Abstract

Williams, Michael Peretzian. Nilpotent N-Lie Algebras. (Dissertation Director: Ernie Stitzinger).

In 1986, Kasymov introduced the concept of nilpotent n-Lie algebras, proved an analogue of Engel's Theorem and later proved an analog of Jacobson's refinement of Engel's Theorem. Despite these achievements, the subject of nilpotency in n-Lie algebras has not been examined in great detail in the literature since. We shall explore the concept of nilpotent n-Lie algebras by examining, and proving where possible, other classical nilpotent group theory and nilpotent Lie algebra results, in the n-Lie algebra setting.

Nilpotent N-Lie Algebras

by

Michael Peretzian Williams

Bachelor of Science, University of South Carolina, 1998

Master of Science, University of South Carolina, 2001

Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in the

Department of Mathematics

North Carolina State University

2004

Dr. Ernie Stitzinger Advisory Committee	Dr. Ronald Fulp Committee Member
Dr. Kwangil Koh Committee Member	Dr. Kailash Misra Committee Member

Biography

I received my under graduate degree in mathematics in December of 1998 from the University of South Carolina. On September 27, 2000 I married Ellison Anne Charping. We both recieved our master's of science in August of 2001 from the University of South Carolina. I worked with Dr. Michael Filaseta in the area of number theory. In the fall of 2001 I began my work towards my PhD at North Carolina State University in the area of Lie algebras under the direction of Dr. Ernest Stitzinger.

Acknowledgements

First, I thank God for making me, making such a marvelous, intricate world and for the mathematical laws that govern it which I will never tire studying. Thank you to Jesus the Christ who hung the stars and also hung on a cross for my treachery. By whose death I may live, love and be who I was meant to be.

I would like to thank my teachers: To Mrs. Pitts who taught me when I multiply and carry to pretend the carried number was her face so that I would erase it. To Mrs. Hamm who taught me that a positive times a negative is a negative, a positive times a negative is a negative, a positive times a negative is a negative, a positive times a negative is a negative... and who also taught me the value of cutting and pasting. To Mrs. Brooks who taught me that balancing an equation is like giving candy to the class, you can't give to one side and not the other. To Mr. Roberts, thank you for passing me in 9^{th} grade algebra which I should have failed miserably. Quite naturally this current work is in algebra. Thank you to Mr. Pearson who taught me to complete the square and that it is possible to do minor body work on your car with your bare hands. Thank you to Mr. Hathaway for being compassionate. Thank you to Mrs. Cain for whispering on that fateful day in the allergist's office, "Major in Math." Thank you to Professor Varner and Dr. Becker for teaching me calculus. Thank you to Dr. Cook for teaching me what little I know about differential equations and more importantly, all of the teaching jokes I know. Such as, when a student comes in late, addressing the class say, "What did we say? The next person to come in buys us all lunch?" This joke works even better in big lecture classes. Thank you to Dr. Oskolkov for teaching me linear algebra the first and second time. Thank you to Dr. Sumner, I got a D+ in his discrete class but learned more about counting than I have since. I know that he's not surprised that I got this far because I had a 50/50 chance... either I would or I wouldn't. Thank you to Dr. Meade for demonstrating that there are

intelligent people who know that Jesus is the Christ. Thank you to "H" for showing me my first glimpse of algebra, how to prove things, and for dubbing me "Skew" and later just "S". Thank you to Dr. Roberts for teaching me complex variables and being a point of humorous hope against the drudgery of graduate analysis. Thank you to Ralph for teaching linear algebra for the third time, that there is no canonicle way to spell cononacle, and for introducing me to the sub-blob. A special thank you to Dr. Filaseta who either directly or indirectly or indirectly taught me everything I know about number theory, particularly thank you for teaching about Bernoulli numbers and polynomials and most importantly for being my Master's Thesis advisor. Thank you to Dr. Luh for teaching me linear algebra for the fourth time. Thank you to Dr. Fulp for teaching me differential geometry, many theoremitos and for being on my committee. Thank you to Dr. Misra for teaching me about Lie algebras and for being on my committee. A huge thank you to Dr. Stitzinger for teaching me algebra and Galois theory, and especially for agreeing to be my thesis advisor. With out his enthusiasm, insight and great patience, this work would not have been possible.

I would like to thank my friends: Thank you to the Gowens sisters (I'm not sure if it was Pam, Cindy, Donna, or Kim), for teaching me to count past 19. Thank you to Sam for encouragement in one of the most difficult times of my life, for your humor, and for being a brother in Christ. Thank you to Joe also for being an encouragement through two bouts with analysis. Be it Vlad's "And that's it," or Pencho's, "Uncountable... ah, ah,ah." Thank you to James; you're terrible at math but you've been about the best friend I've had. Thank you to Garrick; you introduced me to Neil and have been with me through many a tough time dulling pain with rock'n'roll. Fortunately for me this current math project is better than those we did together. Thank you to Danny for helping me to laugh through many hardships with non-linear driving. Again you've disappeared, I hope you come back soon.

My family: Thank you to my father who has passed on to me a love for the natural wonders of God's world governed by mathematics and for God Himself. Thank you

to my mother for instilling in me a love and compassion for everything in God's world that draws breath from chickens, turkeys, peacocks, slugs, toads, bugs, spiders, snails, puppies and people. Thank you to my sister, who loves to multiple, add, subtract, and divide... worms. Thank you to my brother for chess games, drives to Rabons', and for creating "Earl" with me. Finally, thank you to my dear, beloved Ellison Anne who remains, as always, transfinitely cool.

Contents

Introduction 1

Section I: Examples of Nilpotent N-Lie Algebras 7

Section II: Conditions Equivalent To Nilpotentcy 13

Section III: Theorems That Fail To Generalize To N-Lie Algebras 17

Section IV: Some Automorphism and Derivation Theorems 20

Section V: Theorems Of Hall and Chao 29

Section VI: Uncountably Many Non-Isomorphic Real N-Lie Algebras 33

Section VII: Frattini Theory 41

Bibliography 54

Introduction

In 1960 Kurosh introduced the Ω-algebra, an algebra equipped with an n-ary, n-linear product. He also discussed skew-symmetric Ω-algebras and noted that the class of Lie algebras is contained in them. In 1985 Fillipov introduced n-Lie algebras, a skew-symmetric Ω-alebgra with the identity

$$[[x_1, x_2, \ldots, x_n]y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}[x_i, y_2, \ldots, y_n], x_{i+1}, \ldots, x_n]$$

which is called the Jacobian identity for n-Lie algebras. Note that if $n = 2$ this is precisely the familiar Jacobian identity for Lie algebras. Indeed if $n = 2$ we in fact have a Lie algebra.

In his work, Fillipov [FV] gives two main examples of n-Lie algebras.

N-cross product.
Let A have basis $\{x_1, \ldots, x_{n+1}\}$ with product $[x_1, x_2, \ldots, \hat{x_i} \ldots, x_{n+1}] = (-1)^{n+1+i} x_i$. We shall examine this n-Lie algebras to gain insight into Frattini theory for n-Lie algebras in section VII.

Jacobian n-Lie algebra.
Let $A = \mathbb{F}[x_1, \ldots, x_n]$. If $f_1, \ldots, f_n \in A$ then $[f_1, \ldots, f_n]$ is the determinant of the Jacobian matrix of the partial derivatives of $f_1, \ldots, f_n$.

Note that this example is infinite dimensional. In section I we will truncate this and make it finite dimensional to generate examples of nilpotent n-Lie algebras.

To begin a discussion on n-Lie algebras we need some algebraic structures and linear transformations to work with. Some of the first topics discussed in Lie algebras and algebras in general are subalgebras and ideals, homomorphisms, derivations and representations. We now define these structures and transformations for n-Lie algebras. Throughout we let A be an n-Lie algebra.

Subalgebra

If S is a subspace of A that is closed under the n-product, then we call S a *subalgebra*.

Ideal

If I is a subalgebra of A such that $[I, A, \ldots, A] \subset I$ we call I an *ideal* of A and we denote it $I \lhd A$.

Derivations

If δ is a linear transformation such that $([a_1, \ldots, a_n])\delta = \sum_{i=1}^{n} [a_1, \ldots, (a_i)\delta, \ldots, a_n]$ for all $a_j \in A$ then δ is a *derivation* of A.

The n-Lie algebra analogue of the adjoint representation are the right multiplications defined as follows:

Right Multiplications

$R_{(y)} = [__, y_2, \ldots, y_n]$ where (y) will always denote the set $y_2, \ldots, y_n$ of $n-1$ vectors right justified in the n-bracket. Example: $xR_{(y)} = [x, y_2, \ldots, y_n]$.

R(A)

R(A) is the vector space generated by all right multiplications of A.

Associative Algebra of A

A^* is generated by the associative, binary, matrix multiplication on $R(A)$

Representations

If ρ is a linear transformation such that $\rho : A \times A \times \ldots \times A = A^{n-1} \longrightarrow g\ell(k, \mathbb{F})$ and

$$\rho_a\rho_b - \rho_b\rho_a = [\rho_a, \rho_b] = \sum_{i=1}^{n-1}(a_1, \ldots, a_iR_b, \ldots, a_{n-1})\rho$$

$$\rho([a_1, \ldots, a_n], b_2, \ldots, b_n) = \sum_{i=1}^{n}(-1)^{i+1}(a_i, b_2, \ldots, b_n)\rho(a_1, \ldots, \hat{a_i}, \ldots, a_n)\rho$$

where $\rho_a = \rho(a_1, \ldots, a_n)$ and $\rho_b = \rho(b_1, \ldots, b_n)$ then ρ is a *representation* of A.

Homomorphism

If π is a linear transformation such that $[a_1, \ldots, a_n]\pi = [a_1\pi, \ldots, a_n\pi]$ for all $a_i \in A$, then π is a *homomorphism* of A. If in addition, π is non-singular, we say that π is an *isomorphism*. Furthermore, if π is an isomorphism that takes A to A, we say that π is an *automorphism* of A.

Simple

If A has no ideals other than A and 0, we say that A is *simple*.

Lower Central Series

If $A = A^1$, then $A^k = [A^{k-1}, A, \ldots, A]$.

Nilpotent

If $A^s = 0$ for some s we say A is *nilpotent*.

Derived Series

If $A = A^{(1)}$, then $A^{(k)} = [A^{(k-1)}, A^{(k-1)}, A, \ldots, A]$.

Solvable

If $A^{(s)} = 0$ for some s we say A is *solvable*.

The Center of A

Let $z \in A$ such that $[z, A \ldots, A] = 0$. We call the collection of all such elements in A *the center of* A and denote it $Z(A)$.

The Centeralizer

Let $C_A(I) = \{x \in A | [x, I, A, \ldots, A] = 0\}$ and call it *the centralizer of* I *in* A.

The Normalizer

Let $N(H) = \{x \in A | [x, H, A, \ldots, A] \subset H\}$ and call it *the normalizer of* H.

The Frattini Subalgebra

$\phi(A) = \bigcap M$ where M is a maximal proper subalgebra of A.

In 1986, Kasymov [KS1-KS2] introduced the concept of nilpotent n-Lie algebras, proved an analogue of Engel's Theorem and later proved an analogue of Jacobson's refinement of Engel's Theorem. They are as follows:

Engel's Theorem

If A is a finite dimensional n-Lie algebra in which all right multiplications $R_{(a)}$ are nilpotent, then A and A^* are nilpotent.

Jacobson's Refinement of Engel's Theorem

Let S be a spanning n-Lie set of A. If R is nilpotent for all $R \in R(S)$ then A is nilpotent. Where an n-Lie set is a set of vectors that are closed under the n-bracket but not necessarily closed under addition.

Despite these achievements, the subject of nilpotentcy in n-Lie algebras has not been examined in great detail. We shall explore the concept of nilpotent n-Lie algebras by examining, and proving where possible, other classical nilpotent group theory and nilpotent Lie algebra results, in the n-Lie algebra setting.

We shall do so in the following ways:

I) Find examples of nilpotent n-Lie algebras by truncating Fillipov's Jacobian n-Lie algebra.

II) Apply Kasymov's theorems to establish results similar to those of Wielandt, in group theory and Barnes and Chao in Lie algebras. The theorem is as follows:

The following are equivalent:

1) A is a nilpotent Lie algebra (nilpotent group).

2) All maximal subalgebras (subgroups) of A are ideals (normal subgroups).

3) For H a subalgebra (subgroup) of A, H is properly contained $N(H)$.

Where $N(H)$ is the normalizer of H.

4) $\phi(A) = A^2$ ($\phi(A) \supset A^2$ for groups) where $\phi(A)$ is the Frattini subalgebra (subgroup) of A.

III) Give examples of Lie algebra theorems that do not generalize to n-Lie algebras for $n > 3$ and give counterexamples.

IV) Prove n-Lie analogues of J. Thompson's and Jacobson's work in automorphisms and derivations that imply nilpotentcy. In 1959, Thompson [TJ] proved if A admits an automorphism T such that $T^p = I$ and T has no fixed points then A is nilpotent. Jacobson did this for Lie algebras. Jacobson [JN] proved in Lie algebras that if A admits an non-singular derivation over a field of characteristic 0 then A is nilpotent.

V) Prove analogues of Chao and P. Hall's work on groups and Lie algebras respectively. In 1957, P. Hall [HP] proved the following for p-groups. If A/N^2 and N are nilpotent then A is nilpotent. More precisely, if $cl(N) = t$ and $cl(A/N^2) = m$, then $cl(A) \leq \binom{t+1}{2}m - \binom{t}{2}$. In 1967, Chao [CC1] got this same result for Lie algebras.

VI) In 1962 Chao [CC2] proved that there are uncountably many non-isomorphic real Lie algebras of dimension 10 or greater. We will prove an analogue of this result.

VII) Finally, we look at some Frattini theory for n-Lie algebras that parallels Barnes' [BD3] work in Lie algebras.

Section I: Examples of Nilpotent N-Lie Algebras

For Lie algebras the way to generate nilpotent examples is to look at strictly upper triangular matrices. Strictly upper triangular matrices form a nilpotent Lie algebra under the commutator bracket. Unfortunately matrices do not provide examples of n-Lie algebras. We seek a method to generate examples that is to n-Lie algebras what strictly upper triangular matrices are for Lie algebras. To do this we examine the Jacobian n-Lie algebra, an example given by Fillipov.

Let $J = \mathbb{F}[x_1, \ldots, x_n]$. If $f_1, \ldots, f_n \in A$ then $[f_1, \ldots, f_n]$ is the determinant of the Jacobian matrix of the partial derivatives of $f_1, \ldots, f_n$. We noted before that this is an infinite dimensional n-Lie algebra. We seek a nilpotent version of this example. To do this we will truncate the dimension making it finite.

Let $deg(f)$ be the total degree of f for all $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$. Consider the monomials of degree 3 or more and let S be the subspace linearly generated by them. Now consider the monomials of degree r or more where $r \geq 3$ and let I_r be the subspace linearly generated by them. Clearly $I_r \subset S$ and we define $J_r = S/I_r$.

Theorem 1.1

J_r is a nilpotent n-Lie algebra. More precisely, $cl(J_r) = \left\lceil \dfrac{r-3}{2n-3} \right\rceil$, where $cl(J_r) = s$ if s is the smallest integer such that $J_r^{s+1} = 0$.

To prove the theorem we will first need to show that S is a subalgebra of J. Next, to show that J_r is an n-Lie algebra, we will need to show that $I_r \lhd S$. Finally, we

will need to show nilpotentcy and calculate the class of J_r. To do this we will prove the following lemma.

Lemma 1.2

If $f_j = \prod_{i=1}^{n} x_i^{p_{ij}} \in J$ are monomials for $j = 1, 2, \ldots, n$ then

$$[f_1, f_2, \ldots, f_n] = det([p_{ij}]) \prod_{i=1}^{n} x_i^{q_i}$$

where $q_i = p_{i1} + p_{i2} \ldots + p_{in} - 1$. Furthermore, $deg([f_1, \ldots, f_n]) = \left(\sum_{j=1}^{n} deg(f_j) \right) - n$.

Proof of Lemma 1.2

Note that

$$\frac{\partial f_j}{\partial x_i} = p_{ij} x_i^{-1} f_j$$

Let $f_j = \prod_{i=1}^{n} x_i^{p_{ij}}$ where $\sum_{i=1}^{n} p_{ij} = deg(f_j)$. Using the definition of the determinant we see that

$$\begin{aligned}
[f_1, f_2, \ldots, f_n] =& det\left(\left[\frac{\partial f_j}{\partial x_i} \right] \right) \\
&= \sum_{\sigma \in S_n} sgn(\sigma) \left(\frac{\partial f_{\sigma(1)}}{\partial x_1} \frac{\partial f_{\sigma(2)}}{\partial x_2} \cdots \frac{\partial f_{\sigma(n)}}{\partial x_n} \right) \\
&= \sum_{\sigma \in S_n} sgn(\sigma) \left(p_{1\sigma(1)} x_1^{-1} f_{\sigma(1)} p_{2\sigma(2)} x_2^{-1} f_{\sigma(2)} \cdots p_{n\sigma(n)} x_n^{-1} f_{\sigma(n)} \right) \\
&= \sum_{\sigma \in S_n} sgn(\sigma) \left(p_{1\sigma(1)} p_{2\sigma(2)} \cdots p_{n\sigma(n)} x_1^{-1} x_2^{-1} \cdots x_n^{-1} \prod_{i,j=1}^{n} x_i^{p_{i\sigma(j)}} \right) \\
&= \sum_{\sigma \in S_n} sgn(\sigma) \left(p_{1\sigma(1)} p_{2\sigma(2)} \cdots p_{n\sigma(n)} \prod_{i=1}^{n} x_i^{p_{i\sigma(1)} + p_{i\sigma(2)} \cdots + p_{i\sigma(n)} - 1} \right)
\end{aligned}$$

Since $x_i^{p_{i1}+p_{i2}\ldots+p_{in}-1}$ is invariant under S_n, that is to say $x_i^{p_{i1}+p_{i2}\ldots+p_{in}-1} = x_i^{p_{i\sigma(1)}+p_{i\sigma(2)}\cdots+p_{i\sigma(n)}-1}$, we observe,

$$\begin{aligned}
&\sum_{\sigma\in S_n} sgn(\sigma)\left(p_{1\sigma(1)}p_{2\sigma(2)}\cdots p_{n\sigma(n)}\prod_{i=1}^{n} x_i^{p_{i\sigma(1)}+p_{i\sigma(2)}\ldots+p_{i\sigma(n)}-1}\right)\\
=&\sum_{\sigma\in S_n} sgn(\sigma)\left(p_{1\sigma(1)}p_{2\sigma(2)}\cdots p_{n\sigma(n)}\prod_{i=1}^{n} x_i^{p_{i1}+p_{i2}\ldots+p_{in}-1}\right)\\
=&\prod_{i=1}^{n} x_i^{p_{i1}+p_{i2}\ldots+p_{in}-1}\sum_{\sigma\in S_n} sgn(\sigma)\left(p_{1\sigma(1)}p_{2\sigma(2)}\cdots p_{n\sigma(n)}\right)\\
=&det([p_{ij}])\prod_{i=1}^{n} x_i^{q_i}.
\end{aligned}$$

where $q_i = p_{i1} + \ldots + p_{in} - 1$.

This proves the first part of the lemma.

Also,

$$\begin{aligned}
deg([f_1,\ldots,f_n]) =& \sum_{i=1}^{n} q_i\\
=&\sum_{i=1}^{n}(p_{i1}+\ldots+p_{in}-1)\\
=&\left(\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}\right)-n\\
=&\left(\sum_{j=1}^{n}\sum_{i=1}^{n} p_{ij}\right)-n\\
=&\left(\sum_{j=1}^{n} deg(f_j)\right)-n.
\end{aligned}$$

This proves the second part of the lemma.

It is clear that the monomials $f_j = \prod_{i=1}^{n} x_i^{p_{ij}}$ form a basis for J. If $g_1,\ldots,g_n \in S$ and $g_i = \sum_{j=1}^{n} c_j f_{ij}$ where f_j is a monomial whose coefficient is 1, then

$$[g_1, \ldots, g_n] = [\sum_{j_1=1}^{n} c_{j_1} f_{1j_1}, \ldots, \sum_{j_1=1}^{n} c_{j_n} f_{nj_n}]$$
$$= \sum_{j_1=1}^{n} c_{j_1} \ldots \sum_{j_n=1}^{n} c_{j_n} [f_{1j_1}, \ldots, f_{nj_n}]$$

What is important to note is that we can apply second part of lemma 1.2 to any monomial term in the polynomial above and find its total degree.
For any fixed $j_1, \ldots, j_n$

$$deg([f_{1j_1}, \ldots, f_{nj_n}]) = \left(\sum_{r=1}^{n} deg(f_{rj_r})\right) - n \geq \left(\sum_{j=1}^{n} 3\right) - n = 2n \geq 2 \cdot 2 = 4.$$

This proves that $[g_1, \ldots, g_n] \in S$ and that S is a subalgebra of J. Similarly if $g_1, \ldots, \hat{g_i}, \ldots, g_n \in S$ and $g_i \in I_r$ then for any fixed $j_1, \ldots, j_n$

$$deg([f_1, \ldots, f_n]) \geq \left(\sum_{j \neq i} deg(f_j)\right) + deg(f_i) - n$$
$$= 3(n-1) + r - n = 2n + r - 3 \geq 2 \cdot 2 + r - 3 = r + 1.$$

This proves that $[g_1, \ldots, g_n] \in I_r$ and that $I_r \lhd S$. We see that $S/I_R = J_r$ is an n-Lie algebra. Now to show nilpotentcy we utilize the next lemma.

Lemma 1.3

If $f \in J_r^s$ then $deg(f) \geq 2(s-1)n - 3(s-2)$.

Proof Lemma 1.3

We induct on s. If $s = 1$ we get that if $f \in J_r$, then $deg(f) \geq 2(1-1) - 3(1-2) = 3$. Hence the base case holds. Next we assume the result for s and consider $s+1$. Let

$g_1 \in J_r^s$ and $g_2, \ldots, g_n \in J_r$. Using lemma 1.2 we obtain

$$\begin{aligned} deg([g_1, g_2, \ldots, g_n]) & \\ &\geq 2(s-1)n - 3(s-2) + (3(n-1) - n) \\ &= 2sn - 2n - 3s + 6 + 3n - 3 - n = 2sn - 3(s-1). \end{aligned}$$

This proves the lemma.

Proof of Theorem 1.1

By lemma 1.3, it is clear that $J_r^{s+1} \subset J_r^s$ and that $J_r^{k+1} = 0$ for some k and hence J_r is nilpotent of class k. To obtain the class of J_r we set $J_r^{s+1} = 0$ for s minimal. By the lemma 1.3, if $f \in J_r^{s+1}$ then $deg(f) \geq 2sn - 3s + 3 \geq r$ and $\dfrac{r-3}{2n-3} \leq s$. But $\dfrac{r-3}{2n-3}$ need not be an integer. We set $k = \left\lceil \dfrac{r-3}{2n-3} \right\rceil$ (where $\lceil \frac{r-3}{2n-3} \rceil$ denotes the smallest integer greater than $\frac{r-3}{2n-3}$). Then if $f \in J_r^{k+1}$ we obtain $deg(f) \geq k(2n-3) + 3 \geq (2n-3)\dfrac{r-3}{2n-3} + 3 = r$. Hence, $cl(J_r) = \left\lceil \dfrac{r-3}{2n-3} \right\rceil$.

The following is an example. We take J_5 with $n = 2$. A basis for J_5 is $\{xy^2, x^2y, x^3, y^3, x^3y, xy^3, x^2y^2, x^4, y^4\}$ and the multiplication table is

[,]	xy^2	x^2y	x^3	y^3	x^3y	xy^3	x^2y^2	x^4	y^4
xy^2	$0xy^3$	$-3x^2y^2$	$-5x^3y$	$3y^4$	$-5x^3y^2$	$1xy^4$	$-2x^2y^3$	$7x^4y$	$4y^5$
x^2y	$3x^2y^2$	$0x^3y^1$	$-1x^4$	$6x^14y^3$	$-1x^4y^1$	$5x^2y^3$	$2x^3y^2$	$-4x^5$	$8xy^4$
x^3	$5x^3y^1$	$1x^4$	$0x^5$	$9x^2y^2$	$3x^5$	$9x^3y^2$	$6x^4y$	$0x^6y$	$12x^2y^3$
y^3	$-3y^4$	$-6xy^3$	$-9x^2y^2$	$0y^5$	$-9x^2y^3$	$-3y^5$	$-6xy^4$	$-12x^2y^3$	$0y^6$
x^3y	$5x^3y^2$	$1x^4y$	$-3x^5$	$9x^2y^3$	$0x^5y$	$8x^3y^3$	$4x^4y^2$	$-4x^6$	$12x^2y^4$
xy^3	$-1xy^4$	$-5x^2y^3$	$-9x^3y^2$	$3y^5$	$-8x^3y^3$	$0x^{y5}$	$-4x^2y^4$	$-12x^4y^2$	$4y^6$
x^2y^2	$2x^2y^3$	$-2x^3y^2$	$-6x^4y$	$6x^{y4}$	$-4x^4y^2$	$4x^2y^4$	$0x^3y^3$	$-8x^5y$	$8x^{y5}$
x^4	$7x^4y$	$4x^5$	$0x^6$	$12x^2y^3$	$4x^6$	$12x^4y^2$	$8x^5y$	$0x^7$	$16x^3y$
y^4	$-4y^5$	$-8xy^4$	$-12x^2y^3$	$0y^6$	$-12x^2y^4$	$-4y^6$	$-8x^{y5}$	$-16x^3y^3$	$0y^7$

We note that everything in the multiplication is of degree 4 or more. Every entry other than the upper left hand 4×4 is in I_5 and is hence 0.

We obtain

[,]	xy^2	x^2y	x^3	y^3	x^3y	xy^3	x^2y^2	x^4	y^4
xy^2	0	$-3x^2y^2$	$-5x^3y$	$3y^4$	0	0	0	0	0
x^2y	$3x^2y^2$	0	$-1x^4$	$6x^14y^3$	0	0	0	0	0
x^3	$5x^3y^1$	$1x^4$	0	$9x^2y^2$	0	0	0	0	0
y^3	$-3y^4$	$-6xy^3$	$-9x^2y^2$	0	0	0	0	0	0
x^3y	0	0	0	0	0	0	0	0	0
xy^3	0	0	0	0	0	0	0	0	0
x^2y^2	0	0	0	0	0	0	0	0	0
x^4	0	0	0	0	0	0	0	0	0
y^4	0	0	0	0	0	0	0	0	0

We see that J_5^2 is spanned by $\{x^3y, xy^3, x^2y^2, x^4, y^4\}$ so if $f \in J_5^2$ then $deg(f) = 4$. This fits our formula as $deg(f) \geq (2n-3)s+3 = (2 \cdot 2-3)1+3 = 4$. Also we see that $cl(J_5) = 2$. This also fits with our formula as $\left\lceil \frac{r-3}{2n-3} \right\rceil = \left\lceil \frac{5-3}{2 \cdot 2-3} \right\rceil = \lceil 2 \rceil = 2$.

Section II: Conditions Equivalent To Nilpotentcy

Kasymov's [KS1] n-Lie algebra version of Engel's Theorem is as follows:

Theorem 2.1

Let R be the right multiplications on an n-Lie algebra A in a finite dimensional vector space V. Suppose that the operators $R(a_2, \ldots, a_n)$ are nilpotent for all $a_2, \ldots, a_n \in A$, then $R(A)$ is nilpotent.

The following is an n-Lie algebra analogue of famous theorems of Lie algebra and group theory. These results were done by Wielandt (1936-39) in group theory and by Barnes and Chao for Lie algebras in the 1960's. It will follow from Kasymov's work discussed above and Fitting's Lemma.

Theorem 2.2

Let A be a finite dimensional n-Lie Algebra where the dimension of A is k.

The following are equivalent:

1) A is nilpotent.

2) All maximal subalgebras of A are ideals.

3) For H a subalgebra of A, H is properly contained $N(H)$.

Where $N(H)$ is the set such that $[N(H), H, A, \ldots, A] \subset H$

4) $\phi(A) = A^2$.

Proof Theorem 2.2

We will conduct the proof in the following order:

A) $1 \Longrightarrow 3 \Longrightarrow 2 \Longrightarrow 1$

B) $3 \Longleftrightarrow 4$.

For A and most of B we closely follow the group theory proofs in W.R. Scott's group theory book [SW]. To show $\phi(A) = A^2$ we will follow closely Scott [SW] and Chao [CC3].

Proof of $1 \Longrightarrow 3$

We assume A is nilpotent and H is a proper subalgebra of A. By Engel's Theorem for n-Lie Algebras [KS1], A^*, the associative algebra generated by $R(A)$, is nilpotent. If we let $R_1R_2R_3 \ldots R_{t-1}R_t = 0$ for t minimal for all $R_i \in A^*$, then there exist R_i's $\in A^*$ and $v \neq 0 \in A$ such that $vR_1R_2R_3 \ldots R_{t-1} \neq 0$. Let $X = vR_1R_2R_3 \ldots R_{t-1}$. Observe X is a common eigen-vector for all $R \in A^*$ indeed $XR = vR_1R_2R_3 \ldots R_{t-1}R = 0$.

We observe that R acts nilpotently on A/H for all $R \in R(H)$. By the comments above there exists $v \neq 0 \in A/H$ such that $vR = 0$ for all $R \in R(H)$. This means $vR \in H$ and hence $v \in N(H)$ but since $v \neq 0 \in A/H$ we have that v is not in H hence $H \subset N(H)$.

Proof of $3 \Longrightarrow 2$

We observe that if every proper subalgebra H is properly contained in $N(H)$ then any maximal subalgebra M is properly contained in $N(M)$. But since M is a subalgebra, $N(M)$ is also a subalgebra properly containing M. The only way this can happen is if $N(M) = A$ (otherwise we contradict the maximality of M). Since $N(M) = A$, we see that M is an ideal.

Proof of 2 $\Longrightarrow$ 1

Assume A is not nilpotent and all maximal subalgebras are ideals. By Theorem 2.1 there exists $R \in R(A)$ such that $R : A \longrightarrow A$ is a non-nilpotent derivation on A defined $xR \doteqdot [x, a_2, \ldots, a_n]$ where $a_i \in A$.

We observe that the $a_2, \ldots, a_n \in kerR$ as $a_iR = [a_i, a_2, \ldots, a_i, \ldots, a_n] = 0$ for all such a_i where $2 \leq i \leq n$. Now by Fitting's Lemma we obtain

$$K_1 \subset K_2 \subset \ldots \subset K_s = K_{s+1} \doteqdot K.$$

$$I_1 \supset I_2 \supset \ldots \supset I_s = I_{s+1} \doteqdot I.$$

Where $K_i = kerR^i$ and $I_i = ImR^i$ and $A = I \oplus K$. Recall R acts non-singularly on I. Since R is not nilpotent, $K \neq A$.

Now let M be a proper maximal subalgebra of A such that $K \subset M$. By assumption of 2), $M \lhd A$. Since $xR = [x, a_2, \ldots, a_n]$ we see that $a_iR = [a_i, a_2, \ldots, a_n] = 0$ for $1 \leq i \leq n$ and consequently, $a_i \in K_1 \subset K \subset M$. We observe that $R : A \longrightarrow M$ and hence $R : I \longrightarrow I \cap M$. Since R acts non-singularly on I, this implies that $I = I \cap M$. But

$$I+K = A \neq M = A \cap M = (I+K) \cap M = (I \cap M)+(K \cap M) = (I \cap M)+K = I+K.$$

This is obviously a contradiction. Hence, there exist no non-nilpotent R and A is nilpotent by Theorem 2.1.

Proof of 4 $\Longrightarrow$ 3

We need only assume $A^2 \subset \phi(A)$ and M is any maximal subalgebra of A. Then $A^2 \subset M$ by definition of $\phi(A)$. We observe $A^2 = [A, A, \ldots, A] \subset M$ which implies $[M, A, \ldots, A] \subset M$. Hence $M \lhd A$.

To show 3 $\Longrightarrow$ 4 we first show that $dim(M) = k - 1$ for all maximal subalgebras M. Certainly $dim(M) \leq k - 1$ so it is enough to show that $dim(M) \not< k - 1$.

Assume $dim(M) < k-1$. By 3) we get that $M \subset N(M)$ with proper containment and since $3 \implies 2$ we see that $M \lhd A$. Consequently A/M is defined and there exists $0 \neq x \in A/M$ such that $xR(M) \in M$. This means that $x + M$ is a subalgebra whose dimension is greater than that of M contradicting the maximality of M. Hence $dim(M) = k-1$.

Since $M \lhd A$, we note A/M is a subalgebra and it follows from the statements above that $dim(A/M) = 1$. Furthermore A/M is Abelian so $A^2 \subset M$ for all such M and consequently $A^2 \subset \phi(A)$ by definition.

To show $A^2 \supset \phi(A)$ we observe that because A is nilpotent, there exists a nonzero compliment to A^2 in A. Let $u_1, \ldots, u_k$ be a basis for the compliment and let $U_i = \bigoplus_{\ell \neq i} < u_\ell > \oplus A^2$. Obviously the U_i's are maximal subalgebras and $\phi(A) \subset \bigcap_{i=1}^{k} U_i = A^2$.

Section III: Theorems That Fail To Generalize To N-Lie Algebras

The following are in contrast to theorems of Lie algebras that are a consequence of Jacobson's refinement to Engel's theorem. Interestingly, they do not hold for n-Lie algebras with $n > 2$.

Theorem 3.1

Let A be a finite dimensional n-Lie algebra. Then the following do not hold:

1) If A admits an automorphism T of prime order with no fixed points, then A is nilpotent.

2) If A is over a field of characteristic 0 and admits a non-singular derivation D, then A is nilpotent.

Proof of Theorem 3.1

We give counter examples. In both cases we examine an n-Lie algebra A with dimension n and multiplication $[x_1, x_2, \ldots, x_n] = x_2$. Note that if $R_x = [_, x_2, x_3, \ldots, x_n]$, then $AR_x^m = x_2 \neq 0$ for all m. Hence A is not nilpotent. We will show that this algebra admits both an automorphism and a derivation as described above.

Indeed we observe A admits the following automorphism. Let $x_1T = \nu^{p-n+2}x_1$ and $x_iT = \nu x_i$ for $i > 1$ where $\nu = e^{2\pi i/p}$, where p is a prime such that $p > n - 1$. T

is obviously a non-singular linear transformation and is an automorphism as

$$\begin{aligned}
[x_1, x_2, \ldots, x_n]T &= [x_1T, x_2T, \ldots, x_nT] \\
&= [\nu^{p-n+2}x_1, \nu x_2, \ldots, \nu x_n] \\
&= \nu^{p-n+2}\nu^{n-1}[x_1, x_2, \ldots, x_n] \\
&= \nu^{p+1}[x_1, x_2, \ldots, x_n] \\
&= \nu x_2 = x_2T.
\end{aligned}$$

If n is odd A admits the following derivation. Let $x_1D = x_1$, $x_iD = (-1)^i x_i$ for all $i \geq 2$. D is obviously non-singular and derivation as

$$\begin{aligned}
[x_1, x_2, \ldots, x_n]D &= \sum_{i=1}^{n}[x_1, x_2, \ldots, x_iD, \ldots, x_n] \\
&= [x_1D, x_2, \ldots, x_n] + \sum_{i=2}^{n}[x_1, x_2, \ldots, (-1)^i x_i, \ldots, x_n] \\
&= [x_1, x_2, \ldots, x_n] + \sum_{i=2}^{n}(-1)^i[x_1, x_2, \ldots, x_i, \ldots, x_n] \\
&= x_2 + \sum_{i=2}^{n}(-1)^i x_2 \\
&= x_2 + 0 \\
&= x_2 = (-1)^2 x_2 = x_2D
\end{aligned}$$

If n is even, then $n \geq 4$ and A admits the following derivation. Let $x_1D = x_1$, $x_2D = x_2$,$x_3D = -1/2x_3$, $x_iD = (-1)^{i+1}/2x_i$ for all $i \geq 4$. D is obviously non-singular and a derivation as

$$\begin{aligned}
[x_1, x_2, \ldots, x_n]D =& \sum_{i=1}^{n} [x_1, x_2, \ldots, x_iD, \ldots, x_n] \\
=& [x_1, x_2, \ldots, x_n] + [x_1, x_2, \ldots, x_n] \\
&+ [x_1, x_2, -1/2x_3, \ldots, x_n] + \sum_{i=4}^{n} [x_1, x_2, \ldots, (-1)^{i+1}/2x_i, \ldots, x_n] \\
=& 2[x_1, x_2, \ldots, x_n] - 1/2[x_1, x_2, x_3, \ldots, x_n] \\
&+ \sum_{i=4}^{n} (-1)^{i+1}/2[x_1, x_2, \ldots, x_n] \\
=& 3/2x_2 + \sum_{i=4}^{n} (-1)^{i+1}/2x_2 \\
=& 3/2x_2 + -1/2x_2 \\
=& x_2 = x_2D.
\end{aligned}$$

Theorem 3.2

If $A = M_1 + M_2$, $M_1 \lhd A$, $M_2 \lhd A$ and both M_1 and M_2 are nilpotent, then A is not necessarily nilpotent if $n > 2$.

Proof of Theorem 3.2

Let $\{x_1, \ldots, x_n\}$ span A and the only non-zero bracket is $[x_1, \ldots, x_n] = x_n$. Also let $M_1 = \{\hat{x_1}, x_2 \ldots, x_n\}$ and let $M_2 = \{x_1, \hat{x_2}, \ldots, x_n\}$. Clearly $A = M_1 + M_2$, $M_1 \lhd A$, $M_2 \lhd A$ and both M_1 and M_2 are Abelian and hence nilpotent. But A is clearly not nilpotent.

Section IV: Some Automorphism and Derivation Theorems

We begin by proving some theorems about automorphisms and derivations of n-Lie algebras.

Theorem 4.1

Let D be a derivation of A, let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the corresponding characteristic values of D and let $A_{\lambda_1}, A_{\lambda_2}, \ldots, A_{\lambda_r}$ be the characteristic spaces of A with respect to D. Then the n-bracket preserves the characteristic spaces, i.e.

$$[A_{\lambda_{i_1}}, A_{\lambda_{i_2}}, \ldots, A_{\lambda_{i_n}}] = A_{\lambda_{i_1}+\lambda_{i_2}+,\ldots,+\lambda_{i_n}}$$

Here if $A_{\lambda_{i_1}+\lambda_{i_2}+,\ldots,+\lambda_{i_n}}$ is not a a characteristic space then we say $A_{\lambda_{i_1}+\lambda_{i_2}+,\ldots,+\lambda_{i_n}} = 0$.

Proof of Theorem 4.1

$$\begin{aligned}(**)[x_1, x_2, \ldots, x_n]\big(D - \sum_{i=1}^{n} \lambda_i\big) &= [x_1, x_2, \ldots, x_i, \ldots, x_n]D - \sum_{i=1}^{n} \lambda_i [x_1, x_2, \ldots, x_n] \\ &= \sum_{i=1}^{n} [x_1, x_2, \ldots, x_i D, \ldots, x_n] \\ &\quad - \sum_{i=1}^{n} [x_1, x_2, \ldots, \lambda_i x_i, \ldots, x_n] \\ &= \sum_{i=1}^{n} [x_1, x_2, \ldots, x_i(D - \lambda_i), \ldots, x_n].\end{aligned}$$

We claim that

$$[x_1, x_2, \ldots, x_n]\left(D - \sum_{i=1}^{n} \lambda_i\right)^k$$
$$= \sum_{p_1+\ldots+p_n=k} [x_1(D-\lambda_1)^{p_1}, x_2(D-\lambda_2)^{p_2}, \ldots, x_n(D-\lambda_n)^{p_n}].$$

We induct on k. For $k = 1$ we obtain $(**)$. Now we assume the result is true for k and show for $k+1$. We observe

$$[x_1, x_2, \ldots, x_n]\left(D - \sum_{i=1}^{n} \lambda_i\right)^{k+1}$$
$$=[x_1, x_2, \ldots, x_n]\left(D - \sum_{i=1}^{n} \lambda_i\right)^{k}\left(D - \sum_{i=1}^{n} \lambda_i\right)$$
$$= \sum_{p_1+\ldots+p_n=k} [x_1(D-\lambda_1)^{p_1}, x_2(D-\lambda_2)^{p_2}, \ldots, x_n(D-\lambda_n)^{p_n}]\left(D - \sum_{i=1}^{n} \lambda_i\right)$$
$$= \sum_{p_1+\ldots+p_n=k} \sum_{i=1}^{n} [x_1(D-\lambda_1)^{p_1}, x_2(D-\lambda_2)^{p_2}, \ldots, x_i(D-\lambda_i)^{p_i+1}, \ldots, x_n(D-\lambda_n)^{p_n}]$$
$$= \sum_{p_1+\ldots+p_n=k+1} [x_1(D-\lambda_1)^{p_1}, x_2(D-\lambda_2)^{p_2}, \ldots, x_n(D-\lambda_n)^{p_n}].$$

Now let k_i for $1 \leq i \leq n$, be minimal such that $(D-\lambda_i)^{k_i} v = 0$ for all $v \in A_{\lambda_i}$ and let $t = (\sum_{i=1}^{n}(k_i - 1)) + 1$. Then if $x_i \in A_{\lambda_i}$ for $1 \leq i \leq n$, we obtain

$$[x_1, x_2, \ldots, x_n]\left(D - \sum_{i=1}^{n} \lambda_i\right)^{t}$$
$$= \sum_{p_1+\ldots+p_n=t} [x_1(D-\lambda_1)^{p_1}, x_2(D-\lambda_2)^{p_2}, \ldots, x_n(D-\lambda_n)^{p_n}]$$

We claim this product is 0. If we assume to the contrary then

$$0 \neq [x_1, x_2, \ldots, x_n]\left(D - \sum_{i=1}^{n} \lambda_i\right)^{t}$$
$$= \sum_{p_1+\ldots+p_n=t} [x_1(D-\lambda_1)^{p_1}, x_2(D-\lambda_2)^{p_2}, \ldots, x_n(D-\lambda_n)^{p_n}]$$

But this implies that there exist $p_1, p_2, \ldots, p_n$ such that

$$[x_1(D-\lambda_1)^{p_1}, x_2(D-\lambda_2)^{p_2}, \ldots, x_n(D-\lambda_n)^{p_n}] \neq 0.$$

This means that $p_j \leq k_i$ for all i and hence $t = \sum_{i=1}^{n} p_i \leq \sum_{i=1}^{n}(k_i - 1) = t - 1$ But this is a contradiction and thus proves the claim. Now since $[x_1, x_2, \ldots, x_n]\left(D - \sum_{i=1}^{n} \lambda_i\right)^t = 0$ we see that $[x_1, x_2, \ldots, x_n] \in A_{\lambda_1+\ldots+\lambda_n}$. This proves the lemma.

Theorem 4.2

If T is an n-Lie algebra automorphism and A_{λ_i} are the characteristic spaces of A with respect to T, then the n-bracket preserves the characteristic spaces. That is $[A_{\lambda_{i_1}}, A_{\lambda_{i_2}}, \ldots, A_{\lambda_{i_n}}] = A_{\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_n}}$.

Proof of Theorem 4.2

First we claim that

$$\begin{aligned}
&[x_1, x_2, \ldots, x_n](T - \lambda_1\lambda_2\ldots\lambda_n) \\
=&\sum_{i=1}^{n}[\lambda_1 x_1, \lambda_2 x_2, \ldots, x_i(T-\lambda_i), x_{i+1}T, x_{i+2}T, \ldots, x_nT].
\end{aligned}$$

Observe

$$\begin{aligned}
&\sum_{i=1}^{n}[\lambda_1 x_1, \lambda_2 x_2, \ldots, x_i(T-\lambda_i), x_{i+1}T, x_{i+2}T, \ldots, x_nT] \\
=&\sum_{i=1}^{n}[\lambda_1 x_1, \lambda_2 x_2, \ldots, x_iT, x_{i+1}T, \ldots, x_nT] \\
-&\sum_{i=1}^{n}[\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_i x_i, x_{i+1}T, x_{i+2}T, \ldots, x_nT] \\
=&\sum_{i=1}^{n}[\lambda_1 x_1, \lambda_2 x_2, \ldots, x_iT, x_{i+1}T, \ldots, x_nT] \\
-&\left(\sum_{i=1}^{n-1}[\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_i x_i, x_{i+1}T, x_{i+2}T, \ldots, x_nT] + [\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n]\right)
\end{aligned}$$

And we see

$$
\begin{aligned}
&= \sum_{i=1}^{n} [\lambda_1 x_1, \lambda_2 x_2, \ldots, x_i T, x_{i+1} T, \ldots, x_n T] \\
&- \left(\sum_{k=2}^{n} [\lambda_1 x_1, \lambda_2 x_2, \ldots, x_k T, x_{k+1} T, \ldots, x_n T] - [\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n] \right) \\
&= [x_1 T, x_2 T, \ldots, x_n T] \\
&+ \left(\sum_{i=2}^{n} [\lambda_1 x_1, \lambda_2 x_2, \ldots, x_i T, \ldots, x_n T] - \sum_{k=2}^{n} [\lambda_1 x_1, \lambda_2 x_2, \ldots, x_k T, \ldots, x_n T] \right) \\
&- [\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n] \\
&= [x_1 T, x_2 T, \ldots, x_n T] - [\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n] \\
&= [x_1, x_2, \ldots, x_n](T - \lambda_1 \lambda_2 \ldots \lambda_n).
\end{aligned}
$$

We claim that for $x_i \in A_{\lambda_i}$

$$
[x_1, x_2, \ldots, x_n](T - \lambda_1 \lambda_2 \ldots \lambda_n)^k = \sum_{p_1 + \ldots + p_n = k} [y_1, y_2, \ldots, y_n]
$$

where $y_i = \lambda_i^{(k - p_1 - \ldots - p_i)} x_i T^{(p_1 + \ldots + p_{i-1})} (T - \lambda_i)^{p_i}$ and $p_i \in \mathbb{Z}^+$ for all i. We induct on k. If $k = 1$, then

$$
\begin{aligned}
&[x_1, x_2, \ldots, x_n](T - \lambda_1 \lambda_2 \ldots \lambda_n) \\
&= \sum_{i=1}^{n} [\lambda_1 x_1, \lambda_2 x_2, \ldots, (T - \lambda_i) x_i, T x_{i+1}, \ldots, T x_n] \\
&= \sum_{p_1 + \ldots + p_n = 1} [\lambda_1^{1 - p_1} x_1 T^{p_1} (T - \lambda_1)^{p_1}, \ldots, \lambda_i^{1-1} x_n T^1 (T - \lambda_i)^0] \\
&= \sum_{p_1 + \ldots + p_n = 1} [y_1, y_2, \ldots, y_n].
\end{aligned}
$$

Now we assume the result is true for k and show it is true for $k+1$.

$$\begin{aligned}
&[x_1, x_2, \ldots, x_n](T - \lambda_1\lambda_2 \ldots \lambda_n)^{k+1} \\
&= [x_1, x_2, \ldots, x_n](T - \lambda_1\lambda_2 \ldots \lambda_n)(T - \lambda_1\lambda_2 \ldots \lambda_n)^k \\
&= \sum_{p_1+\ldots+p_n=k} [y_1, y_2, \ldots, y_n](T - \lambda_1\lambda_2 \ldots \lambda_n) \\
&= \sum_{p_1+\ldots+p_n=k} \sum_{i=1}^{n} [\lambda_1 y_1, \ldots, y_i(T - \lambda_i), \ldots, y_n T] \\
&= \sum_{p_1+\ldots+p_n=k+1} [y_1, y_2 \ldots, y_n].
\end{aligned}$$

Let k_i for $1 \leq i \leq n$, be minimal such that $A_{\lambda_i}(T - \lambda_i)^{k_i} = 0$ and let $t = (\sum_{i=1}^{n} k_i - 1) + 1$. We claim $[x_1, x_2, \ldots, x_n](T - \lambda_1\lambda_2 \ldots \lambda_n)^t = 0$. Assume to the contrary that it is non-zero. Then

$$[x_1, x_2, \ldots, x_n](T - \lambda_1\lambda_2 \ldots \lambda_n)^t = \sum_{p_1+\ldots+p_n=k+1} [y_1, y_2 \ldots, y_n] \neq 0.$$

This implies that there exist $p_1, \ldots, p_n$ such that the $[y_1, y_2, \ldots, y_n] \neq 0$. Hence the y_i's all are non-zero. This means $p_i \leq k_i - 1$ for all i. We obtain $t = \sum_{i=1}^{n} p_i \leq \sum_{i=1}^{n}(k_i - 1) = t - 1$ which is impossible hence $[x_1, x_2, \ldots, x_n](T - \lambda_1\lambda_2 \ldots \lambda_n)^t = 0$. As a result $[A_{\lambda_1} A_{\lambda_2} \ldots A_{\lambda_n}] \in A_{\lambda_1\lambda_2\ldots\lambda_n}$ proving the lemma.

Theorem 4.3

Let D be a nilpotent derivation of A, an n-Lie algebra over a field $\mathbb{F}$. Then $expD = \sum_{i=0}^{m} \frac{D^i}{i!}$ is an automorphism of A under the following field considerations: either $char(\mathbb{F}) = 0$ or, if $char(\mathbb{F}) = p \neq 0$ and $D^k = 0$ for some minimal k, then $k < p$.

Proof of Theorem 4.3

By Leibnitz's rule for n-Lie algebras we see that

$$(*)\left[x_1, x_2, \ldots, x_n\right]\frac{D^k}{k!} = \frac{1}{k!}\sum_{i_1+\ldots+i_n=k}\binom{k}{i_1, i_2, \ldots, i_n}[x_1D^{i_1}, x_2D^{i_2}, \ldots, x_nD^{i_n}]$$
$$= \sum_{i_1+\ldots+i_n=k}\left[\frac{x_1D^{i_1}}{i_1!}, \frac{x_2D^{i_2}}{i_2!}, \ldots, \frac{x_nD^{i_n}}{i_n!}\right].$$

Hence

$$[expD(x_1), expD(x_2), \ldots, expD(x_n)] = \left[\sum_{i_1=1}^{m}\frac{x_1D^{i_1}}{i_1!}, \sum_{i_2=1}^{m}\frac{x_2D^{i_2}}{i_2!}, \ldots, \sum_{i_n=1}^{m}\frac{x_nD^{i_n}}{i_n!}\right]$$

$$= \sum_{i_1,\ldots,i_n=0}^{m}\left[\frac{x_1D^{i_1}}{i_1!}, \frac{x_2D^{i_2}}{i_2!}, \ldots, \frac{x_nD^{i_n}}{i_n!}\right]$$
$$= \sum_{k=1}^{nm}\sum_{i_1+\ldots+i_n=k}\left[\frac{x_1D^{i_1}}{i_1!}, \frac{x_2D^{i_2}}{i_2!}, \ldots, \frac{x_nD^{i_n}}{i_n!}\right]$$
$$= \sum_{k=1}^{nm}\frac{[x_1, x_2, \ldots, x_n]D^k}{k!}$$
$$= \sum_{k=1} m\frac{[x_1, x_2, \ldots, x_n]D^k}{k!}$$
$$= expD([x_1, x_2, \ldots, x_n])$$

Hence $exp(D)$ is an n-Lie algebra homomorphism. Furthermore

$$\begin{aligned}(exp(D))(exp(-D)) =&(exp(D))\left(\sum_{i=0}^{m}\frac{(-D)^i}{i!}\right)\\ =&\sum_{i,j=0}^{m}\frac{D^j(-D)^i}{j!i!}\\ =&\sum_{k=0}^{2m}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}D^i(-D)^{k-i}\\ =&\sum_{k=0}^{2m}\frac{1}{k!}(D-D)^k\\ =&I\end{aligned}$$

Similarly, $(exp(-D))(exp(D) = I$. Hence $exp(-D) = (exp(D))^{-1}$ and $exp(D)$ is an automorphism. This proves lemma 4.3.

We now look at some applications of theorems 4.1 and 4.2. After we introduce the next theorem and corollary we will be able to find conditions for which the statements of theorem 3.1 will hold. As we have mentioned, Kasymov proved a generalization of an analogue of Jacobson's refinement of Engel's Theorem [KS2].

Theorem 4.4

Let A be a finite dimensional n-Lie algebra over a field $\mathbb{F}$, S and T be subsets such that $[S, T, T, \ldots, T] \subset T$ and $[S, S, T, T \ldots, T] \subset S$ and the operators $R(x, t_2, t_3, \ldots, t_n)$ be nilpotent for arbitrary $x \in S$ and $t_i's \in T$. Then the associative algebra S^* is nilpotent.

From this we obtain the extremely useful n-Lie algebra analogue of Jacobson's refinement of Engel's theorem.

Corollary 4.5

Let S be a spanning n-Lie set of A. If R is nilpotent for all $R \in R(S)$ then A is nilpotent, where an n-Lie set is a set closed under the n-bracket but not necessarily

under addition.

Proof of Corollary 4.5

Let $S = T$ as in theorem 4.4 and span A, an n-Lie algebra. Then $[S, S, \ldots, S] \subset S$ and is hence an n-Lie set. By the theorem, S^* is nilpotent and as a result, A is nilpotent.

Now we introduce conditions that make the statements of theorem 3.1 hold.

Theorem 4.6

Let A be a finite dimensional n-Lie algebra. The following hold:

1) If A admits an automorphism T of prime order with characteristic spaces $A_{\lambda_1}, A_{\lambda_2}, \ldots, A_{\lambda_r}$ and the product of any $n-1$ of the $\lambda_i's$ is not equal to 1 then A is nilpotent.

2) If A is over a field of characteristic 0, admits a non-singular derivation D with characteristic spaces $A_{\lambda_1}, A_{\lambda_2}, \ldots, A_{\lambda_r}$, and the sum of any $n-1$ of the $\lambda_i's$ is non-zero, then A is nilpotent.

Proof of Theorem 4.6

For 1 let $R_\lambda = [_, x_2, x_3, \ldots, x_n]$ for any $x_i \in A_{\lambda_i}$ and any $y \in A_{\lambda_1}$. By theorem 4.2, $[A_\alpha, A_\beta, \ldots, A_\omega] = A_{\alpha\beta\ldots\omega}$. As a result, $yR_\lambda^k \in A_{\lambda_1(\lambda_2\ldots\lambda_n)^k}$. Let $\lambda_2 \ldots \lambda_n = \varepsilon$ and $\lambda_1 = \eta$. Note that $\varepsilon \neq 1$ because the product of any $n-1$ of the $\lambda_i's$ is not equal to 1. Since $\varepsilon, \eta \in \mathbb{F}_p$ there exists i such that $\varepsilon^i \eta = 1$ but this means

$$yR_\lambda^i = A_{\varepsilon^i \eta} = A_1 = 0.$$

This is true for any choice $y's$ and $x_i's$. By theorem 4.2 the union of characteristic spaces of an automorphism form an n-Lie set. Applying corollary 4.5 we see that A is nilpotent.

For 2 let $R_\lambda = [_, x_2, x_3, \ldots, x_n]$ for any $x_i \in A_{\lambda_i}$ and any $y \in A_{\lambda_1}$. By theorem 4.1, $[A_\alpha, A_\beta, \ldots, A_\omega] = A_{\alpha+\beta+\ldots+\omega}$. As a result, $yR_\lambda^k \in A_{\lambda_1+k(\lambda_2+\ldots+\lambda_n)}$. Since each sum of $n-1$ $\lambda_i's$ is non-zero, our field is characteristic 0 and D is non-singular, we see $A_{\lambda_1+k(\lambda_2+\ldots+\lambda_n)} \neq A_{\lambda_1+(k+i)(\lambda_2+\ldots+\lambda_n)}$ for any choice of k and $i > 0$. Furthermore, since A is finite dimensional, it has only finitely many characteristic spaces and there exists an m such that $yR_\lambda^m = 0$ i.e. R_λ is nilpotent. By theorem 4.1 the union of characteristic spaces of a derivation form an n-Lie set. Applying corollary 4.5 we see that A is nilpotent.

Section V: Theorems Of Hall and Chao

We now turn to work done by Hall and later Chao. In 1958 Hall proved the following in group theory. Let $M \lhd G$ be a nilpotent group and M^2 its derived group. If G/M^2 is a nilpotent group, then G is itself nilpotent. Furthermore, if $cl(M) = t$ and $cl(G/M^2) = m$, then $cl(G) \leq \binom{t+1}{2}m - \binom{t}{2}$.

Then in 1968, Chao proved the Lie algebra analogue of this theorem. Let L be a Lie algebra and $N \lhd A$. If N and L/N^2 are nilpotent Lie algebras then L is nilpotent. Furthermore Chao proved that if $cl(L/N^2) = m$ and $cl(N) = t$ then $cl(L) \leq \binom{t+1}{2}m - \binom{t}{2}$ [C1]. We derive an n-Lie algebra analogue of these results.

Theorem 5.1

Let A be an n-Lie algebra with $N \lhd A$. If $N^{t+1} = 0$ and $(A/N^2)^{m+1} = 0$, then $cl(A) \leq tm + \frac{1}{2}t(t-1)(m-1)(n-1)$. Where $cl(A)$ is the smallest natural number such that $A^{cl(A)+1} = 0$.

Before we start the proof we make some definitions and prove a lemma. Let $N^iA = [N^i, A, \ldots, A]$ and $N^iA^j = [N^iA^j, A, \ldots, A]$ for $j > 1$.

Lemma 5.2

Let A/N^2 be nilpotent. If $A^{m+1} \subset N^2$ for some m, minimal then $N^rA^u \subset N^{r+1}$ for $r > 0$ where $u = (r-1)(n-1)(m-1) + m$.

Proof of Lemma 5.2

We induct on r. If $r = 1$ we see that $N^1A^{(1-1)(n-1)(m-1)+m} = NA^m = \subset A^{m+1} \subset N^2$

and the base case is satisfied. We now assume the result holds for r and consider $r+1$.

Let $s = r(n-1)(m-1)+m$ and $u = (r-1)(n-1)(m-1)+m$. By the Leibnitz rule for n-Lie algebras we obtain

$$\begin{aligned} N^{r+1}A^s =& [N^r, N, \ldots, N]A^s \\ &= \sum_{s_1+\ldots+s_n=s} [N^r A^{s_1}, NA^{s_2}, NA^{s_3}, \ldots, NA^{s_n}] \end{aligned}$$

Suppose $s_1 \geq u$. Then by the induction hypothesis, $N^r A^{s_1} \subset N^{r+1}$ and

$$\begin{aligned} \sum_{s_1+\ldots+s_n=s} [N^r A^{s_1}, NA^{s_2}, NA^{s_3}, \ldots, NA^{s_n}] \subset & \sum_{s_1+\ldots+s_n=s} [N^{r+1}, N, N, \ldots, N] \\ & \subset N^{r+2}. \end{aligned}$$

Suppose $s_1 < u$. We claim there exists $s_k \geq m$. Assume to the contrary that $s_j < m$ for all j.

We obtain

$$\begin{aligned} s &= (s_1) + (s_2 + \ldots + s_n) \\ &< u + (n-1)(m-1) \\ &= (r-1)(n-1)(m-1) + m + (n-1)(m-1) \\ &= r(n-1)(m-1) + m = s. \end{aligned}$$

But this is impossible. Hence there exists $s_k \geq m$ for some k.

As a result $NA^{s_k} \subset N^2$ and using the Jacobi identity and skew symmetry we obtain

$$
\begin{aligned}
&[N^r A^{s_1}, NA^{s_2}, NA^{s_3}, \ldots, NA^{s_k}, \ldots, NA^{s_n+1}] \\
&= [N^r, N, N, \ldots, N^2, \ldots, N] \\
&= [N^2, N^r, N, \ldots, N] \\
&= [[N, \ldots, N], N^r, N, \ldots, N] \\
&= [[N, N^r \ldots, N]N, \ldots, N] + [N, [N, N^r \ldots, N]N, \ldots, N] \\
&+ \ldots + [N, \ldots, N, [N, N^r \ldots, N]] \\
&= [[N, N^r \ldots, N]N, \ldots, N] \\
&= [N^{r+1}, N, \ldots, N] \\
&= N^{r+2}.
\end{aligned}
$$

This proves the lemma.

Proof of Theorem 5.1

Using lemma 5.2 we observe that

$$
\begin{aligned}
A^{m+1} &\subset N^2 \\
&\subset N^2 A^{m+(n-1)(m-1)} \subset N^3 \\
&\subset N^3 A^{m+2(n-1)(m-1)} \subset N^4 \\
&\subset N^4 A^{m+3(n-1)(m-1)} \subset N^5 \\
&\vdots \\
&\subset N^t A^{m+(t-1)(n-1)(m-1)} \subset N^{t+1} = 0
\end{aligned}
$$

Adding the exponents on the left hand side we see that $A^w = 0$ where $w = tm + \frac{1}{2}t(t-1)(n-1)(m-1) + 1$ proving the theorem.

Remark: If we set $n = 2$ we observe

$$\begin{aligned} cl(A) &\leq tm + \frac{1}{2}t(t-1)(2-1)(m-1) \\ &= tm + \frac{1}{2}(t^2m - t^2 - tm + t) \\ &= \frac{1}{2}t(t+1)m - \frac{1}{2}t(t-1) \\ &= \binom{t+1}{2}m - \binom{t}{2}. \end{aligned}$$

This matches Chao's Lie algebra result precisely.

Section VI: Uncountably Many Non-Isomorphic Real N-Lie Algebras

Classifying nilpotent, real Lie algebras has been an often studied subject since Engel. In 1962, Chao [CC2] proved that there are uncountably many such Lie algebras of dimension 10 and greater that are non-isomorphic. We shall set about proving an n-Lie algebra analogue of this theorem.

Theorem 6.1

There are uncountably many non-isomorphic n-Lie algebras of dimensions d and nilpotent of length 2 when

1)$n = 2$ and $d = 10$.

2)$n = 3$ and $d = 9$.

3)$n = 4$ and $d = 10$.

4)$n \geq 6$ and $d = n + 4$.

Definition 6.2

Let $\mathbb{F}$ be a subfield of $\mathbb{R}$. An n-Lie algebra A over R is said to be an $\mathbb{F}$-algebra if its structure constants with respect to some basis of A lie in $\mathbb{F}$.

Let $\mathbb{F}$ be a subfield of $\mathbb{R}$ and $C^k_{i_1,i_2,\ldots,i_n}$ be real numbers in $\mathbb{F}$ such that $\sigma(C^k_{i_1,i_2,\ldots,i_n}) = C^k_{i_{\sigma(1)},i_{\sigma(2)},\ldots,i_{\sigma(n)}} = sgn\sigma(C^k_{i_1,i_2,\ldots,i_n})$ for all $\sigma \in S_n$, the symmetric group. Let A be an n-Lie algebra over $\mathbb{F}$ with a basis $(x_1, x_2, \ldots, x_\ell, y_1, y_2, \ldots, y_m)$ where $\ell \geq n$ and multiplication given by $[x_{i_1}, x_{i_2}, \ldots, x_{i_n}] = \sum_{k=1}^{n} C^k_{i_1,i_2,\ldots,i_n} y_k$ and all other products

0. Note that this fits the anti-symmetric condition as,

$$\begin{aligned}
\sigma([x_{i_1}, x_{i_2}, \ldots, x_{i_n}]) &= [x_{i_{\sigma(1)}}, x_{i_{\sigma(2)}}, \ldots, x_{i_{\sigma(n)}}] \\
&= \sigma(\sum_{k=1}^{n} C^k_{i_1,i_2,\ldots,i_n} y_k) \\
&= \sum_{k=1}^{n} \sigma(C^k_{i_1,i_2,\ldots,i_n}) y_k \\
&= \sum_{k=1}^{n} sgn\sigma(C^k_{i_1,i_2,\ldots,i_n}) y_k \\
&= sgn\sigma\left(\sum_{k=1}^{n} C^k_{i_1,i_2,\ldots,i_n} y_k\right) \\
&= sgn\sigma\big([x_{i_1}, x_{i_2}, \ldots, x_{i_n}]\big).
\end{aligned}$$

Lemma 6.3

If the numbers $C^k_{i_1,i_2,\ldots,i_n}$, for $1 \leq i_1 < i_2 < \ldots < i_n \leq \ell$, and $1 \leq k \leq m$ are algebraically independent over $\mathbb{F}$ and if $\binom{\ell}{n} m > m^2 + \ell^2$, then A is not an $\mathbb{F}$-algebra.

Proof of Lemma 6.3

We want to show that $A^2 =< y_1, \ldots, y_m >= Z(A)$. First we show that $A^2 =< y_1, \ldots, y_m >$. Since $\binom{\ell}{n} > m$ we can pick m distinct sets of n integers between 1 and ℓ. Each such set determines a set of vectors from $x_1, \ldots, x_\ell$ and we label these sets S_j, $j = 1, \ldots, m$. Let z_j be the product of the elements in S_j where the indices are arranged in increasing order in the product. As a result $z_j = \sum_{k=1}^{m} C^k_j y_k$ for $j = 1, \ldots, m$ where $C^k_j = C^k_{j_1,j_2,\ldots,j_n}$ if $x_{j_1}, x_{j_2}, \ldots, x_{j_n} \in S_j$. The polynomial $det(x_{ij})$ for $i, j = 1, \ldots, m$ has integer coefficients and thus lies in $\mathbb{F}[x_{11}, \ldots, x_{kj}, \ldots, x_{mm}]$. If $det(C^k_j) = 0$, then the C^k_j's are not algebraically independent, which is a contradiction. Therefore (C^k_j) is a non-singular matrix which generates $< y_1, \ldots, y_m >$ and hence $A^2 =< y_1, \ldots, y_m >$.

Now we show that $Z(A) =< y_1, \ldots, y_m >$. Since the only non-zero products in A are products of the x_i's it is clear that $< y_1, \ldots, y_m > \subset Z(A)$.

Let $z = \sum_{j=1}^{\ell} a_j x_j + \sum_{j=1}^{m} b_k y_k \in Z(A)$ and let $R_\pi = [_, x_{i_2}, x_{i_3}, \ldots, x_{i_n}]$ then,

$$\begin{aligned} 0 &= zR_\pi \\ &= \left(\sum_{j=1}^{\ell} a_j x_j\right) R_\pi + \left(\sum_{k=1}^{m} b_k y_k\right) R_\pi \\ &= \sum_{j=1}^{\ell} a_j (x_j R_\pi) + 0 \\ &= \sum_{j=1}^{\ell} \sum_{k=1}^{m} a_j C_{j\pi}^k y_k \end{aligned}$$

where $C_{j\pi}^k = C_{ji_2i_3\ldots i_n}^k$.

By virtue of the linear independence of the y_k's we obtain $\sum_{j=1}^{\ell} a_i C_{j\pi}^k = 0$. For each $1 \leq t \leq \ell$ choose $\pi_t = t_2, \ldots, t_n$ such that $t_r \neq t_s$ for $r \neq s$ and $t \neq t_2, \ldots, t_n$. Then $\sum_{j=1}^{\ell} a_i C_{j\pi_t}^k = 0$. We observe that $C_{t\pi_t}^k \neq 0$ and $C_{t\pi_t}^k$ is in the algebraically independent set. Repeating this process for each t, $1 \leq t \leq \ell$ gives us a system of ℓ equations and ℓ unknowns. The coefficient matrix C has non-zero elements on the diagonal and hence are algebraically independent. Considering $det(x_{ij})$ as in the last paragraph, gives us a polynomial in ℓ^2 variables with coefficients ± 1. If C is singular then the elements of C satisfy $det(x_{ij})$. The non-zero elements of C satisfy a polynomial obtained from $det(x_{ij})$ by deleting terms if necessary, from any elements of C that are 0. The resulting polynomial is non-zero because of the non-zero diagonal of C. This non-zero polynomial is satisfied by a set of algebraically independent elements. This is a contradiction and hence C is non-singular. As a result $a_1 = a_2 = \ldots = a_\ell = 0$ and $z = \sum_{j=1}^{\ell} a_j x_j + \sum_{j=1}^{m} b_k y_k = \sum_{j=1}^{m} b_k y_k \in < y_1, \ldots, y_m >$. Thus $Z(A) = < y_1, \ldots, y_m > = A^2$.

Now we prove lemma 6.3. Suppose, to the contrary, that A satisfies the conditions of the lemma. Namely, A is an $\mathbb{F}$-algebra with basis $(z_1, \ldots, z_\ell, z_{\ell+1}, \ldots, z_{\ell+m})$ and structure constants $D_{i_1,i_2,\ldots,i_n}^k$'s for $1 \leq i_1, i_2, \ldots, i_n \leq \ell$ and $1 \leq k \leq \ell + m$. We can

assume without loss of generality that $(z_1, \ldots, z_\ell)$ form a basis for C a compliment of A^2. We can write $z_{\ell+i} = v_i + t_i$ for all i where $v_i \in C$ and $t_i \in A^2$ for $i = 1, \ldots, m$. We observe

$$[z_{i_1}, z_{i_2}, \ldots, z_{i_n}] = \sum_{r=1}^{\ell} D^r_{i_1,i_2,\ldots,i_n} z_r + \sum_{s=\ell+1}^{\ell+m} D^s_{i_1,i_2,\ldots,i_n} v_{s-\ell} + \sum_{s=\ell+1}^{\ell+m} D^s_{i_1,i_2,\ldots,i_n} t_{s-\ell}$$

Since $[z_{i_1}, z_{i_2}, \ldots, z_{i_n}] \in A^2$ we see the first two summands must be 0 and we obtain

$$\begin{aligned}[z_{i_1}, z_{i_2}, \ldots, z_{i_n}] &= \sum_{s=\ell+1}^{\ell+m} D^s_{i_1,i_2,\ldots,i_n} t_{s-\ell} \\ &= \sum_{u=1}^{m} D^{u+\ell}_{i_1,i_2,\ldots,i_n} t_u.\end{aligned}$$

As a result $(z_1, \ldots, z_\ell, t_{\ell+1}, \ldots, t_{\ell+m})$ is a new basis for A whose structure coefficients are a subset of the structure coefficients for the old basis. We observe that $(x_1, \ldots, x_\ell)$ is a basis for C' a compliment of A^2. Now let s_i be such that $s_i - z_i \in A^2$ and $s_i \in C'$ for $1 \leq i \leq \ell$. We obtain yet another basis $(s_1, \ldots, s_\ell, t_1, \ldots, t_m)$ which has the same structure coefficients as $(z_1, \ldots, z_\ell, t_{\ell+1}, \ldots, t_{\ell+m})$.
Indeed,

$$\begin{aligned}[s_{i_1}, s_{i_2}, \ldots, s_{i_n}] &= [z_{i_1} + A^2, z_{i_2} + A^2, \ldots, s_{i_n} + A^2] \\ &= [z_{i_1} + Z(A), z_{i_2} + Z(A), \ldots, z_{i_n} + Z(A)] \\ &= [z_{i_1}, z_{i_2}, \ldots, z_{i_n}].\end{aligned}$$

Since $(s_1, \ldots, s_\ell)$ and $(x_1, \ldots, x_\ell)$ both form a basis for C' there exists a non-singular matrix, $B = (b_{ip})$ such that $s_i = \sum_{p=1}^{\ell} b_{ip} x_p$ for all $1 \leq i \leq \ell$. Likewise there exists a non-singular $G = (g_{ur})$ such that $t_u = \sum_{r=1}^{m} g_{ur} y_r$ for all $1 \leq u \leq m$.

Substituting into $[z_{i_1}, \ldots, z_{i_n}] = [s_{i_1}, \ldots, s_{i_n}] = \sum_{u=1}^{m} D^{\ell+u}_{i_1,\ldots,i_n} t_u$ we observe for all $1 \leq i_1, i_2, \ldots, i_n \leq \ell$

$$\begin{aligned}
[s_{i_1}, s_{i_2}, \ldots, s_{i_n}] &= \left[\sum_{p_1=1}^{\ell} b_{i_1 p_1} x_{p_1}, \sum_{p_2=1}^{\ell} b_{i_2 p_2} x_{p_2}, \ldots, \sum_{p_n=1}^{\ell} b_{i_n p_n} x_{p_n}\right] \\
&= \sum_{p_1=1}^{\ell}\sum_{p_2=1}^{\ell}, \ldots, \sum_{p_n=1}^{\ell}\left(b_{i_1 p_1} b_{i_2 p_2}, \ldots, b_{i_n p_n}[x_{p_1}, x_{p_2}, \ldots, x_{p_n}]\right) \\
&= \sum_{p_1=1}^{\ell}\sum_{p_2=1}^{\ell}, \ldots, \sum_{p_n=1}^{\ell}\left(b_{i_1 p_1} b_{i_2 p_2}, \ldots, b_{i_n p_n} \sum_{r=1}^{m} C^{r}_{p_1,p_2,\ldots,p_n} y_r\right) \\
&= \sum_{u=1}^{m} D^{\ell+u}_{i_1,i_2,\ldots,i_n} t_u \\
&= \sum_{u=1}^{m}\sum_{r=1}^{m} D^{\ell+u}_{i_1,i_2,\ldots,i_n} g_{ur} y_r.
\end{aligned}$$

This implies that for fixed $i_1, i_2, \ldots, i_n$ and r we obtain

$$\sum_{p_1=1}^{\ell}\sum_{p_2=1}^{\ell}\ldots, \sum_{p_n=1}^{\ell} b_{i_1 p_1} b_{i_2 p_2}, \ldots, b_{i_n p_n} C^{r}_{p_1,p_2,\ldots,p_n} = \sum_{u=1}^{m} D^{\ell+u}_{i_1,i_2,\ldots,i_n} g_{ur}.$$

We claim that this in turn implies that,

$$C^{r}_{p_1,p_2,\ldots,p_n} = \sum_{p_1=1}^{\ell}\sum_{p_2=1}^{\ell}\ldots, \sum_{p_n=1}^{\ell}\sum_{u=1}^{m} D^{\ell+u}_{i_1,i_2,\ldots,i_n} g_{ur} \overline{b}_{p_1 i_1} \overline{b}_{p_2 i_2}, \ldots, \overline{b}_{p_n i_n}$$

where $B^{-1} = [\overline{b}_{ip}]$.

We show the t^{th} step. Suppose for

$1 \leq p_1, p_2 \ldots p_{t-1} \leq \ell$ and $1 \ \leq \ i_t, \ i_{t+1} \ \ldots \ i_n \ \leq \ \ell$ and r fixed that

$$\sum_{p_t=1}^{\ell} \sum_{p_{t+1}=1}^{\ell}, \ldots, \sum_{p_n=1}^{\ell} b_{i_t p_t} b_{i_{t+1} p_{t+1}}, \ldots, b_{i_n p_n} C^r_{p_1, p_2, \ldots, p_n}$$

$$= \sum_{i_1=1}^{\ell} \sum_{i_2=1}^{\ell} \ldots, \sum_{i_{t-1}=1}^{\ell} \overline{b}_{p_1 i_1} \overline{b}_{p_2 i_2} \ldots \overline{b}_{p_{t-1} i_{t-1}} \sum_{u=1}^{m} D^r_{i_1 i_2 \ldots i_n} g_{ur}.$$

Let

$$A_{p_t} = \sum_{p_{t+1}=1}^{\ell} \sum_{p_{t+2}=1}^{\ell} \ldots, \sum_{p_n=1}^{\ell} b_{i_{t+1} p_{t+1}} b_{i_{t+2} p_{t+2}}, \ldots, b_{i_n p_n} C^r_{p_1, p_2, \ldots, p_n}$$

for $p_t = 1, \ldots, \ell$ and

$$E_{i_t} = \sum_{i_1=1}^{\ell} \sum_{i_2=1}^{\ell} \ldots, \sum_{i_{t-1}=1}^{\ell} \overline{b}_{p_1 i_1} \overline{b}_{p_2 i_2} \ldots \overline{b}_{p_{t-1} i_{t-1}} D^r_{i_1 i_2 \ldots i_n} g_{ur}$$

for $i_t = 1, 2, \ldots, \ell$.

This implies that

$$b_{i_t 1} A_1 + b_{i_t 2} A_2 + \ldots + b_{i_t \ell} A_\ell = E_{i_t}$$

or

$$\begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1\ell} \\ b_{21} & b_{22} & \ldots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{\ell 1} & b_{\ell 2} & \ldots & b_{\ell\ell} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_\ell \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_\ell. \end{bmatrix}$$

So

$$A_{p_t} = \overline{b}_{p_t 1} E_1 + \overline{b}_{p_t 2} E_2 + \ldots + \overline{b}_{p_t n} E_n$$

and

$$A_{p_t} = \sum_{i_1=1}^{\ell} \sum_{i_2=1}^{\ell} \ldots, \sum_{i_t=1}^{\ell} \overline{b}_{p_1 i_1} \overline{b}_{p_2 i_2} \ldots \overline{b}_{p_t i_t} D^r_{i_1 i_2 \ldots i_n}.$$

Finally,

$$\sum_{p_{t+1}=1}^{\ell} \sum_{p_{t+2}=1}^{\ell} \ldots, \sum_{p_n=1}^{\ell} b_{i_{t+1} p_{t+1}} b_{i_{t+2} p_{t+2}}, \ldots, b_{i_n p_n} C^r_{p_1, p_2, \ldots, p_n}$$

$$= \sum_{i_1=1}^{\ell} \sum_{i_2=1}^{\ell} \ldots, \sum_{i_t=1}^{\ell} \overline{b}_{p_1 i_1} \overline{b}_{p_2 i_2} \ldots \overline{b}_{p_t i_t} D^r_{i_1 i_2 \ldots i_n}.$$

This proves the claim.

The claim implies that $C^r_{p_1,p_2,\ldots,p_n} \in \mathbb{E} = \mathbb{F}(b_{ip}, g_{ur})$. But the degree of transcendence of $\mathbb{E}$ over $\mathbb{F}$ is at most $\ell^2 + m^2$ which is less than, $\binom{\ell}{n} m$, the number of $C^r_{p_1,p_2,\ldots,p_n}$'s. This a contradiction and hence A is not an $\mathbb{F}$-algebra, proving the lemma.

Proof of Theorem 6.1

It is known that there exists a set S of uncountably many real numbers that are algebraically independent over $\mathbb{Q}$. We can divide S into uncountably many disjoint subsets $\{C^k_{i_1,i_2,\ldots,i_n}\}_\alpha$ of size $\binom{\ell}{n} m$ where α distinguishes subsets. Define the n-Lie algebra A_α with basis $(x_1, x_2, \ldots, x_\ell, y_1, y_2, \ldots, y_m)$ and multiplication given by $[x_{i_1}, x_{i_2}, \ldots, x_{i_n}] = \sum_{k=1}^{n} C^k_{i_1,i_2,\ldots,i_n} y_k$ and all other products 0 where $C^k_{i_1,i_2,\ldots,i_n} \in \{C^k_{i_1,i_2,\ldots,i_n}\}_\alpha$ for all $1 \leq i_1, i_2, \ldots, i_n \leq \ell$ and $1 \leq k \leq m$. For $\alpha \neq \beta$ we claim that A_α and A_β are non-isomorphic. Indeed, since the $(C^k_{i_1,i_2,\ldots,i_n})_\alpha$'s are algebraically independent over $\mathbb{Q}[\{C^k_{i_1,i_2,\ldots,i_n}\}_\beta]$, if we apply lemma 6.3 to A_α, we see that it is not a $\mathbb{Q}[(C^k_{i_1,i_2,\ldots,i_n})_\beta]$-algebra. Hence A_α and A_β are non-isomorphic as claimed.

To prove the theorem it remains to find for each given n and d in 1-4, an m and k where $d = n + m + k$ such that $f(k, m, n) = \binom{n+k}{n} m - (n + k)^2 + m^2 > 0$. We do this case by case.

1) When $k = m = 4$, we obtain $f(4, 4, n) = 1/6n^4 + 5/3n^3 + 29/6n^2 + 1/3n - 28$. The only positive root is approximately $n = 1.807126451$. Hence $f(4, 4, n) > 0$ if $n \geq 2$. Setting $n = 2$ gives $d = n + m + k = 10$. This coincides with Chao's result.

2) When $k + m = 3 + 4 = 7$, we obtain $f(3, 4, n) = 2/3n^3 + 3n^2 + 4/3n - 21$. The only positive root is approximately $n = 2.046172397$. Hence $f(3, 4, n) > 0$ if $n \geq 3$. Hence $f(3, 4, n) > 0$ if $n \geq 3$. Setting $n = 3$ gives $d = n + m + k = 9$.

3) When $k = 3, m = 2$, we obtain $f(3, 2, n) = 1/3n^3 + n^2 - 7/3n - 11$. The only positive root is $n = 3$. Hence $f(3, 2, n) > 0$ if $n \geq 4$. Setting $n = 2$ gives $d = n + m + k = 10$.

4) When $k = 3, m = 1$, we obtain $f(3, 1, n) = 1/6n^3 - 25/6n - 9$. The only positive root is approximately $n = 5.850622760$. Hence $f(3, 1, n) > 0$ if $n \geq 6$. Thus $d = n + m + k = n + 4$.

Note that if $n \geq 6$, then $d - n$ cannot be less than 4. That is to say, we have found the minimal $k + m$ such that $f(k, m, n) > 0$. If we set $k + m < 4$, we get no solutions. Indeed, if $k = 0$ or $m = 0$, we obtain $f(k, m, n) = m - m^2 - (n)^2 \leq 0$ and $f(k, m, n) = -(n+k)^2 \leq 0$. For $m+k = 1+1 = 2$ we obtain $f(k, m, n) = -n-n^2-1$ which has no real roots. For $m + k = 2 + 1 = 3$ and $m + k = 1 + 2 = 3$ we obtain $f(k, m, n) = -1/2n^2 - 5/2n - 4$ and $f(k, m, n) = -3 - n^2$ neither of which have real roots and are always negative.

Section VII: Frattini Theory

Barnes proved the following in Lie algebras: if $B, C \lhd A$ where $C \subset \phi(A) \bigcap B$ and B/C is nilpotent, then B is nilpotent. A corollary to this is the following: if $\phi(A) \lhd A$, then $\phi(A)$ is nilpotent. This theorem raises an obvious question: when is $\phi(A) \lhd A$? For groups it is known that $\phi(A) \lhd A$ because all automorphisms permute maximal subgroups. As we have mentioned, Barnes and Chao proved that A is nilpotent Lie algebra if and only if $\phi(A) = A^2$. If A is nilpotent, clearly $\phi(A) = A^2 \lhd A$. In 1968, Barnes proved that if A is a solvable Lie algebra, then $\phi(A) \lhd A$. One might believe this to be true in general for Lie algebras, but over $\mathbb{F}_2 = \{0, 1\}$ if A is the cross product n-Lie algebra that introduced in the introduction, then $\phi(A) \not\lhd A$.

In the last section we proved for A an n-Lie algebra, A is nilpotent if and only if $\phi(A) = A^2$. In this section we will prove the n-Lie algebra version of Barnes' theorem and continue to examine when $\phi(A) \lhd A$ for n-Lie algebras. We will prove the following three theorems:

Theorem 7.1

Let A be an n-Lie algebra. If $B, C \lhd A$ where $C \subset \phi(A) \bigcap B$ and B/C is nilpotent, then B is nilpotent.

Theorem 7.2

If A is a solvable n-Lie algebra, then $\phi(A) \lhd A$.

Theorem 7.3

If A is the n-Lie cross product over $\mathbb{F}_2$, then $\phi(A) \not\lhd A$.

Proof of Theorem 7.1

We will prove the theorem by contradiction. Assume that the hypothesis holds but B is not nilpotent. By Engel's theorem there exists and $R \in R(B)$ such that $BR^s \neq 0$ for all s. We apply Fitting's lemma. Let I be the final image and K be the Fitting null component. Recall that $A = I \bigoplus K$ and K is a subalgebra of A. Since $B \lhd A$, we observe that $I \subset B$. Furthermore, since B/C is nilpotent, $I \subset C \subset \phi(A)$. Now since $A = I \bigoplus K$ we see that $A = \phi(A) + K$. Since K is a subalgebra it is contained in some maximal subalgebra M and $A = \phi(A) + M$. Since M is a maximal subalgebra, $\phi(A) \subset M$ and $M = A$ but this is a contradiction as M is a maximal subalgebra. This proves Theorem 7.1.

Corollary 7.4

If $D \lhd A$ and $D \subset \phi(A)$, then D is nilpotent.

Proof of Corollary 7.4

Let$B = C = D$. We note that $B/C = 0$ is certainly nilpotent and if we apply theorem 1, the corollary follows. We note also if $D = \phi(A)$, then the corollary still holds.

Corollary 7.5

If $\phi(A) \lhd A$, then $A/\phi(A)$ is nilpotent if and only if A is nilpotent.

Proof of Corollary 7.5

If A is nilpotent then certainly $A/\phi(A)$ is nilpotent. If $A/\phi(A)$ is nilpotent, then we let $B = A$ and $C = \phi(A)$ and apply the theorem. This proves the corollary.

We begin the proof of the theorem 7.2 with the following lemmas:

Lemma 7.6

If I is a minimal ideal of A a solvable n-Lie algebra, then

1) $[I, I, A \ldots, A] = 0$

2) $I \bigcap M = 0$ or $I \bigcap M = I$ for all M maximal subalgebras of A.

Proof of Lemma 7.6

First we show 1. We observe

$$\begin{aligned}[[I, I, A, \ldots, A], A, \ldots, A] &= [[I, A, A, \ldots, A], I, \ldots, A] + [I, [I, A, \ldots, A]A, \ldots, A] \\ &\quad + [I, I, [A, \ldots, A], A, \ldots, A] \\ &\subset [[I, I, A, \ldots, A]A, \ldots, A]\end{aligned}$$

and $[I, I, A, \ldots, A] \lhd A$. Since A is solvable, $[I, I, A, \ldots, A]$ is properly contained in I. Indeed, if $I = [I, I, A, \ldots, A]$, we can prove that $I = [I, I, A, \ldots, A] \subset A^{(n)}$ for all n. We do so inductively. Obviously $[I, I, A, \ldots, A] \subset A^{(2)}$ and taking the inductive step, we assume that $I = [I, I, A, \ldots, A] \subset A^{(n)}$. Then $I = [I, I, A, \ldots, A] \subset [A^{(n)}, A^{(n)}, A, \ldots, A] = A^{(n+1)}$ and $I = [I, I, A, \ldots, A] \subset A^{(n)}$. Since A is solvable, $I = 0$ contradicting the minimality condition of I. Hence, $[I, I, A, \ldots, A]$ is properly contained in I. But the only way this can happen is if $[I, I, A, \ldots, A] = 0$ otherwise we will again contradict the minimality condition of I. This proves 1.

Now we show 2. Assume that $I \bigcap M$ is properly contained in I. Since I is not contained in M we observe M is properly contained in $I + M$ and $I + M$ is a subalgebra. The only way this can happen is if $I + M = A$. Now we observe that

$$\begin{aligned}[I \bigcap M, A, \ldots, A] =& [I \bigcap M, M + I, \ldots, M + I] \\ =& [I \bigcap M, M, \ldots, M] + 0 + \ldots, +0 \in I \bigcap M\end{aligned}$$

hence $I \bigcap M \lhd A$. As a result $I \bigcap M = 0$, otherwise we contradict the minimality of I. This proves the lemma.

Proof of Theorem 7.2

The proof follows closely that of Barnes' [BD] analogous proof for Lie Algebras. We induct on $dim(A)$. Let I be a minimal ideal of A and let ϕ_I denote the intersection of all maximal subalgebras M such that $I \subset M$.

Then

$$\phi_I/I = (\bigcap_{M|I\subset M} M)/I = \bigcap_{M|I\subset M} M/I = \phi(A/I).$$

By the induction hypothesis we have that $\phi_I/I \lhd A/I$ and hence $\phi_I \lhd A$. If $I \subset \phi(A)$, then $\phi(A) = \phi_I \lhd A$ and we're done.

From here the proof can be broken up and conducted in two cases:
1) I is not a subset of $\phi(A)$ and $C_A(I) \neq I$.
2) I is not a subset of $\phi(A)$ and $C_A(I) = I$.
We define $C_A(I) = \{x \in A | [x, I, A, \ldots, A] = 0\}$ and call it *the centralizer of I in A.*

Note that since $I \lhd A$, then $C_A(I) \lhd A$. Indeed, for $x \in C_I(A)$ we see by using the Jacobian property of n-Lie algebras that

$$[[x, A, \ldots, A], I, \ldots, A] = [[x, I, A \ldots, A], A, \ldots, A] + [x, [A, I, \ldots, A], A \ldots, A]$$
$$= 0 + [x, I, A \ldots, A] = 0 + 0.$$

Hence $C_A(I) \lhd A$. We now resume the proof.

Case 1

Suppose I is not a subset of $\phi(A)$ and $C_A(I) \neq I$. Due to this assumption there exists M, a maximal subalgebra of A such that $I \bigcap M$ is a proper subset of I. By our lemma 7.6 $I \bigcap M = 0$. Recall that $C_A(I) = \{x \in A | [x, I, A, \ldots, A] = 0\}$. Recall also that if $I \lhd A$, then $C_A(I) \lhd A$. Suppose $I \subset C_A(I)$, then $C_A(I) \bigcap M \neq 0$ and

$$\begin{aligned}[C_A(I) \bigcap M, A, \ldots, A] =& [C_A(I) \bigcap M, M + I, \ldots, M + I] \\ &= [C_A(I) \bigcap M, M, \ldots, M] + 0 + \ldots, +0 \subset C_A(I) \bigcap M\end{aligned}$$

because $C_A(I) \lhd A$ and M is a subalgebra. Hence $C_A(I) \bigcap M \lhd A$. As a result for each M a maximal subalgebra there exists $J \subset M$ where J is a minimal ideal of A. We see that

$$\phi(A) = \bigcap M = \bigcap_{J \subset M} \phi_J \lhd A.$$

Case 2

Suppose $C_A(I) = I$ and I is not contained in $\phi(A)$. We will show that $\phi(A) = 0$. As in Case 1, due to this assumption and lemma 7.6 there exists M, a maximal subalgebra of A such that $I \bigcap M = 0$. For all $m \in M$ we prove that $m \notin \phi(A)$. Since $m \notin I = C_A(I)$ there exists $i \in I$ and $a_i's \in A$ such that $[m, i, a_3, a_4, \ldots, a_n] = mR_a \neq 0$. Note that R_a is a derivation on I. Since $[I, I, A \ldots, A] = 0$ we see that $R_a^2 = 0$ and $expR_a = 1 + R_a$ is an automorphism of A. Set $N = expR_a(M)$ a maximal subalgebra. If $m \in N$ then $m = n(1 + R_a) = n + nR_a$ for some $n \in M$. Since $n, m \in M$ we see that $m - n = nR_a \in M$. Hence $nR_a \in I \bigcap M = 0$ and $m = n$. But this means that $mR_a = nR_a = 0$ which contradicts the fact that $mR_a \neq 0$. This implies that $M \bigcap N = 0$ and in turn that $\phi(A) = 0 \lhd A$. This proves the theorem.

To prove the theorem 7.3 we will first prove a few lemmas. Recall that the n-Lie Algebra cross product is defined on an $n+1$ dimensional vector space with basis $\{x_1\}_{i=1}^{n+1}$ and multiplication over $\mathbb{F}_2$ is given by $[x_1, x_2, \ldots, \hat{x_i}, \ldots, x_{n+1}] = (-1)^{n+1+i} x_i = x_i$. We call this basis the *defining basis*.

Lemma 7.7

Either $\phi(A) = 0$ or $\phi(A) =< v >$ where $v = \sum_{i=1}^{n+1} x_i$.

Proof of Lemma 7.7

First we note that all n-dimensional subalgebras are maximal. Let $w = \sum_{i=1}^{n+1} c_i x_i$ be any element of $\phi(A)$. If $c_j = 0$ and $c_k \neq 0$ for some j and k we claim that we will get a contradiction: namely that, w is not in some n-dimensional subalgebra of A and in turn not in $\phi(A)$. This will prove the lemma because the only w that we can not assume that $c_j = 0$ and $c_k \neq 0$ is either $w = \sum_{i=1}^{n+1} 0 \cdot x_i = 0$ or $w = \sum_{i=1}^{n+1} 1 \cdot x_i = v$.

We construct the subalgebra as follows:
Let $S =< x_j + x_k, x_1, x_2, \ldots, \hat{x_j}, \ldots, \hat{x_k}, \ldots x_{n+1} >$. This is an n-dimensional maximal subalgebra as the only non-zero multiplication is

$$\begin{aligned}[x_j + x_k, x_1, x_2, \ldots, \hat{x_j}, \ldots, \hat{x_k}, \ldots, x_{n+1}] &= [x_1, x_2, \ldots, x_j, \ldots, \hat{x_k}, \ldots, x_{n+1}] \\ &\quad + [x_1, x_2, \ldots, \hat{x_j}, \ldots, x_k, \ldots, x_{n+1}] \\ &= x_j + x_k \in S.\end{aligned}$$

We observe $w \in \phi(A) \subset S$. Hence $w = \sum_{i=1}^{n+1} c_i x_i = d_j(x_j + x_k) + \sum_{i \neq j,k} d_i x_i$ and

$$w - w = 0 = (c_j - d_j)x_j + (c_k - d_j)x_k + \sum_{i \neq j,k} (c_i - d_i)x_i$$

By linear independence, $c_j - d_j = 0 = c_k - d_j$ and $c_j = c_k$. But by our assumption $0 = c_j = c_k \neq 0$, which is a contradiction. Hence either $w = \sum_{i=1}^{n+1} 0 \cdot x_i = 0$ or $w = \sum_{i=1}^{n+1} 1 \cdot x_i = v$. This proves the lemma. For the remainder of this section we show that $\phi(A) \neq 0$.

To show $\phi(A) \neq 0$ we need to show that v is in all maximal subalgebras. We begin by examining the most obvious maximal subalgebras, those of dimension n. Once we have information about n-dimensional subalgebras we examine other maximal subalgebras and show that v is in each of these as well.

One difficulty of dealing with a subalgebra is that although we have a basis for the subalgebra we only know how the bracket works on the defining basis. To overcome this impediment we introduce the following lemma:

Lemma 7.8

Let $S_{v_j} = \{x_i | c_i \neq 0\}$ for all $v_j \in S$. If S is an n-dimensional subspace with basis $\{v_1, \ldots, v_n\}$ and $v_j = \sum_{i=1}^{n+1} c_i x_i$, then:

1) S has a basis $\{u_i\}_{i=1}^{r}$ such that $|S_{u_i}| \leq 2$ for all u_i.
2) Allowing for reordering of subindices, $\{x_{n+1} + x_i, x_j\}_{i=1}^{s}$ where $s + 1 \leq j \leq n$, is a basis for S.

Proof of Lemma 7.8

First we prove 1). Suppose to the contrary that there is a v_j such that $|S_{v_j}| > 2$. Let $S' =< v_1, v_2, \ldots, \hat{v_j}, \ldots, v_n >$. If $< S_{v_j} > \bigcap S' = 0$, then the x_i's in S_{v_j} and the v_i's in S' form a linearly independent set.
But

$$(*) \qquad dim(< S_{v_j} > + S') = dim(< S_{v_j} >) + dim(S') - dim(< S_{v_j} > \bigcap S')$$
$$> 2 + (n-1) - 0 = n + 1$$

which is impossible as A is $(n+1)$-dimensional.

Hence there exists a $y_j \neq 0 \in < S_{v_j} > \bigcap S'$ such that $|S_{v_j}| - |S_{y_j}| \leq 2$ otherwise we reach a contradiction similar to the one in $(*)$. Let $u_j = v_j - y_j$. Since $y_j \in S'$ it can be written as a linear combination of the v_i's for $i \neq j$ and $\{v_1, v_2, \ldots, u_j, \ldots, v_n\}$ is a basis for S with $|S_{u_j}| = |S_{v_j}| - |S_{y_j}| \leq 2$. We can repeat this process as necessary until 1) holds.

Now we prove 2). We begin by applying 1). As a result need only consider $u_i = x_{\alpha_i} + c_i x_{\beta_i}$. Let s be the number c_i's that are non-zero. For convenience we reorder and rename the u_i's such that $c_i \neq 0$ for $1 \leq i \leq s$. If $S_{u_k} \bigcap S_{u_j} = u_j \neq 0$ for some $1 \leq k \leq s$ and some $s+1 \leq j \leq n$, then we replace u_k with $u_k - u_j$. The result is still a basis. Let us assume that we have done all such possible reductions. As a result, $(\bigcup_{j=1}^{s} S_{u_j}) \bigcap (\bigcup_{j=s+1}^{n} S_{u_j}) = 0$.

We now make some remarks about $|\bigcup_{j=1}^{s} S_{u_j}|$. Let $P = \bigcup_{j=1}^{s} S_{u_j}$ and $Q = \bigcup_{j=s+1}^{n} S_{u_j}$. Since Q has only singletons we know that $|Q| = n - s$. As a result $|P| + |Q| = (n-s) + |P| \leq n+1$ and $|P| \leq s+1$. Also $|P| \geq s$ otherwise the u_i's would not form a basis. We see that $|P| = s$ or $|P| = s+1$. To get more precise information we define a graph G on P as follows. Let the vertices of G be x_i's $\in P$ and let e_i be the edge connecting x_{α_i} to x_{β_i} where $u_i = x_{\alpha_i} + x_{\beta_i}$. Also let $E(G)$ be the number of edges and $V(G)$ be the number of vertices. We recall a few graph theory definitions. If between all vertices there exists at least one path, a sequence of edges, we say G is *connected*. If there are two paths between any two vertices we

say that G has a cycle; otherwise we say G is *acyclic*. If G is connected and the path between any two vertices is unique we say that G is a *tree*. We claim that G is a tree. To prove this we will the following theorem [BB]:

The Following Are Equivalent:

A) G a tree.

B) G is connected with $E(G) = s$ and $V(G) = s + 1$.

C) G is acyclic with $E(G) = s$ and $V(G) = s + 1$.

We will also use the fact that graphs such that $V(G) \leq E(G)$ have a cycle. First we note that if G has a cycle then there are at least two paths P_1 and P_2 from some x_a to some x_b. As a result the sum of all vertices in the cycle is zero implying a linear dependence which is impossible as the u_i's form a basis for S. Hence G is acyclic. We know that $|P| = V(G) \leq s + 1$ but if $V(G) = s$, then G will have a cycle. Hence $|P| = V(G) = s + 1$. We see C) is satisfied, and therefore G is a tree.

Now we prove 2). Since G satisfies C), by B) it is connected, and therefore there exists a unique set of e_j's in G that connects x_{n+1} to x_i for all i. Indeed, if we take the sum of the u_j's associated with the e_j's we will get a single edge f_i that connects x_{n+1} to x_i.

Example: If the path between x_{n+1} and x_5 has the following edges in it $x_{n+1}+x_4, x_4+x_2, x_2+x_5$, then $f_5 = (x_{n+1} + x_4) + (x_4 + x_2) + (x_2 + x_5) = x_{n+1} + 2x_4 + 2x_2 + x_5 = x_{n+1} + 0x_4 + 0x_2 + x_5 = x_{n+1} + x_5$ as we are over $(F)_2$. Now although this f_i is new and is not in G, we see that $x_{n+1} + x_i$, the vector associated with f_i, is in S. If we do this with each vector we obtain the set $\{x_{n+1} + x_i, x_j | s + 1 \leq j \leq n\}_{i=1}^{s}$. Since these are clearly linearly independent and were obtained by taking sums of existing basis vectors the claim holds.

Lemma 7.9

$[x_{n+1} + x_1, x_{n+1} + x_2, \ldots, x_{n+1} + x_s, x_{s+1}, \ldots, x_n] = x_{n+1} + \sum_{i=1}^{s} x_i$.

Proof of Lemma 7.9

Indeed,

$$
\begin{aligned}
&[x_{n+1} + x_1, x_{n+1} + x_2, \ldots, x_{n+1} + x_s, x_{s+1}, \ldots, x_n] \\
=&[x_{n+1}, x_{n+1} + x_2, \ldots, x_{n+1} + x_s, x_{s+1}, \ldots, x_n] \\
&+ [x_1, x_{n+1} + x_2, \ldots, x_{n+1} + x_s, x_{s+1}, \ldots, x_n] \\
=&[\hat{x_1}, x_2, \ldots, x_s, x_{s+1}, \ldots, x_n, x_{n+1}] + [x_1, x_{n+1} + x_2, \ldots, x_{n+1} + x_s, x_{s+1}, \ldots, x_n] \\
=&x_1 + [x_1, x_{n+1} + x_2, \ldots, x_{n+1} + x_s, x_{s+1}, \ldots, x_n].
\end{aligned}
$$

Repeating the process we see that the claim holds.

Lemma 7.10

For S an n-dimensional subspace the following are equivalent:

1) $v \in S$

2) s is odd

3) S is a maximal subalgebra.

Proof of Lemma 7.10

We note that two things happen because S is n-dimensional. First, if S is a subalgebra, it is automatically a maximal subalgebra. Second, we can use lemma 7.8 and work with the basis $\{x_{n+1} + x_i, x_j | s+1 \leq j \leq n\}_{i=1}^{s}$ throughout the proof. We begin by proving 1) if and only if 2).

Assume s is even and $v \in S$ then

$$\begin{aligned} v + \sum_{i=1}^{s}(x_{n+1} + x_i) + \sum_{i=s+1}^{n} x_i =& v + sx_{n+1} + \sum_{i=1}^{s} x_i + \sum_{i=s+1}^{n} x_i \\ =& v + 0 \cdot x_{n+1} + \sum_{i=1}^{n} x_i \\ =& v + \sum_{i=1}^{n} x_i \\ =& \sum_{i=1}^{n+1} x_i + \sum_{i=1}^{n} x_i \\ =& x_{n+1} \in S. \end{aligned}$$

As a result, $x_{n+1} + (x_{n+1} + x_i) = x_i \in S$ for all i and $S = A$. This is a contradiction. Hence $v \in S$ implies s is odd.

If s is odd, then

$$\begin{aligned} \sum_{i=1}^{s}(x_{n+1} + x_i) + \sum_{i=s+1}^{n} x_i =& sx_{n+1} + \sum_{i=1}^{s} x_i + \sum_{i=s+1}^{n} x_i \\ =& x_{n+1} + \sum_{i=1}^{n} x_i \\ =& v \end{aligned}$$

and $v \in S$.

Now we show 2) if and only if 3).

If s is odd then by lemma 7.9

$$\begin{aligned}[x_{n+1}+x_1, x_{n+1}+x_2, \ldots, x_{n+1}+x_s, x_{s+1}, \ldots, x_n] =& x_{n+1} + \sum_{i=1}^{s} x_i \\ =& s x_{n+1} + \sum_{i=2}^{s} x_i \\ =& \sum_{i=1}^{s} (x_{n+1} + x_i) \in S.\end{aligned}$$

and S is a an n-dimensional subalgebra.

Assume s is even. If S is a subalgebra then again by lemma 7.9

$$\begin{aligned}[x_{n+1}+x_1, x_{n+1}+x_2, \ldots, x_{n+1}+x_s, x_{s+1}, \ldots, x_n] =& x_{n+1} + \sum_{i=1}^{s} x_i \\ =& (s+1) x_{n+1} + \sum_{i=1}^{s} x_i \\ =& x_{n+1} + s x_{n+1} + \sum_{i=1}^{s} x_i \\ =& x_{n+1} + \sum_{i=1}^{s} (x_{n+1} + x_i).\end{aligned}$$

Hence $x_{n+1} + \sum_{i=1}^{s} (x_{n+1} + x_i) \in S$ and $\left(x_{n+1} + \sum_{i=1}^{s} (x_{n+1} + x_i) \right) + \sum_{i=1}^{s} (x_{n+1} + x_i) = x_{n+1} \in S$. As before this means that $x_{n+1} + (x_{n+1} + x_i) = x_i \in S$ for all i and $A = S$ which is a contradiction. As a result, if S is a subalgebra, then s is odd. This concludes the lemma.

We have shown $v \in S$ for all n-dimensional subalgebras. To show that $\phi(A) \neq 0$ it remains to show the following lemma.

Lemma 7.11

v is in every maximal subalgebra.

Proof of Lemma 7.11

Let T be an r-dimensional maximal subalgebra of A with $r < n$. Since any Abelian subalgebra is contained in an $(n-1)$-dimensional Abelian subalgebra, we can assume $r = n - 1$. Let $\{v_j\}_{j=1}^{n-1}$ be a basis for T. If $v \in T$, then we are done. If $v \notin T$, then we extend T by adding v to form S, an n-dimensional subspace. By lemma 7.10, S is a subalgebra. This proves the lemma.

Proof of Theorem 7.3

As a result of the lemma 7.7, $\phi(A) =< v >$ or $\phi(A) = 0$ and as a result of lemmas 7.8-7.11 $\phi(A) \neq 0$ hence $\phi(A) =< v >$.

But

$$\begin{aligned}[v, x_3, x_4, \ldots, x_{n+1}] &= [x_1 + x_2, x_3, x_4, \ldots, x_{n+1}] \\ &= x_1 + x_2 \\ &\notin \phi(A)\end{aligned}$$

and $\phi(A) \not\lhd A$.

Bibliography

[BB] Bela Bollobas

Graph Theory : An Introductory Course

Published: New York : Springer Verlag, c1979.

[BD1] Barnes, Donald W.

Nilpotency of Lie algebras.

Math. Z. 79 1962 237–238.

[BD2] Barnes, Donald W.; Gastineau-Hills, Humphrey M.

On the theory of soluble Lie algebras.

Math. Z. 106 1968 343–354.

[BD3] Barnes, Donald W.; Newell, Martin L.

Some theorems on saturated homomorphs of soluble Lie algebras.

Math. Z. 115 1970 179–187.

[CC1] Chao, Chong-Yun

Some characterizations of nilpotent Lie algebras.

Math. Z. 103 1968 40–42.

[CC2] Chao, Chong-Yun

Uncountably many nonisomorphic nilpotent Lie algebras.

Proc. Amer. Math. Soc. 13 1962 903–906.

[CC3] Chao, Chong-Yun
A nonimbedding theorem of nilpotent Lie algebras.
Pacific J. Math. 22 1967 231–234.

[FV] Filippov, V. T.
n-Lie algebras. (Russian)
Sibirsk. Mat. Zh. 26 (1985), no. 6, 126–140, 191.

[HP] Hall, P.
Some sufficient conditions for a group to be nilpotent.
Illinois J. Math. 2 1958 787–801.

[HG1] Higman, Graham
Groups and rings having automorphisms without
non-trivial fixed elements. J. London Math. Soc. 32 (1957),
321–334

[HG2] Higman, Graham
Lie ring methods in the theory of finite nilpotent groups.
1960 Proc. Internat. Congress Math. 1958 pp. 307–312 Cambridge Univ. Press, New York

[JN] Jacobson, Nathan
Lie algebras.
New York : Dover, 1979, c1962.

[KI] Kaplansky, Irving

Lie algebras and locally compact groups.

Chicago, University of Chicago Press 1971

[KS1] Kasymov, Sh. M.

On a theory of n-Lie algebras. (Russian)

Algebra i Logika 26 (1987), no. 3, 277–297, 398.

[KS2] Kasymov, Sh. M.

Nil-elements and nil-subsets in n-Lie algebras. (Russian)

Sibirsk. Mat. Zh. 32 (1991), no. 6, 77–80, 204; translation in Siberian Math. J. 32 (1991), no. 6, 962–964 (1992)

[SW] Scott, W.R.

Group Theory

Englewood Cliffs, N.J., Prentice-Hall 1964

[TJ] Thompson, John

Finite groups with fixed-point-free automorphisms of prime order.

Proc. Nat. Acad. Sci. U.S.A. 45 1959 578–581.

[TD] Towers, D. A.

A Frattini theory for algebras.

Proc. London Math. Soc. (3) 27 (1973), 440–462.