# ABSTRACT

ATTIOGBE, CYRIL EFOE. On Characterizing Nilpotent Lie algebras by their Multipliers. (Under the direction of Ernest L. Stitzinger.)

Authors have turned their attentions to special classes of nilpotent Lie algebras such as two-step nilpotent and filiform Lie algebras, in particular filiform Lie algebras are classified up to dimension eleven [8]. These techniques have not worked well in higher dimensions. For a nilpotent Lie algebra L, of dimension n, we consider central extensions $0 \to M \to C \to L \to 0$ with $M \subseteq c^2 \cap Z(C)$, where $c^2$ is the derived algebra of C and $Z(C)$ is the center of C. Let $M(L)$ be the M of largest dimension and call it the multiplier of L due to it's analogy with the Schur multiplier. The maximum dimension that M can obtain is $\frac{1}{2} n(n-1)$ and this is met if and only if L is abelian.

Let $t(L) = \frac{1}{2} n(n-1) - \dim M(L)$. Then $t(L) = 1$ if and only if $L = H(1)$, where $H(n)$ is the Heisenberg algebra of dimension $2n + 1$.

A recent technique to classify nilpotent Lie algebra is to use the dimension of the multiplier of L. In particular, to find those algebras whose multipliers have dimension close to the maximum, we call this invariant $t(L)$. Algebras with $t(L) \leq 8$ have been classified [10]. It's the purpose of this work to use this technique on filiform Lie algebras along with three main tools namely: Propositions 0, 3, and theorem 4. All algebras in this work will be taken over any field whereas in previous works, they have been taken over the field of real and complex numbers.

# ON CHARACTERIZING NILPOTENT LIE ALGEBRAS
# BY THEIR MULTIPLIERS

by

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in partial fulfillment of the
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Doctor of Philosophy

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## APPROVED BY:

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Chair of Advisory Committee          Co-chair of Advisory Committee

# DEDICATION

*To my beautiful and loving wife*

**Evonne Onike Cole-Attiogbe,**

*wonderful kids*

**Emilia Elizabeth Kokoe Attiogbe,**

**Cyril Efoe Attiogbe Jr.**

*and my late mother*

**Keziah Elizabeth Dohoo Baker**

*For helping me keep*

*the Ph.D. process*

*in perspective*

# BIOGRAPHY

Cyril Efoe Attiogbe was born in Freetown, Sierra-Leone, West Africa on September 17 1959. He is married to a beautiful and loving wife by the name of Mrs. Evonne Onike Cole-Attiogbe in 1989, together they are blessed with two wonderful kids, Emilia Elizabeth Kokoe Attiogbe and Cyril Efoe Attiogbe Jr.. Emilia is a $9^{th}$ grader at Southeast Raleigh High School and also wants to be a teacher like her dad, grandma and uncle. Cyril Jr., a $4^{th}$ grader and honor roll student at Wilburn Elementary School wants to be an Aerospace Engineer. His favorite game is the Microsoft flight simulator 2004.

Cyril Efoe Attiogbe received his elementary education from the Murray Town Municipal School, Freetown, Sierra-Leone. He received his secondary education from the Sierra-Leone Grammar School, Freetown, Sierra-Leone, graduating in 1977. After completing high school he worked at the British Guardian Royal Exchange Assurance Company as a statistician for four years and was thus able to save up some money to pursue his dream of coming to America for further studies.

In 1982 he gained admission to St-Augustine's College here in Raleigh, North Carolina, and being very intelligent and determined he graduated with honors with a GPA of 3.8(Suma Cum Laude) in 1986 and won a special talent award to graduate school at North Carolina Central University in Durham North Carolina. He graduated from Central in 1988 with masters in pure mathematics and a thesis in the area of Abstract Algebra. He then taught Algebra and Geometry to middle and high school students in the Wake County Public School System while also simultaneously serving as and instructor with the Upward Bound/Talent Search program in Rocky Mount, N.C.. He also taught

Statistics and Calculus for a couple of years at the Wake County Technical Community College and after that was offered a position as an Assistant Professor of Mathematics in 1989 and has since been with St-Augustine's College(15 years).

In 1998 Cyril Efoe Attiogbe was admitted to the Doctoral Program at North Carolina State University. While enrolled at N.C. State he was able to take several courses in the College of Engineering specifically in the department of Electrical and Computer Engineering. He designed a "Pulse Amplitude Modulation Transmitter" ( PAM) capable of transmitting human voice to an AM radio at a frequency of 1450 KHz using a minimal number of operational amplifiers. He also designed an equalizer for an electronics lab project, he also did some signal processing research at the Engineering Graduate Research Center at the Centennial Campus in the area of Global Positioning System(GPS) and also conducted an investigation into the Detection of Radar Signal in Noise and the extraction of Target information contained in the Signal in the Spring of 1999. He also served as a Graduate Assistant Grader in the department of Electrical and Computer Engineering at N.C. State in 1998.

Cyril Efoe Attiogbe earned a second masters in Mathematics from N.C. State University in 2003 upon the successful completion of the PhD written exam in the areas of Functional Analysis, Modern Algebra and Linear/Lie Algebras. He also passed the PhD Language Requirement in French. His dissertation topic is "On Characterizing Nilpotent Lie Algebras by Their multipliers"

Although Cyril Efoe Attiogbe works very hard, he is very thankful to God to have blessed him with a family to spend time with. He also gives thanks to his parents who

believed in him and supported him. One of his main mentors was his late mother Mrs. Keziah Elizabeth Dohoo Baker who was also a teacher and disciplinarian. Another mentor was his dad Mr. Simon Attiogbe who taught him that no pain means no gain. Cyril Efoe Attiogbe has several honors including but not limited to "who's who among students in American Universities and Colleges", Special Talent Award by the graduate dean of North Carolina Central University, an Outstanding Professorship honoree, Delta Sigma Theta Sorority Inc., service award by Dr. Diane B. Suber, President of St-Augustine's College. He is also a member of the Mathematical Society of America and the National Society of Black Engineers.

Cyril Efoe Attiogbe has also conducted several workshops on the General Knowledge and Specialty areas of the National Teachers Examination (NTE) for teacher Ed majors in the NTE lab at St-Aug. He also conducted workshops on Linear Regression and Modeling using the TI-85 Graphing calculator  and the software "Derive" for the Upward Bound students on a model created by the school of Science and Math in Durham, N.C. He also created brochures on his native West African Country (Sierra-Leone) and gave a brief summary on its cultures to the Upward Bound students. In the spring of 2003 he did a Scholarly Presentation at St-Augustine's college on Electronic Circuits and Real-life Application Problems in Linear Algebra.

# ACKNOWLEDGMENTS

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# 1. Introduction

The classification of nilpotent Lie algebra has a long history, beginning with Umlauf, a student of Engel, in 1891. He found there were only finitely many non-isomorphic nilpotent Lie algebras of dimension less than or equal to 6. For dimensions greater than 6, there are infinitely many non-isomorphic Lie algebras [1], [9]. In recent years lists of Lie algebras of dimensions 7 and 8 have been studied [9]. Unfortunately, these lists contain errors as noted by [8], the list of Lie algebra of dimension 7 have been corrected more recently as noted in a complete list of nilpotent Lie Algebras of dimension 7 found in [9], dimensions of 8 and higher have been harder to classify.

Authors have turned their attentions to special classes of nilpotent Lie algebras such as two-step nilpotent [9], and filiform Lie algebras [8]. In particular filiform lie algebras are classified up to dimensions eleven [8]. These techniques have not worked well in higher dimensions. A recent technique to classify nilpotent lie algebras is to use the dimension of the multiplier of L.(Note: We will define The Multiplier in the next section). In particular, to find those algebras whose multipliers have dimension close to the maximum, we call this invariant $t(L)$. The philosophy of this work is to collect the algebras for values of $t(L)$ and use those to find the next case. This is done for a given $t(L)$ by considering $H=L/Z(L)$ which has $t(H) \leq t(L)$ and is filiform. From the previous cases found in table 22.1, we choose candidates for H and compute the central extensions to get candidates for L. Algebras with $t(L) \leq 8$ have been classified in [10]. It's the purpose of this work to use this technique on filiform lie algebras. Note: All algebras in this work will be taken over any field whereas in previous works, they have been taken over the field of real and complex numbers.

**2. Preliminaries.** Now we will  define The Multiplier of L:

Consider all Lie algebras H such that $H/Z(H) \simeq L$. H is called a central extension of L . We want to consider the H of largest dimension. Without further restrictions H can be infinite, take $L \oplus$ (infinite dimensional abelian Lie algebra) so we demand that $Z(H) \subseteq H^2$. An H of largest dimension is called a cover and $Z(H)$ is called the multiplier of L, denote it by $M(L)$. We see that $M(L)$ is abelian since it is the center of some H.

Note: 1. All covers of L are isomorphic (Peggy Batten)

      2. If L is abelian then dim $M(L) = \frac{1}{2}n(n-1)$ where n=dim L.

      3. dim $M(L) \leq \frac{1}{2}n(n-1) \; \forall \; L$.

We let  $t = t(L) = \frac{1}{2}n(n-1) - \dim M(L)$.


**Definition.** Let L be a nilpotent Lie algebra with lower central series

$L^1 = L, L^2 = [L^1, L], \dots , L^j = [L^{j-1}, L] \dots$

with $L^s = 0$, then L is called filiform if dim $L^j/L^{j+1} = 1$

For j = 2,…, s-1, and dim $L/L^2 = 2$.

Note that L is the maximal length a nilpotent Lie algebra can have.

Also note the upper central series

$0 = Z_0 \subset Z_1 \subset \dots \subset Z_{s-1} \subset Z_s = L$ has dimension $Z_{j+1}/Z_j = 1$

when j=0,…, s-2 and dim $L/Z_{s-1} = 2$.

In a nilpotent Lie algebra $Z_1 \cap B \neq O$ for any

ideal B≠0, hence $Z_1 \subseteq B$ since $O \not\subseteq Z_1 \cap B \subseteq Z_1$ and dim $Z_1 = 1$

**The following are examples of filiform lie algebra:**

**Ex 1**

Let $L = (x,y,z)$ where $[x,y] = z$

$$L^2 = (z), \ L^3 = [L^2,L] = 0, \quad Z(L) = (z) \Rightarrow L \supset L^2 \supset L^3 = 0$$

L is H(1) the three dimensional Heisenberg.

**Ex 2**

$L = (x,y,z,c)$ where $[x,y] = z, \ [x,z] = c$

$$L^2 = (z,c), \ L^3 = [L^2,L] = c, \quad L^4 = [L^3, L] = 0 \Rightarrow Z(L) = (c)$$
$$\Rightarrow L \supset L^2 \supset L^3 \supset L^4 = 0$$

**Ex 3**

$L = (x,y,z,c,r)$ where $[x,y] = z, \ [x,z] = c, \ [x,c] = r, \ [y,z] = r$

$$L^2 = (z,c,r), \ L^3 = (c,r), \ L^4 = (r), \ L^5 = 0 \Rightarrow L^1 \supset L^2 \supset L^3 \supset L^4 \supset L^5 = 0$$

**Ex 4**

The following is not filiform. Tn = strictly upper triangular matrices of size $n \geq 4$. Since

$Dim(T/T^2) = n-1$.

We will use the technique in [10] on filiform Lie algebra to classify Lie

Algebras with small t(L). Recall t=t(L).

We will begin by introducing some inequalities necessary to our method.

**Proposition 0**  Let L be a nilpotent Lie algebra of dimension n. Then $t \leq \frac{1}{2} n(n-1)$

Proof: Since $\dim M(L) \leq \frac{1}{2} n(n-1)$, $t = \frac{1}{2} n(n-1) - \dim M(L)$ this implies $t \leq \frac{1}{2} n(n-1)$  ∎

**Proposition 1**  Let L be a nilpotent Lie algebra of dimension n.

Then $\dim L^2 + \dim M(L) \leq \frac{1}{2} n(n-1)$

*Proof:*  see [10]

**Proposition 2**   Let L be a nilpotent Lie algebra with $Z \subseteq Z(L) \cap L^2$ and dim $Z = 1$

then dim $M(L) + 1 \leq$ dim $M(L/Z) +$ dim $(L/L^2)$

*Proof:*  see [10]


**Proposition 3**  Let L be filiform Lie algebra of dimension n.

Let $K = Z(L)$ and $H = L/K$   then $t(H) +$ dim $L^2 \leq t(L)$

*Proof:*    Since L is filiform, dim $K = 1$, we know that $t(L) = \frac{1}{2}n(n-1) -$ dim $M(L)$

$\Rightarrow$ dim $M(L) = \frac{1}{2}n(n-1) - t(L)$

now substitute for dim $M(L)$ in Prop 2 [ie dim $M(L) + 1 \leq$ dim $M(L/Z) +$ dim $(L/L^2)$]

$\Rightarrow \frac{1}{2}n(n-1) - t(L) + 1 \leq$ dim $M(L/Z) +$ dim $L -$ dim $L^2$

$\Rightarrow \frac{1}{2}n(n-1) - t(L) + 1 \leq$ dim $M(H) + n -$ dim $L^2$ $\qquad\qquad$ (i)

note:  dim $M(H) = M(L) - 1$. Thus if dim $M(L) = \frac{1}{2}n(n-1) - t(L)$

then dim $M(H) = \frac{1}{2}(n-1)(n-2) - t(H)$ $\qquad\qquad$ (ii)

now substitute (ii) in (i)

$\Rightarrow \frac{1}{2}n(n-1) - t(L) + 1 \leq \frac{1}{2}(n-1)(n-2) - t(H) + n -$ dim $L^2$

$\Rightarrow t(H) +$ dim $L^2 \leq -\frac{1}{2}n(n-1) + t(L) - 1 + \frac{1}{2}(n-1)(n-2) + n$

$\qquad\qquad = -\frac{1}{2}n^2 + \frac{1}{2}n + t(L) - 1 + \frac{1}{2}[n^2 - 3n + 2] + n$

$\qquad\qquad = -\frac{1}{2}n^2 + \frac{1}{2}n + t(L) - 1 + \frac{1}{2}n^2 - 3/2n + 1 + n$

$\qquad\qquad = \frac{1}{2}n + t(L) - \frac{1}{2}n = t(L)$

thus $t(H) +$ dim $L^2 \leq t(L)$  ■


**Proposition 4**  Let L be nilpotent.  Then dim $L^2 \leq t(L)$

*Proof:*  This follows immediately from Proposition 3.

From Pete Hardy's work and using propositions 1, 2 and 3,

we arrive at the following theorems.

**Theorem 1**  $t(L) = 0$  IFF L is abelian.

*Proof:* see [10]

Note: L is abelian and filiform implies dim L $= 2$.

**Theorem 2**  $t(L) = 1$ IFF $L = H(1)$.

*Proof;* see [10]

Note:  that H(1)  is indeed filiform Lie algebra . See example 1

**Theorem 3**  There are no filiform Lie algebras with $t(L)=2$

*Proof:* see [10]

We will take advantage of the following new results.

**Theorem 4**  Let L be a Lie algebra, then  $c^2 + c \leq 2t$

where  $c = \dim L^2$

Proof: In order to prove theorem 4 we need the following result.

**Theorem 5** Let L be a nilpotent Lie algebra with $H/Z(H) \simeq L$. Then $\dim(H^2 \cap Z(H)) \leq$ $\dim M(L/L^2) + \dim L^2[d(L/Z(L)-1]$ where $d(X)=\dim(X/X^2)$ thus $d(L/Z(L))=\dim\{(L/Z(L))/(L/Z(L))^2\}$.

Proof: In general if $H/Z(H) \simeq L$ but $Z(H) \not\subseteq H^2$, $H= H^* \oplus S$ where S is a central ideal and $Z(H)= Z(H^*) \oplus S$, $H^2 \cap Z(H) = H^{*2} \cap Z(H^*)$ and $L \simeq H/Z(H) = (H^* \oplus S)/(Z(H^*) \oplus S) \simeq$ $H^*/Z(H^*)$. Hence we assume that $H^2 \supseteq Z(H)$

We break the process into a collection of Lemmas. Let H be a nilpotent Lie algebra with

$Z_1= Z_1(H) =\{x \in H/ [x,y]=0 \; \forall \; y \in H\}$

$Z_2= Z_2(H) =\{x \in H/[x,y] \in Z_1(H) \; \forall \; y \in H\}$, $Z_2(H)/ Z(H) \simeq Z(L)$

Since H is nilpotent, if H is not abelian then $0 \subset Z_1 \subset Z_2$ and $0 \neq Z_1 \neq Z_2$.

**Lemma 1**. Let $y \in Z_2$, $y \notin Z_1$, then $ad_y$ is a homomorphism from H onto N where $N \subseteq Z_1$.

Proof: Since $ad_y$ is linear we need to prove that $ad_y([x, z]) = [ad_y(x), ad_y(z)]$

Note $y \in Z_2$ implies $ad_y(h)=[y, h] \in Z_1$ hence $N \subseteq Z_1$

Thus $ad_y([x, z])=[y, [x, z]]= -[x, [y, z]] - [z_1, [x,y]]=0$

Also $ad_y(x)=[y, x] \in Z_1$ and $ad_y(z)=[y, z] \in Z_1$ which implies $[ad_y(x), ad_y(z)]=0$ ∎

**Lemma 2**. $[H^2, Z_2]=0$

Proof: Need to show that if $[x, y] \in H^2$ and $z \in Z_2$ then $[[x, y], z] = 0$

Note: $H^2= <[x, y]>$ where $x, y \in H$

now consider $[[x, y], z] +[[y, z], x] + [[z, x], y] = 0$

this implies $[[x, y], z] = -[[y, z], x] - [[z, x], y] = 0$ since $[y, z] \in Z_1$ and $[z, x] \in Z_1$. ■

**Lemma 3**. Suppose $y \in Z_2 \cap H^2$ , $y \notin Z_1$, then $Z_2 \subseteq \mathrm{Ker}\ ad_y$

Proof: Let $z \in Z_2$ we need to show that $z \in \mathrm{Ker}\ ad_y$ which implies that $ad_y(z) = [y, z] = 0$

$\forall\ y \in H^2 \cap Z_2$.

Since $z \in Z_2$ $ad_y(z) = [y, z]$, $[H^2, Z_2] = 0$ ■

By Lemma 3 there is an induced homomorphism $\sigma: H/Z_2(H) \rightarrow N$ given by

$\sigma(\ell + Z_2(H)) = ad_y(\ell) = [y, \ell]$. Now the theorem holds if $k(L) \equiv \dim L^2 = 0$ for then

$H^2 \subseteq Z(H)$ i.e., $Z(H) \cap H^2 = H^2$. We proceed by induction: Suppose the result holds for all

nilpotent Lie algebras K with $\dim K^2 < \dim L^2$ and $\dim(K/K^2) \leq \dim(L/L^2)$.

**Lemma 4**. $\mathrm{Dim}M(K/K^2) \leq \dim M(L/L^2)$

Proof: Since $\dim K^2 < \dim L^2$ and $\dim(K/K^2) \leq \dim(L/L^2)$ letting $\dim(K/K^2) = n$ and

$\dim(L/L^2) = m$ it follows that $\dim M(K/K^2) = \frac{1}{2}n(n-1) \leq \dim M(L/L^2) = $

$\frac{1}{2}m(m-1)$. ■

Now let y and N be as before (see Lemma 1). Since $L/Z(L) \simeq H/Z_2(H)$ and $\sigma$ is a

homomorphism from $H/Z_2(H)$ onto N, we consider $\sigma$ also to be a homomorphism from

$L/Z(L)$ onto N therefore $\sigma(L/Z(L))^2 = N^2 = 0$ therefore $(L/Z(L))^2 \subseteq \mathrm{Ker}\ (\sigma)$.

**Lemma 5**. $\dim N \leq \dim(L/Z(L)) - \dim(L/Z(L))^2 = \dim\{(L/Z(L))/(L/Z(L))^2\}$

Proof: Since $\sigma$ maps $L/Z(L)$ onto N it follows that $\dim(L/Z(L)) = \dim N + \dim \mathrm{Ker}(\sigma) \geq$

dim N + dim(L/Z(L))$^2$, since $\{(L/Z(L))^2 \leq$ Ker $(\sigma).\}$ ∎

**Lemma 6**. Let E=H/N then, (i) y+ N $\subseteq$ Z(E)  and (ii) y + N $\notin$ Z(H)/N.

(iii) dim Z(H)/N⊊dim(Z(H/N))

Proof: (i) This follows since [x + N, y + N] = [x, y] + N = ad$_y$(x) + N = N

    (Since ad$_y$(x) $\in$ N i.e. N = image of  ad$_y$)

    (ii) Recall that y $\notin$ Z(H) (=Z$_1$)

    Suppose that y + N $\in$ Z(H)/N. This implies that y + N = x + N,  for some

    x $\in$Z(H) .  Therefore  y = x + n for some n $\in$ N. Thus y $\in$ Z(H), a  contradiction.

    (iii)Since Z(H)/N is always contained in Z(H/N), we get Z(H)/N⊊

 Z(H/N) by (i), (ii) of Lemma 6  ∎

 Let A $\equiv$ E /Z(E) then L $\simeq$ H/Z(H) $\simeq$ (H/N)/(Z(H)/N) $\rightarrow$ (H/N)/(Z(H/N)) = E/Z(E)=A

Hence  A  is the homomorphic image of L and dim A  < dim L by the last proof. Hence

A/(A)$^2$ is a homomorphic image of L/L$^2$ and  A/Z(A) is a homomorphic image of L/Z(L)

and (A)$^2$ is a homomorphic image of L$^2$. In fact we use induction on A. Note this

discussion yields:

**Lemma 7**.  dim(A/ (A)$^2$) $\leq$ dim(L/L$^2$) and dim (A)$^2 \leq$ dim L$^2$

Proof : Recall y $\in$ Z$_2 \cap$ H$^2$ , y $\notin$Z$_1$ and y + N $\subseteq$  Z(H/N) since N = image ad$_y$. So in the

homomorphisms  L$\simeq$ H/Z(H) $\rightarrow$ A , the image of y in L is in L$^2$ but it is not 0 but it's

image in A is 0. Hence the homomorphism carries L$^2$ onto A$^2$ and it is not

injective. Thus dim L$^2 >$ dim A$^2$. By induction dim(E$^2 \cap$ Z(E))$\leq$ dimM(A/A$^2$) + dim A$^2$

dim[(A /Z(A))/(A/Z(A))$^2$ – 1].

By a previous " proof " $\dim M(A/A^2) \le \dim M(L/L^2)$ and we have seen $\dim (A)^2 < \dim L^2$.

Since $A/Z(A)$ is the homomorphic image of L,

$\dim[(A/Z(A))/(A/Z(A))^2] \le \dim[(L/Z(L))/(L/Z(L))^2]$. Substitution yields

$\dim [(E)^2 \cap Z(E)] \le \dim M(L/L^2) + [\dim(L^2) - 1] \{\dim[(L/Z(L))/(L/Z(L))^2 - 1]\}$

Now $y + N \in Z(E) \cap (E)^2$ and $y+N \notin H^2 \cap Z(H)/N$ and since the latter

is always contained in the former, $\dim H^2 \cap Z(H)/N < \dim [(E)^2 \cap$

$Z(E)$. Thus $\dim[(E)^2 \cap Z(E)] \le \dim N + \dim((E)^2 \cap Z(E)) - 1$

$\le \dim[(L/Z(L))/(L/Z(L))^2] + \dim M(L/L^2) + [\dim(L^2) - 1]$

$[\dim((L/Z(L))/(L/Z(L))^2 - 1] - 1$

$= \dim M(L/L^2) - \dim L^2 + \dim L^2 \dim[(L/Z(L))/(L/Z(L))^2 - 1]$

$= \dim M(L/L^2) + \dim L^2[\dim[(L/Z(L))/(L/Z(L))^2]$ and the result is shown.

( Note that we have used the fact that $ad_y$ is a homomorphism from H onto N and $N \subseteq Z_1$

by lemma 1 as a consequence N is abelian and by lemma 3 $Z_2 \subseteq \text{Ker } ad_y$ . Consequently

we have an induced homomorphism from $H/Z_2(H) \rightarrow N$ and an induced homomorphism

from $(H/Z_2(H))/(H/Z_2(H)^2) \rightarrow N/N^2 \simeq N$. Therefore $\dim N \le \dim[(L/Z(L))/(L/Z(L))^2]$.) $\blacksquare$


When working with M(L), $H^2 \supseteq Z(H)$ so the formula is $\dim M(L/L^2) = \dim H^2 \cap Z(H) \le$

$\dim M(L/L^2) + \dim L^2[ \dim[(L/Z(L))/(L/Z(L))^2]$ or $\frac{1}{2}n(n-1) - t \le \frac{1}{2} d(d-1) + c[\delta - 1]$

where $c = \dim L^2$, $d = \dim(L/L^2)$ $(c + d=n)$ $\dim L = n$ and $\delta = \dim[(L/Z(L))/(L/Z(L))^2]$.


We can now prove theorem 4.

Proof:  Note: $\dim L^2 = c$, $\dim L = n$ where $L^2 \supseteq Z(L)$ . For Lie algebra $n = c + d$

$\delta = \dim(L/Z(L)) - \dim(L^2/Z(L)) = \dim(L/L^2)$ we also know from the definition of $t = t(L)$

that $\dim M(L) = \frac{1}{2} n(n-1) - t$ and now substituting for $\dim M(L)$ in the inequality

$\dim M(L) \le \dim M(L/L^2) + \dim L^2[d(L/Z(L)) -1]$

$\Rightarrow$ ½ n(n-1) – t $\le$ ½d(d - 1) + c[d – 1]

$\Rightarrow$ ½ (c + d)[c + d –1] – t $\le$ ½ d(d-1)+ c[d – 1]

$\Rightarrow$ ½[c² + cd – c + dc + d² – d] – t $\le$ ½ [d² – d ] + (cd – c)

$\Rightarrow$ c² – c – 2t $\le$ -2c

$\Rightarrow$ c² + c $\le$ 2t  ∎

Note theorem 5 has been proved in [17 ] by cohomological methods.


Henceforth, L will be a filiform Lie algebra, we define

c = dim $L^2$,  t = t(L) and t* = t(H),  H = L/Z(L)

and from proposition 3,    t* + c $\le$ t .

We will now find the filiform Lie algebras for increasing values of t(L).

## 3. t(L) = 3

This case has been computed by Hardy. We will do the work to check that our method is consistent with Hardy's. By proposition 4 (dim $L^2 \leq 3$). If dim $L^2 = 3$ then

dim$H^2 = 2$ {since $L^2 \simeq (H/Z(H))^2 = (H^2 + Z(H))/Z(H) \simeq H^2/Z(H)$} ( and t(H )= 0 by proposition 3 (t(H) + dim $L^2 \leq 3$) which implies H is abelian, a contradiction. If dim $L^2 = 1$, then $L^2 = Z(L)$ and L is Heisenberg. But the only Heisenberg that is filiform is the three dimensional one of example 1. It has t(L) =1 (by Theorem 2), a contradiction. If dim $L^2 = 0$, then t(L) = 0, another contradiction. Hence, consider dim $L^2 = 2$. Then dim $H^2 = 1$ and t(H) $\leq 1$. If t(H) =0 , then H is abelian which implies dim $H^2 = 0$, a contradiction (since dim $H^2 = 1$). If t(H) = 1, then by Theorem 2, H = H(1) and L is a central extension of a one dimensional ideal K by H(1) and K$\subseteq L^2$.

A basis for L is {x,y,z,r} with $[x,y] = z + \alpha_1 r$, $[x,z] = \alpha_2 r$, $[y,z] = \alpha_3 r$ and r $\epsilon$ Z(L). Either $\alpha_2$ or $\alpha_3$ must be non-zero, for otherwise dim $L^2 = 1$ a Contradiction. WOLOG, let $\alpha_2 \neq 0$. Then letting $z' = z + \alpha_1 r$,

$y' = y - \alpha_3/\alpha_2 x$, $r' = \alpha_2 r$ and relabelling yields $[x,y] = z$, $[x,z] = r$ and $[y,z] = 0$. Now to compute the multiplier start with

$[x,y]=z + s_1$   $[y,z]=s_3$   $[y,r]=s_5$

$[x,z]=r + s_2$   $[x,r]=s_4$   $[z,r]=s_6$

where $s_1, s_2, \ldots, s_6$ generate M(L). Next we perform a change of variables i.e. $[x,y]=z + s_1 = z^*$ which implies $[x,y]=z^*$ similarly

$[x,z]=r + s_2 = r^*$ and $[y,z]=s_3$ ,$[x,r]=s_4$   $[y,r]=s_5$ and $[z,r]=s_6$

thus $s_1 = s_2 = 0$. Next we use Jacobi on all possible triples.

$$[[x,y],z] + [[y,z],x] + [[[z,x],y] = 0$$

$$\Rightarrow \quad 0 + [s_3,x] + s_5 = 0$$

$$\Rightarrow \quad s_5 = 0.$$

Similarly, $s_6 = 0$. Therefore we get $s_1 = s_2 = s_5 = s_6 = 0$. Now going back

to the original algebra which was $(s_1, s_2, s_3, s_4, s_5, s_6)$ now becomes

$(s_3, s_4) = 2 = \dim M(L)$. These are free variables and we can vary them and

have different central extensions since $t(L) = \frac{1}{2}n(n-1) - \dim M(L)$,

it follows that $t(L) = 6 - 2 = 4$. Note n=4 since the basis for

L={x,y,z,r} we adopt the notation of Hardy and denote this algebra by L(3,4,1,4)

where the first digit designates it came from the $t(L) = 3$ case, the

second is the dimension of L, the third is dim Z(L) and the last is

t(L). Thus L(3,4,1,4) is the filiform Lie algebra with basis

{x,y,z,r} and non-zero multiplication $[x,y] = z$, $[x,z] = r$.

We will use this algebra when $t(L) = 4$.

**Theorem 6**  There are no filiform Lie algebras with $t(L) = 3$.

## 4.  t(L) = 4

By proposition 4 dim $L^2 \leq 4$ and if dim $L^2 = 4$, then dim $H^2 = 3$ and t (H) = 0,

(by proposition 3) which implies H is abelian, a contradiction.  If dim $L^2 = 3$,

then dim $H^2 = 2$ and t (H) $\leq$ 1 (by proposition 3).There are no such filiform

algebras by previous work(i.e t(H) = 0, t(H) = 1).  If dim $L^2 = 1$, then $L^2 = Z$ (L) and L is

Heisenberg since L is filiform, L is the three dimensional heisenberg. This contradicts

theorem 2.  If dim $L^2 = 0$, then L is abelian and t(L) = 0, a contradiction.  The only

remaining possibility is dim $L^2 = 2$,  then dim $H^2 = 1$ and t(H) $\leq$ 2, by proposition 3.

By previous work, H = H(1). The case H= H(1) has been computed in the last section and

it was found that  L = L (3,4,1,4), which satisfies t(L) = 4.


**Theorem 7** Let L be filiform with t(L) = 4. Then L = L(3,4,1,4).

## 5. t(L) = 5

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$, thus if

$t = t(L) = 5$ then $c^2 + c \leq 10$ which implies that $c \leq 2$ therefore $c = 0, 1$ or $2$. If

$\dim L^2 = 0 = c$, then $t(L) = 0$ and L is abelian, a contradiction (see Theorem 1). If

$\dim L^2 = 1 = c$, then $L^2 = Z(L)$ and L is Heisenberg(see Theorem 2) but the only

Heisenberg Lie algebra that is filiform is the 3 dimensional one of example 1. It has

$t(L) = 1$, a contradiction. If $\dim L^2 = 2$, then $\dim H^2 = 1$ and $t(H) \leq 3$. i.e $t(H) = 0, 1, 2$

or 3. If $t(H) = 1$, then $H = H(1)$, $L = L(3,4,1,4)$ and $t(L) = 4$, a contradiction. If $t(H) = 2$,

then there is no filiform Lie algebra (Theorem 3). Similarly if $t(H) = 3$ there is no

filiform Lie algebra(Thm 6).

If $t(H) = 0$, then H is abelian , a contradiction.(Theorem 1).


**Theorem 8** There are no filiform Lie algebras with $t(L) = 5$

## 6. t(L) = 6

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$, thus if

$t = t(L) = 6$ then $c^2 + c \leq 12$ which implies that $c \leq 3$ therefore $c = 0,1,2$ or 3. Also

from Proposition 0{i.e $t \leq \frac{1}{2}(n)(n-1)$}and since n=c+2 for filiform we get

$t \leq \frac{1}{2}(c+2)(c+1)$. If c=0, then t=6$\leq \frac{1}{2}(2)(1)$=1, a contradiction. If c=1, then

t=6$\leq \frac{1}{2}(3)(2)$=3, a contradiction. If c =2, then t=6$\leq \frac{1}{2}(4)(3)$=6, and

if c=3 t=6$\leq \frac{1}{2}(5)(4)$=10. Thus c=2, 3 are the only cases that satisfy the

inequality(Proposition 0), therefore c$\geq$2.

If $c = \dim L^2 = 2$, then $\dim H^2$=1 and t(H) $\leq$ 4(Prop 3) i.e t(H) = 0,1,2,3 or 4. If

t(H)=0 then H is abelian, a contradiction(Theorem 1). If t(H) = 1, then H=H(1),

H=L(3,4,1,4) and t(L) = 4, a contradiction. By previous work there are no

filiform Lie algebras for t(H)=2 or 3(see theorems 3 and 4 respectively).

If t(H) = 4, then H=L(3,4,1,4) a contradiction(since the dimension of the derived

algebra is 2 not 1). If c=3=$\dim L^2$ then $\dim H^2$ =2 and t(H)$\leq$3(Prop 3)i.e t(H)=0,1,2,3.

If t(H) =0 the H is abelian, a contradiction(Theorem 1). If t(H) =1 then H=H(1),

L=L(3,4,1,4) and t(L)=4, a contradiction. By previous work, if t(H)=2 or 3 there are

no filiform Lie algebras(Theorems 3 ,6). If t(H)=4, then Proposition 3 is contradicted.

**Theorem 9** There are no filiform Lie algebras with t(L)= 6

## 7. t(L) = 7

Again by Theorem 4, $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$, thus if $t = t(L) = 7$,

then $c^2 + c \leq 14$ which implies that $c \leq 3$ therefore $c = 0, 1, 2$ or $3$. Again by

Proposition 0 {i.e. $t \leq \frac{1}{2}(n)(n-1)$} and since $n = c+2$ for filiform we get $t \leq \frac{1}{2}(c+2)(c+1)$.

Similarly if $c=0$, then $t = 7 \leq \frac{1}{2}(2)(1) = 1$, a contradiction. If $c=1$, then $t = 7 \leq \frac{1}{2}(3)(2) = 3$

another contradiction. If $c=2$ then $t = 7 \leq \frac{1}{2}(4)(3) = 6$ another contradiction. Finally if

$c=3$, then $t = 7 \leq \frac{1}{2}(5)(4) = 10$, a true statement, therefore the inequality is true only for

$c=3$, thus if $c = \dim L^2 = 3$, then $\dim H^2 = 2$ and $t(H) \leq 4$ (proposition 3) i.e. $t(H) = 0, 1, 2, 3$

or 4. If $t(H) = 0$, then H is abelian, a contradiction. If $t(H) = 1$, then $H = H(1)$,

$L = L(3,4,1,4)$ and $t(L) = 4$, a contradiction. Again by previous work, if $t(H) = 2$ or $3$

there are no filiform Lie algebras (Theorems 3 and 4). If $t(H) = 4$, then $H = L(3,4,1,4)$.

(Henceforth $t(H) = 4$ and $\dim L^2 = 3$).

Now consider $H = L(3,4,1,4)$. Then L can be described generally by the basis

$\{x, y, z, c, r\}$ and multiplication $[x,y] = z$, $[x,z] = c$, $[y,z] = \alpha_3 r$, $[x,c] = \alpha_4 r$, $[y,c] = \alpha_5 r$ and

$[z,c] = \alpha_6 r$, $r \in Z(L)$. The Jacobi identity shows that $\alpha_5 = \alpha_6 = 0$.

Now, assume $c \notin Z(L)$, which implies $\alpha_4 \neq 0$. If $\alpha_3 = 0$, then replacing $r$ by $(1/\alpha_4)r$ and

relabelling yields non-zero multiplication $[x, y] = z$, $[x, z] = c$, $[x, c] = r$. To compute the

multiplier, start with

$[x,y] = z + s_1 \quad [y,c] = s_5 \quad [z,r] = s_9$

$[x,z] = c + s_2 \quad [z,c] = s_6 \quad [s,r] = s_{10}$

$[y,z] = s_3 \quad\quad [x,r] = s_7$

$[x,c] = r + s_4 \quad [y,r] = s_8$

where $s_1, \ldots, s_{10}$ generate the multiplier. By relabelling we get $s_1 = s_2 = s_4 = 0$.

Using Jacobi identity on all triples gives $s_6 + s_8 = 0$ and $s_5 = s_9 = s_{10} = 0$. Hence the

multiplier has basis $\{s_3, s_6, s_7\}$, dim $M(L) = 3$ and $t(L) = 7$. This filiform algebra

satisfies the requirements and we call it $L(7,5,1,7)$. For the record, it has

basis $\{x,y,z,c,r\}$ and non-zero multiplication $[x, y] = z$, $[x, z] = c$, $[x, c] = r$. If $\alpha_3 \neq 0$,

then replacing r by $(1/\alpha_3)$r and relabelling yields non-zero multiplication

$\quad$ $[x, y] = z$, $[x, z] = c$, $[y, z] = r$, $[x, c] = \beta r$, $\beta = (\alpha_4/\alpha_3) \neq 0$. Then letting $y^* = \beta y$,

$z^* = \beta z$, $c^* = \beta c$ and $r^* = \beta^2 r$ and relabelling yields non-zero multiplication

$[x, y] = z$, $[x, z] = c$, $[y, z] = r$, $[x, c] = r$

To compute the multiplier, start with

$\quad\quad$ $[x, y] = z + s_1$ $\quad$ $[y, c] = s_5$ $\quad\quad$ $[z, r] = s_9$

$\quad\quad$ $[x, z] = c + s_2$ $\quad$ $[z, c] = s_6$ $\quad\quad$ $[s, r] = s_{10}$

$\quad\quad$ $[y, z] = r + s_3$ $\quad$ $[x, r] = s_7$

$\quad\quad$ $[x, c] = r + s_4$ $\quad$ $[y, r] = s_8$

where $s_1, \ldots, s_{10}$ generate the multiplier.

By relabelling we get $s_1 = s_2 = s_3 = 0$. Using the Jacobi identity on all possible triple gives

$s_5 - s_7 = 0$, $s_6 + s_8 = 0$ and $s_9 = s_{10} = 0$. Hence, the multiplier has basis $\{s_4, s, s_6\}$,

dim $M(L) = 3$ and $t(L) = 7$. This filiform algebra satisfies the requirement and call it

$L'(7,5,1,7)$. For the record, it has basis $\{x,y,z,c,r\}$ and non-zero multiplication

$\quad$ $[x,y] = z$, $[x,z] = c$, $[y,z] = r$, $[x,c] = r$.

**Theorem 10** Let L be filiform with $t(L) = 7$ then $L = L(7,5,1,7)$ or $L = L'(7,5,1,7)$

## 8. t(L) = 8

Again by Theorem 4, $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t=t(L)$, thus if $t=t(L)=8$ then

$c^2 + c \leq 16$ which implies that $c \leq 3$ therefore $c=0,1,2$ or $3$ but again by proposition 0

we have $t \leq \frac{1}{2}(c+2)(c+1)$ which implies that if $c=0$, then $t=8 \leq \frac{1}{2}(2)(1)=1$, a

contradiction. If $c=1$, then $t=8 \leq \frac{1}{2}(3)(2)=3$, a contradiction. If $c=2$, then

$t=8 \leq \frac{1}{2}(4)(3)=6$, a contradiction. If $c=3$, then $t=8 \leq \frac{1}{2}(5)(4)=10$, a true statement,

therefore $c=3$ is the only case that satisfies the inequality. Thus if $c=\dim L^2=3$, then

$\dim H^2=2$ and $t(H) \leq 5$ (Prop3). Therefore $t(H)=0,1,2,3,4$ or $5$. If $t(H)=0$ then H is

abelian, a contradiction (Theorem 1). If $t(H)=1$, then $H=H(1)$, $L=L(3,4,1,4)$ and

$t(L)=4$, a contradiction. Again by previous work, if $t(H)=2$ or $3$ there are no filiform

Lie algebras(Theorems 3 and 6). If $t(H)=4$, then $H=L(3,4,1,4)$, a contradiction since we

have found that $t(L) = 7$. If $t(H) = 5$ there are no filiform Lie algebras(Theorem 6).


**Theorem 11**   There are no filiform Lie algebras with $t(L)=8$

## 9. t(L)=9

By Theorem 4, $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$, thus if $t = t(L) = 9$ then $c^2 + c \leq 18$ which implies that $c \leq 3$ therefore $c = 0, 1, 2$ or $3$ but by proposition 0, we have $9 \leq \frac{1}{2}(c+2)(c+1)$ and by solving this inequality we know that $c = 3$ is the only case that satisfies the inequality. Thus if $c = \dim L^2 = 3$, then $\dim H^2 = 2$ and $t(H) \leq 6$(prop 3). By previous work there are no filiform Lie algebras if $t(H) = 2, 3, 5$ or $6$(theorems 3,6,8 and 9). If $t(H) = 0$, then H is abelian, a contradiction(theorem 1). If $t(H) = 1$, then $H = H(1)$, $L = L(3,4,1,4)$ and $t(L) = 4$, a contradiction. If $t(H) = 4$, then $H = L(3,4,1,4)$, a contradiction, since from the $t(L) = 7$ case we know that $L = L(7,5,1,7)$ or $L = L'(7,5,1,7)$ which have $t(L) = 7$ but here $t(L) = 9$.

**Theorem 12**   There are no filiform Lie algebras with $t(L) = 9$.

## 10. t(L)= 10

Similarly, by Theorem 4, if t=t(L)=10 then $c^2+c \leq 20$ which implies that $c \leq 4$,

therefore c=0,1,2,3 or 4 but by proposition 0, we have $10 \leq \frac{1}{2}(c+2)(c+1)$

and by solving this inequality we know that c=3 and c=4 are the two cases that satisfy

the inequality.Thus if c=dim$L^2$=3, then dim $H^2$=2 and t(H)$\leq$7(prop 3). Again by

previous work, there are no filiform Lie algebras if t(H)= 2,3,5 or 6(Theorems 3,6,8,9

respectively). If t(H)=0, then H is abelian, a contradiction(Theorem 1). If t(H)=1, then

H=H(1), L=L(3,4,1,4) and t(L)=4, a contradiction since from the t(L)=7 case we

know that L=L(7,5,1,7) or L=L'(7,5,1,7) which have t(L)=7 but here t(L)=10. If

t(H)=4, then H=L(3,4,1,4), a contradiction since from the t(L)=7 case we know that

L=L(7,5,1,7) or L'(7,5,1,7) which have t(L)=7 but here t(L)=10. If  t(H)=7, then

H=L(7,5,1,7) or L'(7,5,1,7), a contradiction since the algebras H=L(7,5,1,7) or

L'(7,5,1,7) were found to have t(L)=11 and here t(L)= t(L)=10.

Also if c=dim$L^2$=4, then dim $H^2$=3 and t(L)$\leq$6(prop 3).

Again by previous work, there are no filiform Lie algebras if t(H)=2,3,5 or 6. If

t(H)=0, then H is abelian, a contradiction(Theorem1). If t(H)=1, then H=H(1) ,

L=L(3,4,1,4) and t(L)= 4, a contradiction. If t(H)=4, then H=L(3,4,1,4), a

Contradiction by similar argument above.

**Theorem 13** There are no filiform Lie algebras with t(L)= 10.

# 11. t(L)= 11

By Theorem 4 , if t=11 then $c^2+ c \leq 22$ which implies that $c \leq 4$ therefore c=0,1,2,3 or 4, but by proposition 0, we have $11 \leq \frac{1}{2}(c+2)(c+1)$ and by solving this inequality we know that c=4 is the only case that satisfies the inequality. Thus if $c=\dim L^2=4$, then $\dim H^2=3$ and $t(H) \leq 7$(prop 3). By looking at table 22.1, we found that the only time these parameters are satisfied was when t(L)=7.Again by table 22.1, there are no filiform Lie algebras if t(H)=2,3,5 or 6. If t(H)=0, then H is abelian, a contradiction. If t(H) =1, then H=H(1), L=L(3,4,1,4) and t(L)= 4, a contradiction. If t(H)= 7, then L=L(7,5,1,7) or L'(7,5,1,7).

We will consider these two cases. First let's consider Case 1 where H=L(7,5,1,7). {We will later consider Case 2 where H=L'(7,5,1,7), since it has different multiplication table}.This L can be described  generally by the basis {x, y, z, c, r, t} and multiplication

[x,y]=z, [x,z]=c, [x,c]=r,  [x,r]=$\alpha_4$t, [y,z]=$\alpha_5$t, [y,c]= $\alpha_6$t,  [y,r]= $\alpha_7$t, [z,c]= $\alpha_8$t

[z,r]= $\alpha_9$t, [c,r ]=$\alpha_{10}$t,  and t$\in$Z(L). The Jacobi identity shows that $\alpha_6$ =$\alpha_9$ =$\alpha_{10}$=0 and

$\alpha_8$ = - $\alpha_7$.

Relabelling yields the non zero multiplication [x,y]=z, [x, z]=c, [x, c]=r,  [x, r]= $\alpha_4$t, [y, z]=$\alpha_5$t, [y, c]= 0, [y, r]= $\alpha_7$t, [z, c]= -$\alpha_7$t, [z, r]= 0, [c, r]= 0 and  t$\in$Z(L).

Now if $\alpha_7$=0 and $\alpha_4 \neq 0$. (Since if $\alpha_7 = \alpha_4$=0 then r is also in the center Z(L), a contradiction because L is filiform and only one element can be in Z(L) and we have already designated t to be in Z(L) earlier.) Further relabelling yields

[x,y]=z, [x, z]=c, [x, c]=r,  [x, r]=$\alpha_4$t, [y, z]=$\alpha_5$t, [y, c]= 0, [y, r]= $\alpha_7$t, [z, c]= -$\alpha_7$t, [z, r]=0,

[c, r]= 0 and  t∈Z(L).

Now by substituting t´= $\alpha_4$t which implies t=(1/ $\alpha_4$)t´, we get

[x,y]=z, [x, z]=c, [x, c]=r,  [x, r]= t´, [y, z]= ($\alpha_5/\alpha_4$)t´, [y, c]= 0, [y, r]=0, [z, c]=0, [z, r]=0,

[c, r]= 0, now let yz=$\alpha_5$´t this implies

[x,y]=z, [x, z]=c, [x, c]=r,  [x, r]= t´=t, [y, z]=$\alpha_5$ ´t, [y, c]= 0, [y, r]=0, [z, c]=0, [z, r]=0,

[c, r]= 0, thus $\alpha_4$=1 therefore  we will now consider:

**Case 1 (i) where we suppose  $\alpha_7 = 0$ and  $1= \alpha_4 \neq 0$ .** (Since again if $\alpha_7 = \alpha_4$=0 then r is also in the center Z(L), a contradiction.)

Relabelling yields the non zero multiplication [x, y]= z, [x, z]=c, [x, c]=r,  [x, r]= t,

[y, z]=$\alpha_5$t, [y, c]= 0, [y, r]= 0, [z, c]= 0, [z, r]= 0, [c, r]= 0 and  t∈Z(L).

To  compute the  multiplier start  with

| | | | |
|---|---|---|---|
| [x,y]=z+$s_1$ | [y, z]=$\alpha_5$t+$s_5$ | [z,r]=$s_9$ | [z, t]=$s_{13}$ |
| [x, z]=c+$s_2$ | [y, c]=$s_6$ | [c, r]=$s_{10}$ | [c, t]=$s_{14}$ |
| [x, c]=r+$s_3$ | [y, r]= $s_7$ | [x, t]=$s_{11}$ | [r, t]=$s_{15}$ |
| [x, r]=t+$s_4$ | [z,c]= $s_8$ | [y, t]=$s_{12}$ | |

where $s_1$,…,$s_{15}$ generate the multiplier. Now we will use the Jacobi identity on all triples.

• (xy)z+(yz)x+(zx)y = 0

$\Rightarrow$ ($\alpha_5$t)x-cy=0

$\Rightarrow$ - $\alpha_5 s_{11}$+ $s_6$=0

$\Rightarrow$ $s_6$= $\alpha_5 s_{11}$

- $(xy)c+(yc)x+(cx)y = 0$     $\Rightarrow zc-ry=0$     $\Rightarrow s_8- s_7 =0$     $\Rightarrow s_8=- s_7$ .

- $(xy)r+(yr)x+(rx)y = 0$     $\Rightarrow zr+s_7x-ty =0$     $\Rightarrow s_9+ s_{12}=0$     $\Rightarrow - s_9=-s_{12}.$

- $(xy)t+(yt)x+(tx)y = 0$     $\Rightarrow zt =0$     $\Rightarrow s_{13}=0.$

- $(xz)r+(zr)x+(rx)z = 0$     $\Rightarrow cr- (\alpha_4t)z =0$     $\Rightarrow s_{10}=0.$

- $(xz)c+(zc)x+(cx)z = 0$     $\Rightarrow 0+s_8x-rz=0$     $\Rightarrow 0-rz=0$     $\Rightarrow s_9=0.$

- $(xz)t+(zt)x+(tx)z = 0$     $\Rightarrow ct=0$     $\Rightarrow s_{14}=0$ .

- $(xr)c+(rc)x+(cx)r = 0$     $\Rightarrow (\alpha_4t)c- s_{10}x=0$     $\Rightarrow \alpha_4s_{14}=0$     $\Rightarrow s_{14}=0.$

- $(xr)t+(rt)x+(tx)r = 0$     $\Rightarrow (\alpha_4t)t=0$

- $(xc)t+(ct)x+(tx)c = 0$     $\Rightarrow rt=0$     $\Rightarrow s_{15}=0$

- $(yz)c+(zc)y+(cy)z = 0$     $\Rightarrow (\alpha_5t)c+s_8y-s_6z=0$     $\Rightarrow -\alpha_5s_{14}=0$     $\Rightarrow s_{14}=0$

- $(yz)r+(zr)y+(ry)z = 0$     $\Rightarrow (\alpha_5t)r+(-\alpha_7t)z=0$     $\Rightarrow \alpha_50+\alpha_70=0$

- $(yz)t+(zt)y+(ty)z = 0$     $\Rightarrow (\alpha_5t)t=0$     $\Rightarrow 0=0$

- $(yc)r+(cr)y+(ry)c = 0$     $\Rightarrow 0=0$

- $(yc)t+(ct)y+(ty)c = 0$     $\Rightarrow 0=0$

- $(rc)t+(ct)r+(tr)c = 0$     $\Rightarrow 0=0$

From the preceding work we obtain the following equations.

(i) $s_6=\alpha_5 s_{11}$ (ii) $s_7=-s_8$ (iii) $s_9=-s_{12}$ (iv) $s_{13}=0$ (v)$s_{10}=0$

(vi) $s_9=0$ (vii) $s_{14}=0$ (viii) $s_{15}=0$. This implies

$s_9=s_{10}=s_{12}=s_{13}=s_{14}=s_{15}=0$. Further $s_6, s_8 \in (s_4,s_5,s_7, s_{11})$ thus dimM(L)=$(s_4,s_5,s_7, s_{11})$=4.

Therefore t(L)=½n(n-1)-dimM(L)= ½(6)(5) - 4=11, call this L=L(11,6,1,11).

Now by substituting $t´=\alpha_7 t$ which implies $t=(1/\alpha_4)t´$, we get

[x,y]=z, [x, z]=c, [x, c]=r, [x, r]=$\alpha_4 t´$, [y, z]= $(\alpha_5/\alpha_7)t´$, [y, c]= 0, [y, r]=t´, [z, c]= -t´,

[z, r]=0, [c, r]= 0, now let yz=$\alpha_5´t$ this implies

[x,y]=z, [x, z]=c, [x, c]=r, [x, r]= $\alpha_4 t$, [y, z]=$\alpha_5´t$, [y, c]= 0, [y, r]=t´, [z, c]=-t´, [z, r]=0,

[c, r]= 0, thus $\alpha_7=1$ therefore we will now consider:

**Case 1 (ii) where we suppose 1= $\alpha_7 \neq 0$.**

Relabelling yields the non zero multiplication [x, y]= z, [x, z]=c, [x, c]=r, [x, r]= $\alpha_4 t$,

   [y, z]=$\alpha_5 t$, [y, c]= 0, [y, r]= t, [z, c]= -t, [z, r]= 0, [c, r]= 0 and t$\in$Z(L).

To compute the multiplier start with

| | | | |
|---|---|---|---|
| [x,y]=z+$s_1$ | [y, z]=$\alpha_5 t$+$s_5$ | [z,r]=$s_9$ | [z, t]=$s_{13}$ |
| [x, z]=c+$s_2$ | [y, c]=$s_6$ | [c, r]=$s_{10}$ | [c, t]=$s_{14}$ |
| [x, c]=r+$s_3$ | [y, r]=t+$s_7$ | [x, t]=$s_{11}$ | [r, t]=$s_{15}$ |
| [x, r]= $\alpha_4 t$+$s_4$ | [z,c]= - t +$s_8$ | [y, t]=$s_{12}$ | |

where $s_1,…,s_{15}$ generate the multiplier. Now we will use the Jacobi on all triples.

• (xy)z+(yz)x+(zx)y = 0

$\Rightarrow$zz+ ($\alpha_5$t)x+ $s_5$x+(-c)y=0

$\Rightarrow$- $\alpha_5$ $s_{11}$+ $s_6$=0     $\Rightarrow s_6=\alpha_5 s_{11}$

- $(xy)c+(yc)x+(cx)y = 0 \Rightarrow (z)c+ (s_6)x-(r)y=0 \Rightarrow -t-ry=0 \Rightarrow -t+t+ s_7=0 \Rightarrow s_7=0$

- $(xy)r+(yr)x+(rx)y = 0 \Rightarrow zr+tx-( \alpha_4 t-s_4)y=0 \Rightarrow s_9- s_{11}+ \alpha_4 s_{12}=0 \Rightarrow s_{11}=s_9+ \alpha_4 s_{12}$

- $(xy)t+(yt)x+(tx)y = 0 \Rightarrow zt+ s_{12}x- s_{11}y=0 \Rightarrow s_{13}=0$

- $(xz)r+(zr)x+(rx)z = 0 \Rightarrow cr+s_9x- \alpha_4 tz-s_4z=0 \Rightarrow s_{10}+ \alpha_4 s_{13}=0 \Rightarrow s_{10}=- \alpha_4 s_{13}$

- $(xz)c+(zc)x+(cx)z = 0 \Rightarrow cc-tx-rz=0 \Rightarrow s_{11}=-s_9$

- $(xz)t+(zt)x+(tx)z = 0 \Rightarrow ct+s_{13}x-s_{11}z=0 \Rightarrow \alpha_4 s_{14}=0$

- $(xr)c+(rc)x+(cx)r = 0 \Rightarrow (\alpha_4 t)c-s_{10}x-rr=0 \Rightarrow s_{14}=0$

- $(xr)t+(rt)x+(tx)r = 0 \Rightarrow (\alpha_4 t)t+s_4t+s_{15}x-s_{11}r=0 \Rightarrow 0=0$

- $(xc)t+(ct)x+(tx)c = 0 \Rightarrow rt+s_{14}x-s_{11}c=0 \Rightarrow s_{15}=0$

- $(yz)c+(zc)y+(cy)z = 0 \Rightarrow (\alpha_5 t)c -ty-s_6z=0 \Rightarrow -\alpha_5 s_{14}+s_{12}=0 \Rightarrow s_{12}= \alpha_5 s_{14}$

- $(yz)r+(zr)y+(ry)z = 0 \Rightarrow s_5r+s_9y-tz=0 \Rightarrow s_{13}=0$

- $(yz)t+(zt)y+(ty)z = 0 \Rightarrow (\alpha_5 t)t+s_{13}y+(-s_{12})z=0 \Rightarrow 0=0$

- $(yc)r+(cr)y+(ry)c = 0 \Rightarrow s_6r+s_{10}y+tc=0 \Rightarrow s_{14}=0$

- $(yc)t+(ct)y+(ty)c = 0 \Rightarrow s_6t+s_{14}y-s_{12}c=0 \Rightarrow 0=0$

- $(rc)t+(ct)r+(tr)c = 0 \Rightarrow -s_{10}t+s_{14}r-s_{15}c=0 \Rightarrow 0=0$

From the preceding work  we obtain the following equations.

(i) $s_6=\alpha_5 s_{11}$ (ii)$s_7=0$ (iii) $s_{11}=s_9+\alpha_4 s_{12}$ (iv) $s_{13}=0$ (v) $s_{10}=-\alpha_4 s_{13}$ (vi) $s_{11}=-s_9$

(vii) $s_{14}=0$ (viii) $s_{15}=0$ (ix) $s_{12}=\alpha_5 s_{14}$. This implies $s_7=s_8=s_9=s_{10}=s_{11}=s_{12}=s_{13}=s_{14}=s_{15}=0$.

Further $(s_9, s_{11}, s_{12}) \in (s_4, s_5, s_6)$ thus $\dim M(L)=(s_4, s_5, s_6)=3$.

Therefore $t(L)=\frac{1}{2}n(n-1)-\dim M(L)=\frac{1}{2}(6)(5)-3=12$, call this algebra $L=L(11,6,1,12)$.


Now we will consider case 2 where $H=L'(7,5,1,7)$This L can be described  generally by

the basis  $\{x, y, z, c, r, t\}$ and multiplication

$[x,y]=z$, $[x,z]=c$, $[x,c]=r$,  $[x,r]=\alpha_4 t$, $[y,z]=r+\alpha_5 t$, $[y,c]=\alpha_6 t$,  $[y,r]=\alpha_7 t$, $[z,c]=\alpha_8 t$

$[z,r]=\alpha_9 t$, $[c,r]=\alpha_{10}t$,  and $t\in Z(L)$. The Jacobi identity shows that $\alpha_9=\alpha_{10}=0$ and

$\alpha_6=\alpha_4$ and $\alpha_8=-\alpha_7$.

Relabelling yields the non zero multiplication $[x,y]=z$, $[x, z]=c$, $[x, c]=r$,  $[x, r]=\alpha_4 t$,

$[y, z]=r+\alpha_5 t$, $[y, c]=\alpha_4 t$ , $[y, r]=\alpha_7 t$, $[z, c]=-\alpha_7 t$, $[z, r]=0$, $[c, r]=0$ and  $t\in Z(L)$.

Now by substituting $t'=\alpha_7 t$ which implies $t=(1/\alpha_7)t'$, we get

$[x,y]=z$, $[x, z]=c$, $[x, c]=r$,  $[x, r]=(\alpha_4/\alpha_7)t'$, $[y, z]=r+(\alpha_3/\alpha_7)t'$, $[y, c]=(\alpha_4/\alpha_7)t'$, $[y, r]=t'$,

$[z, c]=-t'$, $[z, r]=0$, $[c, r]=0$, now letting  $yc=\alpha_4't$  implies $yc=\alpha_4 t \Rightarrow \alpha_7=1$

also letting $yr=\alpha_7't$ implies $yr=\alpha_7 t \Rightarrow$, also letting $zc=-\alpha_7't$ implies $zc=-\alpha_7 t \Rightarrow \alpha_7=1$

similarly letting $yr=r+\alpha_3't$ implies $yr=r+\alpha_3 t \Rightarrow \alpha_7=1$.

Thus $\alpha_7=1$ therefore  we will now consider:

**Case 2 (i) where we suppose  $1=\alpha_7\neq 0$.**


Relabelling yields the non zero multiplication $[x, y]=z$, $[x, z]=c$, $[x, c]=r$,  $[x, r]=\alpha_4 t$,

$[y, z]=r+\alpha_5 t$, $[y, c]=\alpha_4 t$, $[y, r]=t$, $[z, c]=-t$, $[z, r]=0$, $[c, r]=0$ and  $t\in Z(L)$.

To compute the multiplier start with

$[x,y]=z+s_1$ $\quad$ $[y, z]=r+ \alpha_5 t+s_5$ $\quad$ $[z,r]=s_9$ $\quad$ $[z, t]=s_{13}$

$[x, z]=c+s_2$ $\quad$ $[y, c]= \alpha_4 t +s_6$ $\quad$ $[c, r]=s_{10}$ $\quad$ $[c, t]=s_{14}$

$[x, c]= r+s_3$ $\quad$ $[y, r]=t+ s_7$ $\quad$ $[x, t]=s_{11}$ $\quad$ $[r, t]=s_{15}$

$[x, r]= \alpha_4 t +s_4$ $\quad$ $[z,c]=-t+ s_8$ $\quad$ $[y, t]=s_{12}$

where $s_1,\ldots,s_{15}$ generate the multiplier. Now we will use the Jacobi identity on all

triples.

- $(xy)z+(yz)x+(zx)y = 0$ $\quad \Rightarrow zz+ (r+\alpha_5 t)x-(-c)y=0$ $\quad \Rightarrow -\alpha_4 t_. \alpha_5 s_{11}+ \alpha_4 t =0$ $\quad \Rightarrow -\alpha_5 s_{11}=0$

- $(xy)c+(yc)x+(cx)y = 0$ $\quad \Rightarrow (z)c+ (\alpha_4 t)x-(r)y=0$ $\quad \Rightarrow -t- \alpha_4 s_{11}+t=0$ $\quad \Rightarrow \alpha_4 s_{11}=0$

- $(xy)r+(yr)x+(rx)y = 0$ $\quad \Rightarrow zr+tx-( \alpha_4 t)y=0$ $\quad \Rightarrow s_9- s_{11}+ \alpha_4 s_{12}=0$ $\quad \Rightarrow s_9=s_{11}-$

$\alpha_4 s_{12}$

- $(xy)t+(yt)x+(tx)y = 0$ $\quad \Rightarrow zt+ s_{12}x- s_{11}y=0$ $\quad \Rightarrow s_{13}=0$

- $(xz)r+(zr)x+(rx)z = 0$ $\quad \Rightarrow cr+s_9x- \alpha_4 tz=0$ $\quad \Rightarrow s_{10}- \alpha_4 s_{13}=0$ $\quad \Rightarrow s_{10}= \alpha_4 s_{13}=0$

- $(xz)c+(zc)x+(cx)z = 0$ $\quad \Rightarrow cc-tx-rz=0$ $\quad \Rightarrow s_{11}=-s_9$

- $(xz)t+(zt)x+(tx)z = 0$ $\quad \Rightarrow ct+s_{13}x-s_{11}z=0$ $\quad \Rightarrow s_{14}=0$

- $(xr)c+(rc)x+(cx)r = 0$ $\quad \Rightarrow(\alpha_4 t)c-s_{10}x-rr=0$ $\quad \Rightarrow \alpha_4 s_{14}=0= \alpha_4 s_{11}$ $\quad \Rightarrow s_{14}=s_{11}=0$ $\Rightarrow s_9=0$

- $(xr)t+(rt)x+(tx)r = 0$ $\quad \Rightarrow(\alpha_4 t)t+s_{15}x-s_{11}r=0$ $\quad \Rightarrow 0=0$

• $(xc)t+(ct)x+(tx)c = 0$      $\Rightarrow rt+s_{14}x-s_{11}c=0$      $\Rightarrow s_{15}=0$

• $(yz)c+(zc)y+(cy)z = 0$      $\Rightarrow (r+\alpha_5 t)c -ty +(-\alpha_4 t)z=0$      $\Rightarrow rc+ \alpha_5 tc-ty- \alpha_4 tz =0$

$\Rightarrow - s_{10}- \alpha_5 s_{14}+ s_{12}+ s_{13}=0$      $\Rightarrow s_{12}= 0$      $\Rightarrow s_9=s_{11}$

• $(yz)r+(zr)y+(ry)z = 0$      $\Rightarrow (r+ \alpha_5 t)r+s_9 y-tz=0$      $\Rightarrow rr+ \alpha_5 tr+s_{13}=0$

$\Rightarrow - \alpha_5 s_{15}+s_{13}=0$      $\Rightarrow s_{13}=0$

• $(yz)t+(zt)y+(ty)z = 0$      $\Rightarrow (r+\alpha_5 t)t+s_{13}y+(-s_{12})z=0$      $\Rightarrow s_{15}=0$

• $(yc)r+(cr)y+(ry)c = 0$      $\Rightarrow \alpha_4 tr+s_{10}y-tc=0$      $\Rightarrow - \alpha_4 s_{15}+s_{14}=0$    $\Rightarrow 0=0$

• $(yc)t+(ct)y+(ty)c = 0$      $\Rightarrow (\alpha_4 t)t+s_{14}y-s_{12}c=0$      $\Rightarrow 0=0$

• $(rc)t+(ct)r+(tr)c = 0$      $\Rightarrow -s_{10}t+s_{14}r-s_{15}c=0$      $\Rightarrow 0= 0$


From the preceding work  we obtain the following equations.

(i)- $\alpha_5 s_{11}=0$  (ii) $\alpha_4 s_{11}=0$ (iii) $s_9=s_{11}- \alpha_4 s_{12}$ (iv) $s_{13}=0$ (v)$s_{10}=0$

(vi) $s_{11}=-s_9$ (vii)$s_{14}=0$ (viii) $\alpha_4 s_{14}=0$ (ix) $s_{11}=s_{14}=0$ (x)$s_9=0$ (xi)$s_{15}=0$ (xii) $s_{12}=0$ (xiii) $s_{13}=0$

This implies $s_7=s_8=s_9=s_{10}=s_{11}= s_{12}= s_{13}= s_{14}=s_{15}=0$, thus

dimM(L)=$(s_4, s_5, s_6)$=3.

Therefore t(L)=½n(n-1)-dimM(L)= ½(6)(5) - 3=12, call this algebra L=L′(11,6,1,12).

Now by substituting t′= $\alpha_4$t which implies t=(1/ $\alpha_4$)t′, we get

[x, y]=z, [x, z]=c, [x, c]=r,  [x, r]=1/$\alpha_4$t′, [y, z]=r+($\alpha_5$/$\alpha_4$)t′, [y, c]=(1/ $\alpha_4$)t′, [y, r]=0,

[z, c]=0,[z, r]=0, [c, r]= 0, now letting [x, r]=1/$\alpha_4$t′=t′′  implies t′= $\alpha_4$t′′ thus relabelling

implies $[x, y]=z$, $[x, z]=c$, $[x, c]=r$, $[x, r]= 1/\alpha_4 t'=t''=t$, $[y, z]= r+(\alpha_5/\alpha_4)(\alpha_4 t'')=r+ \alpha_5 t$,

$[y, c]= (1/\alpha_4)(\alpha_4 t'')=t$, $[y, r]=0$, $[z, c]=0$, $[z, r]=0$, $[c, r]= 0$, thus $\alpha_4=1$. Therefore we will now consider:

**Case 2 (ii) where we suppose $\alpha_7 = 0$ and $1= \alpha_4 \neq 0$.** (Again, $\alpha_4$ and $\alpha_7$

cannot both be zero since this implies r is also in the center, a contradiction)

Relabelling yields the non zero multiplication $[x, y]= z$, $[x, z]=c$, $[x, c]=r$, $[x, r]= t$,

$[y, z]=r + \alpha_5 t$, $[y, c]= t$, $[y, r]= 0$, $[z, c]= 0$, $[z, r]= 0$, $[c, r]= 0$ and $t \in Z(L)$.

To compute the multiplier start with

$$[x,y]=z+s_1 \qquad [y, z]=r + \alpha_5 t+ s_5 \qquad [z,r]=s_9 \qquad [z, t]=s_{13}$$

$$[x, z]=c+s_2 \qquad [y, c]=t + s_6 \qquad [c, r]=s_{10} \qquad [c, t]=s_{14}$$

$$[x, c]=r+s_3 \qquad [y, r]= s_7 \qquad [x, t]=s_{11} \qquad [r, t]=s_{15}$$

$$[x, r]= t+s_4 \qquad [z,c]= s_8 \qquad [y, t]=s_{12}$$

where $s_1,\ldots,s_{15}$ generate the multiplier. Now we will use the Jacobi identity on all triples.

- $(xy)z+(yz)x+(zx)y = 0 \qquad \Rightarrow zz+ (r+\alpha_5 t)x+(-c)y=0 \qquad \Rightarrow - \alpha_5 s_{11}=0$

- $(xy)c+(yc)x+(cx)y = 0 \qquad \Rightarrow (z)c+ (\alpha_4 t)x-(r)y=0 \qquad \Rightarrow s_8-s_{11}+s_7 =0 \qquad \Rightarrow s_8 = s_{11}-s_7$

- $(xy)r+(yr)x+(rx)y = 0 \qquad \Rightarrow zr+ \alpha_7 tx-(\alpha_4 t)y=0 \qquad \Rightarrow s_9- \alpha_7 s_{11}+ \alpha_4 s_{12}=0$

$$\Rightarrow s_9= \alpha_7 s_{11}- \alpha_4 s_{12} =- s_{12}$$

- $(xy)t+(yt)x+(tx)y = 0$      $\Rightarrow zt+ s_{12}x- s_{11}y=0$      $\Rightarrow s_{13}=0$

- $(xz)r+(zr)x+(rx)z = 0$      $\Rightarrow cr+s_9x-tz=0$      $\Rightarrow s_{10}+ s_{13}=0$      $\Rightarrow s_{10}= -s_{13}$

- $(xz)c+(zc)x+(cx)z = 0$      $\Rightarrow cc- \alpha_7tx-rz=0$      $\Rightarrow -\alpha_7s_{11}+s_9=0$      $\Rightarrow s_9=0$

- $(xz)t+(zt)x+(tx)z = 0$      $\Rightarrow ct+s_{13}x-s_{11}z=0$      $\Rightarrow s_{14}=0$

- $(xr)c+(rc)x+(cx)r = 0$      $\Rightarrow (\alpha_4t)c-s_{10}x-rr=0$      $\Rightarrow \alpha_4 s_{14}=0$      $\Rightarrow s_{14}=0$

- $(xr)t+(rt)x+(tx)r = 0$      $\Rightarrow (\alpha_4t)t+s_{15}x-s_{11}r=0$      $\Rightarrow 0=0$

- $(xc)t+(ct)x+(tx)c = 0$      $\Rightarrow rt+s_{14}x-s_{11}c=0$      $\Rightarrow s_{15}=0$

- $(yz)c+(zc)y+(cy)z = 0$      $\Rightarrow (r+\alpha_5t)c – \alpha_7ty +(-\alpha_4t)z=0$      $\Rightarrow rc+ \alpha_5tc-tz=0$

$\Rightarrow s_{10}=- \alpha_5s_{14}$

- $(yz)r+(zr)y+(ry)z = 0$      $\Rightarrow (r+ \alpha_5t)r+s_9y-tz=0$      $\Rightarrow rr+ \alpha_5tr+s_{13}=0$

     $\Rightarrow -\alpha_5s_{15}+s_{13}=0$      $\Rightarrow \alpha_5s_{15}=0$

- $(yz)t+(zt)y+(ty)z = 0$      $\Rightarrow (r+\alpha_5t)t+s_{13}y+(-s_{12})z=0$      $\Rightarrow s_{15}=0$

- $(yc)r+(cr)y+(ry)c = 0$      $\Rightarrow tr+s_{10}y- s_7c=0$      $\Rightarrow s_{15}=0$

- $(yc)t+(ct)y+(ty)c = 0$      $\Rightarrow (t)t+s_{14}y-s_{12}c=0$      $\Rightarrow 0=0$

- $(rc)t+(ct)r+(tr)c = 0$      $\Rightarrow -s_{10}t+s_{14}r-s_{15}c=0$      $\Rightarrow 0= 0$

From the preceding work  we obtain the following equations.

(i)- $\alpha_5 s_{11} = 0$  (ii) $s_8 = s_{11} - s_7$ (iii) $s_9 = \alpha_7 s_{11} - \alpha_4 s_{12} = -s_{12}$  (iv) $s_{13} = 0$ (v) $s_{10} = -s_{13}$

(vi) $s_9 = 0$ (vii) $s_{14} = 0$ (viii) $s_{15} = 0$ (ix) $s_{10} = - \alpha_5 s_{14}$ (x) $\alpha_5 s_{15} = 0$ .Now from (i) if $\alpha_5$ is not zero,

this implies $s_8 = s_9 = s_{10} = s_{12} = s_{13} = s_{14} = s_{15} = 0$.  Further $s_{11} \in (s_5, s_6, s_7, s_8)$ thus

$\dim M(L) = (s_5, s_6, s_7, s_8) = 4$.

Therefore $t(L) = \frac{1}{2} n(n-1) - \dim M(L) = \frac{1}{2}(6)(5) - 4 = 11$, we call this algebra $L = L'(11,6,1,11)$.

From this work we have found the following algebras for $t(L) = 11$ and we summarize

them in the following theorem.


**Theorem 14**  Let L be filiform with $t(L) = 11$ then $L = L(11,6,1,11)$ or  $L = L'(11,6,1,11)$

## 12. t(L)= 12

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 12$ then

$c^2 + c \leq 24$ which implies that $c \leq 4$ therefore $c = 0, 1, 2, 3$ or $4$, but by proposition 0

(i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $12 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving

this inequality we know that $c = 4$ is the only case that satisfies the inequality. Thus if $c = $

$\dim L^2 = 4$, then $\dim H^2 = 3$ and $t(H) \leq 8$ (proposition 3). In the $t(L)=11$ case we have

already computed these and we list them in the following theorem.

**Theorem 15**   Let L be filiform with t(L)=12, then L=L(11,6,1,12) or L=L´(11,6,1,12).

## 13. t(L) = 13

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 13$ then

$c^2 + c \leq 26$ which implies that $c \leq 4$ therefore $c = 0, 1, 2, 3$ or $4$, but by proposition 0

(i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $13 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving

this inequality we know that $c = 4$ is the only case that satisfies the inequality. Thus if $c =$

$\dim L^2 = 4$, then $\dim H^2 = 3$ and $t(H) \leq 9$ (proposition 3), but from Table 22.1, we

know that the only time these conditions are satisfied is when $t(L)=7$, but here $t(L)=13$,

hence a contradiction. We thus summarize our findings in the following Theorem.

**Theorem 16** There are no filiform Lie algebras with $t(L) = 13$.

## 14. t(L)=14

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 14$ then

$c^2 + c \leq 28$ which implies that $c \leq 4$ therefore $c = 0, 1, 2, 3$ or $4$, but by proposition 0

(i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $14 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving

this inequality we know that $c = 4$ is the only case that satisfies the inequality. Thus if $c =$

$\dim L^2 = 4$, then $\dim H^2 = 3$ and $t(H) \leq 10$ (proposition 3), but from Table 22.1, we

know that the only time these conditions are satisfied is when $t(L)=7$, but here $t(L)=14$,

hence a contradiction. We thus summarize our findings in the following Theorem.

**<u>Theorem 17</u>** There are no filiform Lie algebras with $t(L) = 14$.

## 15. t(L)=15

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 15$ then

$c^2 + c \leq 30$ which implies that $c \leq 5$ therefore $c = 0, 1, 2, 3, 4$ or $5$ but by proposition 0

(i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $15 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving

this inequality we know that $c = 4$ and $c = 5$ are the two cases that satisfy the inequality.

Thus if $c = \dim L^2 = 4$, then $\dim L = 6$ which implies $\dim H = 5$, but from Table 22.1

This is a contradiction.

Also if $c = \dim L^2 = 5$, then $\dim H^2 = 4$ and $t(H) \leq 10$ (proposition 3), but again from

Table 22.1, we know that there are no filiform Lie algebras satisfying these conditions,

hence a contradiction. We thus summarize our findings in the following Theorem.

**<u>Theorem 18</u>** There are no filiform Lie algebras with $t(L) = 15$.

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## 16. t(L)=16

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 16$ then

$c^2 + c \leq 32$ which implies that $c \leq 5$ therefore $c = 0, 1, 2, 3, 4$ or $5$ but by proposition 0

(i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $16 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving

this inequality we know that $c = 4$ or $c = 5$. Thus (1) if $c = \dim L^2 = 4$, then $\dim H^2 = 3$

and $t(H) \leq 12$ (proposition 3). (2) If $c = \dim L^2 = 5$ then $\dim H^2 = 4$ and $t(H) \leq 11$

(proposition 3), but from previous work we know that the only time these conditions are

satisfied is when $t(L)=11$ or $t(L)=12$. If $t(H)=11$, then $H=L(11,6,1,11)$ or $H=L'(11,6,1,11)$

and if $t(H)=12$, then $H=L(11,6,1,12)$ or $H=L'(11,6,1,12)$ but note that $c = \dim L^2 = 4$ is

impossible since the dimension of c must equal to 5 because c has to be 2 less than the

dimension of L which has 7 elements as defined by the basis $\{x, y, z, c, r, t, u\}$ which is

what we need here for $t(L)=16$. Therefore (1) above is totally eliminated leaving us with

only (2). We will therefore now consider the two cases for $t(L)=11$. The first is when

$H=L(11,6,1,11)$ and the second is when $H=L'(11,6,1,11)$.

**CASE 1** Let $H=L(11,6,1,11)$, this L can be described generally by the basis

$\{x, y, z, c, r, t, u\}$ and multiplication

$[x,y]=z$, $[x,z]=c$, $[x,c]=r$, $[x,r]= \alpha_4 t + \beta_4 u$, $[y,z]=\alpha_5 t + \beta_5 u$, $[y,c]= \beta_6 u$, $[y,r]= t + \beta_7 u$,

$[z,c]= -t + \beta_8 u$, $[z,r]=\beta_9 u$, $[c,r]=\beta_{10} u$, $[x, t]= \beta_{11} u$, $[y, t]= \beta_{12} u$, $[z, t]= \beta_{13} u$, $[c, t]= \beta_{14} u$,

$[r, t]= \beta_{15} u$, and $u \in Z(L)$. Now we will use the Jacobi Identity on all triples.

• $(xy)z+(yz)x+(zx)y = 0$

$\Rightarrow zz + (\alpha_5 t + \beta_5 u)x + (-c)y = 0$

$\Rightarrow \alpha_5 tx + \beta_5 ux - cy = 0$

$\Rightarrow -\alpha_5\beta_{11}u-(-\beta_6 u)=0$ $\Rightarrow \beta_6= \alpha_5\beta_{11}$

• $(xy)c+(yc)x+(cx)y = 0$ $\Rightarrow zc+ \beta_6 ux-(-t- \beta_7 u )=0$ $\Rightarrow zc+ t+ \beta_7 u=0$ $\Rightarrow \beta_7 = - \beta_8$

• $(xy)r+(yr)x+(rx)y = 0$ $\Rightarrow zr+ (t + \beta_7 u)x + (-\alpha_4 t-\beta_4 u)y=0$ $\Rightarrow zr+ tx+ \beta_7 ux-\alpha_4 ty\beta_4 uy=0$

$\Rightarrow \beta_9 u-\beta_{11}u- \alpha_4(-\beta_{12}u)$ $\Rightarrow \beta_9= \beta_{11}- \alpha_4\beta_{12}$

• $(xy)t+(yt)x+(tx)y = 0$ $\Rightarrow zt + \beta_{12}ux-\beta_{11}uy=0$ $\Rightarrow \beta_{13}u=0$

• $(xy)u+(yu)x+(ux)y = 0$ $\Rightarrow 0=0$

• $(xz)c+(zc)x+(cx)z = 0$ $\Rightarrow cc+(-t+ \beta_8 u)x-rz=0$ $\Rightarrow -tx-rz=0$ $\Rightarrow \beta_{11}u+ \beta_9 u=0$

$\Rightarrow \beta_9=- \beta_{11}$

• $(xz)r+(zr)x+(rx)z = 0$ $\Rightarrow cr+ \beta_9 ux- \alpha_4 tz- \beta_4 uz=0$ $\Rightarrow \beta_{10}u+ \alpha_4\beta_{13}u=0$

$\Rightarrow \beta_{10}=- \alpha_4\beta_{13}$

• $(xz)t+(zt)x+(tx)z = 0$ $\Rightarrow ct+ \beta_{13}ux- \beta_{11}uz=0$ $\Rightarrow \beta_{14}u=0$

• $(xz)u+(zu)x+(ux)z = 0$ $\Rightarrow 0=0$

• $(xc)r+(cr)x+(rx)c = 0$ $\Rightarrow rr+ \beta_{10}ux- \alpha_4 tc- \beta_4 uc=0$ $\Rightarrow -\alpha_4 tc=0$ $\Rightarrow \alpha_4\beta_{14}u=0$

• $(xc)t+(ct)x+(tx)c = 0$ $\Rightarrow rt+ \beta_{14}ux+(-\beta_{11}u)c=0$ $\Rightarrow \beta_{15}u=0$

• $(xc)u+(cu)x+(ux)c = 0$ $\Rightarrow 0=0$

• $(xr)t+(rt)x+(tx)r = 0$ $\Rightarrow (\alpha_4 t+ \beta_4 u)t+ \beta_{15} ux+(-\beta_{11} u)r=0$ $\Rightarrow 0=0$

• $(xr)u+(ru)x+(ux)r = 0$ $\Rightarrow 0=0$

• $(xt)u+(tu)x+(ux)t = 0$ $\Rightarrow 0=0$

• $(yz)c+(zc)y+(cy)z = 0$ $\Rightarrow (\alpha_5 t+ \beta_5 u)c+(-t+ \beta_8 u)y+(-\beta_6 u)z=0$

$\Rightarrow \alpha_5 tc+ \beta_5 uc- ty+ \beta_8 uy- \beta_8 uz=0$ $\Rightarrow -\alpha_5 \beta_{14} u+ \beta_{12} u=0$ $\Rightarrow \alpha_5 \beta_{14}= \beta_{12}$

• $(yz)r+(zr)y+(ry)z = 0$ $\Rightarrow (\alpha_5 t+ \beta_5 u)r+ \beta_9 uy+(- t- \beta_7 u)z=0$ $\Rightarrow \alpha_5 tr- tz- \beta_7 uz=0$

$\Rightarrow -\alpha_5 \beta_{15} u+ \beta_{13} u=0$ $\Rightarrow \beta_{13}= \alpha_5 \beta_{15}$

• $(yz)t+(zt)y+(ty)z = 0$ $\Rightarrow (\alpha_5 t+ \beta_5 u)t + \beta_{13} uy- \beta_{12} uz=0$ $\Rightarrow 0=0$

• $(yz)u+(zu)y+(uy)z = 0$ $\Rightarrow 0=0$

• $(yc)r+(cr)y+(ry)c = 0$ $\Rightarrow \beta_6 ur+ \beta_{10} uy+(-t- \beta_7 u)c=0$ $\Rightarrow -tc- \beta_7 uc=0$ $\Rightarrow \beta_{14} u=0$

• $(yc)t+(ct)y+(ty)c = 0$ $\Rightarrow \beta_6 ut+ \beta_{14} uy+ \beta_7 uc=0$ $\Rightarrow 0=0$

• $(yc)u+(cu)y+(uy)c = 0$ $\Rightarrow 0=0$

• $(yr)t+(rt)y+(ty)r = 0$ $\Rightarrow (t+ \beta_7 u)t+ \beta_{15} uy- \beta_{12} ur=0$ $\Rightarrow 0=0$

• $(yr)u+(ru)y+(uy)r = 0$ $\Rightarrow 0=0$

• $(yt)u+(tu)y+(uy)t = 0$ $\Rightarrow 0=0$

• $(zc)r+(cr)z+(rz)c = 0$ $\Rightarrow (-t+ \beta_8 u)r+ \beta_{10}uz-\beta_9 uc=0$ $\Rightarrow -tr=0$ $\Rightarrow \beta_{15}u=0$

• $(zc)t+(ct)z+(tz)c = 0$ $\Rightarrow (-t+ \beta_8 u)t- \beta_{14}uz-\beta_{13}uc=0$ $\Rightarrow 0=0$

• $(zc)u+(cu)z+(uz)c = 0$ $\Rightarrow 0=0$

• $(zr)t+(rt)z+(tz)r = 0$ $\Rightarrow \beta_9 ut+ \beta_{15}uz-\beta_{13}ur=0$ $\Rightarrow 0=0$

• $(zr)u+(ru)z+(uz)r = 0$ $\Rightarrow 0=0$

• $(zt)u+(tu)z+(uz)t = 0$ $\Rightarrow 0=0$

• $(cr)t+(rt)c+(tc)r = 0$ $\Rightarrow \beta_{10}ut+ \beta_{15}uc-\beta_{14}ur=0$ $\Rightarrow 0=0$

• $(cr)u+(ru)c+(uc)r = 0$ $\Rightarrow 0=0$

• $(rt)u+(tu)r+(ur)t = 0$ $\Rightarrow 0=0$

We will now summarize the equations found below:

(1) $\beta_6= \alpha_5 \beta_{11}$ (2) $\beta_7 =- \beta_8$ (3) $\beta_9= \beta_{11}-\alpha_4 \beta_{12}$ (4) $\beta_{13}u=0$ (5) $\beta_9= \beta_{11}$ (6) $\beta_{10}=-\alpha_4 \beta_{13}$

(7) $\beta_{14}u=0$ (8) $\alpha_4\beta_{14}u=0$ (9) $\beta_{15}u=0$ (10) $\alpha_5 \beta_{14}=\beta_{12}$ (11) $\beta_{13}= \alpha_5 \beta_{15}$

From the preceding work we obtain the following equations.

(i)   $\beta_6 = \beta_9 = \beta_{10} = \beta_{11} = \beta_{12}= \beta_{13}= \beta_{14}= \beta_{15}=0$  (ii) $\beta_7=-\beta_8$

(iii) $\beta_9 =-\beta_{11}= \beta_{11}=0$  (iv) $\beta_{10}=-\alpha_4 \beta_{13}=0$  (v) $\beta_{13}= \alpha_5 \beta_{15}=0$.

Relabelling yields the non zero multiplication

[x,y]=z, [x,z]=c, [x,c]=r,  [x,r]= $\alpha_4$t+$\beta_4$u, [y,z]=$\alpha_5$t+$\beta_5$u, [y,c]= 0,  [y,r]= t-$\beta_8$u,

[z,c]= -t+$\beta_8$u , [z,r]=0, [c,r ]=0, [x, t]= 0, [y, t]= 0, [z, t]= 0, [c, t]= 0,

[r, t]=0, and u∈Z(L). This implies that t is also in the center, a contradiction, thus we we

have not found any filiform lie algebra for this case.

CASE 2. We will now consider H=L´(11,6,1,11). This can be described generally

by the basis {x, y, z, c, r, t, u} and multiplication

[x,y]=z, [x,z]=c, [x,c]=r,  [x,r]= $\alpha_4$t+$\beta_4$u, [y,z]=r+$\alpha_5$t+$\beta_5$u, [y,c]= $\alpha_4$t+$\beta_6$u,  [y,r]= t+$\beta_7$u,

[z,c]= -t+$\beta_8$u , [z,r]=$\beta_9$u, [c,r ]=$\beta_{10}$u, [x, t]= $\beta_{11}$u, [y, t]= $\beta_{12}$u, [z, t]= $\beta_{13}$u, [c, t]= $\beta_{14}$u,

[r, t]= $\beta_{15}$u, and u∈Z(L). Now we will use the Jacobi Identity on all triples.


• (xy)z+(yz)x+(zx)y = 0   $\Rightarrow$zz+ (r+$\alpha_5$t+ $\beta_5$u)x +(-c)y=0      $\Rightarrow$rx+ $\alpha_5$tx+ $\beta_5$ux-cy=0

$\Rightarrow$ (-$\alpha_4$t-$\beta_4$u)+$\alpha_5$(-$\beta_{11}$u)-(-$\alpha_4$t-$\beta_6$u)=0   $\Rightarrow$ $\beta_6$=$\beta_4$u+$\alpha_5\beta_{11}$


• (xy)c+(yc)x+(cx)y = 0   $\Rightarrow$ zc+($\alpha_4$t+$\beta_6$u)x-ry=0   $\Rightarrow$ -t+$\beta_8$u+ $\alpha_4$tx+ $\beta_6$ux+t+$\beta_7$u)=0


$\Rightarrow$ $\beta_7$=$\alpha_4\beta_{11}$-$\beta_8$


• (xy)r+(yr)x+(rx)y = 0   $\Rightarrow$ zr+ (t +$\beta_7$u)x + (-$\alpha_4$t- $\beta_4$u)y=0  $\Rightarrow$ $\beta_9$u+tx+ $\beta_7$ux-$\alpha_4$ty-$\beta_4$uy =0


$\Rightarrow$ $\beta_9$u-$\beta_{11}$u- $\alpha_4$(-$\beta_{12}$u)   $\Rightarrow$ $\beta_9$= $\beta_{11}$- $\alpha_4\beta_{12}$


• (xy)t+(yt)x+(tx)y = 0   $\Rightarrow$ zt + $\beta_{12}$ux-$\beta_{11}$uy=0      $\Rightarrow$$\beta_{13}$u=0


• (xy)u+(yu)x+(ux)y = 0  $\Rightarrow$0=0

- $(xz)c+(zc)x+(cx)z = 0 \quad \Rightarrow cc+(-t+\beta_8 u)x-rz=0 \quad \Rightarrow -tx-rz=0 \quad \Rightarrow \beta_{11}u+\beta_9 u=0$

$\Rightarrow \beta_9=-\beta_{11}$

- $(xz)r+(zr)x+(rx)z = 0 \quad \Rightarrow cr+\beta_9 ux-\alpha_4 tz-\beta_4 uz=0 \quad \Rightarrow \beta_{10}u+\alpha_4\beta_{13}u=0 \quad \Rightarrow \beta_{10}=-\alpha_4\beta_{13}$

- $(xz)t+(zt)x+(tx)z = 0 \quad \Rightarrow ct+\beta_{13}ux-\beta_{11}uz=0 \quad \Rightarrow \beta_{14}u=0$

- $(xz)u+(zu)x+(ux)z = 0 \quad \Rightarrow 0=0$

- $(xc)r+(cr)x+(rx)c = 0 \quad \Rightarrow rr+\beta_{10}ux-\alpha_4 tc-\beta_4 uc=0 \Rightarrow -\alpha_4 tc=0 \quad \Rightarrow \alpha_4\beta_{14}u=0$

- $(xc)t+(ct)x+(tx)c = 0 \quad \Rightarrow rt+\beta_{14}ux+(-\beta_{11}u)c=0 \quad \Rightarrow \beta_{15}u=0$

- $(xc)u+(cu)x+(ux)c = 0 \quad \Rightarrow 0=0$

- $(xr)t+(rt)x+(tx)r = 0 \quad \Rightarrow (\alpha_4 t+\beta_4 u)t+\beta_{15}ux+(-\beta_{11}u)r=0 \quad \Rightarrow 0=0$

- $(xr)u+(ru)x+(ux)r = 0 \quad \Rightarrow 0=0$

- $(xt)u+(tu)x+(ux)t = 0 \quad \Rightarrow 0=0$

- $(yz)c+(zc)y+(cy)z = 0 \quad \Rightarrow (\alpha_5 t+\beta_5 u)c+(-t+\beta_8 u)y+(-\beta_6 u)z=0$

$\Rightarrow (\alpha_5 t+\beta_5 u)c+(-t+\beta_8 u)y+(-\beta_6 u)z=0 \quad \Rightarrow \alpha_5 tc+\beta_5 uc-ty+\beta_8 uy-\beta_8 uz=0$

$\Rightarrow -\alpha_5 \beta_{14}u+\beta_{12}u=0 \quad \Rightarrow \alpha_5 \beta_{14}=\beta_{12}$

- $(yz)r+(zr)y+(ry)z = 0 \Rightarrow (\alpha_5 t+\beta_5 u)r+\beta_9 uy+(-t-\beta_7 u)z=0 \quad \Rightarrow \alpha_5 tr-tz-\beta_7 uz=0$

$\Rightarrow -\alpha_5 \beta_{15}u+\beta_{13}u=0 \quad \Rightarrow \beta_{13}=\alpha_5 \beta_{15}$

- $(yz)t+(zt)y+(ty)z = 0 \Rightarrow (\alpha_5 t+ \beta_5 u)t + \beta_{13}uy - \beta_{12}uz=0 \Rightarrow 0=0$

- $(yz)u+(zu)y+(uy)z = 0 \Rightarrow 0=0$

- $(yc)r+(cr)y+(ry)c = 0 \Rightarrow (\alpha_4 t+\beta_6 u)r+ \beta_{10}uy+(-t- \beta_7 u)c=0 \Rightarrow \alpha_4 tr+\beta_6 ur+ -tc- \beta_7 uc=0$

$\Rightarrow \beta_{14}= \alpha_4\beta_{15}$

- $(yc)t+(ct)y+(ty)c = 0 \Rightarrow (\alpha_4+\beta_6 u)t+ \beta_{14}uy+\beta_7 uc=0 \Rightarrow 0=0$

- $(yc)u+(cu)y+(uy)c = 0 \Rightarrow 0=0$

- $(yr)t+(rt)y+(ty)r = 0 \Rightarrow \beta_{12}ut+ \beta_{15}uy-\beta_{12}ur=0 \Rightarrow 0=0$

- $(yr)u+(ru)y+(uy)r = 0 \Rightarrow 0=0$

- $(yt)u+(tu)y+(uy)t = 0 \Rightarrow 0=0$

- $(zc)r+(cr)z+(rz)c = 0 \Rightarrow (-t+ \beta_8 u)r+ \beta_{10}uz-\beta_9 uc=0 \Rightarrow -tr=0 \Rightarrow \beta_{15}u=0$

- $(zc)t+(ct)z+(tz)c = 0 \Rightarrow (-t+ \beta_8 u)t+\beta_{14}uz-\beta_{13}uc=0 \Rightarrow 0=0$

- $(zc)u+(cu)z+(uz)c = 0 \Rightarrow 0=0$

- $(zr)t+(rt)z+(tz)r = 0 \Rightarrow \beta_9 ut+ \beta_{15}uz-\beta_{13}ur=0 \Rightarrow 0=0$

- $(zr)u+(ru)z+(uz)r = 0 \Rightarrow 0=0$

- $(zt)u+(tu)z+(uz)t = 0 \quad \Rightarrow 0=0$

- $(cr)t+(rt)c+(tc)r = 0 \quad \Rightarrow \beta_{10}ut+ \beta_{15}uc-\beta_{14}ur=0 \Rightarrow 0=0$

- $(cr)u+(ru)c+(uc)r = 0 \quad \Rightarrow 0=0$

- $(rt)u+(tu)r+(ur)t = 0 \quad \Rightarrow 0=0$

We will now summarize the equations found below:

(1) $\beta_6=\beta_4u+ \alpha_5 \beta_{11}$  (2) $\beta_7 =\alpha_4\beta_{11}-\beta_8$  (3) $\beta_9=\beta_{11}-\alpha_4 \beta_{12}$  (4) $\beta_{13}u=0$  (5) $\beta_9= \beta_{11}$

(6) $\beta_{10}=-\alpha_4\beta_{13}$  (7) $\beta_{14}u=0$  (8) $\alpha_4\beta_{14}u=0$  (9) $\beta_{15}u=0$  (10) $\alpha_5 \beta_{14}=\beta_{12}$  (11) $\beta_{13}= \alpha_5 \beta_{15}$

(12) $\beta_{14}= \alpha_4 \beta_{15}$

From the preceding work we obtain the following equations.

(i) $\beta_9 = \beta_{10} = \beta_{11} = \beta_{12}= \beta_{13}= \beta_{14}= \beta_{15}=0$  (ii) $\beta_7=-\beta_8$

(iii) $\beta_6 =\beta_4$

Relabelling yields the non zero multiplication

$[x,y]=z$, $[x,z]=c$, $[x,c]=r$,  $[x,r]= \alpha_4t+\beta_4u$, $[y,z]=r+\alpha_5t+\beta_5u$, $[y,c]=\alpha_4t$,  $[y,r]= t-\beta_8u$,

$[z,c]= -t+\beta_8u$ , $[z,r]=0$, $[c,r]=0$, $[x, t]= 0$, $[y, t]= 0$, $[z, t]= 0$, $[c, t]= 0$,

$[r, t]=0$, and $u\in Z(L)$. Again this implies that t is also in the center, a contradiction, thus we again have not found any filiform Lie algebras for this case and now summarize our findings in the following Theorem

**Theorem 19**  There are no filiform Lie algebras for $t(L)=16$

## 17. t(L)=17

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 17$ then

$c^2 + c \leq 34$ which implies that $c \leq 5$ therefore $c = 0, 1, 2, 3, 4$ or $5$ but by proposition 0

(i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $17 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving

this inequality we know that $c = 5$ is the only case that satisfies the inequality. Thus if $c = $

$\dim L^2 = 5$, then $\dim H^2 = 4$ and $t(H) \leq 12$ (proposition 3), but from Table 22.1, we

know that the only time these conditions are satisfied is when $t(H)=12$. We will therefore

consider the two cases for $t(H)=12$. The first case is when $H=L(11, 6, 1, 12)$ and the

second is when $H=L'(11, 6, 1, 12)$.

**CASE 1.** Let $H=L(11, 6, 1, 12)$, this L can be described generally by the basis

$\{x, y, z, c, r, t, u\}$ and multiplication

$[x,y]=z$, $[x,z]=c$, $[x,c]=r$, $[x,r]=t$, $[y,z]=\alpha_5 t+\beta_5 u$, $[y,c]=\beta_6 u$, $[y,r]=\beta_7 u$,

$[z,c]=\beta_8 u$, $[z,r]=\beta_9 u$, $[c,r]=\beta_{10}u$, $[x, t]=\beta_{11}u$, $[y, t]=\beta_{12}u$, $[z, t]=\beta_{13}u$, $[c, t]=\beta_{14}u$,

$[r, t]=\beta_{15}u$, and $u \in Z(L)$. Now we will use the Jacobi Identity on all triples.

• $(xy)z+(yz)x+(zx)y = 0 \Rightarrow zz + \alpha_5 tx + \beta_5 ux +(-c)y=0 \Rightarrow \alpha_5 tx-cy=0$

$\Rightarrow -\alpha_5\beta_{11}u-(-\beta_6 u)=0 \Rightarrow \beta_6 = \alpha_5\beta_{11}$

• $(xy)c+(yc)x+(cx)y = 0 \Rightarrow zc + \beta_6 ux-ry=0 \qquad \Rightarrow \beta_8 u+ \beta_7 u=0 \qquad \Rightarrow \beta_7 = -\beta_8$

• $(xy)r+(yr)x+(rx)y = 0 \Rightarrow zr + (\beta_7 u)x + (-t)y=0 \Rightarrow \beta_9 u - (-\beta_{12}u)=0 \Rightarrow \beta_9=-\beta_{12}$

- $(xy)t+(yt)x+(tx)y = 0$     $\Rightarrow zt + \beta_{12}ux-\beta_{11}uy=0$    $\Rightarrow \beta_{13}u=0$     $\Rightarrow \beta_{13}=0$

- $(xy)u+(yu)x+(ux)y = 0$    $\Rightarrow 0=0$

- $(xz)c+(zc)x+(cx)z = 0$     $\Rightarrow cc+\beta_8ux-rz=0$     $\Rightarrow \beta_9u=0$     $\Rightarrow \beta_9=0$

- $(xz)r+(zr)x+(rx)z = 0$     $\Rightarrow cr+ \beta_9ux-tz=0$     $\Rightarrow cr-(-\beta_{13}u)=0$   $\Rightarrow \beta_{10}u+ \alpha_4\beta_{13}u=0$

$\Rightarrow \beta_{10}=-\beta_{13}$        $\Rightarrow \beta_{10}=0$

- $(xz)t+(zt)x+(tx)z = 0$     $\Rightarrow ct+ \beta_{13}ux- \beta_{11}uz=0$    $\Rightarrow \beta_{14}u=0$     $\beta_{14}=0$

- $(xz)u+(zu)x+(ux)z = 0$    $\Rightarrow 0=0$

- $(xc)r+(cr)x+(rx)c = 0$     $\Rightarrow rr+ \beta_{10}ux-\alpha_4tc=0$     $\Rightarrow -\alpha_4tc=0$    $\Rightarrow \beta_{14}=0$

- $(xc)t+(ct)x+(tx)c = 0$     $\Rightarrow rt+ \beta_{14}ux+(-\beta_{11}u)c=0$    $\Rightarrow \beta_{15}u=0$    $\Rightarrow \beta_{15}=0$

- $(xc)u+(cu)x+(ux)c = 0$    $\Rightarrow 0=0$

- $(xr)t+(rt)x+(tx)r = 0$     $\Rightarrow (t+ \beta_4u)t+ \beta_{15}ux-\beta_{11}ur=0$   $\Rightarrow tt+ \beta_4ut+ \beta_{15}ux-\beta_{11}ur=0$

$\Rightarrow 0=0$

- $(xr)u+(ru)x+(ux)r = 0$    $\Rightarrow 0=0$

- $(xt)u+(tu)x+(ux)t = 0$    $\Rightarrow 0=0$

- $(yz)c+(zc)y+(cy)z = 0$    $\Rightarrow (\alpha_5t+ \beta_5u)c+\beta_8uy+(-\beta_6u)z=0$   $\Rightarrow \alpha_5tc=0$   $\Rightarrow \alpha_5\beta_{14}=0$   $\Rightarrow 0=0$

- $(yz)r+(zr)y+(ry)z = 0$     $\Rightarrow (\alpha_5 t + \beta_5 u)r + \beta_9 uy - \beta_7 uz = 0$     $\Rightarrow \alpha_5 tr = 0 \Rightarrow \alpha_5 \beta_{15} u = 0 \Rightarrow 0 = 0$

- $(yz)t+(zt)y+(ty)z = 0$     $\Rightarrow (\alpha_5 t + \beta_5 u)t + \beta_{13} uy - \beta_{12} uz = 0 \Rightarrow 0 = 0$

- $(yz)u+(zu)y+(uy)z = 0$     $\Rightarrow 0 = 0$

- $(yc)r+(cr)y+(ry)c = 0$     $\Rightarrow \beta_6 ur + \beta_{10} uy + (-\beta_7 u)c = 0$          $\Rightarrow 0 = 0$

- $(yc)t+(ct)y+(ty)c = 0$     $\Rightarrow \beta_6 ut + \beta_{14} uy - \beta_{12} uc = 0$          $\Rightarrow 0 = 0$

- $(yc)u+(cu)y+(uy)c = 0$     $\Rightarrow 0 = 0$

- $(yr)t+(rt)y+(ty)r = 0$     $\Rightarrow (\beta_7 u)t + \beta_{15} uy - \beta_{12} ur = 0$          $\Rightarrow 0 = 0$

- $(yr)u+(ru)y+(uy)r = 0$     $\Rightarrow 0 = 0$

- $(yt)u+(tu)y+(uy)t = 0$     $\Rightarrow 0 = 0$

- $(zc)r+(cr)z+(rz)c = 0$     $\Rightarrow \beta_8 ur + \beta_{10} uz - \beta_9 uc = 0$     $\Rightarrow 0 = 0$

- $(zc)t+(ct)z+(tz)c = 0$     $\Rightarrow \beta_8 ut - \beta_{14} uz - \beta_{13} uc = 0$     $\Rightarrow 0 = 0$

- $(zc)u+(cu)z+(uz)c = 0$     $\Rightarrow 0 = 0$

- $(zr)t+(rt)z+(tz)r = 0$     $\Rightarrow \beta_9 ut + \beta_{15} uz - \beta_{13} ur = 0$     $\Rightarrow 0 = 0$

- $(zr)u+(ru)z+(uz)r = 0$     $\Rightarrow 0 = 0$

• $(zt)u+(tu)z+(uz)t = 0 \Rightarrow 0=0$

• $(cr)t+(rt)c+(tc)r = 0 \Rightarrow \beta_{10}ut+ \beta_{15}uc-\beta_{14}ur=0 \Rightarrow 0=0$

• $(cr)u+(ru)c+(uc)r = 0 \Rightarrow 0=0$

• $(rt)u+(tu)r+(ur)t = 0 \Rightarrow 0=0$

We will now summarize the equations found below:

(1) $\beta_6=\alpha_5\beta_{11}$ (2) $\beta_7=- \beta_8$ (3) $\beta_9=-\beta_{12}$ (4) $\beta_{13}=0$ (5) $\beta_9=0$ (6) $\beta_{10}=0$

(7) $\beta_{14}=0$ (8) $\beta_{15}=0$.

From the preceding work we obtain the following equations.

(i) $\beta_9 = \beta_{10} = \beta_{13}= \beta_{14}= \beta_{15}=0$ (ii) $\beta_7=-\beta_8$ (iii) $\beta_{12}=0$ (iv) $\beta_6 = \alpha_5$

Relabelling yields the non zero multiplication

$[x,y]=z$, $[x,z]=c$, $[x,c]=r$, $[x,r]= t$, $[y,z]= \alpha_5t+\beta_5u$, $[y,c]= \alpha_5u$, $[y,r]=-\beta_8u$,

$[z,c]=\beta_8u$ , $[z,r]=0$, $[c,r]=0$, $[x, t]=u$, $[y, t]= 0$, $[z, t]= 0$, $[c, t]= 0$,

$[r, t]=0$, and $u \in Z(L)$. For t to not be in Z(L), either $\beta_{11}$ or $\beta_{12}$ is not zero, but $\beta_{12}=0$ so

$\beta_{11}\neq0$ thus $[x, t]=u$ by the substitution $u´= \beta_{11}u$ which implies that $\beta_{11}=1$.

To compute the multiplier start with

$[x,y]=z$, $[x,z]=c$, $[x,c]=r$, $[x,r]= t$, $[y,z]= \alpha_5t+\beta_5u+s_5$, $[y,c]= \alpha_5u+s_6$,

$[y,r]=-\beta_8u+s_7$ $[z,c]=\beta_8u+s_8$ , $[z,r]=s_9$, $[c,r]=s_{10}$, $[x, t]=u$, $[y, t]=s_{12}$,

$[z, t]=s_{13}$, $[c, t]=s_{14}$, $[r, t]=s_{15}$, $[x, u]=s_{16}$, $[y, u]=s_{17}$, $[z, u]=s_{18}$,$[c, u]=s_{19}$, $[r, u]=s_{20}$

$[t, u]=s_{21}$, where $s_1,…,s_{21}$ generate the multiplier.

Now we will use the Jacobi identity on all triples.

- $(xy)z+(yz)x+(zx)y = 0$ $\Rightarrow zz+ (\alpha_5 t+ \beta_5 u+s_5)x +(-c)y=0$ $\Rightarrow \alpha_5 tx+ \beta_5 ux+s_5 x-cy=0$

$\Rightarrow \alpha_5(-u)-\beta_5 s_{16}+ \alpha_5 u+s_6=0$ $\Rightarrow s_6= \beta_5 s_{16}$

- $(xy)c+(yc)x+(cx)y = 0$ $\Rightarrow zc+ (\alpha_5 u+s_6)x-ry=0$ $\Rightarrow \beta_8 u+s_8+ \alpha_5 ux+s_6 x-ry=0$

$\Rightarrow \beta_8 u+s_8+ \alpha_5(-s_{16})-(\beta_8 u-s_7)=0$ $\Rightarrow s_8 = \alpha_5 s_{16}-s_7$

- $(xy)r+(yr)x+(rx)y = 0$ $\Rightarrow zr+(-\beta_8 u+s_7)x+(-\alpha_4 t)y=0$ $\Rightarrow s_9-\beta_8 ux+ s_7 x-ty=0$

$\Rightarrow s_9-\beta_8(-s_{16})-\alpha_4(-s_{12})=0$ $\Rightarrow s_9+\beta_8 s_{16}+ s_{12}$ $\Rightarrow s_9=-\beta_8 s_{16}-s_{12}$

- $(xy)t+(yt)x+(tx)y = 0$ $\Rightarrow zt+(s_{12})x+(-u)y=0$ $\Rightarrow s_{13}-(-s_{17})=0$

$\Rightarrow s_{13}=-s_{17}$

- $(xy)u+(yu)x+(ux)y = 0$ $\Rightarrow zu+s_{17}x-s_{16}y=0$ $\Rightarrow s_{18}=0$

- $(xz)c+(zc)x+(cx)z = 0$ $\Rightarrow cc+( \beta_8 u+s_8)x-rz=0$ $\Rightarrow \beta_8 ux+s_8 x+s_9=0$

$\Rightarrow \beta_8(-s_{16})+s_9=0$ $\Rightarrow s_9 =\beta_8 s_{16}$

- $(xz)r+(zr)x+(rx)z = 0$ $\Rightarrow cr+s_9 x+(-t)z=0$ $\Rightarrow s_{10}=-\alpha_4 s_{13}$

- $(xz)t+(zt)x+(tx)z = 0$ $\Rightarrow ct+ s_{13}x+(-u)z=0$ $\Rightarrow s_{14}-uz=0$

$\Rightarrow s_{14}-(-s_{18})=0$ $\Rightarrow s_{14}= -s_{18}$ $\Rightarrow s_{14}=0$

- $(xz)u+(zu)x+(ux)z = 0$ $\Rightarrow cu + s_{18}x- s_{16}z=0$ $\Rightarrow s_{19}=0$

- $(xc)r+(cr)x+(rx)c = 0$     $\Rightarrow rr+s_{10}x+(-t)c=0$     $\Rightarrow -tc=0$     $\Rightarrow s_{14}=0$

- $(xc)t+(ct)x+(tx)c = 0$     $\Rightarrow rt+(tx)c=0$     $\Rightarrow s_{15}=- s_{19}$   $\Rightarrow s_{15}=0$

- $(xc)u+(cu)x+(ux)c = 0$     $\Rightarrow ru + s_{19}x- s_{16}c=0$     $\Rightarrow s_{20}=0$

- $(xr)t+(rt)x+(tx)r = 0$     $\Rightarrow -ur=0$     $\Rightarrow s_{20}=0$

- $(xr)u+(ru)x+(ux)r = 0$     $\Rightarrow tu+s_{20}x- s_{16}r=0$     $\Rightarrow s_{21}=0$

- $(xt)u+(tu)x+(ux)t = 0$     $\Rightarrow s_{21}x- s_{16}t=0$     $\Rightarrow 0=0$

- $(yz)r+(zr)y+(ry)z = 0$     $\Rightarrow (\alpha_5 t+ \beta_5 u+s_5)r+s_9y+ (\beta_8 u-s_7)z=0$

$\Rightarrow (\alpha_5 tr+ \beta_5 ur+s_5r)+\beta_8 uz$    $\Rightarrow -\alpha_5 s_{15}-\beta_5 s_{20}- \beta_8 s_{18}=0$    $\Rightarrow \alpha_5 s_{15}= \beta_8 s_{18}=0$

- $(yz)c+(zc)y+(cy)z = 0$     $\Rightarrow (\alpha_5 t+ \beta_5 u)c+ \beta_8 uy-\alpha_5 uz=0$

$\Rightarrow -\alpha_5 s_{14}-\beta_8 s_{17}-\alpha_5 s_{18}=0$     $\Rightarrow \beta_8 s_{17}=0$

- $(yz)t+(zt)y+(ty)z = 0$     $\Rightarrow (\alpha_5 t+ \beta_5 u+s_5)t+s_{13}y+(-s_{12})z=0$   $\Rightarrow \beta_5 ut =0$

$\Rightarrow -\beta_5 s_{21}=0$     $\Rightarrow \beta_5 s_{21}= 0$

- $(yz)u+(zu)y+(uy)z = 0$     $\Rightarrow (\alpha_5 t+ \beta_5 u+s_5)u+s_{18}y-s_{17}z=0$   $\Rightarrow \alpha_5 tu+ \beta_5 uu=0 \Rightarrow \alpha_5 s_{21}=0$

- $(yc)r+(cr)y+(ry)c = 0$     $\Rightarrow (\alpha_5 u+s_6)r+s_{10}y+(\beta_8 u-s_7)c=0 \Rightarrow \alpha_5 ur+ s_6r+s_{10}y+\beta_8 uc=0$

$\Rightarrow \alpha_5(-s_{20})+ \beta_8(s_{19})=0$     $\Rightarrow -\beta_8 s_{19}=\alpha_5 s_{20}$

• $(yc)t+(ct)y+(ty)c = 0$ $\Rightarrow (\alpha_5 u+s_6)t+ s_{14}y+s_{12}c=0$ $\Rightarrow \alpha_5 ut=0$ $\Rightarrow \alpha_5 s_{21}=0$

• $(yc)u+(cu)y+(uy)c = 0$ $\Rightarrow (\alpha_5 u+s_6)u+ s_{19}y+s_{17}c=0$ $\Rightarrow \alpha_5 uu+s_6 u=0$ $\Rightarrow 0=0$

• $(yr)t+(rt)y+(ty)r = 0$ $\Rightarrow (-\beta_8 u+s_7)t+s_{15}y+(-s_{12})r=0$ $\Rightarrow -\beta_8 ut =0$

$\Rightarrow \beta_8 s_{21}=0$ $\Rightarrow 0=0$

• $(yr)u+(ru)y+(uy)r = 0$ $\Rightarrow (-\beta_8 u+s_7)u+s_{20}y-s_{17}r=0$ $\Rightarrow -\beta_8 uu=0$ $\Rightarrow 0=0$

• $(yt)u+(tu)y+(uy)t = 0$ $\Rightarrow (s_{12})u+ s_{12}y- s_{17}t=0$ $\Rightarrow 0=0$

• $(zc)r+(cr)z+(rz)c = 0$ $\Rightarrow ( \beta_8 u+s_8)r+s_{10}z-s_9 c=0$ $\Rightarrow \beta_8 ur=0$ $\Rightarrow 0=0$

• $(zc)t+(ct)z+(tz)c = 0$ $\Rightarrow (\beta_8 u+s_8)t+s_{14}z+-s_{13}c=0$ $\Rightarrow \beta_8 ut=0$ $\Rightarrow 0=0$

• $(zc)u+(cu)z+(uz)c = 0$ $\Rightarrow ( \beta_8 u+s_8)u+s_{19}z-s_{18}c=0$ $\Rightarrow 0=0$

• $(zr)t+(rt)z+(tz)r = 0$ $\Rightarrow s_9 z+ s_{15}z-s_{13}r=0$ $\Rightarrow 0=0$

• $(zr)u+(ru)z+(uz)r = 0$ $\Rightarrow s_9 u+ s_{20}z-s_{18}r=0$ $\Rightarrow 0=0$

• $(zt)u+(tu)z+(uz)t = 0$ $\Rightarrow s_{13}u+ s_{21}z-s_{18}t=0$ $\Rightarrow 0=0$

• $(cr)t+(rt)c+(tc)r = 0$ $\Rightarrow s_{10}t+ s_{15}c-s_{14}r=0$ $\Rightarrow 0=0$

• $(cr)u+(ru)c+(uc)r = 0$ $\Rightarrow s_{10}u+ s_{20}c-s_{19}r=0$ $\Rightarrow 0=0$

• $(rt)u+(tu)r+(ur)t = 0$ $\Rightarrow s_{15}u+ s_{21}r-s_{20}t=0$ $\Rightarrow 0=0$

We now summarize the equations found below:

(1) $s_6 = \beta_5 s_{16}$ (2) $s_8 = \alpha_5 s_{16} - s_7$ (3) $s_9 = -\beta_8 s_{16} - s_{12}$ (4) $s_{13} = -s_{17}$

(5) $s_{18} = 0$ (6) $s_9 = \beta_8 s_{16}$ (7) $s_{10} = -s_{13}$ (8) $s_{14} = 0$ (9) $s_{19} = 0$ (10) $s_{15} = -s_{19} = 0$ (11) $s_{20} = 0$

(12) $s_{21} = 0$ (13) $\beta_8 s_{17} = 0$

From the preceding work and equation (13) above we have two cases. $\beta_8 s_{17} = 0$ implies

Either Case (A) $s_{17} = 0$ or Case (B) $s_{17} \neq 0$ and $\beta_8 = 0$

In Case (A), if $s_{17} = 0$ then $s_{21} = s_{20} = s_{19} = s_{18} = s_{15} = s_{14} = s_{13} = s_{10} = 0$ and $s_6 = \beta_5 s_{16}$, $s_8 = \alpha_5 s_{16} - s_7$

$s_9 = \beta_8 s_{16}$, $s_{12} = -\beta_8 s_{16} - s_9$ which implies $s_6, s_8, s_9, s_{12} \in (s_5, s_7, s_{16})$ thus dimM(L)=3

which implies $t(L) = \frac{1}{2} n(n-1) - \dim M(L) = \frac{1}{2}(7)(6) - 3 = 18$, we call this algebra

L=L(17, 7, 1,18).

In Case (B), $s_{17} \neq 0$ and $\beta_8 = 0$ then $s_{21} = s_{20} = s_{19} = s_{18} = s_{15} = s_{14} = 0$ and $s_6 = \beta_5 s_{16}$, $s_8 = \alpha_5 s_{16} - s_7$

$s_9 = \beta_8 s_{16}$, $s_{12} = -s_9$, $s_{10} = -s_{13} = s_{17}$ which implies $s_6, s_8, s_9, s_{12}, s_{10}, s_{13} \in (s_5, s_7, s_{16}, s_{17})$ thus

dimM(L)=4 which implies $t(L) = \frac{1}{2} n(n-1) - \dim M(L) = \frac{1}{2}(7)(6) - 4 = 17$, we call this algebra

L=L(17, 7, 1,17).

**CASE 2.** Let H=L´(11, 6, 1, 12), this L can be described generally by the basis

{x, y, z, c, r, t, u} and multiplication

$[x,y]=z$, $[x,z]=c$, $[x,c]=r$, $[x,r]=t$, $[y,z]=r+\alpha_5 t+\beta_5 u$, $[y,c]=t+\beta_6 u$, $[y,r]=\beta_7 u$,

$[z,c]=\beta_8 u$, $[z,r]=\beta_9 u$, $[c,r]=\beta_{10}u$, $[x,t]=\beta_{11}u$, $[y,t]=\beta_{12}u$, $[z,t]=\beta_{13}u$, $[c,t]=\beta_{14}u$,

$[r,t]=\beta_{15}u$, and $u \in Z(L)$. Now we will use the Jacobi Identity on all triples.

- $(xy)z+(yz)x+(zx)y = 0$  $\Rightarrow zz+ (r+\alpha_5 t+\beta_5 u)x +(-c)y=0$  $\Rightarrow rx+\alpha_5 tx+\beta_5 ux-(-t-\beta_6 u)=0$

$\Rightarrow -t-\alpha_5\beta_{11}+t+\beta_6 u=0$  $\Rightarrow \beta_6 =\alpha_5\beta_{11}$

- $(xy)c+(yc)x+(cx)y = 0$  $\Rightarrow zc+(t+\beta_6 u)x-ry=0$  $\Rightarrow \beta_8 u+tx+\beta_6 ux-ry=0$

$\Rightarrow \beta_8 u-\beta_{11}u+\beta_7 u=0$  $\Rightarrow \beta_7=\beta_{11}-\beta_8$

- $(xy)r+(yr)x+(rx)y = 0$  $\Rightarrow zr+\beta_7 ux+(-t)y =0$  $\Rightarrow \beta_9 u-ty =0$  $\Rightarrow \beta_9=-\beta_{12}$

- $(xy)t+(yt)x+(tx)y = 0$  $\Rightarrow zt + \beta_{12}ux-\beta_{11}uy=0$  $\Rightarrow \beta_{13}u=0$  $\Rightarrow \beta_{13}=0$

- $(xy)u+(yu)x+(ux)y = 0$  $\Rightarrow 0=0$

- $(xz)c+(zc)x+(cx)z = 0$  $\Rightarrow cc+(\beta_8 u)x-rz=0$  $\Rightarrow \beta_9=0$

- $(xz)r+(zr)x+(rx)z = 0$  $\Rightarrow cr+\beta_9 ux+(-t-\beta_4 u)z=0$  $\Rightarrow \beta_{10}u-tz =0$  $\Rightarrow \beta_{10}=-\beta_{13}$

- $(xz)t+(zt)x+(tx)z = 0$  $\Rightarrow ct+\beta_{13}ux-\beta_{11}uz=0$  $\Rightarrow \beta_{14}u=0$  $\Rightarrow \beta_{14}=0$

- $(xz)u+(zu)x+(ux)z = 0$  $\Rightarrow 0=0$

- $(xc)r+(cr)x+(rx)c = 0$  $\Rightarrow rr+\beta_{10}ux+(-t)c=0$  $\Rightarrow \beta_{14}=0$

- $(xc)t+(ct)x+(tx)c = 0 \quad \Rightarrow rt+ \beta_{14}ux+(-\beta_{11}u)c=0 \quad \Rightarrow \beta_{15}u=0 \quad \Rightarrow \beta_{15}=0$

- $(xc)u+(cu)x+(ux)c = 0 \quad \Rightarrow 0=0$

- $(xr)t+(rt)x+(tx)r = 0 \quad \Rightarrow (t)t+ \beta_{15}ux+(-\beta_{11}u)r=0 \quad \Rightarrow 0=0$

- $(xr)u+(ru)x+(ux)r = 0 \quad \Rightarrow 0=0$

- $(xt)u+(tu)x+(ux)t = 0 \quad \Rightarrow 0=0$

- $(yz)c+(zc)y+(cy)z = 0 \quad \Rightarrow (r+ \alpha_5 t+ \beta_5 u)c+(\beta_8 u)y+(-t-\beta_6 u)z=0$

$\Rightarrow rc+ \alpha_5 tc+ \beta_5 uc+ \beta_8 uy-tz-\beta_6 uz=0 \quad \Rightarrow -\beta_{10}u- \alpha_5\beta_{14}u+ \beta_{13}u=0 \quad \Rightarrow \beta_{13}= \beta_{10}+ \alpha_5\beta_{14}$

- $(yz)r+(zr)y+(ry)z = 0 \quad \Rightarrow (r+ \alpha_5 t+ \beta_5 u)r+ \beta_9 uy+( \beta_7 u)z=0 \quad \Rightarrow rr + \alpha_5 tr+ \beta_5 ur=0$

$\Rightarrow -\alpha_5\beta_{15}u=0 \quad \Rightarrow \alpha_5 \beta_{15}=0$

- $(yz)t+(zt)y+(ty)z = 0 \quad \Rightarrow (r+ \alpha_5 t+ \beta_5 u)t + \beta_{13}uy- \beta_{12}uz=0 \quad \Rightarrow rt + \alpha_5 tt+ \beta_5 ut=0$

$\Rightarrow \beta_{15}u=0 \quad \Rightarrow \beta_{15}=0$

- $(yz)u+(zu)y+(uy)z = 0 \quad \Rightarrow 0=0$

- $(yc)r+(cr)y+(ry)c = 0 \quad \Rightarrow (t+\beta_6 u)r+ \beta_{10}uy+(-\beta_7 u)c=0 \quad \Rightarrow tr+\beta_6 ur=0$

$\Rightarrow \beta_{15}= 0$

- $(yc)t+(ct)y+(ty)c = 0 \quad \Rightarrow (t+\beta_6 u)t+ \beta_{14}uy-\beta_{12}uc=0 \quad \Rightarrow 0=0$

- $(yc)u+(cu)y+(uy)c = 0 \Rightarrow 0=0$

- $(yr)t+(rt)y+(ty)r = 0 \Rightarrow \beta_7ut+ \beta_{15}uy-\beta_{12}ur=0 \Rightarrow 0=0$

- $(yr)u+(ru)y+(uy)r = 0 \Rightarrow 0=0$

- $(yt)u+(tu)y+(uy)t = 0 \Rightarrow 0=0$

- $(zc)r+(cr)z+(rz)c = 0 \Rightarrow (\beta_8u)r+ \beta_{10}uz-\beta_9uc=0 \Rightarrow 0=0$

- $(zc)t+(ct)z+(tz)c = 0 \Rightarrow (\beta_8u)t+\beta_{14}uz-\beta_{13}uc=0 \Rightarrow 0=0$

- $(zc)u+(cu)z+(uz)c = 0 \Rightarrow 0=0$

- $(zr)t+(rt)z+(tz)r = 0 \Rightarrow \beta_9ut+ \beta_{15}uz-\beta_{13}ur=0 \Rightarrow 0=0$

- $(zr)u+(ru)z+(uz)r = 0 \Rightarrow 0=0$

- $(zt)u+(tu)z+(uz)t = 0 \Rightarrow 0=0$

- $(cr)t+(rt)c+(tc)r = 0 \Rightarrow \beta_{10}ut+ \beta_{15}uc-\beta_{14}ur=0 \Rightarrow 0=0$

- $(cr)u+(ru)c+(uc)r = 0 \Rightarrow 0=0$

- $(rt)u+(tu)r+(ur)t = 0 \Rightarrow 0=0$

We will now summarize the equations found below:

(1)$\beta_6 = \alpha_5\beta_{11}$ (2) $\beta_7 = \beta_{11} - \beta_8$ (3) $\beta_9 = -\beta_{12}$ (4) $\beta_{13} = 0$ (5) $\beta_9 = 0$

(6) $\beta_{10} = -\beta_{13}$ (7) $\beta_{14} = 0$ (8) $\beta_{13} = \beta_{10} + \alpha_5\beta_{14}$ (9) $\beta_{15} = 0$ (10) $\alpha_5\beta_{15} = 0$

From the preceding work we obtain the following equations.

(i) $\beta_9 = \beta_{10} = \beta_{12} = \beta_{13} = \beta_{14} = \beta_{15} = 0$ (ii) $\beta_8 = 1 - \beta_7$ (iii) $\alpha_5 = \beta_6$

Relabelling yields the non zero multiplication

$[x,y] = z$, $[x,z] = c$, $[x,c] = r$, $[x,r] = t$, $[y,z] = r + \alpha_5 t + \beta_5 u$, $[y,c] = t + \beta_6 u$, $[y,r] = \beta_7 u$,

$[z,c] = \beta_8 u$, $[z,r] = 0$, $[c,r] = 0$, $[x, t] = \beta_{11}u$, $[y, t] = 0$, $[z, t] = 0$, $[c, t] = 0$,

$[r, t] = 0$, and $u \in Z(L)$. Hence for $t \notin Z(L)$, $\beta_{11} \neq 0$ and we can use $xt = u$ for multiplier .

To compute the multiplier start with

$[x,y] = z$, $[x,z] = c$, $[x,c] = r$, $[x,r] = t$, $[y,z] = r + \alpha_5 t + \beta_5 u + s_5$, $[y,c] = t + \beta_6 u + s_6$,

$[y,r] = \beta_7 u + s_7$ $[z,c] = \beta_8 u + s_8$, $[z,r] = s_9$, $[c,r] = s_{10}$, $[x, t] = u$, $[y, t] = s_{12}$,

$[z, t] = s_{13}$, $[c, t] = s_{14}$, $[r, t] = s_{15}$, $[x, u] = s_{16}$, $[y, u] = s_{17}$, $[z, u] = s_{18}$, $[c, u] = s_{19}$, $[r, u] = s_{20}$

$[t, u] = s_{21}$, where $s_1, \ldots, s_{21}$ generate the multiplier.

Now we will use the Jacobi identity on all triples.

• $(xy)z + (yz)x + (zx)y = 0$ $\Rightarrow zz + (r + \alpha_5 t + \beta_5 u + s_5)x + (-c)y = 0$ $\Rightarrow rx + \alpha_5 tx + \beta_5 ux + s_5 x - cy = 0$

$\Rightarrow rx + \alpha_5 tx + \beta_5 ux - (-t - \beta_6 u - s_6) = 0$ $\Rightarrow -t + \alpha_5(-u) + t + \beta_6 u + s_6 = 0$

$\Rightarrow s_6 = \beta_5 s_{16}$

- $(xy)c+(yc)x+(cx)y = 0$ $\Rightarrow zc+ (t+\beta_6 u+s_6)x-ry=0$

$\Rightarrow \beta_8 u+s_8-u-\beta_6 s_{16}+\beta_7 u+s_7 =0$ $\Rightarrow s_8+\beta_6(-s_{16})+s_7=0$

$\Rightarrow s_8=\beta_6 s_{16}-s_7$

- $(xy)r+(yr)x+(rx)y = 0$ $\Rightarrow zr+(\beta_7 u+s_7)x-ty=0$

$\Rightarrow s_9 +\beta_7(-s_{16})-s_{12}=0$ $\Rightarrow s_{12}= s_9-\beta_7 s_{16}$

- $(xy)t+(yt)x+(tx)y = 0$ $\Rightarrow zt+ s_{12}x+(-u)y=0$ $\Rightarrow s_{13}+ s_{17}=0$ $\Rightarrow s_{13}=-s_{17}$

- $(xy)u+(yu)x+(ux)y = 0$ $\Rightarrow zu+s_{17}x-s_{16}y=0$ $\Rightarrow s_{18}=0$

- $(xz)c+(zc)x+(cx)z = 0$ $\Rightarrow cc+( \beta_8 u+s_8)x-rz=0$ $\Rightarrow \beta_8 ux+s_8 x+s_9=0$

$\Rightarrow \beta_8(-s_{16})+s_9=0$ $\Rightarrow s_9 =\beta_8 s_{16}$

- $(xz)r+(zr)x+(rx)z = 0$ $\Rightarrow cr+s_9 x-tz=0$ $\Rightarrow s_{10}+ s_{13}=0$ $\Rightarrow s_{10}=-s_{13}$

- $(xz)t+(zt)x+(tx)z = 0$ $\Rightarrow ct+ s_{13}x-uz=0$ $\Rightarrow s_{14}+ s_{18}=0$ $\Rightarrow s_{14}=-s_{18}$

- $(xz)u+(zu)x+(ux)z = 0$ $\Rightarrow cu + s_{18}x- s_{16}z=0$ $\Rightarrow s_{19}=0$

- $(xc)r+(cr)x+(rx)c = 0$ $\Rightarrow -tc=0$ $\Rightarrow s_{14}=0$

- $(xc)t+(ct)x+(tx)c = 0$ $\Rightarrow rt + s_{14}x -uc =0$ $\Rightarrow s_{15}+s_{19}=0$ $\Rightarrow s_{15}=0$

- $(xc)u+(cu)x+(ux)c = 0$ $\Rightarrow ru + s_{19}x- s_{16}c=0$ $\Rightarrow s_{20}=0$

- $(xr)t+(rt)x+(tx)r = 0$     $\Rightarrow -ur=0 \Rightarrow s_{20}=0$

- $(xr)u+(ru)x+(ux)r = 0$     $\Rightarrow s_{21}=0$

- $(xt)u+(tu)x+(ux)t = 0$     $\Rightarrow 0=0$

- $(yz)r+(zr)y+(ry)z = 0$     $\Rightarrow (r+\alpha_5 t+ \beta_5 u+s_5)r+s_9 y+ (-\beta_7 u-s_7)z=0$

$\Rightarrow (rr+\alpha_5 tr+ \beta_5 ur+s_5 r)-\beta_7 uz=0$   $\Rightarrow -\alpha_5 s_{15}-\beta_5 s_{20}+\beta_7 s_{18}=0$   $\Rightarrow \alpha_5 s_{15}= 0$

- $(yz)c+(zc)y+(cy)z = 0$     $\Rightarrow (r+\alpha_5 t+ \beta_5 u+s_5)c+(\beta_8 u+s_8)y+ (-t-\beta_6 u-s_6)z=0$

$\Rightarrow rc+\alpha_5 tc+ \beta_5 uc+s_5 c+\beta_8 uy+s_8 y-tz-\beta_6 uz-s_6 z=0$

$\Rightarrow -s_{10}+ \alpha_5(-s_{14})+ \beta_5(-s_{19})+\beta_8(-s_{17})-(-s_{13})- \beta_6(-s_{18})=0$

$\Rightarrow s_{10}=-\beta_8 s_{17}+s_{13}$

- $(yz)t+(zt)y+(ty)z = 0$     $\Rightarrow (r+\alpha_5 t+ \beta_5 u+s_5)t+s_{13}y+(-s_{12})z=0$

$\Rightarrow s_{15}=0$

- $(yz)u+(zu)y+(uy)z = 0$     $\Rightarrow (r+\alpha_5 t+ \beta_5 u+s_5)u+s_{18}y-s_{17}z=0$   $\Rightarrow ru+ \alpha_5 tu+ \beta_5 uu=0$

$\Rightarrow s_{20}=-\alpha_5 s_{21}$           $\Rightarrow 0=0$

- $(yc)r+(cr)y+(ry)c = 0$     $\Rightarrow (t+\beta_6 u+s_6)r+s_{10}y+(-\beta_7 u-s_7)c=0$

$\Rightarrow tr+\beta_6 ur+ s_6 r+s_{10}y-\beta_7 uc=0$

$\Rightarrow$ -$s_{15}$+ $\beta_6$(-$s_{20}$)+$\beta_7$($s_{19}$)=0    $\Rightarrow$ $s_{19}$ =0

- (yc)t+(ct)y+(ty)c = 0      $\Rightarrow$ (t+$\beta_6$u+$s_6$)t+$s_{14}$y+(-$s_{12}$)c=0

$\Rightarrow$ tt+$\beta_6$ut+$s_6$t-$s_{12}$c=0   $\Rightarrow$ $\beta_6$(-$s_{21}$)=0         $\Rightarrow$ $\beta_6 s_{21}$=0      $\Rightarrow$ 0=0

- (yc)u+(cu)y+(uy)c = 0      $\Rightarrow$ (t+$\beta_6$u+$s_6$)u+ $s_{19}$y-$s_{17}$c=0       $\Rightarrow$ tu=0             $\Rightarrow$ $s_{21}$=0

- (yr)t+(rt)y+(ty)r = 0      $\Rightarrow$ ($\beta_7$u+$s_7$)t+$s_{15}$y+(-$s_{12}$)r=0       $\Rightarrow$ 0=0

$\Rightarrow$ $\beta_7$ut+$s_7$t-$\beta_{12}$ur=0      $\Rightarrow$ $\beta_7$ (-$s_{21}$)-$\beta_{12}$(-$s_{20}$)=0         $\Rightarrow$ $\beta_{12}s_{20}$= $\beta_7 s_{21}$

- (yr)u+(ru)y+(uy)r = 0     $\Rightarrow$ ($\beta_7$u+$s_7$)u+$s_{20}$y-$s_{17}$r=0        $\Rightarrow$ $\beta_7$uu=0      $\Rightarrow$ 0=0

- (yt)u+(tu)y+(uy)t = 0     $\Rightarrow$ ($s_{12}$ )u + $s_{21}$y- $s_{17}$t=0     $\Rightarrow$ 0=0

- (zc)r+(cr)z+(rz)c = 0      $\Rightarrow$ ( $\beta_8$u+$s_8$)r+$s_{10}$z-$s_9$c=0        $\Rightarrow$ $\beta_8$ur=0      $\Rightarrow$ 0=0

- (zc)t+(ct)z+(tz)c = 0      $\Rightarrow$ ($\beta_8$u+$s_8$)t+$s_{14}$z-$s_{13}$c=0        $\Rightarrow$ $\beta_8$ut=0      $\Rightarrow$ 0=0

- (zc)u+(cu)z+(uz)c = 0    $\Rightarrow$ ( $\beta_8$u+$s_8$)u+$s_{19}$z-$s_{18}$c=0      $\Rightarrow$ 0=0

- (zr)t+(rt)z+(tz)r = 0      $\Rightarrow$ $s_9$z+ $s_{15}$z-$s_{13}$r=0        $\Rightarrow$ 0=0

- (zr)u+(ru)z+(uz)r = 0      $\Rightarrow$ $s_9$u+ $s_{20}$z-$s_{18}$r=0        $\Rightarrow$ 0=0

- (zt)u+(tu)z+(uz)t = 0      $\Rightarrow$ $s_{13}$u+ $s_{21}$z-$s_{18}$t=0        $\Rightarrow$ 0=0

- (cr)t+(rt)c+(tc)r = 0      $\Rightarrow$ $s_{10}$t+ $s_{15}$c-$s_{14}$r=0        $\Rightarrow$ 0=0

• $(cr)u + (ru)c + (uc)r = 0$     $\Rightarrow s_{10}u + s_{20}c - s_{19}r = 0$       $\Rightarrow 0 = 0$

• $(rt)u + (tu)r + (ur)t = 0$     $\Rightarrow s_{15}u + s_{21}r - s_{20}t = 0$       $\Rightarrow 0 = 0$

We now summarize the equations found below:

(1) $s_6 = -\beta_5 s_{16}$   (2) $s_8 = \beta_6 s_{16} - s_7$   (3) $s_{12} = s_9 - \beta_7 s_{16}$   (4) $s_{13} = -s_{17}$

(5) $s_9 = \beta_8 s_{16}$   (6) $s_{13} = -s_{10}$   (7) $s_{10} = -\beta_8 s_{17} + s_{13}$   (8) $s_{19} = 0$

(9) $s_{21} = s_{20} = s_{19} = s_{18} = s_{15} = s_{14} = 0$

From the preceding work we have $\dim M(L) = (s_{17}, s_{16}, s_7, s_5) = 4$.

Thus $t(L) = t(L) = \frac{1}{2}n(n-1) - \dim M(L) = \frac{1}{2}(7)(6) - 3 = 17$, we call this algebra $L = L'(17, 7, 1, 17)$,

And we summarize our result in the following theorem.

**<u>Theorem 20</u>** Let L be filiform with $t(L) = 17$ then $L = L(17,7,1,17)$ or $L = L'(17, 7, 1, 17)$.

## 18. t(L)=18

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 18$ then

$c^2 + c \leq 36$ which implies that $c \leq 5$ therefore $c = 0, 1, 2, 3, 4$ or $5$ but by proposition 0

(i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $18 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving

this inequality we know that $c = 5$ is the only case that satisfies the inequality. Thus if $c =$

$\dim L^2 = 5$, then $\dim H^2 = 4$ and $t(H) \leq 13$(proposition 3). In the t(L)=17 case we have

already computed this algebra and we list it in the following theorem.


**Theorem 21** Let L be filiform with t(L)=18, then L=L(17, 7, 1, 18).

## 19. t(L)=19

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 19$ then

$c^2 + c \leq 38$ which implies that $c \leq 5$ therefore $c = 0, 1, 2, 3, 4$ or $5$ but by proposition 0

(i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $19 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving

this inequality we know that $c = 5$ is the only case that satisfies the inequality. Thus if

$c = \dim L^2 = 5$, then $\dim H^2 = 4$ and $t(H) \leq 14$ (proposition 3), but again from Table 22.1,

we know that there are no filiform Lie algebras possible under these conditions,

thus we have the following Theorem.

**<u>Theorem 22</u>** There are no filiform Lie algebras with $t(L) = 19$.

## 20. t(L)=20

By Theorem 4, we know that $c^2 + c \le 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 20$ then

$c^2 + c \le 40$ which implies that $c \le 5$ therefore $c = 0, 1, 2, 3, 4$ or 5 but by proposition 0

(i.e. $t \le \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $20 \le \frac{1}{2}(c + 2)(c + 1)$ and by solving

this inequality we know that $c = 5$ is the only case that satisfies the inequality. Thus if $c =$

$\dim L^2 = 5$, then $\dim H^2 = 4$ and $t(H) \le 15$ (proposition 3), but from Table 22.1, we

know that there are no filiform Lie algebras possible under these conditions, thus we have

the following Theorem.

**<u>Theorem 23</u>** There are no filiform Lie algebras with $t(L) = 20$.

## 21. t(L)=21

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 21$ then

$c^2 + c \leq 42$ which implies that $c \leq 6$ therefore $c = 0, 1, 2, 3, 4, 5$ or $6$ but by proposition $0$

(i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $21 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving

this inequality we know that $c = 5$ and $c=6$ are the two cases that satisfy the inequality.

Thus if $c = \dim L^2 = 5$, then $\dim H^2 = 4$ and $t(H) \leq 16$ (proposition 3), but from Table

22.1, we know that there are no filiform Lie algebras possible under these conditions.

Also if $c = \dim L^2 = 6$, then $\dim H^2 = 5$ and $t(H) \leq 15$ (proposition 3), but again from

Table 22.1, we know that there are no filiform Lie algebras possible under these

conditions, thus we have the following Theorem.

**Theorem 24** There are no filiform Lie algebras with $t(L) = 21$.

## 22. SUMMARY

The following table summarizes the findings of this paper :

**TABLE 22.1 The Filiform Lie Algebras With t(L)=3 Through t(L)=21.**

| t(L) | dimL | Basis | Non Zero Multiplication | Filiform Lie Algebra |
|------|------|-------|-------------------------|----------------------|
| 4 | 4 | {x,y,z,r} | [x,y]=z,  [x,z]=r | L=L(3, 4, 1, 4) |
| 7 | 5 | {x,y,z,c,r} | [x,y]=z,  [x,z]=c,  [x,c]=r   OR<br>[x,y]=z,  [x,z]=c, [y,z]=r [x,c]=r | L=L(7, 5, 1, 7) OR<br>L=L′(7, 5, 1, 7) |
| 11 | 6 | {x,y,z,c,r,t}<br>1= $\alpha_7 \neq 0$ | [x,y]=z,  [x,z]=c,  [x,c]=r,<br>[x,r]=$\alpha_4$t, [y,z]=$\alpha_5$t, [y,c]=0<br>[y,r]=t, [z,c]=-t, [z,r]=0<br>[c,r]=0<br>OR<br>[x,y]=z,  [x,z]=c,  [x,c]=r,<br>[x,r]=$\alpha_4$t, [y,z]=r+$\alpha_5$t, [y,c]=$\alpha_4$t<br>[y,r]=t, [z,c]=-t, [z,r]=0<br>[c,r]=0 | L=L(11,6,1,11) OR<br>L=L′(11,6,1,11) |
| 12 | 6 | {x,y,z,c,r,t} | [x,y]=z,  [x,z]=c,  [x,c]=r<br>[x,r]=t, [y,z]=$\alpha_5$t, [y,c]=0<br>[y,r]=0, [z,c]=0, [z,r]=0<br>[c,r]= 0<br>OR<br>[x,y]=z,  [x,z]=c,  [x,c]=r,<br>[x,r]=t, [y,z]=r+$\alpha_5$t, [y,c]=t<br>[y,r]=0, [z,c]=0, [z,r]=0<br>[c,r]=0 | L=L(11,6,1,12) OR<br>L=L′(11,6,1,12) |
| 17 | 7 | {x,y,z,c,r,t,u} | [x,y]=z,  [x,z]=c,  [x,c]=r<br>[x,r]=t, [y,z]= $\alpha_5$t+ $\beta_5$u, [y,c]= $\beta_6$u<br>[y,r]=0, [z,c]= 0, [z,r]= 0<br>[c,r]= 0,[x,t]= 0, [y,t]=0<br>[z,t]= 0, [c,t]= 0, [r,t]= 0<br>OR<br>[x,y]=z,[x,z]=c,[x,c]=r,[x,r]=t+$\beta_4$u<br>[y,z]=r+$\alpha_5$t+$\beta_5$u, [y,c]=t+$\beta_6$u<br>[y,r]= $\beta_7$u, [z,c]= $\beta_8$u, [z,r]= 0<br>[c,r]= 0,[x,t]= 0, [y,t]=0<br>[z,t]= 0, [c,t]= 0, [r,t]= 0 | L=L(17,7,1,17)<br>$\beta_6$= $\alpha_5$<br><br><br><br><br>OR<br>L=L′(17,7,1,17)<br>$\beta_6$= $\alpha_5$ and<br>$\beta_8$=1- $\beta_7$ |

**TABLE 22.1 (continued)**

| 18 | 7 | {x,y,z,c,r,t,u} | [x,y]=z, [x,z]=c, [x,c]=r, | L=L(17,7,1,18) |
|---|---|---|---|---|
| | | | [x,r]=t, [y,z]= $\alpha_5 t + \beta_5 u$, [y,c]= $\beta_6 u$ | $B_7 = -\beta_8$ |
| | | | [y,r]= $\beta_7 u$, [z,c]= $\beta_8 u$, [z,r]= 0 | |
| | | | [c,r]= 0, [x,t]= 0, [y,t]= 0 | |
| | | | [z,t]= 0, [c,t]= 0, [r,t]= 0 | |

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