Abstract

CABBAGE, BRIAN. The Stone-Čech Compactification of the Plane. (Under the direction of Prof. Gary D. Faulkner)

The motivation for this dissertation came from Franco Obersnel's dissertation *On Compactifications of the Set of Natural Numbers and the Half Line*. In it he proves that any non-degenerate subcontinuum of the Stone-Čech remainder of the half line will map onto any arbitrary continuum of weight $\leq \omega_1$. We are able to prove the same property for many (though not all) non-degenerate subcontinua of the Stone-Čech remainder of the plane, as well as investigating certain algebraic and topological structures on subsets of the remainder.

THE STONE-ČECH COMPACTIFICATION OF THE PLANE

BY

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Biography

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Table of Contents

1 Background **1**

1.1 The Stone-Čech Compactification 1

1.2 Hyperreals 6

1.3 Dedekind Completions 10

2 The Stone-Čech Compactification of the Plane **13**

2.1 Hypercircles in Π^* 13

2.2 Hyperreals in Π^* 19

2.3 A Mapping Property of Certain Subcontinua of Π^* 22

2.4 Open Questions 24

List of References **26**

Chapter 1

Background

1.1 The Stone-Čech Compactification

A general knowledge of topology is assumed. A topological space X is said to be *compact* if every open cover of X has a finite subcover. Any subspace of a compact Hausdorff space is closed iff it is compact. A *compactification* of a topological space X is an ordered pair (K, f) such that K is a compact Hausdorff space and $f : X \to K$ is a dense embedding of X into K.

If $A \subseteq X$, we will often use $\overline{A}$ to denote the closure of A in X. One exception will be in cases where there may be some ambiguity as to what space the closure is being taken in. For example, if we also have $Y \subseteq X$, we may write $Cl_Y(A)$ to indicate the closure of A is being taken in the subspace topology of Y.

It is natural to ask what topological spaces have compactifications. A *Tychonoff* space is a T_1 space X that is *completely regular*, that is, for any closed $F \subseteq X$ and $x \notin F$, there is a continuous function $f : X \to I$ (where I denotes the unit interval $[0, 1]$) such that $f(x) = 0$ and $f(F) = 1$. In this case f is said to *separate* x from F.

A collection $\{f_\lambda | \lambda \in \Lambda\}$ of functions $f_\lambda : X \to X_\lambda$ is said to *separate points from closed sets* if whenever $F \subseteq X$ is closed and $x \notin X$, there exists $\lambda \in \Lambda$ such that $f_\lambda(x) \notin \overline{f_\lambda(F)}$. A well known property of such a collection is that we then have an embedding of X into the product space $\Pi_{\lambda\in\Lambda}X_\lambda$ (given by the evaluation map $e : X \to \Pi_{\lambda\in\Lambda}X_\lambda$ defined by $[e(x)]_\lambda = f_\lambda(x)$).

It follows that any Tychonoff space can be embedded into a *cube*, i.e., a product of unit intervals. The following theorem is a well known equivalent of the axiom of choice:

Theorem 1.1.1. *(Tychonoff) A nonempty product space is compact iff each factor space is compact.*

(Proofs can be found in both [5] and [9]). As a consequence, if X is a Tychonoff space consider the embedding of X in a cube. The closure of this embedding is a compact space with (an embedded copy of) X as a dense subspace, hence it is a compactification of X. Thus any Tychonoff space has a compactification. The converse is true as well (and can also be found in [9]):

Theorem 1.1.2. *A topological space X has a compactification iff X is a Tychonoff space.*

The compactification of X described above is well-known as the the Stone-Čech compactification and is commonly denoted βX. The Stone-Čech remainder of X, $\beta X \setminus X$, is commonly denoted by X^*. The Stone-Čech compactification is the "largest" compactification of X in the sense that any bounded, continuous function $f : X \to \mathbb{R}$ can be continuously extended to all of βX (i.e., X is C^**-embedded* in βX). In fact, any continuous map $f : X \to K$ (where K is any compact Hausdorff space) has a unique continuous extension to all of βX. Such an extension is generally denoted βf; it's restriction to X^* is sometimes denoted f^*.

If X is also *locally compact* (every point in X has a neighborhood base consisting of compact sets), then βX has the additional nice property that X is open in βX. This is implied by the following proposition taken from [9]:

Proposition 1.1.3. *In a locally compact Hausdorff space, the intersection of an open set with a closed set is locally compact. Conversely, a locally compact subset of a Hausdorff space is the intersection of an open set and a closed set.*

Proof. Let X be locally compact and T_2. If X is open in X and $a \in A$, then a has a compact neighborhood K contained in A, and K is then a compact neighborhood of a in A, so A is locally compact. If B is closed in X and $b \in B$, then b has a compact neighborhood K in X and $K \cap B$ is a compact neighborhood of b in B, so B is locally compact. Hence, open subsets and closed subsets of X are locally compact. But the intersection of two locally compact subsets of X is locally compact so, in particular, the intersection of an open set with a closed set in X is locally compact.

Conversely, suppose Y is Hausdorff and X is a locally compact subset of Y. It will suffice to show X is open in $Cl_Y(X)$. Let $x \in X$ and find a neighborhood U of x in X such that $Cl_X(U)$ is compact, by local compactness. Say $U = X \cap V$ where V is open in Y. Then

$$Cl_Y(X \cap V) \cap X = Cl_Y(U) \cap X = Cl_X(U)$$

and the latter is compact. Thus $Cl_Y(X \cap V) \cap X$ is closed in Y. But it contains $X \cap V$ and thus $Cl_Y(X \cap V)$; i.e., $Cl_Y(X \cap V) \cap X \supseteq Cl_Y(X \cap V)$. But then $Cl_Y(X \cap V) \subseteq X$, and hence $Cl_Y(X) \cap V \subseteq X$. Thus $Cl_Y(X) \cap V$ is a neighborhood of x in $Cl_Y(X)$ which is contained in X, so X is open in $Cl_Y(X)$. □

Corollary 1.1.4. *A dense subset of a compact Hausdorff space is locally compact iff it is open.*

Proof. Let X be a compact Hausdorff space with a dense subset D. First suppose D is locally compact. We then know $D = G \cap F$ for some G open in X and F closed in X. Since F is a closed set containing D, we must have $F = X$. Therefore $D = G \cap X = G$ is open.

Now suppose D is open. It is well known that any Hausdorff space is locally compact iff each point in the space has a compact neighborhood, so given $d \in D$, we must find a compact neighborhood of d in D. Since X is a normal Hausdorff space, there is a neighborhood U of d in D such that $Cl_X(U) \subseteq D$, and hence $Cl_X(U)$ is the required neighborhood. $\square$

It follows that X is open in βX iff X is locally compact. Because of this nice property, all spaces will be assumed to be locally compact Hausdorff spaces.

There are many additional unique properties of βX, but first we need some definitions.

Definition 1.1.5. *$Z \subseteq X$ is a* ***zero-set*** *if $Z = f^{-1}(0)$ for some continuous real valued function f on X.*

It's clear that every zero-set is closed. If X is a metric space, then we also have that every closed set is a zero-set: Suppose F is a closed subset of a metric space X. Define $D : X \to \mathbb{R}$ by setting $D(x)$ as the distance from x to F. Then $D^{-1}(0) = F$.

Definition 1.1.6. *A* ***z-filter*** *on X is a filter whose elements are zero-sets of X. A* ***z-ultrafilter*** *is a maximal z-filter.*

Definition 1.1.7. *A* ***prime*** *z-filter u is a filter with the property that whenever Z_1 and Z_2 are zero-sets such that $Z_1 \cup Z_2 \in u$, then either $Z_1 \in u$ or $Z_2 \in u$.*

It should be noted that any z-ultrafilter is prime.

Definition 1.1.8. *A family F of subsets of X is said to have the finite intersection property if any finite subcollection of F has nonempty intersection*

A well known property of such families of sets is that they can always be extended to ultrafilters. This follows from a direct application of Zorn's lemma.

The following theorem is proven in [1] and describes some unique properties of the Stone-Čech Compactification of X:

Theorem 1.1.9. *Suppose X is dense in a compact space K. The following statements are equivalent:*

1. Every continuous mapping from X into any compact space Y has an extension to a continuous mapping from K into Y.

2. X is C^*-embedded in K.

3. Any two disjoint zero-sets in X have disjoint closures in K.

4. For any two zero-sets Z_1 and Z_2 in X,

$$Cl_K(Z_1 \cap Z_2) = Cl_K Z_1 \cap Cl_K Z_2.$$

5. Distinct z-ultrafilters on X have distinct limits in K

6. K is homeomorphic to βX.

It will be convenient to have a nice base for βX. For any open $U \subseteq X$, define $Ex\ U = \beta X \setminus Cl_{\beta X}(X \setminus U)$. It is evident that $Ex\ U$ is the largest open set in βX whose intersection with X equals U. The following proposition and proof were inspired by [3]:

Proposition 1.1.10. *The set $\{Ex\ U | U$ is open in $X\}$ forms a base for βX.*

Proof. Let $x \in \beta X$ and let $U \subseteq \beta X$ be an open neighborhood of x. We need to find $V \in \{Ex\ U | U \text{ is open in } X\}$ such that $x \in V \subseteq U$.

Since βX is a normal Hausdorff space, there is an open $G \subseteq \beta X$ containing x such that $Cl_{\beta X}(G) \subseteq U$. Set $G' = G \cap X$ and let $V = Ex\ G'$.

It's clear that $x \in V$. If $y \in V = Ex\ G' = \beta X \setminus Cl_{\beta X}(X \setminus G')$, then $y \notin Cl_{\beta X}(X \setminus G')$. Therefore, we must have $y \in Cl_{\beta X}(G') = Cl_{\beta X}(G \cap X) \subseteq Cl_{\beta X}(G) \subseteq U$. Hence $x \in V \subseteq U$. $\square$

An alternate construction of βX using z-ultrafilters is given in [8]. Briefly: Define βX as the set of all z-ultrafilters on X. A base for the closed sets of βX is then given by the family of sets of the form $W(Z) = \{u \in \beta X | Z \in u\}$, where Z is any zero-set in X. We will often consider points in X^* as z-ultrafilters on X.

A nice property of this construction is that a z-ultrafilter u on X consists exactly of those closed subsets of X that have $u \in \beta X$ as a limit point.

Definition 1.1.11. *A continuous map $f : X \to Y$ is a **compact map** if $f^{-1}(y)$ is compact for every $y \in Y$. If f is also closed, then f is said to be **perfect**.*

The following theorem and proof come from [8]:

Theorem 1.1.12. *If X and Y are Tychonoff spaces, then the following are equivalent for a map $f : X \to Y$:*

1. f is perfect.

2. If u is an ultrafilter on X and if $f(u)$ converges to y in Y, then u converges (necessarily to some $x \in f^{-1}(y)$).

3. βf takes growth to growth, i.e. $\beta f(X^*) \subseteq Y^*$.

Proof. 1⇒2: Let f be perfect and u be an ultrafilter on X such that $f(u)$ converges to $y \in Y$. Because f is continuous, if u converges it must converge to a point of $f^{-1}(y)$. If u fails to converge, then for all $x \in f^{-1}(y)$, there is an open neighborhood U_x of x such that $U_x \notin u$. Since $f^{-1}(y)$ is compact, it is covered by a finite subfamily $\{U_{x_i}\}$. The open set $V = \bigcup U_{x_i}$ does not belong to u because u is an ultrafilter. Thus, $X \setminus V \in u$ so that $f(X \setminus V) \in f(u)$. Because f is a closed map, $Y \setminus f(x \setminus V)$ is a neighborhood of y which fails to belong to $f(u)$. This contradicts the assumption that $f(u)$ converges to y. Hence, u must converge.

2⇒1: We first show that f has compact fibers. Let v be an ultrafilter on $f^{-1}(y)$ for some $y \in Y$. Let u be an ultrafilter on X which contains v. Then $f(u)$ converges to $y \in Y$. Hence, u and therefore v converge to a point in $f^{-1}(y)$ and $f^{-1}(y)$ is compact because every ultrafilter on $f^{-1}(y)$ converges.

Now we show that f is closed. Let F be a closed subset of X and let v be an ultrafilter on $f(F)$ converging to a point $y \in Y$. For every $V \in v$, $f(f^{-1}(V) \cap F) = V \cap f(F)$ is non-empty. Hence, the family $\{f^{-1} \cap F | V \in v\}$ is contained in an ultrafilter u on X. Then $f(u)$ converges to y, and therefore u converges to a point $x \in f^{-1}(y)$. Since F is closed, $x \in F$. Hence, $y = f(x)$ and $f(F)$ is closed.

2⇒3: Let $p \in \beta X$. Because X is dense in βX, there is an ultrafilter u on X which converges p. Continuity implies that $\beta f(u) = f(u)$ converges to a point $q \in \beta Y$. If $q \in Y$, then u converges to a point in $f^{-1}(y)$. Because βX is Hausdorff, we must have $p = x$. Thus, the only points of βX which are mapped to points of Y are the points of X.

3⇒2: Suppose that u is an ultrafilter on X such that $f(u)$ converges to $y \in Y$. Because X is dense in βX, u converges to a point $p \in \beta X$. Then continuity implies that $\beta f(u) = f(u)$ converges to $y \in Y$ and that $p \in \beta f^{-1}(y)$. Since βf sends X^* into Y^*, p must belong to X. □

Definition 1.1.13. *A continuous map $f : X \to Y$ is **monotone** if $f^{-1}(y)$ is connected for all $y \in Y$.*

We will also need the following proposition (borrowed from [3]):

Proposition 1.1.14. *Let $f : X \to Y$ be a perfect and monotone map. Then the map $\beta f : \beta X \to \beta Y$ is also monotone.*

Proof. It suffices to show that $\beta f^{-1}(z)$ is connected for every $z \in Y^*$. So let $z \in Y^*$ and write $\beta f^{-1}(z)$ as the disjoint union of two closed sets A and B. Using normality of βX find open sets U and V around A and B respectively, whose closures are disjoint. As the open set $U \cup V$ contains $\beta f^{-1}(z)$ there is an open set O in βY containing z such that $\beta f^{-1}(O) \subseteq U \cup V$; after shrinking U and V a bit we may as well assume that $\beta f^{-1}(O) = U \cup V$.

The set $(U \cup V) \cap X$ is saturated with respect to f: it equals $f^{-1}(O \cap Y)$. Since $f^{-1}(y)$ is connected for every $y \in Y$ and since U and V are disjoint open sets in βX we see that $f^{-1}(y)$ is contained in U or V for every $y \in O$. Therefore $U \cap X$ and $V \cap X$ are saturated with respect to f as well. We conclude that $U' = f(U \cap X)$ and $V' = f(V \cap X)$ are disjoint open sets in Y; moreover $O \cap Y = U' \cup V'$.

We claim that $\overline{U'} \cap \overline{V'} \cap O = \emptyset$. To see this consider $x \in O$ and assume $x \in \overline{U'}$. Let $g : \beta Y \to [0, 1]$ be continuous such that $g(x) = 1$ and $g(\beta Y \setminus O) \subseteq \{0\}$. Using g we define $h : Y \to [-1, 1]$ by

$$h(y) = \begin{cases} g(y) & \text{if } y \in Y \setminus V', \\ -g(y) & \text{if } y \in Y \setminus U'. \end{cases}$$

Observe that h is continuous and that $|h|$ equals the restriction of g to Y, hence $|\beta h| = g$. Now $h(y) = g(y) \geq 0$ for $y \in U'$ so that $\beta h(x) = g(x) = 1$; on the other hand $h(y) = -g(y)$ for $y \in V'$, hence $\beta h(y) \leq 0$ for all $y \in \overline{V'}$. We see that $x \notin \overline{V'}$.

Assume for example that $z \in \overline{V'}$. Now $V \subseteq \overline{V \cap X} \subseteq \beta f^{-1}(\overline{V'})$, because $V \cap X = f^{-1}(V')$. But then $B \subseteq V \cap \beta f^{-1}(z) = \emptyset$. It follows that $\beta f^{-1}(z)$ is connected. □

A *continuum* is a compact, connected Hausdorff space. Much of this paper will be regarding continua. It is well known that continuous images of compact/connected spaces are compact/connected, respectively. In general, it is not true that continuous images of Hausdorff spaces are Hausdorff. However, since all spaces to be considered here will be Hausdorff, for our purposes we can assume continuous images of continua to be continua.

We will also have occasion to use Tietze's extension theorem:

Theorem 1.1.15. *X is normal iff whenever A is a closed subset of X and $f : A \to \mathbb{R}$ is continuous, there is an extension of f to all of X.*

A proof can be found in [9].

1.2 Hyperreals

The hyperreals are an extension of the real numbers which include both infinitesimals and infinitely large numbers; they form the foundation for what is known as non-standard analysis. Their first construction in 1961 by Abraham Robinson (see [7]) resolved a centuries old debate regarding the application of infinitesimals in analysis. A brief overview of this debate is given in [2].

We will be primarily interested only in the ordering and algebraic properties of the hyperreals, rather than their applications in non-standard analysis. The following construction of the hyperreals has been taken from [2]

We'll use $\mathbb{R}^\omega$ to denote the set of sequences of real numbers. Let u be a nonprincipal ultrafilter on ω (=the set of natural numbers). Define an equivalence relation on $\mathbb{R}^\omega$ as follows. If $r = \{r_n\}$ and $s = \{s_n\}$ are any two sequences of real numbers, we'll write $r \equiv s$ iff $\{n|r_n = s_n\} \in u$.

Proposition 1.2.1. *$\equiv$ is an equivalence relation on $\mathbb{R}^\omega$.*

Proof. It's clear that $\equiv$ is reflexive and symmetric. To see that it's transitive, suppose $r, s, t \in \mathbb{R}^\omega$ with $r \equiv s$ and $s \equiv t$. Then $\{n|r_n = t_n\} \supseteq \{n|r_n = s_n\} \cap \{n|s_n = t_n\}$. Since $\{n|r_n = s_n\} \cap \{n|s_n = t_n\} \in u$, we have $\{n|r_n = t_n\} \in u$, hence $r \equiv t$. □

If $r \in \mathbb{R}^\omega$, we'll denote the equivalence class of r under $\equiv$ by $[r]$. We'll denote the set of equivalence classes of $\mathbb{R}^\omega$ by ${}^*\mathbb{R}$.

Define an order $\leq$ on ${}^*\mathbb{R}$ by setting $[r] \leq [s]$ iff $\{n|r_n \leq s_n\} \in u$. This is well-defined: If $[r] \leq [s]$ and $r' \in [r]$, $s' \in [s]$, then $\{n|r'_n \leq s'_n\} \supseteq \{n|r_n = r'_n\} \cap \{n|s_n = s'_n\} \cap \{n|r_n \leq s_n\}$. Since $\{n|r_n = r'_n\} \cap \{n|s_n = s'_n\} \cap \{n|r_n \leq s_n\} \in u$, we have $\{n|r'_n \leq s'_n\} \in u$. We'll write $[r] < [s]$ iff $[r] \leq [s]$ and $[r] \neq [s]$ (equivalently, $[r] < [s] \Leftrightarrow \{n|r_n < s_n\} \in u$).

Proposition 1.2.2. *$\leq$ is a linear order on ${}^*\mathbb{R}$.*

Proof. The proof that $\leq$ is transitive is routine and similar to the proofs already presented.

To see that $\leq$ is linear, pick any $[r], [s] \in^* \mathbb{R}$. Note that $\{n|r_n \leq s_n\} \cup \{n|s_n \leq r_n\} = \omega \in u$. Since u is prime, either $\{n|r_n \leq s_n\} \in u$ or $\{n|s_n \leq r_n\}$ u, i.e., either $[r] \leq [s]$ or $[s] \leq [r]$, respectively.

If we have $[r] \leq [s]$ and $[s] \leq [r]$, then $\{n|r_n \leq s_n\} \cap \{n|s_n \leq r_n\} = \{n|r_n = s_n\} \in u$, hence $[r] = [s]$. □

Define two binary operations $+$ and $\times$ on ${}^*\mathbb{R}$ as follows. For any $[r], [s] \in^* \mathbb{R}$, set $[r]+[s] = [r+s]$, where $r+s$ is the sequence $(r+s)_n = r_n + s_n$ $\forall n \in \omega$. Set $[r] \times [s] = [rs]$, where rs is the sequence $(rs)_n = r_n s_n$ $\forall n \in \omega$.

$+$ is well-defined: If $r' \in [r]$, $s' \in [s]$, then $\{n|r'_n + s'_n = r_n + s_n\} \supseteq \{n|r'_n = r_n\} \cap \{n|s'_n = s_n\}$. Since $\{n|r'_n = r_n\} \cap \{n|s'_n = s_n\} \in u$, we must have $\{n|r'_n + s'_n = r_n + s_n\} \in u$, hence $[r+s] = [r'+s']$. A similar argument demonstrates that $\times$ is well-defined.

Let $[0]$ and $[1]$ denote the sequences that are constantly 0 and 1, respectively.

Theorem 1.2.3. *The structure $({}^*\mathbb{R}, +, \times, <)$ is an ordered field with zero $[0]$ and unity $[1]$.*

The following sketch of a proof is taken from [2].

Proof. The proof that ${}^*\mathbb{R}$ is a integral domain with zero $[0]$ and unity $[1]$ is routine and similar to arguments already presented.

If $[r] \in {}^*\mathbb{R}$, it's easy to check that the additive inverse of $[r]$ is $-[r] = [-r]$ (where, of course, $(-r)_n = -r_n \ \forall n \in \omega$).

The multiplicative inverse of $[r] \in {}^*\mathbb{R}$ ($[r] \neq [0]$) is the sequence $[s] \in {}^*\mathbb{R}$ defined by

$$s_n = \begin{cases} \frac{1}{r_n} & \text{if } r_n \neq 0, \\ 0 & \text{if } r_n = 0 \end{cases}$$

Since $\{n | r_n \neq 0\} \in u$, we have $\{n | r_n s_n = 1\} \in u$, hence $[r] \times [s] = [1]$.

We've shown that ${}^*\mathbb{R}$ is a field. To show that ${}^*\mathbb{R}$ is an ordered field, set $P = \{[r] \in {}^*\mathbb{R} \mid [r] > [0]\}$. We need to show that for any $[r], [s] \in P$:

1. $[r] + [s] \in P$
2. $[r] \times [s] \in P$
3. $-[r] \notin P$

We'll prove 2 (the others are similar).

We have $\{n | r_n s_n > 0\} \supseteq \{n | r_n > 0\} \cap \{n | s_n > 0\}$. Since $\{n | r_n > 0\} \cap \{n | s_n > 0\} \in u$, we have $\{n | r_n s_n > 0\} \in u$, and hence $[r] \times [s] \in P$. □

$\mathbb{R}$ is embedded in ${}^*\mathbb{R}$: Each $a \in \mathbb{R}$ corresponds directly with the equivalence class of the constant sequence $\{a_n\}$, $a_n = a \ \forall n \in \omega$. It's easy to check that the order and algebraic structure on this embedded copy of $\mathbb{R}$ are isomorphic to the usual structure on $\mathbb{R}$.

Define an absolute value on $\mathbb{R}$ by setting $|[r]| = [|r|]$, where $|r|$ is the sequence defined by $|r|_n = |r_n|$.

Proposition 1.2.4. *(Triangle Inequality) For any* $[r], [s] \in {}^*\mathbb{R}$, *we have* $|[r] + [s]| \leq |[r]| + |[s]|$.

Proof. This follows directly from the fact that $|r_n + s_n| \leq |r_n| + |s_n|$ for all $n \in \omega$. □

Definition 1.2.5. $[r] \in {}^*\mathbb{R}$ *is said to be an* ***infinitesimal*** *if* $|[r]| < a$ *for all positive real numbers* a.

It's clear that 0 is an infinitesimal. To see that there is a nontrivial infinitesimal, consider the sequence $\epsilon = \{\epsilon_n\}$ defined by $\epsilon_0 = 1$ and $\epsilon_n = \frac{1}{n}$ for $n = 1, 2, 3, ...$ If a is any positive real number, then the set $\{n | \epsilon_n < a\}$ is an element of u since it is cofinite. It follows that $0 < [\epsilon] < a$ for all positive real a, and therefore $[\epsilon]$ is a non-real infinitesimal.

Definition 1.2.6. $[r] \in {}^*\mathbb{R}$ *is said to be an* ***unlimited*** *if* $a < |[r]|$ *for all real numbers* a.

$[\Omega] \in {}^*\mathbb{R}$ defined by $\Omega_n = n$ for all $n \in \omega$ is unlimited: If a is any real number, the set $\{n | \Omega_n > a\}$ is cofinite, hence it is an element of u. It follows that $[\Omega] > a$ for all real numbers a.

Denote the finite (i.e., not unlimited) elements of ${}^*\mathbb{R}$ by $\mathrm{fin}({}^*\mathbb{R})$. $\mathrm{fin}({}^*\mathbb{R})$ is a subgroup of ${}^*\mathbb{R}$ under addition: If $[r], [s] \in \mathrm{fin}({}^*\mathbb{R})$ with $|[r]| < a_r$ and $|[s]| < a_s$, then $|[r]+[s]| \leq |[r]|+|[s]| < a_r + a_s$, so $[r]+[s] \in \mathrm{fin}({}^*\mathbb{R})$ (it's also clear that $|-[r]| = |[r]| < a_r$, and so $-[r] \in \mathrm{fin}({}^*\mathbb{R})$).

Consider the subgroup (under addition) of $\mathrm{fin}({}^*\mathbb{R})$ generated by $\{2\pi\}$, i.e., the subgroup consisting exactly of the integral multiples of 2π. Denote this subgroup by (2π) and the corresponding quotient group by $\mathrm{fin}({}^*\mathbb{R})/(2\pi)$. If r and s are sequences in the same equivalence class of $\mathrm{fin}({}^*\mathbb{R})/(2\pi)$, we'll write $r \equiv s \pmod{2\pi}$.

Now denote the set of sequences on the real number interval $[0, 2\pi)$ by $[0, 2\pi)^\omega$, and denote the set of equivalence classes of those sequences under $\equiv$ by ${}^*[0, 2\pi)$.

Proposition 1.2.7. *There is a bijection between fin(*$^*\mathbb{R})/(2\pi)$ *and* ${}^*[0, 2\pi)$*:*

Proof. Let $r = \{r_n\}$ be a sequence from some equivalence class in $\mathrm{fin}({}^*\mathbb{R})/(2\pi)$. Because $[r]$ is a finite hyperreal, r is bounded on some $F \in u$, i.e., there is a real number a such that $F = \{n| \, |r_n| < a\} \in u$. By primality of u, there is some integer k such that $F_k = \{n | 2\pi k \leq r_n < 2\pi(k+1)\} \in u$. Define $r' \in [0, 2\pi)^\omega$ by

$$r'_n = \begin{cases} r_n - 2\pi k & \text{if } n \in F_k, \\ 0 & \text{if } n \notin F_k \end{cases}$$

It's clear that $r \equiv r' \pmod{2\pi}$. We identify the equivalence class of r in $\mathrm{fin}({}^*\mathbb{R})/(2\pi)$ with the equivalence class of r' in ${}^*[0, 2\pi)$.

This identification is well-defined: If $s \equiv r \pmod{2\pi}$, analogous to the above there is an integer i and a set $G_i = \{n | 2\pi i \leq s_n < 2\pi(i+1)\} \in u$. Define $s' \in [0, 2\pi)^\omega$ by

$$s'_n = \begin{cases} s_n - 2\pi i & \text{if } n \in G_i, \\ 0 & \text{if } n \notin G_i \end{cases}$$

Since $r' \equiv s' \pmod{2\pi}$, we have $\{n | r'_n - s'_n = 0\} \in u$; in other words, $r' \equiv s'$ (and thus r' and s' are in the same equivalence class of ${}^*[0, 2\pi)$), hence this identification from $\mathrm{fin}({}^*\mathbb{R})/(2\pi)$ to ${}^*[0, 2\pi)$ is well-defined.

We now check that this identification is a bijection. We'll keep the same notation as above (using r and s to represent sequences from (not necessarily equal) equivalence classes in $\mathrm{fin}({}^*\mathbb{R})/(2\pi)$ and using r', s' to denote their respective corresponding sequences from equivalence classes of ${}^*[0, 2\pi)$).

The identification is one-to-one. Suppose r' and s' are in the same equivalence class of ${}^*[0, 2\pi)$. Then $r' \equiv s'$, and therefore $r \equiv r' \equiv s' \equiv s \pmod{2\pi}$, so the identification is one-to-one.

The identification is onto: This is trivial: The equivalence class of r' in ${}^*[0, 2\pi)$ is mapped onto by the equivalence class in $\mathrm{fin}({}^*\mathbb{R})/(2\pi)$ containing r'. □

Thus ${}^*[0, 2\pi)$ inherits the group structure of $\mathrm{fin}({}^*\mathbb{R})/(2\pi)$. It is easy to check that this addition inherited by ${}^*[0, 2\pi)$ is the usual addition mod 2π; i.e., if $[r], [s] \in {}^*[0, 2\pi)$, then $[r] + [s]$ is the equivalence class in ${}^*[0, 2\pi)$ of the sequence $\{r_n + s_n \pmod{2\pi}\}$.

We'll have occasion to refer to this group again later on.

On a final note regarding the hyperreals, one might ask whether the order or algebraic structure of ${}^*\mathbb{R}$ depend on the choice of the ultrafilter u. To (partially) answer this question, we must first mention the continuum hypothesis:

Definition 1.2.8. *The **Continuum Hypothesis** (CH) is the assumption that $2^\omega = \omega_1$, where ω_1 denotes the smallest uncountable cardinal and 2^ω denotes the cardinality of the set of binary sequences (or, equivalently, the cardinality of $\mathbb{R}$).*

It is well-known that CH is independent of the standard Zermelo-Fraenkel-Choice axioms of set theory. Under CH, the choice of the ultrafilter u is irrelevant; all constructions of ${}^*\mathbb{R}$ are isomorphic as ordered fields regardless of the choice of u. Without CH the situation is undetermined (see [2]).

1.3 Dedekind Completions

Let L be a set linearly ordered by $\leq$. We begin with a few definitions:

Definition 1.3.1. *$D \subseteq L$ is said to be **dense** in L if for all $a, b \in L$ with $a < b$, there exists $d \in D$ such that $a \leq d \leq b$.*

Definition 1.3.2. *L is said to be **complete** if every nonempty subset of L bounded above has a least upper bound. If M is a set linearly ordered by $\prec$ and N is a dense subset of M, M is said to be the **completion** of N if for any $m \in M$ there are $n_1, n_2 \in N$ such that $n_1 \preceq m \preceq n_2$.*

Definition 1.3.3. *A **Dedekind cut** in L is a pair $< A, B >$ of nonempty, disjoint subsets of L such that $A \cup B = L$ and for all $a \in A$ and $b \in B$, we have $a < b$. A Dedekind cut is a **gap** if A has no largest element and B has no smallest element.*

Dedekind cuts are a natural way of completing linearly ordered sets. A typical example is the standard construction of $\mathbb{R}$ as the Dedekind completion of the rational numbers $\mathbb{Q}$. We'll show that any linearly ordered set L has exactly one completion (up to isomorphism). First we need a lemma:

Lemma 1.3.4. *L has at most one completion (up to isomorphism).*

Proof. Suppose M and N are two completions of L.

We'll define a partial function $h : M \to N$ by setting $h(l) = l$ for all $l \in L$

Let X be the set of gaps in L. Define $f : M \setminus L \to X$ and $g : N \setminus L \to X$ by $f(m) =< A_m, B_m >$ and $g(n) =< A_n, B_n >$, where $A_m = \{l \in L | l < m\}$, $B_m = \{l \in L | m < l\}$, $A_n = \{l \in L | l < n\}$, and $B_n = \{l \in L | n < l\}$. All four of these sets are nonempty (by the definition of completion).

We'll show that the map f is a bijection. Suppose $m, m' \in M \setminus L$ with $m \neq m'$. Without loss of generality, $m < m'$. Since L is dense in M, there is $l \in L$ such that $m < l < m'$. Then $A_m \neq A_{m'}$ and $B_m \neq B_{m'}$, implying $f(m) \neq f(m')$, and hence f is one-to-one.

Now let $< A, B >\in X$. A is a nonempty set bounded above (by any element of B), so it has a least upper bound $m \in M$. Since A does not have a largest element, $m \notin A$ and so we have $a < m$ for all $a \in A$. Since every element of B is an upper bound for A, we have $m \leq b$ for all $b \in B$. Since B does not have a smallest element, we have $m < b$ for all $b \in B$. It follows that $m \notin L$ and $f(m) =< A, B >$. Therefore, f is onto.

Similarly, g is a bijection.

Extend h to all of M by setting $h = g^{-1} \circ f$ on $M \setminus L$. We know h is a bijection; we need to show it preserves the orders $<_M$ and $<_N$ on M and N, respectively.

h is the identity on L, so it preserves the order on L. If $l \in L$ and $m \in M \setminus L$ with $l < m$, then $l \in A_m = A_{h(m)}$, and so $h(l) = l < h(m)$. Similarly, if we had $l > m$ then $h(l) > h(m)$.

Finally, suppose $m, m' \in M \setminus L$ with $m < m'$. By density, there is $l \in L$ such that $m < l < m'$. We then have $h(m) < h(l) < h(m')$, and we're done. □

Theorem 1.3.5. *Any set L linearly ordered by $\leq$ has a completion.*

The following proof is taken directly from [4].

Proof. For $a \in L$, let $\hat{a} = \{b \in L | b < a\}$. The Dedekind cut $< \hat{a}, L \setminus \hat{a} >$ will be denoted by S_a. Let X be the set of all gaps in L. Finally, let $C = \{S_a | a \in L\} \cup X$. Thus C contains all Dedekind cuts $< A, B >$ such that A does not have a largest element or B does have a smallest element.

If $< A_0, B_0 >$ and $< A_1, B_1 >$ are Dedekind cuts such that A_1 is not a subset of A_0, then A_0 is an initial segment of A_1. Thus we can define a linear order relation $\leq_C$ on C as follows: $< A_0, B_0 > \leq_C < A_1, B_1 >$ iff $A_0 \subseteq A_1$.

The function $i : L \to C$ defined by $i(a) = S_a$ is an embedding of L into C, and it is easy to see that $i(L)$ is dense in C.

Now we show that the linearly ordered set $(C, \leq_C)$ is complete. Consider a Dedekind cut $< X, Y >$ in C, where $X = \{< A_i, B_i > | i \in I\}$ and $Y = \{< D_j, E_j > | j \in J\}$. Let

$$A' = \bigcup_{i \in I} A_i, \qquad B' = \bigcap_{i \in I} B_i.$$

Then $< A', B' >\in C$ and $X \leq_C < A', B' > \leq_C Y$. Thus $(C, \leq_C$ is complete. It is not hard to see that C has a minimum iff L does, and that a minimum element of C will necessarily be in the range of i. The same is true for maxima. This proves that $\forall a \in C\ \exists b, c \in L$ such that $i(b) \leq_C a$ and $a \leq_C i(c)$. $\square$

This completion of L is known as the **Dedekind completion** of L.

Chapter 2

The Stone-Čech Compactification of the Plane

2.1 Hypercircles in Π^*

Let Π denote the plane, $\mathbb{H}$ the closed half line of nonnegative reals, ω the natural numbers, and C the circle $\mathbb{R}$ mod 2π, all with the usual topologies. Points in Π will typically be denoted in polar coordinates: $\Pi = \{(r,\theta)|r \geq 0, \theta \in C\}$. Identify $\mathbb{H}$ as a subset of Π by $\mathbb{H} = \{(r,\theta)|r \geq 0,\ \theta = 0\}$.

We will denote the Stone-Čech compactification of a Tychonoff space X by βX and its remainder by X^*. Similarly, if X is a Tychonoff space, K a compact Hausdorff space, and $f : X \to K$ is continuous, we will denote the Stone-Čech extension of f by $\beta f : \beta X \to K$.

$\pi : \Pi \to \mathbb{H}$ will denote the standard norm map. If $X \subseteq \Pi$, we will denote the restriction of π to X by π_X.

Definition 2.1.1. *Any $f \subseteq \Pi$ such that $\pi_f : f \to \mathbb{H}$ is a homeomorphism will be said to be a **line**.*

If f is a line and $r \in \mathbb{H} \setminus \{0\}$, then $\pi_f^{-1}(r) = (r,\theta) \in \Pi$ for some unique $0 \leq \theta < 2\pi$; we set $f(r) = \theta$. If we also set $f(0) = 0$, we can consider f as a function, $f : \mathbb{H} \to C$. Conversely, any function $f : \mathbb{H} \to C$ can be considered as a subset of Π, since $f = \{(r, f(r))|r \in \mathbb{H}\} \subseteq \Pi$ (considering $(r, f(r))$ as the polar representation of a point in the plane).

Proposition 2.1.2. *Let $f : \mathbb{H} \to C$. As just mentioned, we can consider $f \subseteq \Pi$. Then f is a line if and only if $f : \mathbb{H} \to C$ is continuous everywhere except possibly at zero.*

Proof. First, suppose f is a line, that is, $\pi_f : f \to \mathbb{H}$ is a homeomorphism. Pick $x \in \mathbb{H} \setminus \{0\}$ and let $\{x_n\} \subseteq \mathbb{H}$ be a sequence converging to x. Note that $\pi_f^{-1}(r) = (r, f(r))$ for all $r \in \mathbb{H}$. Since π_f^{-1} is continuous, the sequence $\{\pi_f^{-1}(x_n)\}$ converges to $\pi_f^{-1}(x)$. By the previous note, this implies the sequence $\{f(x_n)\}$ converges to $f(x)$, implying f is continuous at x. Since x was arbitrary, f is continuous on $\mathbb{H} \setminus \{0\}$.

Now suppose $f : \mathbb{H} \to C$ is continuous (except possibly at zero). That π_f is a bijection follows from f being a function on $\mathbb{H}$. π_f inherits continuity from π, so we need to show π_f^{-1} is continuous.

Pick $x \in \mathbb{H} \setminus \{0\}$ and let $\{x_n\} \subseteq \mathbb{H}$ be a sequence converging to x. Since f is continuous at x, $\{f(x_n)\}$ converges to $f(x)$. It follows that the sequence $\{\pi_f^{-1}(x_n)\} = \{(x_n, f(x_n))\}$ converges to $(x, f(x)) = \pi_f^{-1}(x)$, and therefore π_f^{-1} is continuous at all nonzero x.

Finally, to show that π_f^{-1} is continuous at 0, let $\{x_n\} \subseteq \mathbb{H}$ be a sequence converging to 0. Then the sequence $\{\pi_f^{-1}(x_n)\} = \{(x_n, f(x_n))\}$ converges to the origin (since the first coordinate converges to 0).

Therefore, π_f^{-1} is continuous on $\mathbb{H}$, and hence π_f is a homeomorphism. □

L will be used to denote the set of lines.

We now fix $u \in H^*$ and (since we can consider u as a z-ultrafilter on H) define an equivalence relation $=_u$ on L relative to u as follows: If $f, g \in L$, we say $f =_u g$ if and only if $\{r \in \mathbb{H} | f(r) = g(r)\} \in u$.

Proposition 2.1.3. *$=_u$ is an equivalence relation on L.*

Proof. It's clear that $=_u$ is reflexive and symmetric.

To show that $=_u$ is transitive, let $f, g, h \in L$ with $f =_u g$ and $g =_u h$. Then $\{r \in \mathbb{H} | f(r) = g(r)\} \in u$ and $\{r \in \mathbb{H} | g(r) = h(r)\} \in u$, and so the intersection of these two sets is in u. The set $\{r \in \mathbb{H} | f(r) = h(r)\}$ is a closed superset of that intersection (it is closed because it is the set $(f - h)^{-1}(0)$). It follows that $\{r \in \mathbb{H} | f(r) = h(r)\} \in u$ implying that $f =_u h$, hence $=_u$ is transitive and therefore an equivalence relation. □

Denote the set of equivalences classes of L under $=_u$ by L/u. If $f \in L$, denote its equivalence class by $[f]_u$.

We now let $L^- = \{f \in L | f(r) \neq 0 \ \forall r > 0\}$. If $f \in L^-$, then $0 < f(r) < 2\pi, \forall r > 0$. This will allow us to order equivalence classes of L^- as follows:

As before, denote the set of equivalence classes of L^- under $=_u$ by L^-/u. This set of equivalence classes admits a natural linear order $\leq_u$ relative to u: If $f, g \in L^-$, we say $[f]_u \leq_u [g]_u$ if and only if $\{r \in \mathbb{H} | f(r) \leq g(r)\} \in u$.

Proposition 2.1.4. *$\leq_u$ is a linear order on L^-/u.*

Proof. We must first show that $\leq_u$ is well defined. Suppose $f, g \in L^-$; $[f]_u \leq_u [g]_u$, and pick $f' \in [f]_u$, $g' \in [g]_u$. We need to show $\{r \in \mathbb{H} | f'(r) \leq g'(r)\} \in u$:

By definitions, the three sets $\{r \in \mathbb{H} | f(r) = f'(r)\}$, $\{r \in \mathbb{H} | g(r) = g'(r)\}$, and $\{r \in \mathbb{H} | f(r) \leq g(r)\}$ are all in u, and so their intersection is in u. The set $\{r \in \mathbb{H} | f'(r) \leq g'(r)\}$ is a closed superset of this intersection (closed because $\{r \in \mathbb{H} | f'(r) \leq g'(r)\} = (g' - f')^{-1}(\mathbb{H})$), and is therefore also in u. This shows $\leq_u$ is well defined.

$\leq_u$ is transitive: Let $f, g, h \in L^-$ with $[f]_u \leq_u [g]_u$ and $[g]_u \leq_u [h]_u$. The set $\{r \in \mathbb{H} | f(r) \leq h(r)\} \in u$, since it's a closed superset of $\{r \in \mathbb{H} | f(r) \leq g(r)\} \cap \{r \in \mathbb{H} | g(r) \leq h(r)\} \in u$. This shows that $[f]_u \leq_u [h]_u$.

$\leq_u$ is anti-symmetric: Let $f, g \in L^-$ with $[f]_u \leq_u [g]_u$ and $[g]_u \leq_u [f]_u$. The set $\{r \in \mathbb{H} | f(r) = g(r)\} \in u$, since it's a closed superset of $\{r \in \mathbb{H} | f(r) \leq g(r)\} \cap \{r \in \mathbb{H} | g(r) \leq f(r)\} \in u$. Therefore, $[f]_u =_u [g]_u$.

$\leq_u$ is linear: Let $f, g \in L^-$. Write $\mathbb{H}$ as a union of three closed sets: $\mathbb{H} = \{r \in \mathbb{H} | f(r) \leq g(r)\} \cup \{r \in \mathbb{H} | f(r) = g(r)\} \cup \{r \in \mathbb{H} | g(r) \leq f(r)\}$. $\mathbb{H} \in u$, and so by primality of u, one of these three sets must be in u, implying $[f]_u \leq_u [g]_u$, $[f]_u =_u [g]_u$, or $[g]_u \leq_u [f]_u$, respectively. □

In the case that $[f]_u \leq_u [g]_u$ and $[f]_u \neq [g]_u$, we will write $[f]_u <_u [g]_u$.

We now turn our attention to Π^* (and various subsets thereof). By Prop. 1.1.14, $\beta\pi^{-1}(u)$ is a subcontinuum of Π^*.

Definition 2.1.5. *For any $u \in H^*$, the continuum $\beta\pi^{-1}(u)$ will be denoted C_u and referred to as a **hypercircle**. We will also set $C_u^- = C_u \setminus \{u\}$.*

By Prop. 1.1.12, Π^* is the disjoint union of all such hypercircles.

If $f \in L$, we know $\pi_f : f \to \mathbb{H}$ is a homeomorphism; it follows that $\beta\pi_f : \beta f \to \beta\mathbb{H}$ is also a homeomorphism. In particular, $\beta f \cap C_u$ is a single point; denote this point by $f(u)$.

Proposition 2.1.6. *Suppose $u \in \mathbb{H}^*$ and $x \in C_u$. Then if F is any closed subset of $\mathbb{H}$ with $F \in u$, we have $\pi^{-1}(F) \in x$. In particular, it follows that x is a limit point of $\pi^{-1}(F)$.*

Proof. Considering x and u as ultrafilters on Π and $\mathbb{H}$, respectively, we have $\pi(x) = u$. If $F \in u$, it follows that $\pi^{-1}(F) \in x$. □

Definition 2.1.7. *If f is any line and $u \in H^*$, $f(u)$ will be called a **linear point** of C_u. The set of linear points will be denoted as L_u. If $x \in C_u \setminus L_u$, x will be called a **nonlinear point**. Since u itself is a linear point ($f(u) = u$ when f is identically zero on $\mathbb{H}$), set $L_u^- = L_u \setminus \{u\}$.*

Proposition 2.1.8. *For a fixed $u \in H^*$, the map which sends the equivalence class $[f]_u \in L/u$ to the linear point $f(u) \in L_u$ is one-to-one.*

Proof. If $[f]_u \in L/u$, we already know $f(u)$ is a single point in L_u. We need to show this map is well defined, i.e., if $f' \in [f]_u$, we need to show $f(u) = f'(u)$:

Since $\{r \in \mathbb{H} | f(r) = f'(r)\} \in u$, there is a net $\{x_\lambda\} \subseteq \{r \in \mathbb{H} | f(r) = f'(r)\}$ converging to u. The nets $\{(x_\lambda, f(x_\lambda))\}$ and $\{(x_\lambda, f'(x_\lambda))\}$ converge to $f(u)$ and $f'(u)$, respectively. However, $\{(x_\lambda, f(x_\lambda))\} = \{(x_\lambda, f'(x_\lambda))\}$, and so $f(u) = f'(u)$.

Now suppose $f, g \in L$ with $f(u) = g(u)$. We need to show $[f]_u = [g]_u$:

$f(u) \in Cl_{\beta\Pi}(f) \cap Cl_{\beta\Pi}(g) = Cl_{\beta\Pi}(f \cap g)$, and hence there is a net $\{x_\lambda\} \subseteq f \cap g$ converging to $f(u)$. The net $\{\pi(x_\lambda)\} \subseteq \pi(f \cap g)$ converges to $\pi(f(u)) = u$. Note that $\pi(f \cap g)$ is the closed set $\{r \in \mathbb{H} | f(r) = g(r)\}$; since there is a net inside of this closed set converging to u, we have $\{r \in \mathbb{H} | f(r) = g(r)\} \in u$, and therefore $[f]_u = [g]_u$, completing the proof. □

Corollary 2.1.9. *The same map establishes a one-to-one correspondence between L^-_u and L^-/u.*

Proof. Suppose $x \in L^-_u$. We need to show there is a $g \in L^-$ such that $g(u) = x$.

Let $f \in L$ with $f(u) = x$. We know $f^{-1}(0) \notin u$ (otherwise, $f(u) = u$, a contradiction), so $\exists Z \in u$ such that $f^{-1}(0) \cap Z = \emptyset$. Define $g_Z : Z \to (0, 2\pi)$ by $g_Z(r) = f(r)\ \forall r \in Z$.

Since Z is a closed subset of the (normal) space $\mathbb{H} \setminus \{0\}$, and since $(0, 2\pi)$ is homeomorphic to $\mathbb{R}$, we can use Tietze's extension theorem to extend g_Z continuously to $g' : \mathbb{H} \setminus \{0\} \to (0, 2\pi)$. Finally, extend g' to $g : \mathbb{H} \to C$ by setting $g(0) = 0$. Then $g \in L^-$ and $g(u) = x$ since g and f agree on $Z \in u$. □

Because of this one-to-one correspondence, L^-_u inherits the linear order from L^-/u; we will continue to use $\leq_u$ for this inherited linear order on L^-_u: For any $f(u), g(u) \in L^-_u$, we have $f(u) \leq_u g(u)$ if and only if $[f]_u \leq_u [g]_u$.

This linear order on L^-_u defines a topology on L^-_u which we will denote by $\tau_{<_u}$. For the moment we will denote the basic open intervals of L^-_u by using standard interval notation: $(a, b) = \{x \in L^-_u | a <_u x <_u b\}$.

We will now show that the topology $\tau_{<_u}$ on L^-_u is the same as the topology that L^-_u inherits from $\beta\Pi$. First we need a definition, lemma, and corollary:

Definition 2.1.10. *For any interval $I \subseteq \mathbb{H}$ and $0 < \epsilon < \pi$, the polar rectangle $\{(r, \theta) | r \in I,\ -\epsilon < \theta < \epsilon\} \subseteq \Pi$ will be denoted by $R(I; \epsilon)$.*

Lemma 2.1.11. *Let $U \subseteq \Pi$ be open with $U \cap \mathbb{H} \neq \emptyset$. If (a, b) is a component of $U \cap \mathbb{H}$, then there is a continuous function $f : \mathbb{H} \to [0, \pi/2)$ such that*

1. *$(r, f(r)) \in U\ \forall r \in (a, b)$,*
2. *$f(r) > 0\ \forall r \in (a, b)$, and*
3. *$f(r) = 0$ elsewhere.*

Proof. Let $I_0 = [a_0, b_0] = [(2a+b)/3, (a+2b)/3]$ (the closed middle third of (a,b)), and, more generally, for $n \in \omega$ set $I_n = [a_n, b_n] = [a + \frac{b-a}{n+3}, b - \frac{b-a}{n+3}]$. Note that $\bigcup_{n\in\omega} I_n = (a,b)$ and that for all $n \in \omega$ we have $I_n \subseteq I_{n+1}$.

For each $x \in I_1$, pick an open interval $J_x \subseteq \mathbb{H}$ and an open polar rectangle $R_x = R(J_x; \epsilon_x)$ such that $x \in R_x \subseteq U$. $\{R_x | x \in I_1\}$ is an open cover of the compact set I_1, so we can take a finite subcover $R_{x_1}, ..., R_{x_n}$. Set $\phi_0 = [min\{\epsilon_{x_1}, ..., \epsilon_{x_n}\}]$. Note that $I_1 \subseteq R([a_1, b_1]; \phi_0) \subseteq U$; label this polar rectangle R_1.

Proceed by induction: Having constructed ϕ_n so that $0 < \phi_n < \pi/2$ and $I_{n+1} \subseteq R([a_{n+1}, b_{n+1}]; \phi_n) \subseteq U$, construct ϕ_{n+1} as follows:

As before, cover I_{n+2} with open polar rectangles: for each $x \in I_{n+2}$, take an open interval $J_x \subseteq \mathbb{H}$ and $R_x = R(J_x; \epsilon_x)$ so that $x \in R_x \subseteq U$. Since I_{n+2} is compact, we can take a finite subcover $R_{x_1}, ..., R_{x_m}$. Now set $\phi_{n+1} = [min\{\phi_n/2, \epsilon_{x_1}, ..., \epsilon_{x_m}\}]$. As before, note that $I_{n+2} \subseteq R([a_{n+2}, b_{n+2}]; \phi_{n+1}) \subseteq U$; label this rectangle R_{n+2}.

Note that the sequence $\{\phi_n\}$ is decreasing and bounded below by zero, so it converges to some $\phi_\omega \in \mathbb{H}$.

Now define $g_0 : I_0 \to (0, \pi/2)$ by $g_0(r) = \phi_0/2 \ \forall r \in I_0$, which is clearly continuous.

Again, proceed by induction: Having constructed $g_n : I_n \to (0, \pi/2)$, we will construct a continuous function $g_{n+1} : I_{n+1} \to (0, \pi/2)$ so that g_{n+1} extends g_n:

$$g_{n+1}(r) = \begin{cases} g_n(r) & \text{if } r \in I_n, \\ \phi_{n+1}/2 & \text{if } r \in [a_{n+1}, a'] \cup [b', b_{n+1}], \\ \text{connected linearly} & \text{elsewhere.} \end{cases}$$

(where a', b' are fixed arbitrary points in $(a_{n+1}, a_n), (b_n, b_{n+1})$, respectively).

Now define $g : (a,b) \to (0, \pi/2)$ by setting $g = \bigcup_{n\in\omega} g_n$. g is continuous on (a,b), and since $\lim_{r\to a^+} g(r) = \lim_{r\to b^-} g(r) = \phi_\omega/2$, we can continuously extend g to the closed interval $[a,b]$ by setting $g(a) = g(b) = \phi_\omega/2$.

Finally, define $f : \mathbb{H} \to [0, \pi/2)$ by

$$f(r) = \begin{cases} g(r) - \phi_\omega/2 & \text{if } r \in [a,b], \\ 0 & \text{elsewhere.} \end{cases}$$

f is clearly continuous. Since the sequence $\{\phi_n\}$ is strictly decreasing, for all $r \in (a,b)$ we have $f(r) = g(r) - \phi_\omega/2 > 0$; elsewhere f(r)=0.

Finally, if $r \in (a,b)$, pick the smallest $n \in \omega \setminus \{0\}$ such that $r \in I_n$. Then $0 < f(r) \leq g_n(r) \leq \phi_{n-1}/2 < \phi_{n-1}$, therefore $(r, f(r)) \in R_n \subseteq U$, completing the proof. $\square$

Corollary 2.1.12. *Let $U \subseteq \Pi$ be open with $U \cap \mathbb{H} \neq \emptyset$. Then there is a continuous function $f : \mathbb{H} \to [0, \pi/2)$ such that*

1. If $r \in U \cap \mathbb{H}$, then $(r, f(r)) \in U$ and $f(r) > 0$.

2. $f(r) = 0$ elsewhere.

Proof. U intersects $\mathbb{H}$ in at most countably many disjoint intervals; order these intervals by ω: $\{I_n\}_{n\in\omega}$.

For each I_n, apply Lemma 2.1.11 to construct $f_n : \mathbb{H} \to [0, \pi/2)$ such that:

1. $(r, f_n(r)) \in U \ \forall r \in I_n$,
2. $f_n(r) > 0 \ \forall r \in I_n$, and
3. $f_n(r) = 0$ elsewhere

Define $f' : \mathbb{H} \to [0, \pi/2)$ by $f'(r) = \Sigma_{n\in\omega} f_n(r) \ \forall r \in \mathbb{H}$ (this is clearly convergent since for any $r \in \mathbb{H}$, we have that $f_n(r) \neq 0$ for at most one $n \in \omega$).

For any $r \in \mathbb{H}$, let $d(r)$ denote the distance from r to the set $\mathbb{H} \setminus U$; it's clear that this is a continuous function on $\mathbb{H}$. Now define $f : \mathbb{H} \to [0, \pi/2)$ by $f(r) = \min\{f'(r), d(r)\} \ \forall r \in \mathbb{H}$.

It's clear that f satisfies properties 1. and 2. stated in the Corollary. It's also clear that f continuous on $U \cap \mathbb{H}$ (since both f' and d are continuous on $U \cap \mathbb{H}$ and $\forall r \in \mathbb{H}$, $f(r) = \frac{f'(r)+d(r)}{2} - \frac{|f'(r)-d(r)|}{2}$).

We need to show that f is continuous on $\mathbb{H} \setminus U$, so pick an arbitrary $r_0 \in \mathbb{H} \setminus U$. Given $\epsilon > 0$, pick $\delta = \epsilon$. Then if $|r - r_0| < \delta$, we have $|f(r) - f(r_0)| = |f(r)| \leq |d(r)| \leq |r - r_0| < \delta = \epsilon$, and we're done. □

Proposition 2.1.13. *The topology $\tau_{<_u}$ on L_u^- is the same as the topology that L_u^- inherits from $\beta\Pi$.*

Proof. Denote the usual topology on $\beta\Pi$ by τ. For any $X \subseteq \Pi$, denote the closure of X in Π by $\overline{X}$.

Suppose $(a, b) \in \tau_{<_u}$. We will construct U such that $U \cap L_u^- = (a, b)$:

Let $f_a, f_b \in L^-$ such that $f_a(u) = a$ and $f_b(u) = b$. Set $U' = \{(r, \theta) \in \Pi | f_a(r) < \theta < f_b(r)\}$ (without loss of generality, assume $f_a(0) = f_b(0) = 0$, so that the origin is not in U'), and define $U = Ex\ U' = \beta\Pi \setminus Cl_{\beta\Pi}(\Pi \setminus U')$.

Let $c \in (a, b)$. Pick $f_c \in L^-$ such that $f_c(u) = c$. Then $A = \{r \in \mathbb{H} : f_a(r) \leq f_c(r) \leq f_b(r)\} \in u$, so there is a net $\{x_\lambda\} \subseteq A$ converging to u. Note that the net $\{\pi_{f_c}^{-1}(x_\lambda)\} \subseteq \overline{U'}$ converges to c, hence c is a limit point of $\overline{U'}$.

Now suppose c is also a limit point of $\Pi \setminus U'$, so of course c is a limit point of $\overline{\Pi \setminus U'}$ as well. Since $\overline{U'}$ and $\overline{\Pi \setminus U'}$ are both zero-sets in Π, we have $c \in Cl_{\beta\Pi}[\overline{U'}] \cap Cl_{\beta\Pi}[\overline{\Pi \setminus U'}] = Cl_{\beta\Pi}[\overline{U'} \cap \overline{\Pi \setminus U'}]$, and so c is a limit point of $\overline{U'} \cap \overline{\Pi \setminus U'}$. But $\overline{U'} \cap \overline{\Pi \setminus U'}$ is the boundary of U', which is contained in

$f_a \cup f_b$. Therefore, c is a limit point of $f_a \cup f_b$. However, the only limit points of $f_a \cup f_b$ contained in L_u^- are a and b, which contradicts the fact that c is strictly between a and b.

Therefore, c is not a limit point of $\Pi \setminus U'$, and so $c \in U = \beta\Pi \setminus Cl_{\beta\Pi}(\Pi \setminus U')$.

Conversely, suppose $c \in U \cap L_u^-$. Again, pick $f_c \in L^-$ such that $f_c(u) = c$.

Suppose $c \notin (a, b)$. Then $\{r \in \mathbb{H} : f_a(r) \leq f_c(r) \leq f_b(r)\} \notin u$, so either $\{r \in \mathbb{H} : f_a(r) \leq f_c(r)\} \notin u$ or $\{r \in \mathbb{H} : f_c(r) \leq f_b(r)\} \notin u$. We will consider the latter case (the former case is similar).

Since u is a prime z-filter, $\{r \in \mathbb{H} : f_c(r) \leq f_b(r)\} \notin u$ implies $B = \{r \in \mathbb{H} : f_b(r) \leq f_c(r)\} \in u$. Therefore, there is a net $\{x_\lambda\} \subseteq B$ converging to u. Note that the net $\{\pi_{f_c}^{-1}(x_\lambda)\} \subseteq \Pi \setminus U'$ converges to c, hence c is a limit point of $\Pi \setminus U'$. Therefore, $c \notin \beta\Pi \setminus Cl_{\beta\Pi}(\Pi \setminus U') = U$, a contradiction. Therefore $c \in (a, b)$.

Therefore, $U \cap L_u^- = (a, b)$, and hence $\tau_{<_u} \subseteq \tau$.

Now let $U \in \tau$ and fix $x \in U \cap L_u^-$ (without loss of generality, assume $u \notin U$). We will construct $(a, b) \in \tau_{<_u}$ such that $x \in (a, b) \subseteq U \cap L_u^-$.

Since $\{Ex\ U' | U' \subseteq \Pi \text{ is open}\}$ is a base for τ (by Proposition 1.1.10), without loss of generality we can assume $U = Ex\ U'$ for some open $U' \subseteq \Pi$.

Let $f_x \in L^-$ such that $f_x(u) = x$. Considering $f_x \subseteq \Pi$ (as in Proposition 2.1.2), we know $f_x \cap U' \neq \emptyset$ (since x is a limit point of f_x).

Consider the homeomorphism $F : \Pi \to \Pi$ defined by $F(r, \theta) = (r, \theta - f_x(r))$. Clearly the images $F(f_x) = \mathbb{H}$ and $F(U') \cap \mathbb{H} \neq \emptyset$.

By Corollary 2.1.12 (and by symmetry), there are functions $f^* : \mathbb{H} \to [0, \pi/2)$ and $f_* : \mathbb{H} \to (-\pi/2, 0]$ such that:

1. If $r \in F(U') \cap \mathbb{H}$, then $(r, f^*(r))$, $(r, f_*(r)) \in F(U')$, $f^*(r) > 0$, and $f_*(r) < 0$,
2. $f^*(r) = f_*(r) = 0$ elsewhere.

Now define the lines f_a and f_b by $f_a(r) = f_*(r) + f_x(r) = F^{-1}(f_*)$ and $f_b(r) = f^*(r) + f_x(r) = F^{-1}(f^*)$. It is then easy to check that f_a and f_b satisfy:

1. If $(r, f_x(r)) \in U' \cap f_x$, then $(r, f_a(r))$, $(r, f_b(r)) \in U'$, $f_b(r) > f_x(r)$, and $f_a(r) < f_x(r)$,
2. $f_a(r) = f_b(r) = f_x(r)$ elsewhere.

Set $V = \{(r, \theta) \in \Pi | f_a(r) < \theta < f_b(r)\}$. Clearly $V \subseteq U'$, and hence $Ex\ V \subseteq Ex\ U' = U$.

Let $a = f_a(u)$ and $b = f_b(u)$. Then $x \in (a, b) = Ex\ V \cap L_u^- \subseteq U \cap L_u^-$, and we are done. □

2.2 Hyperreals in Π^*

In this section we will investigate the structure of the hypercircles further and demonstrate a relationship in certain cases between the linear points L_u and the hyperreals.

We now consider C_u^-. If $x \in C_u^-$ is a nonlinear point and f is a line, then by primality of the z-ultrafilter x, either $\{(r,\theta) \in \Pi | 0 \leq \theta \leq f(r)\} \in x$ or $\{(r,\theta) \in \Pi | f(r) \leq \theta \leq 2\pi\} \in x$ (but not both, by nonlinearity of x). In the former case, we'll say "x lies below f"; in the latter, "x lies above f".

This allows us to define an equivalence relation $\sim_u$ on C_u^- in the following manner. If $a \in L_u^-$, its equivalence class is the singleton set $\{a\}$. If $x, y \in C_u^-$ are nonlinear points, define $x \sim_u y$ if and only if $\{f \in L | x \text{ lies above } f\} = \{f \in L | y \text{ lies above } f\}$. It's clear that this establishes an equivalence relation on C_u^-. If $x \in C_u^-$, denote its equivalence class by $[x]_{\sim_u}$.

We will now define a linear order on these equivalence classes. This order will be identical to $<_u$ on (the singleton set equivalence classes of) L_u; since we are then actually extending $<_u$, we will continue to use the same notation.

If $[x]_{\sim_u}$ is an equivalence class of nonlinear points, $a \in L_u$ with corresponding line f_a, define $a <_u [x]_{\sim_u}$ if and only if x lies above f_a; $[x]_{\sim_u} <_u a$ if and only if x lies below f_a.

This is well defined since if $a <_u [x]_{\sim_u}$ and g is another line limiting on a, we have $\{(r,\theta) \in \Pi | g(r) \leq \theta \leq 2\pi\} \supseteq \{(r,\theta) | f_a(r) = g(r)\} \cap \{(r,\theta) \in \Pi | g(r) \leq \theta \leq 2\pi\} = \{(r,\theta) | f_a(r) = g(r)\} \cap \{(r,\theta) \in \Pi | f_a(r) \leq \theta \leq 2\pi\} \in x$, and so x lies above g. (The other order, $[x]_{\sim_u} <_u a$, is similarly well defined).

From the argument in the previous paragraph, it is also clear that this definition of $<_u$ is consistent with the original definition; if $a, b \in L_u^-$ with $a <_u b$ under the previous definition, we still have $a <_u b$ in the current definition (i.e., b lies above f_a, where f_a is any line limiting on a).

Extend $<_u$ transitively: If $[x]_{\sim_u}, [y]_{\sim_u}$ are two distinct equivalence classes of nonlinear points and $a \in L_u^-$, then $[x]_{\sim_u} <_u a$, $a <_u [y]_{\sim_u}$ together imply $[x]_{\sim_u} <_u [y]_{\sim_u}$.

This (and similar arguments) establish:

Proposition 2.2.1. *This extension of $<_u$ is a linear order on the equivalence classes of C_u^- under $\sim_u$.*

Proposition 2.2.2. *These equivalence classes of C_u^- form the Dedekind completion of L_u^-.*

Proof. It's clear that each equivalence class of nonlinear points corresponds to a unique gap of linear points: Each equivalence class partitions the linear points into two sets–those linear points larger than the equivalence class, and those smaller.

We'll now show that each gap corresponds to a unique equivalence class of nonlinear points. As usual, given $a \in L_u^-$, f_a will be used to denote an arbitrary line corresponding to a. If f, g are lines with $f <_u g$, we will use $[f, g]$ denote the closed region of the plane defined by $\{(r,\theta) | f(r) \leq \theta \leq g(r)\}$.

Let $< A, B >$ be a gap in L_u^-. Note that the family of closed sets $\{[f_a, f_b] \cap \pi^{-1}(F) | a \in A, b \in B, F \in u\}$ has the finite intersection property, so it extends to a z-ultrafilter. Any such z-ultrafilter will be greater than all linear points in A, less than all linear points in B, and contained in C_u^-. Conversely, any z-ultrafilter satisfying these three properties extends this family of closed sets. Therefore, this gap corresponds to the unique equivalence class of z-ultrafilters in C_u^- above the linear points in A and below the linear points in B. □

Proposition 2.2.3. *If $u \in \mathbb{H}^*$ is in the closure of a countable, discrete subset of $\mathbb{H}$, then L_u is dense in C_u.*

Proof. Let D denote the countable discrete subset of $\mathbb{H}$. Pick $x \in C_u^-$ and an open neighborhood $U \subset \beta\Pi$ of x. Without loss of generality, assume $U = Ex\ U'$ for some open $U' \subseteq \Pi$.

$U' \cap \pi^{-1}(D) \neq \emptyset$, so the set $D' = \{r \in D | U' \cap \pi^{-1}(r) \neq \emptyset\}$ is nonempty; in fact, $D' \in u$. Construct a line $f : \mathbb{H} \to C$ as follows: For all $r \in D'$, arbitrarily define $f(r)$ so that $(r, f(r)) \in U'$. Extend f continuously to all of $\mathbb{H}$ in any arbitrary manner (this is possible since D' is discrete).

Clearly $f(u) \notin Cl_{\beta\Pi}(\Pi \setminus U')$, and so $f(u) \in \beta\Pi \setminus Cl_{\beta\Pi}(\Pi \setminus U') = Ex\ U' = U$, hence L_u is dense in C_u. □

We can define a binary operation on L_u as follows: For any $a, b \in L_u$ (with corresponding lines f_a and f_b, respectively), define $a + b = (f_a + f_b)(u)$.

Proposition 2.2.4. *This addition is well-defined and L_u is an abelian group under $+$.*

Proof. Pick $a, b \in L_u$ and corresponding lines $f_a,\ f_b$, respectively. Suppose $f'_a \in [f_a]_u, f'_b \in [f_b]_u$. We know $\{r \in \mathbb{H} | f'_a(r) + f'_b(r) = f_a(r) + f_b(r)\} \in u$ since it is a closed superset of $\{r \in \mathbb{H} | f'_a(r) = f_a(r)\} \cap \{r \in \mathbb{H} | f'_b(r) = f_b(r)\} \in u$. Therefore, $f'_a + f'_b \in [f_a + f_b]_u$ and hence our binary operation is well-defined.

Clearly $a + u = u + a = a$ for any $a \in L_u$ (pick $f : \mathbb{H} \to C$ identically zero for the line corresponding to u), so u is the additive identity.

If $a \in L_u$, set $-a = -f_a(u)$. It is then clear that $a + (-a) = (-a) + a = u$ (since $f_a + (-f_a)$ is identically zero on $\mathbb{H}$).

Finally, if $a, b \in L_u$, then $a + b = b + a$ (since $f_a + f_b = f_b + f_a$). □

Proposition 2.2.5. *L_u is homogeneous.*

Proof. Pick arbitrary $a, b \in L_u$. We will construct a homeomorphism of L_u onto itself taking a to b.

Let f_a and f_b be lines such that $f_a(u) = a,\ f_b(u) = b$. Define $F : \Pi \to \Pi$ by $F(r, \theta) = (r, \theta - f_a(r) + f_b(r))$. F is clearly a homeomorphism, and so its extension $\beta F : \beta\Pi \to \beta\Pi$ is also a homeomorphism.

Let $c \in L_u$ with corresponding line f_c, and let $\{(r_\lambda, f_c(r_\lambda))\} \subseteq f_c$ be a net converging to c. The net $\{F(r_\lambda, f_c(r_\lambda))\} = \{(r_\lambda, f_c(r_\lambda) - f_a(r_\lambda) + f_b(r_\lambda))\}$ converges to $\beta F(c) = \beta F(f_c(u)) =$

$(f_c - f_a + f_b)(u) = c - a + b \in L_u$, and so βF maps L_u to L_u. We need to show the restriction $\beta F' : L_u \to L_u$ is a bijection:

If $x \in L_u$, then $\beta F'(x + a - b) = (x + a - b) - a + b = x$, and so $\beta F'$ is onto.

If $x, y \in L_u$ with $\beta F'(x) = \beta F'(y)$, then $x - a + b = y - a + b$. It follows that $x = y$, and so $\beta F'$ is one-to-one.

Finally, $\beta F'(a) = a - a + b = b$, completing the proof □

Theorem 2.2.6. *If $u \in \mathbb{H}^*$ is in the closure of a countable discrete subset of $\mathbb{H}$, then L_u and fin($^*\mathbb{R}$)$/(2\pi)$ (i.e., the circle of finite hyperreals mod 2π) are isomorphic as groups (for some appropriate ultrafilter u' on ω).*

Proof. Let D denote the countable discrete subset of $\mathbb{H}$. It's easy to check that the collection of sets $u_D = \{F \cap D | F \in u\}$ is an ultrafilter on D.

Let $f : D \to \omega$ be the natural order preserving bijection. Set $u' = f(u_D)$. Then the construction of $^*[0, 2\pi)$ using the ultrafilter u' is clearly equivalent to the construction of L_u: The sequences in $[0, 2\pi)^\omega$ correspond directly to the lines in L with their domain restricted to $D \in u$. Since $^*[0, 2\pi)$ and fin($^*\mathbb{R}$)$/(2\pi)$ are isomorphic (by Prop. 1.2.7), we have that fin($^*\mathbb{R}$)$/(2\pi)$ is isomorphic to L_u. □

2.3 A Mapping Property of Certain Subcontinua of Π^*

We need two final results before proving that many subcontinua of Π^* map onto any arbitrary continuum of weight $\leq \omega_1$.

Proposition 2.3.1. *Let f_π be the line defined by $f_\pi(r) = \pi$ for all $r \in \mathbb{H}$. For any $a, b \in L_u$, there is a homeomorphism $G^* : \Pi^* \to \Pi^*$ such that $G^*(a) = u$ and $G^*(b) = f_\pi(u)$.*

Proof. From the proof of Prop. 2.2.5, there is a homeomorphism $\beta F : \beta\Pi \to \beta\Pi$ fixing L_u such that $\beta F(a) = u$, so without loss of generality, we can assume $a = u$.

We need a homeomorphism of $\beta\Pi$ that fixes u and sends b to $f_\pi(u)$. Define the line f_0 by $f_0(r) = 0$ for all $r \in \mathbb{H}$ (so that $f_0(u) = u$). From the proof of Corollary 2.1.9, there is a line f_b such that $f_b(u) = b$ and $0 < f_b(r) < 2\pi$ $\forall r \in \mathbb{H}$.

Define $G : \Pi \to \Pi$ by

$$G(r, \theta) = \begin{cases} (r, \frac{\pi}{f_b(r)}\theta) & \text{if } 0 \leq \theta < f_b(r), \\ (r, \frac{\pi(\theta - 2\pi)}{2\pi - f_b(r)} + 2\pi) & \text{if } f_b(r) < \theta < 2\pi \end{cases}$$

It's routine to check that G is a homeomorphism, the important thing to note is that G stretches/shrinks the plane (around the origin) so that the line f_0 is fixed, while the line f_b is taken to the line f_π.

Since G preserves distances of points from the origin, if $\{x_\lambda\} \subseteq f_b$ is a net converging to $f_b(u) = b$, then the net $\{G(x_\lambda)\} \subseteq f_\pi$ converges to $f_\pi(u)$, and so the extension $\beta G(b) = G^*(b) = f_\pi(u)$. $G^*(u) = u$ as well, and we are done. □

Proposition 2.3.2. *If a and b are any two distinct linear points of the hypercircle C_u, then $C_u \backslash \{a, b\}$ is disconnected.*

Proof. Note that the map in Prop. 2.2.5 preserves distance from the origin on Π, therefore it maps C_u homeomorphically to itself. Hence, without loss of generality, we can assume $a = u$.

Define f_0 and f_b as before in the proof of Prop. 2.3.1. Using the notation previously defined in Prop. 2.2.2, consider the regions of the plane $A = [f_0, f_b]$ and $B = [f_b, f_0]$. Clearly $A \cup B = \Pi$ and $A \cap B = f_0 \cup f_b$.

Note that the set $F = \beta f_0 \cup \beta f_b$ is closed, hence the sets $U = (\beta A) \setminus F$ and $V = (\beta B) \setminus F$ are both open. It's clear that each of these open sets have nonempty intersection with C_u. But then $(U \cap C_u) \cup (V \cap C_u) = C_u \setminus \{a, b\}$, and so we have disconnected $C_u \setminus \{a, b\}$. □

Theorem 2.3.3. *Let $K \subseteq \Pi^*$ be a non-degenerate subcontinuum. If:*

1. *K intersects two distinct hypercircles, or*
2. *K contains two distinct linear points within the same hypercircle, or*
3. *$K \subseteq C_u$ where $u \in \beta\mathbb{H}$ is in the closure of a countable, discrete subset of $\mathbb{H}$,*

then K maps onto any continuum of weight $\leq \omega_1$.

Proof. In his PhD thesis, Franco Obersnel showed that any non-degenerate subcontinuum of H^* maps onto any continuum of weight $\leq \omega_1$, so it is sufficient to map K onto a non-degenerate subcontinuum of H^*.

If K intersects two distinct hypercircles, this is trivial since in that case $\beta\pi(K)$ itself is a non-degenerate subcontinuum of H^*.

Suppose K contains two distinct linear points within the same hypercircle C_u. By Prop. 2.3.1, without loss of generality we can assume the two points are u and $f_\pi(u)$.

Define $M, N \subseteq \mathbb{H}$ by $M = \bigcup_{j=0}^{\infty} [2^{2j}, 2^{2j+1}]$, $N = \bigcup_{j=0}^{\infty} [2^{2j+1}, 2^{2j+2}]$. By primality of u, either $M \in u$ or $N \in u$. Without loss of generality, suppose $N \in u$ (the other case is similar).

Define a homeomorphism $h : \Pi \to \Pi$ by:

$$h(r, \theta) = \begin{cases} (\frac{\theta+\pi}{2\pi} r, \theta) & \text{if } 0 \leq \theta \leq \pi, \\ (\frac{3\pi-\theta}{2\pi} r, \theta) & \text{if } \pi < \theta < 2\pi \end{cases}$$

Again, it is routine to check that this is homeomorphism.

Extend h to $\beta h : \beta\Pi \to \beta\Pi$. Note that h is the identity on f_π (and hence $\beta h(f_\pi(u)) = f_\pi(u)$), while on f_0, h maps N onto M.

Either $\beta h(u) = u$ or $\beta h(u) \neq u$. In the latter case, we still have $\beta h(u) \in Cl_{\beta\Pi}(f_0)$ (since the image of f_0 under βh is still f_0), implying u and $\beta h(u)$ lie on different hypercircles, and therefore we are done ($\beta h(K)$ is now a subcontinuum of Π^* that intersects two distinct hypercircles).

Suppose $\beta h(u) = u$. Since (along f_0) h maps N onto M, this implies both $M, N \in u$. Therefore, $M \cap N = \{2^j | j = 1, 2, 3, ...\} \in u$.

By primality of u, either $E = \{2^j | j$ is an even positive integer$\}$ or $O = \{2^j | j$ is an odd positive integer$\}$ is in u. Without loss of generality, assume $E \in u$ (the other case is similar).

Since $\beta h(u) = u$ and $h(E) = O$, we have $O \in u$. Therefore, $E \cap O = \emptyset \in u$. This is impossible, hence we must have $\beta h(u) \neq u$, and we are done with case 2.

Finally, suppose $K \subseteq C_u$ where $u \in \beta\mathbb{H}$ is in the closure of a countable, discrete $D \subseteq \mathbb{H}$.

K contains at most one linear point (otherwise we're back in case 2), so without loss of generality, assume $u \notin K$. (This will allow us to use the linear order $<_u$ on (equivalence classes of) C_u^-).

Pick any linear point $b \in C_u^- \setminus K$. Then for all $x \in K$, the equivalence class $[x]_{\sim_u} <_u b$ (or $\forall x \in K$, $[x]_{\sim_u} >_u b$, but without loss of generality we'll assume the former). Otherwise K is disconnected by b and u (by Prop. 2.3.2).

Again, define f_0 and f_b as in Prop. 2.3.1; also, $[f_0, f_b]$ is defined as in Prop. 2.2.2.

For all $r \in D$, define $A_r = C_r \cap [f_0, f_b]$ (i.e., each A_r is the arc of the circle C_r starting at the line f_0 and going counter clockwise to f_b).

Label the points of D by r_n ($n \in \omega$) in the natural way so that $r_0 < r_1 < r_2 < ...$ (we can do this since D is countable and discrete).

Trace out a homeomorphic copy of $\mathbb{H}$ as follows. We'll connect the arcs A_r by pieces of the lines f_0 and f_b:

For any closed interval $[m, n]$ in $\mathbb{H}$ and any line f, let $f[m, n]$ denote the closed segment of f starting a distance of m from the origin and ending a distance of n from the origin.

Our copy of $\mathbb{H}$ will now be $X = [\bigcup_{r \in D} A_r] \cup \{f_0[r_{2n}, r_{2n+1}] | n \in \omega\} \cup \{f_b[r_{2n+1}, r_{2n+2}] | n \in \omega\}$.

Since K lies between u and b, we have $K \subseteq X^*$. Since X^* is homeomorphic to $\mathbb{H}^*$, K maps onto any arbitrary continuum of weight $\leq \omega_1$. $\square$

2.4 Open Questions

Question 2.4.1. *Can the remaining non-degenerate subcontinua of* Π^* *not covered in Theorem 2.3.3 map onto any arbitrary continuum of weight* $\leq \omega_1$*?*

Question 2.4.2. *If* $u, u' \in \mathbb{H}^*$*, under what conditions will* C_u *be homeomorphic to* $C_{u'}$*?*

Question 2.4.3. *Is* C_u *homogeneous for any/all* $u \in \mathbb{H}^*$*?*

Question 2.4.4. *Is L_u dense in C_u if u is not in the closure of any countable discrete subset of $\mathbb{H}$?*

Question 2.4.5. *Can the group structure on L_u be extended to all of C_u?*

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