Abstract

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With increased globalization and competition in the current market, supply chain has become longer and more complicated than ever before. An effective and efficient supply chain is crucial and essential to a successful firm. In a supply chain, inventories are a very important component as the investment in inventories is enormous. Inventory management is always coupled with other functions, for example purchasing, production and marketing. In this dissertation, we study inventory management for both single-stage and multi-echelon systems. Two main streams of research work are summarized. The first is the joint optimization of pricing and inventory control for continuous/periodic review single-stage inventory/production systems. We characterize the optimal policies and further develop efficient computational algorithms to find the optimal control parameters. We also provide insights on the pricing and inventory relationship. The second is the analysis of multi-echelon inventory systems, in which we derive the optimal inventory control policies for several different systems that have not yet been studied in the current literature. Moreover, simple bounds and heuristics for the optimal policies are developed for the serial systems with and without expedited shipping so that the implementability of the optimal policies is improved.

JOINT OPTIMIZATION IN SUPPLY CHAIN

by

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Biography

Sean Xiang Zhou was born to Zhuowei Zhou and Congcong Lin on December 25th, 1978 in the city of Xiamen, Fujian Province, China. He spent the first 19 years of his life there before he left for his undergraduate study at Zhejiang University (ZJU), Hangzhou, China.

He was admitted to Zhejiang University in 1997 and majored in Automation (now called System Science and Engineering) in the College of Electrical Engineering. In the four years of study at ZJU, he not only got the knowledge and skill of engineering but also built strong mathematical background, which helps him greatly in the later graduate study and research.

In August of 2001, he joined the Ph.D. Program of Industrial Engineering Department at North Carolina State University, Raleigh, NC. One semester after that, up the suggestion by Professor Xiuli Chao, his later Ph.D. adviser, he transferred to the Operations Research Program. Initially, the mathematical rigor of the courses was both hard and amusing to him. It took him a while to understand why he was asked to write proofs of so many statements that seem so obvious to the engineer in him and to develop the skills to rigorously prove a statement. From then on, he started to touch the inventory theory and supply chain management and decided to do research in this area. After some hard work, great advice and five years, he finds himself at the end of graduate study.

To my family and my teachers.

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I thank Professor Shu-Cherng Fang for being my co-advisor at NC State University. His personal and academic advice during my whole graduate study benefits me a lot. I also thank Professor Henry Nuttle, Mike Devetsikiotis for being my committee members. I am also thankful to Professor Kevin Shang from Duke University, with whom I had opportunity to get involved in some joint research, which is fruitful and very enjoyable. I would like to thank Professor Chung-Yee Lee and Professor Shaohui Zheng from The Hong Kong University of Science and Technology (HKUST) for their kindly help and support during my visit at HKUST in Spring semester of 2005.

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Chapter 1

Introduction

With information revolution, increased globalization and competition, supply chain has become longer and more complicated than ever before. These developments bring supply chain management to the forefront of the management's attention. Inventories are very important in a supply chain. The total investment in inventories is enormous, and the management of inventory is crucial to avoid shortages or delivery delays for the customers and serious drain on a company's financial resources.

Traditional production and inventory models assume an exogenous selling price and focus on effective replenishment strategies. However, with the increased availability of demand data and new technologies, e.g., electronic data interchange (EDI), electronic funds transfer (EFT), and point of sale (POS) devices, a number of industries have begun to adopt dynamic pricing strategies to effectively regulate demand and manage inventory. For example, Dell sells its products through its website offering promotions every week and even changing prices daily. Anther telling example is websites for selling airline tickets, such as expedia.com. Indeed, pricing has become a useful and powerful tool to balance customer demand and the firm's inventory.

These developments raise the need and interest of developing models that integrate production decisions, inventory control and pricing strategies. In this study, we discuss several models that combine inventory control or production planning with pricing. We characterize the optimal inventory/production and pricing strategies and develop computational algorithms for the optimal policies. Furthermore, we provide some quantitative and qualitative relationship of the optimal policies and system parameters. After reviweing the literature related to this dissertation in Chapter 2, in Chapter 3, we analyze joint optimization of pricing and inventory control for several continuous-review inventory systems, namely, $(Q, r, \mathbf{p})$ system, batch ordering $(R, nQ, \mathbf{p})$ system and $(s, S, \mathbf{p})$ type system. In these systems, demand arrival rate depends on the price and our objective is to maximize the average profit. In contrast to Chapter 3, we discuss two periodic-review inventory/production models with pricing in Chapter 4. One is production model with production smoothing cost; the other is inventory model with two transportation modes. The demand functions in both models are price-dependent and our objective is to maximize the total discounted profit over finite or infinite planning horizon. We characterize the optimal policies and provide some structure properties of the policy parameters and key performance measures.

Supply chains are not always part of a single company. As outcouring becomes a global trend in current industry, more companies get involved in a supply chain. Before reaching the customer, a final product goes through several tiers of suppliers, distributors and retailers. The companies work together to improve the coordination of total material flow in the supply chain. Multi-echelon inventory system conceptualizes and models the production and transportation activities of a multi-location supply chain, which requires joint optimization as well, i.e., jointly optimizing the inventory control for each stage in the system. Not only practically but also theoretically, multi-echelon inventory systems are considered as one of the most important classes of models for supply chains. Multi-echelon inventory system includes three major building blocks: serial system, assembly system and distribution systems. Starting from Clark and Scarf (1960)'s seminal paper, many research works have been conducted in analyzing multi-echelon inventory systems with different structures and settings. In this dissertation, we mainly analyze serial system and assembly system.

Delivery time has increasingly become an important measure of service levels for customers and companies are all trying to respond customer orders in minimum time. But usually quicker response means higher cost. There are different classes of customers who have different delivery time requirements. In Chapter 5, we develop multi-echelon inventory models with multiple classes of demand and guaranteed delivery time and we show that echelon base-stock policy is optimal. We also present efficient computational algorithms for determining the optimal control parameters.

In production/distribution systems, material is usually produced/ordered in batches and inventory replenishment between locations often follows a fixed schedule. Large retail chains, such as Wal-Mart, replenish its items in several base quantities and deliver to retailer sites according to a fixed schedule. In Chapter 6, we derive the optimal policy for a serial system with batch ordering and nested replenishment schedule and develop computational algorithm for the optimal policies.

The optimal inventory policy for classical serial system is known to be an echelon base-stock policy, which can be computed through minimizing N nested convex functions recursively. However, it is not easy to see the dependency of the optimal policy and cost on the system parameters. In Chapter 7, we develop probabilistic solutions of optimal base-stock levels for the classical serial system with both average cost and discounted cost. Based on these solutions, we derive several newsvendor bounds. Based on these bounds, a simple heuristic for the optimal policies is developed.

Supply chain models with multiple transportation modes have gained momentum in recent years due to the increasing popularity of outsourcing. Cost and leadtime are two important measures of the suppliers for outsourcing. A supplier who provides shorter leadtime usually has higher price. To balance this tradeoff, companies often adopt multiple sourcing strategies by sharing its business with multiple suppliers. Consequently, companies need to strategically determine the ordering quantity from each supplier based on its inventory status and demand forecast in order to minimize cost. The strategic importance of utilizing multiple suppliers with long and short lead time was first recognized by the US fashion industry. Many firms in this sector have moved their major manufacturing facilities offshore to take advantage of the lower production cost. However, some still prefer to maintain costly domestic facilities so that they can better respond to changes in market demand. The combination of 'quick-response, or short leadtime' suppliers with 'low cost, long leadtime' suppliers has been viewed by many as an appropriate strategy to meet fickle customer demand. In Chapter 8, we study the infinite horizon problem of serial inventory system with two transportation modes between stages. It is known that the optimal stationary policies of this system can be obtained by solving $2N$ nested convex optimization problems recursively. Despite its simple form, however, it is not easy to see the key determinants of the optimal policy and minimum cost from the recursion. we develop simple newsvendor bounds and heuristics for the optimal inventory control policies that can shed light on the effect of system parameters.

In Chapter 9, we conclude the dissertation and present some interesting problems for future research work.

Chapter 2

Literature Review

2.1 Combined Inventory Control and Pricing for Single-Stage Inventory Models

In this section, we first review the single-stage stochastic inventory models that are closely related to our work. After that, we go over the literature on joint optimization of pricing and inventory.

(Q, r) policy is one of the most commonly used inventory control methods, where an order of fixed quantity Q is placed as soon as the inventory position drops to a fixed reorder point r. The cost function for the (Q, r) system with Poisson demand is derived by Galliher, Morse and Simond (1959) and Hadley and Whitin (1963). Sivazlian (1974) extends the results to renewal process demands. Zipkin (1986a) identifies very general conditions for stochastic leadtimes, under which the cost function can be derived similarly as the deterministic leadtime case. Zipkin (1986b) proves that for the continuous cumulative demand model the steady state inventory position is uniformly distributed and independent of the leadtime demand and the average number of backorder is a convex function of Q and r. Zheng (1992) thoroughly discusses the properties of the (Q, r) system.

Continuous-review (S, s) model is a generalization of (Q, r) system. An (S, s) policy, brings the level of inventory to some point S once inventory level drops to or below s, and orders nothing otherwise. Scarf (1960) proves this policy is optimal for a finite-horizon, periodic-review inventory model with setup cost. He introduces a new class of functions, K-convex functions. To improve the implementability of (S, s) policy, Zheng and Federgruen (1991) propose a simple and efficient algorithm to compute the optimal policy parameters S and s. Feng and Xiao (2000) present a different efficient approach to compute the optimal (S, s) policy.

The (R, nQ) policies, if the inventory level is less than or equal to R, an integer multiple of Q is ordered to bring the inventory level into the interval $[R+1, R+Q]$, have been studied by many researchers. Hadley and Whitin (1961) show that the inventory position is uniformly distributed between $[R+1, R+Q]$. Veinott (1965) proves $(R\ ,\ nQ)$ policy is optimal when the order quantity is restricted to be integer multiples of a given base quantity Q for single-stage inventory model. Zheng and Chen (1992) provide an optimization algorithm and sensitivity analysis.

Production smoothing cost is common in the production systems when the production rate is changed between two consecutive periods, mostly, hiring and training expenses, set-up charges for additional equipment when production is increased, firing costs and overheads for equipments used below normal capacity when production is decreased. Therefore, there is a natural tendency to smooth production in order to reach an economic balance between adjustment, production and inventory cost. Beckman (1961) studies such a problem in which the cost of changing the production rate, i.e., the smoothing cost, is proportional to the amount of change, the inventory holding and demand backlog cost is linear. The objective of the firm is to maximize the short-run or long-run discounted profit. Sobel (1969) generalizes Beckman's results under more general assumption of convex expected inventory holding and shortage cost for finite horizon problem. Sobel (1971) further extends the result to the infinite horizon case.

In all the inventory models we have reviewed, there is only one delivery mode. The earliest work on inventory models with two delivery modes can be traced back to Barankin (1961), who studies a single period problem. Daniel (1963) is regarded as the first work on a multi-period single-stage model with one regular supplier and one emergency supplier, with leadtimes being 1 and 0 respectively. Fukuda (1964) extends the work of Daniel to the case where the leadtimes of the two supply modes are L and $L + 1$ respectively. Whittmore and Saunders (1977) consider the dual-supplier problem with arbitrary length of leadtime and demonstrate that the optimal control policy is very complicated and state-dependent if the difference in leadtimes of different transportation modes is not 1. Because of the complexity of the optimal policy, Scheller-Wolf et al. (2003) and Veeraraghavan and Scheller-Wolf (2004) focus on evaluation and optimization of two classes of policies, i.e., "single index" and "dual index" policies. Other related work on single-stage inventory systems with multiple transportation modes includes Feng et al. (2003) and Feng et al. (2004). The combination of 'quick-response, or short leadtime' suppliers with 'low cost, long leadtime' suppliers has been viewed by many as an appropriate strategy to meet fickle customer demand. Some references in this area are Donohue (2000), Eppen and Iyer (1997), Fisher, et al. (1994), Fisher and Raman (1996), and Haksoz and Seshadri (2004).

The traditional inventory models we have reviewed focus on effective replenishment strategies and assume a commodity's price is exogenously determined. Recently, many researchers develop models that jointly optimize the inventory replenishment and pricing. These models can be divided into two different classes.

The first class of models is Revenue Management which has already been successful applied in the airline, hotel and fashion goods retail industries. At the beginning of the planning horizon, the decision maker has a fixed number of initial inventory, such as seats of the flight, empty rooms in the hotel, etc. He tries to use pricing strategy to maximize his revenue over the finite planning horizon while he cannot replenish inventory during the selling season. For a detail review, we refer to the paper by McGill and Ryzin (1999).

The second class of models is in the coordination of inventory replenishment strategies and pricing policies, such that at every decision epoch, the manager needs to decide how much to order from the supplier and what is the selling price for the commodities. This topic, starting with the work of Whitin (1955) who analyzes the newsvendor problem with price-dependent demand, has been the focus of many papers. Federgruen and Heching (1999) extend Whitin (1955) to a multi-period model and characterize the optimal inventory and pricing policy as base-stock list price policy.

Building on Federgruen and Heching's work, Chen and Simchi-Levi (2002 (a) and (b)) include fixed setup cost for ordering into the model. They derive the optimal pricing and replenishment strategies for both finite and infinite planning horizons. Feng and Chen (2003) study joint pricing and inventory optimization for a periodic review inventory system with the criterion of maximizing the long run average profit. Feng and Chen prove the optimality of $(s, S, \mathbf{p})$ policy and develop an ascent algorithm to compute the optimal policies. Feng and Chen (2002) consider a continuous review model with discrete and prespecified prices set, especially for two prices and establish the optimality of (s, d, D, S) policy.

2.2 Multi-echelon Inventory Systems

After reviewing single location inventory models, in this section, we switch to multi-echelon inventory models.

Clark and Scarf (1960) study a serial inventory system with finite planning horizon. The system incurs linear inventory holding cost and shortage cost. They introduce a concept of echelon stock and prove the optimality of echelon base-stock policies. Their proof technique involves a decomposition of multi-stage problem into single-stage problem. This approach guides most of the subsequent literature on multi-echelon inventory studies. In particular, Federgruen and Zipkin (1984) extend the result to the infinite horizon setting. Chen and Zheng (1994) use a lower bound approach to prove the optimality of echelon base-stock policy for the average cost criterion. For assembly system, Rosling (1989) provides the conditions under which an assembly system can be equivalently converted to a serial system. So under these mild conditions, most results of serial system can be carried over to assembly system.

Lawson and Porteus (2000) extend Clark-Scarf (1960)'s model by considering dual transportation modes in each stage. Under two assumptions that the leadtime difference of regular and expedited supply is one period and linear additive shipping costs, they characterize the form of optimal inventory control policy as top-down echelon base-stock policy. The control parameters of each echelon consist of two numbers, one for regular shipping and the other for expedited shipping. Muharremoglu and Tsitsiklis (2003) generalize Lawson and Porteus' second assumption by introducing "supermodular" shipping costs structure and show that the optimal policy is extended echelon base-stock type. There is another related work by Huggins and Olsen (2001), who formulate a two-stage supply chain in which stage 1's order is always met by stage 2. Under the linear ordering cost for regular production and fixed cost for expedited production, they characterize the optimal policy for both downstream and the whole system.

In Clark and Scarf (1960) model, each stage can order any quantity in every period. Chen (2000) generalizes the model by introducing the batch ordering constraint into each stage and establishes the optimality of echelon (R, nQ) policy. For references on echelon-stock (R, nQ) policies, the reader is referred to De Bodt and Graves (1985), Axsäter and Rosling (1993), Chen and Zheng (1994, 1998). For this type of policy, every stage can only place an order which is an integer multiple of the base quantity. And the base quantity at each stage satisfies and an integer ratio constraint, i.e., except the most downstream stage, each stage's base quantity is an integer multiple of its next downstream stage's base quantity.

More recently, van Houtum et al. (2003) extend Clark and Scarf (1960) model by considering a serial system with periodic batching constraints, and they prove that the echelon base-stock policy is optimal for the multi-echelon system and derive newsvendor-type characteristics for the optimal base-stock levels. For the model with fixed multi-period replenishment cycles, there are papers that consider different system configurations, such as assembly and distribution systems or joint replenishment problems (e.g., Atkins and Iyogun 1988, Eppen and Schrage 1981, Erhun and Tayur 2003, Graves 1996, Hopp and Kuo 1998, Jackson 1988, McGavin et al. 1993, and Yano and Carlson 1998). In this type of system, every stage in the system can only place order for every given reorder interval. The reorder interval of each stage also satisfies an integer ratio constraint, i.e., each stage's reorder interval is an integer multiple of its next downstream's reorder interval.

There is another stream of research work in the multi-echelon system that focuses on developing simple solutions, bounds, and heuristics for the optimal policies. This is motivated by the observation that even though the computational algorithms for optimal policies of the models we have reviewed are known, it is not easy to see the key determinant of the optimal control parameters and cost from the algorithm. Zipkin (2000) introduces a lower bound for a two-stage system by restricting the possibility of holding inventory at the upper stream stage. Dong and Lee (2003) develop lower bounds for optimal policies of serial system with discounted cost criterion, while Shang and Song (2003) obtain simple newsvendor type of bounds and develop simple heuristics for serial systems with average cost criterion, using a different approach than that of Dong and Lee. Another related work on bounds and heuristics for serial systems is Gallego and Ozalp (2004).

Chapter 3

Joint Optimization of Pricing and Inventory Control for Continuous Review Inventory Systems

In this chapter, we study three continuous-review inventory systems. Besides the inventory replenishment decision, there is pricing decision to be made. In §3.1, an inventory model with simple Poisson demand and price dependent arrival rate is analyzed. In §3.2, we study a batch ordering inventory model with compound Poisson demand, in which the inventory policy is (R, nQ) type of policy. In §3.3, We consider $(s, S, \mathbf{p})$ type model, in which there is setup cost and the demand process is compound Poisson. Finally, we summarize this chapter in §3.4.

3.1 $(Q, r, \mathbf{p})$ Model

This section is organized as follows. In §3.1.1, we introduce the notation and present the model. After that, in §3.1.2, we characterize the optimal pricing policy and inventory control strategy. Furthermore, we present efficient computational algorithms to compute the optimal price and inventory control parameters. In §3.1.3 we discuss a structural property of the optimal price. In §3.1.4 the optimality of $(Q, r, \mathbf{p})$ policy is validated.

3.1.1 Model Description

Consider a continuous review inventory system for a single item. The demand arrives according to a Poisson process whose rate $\lambda(p)$ depends on selling price p. Let $u(p) = 1/\lambda(p)$ denote the average demand interarrival time. There is a fixed ordering cost K and unit purchasing cost c. As in Federgruen and Heching (1999), Chen and Simchi-Levi (2002) (2003), and Feng and Chen (2002) (2003), we assume that the supply leadtime is 0. The objective is to determine the selling price and the inventory replenishment policy so that the average profit is maximized.

If a demand occurs and there is no on-hand inventory available, it is backlogged. The holding and shortage cost rate is $G(y)$ when the inventory level is y. A typical form for $G(y)$ is $G(y) = hy^+ + by^-$, where $y^+ = \max\{y, 0\}$, $y^- = \max\{-y, 0\}$, and h and b are the holding cost and shortage cost per unit of time per item, respectively. Clearly, the minimum point of $G(y)$ is 0. In the remainder of this chapter, we do not require that $G(y)$ take this typical form specified above but only that it be a convex function of inventory level y with minimum point at $y = 0$.

We assume that the average interarrival time $u(p)$ is increasing convex in p and that the revenue rate $p\lambda(p)$ is concave in p. That $u(p)$ is increasing in p is clear – the higher the selling price, the longer the average time until the next demand. That $u(p)$ is convex implies that the rate at which the average interarrival time increases is getting higher as the price increases. We shall further assume, though it is not essential, that $\lim_{p\to 0} p\lambda(p) = 0$ and $\lim_{p\to\infty} p\lambda(p) = 0$. The last assumption implies that as selling price goes to infinity, all customers are driven away.

It is known that the optimal policy for this problem is $(Q, r, \mathbf{p})$ (see e.g., Chen and Simchi-Levi (2003)). In this policy the parameter r denotes the reorder point, Q is the order quantity, and $\mathbf{p} = (p(r+1), \ldots, p(r+Q))$ is a Q-dimensional vector of selling prices associated with inventory levels $r+1, r+2, \ldots, r+Q$. The $(Q, r, \mathbf{p})$ policy works as follows: As soon as the inventory level reaches r, the firm places an order of size Q, and when the inventory level is i, the selling price is set at $p(i)$, $i = r+1, r+2, \ldots, r+Q$. Under the $(Q, r, \mathbf{p})$ policy, the length of time the inventory level stays at i is exponentially distributed with mean $u(p(i))$. Thus the inventory level process is a Markov Process. To compute the average profit we use renewal reward theory. A cycle starts every time an order is placed. Then the inventory process constitutes a renewal process with average cycle length $\sum_{i=r+1}^{r+Q} u(p(i))$. The average revenue in each cycle is $\sum_{i=r+1}^{r+Q} p(i)$, the purchase cost is cQ, and the holding and shortage cost per cycle is $\sum_{i=r+1}^{r+Q} u(p(i))G(i)$. Let $v(Q, r, \mathbf{p})$ be the average profit per unit time for the inventory system. It follows from renewal reward theory that

$$v(Q, r, \mathbf{p}) = \frac{-K + \sum_{i=r+1}^{r+Q}(p(i) - c - u(p(i))G(i))}{\sum_{i=r+1}^{r+Q} u(p(i))}. \tag{3.1}$$

Our objective is to find the optimal strategy $(Q, r, \mathbf{p})$ that maximizes the average profit $v(Q, r, \mathbf{p})$.

To find the optimal solution for $v(Q,r,\mathbf{p})$, we introduce an auxiliary function. For a given but arbitrary policy $(Q,r,\mathbf{p})$, we define

$$\begin{aligned}
\ell_\gamma(Q,r,\mathbf{p}) &= -K + \sum_{i=r+1}^{r+Q} (p(i) - c - u(p(i))G(i)) - \gamma \sum_{i=r+1}^{r+Q} u(p(i)) \\
&= -K + \sum_{i=r+1}^{r+Q} (p(i) - c - u(p(i))(G(i)+\gamma)), \qquad (3.2)
\end{aligned}$$

where γ is called the dummy profit. It is a simple result from fractional programming (see e.g., Schaible (1995)) that γ is the maximum average profit for the optimization problem (3.1) if and only if $\max_{Q,r,\mathbf{p}} \ell_\gamma(Q,r,\mathbf{p}) = 0$.

For a given γ let

$$\ell_\gamma = \max_{Q,r,\mathbf{p}} \ell_\gamma(Q,r,\mathbf{p}). \qquad (3.3)$$

Since for a given policy $(Q,r,\mathbf{p})$ the auxiliary function $\ell_\gamma(Q,r,\mathbf{p})$ is a decreasing convex function of γ, so is ℓ_γ.

To obtain the optimal solution for (3.2) for a given γ, we first compute the optimal $\mathbf{p}$ for given r and $r+Q$. For $i = r+1,\ldots,r+Q$, since $p(i) - c - u(p(i))(G(i)+\gamma)$ is a concave function of $p(i)$, the optimal $p(i)$ can be obtained by taking derivative and setting it to 0,

$$1 - u'(p(i))(G(i)+\gamma) = 0.$$

Thus we obtain

$$p(i) = (u')^{-1}\left(\frac{1}{G(i)+\gamma}\right), \qquad i = r+1,\ldots,r+Q, \qquad (3.4)$$

where $(u')^{-1}$ is the inverse function of u', which is well-defined since u' is increasing. Note that $p(i)$ depends on γ even though we have made the dependency implicit. To find the optimal r and $r+Q$, we define a function $f(\gamma, y)$ as

$$f(\gamma, y) \quad = \quad p(y) - c - u(p(y))(G(y) + \gamma) \tag{3.5}$$

where $p(y)$ is given by (3.4). Clearly, y can only take integer values in (8.2). However in what follows we relax this to allow y to take any real value. In the following discussion, we use $f_1'(\gamma, y)$ to denote the derivative of $f(\gamma, y)$ with respect to γ and $f_2'(\gamma, y)$ to denote the derivative of $f(\gamma, y)$ with respect to y. Let $\mathcal{R}$ be the set of real numbers.

Lemma 3.1.1 *(i) For any given $\gamma > 0$, $f(\gamma, y) : \mathcal{R} \rightarrow \mathcal{R}$ is a unimodal function of y, and $y_0 = 0$ is its maximum point.*

(ii) For any given y, $f(\gamma, y)$ is a decreasing concave function of γ.

Proof. We first prove (i). Taking derivative of $f(\gamma, y)$ with respect to y yields

$$\begin{aligned} f_2'(\gamma, y) &= p'(y) - u'(p(y))p'(y)(G(y) + \gamma) - u(p(y))G'(y) \\ &= -u(p(y))G'(y) \end{aligned}$$

where the second equality follows from the fact that $f_2'(\gamma, y)$ is evaluated at the optimal price p which satisfies $u'(p(y)) = 1/(G(y) + \gamma)$. Because $u > 0$, $f_2'(\gamma, y)$ is positive (negative) whenever $G'(y)$ is negative (positive). Thus it follows from the convexity of $G(y)$ and $\lim_{|y| \rightarrow \infty} G(y) = \infty$ that $f(\gamma, y)$ is unimodal in y, and the maximum point of $f(\gamma, y)$ is the minimum point of $G(y)$, which is 0.

We next prove (ii). We denote by p_γ' the derivative of $p(y)$ with respect to γ. Note

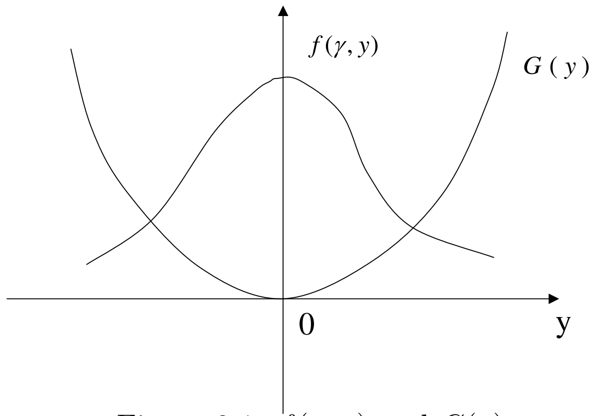


Figure 3.1: $f(\gamma, y)$ and $G(y)$

that the first two derivatives of $f(\gamma, y)$ with respect to γ are

$$
\begin{aligned}
f_1'(\gamma, y) &= p' - u'(p)p'(G(y) + \gamma) - u(p) \\
&= -u(p), \\
f_{11}''(\gamma, y) &= -u'(p)p_\gamma'(y) \\
&= \frac{u'(p)}{u''(p(y))(G(y) + \gamma)^2},
\end{aligned}
$$

where the second equality follows from $u'(p) = 1/(G(y) + \gamma)$ and the last equality follows from

$$
\begin{aligned}
p_\gamma'(y) &= \left((u')^{-1}\left(\frac{1}{G(y) + \gamma}\right)\right)' \\
&= -\frac{1}{u''\left((u')^{-1}\left(\frac{1}{G(y)+\gamma}\right)\right)} \frac{1}{(G(y) + \gamma)^2} \ .
\end{aligned}
$$

Thus it follows from the increasing convexity of $u(p)$ that $f(\gamma, y)$ is decreasing concave in γ. $\square$

The relationship between $f(\gamma, y)$ and $G(y)$ are depicted in Figure 1.

Remark 1. One might expect $f(\gamma, y)$ to be a concave function of y. We point out that this is not true in general.

Let

$$\ell_\gamma(Q, r) = \max_{\mathbf{p}} \ell_\gamma(Q, r, \mathbf{p}).$$

We next characterize the optimal $r(\gamma)$ and $Q(\gamma)$ that maximize $\ell_\gamma(r, Q)$ for a given γ.

Lemma 3.1.2 *(i) Given γ, the optimal $r(\gamma)$ and $r(\gamma) + Q(\gamma)$ are given by*

$$\begin{aligned} r(\gamma) &= \max\{y \le 0 \textit{ and integer} : f(\gamma, y) \le 0\} &(3.6)\\ r(\gamma) + Q(\gamma) &= \max\{y \ge 0 \textit{ and integer} : f(\gamma, y)) \ge 0\} &(3.7)\end{aligned}$$

(ii) $r(\gamma)$ is increasing in γ.

(iii) $r(\gamma) + Q(\gamma)$ is decreasing in γ.

Proof. Note the relationship

$$\ell_\gamma(r, Q) = -K + \sum_{i=r+1}^{r+Q} f(\gamma, i). \tag{3.8}$$

Since $f(\gamma, i)$ is a unimodal function, the optimal solution for $\max_{r,Q} \ell_\gamma(r, Q)$ should be the r and Q such that $f(\gamma, r+1), \ldots, f(\gamma, r+Q)$ are all nonnegative. This proves (i). By Lemma 3.1.1, $f(\gamma, y)$ is a decreasing function of γ. Hence it follows from the definition of $r(\gamma)$ and $r(\gamma) + Q(\gamma)$ that $r(\gamma)$ is increasing in γ, and $r(\gamma) + Q(\gamma)$ is decreasing in γ. $\square$

The pair of points $(r(\gamma), r(\gamma) + Q(\gamma))$ and $(r(\gamma'), r(\gamma') + Q(\gamma'))$ with $\gamma < \gamma'$ are

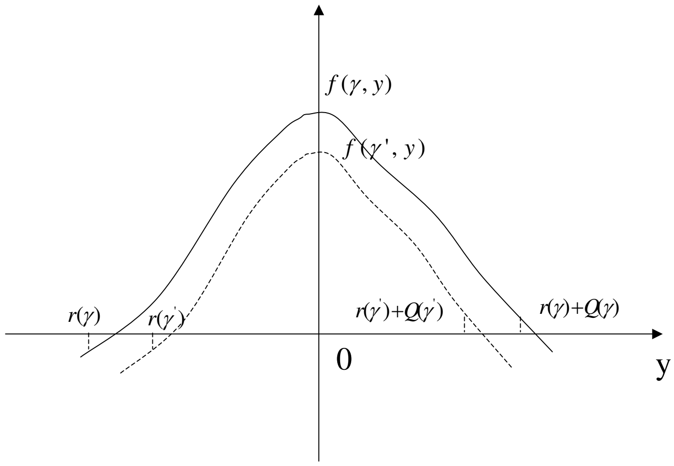


Figure 3.2: $r(\gamma)$ and $r(\gamma) + Q(\gamma)$

visualized in Figure 2.

We are now ready to present the main result of this section.

Theorem 3.1.1 *The optimal inventory and pricing strategy* $(Q^*, r^*, \boldsymbol{p}^*)$ *is*

$$r^* = \max\{y \le 0 \textit{ and integer} : f(\gamma^*, y) \le 0\} \tag{3.9}$$

$$r^* + Q^* = \max\{y \ge 0 \textit{ and integer} : f(\gamma^*, y) \ge 0\} \tag{3.10}$$

and

$$p^*(i) = (u')^{-1}\left(\frac{1}{G(i) + \gamma^*}\right), \quad i = r^* + 1, \ldots, r^* + Q^*, \tag{3.11}$$

where γ^* *is the optimal average profit determined by*

$$\sum_{i=r^*+1}^{r^*+Q^*} f(\gamma^*, i) = K. \tag{3.12}$$

Proof. For any γ, we have obtained the optimal $r(\gamma), Q(\gamma)$ and $\mathbf{p}(\gamma)$ in the previous analysis. Equation (3.3) can be expressed as

$$\ell_\gamma = -K + \sum_{i=r(\gamma)+1}^{r(\gamma)+Q(\gamma)} f(\gamma, i).$$

Since ℓ_γ is a strictly decreasing convex function of γ, $\ell_\gamma = 0$ has a unique solution γ^*, which is given by (3.12). Therefore, from the result of fractional programming (see e.g., Schaible (1995)), the proceeding argument guarantees the optimality of the solution. $\square$

One interesting feature of optimal pricing is its dependency on the inventory level. Federgruen and Heching (1999) prove that for their model the optimal price decreases as the inventory level increases. This does not hold in our model. The following result presents the qualitative relationship between pricing and inventory level in our model.

Theorem 3.1.2 *The optimal selling price $p^*(y)$ is increasing on $r^* + 1 \leq y \leq 0$, and decreasing on $0 \leq y \leq r^* + Q^*$.*

Proof. For convenience we drop the star on p, r, Q and γ. Again we relax y to allow it to take any real value. Recall that $p(y)$ satisfies equation

$$1 - u'(p)(G(y) + \gamma) = 0.$$

Taking derivative with respect to y yields

$$-u''(p)p'(y)(G(y) + \gamma) - u'(p)G'(y) = 0$$

and

$$p'(y) = -\frac{u'(p)G'(y)}{u''(p)(G(y) + \gamma)}, \qquad r + 1 \leq y \leq r + Q.$$

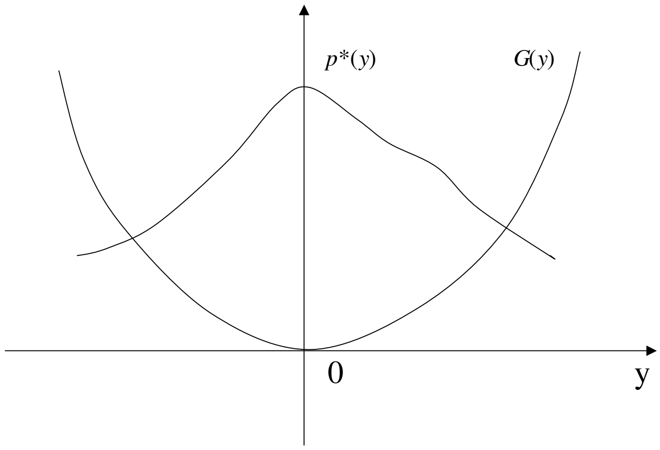


Figure 3.3: $p^*(y)$ and $G(y)$

Since $u(p)$ is increasing convex, p increases as $G(y)$ decreases, and p decreases as $G(y)$ increases. Because $G(y)$ is convex with minimum point 0, the result follows. □

This result shows that the higher the inventory level, the lower the optimal selling price, and the higher the backlog level, the lower the optimal selling price (see Figure 3). An intuitive explanation for this phenomenon is that, when the on-hand inventory is high, the manager should set the selling price low in order to attract more demand to sell the product in stock; while when the backlog level is high, the manager should also set the selling price low but that is for a different reason – to boost demand so that the reorder point can be reached quickly to incur a low penalty cost.

3.1.2 Algorithms

In Theorem 3.1.1, the parameter γ^* represents the maximum average profit for the inventory system. Once γ^* is obtained the optimal strategy follows from (3.9), (3.10) and (3.11). Since γ^* is determined by (3.12) which can be written as ℓ_{γ^*}=0, we need to search for γ that satisfies $\ell_\gamma = 0$.

For any given γ, the $(Q, r, \mathbf{p})$ that maximizes $\ell_\gamma(Q, r, \mathbf{p})$ is easily computed from (3.4), (3.6) and (3.7), and the optimal value ℓ_γ is determined. If $\ell_\gamma = 0$ then we know that the current strategy is optimal. Otherwise, if $\ell_\gamma > 0$ then by the fact that ℓ_γ is decreasing in γ we know that the optimal $\gamma^* > \gamma$, and we should increase γ to search for the optimal strategy. Similarly, if $\ell_\gamma < 0$ then the optimal $\gamma^* < \gamma$, and we should decrease γ to search for the optimal strategy. This analysis leads to the following bisection algorithm for computing the optimal strategy.

Algorithm I

Step 1 (Initialization)

Let $\gamma_1 = \max_p \lambda(p)(p - c)$ and $\gamma_2 = v(1, -1, p(0))$. Then $\ell_{\gamma_1} < 0$ and $\ell_{\gamma_2} \geq 0$. Let $\epsilon > 0$ be the tolerance level for γ^*.

Step 2 (Update γ)

Let $\gamma = (\gamma_1 + \gamma_2)/2$. Compute the corresponding $(Q(\gamma), r(\gamma), \mathbf{p}(\gamma))$ using (3.4), (3.6), and (3.7). Compute $\ell_\gamma(Q(\gamma), r(\gamma), \mathbf{p}(\gamma))$.

If $\ell_\gamma(Q(\gamma), r(\gamma), \mathbf{p}(\gamma)) = 0$, then go to Step 3.

If $\ell_\gamma(Q(\gamma), r(\gamma), \mathbf{p}(\gamma)) < 0$, then $\gamma_1 = \gamma$;

If $\ell_\gamma(Q(\gamma), (r(\gamma), \mathbf{p}(\gamma)) > 0$, then $\gamma_2 = \gamma$.

If $\gamma_1 - \gamma_2 \geq \epsilon$, go back to Step 2; otherwise, set $\gamma = (\gamma_1 + \gamma_2)/2$ and go to Step 3.

Step 3 (Termination)

Stop. $\gamma^* = \gamma$, $r^* = r(\gamma)$, $Q^* = Q(\gamma)$, and $\mathbf{p}^* = \mathbf{p}(\gamma)$.

The algorithm starts with two initial inputs: γ_1 and γ_2; γ_1 is the average profit when the ordering cost $K = 0$, which cannot be achieved by a feasible policy and is an upper bound for the optimal average profit; hence the value of the auxiliary function is negative, i.e., $\ell_{\gamma_1} < 0$; γ_2 is the profit for a feasible policy which is a lower bound of the optimal average profit. So $\ell_{\gamma_2} \geq 0$. Moreover, we specify a small positive number ϵ as the tolerance level of γ^*.

In Step 2, we use bisection search method to locate the optimal γ, and update the corresponding optimal $(Q, r, \mathbf{p})$ using Theorem 1. If the $\ell_\gamma < 0$, then the optimal γ^* is between γ and γ_2, so we replace γ_1 by γ; otherwise, it is between γ_1 and γ and we replace γ_2 by γ. After we update either γ_1 or γ_2, we repeat Step 2. If we find a γ that satisfies $\ell_\gamma = 0$, then we have reached the optimal solution and terminate the algorithm. If the difference between γ_1 and γ_2 is less than ϵ, we also terminate the program and obtain an ϵ-approximate optimal γ. We use this γ to compute the corresponding (Q, r) and $\mathbf{p}$.

Because the problem involves continuous optimization, there exists no algorithm that is guaranteed to stop at the exact optimal solution in a finite number of iterations, as is typical in any continuous optimization problem. However, the algorithm terminates at an ϵ-approximate optimal solution in a finite number of steps.

To develop our second algorithm for determining the optimal policy, we need the following lemma.

Lemma 3.1.3 *If $\ell_\gamma(Q, r, \boldsymbol{p}) > 0$, then $\gamma_1 = v(Q, r, \boldsymbol{p}) > \gamma$.*

Proof. By definition we have

$$\ell_\gamma(Q, r, \mathbf{p}) = -K + \sum_{i=r+1}^{r+Q} (p(i) - c - u(p(i))G(i)) - \gamma \sum_{i=r+1}^{r+Q} u(p(i)) > 0.$$

Thus

$$v(Q,r,\mathbf{p}) - \gamma \quad = \quad \frac{-K + \sum_{i=r+1}^{r+Q}(p(i) - c - u(p(i))G(i))}{\sum_{i=r+1}^{r+Q} u(p(i))} - \gamma > 0.$$

This shows $\gamma_1 > \gamma$ and the lemma follows. $\square$

Lemma 3.1.3 states that, if the average profit γ of a policy is not optimal (because $\ell_\gamma > 0$), then the average profit of the current policy γ_1 is greater than γ. This suggests that if we start with the profit γ of a feasible strategy that is not optimal, then we can improve on it. The process can be continued to generate an increasing sequence which converges. This leads to the following algorithm.

Algorithm II

Step 1 (Initialization)

Let $\gamma_0 = v(1, -1, p(0))$, which satisfies $\ell_{\gamma_0} \geq 0$. Let $\epsilon > 0$ be the tolerance level for γ^* and $n = 0$.

Step 2 (Update γ) Compute $(Q(\gamma_n), r(\gamma_n), \mathbf{p}(\gamma_n))$ based on Lemma 3.1.1, and compute $\ell_{\gamma_n}(Q(\gamma_n), r(\gamma_n), \mathbf{p}(\gamma_n))$.

If $\ell_{\gamma_n}(Q(\gamma_n), r(\gamma_n), \mathbf{p}(\gamma_n)) = 0$, then go to Step 3.

If $\ell_{\gamma_n}(Q(\gamma_n), r(\gamma_n), \mathbf{p}(\gamma_n)) > 0$, $\gamma_{n+1} = v(Q(\gamma_n), r(\gamma_n), \mathbf{p}(\gamma_n))$. Set $n = n + 1$, If $\gamma_n - \gamma_{n-1} \geq \epsilon$, go back to Step 2; otherwise, go to step 3.

Step 3 (Termination)

Stop. $\gamma^* = \gamma_n$, $r^* = r(\gamma_n)$, $Q^* = Q(\gamma_n)$, and $\mathbf{p}^* = \mathbf{p}(\gamma_n)$.

The remaining question is whether the point of convergence of this algorithm is the maximum average profit. This is guaranteed by the following result.

Proposition 3.1.1 *In Algorithm II, γ_n converges to the optimal γ^*.*

Proof. Since $\{\gamma_n\}$ is an increasing sequence bounded from above by $\max_p \lambda(p)(p-c)$, it converges to some finite number, say, γ. We need to prove $\gamma = \gamma^*$. If this is not true, then $\ell_\gamma > 0$. It follows from Lemma 3.1.3 that $\gamma' = v(Q(\gamma), r(\gamma), \mathbf{p}(\gamma)) > \gamma$. Because $v(Q(\gamma), r(\gamma), \mathbf{p}(\gamma))$ is a continuous function of γ, for any $\epsilon > 0$ such that $\gamma' - \epsilon > \gamma$, there exists a positive integer N, such that when $n > N$,

$$\gamma_{n+1} = v(Q(\gamma_n), r(\gamma_n), \mathbf{p}(\gamma_n)) \geq v(Q(\gamma), r(\gamma), \mathbf{p}(\gamma)) - \epsilon = \gamma' - \epsilon > \gamma.$$

This contradicts $\gamma_{n+1} \leq \gamma$. Therefore $\gamma_n \to \gamma^*$ and $\ell_{\gamma_n} \to 0$ as $n \to \infty$. $\square$

Remark 2. We point out that our algorithms are different from that of Feng and Chen (2002) in that we have a continuous optimization problem, while Feng and Chen (2002) have a discrete optimization problem. This is because Feng and Chen have a finite number of possibilities for the price, while our price can take any nonnegative value. For example, for the case with two possible prices, $p_1 < p_2$, they show that a (s, d, D, S) policy is optimal. Their algorithm is mainly the search of optimal control parameters (s, d, D, S). After the optimal (s, d, D, S) are computed, the optimal price is p_1 when inventory level is between s and d or between D and S, and the optimal price is p_2 when inventory level is between d and D. However, in our problem since p can take any value, we have to introduce a stopping criterion to find an ϵ-approximate optimal solution.

3.1.3 Numerical Studies

In this section, we provide several numerical examples to illustrate the properties of the optimal pricing strategies. Two particular forms of $u(p)$ are considered: $u(p) = \alpha + \beta p^2$ and $u(p) = e^{\theta p}$.

In Table 1, $u(p) = \alpha + \beta p^2$ with $\alpha = 0.05$, $\beta = 0.005$ and $K = 20$. We present the optimal r^*, $r^* + Q^*$ and maximum average profit γ^* in the table for the examples generated by varying the parameters h, b, and c one at a time from base case values of $h = 1$, $b = 10$, and $c = 1$. Because the dimension of the optimal price vector changes as r^* and Q^* change, we do not include the optimal price in the table. We list some examples here: When $h = 3$, $r^* = -2$ and $r^* + Q^* = 5$, the optimal price is $\mathbf{p}^*(-1, 5) = (6.50, 18.50, 11.90, 8.80, 6.90, 5.70, 4.90)$; another example, when $b = 4$, $r^* = -4$ and $r^* + Q^* = 12$, the optimal price is $\mathbf{p}^*(-3, 12) = (4.40, 5.40, 6.80, 9.40, 8.60, 7.90, 7.30, 6.80, 6.40, 6.00, 5.70, 5.40, 5.10, 4.80, 4.60, 4.40)$.

Table 3.1: $u(p) = \alpha + \beta p^2$

h	r^*	$r^* + Q^*$	γ^*	b	r^*	$r^* + Q^*$	γ^*	c	r^*	$r^* + Q^*$	γ^*
1	-2	13	10.047671	2	-6	11	11.496069	1	-2	13	10.047671
2	-2	8	6.964651	4	-4	12	10.627030	2	-2	10	7.171130
3	-2	5	5.412796	6	-3	12	10.295968	3	-1	8	5.515960
4	-2	4	4.523065	8	-2	13	10.151128	4	-1	6	4.482702
5	-2	3	3.964397	10	-2	13	10.047671	5	-1	5	3.775697
6	-2	3	3.612643	12	-2	13	9.964905	6	-1	4	3.272959

In Table 2, $u(p)$ takes the form $u(p) = e^{\theta p}$ with $\theta = 0.05$ and $K = 30$. The base case values for the other parameters are again $h = 1$, $b = 10$, and $c = 1$. And again we provide the optimal r^*, $r^* + Q^*$ and maximum average profit γ^* as cost parameter changes. We present the optimal price for several instances: When $h = 1$, $r^* = -1$, $r^* + Q^* = 3$ and $\mathbf{p}^*(0, 3) = (36.60, 31.20, 26.90, 23.40)$; when $b = 2$, $r^* = -2$, $r^* + Q^* = 3$ and $\mathbf{p}^*(-1, 3) = (26.00, 35.20, 30.10, 26.00, 22.70)$.

Table 3.2: $u(p) = e^{\theta p}$

h	r^*	$r^* + Q^*$	γ^*	b	r^*	$r^* + Q^*$	γ^*	c	r^*	$r^* + Q^*$	γ^*
1	-1	3	3.211318	2	-2	3	3.442351	1	-1	3	3.442247
2	-1	2	2.367485	4	-1	3	3.211318	2	-1	3	3.211318
3	-1	1	1.986400	6	-1	3	3.211318	3	-1	3	2.998423
4	-1	1	1.725084	8	-1	3	3.211318	4	-1	3	2.791912
5	-1	1	1.518209	10	-1	3	3.211318	5	-1	3	2.596384
6	-1	0	1.485544	12	-1	3	3.211318	6	-1	3	2.415886

From Tables 1 and 2, the following observations can be easily made and verified. First, the optimal order-up-to level $r^* + Q^*$ is decreasing in the unit holding cost rate h and the linear purchasing cost c, but it is increasing in the unit backlog cost rate b. Second, the optimal reorder point r^* is independent of the holding cost rate h, which is because $r^* < 0$, but r^* is increasing in b. However, since the demand follows a Poisson process and order leadtime is 0, r^* is always less than 0. Hence, r^* remains constant at -1 after some level of b. Third, the optimal profit decreases as the cost parameters increase. Fourth, it can be proved analytically that, in these examples the rate of decrease in $p^*(y)$ gets smaller with positive, increasing values of y, and the rate of increase in $p^*(y)$ gets larger with negative, increasing values of y. This interesting property, however, does not hold for general $u(p)$.

3.1.4 Optimality Verification of $(Q, r, \mathbf{p})$ Policy

The policy $(Q^*, r^*, \mathbf{p}^*)$ obtained in the previous algorithm (we will omit the $*$ in the following proof) is optimal among all the feasible policies if this policy and its long-run average profit R^* together satisfy the following *long-run average profit criterion*:

$$h(x) = \sup_{y \geq x, p \in [c, \infty)} \left\{ -K\delta\{y > x\} - \frac{G(y) + R^*}{\lambda(p_y)} + (p_y - c) + h(y-1) \right\},$$

where $h(x)$ must be a bounded function. So we relax the original problem to the following problem which is allowed to return some unsold goods:

$$h(x) = \sup_{y \neq x, p \in [c, \infty)} \left\{ -K\delta\{y \neq x\} - \frac{G(y) + R^*}{\lambda(p_y)} + (p_y - c) + h(y-1) \right\}$$

where $\delta(A) = 1$ if the A is true, and otherwise it is 0.

As will be shown later in this section, it turns out that the optimal solution to the relaxed formulation stipulates a $(Q, r, \mathbf{p})$ policy for inventory management and pricing. Specifically, when the inventory level of goods is above $r + Q$, the retailer may return the inventory in order to bring the inventory down to $r + Q$. As a result of returning goods, it will never happen more than once since after that the inventory level will always at or below $r + Q$. Therefore, for the criterion of long run average profit, if we follow the same policy for the original problem with no good-returning allowed, the same long run average profit will be achieved and it must be optimal for the original problem too. This technique was first employed in Zheng (1991).

Now we need to construct a bounded function $h(x)$ that satisfies the optimality equation. We will structure the function in relation to the auxiliary function $\ell_v(Q, r, \mathbf{p})$. In the following proof procedure, we will change the notation of $\ell_v(Q, r, \mathbf{p})$ to $\ell_v(r, r+Q, \mathbf{p})$ for convenience.

A function $h(x)$ is defined recursively as follows:

$$h(x) = \begin{cases} -K & \text{if } x \leq r, \\ \ell_v(r, x, p(r+1, x)) & \text{if } r < x \leq r + Q, \\ \max\{-K, \max_p\{-\frac{G(x)+v}{\lambda(p)} + (p-c) + h(x-1)\} & \text{if } x > r + Q. \end{cases}$$

First we prove that $h(x)$ is bounded. And we want to show $-K \leq h(x) \leq 0$.

Now it is obviously true that $-K \leq h(x) \leq 0$ when $x \leq r$. For $x > r+Q$, $h(x) \geq -K$. Consider the case when $x \in (r, r+Q]$. As $(Q, r, \mathbf{p})$ optimizes $\ell_v(r, Q+r, \mathbf{p})$, so $\ell_v(r, x, p(r, r+x)) < \ell_v(r, r+Q, \mathbf{p}) = 0$ when $x > r$.

In the following we show $h(x) \geq -K$ when $r < x \leq r+Q$:

$$\begin{aligned} h(x) = \ell_v(r, x, p) &= -K + \sum_{i=r+1}^{x} (p_i - c - u(p_i)(G(i) + v)) \\ &\geq -K \end{aligned}$$

Because $\lambda(p_{x-i})(p_{x-i} - c) - G(x-i) - v \geq 0$ by the definition of the optimal r and Q.

Now we proceed to the case that $x > r+Q$, we prove it by induction, start from $x = r+Q+1$, if $h(r+Q+1) = -K$, then it is automatically satisfied, otherwise:

$$\begin{aligned} h(r+Q+1) &= \max_p \left\{ -\frac{G(r+Q+1)+v}{\lambda(p)} + (p-c) + h(r+Q+1-1) \right\} \\ &\leq \max_p \left\{ -\frac{G(r+Q+1)+v}{\lambda(p)} + (p-c) \right\} \\ &\leq 0 \end{aligned}$$

The first inequality follows from the previous results and the second is based on the definition of $r+Q$. The induction procedure is simple so we omit it here. So far, $-K \leq h(x) \leq 0$.

After we show $h(x)$ is bounded, we need to show it satisfies the dynamic programming equation above. We will discuss several cases separately.

For $x \leq r$:

We need to show $-K \geq -\frac{G(x)+v}{\lambda(p_x)} + (p_x - c) + h(x-1)$, which can be validated by definition of r:

$$G(r) > \lambda(p)(p-c) - v$$

so with $x-1<r$ and $h(x-1)=-K$

$$-\frac{G(x)+v}{\lambda(p_x)} + (p_x - c) - K \leq -K$$

so $h(x)=-K$ satisfies the optimality equation.

For $r<x\leq r+Q$, We prove the result by induction, let $x=r+1$ then

$$\begin{aligned} h(r+1) &= (p_{r+1}-c) - \frac{G(r+1)-v}{\lambda(p_{r+1})} - K \\ &= \ell_v(r, r+1, p(r+1)) \end{aligned}$$

Suppose the this is true for $x=i$, then for $x=i+1$

$$\begin{aligned} h(i+1) &= -\frac{G(i+1)+v}{\lambda(p_{i+1})} + (p_{i+1}-c) + h(i) \\ &= -\frac{G(i+1)+v}{\lambda(p_i)} + (p_{i+1}-c) + \ell_v(r+1, i, p(r+1, i)) \\ &= \ell_v(r, i+1, p(r+1, i+1)) \end{aligned}$$

Therefore, we finish the induction and prove this case.

Finally, we prove the $h(x)$ we constructed is valid for range $x>r+Q$. We just need to verify in this range, $h(x)$ is equivalent to:

$$\max\left\{-K, \max_p\{-\frac{G(x)+v}{\lambda(p)} + (p-c) + h(x-1)\right\}$$

which can be proved for $x=r+Q+1$ since $h(r+Q)=0$ and we can easily prove this case by induction. Hence, we finish the proof for the optimality of $(Q, r, \mathbf{p})$ policy. $\square$

3.2 Batch Ordering $(R, nQ, \mathbf{p})$ Model

In this section, we consider a continuous-review inventory model, in which demand arrives according to a batch Poisson process. The demand arrival rate depends on the selling price while the demand size does not. In addition, the ordering quantity must be an integer multiple of a base quantity Q. We characterize the optimal inventory and pricing policies. We also present how to calculate the control parameters and obtain a structural property for the optimal price.

3.2.1 Model Description

Consider a continuous-review, single-stage inventory system. Demand arrives according to a batch Poisson process. The demand size is i.i.d. with distribution $\phi(\cdot)$ and mean μ. The interarrival time is exponential distributed with rate $\lambda(p)$, which depends on the selling price. The firm needs to make pricing as well as the inventory replenishment decisions. Assume the supply leadtime is 0 and unsatisfied demand is fully backlogged. Let x denote the initial inventory level and y denote the inventory level after replenishment. The inventory holding and shortage cost function is $G(y)$, which is a function of the inventory level after replenishment. The unit purchasing cost is c. When the firm places an order, the order quantity is an integer multiple of a given base quantity Q.

The time sequence of the events is: First, demand arrives and is satisfied if the inventory is enough, otherwise it is backordered; second, the firm decides whether to place an order and if so, how much to order to replenish its inventory; third, the firm determines the selling price for the product; fourth, all costs and revenue are incured.

Assumption 3.2.1 *$G(\cdot)$ is a convex function and $G(y) \to \infty$ when $|y| \to \infty$ and let x_0*

be the minimum point of $G(\cdot)$;

Assumption 3.2.2 *$u(p) = 1/\lambda(p)$ is a convex function of p and it is strictly increasing in p.*

The objective of the firm is to maximize its long run average profit per unit of time. We consider the following $(R, nQ, \mathbf{p})$ policy: If inventory level drops to or below R, place an order which is an integer multiple of Q to raise inventory level to some point between $R+1$ and $R+Q$; otherwise do not order anything. The optimal price is a function of inventory level.

We first derive the average profit function for a given $(R, nQ, \mathbf{p})$ policy. Let $\pi(i)$ denote the stationary probability for the inventory level to be i if the length of inter arrival time is exact one unit. Then

$$\pi(0) = \pi(0) \sum_{k=0}^{\infty} \phi(kQ) + \sum_{j=1}^{Q-1} \pi(j) \sum_{n=1}^{\infty} \phi(-j + nQ),$$

$$\pi(i) = \sum_{j=0}^{i} \pi(j) \sum_{k=0}^{\infty} \phi(i - j + kQ) + \sum_{j=i+1}^{Q-1} \pi(j) \sum_{n=1}^{\infty} \phi(i - j + nQ),$$

where $\pi(Q-1) = \sum_{j=0}^{Q-1} \pi(j) \sum_{k=0}^{\infty} \phi(Q - 1 - j + kQ)$.

Display it in the matrix form as:

$$\pi = \pi \begin{bmatrix} \sum_{k=0}^{\infty} \phi(kQ) & \sum_{n=1}^{\infty} \phi(-1+nQ) & \sum_{n=1}^{\infty} \phi(-2+nQ) & \cdots & \sum_{n=1}^{\infty} \phi(-(Q-1)+nQ) \\ \sum_{k=0}^{\infty} \phi(1+kQ) & \sum_{k=0}^{\infty} \phi(kQ) & \sum_{n=1}^{\infty} \phi(-1+nQ) & \cdots & \sum_{n=1}^{\infty} \phi(1-(Q-1)+nQ) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \sum_{k=0}^{\infty} \phi(Q-1+kQ) & \cdots & \cdots & \cdots & \sum_{k=0}^{\infty} \phi(kQ) \end{bmatrix}.$$

It can be easily seen that the matrix is doubly stochastic. so

$$\pi(i) = \frac{1}{Q}$$

Hence it follows from the theory of Semi-Markov process (Ross, 2003) that the stationary probability of inventory level at i is:

$$\pi_i = \frac{1/(\lambda(p_i)Q)}{\sum_{j=R+1}^{R+Q}(\lambda(p_j)Q)} \qquad i = R+1, \ldots, R+Q$$

where p_i is the price when the inventory level is i.

Let $v(R, \mathbf{p})$ be the optimal average profit. Since the base quantity Q is given, we suppress it in v to simplify the notation. Thus the average profit per unit of time is given by:

$$\begin{aligned} v(R, \mathbf{p}) &= \sum_{i=R+1}^{R+Q} (\pi_i)(\lambda(p_i)(p_i - c)\mu - G(i)) \\ &= \frac{\sum_{i=R+1}^{R+Q} \frac{1}{\lambda(p_i)}(\lambda(p_i)(p_i - c)\mu - G(i))}{\sum_{j=R+1}^{R+Q} \frac{1}{\lambda(p_j)}} \end{aligned} \tag{3.13}$$

We want to maximize the above function, with respect to both R and $\mathbf{p}$. Since this function is not easy to optimize directly, in the following we again introduce an auxiliary function as the tool to simplify the optimization procedure.

We define the auxiliary profit function as follows:

$$\ell_\gamma(R, \mathbf{p}) = \sum_{i=R+1}^{R+Q} \frac{1}{\lambda(p_i)}(\lambda(p_i)(p_i - c)\mu - G(i) - \gamma) \tag{3.14}$$

where γ is the dummy profit. This is a fractional programming expression of the average profit of function (3.13). Again from the result of fractional programming, $\gamma = v^*$, where v^* is the optimal profit, if and only if $\ell_\gamma(R, \mathbf{p}) = 0$.

When the dummy profit γ is the long-run average profit of a known policy $(R_0, \mathbf{p}_0)$, we can interpret above function (3.14) as follows. As $\gamma = v(R_0, \mathbf{p}_0)$, γ can be regarded as the reference profit per period which will be earned when policy $(R_0, \mathbf{p}_0)$ is implemented. As a result, the auxiliary function can be viewed as an indicator of the comparative performance of policy $(R, \mathbf{p})$.

3.2.2 Optimal $(R, nQ, \mathbf{p})$ Policy

Before we proceed, first we introduce a function $f(y)$:

$$f(y) = \max_p \left\{ \frac{1}{\lambda(p)} (\lambda(p)(p - c)\mu - G(y) - \gamma) \right\}. \tag{3.15}$$

Lemma 3.2.1 *$f(y)$ is a unimodal function of y, so there exists one point y_0 which maximizes $f(y)$.*

Proof. As $u(p) = 1/\lambda(p)$ and

$$g(y, p) = \frac{1}{\lambda(p)} (\lambda(p)(p - c)\mu - G(y) - \gamma)$$

then the optimal p should satisfy the following first order necessary condition:

$$\begin{aligned} & g'_p(y, p) = \mu - u'(p)(G(y) + \gamma) = 0 \\ \Rightarrow \quad & \mu = u'(p)(G(y) + \gamma). \end{aligned}$$

Take derivative of $f(y)$ with respect to y,

$$\begin{aligned} f'(y) &= p'\mu - u'(p)p'(G(y) + \gamma) - u(p)G'(y) \\ &= -u(p)G'(y), \end{aligned}$$

where the second equality follows from the fact that $f'(y)$ is evaluated at the optimal price p and optimal price p needs to satisfy previous equation. Because $G(y)$ is a convex function, $G'(y)$ is first negative then positive. In addition, $u(p) \geq 0$. So $f'(y)$ is first positive then after some point it becomes negative, which means that $f(y)$ is first increasing then after some point it will decrease. So it is a unimodal by the definition. Therefore, there exists a maximum point y_0. $\square$

Proposition 3.2.1 *If $u(p)$ is convex in p, then the optimal price p^* has such property that it is increasing when inventory level is less than x_0 and decreasing when inventory level is greater x_0, where x_0 is the minimum point of $G(\cdot)$.*

Proof. Take derivative of $\mu = u'(p)(G(y) + \gamma)$ with respect to y,

$$u''(p)p'(G(y) + \gamma) + u'(p)G'(y) = 0$$

$$p'G'(y) = -\frac{u'(p)(G'(y))^2}{u''(p)(G(y) + \gamma)}.$$

Because $\lambda(p)$ is strictly decreasing in p, $u(p)$ will increase in p. Therefore, $u'(p) > 0$. In addition, since $u(p)$ is convex in p, $u''(p) > 0$. So that $p'G'(y) < 0$, which means that when $G'(y) > 0$, $p' < 0$, vice versa. $\square$

This property is quite intuitive: When inventory level is positive, the higher the inventory, the lower the selling price. When inventory level is negative (backlog), the higher the inventory level, the higher the selling price. Or, the higher the backlog level, the lower the selling price.

Theorem 3.2.1 *Let R_γ denote the optimal R for given γ, then we can locate the optimal*

R_γ *as follows:*

$$R_\gamma = \max\{x : f(x+Q) \geq f(x)\}$$

Proof. Suppose the price for inventory level in $[R+1, R+Q-1]$ are the same for the two policies $(R, \mathbf{p})$ and $(R-1, \mathbf{p}')$. Then the optimal reorder point can be located in the following way:

$$\begin{aligned}
\ell_\gamma(R) - \ell_\gamma(R-1) &= \frac{1}{\lambda(p_{R+Q})}(\lambda(p_{R+Q})(p_{R+Q}-c)\mu - G(R+Q) - \gamma) \\
&\quad - \frac{1}{(\lambda(p_R)}(\lambda(p_R)(p_R - c)\mu - G(R) - \gamma) \\
&= f(R+Q) - f(R) \geq 0
\end{aligned}$$

Because we already showed that $f(y)$ is unimodal, so above inequality is well defined. Then we can locate the optimal R_γ easily. $\square$

Theorem 3.2.2 *Suppose we already pinpoint the* R *at* R_γ *for a given* γ*, then we can decompose the price optimization procedure of* Q *prices in to* Q *one price optimization problem and we can solve it in real time independently as follows, for* $y = R+1, \ldots, R+Q$*:*

$$p(y) = u^{-1}\left(\frac{\mu}{G(y)+\gamma}\right)$$

Proof. Note that

$$p_\gamma(y) = \arg\max\left\{\frac{1}{\lambda(p)}(\lambda(p)(p-c)\mu - G(y) - \gamma)\right\}.$$

By taking derivative of the objective function with respect to p, we can get the desired result. $\square$

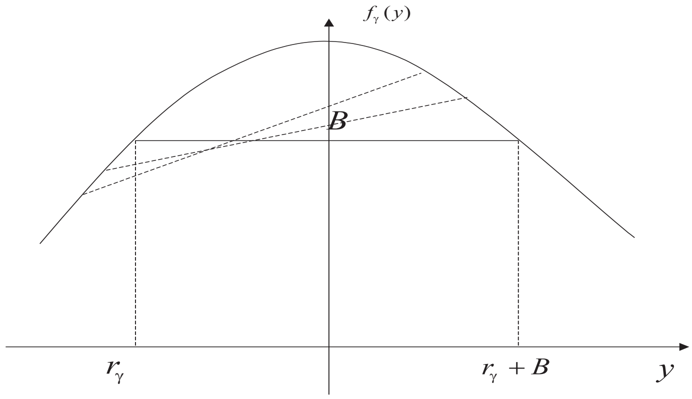


Figure 3.4: Optimal R and $R+Q$

So now we already know how to optimize the auxiliary function for a given γ, we can easily obtain the optimal R_γ and prices for inventory level from $R_\gamma + 1$ to $R_\gamma + Q$. Next we show how to upgrade the dummy profit and finally get the optimal average profit per unit of time v^* , the optimal reorder point R and price vector $\mathbf{p}$.

Clearly, policy $(R_0, \mathbf{p}_0)$ is not optimal if and only if $(R_\gamma, \mathbf{p}(R_\gamma{+}1, R_\gamma{+}Q))$ outperforms it with respect to the objective function $\ell_\gamma(R_\gamma, \mathbf{p})$. Thus the end of searching an optimal $(R_\gamma, \mathbf{p})$ is either a better alternative or a conclusion that current policy we have is optimal. However, the following lemma shows that a better alternative need not to be optimal.

Lemma 3.2.2 *Suppose that* $\ell_\gamma(R, \boldsymbol{p}) > 0$*, then,*$\gamma_1 = v(R, \boldsymbol{p}) > \gamma$*.*

Proof. The proof is parallel to Lemma 3.1.3, so we omit it. $\square$

Finally, we establish the termination rule for the procedure of searching v^* and the corresponding parameters.

Lemma 3.2.3 *Suppose for some γ, $\ell_\gamma(R, \boldsymbol{p}) = 0$, then we already find the optimal policy $(R, \boldsymbol{p})$, and $v^* = \gamma$.*

3.2.3 Algorithm

Since we already know $R^* < \max_p\{\lambda(p)(p-c)\mu\}$, so if we let $\gamma_1 = \max_p\{\lambda(p)(p-c)\mu - G(y)\}$, and $\gamma_2 = c(x_0, \mathbf{p})$. Then,$\ell_{\gamma_1}(R_{\gamma_1}, \mathbf{p}) < 0$ and $\ell_{\gamma_2}(R_{\gamma_2}, \mathbf{p}) > 0$. Therefore we can use bisection method to search the optimal R^* because we can locate the corresponding optimal R_γ and the optimal price easily for any given γ. Let ϵ be the tolerance level.

Algorithm:

- **Step 1**: Initialization, set $\gamma_L = \gamma_1$ and $\gamma_U = \gamma_2$;
- **Step 2**: Set $\gamma = (\gamma^L + \gamma^U)/2$, search the corresponding optimal R, continue;
- **Step 3**: Price optimization: for each $i = R+1, \ldots, R+Q$:

$$p_i = (u')^{-1}\left(\frac{\mu}{G(i)+\gamma}\right).$$

- **Step 4**: Calculate

$$\ell_\gamma = \sum_{i=R+1}^{R+Q} ((p_i - c)\mu - u(p_i)(G(i)+\gamma)).$$

  if $\ell_\gamma > 0$, then $\gamma^L = \gamma$, otherwise $\gamma^U = \gamma$. go to step 2. If $\ell_\gamma = 0$ or $\gamma^u - \gamma^L < \epsilon$, go to Step 5;

- **Step 5**: Stop, γ is the ϵ-optimal average profit, and the R and $\mathbf{p}$ are the ϵ-optimal reorder point and price vector, respectively.

It is the fact that the bisection search is very efficient, so the algorithm converges very fast and get the ϵ-optimal solutions.

3.3 $(s, S, \mathbf{p})$ Model

3.3.1 Model Description

In this section, we study a continuous-review inventory system with setup cost for each order. Again the demand arrives according to a compound Poisson process. The demand size is i with distribution $\phi(i)$ and mean μ. In addition, the demand arrival rate λ depends on the selling price p, i.e. $\lambda(p)$. The firm determines the selling price and the inventory replenishment strategies. Let K denote the setup cost and c denote the unit purchasing cost. And the inventory holding and shortage cost is $G(y)$. The firm wants to determine the policy that maximizes the long run average profit.

If the mean of interarrival time is 1 unit, then the stationary probability of inventory level at $S - i$ is $m(i)$,

$$m(0) = \frac{1}{1 - \phi(0)},$$

$$m(i) = \sum_{j=0}^{i} m(j)\phi(i - j).$$

For the general case when the arrival rate is $\lambda(p_i)$, the stationary probability of inventory level at point i, which depends on the selling price will be:

$$\pi_i = \frac{1/\lambda(p_i)m(i)}{\sum_{j=1}^{S-s-1} 1/\lambda(p_j)m(j)},$$

where $\lambda(p_j)$ is the arrival rate when the inventory level after ordering is j and the price is p_j.

The average profit per unit of time is

$$v(s, S, \mathbf{p}) = \frac{-K + \sum_{i=0}^{S-s-1} \frac{1}{\lambda(p_{S-i})} m(i)(\lambda(p_{S-i})(p_{S-i}\mu - c\mu) - G(S-i))}{\sum_{j=0}^{S-s-1} \frac{1}{\lambda(p_{S-j})} m(j)}. \quad (3.16)$$

Again we construct an auxiliary function

$$\ell_\gamma(s, S, \mathbf{p}) = -K + \sum_{i=0}^{S-s-1} \frac{1}{\lambda(p_{S-i})} m(i)(\lambda(p_{S-i})(p_{S-i}\mu - c\mu) - G(S-i) - \gamma)$$

where γ is the dummy profit. Based on lemma 3.1.1, the optimal profit $R^* = \gamma$ if and only if $\ell_\gamma(s, S, \mathbf{p}) = 0$

3.3.2 Optimal $(s, S, \mathbf{p})$ Policy

In this section, we compute the optimal $(s_\gamma, S_\gamma, \mathbf{p})$ that minimize $l_\gamma(s_\gamma, S_\gamma, \mathbf{p})$ for a given γ by using the auxiliary function.

Theorem 3.3.1 *For any given γ, let $a = \max_p\{(p - c)\lambda(p)\}$*

(a) the optimal reorder point:

$$s_\gamma = \sup\{y : G(y) \geq a\mu - \gamma, y < y_0\}$$

(b) the upper bound for the order up to point S_γ:

$$\overline{S}_\gamma = \sup\{y : G(y) \leq a\mu - \gamma, y \geq y_0\}$$

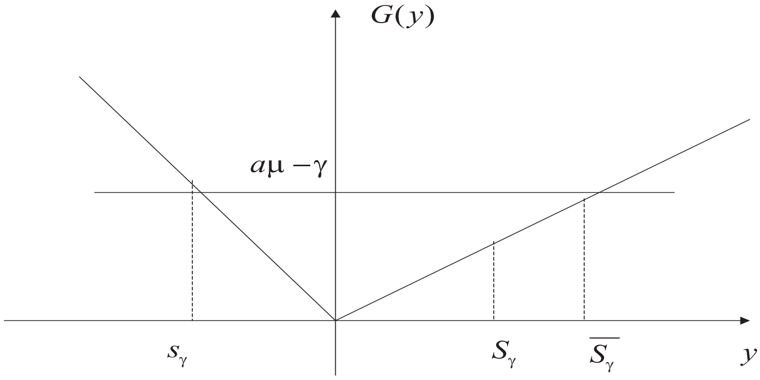


Figure 3.5: Optimal s_γ and $\bar{S}_\gamma$

(c) The optimal price, for $y = s_\gamma + 1, \ldots, S_\gamma$, is given by

$$p_y = u'^{-1}\left(\frac{\mu}{G(y)+\gamma}\right)$$

Proof. For (a), given any fixed S, and suppose $\mathbf{p}(s,S)$ is the same at $s+1,\ldots,S$ for the two policies, then optimal s satisfies the following:

$$\ell_\gamma(s-1,S,\mathbf{p}) - \ell_\gamma(s,S,\mathbf{p}) \quad = \quad m(S-s)(\lambda(p_s)(p_s-c)\mu - G(s) - \gamma) \geq 0$$

which implies

$$\alpha\mu - G(s) - \gamma \quad \geq \quad 0,$$

which follows from the definition of α. The left hand side can be argued, if for some p, which maximizes $\lambda(p)(p-c)$ and such that $\lambda(p)(p-c) \geq G(s)+\gamma$, then setting $p(s,S) = (p, p(s+1,S))$, we should have $\ell_\gamma(s-1,S,\mathbf{p}(s,S)) - \ell_\gamma(s,S,\mathbf{p}(s+1,S)) \geq 0$.

This proves (a).

We prove assertion (b) by contradiction. Note that

$$\begin{aligned}
\ell_\gamma(s,S,p(s+1,S)) &= -K + \sum_{i=0}^{S-s-1} \frac{1}{\lambda(p_{S-i})} m(i)(\lambda(p_{S-i})(p_{S-i}\mu - c\mu) - G(S-i) - \gamma) \\
&= -G(S) - \gamma + \lambda(p_S)(p_S - c)\mu - K(1 - \sum_{j=0}^{S-s-1} \phi(j)) \\
&\quad + \sum_{j=0}^{S-s-1} \phi(j)\ell_\gamma(s, S-j, p(s+1, S-j) \\
&\leq \alpha\mu - \gamma - G(S) - K(1 - \sum_{j=0}^{S-s-1} \phi(j)) \\
&\quad + \sum_{j=0}^{S-s-1} \phi(j)\ell_\gamma(s, S-j, p(s+1, S-j) \\
&< -K(1 - \sum_{j=0}^{S-s-1} \phi(j)) + \sum_{j=0}^{S-s-1} \phi(j)\ell_\gamma(s,S,p(s+1,S)
\end{aligned}$$

The following inequality follows:

$$\ell_\gamma(s,S,p(s+1,S) < -K,$$

as to be shown, an impossible relation. Now, consider a policy $(s, s+1, p)$ for a given γ, such that

$$\lambda(p)(p-c)\mu - G(s+1) - \gamma \geq 0$$

So this policy will be a lower bound on $\ell_\gamma(s,S,\mathbf{p})$ as:

$$\ell_\gamma(s,S,\mathbf{p}) \geq \ell_\gamma(s,s+1,p') = -K + m(0)\frac{1}{\lambda(p')}((p'-c)\mu - \gamma - G(s+1) \geq -K$$

which contradicts with the previous result. Consequently, we invalidate the assumption and prove the result.

Assertion(c) is easy to show because $m(j),\ j = 0, \ldots, S - s - 1$ are independent of p. We assume that $\lambda(p)$ is strictly decreasing of p. Therefore, the demand arrival rate λ and price p is one to one correspondent, for each p, we can find a corresponding λ. Thus we can take derivative with respect to p of the objective function,

$$\mu - u'(p_y)(G(y) - \gamma) = 0$$

which implies

$$p_y = u'^{-1}\left(\frac{\mu}{G(y) + \gamma}\right)$$

Then we can solve for optimal p immediately. $\square$

Theorem 3.3.2 *If $u(p)$ is a convex function, the optimal price $p(y)$ increases as y increases when $y \leq y_0$, and $p(y)$ decreases after $y > y_0$. That is, the path of the function $p(y)$ is opposite to the one period cost function $G(y)$.*

Proof. The proof is similar to the one in the previous section, we skip it here. $\square$

3.3.3 Algorithm

In this algorithm, we assume that a minimizer of $G(y)$, y_0 is known and we have calculated the renewal density $m(i)$ off line. The algorithm starts with a policy $(s_0, S_0, \mathbf{p}_0)$. One choice of $(s_0, S_0, \mathbf{p}_0)$ is $(y_0 - 1, y_0, p_0)$, where p_0 is the price entailing $f(y_0)$. Then γ_0 is the average profit achieved by this policy.

- **Step 0 (Initialization)**

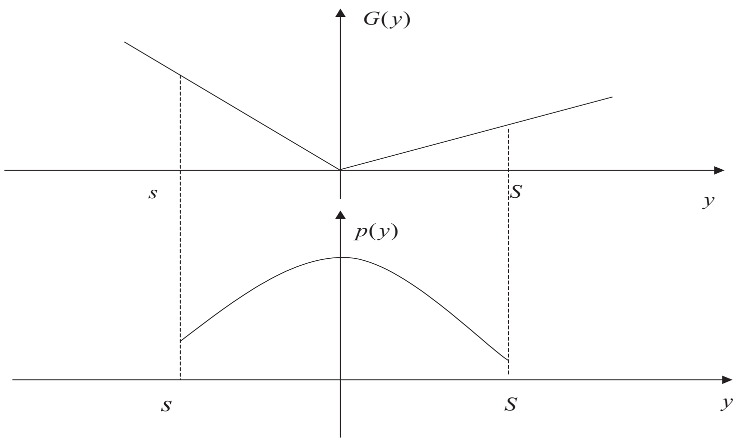


Figure 3.6: Properties of optimal p

Let $\gamma = \gamma_0 = v(y_0 - 1, y_0, p)$. Compute $f(y_0 - 1)$, $f(y_0 - 2)$, $\dots$ until some $f(y_0 - n) < \gamma$. Let $s_\gamma = y_0 - n$, $\overline{y} = \max\{s_\gamma + 1, x_0\}$. Let ϵ be the tolerance level.

Calculate the $p(s_\gamma + 1, \overline{y})$ by previous result and the value of auxiliary function ℓ then go to step 1.1.

- **Step 1(Upgrading the dummy profit)**

  1.1 (**Updating the dummy profit**) If $\ell < 0$, go to step 1.2. If $\ell > 0$, let $\gamma = \frac{\ell}{\sum_{i=0}^{S-s-1} 1/\lambda(p_{S-i})m(i))} + \gamma$, go to step 1.3. If $\frac{\ell}{\sum_{i=0}^{S-s-1} 1/\lambda(p_{S-i})m(i))} \leq \epsilon$, go to Step 2.

  1.2 (**Update order up to point**) If $G(\overline{y}+1) > \alpha\mu - \gamma$, go to step 2. Otherwise $\overline{y} = \overline{y} + 1$, then calculate the $p(\overline{y})$ and ℓ. Go to step 1.1.

  1.3 (**Update reorder point**)

  if $G(s_\gamma + 1) > a\mu - \gamma$, $s_\gamma = s_\gamma + 1$ until $G(s_\gamma + 1) \leq a\mu - \gamma$. If $\overline{y} \leq s_\gamma$ set $\overline{y} = s_\gamma + 1$, again, calculate the optimal price p and ℓ

- **Step 2**(**Termination**)

  Stop with $\gamma = v^*$ and optimal policy $(s, S, \mathbf{p})$.

3.3.4 Optimality Verification

In this section, we verify that $(s, S, \mathbf{p})$ policy is optimal among all other policies. Even though the optimality of the policy has been proved by Chen and Simchi-Levi (2002 a), we prove it through different approach in this section.

The optimal $(s^*, S^*, \mathbf{p}^*)$ policy (we will omit the $*$ in the following proof) is optimal among all the feasible policies if this policy and its long-run average profit v^* together satisfy the following *long-run average profit criterion*:

$$h(x) = \sup_{y \geq x, p \in [c,\infty)} \left\{ -K\delta\{y - x > 0\} - \frac{G(y) + v^*}{\lambda(p_y)} + (p_y - c)\mu + E[h(y - D)] \right\},$$

where $h(x)$ must be a bounded function. We relax the original problem to the following problem which is allowing return the inventory:

$$h(x) = \sup_{y \neq x, p \in [0,\infty)} \left\{ -K\delta\{y \neq x\} - \frac{G(y) + v^*}{\lambda(p_y)} + (p_y - c)\mu + E[h(y - D)] \right\}.$$

If our policy is optimal for the relaxed problem, it must be optimal for the original problem too. Now what we need is to construct a bounded function $h(x)$ to satisfy the criterion above. We structure the function in relation to the auxiliary function $\ell_{v^*}(s, S, \mathbf{p})$. A function $h(x)$ is defined recursively as follows:

$$h(x) = \begin{cases} -K & \text{if } x \leq s, \\ \ell_v(s, x, p(s+1, x)) & \text{if } s < x \leq S \\ \max_p\{-\frac{G(x)+v^*}{\lambda(p)} + (p-c)\mu + E[h(x-D)]\} & \text{if } S < x \leq \overline{S}, \\ \max\{-K, \max_p\{-\frac{G(x)+v^*}{\lambda(p)} + (p-c)\mu + E[h(x-D)]\} & \text{if } x > \overline{S} \end{cases}$$

It is obviously that $h(x) \geq -K$ when $x \leq s$ or $x > \overline{S}$, we want to show $-K \leq h(x) \leq 0$, and which is the case for $x \leq s$ that we can get immediately from above equations. Consider the case when $x \in (s, \overline{S})$. As (s, S, p) optimizes $\ell_v(s, S, p)$, so $\ell_v(s, x, p) <$

$\ell_v(s,S,p) = 0$ when $x > s$. To show $h(x) \geq -K$ in this range, when $s < x \leq \overline{S}$

$$\begin{aligned} h(x) = \ell_v(s,x,p) &= -K + \sum_{i=0}^{x-s-1} \frac{1}{\lambda(p_{x-i})} m(i)(\lambda(p_{x-i})(p_{x-i}\mu - c\mu) - G(x-i) - v^*) \\ &\geq -K, \end{aligned}$$

where $\lambda(p_{x-i})(p_{x-i}\mu - c\mu) - G(x-i) - v^* \geq 0$ by the definition of the optimal s and $\overline{S}$. Now we proceed to the case when $x > \overline{S}$. We prove it by induction starting from $x = \overline{S} + 1$:

$$\begin{aligned} h(\overline{S}+1) &= \max_p \left\{ -\frac{G(\overline{S}+1) + v^*}{\lambda(p)} + (p-c)\mu + E[h(\overline{S}+1-D)] \right\} \\ &\leq \max_p \left\{ -\frac{G(\overline{S}+1) + v^*}{\lambda(p)} + (p-c)\mu \right\} \\ &\leq 0 \end{aligned}$$

where the first inequality is based on the previous results and the second is based on the definition of $\overline{S}$. The induction procedure is simple so we omit it here.

After we show $h(x)$ is bounded, we need to show it satisfies the optimality equation above.

For $x \leq s$, we need to show $-K \geq -\frac{G(x)+v^*}{\lambda(p_x)} + (p_x - c)\mu + E[h(x-D)]$, which can be validated by definition of s:

$$G(s) > \max_p \{\lambda(p)(p_s) - c)\mu\} - v^*.$$

So when $x \leq s$

$$-\frac{G(x) + v^*}{\lambda(p_x)} + (p_x - c)\mu - K \leq -K$$

hence $h(x) = -K$.

For $s < x \le S$, we again prove it by induction, let $x = s+1$ then

$$h(s+1) = (p_{s+1} - c)\mu - \frac{G(s+1) - v^*}{\lambda(p_{s+1}} + \phi(0)h(s+1) + (1-\phi(0))(-K)$$

which implies

$$\begin{aligned} h(s+1) &= -K + m(0)[(p_{s+1} - c)\mu - \frac{G(s+1) - v^*}{\lambda(p_{s+1}}] \\ &= \ell_{v^*}(s, s+1, p(s, s+1)). \end{aligned}$$

Suppose that this is true for $x = j$, then for $x = j+1$

$$\begin{aligned} h(j+1) &= -\frac{G(j+1) + v^*}{\lambda(p_{j+1})} + (p_j - c)\mu + E[h(j+1-D)] \\ &= -\frac{G(j+1) + v^*}{\lambda(p_{j+1})} + (p_{j+1} - c)\mu + \sum_{i=1}^{j-s}(\phi(i)h(j+1-i)) + \phi(0)h(j+1) \\ &\quad + \sum_{i=j-s+1}^{\infty} \phi(i)(-K) \\ &= -\frac{G(j+1) + v^*}{\lambda(p_{j+1})} + (p_{j+1} - c)\mu \\ &\quad + \sum_{i=1}^{j-s}(\phi(i)\ell_{v^*}(s, j+1-i, p)) + \phi(0)h(j+1) + \sum_{i=j-s+1}^{\infty} \phi(i)(-K) \\ \Rightarrow h(j+1) &= m(0)[\frac{G(j+1) + v^*}{\lambda(p_{j+1})} + (p_{j+1} - c)\mu + \sum_{i=1}^{j-s} \phi(i)\ell_{v^*}(s, j+1-i, p) \\ &\quad + \sum_{i=j-s+1}^{\infty} \phi(i)(-K)] \\ &= \ell_{v^*}(s, j+1, p(s+1, j+1)), \end{aligned}$$

where the last equality can be validated by the following:

$$\begin{aligned}
\ell_{v^*}(s, j+1, p(s+1, \ldots, j+1)) &= -K + m(0)((p_{j+1} - c)\mu - \frac{G(j+1) + v^*}{\lambda(p_{j+1})}) \\
&\quad + m(0) \sum_{i=1}^{j-s} \phi(i)(\ell_{v^*}(s, j+1-i, p) + K) \\
&= m(0)((p_{j+1} - c)\mu - \frac{G(j+1) + v^*}{\lambda(p_{j+1})}) \\
&\quad + m(0) \sum_{i=1}^{j-s} \phi(i)(\ell_{v^*}(s, j+1-i, p)) \\
&\quad + \sum_{i=j-s+1}^{\infty} \phi(i)(-K).
\end{aligned}$$

For $S < x < \overline{S}$ we need to show

$$-\frac{G(x) + v^*}{\lambda(p_x)} + (p_x - c)\mu + E[h(x - D)] \geq -K$$

which can be proved by the definition of $\overline{S}$:

$$\begin{aligned}
-\frac{G(x) + v^*}{\lambda(p_x)} + (p_x - c)\mu + E[h(x - D)] &\geq -\frac{G(x) + v^*}{\lambda(p_x)} + (p_x - c)\mu - K \\
&\geq -K
\end{aligned}$$

Finally, we prove the $h(x)$ is valid for range $x > \overline{S}$. We just need to verify that in this range, $h(x)$ is equivalent to:

$$\max\left\{ -K, \max_p \{ -\frac{G(x) + v^*}{\lambda(p)} + (p - c)\mu + E[h(x - D)] \right\}$$

which can be easily proved since $h(S) = 0$. □

3.4 Summary

We have presented three continuous-review inventory models that jointly optimize pricing and inventory control strategies. We characterize the simple structure of their optimal policies. Furthermore, we obtain a structural property for the path of the optimal price, which is that the price decreases when backorder increases and decreases when inventory on hand increases. We also develop efficient algorithms to calculate the optimal control parameters, which improve the implementability of the optimal policy.

Chapter 4

Joint Optimization of Pricing and Inventory Control for Periodic-Review Inventory Systems

We study two periodic-review inventory/production models in this chapter. We characterize optimal inventory and pricing policies for both models. In §4.1, we consider pricing and inventory strategies for a inventory model with dual supply modes. In §4.2, we combine pricing and production decisions for a production smoothing model.

4.1 Single-Stage Inventory Model with Dual Transportation Modes

In this section, we study a periodic-review inventory model with dual supply options. The firm has three decisions to make at the beginning of each period: the quantity of emergency order, the quantity of regular order and the selling price for the product. The leadtime difference between the emergency order and regular order is one period. The objective of the firm is to maximize the total discounted profit over a finite or an infinite horizon.

Time sequence of events is : First, the firm receives the regular order placed in previous period and observes the current inventory level; second, he decides the order quantity by using emergency order and receives it immediately; third, a regular order is placed if needed and the selling price is set; fourth, demand is realized and excess demand is backlogged; fifth, all costs and revenue incur.

4.1.1 Finite Horizon Problem

The following are the notation we need:

x_n = the initial inventory level at the beginning of period n before any decisions are made;

y_n = the inventory level after placing the emergency order;

u_n = the inventory position after placing the regular order;

T = the length of the planning horizon;

c_0 = the unit purchasing cost for regular order;

c_1 = the unit purchasing cost for emergency order, $c_1 > c_0$;

p_n = the unit selling price, $p_n \geq c_1$;

$D_n(p, \epsilon)$ = the demand in period n and $E[D_n(p, \epsilon)] = d_n(p)$, where ϵ is the random perturbation;

$G(p, y)$ = the expected inventory holding and shortage cost, i.e. $G(p, y) = E[h[y - D(p, \epsilon)]$;

p_{max}, p_{min} = the lower bound and upper bound of the selling price, respectively;

α = the discount factor, $\alpha \leq 1$.

The following assumptions are needed for the deduction of the results:

Assumption 4.1.1 *The demand $D_t(p, \epsilon)$ is concave and decreasing in p. Thus, $E[D_t(p, \epsilon)] = d_t(p)$ is concave and decreasing in p.*

Assumption 4.1.2 *The expected inventory holding and shortage cost per period $G_t(y, p) = E[h_t(y - D_t(p, \epsilon))]$ is jointly convex in y and p.*

Assumption 4.1.3 *The expected revenue function $R_t(p) = pd_t(p)$ is concave in p.*

Assumption 4.1.4 $\lim_{y\to\infty} G_t(y, p) = \lim_{y\to\infty}[(c_1 - c_0)y + c_0 u + G_t(y, p)] = \lim_{y\to\infty}[(c_0 - \alpha c_1)u + (c_1 - c_0)c_1)y + G_t(y, p)] = \infty$ *for all* p.

The problem can be formulated as,

$$\begin{aligned} v_n(x_n) &= \max_{u_n \geq y_n \geq x_n, p_n \in [p_{min}, p_{max}]} \{-c_1(y_n - x_n) - c_0(u_n - y_n) - G_n(y_n, p_n) + pd(p) \\ &\quad +\alpha E[v_{n-1}(u_n - D_n(p, \epsilon)]\}. \end{aligned} \tag{4.1}$$

Let

$$f_n(y_n, u_n, p_n) = -c_1 y_n - c_0(u_n - y_n) + R(p) - G_n(y_n, p_n) + \alpha E[v_{n-1}(u_n - D_n(p, \epsilon)] \tag{4.2}$$

and

$$V_n(x_n) = v_n(x_n) - c_1 x_n$$

which is equivalent to shift the $c_1 x_n$ to the previous period, and set $V_0(x) = 0$. We can change (4.1) and (4.2) accordingly,

$$V_n(x_n) = \max_{y_n \geq x_n, p_n, r_n} \{f_n(y_n, u_n, p_n)\} \tag{4.3}$$

where

$$\begin{aligned} f_n(y_n, u_n, p_n) &= (c_0 - c_1)y_n - c_0 u_n + R(p_n) - G_n(y_n, p_n) \\ &\quad + \alpha E[c_1(u_n - D_n(p, \epsilon))] + \alpha E[V_{n-1}(u_n - D_n(p, \epsilon)] \\ &= (c_0 - c_1)y_n + (\alpha c_1 - c_0)u_n + R(p_n) - G_n(y_n, p_n) - \alpha c_1 d(p_n) \\ &\quad + \alpha E[V_{n-1}(u_n - D_n(p, \epsilon)]. \end{aligned} \tag{4.4}$$

4.1.2 Optimal Policies

Theorem 4.1.1 *(a) $f_n(y_n, u_n, p_n)$ is concave with respect to y_n, u_n, p_n;*

(b) $V_n(x_n)$ is concave and nonincreasing in x_n.

Proof. We prove it by induction. Because $V_0(x) = 0$, obviously it is true for $n = 0$. Supposed $V_{n-1}(x)$ is concave in x, then let $y' = \lambda y_1 + (1 - \lambda)y_2$, $u' = \lambda u_1 + (1 - \lambda)u_2$,

$p' = \lambda p_1 + (1-\lambda)p_2$, $\lambda \in [0,1]$ and skip the linear term,

$$\begin{aligned}
f_n(y',u',p') &= -G(y',p') + (p'-\alpha c_1)d(p') + \alpha E[V_{n-1}(u' - D(p',\epsilon)] \\
&= (p'-\alpha c_1)d(\lambda p_1 + (1-\lambda)p_2) - G(\lambda y_1 + (1-\lambda)y_2, \lambda p_1 + (1-\lambda)p_2) \\
&\quad + \alpha E[V_{n-1}(\lambda(u_1) + (1-\lambda)(u_2) - D_n(\lambda p_1 + (1-\lambda)p_2, \epsilon)] \\
&\geq \lambda((p_1 - \alpha c_1)d(p_1) - G(y_1,p_1) + \alpha E[V_{n-1}(u_1 - D(p_1,\epsilon)]) \\
&\quad + (1-\lambda)((p_2 - \alpha c_1)d(p_2) - G(y_2,p_2) + \alpha E[V_{n-1}(u_2 - D(p_2,\epsilon)]) \\
&= \lambda f_n(y_1,u_1,p_1) + (1-\lambda)f_n(y_2,u_2,p_2),
\end{aligned}$$

where the inequality follows from the concavity and monotonicity of $D(p,\epsilon)$, convexity of $G(y,p)$ and the nonincreasingness of $V(x)$. Then by Proposition B4 in Sobel (1984), $V_n(x)$ is concave.

From the optimality equation, $V_n(x_n)$ is nonincreasing in x_n because the feasible domain of y_n becomes smaller as x_n increases and we are trying to maximize the objective function. □

Therefore, the optimal price is

$$p_n(y_n,u_n) = \arg\max_{p\in[p_{min},p_{max}]}\{f_n(y_n,u_n,p_n)\}. \tag{4.5}$$

Lemma 4.1.1 *(a) $f_n(y_n,u_n,p_n)$ is a submodular in y and p;*

(b)$f_n(y_n,u_n,p_n)$ is a submodular in u and p;

Proof. Note that the first two terms of (4.3) only depend on one variable, so they are obviously submodular. For $G_n(y_n,p_n) = E[h(y_n - D(p_n,\epsilon))]$, let $y_1 < y_2$, $p_1 < p_2$, let

$s_1 = y_1 - D(p_1,\epsilon)$, $s_2 = y_1 - D(p_2,\epsilon)$, $s_3 = y_2 - D(p_1,\epsilon)$, $s_4 = y_2 - D(p_2,\epsilon)$, then

$$s_1 < s_2, \quad s_3 < s_4$$

by the monotonicity of $D(p,\epsilon)$. Therefore,

$$\begin{aligned} h(s_1) - h(s_3) &= h(s_3 + (y_1 - y_2)) - h(s_3) \\ &\le h(s_4 + (y_1 - y_2)) - h(s_4) \\ &= h(s_2) - h(s_4). \end{aligned}$$

So $h(\cdot)$ has isotone difference and is supermodular. Then $G(y,p) = E[h(y - D(p,\epsilon))]$ is supermodular too. For (b), we just need to show the submodularity of $V_n(u_n - D_n(p,\epsilon))$. By the concavity of $V_n(x)$, it can be proved by the similar method as we show the supermodularity of $G(\cdot,\cdot)$ to prove the submodularity of V_n. □

Proposition 4.1.1 *$p_n(y_n,u_n)$ is nonincreasing in both y_n and u_n.*

Proof. This proposition immediately follows from the submodularity of f_n. □

Substitute the optimal price $p(y,u)$ into the value function,

$$\begin{aligned} V_n(x_n) &= \max_{u_n \ge y_n \ge x_n} \{(c_0 - c_1)y_n + (\alpha c_1 - c_0)u_n + R(p_n(u_n,y_n)) - G_n(y_n, p_n(u_n,y_n)) \\ &\quad -\alpha c_1 d(p_n(u_n,y_n)) + \alpha E[V_{n-1}(u_n - D_n(p(u_n,y_n),\epsilon)]\}. \end{aligned}$$

We optimize y_n first, Let

$$\begin{aligned} g_n(y_n,u_n) &= (c_0 - c_1)y_n + R(p_n(u_n,y_n)) - G_n(y_n, p_n(u_n,y_n)) \\ &\quad -\alpha c_1 d(p_n(u_n,y_n)) + \alpha E[V_{n-1}(u_n - D_n(p(u_n,y_n),\epsilon)]. \end{aligned}$$

Because $g_n(y_n, r_n)$ is concave, then there exists u_n, such that

$$y_n(u_n) = \arg\max_{y_n}\{g_n(y_n, u_n)\}.$$

The last step is to optimize the regular order-up-to level u_n. Let

$$\begin{aligned} J_n(u_n) &= (c_0 - c_1)y_n(u_n) + (\alpha c_1 - c_0)u_n + (p_n(u_n) - \alpha c_1)d(p_n(u_n)) \\ &\quad -G_n(y_n(u_n), p_n(u_n) + \Gamma_n(u_n) + \alpha E[V_{n-1}(u_n - D_n(p(u_n), \epsilon)]. \end{aligned}$$

where

$$\begin{aligned} \Gamma_n(u_n) &= (c_0 - c_1)u_n + R(p_n(u_n, u_n)) - G_n(y_n, p_n(u_n, u_n)) \\ &\quad -\alpha c_1 d(p_n(u_n, u_n)) + \alpha E[V_{n-1}(u_n - D_n(p(u_n, u_n), \epsilon)] \\ &\quad -\Big((c_0 - c_1)y_n(u_n) + R(p_n(u_n, y_n(u_n))) - G_n(y_n(u_n), p_n(u_n, y_n(u_n))) \\ &\quad -\alpha c_1 d(p_n(u_n, y_n(u_n))) + \alpha E[V_{n-1}(u_n - D_n(p(u_n, y_n(u_n)), \epsilon)]\Big) \end{aligned}$$

if $u_n < y_n(u_n)$; otherwise $\Gamma_n(u_n) = 0$.

Because $J_n(u_n)$ is concave in u_n then there exists one u_n^* which maximizes $J_n(\cdot)$. Let $y_n^* = y_n(u_n^*)$ and $p_n^* = p_n(y_n^*, u_n^*)$.

Theorem 4.1.2 *(a) There exists a set of finite maximizers, denoted by (y_n^*, u_n^*, p^*), of $f_n(y, u, p)$;*

(b) The optimal emergency order policy is base-stock policy, which is

$$y_n = \begin{cases} y_n^* & \textit{if } x_n < y_n^*, \\ x_n & \textit{otherwise.} \end{cases}$$

The optimal regular order policy is also base-stock policy

$$u_n = \begin{cases} u_n^* & u_n^* > y_n, \\ y_n & \textit{otherwise.} \end{cases}$$

(c) The optimal price p_n^* can be determined after computing y_n^* and u_n^*.

Proof. By assumption 4.4 and for $p \in [p_{min}, p_{max}]$, $f_t(y, u, p) \to -\infty$ as $y \to \infty$. This implies that $f_t(y, u, p)$ has finite maximizer. And because of the concavity, the result follows. □

4.1.3 Infinite Horizon Problem

In this section, we extend the problem to the infinite planning horizon case. In analyzing infinite horizon models, it is often useful to have one period reward that is uniformly of the same sign. To achieve this, we subtract a constant $M = \max_{p_{min} \le p \le p_{max}} pd(p)$ from the one period expected profit ($M < \infty$). We then obtain the shifted value function:

$$\bar{V}_t(x) = V(x) - \frac{M(1 - \alpha^{t+1})}{1 - \alpha}$$

and

$$\bar{f}_t(y, u, p) = f(y, u, p) - \frac{M(1 - \alpha^{t+1})}{1 - \alpha}.$$

The optimality equation is given by:

$$V(x) = \max_{u \ge y \ge x, p \in [p_{min}, p_{max}]} f(y, u, p)$$

where

$$f(y,u,p) \;=\; (c_0-c_1)y+(\alpha c_1-c_0)u+(p-\alpha c_1)d(p)-G(y,p)-M$$
$$+\alpha E[V(y+u-D(p,\epsilon)].$$

The following theorem describes the structure of an optimal policy in the infinite horizon model, and its relationship to that of the finite horizon model.

Theorem 4.1.3 *1.* $\bar{V}(x)=\lim_{t\to\infty}\bar{V}_t(x)$, $V(x)=\lim_{t\to\infty}V_t(x)$, $\bar{f}(y,u,p)=\lim_{t\to\infty}\bar{f}_t(y,u,p)$, $f(y,u,p)=\lim_{t\to\infty}f_t(y,u,p)$. *and* $\bar{V}(x)=V(x)-\frac{M}{1-\alpha}$ *and* $\bar{f}(y,u,p)=f(y,u,p)-\frac{M}{1-\alpha}$.

2. $\bar{V}$ *and* $\bar{f}$ *(*V *and* f*) satisfy the transformed (original) optimality equation.*

3. $V(x)$ *is concave and decreasing in* x *and* $f(y,u,p)$ *is concave in* y*,* u *and* p *and* $f(y,u,p)$ *has a finite maximizer* (y^*,u^*,p^*)*.*

4. The optimal inventory policy for the infinite horizon model can be characterized as:

For the emergency order:

$$y=\begin{cases} y^* & \text{if } x<y^*, \\ x & \text{otherwise.} \end{cases}$$

For the regular order,

$$u=\begin{cases} u^* & y<u^*, \\ y & \text{otherwise.} \end{cases}$$

5. The sequence $\{(y_t^*,u_t^*,p_t^*)\}$ *has at least one limit point and such limit point is an optimal policy for the infinite horizon problem.*

Proof. Because after we subtract a finite constant M from the original one period reward function, it will always be negative. We can apply the results in Negative Markov Decision Problem to prove the theorem easily. (See Ross(1983))

Since we only subtract a finite constant from the original problem, so the optimal policy for the transformed problem is also optimal for the original problem. $\square$

4.2 Joint Optimization of Production and Pricing for Production Smoothing Model

In this section, we study a production system, in which random demand depends on price and changes in production rate incur a cost. Besides the production decision, again the firm needs to determine the unit selling price for the product at the beginning of each period.

4.2.1 Finite Horizon Problem

We consider the effects of smoothing costs, i.e., costs that discourage intertemporal volatility of production quantities. Let z_t denote the production level at period t. Suppose that the smoothing costs in period t are $u_t(z_t - z_{t-1})$ if $z_t \geq z_{t-1}$ and $w_t(z_{t-1} - z_t)$ if $z_{t-1} > z_t$. Let $b_t = (u_t + w_t)/2$ and $e_t = (u_t - w_t)/2$, then the smoothing cost can be restated as

$$u_t(z_t - z_{t-1})^+ + w_t(z_{t-1} - z_t)^+ = b_t\,|z_t - z_{t-1}| + e_t(z_t - z_{t-1}).$$

The unit production cost is c_t, and inventory holding and shortage cost is $L_t(y)$. We denote the unit selling price for the product at period t by p_t, which belongs to $[p_{min}, p_{max}]$. The random demand size at period t depends on the price, i.e. $D_t(p, \epsilon_t)$, where ϵ is a random perturbation. Assume the production leadtime is 0 and unsatisfied demand is backlogged. The objective of the firm is to maximize its total discounted profit of T periods. Suppose the salvage value for each unit of the inventory at the beginning of period $T+1$ is c_{T+1} which means the firm can sell its product at c_{T+1} if it has inventory or produce at the same cost to satisfy the backlog.

Let $\beta < 1$ be the discounted factor, when the initial inventory is x_1 and the preceding rate of production is z_0, the total discounted profit of the firm in T periods is given by:

$$\begin{aligned} C &= \sum_{t=1}^{T} \beta^{t-1}\{p_t D_t(p_t, \epsilon) - [b_t\,|z_t - z_{t-1}| + e_t(z_t - z_{t-1}) + c_t z_t + L_t(y_t - D_t(p_t, \epsilon)]\} \\ &\quad +\beta^T c_{T+1} x_{T+1} \end{aligned}$$

The expected present profit is:

$$\begin{aligned} E(C) &= E\{\sum_{t=1}^{T} \beta^{t-1}[p_t D_t(p_t, \epsilon) - (b_t\,|y_t - x_t - z_{t-1}| + G_t(y_t, p)]\} \\ &\quad +[(c_1 + e_1 - \beta e_2)x_1 + e_1 z_0] \end{aligned} \tag{4.6}$$

where $e_t = 0$ when $t > T$ and

$$\begin{aligned} G_t(y, p) &= [c_t + e_t - \beta c_{t+1} - 2\beta e_{t+1} + \beta^2 e_{t+2}]y + E[L_t(y - D_t(p)] \\ &\quad -\beta d_t(p)(-e_{t+1} + \beta e_{t+2} - c_{t+1}). \end{aligned} \tag{4.7}$$

Because the second line of (4.6) depends on neither price nor production policy, both of the policies will be optimal if and only if it is optimal for the following problem with

$f_{T+1}(\cdot,\cdot) \equiv 0$ and $t = 1, \ldots, T$:

$$f_t(x_t, z_{t-1}) \quad = \quad \sup_{y \geq x, p_t \in [p_{min}, p_{max}]} \{-(b_t \left| y_t - x_t - z_{t-1} \right| + J_t(x_t, y_t, p_t)\}, \tag{4.8}$$

where

$$J_t(x_t, y_t, p_t) \quad = \quad p_t d(p_t) - G(y_t, p_t) + \beta E[f_{t+1}(y_t - D_t(p_t, \epsilon), y_t - x_t)]. \tag{4.9}$$

Let $p_t(x, y) \in \arg\max_{p_{max} \geq p_t \geq p_{min}} J_t(x, y, p)$

Then,

$$f_t(x_t, z_{t-1}) = \sup_{y \geq x} \{-b_t \left| y_t - x_t - z_{t-1} \right| + J_t(x_t, y_t, p_t(x_t, y_t))\}. \tag{4.10}$$

Before proceed to the next step, we need some important assumptions here:

Assumption 4.2.1 *For t=1,2,...,T, $D_t(p, \epsilon)$ has the following structure:*

$$D_t(p, \epsilon) = Ad_t(p) + B$$

where $d_t(p) = a - bp$, A and B are independent random variables with $E[A] = 1$ and $E[B] = 0$.

Assumption 4.2.2 *For each t, $t = 1, 2, \ldots, T$, $L_t(y)$ is a convex function of the inventory level y at the end of the period t. $E[L_t(y - D_t(p, \epsilon)]$ is jointly convex in y and p.*

Lemma 4.2.1 *If $D_t(p, \epsilon)$ is linear in p, then $G_t(y, p)$ is jointly convex in y and p*

Assumption 4.2.3 $(-e_{t+1} + \beta e_{t+2} - c_{t+1}) \geq 0$

Theorem 4.2.1 *(a) For any* $t = 1, 2, \ldots, T$*,* $J_t(x_t, y_t, p_t)$ *is continuous in* (x, y, p) *and* $\lim_{|y| \to \infty} J_t(x_t, y_t, p_t) = -\infty$ *for any* p *in* $[p_{min}, p_{max}]$*. Hence, for any fixed* y *and* x*,* $J_t(x, y, p)$ *has a finite maximizer, denote by* $p_t(x, y)$*.*

(b) For any $t = 1, 2, \ldots, T$*,* $J_t(x, y, p)$ *is concave in* x*,* y *and* p*;*

(c) $f_t(x, z)$ *is jointly concave in* x *and* z*.*

Proof. By assumption 4.2.1 and 4.2.2, that $d_t(p)$ is linear in p and $E[L_t(y - D_t(p, \epsilon))]$ is jointly convex in y and p, we can conclude directly that $G_t(y, p)$ is jointly convex in y and p based on assumption 4.2.3. For $t = T$,

$$J_T(x_T, y_T, p_T) = p_T d(p_T) - G(y_T, p_T),$$

which is independent of x_T and jointly concave in y and p. As a result, f_T is jointly concave of x and z by Proposition B4 in Sobel (1983).

By induction, suppose part (a) and (b) hold for $t = n+1$. For any $\lambda \in [0,1]$,

$$
\begin{aligned}
& J_n(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \lambda p_1 + (1-\lambda)p_2) \\
= \;& pd(p) - G(\lambda y_1 + (1-\lambda)y_2, \lambda p_1 + (1-\lambda)p_2) \\
& +\beta E[f_{n+1}(\lambda y_1 + (1-\lambda)y_2 \\
& -D(\lambda p_1 + (1-\lambda)p_2), \lambda y_1 + (1-\lambda)y_2 - \lambda x_1 - (1-\lambda)x_2] \\
\geq \;& pd(p) - [\lambda G(y_1, p_1) + (1-\lambda)G(y_2, p_2)] \\
& +\beta E[f_{n+1}(\lambda(y_1 - D(p_1, \epsilon)) \\
& +(1-\lambda)(y_2 - D(p_2, \epsilon)), \lambda(y_1 - x_1) + (1-\lambda)(y_2 - x_2)] \\
\geq \;& \lambda p_1 d(p_1) + (1-\lambda)p_2 d(p_2) - \lambda G(y_1, p_1) - (1-\lambda)G(y_2, p_2) \\
& +\lambda\beta E[f_{n+1}(y_1 - D(p_1, \epsilon), y_1 - x_1)] + (1-\lambda)\beta E[f_{n+1}(y_2 - D(p_2, \epsilon), y_2 - x_2)] \\
= \;& \lambda J_n(x_1, y_1, p_1) + (1-\lambda)J_n(x_2, y_2, p_2).
\end{aligned}
$$

Thus, J_n is concave in x, y and p. Therefore, $f_n(x_n, z_{n-1})$ is concave in x and z because $-b_n\,|y_n - x_n - z_{n-1}|$ is concave in x and z. $\square$

Lemma 4.2.2 *(a) $J_t(x,y,p)$ is a supermodular function in x and y.*

(b) $J_t(x,y,p)$ is a submodular function in p and y.

Proof. For part (a),

$$J_t(x,y,p) = p_t D(p_t) - G(y_t, p_t) + \beta E[f_{t+1}(y_t - D_t(p_t, \epsilon), y_t - x_t)],$$

in which the first term is constant, it is trivially supermodular. For the second term, when we consider the relationship between x and y, it only depends on one variable, thus the supermodularity of G is obvious. For the last term, Let $y_2 \geq y_1$, $x_2 \geq x_1$,

$$\begin{aligned} f(y_2, y_2 - x_1) - f(y_2, y_2 - x_2) &= f(y_2, y_2 - x_2 + (x_2 - x_1)) - f(y_2, y_2 - x_2) \\ &\leq f(y_1, y_1 - x_2 + (x_2 - x_1)) - f(y_1, y_1 - x_2) \\ &= f(y_1, y_1 - x_1) - f(y_1, y_1 - x_2), \end{aligned}$$

where the second inequality follows from the concavity of $f(\cdot, \cdot)$. Therefore, f is a supermodular function in x and y based on the definition of supermodularity and so as $E[f]$.

For part (b), first term of J only depends on one variable p, it is trivially supermodular. For second term, we can prove it through the method in our previous chapter. The last term, suppose $y_2 > y_1$, $p_2 \geq p_1$

$$\begin{aligned} & f(y_2 - D(p_2, \epsilon), y_2 - x) - f(y_2 - D(p_1, \epsilon), y_2 - x) \\ = \;& f(y_2 - D(p_1, \epsilon) + (D(p_1, \epsilon) - D(p_2, \epsilon), y_2 - x) - f(y_2 - D(p_1, \epsilon), y_2 - x) \\ \leq \;& f(y_1 - D(p_1, \epsilon) + (D(p_1, \epsilon) - D(p_2, \epsilon), y_1 - x) - f(y_1 - D(p_1, \epsilon), y_1 - x) \\ = \;& f(y_1 - D(p_2, \epsilon), y_1 - x) - f(y_1 - D(p_1, \epsilon), y_1 - x). \end{aligned}$$

So the submodularity is proved. □

Lemma 4.2.3 *$J_t(x, x + y, p)$ is a submodular function in x and y.*

Proof. First term of J is constant in this case. Let $y_2 \geq y_1$ and $x_2 \geq x_1$,

$$\begin{aligned} G(y_2 + x_2, p_t) - G(y_2 + x_1, p_t) &= G(y_2 + x_1 + (x_2 - x_1, p_t)) - G(y_2 + x_1, p_t) \\ &\geq G(y_1 + x_1 + (x_2 - x_1, p_t)) - G(y_1 + x_1, p_t) \\ &= G(y_1 + x_2) - G(y_1 + x_1, p_t), \end{aligned}$$

where the second inequality follows from the convexity of G. Based on the definition, $G(x+y,\cdot)$ is supermodular and so $-G$ is submodular. For the last term,

$$\begin{aligned} f(y_2+x_2,y_2)-f(y_2+x_1,y_2) &= f(y_2+x_1+(x_2-x_1),y_2)-f(y_2+x_1,y_2) \\ &\leq f(y_1+x_1+(x_2-x_1),y_1)-f(y_1+x_1,y_1) \\ &= f(y_1+x_2,y_1)-f(y_1+x_1,y_1), \end{aligned}$$

where the second inequality is due to the concavity of f. Again by definition, f is a submodular function in y and x. □

Let J_{it}, $i=1,2,3$, denote the partial derivative with respect to the i-th component of $J_t(x,y,p(x,y))$. We discuss the problem under two cases:

Case 1: if $y \geq x+z$, then

$$\begin{aligned} &(-b_t\,|y_t-x_t-z_{t-1}|+J_t(x_t,y_t,p_t(x_t,y_t))' = -b_t+J_{2t}(x,y,p(x,y)) \\ &+J_{3t}(x,y,p(x,y))p'_y(x,y)=0. \end{aligned}$$

Case 2: if $y < x+z$, then

$$\begin{aligned} &(-b_t\,|y_t-x_t-z_{t-1}|+J_t(x_t,y_t,p_t(x_t,y_t))' = b_t+J_{2t}(x,y,p(x,y)) \\ &+J_{3t}(x,y,p(x,y))p'_y(x,y)=0. \end{aligned}$$

Let

$$y_{kt}(x)=\sup\{y: J_{2t}(x,y,p(x,y))+J_{3t}(x,y,p(x,y))p'_y(x,y)\leq(-1)^{k-1}b_t\}$$

for $k=1,2$, $t=1,2,\ldots,T$ and $y_{2t}(x)\geq y_{1t}(x)$ since J_t is concave in y. We show in the following theorem that $y_{1t}(x)$ and $y_{2t}(x)$ parameterize an optimal policy.

Theorem 4.2.2 *(a) If $G_t(y,p) \to \infty$ as $|y| \to \infty$, $t = 1, 2, \ldots, T$, then for each $x \in \Re$, $y_{kt}(x)$ is finite, and the optimal policy is given by the following:*

$$y_t^* = \begin{cases} y_{1t}(x) & \text{if } x_t + z_{t-1} < y_{1t}(x), \\ x_t + z_{t-1} & \text{if } y_{1t}(x) \leq x_t + z_{t-1} < y_{2t}(x), \\ y_{2t}(x) & \text{if } x \leq y_{2t}(x) \leq x_t + z_{t-1}, \\ x_t & \text{if } y_{2t}(x) < x_t. \end{cases}$$

(b) The optimal price depends on initial inventory level at the beginning of each period, such that, $p_t = p^*(y_t^*(x)) = p^*(x)$.

(c) If for some $\bar{x}$ there are y' and y'' such that

$$J_{2t}(\bar{x}, y') + J_{3t}(\bar{x}, y')p' \leq -b_i \leq b_i < J_{2t}(\bar{x}, y'') + J_{3t}(\bar{x}, y'')p'$$

then

$$0 \leq y'_{1t}(x) \leq 1, \quad 0 \leq y'_{2t}(x) \leq 1.$$

Proof. Part (a) can be proved by the concavity of the value function. Part (b) is immediate. For part (c), the hypothesis yields

$$-\infty < y_{1t}(\bar{x}) \leq y_{2t}(\bar{x}) < \infty$$

hence

$$|y_{kt}(x)| < \infty \qquad k = 1, 2$$

the conclusion follows from

$$(i) \quad y_{kt}(x) \text{ is monotone increasing} \quad k = 1, 2,$$

and

$$(ii) \quad y_{kt}(x) - x \text{ is monotone decreasing} \quad k = 1, 2.$$

a sufficient condition for the statement (i) is $J_n(x, y, p)$ is supermodular function in x and y; based on the definition of y_{kt}, the statement (ii) is equivalent to

$$\begin{aligned} & y_{kt}(x) - x \\ = \; & \sup\{y : J_{2t}(x, y + x, p(x, y + x)) + J_{3t}(x, y + x, p(x, y + x))p'(x, y + x) \leq (-1)^{k-1} b_t\}, \end{aligned}$$

which implies a sufficient condition, $J_{2t}(x, y+x, p(x, y+x)) + J_{3t}(x, y+x, p(x, y+x))p'(y+x)$ is monotone decreasing in x for each y. So we need the following two lemmas:

Lemma 4.2.2 and lemma 4.2.3 prove statements (i) and (ii), thus $0 \leq y_{1t} \leq 1$ and $0 \leq y_{2t} \leq 1$. □

So we have specified the optimal production policy. For the optimal price p_t, we can get the optimal price p of period t after we obtain optimal y_t.

Proposition 4.2.1 *$p(x, y)$ is nonincreasing in y for any given x.*

Proof. According to the submodularity of J, the result follows. □

Remark 4.2.1 *There is no monotonicity relationship between p and x.*

4.2.2 Infinite Horizon Problem

In this section, we consider infinite horizon case and the objective is to maximize the long-run total discounted profit with stationary cost and revenue parameters as well as demand distribution. In discussing infinite horizon model, we always try to make the problem fall into either negative dynamic programming or positive dynamic programming category. Therefore, it is convenient to have one period reward to be uniformly of the same sign. To achieve this, we subtract a constant $M = \max_{p\in[p_{min},p_{max}]} pd(p)$ uniformly from the one period reward function. We thus obtain the shifted value function $\bar{f}_t$ and $\bar{J}_t$:

$$\bar{f}_t = f_t - \frac{M(1-\beta^{t+1})}{1-\beta} \quad \text{and} \quad \bar{J}_t = J_t - \frac{M(1-\beta^{t+1})}{1-\beta}.$$

The infinite horizon optimality equation (for the transformed model) is given by:

$$\bar{f}(x,z) \quad = \quad \sup_{y\geq x, p\in[p_{min},p_{max}]} \{-(b\,|y-x-z| + \bar{J}(x,y,p)\},$$

where

$$\bar{J}(x,y,p) \quad = \quad pd(p) - G(y,p) - M + \beta E[\bar{f}(y - D(p,\epsilon), y-x)].$$

The following theorem describes the structure of an optimal policy in the infinite horizon model, and its relationship to the finite horizon models

Theorem 4.2.3 *(a)* $\bar{f} = \lim_{t\to\infty} \bar{f}_t$*,* $f = \lim_{t\to\infty} f_t$*,* $\bar{J} = \lim_{t\to\infty} \bar{J}_t$*,* $J = \lim_{t\to\infty} J_t$ *and* $\bar{f} = f - M/(1-\beta)$*,* $\bar{J} = J - M/(1-\beta)$ *and* f *and* $\bar{f}$ *equal the maximum infinite horizon discounted profit in the original and transformed models, respectively.*

(b) $\bar{f}$ *and* $\bar{J}$*(*f *and* J*) satisfy the infinite horizon optimality equation in the transformed (original) model.*

(c) J *and* f *are concave functions. In addition,* $J(x,y,p)$ *is a supermodular function in* x*,* y*, submodular function in* p *and* y*.*

Proof. Because for the transformed problem we subtract M from one period reward function, the problem becomes a negative MDP problem. So the method of successive approximation works, all the results can follow from the finite horizon problem. $\square$

Theorem 4.2.4 *(a)* $y_i(x) = \lim_{t\to\infty} y_{it}(x)$, $i = 1, 2$. *Also* $y_i(x)$ *is differentiable with*

$$0 \le y_1'(x) \le 1, \quad 0 \le y_2'(x) \le 1$$

(b) The optimal inventory policy has the following structure:

$$y^*(x) = \begin{cases} y_1(x) & \textit{if } x + z < y_1(x), \\ x + z & \textit{if } y_1(x) \le x + z < y_2(x), \\ y_2(x) & \textit{if } x \le y_2(x) \le x + z, \\ x & \textit{if } y_2(x) < x \end{cases}$$

(c) The stationary optimal price is given by : $p(y^*(x))$

Proof. The results carry over from finite horizon case. $\square$

4.3 Summary

In this chapter, we study two periodic-review inventory/production models with pricing decisions. First, we include the pricing decision into the model with two supply modes. With a general decreasing and concave demand function, we characterize the optimal inventory policy and pricing strategy which provide some managerial insights on how to manage dual supply and the pricing at the same time. The optimal inventory policy is easy to implement since it just depends on two numbers: one for the emergency order and one for the regular order. The optimal selling price depends on both emergency order up to level and regular order up to level.

Second, for the production smoothing model which is a practical problem and was studied extensively during 1960's, we again include pricing decision, which makes the problem become more interesting. Under some mild assumptions, we characterize the optimal inventory control policy, which is determined by two state dependent parameters. We present some structural properties of the optimal price and the cost function.

Chapter 5

Multi-Echelon Inventory Systems with Guaranteed Demand Delivery

In this chapter, we consider a periodic-review serial inventory system. There are N stages. Stage 1 orders from stage 2, stage 2 orders from stage 3, ..., stage N orders from an outside supplier with ample supply. In each period, $N+1$ classes of demand originate at stage 1 simultaneously. Each class of demand has different delivery time requirements. The system incurs linear shortage cost and inventory holding cost at each stage.

In this chaper, we consider the following scenario: A large supply chain system which is composed of $N-1$ warehouses and one retail store faces $N+1$ classes of demand. All classes of demand come to the retail store. Class i demand is guaranteed to be satisfied within $i-1$ periods, for example, class 1 customer has highest priority and must be satisfied immediately (0 period). Class 2 customer can be satisfied immediately only if there is enough inventory left in the system, otherwise, it is backlogged at warehouse N. But the stage N manager will place an order and satisfy these backlog at the beginning of next period (leadtime is 1), so class 2 demand is satisfied within 1 periods. Similarly, class i demand can be satisfied within $i-1$ periods, because the order from stage $N-i$ manager will arrive no longer than $i-1$ periods.

In the above scenario, each class of demand gets time guaranteed delivery based on its priority and the availability of the inventory. The scenario happens in the real life every day. To our knowledge, prior research has not considered this type of modelwith multiple classes of demand and time guaranteed delivery.

5.1 Model I: Single Class Demand with Guaranteed Delivery

In this section, we consider a special case that there is only one class of demand which must be satisfied immediately once it realizes. As an example of the current model, suppose that several warehouses are located serially and each one is the supplier of its downstream warehouse. The products will typically go through a national warehouse, a regional warehouse and then the local warehouse, finally sold to the end customers by a retailer. Then since demand is uncertain and if the retailer runs out of stock, it may be economically desirable to meet the shortage by a special order from retailer to its upstream supplier rather than to wait until the shortage can be supplied by the regular order. Item shipped in this way incurs the transportation cost from warehouse to warehouse. Within each period, events occur in the following sequence using stage i as an example: First, the regular shipment from stage $i+1$ is received. Second, the order decision is made at stage i. Third, the customer demand realizes at stage 1 and is filled immediately. Fourth, all the cost incur at the end of period.

For this case, every period's demand must be satisfied, if stage 1 does not have enough on hand inventory, it first satisfies the demand as much as it can and transmits the remaining demand to stage 2 without delay, if stage 2 has enough on hand inventory, then the demand is satisfied there; otherwise, the remaining demand is further transmitted to stage 3, etc., until the demand is fully filled. Under this setting, the customer demand can always be satisfied since outside supply is ample.

From the model setting, the inventory level (position) is never negative as there is no backlog. Thus, the dynamics of the system variables is given by,

$$IL_i(t-(i-1)) = (IP_i(t-i) - D[t-i, t-i+1])^+, \quad i = 1, 2, \ldots, N,$$

$$IL_i^-(t-(i-1)) = (IP_i(t-i) - D[t-i, t-i+1))^+, \quad i = 1, 2, \ldots, N$$

and

$$IP_i(t-i) \leq IL_{i+1}^-(t-i) \quad i = 1, 2, \ldots, N-1.$$

For simplicity, write IP_i for $IP_i(t-i)$, IL_i for $IL_i(t-i+1)$, IL_i^- for $IL_i^-(t-i+1)$ and D_i for $D[t-i, t-i+1]$. The steady state average cost per unit of time for the system (there is no shortage cost in this case):

$$\sum_{i=1}^{N} E[h_i(IP_i - D_i)^+ + b_i(IP_i - D_i)^-] \tag{5.1}$$

Let

$$g_i(y) = h_i E(y - D_i)^+ + b_i E(y - D_i)^- \qquad i = 1, 2, \ldots, N,$$

where $g_i(y)$ is clearly convex.

5.1.1 Optimality and Algorithm

In this section, we first derive the optimal strategy for the system then we present the algorithm to compute the optimal control parameters.

As in previous section, define

$$G_1(y) = g_1(y)$$

$G_1(y)$ is convex, let s_1^* be the minimum point, then define

$$G_1^1(y) = \begin{cases} G_1(s_1^*) & \text{if } y \le s_1^*, \\ G_1(y) & \text{otherwise.} \end{cases}$$

Let

$$G_1^2(y) = G_1(y) - G_1^1(y),$$

so by lemma 7.1.2, $G_1^1(y)$ is nondecreasing convex and $G_1^2(y)$ is nonincreasing convex function. Let

$$G_2(y) = E[g_2(y) + G_1^2((y - D)^+)].$$

Assume $G_j(\cdot)$ for $j = 2, 3, \ldots, i-1$ is convex with minimizer s_j, then

$$G_i(y) = E[g_i(y) + G_{i-1}^i((y - D)^+)].$$

Let s_i be the minimum point of $G_i(\cdot)$. Then we can define

$$G_i^i(y) = \begin{cases} G_i(s_i) & \text{if } y \le s_i, \\ G_i(y) & \text{otherwise} \end{cases}$$

and

$$G_i^{i+1}(y) = G_i(y) - G_i^i(y).$$

Lemma 5.1.1 *$G_i(y)$ is convex, for $i = 1, \ldots, N$.*

Proof. For $i = 2$, since $G_1^2(y)$ is a decreasing convex function and y^+ is increasing convex, so $G_1^2(y^+)$ is not a convex function anymore. However, we can easily verify that $G_1^2(y^+)$ is a quasi-convex function because it is constant for $y < 0$ and convex for $y \geq 0$. In addition, $g_2(y) = h_2 E(y-D)^+ + b_2 E(y-D)^-$, which is convex. So for $y \geq 0$, take derivative with respect to y of $G_2(y)$,

$$\begin{aligned} G_2'(y) &= \int_0^y h_2 f(t)dt - \int_y^\infty b_2 f(t)dt + \int_0^y G_1^{2'}(y-t)f(t)dt, \\ G_2''(y) &= h_2 f(y) + b_2 f(y) + \int_0^y G_1^{2''}(y-t)f(t)dt + G_1^{2'}(0)f(y) \\ &= (h_2 + b_2 + G_1^{2'}(0))f(y) + \int_0^y G_1^{2''}(y-t)f(t)dt. \end{aligned}$$

As $G_1^{2'}(0) \geq -b_1$, $G_1^{2''} > 0$ by the convexity of $G_1^2(\cdot)$ and $h_2 + b_2 > b_1$, then $G_2(\cdot)$ is convex for $y > 0$. In addition, for $y < 0$, $G_2'(y) = -b_2$, so $G_2'(0^+) = G_2'(0^-)$. Therefore, $G_2(\cdot)$ is convex.

Suppose the lemma is true for $i-1$, then $G_i(y) = E[g_i(y) + G_{i-1}^i((y-D)^+)]$, where $G_{i-1}^i(z^+)$ is a nonincreasing function and is convex for $z \geq 0$. So if we take derivative of $G_i(y)$:

$$\begin{aligned} G_i'(y) &= \int_0^y h_i f(t)dt - \int_y^\infty b_i f(t)dt + \int_0^y G_{i-1}^{i'}(y-t)f(t)dt \\ G_i''(y) &= (h_i + b_i + G_{i-1}^{i'}(0))f(y) + \int_0^y G_{i-1}^{i''}(y-t)f(t)dt \end{aligned}$$

Similarly, $G_{i-1}^{i'}(0) \geq -b_{i-1}$, so $h_i + b_i + G_{i-1}^{i'}(0) > 0$ based on assumption. Also, $G_{i-1}^i(y-D)$ is convex for $y - D \geq 0$, therefore $G_{i-1}^{i''} \geq 0$, so $G_i''(y) > 0$ for $y \geq 0$, in addition, for $y < 0$, $G_i(y)$ is monotone decreasing in y because

$$G_i(y) = E[b_i(D-y) + G_{i-1}^i(0)]$$

and $G_i'(0^+) = G_i'(0^-) = b_i$, so the induction is completed and the lemma is verified. □

After having this lemma, the previous definitions of $G_i(y)$ and $G_i^{i+1}(y)$ are valid.

Theorem 5.1.1 *For the N-stages system, $E[\sum_{i=1}^{N} g_i(IP_i)] \geq E[\sum_{i=1}^{N-1} G_i^i(IP_i)+G_N(IP_N)]$.*

Proof. If $N = 1$, $E[g_1(IP_1)] = G_1(IP_1)$ by definition., then if $N = 2$,

$$\begin{aligned}
E[\sum_{i=1}^{2} g_i(IP_i)] &= E[g_2(IP_2)] + G_1(IP_1)] \\
&= E[g_2(IP_2) + G_1^1(IP_1) + G_1^2(IP_1)] \\
&\geq E[g_2(IP_2) + G_1^1(IP_1) + G_1^2(IL_2^-)] \\
&= E[g_2(IP_2) + G_1^1(IP_1) + G_1^2((IP_2 - D)^+)] \\
&= E[G_2(IP_2) + G_1^1(IP_1)], \qquad (5.2)
\end{aligned}$$

where the second inequality follows from the nonincreasingness of $G_1^2(\cdot)$. Similarly, we can prove for N by induction. □

Lemma 5.1.2 *Let C_i be the minimum value of the function $G_i(\cdot)$, $i = 1, 2 \ldots, N$, then $\sum_{i=1}^{N} C_i$ is the lower bound of the minimum cost of the N-stages serial system.*

Proof. It follows from Theorem 5.1.1 that

$$E[\sum_{i=1}^{N} g_i(IP_i)] \geq \sum_{i=1}^{N-1} C_i + G_N(IP_N).$$

In other words, given $IP_N = y$, the expected systemwide holding and transportation costs charged to period $t-N$ under any policy are bounded below by $\sum_{i=1}^{N-1} C_i + G_N(y)$. By substituting the latter for the former, the original system collapses to a single stage system. Because $\sum_{i=1}^{N-1} C_i$ is constant, and $G_N(y)$ is convex which optimal policy is base stock policy and the optimal cost for this system is $\sum_{i=1}^{N} C_i$. So this cost will be the lower bound of the N stages system. □

Theorem 5.1.2 *The echelon base stock policy is optimal for the system, for which the stage i always tries to order up to s_i and the minimum cost of the system is $\sum_{i=1}^{N} C_i$.*

Proof. Suppose the above policy is used in the N-stage system, we just need to show that given $IP_N = y$, the expected systemwide cost charge to $t - N$ is exactly equal to $\sum_{i=1}^{N-1} C_i + G_N(y)$.

Take any $i < N$. Notice that $IP_i \leq IL_{i+1}^-$. Since the stage i order up to s_i, we have $IP_i = \min\{s_i, IL_{i+1}^-\}$. Thus $G_i^i(IP_i) = C_i$. Therefore, it is sufficient to show that the inequality in (5.2) is equality. To see this, we only need to consider $s_i < IL_i^-$, if so, $G_i^{i+1}(IP_i) = G_i^{i+1}(IL_{i+1}^-) = 0$. $\square$

Remark 5.1.1 *: Our model can easily include the setup cost into the last stage of the system. The optimal policy for the first $N - 1$ stages is still echelon base stock policy while for the last stage it is echelon (s, S) policy.*

The following algorithm can be used to compute the optimal base stock level for each stage recursively.

Algorithm:

- **Step 1.** Set $G_0(y) = 0$, $g_i(y) = h_i E(y - D)^+ + b_i E(y - D)^-$,$i = 1$.
- **Step 2.** $G_i(y) = E[g_i(y) + G_{i-1}(s_{i-1} \wedge (y - D)^+))]$
- **Step 3.** $s_i = \arg\min \ G_i(y)$, if $i < N$, go to step 2, otherwise, stop.

5.2 Model II: Two Classes of Demand

Consider a multi-echelon inventory model with N stages and two classes of demand. One class of demand is guaranteed to be satisfied immediately and another one can be backlogged at stage 1. For convenience, let the outside supplier be stage $N+1$. Suppose the leadtime is 1 unit, which means the order placed at the beginning of period will arrive at the end of the period. During each period, once demand occurs, the manager first satisfies class 1 demand by using the system inventory. The supply policy is described as follows. The manager transmits the class 1 demand to the stage with positive inventory position. And start from that stage, if the positive part of the inventory position is not enough to satisfy the demand, the excess demand is transmitted to upper stage until it is satisfied. After the class 1 demand is satisfied, the class 2 demand is satisfied if stage 1 has inventory left, otherwise the excess class 2 demand is backlogged.

The system dynamics are,

$$IL_i(t-i) = -IP_i^-(t-i) + (IP_i(t-i) - D^1(t-i))^+ - D^2(t-i) \tag{5.3}$$

$$IL_i^-(t-i+1) = -IP_i^-(t-i) + (IP_i(t-i) - D^1(t-i))^+ - D^2(t-i) \tag{5.4}$$

$$IP_i(t-i) \leq IL_{i+1}^-(t-i) = IL_{i+1}(t-i-1) \tag{5.5}$$

We briefly explain the system dynamics above: If $IP_i < 0$ and the second term of (5.3) becomes zero, which means the echelon inventory position at stage i is negative and the manager will not use the echelon i inventory to satisfy class 1 demand while transmitting the demand to upper stage. Hence the ending echelon inventory level at this stage is the current inventory position minus the class 2 demand. Otherwise, if $IP_i \geq 0$, the first term of (5.3) becomes 0. The system manager will use those inventory that are not reserved for the backlog of class 2 demand to satisfy class 1 demand if possible. Therefore, if $IP_i > D^1$, the ending inventory level is the inventory position after satisfied class 1 demand minus class 2 demand. Otherwise, the system inventory position is 0 after satisfying class 1 demand and the manager transmits the excess demand to upper stage. So the ending inventory level is just the negative of the class 2 demand of the current period.

Under the system dynamics above, the average cost function is:

$$\begin{aligned}
& \sum_{i=1}^{N} E[h_i IL_i(t) + b_i(IP_i(t)^+ - D^1(t))^-] + (p + H_1)B(t) \\
= & \sum_{i=1}^{N} E[h_i(-(IP_i(t)^- + (IP_i(t) - D^1(t))^+ - D^2(t)) + b_i(IP_i(t)^+ - D^1(t))^-] \\
& +(p + H_1)E(-(IP_1(t)^- + (IP_1(t) - D^1(t))^+ - D^2(t))^-
\end{aligned}$$

We charge the cost back to $t - N$. The accounting scheme is reasonable because: $IL_i(t - i) = -IP_i^-(t-i) + (IP_i(t-i) - D^1(t-i))^+ - D^2(t-i)$, we see that $IL_i(t-i)$ is statistically determined by $IP_i(t-i)$. Moreover, by definition, $IP_i(t-i)$ is constrained by $IL_{i+1}^-(t-i)$. In turn, $IL_{i+1}^-(t-i)$ is statically determined by $IP_{i+1}(t-i-1)$. A simple induction shows that $IP_N(t - N)$ determines, directly or indirectly, $IL_i(t - i)$ for $i = 1, \ldots, N$.

Assumption 5.2.1 *The installation holding cost rate, transmission cost rate and backlog cost rate satisfy:* $b_i \geq H_{i+1} + p$.

5.2.1 Optimization

Let

$$\begin{aligned} G_1(y) &= E[h_1(-y^- + (y - D^1)^+ - D^2) + b_1(y^+ - D^1)^- \\ &\quad +(p + H_1)(-y^- + (y - D^1)^+ - D^2)^-]. \end{aligned}$$

Lemma 5.2.1 *$G_1(y)$ is a quasi-convex function, specifically ,when $y \geq 0$ it is a convex function and when $y < 0$ it is a linear decreasing function with slope $(-H_2 + p)$.*

Proof. For $y > 0$,

$$\begin{aligned} & G_1(y) \\ = \; & h_1 E(y - D^1)^+ + b_1 E(y - D^1)^- + (p + H_1)E((y - D^1)^+ - D^2)^- \\ = \; & E\Big[h_1(y - D^1)^+ + b_1(y - D^1)^- + (p + H_1)((y - D^1)^+ - D^2)^+ \\ & -(p + H_1)((y - D^1)^+ - D^2)\Big] \\ = \; & E\Big[h_1(y - D^1)^+ + b_1(y - D^1)^- + (p + H_1)((y - D^1)^+ - D^2)^+ \\ & -(p + H_1)((y - D^1 - D^2) - (p + H_1)(y - D^1)^-\Big] \\ = \; & E\Big[h_1(y - D^1)^+ + (b_1 - (p + H_1))(y - D^1)^- + (p + H_1)((y - D^1)^+ - D^2)^+ \\ & -(p + H_1)((y - D^1 - D^2)\Big] \\ = \; & E\Big[h_1(y - D^1) + (b_1 + h_1 - (p + H_1))(y - D^1)^- + (p + H_1)(y - D^1 - D^2)^+ \\ & -(p + H_1)((y - D^1 - D^2)\Big] \end{aligned}$$

So if $b_1 \geq (p + H_2)$, then $G_1(y)$ is convex for $y > 0$.

For $y \leq 0$,

$$G_1(y) = E[h_1(y - D^2) + b_1(D_1) + (p + H_1)(D_2 - y)]$$

which is clearly linear decreasing in y since $p + H_1 > h_1$. $\square$

Define

$$G_1^1(y) = \begin{cases} G_1(s_1) & y \leq s_1 \\ G_1(y) & y > s_1 \end{cases}$$

and

$$G_1^2(y) = G_1(y) - G_1^1(y)$$

$G_1^1(y)$ is convex increasing and $G_1^2(y)$ is decreasing in y and linear decreasing for $y < 0$. Define

$$G_2(y) = E[h_2(-y^- + (y - D^1)^+ - D^2) + b_2(y^+ - D^1)^- + G_1^2(-y^- + (y - D^1)^+ - D^2)]$$

Suppose $G_j(y)$ is well defined for $i = 1, 2, \ldots, j-1$, and s_j is the minimizer for G_j then

$$G_i^i(y) = \begin{cases} G_i(s_i) & y \leq s_i \\ G_i(y) & y > s_i \end{cases}$$

and

$$G_i^{i+1}(y) = G_i(y) - G_i^i(y)$$

$$\begin{aligned} G_{i+1}(y) &= E[h_{i+1}(-y^- + (y - D^1)^+ - D^2) + b_{i+1}(y^+ - D^1)^- \\ &\quad + G_i^{i+1}(-y^- + (y - D^1)^+ - D^2)]. \end{aligned}$$

Theorem 5.2.1 *$G_i(y)$ is quasi-convex. In particular, it is linear decreasing with slope $-(H_{i+1}+p)$ for $y<0$ and convex for $y \geq 0$, $i = 1, 2, \ldots, N$.*

Proof. The theorem is true for $i = 1$ as shown in Lemma 1. Suppose it is true for $i = n-1$, then for $i = n$, for $y < 0$

$$G_n(y) = E[h_n(y - D_2) + b_n D_1 + G^n_{n-1}(y - D^2)]$$

We know $G'^n_{n-1}(t)$ is $-(H_n + p)$ for $t < 0$, so $G_n(y)$ is linear decreasing with slope $-(H_{n+1} + p)$.

For $y \geq 0$,

$$G_n(y) = E[h_n((y - D^1)^+ - D^2) + b_n(y - D^1)^- + G^n_{n-1}((y - D^1)^+ - D^2)]$$

Take derivative with respect to y

$$\begin{aligned} G'_n(y) &= E[h_n\mathbf{1}(y \geq D^1) - b_n 1(y < D_1) + (G^n_{n-1})'(y - D^1 - D^2)\mathbf{1}(y \geq D^1)] \\ &= h_n P(y \geq D^1) - b_n P(y < D_1) + \int_0^\infty \int_0^y (G^n_{n-1})'(y - t^1 - t^2) f_1(t_1) f_2(t_2) dt_1 dt_2 \end{aligned}$$

and the second derivative is

$$\begin{aligned} G''_n(y) &= h_n f_1(y) + b_n f_1(y) + \int_0^\infty \int_0^y (G^n_{n-1})''(y - t^1 - t^2) f_1(t_1) f_2(t_2) dt_1 dt_2 \\ &\quad + EG^{n'}_{n-1}(-D^2) f_1(y) \\ &= E[(b_n + h_n - (p + H_n)) f_1(y) + G^{n''}_{n-1}(y - D^1 - D^2) 1(y \geq D^1) \end{aligned}$$

based on induction, $G^{n''}_{n-1}(y - D^1 - D^2)$ is either 0 or nonnegative. From the assumption that $b_i \geq p + H_{i+1}$, $G''_n(y) \geq 0$ for $y \geq 0$. we finish the induction and complete the proof. $\square$

Lemma 5.2.2

$$\begin{aligned}
&\sum_{i=1}^{N} E[h_i(-(IP_i(t)^- + (IP_i(t) - D^1(t))^+ - D^2(t)) + b_i(IP_i(t)^+ - D^1(t))^-] \\
&+(p + H_1)E(-(IP_1(t)^- + (IP_1(t) - D^1(t))^+ - D^2(t))^- \\
\geq\ & E[G_N(IP_N) + \sum_{i=1}^{N-1} G_i^i(IP_i)].
\end{aligned}$$

Proof. For $N = 2$

$$\begin{aligned}
&\sum_{i=1}^{2} E[h_i((-IP_i^- + (IP_i - D^1)^+ - D^2) + b_i(IP_i^+ - D^1)^-] \\
&+(p + H_1)E(-(IP_1^- + (IP_1 - D^1)^+ - D^2)^- \\
=\ & E[h_2(((-IP_2^- + (IP_2 - D^1)^+ - D^2) + b_2(IP_2^+ - D^1)^-] + G_1(IP_1) \\
=\ & E[h_2(((-IP_2^- + (IP_2 - D^1)^+ - D^2) + b_2(IP_2^+ - D^1)^-] + G_1^1(IP_1) + G_1^2(IP_1) \\
\geq\ & E[h_2((-(IP_2^- + (IP_2 - D^1)^+ - D^2) + b_2(IP_2^+ - D^1)^-] + G_1^1(IP_1) + G_1^2(IL_2^-) \\
=\ & E[h_2(((-IP_2^- + (IP_2 - D^1)^+ - D^2) + b_2(IP_2^+ - D^1)^-] + G_1^1(IP_1) + G_1^2(-IP_2^- \\
& +(IP_2 - D^1)^+ - D^2) \\
=\ & G_2(IP_2) + G_1^1(IP_1)
\end{aligned}$$

By simple induction, it can be proved for any N. □

Lemma 5.2.3 *Let $C_i = G_i(s_i)$, then $\sum_{i=1}^{N} C_i$ is the lower bound on the system average cost.*

Theorem 5.2.2 *The echelon base stock policy is optimal.*

Proof. The proof is similar to the one in the previous section. □

Algorithm:

- **Step 1.** Set $G_1(y) = E[h_1(-y^- + (y-D^1)^+ - D^2) + b_1(y^+ - D^1)^- + (p+H_1)(-y^- + (y-D^1)^+ - D^2)^-]$, $i = 1$.

- **Step 2.** $G_i(y) = E[h_i((y - D^1)^+ - D^2) + b_i(y - D^1)^- + G_{i-1}(s_{i-1} \wedge (-y^- + (y - D^1)^+ - D^2)]$

- **Step 3.** $s_i = \arg\min G_i(y)$, $i = i + 1$, if $i < N$, go to step 2; otherwise, stop.

Through recursive optimization procedure above, the optimal echelon base-stock level s_i can be obtained and the optimal average cost is given by $C(\mathbf{s})$.

5.3 The General Model

In this section, we analyze the general multi-echelon inventory model with multiple demand classes and present the results for this general model.

As indicated earlier, there are N stages in the serial system, denoted by $i = 1, 2, \ldots, N$. There are $N + 1$ classes of demand, denoted by $D^1, \ldots, D^{N+1}$ which are independently distributed with density $f_i(\cdot)$ and distribution $F_i(\cdot)$. Without loss of generality, suppose class 1 has highest priority and class $N + 1$ has lowest priority.

The time sequence of events for this model is: First, at the beginning of period t, stage i places an order from stage $i+1$. Second, the order placed in previous period is received and backlog is filled if have any. Third, the $N + 1$ classes of demand realize and class 1 demand is first satisfied. After that, the system manager uses up stage N's left inventory to satisfy class 2 demand if possible, otherwise the excess class 2 demand is backlogged at stage N. Sequentially, he uses up stage $N - 1$'s inventory to satisfy class 3 demand, otherwise class 3 demand is backlogged at stage $N - 1$, etc. This procedure continues until either class $N+1$ demand is satisfied by stage 1's inventory or backlogged at stage 1.

Without loss of generality, we assume the leadtime of regular order between stages is one period, the Clark-Scarf model allows for the leadtime to be more than one period. The more general Clark-Scarf setting can be converted into an equivalent formulation of our model. The idea is simple and can be illustrated by one example if the leadtime between stage 1 and 2 are 3 periods, then we insert three psuedostages between them which will has exactly one unit of leadtime and set their inventory holding cost to be very high to make sure they will not keep inventory (Heching and Porteus (2000)).

From the model description, if stage i has backlog, then stages $1, 2, \ldots, i-1$ must all have backlogs, too. Furthermore, the backlog at stage i (which is demand class $N-i+2$) will be satisfied before the backlog at stage $1, 2, \ldots, i-1$ if possible, so the priority rule still holds even in the backlog. In addition, the inventory reserves in upper streams for the downstream's backlogs cannot be used to satisfy the higher priority classes demand because of the delivery time guaranteed.

Notation:

$IP_i(t)$ = echelon i inventory position at the beginning of period t,

$IL_i(t)$ = echelon i inventory level at the end of period t,

$IL_i^-(t)$ = initial echelon i inventory level at the beginning of period t,

h_i = echelon i inventory holding cost rate, $i = 1, 2 \ldots, N$,

H_i = installation i inventory holding cost rate, i.e. $H_i = \sum_{j=i}^{N} h_j$,

b_i = unit transmitting cost from stage i to stage $i+1$, $i = 1, 2, \ldots, N$,

p_i = echelon class i demand shortage cost rate, $i = 2, \ldots, N+1$ (no backlog for class 1).

For $t_1 \leq t_2$, let $[t_1, t_2]$ denote the periods $t_1, \ldots, t_2$ and $[t_1, t_2)$ denote the periods $t_1, \ldots, t_2 - 1$. Let $D_i[[t_1, t_2]$ and $D^i[[t_1, t_2)$ be the class i demand in $[t_1, t_2]$ and $[t_1, t_2)$, respectively, $i = 1, 2, \ldots, N+1$.

Based on the model description, the system dynamics is:

$$\begin{aligned} IL_i(t-i+1) &= -(IP_i(t-i))^- + \left(IP_i(t-i) - \sum_{j=1}^{i} D^j[t-i, t-i+1] \right)^+ \\ &\quad - \sum_{j=i+1}^{N+1} D^j[t-i, t-i+1] \end{aligned}$$

$$IL_i^-(t-i) = IL_i(t-i+1)$$

and

$$IL_i^-(t) \leq IP_i(t) \leq IL_{i+1}^-(t) \qquad i = 1, 2, \ldots, N-1.$$

Assumption 5.3.1 *For $i = 1, 2, \ldots, N$, $b_i \geq H_{i+1} + p_{N-i+2}$.*

For this assumption, it means that the unit transmission cost from stage i to stage $i+1$ is greater than or equal to the unit holding cost at stage $i+1$ and shortage cost for class $N-i+2$ demand which can be regarded as the backlog cost at stage i. This is reasonable because it is more economical not to ship the unit demand to stage $i+1$ for class $N-i+2$.

Assumption 5.3.2 *For* $i = 1, 2, \ldots, N$, $b_i \geq b_{i-1}$, *suppose* $b_0 = 0$.

This is a technical assumption, but it is also reasonable because usually the transmission cost becomes higher as going further upstream in a supply chain.

Let IP_i be the steady stage inventory position and D^i be the class i demand, the average cost of the system can be expressed as:

$$\begin{aligned}
C(\mathbf{IP}) &= E\Bigg[h_1[-IP_1^- + (IP_1 - D^1)^+ - D^2 -, \ldots, -D^{N+1}] \\
&\quad + h_2[-IP_2^- + ((IP_2 - D^1)^+ - D^2)^+ -, \ldots, -D^{N+1}] \\
&\quad + \ldots + h_N\left[-IP_N^- + (IP_N - \sum_{i=1}^{N} D^i)^+ - D^{N+1}\right] \\
&\quad + b_1(IP_1^+ - D^1 - D^2 - \ldots, -D^N)^- \\
&\quad + \ldots + b_N(IP_N^+ - D^1)^- + (H_1 + p_2)(-IP_1^- + (IP_1 - D^1)^+ - D^2 -, \ldots, -D^{N+1})^- \\
&\quad + p_3(-IP_2^- + (IP_2 - D^1 - D^2)^+ -, \ldots, -D^{N+1})^- + \ldots \\
&\quad + p_{N+1}\left(-IP_N^- + \left(IP_N - \sum_{j=1}^{N} D^j\right)^+ - D^{N+1}\right)^-\Bigg] \\
&= \sum_{i=1}^{N} h_i E\left[-IP_i^- + \left(IP_i(t) - \sum_{j=1}^{i} D^j\right)^+ - \sum_{j=i+1}^{N+1} D^j\right] \\
&\quad + \sum_{i=1}^{N} b_i E\left(IP_i^+ - \sum_{j=1}^{N-i+1} D^j\right)^- \\
&\quad + \sum_{i=2}^{N} p_{i+1} E\left(-IP_i^- + (IP_i - \sum_{j=1}^{i} D^j)^+ - \sum_{j=i+1}^{N+1} D^j\right)^- \\
&\quad + (H_1 + p_2) E\left(-IP_1^- + (IP_1 - D^1)^+ - \sum_{j=2}^{N+1} D^j\right)^-
\end{aligned} \tag{5.6}$$

Let

$$
\begin{aligned}
g_1(y) &= h_1 E\left[-y^- + (y - D^1)^+ - \sum_{j=2}^{N+1} D^j\right] + b_1 E\left(y^+ - \sum_{j=1}^{N} D^j\right)^- \\
&\quad + (H_1 + p_2) E\left(-y^- + (y - D^1)^+ - \sum_{j=2}^{N+1} D^j\right)^- .
\end{aligned}
$$

For $i = 2, \ldots, N$ Let

$$
\begin{aligned}
g_i(y) &= h_i E\left[-y^- + (y - \sum_{j=1}^{i} D^j)^+ - \sum_{j=i+1}^{N+1} D^j\right] + b_i E\left(y^+ - \sum_{j=1}^{N-i+1} D^j\right)^- \\
&\quad + p_{i+1} E\left(-y^- + (y - \sum_{j=1}^{i} D^j)^+ - \sum_{j=i+1}^{N+1} D^j\right)^- .
\end{aligned}
$$

Before we proceed, the following lemma is presented first since all the following results count on it.

Lemma 5.3.1 *(Karush 1959) (a) If a function $f(y)$ is convex on $(-\infty, \infty)$ and attains its minimum at y^*, then*

$$\min_{a \le y \le b} f(y) = f^L(a) + f^U(b)$$

where $f^L(a) := min_{a \le y} f(y) = f(max(a, y^*))$ is convex nondecreasing in a and $f^U(b) := f(b) - min_{b \le y} f(y) = f(b) - f(max(b, y^*))$ is convex nonincreasing.

(b)If a function $f(y)$ is quasi-convex on $(-\infty, \infty)$ and attains its minimum at y^*, then

$$\min_{a \le y \le b} f(y) = f^L(a) + f^U(b)$$

where $f^L(a) := min_{a \le y} f(y) = f(max(a, y^*))$ is nondecreasing and $f^U(b) := f(b) - min_{b \le y} f(y) = f(b) - f(max(b, y^*))$ is nonincreasing.

Let

$$G_1(y) = g_1(y).$$

Define

$$G_1^1(y) = \begin{cases} G_1(s_1) & \text{If } y \leq s_1 \\ G_1(y) & \text{Otherwise} \end{cases}$$

where s_1 is the minimum point of $G_1(y)$ and

$$G_1^2(y) = G_1(y) - G_1^1(y).$$

Let

$$G_2(y) = E\left[g_2(y) + G_1^2\left(-y^- + (y - \sum_{j=1}^{2} D^j)^+ - \sum_{j=3}^{N+1} D^j\right)\right].$$

Assume $G_j(y)$ is defined for $j = 1, \ldots, i-1$, then

$$G_i(y) = E\left[g_i(y) + G_{i-1}^i\left(-y^- + (y - \sum_{j=1}^{i} D^j)^+ - \sum_{j=i+1}^{N+1} D^j\right)\right]$$

and let s_i be the minimum point of $G_i(\cdot)$,

$$G_i^i(y) = \begin{cases} G_i(s_i) & \text{If } y \leq s_i \\ G_i(y) & \text{Otherwise} \end{cases}$$

and

$$G_i^{i+1}(y) = G_i(y) - G_i^i(y).$$

Lemma 5.3.2 *If assumptions 5.3.1 and 5.3.2 are satisfied, $G_i(y)$ is quasi convex for all i.*

Proof. The case $i = 1$ has been proved in Lemma 5.2.1. For general i, it can be similarly proved by induction. So we skip the proof here. □

Theorem 5.3.1 *The echelon base stock policy is optimal, in which echelon i manager will order up to s_i, where s_i is defined recursively as above, $i = 1, 2, \ldots, N$.*

Proof. It is similar to the proof in previous section, so we omit them here. $\square$

Algorithm:

- **Step 1.** Set $G_1(y) = h_1 E\left[-y^- + (y - D^1)^+ - \sum_{j=2}^{N+1} D^j\right] + b_1 E\left(y^+ - \sum_{j=1}^{N} D^j\right)^- + (H_1 + p_2) E\left(-y^- + (y - D^1)^+ - \sum_{j=2}^{N+1} D^j\right)^-$, $i = 1$.
- **Step 2.** $G_i(y) = E[g_i(y) + G_{i-1}(s_{i-1} \wedge \left(-y^- + (y - \sum_{j=1}^{i} D^j)^+ - \sum_{j=i+1}^{N+1} D^j\right)]$
- **Step 3.** $s_i = \arg\min G_i(y)$, $i = i + 1$, if $i < N$, go to step 2; otherwise, stop.

5.4 Summary

In this chapter, we have extended Clark-Scarf model to include multiple classes of demand. Each class of demand can be satisfied by the inventory up to some stage in the system depending on its priority, otherwise, it is backlogged. We first show the echelon base-stock policy is optimal for two models: single class demand but the demand delivery is guaranteed and two classes of demand of which one can be backlogged, the other is guaranteed delivery. Finally, we present the general model with multiple classes of demand and show the echelon base-stock policy is optimal. For all models, we develop the computational algorithms for computing the optimal base-stock levels.

Chapter 6

Optimal Policy for Multi-Echelon Inventory System with Batch Ordering and Nested Replenishment Schedule

In many production/distribution systems materials flow in fixed lot sizes (e.g., in full truckloads or full containers) and under regular schedules (e.g., delivery every week). In this chapter we derive the optimal policies for multi-echelon serial system with batch ordering and nested replenishment schedule and present an efficient computational algorithm for the optimal control parameters. Furthermore, we show that the optimal expected system cost is minimized when the ordering times for different stages are synchronized. In contrast to Chen and Zheng (1994), who develop a lower bound for the average cost of a given period, we develop a lower bound for the average total cost over an appropriately defined cycle, and then construct a policy which reaches the lower bound. This note generalizes the recent work of Chen (2000) and van Houtum et al. (2003).

6.1 The Model

Since each stage can order only after the predetermined reorder interval, when one ordering instant of a stage is known, it determines all the ordering instants for that stage. Suppose the ordering instants of each stage are known.

The time sequence of events is as follows. For any stage $i > 1$, at the beginning of any period, the order placed from downstream stage $i - 1$, if any, is received; an order for stage $i + 1$, if stage i is allowed to order in this period, is placed; an in-transit shipment to stage i is received; and a shipment to stage $i - 1$ is sent out. For stage 1, order is placed at the beginning of the period if stage 1 is allowed to order and customer demand arrives during the period. All costs are charged at the end of the period.

Some notation is defined below. Some of the notation is illustrated in Figure 1. The subscript i denotes the stage number. Whenever possible, we stick to the notation of Chen and Zheng (1994).

D_t = customer demand in period t, an integer-valued random variable,

μ = average demand per period, i.e., $E[D_t]$,

$IL_i^-(t)$ = echelon inventory level of stage i at the beginning of period t,

$IP_i(t)$ = echelon inventory position of stage i at the beginning of period t,

$IL_i(t)$ = echelon inventory level of stage i at the end of period t,

$B(t)$ = backorder level at stage 1 at the end of period t,

$T_i =$ the given reorder interval of stage i and $T_{i+1}/T_i = r_i$, i.e., $T_i = \prod_{j=1}^{i-1} r_j T_1$,

$Q_i =$ the given base quantity for stage i and $Q_{i+1}/Q_i = q_i$, i.e., $Q_i = \prod_{j=1}^{i-1} q_j Q_1$,

$l_i =$ transportation leadtime between stage $i+1$ and stage i,

$L_i =$ transportation leadtime from outside supplier to stage i, i.e., $L_i = \sum_{j=i}^{N} l_j$,

$a_i =$ non-synchronization factor, it is defined as the number of periods between stage $i+1$ receiving

a shipment and the first period following that at which stage i is allowed to order, $0 \leq a_i < T_i$,

$A_i = \sum_{j=i}^{N-1} a_j$,

$h_i =$ echelon inventory holding cost per unit per period,

$H_i =$ installation inventory holding cost per unit per period, i.e., $H_i = \sum_{j=i}^{N} h_j$,

$b =$ backorder cost per unit per period,

$x^+ = \max\{x, 0\}$,

$x^- = \max\{-x, 0\}$.

Let $D[t_1, t_2)$ denote the total demand in periods $t_1, \ldots, t_2 - 1$ and $D[t_1, t_2]$ denote the total demand in periods $t_1, \ldots, t_2$. Whenever possible we ignore the actual time of demand and use $D(k)$ to represent a k-period demand.

If all the a_i are equal to 0, then in the period a stage receives an order from its upstream stage, the order can be shipped in the same period to the downstream stage. If this is the case, the ordering times of the different stages are said to be synchronized. For convenience, let $\gamma_i = \prod_{j=i}^{N-1} r_j$, with $\gamma_N = 1$.

Figure 1 illustrates a three-stage system with leadtimes l_1, l_2 and l_3 and reorder intervals $T_3 = 2T_2 = 4T_1$. Time t is an ordering instant of stage 3, thus stage 3 can order in periods t, $t + T_3$, $t + 2T_3$, ... Stage 2 can order in periods $t + l_3 + a_2$, $t + l_3 + a_2 + T_2$, $t + l_3 + a_2 + 2T_2$, ..., and stage 1 can order in periods $t + L_2 + A_1$, $t + L_2 + A_1 + T_1$, $t + L_2 + A_1 + 2T_1$, ...

Suppose the system starts with a plausible initial state that the initial on-hand inventory at stage $i+1$ is a nonnegative integer multiple of Q_i, $i = 1, 2, \ldots, N-1$. This initial condition with the integer-ratio constraint implies that for all t we have

$$IL_{i+1}^{-}(t) - IP_i(t) = mQ_i, \qquad i = 1, 2, \ldots, N-1, \tag{6.1}$$

where m is a nonnegative integer. The system dynamics are,

$$IL_i^{-}(t + l_i) = IP_i(t) - D[t, t + l_i) \quad i = 1, \ldots, N, \tag{6.2}$$

$$IL_i(t + l_i) = IP_i(t) - D[t, t + l_i] \quad i = 1, \ldots, N, \tag{6.3}$$

$$IP_i(t) \leq IL_{i+1}^{-}(t) \qquad i = 1, \ldots, N-1. \tag{6.4}$$

To compute the system cost, we define a *cycle* of length T_N for each stage as follows. A cycle for stage N is defined as the time between two consecutive periods at which stage N receives its orders. Suppose at time t stage N places an order which arrives at $t + l_N$ and initiates a cycle in the supply chain (see Figure 1). The first period after $t + l_N$ that stage $N-1$ can order is $t + l_N + a_{N-1}$. Therefore, during the cycle between period $t + l_N$ and period $t + l_N + T_N$, the feasible ordering times for stage $N-1$ are $t + l_N + a_{N-1} + kT_{N-1}$, where $k = 0, 1, \ldots, r_{N-1} - 1$. We define a cycle for stage $N-1$

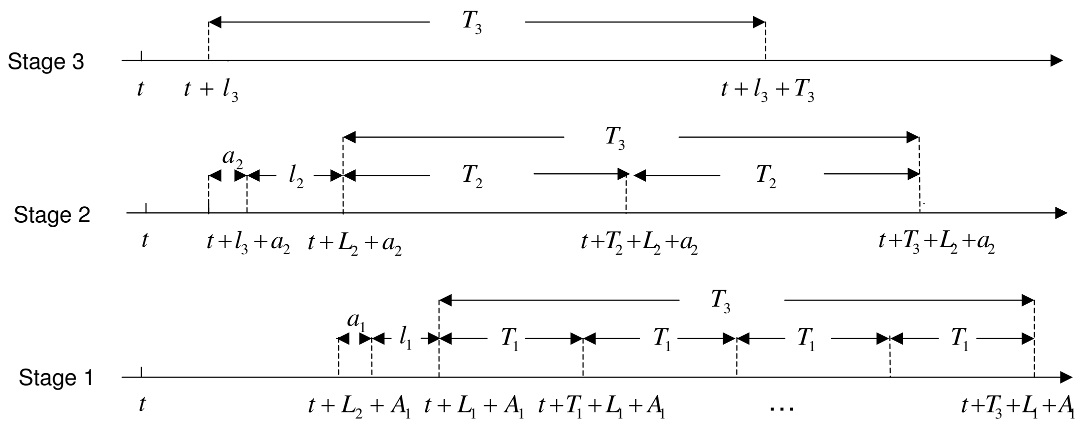


Figure 6.1: Time line of a three-stage system with $T_3 = 2T_2 = 4T_1$

as from period $t + L_{N-1} + a_{N-1}$, at which stage $N-1$ receives the order it placed at time $t + l_N + a_{N-1}$, to period $t + L_{N-1} + a_{N-1} + T_N$, i.e., shifting the cycle of stage N to the right by $l_{N-1} + a_{N-1}$ time periods. Similarly, we define a cycle for stage $N-2$ as starting from period $t + L_{N-2} + A_{N-2}$ to period $t + L_{N-2} + A_{N-2} + T_N$, and in general, a cycle of stage i starts from period $t + L_i + A_i$ to period $t + L_i + A_i + T_N$, which shifts the cycle of stage N to the right by $L_i - l_N + A_i$.

From the definitions above it can be seen that stage N can only order once in a cycle; stage $N-1$ can order Γ_{N-1} times in a cycle, and in general, stage i ($i = 1, 2, \ldots, N$) can order γ_i times in the cycle.

For any feasible policy, we compute the total expected cost over a cycle of length T_N by adding up the cost for each stage during the cycle defined above. With the understanding that $L_{N+1} = 0$ and $A_N = 0$, the total expected cost over a cycle for an arbitrary policy is

$$
\begin{aligned}
& E\left[\sum_{\ell=0}^{T_N-1}\Big(\sum_{i=1}^{N} h_i IL_i(t+L_i+A_i+\ell)+(b+H_1)B(t+L_1+A_1+\ell)\Big)\right] \qquad (6.5)\\
= \; & E\Big[\sum_{\ell=0}^{T_N-1}\Big(\sum_{i=1}^{N} h_i(IP_i(t+L_{i+1}+A_i+\ell)-D[t+L_{i+1}+A_i+\ell,t+L_i+A_i+\ell])\\
& +(b+H_1)(IP_1(t+L_2+A_1+\ell)-D[t+L_2+A_1+\ell,t+L_1+A_1+\ell])^-\Big)\Big]\\
= \; & E\Big[\sum_{i=1}^{N}\sum_{k=0}^{\gamma_i-1}\sum_{\ell=0}^{T_i-1} h_i(IP_i(t+L_{i+1}+A_i+kT_i+\ell)\\
& \qquad -D[t+L_{i+1}+A_i+kT_i+\ell,t+L_i+A_i+kT_i+\ell])\\
& \qquad +(b+H_1)\sum_{k=0}^{\gamma_1-1}\sum_{\ell=0}^{T_1-1}(IP_1(t+L_2+A_1+kT_1+\ell)\\
& \qquad -D[t+L_2+A_1+kT_1+\ell,t+L_1+A_1+kT_1+\ell])^-\Big]\\
= \; & E\Big[\sum_{i=1}^{N}\sum_{k=0}^{\gamma_i-1}\sum_{\ell=0}^{T_i-1} h_i(IP_i(t+L_{i+1}+A_i+kT_i)\\
& \qquad -D[t+L_{i+1}+A_i+kT_i,t+L_i+A_i+kT_i+\ell])\\
& \qquad +(b+H_1)\sum_{k=0}^{\gamma_1-1}\sum_{\ell=0}^{T_1-1}(IP_1(t+L_2+A_1+kT_1)\\
& \qquad -D[t+L_2+A_1+kT_1,t+L_1+A_1+kT_1+\ell])^-\Big],
\end{aligned}
$$

where the first equality follows from the fundamental relationship between echelon inventory positions and echelon inventory levels (6.2), the second equality follows from the fact that a cycle for stage i consists of γ_i ordering decisions, and the third equality follows from the constraint that stage i cannot order between periods $t+L_{i+1}+A_i+kT_i+1$ and $t+L_{i+1}+A_i+kT_i+\ell$ ($\ell<T_i$), therefore the echelon inventory positions of stage i in periods $t+L_{i+1}+A_i+kT_i$ and $t+L_{i+1}+A_i+kT_i+\ell$ satisfy

$$
\begin{aligned}
& IP_i(t+L_{i+1}+A_i+kT_i+\ell)\\
= \; & IP_i(t+L_{i+1}+A_i+kT_i)-D[t+L_{i+1}+A_i+kT_i,t+L_{i+1}+A_i+kT_i+\ell).
\end{aligned}
$$

To simplify the notation, in the following we write

$$
\begin{aligned}
IP_i(k) &\stackrel{\text{def}}{=} IP_i(t + L_{i+1} + A_i + kT_i), \\
IL^-_{i+1}(k) &\stackrel{\text{def}}{=} IL^-_{i+1}(t + L_{i+1} + A_i + kT_i)
\end{aligned}
$$

Then, the average cost over a cycle can be rewritten as

$$
\begin{aligned}
& E\Big[\sum_{i=1}^{N}\sum_{k=0}^{\gamma_i-1}\sum_{\ell=0}^{T_i-1} h_i(IP_i(k) - D(l_i+\ell+1)) \\
& \qquad\qquad +(H_1+b)\sum_{k=0}^{\gamma_1-1}\sum_{\ell=0}^{T_1-1}(IP_1(k) - D(l_1+\ell+1))^-\Big] \\
= \ & E\Big[\sum_{i=1}^{N}\sum_{k=0}^{\gamma_i-1} h_iT_iIP_i(k) + (H_1+b)\sum_{k=0}^{\gamma_1-1}\sum_{\ell=0}^{T_1-1}(IP_1(k) - D(l_1+\ell+1))^-\Big] \\
& -\sum_{i=1}^{N}\sum_{\ell=0}^{T_i-1} h_i\gamma_i E[D(l_i+\ell+1)],
\end{aligned}
$$

where $IP_i(k)$ and $D(l_i+\ell+1)$ are independent random variables. Since the last term is a constant which is independent of inventory strategy, it will be ignored in the subsequent analysis.

Remark If $T_i = 1$ for all i, then the model is reduced to the multi-echelon inventory model with batch ordering studied thoroughly in Chen (2000). If $Q_i = 1$ for every i, the problem has been analyzed in van Houtum et al. (2003). Moreover, if $T_i = 1$ and $Q_i = 1$, then the model collapses down to the classical Clark-Scarf model (Clark and Scarf 1960, and Chen and Zheng 1994).

6.2 The Main Result

In this section, we find the optimal ordering policy for each stage based on the cost function derived in the previous section. First, we need the following result which is due to Chen (2000). Let $\mathcal{Z}$ be the set of integers and $\mathcal{R}$ be the set of real numbers.

Lemma 6.2.1 *Let $G(\cdot) : \mathcal{Z} \to \mathcal{R}$ be a function and Q a positive integer. Define $\bar{G}(y) = \sum_{x=1}^{Q} G(y + x)$ and suppose $\bar{G}(y)$ is quasiconvex with finite minimum point R.*

(a) For any given z, $G(z + xQ)$ is quasiconvex in $x \in \mathcal{Z}$. Let x_z be the unique integer so that $R+1 \leq z + x_z Q \leq R + Q$, then $G(z + xQ)$ as a function of x is minimized at x_z.

(b) For any x, define

$$O[x] = \begin{cases} x, & \text{if } x \leq R + Q \\ x - nQ & \text{if } x > R + Q \end{cases}$$

where n is the largest integer that $x - nQ > R$. Then $\sum_{x=1}^{Q} G(O[y + x]) = \bar{G}(\min\{R, y\})$ which is quasiconvex and nonincreasing in $y \in \mathcal{Z}$.

Define a sequence of functions recursively as follows. Let

$$G_1(y) = T_1 h_1 y + (H_1 + b) \sum_{\ell=0}^{T_1 - 1} E[(y - D(l_1 + \ell + 1))^-]. \tag{6.6}$$

For $i = 1, 2, \ldots, N$, assume that $G_i(y)$ have been defined and $\bar{G}_i(y) \equiv \sum_{x=1}^{Q_i} G_i(y + x)$ is convex and minimized at $y = R_i$, a finite integer. Thus, $G_i(\cdot)$ satisfies the condition in Lemma 6.2.1. Define $O_i[y]$ as in Lemma 6.2.1 after replacing R and Q by R_i and Q_i

respectively. Define

$$G_{i+1}(y) = T_{i+1}h_{i+1}y + \sum_{k=0}^{r_i-1} E\big[G_i(O_i[y - D(l_{i+1} + kT_i + a_i)])\big], \qquad i = 1, \dots, N-1. \quad (6.7)$$

The assumption used in the definition above is guaranteed by the following lemma.

Lemma 6.2.2 *$\bar{G}_i(y)$ is convex and is minimized at a finite point R_i, $i = 1, 2, \dots, N$.*

Proof. We prove Lemma 2 by induction. We first prove convexity. That $\bar{G}_1(y)$ is convex follows from its definition. Suppose $\bar{G}_i(y)$ has been shown to be convex and we proceed to prove $i+1$. By definition,

$$\bar{G}_{i+1}(y) = \sum_{x=1}^{Q_{i+1}} T_{i+1}h_{i+1}(y+x) + \sum_{x=1}^{Q_{i+1}} \sum_{k=0}^{r_i-1} E\big[G_i(O_i[y + x - D(l_{i+1} + kT_i + a_i)])\big].$$

The first term is obviously convex. From (6.12) we have

$$\begin{aligned} & \sum_{x=1}^{Q_{i+1}} \sum_{k=0}^{r_i-1} E\big[G_i(O_i[y + x - D(l_{i+1} + kT_i + a_i)])\big] \\ = \ & \sum_{z=0}^{q_i-1} \sum_{k=0}^{r_i-1} E\big[\bar{G}_i(\min\{R_i, y + zQ_i - D(l_{i+1} + kT_i + a_i)\})\big] \end{aligned}$$

which is convex in y because R_i is the minimum of $\bar{G}_i(\cdot)$. Therefore, $\bar{G}_{i+1}(\cdot)$ is convex.

We next show that $\bar{G}_i(y)$ is minimized at a finite point. We first prove that, $G_i(y) \geq G_i^d(y)$ for all i where $G_1^d(y) = T_1h_1y + (H_1 + b)\sum_{\ell=0}^{T_1-1}(y - (l_1 + \ell + 1)\mu)^-$, and for

$i = 2, \ldots, N$,

$$\begin{aligned}
G_i^d(y) &= \sum_{k=0}^{r_{i-1}-1} T_{i-1}\Big(h_i\big(y-(l_i+kT_{i-1}+a_{i-1}+1)\mu\big)^+ \\
&\quad +(b+H_{i+1})\big(y-(l_i+kT_{i-1}+a_{i-1})\mu\big)^-\Big) \\
&\quad -\Big(\prod_{\ell=1}^{i-1} r_\ell T_1 h_2 + \prod_{\ell=2}^{i-1} r_\ell T_2 h_3 + \cdots + r_{i-1}T_{i-1}h_i\Big)\mu.
\end{aligned}$$

Again we prove it by induction. This is clearly true for $i = 1$ since $G_1(y)$ is convex and $G_1(y) \geq G_1^d(y)$ follows from Jensen's inequality. Suppose the result has been established for i, and we prove $G_{i+1}(y) \geq G_{i+1}^d(y)$ for all y. Note that

$$\begin{aligned}
G_i^d(y) &\geq \sum_{k=0}^{r_{i-1}-1} T_i(b+H_{i+1})\big(y-(l_i+kT_{i-1}+a_{i-1})\mu\big)^- \\
&\quad -\Big(\prod_{\ell=1}^{i-1} r_\ell T_1 h_2 + \cdots + r_{i-1}T_{i-1}h_i\Big)\mu \\
&\geq \sum_{k=0}^{r_{i-1}-1} T_{i-1}(b+H_{i+1})(y)^- - (\prod_{\ell=1}^{i-1} r_\ell T_1 h_2 + \cdots + r_{i-1}T_{i-1}h_i)\mu \\
&= T_i(b+H_{i+1})(y)^- - \Big(\prod_{\ell=1}^{i-1} r_\ell T_1 h_2 + \cdots + r_{i-1}T_{i-1}h_i\Big)\mu.
\end{aligned}$$

The inductive assumption, the fact that $O_i[y] \leq y$ and the previous inequality lead to

$$G_i(O_i[y]) \;\geq\; G_i^d(O_i[y]) \geq T_i(b+H_{i+1})(y)^- - \Big(\prod_{\ell=1}^{i-1} r_\ell T_1 h_2 + \cdots + r_{i-1}T_{i-1}h_i\Big)\mu.$$

Based on the definition of $G_{i+1}(y)$ together with above inequality,

$$\begin{aligned}
G_{i+1}(y) &= T_{i+1}h_{i+1}y + \sum_{k=0}^{r_i-1} E\big[G_i(O_i[y - D(l_{i+1} + kT_i + a_i)])\big] \\
&\geq \sum_{k=0}^{r_i-1} T_i h_{i+1}\big(y - (l_{i+1} + kT_i + a_i)\mu - \mu\big) \\
&\quad + \sum_{k=0}^{r_i-1} \Bigg[T_i(b + H_{i+1})E\big[\big(y - D(l_{i+1} + kT_i + a_i)\big)^-\big] \\
&\quad - \Big(\prod_{\ell=1}^{i-1} r_\ell T_1 h_2 + \cdots + \prod_{\ell=i-1}^{i-1} r_\ell T_{i-1} h_i\Big)\mu\Bigg] \\
&\geq \sum_{k=0}^{r_i-1} T_i h_{i+1}(y - (l_{i+1} + kT_i + a_i)\mu) \\
&\quad + \sum_{k=0}^{r_i-1} T_i(b + H_{i+1})\big(y - (l_{i+1} + kT_i + a_i)\mu\big)^- \\
&\quad - \Big(\prod_{\ell=1}^{i} r_\ell T_1 h_2 + \cdots + \prod_{\ell=i-1}^{i} r_\ell T_{i-1} h_i + r_i T_i h_{i+1}\Big)\mu \\
&= \sum_{k=0}^{r_i-1} T_i h_{i+1}\big(y - (l_{i+1} + kT_i + a_i)\mu\big)^+ \\
&\quad + \sum_{k=0}^{r_i-1} T_i(b + H_{i+2})\big(y - (l_{i+1} + kT_i + a_i)\mu\big)^- \\
&\quad - \Big(\prod_{\ell=1}^{i} r_\ell T_1 h_2 + \cdots + \prod_{\ell=i-1}^{i} r_\ell T_{i-1} h_i + r_i T_i h_{i+1}\Big)\mu \\
&= G_{i+1}^d(y),
\end{aligned}$$

where the second inequality follows from Jensen's inequality, and the second equality follows, for any number x, from $x + x^- = x^+$.

Because for any $i = 1, 2, \ldots, N$, $\lim_{|y|\to\infty} G_i^d(y) = \infty$, we must have $\lim_{|y|\to\infty} G_i(y) = \infty$ and $\lim_{|y|\to\infty} \bar{G}_i(y) = \infty$, thus $\bar{G}_i(y)$ is minimized at a finite point R_i. This completes the proof. $\square$

The proof of the following lemma mimics that of Lemma 2 in Chen (2000), thus it is omitted.

Lemma 6.2.3 *For any given integer* $k = 0, 1, \ldots, \Gamma_i - 1$,

$$G_i(IP_i(k)) \geq G_i(O_i[IL_{i+1}^-(k)]), \qquad i = 1, 2, \ldots, N-1. \tag{6.8}$$

Lemma 6.2.4 *For all* i, *we have*

$$\sum_{k=0}^{\gamma_i - 1} G_i(O_i[IL_{i+1}^-(k)]) = \sum_{k=0}^{\gamma_{i+1}-1} \sum_{\ell=0}^{r_i - 1} G_i(O_i[IP_{i+1}(k) - D(l_{i+1} + \ell T_i + a_i)]). \tag{6.9}$$

Proof. We only prove it for $i = 1$. The proof for other i is identical. Consider the summation of k that goes from 0 to $\Gamma_1 - 1$ on the left hand side of (6.9). That is, consider

$$IL_2^-(k) = IL_2^-(t + L_2 + A_1 + kT_1)$$

as k goes from 0 to $\Gamma_1 - 1$. First, when k increases from 0 to ℓ with $0 \leq \ell < r_1$, we have

$$IL_2^-(t+L_2+A_1+\ell T_1) = IP_2(t+L_3+A_1+\ell T_1) - D[t+L_3+A_1+\ell T_1, t+L_2+A_1+\ell T_1).$$

However, note that stage 2 cannot order in period $t + L_3 + A_1 + \ell T_1$. A moment of reflection shows that the last period before $t + L_3 + A_1 + \ell T_1$ that stage 2 can order is $t + L_3 + A_2$. This implies that

$$IP_2(t + L_3 + A_1 + \ell T_1) = IP_2(t + L_3 + A_2) - D[t + L_3 + A_2, t + L_3 + A_1 + \ell T_1).$$

Therefore,

$$\begin{aligned}
& IL_2^-(t + L_2 + A_1 + \ell T_1) \\
= \;& IP_2(t + L_3 + A_2) - D[t + L_3 + A_2, t + L_2 + A_1 + \ell T_1) \\
= \;& IP_2(0) - D(l_2 + \ell T_1 + a_1).
\end{aligned}$$

Then consider k that increases from r_1 to $r_1+\ell$ with $0 \leq \ell < r_1$. The last period before $t+L_3+A_1+(r_1+\ell)T_1$ that stage 2 can order is $t+L_3+A_1+r_1T_1-a_1 = t+L_3+A_2+T_2$. Thus

$$\begin{aligned}
IP_2(t + L_3 + A_1 + (r_1 + \ell)T_1) \;=\;& IP_2(t + L_3 + A_2 + T_2) \\
& -D[t + L_3 + A_2 + T_2, t + L_3 + A_1 + (r_1 + \ell)T_1),
\end{aligned}$$

and as a result,

$$\begin{aligned}
& IL_2^-(t + L_2 + A_1 + (r_1 + \ell)T_1) \\
= \;& IP_2(t + L_3 + A_1 + (r_1 + \ell)T_1) \\
& \qquad -D[t + L_3 + A_1 + (r_1 + \ell)T_1, t + L_2 + A_1 + (r_1 + \ell)T_1) \\
= \;& IP_2(t + L_3 + A_2 + T_2) - D[t + L_3 + A_2 + T_2, t + L_2 + A_1 + (r_1 + \ell)T_1) \\
= \;& IP_2(1) - D(l_2 + \ell T_1 + a_1).
\end{aligned}$$

More generally, as k increases from ur_1 to $ur_1 + \ell$, where $0 \leq u < \Gamma_2$ and $0 \leq \ell < r_1$, we have

$$IL_2^-(t + L_2 + A_1 + (ur_1 + \ell)T_1) = IP_2(u) - D(l_2 + \ell T_1 + a_1).$$

Combining the equations above, we obtain, as k increases from 0 to $\Gamma_1 - 1$, that

$$\sum_{k=0}^{\gamma_1 - 1} G_1(O_1[IL_2^-(k)]) = \sum_{k=0}^{\Gamma_2 - 1} \sum_{\ell=0}^{r_1 - 1} G_1(O_1[IP_2(k)) - D(l_2 + \ell T_1 + a_1)]),$$

proving Lemma 4. □ In what follows we derive a lower bound on the expected total cost for the system *over an ordering cycle.* Remember that we ignore the constant terms.

Theorem 6.2.1 *Let*

$$C^* = \frac{\bar{G}_N(R_N)}{Q_N T_N}. \tag{6.10}$$

Then C^* is a lower bound for the average cost per period for the serial system with batch ordering and periodic batching.

Proof. Let TC be the expected total cost for the system over an ordering cycle, then

$$\begin{aligned}
TC &= E\Big[\sum_{i=1}^{N}\sum_{k=0}^{\gamma_i-1} h_iT_iIP_i(k) + (H_1+b)\sum_{k=0}^{\gamma_1-1}\sum_{\ell=0}^{T_1-1} E[(IP_1(k)-D(l_1+\ell+1))^-]\Big] \\
&= E\Big[\sum_{i=2}^{N}\sum_{k=0}^{\gamma_i-1} h_iT_iIP_i(k) + \sum_{k=0}^{\gamma_1-1}\Big(h_1T_1IP_1(k) + \sum_{\ell=0}^{T_1-1} E[(IP_1(k)-D(l_1+\ell+1))^-\Big)\Big] \\
&= E\Big[\sum_{i=2}^{N}\sum_{k=0}^{\gamma_i-1} h_iT_iIP_i(k) + \sum_{k=0}^{\gamma_1-1} G_1(IP_1(k))\Big] \\
&\geq E\Big[\sum_{i=2}^{N}\sum_{k=0}^{\gamma_i-1} h_iT_iIP_i(k) + \sum_{k=0}^{\gamma_1-1} G_1(O_1[IL_2^-(k)])\Big] \\
&= E\Big[\sum_{i=2}^{N}\sum_{k=0}^{\gamma_i-1} h_iT_iIP_i(k) + \sum_{k=0}^{\gamma_2-1}\sum_{\ell=0}^{r_1-1} E\Big[G_1(O_1[IP_2(k)-D(l_2+\ell T_1+a_1)])\Big]\Big] \\
&= E\Big[\sum_{i=3}^{N}\sum_{k=0}^{\gamma_i-1} h_iT_iIP_i(k) \\
&\qquad + \sum_{k=0}^{\gamma_2-1}\Big(h_2T_2IP_2(k) + \sum_{\ell=0}^{r_1-1} E\Big[G_1(O_1[IP_2(k)-D(l_2+\ell T_1+a_1)])\Big]\Big)\Big] \\
&= E\Big[\sum_{i=3}^{N}\sum_{k=0}^{\gamma_i-1} h_iT_iIP_i(k) + \sum_{k=0}^{\gamma_2-1} G_2(IP_2(k))\Big] \qquad (6.11)
\end{aligned}$$

where the second equality follows from (6.6), the inequality follows from Lemma 6.2.3, the third equality follows from Lemma 4, and the last equality follows from (6.7).

By repeating the argument in (6.11), we obtain

$$TC \geq E[G_N(IP_N(t))].$$

Note that, stage N can order in every T_N periods of time and has to order an integer multiple of Q_N. Thus, we can interpret $IP_N(t)$ as the inventory position of a single-stage problem with demand distribution being the T_N-fold convolution of the original single period demand. It is well-known that under very mild conditions the optimal inventory position of the single stage inventory model with batch ordering is uniformly distributed on $\{R_N+1,\ldots,R_N+Q_N\}$ (see Veinott 1965 and Chen 2000). Thus, the average cost is

$$E[G_N(IP_N(t))] = \frac{1}{Q_N}\sum_{y=R_{N+1}}^{R_N+Q_N} G_N(y) = \frac{\bar{G}_N(R_N)}{Q_N}.$$

Because the cycle length is T_N, the average cost per period of time is TC/T_N. This shows that the average cost per period of time for the serial system under an arbitrary feasible policy is bounded from below by C^*. $\square$

The lower bound in Theorem 1 can be achieved. Consider the following policy. For $i = 1, 2, \ldots, N$, at any period that stage i can order, if the echelon inventory position of stage i is at or below R_i, stage i orders an integer multiple of Q_i to bring the echelon inventory position into $\{R_i + 1, \ldots, R_i + Q_i\}$; and if the on hand inventory at stage $i+1$ is insufficient, then stage $i+1$ ships as much as possible. This is the so-called echelon (R_i, nQ_i) policy.

Theorem 6.2.2 *For each stage i, the echelon (R_i, nQ_i) policy is optimal for the serial system with batch ordering and periodic batching. The minimum average cost per period is C^*.*

Proof. It is sufficient to prove that the average system cost under echelon (R_i, nQ_i) policy reaches the lower bound C^*. If stage i follows an (R_i, nQ_i) policy, then $IP_i(k) = O_i[IL_{i+1}^-(k)]$ after sometime for every integer k, i.e., at every period that stage i can order. Thus, the inequality in Lemma 6.2.3 becomes equality. Hence, from Theorem 1, the (R_N, nQ_N) policy achieves the lower bound C^*. $\square$

We remark that, for the case $Q_i = 1$ for all i, van Houtum et al. (2003) proved the optimality of echelon base-stock policy for the system by using a different but rather complicated method. The extension of Chen and Zheng (1994)'s lower bound approach not only gives a very simple and straightforward proof for the result of van Houtum et al. (2003), but it also generalizes the result to the model with batch ordering.

In the following, we discuss how the optimal policy and the optimal expected system cost are affected by a_i, the non-synchronization factors. To emphasize the dependency of the optimal average cost on the non-synchronization factors, let $C(a_1, \ldots, a_{N-1})$ denote the minimum average cost. For simplicity let $\bar{G}_1^*(y) = \bar{G}_1(y)$ and $\bar{G}_{i+1}^*(y)$ be the $\bar{G}_{i+1}(y)$ associated with $a_j = 0$ for $j = 1, 2, \ldots, i$, $i = 1, \ldots, N-1$, and let R_{i+1}^* be the corresponding optimal reorder point, the minimizer of $\bar{G}_{i+1}^*(y)$.

First, note that it follows from Lemma 1(b) and (6.7) that, once $\bar{G}_i$ is known, $\bar{G}_{i+1}$ can be computed as follows.

$$\begin{aligned}
&\bar{G}_{i+1}(y) \\
= \;& \sum_{x=1}^{Q_{i+1}} G_{i+1}(y+x) \\
= \;& T_{i+1}h_{i+1}\sum_{x=1}^{Q_{i+1}}(y+x) + \sum_{k=0}^{r_i-1} E\Big[\sum_{x=1}^{q_iQ_i} G_i(O_i[y+x-D(l_{i+1}+kT_i+a_i)])\Big] \\
= \;& T_{i+1}h_{i+1}\sum_{x=1}^{Q_{i+1}}(y+x) + \sum_{k=0}^{r_i-1} E\Big[\sum_{z=0}^{q_i-1}\sum_{x=1}^{Q_i} G_i(O_i[y+zQ_i+x-D(l_{i+1}+kT_i+a_i)])\Big] \\
= \;& T_{i+1}h_{i+1}\sum_{x=1}^{Q_{i+1}}(y+x) + \sum_{k=0}^{r_i-1}\sum_{z=0}^{q_i-1} E\big[\bar{G}_i(\min\{R_i, y+zQ_i-D(l_{i+1}+kT_i+a_i)\})\big].
\end{aligned} \tag{6.12}$$

The first result shows that, the optimal reorder point of each stage is minimized when the ordering times of all stages are synchronized.

Proposition 6.2.1 *$R_i^* \le R_i$, $i = 1, 2, \ldots, N$.*

Proof. By induction on i. Let Δ denote the difference operator, i.e., $\Delta\bar{G}_i(y) = \bar{G}_i(y+1) - \bar{G}_i(y)$. We prove $\Delta\bar{G}_i(y) \leq \Delta\bar{G}_i^*(y)$ for all y and all i, which implies $R_i^* \leq R_i$ because of convexity.

The proposition holds true for $i = 1$ as $\bar{G}_1^*(y) = \bar{G}_1(y)$. Suppose the result is true for i. Then, it follows from (6.12) that, it suffices to prove

$$\Delta(E[\bar{G}_i^*(\min\{R_i^*, y+zQ_i-D(l_{i+1}+kT_i)\})] \geq \Delta(E[\bar{G}_i(\min\{R_i, y+zQ_i-D(l_{i+1}+kT_i+a_i)\})].$$

Let $p_\ell, \ell = 0, 1, \ldots$ denote the probability mass function of $D(l_{i+1} + kT_i + a_i)$. We know

$$\Delta(E\bar{G}_i^*(\min\{R_i^*, y+zQ_i-D(l_{i+1}+kT_i)\}) \geq \Delta(E\bar{G}_i^*(\min\{R_i^*, y+zQ_i-D(l_{i+1}+kT_i+a_i)\})$$

since $\bar{G}_i^*(\cdot)$ is convex and minimized at R_i^*. Then, we can show that, after some algebra,

$$\begin{aligned}
& \Delta(E[\bar{G}_i^*(\min\{R_i^*, y + zQ_i - D(l_{i+1} + kT_i + a_i)\})] \\
& \qquad -\Delta(E[\bar{G}_i(\min\{R_i, y + zQ_i - D(l_{i+1} + kT_i + a_i)\})] \\
= \; & \sum_{\ell=y+1+zQ_i-R_i^*}^{\infty} \Delta\bar{G}_i^*(y + 1 + zQ_i - \ell)p_\ell - \sum_{\ell=y+1+zQ_i-R_i}^{\infty} \Delta\bar{G}_i(y + 1 + zQ_i - \ell)p_\ell \\
= \; & \sum_{\ell=y+1+zQ_i-R_i^*}^{\infty} \Big(\Delta\bar{G}_i^*(y + 1 + zQ_i - \ell) - \Delta\bar{G}_i(y + 1 + zQ_i - \ell)\Big)p_\ell \\
& - \sum_{\ell=y+1+zQ_i-R_i}^{y+1+zQ_i-R_i^*} \Delta\bar{G}_i(y + 1 + zQ_i - \ell)p_\ell \\
\geq \; & 0,
\end{aligned}$$

where the inequality follows from the induction assumption and the fact that $\Delta\bar{G}_i(y)$ is non-positive on $y < R_i$. Therefore, $\Delta\bar{G}_{i+1}(y) \leq \Delta\bar{G}_{i+1}^*(y)$ and we conclude $R_{i+1}^* \leq R_{i+1}$.

The induction proof is completed. $\square$

The following result states that the average cost of the system is minimized when the different stages' ordering times are synchronized, i.e., every stage except the most upstream stage places order at the instants its upstream stage receives order and at the equidistant time instants in between. Its proof is provided in the Appendix.

Theorem 6.2.3 *The optimal average cost function satisfies*

$$C(0,\ldots,0) \leq C(a_1,\ldots,a_{N-1}).$$

Proof. We first prove by induction that for all y,

$$\bar{G}_i^*(y) \leq \bar{G}_i(y), \qquad i = 1, 2, \ldots, N. \tag{6.13}$$

This is clearly true for $i = 1$ since $\bar{G}_1^*(y) = \bar{G}_1(y)$. Suppose it has been proved for $i \geq 1$, and we proceed to prove $i + 1$. By (6.12), it suffices to prove

$$\bar{G}_i^*(\min\{R_i^*, y + zQ_i - D(l_{i+1} + kT_i)\}) \leq \bar{G}_i(\min\{R_i, y + zQ_i - D(l_{i+1} + kT_i + a_i)\})$$

for any realization of demand. Note that $D(l_{i+1} + kT_i) \leq D(l_{i+1} + kT_i + a_i)$.

We consider several cases. If $y + zQ_i - D(l_{i+1} + kT_i + a_i) \geq R_i$, by Proposition 1 we have $y + zQ_i - D(l_{i+1} + kT_i) \geq R_i^*$, then

$$\bar{G}_i^*(R_i^*) \leq \bar{G}_i^*(R_i) \leq \bar{G}_i(R_i),$$

where the first inequality follows from the optimality of R_i^* for $\bar{G}_i^*(\cdot)$ and the second inequality from induction assumption.

If $R_i \geq y + zQ_i - D(l_{i+1} + kT_i + a_i)$ and $y + zQ_i - D(l_{i+1} + kT_i) \geq R_i^*$, then

$$\bar{G}_i^*(R_i^*) \leq \bar{G}_i^*(R_i) \leq \bar{G}_i(R_i) \leq \bar{G}_i(y + zQ_i - D(l_{i+1} + kT_i + a_i)).$$

The last inequality follows from the optimality of R_i for $\bar{G}_i(\cdot)$.

If $R_i \geq y+zQ_i-D(l_{i+1}+kT_i+a_i)$ and $R_i^* \geq y+zQ_i-D(l_{i+1}+kT_i)$, then by convexity of $\bar{G}(\cdot)$ and induction assumption, we obtain

$$\begin{aligned}
\bar{G}_i^*(y + zQ_N - D(l_{i+1} + kT_i)) &\leq \bar{G}_i^*(y + zQ_i - D(l_{i+1} + kT_i) + a_i) \\
&\leq \bar{G}_i(y + zQ_i - D(l_{i+1} + kT_i + a_i)).
\end{aligned}$$

So $\bar{G}_{i+1}^*(y) \leq \bar{G}_{i+1}(y)$.

It follows from Theorem 1 and Theorem 2 that

$$C(a_1, \ldots, a_{N-1}) = \frac{\bar{G}_N(R_N)}{Q_N T_N}, \tag{6.14}$$

$$C(0, \ldots, 0) = \frac{\bar{G}_N^*(R_N^*)}{Q_N T_N}. \tag{6.15}$$

Since

$$\bar{G}_N^*(R_N^*) \leq \bar{G}_N^*(R_N) \leq \bar{G}_N(R_N),$$

where the first inequality follows from that R_N^* is the minimizer of $\bar{G}_N^*(y)$, and the second inequality follows from (6.13). Thus, by (6.14) and (6.15), Theorem 3 is proved. □ We conclude this section with an efficient computational algorithm for the optimal reorder points R_i, which is the minimum point of the convex function $\bar{G}_i(\cdot)$, $i = 1, 2, \ldots, N$. The procedure is bottom-up. We first compute R_1 that is used to compute $G_2(\cdot)$ and $\bar{G}_2(\cdot)$; then compute R_2, and so on so forth until R_N is computed. Since $\bar{G}_i$ is convex, the search for R_i in Step 2 can be done by evaluating $\Delta\bar{G}_i(y)$.

Computational Algorithm for Optimal Policy:

Step 1. Let $i = 1$, $\bar{G}_1(y) = \sum_{x=1}^{Q_1}[T_1 h_1(y+x) + \sum_{\ell=0}^{T_1-1}(H_1+b)E(y+x-D(l_1+\ell+1))^-]$.

Step 2. $R_i = \arg\min_{y\in\mathcal{Z}} \bar{G}_i(y)$.

Step 3. If $i < N$, compute $\bar{G}_{i+1}(y)$ using (6.12), $i = i+1$ and go to Step 2; otherwise, stop.

6.3 Numerical Studies

We consider a three-stage system with basic parameters: $T_1 = 2, T_2 = 8, T_3 = 16$, $Q_1 = 2, Q_2 = 8, Q_3 = 16$ $b = 30$ $h_1 = 2, h_2 = 1, h_3 = 1$, $a_1 = 0, a_2 = 0$. By alternating some parameter, we generate numerical examples summarized in the table.

T_1	R_1	R_2	R_3	c^*	Q_1	R_1	R_2	R_3	c^*	a_2	R_1	R_2	R_3	c^*
1	11	41	71	85.72	1	15	41	71	89.05	1	15	41	75	93.51
2	15	41	71	89.14	2	15	41	71	89.14	2	15	41	80	97.87
4	22	40	71	95.63	4	14	41	71	89.50	3	15	41	84	102.21
8	37	37	69	106.90	8	13	41	71	90.88	4	15	41	88	106.54

Table 6.1: Three-stage model with batch ordering and fixed replenishment schedule

6.4 Summary

In this chapter we study a periodic-review multi-echelon inventory system with batch ordering and periodic batching and derive its optimal control policy. We show that the system achieves minimum expected average cost when the ordering times for all stages are synchronized, the optimal reorder point for each stage is also the smallest as all stages are synchronized. This work generalizes the recent results of Chen (2000) and van Houtum et al. (2003).

Our results can be extended to an assembly system described as follows. There are N distinct items: the components, subassemblies, and the end item. The assembly system has a tree structure where the root is the end item and the leaves are the components, and each item except the end item has only one successor. The customer demand is only for the end item. Assume that the suppliers for the components have ample supply and the supply leadtime is fixed. The production leadtime for other items is also fixed. We rearrange these N items of the assembly system according to their total leadtimes that is the sum of the leadtimes of the item and all its successor items. Thus the end item is item 1 and the item with longest leadtime is item N. We consider this assembly system as an N-stage serial system in which stage i is associated with item i for all i. If both the base quantities Q_i and reorder interval T_i for the items satisfy the integer-ratio relationships, then we can apply the approach discussed in this note to prove the optimality of echelon (R_i, nQ_i) policies for this assembly system. Refer to Rosling (1989) and Chen (2000) on transforming an assembly system to a serial system.

We remark that when $Q_i = 1$ for all i, stage N can include a setup cost. In that case, the optimal policy for each downstream stage is still echelon base-stock policy, but the optimal policy for stage N becomes echelon (s, S) policy. We can also impose a finite capacity constraint at stage N, and in that case the optimal policy at stage N becomes modified echelon base-stock policy, that is, at the beginning of each period raise the echelon inventory position to the base-stock level if possible, and otherwise raise the echelon inventory position to as close to the echelon base-stock level as possible.

Chapter 7

Probabilistic Solution and Bounds for Classical Serial Inventory Systems

In this chapter, we consider the infinite horizon multi-echelon inventory model of Clark-Scarf with both average cost and discounted cost criteria. The optimal echelon base-stock levels are obtained in terms of only probabilistic distributions of leadtime demand. The results offer insights on the key determinants of the optimal policy and provide a unified approach for developing bounds and simple heuristics. In addition to deriving the known bounds, we develop an upper bound for average cost case and two upper bounds for the discounted cost case. Simple heuristic is developed based on these bounds and numerical examples show that the heuristic performs very well.

7.1 Probabilistic Solution

Consider a single item serial inventory system with N stages. Compound Poisson demand arrives at stage 1, which orders from stage 2, stage 2 orders from stage 3, etc., and stage N orders from the outside with ample supply. There are constant transportation time between stages, and unsatisfied demand is fully backlogged at stage 1.

The following notation will be used for stage $i = 1, 2, \ldots, N$:

L_i = the leadtime between stage i to stage $i + 1$.

D_i = the leadtime demand during L_i units of leadtime.

$F_i(.)$ = the distribution function of D_i.

y_i = echelon inventory position at stage i after ordering.

h_i = echelon i inventory holding cost rate.

H_i = installation i inventory holding cost rate, i.e., $H_i = \sum_{j=i}^{N} h_j$.

IP_i = echelon inventory position of stage i.

b = backorder cost rate at stage 1.

For convenience we let $L_{i,j}$ represent the leadtime between stage i and stage $j+1$, i.e., $L_{i,j} = \sum_{k=i}^{j} L_k$, let $D_{i,j}$ represent the demand during leadtime $L_{i,j}$, i.e., $D_{i,j} = \sum_{k=i}^{j} D_k$, let $F_{i,j}$ be the distribution function of $D_{i,j}$, and $\bar{F}_{i,j} = 1 - F_{i,j}$. Clearly, $L_{i,i} = L_i, D_{i,i} = D_i, F_{i,i} = F_i$ and they will be used interchangeably, and $L_{i,j} = 0, D_{i,j} = 0$ for $j < i$.

Echelon base-stock policy is optimal for this system for both average cost and discounted total cost criteria (see Federgruen and Zipkin (1984) and Chen and Zheng (1994)).

We then turn to the case of minimizing total discounted cost with discount factor α, the algorithm for computing the optimal base-stock levels is: Let $G_0^1(x) = (H_1 + b)x^-$. For $j = 1, 2, \ldots, N$, compute

$$
\begin{aligned}
G_i(x) &= \alpha^{L_i} h_i E(x - D_i) + \alpha^{L_i} E[G_{i-1}^i(x - D_i)], && (7.1)\\
s_i^* &= \arg\min G_i(x), && (7.2)\\
G_i^{i+1}(x) &= G_i(x \wedge s_i^*). && (7.3)
\end{aligned}
$$

And the optimum total discounted cost is given by $C^* = G_N(s_N^*)$.

The following result gives the optimal base-stock levels for discounted cost criterion in terms of the leadtime demand distributions. To the best of our knowledge, no recursive equations have ever been reported in the literature for Clark-Scarf model with discounted cost criterion.

Proposition 7.1.1 *Assuming $s_1^*, \ldots, s_{i-1}^*$, the optimal echelon base-stock level for stage i, s_i^*, for $i = 1, 2, \ldots, N$, is the solution of*

$$
\begin{aligned}
&h_i + \sum_{j=1}^{i-1} \alpha^{L_{j,i-1}} h_j P(D_{j+1,i} \geq y - s_j^*, D_{j+2,i} \geq y - s_{j+1}^*, \ldots, D_{i,i} \geq y - s_{i-1}^*)\\
&-\alpha^{L_{1,i-1}} (H_1 + b) P(D_{1,i} \geq y, D_{2,i} \geq y - s_1^*, \ldots, D_{i,i} \geq y - s_{i-1}^*) = 0.
\end{aligned}
$$

The left hand side of the equation is increasing in y.

Thus, for stage 1, the optimal base-stock level s_i^* is the solution of

$$h_1 - (H_1 + b)P(D_1 > y) = 0. \tag{7.4}$$

After s_1^* is computed, the optimal base-stock level for stage 2, s_2^*, is determined by

$$h_2 + \alpha^{L_1} h_1 P(D_2 > y - s_1^*) - \alpha^{L_1}(H_1 + b)P(D_2 \geq y - s_1^*, D_1 + D_2 > y) = 0.$$

And finally, the optimal base-stock level for stage N, s_N^*, is the solution of

$$\begin{aligned} & h_N + \sum_{j=1}^{N-1} \alpha^{L_{j,N-1}} h_j P(D_{j+1,N} \geq y - s_j^*, \ldots, D_N \geq y - s_{N-1}^*) \\ & -\alpha^{L_{1,N-1}}(H_1 + b)P(D_{1,N} \geq y, D_{2,N} \geq y - s_1^*, \ldots, D_N \geq y - s_{N-1}^*) = 0. \end{aligned}$$

Proof of Proposition 7.1.1. We prove by induction that for $i = 1, \ldots, N$, we have

$$\begin{aligned} & G_i'(y) \\ = \; & E\Big[\alpha^{L_i} h_i \\ & +\alpha^{L_i} \sum_{j=1}^{i-1} \alpha^{L_{j,i-1}} h_j 1[D_{j+1,i} \geq y - s_j^*, D_{j+2,i} \geq y - s_{j+1}^*, \ldots, D_i \geq y - s_{i-1}^*] \\ & -\alpha^{L_{1,i}}(H_1 + b)1[D_{1,i} \geq y, D_{2,i} \geq y - s_1^*, \ldots, D_i \geq y - s_{i-1}^*]\Big]. \end{aligned}$$

Since s_i^* is the solution of $G_i'(y) = 0$, the result of Proposition 2 then follows.

First, consider $i = 1$. The optimal base-stock level s_1^* is the minimizer of

$$G_1(y) = \alpha^{L_1} h_1 E(y - D_1) + \alpha^{L_1}(H_1 + b)E[(y - D_1)^-].$$

This is a convex function, taking derivative and set it to 0 we obtain (7.4).

Suppose the result has been proved for i and we proceed to prove $i+1$. It follows from equation (7.1) that

$$\begin{aligned} G_{i+1}(y) &= \alpha^{L_{i+1}} h_{i+1} y + \alpha^{L_{i+1}} E[G_i^{i+1}(y - D_{i+1})] \\ &= \alpha^{L_{i+1}} h_{i+1} y + \alpha^{L_{i+1}} E[G_i((y - D_{i+1}) \wedge s_i^*)] \\ &= \alpha^{L_{i+1}} E\Big[h_{i+1} y + G_i(y - D_{i+1}) 1[y - D_{i+1} \leq s_i^*] + G_i(s_i^*) 1[y - D_{i+1} > s_i^*]\Big]. \end{aligned}$$

We have identities

$$\begin{aligned} & (G_i(y - D_{i+1}) 1[y - D_{i+1} \leq s_i^*])' \\ = \; & G_i'(y - D_{i+1}) 1[y - D_{i+1} \leq s_i^*] - G_i(y - D_{i+1}) \delta(y - D_{i+1} + s_i^*), \end{aligned}$$

and

$$(G_i(s_i^*) 1[y - D_{i+1} > s_i^*])' = G_i(s_i^*) \delta(y - D_{i+1} - s_i^*).$$

Taking derivative of $G_{i+1}(y)$ and substituting the two identities above yield, after canceling common terms,

$$\begin{aligned} & G_{i+1}'(y) \\ = \; & \alpha^{L_{i+1}} E\Big[h_{i+1} + G_i'(y - D_{i+1}) 1[y - D_{i+1} \leq s_i^*]\Big] \\ = \; & \alpha^{L_{i+1}} E\Big[h_{i+1} + \alpha^{L_i} h_i 1[D_{i+1} \geq y - s_i^*] \\ & + \sum_{j=1}^{i-1} \alpha^{L_{j,i}} h_j 1[D_{j+1,i} \geq y - D_{i+1} - s_j^*, \ldots, D_i \geq y - D_{i+1} - s_{i-1}^*] 1[D_{i+1} \geq y - s_i^*] \\ & - \alpha^{L_{1,i}} (H_1 + b) 1[D_{1,i} \geq y - D_{i+1}, \ldots, D_i \geq y - D_{i+1} - s_{i-1}^*] 1[D_{i+1} \geq y - s_i^*]\Big] \\ = \; & \alpha^{L_{i+1}} E\Big[h_{i+1} + \sum_{j=1}^{i} \alpha^{L_{j,i}} h_j 1[D_{j+1,i+1} \geq y - s_j^*, \ldots, D_{i,i+1} \geq y - s_{i-1}^*, D_{i+1} \geq y - s_i^*)] \\ & - \alpha^{L_{1,i}} (H_1 + b) 1[D_{1,i+1} \geq y, \ldots, D_{i,i+1} \geq y - s_{i-1}^*, D_{i+1} \geq y - s_i^*]\Big], \end{aligned}$$

where the second equality follows from the induction hypothesis. This completes the proof of Proposition 7.1.1. $\square$

Proposition 7.1.1 presents explicit form dependency of the optimal inventory control strategies on its determinants. In the next section we will see how these results can be applied to develop simple upper and lower bounds for the optimal control parameters.

7.2 Bounds

The explicit results of optimal echelon base-stock levels given in the last section can be used to derive simple bounds for optimal control parameters. The idea is simple: If we approximate the left hand side of (**??**) by another function which yields a simple solution, then the solution serves as an approximation for s_i^*. Furthermore, since the left hand side of (**??**) is increasing in y, if we approximate the left hand side by a smaller function, say $g(y)$, then $g(y) = 0$ will give an upper bound for s_i^*; while if we approximate the left hand side of (**??**) by a larger function $g(y)$, then $g(y) = 0$ will give a lower bound for s_i^*.

We illustrate this by considering $i = 2$ for the average cost case since the result for $i = 1$ is exact. For $i = 2$, the optimal base-stock level s_2^* is determined by

$$h_2 + h_1 P(D_2 \geq y - s_1^*) - (H_1 + b) P(D_1 + D_2 \geq y, D_2 \geq y - s_1^*) = 0. \tag{7.5}$$

Because

$$\begin{aligned} P(D_2 \geq y - s_1^*) &\geq P(D_1 + D_2 \geq y, D_2 \geq y - s_1^*), \\ P(D_1 + D_2 \geq y) &\geq P(D_1 + D_2 \geq y, D_2 \geq y - s_1^*), \end{aligned}$$

it follows that

$$
\begin{aligned}
& h_2 + h_1 P(D_2 \geq y - s_1^*) - (H_1 + b)P(D_1 + D_2 \geq y, D_2 \geq y - s_1^*) \\
\geq\ & h_2 + h_1 P(D_1 + D_2 \geq y, D_2 \geq y - s_1^*) - (H_1 + b)P(D_1 + D_2 \geq y, D_2 \geq y - s_1^*) \\
=\ & h_2 - (H_2 + b)P(D_1 + D_2 \geq y, D_2 \geq y - s_1^*) \\
\geq\ & h_2 - (H_2 + b)P(D_1 + D_2 \geq y).
\end{aligned}
$$

Hence $h_2 - (H_2 + b)P(D_1 + D_2 \geq y)$ is a lower bound for the left hand side of (7.5), and the solution of

$$
h_2 - (H_2 + b)P(D_1 + D_2 \geq y) = 0,
$$

or

$$
s_2^u = \bar{F}_{1,2}^{-1}\left(\frac{h_2}{H_2 + b}\right) \tag{7.6}
$$

gives an upper bound for s_2^*. This is exactly the upper bound obtained by Shang and Song (2003).

Another upper bound for s_2^* can be obtained as follows. Since

$$
\begin{aligned}
& h_2 + h_1 P(D_2 \geq y - s_1^*) - (H_1 + b)P(D_1 + D_2 \geq y, D_2 \geq y - s_1^*) \\
\geq\ & h_2 + h_1 P(D_2 \geq y - s_1^*) - (H_1 + b)P(D_2 \geq y - s_1^*) \\
=\ & h_2 - (H_2 + b)(D_2 \geq y - s_1^*),
\end{aligned}
$$

we conclude that another upper bound for s_2^* is determined by

$$
h_2 - (H_2 + b)P(D_2 \geq y - s_1^*) = 0,
$$

yielding

$$\hat{s}_2^u = s_1^* + \bar{F}_2^{-1}\left(\frac{h_2}{H_2+b}\right) = \bar{F}_1^{-1}\left(\frac{h_1}{H_1+b}\right) + \bar{F}_2^{-1}\left(\frac{h_2}{H_2+b}\right). \tag{7.7}$$

Note that this upper bound presents an expression on the additional installation inventory level that stage 2 should maintain, since it gives the level for stage 2 beyond the base-stock level for stage 1, s_1^*.

Remark

The following example demonstrates that the two upper bounds, i.e. (7.6) and (7.7), do not have a dominating relationship, hence anyone can be a better bound, depending on the instance.

We next present the lower and upper bounds for the serial system with discounted cost criterion.

Proposition 7.2.1 *An upper bound for the optimal echelon base-stock level of Clark-Scarf model with total discounted cost criterion is, if*

$$\alpha^{L_{1,i-1}}(H_1+b) - \sum_{j=1}^{i-1} \alpha^{L_{j,i-1}} h_j > h_i, \tag{7.8}$$

$$s_i^u = \bar{F}_{1,i}^{-1}\left(\frac{h_i}{\alpha^{L_{1,i-1}}(H_1+b) - \sum_{j=1}^{i-1} \alpha^{L_{j,i-1}} h_j}\right), \quad i = 1, \ldots, N, \tag{7.9}$$

and otherwise

$$s_i^u = 0, \qquad i = 1, \ldots, N.$$

Another upper bound for the discounted cost case is, if (7.8) is satisfied then

$$\hat{s}_i^u = s_i^* + \bar{F}_i^{-1}\left(\frac{h_i}{\alpha^{L_{1,i-1}}(H_1+b) - \sum_{j=1}^{i-1}\alpha^{L_{j,i-1}}h_j}\right), \quad i = 1,\ldots,N, \tag{7.10}$$

and (7.8) is not satisfied then $\hat{s}_i^u = 0$. *Inductively we obtain upper bound,*

$$\tilde{s}_i^u = \sum_{j=1}^{i} \bar{F}_j^{-1}\left(\frac{h_j}{\alpha^{L_{1,j-1}}(H_1+b) - \sum_{k=1}^{j-1}\alpha^{L_{k,j-1}}h_k}\right), \quad i = 1,\ldots,N,$$

where $\bar{F}_j^{-1}(x)$, $j = 1,\ldots,i$, *is understood as 0 if either* $x \geq 1$ *or* $x \leq 0$. *And a lower bound for the optimal echelon base-stock level is*

$$s_i^l = \bar{F}_{1,i}^{-1}\left(\frac{\sum_{j=1}^{i}\alpha^{-L_{1,j-1}}h_j}{H_1+b}\right), \qquad i = 1,\ldots,N. \tag{7.11}$$

Proof. We first prove upper bounds. Assuming (7.8) and applying (**??**) we obtain

$$\begin{aligned}
& h_i + \sum_{j=1}^{i-1}\alpha^{L_{j,i-1}}h_j P(D_{j+1,i} \geq y - s_j^*, D_{j+2,i} \geq y - s_{j+1}^*, \ldots, D_{i,i} \geq y - s_{i-1}^*) \\
& -\alpha^{L_{1,i-1}}(H_1+b)P(D_{1,i} \geq y, D_{2,i} \geq y - s_1^*, \ldots, D_{i,i} \geq y - s_{i-1}^*) \\
\geq\ & h_i - \Big(\alpha^{L_{1,i-1}}(H_1+b) - \sum_{j=1}^{i-1}\alpha^{L_{j,i-1}}h_j\Big)P(D_{1,i} \geq y, \ldots, D_{i,i} \geq y - s_{i-1}^*) \\
\geq\ & h_i - \Big(\alpha^{L_{1,i-1}}(H_1+b) - \sum_{j=1}^{i-1}\alpha^{L_{j,i-1}}h_j\Big)P(D_{1,i} \geq y),
\end{aligned}$$

where the second inequality follows from the fact that $\alpha^{L_{1,i-1}}(H_1+b) - \sum_{j=1}^{i-1}\alpha^{L_{j,i-1}}h_j$

due to assumption (7.8). Thus the first upper bound (7.9) follows. Same argument shows

$$\begin{aligned} & h_i + \sum_{j=1}^{i-1} \alpha^{L_{j,i-1}} h_j P(D_{j+1,i} \geq y - s_j^*, D_{j+2,i} \geq y - s_{j+1}^*, \ldots, D_{i,i} \geq y - s_{i-1}^*) \\ & -\alpha^{L_{1,i-1}}(H_1 + b) P(D_{1,i} \geq y, D_{2,i} \geq y - s_1^*, \ldots, D_{i,i} \geq y - s_{i-1}^*) \\ \geq \; & h_i - \Big(\alpha^{L_{1,i-1}}(H_1 + b) - \sum_{j=1}^{i-1} \alpha^{L_{j,i-1}} h_j\Big) P(D_{i,i} \geq y - s_{i-1}^*), \end{aligned}$$

yielding the second upper bound (7.10).

Suppose (7.8) is not satisfied. That is,

$$\alpha^{L_{1,i-1}}(H_1 + b) - \sum_{j=1}^{i-1} \alpha^{L_{j,i-1}} h_j \leq h_i. \tag{7.12}$$

Since the optimal echelon base-stock level s_i^* is determined by (**??**), and that the left hand side of (**??**) is increasing in y. Setting $y = 0$ to the left hand side of (**??**) yields

$$\begin{aligned} & h_i + \sum_{j=1}^{i-1} \alpha^{L_{j,i-1}} h_j P(D_{j+1,i} \geq -s_j^*, D_{j+2,i} \geq y - s_{j+1}^*, \ldots, D_{i,i} \geq -s_{i-1}^*) \\ & -\alpha^{L_{1,i-1}}(H_1 + b) P(D_{1,i} \geq, D_{2,i} \geq -s_1^*, \ldots, D_{i,i} \geq -s_{i-1}^*) \\ = \; & h_i + \sum_{j=1}^{i-1} \alpha^{L_{j,i-1}} h_j - \alpha^{L_{1,i-1}}(H_1 + b) \\ \geq \; & 0, \end{aligned}$$

where the inequality follows from (7.12). This shows that the left hand side of (**??**) is positive for all $y \geq 0$ hence $s_i^* \leq 0$ and $s_i^u = 0$ is an upper bound for s_i^*.

We prove the lower bound by induction. We show that, for all i the following inequality is satisfied:

$$\begin{aligned}
& h_i + \sum_{j=1}^{i-1} \alpha^{L_{j,i-1}} h_j P(D_{j+1,i} \geq y - s_j^*, \ldots, D_i \geq y - s_{i-1}^*) \\
& -\alpha^{L_{1,i-1}} (H_1 + b) P(D_{1,i} \geq y, \ldots, D_i \geq y - s_{i-1}^*) \\
\leq \; & \sum_{j=1}^{i} \alpha^{L_{j,i-1}} h_j - \alpha^{L_{1,i-1}} (H_1 + b) P(D_{1,i} \geq y).
\end{aligned}$$

This is exact for $i = 1$. Suppose that it has been proved for i and we proceed to prove $i + 1$:

$$\begin{aligned}
& h_{i+1} + \sum_{j=1}^{i} \alpha^{L_{j,i}} h_j P(D_{j+1,i+1} \geq y - s_j^*, \ldots, D_{i+1} \geq y - s_i^*) \\
& -\alpha^{L_{1,i}} (H_1 + b) P(D_{1,i+1} \geq y, \ldots, D_{i+1} \geq y - s_i^*) \\
= \; & h_{i+1} + \alpha^{L_i} \int_{y-s_i^*}^{\infty} \Big\{ h_i + \sum_{j=1}^{i-1} \alpha^{L_{j,i-1}} h_j P(D_{j+1,i} + t \geq y - s_j^*, \ldots, D_i + t \geq y - s_{i-1}^*) \\
& -\alpha^{L_{1,i-1}} (H_1 + b) P(D_{1,i} + t \geq y, \ldots, D_i + t \geq y - s_{i-1}^*) \Big\} dF_{i+1}(t) \\
\leq \; & h_{i+1} + \alpha^{L_i} \int_{y-s_i^*}^{\infty} \Big\{ \sum_{j=1}^{i} \alpha^{L_{j,i-1}} h_j - \alpha^{L_{1,i-1}} (H_1 + b) P(D_{1,i} + t \geq y) \Big\} dF_{i+1}(t) \\
\leq \; & h_{i+1} + \int_{0}^{\infty} \Big\{ \sum_{j=1}^{i} \alpha^{L_{j,i}} h_j - \alpha^{L_{1,i}} (H_1 + b) P(D_{1,i} + t \geq y) \Big\} dF_{i+1}(t) \\
= \; & \sum_{j=1}^{i+1} \alpha^{L_{j,i}} h_j - \alpha^{L_{1,i}} (H_1 + b) P(D_{1,i+1} \geq y),
\end{aligned}$$

where the first inequality follows from induction hypothesis, and the second inequality follows the same argument as that in proving Proposition 1. Thus we complete the induction proof for the lower bound. □

We note that the lower bound for the discounted cost case has been obtained by Dong and Lee (2003). The upper bound is, to the best of our knowledge, new.

For average cost case, we can extend the results by let $\alpha = 1$ and we just summarize the results and omit the proof.

Proposition 7.2.2 *(Shang and Song (2003)) An upper bound for the optimal echelon base-stock level of Clark-Scarf model with average cost criterion is*

$$s_i^u = \bar{F}_{1,i}^{-1}\left(\frac{h_i}{\sum_{j=i}^N h_j + b}\right), \qquad i = 1, \ldots, N, \tag{7.13}$$

and a lower bound for the optimal echelon base-stock level is

$$s_i^l = \bar{F}_{1,i}^{-1}\left(\frac{\sum_{j=1}^i h_j}{\sum_{j=1}^N h_j + b}\right), \qquad i = 1, \ldots, N. \tag{7.14}$$

Proposition 7.2.3 *An upper bound for s_i^* is*

$$\hat{s}_i^u = s_{i-1}^* + \bar{F}_i^{-1}\left(\frac{h_i}{\sum_{j=i}^N h_j + b}\right), \qquad i = 1, \ldots, N. \tag{7.15}$$

Inductively we obtain another simple upper bound for s_i^*

$$\tilde{s}_i^u = \sum_{j=1}^{i} \bar{F}_j^{-1}\left(\frac{h_j}{\sum_{k=j}^N h_k + b}\right), \qquad i = 1, \ldots, N. \tag{7.16}$$

Remark 7.2.1 One might also wish to obtain a lower bound for s_i^* in the form of s_{i-1}^* plus a nonnegative number. We argue that this is not possible. It is known that the solution obtained from the computational algorithm (7.1), (7.2) and (7.3) may not satisfy relationship $s_1^* \leq s_2^* \leq \cdots \leq s_N^*$, see for example Gallego and Ozer (2004). Hence in case $s_i^* < s_{i-1}^*$ then s_{i-1}^* is already an upper bound for s_i^* thus it is not possible to give a lower bound of s_i^* in the form s_{i-1}^* plus a nonnegative number. It is also well-known that, in that case, we can define $\bar{s}_i^* = \min\{s_i^*, s_{i+1}^*, \ldots, s_N^*\}$ to give an optimal policy that satisfies $\bar{s}_1^* \leq \bar{s}_2^* \leq \cdots \leq \bar{s}_N^*$.

Remark 7.2.2 It is easily seen from the proof of Proposition 2 that, for any $1 \leq k \leq i$, we have

$$
\begin{aligned}
& h_i + \sum_{j=1}^{i-1} h_j P(D_{j+1,i} \geq y - s_j^*, D_{j+2,i} \geq y - s_{j+1}^*, \ldots, D_i \geq y - s_{i-1}^*) \\
& -(H_1 + b) P(D_{1,i} \geq y, D_{2,i} \geq y - s_1^*, \ldots, D_i \geq y - s_{i-1}^*) \\
\geq \ & h_i - \Big(\sum_{j=i}^{N} h_j + b \Big) P(D_{1,k} \geq y - s_{k-1}^*).
\end{aligned}
$$

Therefore we have a sequence of upper bounds for s_i^*: For $k = 1, \ldots, i$,

$$
\hat{s}_i^u = s_{k-1}^* + \bar{F}_{1,k}^{-1} \left(\frac{h_i}{\sum_{j=i}^{N} h_j + b} \right), \qquad i = 1, \ldots, N,
$$

where s_0^* is understood as 0.

7.3 Numerical Results

In this section we present some numerical examples. Since for the average cost case numerical studies have been done extensively in Shang and Song (2003) and Gallego and Ozer (2004), in this study we only numerically compare the two upper bounds for the average cost case. For the discounted cost case, we compare the optimal solutions with both the lower and upper bounds. We also develop a simple heuristic for the discounted cost case using the lower and upper bounds and demonstrate the performance of the heuristic.

Table 1 compares the optimal base stock levels, the upper bound of Shang and Song (2003) and the upper bound we develop for a four-stage system with $L_2 = L_3 = L_4 = 0.25$, $h_2 = h_3 = h_4 = 0.25$, $b = 9$ and Poisson demand with rate $\lambda = 16$. The echelon holding cost takes values 5 and 35. The transportation leadtime of stage 1 increases at the increments of 0.5, it goes from 0.5 to 3. As we observed earlier, none of these two bounds dominate the other. In all these examples our upper bound for optimal policy of stage 2 is better than that of Shang and Song (2003). Furthermore, it appears that our upper bound is getting tighter than Shang and Song (2003) as transportation leadtime/holding cost of stage 1 increases.

Table 7.1: Optimal and Two Upper Bound Policies: Average Cost

h_1	L_1	s_1^*	s_2^*	s_3^*	s_4^*	s_1^u	s_2^u	s_3^u	s_4^u	$\hat{s}_1^u$	$\hat{s}_2^u$	$\hat{s}_3^u$	$\hat{s}_4^u$	$\tilde{s}_1^u$	$\tilde{s}_2^u$	$\tilde{s}_3^u$	$\tilde{s}_4^u$
5	0.5	9	16	20	25	9	19	24	29	9	17	24	28	9	17	25	33
	1	18	24	28	33	18	30	34	39	18	27	33	37	18	27	34	42
	1.5	26	32	36	41	26	38	43	48	26	34	40	44	26	34	41	49
	2	34	40	44	49	34	48	52	57	34	42	48	52	34	42	50	58
	2.5	42	48	52	56	42	57	61	66	42	50	56	60	42	50	58	66
	3	51	56	60	65	51	66	70	75	51	60	65	69	51	60	68	75
35	0.5	6	13	19	24	6	19	24	29	6	14	21	27	6	14	21	29
	1	13	20	25	29	13	29	33	38	13	21	28	33	13	21	29	36
	1.5	20	27	32	36	20	38	43	48	20	28	35	40	20	28	36	44
	2	28	34	39	43	28	48	52	57	28	36	42	47	28	36	43	51
	2.5	35	42	46	51	35	57	61	66	35	43	50	54	35	42	50	58
	3	43	49	54	58	43	66	70	75	43	51	57	62	43	51	58	66

In Table 2, we provide the lower and upper bounds for the discounted cost case for a four-stage system with $\lambda = 16$, $\alpha = 0.9$, $L_1 = L_2 = L_3 = L_4 = 0.25$. The upper bound showed in the table is the smaller one of two upper bounds we construct, i.e. $\min\{s_i^u, \tilde{s}_i^u\}$. The shortage cost takes two values, 9 and 99 respectively, and the echelon holding costs take values between 1 and 5.

Several heuristics can be constructed for the optimal base-stock levels, using the approaches of Shang and Song (2003) and Gallego and Ozer (2004). For example, any convex combination of s_i^l and s_i^u,

$$as_i^l + (1-a)s_i^u \qquad 0 \le a \le 1$$

Table 7.2: Lower and Upper Bound/Optimal Policies: Discounted Cost

b	h_1	h_2	h_3	h_4	s_1^*	s_2^*	s_3^*	s_4^*	s_1^l	s_2^l	s_3^l	s_4^l	s_1^u	s_2^u	s_3^u	s_4^u
9	1	1	1	1	7	11	15	19	7	10	14	18	7	12	16	21
	1	1	1	5	7	12	16	16	7	11	15	16	7	12	17	16
	1	1	5	1	7	12	13	17	7	11	12	16	7	12	13	20
	1	1	5	5	8	12	13	15	8	11	13	15	8	12	13	16
	1	5	1	1	7	9	14	18	7	9	12	16	7	9	16	20
	1	5	1	5	8	9	15	15	8	9	13	15	8	9	17	16
	1	5	5	1	8	9	12	16	8	9	11	15	8	9	13	20
	1	5	5	5	8	10	12	14	8	9	12	14	8	10	13	16
	5	1	1	1	5	10	14	18	5	8	12	16	5	10	16	20
	5	1	1	5	5	11	15	16	5	9	13	15	5	12	17	16
	5	1	5	1	5	11	12	17	5	9	11	15	5	12	13	20
	5	1	5	5	6	11	13	15	6	9	12	14	6	12	13	16
	5	5	1	1	5	8	13	17	5	8	11	15	5	9	15	20
	5	5	1	5	6	9	14	15	5	8	12	14	6	9	17	16
	5	5	5	1	6	9	11	16	5	8	11	14	6	9	13	20
	5	5	5	5	6	9	12	14	5	9	11	14	6	10	13	16
99	1	1	1	1	9	15	20	24	9	14	19	23	9	15	20	25
	1	1	1	5	9	15	20	22	9	14	19	22	9	15	20	22
	1	1	5	1	9	15	17	23	9	14	17	22	9	15	17	25
	1	1	5	5	9	15	18	21	9	14	17	21	9	15	18	22
	1	5	1	1	9	13	19	24	9	12	17	22	9	13	20	25
	1	5	1	5	9	13	19	22	9	12	17	21	9	13	20	22
	1	5	5	1	9	13	17	23	9	12	16	21	9	13	17	25
	1	5	5	5	9	13	17	21	9	12	16	20	9	13	18	22
	5	1	1	1	8	14	19	24	8	12	17	22	6	15	20	25
	5	1	1	5	8	14	19	22	8	12	17	21	8	15	20	22
	5	1	5	1	8	14	17	23	8	12	16	21	8	15	17	25
	5	1	5	5	8	14	17	21	8	12	16	20	8	15	18	22
	5	5	1	1	8	12	19	24	8	11	16	21	8	13	20	25
	5	5	1	5	8	12	19	21	8	12	16	20	8	13	20	22
	5	5	5	1	8	12	17	23	8	12	16	20	8	13	17	25
	5	5	5	5	8	12	17	21	8	12	16	19	8	13	18	22

can be used to approximate s_i^*. In Table 3, we show the effective of the heuristic for the discounted cost case with a 4-stage system with $L_1 = L_2 = L_3 = L_4 = 1$, $\lambda = 16$, $\alpha = 0.9$. The initial inventory level for each stage is 0. Again the shortage cost value is either 9 or 99, and the echelon holding costs range from 1 to 5. We use the simple average

$$s_i^h = \frac{s_i^u + s_i^l}{2},$$

where s_i^u is again the smaller one of the two upper bounds. If the average is not an integer, we can either round up or round down the value to obtain the nearest integer value for s_i^h. In our numerical examples, we round down the noninteger value. The relative error is defined as

$$Error = \frac{C(\mathbf{s}^h) - C(\mathbf{s}^*)}{C(\mathbf{s}^*)}.$$

From the results in Table 3, the largest relative error among 32 examples is 0.4%, and for the majority of the cases the relative error is 0.1% or less. Thus the heuristic policy performs extremely well and it appears to be quite robust to the system parameters.

7.4 Summary

In this chapter we present an explicit form solution for the optimal echelon base-stock policy of Clark-Scarf model for both average cost and discounted cost criteria. These simple expressions clearly identify the key determinants of the optimal optimal policy, and they provide a unified framework to construct simple bounds and approximations of the optimal solutions. We illustrate the lower bound of Dong and Lee (2003) for the discounted cost and the lower and upper bounds of Shang and Song (2003) for the average cost. We also present new upper bounds for both average cost and discounted cost criteria. Some simple heuristics are developed using the lower and upper bounds, and our numerical examples show that the heuristic performs extremely well.

Similar results can be obtained for assembly systems by following the approach of

Table 7.3: Optimal and Heuristic Policies: Discounted Cost

b	h_1	h_2	h_3	h_4	s_1^*	s_2^*	s_3^*	s_4^*	s_1^h	s_2^h	s_3^h	s_4^h	$C(\mathbf{s}^*)$	$C(\mathbf{s}^h)$	*Error*
9	1	1	1	1	22	38	54	68	22	38	54	70	1828.201	1833.528	0.003
	1	1	1	5	22	39	55	62	22	39	55	64	4390.518	4401.756	0.003
	1	1	5	1	22	39	48	64	22	39	49	66	5819.664	5826.200	0.001
	1	1	5	5	23	40	50	59	22	40	50	63	9534.493	9573.647	0.004
	1	5	1	1	22	34	51	66	22	34	52	67	9541.899	9551.240	0.001
	1	5	1	5	23	35	52	60	23	35	54	63	13350.400	13383.102	0.002
	1	5	5	1	23	35	46	63	23	35	48	64	15995.961	16019.823	0.001
	1	5	5	5	23	36	48	58	23	36	49	62	20946.339	21018.448	0.003
	5	1	1	1	18	36	52	67	18	36	52	68	19159.780	19187.441	0.001
	5	1	1	5	19	38	54	61	19	37	54	63	22060.981	22099.775	0.002
	5	1	5	1	19	38	47	63	19	37	48	64	23807.288	23843.264	0.002
	5	1	5	5	19	38	49	59	19	38	49	61	27873.758	27922.995	0.002
	5	5	1	1	19	32	50	65	18	32	49	66	28156.790	28198.230	0.001
	5	5	1	5	19	33	51	60	19	34	51	62	32272.309	32327.803	0.002
	5	5	5	1	19	33	45	62	19	34	47	63	35204.752	35254.394	0.001
	5	5	5	5	20	34	47	58	20	34	49	60	40475.164	40556.118	0.002
99	1	1	1	1	26	45	63	80	26	44	62	80	2290.485	2293.122	0.001
	1	1	1	5	26	45	63	75	26	44	63	76	5399.932	5410.304	0.002
	1	1	5	1	26	45	58	78	26	44	59	79	7251.390	7263.704	0.002
	1	1	5	5	26	45	58	73	26	44	59	75	11748.364	11773.185	0.002
	1	5	1	1	26	41	61	79	26	41	61	79	11984.222	11997.709	0.001
	1	5	1	5	26	41	61	74	26	41	61	75	16698.802	16717.228	0.001
	1	5	5	1	26	41	57	77	26	41	58	78	20137.901	20158.278	0.001
	1	5	5	5	26	41	57	73	26	41	58	74	26210.046	26240.049	0.001
	5	1	1	1	23	44	62	79	23	43	61	79	24391.987	24416.964	0.001
	5	1	1	5	23	44	62	74	23	43	61	75	28190.038	28220.166	0.001
	5	1	5	1	23	44	58	77	23	43	58	78	30721.910	30755.374	0.001
	5	1	5	5	23	44	58	73	23	43	58	74	35892.483	35933.339	0.001
	5	5	1	1	23	40	61	78	23	40	60	78	36794.493	36830.219	0.001
	5	5	1	5	23	40	61	74	23	40	60	74	42159.527	42195.904	0.001
	5	5	5	1	23	40	57	76	23	40	57	77	46243.995	46280.688	0.001
	5	5	5	5	23	41	57	72	23	40	57	74	52955.831	53007.861	0.001

Rosling (1989) to convert assembly system to serial systems.

We point out that the results in this chapter apply to periodic-review systems with i.i.d. demand. However, we need to change $\bar{F}_{1,i}$ to $\bar{F}_{(1,i)+1}$. This is because, for periodic review systems we make ordering decision at the beginning of a period and charge holding and backlog cost at the end of the period.

Chapter 8

Newsvendor Bounds and Heuristics for Optimal Policy of Serial Inventory System with Regular and Expedited Shippings

In this chapter, we consider an N-stage serial production/distribution system with stationary demand. There are two transportation modes between stages: the regular and expedited shippings. The optimal inventory policy for this system is known to be echelon base-stock policy, which can be computed through minimizing $2N$ nested convex functions recursively. To identity the key determinants of the optimal policy, we develop simple newsvendor type of lower and upper bounds for the control parameters, as well as simple near optimal heuristics. Extensive numerical results show that the heuristic performs well. The bounds and heuristic enhance the accessibility and implementability of the optimal policies in supply chains with multiple transportation modes.

8.1 Probabilistic Solution

Consider an infinite-horizon periodic-review serial inventory system with dual transportation modes. There are N stages, denoted by $1, 2, \ldots, N$, stage i orders from stage $i+1$ ($i = 1, \ldots, N-1$), and stage N orders from an external supplier with unlimited stock. Intuitively, we can imagine that there are two managers at each stage: The expedited order manager is responsible for the expedited ordering and the regular order manager is responsible for the regular ordering. Demand occurs only at stage 1, and excess demand is fully backlogged at stage 1. The regular order has leadtime 1, and the expedited order has leadtime 0. The demands in different periods are i.i.d. random variables. At the beginning of each period, the firm decides the ordering quantities for two supply options at each stage. The objective is to minimize the total discounted cost over an infinite planning horizon.

The events sequence is as follows: First at the beginning of the period, each stage receives the regular order placed in the previous period; second, emergency order is placed from its upstream stage which is delivered immediately; third, regular order is placed from the upstream which will be delivered at the beginning of next period; finally, demand is realized at stage 1 and all costs are calculated.

For $i = 1, 2, \ldots, N$, define:

x_i = initial echelon inventory level at stage i;

y_i^E = echelon inventory level at stage i after placing the expedited order;

y_i^R = echelon inventory position at stage i after placing the regular order;

$\bar{c}_i^E$ = unit expedited shipping cost from stage $i+1$ to stage i;

$\bar{c}_i^R$ = unit regular shipping cost from stage $i+1$ to stage i, i.e., $\bar{c}_i^R < \bar{c}_i^E$;

h_i = unit echelon i inventory holding cost per period;

H_i = unit installation i inventory holding cost per period, s.t., $H_i = \sum_{j=i}^N h_j$.

b = unit demand backlog cost per period;

D_j = demand in period j, $j = 1, 2, \ldots$;

D = generic one-period demand;

$F(.)$ = cumulative distribution function of D;

$\bar{F}(.) = 1 - F(.)$;

$D(j)$ = j-period demand, $j = 1, 2, \ldots$;

$F_j(.)$ = cumulative distribution function of $D(j)$, $j = 1, 2, \ldots$;

$\bar{F}_j(.) = 1 - F_j(.)$, $j = 1, 2, \ldots$.

As explained in Lawson and Porteus (2000), x_i denotes the sum of on-hand stock at stages 1 and i, less the backlog at stage 1; y_i^E is the echelon stock level of stage i after all expediting at stage i and upstream stages, but before expediting into stage stage $i-1$, have taken place. Similarly, y_i^R is the echelon inventory level of stage i after both expedited order and regular order from stage $i+1$. Clearly, $y_i^R - y_i^E \geq 0$ represents the number of regular units placed into the regular flow from stage $i+1$, while $y_i^E - x_i \geq 0$ represents the number of units expedited to stage i from stage $i+1$. Since a product can be expedited to stage i in no time through expedition between stages, $y_i^E - x_i$ has no upper limit. Also, note that $D(1) = D$ and $F_1 = F$. The state of the system at the beginning of a period, before any decision is made, is $\mathbf{x} = (x_1, \ldots, x_N)$.

The following result is easily verified, and it is originally due to Karush (1958).

Lemma 8.1.1 *Let $g(x)$ be a convex function with a minimizer s, then for any $x \leq y$,*

$$\min_{x \leq z \leq y} g(z) = g(x \vee s) + g(y \wedge s) - g(s).$$

Note that $g(x \vee R)$ is an increasing convex function of x while $g(y \wedge R)$ is a decreasing convex function of y.

Let $f(\mathbf{x})$ be the minimum expected total discounted cost given initial echelon inventory level $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. Let $L_1(x) = h_1 E[x - D] + (H_1 + b)E[(x - D)^-]$ and $L_i(x) = h_i[x - D]$ for $i > 1$. The optimality equation is

$$f(\mathbf{x}) = \min_{x_i \leq y_i^E \leq y_i^R \leq y_{i+1}^E} \left\{ \sum_{i=1}^{N} (\bar{c}_i^E (y_i^E - x_i) + \bar{c}_i^R (y_i^R - y_i^E) + L_i(y_i^E)) + \alpha E[f(\mathbf{y}^R - D)] \right\} \quad (8.1)$$

where $\mathbf{y}^R = (y_1^R, \ldots, y_N^R)$ and $\mathbf{y}^R - D = (y_1^R - D, \ldots, y_N^R - D)$. For ease of exposition, we shift the cost $-\bar{c}_i^E x$ to the previous period and after some simple algebra, we obtain

$$\begin{aligned} f(\mathbf{x}) \quad = \quad & \min_{x_i \leq y_i^E \leq y_i^R \leq y_{i+1}^E} \left\{ \sum_{i=1}^{N} ((\bar{c}_i^E - \bar{c}_i^R) y_i^E + \alpha c_i^E E[D] + (\bar{c}_i^R - \alpha \bar{c}_i^E) y_i^R + h_i E[(y_i^E - D)]) \right. \\ & \left. + L(y_1^E) + \alpha E[f(\mathbf{y}^R - D)] \right\}. \end{aligned}$$

Let $c_i^E = \bar{c}_i^E - \bar{c}_i^R + h_i > 0$ and $c_i^R = \alpha \bar{c}_i^E - \bar{c}_i^R > 0$, the reason for $c_i^R > 0$ is that otherwise the regular order will never be used and the model collapse to the model with single supply mode which is not the interest of this chapter. We call c_i^E the relative unit expedited ordering cost and c_i^R the relative unit regular ordering cost.

By suppressing the terms that do not affect the optimization, we can finally rewrite the optimality equation as,

$$
\begin{aligned}
f(\mathbf{x}) &= \min_{x_i \le y_i^E \le y_i^R \le y_{i+1}^E} \left\{ \sum_{i=1}^{N} (c_i^E y_i^E - c_i^R y_i^R) + (b + H_1) E[(y_1^E - D)^-] \right. \\
&\qquad \left. + \alpha E[f(\mathbf{y}^R - D)] \right\} \qquad (8.2)
\end{aligned}
$$

It can be shown that $f(\mathbf{x})$ is additively convex,

$$
f(\mathbf{x}) = \sum_{i=1}^{N} f_i(x_i), \qquad (8.3)
$$

where each $f_i(x_i)$ is a convex function (see Lawson and Porteus (2000)). Let $G_1^E(y) = c_1^E y + (H_1 + b)E[(y - D)^-]$, which is a convex function with minimizer s_1^E, a finite number. Applying Lemma 1 to (8.2) yields $f_1(x_1) = G_1^E(x_1 \vee s_1^E)$. Let

$$
G_{1,1}(y) = G_1^E(y \wedge s_1^E) - G_1^E(s_1^E) + \alpha E[G_1^E((y - D) \vee s_1^E)],
$$

referred to as the induced penalty cost, and

$$
G_1^R(y) = G_{1,1}^E(y) - c_1^R y.
$$

Let s_1^R be the minimizer of convex function $G_1^R(.)$. Substituting (8.3) into (8.2) and applying Lemma 1 yields,

$$
\begin{aligned}
& \min_{x_1 \le y_1^E \le y_1^R \le y_2^E} \{G_1^E(y_1^E) - c_1^R y_1^R + \alpha E[f_1(y_1^R - D)]\} \\
= \; & G_1^E(x_1 \vee s_1^E) + \min_{y_1^R \le y_2^E} \{-c_1^R y_1^R + G_{1,1}^E(y_1^R)\} \\
= \; & G_1^E(x_1 \vee s_1^E) + \min_{y_1^R \le y_2^E} G_1^R(y_1^R) \\
= \; & G_1^E(x_1 \vee s_1^E) + G_1^R(y_1^R \wedge s_1^R).
\end{aligned}
$$

Let $G_{1,2}(y) = G_1^R(y_1^R \wedge s_1^R)$, also called reduced cost, and

$$G_2^E(y) = c_2^E y + G_{1,2}^R(y).$$

Let s_2^E be the minimizer of convex function $G_2^E(.)$. This process can be continued and we obtain, in general for $i \geq 1$, after G_i^E is defined with minimizer s_i^E, that

$$\begin{aligned} G_{i,i}(y) &= G_i^E(y \wedge s_i^E) - G_i^E(s_i^E) + \alpha E[G_i^E((y-D) \vee s_i^E)], && (8.4) \\ G_i^R(y) &= G_{i,i}(y) - c_i^R y, && (8.5) \\ G_{i,i+1}(y) &= G_i^R(y \wedge s_i^R), && (8.6) \\ G_{i+1}^E(y) &= c_{i+1}^E y + G_{i,i+1}(y). && (8.7) \end{aligned}$$

And that, for all $i \geq 1$,

$$f_i(x_i) = G_i^E(x_i \vee s_i^E).$$

Note that all these functions are convex, and in particular, $G_{i,i+1}$ is decreasing convex and f_i is increasing convex.

The optimal policy for this system is top-down echelon base-stock policies (Lawson and Porteus (2000)). The top-down base-stock policy works as follows. Each stage tries to raise its echelon inventory position to the expedited order-up-to level S_i^E and regular order-up-to level S_i^R, taking upstream decisions as fixed and ignoring downstream decisions. More formally, a policy with $2N$ base-stock levels is a top-down base-stock policy if the actual decisions can be constructed from the base-stock levels as follows.

$$\begin{aligned} &y_N^R = s_N^R \vee x_N, \\ &y_i^E = s_i^E \vee x_i \wedge y_i^R \qquad i = 1, 2, \ldots, N, \\ &y_i^R = s_i^R \vee x_i \wedge y_{i+1}^E \qquad i = 1, 2, \ldots, N-1. \end{aligned}$$

Lemma 8.1.2 *(1) $s_i^E \le s_{i-1}^R$, for $i = 2, \ldots, N$.*
(2) $s_i^E \le s_i^R$, for $i = 1, \ldots, N$.

Proof. As s_i^E is determined by

$$c_i^E + G'_{i-1,i}(y) = 0, \tag{8.8}$$

it follows from $G'_{i-1,i}(y) = 0$ on $y \ge s_{i-1}^R$ and the increasingness of the left hand side of (8.8) that $s_i^E \le s_{i-1}^R$.

Similarly, s_i^R is the solution of

$$-c_i^R + G'_{i,i}(y) = 0. \tag{8.9}$$

It can be seen that $EG'_{i,i}(y - D) \le 0$ on $y \le s_i^E$, thus $-c_i^R + G'_{i,i}(y) < 0$, implying $s_i^R \ge s_i^E$. □

We can develop probabilistic solutions for the optimal base-stock levels s_i^E and s_i^R. From equations (8.4)-(8.7), the optimal control parameters s_i^E and s_i^R are, respectively, the solution of $(G_i^E(y))' = 0$ and $(G_i^R(y))' = 0$. Let $D_1, D_2, \ldots$ demands in periods be $1, 2, \ldots$. For stage 1, taking derivative of $G_1^E(y)$ with respect to y yields

$$c_1^E - (H_1 + b)P(D_1 > y) = 0,$$

hence the optimal expedited base-stock level for stage 1 is

$$s_1^E = \bar{F}^{-1}\left(\frac{c_1^E}{H_1 + b}\right). \tag{8.10}$$

Note that if $c_1^E \ge H_1 + b$, then $s_1^E = -\infty$ and expedited shipping is never used at stage 1. To solve for s_1^R, it follows from Lemma 8.1.2 that we only need to consider the solution of $(G_1^E(y))' = 0$ on $y \geq s_1^E$. It follows from (8.4) that s_1^R is the solution of

$$-c_1^R + \alpha c_1^E P(D_2 \leq y - s_1^E) - \alpha(H_1 + b)P(D_2 \leq y - s_1^E, D_1 + D_2 > y) = 0. \quad (8.11)$$

Some further algebraic derivations yield that s_2^E is the solution of

$$\begin{aligned}
& c_2^E - c_1^R + c_1^E \mathbf{1}[y < s_1^E] + \alpha c_1^E P(D \leq y - s_1^E) \\
& -(H_1 + b)P(D > y)\mathbf{1}[y < s_1^E] - \alpha(H_1 + b)P(D \leq y - s_1^E, D(2) > y) \\
= \; & 0,
\end{aligned}$$

that s_2^R is the solution of

$$\begin{aligned}
& -c_2^R + \alpha c_2^E P(D \leq y - s_2^E) - \alpha c_1^R(D \leq y - s_2^E, D > y - s_1^R) \\
& +\alpha c_1^E P(D \leq y - s_2^E, D > y - s_1^R, D > y - s_1^E) \\
& +\alpha^2 c_1^E P(D \leq y - s_2^E, D > y - s_1^R, D(2) \leq y - s_1^E) \\
& -\alpha(H_1 + b)P(D \leq y - s_2^E, D > y - s_1^E, D > y - s_1^R, D(2) > y)) \\
& -\alpha^2(H_1 + b)P(D \leq y - s_2^E, D > y - s_1^R, D(2) \leq y - s_1^E, D(3) > y) \\
= \; & 0,
\end{aligned}$$

and that s_3^E is the solution of

$$
\begin{aligned}
& c_3^E - c_2^R + (c_2^E - c_1^R)\mathbf{1}[y < s_2^E] + c_1^E\mathbf{1}[y < s_1^E, y < s_2^E] + \alpha c_1^E P(D \le y - s_1^E)\mathbf{1}[y < s_2^E] \\
& +\alpha c_2^E P(D \le y - s_2^E) - \alpha c_1^R(D \le y - s_2^E, D > y - s_1^R) \\
& +\alpha c_1^E P(D \le y - s_2^E, D > y - s_1^R, D > y - s_1^E) \\
& +\alpha^2 c_1^E P(D \le y - s_2^E, D > y - s_1^R, D(2) \le y - s_1^E) \\
& -(H_1 + b)P(D > y)\mathbf{1}[y < s_1^E, y < s_2^E] \\
& -\alpha(H_1 + b)P(D \le y - s_1^E, D(2) > y)\mathbf{1}[y < s_2^E] \\
& -\alpha(H_1 + b)P(D \le y - s_2^E, D > y - s_1^R, D > y - s_1^E, D(2) > y)) \\
& -\alpha^2(H_1 + b)P(D \le y - s_2^E, D > y - s_1^R, D(2) \le y - s_1^E, D(3) > y) \\
= \; & 0.
\end{aligned}
$$

This process can be continued to give a probability solution for s_i^E and s_i^R for any i. As can be expected, the expression becomes extremely complicated as i increases. The detailed derivation of these equations are in the appendix.

The following lemma presents the dependency of policy parameters on system parameters.

Lemma 8.1.3 *(1) s_i^E is decreasing in c_j^E for $j \le i$, independent of c_j^E for $j > i$, increasing in c_j^R for $j < i$, independent of c_j^R for $j \ge i$ and increasing in b.*

(2) s_i^R is decreasing in c_j^E for $j \le i$,independent of c_j^E for $j > i$, increasing in c_j^R for $j \le i$, independent of c_j^R for $j > i$ and b.

Proof. From equations (8.8) and (8.9) and the convexity of G_i^E and G_i^R, it is easy to see that s_i^E is decreasing in c_i^E and s_i^R is increasing in c_i^R. To prove s_i^R is decreasing in c_i^E, it suffices to prove that the left hand side of (8.9) is increasing in c_i^E. To see this, let (8.9) be written as $g(y, c_i^E) = 0$, and let $s_i^R(c_i^E)$ its solution. Recall that $g(y, c_i^E)$ is increasing in y. Suppose $c_i^E \le \bar{c}_i^E$. If $g(y, c_i^E)$ is increasing in c_i^E, then

$$g(s_i^R(\bar{c}_i^E), c_i^E) \le g(s_i^R(\bar{c}_i^E), \bar{c}_i^E) = 0.$$

Hence it follows from $g(y, c_i^E)$ is increasing in y that, $s_i^R(c_i^E)$, determined by $g(y, c_i^E) = 0$, satisfies $s_i^R(c_i^E) \ge s_i^R(\bar{c}_i^E)$. This shows that s_i^R is decreasing in c_i^E.

The left hand side of (8.9) is

$$g(y, c_i^E) = (G_i^E)'(y)\mathbf{1}[y \le s_i^E] + \alpha \int_0^{y-s_i^E} (G_i^E)'(y-t)dF(t).$$

Noting $(G_i^E)''_{y,c_i^E} = 1$, we obtain

$$\begin{aligned} g'_{c_i^E}(y, c_i^E) &= \mathbf{1}[y < s_i^E] + (G_i^E)'(y)\left(\mathbf{1}(y < s_i^E)\right)'_{c_i^E} \mathbf{1}(y < s_i^E) \\ &\quad + \alpha F(y - s_i^E) - \alpha (G_i^E)'(s_i^E) f(y - s_i^E)(s_i^E)'_{c_i^E} \\ &= \mathbf{1}[y < s_i^E] + (G_i^E)'(y)\left(\mathbf{1}[y < s_i^E]\right)'_{c_i^E} \mathbf{1}[y < s_i^E] + \alpha F(y - s_i^E) \\ &\ge 0, \end{aligned}$$

where the last inequality follows from s_i^E being the minimizer of G_i^E, and the last inequality follows from

$$(G_i^E)'(y) \le 0 \quad \text{when } y < s_i^E, \quad \text{and} \quad \left(\mathbf{1}[y < s_i^E]\right)'_{c_i^E} \le 0,$$

since s_i^E is decreasing in c_i^E. We next show that both s_i^E abd s_i^R are decreasing in c_j^E for $j < i$. Suppose $(G_i^E(y))''_{y,c_j^E} \ge 0$ for i, then for $i+1$, first take derivative of G_i^R with respect to y,

$$(G_i^R(y))' = -c_i^R + \alpha \int_0^{y-s_i^E} (G_i^E(y-\xi))' df(\xi)$$

and

$$(G_i^R(y))''_{y,c_j^E} = \alpha \int_0^{y-s_i^E} (G_i^E(y-\xi))''_{y,c_j^E} df(\xi) \geq 0$$

which imply that s_i^R is decreasing in c_j^E for $j < i$. For s_{i+1}^E, which is the solution of

$$(G_{i+1}^E(y))' = c_{i+1}^E + (G_i^R(y)))' = 0$$

and from previous result, it is clear that

$$(G_{i+1}^E(y))''_{y,c_j^E} = (G_i^R(y)))''_{y,c_j^E} \geq 0$$

which implies that s_{i+1}^E is decreasing in c_j^E for $j < i$.

As s_1^E is independent of c_1^R, we first show s_2^E is increasing in c_1^R. From (**??**),

$$(G_2^E(y))''_{y,c_1^R} = -1 < 0$$

which implies that s_2^E is increasing in c_1^R. Now suppose $(G_i^E(y))''_{y,c_j^R} < 0$ for $j < i$, then for s_i^R,

$$(G_i^R(y))'_{y,c_j^R} = \alpha \int_0^{y-s_i^E} (G_i^E(y-\xi))''_{y,c_j^R} df(\xi) < 0$$

and

$$(G_{i+1}^E(y))'_{y,c_j^R} = (G_i^R(y))''_{y,c_j^R} < 0.$$

Therefore, both both s_i^E and s_i^R are increasing in c_j^R for $j < i$.

Next, we show both s_i^E and s_i^R are increasing in b by induction. It is clear that s_1^E is increasing in b from (8.10). To show s_1^R is increasing in b, it suffices to show that the left hand side of (8.11) is decreasing in b. Let

$$g(y,b) = c_1^E P(D \le y - s_1^E) - (H_1 + b)P(D \le y - s_1^E, D(2) \ge y)$$

and it suffices to prove $g_b'(y,b) \le 0$. Taking derivative with respect to b yields

$$\begin{aligned}
g_b'(y,b) &= \left(\int_0^{y-s_1^E} (c_1^E - (H_1+b)\bar{F}(y-t)dF(t)\right)' \\
&= (y-s_1^E)'(c_1^E - (H_1+b)\bar{F}(s_1^E))F'(y-s_1^E) \\
&\quad - \int_0^{y-s_1^E} \bar{F}(y-t)dF(t) \\
&= -\int_0^{y-s_1^E} \bar{F}(y-t)dF(t) \\
&\le 0.
\end{aligned}$$

Suppose we have proved $(G_i^E(y))'$ and $G_i^R(y))'$ have been proved to be increasing in b, thus s_i^E and s_i^R are increasing in b. We need to show that $(G_{i+1}^E)'$ and $(G_{i+1}^R(y)$ are also decreasing in b. We have

$$(G_{i+1}^E(y))_b' = c_{i+1}^E + G_{i,i+1}'(y) = c_{i+1}^E + (G_i^R)'(y)\mathbf{1}[y < s_i^R].$$

Take derivative with respect to b,

$$\begin{aligned}
(G_{i+1}^E(y))_{y,b}'' &= \left(G_i^R(y)\right)_{y,b}'' \mathbf{1}(y < s_i^R) + (G_i^R)'(y)\left(\mathbf{1}[y < s_i^R]\right)_b' \mathbf{1}[y < s_i^R], \\
&\le 0
\end{aligned}$$

where the inequality follows from the inductive assumption, $G_i^{R'}(y) \le 0$ when $y \le s_i^R$, and

$$\left(\mathbf{1}[y < s_i^R]\right)'_b \geq 0$$

because s_i^R is increasing in b. Thus, s_{i+1}^E is decreasing in b.

Similarly, we have

$$\begin{aligned}
(G_{i+1}^R(y))'_y &= -c_{i+1}^R + G'_{i+1,i+1}(y) \\
&= -c_{i+1}^R + (G_{i+1}^E)'(y)\mathbf{1}[y < s_{i+1}^E] + \alpha E\Big[(G_{i+1}^E)'((y-D))\mathbf{1}[y-D > s_{i+1}^E]\Big].
\end{aligned}$$

Taking derivative with respect to b yields

$$\begin{aligned}
&\left(G_{i+1}^E(y)\right)''_{y,b}\mathbf{1}[y < s_{i+1}^E] + (G_{i+1}^E)'(y)\left(\mathbf{1}[y < s_{i+1}^E]\right)'_b \\
&+\alpha\left[\int_0^{y-s_{i+1}^E}\left(G_{i+1}^E((y-t))\right)''_{y,b} dF(t) - (G_{i+1}^E)'(s_{i+1}^E)f(y-s_{i+1}^E)(s_{i+1}^E)'_b\right] \\
= \; &\left(G_{i+1}^E(y)\right)''_{y,b}\mathbf{1}[y < s_{i+1}^E] + (G_{i+1}^E)'(y)\left(\mathbf{1}[y < s_{i+1}^E]\right)'_b \\
&+\alpha E\left[\left(G_{i+1}^E((y-D))\right)''_{y,b}\mathbf{1}[y-D > s_{i+1}^E]\right] \\
\leq \; & 0
\end{aligned}$$

where the equality follows from the optimality of s_{i+1}^E and the inequality follows from the similar analysis as in the previous discussion. Thus, we complete the induction proof that s_i^E and s_i^R are increasing in b. $\square$

Thus, for each stage, if the expedition cost gets higher, then less expedition will be used and the base-stock level for expedited order becomes lower. If the expedition cost of any downstream stage gets higher, then less expedition will be used in that stage and so the expedited base-stock level of current stage becomes lower. But if the regular shipping cost gets higher, then the regular echelon base stock level becomes higher. The explanation for the latter is the same as that of a single-stage infinite horizon inventory model with periodic review and one ordering opportunity in each period, for which it is well-known that the base-stock level is increasing in purchasing cost. As a result, if downstream stage's regular shipping cost becomes higher, then the downstream stage tends to order more so both the expedited and regular order of current stage becomes higher. That s_i^R is also decreasing in c_j^E can be explained as follows: As we can consider the expedited manager is the downstream of the regular order manager, when the expedition cost is higher, less expedited are placed and consequently, the echelon base-stock level for regular shipping will also become lower. Finally, both s_i^E and s_i^R are increasing in b is intuitively clear: with higher shortage cost, then each stage should keep higher (echelon) inventory to reduce shortage cost.

Lemma 8.1.4 *For $i = 2, \ldots, N$, if $c_{i-1}^R > c_i^E$, $s_i^E \geq s_{i-1}^E$; otherwise $s_i^E \leq s_{i-1}^E$.*

Proof. Recall that s_i^E is the solution of

$$c_i^E - c_{i-1}^R + (G_{i-1}^E(y))'\mathbf{1}(y < s_{i-1}^E) + \alpha E[(G_{i-1}^E(y-D))'\mathbf{1}(y - D > s_{i-1}^E)] = 0.$$

If $c_{i-1}^R > c_i^E$ and $s_i^E < s_{i-1}^E$,

$$c_i^E - c_{i-1}^R + (G_{i-1}^E(y))' < 0$$

which implies that $s_i^E = \infty$, which is a contradiction. So $s_i^E \geq s_{i-1}^E$.

If $c_{i-1}^R \leq c_i^E$ and $s_i^E > s_{i-1}^E$, then

$$c_i^E - c_{i-1}^R + \alpha E[(G_{i-1}^E(y-D))'\mathbf{1}(y - D > s_{i-1}^E)] \geq 0$$

which implies $s_i^E = -\infty$, again a contradiction. So $s_i^E \leq s_{i-1}^E$. □

Lemma 8.1.5 *For $i = 2, \ldots, N$, $s_i^R \geq s_{i-1}^R$, if $s_{i-1}^R - s_i^E \leq F^{-1}\left(\frac{c_i^R}{\alpha c_i^E}\right)$.*

Proof. Recall that s_i^R is the solution of

$$-c_i^R + \alpha E[(c_i^E + (G_{i-1}^R(y - D))'\mathbf{1}(y - D < s_{i-1}^R)\mathbf{1}(y - D > s_i^E)] = 0.$$

So it is sufficient to show that

$$-c_i^R + \alpha E[(c_i^E + (G_{i-1}^R(s_{i-1}^R - D)))'\mathbf{1}(s_{i-1}^R - D > s_i^E)] \leq 0,$$

which is true if $-c_i^R + \alpha c_i^E P(s_{i-1}^R - D > s_i^E) \leq 0$. Thus, if

$$P(s_{i-1}^R - D > s_i^E) \leq \frac{c_i^R}{\alpha c_i^E},$$

then $s_i^R \geq s_{i-1}^R$. $\square$

8.2 Bounds

In this section, we develop several sets of newsvendor lower bounds and upper bounds for the optimal echelon base stock levels. All the proofs are provided in the Appendix.

The basic ideas used in developing upper and lower bounds are as follows: The optimal base-stock level for emergency shipping s_i^E is determined by $(G_i^E(y))' = 0$. Since $(G_i^E(y))'$ is an increasing function of y, if we can find a function g such that $G_i^E(y))' \leq g(y)$, then the solution of $g(y) = 0$ is a lower bound for s_i^E. Similarly, if we can find a function g such that $G_i^E(y))' \geq g(y)$, then the solution of $g(y) = 0$ is an upper bound for s_i^E. The same argument applies to s_i^R which is determined by $(G_i^R(y))' = 0$.

Theorem 8.2.1 *For $i = 1, \dots, N$, the lower bounds for s_i^E and s_i^R are, respectively,*

$$\underline{s}_i^{E1} = \max\left\{ \bar{F}^{-1}\left(\frac{\sum_{j=1}^{i}(c_j^E - c_{j-1}^R)}{H_1 + b} \right), \bar{F}^{-1}\left(\frac{\sum_{j=1}^{i} \alpha^{i-j}(c_j^E - c_{j-1}^R)}{\alpha^{i-1}(H_1 + b)} \right) \right\}, \tag{8.12}$$

and

$$\underline{s}_i^{R1} = \max\left\{ \bar{F}^{-1}\left(\frac{-c_i^R + \sum_{j=1}^{i}(c_j^E - c_{j-1}^R)}{H_1 + b} \right), \bar{F}^{-1}\left(\frac{-c_i^R + \sum_{j=1}^{i} \alpha^{i-j+1}(c_j^E - c_{j-1}^R)}{\alpha^{i}(H_1 + b)} \right) \right\} \tag{8.13}$$

Proof. It is sufficient to show that for any i,

$$(G_i^E(y))' \leq \sum_{j=1}^{i}(c_j^E - c_{j-1}^R) - (H_1 + b)P(D > y), \tag{8.14}$$

$$(G_i^R(y))' \leq -c_i^R + \sum_{j=1}^{i}(c_j^E - c_{j-1}^R) - (H_1 + b)P(D > y), \tag{8.15}$$

and

$$(G_i^E(y))' \leq \sum_{j=1}^{i} \alpha^{i-j}(c_j^E - c_{j-1}^R) - \alpha^{i-1}(H_1 + b)P(D > y), \tag{8.16}$$

$$(G_i^R(y))' \leq -c_i^R + \sum_{j=1}^{i} \alpha^{i-j+1}(c_j^E - c_{j-1}^R) - \alpha^{i}(H_1 + b)P(D > y). \tag{8.17}$$

We prove these inequalities by induction. It is clear true for s_1^E. For (8.15), we have,

$$\begin{aligned}
& -c_1^R + (c_1^E - (H_1+b)P(D_2 > y))\mathbf{1}(y < s_1^E) \\
& +\alpha c_1^E P(D_1 < y - s_1^E) - \alpha(H_1+b)P(D_1 < y - s_1^E, D(2) > y) \\
\leq\; & -c_1^R + (c_1^E - (H_1+b)P(D_2 > y))\mathbf{1}(y < s_1^E) \\
& +c_1^E P(D_1 < y - s_1^E) - (H_1+b)P(D_1 < y - s_1^E, D(2) > y) \\
\leq\; & -c_1^R + E[c_1^E - (H_1+b)P(D_2 > y))\mathbf{1}(y - D_1 < s_1^E)] \\
& +c_1^E P(D_1 < y - s_1^E) - (H_1+b)P(D_1 < y - s_1^E, D(2) > y) \\
\leq\; & -c_1^R + c_1^E - H_1 + b)P(D_2 > y))\mathbf{1}(y - D_1 < s_1^E)) \\
& \qquad\qquad -(H_1+b)P(D_1 < y - s_1^E, D_2 > y) \\
\leq\; & -c_1^R + c_1^E - (H_1+b)P(D > y).
\end{aligned}$$

Assume (8.14) and (8.15) holds for i, and we prove that it holds for $i+1$. It follows from (8.4) to (8.7) that

$$\begin{aligned}
& (G_{i+1}^E(y))' \\
=\; & c_{i+1}^E + (G_i^R(y \wedge s_i^R))' \\
\leq\; & c_{i+1}^E + (G_i^R(y))' \\
\leq\; & c_{i+1}^E - c_i^R + \sum_{j=1}^{i}(c_j^E - c_{j-1}^R) - (H_1+b)P(D > y) \\
=\; & \sum_{j=1}^{i+1}(c_j^E - c_{j-1}^R) - (H_1+b)P(D > y),
\end{aligned}$$

where the first inequality follows from that $G'_{i,i+1}$ is 0 on $y \leq s_i^R$ and increasing on $y \geq s_i^R$, the second inequality follows from inductive assumption.

We then prove (8.15) for $i+1$. Note that

$$\begin{aligned}
& (G^R_{i+1}(y))' \\
= \; & -c^R_{i+1} + G'_{i+1,i+1}(y) \\
= \; & -c^R_{i+1} + (G^E_{i+1})'(y)\mathbf{1}(y < s^E_{i+1}) + \alpha E[(G^E_{i+1})'(y-D)\mathbf{1}(y-D > s^E_{i+1})] \\
\leq \; & -c^R_{i+1} + E(G^E_{i+1})'(y)\mathbf{1}(y-D < s^E_{i+1}) + E[(G^E_{i+1})'(y-D)\mathbf{1}(y-D > s^E_{i+1})] \\
\leq \; & -c^R_{i+1} + \sum_{j=1}^{i+1}(c^E_j - c^R_{j-1})P(y-D_i < s^E_{i+1}) - (H_1+b)P(D_{i+1} > y, y-D_i < s^E_{i+1}) \\
+ \; & \left(\sum_{j=1}^{i+1}(c^E_j - c^R_{j-1}))P(y-D_i > s^E_{i+1}) - (H_1+b)P(D_{i+1}+D_i > y, y-D_i > s^E_{i+1})\right) \\
\leq \; & -c^R_{i+1} + \left(\sum_{j=1}^{i+1}(c^E_j - c^R_{j-1})\right) - (H_1+b)P(D > y),
\end{aligned}$$

where the first inequality follows from $E[(G^E_{i+1})'(y-D)\mathbf{1}(y-D > s^E_{i+1}) \geq 0$ and the second inequalities follows the same lines of the previous arguments.

For (8.16) and (8.17), it is also clearly true for s^E_1, then for s^R_1

$$\begin{aligned}
& -c^R_1 + (c^E_1 - (H_1+b)P(D_2 > y))\mathbf{1}(y < s^E_1) \\
& +\alpha c^E_1 P(D_1 < y - s^E_1) - \alpha(H_1+b)P(D_1 < y - s^E_1, D(2) > y) \\
\leq \; & -c^R_1 + \alpha E[c^E_1 - (H_1+b)P(D_2 > y))\mathbf{1}(y - D_1 < s^E_1)] \\
& +\alpha c^E_1 P(D_1 < y - s^E_1) - \alpha(H_1+b)P(D_1 < y - s^E_1, D(2) > y) \\
\leq \; & -c^R_1 + \alpha c^E_1 - \alpha(H_1+b)P(D_2 > y))\mathbf{1}(y - D_1 < s^E_1)) \\
& \qquad -\alpha(H_1+b)P(D_1 < y - s^E_1, D_2 > y) \\
\leq \; & -c^R_1 + \alpha c^E_1 - \alpha(H_1+b)P(D > y).
\end{aligned}$$

Assume the (8.16) and (8.17)hold for i, and we prove that they hold for $i+1$. It follows from (8.4) to (8.7) that

$$
\begin{aligned}
& (G^E_{i+1}(y))' \\
= \; & c^E_{i+1} + (G^R_i(y \wedge s^R_i))' \\
\leq \; & c^E_{i+1} + (G^R_i(y))' \\
\leq \; & c^E_{i+1} - c^R_i + \sum_{j=1}^{i} \alpha^{i-j+1}(c^E_j - c^R_{j-1}) - \alpha^i (H_1 + b) P(D > y) \\
= \; & \sum_{j=1}^{i+1} \alpha^{i-j+1}(c^E_j - c^R_{j-1}) - \alpha^i (H_1 + b) P(D > y),
\end{aligned}
$$

where the first inequality follows from that $G'_{i,i+1}$ is 0 on $y \leq s^R_i$ and increasing on $y \geq s^R_i$, the second inequality follows from inductive assumption.

We then prove (8.17) for $i+1$. Note that

$$
\begin{aligned}
& (G^R_{i+1}(y))' \\
= \; & -c^R_{i+1} + G'_{i+1,i+1}(y) \\
= \; & -c^R_{i+1} + (G^E_{i+1})'(y)\mathbf{1}(y < s^E_{i+1}) + \alpha E[(G^E_{i+1})'(y - D)\mathbf{1}(y - D > s^E_{i+1})] \\
\leq \; & -c^R_{i+1} + \alpha (G^E_{i+1})'(y)\mathbf{1}(y < s^E_{i+1}) + \alpha E[(G^E_{i+1})'(y - D)\mathbf{1}(y - D > s^E_{i+1})] \\
\leq \; & -c^R_{i+1} + \alpha E (G^E_{i+1})'(y)\mathbf{1}(y - D < s^E_{i+1}) + \alpha E[(G^E_{i+1})'(y - D)\mathbf{1}(y - D > s^E_{i+1})] \\
\leq \; & -c^R_{i+1} + \sum_{j=1}^{i+1} \alpha^{i-j+2}(c^E_j - c^R_{j-1}) P(y - D_i < s^E_{i+1}) \\
& \qquad\qquad -\alpha^{i+1}(H_1 + b) P(D_{i+1} > y, y - D_i < s^E_{i+1}) \\
+ \; & \left(\sum_{j=1}^{i+1} \alpha^{i-j+2}(c^E_j - c^R_{j-1})) P(y - D_i > s^E_{i+1}) \right. \\
& \qquad\qquad \left. -\alpha^{i+1}(H_1 + b) P(D_{i+1} + D_i > y, y - D_i > s^E_{i+1}) \right) \\
\leq \; & -c^R_{i+1} + \left(\sum_{j=1}^{i+1} \alpha^{i-j+2}(c^E_j - c^R_{j-1}) \right) - \alpha^{i+1}(H_1 + b) P(D > y),
\end{aligned}
$$

where the inequalities follow from the same lines of the previous arguments. This completes the proof of Theorem 8.2.1. $\square$

For notational convenience, for $i = 1, \ldots, N,\ j = 1, \ldots, i$, let

$$
\begin{aligned}
A_{i,i} &= 0 \\
B_{i,j} &= c_i^R + \alpha A_{i,j}^+, \\
A_{i,j} &= -c_i^E + B_{i-1,j}.
\end{aligned}
$$

In the next theorem, we give other sets of lower bounds for s_i^E and s_i^R.

Theorem 8.2.2 *For $i = 1, 2, \ldots, N$, if $\sum_{j=1}^{i} \alpha^{i-j}(c_j^E - c_{j-1}^R)) \leq \alpha^{i-1}(H_1 + b)$, then the lower bounds for s_i^E and s_i^R are,*

$$
\underline{s}_i^{E2} = \max\left\{ F_k^{-1}\left(\frac{A_{i,i-k+1}}{\sum_{l=1}^{i-k+1} \alpha^{i-l}(c_l^E - c_{l-1}^R)}\right), k = 2, \ldots, i \right\}, \tag{8.18}
$$

$$
\underline{s}_i^{R2} = \max\left\{ F_{k+1}^{-1}\left(\frac{B_{i,i-k+1}}{\sum_{l=1}^{i-k+1} \alpha^{i-l+1}(c_l^E - c_{l-1}^R)}\right), k = 1, \ldots, i \right\}. \tag{8.19}
$$

Proof. For (8.19), we need to show for $i = 2, \ldots, N$

$$
(G_i^E(y))' \leq -A_{i,i-k+1} + \sum_{\ell=1}^{i-k+1} \alpha^{i-l}(c_l^E - c_{l-1}^R)P(D(k) \leq y), \qquad k = 2, \ldots, i-1
$$

and $i = 1, \ldots, N$

$$(G_i^R(y))' \le -B_{i,i-k+1} + \sum_{j=1}^{i} \alpha^{i-l+1}(c_j^E - c_{j-1}^R)P(D(k+1) \le y), \qquad k = 1, \ldots, i-1.$$

From Theorem 8.2.1, for $i = 1, \ldots, N$,

$$\begin{aligned}
(G_i^R(y))' &= -c_i^R + (G_i^E(y))'\mathbf{1}(y < s_i^E) + \alpha E[(G_i^E)'(y-D)\mathbf{1}(y-D > s_i^E)] \\
&\le -c_i^R + \alpha E[(G_i^E)'(y-D)\mathbf{1}(y-D > s_i^E)] \\
&\le -c_i^R + \alpha E\Big[\sum_{l=1}^{i} \alpha^{i-l}(c_l^E - c_{l-1}^R)P(y-D > s_i^E) \\
&\quad -\alpha^{i-1}(H_1+b)P(D(2) > y, y-D > s_i^E)\Big] \\
&\le -c_i^R + \alpha E\Big[\sum_{l=1}^{i} \alpha^{i-j}(c_l^E - c_{l-1}^R)P(y-D > s_i^E) \\
&\quad -\sum_{j=1}^{i} \alpha^{i-l}(c_l^E - c_{l-1}^R)P(D(2) > y, y-D > s_i^E)\Big] \\
&= -c_i^R + \alpha E\Big[\sum_{l=l}^{i} \alpha^{i-l}(c_l^E - c_{l-1}^R)P(y-D > s_i^E, D(2) \le y)\Big] \\
&\le -c_i^R + \sum_{l=1}^{i} \alpha^{i-l}(c_l^E - c_{l-1}^R)P(D(2) \le y) \\
&= -B_{i,i} + \sum_{l=1}^{i} \alpha^{i-l}(c_l^E - c_{l-1}^R)P(D(2) \le y),
\end{aligned}$$

where the third inequality follows from that $\sum_{l=1}^{i} \alpha^{i-l}(c_l^E - c_{l-1}^R) \le \alpha^{i-1}(H_1+b)$. This validates the case $k = 1$ for G_i^R. The lower bound (8.18) can be easily obtain from above derivation as

$$\begin{aligned}(G_{i+1}^E(y))' &= c_{i+1}^E + (G_i^R(y))'\mathbf{1}(y < s_i^R) \\ &\leq c_{i+1}^E + (G_i^R(y))' \\ &\leq c_{i+1}^E - c_i^R + \sum_{l=1}^{i} \alpha^{i-l}(c_j^E - c_{j-1}^R)P(D(2) \leq y) \\ &\leq -A_{i+1,i} + \sum_{l=1}^{i} \alpha^{i-l}(c_j^E - c_{j-1}^R)P(D(2) \leq y).\end{aligned}$$

This shows the case for $k = 2$ of G_i^E.

Based on these, suppose it is true for some $k = j$ for $G_i^E(\cdot)$, so

$$\begin{aligned}(G_i^R(y))' &\leq -c_i^R + \alpha E[(G_i^E)'(y - D)\mathbf{1}(y - D > s_i^E)] \\ &\leq -c_i^R + \alpha E\Big[A_{i,i-j+1}P(y - D > s_i^E) \\ &\quad + \sum_{l=1}^{i-j+1} \alpha^{i-l}(c_l^E - c_{l-1}^R)P(D(j+1) \leq y, y - D > s_i^E)\Big] \\ &\leq -B_{i,i-j+1} + \sum_{l=1}^{i-j+1} \alpha^{i-l+1}(c_l^E - c_{l-1}^R)P(D(j+1) \leq y)\end{aligned}$$

and

$$\begin{aligned}(G_{i+1}^E(y))' &= c_{i+1}^E + (G_i^R(y))'\mathbf{1}(y < s_i^R) \\ &\leq c_{i+1}^E + (G_i^R(y))' \\ &\leq c_{i+1}^E - c_i^R + \alpha A_{i,i-j+1}^{+} + \sum_{l=1}^{i} \alpha^{i-l+1}(c_l^E - c_{l-1}^R)P(D(j+1) \leq y) \\ &= A_{i+1,i-j+1} + \sum_{l=1}^{i-j+1} \alpha^{i-l+1}(c_l^E - c_{l-1}^R)P(D(j+1) \leq y).\end{aligned}$$

which implies that $(G_i^E(y))' \leq A_{i,i-j} + \sum_{l=1}^{i-j} \alpha^{i-l}(c_l^E - c_{l-1}^R)(H_1 + b)P(D(j+1) \leq y)$.

Then we use this for G_i^R,

$$\begin{aligned}(G_i^R(y))' &\leq -c_i^R + \alpha E[(G_i^E)'(y-D)\mathbf{1}(y-D > s_i^E)] \\ &\leq -c_i^R + \alpha E\Big[A_{i,i-j}P(y-D > s_i^E) \\ &\quad + \sum_{l=1}^{i-1} \alpha^{i-l}(c_l^E - c_{l-1}^R)P(D(j+2) \leq y, y-D > s_i^E)\Big] \\ &\leq -c_i^R + \alpha A_{i,i-j}^+ + \sum_{l=1}^{i-j} \alpha^{i-l+1}(c_l^E - c_{l-1}^R)(H_1+b)P(D(j+2) \leq y) \\ &= B_{i,i-j} + \sum_{l=1}^{i-j} \alpha^{i-l+1}(c_l^E - c_{l-1}^R)(H_1+b)P(D(j+2) \leq y).\end{aligned}$$

Hence, it is true for $k = j+1$. So we complete the proof. $\square$

Theorem 8.2.3 *For* $i = 2, \ldots, N$

$$\underline{s}_i^{E3} = s_{i-1}^E + \max\left\{F^{-1}\left(\frac{c_{i-1}^R - c_i^E}{\sum_{j=1}^{i-1}\alpha^{i-j}(c_j^E - c_{j-1}^R)}\right), F^{-1}\left(\frac{c_{i-1}^R - c_i^E}{\alpha c_{i-1}^E}\right)\right\} \tag{8.20}$$

if $c_{i-1}^R - c_i^E >= 0$. *And* $i = 1, \ldots, N$

$$\underline{s}_i^{R3} = s_i^E + \min\left\{F^{-1}\left(\frac{c_i^R}{\sum_{j=1}^{i}\alpha^{i-j+1}(c_j^E - c_{j-1}^R)}\right), F^{-1}\left(\frac{c_i^R}{\alpha c_i^E}\right)\right\}. \tag{8.21}$$

Proof.

For (8.21), it suffices to show that for $i = 1, 2, \ldots, N$,

$$\begin{aligned}
(G_i^E(y))' &\leq c_i^E - c_{i-1}^R + \alpha c_{i-1}^E P(D < y - s_{i-1}^E), && (8.22)\\
(G_i^R(y))' &\leq -c_i^R + \alpha c_i^E P(D < y - s_i^E), && (8.23)\\
&\text{and}\\
(G_i^R(y))' &\leq -c_i^R + \sum_{j=1}^{i} \alpha^{i-j+1}(c_j^E - c_{j-1}^R) P(D < y - s_i^E). && (8.24)
\end{aligned}$$

For (8.22), note that

$$\begin{aligned}
(G_i^E(y))' &= c_i^E + (G_{i-1}^R(y))' \mathbf{1}(y < s_i^R)\\
&\leq c_i^E + (G_{i-1}^R(y))'\\
&\leq c_i^E + (-c_{i-1}^R + \alpha E[(G_{i-1}^E(y - D))' \mathbf{1}(y - D > s_{i-1}^E)]\\
&\leq c_i^E - c_{i-1}^R + \alpha c_{i-1}^E P(y - D > s_{i-1}^E)
\end{aligned}$$

where the last inequality follows from that $(G_{i-1}^E(y))' \leq c_i^E$. For (8.23)

$$\begin{aligned}
(G_i^R(y))' &\leq -c_i^R + \alpha E[(G_i^E(y - D))' \mathbf{1}(y - D \geq s_i^E)]\\
&\leq -c_i^R + \alpha c_i^E P(y - D \geq s_i^E).
\end{aligned}$$

We prove (8.24) by induction. For $i = 1$, $\underline{s}_1^E = s_1^E$ and for $\underline{s}_1^R$, it follows from

$$\begin{aligned}
&-c_1^R + \alpha c_1^E P(D < y - s_1^E) - \alpha(H_1 + b) P(D < y - s_1^E, D(2) \geq y)\\
\leq\ &-c_1^R + \alpha c_1^E P(D < y - s_1^E).
\end{aligned}$$

Again follow from Theorem 8.2.1, for any $i = 1, 2 \ldots, N$

$$\begin{aligned}
& (G_i^R(y))' \\
= \;& -c_i^R + G'_{i,i}(y) \\
= \;& -c_i^R + (G_i^E(y))' 1[y \le s_i^E] + \alpha E[(G_i^E(y - D_i))' 1[y - D_i \ge s_i^E] \\
\le \;& -c_i^R + \alpha E\Big[(G_i^E(y - D_i))' 1[y - D_i \ge s_i^E]\Big] \\
\le \;& -c_i^R + \alpha \Big(\sum_{j=1}^{i} \alpha^{i-j} (c_j^E - c_{j-1}^R) P(D_i \le y - s_i^E) \\
& -\alpha^{i-1} (H_1 + b) P(D + D_i > y, D_i \le y - s_i^E) \Big) \\
\le \;& -c_i^R + \sum_{j=1}^{i} \alpha^{i-j+1} (c_j^E - c_{j-1}^R) P(D_i \le y - s_i^E),
\end{aligned}$$

where the first inequality follows from that G_{i+1}^E is nonnegative on $y \ge s_{i+1}^E$. For (8.20), from the result of $G_i^R(y)$,

$$\begin{aligned}
(G_{i+1}^E)'(y) &= c_{i+1}^E + (G_i^R(y))' \mathbf{1}(y < s_i^R) \\
&\le c_i^E + (G_i^R(y))' \\
&\le c_{i+1}^E - c_i^R + \sum_{j=1}^{i} \alpha^{i-j+1} (c_j^E - c_{j-1}^R) P(D_i \le y - s_i^E).
\end{aligned}$$

So we complete the proof. □

From this lower bound, note that if the relative unit regular order cost of downstream stage is greater than the relative unit expedited cost of current stage, i.e., $c_{i-1}^R > c_i^E$, then the expedited order-up-to level of the current stage is higher than that of its downstream stage.

Note that some of the lower bounds for the optimal expedited order-up-to level are equal to the lower bound of the optimal expedited order-up-to level of its downstream plus a number, which can be either positive or negative infinity; some of the lower bounds for optimal regular order-up-to level are equal to the lower bound of the optimal expedited order-up-to level plus a nonnegative number. Furthermore, we remark that the lower bounds derived above do not have a dominating relationship. That is, any lower bound can be a better, depending on the problem instance.

We next develop three upper bounds for the optimal echelon base-stock levels of each stage.

Proposition 8.2.1 *(1) If* $c_i^E + \sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R) > H_1 + b$, $s_j^E = -\infty$ *for* $j \geq i$. *(2) If* $\sum_{j=1}^{i}(\alpha c_j^E - c_j^R) > H_1 + b$, $s_j^R = -\infty$ *for* $j \geq i$.

Proof. The proposition can be easily seen from the proof of Theorem 8.2.4. □

Theorem 8.2.4 *For* $i = 1, \ldots, N$, *the upper bound for* s_i^E *and* s_i^R *is*

$$\bar{s}_i^{E1} = \bar{F}_i^{-1}\left(\frac{c_i^E - c_{i-1}^R + \alpha c_{i-1}^E}{H_1 + b - \left(\sum_{j=1}^{i-2}(\alpha c_j^E - c_j^R)\right)}\right), \tag{8.25}$$

if $c_i^E + \sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R) \leq H_1 + b$; *otherwise* $\bar{s}_i^{E1} = -\infty$; *and*

$$\bar{s}_i^{R1} = \bar{F}_{i+1}^{-1}\left(\frac{\alpha c_i^E - c_i^R}{\alpha\left(H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R)\right)\right)}\right), \tag{8.26}$$

if $\sum_{j=1}^{i}(\alpha c_j^E - c_j^R) \leq H_1 + b$; *otherwise,* $s_i^R = -\infty$.

Proof. To prove the theorem, it suffices to show

$$(G_i^E(y))' \geq c_i^E - \left(H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R)\right)\right)P(D(i) \geq y), \tag{8.27}$$

$$\begin{aligned} & E\Big[G_i^E(y-D))'1[y-D \geq s_i^E]\Big] \\ \geq \;& c_i^E - \left(H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R)\right)\right)P(D(i+1) \geq y). \end{aligned} \tag{8.28}$$

The first inequality (8.27) clearly implies (8.25). If (8.27) and (8.28) are satisfied, then

$$\begin{aligned} & (G_i^R(y))' \\ = \;& -c_i^R + G'_{i,i}(y) \\ = \;& -c_i^R + (G_i^E(y))'1[y \leq s_i^E] + \alpha E[(G_i^E(y-D_i))'1[y - D_{i+1} > s_i^E] \\ \geq \;& -c_i^R + E[G_i^E(y-D_{i+1}))'1[y-D_{i+1} \leq s_i^E] + \alpha E[(G_i^E(y-D_i))'1[y-D_{i+1} > s_i^E]] \\ = \;& -c_i^R + c_i^E P[y - D_{i+1} \leq s_i^E] \\ & - \left(H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R)\right)\right) P(D(i+1) \geq y, y - D_{i+1} \leq s_i^E] \\ & + \alpha c_i^E P(y - D_{i+1} > s_i^E) \\ & -\alpha \left(H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R)\right)\right) P(D(i+1) \geq y, y - D_{i+1} > s_i^E) \\ \geq \;& -c_i^R + \alpha c_i^E - \left(H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R)\right)\right)P(D(i+1) \geq y) \\ = \;& -c_i^R + \alpha c_i^E - \left(H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R)\right)\right)P(D(i+1) \geq y), \end{aligned}$$

where the first inequality follows from the fact that $(G_i^E(y))'1[y \leq s_i^E] \geq E[G_i^E(y - D_{i+1}))'1[y - D_{i+1} \leq s_i^E]]$.

Moreover, as $s_i^R \geq s_i^E$, $(G_i^E(y))'1[y \leq s_i^E] = 0$ when we derive the upper bound for s_i^R.

Hence, by (8.28),

$$\begin{aligned}
& -c_i^R + \alpha E[(G_i^E(y - D_i))' 1[y - D_{i+1} > s_i^E] \\
\geq \quad & -c_i^R + \alpha c_i^E - \alpha \left(H_1 + b - \left(\sum_{j=1}^{i-1} (\alpha c_j^E - c_j^R) \right) \right) P(D(i+1) \geq y),
\end{aligned}$$

which implies (8.31).

We prove (8.27) and (8.28) by induction. First, for $i = 1$, (8.27) is equality. For s_1^R, we have

$$\begin{aligned}
& -c_1^R + (c_1^E - (H_1 + b) P(D_2 > y)) \mathbf{1}(y < s_1^E) \\
& + \alpha c_1^E P(D < y - s_1^E) - \alpha (H_1 + b) P(D_1 < y - s_1^E, D_1 + D_2 \geq y) \\
\geq \quad & -c_1^R + (c_1^E P(y - D_1 < s_1^E) - (H_1 + b) P(D_1 + D_2 > y, y - D_1 < s_1^E)) \\
& + \alpha c_1^E P(D < y - s^E) - \alpha (H_1 + b) P(D_1 < y - s_1^E, D_1 + D_2 > y) \\
\geq \quad & -c_1^R + \alpha c_1^E - (H_1 + b) P(D_1 + D_2 > y) \\
= \quad & -c_1^R + \alpha c_1^E - (H_1 + b) P(D(2) \geq y).
\end{aligned}$$

Moreover, as $s_1^R \geq s_1^E$,

$$
\begin{aligned}
& -c_1^R + \alpha c_1^E P(D < y - s_1^E) - \alpha(H_1 + b)P(D_1 < y - s_1^E, D_1 + D_2 \geq y) \\
= \; & -c_1^R + \alpha c_1^E(1 - P(D_1 > y - s^E)) - \alpha(H_1 + b)(P(D_1 + D_2 \geq y) \\
& -P(D_1 > y - s_1^E, D(2) \geq y) \\
= \; & -c_1^R + \alpha c_1^E - \alpha(H_1 + b)P(D_1 + D_2 \geq y) - \alpha c_1^E P(D_1 > y - s_1^E) \\
& +\alpha(H_1 + b)P(D_1 > y - s_1^E, D_1 + D_2 \geq y) \\
= \; & -c_1^R + \alpha c_1^E - \alpha(H_1 + b)P(D_1 + D_2 \geq y) - \alpha(H_1 + b)P(D_2 > s^E)P(D_1 > y - s^E) \\
& +\alpha(H_1 + b)P(D_1 > y - s^E, D_1 + D_2 \geq y) \\
= \; & -c_1^R + \alpha c_1^E - \alpha(H_1 + b)P(D_1 + D_2 \geq y) - \alpha(H_1 + b)\Big(P(D_2 > s^E, D_1 > y - s^E) \\
& -P(D_1 > y - s^E, D_1 + D_2 \geq y)\Big) \\
\geq \; & -c_1^R + \alpha c_1^E - \alpha(H_1 + b)P(D(2) \geq y)
\end{aligned}
$$

which implies the result holds for s_1^R.

Suppose the result is true for i. For $i + 1$, note that

$$
\begin{aligned}
& (G_{i+1}^E(y))' \\
= \; & c_{i+1}^E + (G'_{i,i}(y) - c_i^R)\mathbf{1}(y < s_i^R) \\
\geq \; & c_{i+1}^E + \left[-c_i^R + \alpha c_i^E - \left(H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R)\right)\right)P(D(i+1) \geq y)\right]\mathbf{1}(y < s_i^R) \\
\geq \; & c_{i+1}^E - \left(H_1 + b - \left(\sum_{j=1}^{i}(\alpha c_j^E - c_j^R)\right)\right)P(D(i+1) \geq y)
\end{aligned}
$$

where the first inequality follows from the inductive assumption and the second inequality follows from that $[-c_i^R + \alpha c_i^E \geq 0$. Thus, as $s_{i+1}^E \leq s_i^R$, the solution of

$$
c_{i+1}^E + \left[-c_i^R + \alpha c_i^E - \left(H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R)\right)\right)P(D(i+1) \geq y)\right] = 0
$$

is an upper bound of s_{i+1}^E, i.e.,

$$\bar{s}_{i+1}^E = \bar{F}_{i+1}^{-1}\left(\frac{c_{i+1}^E - c_i^R + \alpha c_i^E}{H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R)\right)}\right) \geq s_{i+1}^E.$$

We then prove (8.28) for $i+1$:

$$\begin{aligned}
& (E[G_{i+1}^E(y-D)])' \\
= \;& c_{i+1}^E + E[(G'_{i,i+1}(y-D))'] \\
= \;& c_{i+1}^E + E[(G_i^R(y-D))'\mathbf{1}[y-D \leq s_i^R]] \\
\geq \;& c_{i+1}^E + E\left[\left(-c_i^R + \alpha c_i^E - \left(H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R)\right)\right)\mathbf{1}(D(i+2) \geq y)\right)\right. \\
& \left.\times 1[y-D \leq s_i^R]\right] \\
\geq \;& c_{i+1}^E - \left(H_1 + b - \left(\sum_{j=1}^{i}(\alpha c_j^E - c_j^R)\right)\right)P(D(i+2) \geq y), \qquad (8.29)
\end{aligned}$$

where the second inequality follows from $\alpha c_i^E - c_i^R \geq 0$ and

$$P(D(i+2) \geq y, D \leq y - s_i^R) \leq P(D(i+2) \geq y).$$

$\square$

The second set of upper bounds is, $i = 1, \ldots, N$,

Theorem 8.2.5 *The upper bound for s_i^E and s_i^R is*

$$\bar{s}_i^{E2} = \bar{s}_{i-1}^R, \qquad (8.30)$$

and

$$\bar{s}_i^{R2} = s_{i-1}^R + \min\left\{ \bar{F}^{-1}\left(\frac{\alpha c_i^E - c_i^R}{\alpha\left(H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R) \right) \right)} \right), F^{-1}\left(\frac{c_i^R}{\alpha c_i^E} \right) \right\}, \quad (8.31)$$

Proof. The (8.30) is valid from Lemma 2. For (8.26), from (8.29), note that

$$\begin{aligned}
& (E[G_i^E(y - D)])' \\
\geq\ & c_i^E + E\Bigg[\left(-c_{i-1}^R + \alpha c_{i-1}^E - \left(H_1 + b - \left(\sum_{j=1}^{i-2}(\alpha c_j^E - c_j^R) \right) \right) \mathbf{1}(D(i+1) \geq y) \right) \\
& \times \mathbf{1}[y - D \leq s_{i-1}^R] \Bigg] \\
\geq\ & c_i^E - \left(H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R) \right) \right) P(D \leq y - s_{i-1}^R).
\end{aligned}$$

Moreover, as $s_i^R > s_i^E$ and for $y > s_i^E$,

$$\begin{aligned}
& (G_i^R(y))' \\
=\ & -c_i^R + G'_{i,i}(y) \\
=\ & -c_i^R + \alpha E[(G_i^E(y - D_i))' 1[y - D_{i+1} > s_i^E] \\
\geq\ & -c_i^R + \alpha E[(G_i^E(y - D_i))'] \\
\geq\ & -c_i^R + \alpha c_i^E - \alpha\left(H_1 + b - \left(\sum_{j=1}^{i-1}(\alpha c_j^E - c_j^R) \right) \right) P(D \leq y - s_{i-1}^R)
\end{aligned}$$

which implies the first term in the bracket of (8.26). For the second term, note that for

$y > s_i^E$

$$\begin{aligned} & (G_i^R(y))' \\ = & \ -c_i^R + \alpha E[(G_i^E(y-D_i))' 1[y - D_{i+1} > s_i^E] \\ \geq & \ -c_i^R + \alpha E[(G_i^E(y-D_i))' 1[y - D_{i+1} > s_{i-1}^R] \\ \geq & \ -c_i^R + \alpha c_i^E P(y - D_{i+1} > s_{i-1}^R). \end{aligned}$$

Therefore, (8.26) is valid. □

In the following we develop another set of newsvendor upper bounds for the optimal base-stock levels. Let

$$\begin{aligned} & C_0 = 0 \\ & C_i = c_i^E - c_{i-1}^R - C_{i-1}^-, i = 1, \ldots, N. \end{aligned}$$

Theorem 8.2.6 *The third set of upper bounds is*

$$\bar{s}_i^{E3} = \min\left\{ \bar{F}^{-1}\left(\frac{C_i}{H_1 + b}\right), \bar{F}_2^{-1}\left(\frac{C_i + \alpha C_{i-1}^+}{H_1 + b}\right) \right\}, \quad i = 1, \ldots, N, \tag{8.32}$$

and

$$\bar{s}_i^{R3} = \bar{F}_2^{-1}\left(\frac{-c_i^R + \alpha C_i}{\alpha(H_1 + b)}\right), \quad i = 1, \ldots, N. \tag{8.33}$$

Proof. It suffices to verify, for $i = 1, \ldots, N$,

$$(G_i^E(y))' \geq C_i - (H_1 + b)P(D > y). \tag{8.34}$$

If (8.34) is satisfied, then

$$\begin{aligned}
& (G_i^R(y))' \\
= \;& -c_i^R + G'_{i,i}(y) \\
= \;& -c_i^R + (G_i^E(y))'\mathbf{1}[y \le s_i^E] + \alpha E[(G_i^E(y-D_i))'1[y-D \ge s_i^E] \\
\ge \;& -c_i^R + E(G_i^E(y-D))'\mathbf{1}[y-D \le s_i^E] + \alpha E[(G_i^E(y-D_i))'1[y-D \ge s_i^E]] \\
\ge \;& -c_i^R + C_i P(y-D \le s_i^E) - (H_1+b)P(D(2) > y, y-D \le s_i^E) \\
& +\alpha C_i P(y-D \ge s_i^E) - \alpha(H_1+b)P(D(2) > y, y-D \ge s_i^E) \\
\ge \;& -c_i^R + \min\{\alpha C_i, C_i\} - (H_1+b)P(D(2) > y).
\end{aligned}$$

Moreover, as $s_i^R \ge s_i^E$,

$$\begin{aligned}
& (G_i^R(y))' \\
= \;& -c_i^R + G'_{i,i}(y) \\
= \;& -c_i^R + (G_i^E(y))'1[y \le s_i^E] + \alpha E[(G_i^E(y-D_i))'1[y-D \ge s_i^E] \\
= \;& -c_i^R + \alpha E[(G_i^E(y-D_i))'1[y-D \ge s_i^E]] \\
\ge \;& -c_i^R + \alpha E[(G_i^E(y-D_i))'] \\
\ge \;& -c_i^R + \alpha C_i - \alpha(H_1+b)P(D(2) > y)
\end{aligned}$$

which implies (8.33).

We prove (8.34) by induction. The case of $i = 1$ is similar to that of Theorem 3.

Suppose it is true for i, and we proceed to prove $i+1$.

$$\begin{aligned}
& (G^E_{i+1}(y))' \\
= \;& c^E_{i+1} + (G'_{i,i}(y) - c^R_i)1[y \le s^R_i] \\
= \;& c^E_{i+1} + (G^E_i(y))'1[y \le s^E_i]1[y \le s^R_i] \\
& +\Big(-c^R_i + \alpha E\Big[G^E_i(y-D))1[y-D \ge s^E_i]\Big]\Big)1[y \le s^R_i] \\
\ge \;& c^E_{i+1} + (C_i - (H_1+b)P(D>y)\Big)1[y \le s^E_i] - c^R_i \\
\ge \;& c^E_{i+1} - c^R_i - C^-_i - (H_1+b)P(D>y) \\
= \;& C_{i+1} - (H_1+b)P(D>y)
\end{aligned}$$

where the first inequality follows from the inductive assumption and $G^E_i(y-D))1[y-D \ge s^E_i] \ge 0$. Moreover, as $s^E_{i+1} \le s^R_i$, we can obtain another upper bound which is the solution of,

$$c^E_{i+1} - c^R_i + \min\{\alpha C_i, C_i\} - (H_1+b)P(D(2)>y) = 0$$

Thus, we complete the proof. □

We remark that none of these upper bounds dominate the other. That is, any of these upper bounds can be sharper, depending on the problem instance.

8.3 Heuristics and Numerical Results

In this section, we develop a simple heuristic based on the newsvendor lower and upper bounds for the echelon base-stock levels for each stage. We also present numerical studies to demonstrate the effectiveness of the heuristic method.

For $i = 1, 2, \ldots, N$ and $0 \leq \beta \leq 1$, let

$$\underset{\sim}{s_i}^E = \max\{\underline{s}_i^{Ej}, j = 1, 2, 3\}, \quad \underset{\sim}{s_i}^R = \max\{\underline{s}_i^{Rj}, j = 1, 2, 3\};$$
$$\tilde{s}_i^E = \min\{\bar{s}_i^{Ej}, j = 1, 2, 3\}, \quad \tilde{s}_i^{Ru} = \min\{\bar{s}_i^{Rj}, j = 1, 2, 3\},$$

Note that from Lemma 8.1.4, if $c_{i-1}^R > c_i^E$, $s_i^E \leq s_{i-1}^E$. Thus, we can set $\tilde{s}_i^E = \tilde{s}_{i-1}^E$, if $\tilde{s}_i^E > \tilde{s}_{i-1}^E$.

and

$$s_i^{Eh} = \left[\beta \underset{\sim}{s_i}^E + (1-\beta)\tilde{s}_i^E\right], \tag{8.35}$$
$$s_i^{Rh} = \left[\beta \underset{\sim}{s_i}^R + (1-\beta)\tilde{s}_i^R\right], \tag{8.36}$$

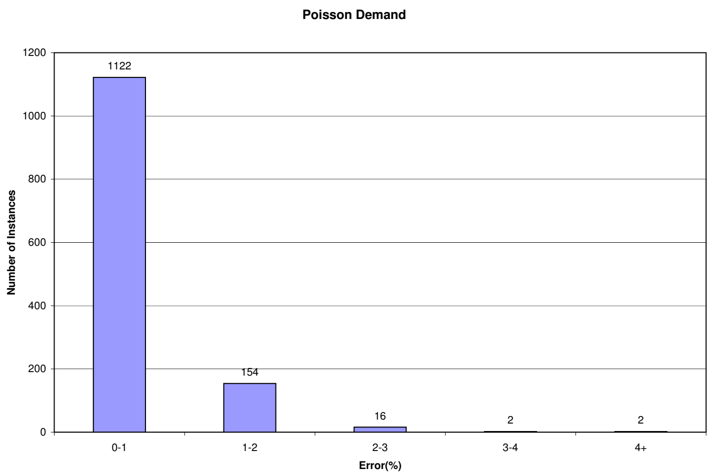
in which [ ] is the round off operator. We choose $\beta = 0.5$ as the heuristic policy. The heuristic policy works in exactly the same manner as the original top-down echelon base-stock policy but using s_i^{Eh} and s_i^{Rh} as the echelon base-stock levels for stage i.

In the following we present two groups of numerical examples classified by the demand distribution to illustrate the effectiveness of this heuristic.

We use the percentage error on the optimal system cost as the measure for effectiveness of the heuristic. Let $\widehat{f}(\mathbf{x})$ denote the cost of heuristic policy, the percentage error of the heuristic is defined as

$$Error\% = \frac{\widehat{f}(\mathbf{x}) - f(\mathbf{x})}{f(\mathbf{x})} \times 100\%.$$

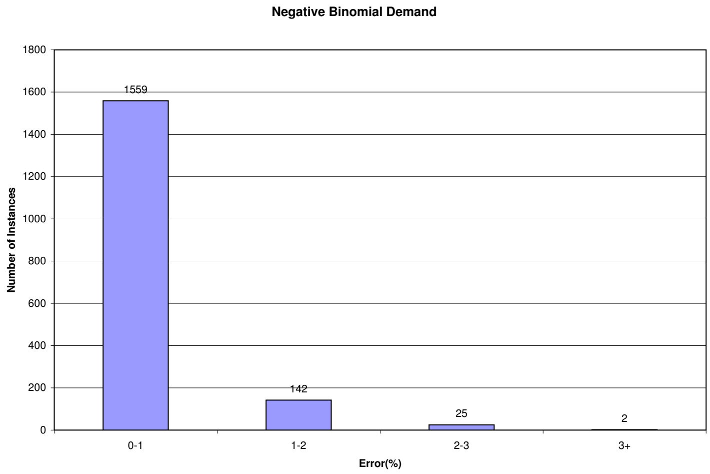
In Group 1, we use Poisson demand with arrival rate $\lambda = 5, 10, 50$. We compare the optimal and heuristic policies for a three-stage system. The parameters for the examples



are $b = 30, 60$, $h_i = 0.1, 1$, $c_i^E = 4, 10$, $c_i^R = 2, 6$, for $i = 1, 2, 3$ and $\alpha = 0.95, T = 100$. By restricting $c_i^E > c_i^R$, we generate 432 instances for each demand arrival rate. The average percentage error among 432 instances for $\lambda = 5$ is 0.57% with the maximum 3.06%, for $\lambda = 10$ is 0.52% with the maximum 4.28% and for $\lambda = 50$ is 0.33% with the maximum 1.70%. The average error for all 1296 instances is 0.47%.

To show that the heuristic is robust under larger demand variance comparing to mean, in Group 2, we use Negative Binomial demand with four sets of mean and variance (30, 120), (30, 40), (6, 24), and (6, 8) while keep other parameters the same as Group 1. These demand parameters generate 4 set of numerical examples and each set includes 432 instances. The average percentage error among 432 instances for the first set is 0.42% with the maximum 3.62%, for the second is 0.37% with the maximum 2.65% and for the third is 0.49% with the maximum 2.64% and for the fourth is 0.48% with the maximum 2.88%. The average error for all 1728 instances is 0.44%. The numerical results indeed show that the effectiveness of our heuristic under larger demand variance.

From our numerical studies, we find that it is more cost efficient to reduce the holding cost at upstream stages and the expedited ordering cost at downstream stages. In addition, the downstream's optimal echelon base-stock levels are independent of up-



stream's cost parameters and upstream's optimal echelon base stock levels are increasing as downstream's ordering costs increase and decreasing when downstream's holding cost increases. Moreover, the increase of backlog cost rate has larger impact on the order-up-to level of expedited shipment (regular shipment) when the unit expedited shipping cost is relatively small (large).

8.4 Summary

In this chapter, we derive newsvendor-type lower bounds and upper bounds for infinite horizon, periodic review, serial inventory system with expedited and regular supply, and based on which we develop a simple and effective heuristic. We use numerical examples to show the effectiveness of the heuristics.

The leadtimes for regular and expedited ordering are 1 and 0 respectively. By inserting stages to stand for leadtime, we can obtain models where leadtimes between stages $i+1$ and i is l_i, and the firm is allowed by expedite shipping between any two stages. Through expedition, the firm can shipping the product from stage $i+1$ to i in ℓ units of time for any $\ell = 0, 1, \ldots, l_i - 1$. The cost for such expedition will have to satisfy the relationship entailed by the model. Indeed, this is exactly the model studied by Muharremoglu and Tsitsilkis (2003) in which the authors obtain the form of optimal inventory control policy. In particular, the result for the case where the leadtimes for stage $i+1$ and i are l_i+1 and l_i can be presented in similar fashion to those in the chapter. This is a natural expansion of the Fukuda model (1964) to serial supply chains. In that case the recursive algorithms for computing the optimal echelon base-stock levels are as follows. For convenience, let $D(l)$ be the leadtime over l periods. Let $G_1^E(y) = c_1^E y + (H_1 + b)E[(y - D(l_1))^-]$, and let s_1^E be the minimizer of G_1^E, for $i \geq 1$, compute:

$$
\begin{aligned}
G_{i,i}(y) &= G_i^E((y - D(l_i) \wedge s_i^E) - G_i^E(s_i^E) + \alpha E[G_i^E((y - D(l_i + 1)) \vee s_i^E)], \\
G_i^R(y) &= G_{i,i}(y) - c_i^R y, \\
G_{i,i+1}(y) &= G_i^R(y \wedge s_i^R), \\
G_{i+1}^E(y) &= c_{i+1}^E y + G_{i,i+1}(y).
\end{aligned}
$$

The bounds developed need to be accordingly revised to reflect the arbitrary leadtimes. For instance, the F_i and F_{i+1} in Theorems 2 and 3 should represent the leadtime demand distributions over $\sum_{j=1}^{i} l_j$ and $\sum_{j=1}^{i+1} l_j$ periods, respectively.

The results reported in this chapter can also be extended to the case where, for some stages, there is only one transportation mode, while the others have two transportation modes. In that case there is one echelon base-stock level for those stations with only one transportation mode, and two echelon base-stock levels for those stages with two transportation modes.

Chapter 9

Conclusion

In this dissertation, we make several contributions to the literature of supply chain management. More specifically, this work develops and analyzes several single-stage inventory/production models with pricing decisions, extends some existing results in multi-echelon inventory systems and provides a unified approach to develop bounds and simple heuristics for multi-echelon inventory systems with and without expedited shipping.

First, we include the optimal pricing into several classical, continuous-review inventory models, characterize the optimal policies and present efficient algorithms to solve the optimization problem. We show that optimal price has a unimodal structure of the inventory level. Then, for two periodic review inventory models, one with two supply modes and the other one with smoothing production cost, we again include the pricing optimization and analyze the optimal strategies.

Our second contribution is the analysis of optimal policies for the serial system with demand guaranteed delivery and multi-echelon system with batch ordering and nested replenishment schedule. We show that the echelon base-stock policy is optimal for the serial system with demand guaranteed delivery. With given base order quantity and replenishment cycles for each stage, we show that echelon (R, nQ) policy is optimal, which generalizes results of Chen (2000) and van Houtum et. al. (2003). In addition, we provide the computational algorithm for the optimal reorder points.

The last contribution is that we provide a unified approach to develop lower bounds and upper bounds for optimal policies of the classical serial system. Based on the bounds, we further develop a simple heuristic for the optimal echelon base-stock levels. Indeed, this approach can also be applied in the serial system with dual supply modes. Extensive numerical studies show that our bounds are tight and the heuristics generate near-optimal solutions.

Distribution system is a very important class of multi-echelon models. Any progress made in the analysis or development of optimal policies or heuristics for the distribution systems will have practical and theoretical values. I hope that the methodologies and results in this dissertation can pave the way for my future research in this direction.

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