ALGEBRAIC DEGREES OF STRETCH FACTORS IN MAPPING CLASS GROUPS

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Soli Deo Gloria

To Sara and my family

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SUMMARY

Given a closed surface S_g of genus g, a mapping class f in $\mathrm{Mod}(S_g)$ is said to be pseudo-Anosov if it preserves a pair of transverse measured foliations such that one is expanding and the other one is contracting by a number $\lambda(f)$. The number $\lambda(f)$ is called a stretch factor (or dilatation) of f. Thurston showed that a stretch factor is an algebraic integer with degree bounded above by $6g-6$. However, little is known about which degrees occur.

Using train tracks on surfaces, we explicitly construct pseudo-Anosov maps on S_g with orientable foliations whose stretch factor λ has algebraic degree $2g$. Moreover, the stretch factor λ is a special algebraic number, called Salem number. Using this result, we show that there is a pseudo-Anosov map whose stretch factor has algebraic degree d, for each positive even integer d such that $d \leq g$. Our examples also give a new approach to a conjecture of Penner.

CHAPTER I

INTRODUCTION

Let S_g be a closed surface of genus $g \geq 2$. The *mapping class group* of S_g, denoted $\mathrm{Mod}(S_g)$, is the group of isotopy classes of orientation preserving homeomorphisms of S_g. An element $f \in \mathrm{Mod}(S_g)$ is called a *pseudo-Anosov* mapping class if there are transverse measured foliations $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$, a number $\lambda(f) > 1$, and a representative homeomorphism ϕ such that

$$\phi(\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda(f)\mu_u) \quad \text{and} \quad \phi(\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda(f)^{-1}\mu_s).$$

In other words, ϕ stretches along one foliation $\mathcal{F}^u$ by $\lambda(f)$ and the other one $\mathcal{F}^s$ by $\lambda(f)^{-1}$. Foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ are called *unstable* and *stable* foliations, respectively. The number $\lambda(f)$ is called the *stretch factor* (or *dilatation*) of f.

A pseudo-Anosov mapping class is said to be orientable if its invariant foliations are orientable. Let $\lambda_H(f)$ be the spectral radius of the action of f on $H_1(S_g; \mathbb{R})$. Then

$$\lambda_H(f) \leq \lambda(f),$$

and the equality holds if and only if the invariant foliations for f are orientable (see [11]). The number $\lambda_H(f)$ is called the *homological stretch factor* of f.

Question. *Which real numbers can be stretch factors?*

It is a long-standing open question. Fried conjectured that $\lambda > 1$ is a stretch factor if and only if all conjugate roots of λ and $1/\lambda$ are strictly greater than $1/\lambda$ and strictly less than λ in magnitude.

Thurston [20] showed that a stretch factor λ is an algebraic integer whose algebraic degree has an upper bound $6g - 6$. More specifically, λ is the largest root in absolute value of a monic palindromic polynomial. Thurston gave a construction of mapping classes of $\mathrm{Mod}(S_g)$ generated by two multitwists and he mentioned that his construction can make a

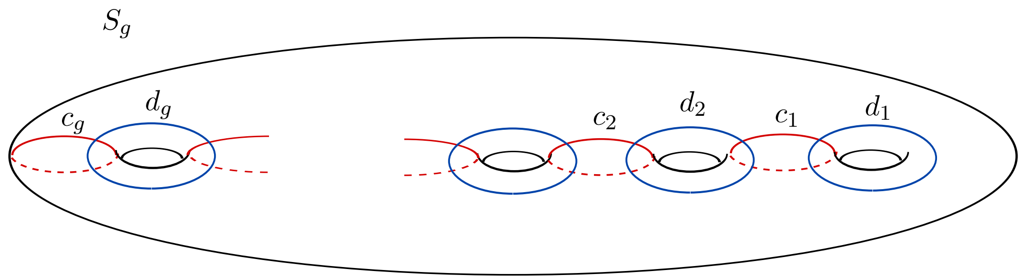

Figure 1: Simple closed curves on S_g

pseudo-Anosov mapping class whose stretch factor has algebraic degree $6g-6$. However, he did not give specific examples.

What happens if we fix the genus g? To simplify the question, we may ask which algebraic degrees are possible on S_g.

Question. *What degrees of stretch factors can occur on S_g?*

Very little is known about this question. Using Thurston's construction, it is easy to find quadratic integers as stretch factors. Neuwirth and Patterson [16] found non-quadratic examples, which are algebraic integers of degree 4 and 6 on surfaces of genus 4 and 6, respectively. Using interval exchange maps, Arnoux and Yoccoz [2] gave the first generic construction of pseudo-Anosov maps whose stretch factor has algebraic degree g on S_g for each $g \geq 2$.

In this thesis, we give a generic construction of pseudo-Anosov mapping classes with stretch factor of algebraic degree $2g$.

1.1 Construction of mapping classes

Let us define a mapping class $f_g \in \mathrm{Mod}(S_g)$ by

$$f_g = (T_{c_g})^3(T_{c_g}T_{d_g}\cdots T_{c_2}T_{d_2}T_{c_1}T_{d_1}),$$

where c_i and d_i are simple closed curves as in Figure 1 and T_c is the Dehn twist about c. We will show that f_g is a pseudo-Anosov mapping class and its stretch factor $\lambda(f_g)$ is a special algebraic integer, called Salem number. A *Salem number* is an algebraic integer $\alpha > 1$ whose Galois conjugates other than α have absolute value less than or equal to 1 and at least one conjugate lies on the unit circle.

Theorem 1 (Main Theorem)**.** *For each $g \geq 2$, f_g is a pseudo-Anosov mapping class and satisfies the following properties:*

1. $\deg \lambda(f_g) = 2g$,
2. $\lambda(f_g) = \lambda_H(f_g)$, *and*
3. $\lambda(f_g)$ *is a Salem number.*

The degree of the stretch factor of a pseudo-Anosov mapping class $f \in \mathrm{Mod}(S_g)$ with orientable foliations is bounded above by $2g$ (see [20]). Therefore our examples give the maximum degrees of stretch factors for orientable foliations in $\mathrm{Mod}(S_g)$ for each g.

The hard part is to show the irreducibility of the minimal polynomial of $\lambda(f_g)$. We will show that all conjugate roots of $\lambda(f_g)$ except $\lambda(f_g)^{-1}$ are on the unit circle and none of them are roots of unity.

Using a branched cover construction, we use the Main Theorem to deduce the following partial answer to our question about algebraic degrees.

Corollary 8. *For each positive integer $h \leq g/2$, there is a pseudo-Anosov mapping class $\tilde{f}_h \in \mathrm{Mod}(S_g)$ such that $\deg(\lambda(\widetilde{f_h})) = 2h$ and $\lambda(\widetilde{f_h})$ is a Salem number.*

1.2 Obstructions

There are three known obstructions for the existence of algebraic degrees. For any pseudo-Anosov mapping class $f \in \mathrm{Mod}(S_g)$, we have:

1. $\deg \lambda(f) \geq 2$,
2. $\deg \lambda(f) \leq 6g - 6$, and
3. if $\deg \lambda(f) > 3g - 3$, then $\deg \lambda(f)$ is even.

The third obstruction is due to Long [13] and we have another proof in section 3.4. It turns out these are the only obstructions for $g = 2$. However it is not known whether there are other obstructions of algebraic degrees for $g \geq 3$. By computer search, odd degree stretch factors are rare compared to even degrees. We conjecture that every even degree $d \leq 6g-6$ can be realized as the algebraic degree of stretch factors.

Conjecture. *On S_g, there exists a pseudo-Anosov mapping class with a stretch factor of algebraic degree d for each positive even integer $d \leq 6g-6$.*

In chapter 4, we show that the conjecture is true for $g = 2, 3, 4$, and 5.

1.3 Outline

In chapter 2, we will give the basic definitions and results about train tracks and Perron–Frobenius theory. Chapter 3 contains the main result; in section 3.1 we will prove the main theorem by finding a train track and describing its action. In section 3.2, we construct pseudo-Anosov mapping classes via branched covers. Section 3.3 contains a new approach to Penner's conjecture using our examples. In section 3.4, we explain some properties of odd degree stretch factors. Section 3.5 is where we prove that the minimal polynomial of $\lambda(f_g)$ has degree $2g$. Chapter 4 contains examples of even degree stretch factors for $g = 2, 3, 4$ and 5.

CHAPTER II

BACKGROUND

We begin by recalling the definitions and results about train tracks and Perron–Frobenius theorem. See [18], [19], or [17] for more discussion.

2.1 Train tracks and Perron–Frobenius theorem.

A *weighted train track* on a surface S is a smooth 1-complex τ whose vertices are called *switches* and whose edges are called *branches* satisfying the following conditions.

1. Each branch is smooth and and each switch of τ is at least tri-valent.

2. At each switch s there is a well-defined unique tangent line. The set of edges incident on the switch is partitioned into two disjoint sets depending on the direction of the one-side tangent at the switch. We arbitrarily choose to call one set of this partition *incoming* and the other *outgoing*.

3. Each component C of $S - \tau$ has negative cusped Euler index, i.e.,

$$\chi(C) - \frac{1}{2}\#(\text{cusps}) < 0,$$

   where a *cusp* is a corner at a vertex that is formed by two edges having the same one-side tangent direction.

4. Each branch of τ is labelled with a nonnegative integer called a *weight*. The weights satisfy the switch conditions; at each switch, the sum of weights on incoming edges is equal to the sum of weights on outgoing edges.

The set μ of weights on τ is called a *measure*. We can think of μ as a function μ : $\{\text{branches of } \tau\} \to [0, \infty)$ satisfying the switch conditions. We say a train track is *recurrent* if it supports a positive measure. Some examples of recurrent train tracks are in Figure 2.

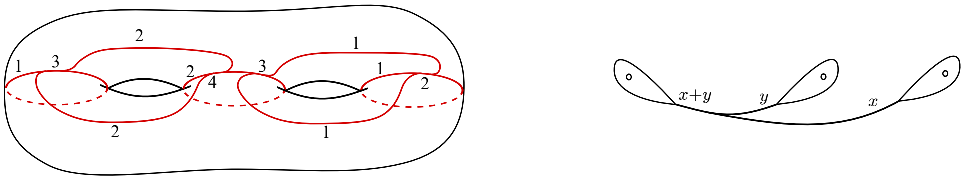

Figure 2: Weighted train tracks on S_2 and 3-punctured plane

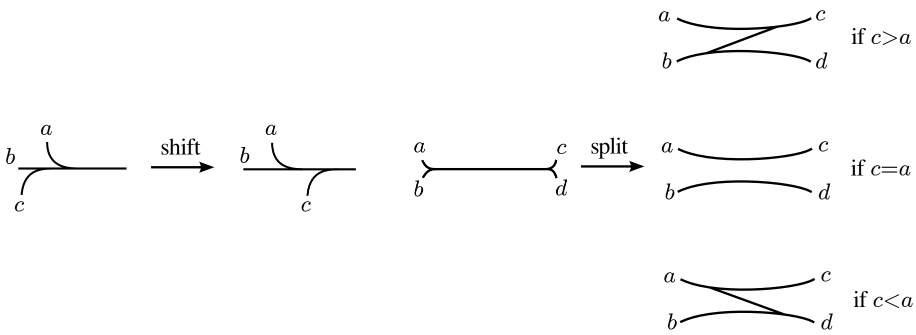

Figure 3: Moves on train tracks

Shifting and *splitting* are moves that can be performed on weighted train tracks as in Figure 3. The inverse of a split is called a *collapse*. A train track is said to be *transversely recurrent* if for any branch b of τ there is an essential simple closed curve α so that α intersects b and no component of $S - (\alpha \cup \tau)$ is a bigon. (A simple closed curve is said to be *essential* if it is not null homotopic.) We say a train track is *birecurrent* if it is both recurrent and transversely recurrent. It turns out we can always split a recurrent train track to a birecurrent one (see [17, Corollary 2.7.3]).

Let $f \in \mathrm{Mod}(S)$ be a mapping class and let τ be a weighted train track such that $f(\tau)$ collapses to τ. Then the action of f on weights can be described by an nonnegative integral matrix called the *transition matrix*. In this case, the spectral radius of the transition matrix gives information about the mapping class. In particular, Penner [18] states a criterion for pseudo-Anosov mapping classes. It requires one more ingredient, the Perron–Frobenius theorem.

A matrix M is said to be *positive* (or *nonnegative*) if each of its entries is positive (or nonnegative). We write $M > 0$ (or $M \geq 0$). A nonnegative matrix is said to be *primitive* or *Perron–Frobenius* if it has a power that is a positive matrix.

Theorem 2 (Perron–Frobenius theorem)**.** *Let M be an $n \times n$ nonnegative integral matrix.*

If M is Perron–Frobenius, then M has a unique positive real eigenvalue λ that is strictly bigger than all other eigenvalues in magnitude.

The eigenvalue in the theorem is called the *Perron–Frobenius eigenvalue* (or PF–eigenvalue) for M.

We are now ready to state Penner's criterion.

Theorem 3. *[18, Corollary 3.2] Given a mapping class f of a surface S of negative Euler characteristic, let ϕ be a representative of f. If there is a birecurrent train track $\tau \subset S$ filling S so that τ splits to $\phi(\tau)$ with Perron–Frobenius transition matrix, then f is a pseudo-Anosov mapping class.*

2.2 Nonnegative matrices and directed graphs

To apply Theorem 3, we need a criterion for showing that the transition matrix of a train track is Perron–Frobenius. We will introduce an equivalent condition using directed graphs.

The *directed graph associated to the nonnegative matrix M* of size $n \times n$ is a directed graph that has the vertex set $V = \{v_1, \cdots, v_n\}$ and for each i and j, there are m_{ij} directed edges from v_i to v_j.

A directed *path* from a vertex v to a vertex w is a finite sequence $v_0, \cdots, v_k$ with $v_0 = v, v_k = w$ where each (v_i, v_j) is an directed edge. The number k, i.e., the number of directed edges in the directed path is called the *length* of the directed path. The directed graph is said to be *strongly connected* if for any two vertices v_i and v_j, there is a directed path joining v_i and v_j. Now we give a graph-theoretical description of the Perron–Frobenius condition.

Theorem 4. *A nonnegative matrix M is Perron–Frobenius if and only if the graph associated to M is strongly connected and has two cycles of relatively prime lengths.*

CHAPTER III

CONSTRUCTION OF PSEUDO-ANOSOV MAPPING CLASSES

In this chapter, we will prove our main theorem. Let us recall Theorem 1.

Let $f_g \in \mathrm{Mod}(S_g)$ be defined by

$$f_g = (T_{c_g})^3 (T_{c_g} T_{d_g} \cdots T_{c_2} T_{d_2} T_{c_1} T_{d_1}),$$

where c_i and d_i are simple closed curves as in Figure 1 and T_c is the Dehn twist about c.

Theorem 1. *For each $g \geq 2$, f_g is a pseudo-Anosov mapping class and satisfies the following properties:*

1. $\deg \lambda(f_g) = 2g$,
2. $\lambda(f_g) = \lambda_H(f_g)$, *and*
3. $\lambda(f_g)$ *is a Salem number.*

3.1 Proof of Main Theorem

3.1.1 Proof that f_g is pseudo-Anosov

We will first find a train track for $f_g \in \mathrm{Mod}(S_g)$ and show that f_g is a pseudo-Anosov mapping class using Theorem 3.

Let us consider the hyperelliptic involution of S_g, that is, the rotation by π about the axis indicated in Figure 4. The quotient of S_g with respect to this involution is topologically a sphere $\bar{S}$ with $2g+2$ marked points (branch points). Furthermore the Dehn twists about the simple closed curves c_i and d_i fixed by the involution correspond to the half twists γ_i and δ_i about two marked points on the sphere as in Figure 4.

Hence the mapping class $\bar{f}_g$ on $\bar{S}$ corresponding to f_g is the product of the half twists:

$$\bar{f}_g = (\gamma_g)^3 (\gamma_g \delta_g \cdots \gamma_2 \delta_2 \gamma_1 \delta_1).$$

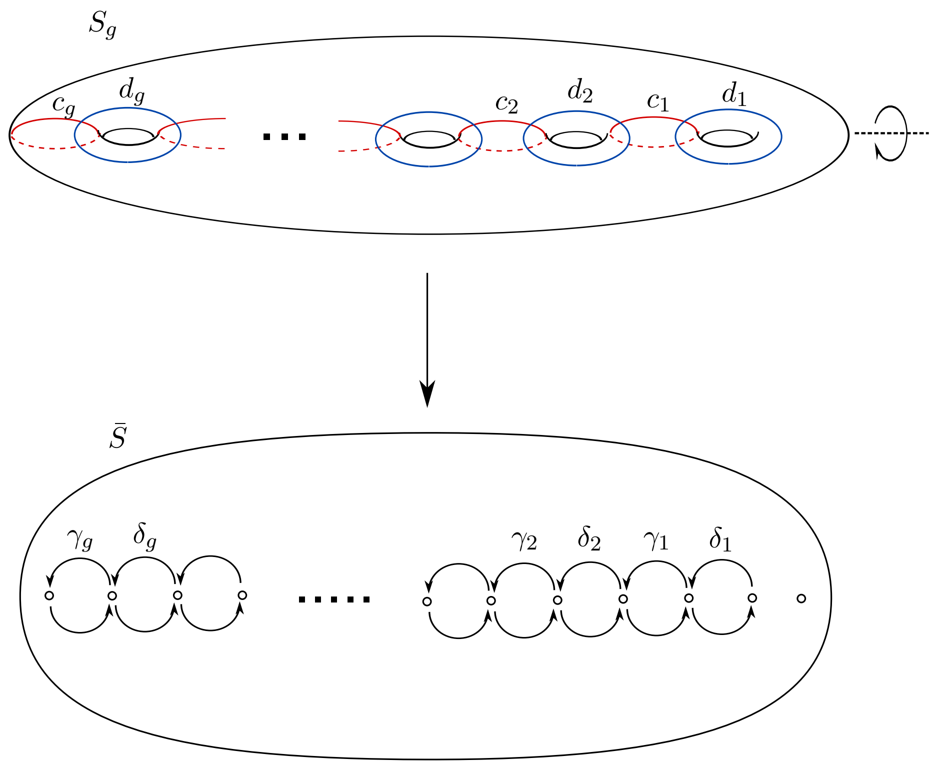

Figure 4: Hyperelliptic involution of S_g

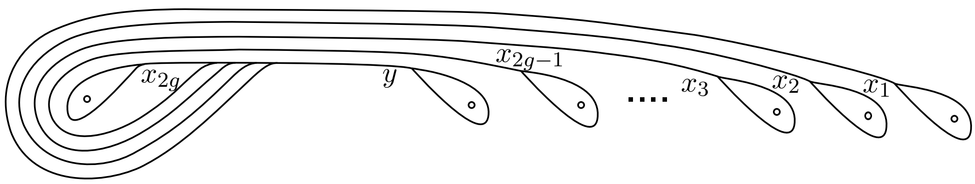

Figure 5: Train track τ of $\bar{f}_g$ on $\bar{S}$

We will show $\bar{f}_g$ is a pseudo-Anosov mapping class on $\bar{S}$. If $\bar{f}_g$ is a pseudo-Anosov mapping class, there is a representative homeomorphism $\phi \in \bar{f}_g$ with two invariant foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ with stretch factor λ. Then ϕ lifts to $\tilde{\phi} \in \mathrm{Homeo}^+(S_g)$ and the two foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ for ϕ lift to $\widetilde{\mathcal{F}}^u$ and $\widetilde{\mathcal{F}}^s$ for $\tilde{\phi}$ with the same stretch factor λ. (A k-pronged singularity lifts to a $2k$-pronged singularity.) So the lift $[\tilde{\phi}] = f_g$ is a pseudo-Anosov mapping class in $\mathrm{Mod}(S_g)$ with stretch factor $\lambda(\bar{f}_g)$.

By deleting the rightmost marked point in Figure 4, we consider $\bar{S}$ as plane with $(2g+1)$ branch points. A train track for $\bar{f}_g$ is given in Figure 5. There are $2g$ branches with integral weights $x_i, 1 \leq i \leq 2g$, and one more branch with weight $y = x_1 + x_2 + \cdots + x_{2g}$. By

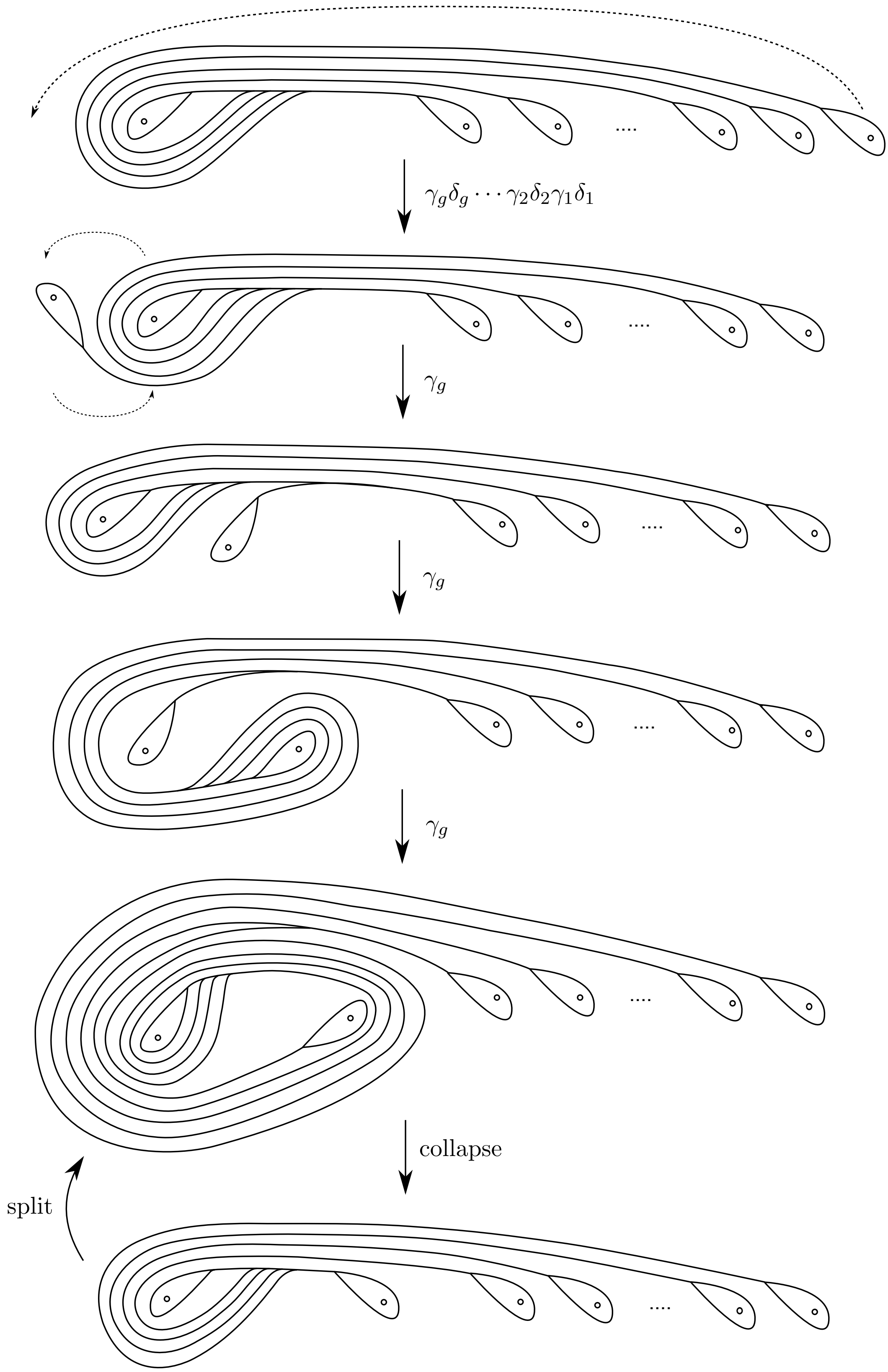

Figure 6: The action of $\bar{f}_g$ on τ

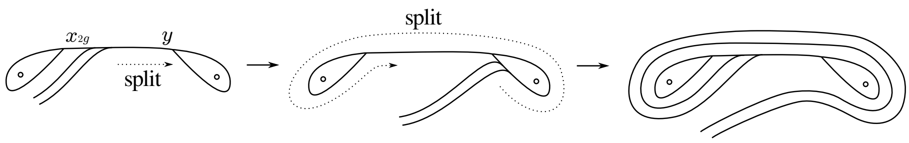

Figure 7: τ splits to $\bar{f}_g(\tau)$

assigning any positive even integers to each x_i, one can see τ supports a positive measure. Hence τ is a recurrent train track.

The action of $\bar{f}_g$ on τ is as in Figure 6. At the final step, $\bar{f}_g(\tau)$ collapses into the original train track τ with new integral weights obtained by zipping the parallel edges. The transition matrix M_g of $\bar{f}_g$ with respect to the basis $(x_1, x_2, \cdots, x_{2g})$ is a $2g \times 2g$ matrix such that

$$M_g = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 2 & 2 & 2 & \cdots & 2 & 1 \end{pmatrix}$$

By applying consecutive splits, one can show that τ splits to $\bar{f}_g(\tau)$ as in Figure 7.

To prove $\bar{f}_g$, hence f_g, is pseudo-Anosov it remains to show the transition matrix M_g is Perron–Frobenius. We will show the equivalent condition that the graph associated to M_g is strongly connected and has two cycles of relatively prime lengths. First, the vertex v_{2g-1} is connected to every vertex v_i because each entry in $(2g-1)$st row is positive. In the other direction, for $i < 2g-1$, v_i is connected to v_{i+1} because $(i, i+1)$-entry of M_g is positive and hence every vertex v_i is connected to v_{2g-1} via the path $v_i, v_{i+1}, \ldots, v_{2g-1}$. The last vertex v_{2g} is also connected to v_{2g-1}. Therefore the graph associated to M_g is strongly connected since for every pair of vertices v_i and v_j there is a path joining v_i and v_j via the vertex v_{2g-1}. Second, we can easily find two cycles of relatively prime lengths. For instance, A cycle $(v_{2g-2}, v_{2g-1}, v_{2g-2})$ has length 2 and a cycle $(v_{2g-3}, v_{2g-2}, v_{2g-1}, v_{2g-3})$

has length 3. Therefore the transition matrix M_g is Perron–Frobenius. By Theorem 3, $\bar{f}_g$ is a pseudo-Anosov mapping class on $\bar{S}$ and so is f_g on S_g.

3.1.2 Characteristic polynomials and Salem numbers

Now we will compute the characteristic polynomial $p_g(x)$ of M_g. Using the cofactor expansion of the first row,

$$p_g(x) = \det(M_g - xI) = \begin{vmatrix} -x & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -x & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1-x & 1 \\ 0 & 2 & 2 & 2 & \cdots & 2 & 1-x \end{vmatrix}$$

$$= (-x) \begin{vmatrix} -x & 1 & 0 & \cdots & 0 & 0 \\ 0 & -x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1-x & 1 \\ 2 & 2 & 2 & \cdots & 2 & 1-x \end{vmatrix} + (-1) \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1-x & 1 \\ 0 & 2 & 2 & \cdots & 2 & 1-x \end{vmatrix}$$

Let us denote the first determinant by $q_{2g-1}(x)$. By the consecutive cofactor expansions of the first row, the second determinant is

$$(-1) \begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -x & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1-x & 1 \\ 0 & 2 & 2 & 2 & \cdots & 2 & 1-x \end{vmatrix} = (-1)^2 \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1-x & 1 \\ 0 & 2 & 2 & \cdots & 2 & 1-x \end{vmatrix}$$

$$= \cdots$$

$$= (-1)^{2g-2} \begin{vmatrix} 1 & 1 \\ 0 & 1-x \end{vmatrix} = -x + 1$$

Hence we can write $p_g(x) = (-x) \cdot q_{2g-1}(x) - x + 1$. We claim that

$$q_k(x) = (-1)^k (x^{k-1} - 2x^{k-2} - 2x^{k-3} - \cdots - 2x^2 - 2x - 1),$$

for each $k \geq 3$. We prove this claim by induction on k.

The base case is when $k = 3$. By direct calculation,

$$q_3(x) = \begin{vmatrix} -x & 1 & 0 \\ 1 & 1-x & 1 \\ 2 & 2 & 1-x \end{vmatrix} = (-1)^3(x^3 - 2x^2 - 2x - 1).$$

Suppose the claim is true for k. By the cofactor expansions of the first row again, we can write

$$\begin{aligned} q_{k+1}(x) &= (-x) \cdot q_k(x) + (-1)^{k-1} \begin{vmatrix} 1 & 1 \\ 0 & 1-x \end{vmatrix} \\ &= (-x) \cdot (-1)^k (x^{k-1} - 2x^{k-2} - \cdots - 2x - 1) + (-1)^{k+1}(-x-1) \\ &= (-1)^{k+1}(x^k - 2x^{k-1} - \cdots - 2x^2 - 2x - 1). \end{aligned}$$

Finally we can conclude that

$$\begin{aligned} p_g(x) &= (-x) \cdot q_{2g-1}(x) - x + 1 \\ &= x^{2g} - 2x^{2g-1} - \cdots - 2x^2 - 2x + 1 \\ &= x^{2g} - 2\left(\sum_{k=1}^{2g-1} x^k\right) + 1. \end{aligned}$$

The largest root of $p_g(x)$ in magnitude is the stretch factor $\lambda(f_g)$. The fact that $\lambda(f_g)$ is a Salem number results in the following theorems in Akiyama–Kwon's paper. More specifically, all roots of $p_g(x)$ except $\lambda(f_g)$ and $\lambda(f_g)^{-1}$ are on the unit circle.

Theorem 5 (Theorem 2.1 [1])**.** *Let*

$$p(x) = x^d - a_{d-1}x^{d-1} - a_{d-2}x^{d-2} - \cdots - a_1 x + 1 \in \mathbb{Z}[x]$$

be a palindromic polynomial, and assume

$$p\left(e^{2k\pi i/d}\right) \geq 0,$$

for all $k = 1, 2, \ldots, d-1$*. If* p *is non-cyclotomic, then there is a root* $\beta > 1$ *of* f *such that* $1/\beta$ *is also a root and all the other roots have modulus 1, whence* β *is a Salem number.*

Lemma 6 (Lemma 2.2 [1])**.** *Let* $b \in \mathbb{Z}$ *and let*

$$p(x) = x^d - b(x^{d-1} + x^{d-2} + \cdots + x) + 1.$$

Then we have

$$p\left(e^{2k\pi i/d}\right) = b + 2,$$

for every $k = 1, 2, \ldots, d-1$.

We will prove the following proposition in section 3.5.

Proposition 7. *None of unimodular roots of* $p_g(x)$ *are roots of unity and hence* $p_g(x)$ *is irreducible over* $\mathbb{Z}$ *for each* $g \geq 2$.

Proposition 7 implies that $p_g(x)$ is the minimal polynomial of $\lambda(f_g)$ and $\deg(\lambda(f_g)) = 2g$.

3.1.3 Orientability

To compute the action of f_g on the first homology, let us choose the following homology classes as a basis for $H_1(S_g)$ as in Figure 8.

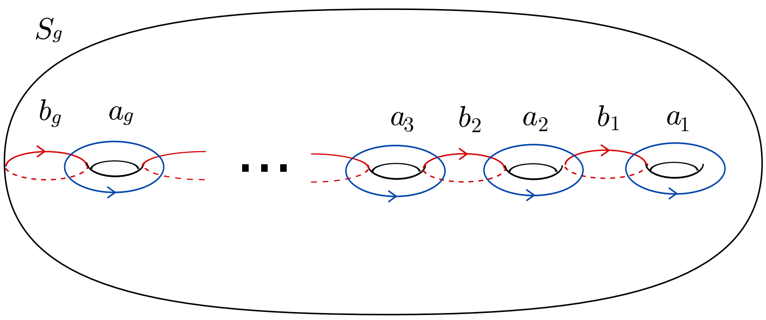

Figure 8: A basis for $H_1(S_g)$.

By computing images of each basis element under f_g, we can get the following action on $H_1(S_g)$.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & -1 \\ 4 & 0 & 0 & 0 & \cdots & -3 \end{pmatrix}$$

By induction again, the characteristic polynomial $h_g(x)$ of the homological action is

$$h_g(x) = x^{2g} + 2\left(\sum_{k=1}^{2g-1} (-1)^k x^k\right) + 1.$$

Since $h_g(-x) = p_g(x)$, the largest root of $h_g(x)$ in magnitude is $-\lambda(f_g)$. Therefore $\lambda_H(f_g) = \lambda(f_g)$. This implies every mapping class f_g has orientable invariant foliations.

It is also possible to see directly that the foliation on S_g is orientable, for instance one can show that the cover $S_g \to \bar{S}$ is the orientation double cover for the foliation on $\bar{S}$.

3.1.4 A family of pseudo-Anosov mapping classes on a given closed surface

We can construct infinitely many pseudo-Anosov mapping classes whose stretch factors have algebraic degree $2g$ on S_g. For $n \geq 3$, define a family of mapping classes $f_{g,n}$ on S_g by

$$f_{g,n} = (T_{c_g})^n (T_{c_g} T_{d_g} \cdots T_{c_2} T_{d_2} T_{c_1} T_{d_1}).$$

Then the minimal polynomial of $\lambda(f_{g,n})$ is

$$p_{g,n}(x) = x^{2g} - (n-1)\left(\sum_{k=1}^{2g-1} (-1)^k x^k\right) + 1.$$

Using the same argument as in the proof of Theorem 1, we can show that stretch factors $\lambda(f_{g,n})$ are Salem numbers and $\deg(\lambda(f_{g,n})) = 2g$. These are not conjugate to each other because $f_{g,n}$ has a different stretch factor for each n.

For mapping classes $f_{g,n}$ with even n, we may use the same train track as in Theorem 1, but for those with odd n, we need a different train track.

3.2 Branched Covers

Lifting a pseudo-Anosov mapping class via a covering map is one way to construct another pseudo-Anosov mapping class. If there is a branched cover $\widetilde{\Sigma} \to \Sigma$ and a pseudo-Anosov mapping class $f \in \mathrm{Mod}(\Sigma)$, then there is some $k \in \mathbb{N}$ such that $\mathrm{Mod}(\widetilde{\Sigma})$ has a pseudo-Anosov element $\tilde{f}$ which is a lift of f^k and hence $\lambda(\tilde{f}) = \lambda(f)^k$.

Corollary 8. *Let $g \geq 2$. For each positive integer $h \leq g/2$, there is a pseudo-Anosov mapping class $\tilde{f}_h \in \mathrm{Mod}(S_g)$ such that $\deg(\lambda(\widetilde{f_h})) = 2h$ and $\lambda(\widetilde{f_h})$ is a Salem number.*

Proof. Let

$$h = \begin{cases} \frac{g-2m}{2}, & \text{if } g \text{ is even, } m = 0, 1, \ldots, (g-2)/2, \\ \\ \frac{g-1-2m}{2}, & \text{if } g \text{ is odd, } m = 0, 1, \ldots, (g-3)/2. \end{cases}$$

Then h is an integer such that $1 \leq h \leq g/2$ if g is even, and $1 \leq h \leq (g-1)/2$ if g is odd.

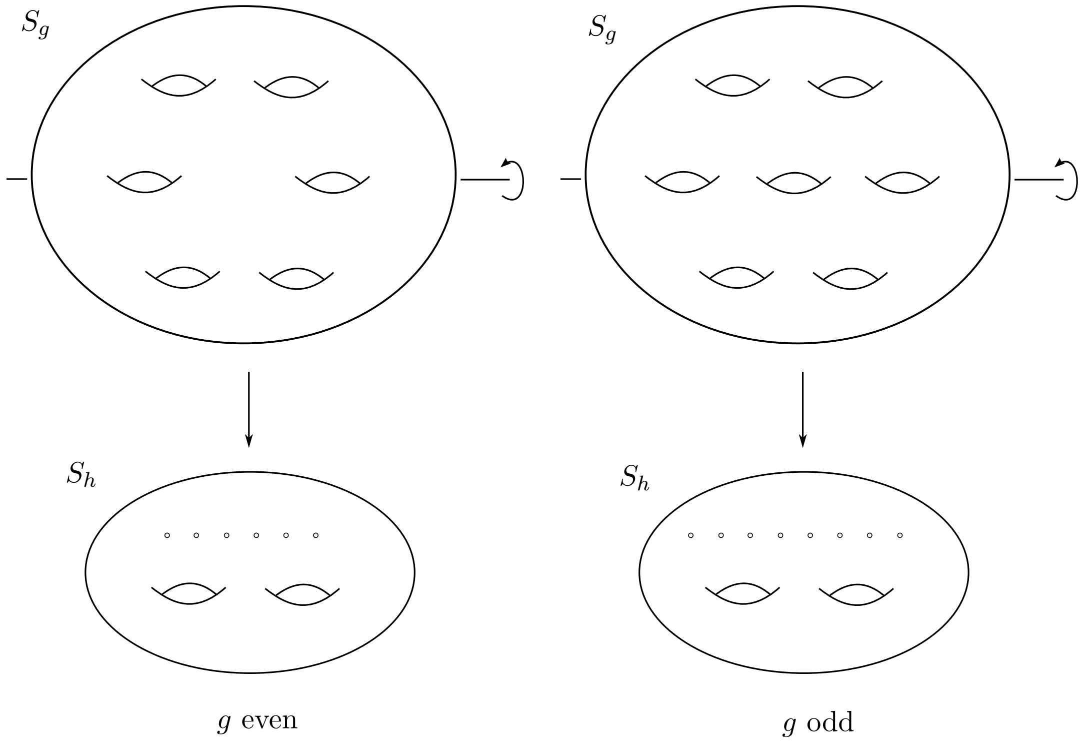

Figure 9: A branched cover

Construct a branched cover $S_g \to S_h$ as in Figure 9. For $h \geq 2$, S_h has a pseudo-Anosov mapping class $f_h \in \mathrm{Mod}(S_h)$ as in Theorem 1 such that its stretch factor has

$\deg(\lambda(f_h)) = 2h$. For some k, ${f_h}^k$ lifts to S_g and the lift has stretch factor $\lambda(f_h)^k$. We claim that $\deg\left(\lambda(f_h)^k\right) = 2h$. Let λ_i, $1 \leq i \leq 2h$, be the roots of the minimal polynomial of $\lambda(f_h)$. Let us consider the polynomial

$$p(x) = \prod_{i=1}^{2h} \left(x - \lambda_i^k\right).$$

Then $p(x)$ is an integral polynomial because the following elementary symmetric polynomials in $\lambda_1, \dots, \lambda_{2h}$

$$\sum \lambda_i, \ \sum_{i<j} \lambda_i \lambda_j, \ \sum_{i<j<l} \lambda_i \lambda_j \lambda_l, \ \cdots, \ \lambda_1 \lambda_2 \cdots \lambda_{2h}$$

are all integers and hence one can deduce that coefficients of $p(x)$

$$\sum \lambda_i^k, \ \sum_{i<j} \lambda_i^k \lambda_j^k, \ \sum_{i<j<l} \lambda_i^k \lambda_j^k \lambda_l^k, \ \cdots, \ \lambda_1^k \lambda_2^k \cdots \lambda_{2h}^k$$

are also integers. Therefore $p(x)$ is divided by the minimal polynomial of $\lambda(f_h)^k$. Due to Proposition 7, all λ_i^k except $\lambda(f_h)^k$ and $\lambda(f_h)^{-k}$ are on the unit circle and they are not roots of unity. This implies that $p(x)$ is irreducible and $\lambda(f_h)^k$ is also a Salem number with $\deg\left(\lambda(f_h)^k\right) = 2h$.

If $h = 1$, S_h is a torus and it admits a Anosov mapping class f whose stretch factor $\lambda(f)$ has algebraic degree 2. Then similar arguments as above tells us that there is a lift of some power of f to S_g whose stretch factor has $\deg(\lambda(f^k)) = 2$.

Therefore there is a pseudo-Anosov map $\widetilde{f_h} \in \mathrm{Mod}(S_g)$ with $\deg(\lambda(\widetilde{f_h})) = 2h$ for each $h \leq g/2$. In other words, every positive even degree $d \leq g$ is realized as the algebraic degree of a stretch factor on S_g. $\square$

3.3 Penner's conjecture

Penner gave a general construction of pseudo-Anosov mapping classes.

Theorem 9 (Penner [19]). *Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ be multicurves on the surface S such that $A \cup B$ fills S. Let f be any product of positive Dehn twists about a_j and negative Dehn twists about b_k, where each a_j and each b_k appears at least once. Then f is a pseudo-Anosov mapping class.*

Penner conjectured that every pseudo-Anosov map has a power that is given by this construction.

Each Dehn twist in this setting has a simple matrix representation. A train track can be obtained by smoothing edges of $A \cup B$ at each intersection (see [19, p.187]). The action of each T_{a_j} and each $T_{b_k}^{-1}$ on the train track can be written as the following block matrices of size $|A \cup B|$.

$$T_{a_j} \leftrightarrow \begin{pmatrix} I_n & U_j \\ 0 & I_m \end{pmatrix}, \quad T_{b_k}^{-1} \leftrightarrow \begin{pmatrix} I_n & 0 \\ L_k & I_m \end{pmatrix},$$

where each U_j and each L_k have only one nonzero row. The jth row in U_j is the only nonzero row , and entries in the jth row are the intersection numbers of a_j with $b_k \in B$, i.e., $(U_j)_{jk} = i(a_j, b_k)$ for $1 \leq k \leq m$. Similarly, the kth row of L_k is the only nonzero row whose entries are intersection numbers of b_k with $a_j \in A$, i.e., $(L_k)_{kj} = i(b_k, a_j)$ for $1 \leq j \leq n$. Hence U_k and L_j are symmetric in a sense that intersection numbers are matching; more precisely, $(U_j)_{jk} = (L_k)_{kj} = i(a_j, b_k)$ for each j and k.

Penner's pseudo-Anosov map has a transition matrix that is the product of these intersection matrices, where each one appears at least once. We conjecture that these matrices does not have eigenvalues on the unit circle except 1.

Conjecture. *Let λ be the stretch factor of a pseudo-Anosov mapping class given by Penner's construction. Then no conjugates of λ are on the unit circle. In particular, a Salem number cannot be the stretch factor of pseudo-Anosov mapping classes from Penner's construction.*

If the above conjecture is true, then we have a counterexample to Penner's conjecture. The stretch factor of the mapping class $f_g \in \mathrm{Mod}(S_g)$ in Theorem 1 is a Salem number such that all other conjugate roots are on the unit circle and none of them are roots of unity. Hence for each $k \in \mathbb{N}$, the stretch factor of f_g^k is also a Salem number, and therefore each power of f_g cannot come from Penner's construction.

We have slightly more general linear algebra problem without geometrical configuration that may give the answer to Penner's conjecture.

Problem 10. *For given integers n and m, let us consider the set of nonnegative integral matrices*

$$P = \left\{ \begin{pmatrix} I_n & U_j \\ 0 & I_m \end{pmatrix}, \begin{pmatrix} I_n & 0 \\ L_k & I_m \end{pmatrix} \right\},$$

such that for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, the jth row of U_j and the kth row of L_k are the only nonzero rows in U_j and L_k, respectively, satisfying $(U_j)_{jk} = (L_k)_{kj}$. Then any product of matrices in P, where each one appears at least once, does not have an eigenvalue on the unit circle except 1.

3.4 Stretch factors of odd degrees

Long proved the following degree obstruction and McMullen communicated to us the following proof. First we will give a definition of the reciprocal polynomial. Given a polynomial $p(x)$ of degree d, we define the reciprocal polynomial $p^*(x)$ of $p(x)$ by $p^*(x) = x^d p(x)$. It is a well-known property that $p^*(x)$ is irreducible if and only if $p(x)$ is irreducible.

Theorem 11 ([13])**.** *Let $f \in \mathrm{Mod}(S_g)$ be a pseudo-Anosov mapping class having stretch factor $\lambda(f)$. If $\deg(\lambda(f)) > 3g - 3$, then $\deg(\lambda(f))$ is even.*

Proof. Since f acts by a piecewise integral projective transformation on the $6g - 6$ dimensional space $\mathcal{PMF}$ of projective measured foliations on S_g, and since $\lambda(f)$ is an eigenvalue of this action, $\lambda(f)$ is an algebraic integer with $\deg(\lambda(f)) \leq 6g - 6$. Also, since f preserves the symplectic structure on $\mathcal{PMF}$, it follows that $\lambda(f)$ is the root of palindromic polynomial $p(x)$ whose degree is bounded above by $6g - 6$.

Let $q(x)$ be the minimal polynomial of $\lambda(f)$ and let $q^*(x)$ be the reciprocal polynomial of $q(x)$. Then either $q(x) = q^*(x)$ or they have no common roots, because if there is at least one common root ζ of $q(x)$ and $q^*(x)$, then both $q(x)$ and $q^*(x)$ are the minimal polynomial of ζ and hence $q(x) = q^*(x)$. Suppose $\deg(q(x)) > 3g - 3$. If $q(x)$ and $q^*(x)$ have no common roots, then their product $q(x)\, q^*(x)$ is a factor of $p(x)$ since $q^*(x)$ is the minimal polynomial of $1/\lambda(f)$. This is a contradiction because $\deg(p(x)) \leq 6g - 6$ but $\deg\big(q(x)\, q^*(x)\big) > 6g - 6$. Therefore we must have $q(x) = q^*(x)$ and this implies $q(x)$ is an irreducible palindromic polynomial. Hence $\deg(q(x))$ is even since roots of $q(x)$ comes in pairs, λ_i and $1/\lambda_i$. □

It follows from the previous proof that if the minimal polynomial $p(x)$ of λ has odd degree, then $p(x)$ is not palindromic and in fact the minimal palindromic polynomial containing λ as a root is $p(x)p^*(x)$.

We will now show that the stretch factors of degree 3 have an additional special property. A *Pisot number*, also called a *Pisot–Vijayaraghavan number* or a *PV number*, is an algebraic integer greater than 1 such that all its Galois conjugates are strictly less than 1 in absolute value.

Proposition 12. *Let $f \in \mathrm{Mod}(S_g)$. If $\deg(\lambda(f)) = 3$, then $\lambda(f)$ is a Pisot number.*

Proof. Let $\lambda_1 > 1$ be the stretch factor of a pseudo-Anosov mapping class with algebraic degree 3, and let $p(x)$ be the minimal polynomial of λ_1. Let λ_1, λ_2, and λ_3 be the roots of $p(x)$. Then the degree of $p(x)p^*(x)$ is 6 and it has pairs of roots $(\lambda_1, 1/\lambda_1), (\lambda_2, 1/\lambda_2), (\lambda_3, 1/\lambda_3)$, where λ_1 is the largest root in absolute value. We claim that the magnitude of λ_2 and λ_3 are strictly less than 1.

Suppose one of them has magnitude greater than or equal to 1, say $|\lambda_2| \geq 1$. The constant term $\lambda_1\lambda_2\lambda_3$ of $p(x)$ is ± 1 since it is the factor of a palindromic polynomial with constant term 1. So $|\lambda_1\lambda_2\lambda_3| = 1$ and we have

$$\frac{1}{|\lambda_3|} = |\lambda_1\lambda_2| \geq |\lambda_1|,$$

which is a contradiction to the fact that the stretch factor λ_1 is strictly greater than all other Galois conjugates. This proves the claim and hence the stretch factor of degree 3 is a Pisot number. □

We now explain two constructions of mapping classes $f \in \mathrm{Mod}(S_g)$ whose degree of $\lambda(f)$ is odd.

1. As we mentioned, Arnoux–Yoccoz gave examples of a pseudo-Anosov mapping class on S_g whose stretch factor has algebraic degree g. In particular for odd g, this gives examples of mapping classes with odd degree stretch factors. They proved that these stretch factors are all Pisot numbers.

2. For genus 2, there is a pseudo-Anosov mapping class f whose stretch factor has algebraic degree 3 (see chapter 4). This is the only possible odd degree on S_2 by Long's obstruction. It is also true that $\deg(f^k) = 3$ for each k because the stretch factor is a Pisot number (Proposition 12). There is a cover $S_g \to S_2$ for each g, so the lift of some power of f has a stretch factor with algebraic degree 3 on S_g.

Proposition 13. *For each genus g, the stretch factor with algebraic degree 3 can occur on S_g.*

Question. *Are there stretch factors with odd algebraic degree that are not Pisot numbers?*

3.5 Irreducibility of Polynomials

In this section, we will prove Proposition 7. For $n \geq 2$, let

$$p_n(x) = x^{2n} - 2\left(\sum_{k=1}^{2n-1} x^k\right) + 1.$$

We will show $p_n(x)$ does not have a cyclotomic polynomial factor. It then follows from Kronecker's theorem that $p_n(x)$ is irreducible.

proof of Proposition 7. Suppose $p_n(x)$ has the mth cyclotomic polynomial factor. Then $e^{2\pi i/m}$ is a root of $p_n(x)$.

$$p_n(e^{2\pi i/m}) = e^{4n\pi i/m} - 2\Big(\frac{e^{4n\pi i/m} - 1}{e^{2\pi i/m} - 1} - 1\Big) + 1 = 0$$

$$\implies e^{2(2n+1)\pi i/m} - 3e^{4n\pi i/m} + 3e^{2\pi i/m} - 1 = 0 \tag{1}$$

Consider the real part and the complex part of (1). Then we have the system of equations

$$\begin{cases} \cos\left(\frac{2(2n+1)\pi}{m}\right) - 3\cos\left(\frac{4n\pi}{m}\right) + 3\cos\left(\frac{2\pi}{m}\right) - 1 = 0 \\ \sin\left(\frac{2(2n+1)\pi}{m}\right) - 3\sin\left(\frac{4n\pi}{m}\right) + 3\sin\left(\frac{2\pi}{m}\right) = 0 \end{cases}$$

Using double-angle formula for the first cosine and sum-to-product formula for the last two cosines, the first equation gives

$$2\sin\Big(\frac{(2n+1)\pi}{m}\Big)\Big[3\sin\Big(\frac{(2n-1)\pi}{m}\Big) - \sin\Big(\frac{(2n+1)\pi}{m}\Big)\Big] = 0.$$

Similarly the second equation gives

$$2\cos\left(\frac{(2n+1)\pi}{m}\right)\left[\sin\left(\frac{(2n+1)\pi}{m}\right) - 3\sin\left(\frac{(2n-1)\pi}{m}\right)\right] = 0.$$

Since sine and cosine have no common zeros, we must have

$$\sin\left(\frac{(2n+1)\pi}{m}\right) - 3\sin\left(\frac{(2n-1)\pi}{m}\right) = 0.$$

For $m \leq 5$, by direct calculation we can see $p_n(e^{2\pi i/m}) \neq 0$. So we may assume that $m \geq 6$. Let $\varphi = (2n-1)\pi/m$ and then we can write the above equation as

$$\sin\left(\varphi + \frac{2\pi}{m}\right) - 3\sin(\varphi) = 0. \tag{2}$$

Since $\sin\big(\varphi + 2\pi/m\big)$ is a real number between -1 and 1, we have

$$-\frac{1}{3} \leq \sin(\varphi) \leq \frac{1}{3}. \tag{3}$$

Let $\psi = \sin^{-1}(1/3)$. Note that $\psi < \pi/6$. Equation (3) gives the restriction on φ, that is,

$$-\psi \leq \varphi \leq \psi \quad \text{or} \quad \pi - \psi \leq \varphi \leq \pi + \psi.$$

Another observation from (2) is that both $\sin\big(\varphi + 2\pi/m\big)$ and $\sin(\varphi)$ must have the same sign, so both $\varphi + 2\pi/m$ and φ are on the above the $x-$axis or below the x-axis.

We claim that φ has to be on the either first or third quadrant. Suppose φ is on the second quadrant, that is, $\pi - \psi < \varphi < \pi$. Note that $m \geq 6$ implies $2\pi/m \leq \pi/3$. Since φ is above the x-axis, $\varphi + 2\pi/m$ also has to be above the x-axis due to (2) and hence the only possibility is that $\varphi + 2\pi/m$ is between φ and π. Then

$$0 < \sin\left(\varphi + \frac{2\pi}{m}\right) < \sin(\varphi) \implies \sin\left(\varphi + \frac{2\pi}{m}\right) < 3\sin(\varphi),$$

which is a contradiction to (2). Similar arguments holds if φ is on the fourth quadrant. Therefore the possible range for φ is

$$0 < \varphi \leq \psi \quad \text{or} \quad \pi < \varphi \leq \pi + \psi.$$

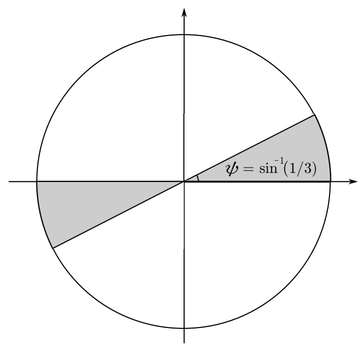

Figure 10: Possible range for φ

Suppose φ is on the first quadrant. Then so is $\varphi + 2\pi/m$ because

$$0 < \varphi + \frac{2\pi}{m} \leq \psi + \frac{\pi}{3} < \frac{\pi}{2}.$$

We can write

$$\varphi = \frac{(2n-1)\pi}{m} \equiv \frac{j\pi}{m}. \quad (\mathrm{mod}\ 2\pi)$$

for some positive integer j, i.e., $0 < j\pi/m < \pi/2$.

If $j \geq 2$, Using the subadditivity of $\sin(x)$ on the first quadrant

$$\sin(x+y) \leq \sin(x) + \sin(y),$$

we have

$$\begin{aligned}
\sin\left(\varphi + \frac{2\pi}{m}\right) - 3\sin(\varphi) &\leq \left(\sin(\varphi) + \sin\left(\frac{2\pi}{m}\right)\right) - 3\sin(\varphi) \\
&= \sin\left(\frac{2\pi}{m}\right) - 2\sin(\varphi) \\
&= \sin\left(\frac{2\pi}{m}\right) - 2\sin\left(\frac{j\pi}{m}\right) < 0,
\end{aligned}$$

which contradicts (2).

If $j = 1$, using triple-angle formula

$$\begin{aligned}
\sin\left(\varphi + \frac{2\pi}{m}\right) - 3\sin(\varphi) &= \sin\left(\frac{3\pi}{m}\right) - 3\sin\left(\frac{\pi}{m}\right) \\
&= \left(3\sin\left(\frac{\pi}{m}\right) - 4\sin^3\left(\frac{\pi}{m}\right)\right) - 3\sin\left(\frac{\pi}{m}\right) \\
&= -4\sin^3\left(\frac{\pi}{m}\right) < 0,
\end{aligned}$$

which contradicts (2) again. Therefore there is no possible φ on the first quadrant. By using the same arguments, the fact that φ is on the third quadrant gives a contradiction. Therefore we can conclude that $p(x)$ does not have a cyclotomic factor.

We now show that $p_n(x)$ is irreducible over $\mathbb{Z}$. Suppose $p_n(x)$ is reducible and write $p_n(x) = g(x)h(x)$ with non-constant functions $g(x)$ and $h(x)$. There is only one root of $p_n(x)$ whose absolute value is strictly greater than 1. Therefore one of $g(x)$ or $h(x)$ has all roots inside the unit disk. By Kronecker's theorem, this polynomial has to be a product of cyclotomic polynomials, which is a contradiction because $p_n(x)$ does not have a cyclotomic polynomial factor. Therefore $p_n(x)$ is irreducible. $\square$

CHAPTER IV

EXAMPLES OF EVEN DEGREES

Tables 1 through 4 give explicit examples of pseudo-Anosov mapping classes whose stretch factors realize various degrees. We will follow the notation of software `Xtrain` by Brinkmann. More specifically, a_i, b_i, c_i, and d_i are Dehn twists along standard curves and A_i, B_i, C_i, and D_i are the inverse twists as in [5]. The only missing degree on S_3 is degree 5. We do not know if there is a degree 5 example or there is another degree obstruction.

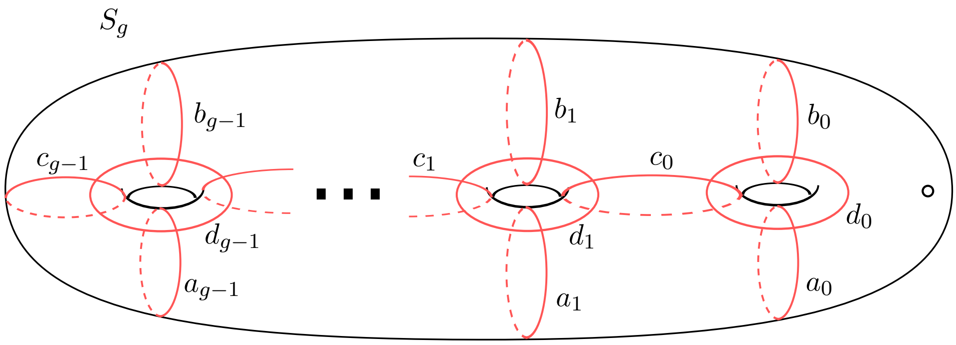

Figure 11: Standard curves in `Xtrain`

Table 1: Examples of genus 2

deg	$f \in \mathrm{Mod}(S_2)$	Minimal polynomial	$\lambda(f)$
2	$a_0a_0d_0C_0D_1C_0$	$x^2 - 3x + 1$	$\lambda = 2.618$
3	$a_0d_0d_0C_0C_0D_1$	$x^3 - 3x^2 - x - 1$	$\lambda = 3.383$
4	$a_0d_0d_0d_1c_0d_0$	$x^4 - x^3 - x^2 - x + 1$	$\lambda = 1.722$
6	$a_0a_0d_0A_0C_0D_1$	$x^6 - x^5 - 4x^3 - x + 1$	$\lambda = 2.015$

Table 2: Examples of genus 3

deg	$f \in \mathrm{Mod}(S_3)$	Minimal polynomial	$\lambda(f)$
2	$a_1c_0d_0c_0d_2C_1D_1$	x^2-4x+1	3.732
3	$a_0c_0d_0C_1D_1D_2$	x^3-2x^2+x-1	1.755
4	$a_1c_0d_0a_1c_1d_1d_2$	$x^4-x^3-2x^2-x+1$	1.722
6	$a_0c_0d_0d_2C_1D_1$	$x^6-3x^5+3x^4-7x^3+3x^2-3x+1$	2.739
8	$a_0c_0d_0d_1C_1D_2$	$x^8-x^7-2x^5-2x^3-x+1$	1.809
10	$a_1c_0d_0d_1C_1A_2D_2$	$x^{10}-x^9-2x^8+2x^7-2x^5+2x^3-2x^2-x+1$	1.697
12	$a_1c_1c_0d_1d_2A_0D_0$	$x^{12}-x^{11}-x^9-x^8+x^7+x^5-x^4-x^3-x+1$	1.533

Table 3: Examples of genus 4

deg	$f \in \mathrm{Mod}(S_4)$	deg	$f \in \mathrm{Mod}(S_4)$
4	$a_0a_0a_1c_0d_0c_1d_1c_2d_2c_3d_3$	12	$a_0B_1d_0c_0d_1c_1d_2c_2d_3c_3$
6	$a_0B_2A_3d_0c_0d_1c_1d_2c_2d_3c_3$	14	$a_0d_0B_0d_0c_0d_1c_1d_2c_2d_3c_3$
8	$a_0A_1d_0c_0d_1c_1d_2c_2d_3c_3$	16	$A_0d_0c_0d_1c_1d_2c_2d_3c_3$
10	$a_0b_1A_2d_0c_0d_1c_1d_2c_2d_3c_3$	18	$a_0B_1A_2d_0c_0d_1c_1d_2c_2d_3c_3$

†Degree 2 on S_4 can be realized by a branched cover $S_4 \to S_2$ as in 3.2.

Table 4: Examples of genus 5

deg	$f \in \mathrm{Mod}(S_5)$	deg	$f \in \mathrm{Mod}(S_5)$
6	$b_3d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	16	$a_1B_2d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$
8	$a_0a_1d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	18	$a_1B_0d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$
10	$a_1A_4d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	20	$a_1A_0d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$
12	$b_2C_2d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	22	$a_2A_1d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$
14	$a_1B_1d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	24	$c_2A_2d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$

†Degree 2 and 4 on S_5 can be realized by a branched cover $S_5 \to S_2$.

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