Abstract

BEIER, JULIE CATHERINE. Crystals for Demazure Modules of Special Linear Quantum Affine Algebras. (Under the direction of Kailash C. Misra.)

Kac-Moody Lie algebras, independently discovered in the 1960's by Victor Kac and Robert Moody, are infinite dimensional analogs of finite dimensional semisimple Lie algebras. Affine Lie algebras form an important class of infinite dimensional Kac-Moody Lie algebras with numerous applications in different areas of mathematics and physics.

Quantum groups, discovered by both Drinfeld and Jimbo in the 1980's, are q-deformations of universal enveloping algebras of symmetrizable Kac-Moody Lie algebras. The quantum groups associated with affine Lie algebras are called quantum affine algebras. For 'q' generic and λ a dominant weight there exists a unique (up to isomorphism) irreducible highest weight module $V(\lambda)$ for the quantum affine algebra $U_q(\mathfrak{g})$. For each $w \in \mathcal{W}$, the Weyl group of $\mathfrak{g}$, there is a finite dimensional subspace $V_w(\lambda)$ of $V(\lambda)$ called a Demazure module generated from the extremal weight vector $u_{w\lambda}$ by the positive part of $U_q(\mathfrak{g})$.

The crystal $B(\lambda)$ associated with $V(\lambda)$ was introduced by Kashiwara and Lusztig

in the 1990's. $B(\lambda)$ provides an important tool to study the combinatorics of $V(\lambda)$. In 1993, Kashiwara showed that a suitable subset $B_w(\lambda)$ of $B(\lambda)$ is the crystal for the Demazure module $V_w(\lambda)$.

In this thesis we give an explicit realization of the Demazure crystals $B_w(\lambda)$ for the special linear quantum affine Lie algebra $U_q(\widehat{sl}(n))$ where $w = w(k)$, $k > 0$, is a suitable linear chain of Weyl group elements. This realization is given in terms of certain combinatorial objects called extended Young diagrams.

Crystals for Demazure Modules of Special Linear Quantum Affine Algebras

by
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Biography

Julie Catherine Beier was born in Henderson, North Carolina on May 1, 1981. At a young age she moved from Oxford, NC to Durham County. After finishing high school, she attended Barton College (formerly Atlantic Christian College) in Wilson, NC where she majored in both Mathematics and Education of the Deaf and Hearing Impaired. After graduating top of her class in May 2003, she moved to Raleigh to attend North Carolina State University where she received her masters degree in December of 2005. She stayed at North Carolina State University and completed her Doctorate of Philosophy in Mathematics with an interdisciplinary minor in Mathematics Education and Statistics in May of 2008. Julie is leaving North Carolina in August of 2008 to begin as an Assistant Professor at Mercer University in Macon, Georgia.

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Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Symmetrizable Kac-Moody Lie algebras (cf. [9]) discovered independently by Victor Kac [8] and Robert Moody [19] are the infinite dimensional analog of finite dimensional semisimple Lie algebras. Affine Lie algebras form an important class of infinite dimensional Kac-Moody Lie algebras with numerous applications in different areas of mathematics and mathematical physics.

An affine Lie algebra, $\widehat{\mathfrak{g}}$, associated with an indecomposable generalized Cartan matrix $\mathcal{C} = (a_{ij}),\ i, j \in I = \{0, 1, \ldots, n-1\}$, can be constructed explicitly from the finite dimensional simple Lie algebra $\mathfrak{g}$ with Cartan matrix $\overline{\mathcal{C}} = (a_{ij}),\ i, j \in \overline{I} = \{1, 2, \ldots, n-1\}$. For example, let $\mathfrak{g} = sl(n, \mathbb{C})$ be the simple Lie algebra of $n \times n$ trace zero matrices. Its Cartan matrix $\overline{\mathcal{C}} = (\overline{a}_{ij}),\ i, j \in \overline{I} = \{1, 2, \ldots, n-1\}$, is given by $\overline{a}_{ii} = 2$, $\overline{a}_{ij} = -1$ for $|j - i| = 1$, and $\overline{a}_{ij} = 0$ otherwise. The associated affine Lie algebra, $\widehat{sl}(n, \mathbb{C})$, is $\widehat{g} = sl(n, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ where c is central and d is the derivation $1 \otimes t\frac{d}{dt}$ with relations:

$$[x \otimes t^i, y \otimes t^j] := [x,y] \otimes t^{i+j} + (i)(tr(xy))\delta_{i+j,0}c \quad \forall x, y \in \mathfrak{g} \quad i,j \in \mathbb{Z}$$

$$[d, x \otimes t^i] := d(x \otimes t^i) = \left(1 \otimes t\left(\frac{d}{dt}\right)\right)(x \otimes t^i) = i(x \otimes t^i)$$

$$[c, g] := 0 \quad \forall g \in \widehat{\mathfrak{g}}.$$

The Cartan matrix for $\widehat{\mathfrak{g}}$ is $\mathcal{C} = (a_{ij})$, $i,j \in I = \{0, 1, \ldots, n-1\}$, given by $a_{ii} = 2$, $a_{ij} = -1$ for $|i-j| = 1$, $a_{0,n-1} = -1 = a_{n-1,0}$ and $a_{ij} = 0$ otherwise. Notice that $\overline{\mathcal{C}}$ sits inside of $\mathcal{C}$.

In this thesis we will focus specifically on the affine Lie algebra $\widehat{\mathfrak{g}} = \widehat{sl}(n)$, which has the triangular decomposition $\widehat{\mathfrak{g}} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{h}$ is the Cartan subalgebra of $\widehat{\mathfrak{g}}$. The subalgebra $\underline{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is called the Borel subalgebra. This subalgebra plays an important role in the construction of Demazure modules as discussed in Chapter 4.

Around 1985, Drinfeld [2] and Jimbo [6] independently introduced the quantum group (also called quantized universal enveloping algebra) $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. These quantum groups are neither groups nor Lie algebras. They are q-deformations of the universal enveloping algebra $U(\mathfrak{g})$. Lusztig [14] showed that for generic q (q not a root of unity), the combinatorial properties of the integrable representations of $\mathfrak{g}$ remain invariant under this deformation. In particular, the weight space dimensions, and hence the characters, of the integrable representations of $\mathfrak{g}$ are the same as that of $U_q(\mathfrak{g})$.

Kashiwara [11, 12] and Lusztig [15, 16] independently introduced a basis called "crystal basis" for the integrable representations of $U_q(\mathfrak{g})$ and proved their existence.

In particular, crystal basis provide a nice combinatorial tool to study invariants in the representation space. Roughly, crystal basis can be thought of as a base at the $q = 0$ limit.

In 1990, Misra and Miwa [17] gave an explicit realization of the crystal basis for the level one integrable highest weight representations of $U_q(\widehat{sl}(n))$ in terms of certain combinatorial objects called "extended Young diagrams". This construction was generalized to arbitrary level integrable highest weight representations of $U_q(\widehat{sl}(n))$ in [7]. An extended Young diagram can be thought of as a "colored Young diagram" with a certain charge.

Let $\mathcal{W} =< r_0, r_1, \ldots, r_{n-1} >$ denote the Weyl group for $\mathfrak{g}$ generated by the simple reflections $r_0, r_1, \ldots, r_{n-1}$. For a dominant weight $\lambda \in P^+$, let $V(\lambda)$ denote the integrable highest weight $U_q(\mathfrak{g})$-module with highest weight λ. For $w \in \mathcal{W}$, it is known (see [9]) that $\dim V(\lambda)_{w\lambda} = \dim V(\lambda)_\lambda = 1$. Let $V(\lambda)_{w\lambda}$ be spanned by $u_{w\lambda}$. The Demazure module $V_w(\lambda)$ is the finite dimensional $U_q(\underline{b})$-submodule generated by the extremal vector $u_{w\lambda}$. For $w, w' \in \mathcal{W}$ and $w \prec w'$ (Bruhat order), we have $V_w(\lambda) \subseteq V_{w'}(\lambda)$. Further we have $V(\lambda) = \bigcup_{w \in \mathcal{W}} V_w(\lambda)$.

In 1993, Kashiwara [13] showed the existence of the crystal $B_w(\lambda)$ for the Demazure module $V_w(\lambda)$ as a subset of the crystal $B(\lambda)$ for the highest weight module $V(\lambda)$. In [3], Foda, Misra, and Okado gave an explicit realization of all Demazure crystals, $B_w(\lambda)$, for the quantum affine Lie algebra $U_q(\widehat{sl}(2))$ in terms of extended Young diagrams. In [18], Demazure crystals, $B_{w(L)}(\Lambda_0)$, were constructed in terms of extended Young diagrams for a certain linear chain of Weyl group elements $w(L)$ and the level one dominant weight Λ_0.

In this thesis we extend this work. In Chapter 3, we will review the concepts of crystal base and the realization given in [17] and [7] in terms of extended Young diagrams. Then in Chapter 4, we extended the work from [18] to give an explicit realization of Demazure crystals, $B_{w(L)}(\lambda)$, for certain linear chains of Weyl group elements $w(L)$ and any dominant weight $\lambda \in P^+$ in terms of extended Young diagrams. In this process we give an explicit description for each extremal vector $u_{w(L)}$ in terms of these diagrams.

Chapter 2

The Special Linear Quantum Affine Algebra

In this chapter we discuss Kac-Moody Lie algebras and their representations. After discussing each in general, we look specifically at the realizations of the affine special linear Lie algebra and of its associated quantum group.

2.1 Lie Algebras, Universal Enveloping Algebras and Representations

Definition 2.1.1. *A vector space L over $\mathbb{C}$ is a Lie algebra if there is a bilinear operation, called bracket, $[\ ,\ ] : L \times L \to L$ such that:*

1. $[x, x] = 0$ *for all* $x \in L$

2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *for all* $x, y, z \in L$. *This property is called the*

Jacobi identity.

In particular, the special linear Lie algebra, denoted as $sl(n, \mathbb{C})$, is the set of $n \times n$ traceless matrices with entries in $\mathbb{C}$. The bracket for $sl(n, \mathbb{C})$ is the commutator bracket; i.e. $[A, B] = AB - BA$. Before considering this general case, consider Example 2.1.2.

Example 2.1.2. *Consider $sl(2, \mathbb{C})$, the set of 2×2 traceless complex matrices with the commutator bracket. The standard basis for $sl(2, \mathbb{C})$ is:*

$$\left\{ E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

and the relations are:

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F$$

Similarly we have a standard basis for $sl(n, \mathbb{C})$. Let $E_{i,j}$ denote the n by n matrix with a one in the (i, j)-entry and zeros elsewhere. We define the following basis vectors:

$$E_i = E_{i,i+1}, \quad F_i = E_{i+1,i}, \quad H_i = E_{i,i} - E_{i+1,i+1}.$$

Then $sl(n, \mathbb{C}) =< E_i, F_i, H_i \big| i = 1, 2, \ldots, n-1 >$.

The Jacobi identity demonstrates that in general L is not associative. However, given L we can uniquely construct an associative algebra called the universal enveloping algebra of L. This algebra is denoted as $U(L)$.

Definition 2.1.3. *Let L be a Lie algebra. A universal enveloping algebra of L is a* *pair $(U(L), i)$ where $U(L)$ is an associative algebra (and hence U(L) is a Lie algebra) and $i : L \to U(L)$ is a Lie algebra homomorphism satisfying the universal property: If (A, j) is another such pair (i.e. A is an associative algebra and $j : L \to A$ is a Lie algebra homomorphism) then there exists a unique associative algebra homomorphism $\phi : U(L) \to A$ such that $j = \phi \circ i$. Pictorially:*

$$
\begin{tikzcd}
\mathrm{U(L)} \arrow[rr, dashed, "\exists!\ \phi"] & & \mathrm{A} \\
& \mathrm{L} \arrow[ul, "i"] \arrow[ur, "j"'] \arrow[u, phantom, "\circlearrowright"] &
\end{tikzcd}
$$

Figure 2.1: Universal enveloping algebra

The universal enveloping algebra $U(L)$ can be constructed as $T(L)/ < x \otimes y - y \otimes x - [x, y] >$ where $T(L)$ is the tensor algebra of L and $< x \otimes y - y \otimes x - [x, y] >$ is the ideal of $T(L)$ generated by elements of the form $x \otimes y - y \otimes x - [x, y]$ for all x, y in L. However, for a Kac-Moody Lie algebra we can realize $U(L)$ in terms of generators and relations. This is shown later in Proposition 2.2.3.

The following Poincare-Birkhoff-Whitt theorem, or PBW theorem, gives a basis for $U(L)$.

Theorem 2.1.4. *[5] Let L be a Lie algebra with ordered basis $\{x_\alpha \big| \alpha \in \Omega\}$. Let $U(L)$ paired with $j : L \to U(L)$ be its universal enveloping algebra. Then $\{j(x_{\alpha_1}) \cdots j(x_{\alpha_n}) \big| n \geq 0, \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n, \alpha_i \in \Omega\}$ is a basis for $U(L)$.*

In particular, the map $j : L \to U(L)$ is an injective homomorphism.

Corollary 2.1.5 illuminates how the direct sum of Lie algebras manifests in the universal enveloping algebra.

Corollary 2.1.5. *Suppose that L is the direct sum of L_1 and L_2 as Lie algebras (i.e. $g_1 + g_2, h_1 + h_2 \in L_1 \oplus L_2 = L$ then $[g_1 + g_2, h_1 + h_2] = [g_1, h_1] + [g_2, h_2]$). Then we have the following natural isomorphism of associative algebras:*

$$U(L) \cong U(L_1) \otimes U(L_2).$$

We now investigate the representation theory of Lie Algebras.

Definition 2.1.6. *Let L be a Lie algebra over $\mathbb{C}$ and V a vector space over $\mathbb{C}$. Then:*

1. *A representation of L on V is a Lie algebra homomorphism $\phi : L \to gl(V)$.*
2. *V is an L-module if there is a bilinear operation from $L \times V$ into V given by $(x, v) \mapsto x \cdot v$ such that $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ for all $x, y \in L$ and $v \in V$.*

Given a representation of L on V, namely ϕ, we can view V as an L-module by $x \cdot v = \phi(x)v$ for all $x \in L$ and $v \in V$. Similarly if V is an L-module we can also say that there is a representation of L on V using $\phi : L \to gl(V)$ defined by $\phi(x)v = x \cdot v$ for all $x \in L$ and $v \in V$.

Example 2.1.7. *Let $L = sl(2, \mathbb{C}) =< E, F, H >$ and $V = \mathbb{C}^2 =< v_1, v_2 >$. Then V is an L-module under matrix multiplication.*

$$\begin{array}{lll} E \cdot v_1 = Ev_1 = \ 0 & F \cdot v_1 = Fv_1 = \ v_2 & H \cdot v_1 = Hv_1 = \ v_1 \\ E \cdot v_2 = Ev_2 = \ v_1 & F \cdot v_2 = Fv_2 = \ 0 & H \cdot v_2 = Hv_2 = \ -v_2 \end{array}$$

Similarly, we can view this as a representation of L on V by $\phi : L \to gl(V)$ where:

$$\begin{array}{lll} \phi(E)v_1 = 0 & \phi(F)v_1 = v_2 & \phi(H)v_1 = v_1 \\ \phi(E)v_2 = v_1 & \phi(F)v_2 = 0 & \phi(H)v_2 = -v_2 \end{array}$$

There is a natural way to extend a representation of L to a representation of $U(L)$. Similarly a representation of $U(L)$ induces a representation on L. Thus the representation theory of L is similar to that of $U(L)$.

2.2 Affine Kac-Moody Lie Algebras

Affine Lie algebras are a generalization of finite dimensional semisimple Lie algebras. These affine Lie algebras are constructed from generators and relations which are based on a special matrix, called a generalized Cartan matrix (GCM).

Definition 2.2.1. *Let $\mathcal{C} = (a_{ij})$ be an $(n-1) \times (n-1)$ matrix (indexed by $I = \{1, 2, \ldots, n-1\}$) with integer entries. We call $\mathcal{C}$ a generalized Cartan matrix if it satisfies the following:*

1. $a_{ii} = 2$ *for all* $i \in I$
2. $a_{ij} \leq 0$ *for* $i \neq j,\ i, j \in I$
3. $a_{ij} = 0$ *iff* $a_{ji} = 0$ *for all* $i, j \in I$

If $\mathcal{C}$ satisfies the additional requirement that it is positive definite then the matrix is called a Cartan matrix.

If there exists a permutation σ of the indices of $\mathcal{C}$ such that

$$\mathcal{C}' = (a_{\sigma(i)\sigma(j)}) = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$$

then $\mathcal{C}$ is called decomposable. If no such σ exists, $\mathcal{C}$ is called indecomposable. Since any decomposable matrix can be realized as the direct sum of indecomposable matrices, it suffices to consider only generalized Cartan matrices that are indecomposable.

Theorem 2.2.2. *[9] Let $\mathcal{C}$ be an $(n-1)\times(n-1)$ indecomposable GCM. Then $\mathcal{C}$ belongs to exactly one of the following categories:*

Finite Type *There exists a column vector $\theta \in \mathbb{Z}^{n-1}$ of positive integers such that $\mathcal{C}\theta$ is a column vector of positive integers. In particular, $\mathcal{C}$ is positive definite.*

Affine Type *There exists a column vector $\theta \in \mathbb{Z}^{n-1}$ of positive integers such that $\mathcal{C}\theta = 0$. In particular, $\mathcal{C}$ is semi-positive definite with corank 1.*

Indefinite Type *There exists a column vector $\theta \in \mathbb{Z}^{n-1}$ of positive integers such that $\mathcal{C}\theta$ is a column vector of negative integers. In particular, $\mathcal{C}$ is negative definite.*

A generalized Cartan matrix, C, is symmetrizable if there exists an invertible diagonal matrix, $D = diag(s_1, s_2, \ldots, s_{n-1})$, (all $s_i > 0$) and a symmetric matrix B such that $C = DB$. All GCMs of finite and affine type are symmetrizable. Here after we assume all GCMs to be both symmetrizible and indecomposable.

There is a correspondence between Cartan matrices and Dynkin diagrams. All of the Dynkin diagrams are classified for symmetrizable generalized Cartan matrices of finite and affine type (see [9]).

Let $\mathcal{C} = (a_{ij})$, $i, j \in I = \{1, \ldots, n-1\}$, be a Cartan matrix (i.e. a positive definite GCM). The triple $(\mathfrak{h}, \Pi, \check{\Pi})$ is a realization of $\mathcal{C}$ if:

- $\mathfrak{h}$ is a complex vector space of dimension n.
- $\Pi = \{\alpha_i \big| i \in I\} \subset \mathfrak{h}^*$ is a basis for $\mathfrak{h}^*$.
- $\check{\Pi} = \{\check{\alpha}_i \big| i \in I\} = \{H_i \big| i \in I\} \subset \mathfrak{h}$ is a basis for $\mathfrak{h}$.
- $\alpha_j(H_i) =< \alpha_j, H_i >= a_{ij}$ for $i, j \in I$

We can now construct the appropriate Lie algebra. Let $\mathcal{C}$ be a Cartan matrix and $(\mathfrak{h}, \Pi, \check{\Pi})$ be a realization of $\mathcal{C}$, as defined previously. Let $\mathfrak{g}(\mathcal{C})$ be the Lie algebra generated by E_i, F_i ($i \in I = \{1, \ldots, n-1\}$), and $\mathfrak{h}$ with relations:

- $[E_i, F_j] = \delta_{ij} H_i$ for $i, j \in I$
- $[H, H'] = 0$ for $H, H' \in \mathfrak{h}$
- $[H, E_i] = \alpha_i(H) E_i$ for $H \in \mathfrak{h}$, $i \in I$
- $[H, F_i] = -\alpha_i(H) F_i$ for $H \in \mathfrak{h}$, $i \in I$
- $(ad(E_i))^{(1-a_{ij})}(E_j) = 0$ for $i, j \in I$, $i \neq j$
- $(ad(F_i))^{(1-a_{ij})}(F_j) = 0$ for $i, j \in I$, $i \neq j$

Then $\mathfrak{g}(\mathcal{C})$ is a finite dimensional Lie algebra. This Lie algebra is also called the Kac-Moody Lie algebra of finite type associated with $\mathcal{C}$. The generators E_i and F_i ($i \in I$) are called the Chevalley generators, and the subalgebra $\mathfrak{h}$ is called the Cartan subalgebra (CSA). The rank of the Cartan matrix is called the rank of $\mathfrak{g}(\mathcal{C})$.

Similarly, there is a realization for generalized Cartan matrices of affine type. Let $\mathcal{C} = (a_{ij}),\ i, j \in I = \{0, 1, \ldots, n-1\}$, be a generalized Cartan matrix of rank $n-1$. The triple $(\mathfrak{h}, \Pi, \check{\Pi})$ is a realization of $\mathcal{C}$ if:

- $\mathfrak{h}$ is a complex vector space with dimension $n+1$.
- $\Pi = \{\alpha_i \big| i \in I\} \subset \mathfrak{h}^*$ is linearly independent.
- $\check{\Pi} = \{\check{\alpha}_i \big| i \in I\} = \{h_i \big| i \in I\} \subset \mathfrak{h}$ is linearly independent.
- $\alpha_j(h_i) =< \alpha_j, h_i >= a_{ij}$ for $i, j \in I$

Again we can construct the Lie algebra associated with $\mathcal{C}$ using the triple $(\mathfrak{h}, \Pi, \check{\Pi})$ as defined in the last realization. Let $\mathfrak{g}(\mathcal{C})$ be the Lie algebra generated by e_i, f_i $(i \in I = \{0, 1, \ldots, n-1\})$, and $\mathfrak{h}$ with relations:

- $[e_i, f_j] = \delta_{ij} h_i$ for $i \in I$
- $[h, h'] = 0$ for $h, h' \in \mathfrak{h}$
- $[h, e_i] = \alpha_i(h) e_i$ for all $h, h' \in \mathfrak{h}$ (2.2)
- $[h, f_i] = -\alpha_i(h) f_i$ for all $h, h' \in \mathfrak{h}$
- $(ad(e_i))^{(1-a_{ij})}(e_j) = 0$ for $i, j \in I$, $i \neq j$
- $(ad(f_i))^{(1-a_{ij})}(f_j) = 0$ for $i, j \in I$, $i \neq j$

Then $\mathfrak{g}(\mathcal{C})$ is called the Kac-Moody Lie algebra of affine type associated with $\mathcal{C}$. Since $\mathcal{C}$ is affine, $\mathfrak{g}(\mathcal{C})$ is also called an affine Lie algebra. Again the e_i and f_i $(i \in I)$ are called the Chevalley generators, and the subalgebra $\mathfrak{h}$ is called the Cartan subalgebra. Also, as before, the rank of the generalized Cartan matrix is called the rank of $\mathfrak{g}(\mathcal{C})$.

Notice that the constructions are similar for the Cartan matrix and for the GCM of affine type. It is important to note that we have chosen to label the finite type generators with capital letters, as we did when discussing $sl(n)$ previously, while we are labeling the affine type generators with lower case letters. Also the affine type Cartan matrix has size $n \times n$ with generators indexed by 0 to $n-1$, while the finite type Cartan matrix has size $(n-1) \times (n-1)$ with generators indexed by 1 to $n-1$.

More details about this construction can be found in [9]. In fact, this same construction can be carried out for any symmetrizable GCM.

For ease of notation, we simply denote $\mathfrak{g}(\mathcal{C})$ as $\mathfrak{g}$.

Once we have the GCM associated with a Lie algebra $\mathfrak{g}$ we can explicitly realize the universal enveloping algebra $U(\mathfrak{g})$ in terms of generators and relations.

Proposition 2.2.3. *[4] The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ with generalized Cartan matrix $\mathcal{C} = (a_{ij})_{i,j \in I}$ and realization $(\mathfrak{h}, \Pi, \check{\Pi})$ is the associative algebra over $\mathbb{C}$ with unity generated by E_i, F_i $(i \in I)$ and $T = span\{H_i\}$ satisfying the relations:*

1. $HH' = H'H$ *for* $H, H' \in T$

2. $E_iF_j - F_jE_i = \delta_{ij}H_i$ *for* $i, j \in I$

3. $HE_i - E_iH = \alpha_i(H)E_i$ *for* $H \in T,\ i \in I$

4. $HF_i - F_iH = -\alpha_i(H)F_i$ *for* $H \in T,\ i \in I$

5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} E_i^{1-a_{ij}-k} E_j E_i^k = 0$ *for* $i \neq j$

6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} F_i^{1-a_{ij}-k} F_j F_i^k = 0$ *for* $i \neq j$

Recall the realization $(\mathfrak{h}, \Pi, \check{\Pi})$ of $\mathcal{C}$. The set Π is the root basis and its elements are called simple roots. Similarly, $\check{\Pi}$ is the co-root basis and its elements are called simple co-roots. The root lattice and the positive root lattice are respectively defined as:

$$Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \quad \text{and} \quad Q^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$$

There is a partial ordering, $\geq$, on Q defined by $\alpha \geq \beta$ if and only if $\alpha - \beta \in Q^+$ for any $\alpha, \beta \in Q$.

Using the Chevalley generators we can see $\mathfrak{g}$ contains $|I|$ copies of $sl(2, \mathbb{C})$. For each $i \in I$, let $\mathfrak{g}_{(i)} = \mathbb{C}e_i + \mathbb{C}f_i + \mathbb{C}h_i$. Then $\mathfrak{g}_{(i)}$ is isomorphic to $sl(2, \mathbb{C})$ with basis $\{e_i, f_i, h_i\}$ and relations: $[e_i, f_i] = h_i$, $[h_i, e_i] = 2e_i$, $[h_i, f_i] = -2f_i$.

The roots give a useful decomposition of $\mathfrak{g}$ as described below.

Definition 2.2.4. *Let* $\mathfrak{g}$ *be a Kac-Moody Lie algebra with realization* $(\mathfrak{h}, \Pi, \check{\Pi})$*. For* $\alpha \in \mathfrak{h}^*$ *define*

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \big| [h, g] = \alpha(h)g \text{ for all } h \in \mathfrak{h}\}.$$

If $\mathfrak{g}_\alpha \neq \{0\}$ *and* $\alpha \neq 0$*, then* α *is called a root and* $\mathfrak{g}_\alpha$ *is a root space. Notice that* $\mathfrak{g}_0 = \mathfrak{h}$*. The dimension of* $\mathfrak{g}_\alpha$ *is the multiplicity of the root* α*. The height of* $\alpha = \sum_{\alpha_i \in \Pi} b_i\alpha_i \in Q$ *is* $\sum_{i \in I} |b_i|$ *and is denoted as* $ht(\alpha)$*.*

Let Δ denote the set of roots of $\mathfrak{g}$. Since any root is a linear combination of the simple roots with either all positive or all negative coefficients, the set of roots splits into the set of positive roots, $\Delta_+ = \Delta \cap Q^+$, and the set of negative roots

$\Delta_- = \Delta \cap -Q^+ = -\Delta_+$. The Lie algebra $\mathfrak{g}$ has a root space decomposition:

$$\mathfrak{g} = \left(\bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha\right) \oplus \mathfrak{h}$$

Further we know that $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$ and $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$.

Additionally there exists an automorphism $\eta : \mathfrak{g} \to \mathfrak{g}$ called the Chevalley involution given by $\eta(e_i) = -f_i$, $\eta(f_i) = -e_i$ and $\eta(h) = -h$ for all $h \in \mathfrak{h}$. Thus for all $\alpha \in \Delta_+$, $\eta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, $mult(\alpha) = mult(-\alpha)$ and $\Delta_- = -\Delta_+$.

Define the following subalgebras of $\mathfrak{g}$:

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$$

$$\mathfrak{n}^- = \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha$$

Then the triangular decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. The Borel subalgebra $\underline{b}$ is $\underline{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$. Thus $\mathfrak{g} = \underline{b} \oplus \mathfrak{n}^-$. Further we see that $\mathfrak{n}^+$ is generated by the e_i's and $\mathfrak{n}^-$ is generated by the f_i's.

Given a symmetrizable generalized Cartan matrix, $\mathcal{C} = DB$ where $D = diag\{s_1, s_2, \ldots, s_{n-1}\}$ (or $D = diag\{s_0, s_1, \ldots, s_{n-1}\}$ for affine type), there is an associated bilinear form on $\mathfrak{h}$.

Let $\mathcal{C} = (a_{ij})$, $i, j \in I = \{1, \ldots, n-1\}$, be a symmetrizable GCM of finite type. Define the bilinear form $(\cdot|\cdot)$ on $\mathfrak{h} = span\{H_i \big| i \in I\}$ by:

$$(H_i|H) = s_i\alpha_i(H) \text{ for } i \in I,\ H \in \mathfrak{h}$$

Now let $\mathcal{C} = (a_{ij}),\ i, j \in I = \{0, 1, \ldots n-1\}$, be a symmetrizable GCM of affine type. Define the bilinear form $(\cdot|\cdot)$ on $\mathfrak{h} = span\{h_0, h_1, \ldots, h_{n-1}, d\}$ by:

$$(h_i|h) = s_i \alpha_i(h) \text{ for } i \in I,\ h \in \mathfrak{h}$$

$$(h_i|d) = \alpha_i(d) \text{ for } i \in I$$

In each case, $(\cdot|\cdot)$ is a non-degenerate, invariant, symmetric, bilinear form. This form induces a non-degenerate, invariant, symmetric, bilinear form on $\mathfrak{h}^*$ through the vector space isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ defined by $\nu(h)(h') = (h|h')$. We use $(\cdot|\cdot)$ to denote the bilinear form on both $\mathfrak{h}$ and $\mathfrak{h}^*$. We also have that $(\alpha_i|\alpha_j) = b_{ij} = a_{ij}/s_i$ for $\alpha_i, \alpha_j \in \Pi$.

Additionally, $(\cdot|\cdot)$ can be uniquely extended to a non-degenerate, invariant, symmetric, bilinear form on $\mathfrak{g}$. This form on $\mathfrak{g}$ has the following properties:

$$\text{For } \alpha, \beta \in \Delta, g \in \mathfrak{g}_\alpha, g' \in \mathfrak{g}_\beta, \quad (g, g') = 0 \text{ whenever } \alpha + \beta = 0$$

$$and \quad (e_i, f_j) = \delta_{ij} s_i$$

Given a Kac-Moody Lie algebra $\mathfrak{g}$ there is an associated group of reflections called the Weyl group, denoted by $\mathcal{W}$. For $i \in I$, we define r_i on $\mathfrak{h}^*$ by $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$. r_i is called a simple reflection. The Weyl group, $\mathcal{W}$, is the subgroup of $Aut(\mathfrak{h}^*)$ generated by the set of simple reflections. Notice that $r_i(\alpha_i) = -\alpha_i$. The Weyl group is particularly important in Chapter 4.

Let w be any Weyl group element. Then w can be written as the product of simple reflections, i.e. $w = \prod_{j=1}^{t} r_{i_j}$ for some positive integer t. w is a reduced expression if t is minimal and t is then called the length of w. We let $l(w)$ denote the length of w.

If $\mathfrak{g}$ is a Kac-Moody Lie algebra of finite type, then all of the roots are in the Weyl orbit of simple roots. In other words, for all $\alpha \in \Delta$, there exits a $w \in W$ such that $w(\alpha) = \alpha_i$ for some $i \in I$. Additionally, in this case, we have that if α is a root, $k\alpha$ is a root only if k is 1 or -1. Notice that this says that the multiplicity of any root is one. However, this is not true in either the affine or indefinite case.

If $\mathfrak{g}$ is a Kac-Moody Lie algebra that is not of finite type, we have some roots that are Weyl conjugate to a simple root and some that are not. If a root is $\mathcal{W}$-conjugate to α_i, for some i, we say it is a real root. Otherwise, it is an imaginary root. Imaginary roots may have multiplicity greater than one. In addition, integer multiples of imaginary roots may also be roots. The set of real roots is denoted by Δ^{re} and the set of imaginary roots as Δ^{im}. Notice that $\Delta = \Delta^{re} \sqcup \Delta^{im}$.

If $\mathfrak{g}$ is of affine type, there is a root δ such that $\delta(h) = 0$ for all $h \in \mathfrak{h}$. This root is called the null root. The null root is imaginary. In fact in the affine case $\Delta^{im} = \{k\delta | k \in \mathbb{Z}\}$.

The PBW theorem tells us that $U(\mathfrak{g})$ also has a triangular decomposition. Namely,

$$U(\mathfrak{g}) = U^-(\mathfrak{g}) \otimes U^0 \otimes U^+(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$$

Similar to the root lattice we define the weight lattice as:

$$P = \{\lambda \in \mathfrak{h}^* \big| \lambda(h_i) \in \mathbb{Z}, h_i \in \check{\Pi}\}$$

The set of dominant weights is defined as :

$$P^+ = \{\lambda \in P | \lambda(h_i) \in \mathbb{Z}_{\geq 0}\}$$

Finally we define the coweight lattice as:

$$\check{P} = Hom_{\mathbb{Z}}(P, \mathbb{Z})$$

Note that $h_i \in \check{P}$ by defining $h_i(\lambda) = \lambda(h_i)$ for all $i \in I$.

Let V be a $\mathfrak{g}$-module. Then, similar to the root spaces, we define weight spaces.

Definition 2.2.5. *For $\lambda \in \mathfrak{h}^*$, define*

$$V_\lambda = \{v \in V \big| h \cdot v = \lambda(h)v \textit{ for all } h \in \mathfrak{h}\}$$

V_λ is called the λ weight space. If $V_\lambda \neq \{0\}$, we call λ a weight. The dimension of V_λ is called the weight multiplicity of λ in V and is denoted as $mult_V(\lambda)$. We denote the set of weights of V as $wt(V)$.

If V has a weight space decomposition, i.e. $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, it is called a weight module. If each of the weight spaces is finite dimensional, the character of V is defined as:

$$ch\ V = \sum_{\lambda \in \mathfrak{h}^*} (dim V_\lambda) e^\lambda$$

For $\lambda, \mu \in P$, we say $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$. We use this partial ordering of P to define the set $D(\lambda)$. For $\lambda \in P^+$, $D(\lambda) = \{\mu \in P \big| \mu \leq \lambda\}$. We say that $D(\lambda)$ is dominated by λ.

We are interested in modules from a certain category. The category $\mathcal{O}$ is the set of weight modules V over $\mathfrak{g}$ for which there exists a finite number of elements, $\lambda_1, \ldots, \lambda_s$, such that $wt(V) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s)$. The morphisms for $\mathcal{O}$ are homomorphisms. Category $\mathcal{O}$ is closed under finite direct sums, finite tensor products and quotients.

Of particular interest are elements of category $\mathcal{O}$ called highest weight modules.

Definition 2.2.6. *(cf. [4]) A weight module V is a highest weight module of highest weight $\lambda \in \mathfrak{h}^*$ if there exists a nonzero vector $u_\lambda \in V$, called a highest weight vector, such that:*

$$e_i(u_\lambda) = 0 \textit{ for all } i \in I$$

$$hu_\lambda = \lambda(h)u_\lambda \textit{ for all } h \in \mathfrak{h}$$

$$V = U(\mathfrak{g})u_\lambda$$

We denote this highest weight module as $V(\lambda)$.

In fact, if $V(\lambda)$ is a highest weight module then $V(\lambda) = U(\mathfrak{g})v_\lambda \cong U^-(\mathfrak{g})v_\lambda$. Further the dimension of $V(\lambda)_\lambda$ is 1 and the dimension of $V(\lambda)_\mu < \infty$ for all μ. $V(\lambda)$ has its own weight space decomposition, namely:

$$V(\lambda) = \bigoplus_{\mu \leq \lambda} V(\lambda)_\mu$$

Example 2.2.7. *Let $L = sl(2, \mathbb{C})$ as in Example 2.1.2. Let $V(\lambda)$ be the irreducible highest weight module of L with highest weight $\lambda \in \mathbb{C}$ and highest weight vector v_λ.*

Then $\lambda \in \mathbb{Z}_{\geq 0}$ and $V(\lambda)$ is $\lambda + 1$ dimensional with basis $\{v_i\}_{i=0,\dots,\lambda}$ where

$$v_i = \frac{f^i(v_\lambda)}{k!}.$$

Now let $\mathfrak{g}$ again be any Kac-Moody Lie algebra. Let $V(\lambda)$ be an integrable highest weight $\mathfrak{g}$-module with highest weight λ and highest weight vector u_λ. For each $\lambda \in \mathfrak{h}^*$, there exits a unique (up to isomorphism) highest weight module called a Verma module, denoted by $M(\lambda)$. The Verma module $M(\lambda)$ has the property that any highest weight $\mathfrak{g}$-module is a quotient of $M(\lambda)$. $M(\lambda)$ has a unique maximal submodule, $N(\lambda)$. $M(\lambda)/N(\lambda)$ is the irreducible highest weight module $V(\lambda)$ with highest weight λ. In fact, every irreducible $\mathfrak{g}$-module in category $\mathcal{O}$ is isomorphic to $M(\lambda)/N(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.

Given a weight module V over $\mathfrak{g}$, we call V integrable if all e_i and f_i, $(i \in I)$, are locally nilpotent on V. We also say that a weight is integral if $\lambda(h_i) \in \mathbb{Z}$ for all $i \in I$. Recall that the set of integral weights is contained in the weight lattice P. Now we can define an important subcategory of $\mathcal{O}$ called $\mathcal{O}_{int}$.

Definition 2.2.8. *[4] The category $\mathcal{O}_{int}$ consists of integrable $\mathfrak{g}$-modules in the category $\mathcal{O}$ such that $wt(V) \subset P$.*

Note that Verma modules, although in category $\mathcal{O}$, are not in $\mathcal{O}_{int}$. However, each irreducible integrable module is a highest weight module. If $\mathfrak{g}$ is of affine type, the integrable modules are direct sums of irreducible integrable modules.

Another important property of $\mathcal{O}_{int}$ concerns the Weyl group. If $V \in \mathcal{O}_{int}$ and λ is a weight of V, then for any Weyl element, $w \in \mathcal{W}$:

$$\dim(V_\lambda) = \dim(V_{w\lambda})$$

From now on we use $V(\lambda)$ to refer to these irreducible highest weight modules of $\mathfrak{g}$ in $\mathcal{O}_{int}$.

2.2.1 The Affine Special Linear Case

We now consider the realization of the affine special linear Lie algebra, $\widehat{sl}(n)$. Note that this is also denoted by $A^{(1)}_{n-1}$. We consider the base field to be $\mathbb{C}$ unless otherwise specified.

Consider the Cartan matrix $\mathcal{C} = (a_{ij})$, $i, j \in I = \{0, 1, \ldots, n-1\}$, with:

$$a_{ii} = 2 \quad \text{for all } i \in I$$

$$a_{i,j} = -1 \quad \text{for all } i, j \in I \text{ such that } |i-j| = 1$$

$$a_{0,n-1} = -1 = a_{n-1,0}$$

$$0 \quad otherwise$$

$\mathcal{C}$ is the generalized Cartan matrix for $\widehat{sl}(n)$. Notice we use the index set $I = \{0, 1, \ldots, n-1\}$ for $\mathcal{C}$. It is clear that $\mathcal{C}$ is indecomposable. Also we see that $\mathcal{C}$ is of affine type with $\theta = (1, 1, \ldots, 1)^T$.

The Dynkin diagram for $\widehat{sl}(n)$ is:

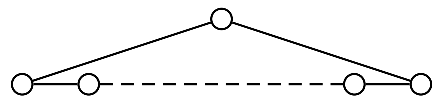


Figure 2.2: Dynkin diagram for $\widehat{sl}(n)$

Now we construct the triple $(\mathfrak{h}, \Pi, \check{\Pi})$ as previously discussed. So the root basis is $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ and $\alpha_j(h_i) = a_{ij}$ for all $i, j \in I$. The coroot basis is $\check{\Pi} = \{h_0, h_1, \ldots, h_{n-1}\} \subset \mathfrak{h}$. $\mathfrak{g} = \widehat{sl}(n)$ can then be constructed from the generators e_i and f_i $(i \in I)$ with relations 2.2. For example, $[h_1, e_2] = \alpha_2(h_1)e_2 = a_{12}e_2 = -e_2$.

Alternatively $\mathfrak{g} = \widehat{sl}(n)$ can be constructed from $L = sl(n)$. Recall that $L = \, < E_i, F_i, H_i \big| i = 1, 2, \ldots, n-1 >$ and is endowed with the commutator bracket. The affine Lie algebra $\widehat{sl}(n)$ is:

$$\mathfrak{g} = \widehat{sl}(n) = sl(n) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where c is central and d is the derivation $1 \otimes t\frac{d}{dt}$. The bracket structure on $\mathfrak{g}$ is defined as follows:

$$[x \otimes t^i, y \otimes t^j] := [x, y] \otimes t^{i+j} + (i)(tr(xy))\delta_{i+j,0}c \quad \forall x, y \in L \quad i, j \in \mathbb{Z}$$

$$[d, x \otimes t^i] := d(x \otimes t^i) = \left(1 \otimes t\left(\frac{d}{dt}\right)\right)(x \otimes t^i) = i(x \otimes t^i)$$

$$[c, g] := 0 \quad \forall g \in \mathfrak{g}$$

In this realization we have the following generating elements for $\mathfrak{g}$:

$$\begin{aligned} e_0 &= F_0 \otimes t & e_i &= E_i \otimes 1 \\ f_0 &= E_0 \otimes t^{-1} & f_i &= F_i \otimes 1 \\ h_0 &= -H_0 \otimes 1 + c & h_i &= H_i \otimes 1 \end{aligned}$$

The central element is $c = h_0 + h_1 + \ldots h_{n-1}$. The CSA, or maximal toral subalgebra, is $\mathfrak{h} = span_{\mathbb{C}}\{h_i, d | i \in I\}$. The simple roots then act on $\mathfrak{h}$ as follows, $\alpha_j(h_i) = a_{ij}$ for $i, j \in I$ and $\alpha_j(d) = \delta_{j0}$. Notice that the null root is $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$. Now $\mathfrak{g}$ is affine and has the imaginary roots $\Delta^{im} = \{k\delta \big| k \in \mathbb{Z}\}$.

As before the root lattice is

$$Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i = \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_{n-1}$$

and the positive root lattice is

$$Q^+ = \bigoplus_{i \in I}^{n-1} \mathbb{Z}_{\geq 0}\alpha_i.$$

Let Δ be the set of all roots. Again, $\mathfrak{g}_{\alpha_i} = span\{e_i\}$ and $\mathfrak{g}_{-\alpha_i} = span\{f_i\}$. Additionally we know that $\widehat{sl}(n)$ has the triangular decomposition $\widehat{sl}(n) = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, as discussed previously.

Notice that $\mathcal{C}$ is symmetric. So the invertible diagonal matrix D discussed previously is the identity. In other words, $s_i = 1$ for all $i \in I$. Thus the non-degenerate, invariant, symmetric, bilinear form, $(\cdot|\cdot)$ on $\mathfrak{h}$ is defined by:

$$(h_i|h) = \alpha_i(h) \text{ for all } i \in I, h \in \mathfrak{h}$$

$$(h_i|d) = \alpha_i(d) = \delta_{i0} \text{ for all } i \in I$$

This form is extended to a bilinear form on $\mathfrak{h}^*$. Then $(\alpha_i|\alpha_j) = a_{ij}$ for all $\alpha_i, \alpha_j \in \Pi$. Extending the form further to a non-degenerate, invariant, symmetric, bilinear form on $\mathfrak{g}$, we have that

$$\text{For } \alpha, \beta \in \Delta,\ g \in \mathfrak{g}_\alpha, g' \in \mathfrak{g}_\beta, \quad (g, g') = 0 \text{ whenever } \alpha + \beta = 0$$

$$\textit{and} \quad (e_i, f_j) = \delta_{ij}$$

Denote the fundamental weights of $\widehat{sl}(n)$ by $\Lambda_i \in \mathfrak{h}^*$ where $\Lambda_i(h_j) = \delta_{i,j}$ and $\Lambda_i(d) = 0$, for all $i, j \in I$. Then $P = \mathbb{Z}\Lambda_0 \oplus \ldots \oplus \mathbb{Z}\Lambda_{n-1} \oplus \mathbb{Z}\delta$ is the weight lattice and $P^+ = \{\lambda \in P \big| \lambda(h_i) \in \mathbb{Z}_{\geq 0}\} \subset P$ is the set of dominant weights. The coweight lattice is $\check{P} = span_{\mathbb{Z}}\{h_i, d \big| i \in I\} \subsetneq \mathfrak{h}$.

For any dominant weight λ we can assume, without loss of generality, that $\lambda = \sum_{i=0}^{n-1} k_i\Lambda_i$ since $V(\lambda + l\delta) \cong V(\lambda) \otimes V(l\delta)$ and $\dim V(l\delta) = 1$.

2.3 Quantum Groups

Now we will look at the quantum group associated with $\mathfrak{g}$ or the quantized universal enveloping algebra, $U_q(\mathfrak{g})$, corresponding to the symmetrizable generalized Cartan matrix $\mathcal{C} = (a_{ij})$. Recall that $\mathcal{C} = DB$, where $D = diag(s_0, s_1, \ldots, s_{n-1})$ and B is symmetric. There are many similarities between the universal enveloping algebra and $U_q(\mathfrak{g})$, particularly when 'q' is generic (i.e. not a root of unity).

Assume q to be a generic parameter that is not a root of unity (i.e. $q^s = 1$ if and only if $s = 0$). For each $n \in \mathbb{Z}$, we define the corresponding q-integer, $[n]_q$, as:

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Next we define the factorial of a q-integer to be:

$$[0]_q! = 1 \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q$$

For every $n, m \in \mathbb{Z}$ where $0 \leq n \leq m$, we define the corresponding q-binomial coefficient, $\begin{bmatrix} m \\ n \end{bmatrix}_q$, as:

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}$$

Observe that as q approaches 1, $[n]_q \to n$, $[n]_q! \to n!$ and $\begin{bmatrix} m \\ n \end{bmatrix}_q \to \binom{m}{n}$.

Definition 2.3.1. *(cf. [4]) Define the quantum group $U_q(\mathfrak{g})$ associated with the symmetrizable generalized Cartan matrix $\mathcal{C}$ to be the associative algebra over $\mathbb{Q}(q)$ with* *unity generated by* $\{q^t, e_i, f_i \big| i \in I,\ t \in \check{P}\}$ *with the following defining relations:*

$$\begin{array}{rll}
(i) & q^0 = 1, q^t q^{t'} = q^{t+t'} & \text{for all } t, t' \in \check{P} \\
(ii) & q^t e_i q^{-t} = q^{\alpha_i(t)} e_i & \text{for all } t \in \check{P},\ i \in I \\
(iii) & q^t f_i q^{-t} = q^{-\alpha_i(t)} f_i & \text{for all } t \in \check{P},\ i \in I \\
(iv) & e_i f_j - f_j e_i = \delta_{i,j} \left(\frac{K_i - K_i^{-1}}{q - q^{-1}} \right) & i, j \in I \\
(v) & \sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} e_i^{1-a_{ij}-n} e_j e_i^n = 0 & i \neq j \\
(vi) & \sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} f_i^{1-a_{ij}-n} f_j f_i^n = 0 & i \neq j
\end{array}$$

where $q_i = q^{s_i}$ *and* $K_i = q^{s_i h_i}$*.*

Note that $\lim_{q \to 1} U_q(\mathfrak{g}) = U(\mathfrak{g})$.

Example 2.3.2. *Again consider the Lie algebra* $L = sl(2, \mathbb{C})$ *as in Example* 2.1.2. $U_q(sl(2, \mathbb{C}))$ *is the associative algebra over* $\mathbb{Q}(q)$ *with generators* $\{e, f, q^{\pm h}\}$ *and relations:*

$$q^h e q^{-h} = q^2 e, \quad q^h f q^{-h} = q^{-2} f, \quad [e, f] = \frac{q^h - q^{-h}}{q - q^{-1}}.$$

Recall that $\mathfrak{g}_{(i)} = \mathbb{C}e_i + \mathbb{C}h_i + \mathbb{C}f_i$ is isomorphic to $sl(2, \mathbb{C})$. Similarly, $U_q(\mathfrak{g}_{(i)})$ is isomorphic to $U_q(sl(2, \mathbb{C}))$.

$U_q(\mathfrak{g})$ has both root space and triangular decompositions (see [4]):

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} (U_q)_\alpha \text{ where } (U_q)_\alpha = \{u \in U_q(\mathfrak{g}) \big| q^h u q^{-h} = q^{\alpha(h)} u \text{ for all } h \in \check{P}\}$$

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^+) = U_q(\mathfrak{n}^-) U_q(\underline{b})$$

$U_q(\mathfrak{g})$ has a Hopf algebra structure (as does $U(\mathfrak{g})$). Let ρ denote the comultiplication, ϵ the counit, and S the antipode. The structure is defined as follows:

- $\rho(q^h) = q^h \otimes q^h$ for all $h \in \check{P}$
- $\rho(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i$ for all $i \in I$
- $\rho(f_i) = f_i \otimes 1 + K_i \otimes f_i$ for all $i \in I$ (2.3)
- $\epsilon(q^h) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0$ for all $i \in I,\ h \in \check{P}$
- $S(q^h) = q^{-h}, \quad S(e_i) = -e_i K_i, \quad S(f_i) = -K_i^{-1} f_i$ for all $i \in I,\ h \in \check{P}$

As discussed in [4], the representation theory of $U_q(\mathfrak{g})$ is similar to that of the Kac-Moody Lie algebra $\mathfrak{g}$. Below we will discuss a few of the important features of $U_q(\mathfrak{g})$.

Let V^q be a $U_q(\mathfrak{g})$-module.

Definition 2.3.3.

$$V^q_\lambda = \{v \in V^q \big| q^h v = q^{\lambda(h)} v \text{ for all } h \in \check{P}\}$$

If $V^q_\lambda \neq \{0\}$, λ is a weight of V^q and V^q_λ is a weight space. $v \in V^q_\lambda$ is called a weight vector. If $e_i v = 0$ for all $i \in I$, v is called a maximal vector.

V^q, a $U_q(\mathfrak{g})$-module, is a weight module if it admits a weight space decomposition. Namely,

$$V^q = \bigoplus_{\lambda \in P} V^q_\lambda.$$

Theorem 2.3.4. *(cf. [4]) If $\lambda \in P^+$ and $V^q(\lambda)$ is the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight λ, then as $q \to 1$, $V^q(\lambda) \to V^1(\lambda) = V(\lambda)$, the irreducible highest weight module $V(\lambda)$ over $U(\mathfrak{g})$ with highest weight λ.*

The category $\mathcal{O}^q$ consists of weight modules V^q over $U_q(\mathfrak{g})$ with finite dimensional weight spaces for which there exists a finite number of elements $\lambda_1, \ldots, \lambda_s \in P$ such that

$$wt(V^q) \subset D(\lambda_1) \cup \ldots \cup D(\lambda_s).$$

A weight module $V^q(\lambda)$ is a highest weight module with highest weight $\lambda \in P$ if there exists a nonzero vector $u_\lambda \in V^q$ such that

$$e_i u_\lambda = 0 \text{ for all } i \in I$$

$$q^t u_\lambda = q^{\lambda(t)} u_\lambda \text{ for all } h \in \check{P}$$

$$V^q(\lambda) = U_q(\mathfrak{g}) u_\lambda$$

u_λ is still called the highest weight vector and is unique up to scalar multiple. Again $\dim V^q(\lambda)_\lambda = 1$, $\dim V^q(\lambda)_\mu < \infty$ for all μ and $V^q(\lambda)$ admits its own weight space decomposition as a direct sum of weight spaces ,$V^q(\lambda) = \bigoplus_{\mu < \lambda} V^q(\lambda)_\mu$.

We again can define Verma modules and their maximal submodules in the same manner as before.

Now we define the category $\mathcal{O}^q_{int}$.

Definition 2.3.5. *(cf. [4]) The category $\mathcal{O}^q_{int}$ consists of $U_q(\mathfrak{g})$-modules V^q satisfying the following conditions:*

- *V^q has a weight space decomposition $V^q = \bigoplus_{\lambda \in P} V^q_\lambda$, where $V^q_\lambda = \{v \in V^q | q^h v = q^{\lambda(h)} v\}$ and $dim V^q_\lambda < \infty$ for all $\lambda \in P$*

- *There exist a finite number of elements $\lambda_1, \ldots, \lambda_s \in P$ such that*

$$wt(V^q) \subset D(\lambda_1) \cup \ldots \cup D(\lambda_s)$$

- *All e_i and f_i, $i \in I$ are locally nilpotent on V^q*

The morphisms are $U_q(\mathfrak{g})$-module homomorphisms.

Again we are interested in irreducible integrable highest weight modules, $V^q(\lambda)$.

Set $e_i^{(k)} = e_i^k/[k]_{q_i}!$ and $f_i^{(k)} = f_i^{(k)}/[k]_{q_i}!$. These are called the divided powers of e_i and f_i. For the case of $U_q(\widehat{sl}(n))$, $q_i = q$ since $s_i = 1$. Thus $e_i^{(k)} = e_i^k/[k]_q!$ and $f_i^{(k)} = f_i^{(k)}/[k]_q!$. Let $V^q(\lambda)$ be an irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight λ. Then, by $U_q(sl(2))$-representation theory, for any element v in the μ weight space $V^q_\lambda = V^q(\lambda)_\mu$, we can uniquely write v as

$$v = \sum_{k \geq 0} f_i^{(k)} v_k$$

where $v_k \in ker(e_i) \cap V^q_{\mu + k\alpha_i}$. We use this to define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$, called the Kashiwara operators, on $V(\lambda)$ as:

$$\tilde{e}_i(v) = \sum_{k \geq 1} f_i^{(k-1)} v_k$$

$$\tilde{f}_i(v) = \sum_{k \geq 0} f_i^{(k+1)} v_k$$

These Kashiwara operators have great significance in later chapters.

2.3.1 The Quantum Affine Special Linear Case

Now consider this in light of the quantum affine special linear algebra, $U_q(\widehat{sl}(n))$. Recall Definition 2.3.1 of quantum group. Since the s_i are the entries of the invertible diagonal matrix D which symmetrizes the Cartan matrix, each $s_i = 1$ for $\widehat{sl}(n)$. So $q_i = q$ and $K_i = q^{h_i}$. Thus in the case of $U_q(\widehat{sl}(n))$, conditions (iv), (v) and (vi) in 2.3.1 simplify to:

$$\begin{array}{rll} (iv) & e_i f_j - f_j e_i = \delta_{i,j} \left(\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \right) & i, j = 0, 1, \ldots, n-1 \\ (v) & \sum\limits_{n=0}^{1-a_{ij}} (-1)^n \left[{1-a_{ij} \atop n} \right]_q e_i^{1-a_{ij}-n} e_j e_i^n = 0 & i \neq j \\ (vi) & \sum\limits_{n=0}^{1-a_{ij}} (-1)^n \left[{1-a_{ij} \atop n} \right]_q f_i^{1-a_{ij}-n} f_j f_i^n = 0 & i \neq j \end{array}$$

Similarly the relations for the Hopf algebra structure labeled 2.3 simplify to:

- $\rho(q^h) = q^h \otimes q^h$ for all $h \in \check{P}$
- $\rho(e_i) = e_i \otimes q^{-h_i} + 1 \otimes e_i$ for all $i \in I$
- $\rho(f_i) = f_i \otimes 1 + q^{h_i} \otimes f_i$ for all $i \in I$
- $\epsilon(q^h) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0$ for all $i \in I, \ h \in \check{P}$
- $S(q^h) = q^{-h}, \quad S(e_i) = -e_i q^{h_i}, \quad S(f_i) = -q^{-h_i} f_i$ for all $i \in I, \ h \in \check{P}$

The representation theory discussed previously carries over with no modification needed. For ease of notation, we supress the superscript q in subsequent chapters.

Additionally, we now only consider the cases where $L = sl(n)$ and $\mathfrak{g} = \widehat{sl}(n)$. The quantum group that we consider is $U_q(\widehat{sl}(n))$.

Chapter 3

Crystal Base

In this chapter we give the definition of crystal base then realize crystals for irreducible highest weight modules of $U_q(\widehat{sl}(n))$ in terms of combinatorial objects called extended Young diagrams. Recall that a crystal basis can be roughly thought of as a base at the $q = 0$ limit of $U_q(\mathfrak{g})$.

3.1 Crystal Base

Let V be an integrable $U_q(\mathfrak{g})$-module. Recall, by $U_q(sl(2))$-representation theory, for each i, any $v \in V_\lambda$ can be written as $v = \sum_{k \geq 0} f_i^{(k)} v_k$ where $f_i^{(k)} = f_i^k / [k]_q!$ and $v_k \in$ $ker\ e_i \cap V_{\lambda + k\alpha_i}$. Also recall that the Kashiwara operators are the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ on V defined by :

$$\tilde{e}_i(v) = \sum_{k \geq 1} f_i^{(k-1)} v_k \text{ and } \tilde{f}_i(v) = \sum_{k \geq 0} f_i^{(k+1)} v_k.$$

Define A to be the set of rational functions without poles in $\mathbb{Q}(q)$. i.e.

$$A = \left\{ \frac{f(q)}{g(q)} \in \mathbb{Q}(q) \big| g(0) \neq 0 \right\}$$

Kashiwara [11, 13] defines a pair (L, B) to be a crystal base of the integrable $U_q(\mathfrak{g})$-module V if L is an A-lattice of V and B is a $\mathbb{Q}$-base of L/qL satisfying:

- $L = \bigoplus_{\lambda \in P} L_\lambda$ where $L_\lambda = L \cap V_\lambda$ and $V = \mathbb{Q}(q) \otimes_A L$
- $B = \bigcup_{\lambda \in P} B_\lambda$ where $B_\lambda = B \cap (L_\lambda / qL_\lambda) \neq \{0\}$
- $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$
- $\tilde{e}_i B \subset B \sqcup \{0\}$ and $\tilde{f}_i B \subset B \sqcup \{0\}$
- For $b, b' \in B$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$

The crystal graph associated with (L, B) is a color oriented graph with B the set of vertices and i-colored arrows defined by $b \xrightarrow{i} b'$ if and only if $\tilde{f}_i b = b'$. B is called the crystal for V.

Note that A is an integrable domain with its fraction field $\mathbb{Q}(q)$.

Let $V(\lambda)$ be an irreducible highest weight module of $U_q(\widehat{sl}(n))$ with highest weight vector u_λ. Then the crystal base $(L(\lambda), B(\lambda))$ is given by [18]:

$$L(\lambda) = \sum_{\substack{l > 0 \\ i_1, \ldots, i_l \in I}} A \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda,$$

$$B(\lambda) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda \ mod\ qL(\lambda) \big| l \geq 0,\ i_1, \ldots, i_l \in I \} \backslash \{0\}.$$

$B(\lambda)$ can be explicitly realized in terms of some combinatorial objects called extended Young diagrams [7]. This realization is discussed later in the chapter.

Let $V(\lambda)$ be a highest weight $U_q(\widehat{sl}(n))$-module with crystal base $(L(\lambda), B(\lambda))$. Then for $b \in B(\lambda)$, set

$$\epsilon_i(b) = \max\{k \geq 0 \big| \tilde{e}_i^k b \neq 0\}$$

$$\phi_i(b) = \max\{k \geq 0 | \tilde{f}_i^k b \neq 0\}$$

If $\epsilon_i(b) = 0$ for all $i \in I$, then $\tilde{e}_i(b) = 0$ for all $i \in I$. We call such a $b \in B(\lambda)$ a highest weight element.

Crystal bases behave well over tensor product as seen in the following theorem.

Theorem 3.1.1. *[11] Let (L_j, B_j) be a crystal base of an integrable $U_q(\mathfrak{g})$-module V_j for $j = 1, 2$. Set $L = V_1 \otimes_A V_2$, $B = \{b_1 \otimes b_2 | b_j \in B_j\}$. Then (L, B) is a crystal base of $V_1 \otimes V_2$ where the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ act as follows.*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 \ if \phi_i(b_1) \geq \epsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 \ if \phi_i(b_1) < \epsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 \ if \phi_i(b_1) > \epsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 \ if \phi_i(b_1) \leq \epsilon_i(b_2) \end{cases}$$

Now abstracting the properties of crystal base, we define a crystal as follows.

Definition 3.1.2. *[10] A crystal is a set B with maps $\tilde{e}_i$, $\tilde{f}_i : B \cup \{0\} \to B \cup \{0\}$ with* *the following properties for each $i \in I$.*

$$\tilde{e}_i 0 = 0 = \tilde{f}_i 0$$

$$\textit{For all } b \in B, i \in I, \textit{ there is an } n \in \mathbb{Z}_{>0} \textit{ such that } \tilde{e}_i^n b = 0 = \tilde{f}_i^n b$$

$$\textit{For all } b, b' \in B,\ i \in I,\ b' = \tilde{f}_i b \textit{ if and only if } b = \tilde{e}_i b'$$

We say that a crystal B is P weighted if $B = \bigcup_{\lambda \in P} B_\lambda$ and for each $i \in I, b \in B_\lambda$ the following hold:

$$\tilde{e}_i b \in B_{\lambda+\alpha_i} \cup \{0\}$$
$$\tilde{f}_i b \in B_{\lambda-\alpha_i} \cup \{0\}$$
$$\lambda(h_i) = \phi_i(b) - \varepsilon_i(b)$$

In the next subsection we discuss extended Young diagrams and use these to realize $B(\lambda)$, the crystal for the highest weight module $V(\lambda)$.

3.2 Extended Young Diagram Realization

Extended Young diagrams are basically colored Young tableau and are associated with a sequence rather than a partition. More formally, we have the following definition:

Definition 3.2.1. *[18] An extended Young diagram, $Y = (y_k)_{k \geq 0}$, is a weakly increasing sequence with integer entries such that there exists some fixed y_∞ with $y_k = y_\infty$ for $k >> 0$. We call y_∞ the charge of the extended Young diagram Y.*

Example 3.2.2.

$$Y = (-4, -3, -1, -1, 1, 1, \ldots)$$

$$-4 \leq -3 \leq -1 \leq -1 \leq 1 \leq 1 \ldots \quad \textit{and charge } y_\infty = 1$$

With each sequence, we associate a diagram in the following way. Draw Y in the $\mathbb{Z} \times \mathbb{Z}$ right half-plane as a diagram of connected columns where each column has depth y_i. Then truncate the diagram at the charge height and fill in boxes. A general illustration of this process is in Figure 3.1 and a specific example follows. At this point, since color has not yet been assigned, it is important to keep track of the charge.

$$
\begin{aligned}
Y &= (y_1, y_2, \ldots, y_{k-1}, y_k, \ldots) \\
&= (y_1, y_2, \ldots, y_{k-1}, y_\infty, y_\infty, \ldots)
\end{aligned}
$$

$$\updownarrow$$

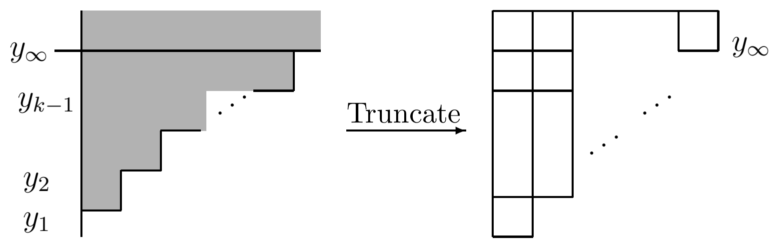


Figure 3.1: General extended Young diagram construction

Example 3.2.3.

$$Y = (-4, -3, -1, -1, 1, 1, \ldots)$$

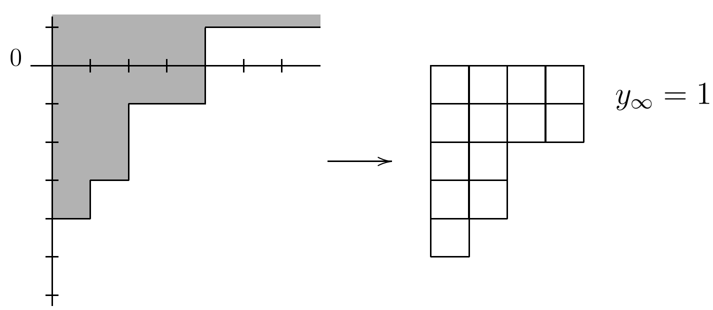


Figure 3.2: Extended Young diagram construction for $Y = (-4, -3, -1, -1, 1, 1, \ldots)$

Denote a k-tuple of extended Young diagrams, $(Y_1, Y_2, \ldots, Y_k)$, by $\mathcal{Y}$.

Let $|Y|$, called the width of Y, denote the number of boxes in the top, or widest, row of Y. $|\mathcal{Y}|$ is the maximum of the $|Y_i|$ for $i = 1, 2, \ldots, k$.

We define the diagonal number of each box to be $d = a + b$ where (a,b) is the coordinate of either the upper-left or lower-right corner of the box. Then we color each box with the i-color if $d \cong i \ mod \ n$. There are n colors labeled $0, 1, \ldots, n-1$, one corresponding to each simple root. If $n > 2$, we will forgo coloring the box and denote an i-colored box as $\boxed{i}$. Notice that the charge color will always be the color of the top left box since this box has coordinates $(0, y_\infty)$. In addition, all boxes on the same diagonal will have the same color.

Example 3.2.4.

$$Y = (-2, -1, -1, 1, 1, \ldots) \ and \ n = 3$$

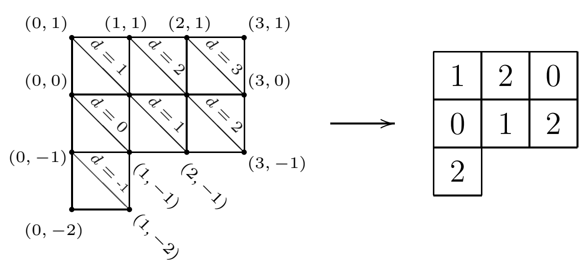


Figure 3.3: Extended Young diagram for $Y = (-2, -1, -1, 1, 1, \ldots)$ and $n = 3$

We define the weight of Y, denoted by $wt(Y)$, by

$$wt(Y) = \Lambda_{charge} - \sum_{i=0}^{n-1} m_i \alpha_i$$

where m_i is the number of i-colored boxes in Y. So in the previous example, the $wt(Y)$ is $\Lambda_1 - 2\alpha_0 - 2\alpha_1 - 3\alpha_2$. The weight of a k-tuple, $\mathcal{Y}$, is defined as: $wt(\mathcal{Y}) = \sum_{i=1}^{k} wt(Y_i)$.

In order to describe the action of the Kashiwara operators, we define corners and i-signatures. A corner occurs in a diagram, $Y = (y_i)_{i\geq 0}$, whenever $y_k \neq y_{k+1}$. A corner may be convex, $\lrcorner$, meaning that it is occupied by a box, or concave, $\ulcorner$, meaning that it is empty. Now we construct i-signatures, one corresponding to each color.

Definition 3.2.5. *[18] The (reduced) i-signature of a k-tuple of extended Young diagrams, $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_k)$, is a sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m)$ such that:*

1. $\sum_{j=1}^{k} \#\{i - corners\ of\ Y_j\} = m$
2. *Each corner, r, is assigned a pair $(d(r), j(r))$, where $d(r) =$ the diagonal number and $j(r) =$ the index of the diagram containing the corner.*
3. *The corners, r_i, are ordered in the following way: $r_1 < r_2$ iff $(d(r_1), j(r_1)) > (d(r_2), j(r_2))$ where we define $(d, j) > (d', j')$ iff either (i) $d > d'$ or (ii) $d = d'$ and $j < j'$.*
4. *Each $\varepsilon_r = 0$ or 1; $\varepsilon_r = 0$ if the corresponding corner is concave and $\varepsilon_r = 1$ if the corner is convex.*
5. *All $(0, 1)$ pairs are recursively deleted.*

Example 3.2.6. *Let $n = 3$. Consider $\mathcal{Y} = (Y_1, Y_2, Y_3)$ where $Y_1 = (-1, 0, 0, \ldots)$, $Y_2 = (-1, 0, 1, 1, \ldots)$, and $Y_3 = (0, 0, 2, 2, \ldots)$. $\mathcal{Y}$ is in $B(\Lambda_0 + \Lambda_1 + \Lambda_2)$. Notice that the diagram charges 'agree' with $\lambda = \Lambda_0 + \Lambda_1 + \Lambda_2$; i.e. the charge of Y_1 is 0, the charge* *of* Y_2 *is 1 and the charage of* Y_3 *is 2. This notion of agreement between the charges of the* Y_i *and the weight* λ *will be discussed in more detail later.*

$$\mathcal{Y} = \left(\begin{array}{|c|}\hline 0 \\ \hline \end{array}\ , \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 0 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}\ , \begin{array}{|c|c|}\hline 2 & 0 \\ \hline 1 & 2 \\ \hline \end{array} \right)$$

Figure 3.4: $\mathcal{Y}$ as in Example 3.2.6; $\mathcal{Y} \in B(\Lambda_0 + \Lambda_1 + \Lambda_2)$

Labeling each corner with (d, j) *we have:*

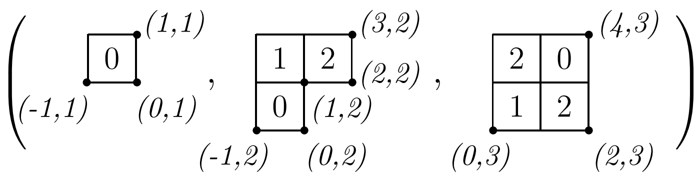


Figure 3.5: $\mathcal{Y}$, as in Example 3.2.6, labeled with pairs (d, j)

The ordering defined previously gives the following order for each set of colored corners in this example:

$$\begin{aligned} 0-corners: &\quad (3,2) > (0,1) > (0,2) > (0,3) \\ 1-corners: &\quad (4,3) > (1,1) > (1,2) \\ 2-corners: &\quad (2,2) > (2,3) > (-1,1) > (-1,2) \end{aligned}$$

Completing the i-signatures by assigning the appropriate 0*'s and* 1*'s, we get:*

$$\begin{aligned} \varepsilon_0(\mathcal{Y}) &= (0,1,1,0) \rightsquigarrow (1,0) \\ \varepsilon_1(\mathcal{Y}) &= (0,0,0) \\ \varepsilon_2(\mathcal{Y}) &= (1,1,0,0) \end{aligned}$$

We now define the actions of the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathcal{Y}$. Let $\mathcal{Y} =$

$(Y_1, Y_2, \ldots, Y_k)$ and $\mathcal{Y}' = (Y_1', Y_2', \ldots, Y_k')$ be two k-tuples of extended Young diagrams. Then we define the actions of $\tilde{e}_i\mathcal{Y} = \mathcal{Y}'$ and $\tilde{f}_i\mathcal{Y} = \mathcal{Y}'$ if and only if [18]:

- The charge of Y_j is the same as the charge of Y_j' for $j = 1, 2, \ldots k$.
- $\varepsilon_i = (\varepsilon_1, \ldots, \varepsilon_m)$ and $\varepsilon_i' = (\varepsilon_1', \ldots, \varepsilon_m')$ are the i-signatures of Y and Y' respectively.
- The number of reductions (or # of (0,1) pairs deleted) to achieve the signature for ε_i and ε_i' are the same. Additionally the index of each deletion should be the same. In other words, if $(\varepsilon_j, \varepsilon_{j+1})$ was deleted during the reduction of ε_i, the corresponding pair, $(\varepsilon_j', \varepsilon_{j+1}')$, must have been deleted during the reduction of ε_i'.
- To define $\tilde{e}_i(\mathcal{Y}) = \mathcal{Y}'$, $\exists k$ such that $\varepsilon_k' = 0$ and $\varepsilon_k = 1$ or to define $\tilde{f}_i(\mathcal{Y}) = \mathcal{Y}'$, $\exists k$ such that $\varepsilon_k' = 1$ and $\varepsilon_k = 0$. In addition, for either case, the following is satisfied:
$$\varepsilon_j = \varepsilon_j' = 1 \text{ for all } j < k$$
$$\varepsilon_j = \varepsilon_j' = 0 \text{ for all } j > k.$$

If no such $\mathcal{Y}'$ exists, we define the action to be 0. If $\mathcal{Y}'$ does exist, the actions are defined as follows:

- $\tilde{e}_i(\mathcal{Y})$ deletes an i-colored box of $\mathcal{Y}$ such that the last 1 in the i-signature becomes a 0
- $\tilde{f}_i(\mathcal{Y})$ adds an i-colored box to $\mathcal{Y}$ such that the first 0 in the i-signature becomes a 1

It is clear that the action of $\tilde{e}_i$ increases the weight of $\mathcal{Y}$ by α_i and the action of $\tilde{f}_i$ decreases the weight of $\mathcal{Y}$ by α_i.

Example 3.2.7. *Let $\mathcal{Y}$ be defined as in the previous example. It is easy to verify that $\tilde{e}_i$ and $\tilde{f}_i$ act as follows:*

$$\tilde{e}_0(\mathcal{Y}) = \left(\ytableaushort{0}\, , \ytableaushort{12}\, , \ytableaushort{20,12} \right) \qquad \tilde{f}_0(\mathcal{Y}) = \left(\ytableaushort{0}\, , \ytableaushort{12,0}\, , \ytableaushort{20,12,0} \right)$$

$$\varepsilon_0 = (0,1,0,0) \rightsquigarrow (0,0) \qquad \varepsilon_0 = (0,1,1,1) \rightsquigarrow (1,1)$$

$$\tilde{e}_1(\mathcal{Y}) = 0 \qquad \tilde{f}_1(\mathcal{Y}) = \left(\ytableaushort{0}\, , \ytableaushort{12,0}\, , \ytableaushort{201,12} \right)$$

$$\varepsilon_1 = (1,0,0)$$

$$\tilde{e}_2(\mathcal{Y}) = \left(\ytableaushort{0}\, , \ytableaushort{12,0}\, , \ytableaushort{20,1} \right) \qquad \tilde{f}_2(\mathcal{Y}) = \left(\ytableaushort{0,2}\, , \ytableaushort{12,0}\, , \ytableaushort{20,12} \right)$$

$$\varepsilon_2 = (1,0,0,0) \qquad \varepsilon_2 = (1,1,1,0)$$

Figure 3.6: Action of the Kashiwara operators on $\mathcal{Y}$ as in Example 3.2.6

Using the Kashiwara operators, we can construct the crystal $B(\lambda)$ corresponding to the highest weight module $V(\lambda)$. Say $\lambda = k_0\Lambda_0 + k_1\Lambda_1 + \ldots + k_{n-1}\Lambda_{n-1}$. Since the crystal is a highest weight crystal, and the $\tilde{f}_i$ lower the weight by adding boxes, we begin with a sequence of $k = k_0 + k_1 + \ldots + k_{n-1}$ (empty) concave corners. The first k_0 corners have charge 0, the next k_1 corners have charge 1 and so on. Next we apply each of the $\tilde{f}_i$. This repeated application creates $B(\lambda)$.

Example 3.2.8. *Following is a partial crystal graph of $B(\Lambda_0 + \Lambda_1)$ when $n = 2$. In this example the zero color is white and the one color is black. The action of $\tilde{f}_0$ is indicated by a down-left arrow and the action of $\tilde{f}_1$ is indicated by a down-right arrow.*

We say that two extended Young diagrams, Y and Y', are contained in each other if they are contained as diagrams. i.e. $Y \subseteq Y'$ if $y_i \geq y_i'$ for all i. For k-tuples, $\mathcal{Y} \subseteq \mathcal{Y}'$ if $Y_j \subseteq Y_j'$ for $j = 1, 2, \ldots, k$.

Define $Y[n]$ to be a shift of the diagram Y by n units. If $Y = (y_1, y_2, \ldots)$ then $Y[n] = (y_1 + n, y_2 + n, \ldots)$. Note that for any extended Young diagram Y, $Y[n] \subset Y$ if $n > 0$, $Y[n] = Y$ if $n = 0$, and $Y[n] \supset Y$ is $n < 0$.

Given the need for the charge of Y_i to appropriately correspond to the weight, as disccussed in Example 3.2.6, we define the set of appropriate diagrams $\mathcal{Y}(\lambda)$.

Definition 3.2.9. *Let $\lambda = \Lambda_{\gamma_1} + \Lambda_{\gamma_2} + \ldots + \Lambda_{\gamma_k}$ with $0 \leq \gamma_1 \leq \ldots \leq \gamma_k \leq n-1$. Define $\mathcal{Y}(\lambda) = \{\mathcal{Y} = (Y_1, Y_2, \ldots, Y_k) \big| Y_i$ has charge $\gamma_i\}$.*

Jimbo, Misra, Miwa and Okado used these defnitions to give an explicit realization for all of the $\mathcal{Y}$ in the highest weight crystal.

Theorem 3.2.10. *[7] Let $B(\lambda)$ be the crystal for the highest weight $U_q(\widehat{sl}(n))$ module $V(\lambda)$. Then*

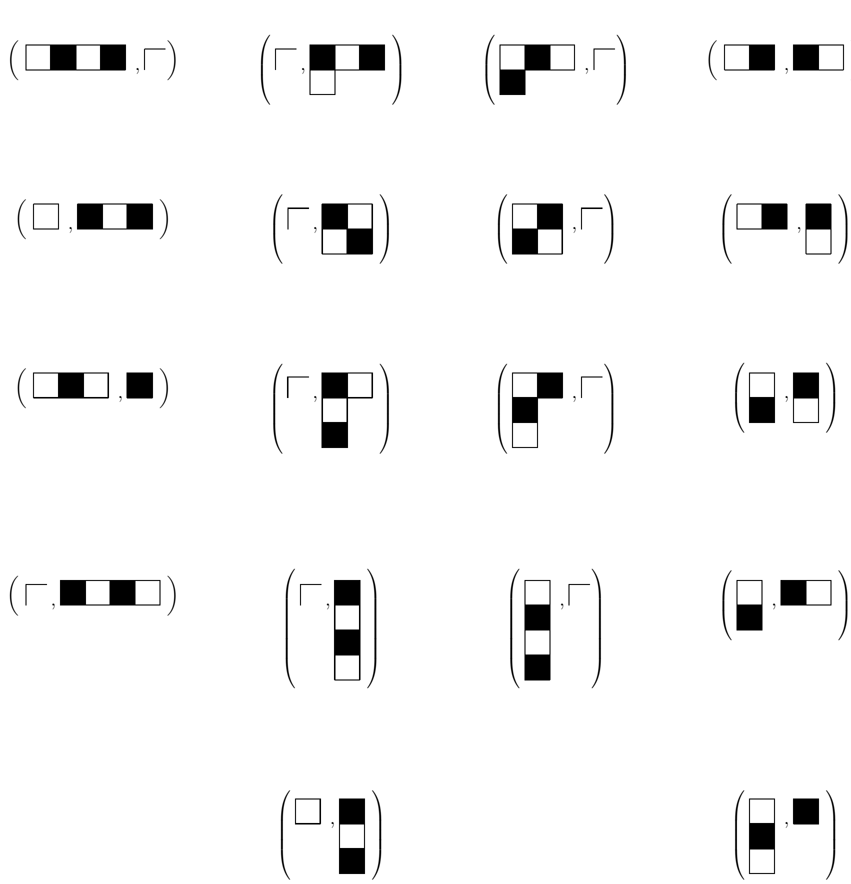
$$B(\lambda) = \{\mathcal{Y} \in \mathcal{Y}(\lambda) \big| Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_k \supseteq Y_{k+1} = Y_1[n],$$

$$and\ for\ each\ j > 0,\ \exists\ some\ 1 \leq i \leq k\ s.t.\ (Y_{i+1})_j \gneq (Y_i)_{j+1}\}$$

Example 3.2.11. *We can use this theorem to find all possible tuples in a specific $B(\lambda)$. For example, let $n = 2$ and $\lambda = \Lambda_0 + \Lambda_1$. Consider the weight space for $\mu = \lambda - 2\alpha_0 - 2\alpha_1$. Following is a table containing the potential 2-tuples of diagrams for $B(\lambda)$ with weight μ.*

Table 3.1: All possible tuples of weight $\Lambda - 2\alpha_0 - 2\alpha_1$ in $B(\Lambda_0 + \Lambda_1)$

I *II* *III* *IV*



Column (I) contains all diagrams of weight μ that are indeed in $B(\lambda)$, so they satisfy both of the conditions of Theorem 3.2.10. Column (II) contains all of the diagrams which fail the first required containment, $Y_1 \supseteq Y_2$. Column (III) contains diagrams which fail the second required containment, $Y_2 \supseteq Y_3 = Y_1[2]$. The final column, (IV), contains all diagrams which fail the inequality condition of Theorem 3.2.10.

Since these are all of the possible diagrams constructed from two white and two black boxes with the charge of $Y_1 = 0$ and the charge of $Y_2 = 1$, we see that the dimension of $V(\lambda)_\mu$ is four.

Let $B(\lambda)$ denote the crystal for the highest weight $U_q(\widehat{sl}(n))$-module $V(\lambda)$. We define the subset $B_L(\lambda)$ as:

$$B_L(\lambda) = \{\mathcal{Y} \in B(\lambda) \big| |\mathcal{Y}| \leq L\}$$

This subset proves to be important in Chapter 4.

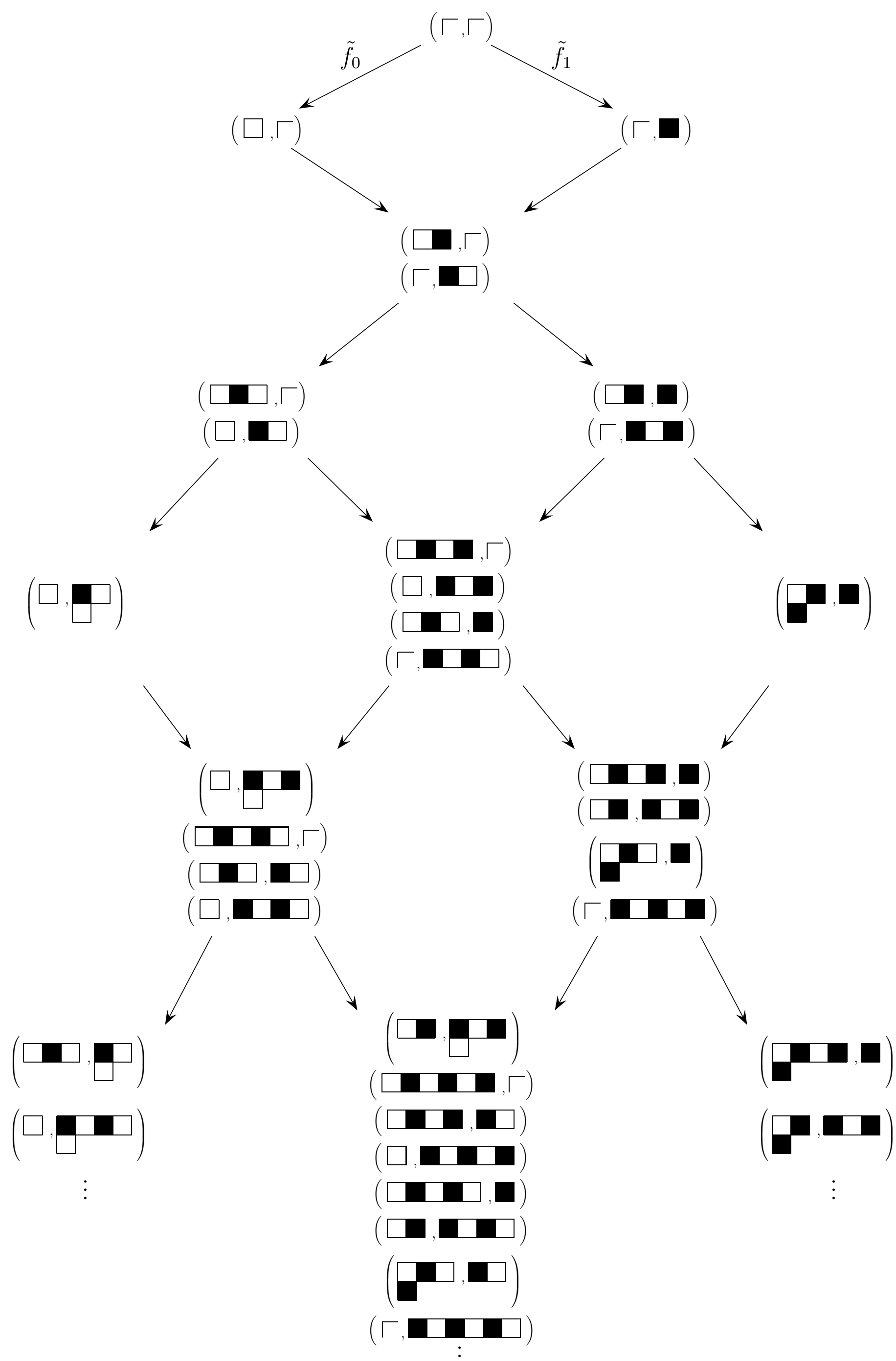


Figure 3.7: Partial crystal graph of $B(\Lambda_0 + \Lambda_1)$ when $n = 2$

Chapter 4

Demazure Crystals

In this chapter we discuss Demazure modules and their corresponding crystals. Then we present new results which provide an explicit realization of a large class of these modules in terms of extended Young diagrams.

Recall that $\mathfrak{g}$ can be decomposed into a Borel subalgebra and a negative piece, $\mathfrak{g} = \underline{b} \oplus \mathfrak{n}^-$. Let $\mathcal{W}$ be the Weyl group of $\mathfrak{g}$, $\mathcal{W} =< r_0, r_1, \ldots, r_{n-1} >$. It is known that $1 = \dim V(\lambda)_\lambda = \dim V(\lambda)_{w\lambda}$ for any $w \in \mathcal{W}$ (see [9]). Since the dimension is one, let $u_{w\lambda}$ be the basis vector for $V(\lambda)_{w\lambda}$. This vector is called the extremal vector. Then the Demazure module associated with w is $V_w(\lambda) = U_q(\underline{b})u_{w\lambda}$.

These Demazure modules, $V_w(\lambda)$, are finite dimensional subspaces of $V(\lambda)$. Further they satisfy the following:

$$V(\lambda) = \bigcup_{w \in \mathcal{W}} V_w(\lambda)$$

$$V_w(\lambda) \subseteq V_{w'}(\lambda) \text{ for all } w \preceq w' \text{ (Bruhat order) , } w, w' \in \mathcal{W}$$

In 1993, Kashiwara [13] proved that the crystal for $V_w(\lambda)$, denoted as $B_w(\lambda)$, is a subset of the highest weight crystal $B(\lambda)$ for $V(\lambda)$. Specifically, he showed that for $w \in \mathcal{W}$, there exits a subset $B_w(\lambda) \subset B(\lambda)$ such that

$$\frac{V_w(\lambda) \cap L(\lambda)}{V_w(\lambda) \cap qL(\lambda)} = \bigoplus_{b \in B_w(\lambda)} \mathbb{Q}b$$

where $(L(\lambda), B(\lambda))$ is the crystal base for $V(\lambda)$. The set $B_w(\lambda)$ is the crystal for the Demazure module $V_w(\lambda)$.

Further, the Demazure crystal $B_w(\lambda)$ has the following recursive property: If $w \prec r_i w$ (Bruhat order), then

$$B_{r_i w}(\lambda) = \{\tilde{f}_i^m b \big| m \geq 0,\ b \in B_w(\lambda),\ \tilde{e}_i b = 0\} \backslash \{0\}.$$

The condition that $\tilde{e}_i b = 0$ can be removed from the requirements for this recursive property.

The crystal $B_w(\lambda)$ can be constructed in terms of exended Young diagrams using two different methods. The first, which comes from the construction of the Demazure module, is to find the extremal vector $u_{w\lambda}$ then act on it by the $\tilde{e}_i$'s. In order to use this construction one must know the extremal vector explicitly.

The second construction is due to the recursive property mentioned above. Using this property, we can construct the crystal using only the $\tilde{f}_i$'s. Recall that for $w \in \mathcal{W}$, $w = r_{i_m} \ldots r_{i_2} r_{i_1}$ is a reduced expression for some simple reflections and some m. Thus, by the recursive property, to find $B_w(\lambda)$ we must act by $\tilde{f}_{i_1}$ exhaustively, then by $\tilde{f}_{i_2}$ and so on.

We define ϕ^i to be a concave (or empty) corner of color i. The superscript i will be supressed when the color is understood. A k-tuple of ϕ^i in $\mathcal{Y}(\lambda)$ is denoted as Φ. Note that Φ is the top tuple in the higheset weight crystal $B(\lambda)$ and corresponds to the highest weight vector u_λ.

We use the notation $w\Phi$ to mean the k-tuple of extended Young diagrams $\mathcal{Y} \in \mathcal{Y}(\lambda)$ which corresponds to $u_{w\lambda}$, where $u_{w\lambda}$ is the basis vector for $V(\lambda)_{w\lambda}$. Similarly, we say that w acting on Φ creates the set of diagrams in the crystal $B_w(\lambda)$.

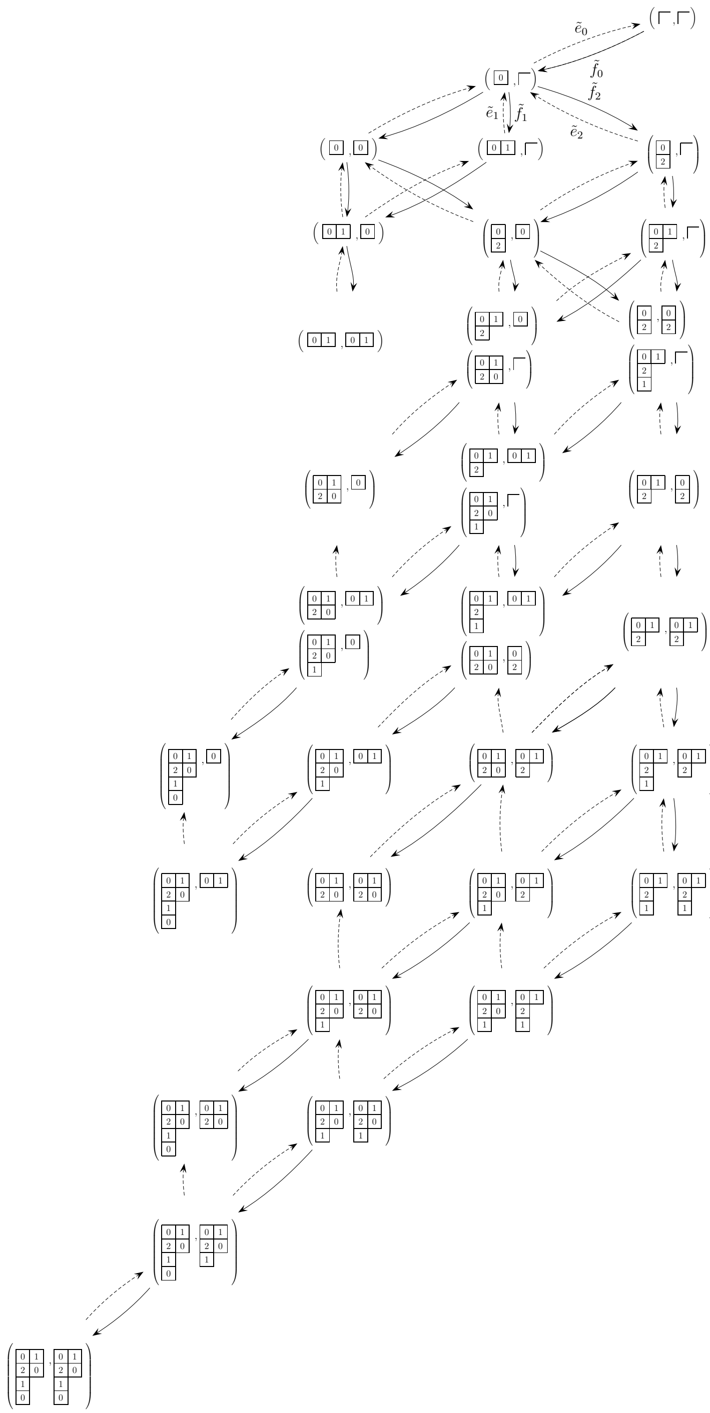
Example 4.0.12. *Let $n = 3$ and $\lambda = 2\Lambda_0$. Consider the Demazure crystal $B_{r_0r_1r_2r_0}(\lambda)$. The recursive property tells us that the Demzaure crystal can be constructed by letting $\tilde{f}_0$ act exhaustively on Φ, then $\tilde{f}_2$ acting exhaustively on all existing diagrams, followed by $\tilde{f}_1$ and finally finished with $\tilde{f}_0$. This construction is indicated in Figure 4.1 with solid downward arrows.*

By the construction of Demazure modules, the Demazure crystal can also be realized by letting the $\tilde{e}_i$ act on the extremal element $r_0r_1r_2r_0\Phi$. It turns out that

$$r_0r_1r_2r_0\Phi = \left(\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 2 & 0 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 0 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \, , \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 2 & 0 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 0 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right).$$

This construction is also indicated in Figure 4.1. The extremal element is located in the bottom left and the dashed upward arrows represent the action of the $\tilde{e}_i$'s.

Figure 4.1: The crystal graph for the Demazure module $V_{r_0 r_1 r_2 r_0}(2\Lambda_0)$ when $n = 3$



Recall that the Weyl group is $\mathcal{W} =< r_0, r_1, \ldots, r_{n-1} >$. For $j \in \mathbb{Z}$, define $r_j = r_{j'}$ when $j' = j \mod n$. The Dynkin diagram (see Figure 2.2) of $\mathfrak{g}$ has an automorphism which rotates the nodes. This induces an automorphism, σ, on the Weyl group which acts as follows: $\sigma(r_j) = r_{j+1}$.

Define $w^{(k)}$ for $k \in \mathbb{Z}_{\geq 0}$ as follows:

$$w^{(0)} = 1 \quad w^k = r_{n-k+1} \ldots r_{n-2} r_{n-1} r_0$$

We refer to Weyl group elements consisting of linear chains of simple reflections in this order as reverse ordered. Let $w(L) = w^{L(n-1)}$. Notice that the length of $w(L)$ is $L(n-1)$. We also have that the length of $w(L)$ equals the length of $\sigma(w(L))$.

Let $B_w(\lambda)$ be the crystal for the Demazure module $U_q(\underline{b})u_{w\lambda}$ where $u_{w\lambda}$ is the basis vector for $V(\lambda)_{w\lambda}$. We use the remainder of this chapter to describe Demazure modules of the form $B_{w(L)}(\lambda)$ in terms of extended Young diagrams.

Recall the notation conventions established in Chapter 3. $\mathcal{Y} = (Y_i)$, $i \in I = \{1, 2, \ldots, k\}$, is a k-tuple of extended Young diagrams. Each $Y_i = (y_1, y_2, \ldots, y_m, y_\infty, y_\infty, \ldots)$ where the charge of Y_i is y_∞.

Given Theorem 3.2.10, we realize that successive columns of an extended Young diagram in $B(\lambda)$ can differ in height by no more than $n-1$ boxes. This corollary is useful later.

Corollary 4.0.13. *Let $\mathcal{Y} \in B(\lambda)$. Then $0 \leq (Y_i)_{j+1} - (Y_i)_j \leq n-1$ for $1 \leq i \leq k,\ j > 0$.*

Proof.

Notice $(Y_i)_{j+1} - (Y_i)_j \leq n-1$ if and only if $(Y_i)_{j+1} - (Y_i)_j < n$ if and only if $(Y_i)_{j+1} < (Y_i)_j + n$, for all i, j. Assume there exists an $1 \leq i_0 \leq k$ and j_0 such that $(Y_{i_0})_{j_0+1} \geq (Y_{i_0})_{j_0} + n$.

Clearly we have the following containments:

$$\ldots \supseteq Y_k[-n] \supseteq Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_k \supseteq Y_1[n] \supseteq Y_2[n] \supseteq \ldots \supseteq Y_k[n] \supseteq Y_1[2n] \supseteq \ldots$$

Define Y_i for all $i \in \mathbb{Z}$ by $Y_i = Y_{p+mk} = Y_p[mn]$ where $1 \leq p \leq k$.

Clearly $(Y_i)_{j+1} - (Y_i)_j \geq 0$, else $(Y_i)_{j+1} < (Y_i)_j$ which contradicts Y an extended Young diagram.

Observe that the conditions in Theorem 3.2.10 are as follows:

$$(*)\ Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_1[n] \Leftrightarrow (Y_{i+1})_j \geq (Y_i)_j \ \ \forall i, j$$

$$(**)\ \textit{For each } j,\ \exists\ i_j \textit{ s.t. } (Y_{i_j+1})_j > (Y_{i_j})_{j+1}\ 1 \leq i_j \leq k$$

Fix $j = j_0$. Consider each of the three cases.

Case 1: $i_0 < i_j$

$$\begin{aligned} (Y_{i_0})_{j_0+1} &\geq (Y_{i_0})_{j_0} + n \ = (Y_{i_0+k})_{j_0} \text{ by assumption} \\ &\geq (Y_{i_0+k-1})_{j_0} \geq \ldots \geq (Y_{i_j+1})_{j_0} \text{ by } (*) \\ &\gneqq (Y_{i_j})_{j_0+1} \text{ by } (**) \\ &\geq (Y_{i_j-1})_{j_0+1} \geq \ldots \geq (Y_{i_0})_{j_0+1} \text{ by } (*) \end{aligned}$$

So $(Y_{i_0})_{j_0+1} \gneqq (Y_{i_0})_{j_0+1}$, which is a contradiction.

Case 2: $i_0 = i_j$

$$\begin{aligned}(Y_{i_0})_{j_0+1} &\geq (Y_{i_0})_{j_0} + n \ = (Y_{i_0+k})_{j_0} \text{ by assumption}\\ &\geq (Y_{i_0+k-1})_{j_0} \geq \ldots \geq (Y_{i_j+1})_{j_0} \text{ by } (*)\\ &\gneqq (Y_{i_j})_{j_0+1} = (Y_{i_0})_{j_0+1} \text{ by } (**)\end{aligned}$$

So $(Y_{i_0})_{j_0+1} \gneqq (Y_{i_0})_{j_0+1}$, which is a contradiction.

Case 3: $i_0 > i_j$

We have assumed $(Y_{i_0})_{j_0+1} \geq (Y_{i_0})_{j_0} + n$, so $(Y_{i_0})_{j_0+1} - n = (Y_{i_0-k})_{j_0+1} \geq (Y_{i_0})_{j_0}$.

$$\begin{aligned}(Y_{i_0-k})_{j_0+1} &\geq (Y_{i_0})_{j_0} \text{ by assumption}\\ &\geq (Y_{i_0-1})_{j_0} \geq \ldots \geq (Y_{i_j+1})_{j_0} \text{ by } (*)\\ &\gneqq (Y_{i_j})_{j_0+1} \text{ by } (**)\\ &\geq (Y_{i_j-1})_{j_0+1} \geq \ldots \geq (Y_{i_0-k})_{j_0+1} \text{ by } (*)\end{aligned}$$

So $(Y_{i_0-k})_{j_0+1} \gneqq (Y_{i_0-k})_{j_0+1}$, which is a contradiction.

Therefore, $(Y_i)_{j+1} - (Y_i)_j \leq n-1 \quad for\, 1 \leq i \leq k,\ j > 0.$ $\square$

Recall the construction of $B_{w(L)}(\lambda)$ based on the construction of Demazure modules requires knowledge of the extremal element. We can explicitly describe the extremal element $w(L)\Phi$ for the case of $\lambda = k\Lambda_0$ using the following lemma.

Lemma 4.0.14. *The extremal vector $w^{L(n-1)}\Phi$ in $B(k\Lambda_0)$ is $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_k)$ where $Y_i = \Big(-L(n-1), -(L-1)(n-1), \ldots, -(n-1), 0, 0, \ldots\Big)$, $i = 1, 2, \ldots, k$.*

Proof.

Let $\Phi = (\phi_1, \phi_2, \ldots, \phi_k)$ with the charge of each ϕ_i equal to zero. Define $b = n-(L-1)(n-1)$ mod n.

Let L $= 1$. Then $b = 0$. Consider $w^{(n-1)}\Phi = r_2 \ldots r_{n-1} r_0 \Phi$.

Notice the signatures of Φ are non-existent except for when $i = 0$. Further, $\varepsilon_0 = \varepsilon_b = (\underbrace{0, 0, \ldots, 0}_{k})$ where each zero corresponds to a concave 0-corner ϕ_i. Note that

$0 = b$ where $b = n - (L-1)(n-1)\ mod\ n$. It is clear from ε_0 that $r_2 \dots r_{n-1} r_0 \Phi = r_2 \dots r_{n-2} r_{n-1}\left(\tilde{f}_0^k \Phi\right)$.

Now, $\tilde{f}_0^k \Phi = (\tilde{f}_0 \phi_1, \tilde{f}_0 \phi_2, \dots, \tilde{f}_0 \phi_k)$ and each $\tilde{f}_0 \phi_i = (-1, 0, 0, \dots)$. Notice the following about $\tilde{f}_0^k \Phi$:

- $r_0 \Phi = \tilde{f}_0^k \Phi = \left(\boxed{0}\,, \boxed{0}\,, \dots\,, \boxed{0}\right)$
- $\varepsilon_0 = (\underbrace{1, 1, \dots, 1}_{k}) = \varepsilon_b$
- $\varepsilon_1 = (\underbrace{0, 0, \dots, 0}_{k}) = \varepsilon_{b+1}$; And since $r_1 = r_{b+1}$ does not appear in the remaining chain of reflections, $r_2 \dots r_{n-1}$, ε_1 will remain all 0's (although the signature length may increase).
- $\varepsilon_{n-1} = (\underbrace{0, 0, \dots, 0}_{k}) = \varepsilon_{b-1}$
- All other ε_i are non-existent.

By the ε_1 comments it is clear that $|w(1)\Phi| = 1$, thus we know by the previous corollary, Corollary 4.0.13, that each Y_i can have no more than $n-1$ boxes. In fact we now show that each Y_i has precisely depth $n-1$.

The $(n-1)$-signature makes it clear that $r_2 \dots r_{n-1} r_0 \Phi = r_2 \dots r_{n-2} r_{n-1}\left(\tilde{f}_0^k \Phi\right) = r_2 \dots r_{n-2}\left(\tilde{f}_{n-1}^k \tilde{f}_0^k \Phi\right)$. Now $\tilde{f}_{n-1}^k \tilde{f}_0^k \Phi = \left(\tilde{f}_{n-1} \tilde{f}_0 \phi_1, \tilde{f}_{n-1} \tilde{f}_0 \phi_2, \dots, \tilde{f}_{n-1} \tilde{f}_0 \phi_k\right)$ and each $\tilde{f}_{n-1} \tilde{f}_0 \phi_i = (-2, 0, 0, \dots)$.

Similar to above we observe the following about $\tilde{f}_{n-1}^k \tilde{f}_0^k \Phi$:

- $r_{n-1} r_0 \Phi = \tilde{f}_{n-1}^k \tilde{f}_0^k \Phi = \left(\begin{array}{|c|}\hline 0 \\ \hline \text{n-1} \\ \hline\end{array}\,, \begin{array}{|c|}\hline 0 \\ \hline \text{n-1} \\ \hline\end{array}\,, \dots\,, \begin{array}{|c|}\hline 0 \\ \hline \text{n-1} \\ \hline\end{array}\right)$

- $\varepsilon_1 = \varepsilon_{b+1} = (\underbrace{0, 0, \ldots, 0}_{k})$
- $\varepsilon_{n-1} = \varepsilon_{b-1} = (\underbrace{1, 1, \ldots, 1}_{k})$
- $\varepsilon_{n-2} = \varepsilon_{b-2} = (\underbrace{0, 0, \ldots, 0}_{k})$
- All other ε_i are non-existent.

Repeating this process to the $n-1$ step, we see:

$$r_2 \ldots r_{n-1} r_0 \Phi = \tilde{f}_2^k \ldots \tilde{f}_{n-1}^k \tilde{f}_0^k \Phi = \Big(\tilde{f}_2 \ldots \tilde{f}_{n-1} \tilde{f}_0 \phi_1, \tilde{f}_2 \ldots \tilde{f}_{n-1} \tilde{f}_0 \phi_2, \ldots, \tilde{f}_2 \ldots \tilde{f}_{n-1} \tilde{f}_0 \phi_k\Big)$$

$$\text{and } \tilde{f}_2 \ldots \tilde{f}_{n-1} \tilde{f}_0 \phi_i = \big(-(n-1), 0, 0, \ldots\big) \text{for each } i.$$

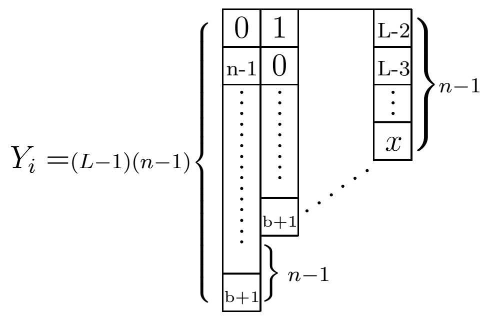
Representing this visually,

$$r_2 \ldots r_{n-1} r_0 \Phi = \tilde{f}_2^k \ldots \tilde{f}_{n-1}^k \tilde{f}_0^k \Phi = \left(\begin{array}{|c|} \hline 0 \\ \hline \text{n-1} \\ \hline \vdots \\ \hline 2 \\ \hline \end{array} \, , \, \begin{array}{|c|} \hline 0 \\ \hline \text{n-1} \\ \hline \vdots \\ \hline 2 \\ \hline \end{array} \, , \, \ldots \, , \, \begin{array}{|c|} \hline 0 \\ \hline \text{n-1} \\ \hline \vdots \\ \hline 2 \\ \hline \end{array} \right)$$

Thus $w^{(n-1)}\Phi = w(1)\Phi = (Y_1, Y_2, \ldots, Y_k)$ where each $Y_i = \big(-(n-1), 0, 0, \ldots\big)$ and the lemma holds for $L = 1$.

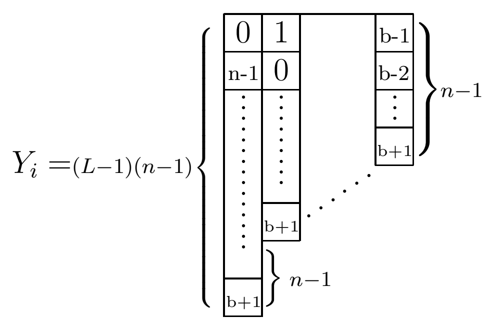
Assume that it holds for $L-1$. i.e. Assume that $w^{(L-1)(n-1)}\Phi = w(L-1)\Phi = (Y_1, Y_2, \ldots, Y_k)$ where $Y_i = \Big(-(L-1)(n-1), -(L-2)(n-1), \ldots, -(n-1), 0, 0, \ldots\Big)$

for each i. Graphically,



where $x = (L-2) - (n-2)$

Notice that $b + 1 \; mod \; n = n - (L - 1)(n - 1) + 1 \; mod \; n = L \; mod \; n = (L - 2) - (n - 2) \; mod \; n$ and in general the i^{th} column ends in a box of color $(i - 1) - ((L - i)(n - 1) - 1) \; mod \; n = L \; mod \; n$. Thus each column of Y_i ends in a $b + 1$-colored box. Also, $L - 2 \; mod \; n = b - 1 \; mod \; n$, so there exists a concave b-corner at the end of the first row of each Y_i. So we see that:



Show that the lemma holds for L.

Observe $w^{L(n-1)}\Phi = r_{b-n+2} \dots r_{b-1} r_b w^{(L-1)(n-1)}\Phi$ where $b = n - (L - 1)(n - 1) \; mod \; n$, as before. Notice the following about $w^{(L-1)(n-1)}\Phi$:

- $\varepsilon_b = (\underbrace{0, 0, \dots, 0}_{Lk})$
- $\varepsilon_{b+1} = (\underbrace{1, 1, \dots, 1}_{(L-1)k})$

- All other ε_i are non-existent

So $w^{L(n-1)}\Phi = r_{b-n+2}\dots r_{b-2}r_{b-1}\left(\tilde{f}_b^{Lk}\left(w^{(L-1)(n-1)}\Phi\right)\right)$ and $\tilde{f}_b^{Lk}\left(w^{(L-1)(n-1)}\Phi\right) = (Y_1, Y_2, \dots, Y_k)$ where each $Y_i = \Big(-(L-1)(n-1)-1, -(L-2)(n-1)-1, \dots,$-$(n-1)-1, -1, 0, 0, \dots\Big)$ because $\tilde{f}_b^{Lk}$ adds a b-colored box to the bottom of each column and to the end of the first row of each Y_i.

Similar to before, the following is true of $\tilde{f}_b^{Lk}w^{(L-1)(n-1)}\Phi$:

- $\varepsilon_b = (\underbrace{1, 1, \dots, 1}_{Lk})$
- $\varepsilon_{b+1} = (\underbrace{0, 0, \dots, 0}_{k})$ Notice that r_{b+1} does not occur in the remaining chain of reflections.
- $\varepsilon_{b-1} = (\underbrace{0, 0, \dots, 0}_{Lk})$
- All other ε_i are non-existent.

So we know:

$$\begin{aligned} r_{b-n+2}\dots r_{b-1}r_b w^{(L-1)(n-1)}\Phi &= r_{b-n+2}\dots r_{b-2}r_{b-1}\left(\tilde{f}_b^{Lk}w^{(L-1)(n-1)}\Phi\right) \\ &= r_{b-n+2}\dots r_{b-2}\left(\tilde{f}_{b-1}^{Lk}\tilde{f}_b^{Lk}w^{(L-1)(n-1)}\Phi\right) \end{aligned}$$

and $\tilde{f}_{b-1}^{Lk}\tilde{f}_b^{Lk}w^{(L-1)(n-1)}\Phi = (Y_1, Y_2, \dots, Y_k)$ where for each i,

$$Y_i = (-(L-1)(n-1)-2, -(L-2)(n-2)-2, \dots, -(n-1)-2, -2, 0, 0, \dots).$$

Repeating this process to the $n-1$ step we see:

$$\begin{aligned} w^{L(n-1)}\Phi &= r_{b-n+2}\dots r_{b-1}r_b w^{(L-1)(n-1)}\Phi \\ &= \tilde{f}_{b-n+2}^{Lk}\dots \tilde{f}_{b-1}^{Lk}\tilde{f}_b^{Lk} w^{(L-1)(n-1)}\Phi \\ &= (Y_1, Y_2, \dots, Y_k) \end{aligned}$$

where for each i,

$$\begin{aligned} Y_i &= \Big(-(L-1)(n-1)-(n-1), -(L-2)(n-1)-(n-1), \dots, \\ &\qquad (n-1)-(n-1), -(n-1), 0, 0, \dots \Big) \\ &= \Big(-L(n-1), -(L-1)(n-1), \dots, -2(n-1), -(n-1), 0, 0, \dots \Big). \end{aligned}$$

□

Note that for the case when $\lambda = \Lambda_0$, this has been shown in [18].

Given this realization of the extremal element we can now realize the Demazure crystals $B_{w(L)}(k\Lambda_i)$ in terms of extended Young diagrams as follows.

Theorem 4.0.15. *For $k \in \mathbb{Z}_{>0}$, $B_{w^{L(n-1)}}(k\Lambda_0) = \big\{\mathcal{Y} \in B(k\Lambda_0) \big| |\mathcal{Y}| \leq L\big\}$.*

Proof.

Let $\Phi = (\phi_1, \phi_2, \dots, \phi_k) \in B(k\Lambda_0)$. Assume $\mathcal{Y} \neq \Phi$.

Let $B_L(k\Lambda_0) = \big\{\mathcal{Y} \in B(k\Lambda_0) \big| |\mathcal{Y}| \leq L\big\}$.

Begin by showing $B_{w^{L(n-1)}}(k\Lambda_0) \subseteq B_L(k\Lambda_0)$. Clearly $\mathcal{Y} \in B_{w^{L(n-1)}}(k\Lambda_0)$ gives that $\mathcal{Y} \in B(k\Lambda_0)$, so it is sufficient to show $|\mathcal{Y}| \leq L$.

Let $L = 1$ and $\mathcal{Y} \in B_{w^{(n-1)}}(k\Lambda_0)$. Then $\mathcal{Y}$ is one of the diagrams created by the action of $r_2 \ldots r_{n-1} r_0$ on Φ. Thus $\mathcal{Y} = \tilde{f}_1^{s_{n-1}} \ldots \tilde{f}_{n-1}^{s_2} \tilde{f}_0^{s_1} \Phi$ for some $s_i \in \mathbb{Z}_{\geq 0}$, $s_1 \neq 0$. and for each $m = 1, 2, \ldots, k$, $Y_m = \tilde{f}_2^{t_{n-1}} \ldots \tilde{f}_{n-1}^{t_2} \tilde{f}_0^{t_1} \phi_m$ for some $t_i \in \mathbb{Z}_{\geq 0}$, $t_i \leq s_i$, $Y_1 \neq \phi_1$. More specifically, we know:

$$0 \leq s_{n-1} \leq s_{n-2} \leq \ldots \leq s_2 \leq s_1 \leq k$$

$$\text{and } 0 \leq t_{n-1} \leq t_{n-2} \leq \ldots \leq t_2 \leq t_1 \leq 1 \quad (i.e.\ t_i \in \{0, 1\} \text{ and } t_i = 0 \Rightarrow t_{i+1} = 0).$$

This is clear from the proof of Lemma 4.0.14, but for clarity we repeat some of this argument below.

The 0-signature of Φ is $(\underbrace{0, 0, \ldots, 0}_{k})$, where each 0 corresponds to a concave 0-corner ϕ_i so $s_1 \leq k$ and $t_1 \leq 1$. It is easily observed that the $(n-1)$-signature of $\tilde{f}_0^{s_1}\Phi$ is $(\underbrace{0, 0, \ldots, 0}_{s_1})$, where each 0 corresponds to the concave $(n-1)$-corner of the first s_1 diagrams in $\mathcal{Y}$ (this corner is located below the 0-box). So $s_2 \leq s_1$ and $t_2 \leq t_1$. We continue in this fashion with the $(n-i)$-signature of $\tilde{f}_{n-i+1}^{s_i} \cdots \tilde{f}_0^{s_1}\Phi$ becoming $(\underbrace{0, 0, \ldots, 0}_{s_i})$, so $0 \leq s_{n-1} \leq \ldots \leq s_2 \leq s_1 \leq k$ and $0 \leq t_{n-1} \leq \ldots \leq t_2 \leq t_1 \leq 1$.

Notice that the action of $\tilde{f}_0$ increases the width of the ϕ_m to one. However, the width is not further increased to two since this can only happen by filling the concave 1-corners (there are s_1 width-increasing 1-corners, s_{n-1} depth increasing 1-corners and the 1-signature of $\tilde{f}_2^{s_{n-1}} \ldots \tilde{f}_{n-1}^{s_2} \tilde{f}_0^{s_1}\Phi$ is $(\underbrace{0, 0, \ldots, 0}_{s_1+s_{n-1}})$) of the Y_m, which requires action by $\tilde{f}_1$, which does not occur in the chain of $\tilde{f}_i$'s.

In other words it is clear that each Y_m is contained in $Y_m^* = (-(n-1), 0, 0, \ldots) =$

$w^{(n-1)}\Phi$. So each $Y_m = (-a_m, 0, 0, \ldots)$ for some a_m with $0 \le a_m \le n-1$, $a_1 \neq 0$. Furthermore since the s_i's and the t_i's weakly decrease, $a_m \ge a_{m+1}$. [Note this also comes from the fact that $\mathcal{Y} \in B(k\Lambda_0)$.] More precisely, we know that a_m is the value such that $t_{a_m} = 1$ and $t_{a_m+1} = 0$ (where we say $t_n = 0$ since $\tilde{f}_1$ does not occur in the sequence). So clearly, $|Y_m| \le 1$ for $m = 1, 2, \ldots, k$ (and specifically $|Y_1| = 1$), thus $|\mathcal{Y}| \le 1$ (in fact $|\mathcal{Y}| = 1$ since $\mathcal{Y} \neq \Phi$). Therefore $B_{w(1)}(k\Lambda_0) \subseteq B_1(k\Lambda_0)$.

Assume the result holds for $L-1$. i.e. Assume $B_{w^{(L-1)(n-1)}}(k\Lambda_0) \subseteq B_{L-1}(k\Lambda_0)$. Show holds for L.

Let $\mathcal{Y} \in B_{w^{L(n-1)}}(k\Lambda_0)$. Observe that $w^{L(n-1)} = r_{b-n+2} \ldots r_{b-1} r_b w^{(L-1)(n-1)}\Phi$ where $b = n - (L-1)(n-1)$ *mod* n. So $\mathcal{Y}$ is one of the diagrams created by the action of $r_{b-n+2} \ldots r_{b-1} r_b$ on $w^{(L-1)(n-1)}\Phi$.

More specifically, there exists a $\mathcal{Y}' = (Y_1', Y_2', \ldots Y_k') \in B_{w^{(L-1)(n-1)}}(k\Lambda_0)$, $\mathcal{Y}' \neq \Phi$, such that $\mathcal{Y}$ is one of the diagrams created by the action of $r_{b-n+2} \ldots r_{b-1} r_b$ on $\mathcal{Y}'$. In other words, $\mathcal{Y} = \tilde{f}_{b-n+2}^{s_{n-1}} \ldots \tilde{f}_{b-1}^{s_2} \tilde{f}_b^{s_1} \mathcal{Y}_m$ for some $s_i \in \mathbb{Z}_{\ge 0}$, so for $m = 1, 2, \ldots, k$, $Y_m = \tilde{f}_{b-n+2}^{t_{n-1}} \ldots \tilde{f}_{b-1}^{t_2} \tilde{f}_b^{t_1} Y_m'$ for some $t_i \in \mathbb{Z}_{\ge 0}$, $t_i \le s_i$, with the sum of the t_i equal to s_i.

Now consider $|Y_m|$. If $|Y_m'| = L-1$, then there is a concave b-corner at the end of the first row that contributes a 0 to the b-signature of Y_m'. So if the 0 remains in the relevant signature, and t_1 is large enough to fill this corner, $|\tilde{f}_b^{t_1} Y_m'| = L$. Else, $|\tilde{f}_b^{t_1} Y_m'| = L-1$ and its width can not be increased by the remaining chain of actions. Now if $|\tilde{f}_b^{t_1} Y_m'| = L$ the only way to increase the width of $\tilde{f}_b^{t_1} Y_m'$ is to fill the $(b+1)$-concave corner at the end of the first row. However the action of $\tilde{f}_{b+1}$ does not occur in our remaining chain so $|Y_m|$ is L or $L-1$.

If $|Y_m'| < L-1$, then there is a convex corner of some color, say q, at the end of the first row. The action by $\tilde{f}_q$ may increase the width by one. However, the next action that can widen Y'_m is now $\tilde{f}_{q+1}$. But this action is $n-1$ away in any reverse ordered chain of actions and we only have, at most, $n-2$ actions remaining in our chain. Thus $|Y_m| \leq L-1$.

Therefore, for each m, $|Y_m| \leq L$. We can also observe this fact by realizing that $\mathcal{Y}$ must be contained in the extremal element $\mathcal{Y}^* = w^{L(n-1)}\Phi = (Y_1^*, Y_2^*, \ldots Y_k^*)$ where $Y_i^* = (-L(n-1), -(L-1)(n-1), \ldots, -(n-1), 0, 0, \ldots)$. So $Y_m \subseteq Y_m^*$ and $|Y_m^*| = L$, so $|Y_m| \leq L$.

Thus $|\mathcal{Y}| \leq L$ and $\mathcal{Y} \in B_L(k\Lambda_0)$. Since $\mathcal{Y}$ is arbitrary, $B_{w^{L(n-1)}}(k\Lambda_0) \subseteq B_L(k\Lambda_0)$.

A shorter argument for $B_{w(n-1)}(k\Lambda_0) \subseteq B_L(k\Lambda_0)$:

Let $\mathcal{Y} \in B_{w^{L(n-1)}}(k\Lambda_0)$. Then $\mathcal{Y}$ is one of the diagrams created by $w^{L(n-1)}$ acting on Φ. i.e. $\mathcal{Y}$ is one of the diagrams created by $\underbrace{\ldots r_{n-2}r_{n-1}r_0}_{L(n-1)}$ acting on Φ. So $\mathcal{Y} \subseteq w^{L(n-1)}\Phi$. But by 4.0.14, $|w^{L(n-1)}\Phi| = L$. Thus $|\mathcal{Y}| \leq L$, so $\mathcal{Y} \in B_L(k\Lambda_0)$.

Next, show $B_L(k\Lambda_0) \subseteq B_{w^{L(n-1)}}(k\Lambda_0)$. Let $L = 1$ and $\mathcal{Y} \in B_1(k\Lambda_0)$. So $|\mathcal{Y}| \leq 1$ and $\mathcal{Y} \in B(k\Lambda_0)$. In other words we know:

- $|Y_m| \leq 1$ for $m = 1, 2, \ldots, k$
- $Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_k \supseteq Y_1[n]$
- $\forall q \geq 0$, $\exists\, p$ such that $(Y_{p+1})_q > (Y_p)_{q+1}$.

In particular, for every $m = 1, 2, \ldots, k$, $Y_m = (-a_m, 0, 0, \ldots)$ where:

- $a_m \in \mathbb{Z}_{\geq 0}$, $a_1 \neq 0$
- $a_m < n$ (by Corollary 4.0.13)

- $a_1 \geq a_2 \geq \ldots \geq a_k \geq 0$
- $a_k > a_1 - n$ (Since $a_1 - n < 0$).

So it is clear that for each m, $Y_m = \tilde{f}_{n-a_m+1} \ldots \tilde{f}_{n-1}\tilde{f}_0\phi_m$ where $n - a_m + 1 \ mod \ n \geq 2$. If the required use of reverse order for the $\tilde{f}_i$ is unclear, refer to the argument below.

Consider the weight of $\mathcal{Y}$, $wt\ \mathcal{Y} = k\Lambda_0 - \sum_{j=0}^{n-1} c_j\alpha_j$. Notice: $c_1 = 0$ since no $\tilde{f}_1$ action appears in the sequences for Y_m. Observe that $c_0 \geq c_{n-1} \geq \ldots \geq c_2$ (clear from actions creating Y_m and $a_1 \geq a_2 \geq \ldots \geq a_k$) and $c_0 \neq 0$. Furthermore, $k \geq c_i \geq 0$ since $|\mathcal{Y}| = 1$. Then we can write $\mathcal{Y}$ specifically as: $\mathcal{Y} = \tilde{f}_2^{c_2} \ldots \tilde{f}_{n-1}^{c_{n-1}}\tilde{f}_0^{c_0}\Phi$.

Although the discussions above should be clear, the details are presented here for rigor. Notice the 0-signature for Φ is $(\underbrace{0, 0, \ldots, 0}_{k})$ where each zero corresponds to a concave 0-corner created by ϕ_i and all other signatures are non-existent. Thus $\tilde{f}_0^{c_0}\Phi$ places a 0-colored box in the first c_0 corners (so all components of $\tilde{f}_0^{c_0}\Phi$ are either $(-1, 0, 0, \ldots)$ or $(0, 0, 0, \ldots)$). Next observe that c_0 is precisely $|\{a_m \big| a_m \geq 1\}|$. Observe the signatures of $\tilde{f}_0^{c_0}\Phi$. The 0-signature is $(\underbrace{1, 1, \ldots, 1}_{c_0}, \underbrace{0, 0, \ldots, 0}_{k-c_0})$. The $(n-1)$- signature is $(\underbrace{0, 0, \ldots, 0}_{c_0})$ where each zero corresponds to a concave $(n-1)$-corner located directly below one of the 0-boxes just created. All other signatures are non-existent. So we see $\tilde{f}_{n-1}^{c_{n-1}}\tilde{f}_0^{c_0}\Phi$ has a $(n-1)$-signature of $(\underbrace{1, 1, \ldots, 1}_{c_{n-1}}, \underbrace{0, 0, \ldots, 0}_{c_0 - c_{n-1}})$, a $(n-2)$-signature of $(\underbrace{0, 0, \ldots, 0}_{c_{n-1}})$ and all other signatures are non-existent. We continue in this fashion to create $\mathcal{Y}$.

Thus $\mathcal{Y} \in B_{w^{n-1}}(k\Lambda_0)$ so $B_1(k\Lambda_0) \subseteq B_{w(1)}(k\Lambda_0)$.

Assume this result holds for $L-1$. i.e. $B_{L-1}(k\Lambda_0) \subseteq B_{w^{(L-1)(n-1)}}(k\Lambda_0)$. Show holds for L.

Let $\mathcal{Y} \in B_L(k\Lambda_0)$. So $|\mathcal{Y}| \leq L$ and $\mathcal{Y} \in B(k\Lambda_0)$. In other words we know:

- $|Y_m| \leq L$ for $m = 1, 2, \ldots, k$
- $Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_k \supseteq Y_1[n]$
- $\forall q \geq 0$, $\exists p$ such that $(Y_{p+1})_q > (Y_p)_{q+1}$

Notice that if $|\mathcal{Y}| < L$, $\mathcal{Y} \in B_{L-1}(k\Lambda_0)$ so by the induction step, $\mathcal{Y} \in B_{w^{(L-1)(n-1)}}(k\Lambda_0) \subseteq B_{w^{L(n-1)}}$. So it is sufficient to only consider $\mathcal{Y}$ for which $|\mathcal{Y}| = L$.

If —$\mathcal{Y}| = L$, $|Y_m| = L$ for at least one m. Notice that due to the containment property, $|Y_m| < L$ gives that $|Y_{m+1}| < L$ (or that $|Y_m| = L \Rightarrow |Y_{m-1}| = L$). Observe the following: The L^{th} box on row 1 of Y_m, if it exists, is of color b ($b = n-(L-1)(n-1)$ mod n as previously defined). The L^{th} box on row 2 of Y_m is of color $b-1 = b+(n-1)$. And so on with the L^{th} box on row i of Y_m having color $b+n-i$. Also realize (by Corollary 4.0.13) the depth of the L^{th} column is less than n.

Let ξ_i be the number of Y_m in $\mathcal{Y}$ such that the L^{th} column has height at least i, $i = 1, 2, \ldots, n-1$. [i.e. If $\tilde{\mathcal{Y}} = (\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_k)$ where each $\tilde{Y}_m$ is the L^{th} column of Y_m, then $\xi_i = \tilde{c}_{b+i-1 \ mod \ n}$ where $wt\ \tilde{\mathcal{Y}} = k\Lambda_0 - \sum_{j=0}^{n-1} \tilde{c}_j \alpha_j$.] Notice that $\xi_n = \xi_{n+1} = \ldots = 0$.

If $\xi_{n-1} \neq 0$, then $Y_1, Y_2, \ldots, Y_{\xi_{n-1}}$ all have a $(b+2)$-colored box at the end of the L^{th} column. So there exists a $\mathcal{Y}' \in B(k\Lambda_0)$ and $\beta_{n-1} \geq \xi_{n-1}$ such that $\tilde{e}_{b+2}^{\beta_{n-1}}\mathcal{Y} = \mathcal{Y}'$ and $|\mathcal{Y}'| = L$.

It is clear that $\xi_{n-1} \leq \xi_{n-2} \leq \ldots \leq \xi_2 \leq \xi_1 > 0$. Similar to the argument above, for every $\xi_i \neq 0$, we can find an appropriate β_i such that we get:

$$\underbrace{\tilde{e}_b^{\beta_2} \ldots \tilde{e}_{b+3}^{\beta_{n-2}} \tilde{e}_{b+2}^{\beta_{n-1}}}_{n-1} \mathcal{Y} = \mathcal{Y}'.$$

However, now $|\mathcal{Y}'| = L - 1$. And:

$$\tilde{f}_{b+2}^{\beta_{n-1}} \tilde{f}_{b+3}^{\beta_{n-2}} \ldots \tilde{f}_{b-1}^{\beta_2} \tilde{f}_{b}^{\beta_1} \mathcal{Y}' = \mathcal{Y}.$$

Since $|\mathcal{Y}'| = L - 1$ and $\mathcal{Y}' \in B(k\Lambda_0)$, $\mathcal{Y}' \in B_{L-1}(k\Lambda_0)$, which means that it is in $B_{w^{(L-1)(n-1)}}(k\Lambda_0)$ by the induction step. But this makes it clear, since there are no more than $n-1$ of the $\tilde{f}_i$ acting on $\mathcal{Y}'$ in reverse order needed to create $\mathcal{Y}$, that $\mathcal{Y} \in B_{w^{L(n-1)}}(k\Lambda_0)$. Since $\mathcal{Y}$ is arbitrary, we have that $B_L(k\Lambda_0) \subseteq B_{w^{L(n-1)}}(k\Lambda_0)$. □

This result again can be found in [18] for the case of $k = 1$.

Interestingly a quick corollary to this theorem can tell us exactly how many diagrams are in $B_{w(L)}(\Lambda_i)$.

Corollary 4.0.16. *In* $U_q(\widehat{sl}(n))$, $\Big|\{Y \in B(\Lambda_0)\big||Y| \leq L\}\Big| = \Big|B_L(\Lambda_0)\Big| = \Big|B_{w(L)}(\Lambda_0)\Big| = n^L$.

Proof.

First observe that this corollary holds for $L = 0$. The only diagram in $B_0(\Lambda_0)$ is ϕ. Thus $|B_0(\Lambda_0)| = 1 = n^0$.

Now note that for $L > 0$ the corollary holds if and only if, $\Big|\{Y \in B(\Lambda_0)\big||Y| = L\}\Big| = (n-1)n^{L-1}$, since $n^L = n(n^{L-1}) = \underbrace{n^{L-1} + \ldots + n^{L-1}}_{n \text{ times}} = n^{L-1} + (n-1)n^{L-1}$ and $n^{L-1} = \{Y \in B(\Lambda_0)\big||Y| \leq L-1\}$.

Recall that $Y \in B(\Lambda_0)$ if and only if $y_{j+1} - y_j < n \quad \forall j$, in other words if and only if consecutive columns in the diagram of Y differ by no more than $n-1$ boxes (Corollary 4.0.13).

Let $L = 1$. Then the diagram Y consists of only one column. If $Y \neq \phi$, the height of this column must be an integer strictly greater than 0 and less than n by Corollary 4.0.13. Thus there are $n-1$, or $(n-1)n^{L-1}$, possibilities.

Let $L = 2$. Assume $|Y| = 2$. Then again the far right column (the second column) can have up to $n-1$ boxes and must have at least one box. Assume there are c boxes in this column. Then the adjacent preceeding column, the first column, must have at least c boxes and can have up to $n-1$ additional boxes (again by Corollary 4.0.13). Thus there are $(n-1)(1+n-1) = (n-1)n$, or $(n-1)n^{L-1}$, possible such Y.

Assume that this holds for diagrams of width $L-1$. Show for width L. Assume that $|Y| = L$. By the induction step, there are $(n-1)n^{L-1}$ options for the diagrams of width $L-1$. So there are $(n-1)n^{L-1}$ options for columns 2 through L of Y. Let the height of the second column of Y be c. Then the first column of Y must have at least c boxes and may have up to $n-1$ additional boxes (Corollary 4.0.13). Thus there are n choices for this column. Therefore there are $n \times ((n-1)n^{L-1})$, or $(n-1)n^L$,options for Y.

Further it is clear that this is the entire set of Y satisfying $y_{j+1} - y_j < n$ and that the Y are all distinct. It is easy to check that all of these diagrams are in $B(\Lambda_0)$. It is clear that condition one of Theorem 3.2.10 is met since $Y_1 \supseteq Y_1[n]$. Condition two of this theorem is also met since $y_{j+1} - y_j < n \Rightarrow y_{j+1} < y_j + n \Rightarrow (Y_1)_{j+1} < (Y_1[n])_j$. $\square$

Now we will generalize the extremal element description to any dominant weight $\lambda \in P^+$.

Theorem 4.0.17. *Let* $\lambda = k_0\Lambda_0 + k_1\Lambda_1 + \ldots + k_{n-1}\Lambda_{n-1}$, $\sum_{i=0}^{n-1} k_i = k$, $\kappa_i = \sum_{j=0}^{i} k_j$ *for* $i = 0, 1, \ldots n-1$. *The extremal vector* $w^{L(n-1)}\Phi$ *in* $B(\lambda)$ *is* $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_k)$ *where:*

(i) $Y_1 = Y_2 = \cdots = Y_{\kappa_0} = \Big(-L(n-1), -(L-1)(n-1), \ldots, -(n-1), 0, 0, \ldots \Big)$

(ii) $Y_{\kappa_{i-1}+1} = Y_{\kappa_{i-1}+2} = \cdots = Y_{\kappa_i} =$
$$\Big(-(L-1)(n-1)+1, -(L-2)(n-1)+1, \ldots, -(n-1)+1, 1, i, i, \ldots \Big)$$

[Notice the depth of the columns of the diagrams with charge 0 is a multiple of $n-1$ and is one more than the depth of the columns of the diagrams with charge $n-1$. Additionally the depth of the columns of the diagrams with charge $n-1$ is one more than those in diagrams with charge $n-2$, which are one more than those with charge $n-3$ and so on.]

Proof.

Let $L = 1$. Consider $w^{n-1}\Phi = r_2 \ldots r_{n-1} r_0 \Phi$. Let ϕ^i be a concave i-corner. So $\Phi = \big(\phi_1^0, \phi_2^0, \ldots, \phi_{\kappa_0}^0, \phi_{\kappa_0+1}^1, \ldots, \phi_{\kappa_1}^1, \ldots, \phi_{\kappa_{n-2}+1}^{n-1}, \ldots, \phi_{\kappa_{n-1}}^{n-1}\big)$. Observe that the signatures of Φ are $\varepsilon_i = (\underbrace{0, 0, \ldots, 0}_{k_i})$, $i = 1, 2, \ldots, n-1$, where each zero corresponds to a ϕ^i. Now using the 0-signature we see $r_0\Phi = \tilde{f}_0^{k_0}\Phi$. Observe the following about $\tilde{f}_0^{k_0}\Phi$:

- $\tilde{f}_0^{k_0}\Phi = \Big(\underbrace{\boxed{0}\,, \boxed{0}\,, \ldots, \boxed{0}}_{k_0}\,, \phi_{\kappa_0+1}^1, \ldots \phi_{\kappa_1}^1, \ldots, \phi_{\kappa_{n-1}}^{n-1}\Big)$

  i.e. $Y_1 = Y_2 = \cdots = Y_{\kappa_0} = (-1, 0, 0, \ldots)$ and all other diagrams of $\tilde{f}_0^{k_0}$ are empty corners.

- $\varepsilon_0 = (\underbrace{1, 1, \ldots, 1}_{k_0})$
- $\varepsilon_1 = (\underbrace{0, 0, \ldots, 0}_{k_0+k_1})$

- $\varepsilon_{n-1} = (\underbrace{0, 0, \ldots, 0}_{k_{n-1}+k_0})$

- $\varepsilon_i = (\underbrace{0, 0, \ldots, 0}_{k_i})$ for $i \neq 0, 1, n-1$

Now the $(n-1)$-signature shows that $r_{n-1} r_0 \Phi = \tilde{f}_{n-1}^{k_{n-1}+k_0} \tilde{f}_0^{k_0} \Phi$. Observe the following about $\tilde{f}_{n-1}^{k_{n-1}+k_0} \tilde{f}_0^{k_0} \Phi$:

- $\tilde{f}_{n-1}^{k_{n-1}+k_0} \tilde{f}_0^{k_0} \Phi =$

$$\Bigg(\underbrace{\begin{array}{|c|}\hline 0 \\ \hline \text{n-1} \\ \hline\end{array}\,, \begin{array}{|c|}\hline 0 \\ \hline \text{n-1} \\ \hline\end{array}\,, \ldots, \begin{array}{|c|}\hline 0 \\ \hline \text{n-1} \\ \hline\end{array}}_{k_0}\,, \phi^1_{\kappa_0+1}, \ldots, \phi^1_{\kappa_1}, \phi^2_{\kappa_1+1}, \ldots, \phi^{n-2}_{\kappa_{n-2}}, \underbrace{\boxed{\text{n-1}}\,, \boxed{\text{n-1}}\,, \ldots, \boxed{\text{n-1}}}_{k_{n-1}}\Bigg)$$

  i.e. $Y_1 = Y_2 = \cdots = Y_{\kappa_0} = (-2, 0, 0, \ldots)$

  $Y_{\kappa_{n-2}+1} = Y_{\kappa_{n-2}+2} = \cdots = Y_{\kappa_{n-1}} = (n-2, n-1, n-1, \ldots)$ and all other diagrams of $\tilde{f}_{n-1}^{k_{n-1}+k_0} \tilde{f}_0^{k_0} \Phi$ are empty corners.

- $\varepsilon_0 = (\underbrace{0, 0, \ldots, 0}_{k_{n-1}})$ [Note: r_0 does not occur in the remaining chain of actions thus no ones will be added to this signature.]

- $\varepsilon_1 = (\underbrace{0, 0, \ldots, 0}_{k_0+k_1})$

- $\varepsilon_{n-2} = (\underbrace{0, 0, 0, \ldots, 0, 0}_{k_0+k_{n-1}+k_{n-2}})$

- $\varepsilon_{n-1} = (\underbrace{1, 1, \ldots, 1}_{k_0+k_{n-1}})$

- $\varepsilon_i = (\underbrace{0, 0, \ldots, 0}_{k_i})$ for $i \neq 0, 1, n-2, n-1$

Repeating this process to the n^{th} step we see that

$r_2 \dots r_{n-1} r_0 \Phi = \tilde{f}_2^{k_0+(k_{n-1}+\dots+k_3+k_2)} \dots \tilde{f}_{n-1}^{k_0+k_{n-1}} \tilde{f}_0^{k_0} \Phi$ and observe the following:

- $\tilde{f}_2^{k_0+(k_{n-1}+\dots+k_3+k_2)} \dots \tilde{f}_{n-1}^{k_0+k_{n-1}} \tilde{f}_0^{k_0} \Phi =$

$$\Bigg(\underbrace{\begin{array}{|c|}\hline 0\\\hline \text{n-1}\\\hline \vdots\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 0\\\hline \text{n-1}\\\hline \vdots\\\hline 2\\\hline\end{array}, \dots, \begin{array}{|c|}\hline 0\\\hline \text{n-1}\\\hline \vdots\\\hline 2\\\hline\end{array}}_{k_0}, \underbrace{\phi^1, \phi^1, \dots, \phi^1}_{k_1}, \underbrace{\boxed{2}, \boxed{2}, \dots, \boxed{2}}_{k_2}, \underbrace{\begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array}, \dots, \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array}}_{k_3},$$

$$\dots, \underbrace{\begin{array}{|c|}\hline \text{n-1}\\\hline \text{n-2}\\\hline \vdots\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline \text{n-1}\\\hline \text{n-2}\\\hline \vdots\\\hline 2\\\hline\end{array}, \dots, \begin{array}{|c|}\hline \text{n-1}\\\hline \text{n-2}\\\hline \vdots\\\hline 2\\\hline\end{array}}_{k_{n-1}} \Bigg)$$

i.e. $Y_1 = Y_2 = \dots = Y_{\kappa_0} = \big(-(n-1), 0, 0, \dots\big)$

$Y_{\kappa_{i-1}+1} = Y_{\kappa_{i-1}+2} = \dots = Y_{\kappa_i} = (1, i, i, \dots)$ for $i = 1, 2, 3, \dots, n-1$

$Y_{\kappa_0+1} = Y_{\kappa_0+2} = \dots = Y_{\kappa_1} = \phi^1 = (1, 1, 1, \dots)$ [which agrees with above]

- $\varepsilon_0 = (\underbrace{0, 0, \dots, 0}_{k_{n-1}})$
- $\varepsilon_1 = (\underbrace{0, 0, 0, 0, \dots, 0, 0, 0}_{(k_0+k_1+\dots+k_{n-1})+k_0}) = (\underbrace{0, 0, \dots, 0}_{k+k_0})$
- $\varepsilon_2 = (\underbrace{1, 1, 1, 1, \dots, 1, 1, 1}_{k_0+(k_2+k_3+\dots+k_{n-1})}) = (\underbrace{1, 1, \dots, 1}_{k-k_1})$
- $\varepsilon_i = (\underbrace{0, 0, \dots, 0}_{k_{i-1}})$ for $i = 3, 4, \dots, n-1$

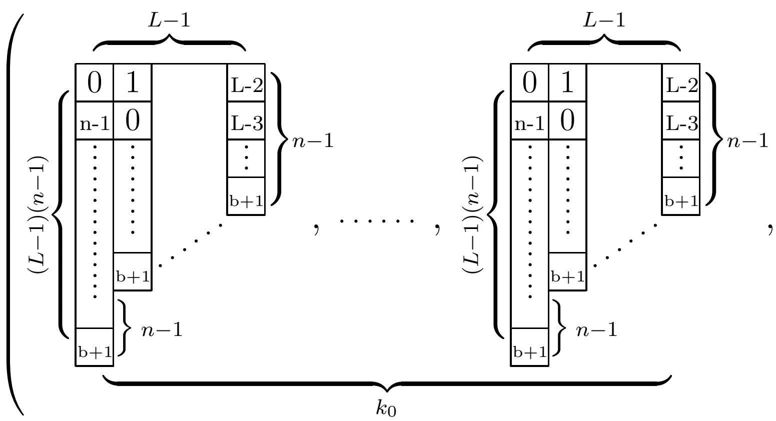
Thus the statement holds for $L = 1$. Assume it holds for $L-1$. i.e. Assume $w^{(L-1)(n-1)}\Phi = (Y_1, Y_2, \dots, Y_k)$ where:

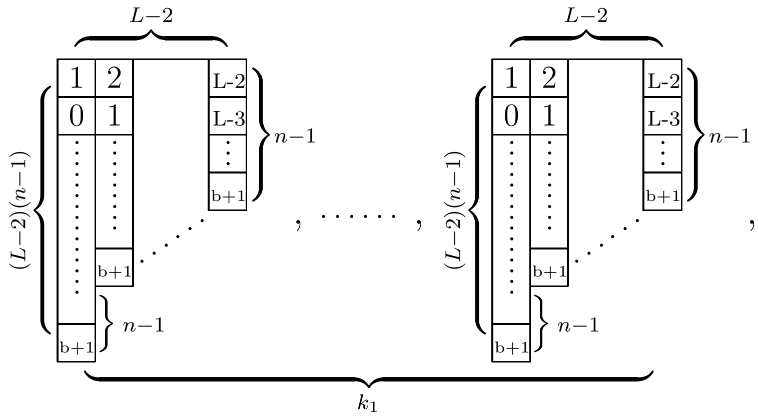
$$Y_1 = Y_2 = \cdots = Y_{\kappa_0} = \Big(-(L-1)(n-1), -(L-2)(n-1), \dots, -(n-1), 0, 0, \dots \Big)$$

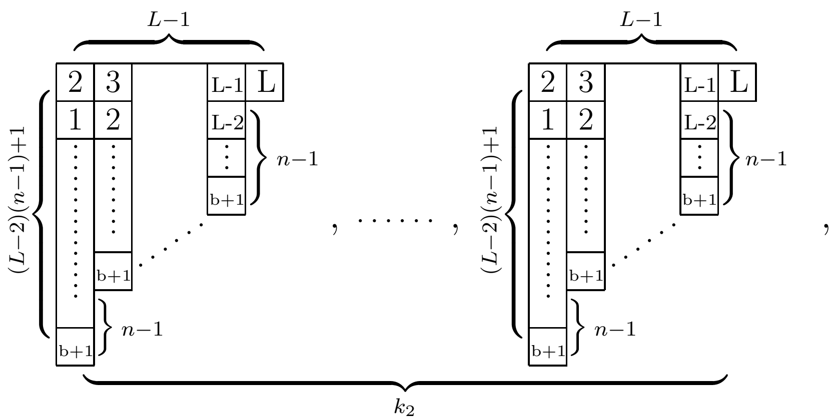
and

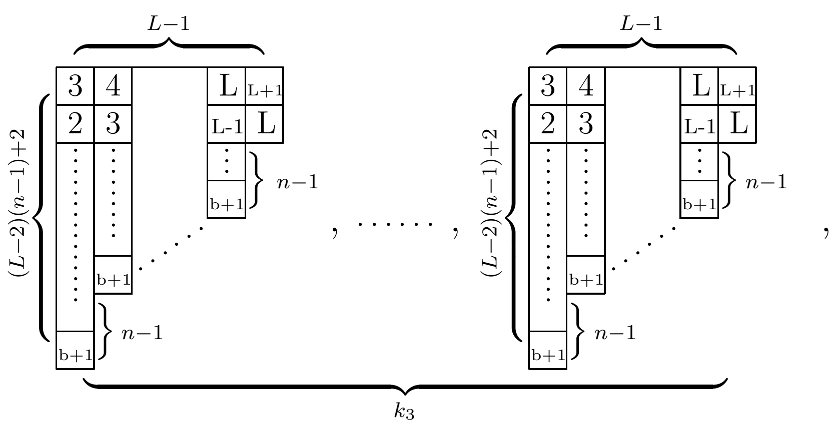
$$Y_{\kappa_{i-1}+1} = Y_{\kappa_{i-1}+2} = \cdots = Y_{\kappa_i} = \Big(-(L-2)(n-1)+1, -(L-3)(n-1)+1, \dots, 1, i, i, \dots \Big)$$

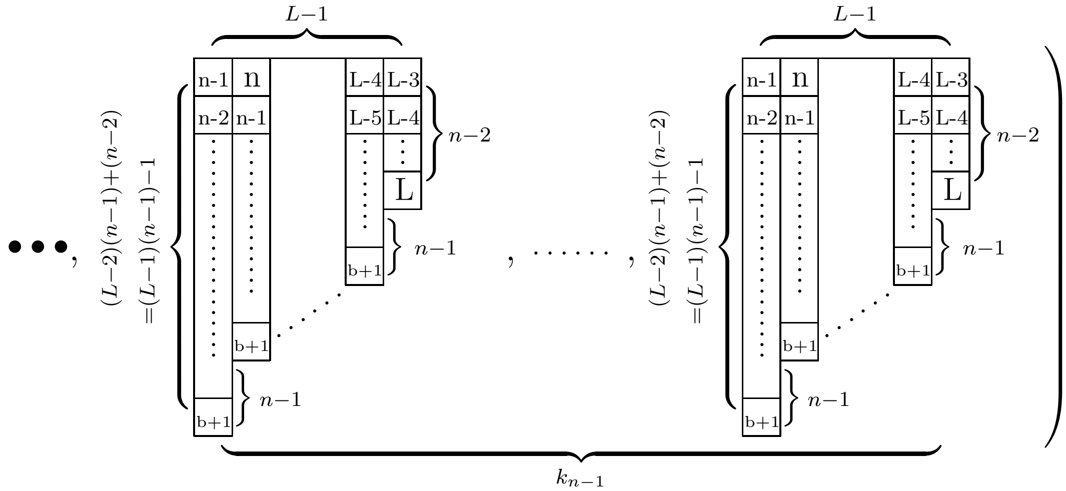
Show that the statements holds for L. Define $b = n - (L-1)(n-1)$ *mod* n. Notice that $b = L-1$. Observe the following about $w^{(L-1)(n-1)}\Phi$:

- $w^{(L-1)(n-1)}\Phi =$ 


L−2
(L−2)(n−1)
n−1
, ⋯⋯ ,
k1


L−1
(L−2)(n−1)+1
n−1
, ⋯⋯ ,
k2

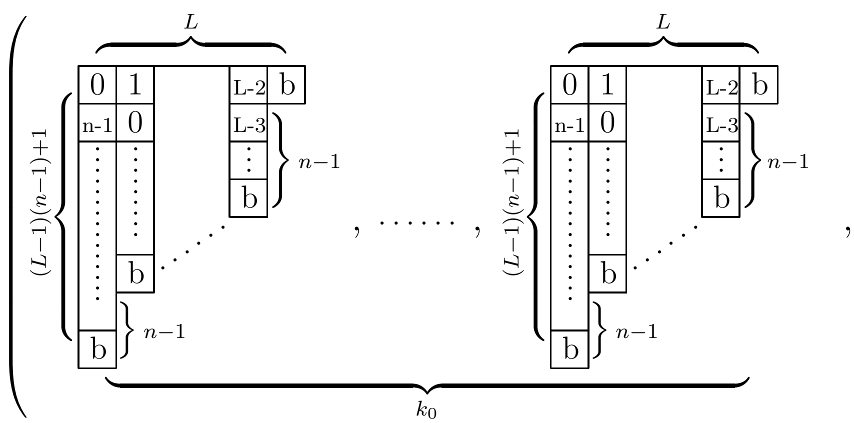

L−1
(L−2)(n−1)+2
n−1
, ⋯⋯ ,
k3

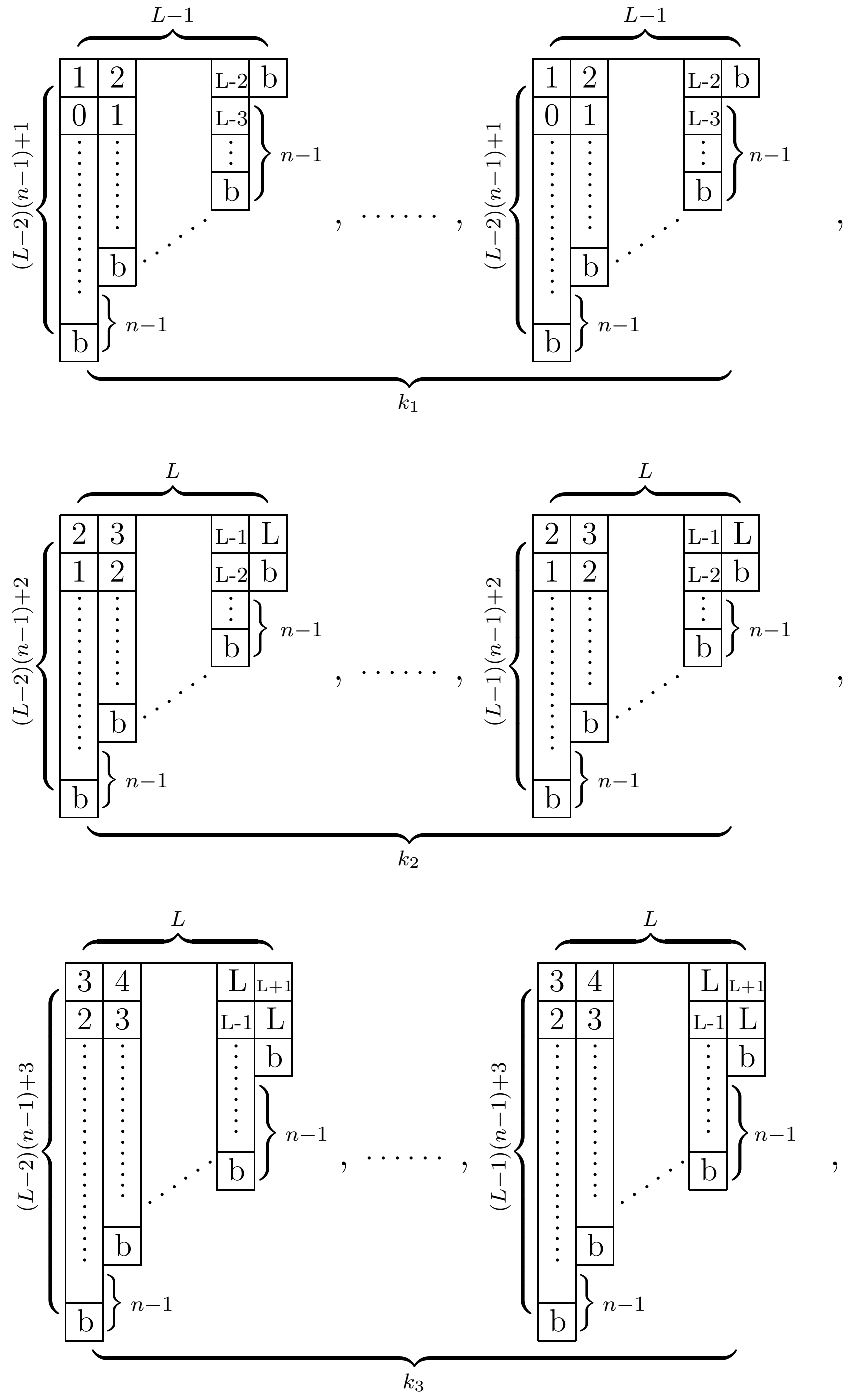


- $\varepsilon_b = (\underbrace{0,0,0,\ldots,0,0}_{(L-1)k+k_0})$ [It is easier to see the length of the signature in this way: $(L-1)k+k_0 = (L-1)(k_0+k_2+k_3+\ldots+k_{n-1})+(L-2)k_1+k_0+k_1$.]

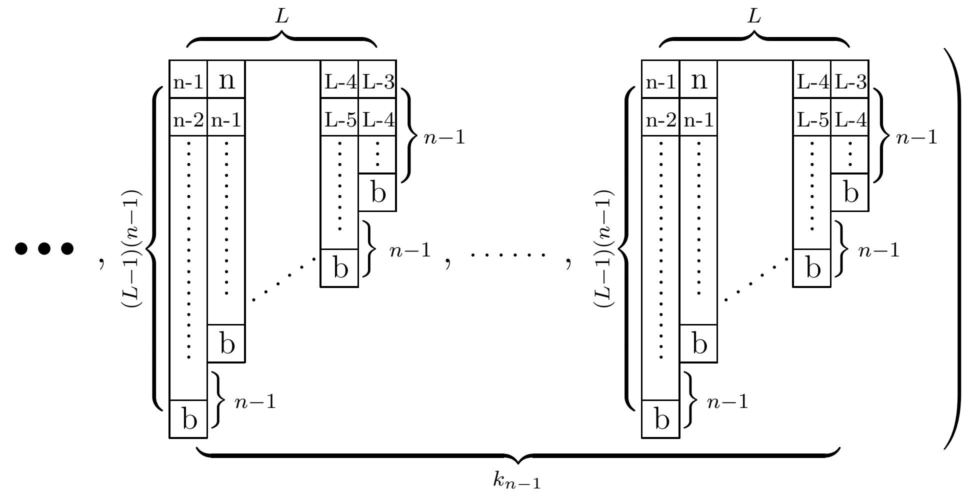
So $w^{L(n-1)}\Phi = r_{b-n+2}\ldots r_{b-1}r_b\left(w^{(L-1)(n-1)}\Phi\right) = r_{b-n+2}\ldots r_{b-2}r_{b-1}\left(\tilde{f}_b^{(L-2)k+k_0}\Phi\right)$. And we observe the following:

- $\tilde{f}_b^{(L-2)k+k_0}\Phi =$





$L-1$
$(L-2)(n-1)+1$
$n-1$
k_1
L
$(L-2)(n-1)+2$
$(L-1)(n-1)+2$
k_2
$(L-2)(n-1)+3$
$(L-1)(n-1)+3$
k_3



i.e. $Y_1 = Y_2 = \cdots = Y_{\kappa_0} =$

$$\Big(-(L-1)(n-1)-1, -(L-2)(n-1)-2, \ldots, -(n-1), -1, 0, 0, \ldots \Big)$$

$Y_{\kappa_0+1} = Y_{\kappa_0+2} = \cdots = Y_{\kappa_1} =$

$$\Big(-(L-2)(n-1), -(L-2)(n-1), \ldots, -(n-1), 0, 1, 1, \ldots \Big)$$

and $Y_{\kappa_{i-1}+1} = Y_{\kappa_{i-1}+2} = \cdots = Y_{\kappa_i} =$

$$\Big(-(L-2)(n-1), -(L-3)(n-1), \ldots, -(n-1), 0, i, i, i, \ldots \Big)$$

for $i = 1, 2, \ldots, n-1$

- $\varepsilon_{b-1} = (\underbrace{0, 0, 0, \ldots, 0, 0}_{Lk-k_1+k_{n-1}})$ [It is easier to see the length of the signature in this way: $Lk - k_1 + k_{n-1} = (L)(k_0 + k_2 + k_3 + \ldots + k_{n-1}) + (L-1)k_1 + k_{n-1}$.]

So $w^{L(n-1)}\Phi = r_{b-n+2} \ldots r_{b-3} r_{b-2} \left(\tilde{f}_{b-1}^{Lk-k_1+k_{n-1}} \tilde{f}_b^{(L-1)k+k_0} \Phi \right)$.

Continue this process to the $(n-1)$ step. Then we find that:

$$\begin{aligned} w^{L(n-1)}\Phi &= r_{b-n+2}\dots r_{b-1}r_b\Phi \\ &= \tilde{f}_{b-n+2}^{Lk-k_1+k_2}\dots\tilde{f}_{b-1}^{Lk-k_1+k_{n-1}}\,\tilde{f}_b^{(L-1)k+k_0}\Phi \end{aligned}$$

and

$$\begin{aligned} Y_1 &= Y_2=\dots=Y_{\kappa_0} \\ &= \Big(-(L-1)(n-1)-(n-1),-(L-2)(n-1)-(n-1),\dots, \\ &\qquad -(n-1)-(n-1),-(n-1),0,0,\dots\Big) \\ &= \Big(-L(n-1),-(L-1)(n-1),\dots,-2(n-1),-(n-1),0,0,\dots\Big) \\ Y_{\kappa_{i-1}+1} &= Y_{\kappa_{i-1}+2}=\dots=Y_{\kappa_i} \\ &= \Big(-(L-1)(n-1)+1-(n-1),-(L-3)(n-1)+1-(n-1),\dots, \\ &\qquad -(n-1)+1-(n-1),1-(n-1),1,i,i,\dots\Big) \\ &= \Big(-(L-1)(n-1)+1,-(L-2)(n-1)+1,\dots, \\ &\qquad -2(n-1)+1,-(n-1)+1,1,i,i,\dots\Big) \\ &\text{for } i=1,2,\dots,n-1 \end{aligned}$$

□

If we change the index set I from our constructions to be $\{1,2,\dots,n\}$ instead of $\{0,1,\dots,n-1\}$, the two descriptions (i) and (ii) collapse into one description.

Now we want to similarly generalize the explicit realization of $B_{w(L)}(k\Lambda_i)$ for any dominant weight $\lambda \in P^+$. In order to accomplish this we need the following lemma, which essentially states that any reverse ordered string of n reflections with the first one having non-trivial action will increase the width of $\mathcal{Y}$ by one.

Lemma 4.0.18. *Let* $w^{(l)} = r_{n-l+1} \ldots r_{n-1} r_0$ *and* $\lambda = k_0\Lambda_0 + k_1\Lambda_1 + \ldots + k_{n-1}\Lambda_{n-1}$ *with* $k_0 \neq 0$. *Then max* $\left\{ |\mathcal{Y}| \middle| \mathcal{Y} \in B_{w^{(l)}}(\lambda) \right\} = \left\lceil \frac{l}{n-1} \right\rceil$.

Proof.

Case 1: l is a multiple of $n-1$

If l is a multiple of $n-1$, then $w^{(l)}\Phi = w^{p(n-1)} = w(p)$ for some p. Then $\mathcal{Y} \in B_{w^{(l)}}(\lambda)$ implies that $\mathcal{Y} \subseteq w^{(l)}\Phi = w(p)\Phi$, since $w(p)\Phi$ is extremal. By Therorem 4.0.17, $|w(p)\Phi| = p$, so $|\mathcal{Y}| \leq p$. But since $w(p)\Phi$ is itself in $B_{w^{(l)}}(\lambda)$, we know *max* $\left\{ |\mathcal{Y}| \middle| \mathcal{Y} \in B_{w^{(l)}}(\lambda) \right\} = p$. Notice that $p = \frac{p(n-1)}{n-1} = \frac{l}{n-1} = \left\lceil \frac{l}{n-1} \right\rceil$. So the statement holds.

Case 2: l is not a multiple of $n-1$

If l is not a multiple of $n-1$, then l is between some multiples of $n-1$. Let L be the smallest integer such the $L(n-1)$ is larger than l. So $(L-1)(n-1) \leq l \leq L(n-1)$. Then we have the following:

$$\begin{aligned} L-1 &= max \left\{ |\mathcal{Y}| \middle| \mathcal{Y} \in B_{w(L-1)}(\lambda) \right\} \\ &\leq max \left\{ |\mathcal{Y}| \middle| \mathcal{Y} \in B_{w^{(l)}}(\lambda) \right\} \\ &\leq max \left\{ |\mathcal{Y}| \middle| \mathcal{Y} \in B_{w(L)}(\lambda) \right\} \quad = L \end{aligned}$$

Let $b = n - (L-1)(n-1) \ mod \ n$ as before. Consider $r_b w(L-1)\Phi$. Clearly this is in $B_{w^{(l)}}(\lambda)$. Now ε_b for $w(L-1)\Phi$ is $(\underbrace{0,0,0,\ldots,0,0}_{(L-1)k+k_0})$ by the argument in Theorem 4.0.17, and k_0 of these zeros correspond to b-corners located at the end of the first row of the first k_0 diagrams, whose current width is $L-1$ (again see argument in Theorem 4.0.17). Then $r_b w(L-1)\Phi = \tilde{f}_b^{(L-1)k+k_0}\Phi$, which fills these empty b-corners (see Theorem 4.0.17). Thus $|r_b w(L-1)\Phi| = L-1+1 = L$. So $max\left\{|\mathcal{Y}|\middle|\mathcal{Y}\in B_{w^{(l)}}(\lambda)\right\} = L = \left\lceil \frac{l}{n-1}\right\rceil$. $\square$

Notice that if $k_0 = 0$, then $max\left\{|\mathcal{Y}|\middle|\mathcal{Y}\in B_{w^{(l)}}(\lambda)\right\} \leq \left\lceil \frac{l}{n-1}\right\rceil$.

Finally we have the following description for $B_{w(L)}(\lambda)$, the Demazure crystal for the Demazure $U_q(\widehat{sl}(n))$-module $V_{w(L)}(\lambda)$.

Theorem 4.0.19. ***Main Theorem*** *Let $B_{w(L)}(\lambda)$ be the Demazure crystal for the Demazure $U_q(\widehat{sl}(n))$-module $V_{w(L)}(\lambda)$ generated by $w(L) = r_{n-L(n-1)+1}\ldots r_{n-1}r_0$ with highest weight $\lambda \in P^+$. Let σ be the Dynkin diagram automorphism whose effect on the simple reflections is: $\sigma(r_i) = r_{i+1}$. If:*

I . $\lambda = k\Lambda_0$, *then* $B_{w(L)}(\lambda) = B_L(\lambda)$

II . $\lambda \in P^+$, *then* $\displaystyle\bigcup_{j=0}^{n-1} B_{\sigma^j(w(L))}(\lambda) = B_L(\lambda)$

III . $\lambda \in P^+$, *then* $\displaystyle\bigcap_{j=0}^{n-1} B_{\sigma^j(w(L))}(\lambda) = B_{L-1}(\lambda)$

Proof.

(I): This is proven previously as Theorem 4.0.15. Note that this statement also agrees with Part II and Part III. Since $\lambda = k\Lambda_0$, any initial action other than r_0 will leave Φ unchanged. So the effect of $\sigma^j(w(L))$ acting on Φ is the same as some subsequence of $w(L)$ acting on Φ. Thus $\bigcup_{j=0}^{n-1} B_{\sigma^j(w(L))}(\lambda) = B_{w(L)}(k\Lambda_0) = B_L(k\Lambda_0)$. Similarly, it is clear that $\bigcap_{j=0}^{n-1} B_{\sigma^j(w(L))}(\lambda) = B_{w(L-1)}(k\Lambda_0) = B_{L-1}(k\Lambda_0)$ since $\bigcap_{j=0}^{n-1} w(L) = r_{n-(L(n-1))+1} \dots r_{n-1}r_0\Phi = w(L-1)\Phi$.

(II): Let λ be any dominant weight and $\mathcal{Y} \in \bigcup_{j=0}^{n-1} B_{\sigma^j(w(L))}(\lambda)$. Immediately we have $\mathcal{Y} \in B(\lambda)$. Now $\mathcal{Y}$ in the union gives that $\mathcal{Y}$ is contained in $B_{\sigma^j(w(L))}(\lambda)$ for some j. Recall $\sigma^j(w(L))$ is $\underbrace{r_{j-(L(n-1))+1} \dots r_{j-1}r_j}_{L(n-1)}$, a sequence of $L(n-1)$ reverse order reflections. So by Lemma 4.0.18, $|\mathcal{Y}| \leq \frac{L(n-1)}{n-1} = L$. Thus $\mathcal{Y} \in B_L(\lambda)$.

Next, we show $B_L(\lambda) \subseteq \bigcup_{j=0}^{n-1} B_{\sigma^j(w(L))}(\lambda)$. Let $\mathcal{Y} \in B_L(\lambda)$. Then $\mathcal{Y} \in B(\lambda)$ and $|\mathcal{Y}| = max\{|Y_i| \big| i = 1, 2, \dots, k\} = L$.

Let $L = 1$ and assume $\mathcal{Y} \neq \Phi$. Then $|Y_i| = 1$ or 0 for each i. We know that each Y_i is created by some sequence of $\tilde{f}_i^{\rho_i}$ acting on ϕ_i ($\rho_i \in \mathbb{Z}_{\geq 0}, \quad 0 \leq i \leq n-1$) and similarly for $\mathcal{Y}$. So we know $\mathcal{Y}$ is in the crystal generated by some sequence of reflections.

Consider $\mathcal{Y}$ in sections by charge: $Y_1, \dots, Y_{\kappa_0}$ is the 0-section; $Y_{\kappa_0+1}, \dots, Y_{\kappa_1}$ is the 1-section, and so on. By determining the reflections needed to generate each section, we can learn those needed for $\mathcal{Y}$.

Look at the 0-section, $(\phi_1^0, \phi_2^0, \dots, \phi_{\kappa_0}^0) = \Phi^0$. ε_0 here is $(\underbrace{0, 0, \dots, 0}_{k_0})$ and all other ε_i are non-existant. Thus the first action to create this section must be $\tilde{f}_0^{\rho_0}$ so we must start with r_0. Now there are three non-trivial signatures for $r_0\Phi^0$: ε_0, ε_1, and ε_{n-1}. These are:

- $\varepsilon_0 = (\underbrace{\underbrace{1, 1, \ldots, 1}_{\rho_0}, 0, 0, \ldots, 0}_{k_0})$
- $\varepsilon_1 = (\underbrace{0, 0, \ldots, 0}_{\rho_0})$ and notice that action of $\tilde{f}_1$ will increase the width of $\mathcal{Y}$ to greater than one.
- $\varepsilon_{n-1} = (\underbrace{0, 0, \ldots, 0}_{\rho_0})$.

So the next action to create $\mathcal{Y}$ must be $\tilde{f}_{n-1}^{\rho_{n-1}}$ since it is the only action that will result in a change that does not increase the width. Thus we act by r_{n-1} and consider the signatures of $r_{n-1}r_0\Phi_0$. We have:

- $\varepsilon_0 = (\underbrace{\underbrace{1, 1, \ldots, 1}_{\rho_0+\rho_{n-1}}, 0, 0, \ldots, 0}_{k_0-\rho_{n-1}})$
- $\varepsilon_1 = (\underbrace{0, 0, \ldots, 0}_{\rho_0})$
- $\varepsilon_{n-1} = (\underbrace{\underbrace{1, 1, \ldots, 1}_{\rho_{n-1}}, 0, 0, \ldots, 0}_{\rho_0})$
- $\varepsilon_{n-2} = (\underbrace{0, 0, \ldots, 0}_{\rho_{n-1}})$

We continue this process to the $n-1$ step. We stop here because Lemma 4.0.18 tells us that the n^{th} step will increase the width of $\mathcal{Y}$ to two. Thus we see that the 0-section is generated by at most $r_2 \ldots r_{n-1}r_0 = w(1)$.

Repeat this process for each section. So the i-section is generated by at most $r_{i-(n-1)+1} \ldots r_{i-1}r_i = \sigma^i(w(1))$.

To generate $\mathcal{Y}$ we must have some subset of these generating sequences (or subsequences of them). Now by Theorem 4.0.17, we know that we need at least $n-1$ reverse order reflections since $w(1) \in B_1(\lambda)$. Each of the pieces above fits this requirement and nothing shorter. But by Lemma 4.0.18, we know that there is at most $n-1$ reflections used to generate $\mathcal{Y}$, else $max\{|Y|\} = 2 > 1$. Thus $\mathcal{Y}$ must be generated by precisely one of the generating sequences above! So $\mathcal{Y} \in \bigcup_{j=0}^{n-1} B_{\sigma^j(w(1))}(\lambda)$.

It is easy to reason that this extends to L. Assume it holds for $L-1$. Show for L.

Now $\mathcal{Y} \in B_L(\lambda)$ and the argument follows directly from Theorem 4.0.17 and Lemma 4.0.18 as before. $\mathcal{Y}$ must be, by argument used previously, generated by some sequence of simple reflections in reverse order: $\ldots r_{n-2}r_{n-1}r_0$. This is assumed for $L-1$ in induction. Now consider the signatures. Action by $\tilde{f}_b$ is the only pertinent one and the b-signature contains at least one zero (if $|\mathcal{Y}| = L$, else $|\mathcal{Y}| < L$ and we are in the induction case), so act by r_b. Next the only non-width-increasing action to use is $b-1$ and so on until we have accumulated up to $n-1$ additional reflections. So each segment is generated by $r_{i-L(n-1)+1} \ldots r_{i-2}r_{i-1}r_i$, $i = 0, 1, \ldots, n-1$. Consider the reflections needed to generate $\mathcal{Y}$. By Lemma 4.0.18, we know the length of this string can not exceed $L(n-1)$, else $max\{|Y|\} = L+1$. However, Theorem 4.0.17 tells us that $w(L)\Phi$ is in $B_L(\lambda)$. Thus we must have at least $L(n-1)$ actions in our sequence if we are to generate any arbitrary piece. Therefore $\mathcal{Y}$ itself must be generated by precisely one of these. So $\mathcal{Y} \in \bigcup_{j=0}^{n-1} B_{\sigma^j(w(L))}(\lambda)$.

(III): Let λ be any dominant weight. Recall we have defined $r_j = r_{j'}$ if $j = j'$ mod n. However, we will avoid taking mod n for large portions of this proof and leave j' in order to be more precise. Essentially we will consider any sequence of reverse order reflections to be portions of:

$$\dots r_{-n-1}r_{-n}\dots r_{-3}r_{-2}r_{-1}r_0r_1r_2r_3\dots r_nr_{n+1}\dots$$

First we show $\bigcap_{j=0}^{n-1} B_{\sigma^j(w(L))}(\lambda) \subseteq B_{L-1}(\lambda)$

Let $\mathcal{Y} \in \bigcap_{j=0}^{n-1} B_{\sigma^j(w(L))}(\lambda)$. Then $\mathcal{Y}$ is generated by *each* of the following sequences of reflections acting on Φ (This is set 4.1 consisiting of $n-1$ linear chains.):

$$\begin{array}{ccccc} r_{-L(n-1)+1} & \dots & r_{-2} & r_{-1} & r_0 \\ r_{-L(n-1)+2} & \dots & r_{-1} & r_0 & r_1 \\ \vdots & & & & \vdots \\ r_{-(L-1)(n-1)+1} & \dots & r_{n-3} & r_{n-2} & r_{n-1} \end{array} \qquad \begin{array}{r} (1:4.1) \\ (2:4.1) \\ \\ (n-1:4.1) \end{array}$$

Furthermore, if we shift the indices by n for any combination of these rows we get that $\mathcal{Y}$ must also be generated by each of the new sequence of reflections acting on Φ. For example, in (2:4.1), r_1 comes before the initial r_0 in (1:4.1), but it could be viewed as coming after it instead by simply shifting the indices $-n$. In other words, we could write (2:4.1) as $r_{-(L+1)(n-1)+1}\dots r_{-n}r_{-n+1}$. Notice that although this does not result in a change in the actual sequence of actions, it does change the intersection of the rows. Since $\mathcal{Y}$ is in the intersection of the $B_{\sigma^j(w(L))}(\lambda)$, it must be generated by some subsequence of reflections common to each sequence $(1:4.1)-(n-1:4.1)$, or common to the same set with one or more index shifted rows.

By examination, the maximal common subsequences are:

$$
\begin{aligned}
S_0 &= r_{-(L-1)(n-1)+1}\dots r_{-1}r_0 \\
\textit{and} & \\
S_j &= \sigma^j(S_0) \quad j = 1, 2, \dots, n-1
\end{aligned}
$$

Notice that the length of S_0 is $(L-1)(n-1)$. Thus we know $S_0 = w(L-1)$ and $S_j = \sigma^j(w(L-1))$. Since $\mathcal{Y}$ is generated by one of these, $\mathcal{Y} \in \bigcup_{j=0}^{n-1} B_{\sigma^j(w(L-1))}(\lambda)$. By Part II of this theorem, we know $\bigcup_{j=0}^{n-1} B_{\sigma^j(w(L-1))}(\lambda) = B_{L-1}(\lambda)$, thus $\mathcal{Y} \in B_{L-1}(\lambda)$.

Now, show $B_{L-1}(\lambda) \subseteq \bigcap_{j=0}^{n-1} B_{\sigma^j(w(L))}(\lambda)$. Let $\mathcal{Y} \in B_{L-1}(\lambda)$. Then by Part II of this theorem, $\mathcal{Y} \in \bigcup_{j=0}^{n-1} B_{\sigma^j(w(L-1))}(\lambda)$. So $\mathcal{Y}$ is generated by at least one of the following sequences of reflections acting on Φ (This is set 4.2 which also consists of $n-1$ linear chains.):

$$
\begin{array}{ccccc}
r_{-(L-1)(n-1)+1} & \dots & r_{-2} & r_{-1} & r_0 \\
r_{-(L-1)(n-1)+2} & \dots & r_{-1} & r_0 & r_1 \\
 & \vdots & & & \vdots \\
r_{-(L-2)(n-1)+1} & \dots & r_{n-3} & r_{n-2} & r_{n-1}
\end{array}
\qquad
\begin{array}{r}
(1:4.2) \\
(2:4.2) \\
\\
(n-1:4.2)
\end{array}
$$

Note that if $\mathcal{Y}$ is generated by $(1 : 4.2)$, we are done since $(1 : 4.2)$ is S_0, a subsequence of each $(1 : 4.1) - (n-1 : 4.1)$. Now look at $(2 : 4.2)$. Clearly it is in each of $(2 : 4.1) - (n-1 : 4.1)$. Since $(1 : 4.1)$ can be rewritten as $r_{n-(L(n-1)+1}\dots r_{n-2}r_{n-1}r_n$, by shifting the indices up n (i.e. moving the section right). Now it is clear that $(2:4.2)$ is contained in $(1:4.1)$. In fact, $(2:4.2)$ is $S_1 = \sigma^1(S_0)$. Similarly, $(i:4.2)$ is in each of $(i:4.1)-(n-1:4.1)$ and in $(1:4.1)-(i-1:4.1)$ after shifting the indices of $(1:4.1)-(i-1:4.1)$ by n. Again we see that $(i:4.2)$ is actually $S_i = \sigma^i(S_0)$.

Since $\mathcal{Y}$ is generated by some $(i:4.2) = S_i$, and *every* S_i is in each $(j:4.1)$, $\mathcal{Y}$ is generated by the actions of *each* $(j:4.1)$. Thus $\mathcal{Y} \in B_{\sigma^j(w(L))}(\lambda)$ for each $j = 0, 1, \ldots, n-1$. Therefore $\mathcal{Y} \in \bigcap_{j=0}^{n-1} B_{\sigma^j(w(L))}(\lambda)$. □

The following example illustrates the main theorem. The first three figures give examples of $B_{\sigma^j(w(L))}(\lambda)$ while the fourth gives the union of the crystals and the final diagram illustrates their intersection.

Example 4.0.20. *Let $n = 3$ and $\lambda = \Lambda_0 + \Lambda_1 + \Lambda_2$. Consider $L = 2$. Then $w(2)$ contains 4 reverse ordered reflections and, as illustrated in the following figures:*

$$\bigcup_{j=0}^{2} B_{\sigma^j(w(L))}(\Lambda_0 + \Lambda_1 + \Lambda_2) =$$

$$B_{r_0r_1r_2r_0}(\Lambda_0+\Lambda_1+\Lambda_2) \cup B_{r_1r_2r_0r_1}(\Lambda_0+\Lambda_1+\Lambda_2) \cup B_{r_2r_0r_1r_2}(\Lambda_0+\Lambda_1+\Lambda_2) =$$

$$B_2(\Lambda_0+\Lambda_1+\Lambda_2)$$

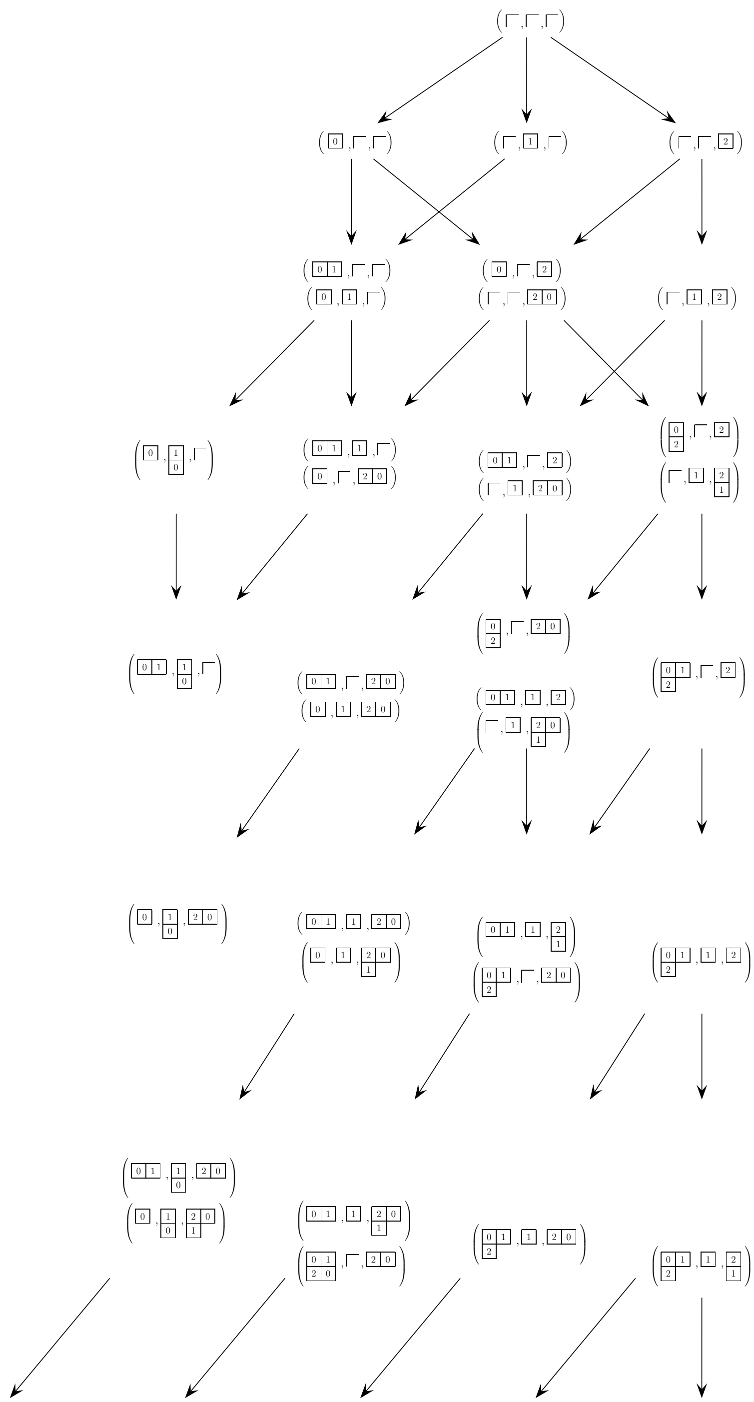
and

$$\bigcap_{j=0}^{2} B_{\sigma^j(w(L))}(\Lambda_0 + \Lambda_1 + \Lambda_2) =$$

$$B_{r_0r_1r_2r_0}(\Lambda_0+\Lambda_1+\Lambda_2) \cap B_{r_1r_2r_0r_1}(\Lambda_0+\Lambda_1+\Lambda_2) \cap B_{r_2r_0r_1r_2}(\Lambda_0+\Lambda_1+\Lambda_2) =$$

$$B_1(\Lambda_0+\Lambda_1+\Lambda_2)$$

Figure 4.2: Crystal graph for the Demazure module $V_{r_0 r_1 r_2 r_0}(\Lambda_0 + \Lambda_1 + \Lambda_2)$ when $n = 3$



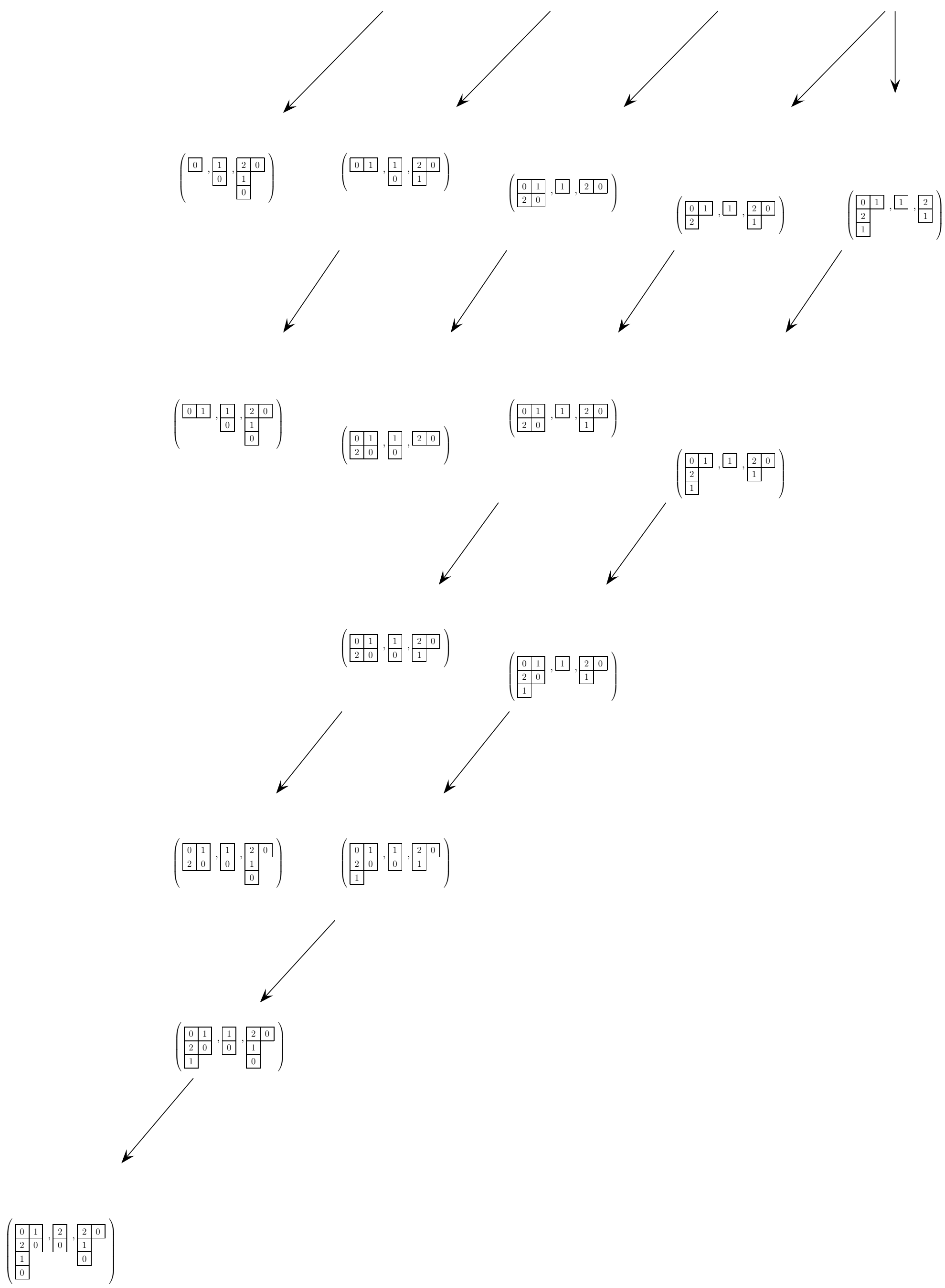
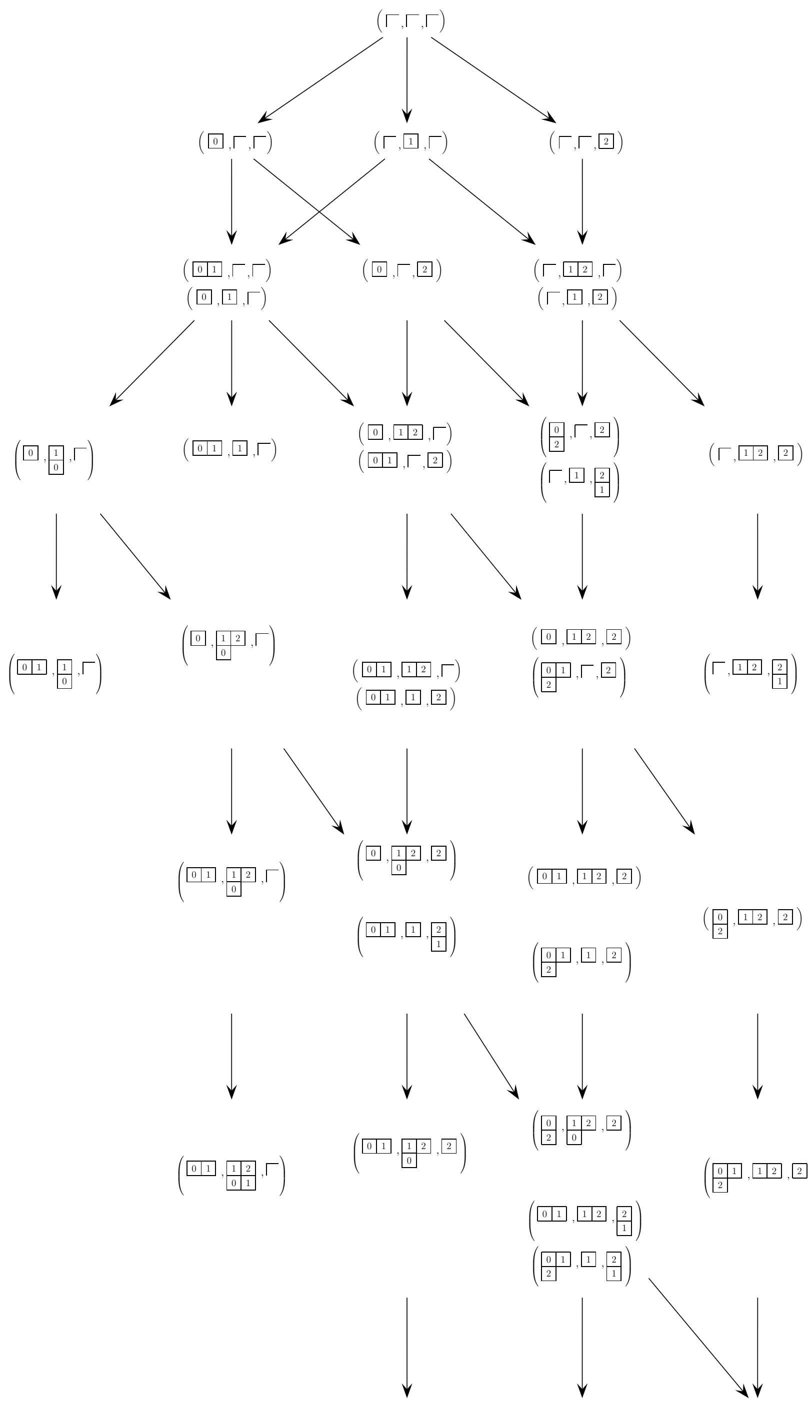


Figure 4.3: Crystal graph for the Demazure module $V_{r_1 r_2 r_0 r_1}(\Lambda_0 + \Lambda_1 + \Lambda_2)$ when $n = 3$



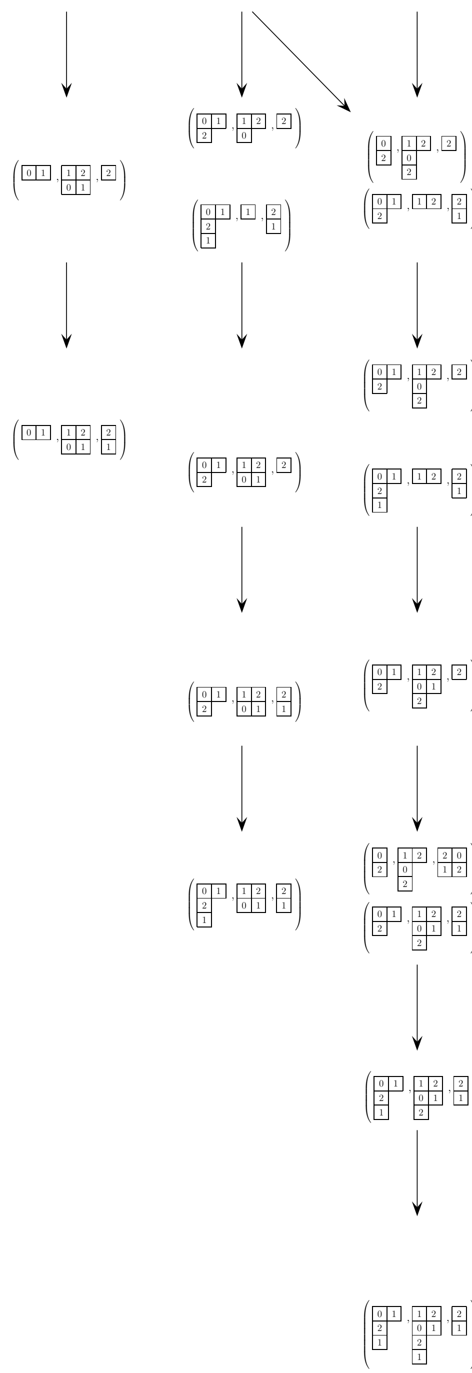
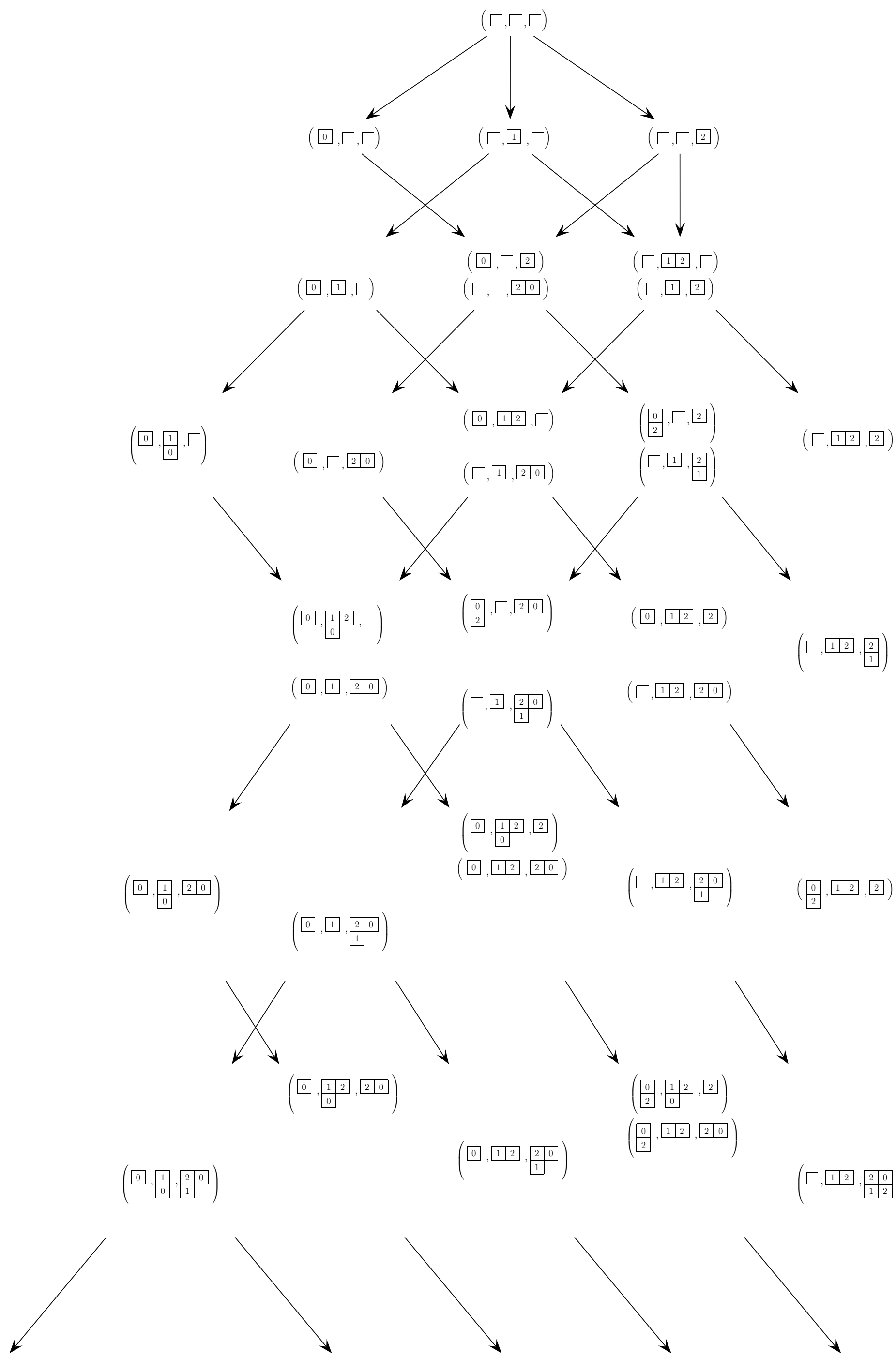


Figure 4.4: Crystal graph for the Demazure module $V_{r_2r_0r_1r_2}(\Lambda_0+\Lambda_1+\Lambda_2)$ when $n = 3$



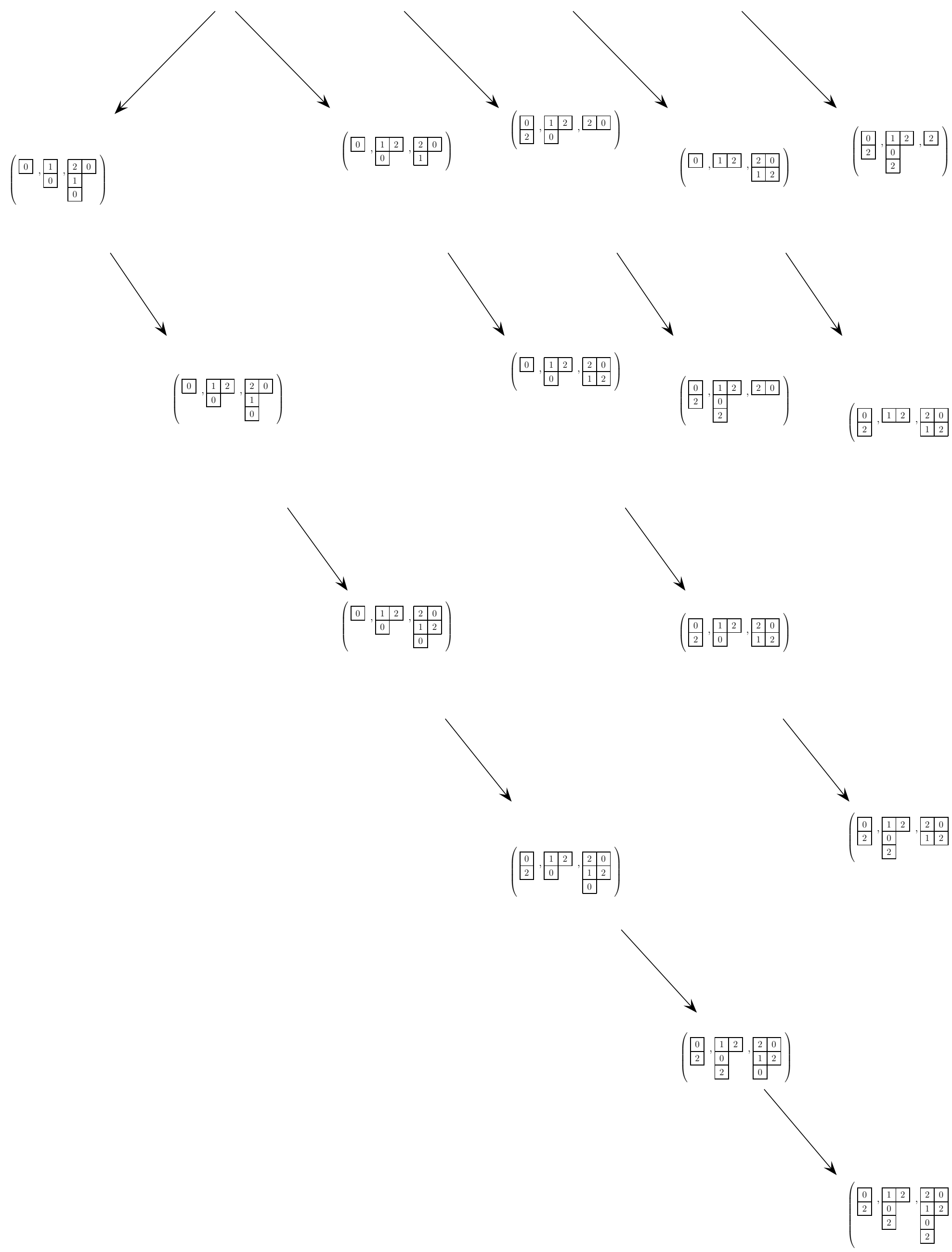
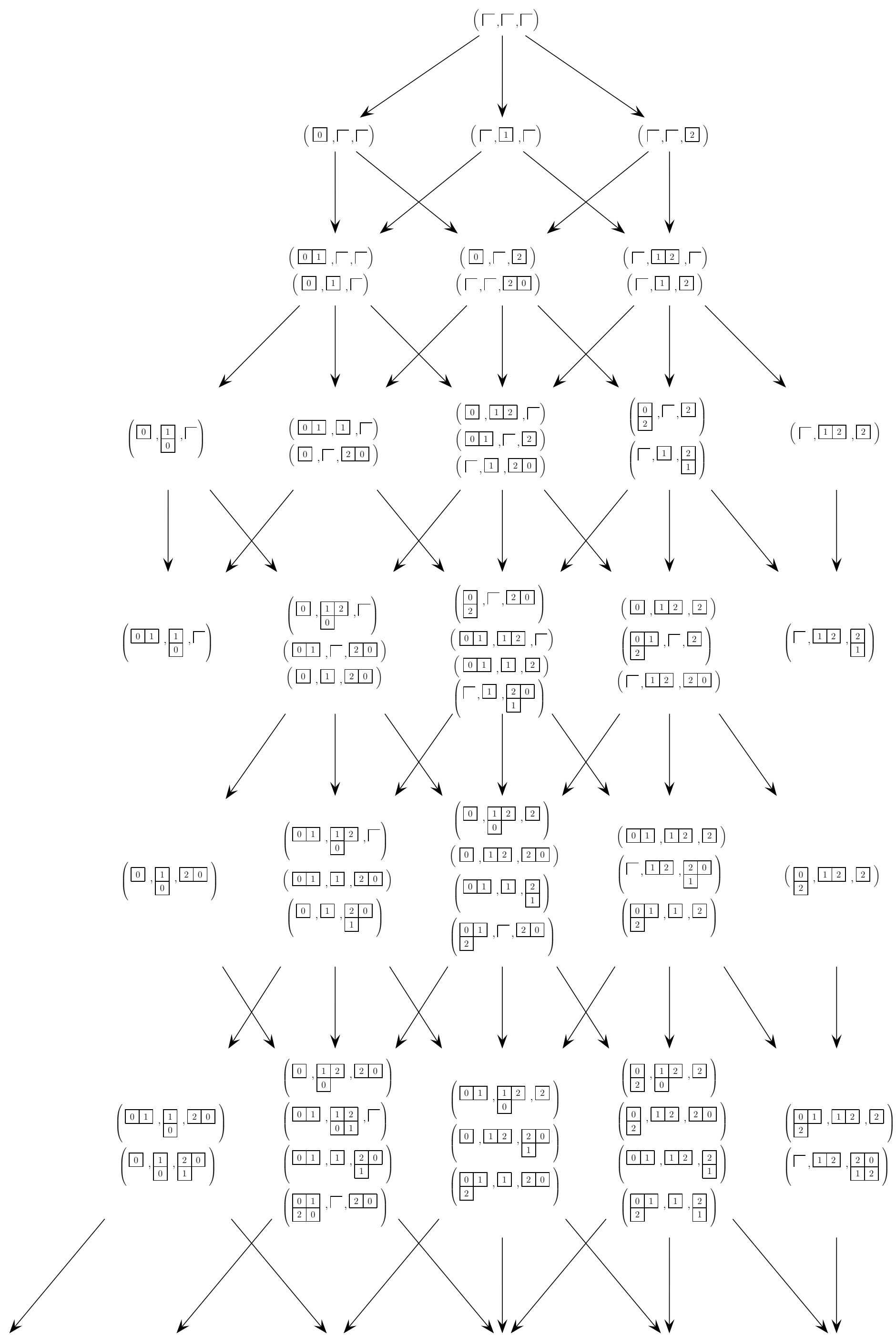
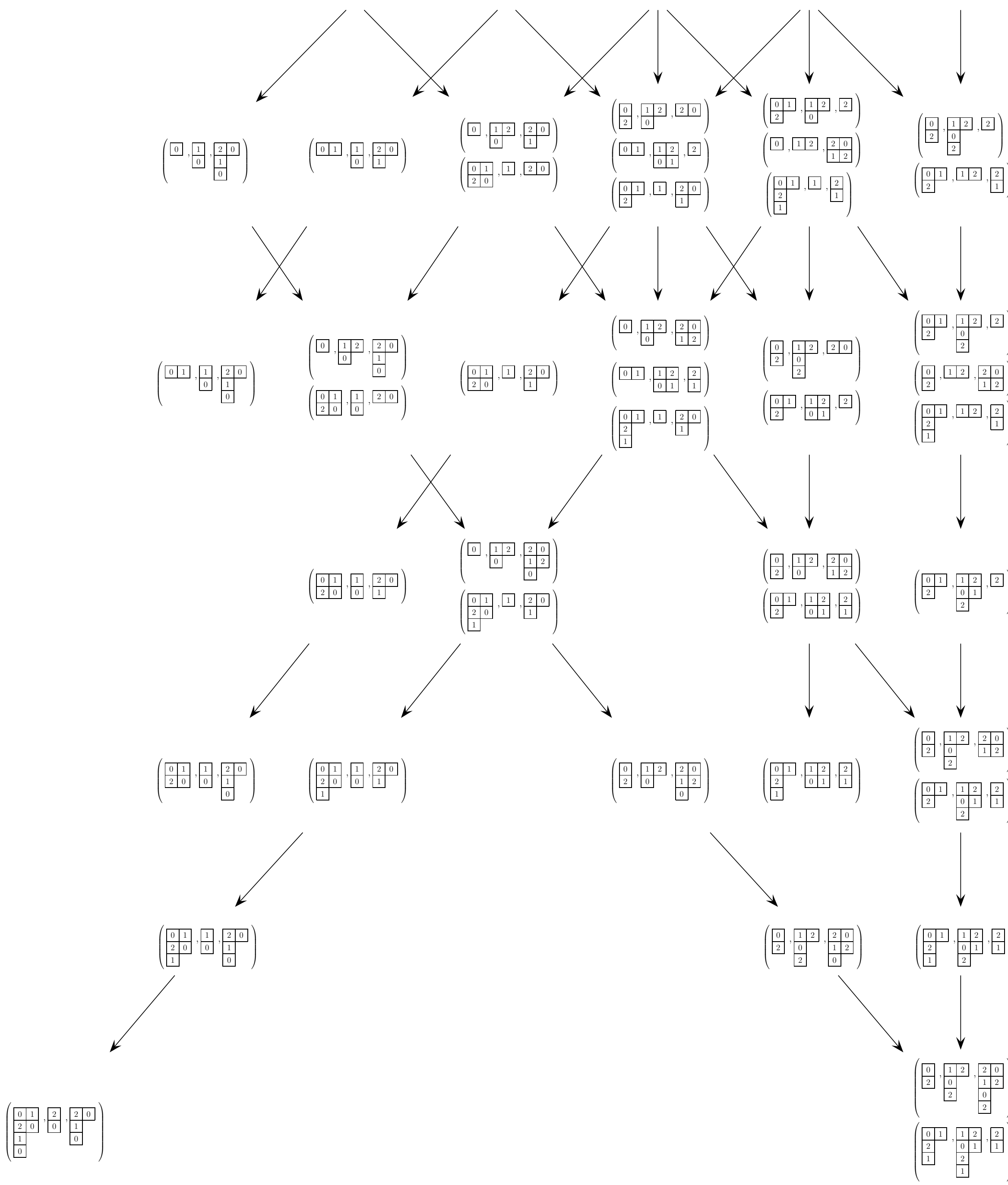


Figure 4.5: $\bigcup_{j=0}^{2} B_{\sigma^j(w(L))}(\Lambda_0 + \Lambda_1 + \Lambda_2) = B_2(\Lambda_0 + \Lambda_1 + \Lambda_2)$





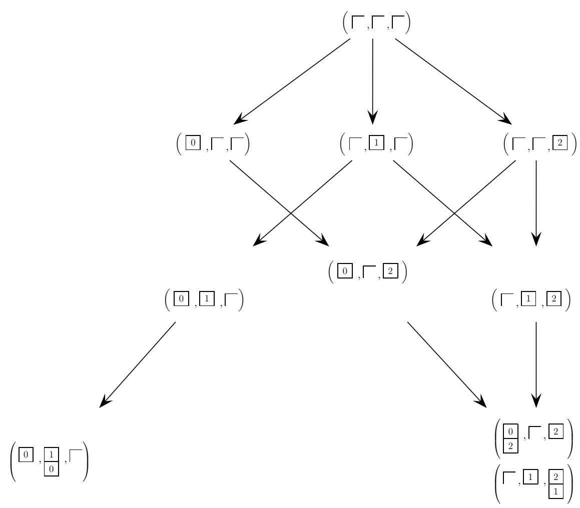


Figure 4.6: $\bigcap_{j=0}^{2} B_{\sigma^j(w(L))}(\Lambda_0 + \Lambda_1 + \Lambda_2) = B_1(\Lambda_0 + \Lambda_1 + \Lambda_2)$

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