;\bar{\phi}_0)\sum_{j=1}^{J} E_0\left\{N_j - N_{j-1}, D_{j-1} = 0\right\} \\
&\quad + \sum_{m=1}^{M-1}\sum_{j=1}^{J-1} I(0,1;\bar{\phi}_m)E_0\left\{N_j - N_{j-1}, D_{j-1} = m\right\} \\
&\quad + \sum_{m=1}^{M-1} I(0,1;\bar{\phi}_m)E_0\left\{N' - N_{J-1}, D_{J-1} = m\right\} \\
&= I(0,1;\bar{\phi}_0)E_0\left\{N'\right\} + \left(I(0,1;\bar{\phi}^0) - I(0,1;\bar{\phi}_0)\right)E_0\left\{N_0\right\} \\
&\quad + \sum_{m=1}^{M-1}\sum_{j=1}^{J-1}\left(I(0,1;\bar{\phi}_m) - I(0,1;\bar{\phi}_0)\right)E_0\left\{N_j - N_{j-1}, D_{j-1} = m\right\} \\
&\quad + \sum_{m=1}^{M-1}\left(I(0,1;\bar{\phi}_m) - I(0,1;\bar{\phi}_0)\right)E_0\left\{N' - N_{J-1}, D_{J-1} = m\right\}
\end{aligned}$$

With a similar argument as we obtain the estimate in (65), there exists constant C_4 which only depending on I_*, I^*, V^* such that for any $m \neq 0$,

$$\begin{aligned}
E_0\left\{N_0\right\} &\leq C_4(A_0 + 1) \\
E_0\left\{N_j - N_{j-1}, D_{j-1} = m\right\} &\leq C_4(A_j + 1)e^{-A_{j-1}}, \quad j = 1, \ldots, J-1 \\
E_0\left\{N' - N_{J-1}, D_{J-1} = m\right\} &\leq C_4(A_J + 1)e^{-A_{J-1}}
\end{aligned}$$

so

$$E_0\left\{Z_{N'}(0,1)\right\} = I(0,1;\bar{\phi}_0)E_0\left\{N'\right\} + O\left(\sum_{j=0}^{J}(A_j+1)e^{-A_{j-1}}\right) = I(0,1;\bar{\phi}_0)E_0\left\{N'\right\} + O(1)$$

and the $O(1)$ term only depends on $I_*, I^*, V^*, \lambda, \rho(\lambda), A^*$.

Therefore, to prove the theorem, we only need to show that at the stopping time N', $E_0\{Z_{N'}(0,1) - A_J\} = O(1)$ with the $O(1)$ term only depending on I_*, I^*, V^*, λ, $\rho(\lambda)$, A^*

and A_{J-1}/A_J. To do so, define

$$\varepsilon = \frac{I_{gap}}{4I^*}.$$

The definition of ε guarantees that for $2 \le m \le M-1$

$$(1-\varepsilon)I(0,m;\bar{\phi}_0) - I(0,1;\bar{\phi}_0) \ge \frac{I_*}{4I^*} I_{gap}.$$

Let the positive constants $C_5 \ge 1$, C_6, δ_1 be as in Proposition 4.5.1 then when c is sufficiently small such that $C_5 A_{J-1} \le \varepsilon A_J$, we have

$$P_0\left\{\max_{1\le m\le M-1} \left|Z_{N_{J-1}}(0,m)\right| \ge L\right\} \le C_6 e^{-\delta_1 L} \tag{66}$$

for any $L \ge \varepsilon A_J$.

When $D_{J-1} = 0$ and $\max_{1\le m\le M-1} |Z_{N_J}(0,m)| \le \varepsilon A_J$, define a stopping time

$$T = \min\Big\{ n \ge N_{J-1} : Z_n(0,1) - Z_{N_{J-1}}(0,1) \ge A_J - Z_{N_{J-1}}(0,1),$$
$$\min_{2\le m\le M-1} (1-\varepsilon)(Z_n(0,m) - Z_{N_{J-1}}(0,m)) \ge A_J - Z_{N_{J-1}}(0,1)\Big\}$$

then because of the definition of ε, at time T, $Z_T(0,1) \ge A_J$ and $\min_{2\le m\le M-1} Z_T(0,m) \ge A_J$. In other words, $T \ge N'$. By Lemma 4.5.3, when $D_{J-1} = 0$ and $\max_{1\le m\le M-1} Z_{N_{J-1}} \le \varepsilon A_J$

$$E\left\{ Z_T(0,1) - A \middle| \mathcal{F}_N \right\} \le C_7$$

where C_7 is a constant which only depends on I_{gap}, I_*, I^*, V^* and A_{J-1}/A_J. By Lemma 4.5.6,

$$E\left\{ Z_{N'}(0,1) - Z_T(0,1) \middle| \mathcal{F}_{N'} \right\} \le 1$$

so

$$E\left\{ Z_{N'}(0,1) - A_J, D_{J-1} = 0, \max_{1\le m\le M-1} \left|Z_{N_{J-1}}(0,m)\right| \le \varepsilon A_J \right\} \le C_7 + 1.$$

Similarly, by using (75) of Lemma 4.5.3

$$E\left\{ Z_{N'}(0,1) - A_J, D_{J-1} = m' \ne 0, \max_{1\le m\le M-1} \left|Z_{N_{J-1}}(0,m)\right| \le \varepsilon A_J \right\} \le C_8 e^{-A_{J-1}}(A_J+1)$$

and for $L \ge \varepsilon A_J$,

$$E_0\left\{ Z_{N'}(0,1) - A_J \middle| \max_{1\le m\le M} \left|Z_{N_{J-1}}\right| = L \right\} \le C_9(A_J + L)$$

where C_8, C_9 are positive constants only depending on I_*, I^*, V^*. By (66),

$$E_0\left\{Z_{N'}(0,1) - A_J, \max_{1\le m\le M}\left|Z_{N_{J-1}}\right| \ge \varepsilon A_J\right\} \le C_{10}$$

with $C_{10} > 0$ a constant depending on $I_*, I^*, V^*, \lambda, \rho(\lambda)$. Together we have

$$E_0\left\{Z_{N'} - A_J\right\} \le C_{11}$$

with $C_{11} > 0$ a constant depending on $I_{gap}, I_*, I^*, V^*, \lambda, \rho(\lambda)$ and A_{J-1}/A_J. This completes our proof. □

Lemma 4.5.1. *Suppose in a test procedure, the fusion center always uses a stationary, though possibly randomized quantizer $\bar{\phi}$ for data compression. Let the stopping time of the procedure be*

$$T = \min\left\{n \ge 1 : \min_{1\le m\le M-1} Z_n(0,m) \ge A\right\}$$

where $A > 0$ is a threshold. Then if $I(0;\bar{\phi}) \ge I_*$

$$E_0\{T\} = A/I(0;\bar{\phi}) + O(\sqrt{A})$$

and the $O(\sqrt{A})$ term only depends on I_*, I^*, V^*.

Proof. Without loss of generality, suppose $I(0,1;\bar{\phi}) = I(0;\bar{\phi})$. Let T' be the first time $Z_n(0,1)$ crosses the threshold A, then $T' \le T$. By optional stopping, $E_0\{T\} \ge E_0\{T'\} \ge A/I(0;\bar{\phi})$. Therefore we only need to prove

$$E_0\{T\} \le A/I(0;\bar{\phi}) + O(\sqrt{A}).$$

To do so, first we prove the inequality for the case $I(0,m;\bar{\phi}) = I(0;\bar{\phi})$ for any $1 \le m \le M-1$. When this is the case, let $n_0 = A/I(0;\bar{\phi})$. By Lemma 4.5.8,

$$P_0\left\{Z_{n_0}\{0,m\} - A \le -a\sqrt{A}\right\} \le \exp\left\{V^*/2 - a/\sqrt{I(0;\bar{\phi})}\right\}$$

so

$$P_0\left\{\min_{1\le m\le M-1} Z_{n_0}\{0,m\} - A \le -a\sqrt{A}\right\} \le (M-1)\exp\left\{V^*/2 - a/\sqrt{I(0;\bar{\phi})}\right\}.$$

As before, let $\mathcal{F}_n$ be the σ-algebra generated by all the information available to the fusion center up to time step n. For $a > 0$, by Lemma 4.5.3, exists constant C which only depends on I_*, I^*, V^* such that

$$E_0\left\{T - n_0 \,\middle|\, \mathcal{F}_{n_0}, \min_{1\le m\le M-1} Z_{n_0}(0,m) - A = -a\right\} \le C(a+1).$$

Together we have

$$E_0\{T\} \le A/I(0;\bar{\phi}) + C(M-1)\exp\{V^*/2\}\sqrt{I^*A} + C$$

completing the proof. □

Proof of Theorem 4.3.2. By Lemma 4.5.2 and optional stopping, there exists constant C which only depends on $I_*, V^*, \lambda, \rho(\lambda)$ such that

$$E_0\{N_{J-1}\} \le CA_{J-1}.$$

By Lemma 4.5.1,

$$E_0\left\{N_J - N_{J-1} \,\middle|\, \mathcal{F}_{N_{J-1}}, D_{J-1} = 0\right\} \le A_J/I(0;\bar{\phi}_0) + C_1(\sqrt{A_J}+1)$$

so

$$E_0\{N_J - N_{J-1}, D_{J-1} = 0\} \le A_J/I(0;\bar{\phi}_0) + C_1(\sqrt{A_J}+1).$$

By a change of measure, we can similarly show that

$$E_0\{N_J - N_{J-1}, D_{J-1} \ne 0\} \le C_2 A_J e^{-A_{J-1}}.$$

Combining them together the theorem is proved. □

Proof of Theorem 4.3.3. In the proof we work with the Bayes test $\delta_B^*(c)$. For state $m = 0,\ldots,M-1$, denote by $\tilde{m}$ one state satisfying $I(m,\tilde{m};\bar{\phi}_m^*) = I(m;\bar{\phi}_m^*)$. Also let $P_m\{D = m'\} = \alpha(m,m')$ and $P_m\{D \ne m\} = \alpha^*(m)$. Obviously, $\alpha^*(m) = \sum_{m'\ne m}\alpha(m,m') = O(c\log c)$. By Wald's likelihood ratio identity,

$$E_m\{Z_N(m,\tilde{m})\} \ge (1-\alpha^*(m))\log\frac{1-\alpha^*(m)}{\alpha(\tilde{m},m)} + \alpha^*(m)\log\frac{\alpha^*(m)}{1-\alpha(\tilde{m},m)}.$$

When $c \to 0$,

$$E_0 \{Z_N(m, \tilde{m})\} \geq -(1 - \alpha^*(m)) \log \alpha(\tilde{m}, m) + O(1).$$

We claim that $\alpha^*(m) \log \alpha(\tilde{m}, m) \to 0$ when $c \to 0$. If otherwise, there exists a sequence $c_i \to 0$ but

$$-\alpha^*(m; c_i) \log \alpha(\tilde{m}, m; c_i) \geq b > 0.$$

Thus, when c_i sufficiently small so that $\alpha^*(m; c_i) \leq 1/2$, we have

$$E_m \{Z_N(m, \tilde{m})\} \geq \frac{b}{2\alpha^*(m; c_i)} + O(1) \geq \frac{b'}{c_i |\log c_i|} + O(1)$$

where $b' > 0$ is another constant not depending on c_i. However, by optional stopping, $E_m\{Z_N(m, \tilde{m})\} \leq I_m^* E_m\{N\}$, therefore

$$E_m\{N\} \geq \frac{b'}{I(m) c_i |\log c_i|} + O(1)$$

which contradicts the optimality of $\delta_B^*(c)$ because it requires that $E_m \{N\} = O(|\log c_i|)$, and the claim is proved. Therefore

$$E_m \{Z_N(m, \tilde{m})\} \geq -\log \alpha(\tilde{m}, m) + O(1).$$

Denote $\alpha'(\tilde{m}, m; c) = \alpha(\tilde{m}, m; c)/c$, we claim that $\alpha'(\tilde{m}, m; c) = O(1)$ for any m. If the claim is true, the theorem is proved because then for any m,

$$E_m \{Z_N(m, \tilde{m})\} \geq |\log c| + O(1)$$

and

$$E_m \{N\} \geq |\log c| / I(m) + O(1).$$

So it remains to prove the claim. Without loss of generality, we only need to prove it for $m = 0$, suppose $c_i \to 0$ such that $\alpha'(\tilde{0}, 0; c_i) \to \infty$, then by possibly going to a subsequence, we can assume that for $m = 1, \ldots, M-1$, $\alpha'(\tilde{m}, m; c_i)$ either goes to infinite or remains bounded. By changing the label of the state, we are free to suppose that exist $1 \leq M' \leq M$ such that for $m = 0, \ldots M'-1$, $\alpha'(\tilde{m}, m; c_i) \to \infty$ while for $m = M', \ldots, M-1$, $\alpha'(\tilde{m}, m; c_i) = O(1)$.

Hence when the cost per step is $c_i \to 0$, for $m = 0, \ldots, M' - 1$,

$$E_m \{Z_N(m, \tilde{m})\} \geq |\log c_i| - \log \alpha'(\tilde{m}, m; c_i) + O(1)$$

and

$$E_m \{N\} \geq |\log c|/I(m) - \log \alpha'(\tilde{m}, m; c_i)/I(m) + O(1)$$

while for $m = M', \ldots, M - 1$,

$$E_m \{N\} \geq |\log c|/I(m) + O(1).$$

Consequently by definition

$$\begin{aligned} & \mathcal{R}_{c_i}(\delta_B^*(c_i)) \\ = & \sum_m \pi_m E_m \{N\} + \sum_{m \neq m'} \pi_m W(m, m') \alpha(m, m') \\ \geq & c_i |\log c_i| \sum_{m=0}^{M-1} \pi_m / I(m) + c_i \sum_{m=0}^{M'-1} \left(\pi_{\tilde{m}} W(\tilde{m}, m) \alpha'(\tilde{m}, m) - \pi_m \log \alpha'(\tilde{m}, m)/I(m) \right) + O(c_i). \end{aligned}$$

For $m = 0, \ldots, M' - 1$, when $c_i \to 0$, $\alpha'(\tilde{m}, m) \to \infty$, so

$$\pi_{\tilde{m}} W(\tilde{m}, m) \alpha'(\tilde{m}, m) - \pi_m \log \alpha'(\tilde{m}, m)/I(m) \to \infty.$$

When c_i is sufficiently small, we must have

$$\mathcal{R}_{c_i}(\delta_B^*(c_i)) > \mathcal{R}_{c_i}(\delta_{MS}^*(c))$$

which is a contradiction. This completes our proof. □

Lemma 4.5.2. *Let $m = 0$ be the true state of nature, and $\underline{\mu} > 0$ be a constant such that $I(0, m; \bar{\phi}) \geq \underline{\mu}$ for any $m = 1, \ldots, M - 1$ and $\bar{\phi} = \bar{\phi}^0, \bar{\phi}_0, \ldots, \bar{\phi}_{M-1}$. Consider a procedure which, for each time step n, the fusion center always chooses one quantizer from the set $\{\phi^0, \bar{\phi}_0, \ldots, \bar{\phi}_{M-1}\}$ to implement and the choice is adaptive to $\mathcal{F}_{n-1}$, i.e., the σ-algebra generated by all the information at fusion center up to time $n - 1$. Let T be the first time that, for each $m = 1, \ldots, M-1$, $Z_n(0, m) \geq A$, where $A > 0$ is a given constant. Then exist positive constants $C_5 \geq 1, C_6, \delta_1$ which only depending on I_*, V^* and $\lambda, \rho(\lambda)$ in Assumption 2, such that for every $L \geq C_5 A$*

$$P_0 \{Z_T(0, m) \geq L\} \leq C_6 e^{-\delta_1 L}, \quad m = 1, \ldots, M - 1.$$

Proof. Let $\lambda > 0$ and $\rho(\lambda)$ as defined in Assumption 2. Then because $E_0\{\Delta Z_n(0,m)\} > 0$

$$1 \le E_0\left\{ e^{\lambda \Delta Z_n(0,m)} \middle| \mathcal{F}_{n-1} \right\} \le \rho(\lambda)$$

and $\{\exp\{\lambda Z_n(0,m)\}, \mathcal{F}_n\}$ is a submartingale. So by Doob's inequality, for any $n \ge 1$,

$$\begin{aligned} & P_0\left\{ \max_{i\le n} Z_i(0,m) \ge L \right\} = P_0\left\{ \max_{i\le n} e^{\lambda Z_i(0,m)} \ge e^{\lambda L} \right\} \\ \le \;& e^{-\lambda L} e^{n\rho(\lambda)} = e^{-L(n\rho(\lambda)/L - \lambda)}. \end{aligned}$$

Take $\delta_1' = \lambda/2$ and $n_0 = \frac{\lambda L}{2\rho(\lambda)}$, then

$$P_0\left\{ \max_{i\le n_0} Z_i(0,m) \ge L \right\} \le e^{-\delta_1' L}. \tag{67}$$

Meanwhile, the process $\{\Delta Z_n(0,m), \mathcal{F}_n\}$ obviously satisfies the condition of Lemma 4.5.7 with $I_* = \underline{\mu}$ and $V = V^* + 2/e$. So exist positive μ_* and δ_3 only depending on I_* and V^* such that for $n = 1, 2, \ldots$

$$P_0\{Z_n(0,m) \le \mu_* n\} \le e^{-\delta_3 n}.$$

Consequently, for $n \ge A/\mu_*$

$$P_0\{Z_n(0,m) \le A\} \le e^{-\delta_3 n}$$

and when $L \ge \frac{2\rho(\lambda)A}{\mu_*\lambda}$, with the n_0 we just defined,

$$P_0\{Z_{n_0}(0,m) \le A\} \le e^{-\delta_3' L}$$

where $\delta_3' = \frac{\delta_3 \lambda}{2\rho(\lambda)} > 0$. So

$$P_0\left\{ \min_{1\le m\le M-1} Z_{n_0}(0,m) \le A \right\} \le (M-1)e^{-\delta_3' L}.$$

Hence

$$P_0\{T \ge n_0\} \le (M-1)e^{-\delta_3' L} \tag{68}$$

because $\{T \ge n_0\} \subset \{\min_{1\le m\le M-1} Z_{n_0}(0,m) \le A\}$. Let $\delta_1 = \min\{\delta_1', \delta_3'\}$, $C_5 = \frac{2\rho(\lambda)}{\mu_*\lambda}$, and $C_6 = M$, then because

$$\{Z_T(0,m) \ge L\} \subset \{T \ge n_0\} \bigcup \left\{ \max_{i\le n_0} Z_i(0,m) \ge L \right\}$$

combining (67) and (68) we complete the proof of the lemma. □

Based on Lemma 4.5.2, we have the following estimation about $Z_n(0,m)$ at time $n = N_{J-1}$, i.e., the time $\tilde{Z}_n$ reaches stopping region $\tilde{\mathcal{O}}_{J-1}$.

Proposition 4.5.1. *Exist positive constants $C_5 \geq 1, C_6, \delta_1$ which only depending on $I_*, I^*, \lambda, \rho(\lambda)$ such that for every $L \geq C_5 A_{J-1}$,*

$$P_0\left\{Z_{N_{J-1}}(0,m) \geq L, D = 0\right\} \leq C_6 e^{-\delta_1 L}$$

for every $L \geq A_{J-1}$,

$$P_0\left\{Z_{N_{J-1}}(0,m) \leq -L, D = m\right\} \leq e^{-L}$$

and for $L \geq C_5 A_{J-1}$, $m' \neq m$

$$P_0\left\{\left|Z_{N_{J-1}}(0,m')\right| \geq L, D = m\right\} \leq 2C_6 e^{-\delta_1 L - A_{J-1}}$$

So with possibly an adjustment of the constants C_5, C_6, δ_1, we have that for any $L \geq C_5 A_{J-1}$,

$$P_0\left\{|Z_{N_{J-1}}(0,m)| \geq L\right\} \leq C_6 e^{-\delta_1 L}. \tag{69}$$

4.5.2 Proofs for Section 4.4

Let $L^* = (V^* + 2/e)/I_* < \infty$.

Proof of Theorem 4.4.1. For convenience, denote $I(0,1;\phi^0) = I_0$, $I(0) = I(0,1;\bar{\phi}_0^*)$, $I_0' = I(0,1;\bar{\phi}_1^*)$. Let's suppose $I^0 < I(0)$, otherwise we only need to exchange state 0 with state 1. Because $\bar{\phi}_0^* \neq \bar{\phi}_1^*$, $I_0' < I(0)$.

Let $L_j \geq 0$ be the overshoot over A_j or $-A_j$ at time N_j, i.e., $L_j = Z_{N_j} - A_j$ ($L_j = -Z_{N_j} - A_j$) when $D_j = 0$ ($D_j = 1$). Also define N_j', N_j'' be the first time after N_{j-1} that Z_n cross the upper bound A_j or lower bound $-A_j$, respectively, if the local sensor did not switch quantizer. Let $L_j' \geq 0, L''_j \geq 0$ be the overshoots at time N_j', N_j''. So when $D_j = 0$, $N_j' = N_j$, $L_j' = L_j$, and when $D_j = 1$, $N_j'' = N_j$, $L_j'' = L_j$.

Like in the proof of Theorem 4.3.1, we have the following decomposation

$$E_0\{Z_N\} = I_0^* E_0\{N\} - (I_0^* - I^0)E_0\{N_0\} - (I(0) - I_0')E_0\{N_j - N_{j-1}, D_{j-1} = 1\}. \tag{70}$$

Easy to show $E_0\{Z_N\} \geq A_J - 1$. To guarantee second order asymptotic optimality, we need $E_0\{N_0\} = O(1)$. Because $E_0\{N_0\} \geq (I^0)^{-1} A_0$, so $A_0 = O(1)$ when $c \to 0$.

Similarly

$$\sum_{j=1}^{J} E_0\left\{N_j - N_{j-1}, D_{j-1} = 1\right\} = O(1). \tag{71}$$

Next we try to estimate a lower bound for $\sum_{j=1}^{J} E_0\left\{N_j - N_{j-1}, D_{j-1} = 1\right\}$. For that, we first obtain a lower bound for making incorrect intermediate decisions, i.e.,

$$P_0\left\{D_j = 1\right\} = e^{-A_j} E_1\left\{D_j = 1, e^{-L_j}\right\}$$

By Jensen inequality,

$$E_1\left\{e^{-L_j''}\right\} \geq \exp\left\{-E_1\left\{L_j''\right\}\right\} \geq e^{-L^*}$$

so

$$E_1\left\{e^{-L_j}, D_j = 1\right\} = E_1\left\{e^{-L_j''}\right\} - E_1\left\{D_j = 0, e^{-L_j''}\right\} \geq e^{-L^*} - P_1\left\{D_j = 0\right\} \geq e^{-L^*} - e^{-A_j}$$

and

$$P_0\left\{D_j = 1\right\} \geq e^{-A_j - L^*} - e^{-2A_j}. \tag{72}$$

Now, on the event $D_{j-1} = 1$,

$$E_0\left\{N_j' - N_{j-1} \middle| \mathcal{F}_{N_{j-1}}\right\} \geq {I_0'}^{-1}(A_j + A_{j-1})$$

so

$$E_0\left\{N_j' - N_{j-1}, D_{j-1} = 1\right\} \geq {I_0'}^{-1}(A_j + A_{j-1}) P_0\left\{D_{j-1} = 1\right\} \geq {I_0'}^{-1}(A_j + A_{j-1})(e^{-A_{j-1} - L^*} - e^{-2A_{j-1}}). \tag{73}$$

On the other hand,

$$\begin{aligned} & E_0\left\{e^{L_{j-1}}, D_{j-1} = 1\right\} = E_1\left\{e^{-A_{j-1}}, D_{j-1} = 1\right\} \\ \leq \quad & e^{-A_{j-1}}\left(1 - e^{-A_{j-1} - L^*} + e^{-2A_{j-1}}\right) \leq 2e^{-A_{j-1}}. \end{aligned}$$

While on the event $\{D_{j-1} = 1\} \cup \{D_j = 1\}$,

$$E_0\left\{N_j' - N_j \middle| \mathcal{F}_{N_j}\right\} \leq {I_0'}^{-1}(L_j + 2A_j + L^*).$$

so on $D_{j-1}=1$

$$\begin{aligned}
& E_0\left\{N_j'-N_j, D_j=1\middle|\mathcal{F}_{N_{j-1}}\right\} \\
\leq\ & {I_0'}^{-1}(2A_j+L^*)P_0\left\{D_j=1|\mathcal{F}_{N_{j-1}}\right\}+{I_0'}^{-1}E_0\left\{L_j, D_j=1|\mathcal{F}_{N_{j-1}}\right\} \\
\leq\ & {I_0'}^{-1}(2A_j+L^*)e^{-A_j+A_{j-1}+L_{j-1}}+{I_0'}^{-1}e^{-A_j+A_{j-1}+L_{j-1}}E_1\left\{D_j=1, L_je^{-L_j}\middle|\mathcal{F}_{N_{j-1}}\right\} \\
\leq\ & 2{I_0'}^{-1}e^{-A_j+A_{j-1}}(A_j+L^*)e^{L_{j-1}}.
\end{aligned}$$

Because $N_j'=N_j$ when $D_j=0$, consequently

$$E_0\left\{N_j'-N_j, D_{j-1}=1\right\}\leq 2{I_0'}^{-1}e^{-A_j+A_{j-1}}(A_j+L^*)E_0\left\{e^{L_{j-1}}, D_{j-1}=1\right\}\leq 4{I_0'}^{-1}e^{-A_j}(A_j+L^*)$$

and together with (73)

$$E_0\left\{N_j-N_{j-1}, D_{j-1}=1\right\}\geq (I_0')^{-1}(A_j+A_{j-1})(e^{-A_{j-1}-L^*}-e^{-2A_{j-1}})-4e^{-A_j}(I_0')^{-1}(A_j+L^*). \tag{74}$$

Summing up (74) for $j=1,\ldots,J$, and using the fact $\sum_{j=1}^{J}A_je^{-A_j}=O(1)$,

$$\sum_{j=1}^{J}E_0\left\{N_j-N_{j-1}, D_{j-1}=1\right\}\geq e^{-L^*}(I_0')^{-1}\sum_{j=1}^{J}A_je^{-A_{j-1}}+O(1).$$

Together with (71), $\sum_{j=1}^{J}A_je^{-A_{j-1}}=O(1)$. This completes our proof. □

4.5.3 Auxiliary Results.

Lemma 4.5.3. *Let* $\Delta\tilde{X}_n=\left(\Delta X_n^{(1)},\ldots,\Delta X_n^{(M-1)}\right)$ *be a sequence of i.i.d. random vectors w.r.t. time step* $n=1,2,\ldots$. *Denote* $\tilde{X}_n=\sum_{i=1}^{n}\Delta\tilde{X}_i$ *and* $X_n^{(m)}=\sum_{i=1}^{n}\Delta X_i^{(m)}$. *Suppose* $\exp\left\{-\Delta X_n^{(1)}\right\}=1$ *and exist* $0<\underline{\mu}<\bar{\mu}$ *and* $V>0$ *such that*

$$\underline{\mu}\leq\min_m E\left\{\Delta X_n^{(m)}\right\}\leq\max_m E\left\{\Delta X_n^{(m)}\right\}\leq\bar{\mu}$$

and

$$\max_m E\left\{\left(\Delta X_n^{(m)}\right)^2\right\}\leq V.$$

Let $A\geq 0$ *be a constant, and stopping time*

$$T=\min\left\{n:\min_m X_n^{(m)}\geq A\right\}.$$

Then exist positive constants C_1 which only depends on $\underline{\mu}$, $\bar{\mu}$ and V such that

$$E\left\{X_T^{(1)}\right\} \leq C_1(A+1) \tag{75}$$

and

$$E\{T\} \leq C_1(A+1). \tag{76}$$

Moreover, if for $m = 2, \ldots, M-1$, exist $\delta > 0$ such that

$$E\left\{\Delta X_n^{(1)} - \Delta X_n^{(m)}\right\} \geq \delta \tag{77}$$

then exists $C_2 > 0$ which only depends on $\underline{\mu}, \bar{\mu}, V$ and δ such that

$$E\left\{X_T^{(1)} - A\right\} \leq C_2. \tag{78}$$

Proof. Let's first prove (78) under (77). For that, define

$$\Delta Y_n^{(m)} = \Delta X_n^{(1)} - \Delta X_n^{(m)}, \quad m = 2, \ldots, M-1$$

and

$$Y_n^{(m)} = \sum_{i=1}^{n} \Delta Y_n^{(m)}, \quad m = 2, \ldots, M-1.$$

Then for $2 \leq m \leq M-1$,

$$\delta \leq E\left\{\Delta Y_n^{(m)}\right\} \leq \bar{\mu}$$

and

$$E\left\{\left(\Delta Y_n^{(m)}\right)^2\right\} \leq 4V.$$

Let $T_0 = 0$ and

$$T_n = \min\left\{i \geq T_{n-1} : \min_{2 \leq m \leq M-1} Y_i^{(m)} - Y_{T_{n-1}}^{(m)} > 0\right\}.$$

Then $\{T_n - T_{n-1}\}$ is a positive i.i.d. sequence. By Lemma 4.5.5, exists $C_4 > 0$ which only depends on $\delta, \bar{\mu}, V$ such that

$$1 \leq E\{T_n - T_{n-1}\} \leq E\left\{(T_n - T_{n-1})^2\right\} \leq C_4.$$

Now $\left\{X_{T_n}^{(1)} - X_{T_{n-1}}^{(1)}\right\}$ is also i.i.d. sequence with

$$X_{T_1}^{(1)} - X_{T_0}^{(1)} = \sum_{i=1}^{T_1} \Delta X_i^{(1)}.$$

Hence

$$E\left\{X_{T_n}^{(1)} - X_{T_{n-1}}^{(1)}\right\} \geq E\left\{\Delta X_1^{(1)}\right\} \geq \underline{\mu}$$

and with $\mu^{(1)} = E\left\{\Delta X_n^{(1)}\right\}$,

$$\begin{aligned}
E\left\{\left(X_{T_n}^{(1)} - X_{T_{n-1}}^{(1)}\right)^2\right\} &= E\left\{\left(\sum_{i=1}^{T_1} \Delta X_i^{(1)}\right)^2\right\} \\
&= E\left\{\left(\sum_{i=1}^{T_1}\left(\Delta X_i^{(1)} - \mu^{(1)}\right) + T_1\mu^{(1)}\right)^2\right\} \\
&\leq E\left\{\left(\sum_{i=1}^{T_1}\left(\Delta X_i^{(1)} - \mu^{(1)}\right)\right)^2\right\} + 2(\mu^{(1)})^2 E\left\{T_1^2\right\} \\
&\leq 2E\left\{T_1\right\} E\left\{\left(\Delta X_1^{(1)} - \mu^{(1)}\right)^2\right\} + 2(\mu^{(1)})^2 E\left\{T_1^2\right\} \\
&\leq 2E\left\{T_1\right\} V + 2\bar{\mu}^2 E\left\{T_1^2\right\} \\
&\leq 2C_4(V + \bar{\mu}^2).
\end{aligned}$$

Define $N = \min\left\{n : X_{T_n}^{(1)} \geq A\right\}$, by Lorden [19],

$$E\left\{X_{T_N}^{(1)} - A\right\} \leq \frac{E\left\{\left(X_{T_n}^{(1)} - X_{T_{n-1}}^{(1)}\right)^2\right\}}{E\left\{X_{T_n}^{(1)} - X_{T_{n-1}}^{(1)}\right\}} \leq 2C_4\left(V + \bar{\mu}^2\right)/\underline{\mu}.$$

By definition, for $2 \leq m \leq M-1$, $0 < Y_{T_N}^{(m)} = X_{T_N}^{(m)} - X_{T_N}^{(1)}$, so

$$X_{T_N}^{(m)} \geq A, \quad , m = 1, \ldots, M$$

and $T \leq T_N$. By Lemma 4.5.6,

$$E\left\{X_T^{(1)} - X_{T_N}^{(1)}\right\} \leq 1$$

so finally we have

$$E\left\{X_T^{(1)} - A\right\} \leq C_3 = 1 + 2C_4\left(V + \bar{\mu}^2\right)/\underline{\mu}.$$

Now we prove (75) and (76). By optional stopping, (76) is a direct consequence if we can prove (75). To do so, define $C_5 = 2\bar{\mu}/\underline{\mu} \geq 1$, and $\Delta X_n'^{(m)} = C_5 \Delta X_n^{(m)}$ for $2 \leq m \leq M-1$. Then

$$E\left\{\Delta X_n'^{(m)} - \Delta X_n^{(1)}\right\} \geq \bar{\mu}/2 \quad m = 2, \ldots, M-1.$$

Let $X_n'^{(m)} = \sum_{i=1}^n \Delta X_n'^{(m)}$, define the stopping time

$$T' = \min\left\{n : X_n^{(1)} \geq C_5 A, \min_{2\leq m\leq M-1} X_n'^{(m)} \geq C_5 A\right\}.$$

By what we have just proved, exists constant $C_6 > 0$ which only depends on $\underline{\mu}, \bar{\mu}, V$ such that

$$E\left\{X_{T'}^{(1)} - C_5 A\right\} \leq C_6.$$

Hence

$$E\left\{X_{T'}^{(1)}\right\} \leq C_5 A + C_6.$$

Meanwhile, because $C_5 \geq 1$, $T \leq T'$ and again by Lemma 4.5.6,

$$E\left\{X_T^{(1)} - X_{T'}^{(1)}\right\} \leq 1$$

and

$$E\left\{X_T^{(1)}\right\} \leq C_5 A + C_6 + 1$$

completing our proof. □

Lemma 4.5.4. *Let $\Delta X_1, \Delta X_2, \ldots$ be i.i.d. sequence with positive mean $E\{\Delta X_1\} = \mu > 0$ and $X_n = \sum_{i=1}^n \Delta X_i$ be the random walk. Define $T = \min\{n : X_n > 0\}$ be the first strong ascending ladder epoch of X_n, then*

$$E\{T\} \leq E\left\{\Delta X_1^2\right\}/\mu^2$$

and

$$E\left\{T^2\right\} \leq 4\frac{E^2\left\{\Delta X_1^2\right\}}{\mu^4}.$$

Proof. By Theorem 1 of Lorden [19], $E\{X_T\} \leq E\left\{\Delta X_1^2\right\}/\mu^2$. By optional stopping, $\mu E\{T\} = E\{X_T\}$, thus the first inequality follows.

To proof the second, for any random variable Y denote $\|Y\|_2 = \sqrt{E\{Y^2\}}$. Note that $\mu\|T\|_2 = \|X_T - \mu T\|_2 + \|X_T\|_2$. By Lemma 1.8.1 of Gut [7],

$$\|X_T\|_2 \leq \|\Delta X_T\|_2 \leq \sqrt{E\{T\}}\|\Delta X_1\|_2.$$

Meanwhile, use optional stopping to variance, we have

$$E\left\{(X_T - T\mu)^2\right\} = E\{T\} Var\{\Delta X_1\} \leq E\{T\} E\left\{\Delta X_1^2\right\}.$$

Hence

$$\mu\|T\|_2 \leq 2\sqrt{E\{T\}}\|\Delta X_1\|_2$$

and

$$E\left\{T^2\right\} \leq 4E\{T\} E\left\{\Delta X_1^2\right\}/\mu^2 \leq 4\frac{E^2\left\{\Delta X_1^2\right\}}{\mu^4}.$$

□

Lemma 4.5.5. *Let $\Delta X_n = (\Delta X_n^{(2)}, \ldots, \Delta X_n^{(M-1)})$ be a seqence of i.i.d. random vectors, and $X_n^{(m)} = \sum_{i=1}^n \Delta X_i^{(m)}$ for $2 \leq m \leq M-1$. Suppose exist $\underline{\mu} > 0$, $\bar{\mu} > 0$, $\nu_2 > 0$ such that for $2 \leq m \leq M-1$, $\underline{\mu} \leq E\{\Delta X_1^m\} \leq \bar{\mu}$ and $E\left\{\left(\Delta X_1^{(m)}\right)^2\right\} \leq V$. Define a stopping time T as*

$$T = \min\{n : \min_{2\leq m\leq M-1} X_n^{(m)} > 0\}.$$

Then

$$E\{T\} \leq E\left\{T^2\right\} \leq C < \infty$$

where $C > 0$ is a constant only depending on $\underline{\mu}, \bar{\mu}, V$.

Proof. Recursively use Lemma 4.5.4. □

Lemma 4.5.6. *Let $\Delta X_1, \Delta X_2, \ldots$ be i.i.d. sequence with positive mean $E\{\Delta X_1\} > 0$ and $E\{e^{-\Delta X_1}\} = 1$. Let $X_n = \sum_{i=1}^n \Delta X_i$ be the random walk, then for any $L \geq 0$,*

$$P\left\{\min_{n\geq 1} X_n \leq -L\right\} \leq e^{-L}.$$

Proof. Let $\mathcal{F}_n = \sigma\{\Delta X_1, \ldots, \Delta X_n\}$, then $\{e^{-X_n}, \mathcal{F}_n\}$ is a martingale, and the conclusion is a direct consequence of Doob's inequality. □

Lemma 4.5.7. *Let $\Delta X_1, \Delta X_2, \ldots$ be a sequence of random variables, and the σ-algebra $\mathcal{G}_n = \sigma\{\Delta X_1, \ldots, \Delta X_n\}$ for $n = 1, 2, \ldots$. Denote $X_n = \sum_{i=1}^n \Delta X_i$. Suppose that*

$$E\{\exp(-\Delta X_n)|\,\mathcal{G}_{n-1}\} = 1$$

and exist $\underline{\mu} > 0$ *and* $V > 0$ *such that for any* n*,*

$$E\left\{\Delta X_n \middle| \mathcal{G}_{n-1}\right\} \geq \underline{\mu}, \quad , E\left\{\Delta X_n^2 \middle| \mathcal{G}_{n-1}\right\} \leq V.$$

Then exist positive constants δ_3 *and* μ_* *which only depend on* $\underline{\mu}$ *and* V *such that*

$$E\left\{X_n \leq \mu_* n\right\} \leq e^{-\delta_3 n}.$$

Proof. For any $n \geq 1$, consider the conditional moments generating function

$$\varphi_n(t) = E\left\{e^{-t\Delta X_n} \middle| \mathcal{G}_{n-1}\right\}.$$

Then $\varphi_n(t)$ is well defined and smooth for $0 \leq t \leq 1$, with $\varphi_n(0) = \varphi_n(1) = 1$. Moreover, we have that

$$\varphi_n'(0) = E\left\{-\Delta X_n \middle| \mathcal{G}_{n-1}\right\} \leq -\underline{\mu}$$

and for $0 \leq t \leq 1$,

$$\varphi_n''(t) = E\left\{\Delta X_n^2 e^{-t\Delta X_n} \middle| \mathcal{G}_{n-1}\right\} \leq E\left\{\Delta X_n^2 \middle| \mathcal{G}_{n-1}\right\} \leq V.$$

Consequently, for $0 \leq t \leq 1$,

$$\varphi_n''(t) \leq 1 - \underline{\mu} t + V t^2/2.$$

Take $\tau = \underline{\mu}/V$,

$$\mu_* = -\frac{V}{2\underline{\mu}} \log(1 - \frac{\underline{\mu}^2}{2V}) > 0$$

and

$$\delta_3 = -\frac{1}{2} \log\left(1 - \frac{\underline{\mu}^2}{2V}\right) > 0$$

then

$$E\left\{e^{-\tau(\Delta X_n - \mu_*)} \middle| \mathcal{G}_{n-1}\right\} \leq e^{-\delta_3}.$$

As a result,

$$E\left\{e^{-\tau X_n} e^{n\tau\mu_*}\right\} \leq e^{-\delta_3 n}$$

and

$$P\left\{X_n - \mu_* n \leq 0\right\} \leq E\left\{e^{-\tau X_n + n\tau\mu_*}\right\} \leq e^{-\delta_3 n}$$

with δ_3 and μ_* only depending on $\underline{\mu}$ and V. □

Lemma 4.5.8. *Let $\Delta X_1, \Delta X_2, \ldots$ be a sequence of i.i.d. random variable such that $E\{\Delta X_1\} = \mu$, $E\left\{\Delta X_1^2\right\} \leq V$, and $E\left\{e^{-\Delta X_1}\right\} < \infty$. Denote $X_n = \sum_{i=1}^n \Delta X_i$. Then for any $a > 0$,*

$$P\left\{X_n - \mu n \leq -\sqrt{n}a\right\} \leq e^{\frac{V}{2}} e^{-a}.$$

Proof. Without loss of generality, assume that $\mu = 0$. Then $\varphi(t) = e^{-t\Delta X_1}$ is well defined and smooth on $0 \leq t \leq 1$. In particular, $\varphi(0) = 1$, $\varphi'(0) = 0$ and

$$0 \leq \varphi''(t) = E\left\{\Delta X_1^2 e^{-t\Delta X_1}\right\} \leq V.$$

Hence for $0 \leq t \leq 1$, $\varphi(t) \leq \exp\left\{\frac{Vt^2}{2}\right\}$. As a result, for $n = 1, 2, \ldots,$

$$E\left\{\exp\left\{-\frac{X_n}{\sqrt{n}}\right\}\right\} \leq \exp\left\{\frac{V}{2}\right\}$$

and

$$P\left\{X_n \leq -\sqrt{n}a\right\} = P\left\{\exp\left\{-X_n/\sqrt{n}\right\} \geq e^a\right\} \leq E\left\{\exp\left\{-X_n/\sqrt{n}\right\}\right\} e^{-a} \leq e^{V/2-a}.$$

□

CHAPTER V

SEQUENTIAL MINIMUM ENERGY DESIGN ALGORITHM

This chapter deals with a new topic on the theoretical properties of the sequential minimum energy design (SMED) proposed in Joseph, Dasgupta and Wu [11].

5.1 Introduction and Background

In a typical computer or laboratory experimental design problem, experimenters aim at finding the global optimum of a black-box response function $p(x)$ over a given design region $\mathcal{X}$, where the function $p(x)$ represents the experimental yields which experimenters often have very little prior knowledge of. In order to find the global optimum, the experimenters typically need to select a set of design points and evaluate the response function over them. In many applications, the evaluation of $p(x)$ over a single design point $x \in \mathcal{X}$ can be expensive and time consuming, and some may need to run a computer simulation for weeks for finite element models. Therefore, a good design should be able to estimate the global optimum accurately with as few design points as possible. Below we will simply state that we want as few runs as possible by referring a run as the evaluation of $p(x)$ at a single design point.

When the design region $\mathcal{X}$ is of low dimension, the selection of design points can be done by traditional methods like fractional factorial designs and orthogonal arrays, see Wu and Hamada [46]. When the dimension of $\mathcal{X}$ becomes slightly higher, say, 10, these methods quickly become infeasible because they yield a prohibitively large number of runs. Other more efficient space-filling designs include Latin hypercule designs (McKay, Beckman and Conover [22]), uniform designs (Fang [5]), and scrambled nets (Owen [28]). Unfortunately, these designs are still not efficient enough for many cases, mainly because they are fixed designs and do not incorporate information obtained from previous evaluated points into subsequent choice of test points.

To overcome such weakness it is better to adopt a sequential design that takes advantage

of the knowledge obtained in already evaluated points. The most famous existing designs of this sort are PI and EI algorithms. Interesting readers are referred to Jones [10] or Zhigljavsky and Zilinskas [48] (Ch 4) for more details. It is shown that, under certain conditions, both methods asymptotically attain the global optimum when the numbers of runs tend to infinite, see for example Guttman [8], Torn and Zilinskas [39], Locateli [17] etc.

However, both PI and EI algorithms are still only "semi-sequential" in the sense that they all include a "preliminary stage" where a fixed design is needed to pick up a promising sub-region to be searched over. As a rule of thumb proposed by Loeppky, Sacks and Welch [18], the sizes of such preliminary designs are as large as $10m$, where m is the dimension of the design region. When $m \sim 10$, the sizes of the preliminary stages can be already unaffordable to experimenters.

In Joseph, Dasgupta and Wu in [11], the authors proposed the Sequential Minimum Energy Design (SMED), an innovative and nature-inspired sequential algorithm that addresses the problems stated above. In SMED, one visualizes the process of selecting design points as laying down particles with positive charges related to the experimental yields $p(x)$, with particles in more promising regions receiving lower charges and vice versa. Optimum design points are calculated by sequentially minimizing the potential energy generated by the corresponding charged particles. As a result, it is very unlikely for the design points to stay over-crowded. Indeed, because of the charge they carry, the design points have a natural tendency to expel each other, thereby achieving good space-filling. More importantly, since the charges are related to the experimental yield $p(x)$, design points tend to gather in regions with higher yields. Because of these two features, the SMED automatically incorporates the information obtained from evaluated points into subsequent steps.

To be more specific, in [11], the potential energy between two particles with unit charges is defined as the inverse of their distance, and the experimental yield over each next design point is predicted through inverse distance weighting extrapolation. In real applications, the SMED algorithm can be used in combination with other efficient space filling designs such as Latin hypercubes to avoid over-concentration in the corners of the design regions. In [11], the SMED is applied to a nano-wires synthesis problem as well as the optimization

of some specially designed test functions, and it illustrates that the SMED mostly hit global optimums in much less number of runs as compared to the EI or PI algorithms.

There are still many open problems regarding the SMED algorithm. Specifically, we know very little about its asymptotic behaviors, and it is desirable to know whether the SMED design points eventually become dense in the design region. This was raised as an unresolved conjecture in [11]. Furthermore, it is helpful to know the long term distribution of the design points because it is directly related to the efficiency of the algorithm. In addition, the algorithm itself can be improved or simplified. In [11], the response surface $p(x)$ need to be re-estimated at each step, this lowers the computational efficiency and we wish to propose a simplified version without the re-estimations.

The aim of the present chapter is to tackles these open problems on the SMED algorithm. We provide useful insights into the theoretical properties of SMED and propose modifications to the algorithm itself to facilitate implementations. The organization of this chapter is as follows. In Section 5.2, we provide a rigorous definition of the SMED algorithm and introduce relevant notation. In Section 5.3, we investigate the asymptotic properties of the SMED design points. In Section 5.4, we propose a simplification to SMED so that the step-wise estimation of the response surface $p(x)$ is no longer necessary. Then numerical simulations are reported in Section 5.5.

5.2 Definition and Notation

To facilitate the discussion, we scale each design variable so that they all have the same range $[0, 1]$. In other words, the experimental region under consideration is $\mathcal{X} = [0, 1]^m$, where m is the dimension of the design region. Let the experimental yield (or response surface) $p(x)$ be a non-negative piecewise continuous function over $\mathcal{X}$. Ideally, one would like to find a design point $x_g \in \mathcal{X}$ (in other words, a specific combination of experimental variables) such that $p(x)$ is globally maximized, i.e., $x_g = \arg\max_{x\in\mathcal{X}} p(x)$. In practice, we often select a set of test points $x_1, \ldots, x_N$ and evaluate the yield function $p(x)$ over them. Then the test point with the largest yield is chosen as an approximation of the global optimum. Hence, the objective is to design an efficient scheme that assigns the test points

in such a way that we can get as close to the true global optimum as possible. Moreover, we want the total number of runs of the experiment to be minimized, since the evaluation of $p(x)$ over every single test point can be costly.

The basic idea of SMED is to visualize each selected point x as a particle with a non-negative charge $q(x)$ that will be defined later. Given the charge function $q(x)$, the SMED algorithm defines the potential energy between two charged particles at x and y as

$$E(x, y) = \frac{q(x)q(y)}{[d(x, y)]^\beta}, \tag{79}$$

where the $\beta \geq 1$ is a pre-specified parameter and $d(x, y)$ is the Euclidean distance between the two points.

There are many different ways to define the charge function $q(x)$ as long as $q(x)$ is a non-negative decreasing function of the experimental yield $p(x)$. Below we focus on the one proposed in [11]:

$$q(x) = (1 - \alpha p(x))^\gamma \tag{80}$$

where α and γ are two positive parameters to be determined. Note that the non-negative assumption of the particle charges implies that

$$\alpha \leq \frac{1}{\max_x p(x)}.$$

Hence, we may want to choose $\alpha \leq 1/M$ if we know that M is a crude upper bound on $p(x)$.

Now let us describe the SMED algorithm from the viewpoint of sequential black-box designs. Suppose that the initial points $\{x_1, \ldots, x_{n_0}\}$ are chosen randomly or according to certain efficient designs (e.g. Latin Hypercube or other space-filling designs), and assume that $n \geq n_0$ design points have already been selected. Then the total potential energy for the selected design points is

$$E_n = \sum_{1 \leq i \neq j \leq n} E(x_i, x_j) \tag{81}$$

where $E(\cdot, \cdot)$ is defined as in (79). To select the next design point x_{n+1}, the SMED algorithm wants to minimize the increase to the total potential energy of the design points, i.e.,

$$x_{n+1} = \arg\min_x \Delta E_{n+1}(x) = \arg\min_x \sum_{i=1}^{n} E(x, x_i). \tag{82}$$

Unfortunately, $\Delta E_{n+1}(x)$ is unobservable since it involves the values of yield $p(x)$ at unobserved points. To overcome the difficulty, it is natural to substitute the unknown value $p(x)$ by a prediction based on the observed points. In [11], an inverse distance weighting method (see Shepard [31]) is used to estimate $p(x)$ based on first n evaluated points $x_1, \ldots, x_n$:

$$\hat{p}^{(n)}(x) = \frac{\sum_{i=1}^{n} d^{-2}(x, x_i) p(x_i)}{\sum_{i=1}^{n} d^{-2}(x, x_i)}. \tag{83}$$

Correspondingly, define $\hat{q}^{(n)}(x) = (1 - \alpha \hat{p}^{(n)(x)})^{\gamma}$ as the predicted charge function at step n and let $\Delta \hat{E}_{n+1}(x)$ be the estimation $\Delta E_{n+1}(x)$ by replacing the charge function $q(x)$ with $\hat{q}^{(n)}(x)$. Then the SMED algorithm select the next design point x_{n+1} as follows.

$$x_{n+1} = \arg\min_{x} \Delta \hat{E}_{n+1}(x). \tag{84}$$

5.3 Asymptotic Properties of SMED

In this section we discuss the asymptotic properties of SMED algorithm. That is, suppose now that we run the algorithm without stopping and thus generate an infinite sequence of design points $\mathcal{E} = \{x_1, x_2, \ldots\}$. The denseness and distributions of $\mathcal{E}$ will be explored in the present section.

Let $\underline{q} = \min_{x \in \mathcal{X}} q(x)$ and $\bar{q} = \max_{x \in \mathcal{X}} q(x)$ be the lower and upper bounds of the charge function. By definition $0 \leq \underline{q} \leq \bar{q} \leq 1$. To eliminate singularities, we first assume $\underline{q} > 0$, which is equivalent to $\alpha < 1/p(x_g)$. The boundaries also apply to the predicted charge function $\hat{q}^{(n)}(x)$, i.e., $\underline{q} \leq \hat{q}^{(n)}(x) \leq \bar{q}$.

To investigate the denseness of $\mathcal{E}$, it turns out that the parameter β in (79) plays an important rule. The following theorem shows that $\mathcal{E}$ is dense when $\beta > m$.

Theorem 5.3.1. *If $\beta > m$, then $\overline{\mathcal{E}} = \mathcal{X}$.*

Proof. Suppose $\mathcal{E}$ is not dense in $\mathcal{X}$, then exist $x_0 \in \mathcal{X}$ and $r_0 > 0$ such that for any $x \in \mathcal{E}$, $d(x, x_0) > r_0$. Thus, at each step n, if x_0 had been picked as the $(n+1)^{th}$ design point, the potential energy increase would have been

$$\Delta \hat{E}_{n+1}(x_0) = \sum_{i=1}^{n} \frac{q(x_i)\hat{q}^{(n)}(x_0)}{d^{\beta}(x_0, x_i)} \leq \frac{n\bar{q}^2}{r_0^{\beta}}. \tag{85}$$

Take $K \in \mathbb{N}$ and evenly divide $\mathcal{X}$ into K^m m-dimensional small boxes with the form

$$\prod_{i=1}^{m} \left[\frac{k_i}{K}, \frac{k_i + 1}{K} \right], \quad k_i = 0, \ldots, K-1.$$

For any $N_0 \in \mathbb{N}$, there must exist one small box $\mathcal{X}_0$ such that more than $[N_0/K^m]$ design points in $\{x_1, \ldots, x_{N_0}\}$ fall into it. Denote these points by $x_{n_1}, x_{n_2}, \ldots, x_{n_j}$ where $j \geq [N_0/K^m]$ and $n_1 < n_2 < \cdots < n_j$. Therefore, when x_{n_j} is added into the set of design points, the total potential energy is increased by

$$\Delta \hat{E}_{n_j}(x_{n_j}) = \sum_{i=1}^{n_j - 1} \frac{q(x_i)\hat{q}^{(n_j-1)}(x_{n_j})}{d^\beta(x_i, x_{n_j})} \geq \sum_{i=1}^{j-1} \frac{q(x_{n_i})\hat{q}^{(n_j-1)}(x_{n_j})}{d^\beta(x_{n_i}, x_{n_j})} \geq m^{-\beta/2} \underline{q}^2 K^\beta ([N_0/K^m] - 1).$$

Compared with (85), because x_{n_j} is required to minimize $\Delta \hat{E}_{n_j}(x)$,

$$m^{-\beta/2} \underline{q}^2 K^\beta ([N_0/K^m] - 1) \leq (n_j - 1)\bar{q}^2 / r_0^\beta \leq N_0 \bar{q}^2 / r_0^\beta$$

which yields

$$[N_0/K^m] - 1 \leq A N_0 / K^\beta \tag{86}$$

where $A = \frac{m^{\beta/2}\bar{q}^2}{\underline{q}^2 r_0^\beta}$ is a finite constant. When $\beta > m$, pick $N_0 = K^{2\beta}$ and when K become sufficiently large, (86) will be violated. This completes the proof. □

The problem of the denseness of $\mathcal{E}$ becomes much more challenging when $\beta = m$. The following theorem shows that the conclusion of Theorem 5.3.1 still holds if $\beta = m = 1$. When $\beta = m > 1$, it is still an open problem whether $\mathcal{E}$ is dense, although we will provide a heuristic proof later.

Theorem 5.3.2. *If $\underline{q} > 0$ and $\beta = m = 1$ then $\overline{\mathcal{E}} = \mathcal{X}$.*

Proof. Now $\mathcal{X} = [0, 1]$. If $\overline{\mathcal{E}} \neq \mathcal{X}$, we can similarly find $x_0 \in \mathcal{X}$ and $r_0 > 0$ such that $|x - x_0| > r_0$ for any $x \in \mathcal{E}$. As in the proof of Theorem 5.3.1,

$$\Delta \hat{E}_i(x_i) \leq \Delta \hat{E}_i(x_0) \leq \frac{(i-1)\bar{q}^2}{r_0}$$

and

$$\sum_{i=2}^{n} \Delta \hat{E}_i(x_i) \leq \frac{n^2 \bar{q}^2}{2 r_0}. \tag{87}$$

Rearrange the first n design points $x_1, \ldots, x_n$ into an increasing sequence $y_1, \ldots, y_n$ such that $y_1 \le y_2 \le \cdots \le y_n$. Define $a_i = y_{i+1} - y_i \ge 0$ for $i = 1, \ldots, n-1$. Because $\underline{q}$ is the lower bound of both the charge function and the predicted charge functions

$$\sum_{i=2}^{n} \Delta \hat{E}_i(x_i) \ge \underline{q}^2 \sum_{1 \le i < j \le n} \frac{1}{|x_i - x_j|} = \underline{q}^2 \sum_{1 \le i < j \le n} \frac{1}{|y_i - y_j|}$$

Meanwhile

$$\sum_{1 \le i < j \le n} \frac{1}{|y_i - y_j|} = \sum_{k=1}^{n} \left(\frac{1}{a_1 + \cdots + a_k} + \frac{1}{a_2 + \cdots + a_{k+1}} + \cdots + \frac{1}{a_{n-k+1} + \cdots + a_n} \right).$$

For $1 \le k \le n$ and $1 \le j \le n - k + 1$ let $b_j = a_j + \cdots + a_{j+k-1}$, then easy to see $b_1 + \cdots + b_{n-k+1} \le k$ and as a result

$$\frac{1}{b_1} + \cdots + \frac{1}{b_{n-k+1}} \ge \frac{(n-k+1)^2}{k}$$

taking the sum for $k = 1, \ldots, n$

$$\begin{aligned} \sum_{1 \le i < j \le n} \frac{1}{|y_i - y_j|} &= \sum_{k=1}^{n} \frac{1}{b_1} + \cdots + \frac{1}{b_{n-k+1}} \\ &\ge \sum_{k=1}^{n} \frac{(n-k+1)^2}{k} = n^2 \sum_{k=1}^{n} (\frac{n-k+1}{n})^2 \frac{1}{k} \\ &\ge n^2 \sum_{k=1}^{[n/2]} \frac{1}{4k} \ge n^2 \log(n/2 - 1)/4. \end{aligned}$$

So

$$\sum_{i=2}^{n} \Delta \hat{E}_i(x_i) \ge \underline{q}^2 n^2 \log(n/2 - 1)/4.$$

Compared with (87),

$$\underline{q}^2 n^2 \log(n/2 - 1)/4 \le \frac{n^2 \bar{q}^2}{2r_0}.$$

The inequality will be violated when n becomes sufficiently large, which completes the proof. □

Finally, we provide some heuristic arguments on the asymptotic distribution of the design points in $\mathcal{E}$ and the denseness of $\mathcal{E}$ when $\beta = m > 1$. Let us begin with a heuristic assumption that is fundamental to our following up analysis.

Assumption 4. *The set $\mathcal{E}$ of SMED design points has a positive density function $\rho(x)$ such that for any open subset $B \subset \mathcal{X}$ the ratio of design points falling into B converges to the integration of $\rho(\cdot)$ over B. To be more specific, let*

$$cnt(B, n) = \sum_{i=1}^{n} I\{x_i \in B\}.$$

Then

$$\lim_{n\to\infty} \frac{cnt(B, n)}{n} = \int_B \rho(x)dx.$$

Assumption 4 itself is hard to prove rigorously but can be checked numerically via simulations. Below we illustrate that Assumption 4 can lead to other useful conclusions. We will consider two scenarios, depending on whether $\beta > m$ or $\beta = m$.

The following heuristic property deals with the density $\rho(x)$ for the SMED design points when $\beta > m$, and it is consistent with the intuition that SMED algorithm puts more points into regions with high yields, and the closer α is to $p(x_g)$, the more concentrated the design points get around the global optimum x_g.

Property 1. *Under Assumption 4, when $\underline{q} > 0$ and $\beta > m$, the density function $\rho(x)$ is proportional to $q(x)^{-\frac{2m}{\beta}} = (1 - \alpha p(x))^{-\frac{2m\gamma}{\beta}}$.*

Heuristic Proof. First we argue heuristically that, using the predictor $\hat{p}^{(n)}(x)$ to estimate the yield function $p(x)$ is irrelevant to the long term behaviour of $\mathcal{E}$ when $\beta > m$. That is, even if we use the original definition (82) to search for the design points, the truthfulness of a claim like Property 1 remains the same. The reason is that because when $\beta > m$ the set $\mathcal{E}$ is dense, after a large but finite amount of steps, such estimation eventually becomes so precise that its difference from the true yield function $p(x)$ can be safely ignored. Thus we assume that in every step of SMED, the next design point is selected by (82) instead of (84).

Let n be a large number. Suppose the $(n + 1)^{th}$ point is placed at position y. We first make a heuristic estimation to $\Delta E_{n+1}(y)$. Pick $\delta > 0$ as a small but fixed positive number such that when $\|y - x\| \leq \delta$, the differences $\rho(y) - \rho(x)$ and $q(x) - q(y)$ are negligibly small. By definition of $\rho(y)$, the number of design points in the open neighborhood

$B(y;\delta) = \{x : \|x - y\| < \delta\}$ and selected before step n can be estimated as $Cn\rho(y)\delta^\beta$ where C is the universal constant such that $C\delta^\beta$ gives the volume of $B(y;\delta)$. When n is large, because these design points are distributed roughly evenly in this small region, their smallest distance to point y can be heuristically estimated as

$$\varepsilon(y,n) \sim (C_1 n\rho(y))^{-\frac{1}{m}}. \tag{88}$$

Here C_1 is another positive constant. Now

$$\begin{aligned}\Delta E_{n+1}(y) &= \sum_{i=1}^{n} \frac{q(x_i)q(y)}{\|x_i - y\|^\beta} \\ &\sim \sum_{1\le i\le n, x_i\in B(y;\delta)} \frac{q(y)^2}{\|x_i - y\|^\beta} + \sum_{1\le i\le n, x_i\notin B(y;\delta)} \frac{q(y)q(x_i)}{\|x_i - y\|^\beta}.\end{aligned}$$

An upper bound of the second term is $nq(y)\bar{q}/\delta^\beta$, and the first term can be estimated by the following integration when n becomes large

$$\begin{aligned}\sum_{1\le i\le n, x_i\in B(y;\delta)} \frac{q(y)^2}{\|x_i - y\|^\beta} &= \int_{\varepsilon(y,n)}^{\delta} \frac{q(y)^2 n\rho(y)}{r^\beta} mCr^{m-1}dr \\ &= C_2 n^{\frac{\beta}{m}}\rho(y)^{\frac{\beta}{m}}q(y)^2 - \frac{C_3}{\delta^{\beta-m}} n\rho(y)q(y)^2.\end{aligned}$$

Here C_2 and C_3 are both positive constants. When $\beta > m$ and $n \to \infty$, the first term eventually dominates, so

$$\Delta E_{n+1}(y) \sim C_2\rho(y)^{\frac{\beta}{m}}q(y)^2 n^{\frac{\beta}{m}}.$$

Such heuristic estimation implies that to lower the overall potential energy, SMED must put new design points to a place where $\rho(x)^{\frac{\beta}{m}}q(x)^2$ is small. Eventually, when equilibrium is reached, $\rho(x)^{\frac{\beta}{m}}q(x)^2$ will be a constant over $\mathcal{X}$, which means $\rho(x) \sim q(x)^{-\frac{2m}{\beta}}$. □

Now let us consider our heuristic results for the case of $\beta = m$ under Assumption 4.

Property 2. *If Assumption 4 holds, then the set $\mathcal{E}$ is dense in $\mathcal{X}$ when $\beta = m \ge 1$. Furthermore, $\rho(x) \sim q(x)^2$.*

Heuristic Proof. Here we only explain the denseness of $\mathcal{E}$ because the techniques for computing $\rho(x)$ are the same as in the heuristic proof of Property 1.

Let's suppose $\overline{\mathcal{E}} \neq \mathcal{X}$. Then as in proof of Theorem 5.3.1, exist a point $x_0 \notin \mathcal{E}$ and $r_0 > 0$ such that for any $x \in \mathcal{E}$, $\|x - x_0\| > r_0$. This implies that for any $n \geq 1$

$$\Delta \hat{E}_{n+1}(x_0) \leq \frac{n\bar{q}^2}{r_0^\beta}.$$

Pick $y \in \mathcal{X}$ such that $\rho(y) > 0$, $\delta > 0$ and when $\|x - y\| \leq \delta$, $\rho(x) - \rho(y)$ is negligibly small. Then for any large N, exists $n \geq N$ such that $\|x_{n+1} - y\| < \delta' << \delta$. Let $\varepsilon = \varepsilon(y, n)$ as defined in (88). For such n,

$$\begin{aligned}
\Delta \hat{E}_{n+1}(x_{n+1}) &\sim \Delta \hat{E}_{n+1}(y) \sim \int\limits_{\|x-y\| \geq \varepsilon} \frac{n\rho(x)q(x)q(y)}{\|x-y\|^\beta} dx \\
&\geq \int\limits_{\varepsilon \leq \|x-y\| \leq \delta} \frac{n\rho(x)q(x)q(y)}{\|x-y\|^\beta} dx \sim C \int\limits_{\varepsilon}^{\delta} \frac{n\rho(y)q(y)^2}{r^\beta} r^{m-1} dr \\
&= Cn\rho(y)q(y)^2(\log \delta - \log \varepsilon) \sim Cn\rho(y)q(y)^2 \left(\log n + \log \rho(y) + \log \delta\right)/m \\
&\sim C_1 n \log n \rho(y) q(y)^2 \leq \frac{n\bar{q}^2}{r_0^\beta} = \Delta \hat{E}_{n+1}(x_0)
\end{aligned}$$

when n gets sufficiently large, there is a contradiction. □

It is interesting that such arguments can not be applied to the case $\beta < m$. The fundamental reason is the curse of dimensionality. The basic idea of SMED is that when the design points become very "crowded" in certain places, the potential energy they generate will be sufficiently large to expel away any new points. However, when $m > \beta$, the volumes of such crowed sub-regions are so small that, they actually contain relatively few design points. Consequently, if the yields in these regions are sufficiently high, it is still optimum for new design points to drop in. Meanwhile, in a sub-region with relatively low yields, it may never be beneficial for SMED to select design points from it. The result is that the set $\mathcal{E}$ may not be dense. This phenomenon can be observed via simulation.

The non-denseness of SMED design points can be viewed as either positive or negative. Theoretically, if the points are not dense, then there are certain regions ignored by the algorithm, which means that the method can be "biased", i.e., it may never find the global optimum in some circumstances. However, by our above analysis, the regions omitted by SMED tend to be those with low yields. Unless there are high yield regions hiding in large

chunks of low yield ones, which is uncommon in real applications, the non-denseness of SMED makes it not waste time in non-promising regions, which is often desirable.

5.4 A Simplification of the SMED Algorithm

From Section 5.2 and Section 5.3, a complication in SMED algorithm is that the yield function $p(x)$ must be re-estimated at each step, which may increase the computational complexity. In this section, we present a simplified algorithm (which we refer to as the adjusted SMED) where such prediction is not involved. We will refer to the original algorithm as ordinary SMED.

There is only a small difference between the adjusted SMED and the ordinary SMED. For $n = n_0, n_0 + 1, \ldots$, when seeking for a position to lay down the design point x_{n+1}, we assign it the unit charge $q = 1$ instead of the position-dependent charge $q(x)$. Then we minimize the increase of total potential energy as before. Only after the optimal position is found for the new point we restore its charge by evaluating the yield function at the chosen place.

Specifically, if point x is added as the $(n+1)^{\text{th}}$ design point, define the increase of total potential energy by

$$\Delta\tilde{E}_{n+1}(x) = \sum_{i=1}^{n} \frac{q(x_i)}{d^{\beta}(x_i, x)} \tag{89}$$

and let

$$x_{n+1} = \arg\min_{x} \Delta\tilde{E}_{n+1}(x).$$

Compared with the ordinary SMED, the adjusted version never evaluates or estimates $p(x)$ at any untested points. The computational costly interpolation (83) is excluded from the algorithm.

The adjusted SMED algorithm has very similar asymptotic properties with the ordinary version. It is easy to see that Theorem 5.3.1 and Theorem 5.3.2 still apply. Indeed, for ordinary SMED algorithm, the only property we used in proving the two Theorems is that $\overline{q} \leq \hat{q}^{(n)}(x) \leq \bar{q}$, where $\hat{q}^{(n)}(x)$ is the charge function we use when looking for the $(n+1)^{\text{th}}$ design point. In adjusted SMED, the charge function we use is a constant $q \equiv 1$, and actually we are free to choose any constant function as long as it takes a positive value.

Therefore, suppose we use a constant charge function $q \equiv C$ such that $\overline{q} \leq C \leq \bar{q}$, then the same proofs go through.

For Property 1 and Property 2, we only need minor modifications for the adjusted SMED.

Property 3. *With the adjusted SMED algorithm, when $\beta \geq m \geq 1$ and $\underline{q} > 0$, the density function $\rho(x)$ is proportional to $q(x)^{-\frac{m}{\beta}} = (1 - \alpha p(x))^{-\frac{m\gamma}{\beta}}$.*

The heuristic proof is the same as that of Property 1 and thus omitted.

5.5 Simulation Study

In this section we validate the heuristic properties we proposed in Section 5.3 and Section 5.4 via numerical simulations. We focus on case where the design region $\mathcal{X} = [0, 1] \times [0, 1]$ is the two dimensional unit square, i.e., $m = 2$. We place the initial point $x_1 = (0.5, 0.5)$ in the center. To simplify the computation, we evenly divide the region into 1000×1000 lattices and will pick each design point on a lattice point.

By Property 1 and Property 2, when $\beta \geq m$, the design points are asymptotically distributed according to a density function $\rho(x) \sim q(x)^{-\frac{2m}{\beta}} = (1 - \alpha p(x))^{-\frac{2m\gamma}{\beta}}$. To make notation clear, let's continue our presentation in terms of the charge function $q(x)$ instead of the yield function $p(x)$. Therefore, for a constant charge $q(x) \equiv 1$, the design points should distribute evenly. Note that in this case the ordinary SMED and the adjusted SMED are equivalent. To verify this, we compute the first $N = 10^4$ design points with constant charge and the parameter $\beta = 2, 3$. Then we construct a 2-dimensional histogram by dividing $\mathcal{X}$ into 10×10 bins and count the number of design points falling into each of them. The numerical results are reported in Table 2 and Table 3. Each item of the tables gives number of design points falling into the corresponding bin.

From the histograms, when the charge functions are constant, the design points are distributed fairly uniformly. Deviations from the uniform distribution are observed only at bins along the boundaries. This is reasonable because a point close to a boundary is surounded by much less other points if compared with center points. It can be predicted that if we use a periodic boundary condition, such deviations will disappear. Indeed, this

Table 2: Histogram: constant charge, $\beta = 3, m = 2, N = 10^4$

119	109	111	111	106	114	105	98	103	116
104	96	98	101	99	100	94	89	97	106
98	92	103	101	100	98	94	97	97	114
100	94	99	97	100	103	97	104	95	104
113	94	101	102	97	103	97	99	96	104
102	95	91	96	94	103	97	99	91	103
104	99	100	101	101	98	95	92	93	96
106	99	102	94	100	89	92	98	95	97
106	91	92	95	94	95	91	98	95	97
117	98	102	105	109	103	103	103	102	113

Table 3: Histogram: constant charge, $\beta = 2, m = 2, N = 10^4$

141	118	117	115	114	117	114	115	118	144
118	94	91	94	90	93	93	92	93	120
116	91	88	86	88	87	85	91	93	114
117	91	88	90	85	86	87	85	92	116
114	91	89	83	84	86	90	83	92	117
113	92	86	88	85	87	86	89	91	115
119	88	90	85	84	85	89	82	91	118
113	93	89	90	82	87	85	90	93	114
122	98	94	92	92	89	92	96	97	120
140	119	114	116	116	116	117	116	120	139

Table 4: Histogram: constant charge, periodic boundary, $\beta = 3, m = 2, N = 10^4$

96	92	88	80	99	90	107	100	103	107
104	84	105	91	96	110	110	90	83	106
108	110	91	95	91	92	114	94	110	108
104	106	102	80	98	94	105	97	106	110
93	116	113	103	101	110	96	118	103	123
104	110	104	100	117	78	103	97	86	84
110	99	86	103	107	92	106	102	98	116
123	99	108	101	102	90	121	111	93	67
97	100	107	106	91	105	90	97	96	95
105	102	92	103	97	81	97	94	93	103

Table 5: Histogram: constant charge, periodic boundary, $\beta = 2, m = 2, N = 10^4$

104	96	87	92	114	78	93	98	110	101
102	95	90	107	103	99	111	111	90	103
116	94	102	87	92	116	107	103	107	98
107	93	103	93	130	109	108	100	114	97
96	110	103	84	96	99	85	111	96	98
97	118	91	105	108	102	120	94	98	100
116	85	102	97	79	100	118	86	87	117
92	109	98	107	72	68	107	99	100	91
84	99	104	105	117	113	102	103	93	96
95	87	112	88	110	88	113	92	108	90

can be seen from Table 4 and Table 5, which report the corresponding results with periodic boundary conditions.

Next we check the distributions of design points as related to the charge functions. By heuristic Property 1, Property 2, when $\beta \geq m$, the density functions $\rho(x)$ is proportional to $q(x)^{-\frac{2m}{\beta}}$. To check this, suppose we divide the 2-dim design region $\mathcal{X}$ into two parts, the upper part $y < 0.5$ and the lower part $y > 0.5$. Then we assign a unit charge function to any design point selected from the lower part, i.e., $q_2 = 1$ but a constant positive charge $q_1 \neq 1$ to the upper part. By Property 1 and Property 2, the distribution functions should be constant in either of the two parts except along the boundaries. In fact, this feature can be easily seen from the histogram presented below in Table 6 and Table 7, where $q_1 = 2$, $\beta = 2, 3$. Note that here we are presenting the simulation results for ordinary SMED, and to reduce the computation complexity we do not predict the yield function but directly use (82) to select the design points. As we argued in the heuristic proof of Property 1, this does

Table 6: Histogram: two charges, $q_1 = 2, \beta = 3, m = 2, N = 10^4$

64	61	59	61	61	58	60	55	60	63
59	54	52	55	55	53	53	57	52	62
61	54	52	51	53	55	55	53	56	57
61	56	56	51	54	53	52	54	53	62
55	46	51	50	50	47	48	51	49	53
151	150	138	143	146	159	159	149	143	150
144	145	139	139	150	155	141	137	132	142
157	141	137	141	147	141	137	134	136	149
154	139	141	142	135	138	137	137	144	150
157	150	153	147	143	140	143	149	157	159

Table 7: Histogram: constant charge, periodic boundary, $\beta = 2, m = 2, N = 10^4$

66	54	53	50	53	51	50	54	54	66
52	43	38	39	36	38	38	39	42	55
52	39	38	38	36	37	36	37	38	50
49	39	33	35	34	32	36	33	38	50
37	26	21	21	23	19	23	22	24	38
192	171	160	159	157	165	156	167	165	196
180	144	140	143	137	142	140	138	147	177
176	139	142	142	139	137	137	140	147	174
180	151	145	142	141	142	145	141	149	182
211	177	175	173	177	175	172	176	185	207

not affect the long term properties of the algorithm.

Let the densities of design points in this two sub-regions be ρ_1 and ρ_2, respectively. By the law of distribution we developed in Property 1 and Property 2, we should have

$$\log \frac{\rho_2}{\rho_1} = \frac{2m}{\beta} \log \frac{q_1}{q_2} = \frac{2m}{\beta} \log q_1.$$

Therefore, one way to check the law is as follows. For different values of $q_{1,i}$, do the simulation and then calculate the log-ratio $\log r_i = \log \frac{\rho_{2,i}}{\rho_{1,i}}$, and see if the points $\{(\log q_{1,i}, \log r_i)\}$ are aligned on a line with zero intercept and a slope close to $\frac{2m}{\beta}$.

Fig. 3 depicts the relevant simulation results. For $\beta = 3$ and $\beta = 2$, the distribution densities $\rho_{1,i}$ and $\rho_{2,i}$ are computed via counting the numbers of design points in each of the two sub-regions. Then a line through the origin is fit with simulated $\{(\log q_{1,i}, \log r_i)\}$ for each case, the slopes of the fitted lines are reported and compared to $\frac{2m}{\beta}$. From the figures, it is clear that the points $\{(\log q_{1,i}, \log r_i)\}$ align very well along the fitted lines with slopes quite close to the theoretical values predicted by Property 1 and Property 2.

Table 8: Histogram: high charge in the center, $\beta = 1, m = 2, N = 10^4$

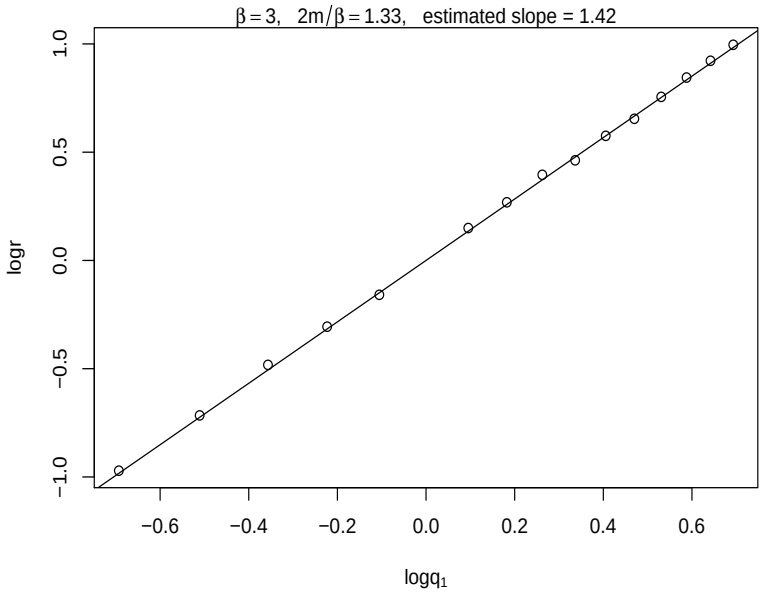
313	183	166	158	155	152	160	163	184	314
183	81	69	69	64	67	68	70	82	182
164	69	58	76	80	84	76	59	70	166
159	70	75	0	0	0	0	75	69	160
155	63	82	0	0	0	0	83	62	154
153	64	85	0	0	1	0	84	66	154
160	69	75	0	0	0	0	76	68	160
164	68	60	76	80	84	76	59	70	166
183	82	70	69	64	67	67	69	83	183
314	182	166	160	152	157	161	165	182	314

For the adjusted SMED algorithm we can use the same method to check the validity of Property 3. The result is shown by Fig. 4, which again supports our result.

Now we provide a simulated example to show that when $\beta < m$, it is indeed possible that the set $\mathcal{E}$ of design points is not dense in $\mathcal{X}$. To do so, we work with ordinary SMED and $m = 2$, i.e., $\mathcal{X} = [0, 1] \times [0, 1]$. We set $\beta = 1$ and pick a charge function as follows.

$$q(x) = \begin{cases} 2, & \text{if } 0.3 < x < 0.7, \quad 0.3 < y < 0.7 \\ 1, & \text{otherwise.} \end{cases} \tag{90}$$

Now we run the simulation to lay down $N = 10^4$ design points, and report the histogram in Table 8. Except the initial point, there's not a single point dropping into the region with high charge (thus low yield). This is a strong evidence that the design points $\mathcal{E}$ is not dense in $\mathcal{X}$.



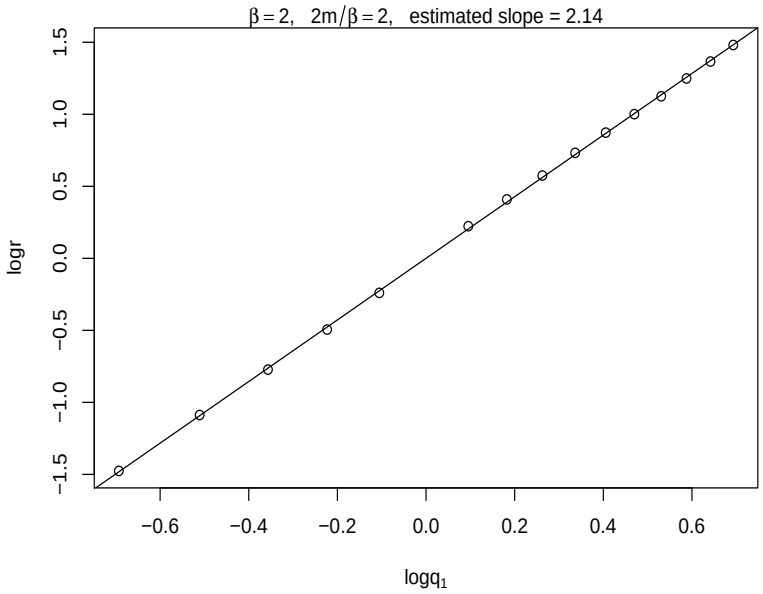


Figure 3: Distribution Law for Ordinary SMED

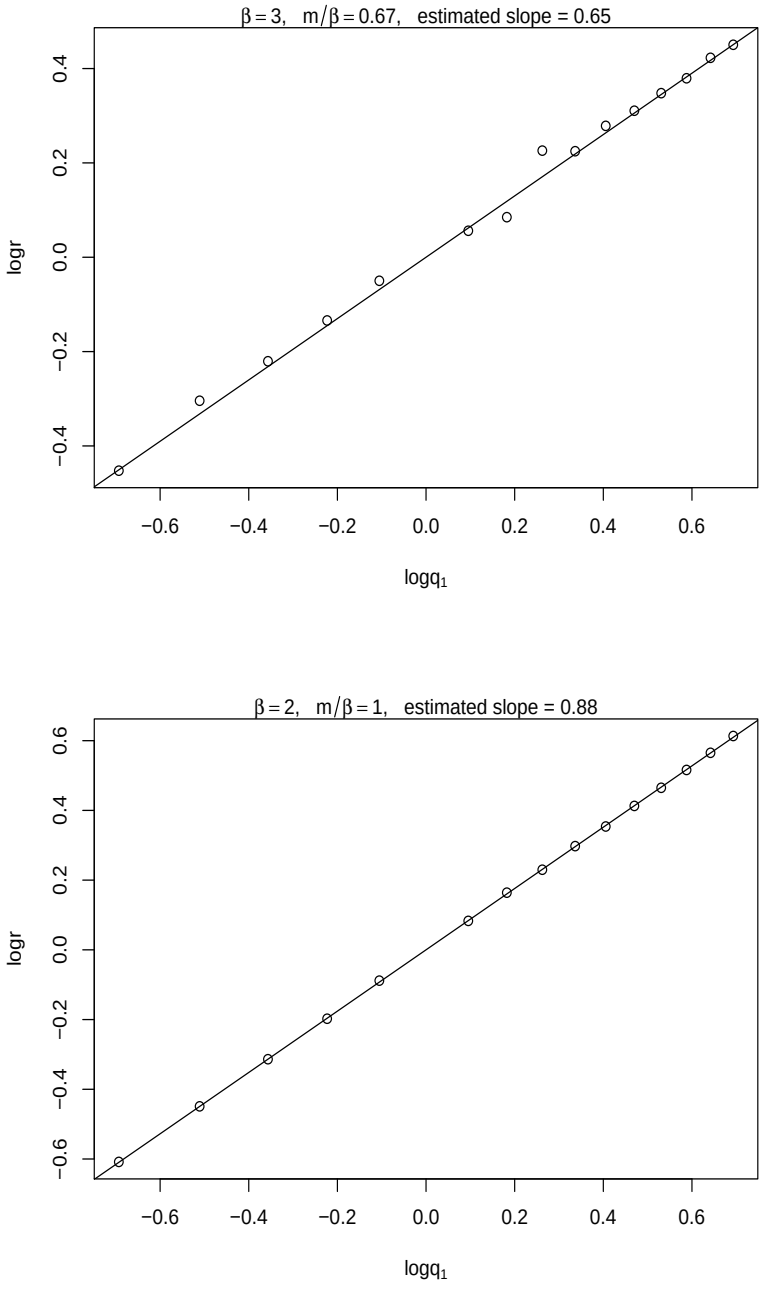


Figure 4: Distribution Law for Adjusted SMED

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