

ABSTRACT

ZHENG,QIAN. Sum of Squares Based Nonlinear Control Design Techniques. (Under the direction of Dr. Fen Wu).

Many practical control systems have inherent nonlinear characteristics that cannot be ignored. However, the analysis and control of nonlinear systems are among the most challenging problems in systems and control theory. In this dissertation, we focus on Sum of Squares (SOS)-based nonlinear control designs for a class of nonlinear systems with polynomial vector fields to achieve stabilization and optimal control criteria.

SOS is a powerful and promising technique which has been widely used in recent years. It provides an efficient way for researchers to explore polynomial nonlinear systems, which can be used to approximate many nonlinear systems. The SOS decomposition has been used for nonlinear system analysis successfully; however, its potential for control system synthesis and design has not been fully explored. Therefore, in this research, the usage of newly emerging SOS decomposition for optimal and adaptive controller design is a unique feature, which has not been addressed in the control literature to date.

In the first part of this dissertation, we study the stabilization of state-dependent /output-dependent polynomial nonlinear systems using rational Lyapunov Functions. A direct consequence of this approach is that the Lyapunov functions can be generalized to pre-specified forms. Therefore, compared to quadratic Lyapunov function, it provides great potential to find Lyapunov certificates through systematic SOS programming and expands the domains of attraction for nonlinear systems. This has been demonstrated on a controlled Van der Pol nonlinear oscillator.

Beyond stabilization problems, we further investigate the performance improvements of nonlinear control systems in terms of $\mathcal{H}_\infty$ control and adaptive control. As is known, the nonlinear $\mathcal{H}_\infty$ control problem is formulated as Hamilton-Jacobi-Isaacs (HJI) equations/inequalities. However, their solutions are extremely difficult not only to compute but also to represent. In the second part, we relax the stringent orthonormal and decoupling assumptions on the plant structure and develop the generalized HJI inequalities, then we focus on materializing $\mathcal{H}_\infty$ theory into an algorithmic procedure for polynomial nonlinear systems. Based on the idea of SOS programming, the nonlinear $\mathcal{H}_\infty$ control problem can be solved by reformulating HJI inequalities into convex optimization conditions. Direct

applications include high-precision spacecraft attitude regulation and command tracking problems.

Finally, we study SOS-based Lyapunov redesign of adaptive controllers for uncertain polynomial nonlinear systems with matching conditions. By developing adaptive control schemes to attenuate the effects of unknown parameters on the controlled output, the Lyapunov functions are extended from quadratic to higher order and the resulting control gains are generalized from constant to parameter dependent. The synthesis conditions could be formulated as polynomial matrix inequalities and are thus solvable by recasting the resulting conditions into an SOS optimization problem. Resorting to an iterative algorithm and vector projection technique, parallel results are also developed for the un-matched parameter uncertainty case. It turns out that for both cases, the constructed adaptive control law as well as the parameter adaptation law guarantee that the states of uncertain systems are finally stabilized to some desired positions and the parameter estimation error converges to zero.

Sum of Squares Based Nonlinear Control Design Techniques

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DEDICATION

To my parents *Yunqiang Zheng and Anping Chen*
my beloved husband *Qing Zhang*
and
my dear daughter *Elisa Zhang*

BIOGRAPHY

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LIST OF NOTATIONS

$0_{n \times m}$	the zero element of $\mathbf{R}^{n \times m}$
$\text{diag}(a_1, a_2, \dots, a_n)$	the n by n diagonal matrix with elements $a_1, a_2, \dots, a_n$ on the diagonal line
$\mathcal{F}_l(\cdot, \cdot)$	linear fractional transformation
I_n	the n -dimensional identity matrix
$\mathcal{L}_2$	space of square integrable functions
M^T	the transpose of the matrix M
M^{-1}	the inverse of the invertible matrix M
$M > 0$ ($M \geq 0$)	the matrix M is positive definite (positive semi-definite)
$M < 0$ ($M \leq 0$)	the matrix M is negative definite (negative semi-definite)
$\mathbf{R}$	set of real numbers
$\mathbf{R}_+$	set of non-negative real numbers
$\mathbf{R}^n$	set of n -dimensional real vectors
$\mathbf{R}^{m \times n}$	set of real $m \times n$ matrices
$\mathbf{S}^{n \times n}$	set of symmetric matrices in $\mathbf{R}^{n \times n}$
$\mathbf{S}_+^{n \times n}$	set of positive definite matrices in $\mathbf{R}^{n \times n}$
$\ u\ _2$	$= [\int_0^\infty u^T(t)u(t)dt]^{\frac{1}{2}}$ for $u \in \mathcal{L}_2$
$\ x\ $	$= (x^T x)^{\frac{1}{2}}$ for vector x
$\frac{\partial V}{\partial x}$	$= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix}$
$M^{[r]}(x)$	a monomial vector of x with all components having degree no more than r
Φ_{SOS}	the set of SOS polynomials
$\ z\ _2^2$	$= \int_{-\infty}^\infty \text{trace}[z^*(t)z(t)]dt$
$\max_{x(0), w} \frac{\ z\ _2}{\ w\ _2}$	the induced $\mathcal{L}_2/\mathcal{H}_\infty$ norm from w to z

Chapter 1

Introduction

1.1 Nonlinear Models and Nonlinear Phenomena

Many practical control systems have inherent nonlinear characteristics that cannot be ignored. Generally, the dynamics of the nonlinear systems could be represented as the following n-dimensional first-order differential equation in the vector form

$$\dot{x} = f(t, x, u)$$

where $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$ is the state, $u = \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix}^T$ is the input and $f(t, x, u) = \begin{bmatrix} f_1(t, x, u) & f_2(t, x, u) & \cdots & f_n(t, x, u) \end{bmatrix}^T$ is the state space matrix.

Comparing with linear systems, the nonlinear systems is more difficult to deal with since the superposition principle does not hold any longer, and analysis tools involve more advanced mathematics. Because of the powerful tools we know for linear systems, the first step in analyzing a nonlinear system is usually to linearize it about some nominal operating point and analyze the resulting linear model. This is a common practice in engineering, and it is a useful one. There is no question that, whenever possible, we should make use of linearization to learn as much as we can about the behavior of a nonlinear system. However, there are two basic limitations of linearization. First, since linearization is an approximation in the neighborhood of an operating point, it can only predict the "local" behavior of the nonlinear system in the vicinity of that point. It cannot predict the "nonlocal" behavior far from the operating point and certainly not the "global" behavior throughout the state space. Second, the dynamics of a nonlinear system are much richer than the dynamics of a

linear system. There are "essentially nonlinear phenomena" that can take place only in the presence of nonlinearity. Therefore, linearization alone will not be sufficient and we must develop tools for the analysis of nonlinear systems.

The following are some examples of essentially nonlinear phenomena:

◇ Finite escape time.

The state of an unstable linear system goes to infinity as time approaches infinity; a nonlinear system's state, however, can go to infinity in finite time.

◇ Multiple isolated equilibria.

A linear system can have only one isolated equilibrium point; thus, it can have only one steady-state operating point that attracts the state of the system irrespective of the initial state. A nonlinear system can have more than one isolated equilibrium point. The state may converge to one of several steady-state operating points, depending on the initial state of the system.

◇ Limit cycles.

For a linear time-invariant system to oscillate, it must have a pair of eigenvalues on the imaginary axis, which is a non-robust condition that is almost impossible to maintain in the presence of perturbations. Even if we do, the amplitude of oscillation will be dependent on the initial state. In real life, stable oscillation must be produced by nonlinear systems. There are nonlinear systems that can go into an oscillation of fixed amplitude and frequency, irrespective of the initial state. This type of oscillation is known as a limit cycle.

◇ Subharmonic, harmonic, or almost-periodic oscillations.

A stable linear system under a periodic input produces an output of the same frequency. A nonlinear system under periodic excitation can oscillate with frequencies that are submultiples or multiples of the input frequency. It may even generate an almost-periodic oscillation, an example is the sum of periodic oscillations with frequencies that are not multiples of each other.

◇ Chaos.

A nonlinear system can have a more complicated steady-state behavior that is not

equilibrium, periodic oscillation, or almost-periodic oscillation. Such behavior is usually referred to as chaos. Some of these chaotic motions exhibit randomness, despite the deterministic nature of the system.

1.2 Motivations and Objectives

In many practical control problems, the controlled dynamics of the systems are dominated by nonlinear effects. Moreover, the increasingly complex nature of nonlinear systems also highlights the need for new nonlinear control techniques to overcome the challenges in control designs. Unfortunately, current nonlinear control techniques cannot satisfy stability and performance requirements simultaneously and often result in compromised designs.

Stability of dynamical systems plays a very important role in control system analysis and design. Unlike the case of linear systems, proving stability of equilibria of nonlinear systems is more complicated. The most practical way to approach the stabilization problem for nonlinear systems is to appeal to the neat results available in the linear case, that is, via linearization. A feedback control law is designed by linearizing the system about the desired equilibrium point and designing a stabilizing linear feedback control for the linearization [35]. The validity of this idea comes from Lyapunov's indirect method. Clearly, this approach is local; that is, it can only guarantee asymptotic stability, but cannot, in general prescribe a region of attraction nor achieve global asymptotic stability. Motivated by the gain scheduling methodology, linear parameter-varying (LPV) control technique [7, 86, 1] aims at extending the region of validity of linearization by solving the stabilization problem at different operating points and allowing the controller to move from one design to another in a smooth or abrupt way. Another linearization idea is presented by transforming a special class of nonlinear systems that can be transformed into linear systems via feedback and (possibly) a change of variables [73]. After this transformation, a stabilizing linear state feedback control is designed for the linear system. This linearization approach is different from the first one in that no approximation is used; it is exact. However, exactness assumes perfect knowledge of the state equation and uses that knowledge to cancel the nonlinearities of the system. Since perfect knowledge of the state equation and exact mathematical cancellation of terms are almost impossible, the implementation of this approach will almost

always result in a closed-loop system, which is a perturbation of a nominal system whose origin is exponentially stable. As a sufficient condition to prove the stability of nonlinear system, Lyapunov's direct method requires the existence of a positive definite function defined in some region of the state space containing the equilibrium point whose derivative along the system trajectories is negative semi-definite [35]. Although it addresses exactly and in a simple way the important issue of stability, it does not provide any coherent methodology for constructing such a function. Lyapunov's indirect method that investigates the local stability of the equilibria, is inconclusive when the linearized system has imaginary axis eigenvalues. Other methodologies to determine the stability of nonlinear systems include sliding mode control [35, 15], by which trajectories are forced to reach a sliding manifold in finite time and to stay on the manifold for all future time. By using a lower order model, the sliding manifold is designed to achieve the control objective. The Lyapunov redesign [19, 35] uses a Lyapunov function of a nominal system to design an additional control component that makes the design robust to large matched uncertainties. Both sliding mode control and Lyapunov redesign produce discontinuous controllers, which could suffer from chattering in the presence of delays or unmodeled high-frequency dynamics. On the other hand, backstepping [40] is a recursive procedure that interlaces the choice of a Lyapunov function with the design of feedback control. It breaks a design problem for the full system into a sequence of design problems for lower order (even scalar) subsystems. By exploiting the extra flexibility that exists with lower order and scalar subsystems, backstepping can often solve stabilization problems under conditions less restrictive than those encountered in other methods. However, a potential drawback of backstepping control designs was the lack of systematic procedures for the construction of Lyapunov functions. This arises mainly from the difficulty of choosing a sufficiently general class of Lyapunov functions and is, of course, common to almost all constructive Lyapunov methods [37]. Alternatively, passivity-based control exploits passivity of the open-loop system in the design of feedback control [35]. Stabilizing a passive system at an equilibrium point amounts to damping injection. For non-passive system, feedback passivation technique is capable of converting it into a passive one via feedback control. Considering that most of the design tools presented above require state feedback, to meet the practical necessity that state variables are not always measurable, high-gain observers are introduced to estimate the state from output measurements [35, 16]. The main idea embed is that the performance under globally bounded state

feedback control can be recovered by output feedback control when the observer gain is sufficiently high.

In this dissertation, we will firstly propose a new nonlinear stabilization approach based on the popular linear parameter-varying control techniques. The regional state/output feedback control problems of polynomial nonlinear systems will be studied using rational Lyapunov functions. The usage of rational Lyapunov functions will be shown as flexible to exploit nonlinear dynamics of the given system and leads to a systematic control design procedure for polynomial nonlinear systems, which can be solved efficiently by Sum-Of-Squares (SOS) programming.

Beyond stabilization problem, we are also interested in improving performance of nonlinear control systems, such as $\mathcal{H}_\infty$ control and adaptive control.

In the past decade, a theoretical framework for exploring $\mathcal{H}_\infty$ control of nonlinear systems has been proposed in [81, 27, 82]. Interpreting nonlinear $\mathcal{H}_\infty$ control in terms of dissipativity and differential game [5], the solution to this problem has been related to an appropriate Hamilton-Jacobi-Isaacs (HJI) equation. For hyperbolic nonlinear systems whose linearized plant are stabilizable, the solution of HJI equation was characterized [81, 82] by an invariant manifold of Hamiltonian vector fields using differential geometric theory. Later on, the result has been generalized to non-hyperbolic nonlinear systems via output feedback control [27]. It was further shown that the solution to output feedback control problem is determined by a pair of coupled HJI equations. Parallel to linear $\mathcal{H}_\infty$ control theory, a separation principle was also established under a detectability hypothesis [26]. In standard HJI equations, some orthonormal and decoupling restrictions are enforced on the plant structure. Although these restrictions help simplify the HJI equations, their solutions are still extremely difficult not only to compute but also to represent. Therefore, in a long term, the key issue of nonlinear optimal control theory is the solvability of HJI equation.

It is well known that the HJI partial differential equation (PDE) reduces to Riccati algebraic equations for linear systems, which can be solved easily by efficient numerical algorithms. In the nonlinear context, however, there is no systematic numerical algorithm currently available for the solution of this PDE. Therefore, the key of nonlinear $\mathcal{H}_\infty$ control theory is the solvability of HJI equations. To this end, various approaches have been proposed to solve HJI equation numerically. In [34, 77], Taylor series expansion of the storage function $V(x)$ were considered to solve HJI equation term by term in an iterative

fashion, provided that the linearized model of nonlinear systems has a solution. However, the solutions from these approaches do not have a closed form and they may not converge to an analytic solution. On the other hand, a convex parametrization of nonlinear $\mathcal{H}_\infty$ control problem was derived in [46] based on a pair of positive definite matrix functions $P(x), Q(x)$. Unfortunately, it is difficult, if not impossible, to specify the form of $P(x)$ such that $\partial V(x)/\partial x = 2x^T P(x)$ except for the trivial case when $P(x)$ is a constant matrix. Therefore, the proposed formulation does not naturally lead to computationally tractable solution algorithms for nonlinear $\mathcal{H}_\infty$ control. In [63], the $\mathcal{L}_2$ gain analysis problem for polynomial nonlinear systems was formulated as a convex state-dependent linear matrix inequality (LMI), which can be recast as a SOS optimization problem. This approach was shown promising to overcome the numerical difficulty in solving HJI inequality and provides an analytic solution at the same time. Reference [85] proposed an iterative method based on SOS programming [60, 24] to solve a special state feedback $\mathcal{H}_\infty$ control problem.

In this dissertation, we will generalize the existing HJI equations by relaxing the stringent constraints on the plant structure and propose a numerically efficient approach to solve the challenging nonlinear output feedback $\mathcal{H}_\infty$ control problem. For polynomial nonlinear systems, it is advantageous to convert the HJIs associated with nonlinear $\mathcal{H}_\infty$ control to polynomial matrix inequalities, and solve the resulting matrix inequalities using SOS programming techniques. The proposed approach will extend from trivial quadratic Lyapunov function case to higher-order Lyapunov functions, therefore it will help improve controlled performance and expand stability region of nonlinear systems.

Another important issue in the nonlinear system control scenario is the adaptive control. Adaptive control estimates unknown constant or slow varying parameters from on-line operating adaptive law and implements the controller with estimated parameters [67, 2, 50, 40].

Currently, there are two commonly used approaches for the design of nonlinear adaptive controllers: Lyapunov-based and estimation-based. With the development of back-stepping techniques, significant progress has been made on Lyapunov-based designs. On the other hand, estimation-based designs are more broadly applicable and allow a choice of parameter update laws from a wide repertoire of gradient and least-squares optimization techniques.

In spite of major development, adaptive control schemes have not yet become

systematic engineering design tools. A possible drawback of adaptive control based on backstepping is lack of optimality and robustness. Moreover, its somewhat complicated design process, restrictive modeling assumption (such as linear parameterization), and poor transient performance often hinder its wide applications to practical problems and make it unfavorable from robustness consideration. To improve robustness property of adaptive control, recent studies in deterministic robust control (DRC) area [98, 79] and adaptive or robust adaptive control (RAC) area [20] have been proposed to integrate robust and adaptive control techniques together for nonlinear systems. Adaptive robust control (ARC) was eventually developed to preserve the theoretical performance results of both RAC and DRC, and prompt them to complement each other so that the well-known practical performance limitations of each were overcome [87, 88, 48]. Nevertheless, the developed robust adaptive techniques have only emphasized on synthesizing an adaptive control law and a parameter adaptation law based on some simple types of Lyapunov function, whereas no optimality involved in.

Motivated by these limitations, **in this dissertation**, the adaptive $\mathcal{H}_\infty$ control problem for a class of polynomial nonlinear systems with matching or un-matched parametric uncertainties will be considered. The $\mathcal{H}_\infty$ control problem for dynamical systems with external disturbances is to design feedback controllers which make the resulting systems to have small $\mathcal{L}_2$ gains (or $\mathcal{H}_\infty$ norms for linear systems) such that the external disturbances are attenuated. Although in this dissertation, what we will consider is a special class of disturbances - disturbances internally produced by the parameter estimation error, the constructed controller will be an optimal one as it is optimized with respect to given cost functional and the disturbances could be proved as converging to zero. Specifically, the techniques from dissipation theory as well as the vector and parameter projection methods will be employed to establish nonlinear state feedback control law and parameter adaptation law. The Lyapunov function will be relaxed from quadratic to higher order and the resulting control gain will be generalized from constant to parameter dependent. By adopting higher-order Lyapunov functions and computationally effective optimization approach to redesign adaptive control laws, it will provide more freedom to achieve a better performance level.

Lastly, as is known, quadratic Lyapunov functions have been widely used to stabilize linear systems in the past. However, for more complicated system such as nonlinear systems or linear systems with structured uncertainties, quadratic Lyapunov function is

known not adequate and often leads to conservative results. For this reason, non-quadratic Lyapunov functions have been proposed in the literature. Using a polyhedral Lyapunov function, linear system with structured time-varying parameter uncertainties and input disturbances was stabilized to a bounded neighborhood of the origin by a continuous or piecewise linear controller [8]. Nevertheless, computational cost of polyhedral functions is very high and remains an issue for large dimensional systems. Another powerful class of non-quadratic Lyapunov functions in analysis and control of nonlinear and linear uncertain systems is homogeneous polynomial Lyapunov functions. In [90], power transformation has been used to convert the original nonlinear system to an equivalent higher order system. Then a quadratic Lyapunov function designed for the higher order system leads to a homogeneous polynomial Lyapunov function for the original system. An LMI-based approach was also pursued in [11] to provide sufficient condition for the existence of an homogeneous polynomial Lyapunov function of given degrees for linear uncertain systems. For a class of nonlinear systems with rational vector fields, Papachristodoulou and Prajna [55] proposed to convert them into equivalent systems with polynomial vector fields under equality and inequality constraints. Then Lyapunov function was constructed in terms of the non-polynomial terms and SOS programming was employed to find a tractable solution. Nonlinear synthesis problem of polynomial nonlinear systems was addressed in [61], where nonlinear state feedback designs have been formulated as state dependent LMIs. However, a special form of polynomial Lyapunov functions has to be chosen to render convex solutions. Other types of non-quadratic Lyapunov functions include maximum and convex hull Lyapunov functions, which have been proposed in [21] to study stability of linear differential inclusions (LDIs).

One of the essences deserved to highlight here is that, **throughout this dissertation**, we will utilize two types of non-quadratic Lyapunov functions. One is the rational Lyapunov functions, the other is the higher order Lyapunov functions. Theoretically, comparing with quadratic Lyapunov functions, these generalized Lyapunov functions will help provide additional freedom to search for a best fit Lyapunov function and construct the resulting nonlinear control law so that the design objective is optimized. Computationally, they are able to lead to systematic control design procedures by formulating a set of polynomial matrix inequalities that solvable through SOS programming.

1.3 Dissertation Outline

The detailed outline of this dissertation is as follows:

Chapter 1 introduces the importance of nonlinear control problem and reviews the development of research work in this field. This chapter also emphasizes related literature that motivates our work and describes the objectives of the dissertation.

Chapter 2 contains definitions, concepts and information about the mathematical background and related tools. They are useful throughout the work and provide foundations for the derivation of theoretical results in the later chapters.

Chapter 3 establishes a nonlinear state/output feedback stabilization strategy for polynomial nonlinear systems using rational Lyapunov functions. By imposing bounds on the derivatives of each state or output variables, the stability region was embedded in the area specified by nonlinear vector field. The regional stabilization conditions are formulated as polynomial matrix inequalities and solved by SOS programming techniques in polynomial-time complexity. The resulting Lyapunov matrix and the controller gains are typically state/output-dependent rational matrix functions. This could provide a powerful tool in systematic construction of different forms of Lyapunov functions and simplify nonlinear control design tasks.

Chapter 4 presents the $\mathcal{H}_\infty$ synthesis condition in terms of HJI inequalities for a class of polynomial nonlinear systems without orthonormal and decoupling assumptions on the plant structure. And a computationally efficient approach based on SOS programming is proposed to solve the generalized HJI inequalities. By applying this $\mathcal{H}_\infty$ control design procedure, attitude stabilization and command tracking problems of an axi-symmetric spacecraft with two control torques are solved locally. The proposed nonlinear $\mathcal{H}_\infty$ control approach utilizes higher order Lyapunov functions and reformulates the difficult Hamilton-Jacobian-Isaacs (HJI) inequalities as semi-definite optimization conditions. SOS programming techniques are then applied to obtain computationally tractable solutions, from which nonlinear control laws will be constructed. The nonlinear $\mathcal{H}_\infty$ control designs for spacecraft are capable of exploiting the most suitable forms of Lyapunov functions for performance improvement and the resulting controller is shown to perform better than existing nonlinear control laws.

In Chapter 5, we study the adaptive $\mathcal{H}_\infty$ control problem for polynomial nonlin-

ear systems with matching and un-matched parametric uncertainties. The objective is to attenuate the effect of the unknown parameters on the controlled output, which is composed of tracking errors and control efforts. Once the parametric uncertainties satisfy the matching condition, the adaptive control law and the parameter adaptation law are constructed through the iterative algorithm simultaneously, while for un-matched case, they are respectively governed by the iterative algorithm and the vector projection. For both cases, the synthesis conditions could be formulated as polynomial matrix inequalities and are thus solvable by recast the resulting conditions into an SOS optimization problem. It is shown that the Lyapunov functions are relaxed from quadratic to higher order iteratively and the resulting control law is generalized from constant to parameter dependent. The system performance is gradually improved and the states are finally posed at some desired position with zero parameter estimation error.

Chapter 6 concludes the research contribution in this dissertation as well as provides a remark on the future research work.

Chapter 2

Preliminaries

In this chapter, we provide some background knowledge which forms the basis for the later study. Convex analysis approach and Linear Matrix Inequalities, are discussed in Sections 2.1 and 2.2, which will be useful in formulating the control problem into mathematical form. Since Sum-of-Squares (SOS) and semi-definite programming play a very important role throughout this research, some background material will be provided in Section 2.3.

2.1 Convex Analysis

In this thesis, the design problem will finally be formulated as a convex optimization problem. We are interested in finding conditions for optimal solutions to exist by using convex analysis. Following are some definitions and elementary properties which make the convexity of sets and functions is a desirable property for optimization.

1. Convex Set

A set Θ in a linear vector space is said to be convex if for all $x_1, x_2 \in \Theta$, we have

$$\alpha x_1 + (1 - \alpha)x_2 \in \Theta, \text{ for all } \alpha \in (0, 1)$$

In geometric terms, this states that for any two points of a convex set also the line segment connecting these two points belongs to the set.

2. Convex Combination

Let Θ be a set of a vector space, $x_1, x_2, \dots, x_n \in \Theta$ and $\alpha_1, \alpha_2, \dots, \alpha_n$ are non-negative

real numbers with $\sum_{i=1}^n \alpha_i = 1$, then

$$x := \sum_{i=1}^n \alpha_i x_i$$

is called a convex combination of $x_1, x_2, \dots, x_n$.

3. Convex Function

A function $f : \Theta \rightarrow \mathbb{R}$ is called convex if Θ is convex and for all $x_1, x_2 \in \Theta$ and $\alpha \in (0, 1)$, we have

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2)$$

4. Affine Function

A function $f : \Theta \rightarrow \Psi$ is affine if $f(x) = f_0 + T(x)$ where $f_0 \in \Psi$ and $T : \Theta \rightarrow \Psi$ is a linear map, i.e.,

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2), \text{ for all } x_1, x_2 \in \Theta, \alpha_1, \alpha_2 \in \mathbb{R}$$

5. Local and Global Minima of Convex Function

A function $f : \Theta \rightarrow \mathbb{R}$ is said to have a local minimum at $x_0 \in \Theta$ if there exists $\epsilon > 0$ such that

$$f(x_0) \leq f(x), \text{ for all } x \in \Theta \text{ with } \|x - x_0\| < \epsilon$$

If it holds for all $x \in \Theta$, then $f(x_0)$ is a global minimum.

Suppose that f is convex, then the local minimum at $x_0 \in \Theta$ is also the global minimum.

6. Convex Optimization

As we will see, many optimization problems in control can be formulated (or reformulated) using linear matrix inequalities and then be solved in an efficient and reliable way. Since the linear matrix inequality defines a convex constraint on the variable vector, optimization problems involving the minimization (or maximization) of a performance function with the convex constraint belongs to the class of convex optimization problems.

2.2 Linear Matrix Inequality (LMI)

In recent years, Linear Matrix Inequality(LMI) has emerged as a powerful tool to approach control problems that appear hard to solve. This is because in many situations, control synthesis and analysis requirements can be (and will be) written as LMIs and powerful numerical interior point techniques have been developed to solve LMIs in a practically efficient manner. Several Matlab software packages are available that allow a simple coding of general LMI problems and provide efficient tools to solve typical control problems. Here we give out the definition of LMIs and provide several important facts which form the basis for the system analysis of the thesis.

1. Linear Matrix Inequality (LMI)

A linear matrix inequality is an inequality that defines a convex constraint on a vector $x \in \mathbb{R}^n$, i.e,

$$F(x) = F_0 + \sum_{i=1}^n x_i F_i > 0$$

where F is an affine function mapping an n dimensional vector space x to the set of $n \times n$ real symmetric matrices.

2. Schur Complement

Schur complement is widely used to convert nonlinear (convex) inequalities to linear matrix inequalities, i.e, a block matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} < 0$$

if and only if

$$\begin{cases} M_{11} < 0 \\ M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0 \end{cases}$$

3. Lyapunov Stability

Lyapunov stability concerns the asymptotic behavior of the state of an autonomous dynamical system. The stability of such systems can be verified in terms of the existence of functions, called Lyapunov functions. For linear systems the problem of finding Lyapunov functions turns can be solved adequately as a feasibility test of a

linear matrix inequality, i.e,

Consider a linear time invariant system

$$\dot{x} = Ax$$

and a quadratic Lyapunov function $V(x) = x^T Px$, where P is symmetric positive definite. The system is asymptotically stable at the origin if and only if

$$A^T P + PA < 0$$

2.3 SOS Programming

Consider a vector $x = [x_1 \ \cdots \ x_n]^T \in \mathbf{R}^n$ and the multi-index vector $k = [k_1 \ \cdots \ k_n]^T \in \mathbf{Z}^n$ with $\bar{k} = \sum_{i=1}^n k_i$. For integers $\bar{k}$, we will define

1. $m_{\bar{k}}(x)$: monomial of x with degree $\bar{k}$

$$m_{\bar{k}}(x) = x^{\bar{k}} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}, \quad \sum_{i=1}^n k_i = \bar{k},$$

2. $f(x)$: polynomial function of x composed of combination of monomials with coefficients $c_{\bar{k}}$'s

$$f(x) = \sum_{\bar{k}} c_{\bar{k}} m_{\bar{k}}(x).$$

A multivariate polynomial $f(x_1, \dots, x_n)$ is an SOS, if there exist polynomials $f_1(x), \dots, f_m(x)$ such that

$$f(x) = \sum_{i=1}^m f_i^2(x). \tag{2.1}$$

This clearly implies $f(x) \geq 0$ for any $x \in \mathbf{R}^n$. SOS decomposition provides a sufficient condition for non-negativity of a multivariate polynomial. The SOS decomposition condition (2.1) is equivalent to the existence of a positive semi-definite matrix G , such that

$$f(x) = M^T(x)GM(x),$$

where $M(x)$ is some vector of monomials and G is called Gramian matrix. Therefore, checking if a given multivariate polynomial $f(x)$ is an SOS is related to the non-negativity of Gramian matrix associated with the polynomial [56], which can be solved efficiently as a

semi-definite programming problem [9, 80, 60]. SOS programming has also been generalized to polynomial matrix forms in [24].

The main advantages of SOS programming are the resulting computational tractability and the algorithmic character of its solution procedure [56]. This could help to provide coherent methodology of synthesizing Lyapunov functions for nonlinear systems. In addition, the importance of SOS technique also lies in its ability to provide tractable relaxations for many difficult optimization problems. As is known, many standard optimization problems (such as dissipativity and passivity) in system and control theory are in the form of indefinite quadratic problems [80]. The maximization of these functionals subject to indefinite quadratic constraints is often computationally complicated, sometimes even NP-hard. Consider now the following constrained feasibility problem:

$$f_0(x) = x^T M_0 x < 0 \quad \forall x \text{ satisfying } f_i(x) = x^T M_i x \geq 0, \quad i \in [1, m]. \quad (2.2)$$

Using the S -procedure [9], a sufficient condition to the assertion (2.2) is given by

$$\exists \tau_1 > 0, \dots, \tau_m > 0 \text{ such that } M_0 + \sum_{i=1}^m \tau_i M_i < 0. \quad (2.3)$$

A geometric interpretation of the relaxation (2.3) is the separation of functional space using linear hyperplane as multipliers. Based on this observation, more powerful tests can be derived [56] by lifting the degree restriction of multipliers. For example, a computationally tractable and less conservative relaxation of the original statement will be

$$f_0(x) + \sum_{i=1}^m Q_i(x) f_i(x) + \sum_{0 \leq i < j \leq m} r_{ij} f_i(x) f_j(x) < 0 \quad \forall x, \quad (2.4)$$

where $Q_i(x) = x^T Q_i x$, $Q_i > 0$ and $r_{ij} \geq 0$. The new relaxation is always at least as powerful as the standard one: this can be easily verified by taking $Q_i = \tau_i I$ and $r_{ij} = 0$. Moreover, feasibility of the resulting problem (2.4) can often be checked using SOS programming tools.

As a powerful and promising technique, SOS programming provides an efficient way to exploit the polynomial nonlinear systems, which can be used to approximate many nonlinear systems. Recent studies have shown the strength of SOS programming in solving nonlinear analysis [55, 74] and stabilization [59, 61, 96] problems.

Chapter 3

Regional Stabilization of Polynomial Nonlinear Systems using Rational Lyapunov Functions

In this chapter, we will address regional stabilization problem for state-dependent polynomial nonlinear systems using state feedback control laws. The control design objective is to construct a Lyapunov certificate for stabilization of such nonlinear systems, pertaining to stability region estimates. To expand stability regions, we will employ rational Lyapunov functions that are more general than polynomial forms and are flexible to exploit nonlinear dynamics of the given system. Moreover, rational type of Lyapunov functions will also lead to a systematic control design procedure for polynomial nonlinear systems. Restricting the state variation rates within a bounded interval, regional stabilization will be achieved by embedding the stability domain within the region specified by the nonlinear vector fields. Consequently, the state feedback synthesis condition will be formulated as a set of state-dependent polynomial matrix inequalities and can be solved efficiently using SOS programming techniques [56, 60]. From which, the state feedback control law will be constructed as a rational function of state variables. The proposed approach will also be extended to regional output feedback control for a class output-dependent nonlinear systems using rational Lyapunov functions.

Section 3.1 covers a review of the popular stabilization control design techniques in

nonlinear control theory. In Section 3.2, we will employ rational Lyapunov functions to solve nonlinear state feedback stabilization problems for polynomial nonlinear systems. Parallel results for output feedback control will be provided in Section 3.3. Several examples will be given in Section 3.4 to demonstrate the proposed nonlinear control approach. Finally, we will conclude the chapter in Section 3.5.

3.1 Introduction

One of the basic problems for nonlinear control is stabilization. In terms of stabilizability, current research can be classified as: global, semi-global and local or regional [35]. Nevertheless, most of existing nonlinear control research considered global stabilization problem. For some nonlinear systems, it is sometimes difficult or impossible to synthesize a global stabilizing controller. Moreover, in a restricted region, local controllers will often perform better than global controllers.

To achieve nonlinear stabilization, control techniques such as feedback linearization and backstepping are often used. As a geometric method, feedback linearization has been applied to many practical nonlinear control problems [73]. The basic idea is to transform the nonlinear system into an equivalent single form which can be controlled by standard linear control techniques. While this approach very effectively exploits the structures of nonlinear systems, beneficial nonlinearities are typically cancelled in control design. Moreover, it is unable to handle parametric and modeling uncertainties. Alternatively, backstepping control technique [40] is a merger of classical Lyapunov stability theory and geometric method. It provides a recursive design procedure to incorporate time-varying uncertainties and unknown constant parameters into nonlinear control systems. As a result, this control technique has significant impacts on a large variety of problems in the field of nonlinear adaptive and robust control. However, a potential drawback of backstepping control designs was the lack of systematic procedures for the construction of Lyapunov functions. Furthermore, although this recursive design procedure is very useful in establishing global stability with uncertainties, it fails to optimize such controller with respect to a given cost functional for good performance.

On the other hand, gain-scheduling control approach is another popular nonlinear control design technique which has been widely used in the fields ranging from aerospace to

process control for long time [66]. The primary advantage of gain-scheduling for nonlinear control design is that it is usually possible to meet performance objectives over a wide range of operating conditions while still taking advantage of the wealth of tools and designers' experience from linear controller synthesis. Motivated by the gain-scheduling control methodology, a systematic control design technique has been developed recently in the framework of linear parameter-varying (LPV) system with guaranteed stability and performance properties [7, 86, 1]. The synthesis of LPV control law and associated single/parameter-dependent quadratic Lyapunov functions can be solved using convex optimization techniques [7, 86]. However, the stability and performance properties of LPV systems only hold in a local sense and the application of LPV control techniques often requires converting nonlinear systems into its quasi-LPV forms.

3.2 State Feedback Control of Polynomial Nonlinear Systems

Consider a polynomial nonlinear autonomous system of the following form

$$\dot{x} = A(x)x + B(x)u, \quad (3.1)$$

where the system state $x \in \mathbf{R}^n$, control input $u \in \mathbf{R}^{n_u}$. It is assumed that the state-space entries $A(\cdot), B(\cdot)$ are polynomial functions of the state x . $(x, u) = (0, 0)$ is an equilibrium of the nonlinear system (3.1). We would like to design a static nonlinear state feedback controller $u = F(x)x$ such that the nonlinear system (3.1) is stabilized asymptotically in a finite state region. In general, $F(x)$ will be a rational function of x .

For nonlinear dynamical systems, Lyapunov function based methods play a vital role in both stability analysis and control synthesis. The synthesis typically involves co-design of the control Lyapunov function and the controllers. Nevertheless, past nonlinear control designs often relied on quadratic Lyapunov functions that would lead to conservative results. In this research, we will relax the form of Lyapunov functions to rational matrix functions of states in the form of

$$V(x) = x^T P(x)x, \quad P(x) > 0.$$

For regional stabilization, a Lyapunov level set will also be introduced to estimate the domain of attraction (DOA). Consequently, the nonlinear system (3.1) will be stabilized by

a state feedback control law $u = F(x)x$ in the state region

$$\Omega(P, 1) = \{x \in \mathbf{R}^n : x^T P(x)x \leq 1\}.$$

Note that the Lyapunov matrix $P(x)$ is not constant but state-dependent. Therefore the resulting DOAs are not necessarily standard ellipsoids but could be more general to facilitate exploiting stability region of nonlinear systems.

For the given state feedback control law, the closed-loop system becomes

$$\dot{x} = [A(x) + B(x)F(x)]x.$$

From Lyapunov stability theory [35], the closed-loop system is asymptotically stable if there exists a Lyapunov function $V(x) = x^T P(x)x > 0$ such that

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P(x)x + x^T P(x)\dot{x} + x^T \sum_{i=1}^n \left(\dot{x}_i \frac{\partial P(x)}{\partial x_i} \right) x \\ &= 2x^T P(x)[A(x) + B(x)F(x)]x + x^T \sum_{i=1}^n \left(\dot{x}_i \frac{\partial P(x)}{\partial x_i} \right) x < 0. \end{aligned} \quad (3.2)$$

The stability condition (3.2) is very similar to the one derived for LPV systems using parameter-dependent Lyapunov functions [86]. However, the solution to inequality (3.2) requires parameter variation rates information which is often difficult to obtain *a priori* for nonlinear stabilization. To overcome such a difficulty, we will specify the state derivative bound as $\nu = [\nu_1 \ \nu_2 \ \cdots \ \nu_n]^T$, and enforce the state derivative constraints during the control design process. Consequently, the nonlinear stabilization problem will be reformulated to

$$2x^T P(x)[A(x) + B(x)F(x)]x \pm x^T \sum_{i=1}^n \left(\nu_i \frac{\partial P(x)}{\partial x_i} \right) x < 0$$

subject to the bounded state derivative condition

$$|\dot{x}_i| = |e_i^T [A(x) + B(x)F(x)]x| \leq \nu_i, \quad i \in \mathbf{I}[1, n]$$

where e_i is the i th column of the unity matrix I_n . Moreover, to guarantee the proposed controller working in the local region $\Omega(P, 1)$, it also requires

$$\Omega(P, 1) \subset \bigcap_{i=1}^n \{ |e_i^T [A(x) + B(x)F(x)]x| \leq \nu_i \}. \quad (3.3)$$

Condition (3.3) implies that the magnitude of state derivative $\dot{x}_i$ of any point within the stability region $\Omega(P, 1)$ is no more than ν_i .

By summarizing the above discussions, we have the following theorem for synthesis of a nonlinear state feedback control law using rational Lyapunov functions.

Theorem 1 *Given the state derivative bounds ν and a small positive number ϵ , the nonlinear system (3.1) is asymptotically stabilizable within the region $\Omega(Q^{-1}, 1)$ by a static, nonlinear state feedback controller $u = F(x)x$ if there exist polynomial matrix functions $Q(x) \in \mathbf{S}_+^{n \times n}$, $K(x) \in \mathbf{R}^{n_u \times n}$ such that*

$$Q(x) < \frac{1}{\epsilon} I \quad (3.4)$$

$$Q(x)A^T(x) + A(x)Q(x) + B(x)K(x) + K^T(x)B^T(x) \mp \sum_{i=1}^n \left(\nu_i \frac{\partial Q(x)}{\partial x_i} \right) < 0 \quad (3.5)$$

$$\begin{bmatrix} Q(x) & [Q(x)A^T(x) + K^T(x)B^T(x)]e_i \\ \star & \nu_i^2 \end{bmatrix} > 0, \quad i \in \mathbf{I}[1, n]. \quad (3.6)$$

Furthermore, the stabilizing rational Lyapunov function is $V(x) = x^T Q^{-1}(x)x$ and its associated nonlinear controller gain is given by $F(x) = K(x)Q^{-1}(x)$.

Proof: Clearly, the existence of $Q^{-1}(x)$ is guaranteed by $Q(x) > 0$. Since $Q^{-1}(x) > \epsilon I$, the set $\Omega(Q^{-1}, 1)$ will specify a finite state region containing origin. Moreover, $V(x) = x^T Q^{-1}(x)x > \epsilon x^T x$ is indeed positive definite.

Given the Lyapunov function $V(x) = x^T Q^{-1}(x)x$ and the state feedback control law as $u = F(x)x$, we have the local stabilization conditions for the open loop system (3.1) to be

$$V(x) > 0 \quad (3.7)$$

$$\dot{V}(x) < 0 \quad (3.8)$$

$$\{x^T Q^{-1}(x)x \leq 1\} \subset \bigcap_{i=1}^n \{ |e_i^T [A(x) + B(x)F(x)]x| \leq \nu_i \}. \quad (3.9)$$

From eqn. (3.5), we have

$$QA^T + AQ + BK + K^T B^T - \sum_{i=1}^n \left(\dot{x}_i \frac{\partial Q(x)}{\partial x_i} \right) < 0$$

for any $|\dot{x}_i| \leq \nu_i$. Multiplying $Q^{-1}(x)$ from left and right hand sides, then the above inequality becomes

$$(A + BF)^T Q^{-1}(x) + Q^{-1}(A + BF) + \sum_{i=1}^n \left(\dot{x}_i \frac{\partial Q^{-1}(x)}{\partial x_i} \right) < 0,$$

where $K(x) = F(x)Q(x)$. Then by substituting the nonlinear dynamics into (3.8), we confirm that

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T Q^{-1}x + x^T Q^{-1}\dot{x} + x^T \sum_{i=1}^n \left(\dot{x}_i \frac{\partial Q^{-1}(x)}{\partial x_i} \right) x \\ &= 2x^T (A + BF)^T Q^{-1}x + x^T \sum_{i=1}^n \left(\dot{x}_i \frac{\partial Q^{-1}(x)}{\partial x_i} \right) x \\ &< 0. \end{aligned}$$

To verify condition (3.9), we multiply $\text{diag}\{Q^{-1}(x), 1\}$ from left and right hand sides of (3.6) and substituting $K(x)$ by $F(x)Q(x)$ again. This leads to

$$\begin{bmatrix} Q^{-1} & (A + BF)^T e_i \\ e_i^T (A + BF) & \nu_i^2 \end{bmatrix} > 0, \quad i \in \mathbf{I}[1, n].$$

Then applying Schur complement to above inequality will yield

$$Q^{-1} - \frac{1}{\nu_i^2} (A + BF)^T e_i e_i^T (A + BF) > 0, \quad i \in \mathbf{I}[1, n],$$

which is the same as the set inclusion relation (3.9).

Q.E.D.

Note that conditions (3.4)-(3.6) are state-dependent polynomial matrix inequalities. Using SOS decomposition and semi-definite programming [60], they can be solved by parameterizing $Q(x)$ and $K(x)$ in proper polynomial forms. Nevertheless, choosing different Lyapunov function forms could result in different shapes and sizes of Lyapunov level set, as shown later in our example study. To render (3.4)-(3.6) into computationally tractable SOS conditions, it often necessary to add some regional constraints in the form of $R_j(x) < 0$, $j \in \mathbf{I}[1, r]$. Then the modified state feedback stabilization condition in terms of

SOS will be

$$-z_1^T \left(Q(x) - \frac{1}{\epsilon} I \right) z_1 + \sum_{j=1}^r \lambda_{1j}(z_1, x) R_j(x) \in \Phi_{\text{SOS}} \quad (3.10)$$

$$\begin{aligned} & -z_2^T \left[Q(x) A^T(x) + A(x) Q(x) + B(x) K(x) + K^T(x) B^T(x) \mp \sum_{i=1}^n \left(\nu_i \frac{\partial Q(x)}{\partial x_i} \right) \right] z_2 \\ & + \sum_{j=1}^r \lambda_{2j}(z_2, x) R_j(x) \in \Phi_{\text{SOS}} \end{aligned} \quad (3.11)$$

$$z_3^T \begin{bmatrix} Q(x) & [Q(x) A^T(x) + K^T(x) B^T(x)] e_i \\ \star & \nu_i^2 \end{bmatrix} z_3 + \sum_{j=1}^r \lambda_{3j,i}(z_3, x) R_j(x) \in \Phi_{\text{SOS}}, \quad i \in \mathbf{I}[1, n] \quad (3.12)$$

where z_1, z_2, z_3 are free variables. $\lambda_{1j}, \lambda_{2j}, \lambda_{3j,i} > 0$ are SOS multipliers. Clearly, this will provide a sufficient condition for eqns. (3.4)-(3.6) and its solution will be valid in a local region specified by the intersection of constraints $R_j(x) < 0$, $j \in \mathbf{I}[1, r]$.

Note that $R_j(x) < 0$ could be any user-specified function of x which characterizes a closed region of states. It will be used to confine the control synthesis to a local region in obtaining a better performed local controller. Moreover, by adding constraints $R_j(x) < 0$, it helps enrich the polynomial representation of the original conditions so that the resulting SOS conditions (3.10)-(3.12) become feasible. Similar considerations will be given to the choices of multipliers $\lambda_{1j}, \lambda_{2j}, \lambda_{3j,i}$. The freedom in selecting different forms of regional constraints as well as their associated multipliers can be explored to optimize solution of the regional stabilization problem.

The resulting Lyapunov function as well as the controller gains are of rational forms unless $Q(x)$ is chosen as a constant matrix. In addition, it is also possible to specify derivative bound v_i as a function of x , i.e., $v_i(x)$. This generalization, however, may lead to more complicated condition than (3.5).

Remark 1 *The size of DOA can be maximized as the largest $\alpha > 0$, with respect to a given shape reference set $\Omega(X_{\mathbf{R}}, 1)$, $X_{\mathbf{R}} \in \mathbf{S}_+^{n \times n}$, such that*

$$\alpha \Omega(X_{\mathbf{R}}, 1) \subset \Omega(Q^{-1}(x), 1).$$

This additional requirement can be converted into an SOS constraint

$$z_4^T (Q(x) - \alpha^2 X_{\mathbf{R}}^{-1}) z_4 + \sum_{j=1}^r \lambda_{4j}(z_4, x) R_j(x) \in \Phi_{\text{SOS}}, \quad \lambda_{4j} > 0 \quad (3.13)$$

and amend to SOS conditions (3.10)-(3.12) to expand the estimated domain of attraction.

3.3 Output Feedback Stabilization

Next, we would like to extend the proposed stabilization approach to output feedback stabilization problem for a class of output-dependent nonlinear systems

$$\begin{cases} \dot{x} &= A(y)x + B(y)u \\ y &= Cx \end{cases} \quad (3.14)$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^{n_u}$ and $y \in \mathbf{R}^{n_y}$. As can be seen, the nonlinear dynamics is captured by the output variable y . To simplify the presentation, C is assumed to be a constant matrix. We would like to design an output feedback control law in the form of

$$\begin{cases} \dot{x}_c &= A_c(y, \dot{y})x_c + B_c(y)y \\ u &= C_c(y)x_c + D_c(y)y \end{cases} \quad (3.15)$$

with its associated rational Lyapunov function of $x_{cl} = [x^T \ x_c^T]^T$

$$V(x_{cl}) = x_{cl}^T W(y) x_{cl}, \quad W(y) > 0$$

to stabilize the nonlinear plant (3.14) within a state region

$$\Omega(W, 1) = \{x_{cl} : x_{cl}^T W(y) x_{cl} \leq 1\}.$$

Using output feedback controller (3.15), it is straightforward to derive the closed loop dynamics as

$$\dot{x}_{cl} = A_{cl}(y, \dot{y})x_{cl},$$

in which

$$A_{cl}(y, \dot{y}) = \begin{bmatrix} A(y) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B(y) \\ I & 0 \end{bmatrix} \begin{bmatrix} A_c(y, \dot{y}) & B_c(y) \\ C_c(y) & D_c(y) \end{bmatrix} \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}.$$

Note that A_{cl} and $\dot{W}$ will depend on not only y but also $\dot{y}$. Defining the output derivative bound vector as $\mu = [\mu_1 \ \mu_2 \ \cdots \ \mu_{n_y}]^T$ and noting

$$\dot{y} = C\dot{x} = \begin{bmatrix} CA(y) + CB(y)D_c(y)C & CB(y)C_c(y) \end{bmatrix} x_{cl},$$

we then have the following sufficient condition to guarantee the closed loop stability

$$W(y) > 0 \quad (3.16)$$

$$A_{cl}^T(y, \dot{y})W(y) + W(y)A_{cl}(y, \dot{y}) + \sum_{i=1}^{n_y} \left(\dot{y}_i \frac{\partial W(y)}{\partial y_i} \right) < 0, \quad \forall |\dot{y}_i| \leq \mu_i. \quad (3.17)$$

Similar to the state feedback case, we are also interested in characterizing a stability region $\Omega(W, 1)$ in which the designed controller would work. To this end, we will enforce

$$\{x_{cl}^T W(y) x_{cl} \leq 1\} \subset \bigcap_{i=1}^{n_y} \left\{ \left| \ell_i^T \begin{bmatrix} CA(y) + CB(y)D_c(y)C & CB(y)C_c(y) \end{bmatrix} x_{cl} \right| \leq \mu_i \right\}, \quad (3.18)$$

where ℓ_i is the i th column of the unity matrix I_{n_y} .

The output feedback stabilization conditions will then be provided in the following theorem, in which eqns. (3.16)-(3.18) are in the form of output-dependent polynomial matrix inequalities.

Theorem 2 *Given the output derivative bounds μ and a small scalar ϵ , the nonlinear system (3.14) is asymptotically stabilizable within the region $\Omega(W, 1)$ by the output feedback controller (3.15) if there exist output-dependent matrix functions $R(y), S(y) \in \mathbf{S}_+^{n \times n}$, and matrix functions $\hat{A}_c(y), \hat{B}_c(y), \hat{C}_c(y), \hat{D}_c(y)$ of compatible dimensions such that*

$$\begin{bmatrix} R(y) & I \\ I & S(y) \end{bmatrix} > \epsilon I \quad (3.19)$$

$$\begin{bmatrix} \begin{bmatrix} A(y)R(y) + B(y)\hat{C}_c(y) \\ +R(y)A^T(y) + \hat{C}_c^T(y)B^T(y) \\ \mp \sum_{i=1}^{n_y} \left(\mu_i \frac{\partial R}{\partial y_i} \right) \end{bmatrix} & \hat{A}_c^T(y) + A(y) + B(y)\hat{D}_c(y)C \\ \star & \begin{bmatrix} S(y)A(y) + \hat{B}_c(y)C \\ +A^T(y)S(y) + C^T\hat{B}_c^T(y) \\ \pm \sum_{i=1}^{n_y} \left(\mu_i \frac{\partial S}{\partial y_i} \right) \end{bmatrix} \end{bmatrix} < 0 \quad (3.20)$$

$$\begin{bmatrix} R(y) & I & [CA(y)R(y) + CB(y)\hat{C}_c(y)]^T \ell_i \\ \star & S(y) & [CA(y) + CB(y)\hat{D}_c(y)C]^T \ell_i \\ \star & \star & \mu_i^2 \end{bmatrix} > 0, \quad i \in \mathbf{I}[1, n_y]. \quad (3.21)$$

The gains of such a stabilizing output feedback controller are given by

$$\begin{bmatrix} A_c(y, \dot{y}) & B_c(y) \\ C_c(y) & D_c(y) \end{bmatrix} = \begin{bmatrix} N(y) & S(y)B(y) \\ 0 & I \end{bmatrix}^{-1} \times \begin{bmatrix} \left\{ \begin{array}{l} \hat{A}_c(y) - S(y)A(y)R(y) \\ +S(y)\dot{R}(y) + N(y)\dot{M}^T(y) \end{array} \right\} & \hat{B}_c(y) \\ \hat{C}_c(y) & \hat{D}_c(y) \end{bmatrix} \begin{bmatrix} M^T(y) & 0 \\ CR(y) & I \end{bmatrix}^{-1} \quad (3.22)$$

with the rational Lyapunov matrix

$$W(y) = \begin{bmatrix} I & S(y) \\ 0 & N^T(y) \end{bmatrix} \begin{bmatrix} R(y) & I \\ M^T(y) & 0 \end{bmatrix}^{-1},$$

where $M(y)N^T(y) = I - R(y)S(y)$.

Proof: We will define two transformation matrix functions $Z_1(y), Z_2(y)$

$$Z_1(y) = \begin{bmatrix} R(y) & I \\ M^T(y) & 0 \end{bmatrix} \quad Z_2(y) = \begin{bmatrix} I & S(y) \\ 0 & N^T(y) \end{bmatrix},$$

where matrices R, S are y -dependent and positive definite. Then it is clear that $WZ_1 = Z_2$ and

$$Z_1^T W Z_1 = Z_1^T Z_2 = \begin{bmatrix} R(y) & I \\ I & S(y) \end{bmatrix} > \epsilon I,$$

which implies $W(y)$ is positive definite within $\Omega(W, 1)$. By congruent transformation [69], we also have

$$\begin{aligned} Z_1^T (W A_{cl}) Z_1 &= Z_2^T A_{cl} Z_1 \\ &= \begin{bmatrix} AR & A \\ S\dot{R} + N\dot{M}^T & SA \end{bmatrix} + \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} \begin{bmatrix} \hat{A}_c & \hat{B}_c \\ \hat{C}_c & \hat{D}_c \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \\ Z_1^T \dot{W} Z_1 &= Z_1^T (\dot{Z}_2 - W \dot{Z}_1) \\ &= \begin{bmatrix} -\dot{R} & -(\dot{R}S + \dot{M}N^T) \\ -(S\dot{R} + N\dot{M}^T) & \dot{S} \end{bmatrix}, \end{aligned}$$

where the transformed controller data is defined as follows

$$\begin{bmatrix} \hat{A}_c & \hat{B}_c \\ \hat{C}_c & \hat{D}_c \end{bmatrix} = \begin{bmatrix} SAR - S\dot{R} - N\dot{M}^T & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} N & SB \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} M^T & 0 \\ CR & I \end{bmatrix} \quad (3.23)$$

Multiplying Z_1^T and Z_1 on the left and right hand sides of (3.17) and applying derived congruent transformation, it can be seen that condition (3.17) is guaranteed by eqn. (3.20).

To verify the set inclusion relation, we represent condition (3.18) in the form of

$$W - \frac{1}{\mu_i^2} \begin{bmatrix} CA + CBD_c C & CBC_c \end{bmatrix}^T \ell_i \ell_i^T \begin{bmatrix} CA + CBD_c C & CBC_c \end{bmatrix} > 0,$$

which is equivalent to the following matrix condition

$$\begin{bmatrix} W & \begin{bmatrix} CA + CBD_c C \\ CBC_c \end{bmatrix}^T \ell_i \\ \star & \mu_i^2 \end{bmatrix} > 0, \quad i \in \mathbf{I}[1, n_y].$$

Multiplying $\text{diag} \{Z_1^T, I\}$ and its transpose from left and right hand sides of above equation and note that

$$\ell_i^T \begin{bmatrix} CA + CBD_c C & CBC_c \end{bmatrix} Z_1 = \ell_i^T \begin{bmatrix} CAR + CB\hat{C}_c & CA + CB\hat{D}_c C \end{bmatrix},$$

then it is clear that condition (3.18) is equivalent to eqn. (3.21). Finally, the controller gain formula (3.22) can be verified by inverting relation (3.23). *Q.E.D.*

Note that all the optimization variables in the synthesis conditions (3.19)-(3.21) are functions of y , hence the output feedback controller gains (3.22) will depend on y and $\dot{y}$ in general. However, by choosing R, S properly, one can eliminate the derivative $\dot{y}$ from the controller gain formula. Possible choices include

1. $R = R_0, S = S(y), M = R_0, N = R_0^{-1} - S(y);$
2. $R = R(y), S = S_0, M = I - R(y)S_0, N = I.$

By parameterizing $R(y), S(y)$ and $\hat{A}_c(y) - \hat{D}_c(y)$ in polynomial function forms, eqns. (3.19)-(3.21) will be transformed to a set of polynomial matrix inequalities with linear dependency on decision variables. Parallel to the discussion near the end of Section 3.2, eqns. (3.19)-(3.21) could be reformulated to SOS conditions by adding regional constraints and solved using SOS programming.

Remark 2 *To improve controlled convergence, one can modify condition (3.17) to*

$$\beta W(y) + A_{cl}^T(y, \dot{y})W(y) + W(y)A_{cl}(y, \dot{y}) + \sum_{i=1}^{n_y} \left(\dot{y}_i \frac{\partial W}{\partial y_i} \right) < 0, \quad \beta > 0.$$

This will provide exponential convergence rate $\beta/2$ for closed loop systems. Consequently, condition (3.20) in Theorem 2 needs to be replaced by

$$\left[\begin{array}{c} \left\{ \begin{array}{c} \beta R(y) + A(y)R(y) + B(y)\hat{C}_c(y) \\ +R(y)A^T(y) + \hat{C}_c^T(y)B^T(y) \mp \sum_{i=1}^{n_y} \left(\mu_i \frac{\partial R}{\partial y_i} \right) \end{array} \right\} \quad \beta I + \hat{A}_c^T(y) + A(y) + B(y)\hat{D}_c(y)C \\ \star \quad \left\{ \begin{array}{c} \beta S(y) + S(y)A(y) + \hat{B}_c(y)C \\ +A^T(y)S(y) + C^T \hat{B}_c^T(y) \\ \pm \sum_{i=1}^{n_y} \left(\mu_i \frac{\partial S}{\partial y_i} \right) \end{array} \right\} \end{array} \right] < 0. \quad (3.24)$$

3.4 Examples

Example I

In this example, we will solve state feedback stabilization problem for a 3rd-order nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -x_1^2 x_2^2 & -3.5 & 1.5 \\ 0.5 & -(0.5 + x_1^2 x_2^2 + 5x_1) & 0 \\ -x_1 x_2^2 x_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -0.3x_1 \\ -0.5x_2 \\ -0.8x_3 \end{bmatrix} u.$$

The stabilization of this system can not be achieved using backstepping control technique for lack of desired triangular structure. On the other hand, calculating the following condition

$$\begin{aligned} G(x) &= \begin{bmatrix} g(x) & ad_f g(x) & ad_f^2 g(x) \end{bmatrix} \\ &= \begin{bmatrix} -0.3x_1 & -0.6x_1^3 x_2^2 - x_1^3 x_2 - 0.7x_2 + 0.75x_3 & (*) \\ -0.5x_2 & -0.1x_1 + 3.5x_1 x_2 - 1.6x_1^2 x_2^3 & (*) \\ -0.8x_3 & -1.6x_1^2 x_2^2 x_3 & (*) \end{bmatrix} \end{aligned}$$

where $(*)$ represent elements of the third column which are irrelevant and are polynomials of x_1, x_2, x_3 . When $G(x)$ is singular at the origin, it implies that the equivalent control gain to achieve feedback linearization will go to infinity when $x \rightarrow 0$. Furthermore, the linear combination of vector column 1 and vector column 2 will not belong to any of them. So the involutive condition does not meet. Therefore, we conclude that the nonlinear system is not linearizable and the feedback linearization technique is not applicable either.

For regional stabilization, we will set state derivative bounds $\nu_1 = 4, \nu_2 = 2, \nu_3 = 1$. Assuming that $Q(x)$ and $K(x)$ have the following forms

$$Q(x) = Q_0 + Q_1x_1 + Q_2x_2 + Q_3x_3$$

$$K(x) = K_0 + K_1x_1 + K_2x_2 + K_3x_3,$$

we solve Theorem 1 by SOS programming tool such as SOSTOOL [60] and obtain

$$\begin{aligned} Q_0 &= 10^{-4} \times \begin{bmatrix} 0.1116 & 0.0101 & -0.0309 \\ 0.0101 & 0.0144 & 0.0015 \\ -0.0309 & 0.0015 & 0.0486 \end{bmatrix} \\ Q_1 &= 10^{-6} \times \begin{bmatrix} 0.6552 & -0.7728 & -0.1426 \\ -0.7728 & 0.8244 & 0.2112 \\ -0.1426 & 0.2112 & 0.0518 \end{bmatrix} \\ Q_2 &= 10^{-8} \times \begin{bmatrix} -0.2493 & -0.3653 & -0.0798 \\ -0.3653 & 0.4725 & -0.0555 \\ -0.0798 & -0.0555 & 0.0130 \end{bmatrix} \\ Q_3 &= 10^{-7} \times \begin{bmatrix} 0.1321 & 0.2050 & 0.0233 \\ 0.2050 & -0.2686 & 0.0164 \\ 0.0233 & 0.0164 & -0.0027 \end{bmatrix} \\ K_0 &= 10^{-5} \times \begin{bmatrix} -0.1013 & -0.1543 & -0.0085 \end{bmatrix} \\ K_1 &= 10^{-4} \times \begin{bmatrix} 0.1704 & -0.0599 & 0.0120 \end{bmatrix} \\ K_2 &= 10^{-5} \times \begin{bmatrix} 0.0540 & 0.6419 & 0.2446 \end{bmatrix} \\ K_3 &= 10^{-4} \times \begin{bmatrix} -0.0088 & 0.0271 & 0.2541 \end{bmatrix}. \end{aligned}$$

As a result, the stabilizing state feedback control law is $u = K(x)Q^{-1}(x)x$.

Starting from an initial condition $x_0 = \begin{bmatrix} 0.5 & -0.5 & 0.5 \end{bmatrix}^T$, we plot three states separately and compare the open loop and closed loop dynamic behavior in Fig. 3.1. Note that the uncontrolled states drift to some point in state space; while the controlled dynamics converges to the origin. Fig. 3.2(a) provides phase portrait of the closed loop system in 3-dimensional space and Fig. 3.2(b) is the profile of corresponding control input.

Different from backstepping control technique which manually constructs a Lyapunov function and its associated nonlinear controller, our proposed regional stabilization

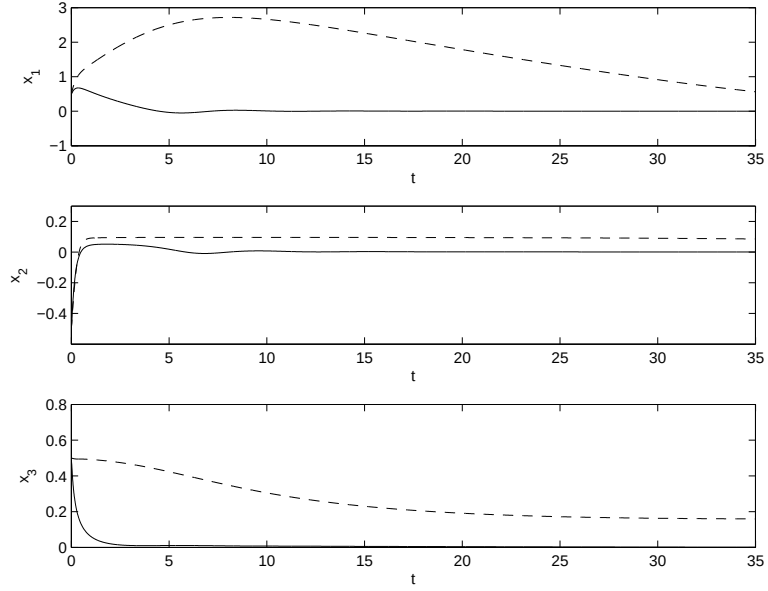


Figure 3.1: Comparison of open loop and closed loop state trajectories of Example I.

approach provides a systematic way to determine Lyapunov certificates using SOS programming techniques. It could also help to achieve some optimization criteria such as enlarged domain of attraction and desired convergence rate by searching for a better Lyapunov function within a local region.

Example II

We will then consider the controlled Van der Pol equation using both state feedback and output feedback controls.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 - x_1^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

It is well known that uncontrolled Van der Pol equation has an unstable origin and a stable limit cycle.

We first solve the state feedback stabilization problem by setting the state derivative bounds $\nu_1 = 4, \nu_2 = 2$. The Lyapunov matrix $Q(x)$ will be parameterized in different

forms as

$$\begin{aligned} Q^{[0]}(x) &= Q_0 \\ Q^{[1]}(x) &= Q_0 + Q_1x_1 + Q_2x_2 \\ Q^{[2]}(x) &= Q_0 + Q_1x_1 + Q_2x_2 + Q_{11}x_1^2 + Q_{12}x_1x_2 + Q_{22}x_2^2 \end{aligned}$$

and $K(x)$ is in the form of quadratic function of state x . Applying Theorem 1, we obtain different Lyapunov level sets as shown in Fig. 3.3(a). It is observed that the estimated domain of attraction tends to be larger when $Q(x)$ becomes more complicated. For the Lyapunov function $V(x) = x^T(Q^{[1]}(x))^{-1}x$, one can use backwards simulation for this simple 2nd-order nonlinear system to determine the actual domain of attraction. As shown in Fig. 3.3(b), the estimated DOA captures a large portion of actual DOA.

Then we simulate the Van der Pol equation with or without state feedback control law. Fig. 3.4(a) provides the comparison of the open loop and closed loop phase portraits. With an initial condition $x_0 = [5 \quad -2.5]^T$, it can be seen that uncontrolled open loop dynamics settles onto a limit cycle. On the other hand, the closed loop system is asymptotically stable due to the nonlinear control action. Fig. 3.4(b) provides the profile of nonlinear control input.

Secondly, we apply Theorem 2 to the Van der Pol equation to achieve output feedback stabilization. To transform eqns. (3.19)-(3.21) into SOS conditions, we will set output variation rate $\mu = 5$ and add a local region constraint $R_1(y) = y^2 - 1.2 < 0$. In addition, a closed-loop convergence rate $\beta = 0.05$ is specified through eqn. (3.24). The polynomial forms of optimization variables in Theorem 2 are chosen as

$$\begin{aligned} R(y) &= R_0 + R_1y^2 & S(y) &= S_0 \\ \hat{B}_c(y) &= \hat{B}_{c0} + \hat{B}_{c1}y^2 & \hat{C}_c(y) &= \hat{C}_{c0} & \hat{D}_c(y) &= \hat{D}_{c0} \end{aligned}$$

and $\hat{A}_c(y) = -\beta I - A^T(y) - C^T \hat{D}_c^T(y) B^T(y)$. This special selected $\hat{A}_c(y)$ will zero out the off-diagonal element of eqn. (3.24) and help reduce the computational burden in solving output feedback stabilization problem. Using SOS programming, we obtain the following

results by solving Theorem 2 with modified condition (3.24)

$$\begin{aligned}
 R_0 &= \begin{bmatrix} 63.249 & -16.414 \\ -16.414 & 33.971 \end{bmatrix} & R_1 &= \begin{bmatrix} 0.1639 & -0.1803 \\ -0.1803 & 0.6722 \end{bmatrix} \\
 S_0 &= \begin{bmatrix} 69.841 & -22.358 \\ -22.358 & 12.907 \end{bmatrix} \\
 \hat{B}_{c0} &= \begin{bmatrix} -76.413 \\ -34.576 \end{bmatrix} & \hat{B}_{c1} &= \begin{bmatrix} 15.899 \\ -22.358 \end{bmatrix} \\
 \hat{C}_{c0} &= \begin{bmatrix} 40.062 & -67.872 \end{bmatrix} & \hat{D}_{c0} &= 0.
 \end{aligned}$$

Letting $M(y) = I - R(y)S_0$ and $N = I$, the output feedback controller gains could be constructed from (3.22) as rational functions of output y only. Moreover, the resulting Lyapunov matrix is

$$W(y) = \frac{1}{\Delta} \begin{bmatrix} 69.841\Delta & -22.358\Delta & 949.58 + 21.427y^2 + 0.0396y^4 & & & & \\ \star & 12.907\Delta & -1375.4 - 30.854y^2 - 0.0568y^4 & & & & \\ \star & \star & 30.666 + 0.6897y^2 + 0.0013y^4 & & & & \\ \star & \star & & \star & & & \\ & & & & 0 & & \\ & & & & -1412.0 - 31.559y^2 - 0.05809y^4 & & \\ & & & & 53.281 + 1.1952y^2 + 0.0022y^4 & & \\ & & & & 30.666 + 0.6897y^2 + 0.0013y^4 & & \end{bmatrix}$$

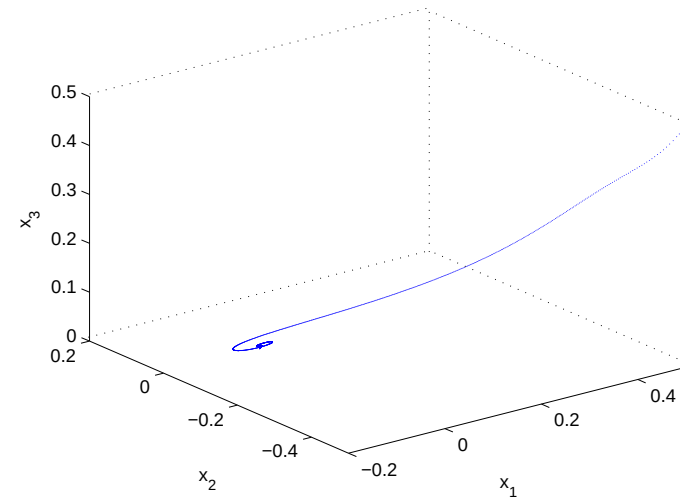
with $\Delta = 949.51 + 21.427y^2 + 0.0396y^4$. The cross-section of Lyapunov level set $V(x_{cl}) \leq 1$ in x_1 - x_2 plane and the state region constraint is shown in Fig. 3.5. It is clear that the estimated domain of attraction lies within the constrained state region described by $|\dot{y}| \leq 5$ and $y^2 - 1.2 < 0$. However, the estimated DOA is smaller than the one derived from state feedback case. Using the output feedback control law, we also simulate the state trajectories starting from initial point $x_0 = [1 \ 1]^T$. The controlled Van der Pol dynamics is stabilized within the constrained region and converges to the origin (see Fig. 3.6(a)). Note that the convergence rate in this case is relatively slow in comparison to state feedback control case. Fig. 3.6(b) provides the profile of corresponding control input.

Again, the proposed control design approach provides an effective way to construct an output feedback controller as well as characterize a Lyapunov level set in which the

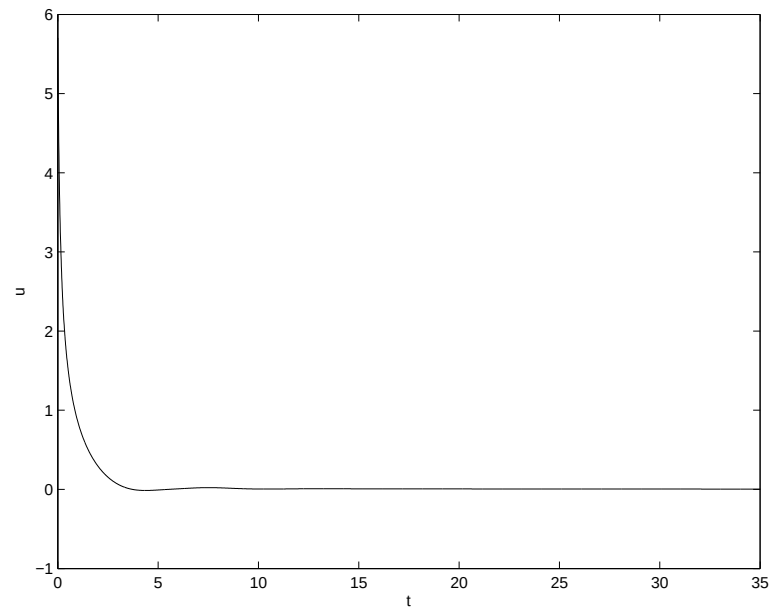
controller works. Note that we only solve one feasible Lyapunov level set in this example. Using optimization technique, one could maximize the domain of attraction to cover a larger portion of the state region.

3.5 Summary

In this chapter, we proposed a new nonlinear stabilization approach based on the popular linear parameter-varying control techniques. The regional state feedback control problem of polynomial nonlinear systems was studied using rational Lyapunov functions of states. By bounding the variation rates of each state, the domain of attraction was embedded in the region specified by the nonlinear vector field. As a result, the state feedback stabilization conditions was formulated as a set of polynomial matrix inequalities and can be solved efficiently by SOS programming. The resulting Lyapunov matrix and state feedback gains were typically state-dependent rational matrix functions. This approach was also extended to a class of output-dependent nonlinear systems where the stabilizing output feedback controller can be synthesized using rational Lyapunov functions of outputs. This generalization will make the proposed approach applicable to situations when only part of the states are measurable. The proposed stabilization approach could greatly simplify the solution to nonlinear stabilization problems with a prescribed stability region, and also shed a light on nonlinear optimal/robust control problems for polynomial nonlinear systems.

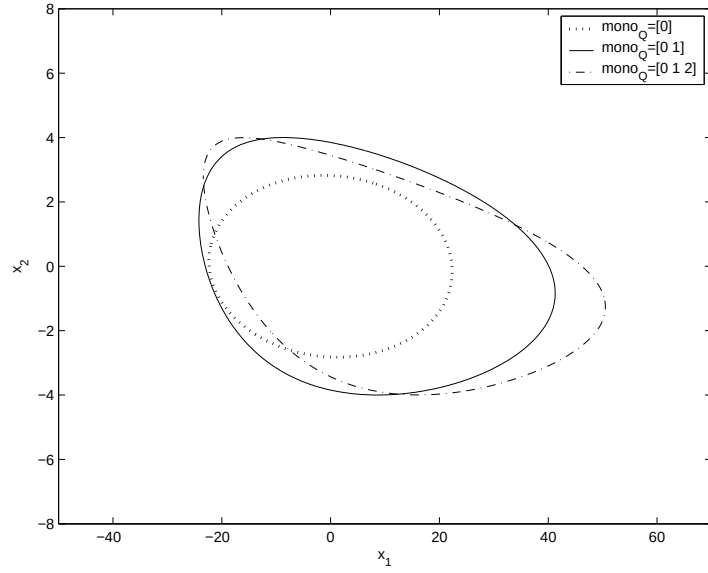


(a) state trajectory

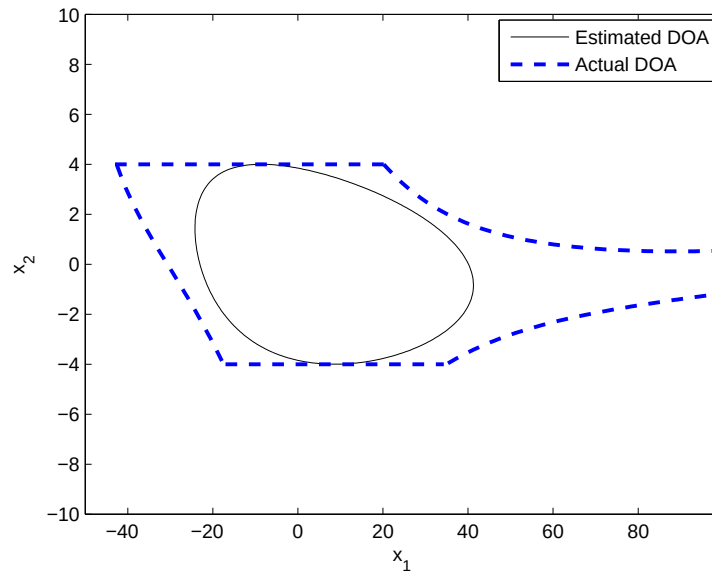


(b) control force

Figure 3.2: State trajectory and control input of Example I.

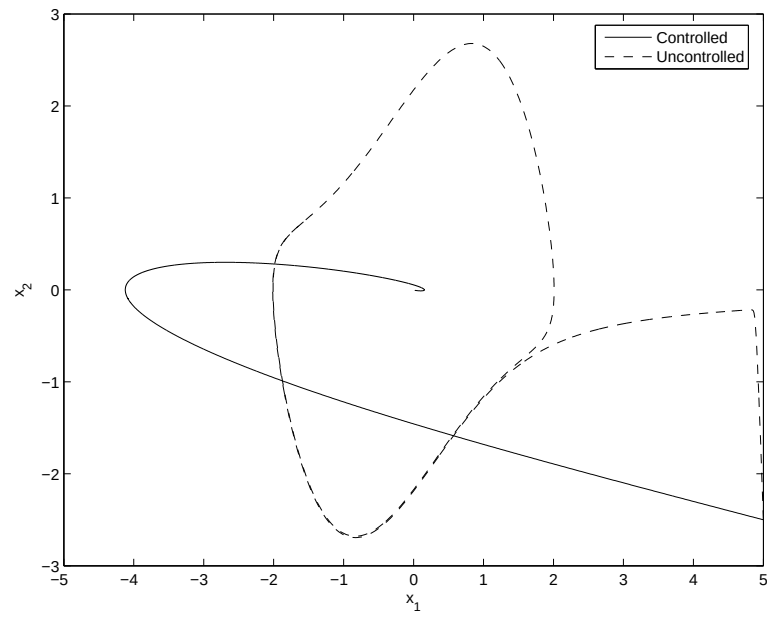


(a) Comparison of different Lyapunov level sets

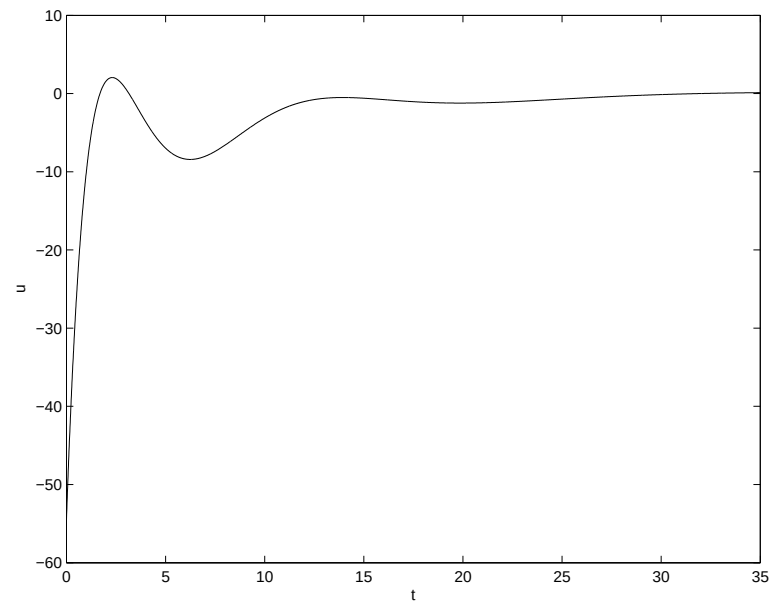


(b) Estimated and actual DOA

Figure 3.3: Characteristics of Lyapunov level set of Example II: state feedback case.



(a) Phase trajectory



(b) Control input

Figure 3.4: State trajectory and control input of Example II: state feedback case.

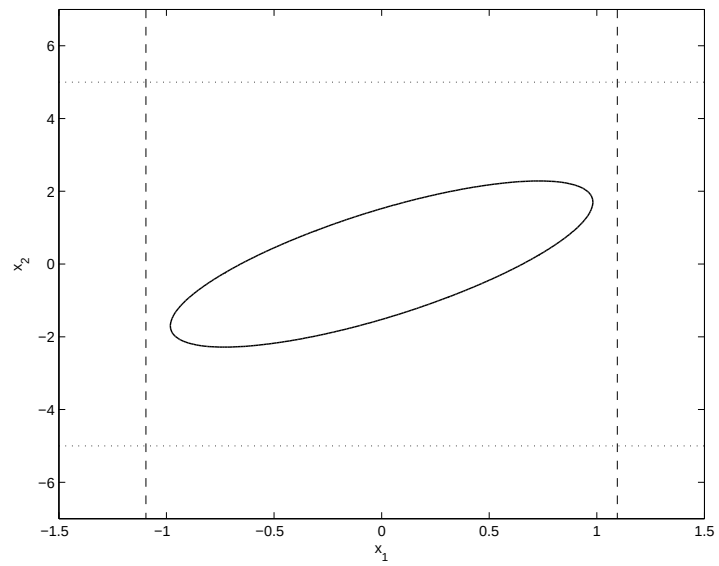
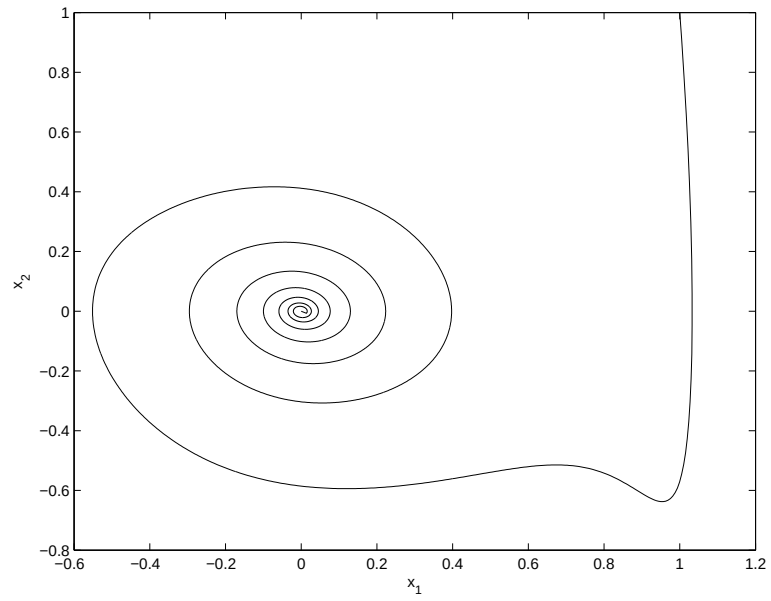
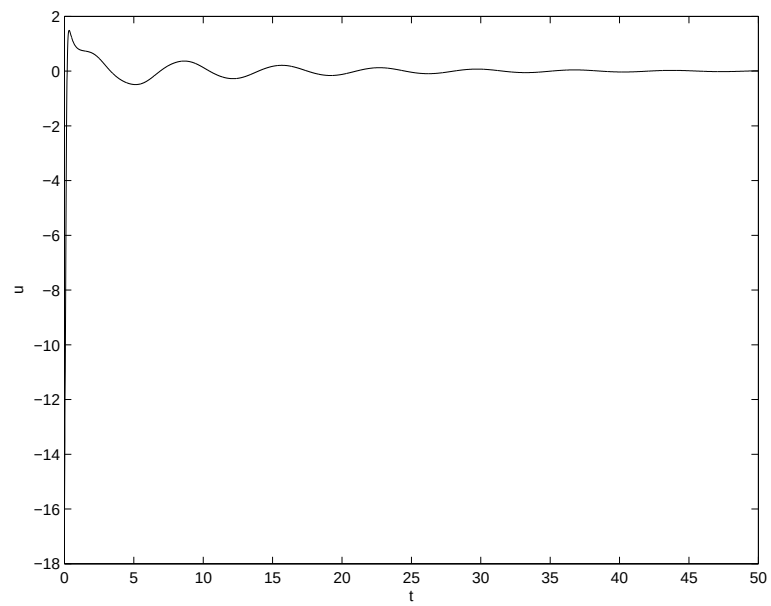


Figure 3.5: Lyapunov level set of Example II: output feedback case.



(a) Phase trajectory



(b) Control input

Figure 3.6: State trajectory and control input of Example II: output feedback case.

Chapter 4

Nonlinear Output Feedback $\mathcal{H}_\infty$ Control for Polynomial Nonlinear Systems

One of the most important contribution of this thesis is that by investigating comprehensive nonlinear control techniques for polynomial nonlinear systems, we overcome the bottleneck existing in current optimal control literature and proposed the nonlinear output feedback $\mathcal{H}_\infty$ control design scheme. Specifically, we will firstly present $\mathcal{H}_\infty$ synthesis condition in terms of HJI inequalities for a class of polynomial nonlinear systems without orthonormal and decoupling assumptions on the plant structure. Then a computationally efficient approach based on SOS programming [56, 60] will be proposed to solve the generalized HJI inequalities, which helps provide a coherent methodology of synthesizing higher order Lyapunov functions and their associated nonlinear control laws. After that, we will demonstrate the effective $\mathcal{H}_\infty$ control design approach and facilitate the application of advanced control techniques to spacecraft problem. Equipped with effective nonlinear $\mathcal{H}_\infty$ control design tool, we will study both of the attitude regulation and trajectory tracking problems for an axi-symmetric rigid spacecraft with two controls. A state feedback $\mathcal{H}_\infty$ controller will be first designed to stabilize the axi-symmetric spacecraft on a rotating plane and compare our results with the existing nonlinear control design in [78]. In addition, we will develop nonlinear output feedback $\mathcal{H}_\infty$ controller, which enables the spacecraft to track

some desired trajectory in the presence of external disturbances.

This chapter is organized as follows: In Section 4.1, we will summarize the existing methods dedicated for the solution of the HJI equations. Section 4.2-4.4 generalize nonlinear $\mathcal{H}_\infty$ synthesis conditions by removing orthonormal and decoupling assumptions and provide the solution procedure in solving nonlinear $\mathcal{H}_\infty$ analysis and synthesis conditions using SOS programming approach. In Section 4.5, we will firstly review the research development in the field of high performance spacecraft control, then the nonlinear model of an axis-symmetric spacecraft will be set up. The rest part of Section 4.5 is devoted to the nonlinear $\mathcal{H}_\infty$ control designs for the axis-symmetric spacecraft to achieve attitude regulation and trajectory tracking objectives. Simulation results are also provided to compare with existing nonlinear spacecraft control laws and demonstrate new $\mathcal{H}_\infty$ control designs. Finally, the chapter concludes in Section 4.6.

4.1 Introduction

Beyond the stabilization problem, to address modeling uncertainties and seek optimal performance, nonlinear robust control theory has been developed [73, 50, 19]. In the past decade, a theoretical framework for exploring $\mathcal{H}_\infty$ control of nonlinear systems has been proposed in [81, 27, 82]. Interpreting nonlinear $\mathcal{H}_\infty$ control in terms of dissipativity and differential game [5], the solution to this problem has been related to an appropriate Hamilton-Jacobi-Isaacs (HJI) equation. It was further shown that the solution to output feedback control problem is determined by a pair of coupled HJI equations. Parallel to linear $\mathcal{H}_\infty$ control theory, a separation principle was also established under a detectability hypothesis [26]. However, how to solve nonlinear $\mathcal{H}_\infty$ control problem in a numerically efficient way to make it practically useful to physical world remains an unsolved issue.

To this end, various approaches have been proposed to solve HJI equation numerically. In [34, 77], Taylor series expansion of the storage function $V(x)$ were considered to solve HJI equation term by term in an iterative fashion, provided that the linearized model of nonlinear systems has a solution. However, the solutions from these approaches do not have a closed form and they may not converge to an analytic solution. Furthermore, Galerkin successive approximation was proposed in [6] to reduce HJI equations to a sequence of linear partial differential equations. Then Galerkin method was applied to solve

control input and worst case disturbance iteratively to find an approximated solution. As a result, the computation cost increases exponentially as the number of states increase. On the other hand, a convex parametrization of nonlinear $\mathcal{H}_\infty$ control problem was derived in [46] based on a pair of positive definite matrix functions $P(x), Q(x)$. Unfortunately, it is difficult, if not impossible, to specify the form of $P(x)$ such that $\partial V(x)/\partial x = 2x^T P(x)$ except for the trivial case when $P(x)$ is a constant matrix. In [63], the $\mathcal{L}_2$ gain analysis problem for polynomial nonlinear systems was formulated as a convex state-dependent linear matrix inequality (LMI), which can be recast as a SOS optimization problem. This approach was shown promising to overcome the numerical difficulty in solving HJI inequality and provides an analytic solution at the same time. As a powerful and promising technique, SOS programming has also been applied to solve nonlinear analysis [55, 74] and stabilization [59, 61] problems.

4.2 Nonlinear Output Feedback $\mathcal{H}_\infty$ Synthesis Problem

Now, we consider a polynomial nonlinear system that is affine in exogenous disturbance and control input

$$\begin{cases} \dot{x} &= A(x)x + B_1(x)d + B_2(x)u \\ e &= C_1(x)x + D_{12}(x)u \\ y &= C_2(x)x + D_{21}(x)d \end{cases} \quad (4.1)$$

where the system state $x \in \mathbf{R}^n$, control input $u \in \mathbf{R}^{n_u}$, exogenous disturbance $d \in \mathbf{R}^{n_d}$, controlled output $e \in \mathbf{R}^{n_e}$, and measured output $y \in \mathbf{R}^{n_y}$. All of the state space entries are polynomial functions of the state x with compatible dimensions. In the past, most of literature [27, 82, 45] derived the nonlinear output feedback $\mathcal{H}_\infty$ synthesis condition by enforcing the plant with orthonormal and decoupling structures like

$$D_{12}^T(x) \begin{bmatrix} C_1(x) & D_{12}(x) \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \quad \begin{bmatrix} B_1(x) \\ D_{21}(x) \end{bmatrix} D_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

These assumptions are quite restrictive and will be removed in this study. Specifically, we assume that D_{12} and D_{21} are constant matrices with full column and row ranks respectively.

The objective of nonlinear output feedback $\mathcal{H}_\infty$ control is to design an output feedback controller such that the nonlinear closed-loop system is asymptotically stable and

its $\mathcal{H}_\infty$ norm is less than γ , i.e.

$$\|e\|_2 \leq \gamma \|d\|_2 \quad \text{when} \quad x(0) = 0.$$

It turns out that the nonlinear output feedback $\mathcal{H}_\infty$ control problem can be decomposed into a state feedback and an output estimation problems. The solvability conditions will be given in the form of HJI inequalities, which are partial differential inequalities and extremely difficult to solve. For the general case without the above orthonormal and decoupling constraints, the problem becomes more complicated and there is currently no systematic solution procedure available. In [96], we proposed a computationally efficient scheme to solve the standard $\mathcal{H}_\infty$ control problem by reformulating HJI inequalities as semi-definite optimization conditions and solved using SOS programming. In this paper, we will relax stringent plant constraints and develop the solution procedure for the resulting generalized HJI inequalities. The relaxation would help provide additional freedom to search for a suitable Lyapunov function and result in better nonlinear $\mathcal{H}_\infty$ control law so that the design objective is optimized. In addition, the applicability of the proposed design approach would be greatly enhanced since a wider class of physical systems will be covered and solvable within this framework.

Theorem 3 *Given a scalar $\gamma > 0$, the nonlinear output feedback $\mathcal{H}_\infty$ control problem is (locally) solvable for the nonlinear system (4.1) if there exist two positive definite matrix functions $U(x), V(x)$ satisfying $U(0) = V(0) = 0$ such that*

$$\mathcal{H}_{FI}(U, \gamma, x) < 0 \tag{4.2}$$

$$\mathcal{H}_{FC}(V, \gamma, x) - \gamma^{-2} \mathcal{H}_{FI}(U, \gamma, x) < 0 \tag{4.3}$$

$$\left. \frac{\partial^2}{\partial x^2} (\mathcal{H}_{FC} - \gamma^{-2} \mathcal{H}_{FI}) \right|_{x=0} < 0 \tag{4.4}$$

$$W(x) := V(x) - \gamma^{-2} U(x) \geq 0, \tag{4.5}$$

where

$$\begin{aligned}
\mathcal{H}_{FI}(U, \gamma, x) &:= \frac{\partial U}{\partial x} A(x)x + x^T C_1^T(x) [I - D_{12}(D_{12}^T D_{12})^{-1} D_{12}^T] C_1(x)x \\
&\quad - \frac{1}{2} \frac{\partial U}{\partial x} B_2(x)(D_{12}^T D_{12})^{-1} D_{12}^T C_1(x)x - \frac{1}{2} x^T C_1^T(x) D_{12}(D_{12}^T D_{12})^{-1} B_2^T(x) \frac{\partial U^T}{\partial x} \\
&\quad + \frac{1}{4} \frac{\partial U}{\partial x} \left[\frac{1}{\gamma^2} B_1(x) B_1^T(x) - B_2(x)(D_{12}^T D_{12})^{-1} B_2^T(x) \right] \frac{\partial U^T}{\partial x} \\
\mathcal{H}_{FC}(V, \gamma, x) &:= \frac{\partial V}{\partial x} A(x)x + \frac{1}{4} \frac{\partial V}{\partial x} B_1(x) [I - D_{21}^T (D_{21} D_{21}^T)^{-1} D_{21}] B_1^T(x) \frac{\partial V^T}{\partial x} \\
&\quad - \frac{1}{2} x^T C_2^T(x) (D_{21} D_{21}^T)^{-1} D_{21} B_1^T(x) \frac{\partial V^T}{\partial x} - \frac{1}{2} \frac{\partial V}{\partial x} B_1(x) D_{21}^T (D_{21} D_{21}^T)^{-1} C_2(x)x \\
&\quad + x^T \left[\frac{1}{\gamma^2} C_1^T(x) C_1(x) - C_2^T(x) (D_{21} D_{21}^T)^{-1} C_2(x) \right] x.
\end{aligned}$$

Moreover, one of the n th-order output feedback controllers is given by

$$\begin{cases} \dot{x}_c &= A(x_c)x_c + B_1(x_c)F_1(x_c) + B_2(x_c)F_0(x_c) + L_0(x_c) [C_2(x_c)x_c + D_{21}F_1(x_c) - y] \\ u &= F_0(x_c) \end{cases} \quad (4.6)$$

where the matrix functions $F_0(x)$, $F_1(x)$ and $L_0(x)$ are defined as

$$F_0(x) := -(D_{12}^T D_{12})^{-1} \left[\frac{1}{2} B_2^T(x) \frac{\partial U^T}{\partial x} + D_{12}^T C_1(x)x \right] \quad (4.7)$$

$$F_1(x) := \frac{1}{2\gamma^2} B_1^T(x) \frac{\partial U^T}{\partial x} \quad (4.8)$$

$$\frac{\partial W(x)}{\partial x} L_0(x) := -2 [C_2(x)x + D_{21}F_1(x)]^T (D_{21} D_{21}^T)^{-1}. \quad (4.9)$$

Note that solvability conditions (4.2)-(4.3) are also given in the form of generalized HJI inequalities, which remain difficult to solve. For nonlinear state feedback problem, the $\mathcal{H}_\infty$ synthesis condition degenerates to a single HJI inequality (4.2). In the special case when both $V(x)$, $U(x)$ are quadratic functions of state x , the solvability conditions (4.2)-(4.5) become a set of state-dependent LMIs. However, for general polynomial Lyapunov functions, the bilinear term $-\frac{1}{4} \frac{\partial U}{\partial x} B_2(x)(D_{12}^T D_{12})^{-1} B_2^T(x) \frac{\partial U^T}{\partial x}$ in condition (4.2) will render a non-convex problem. SOSTOOL [60] provides an efficient way to obtain the tractable solutions by reformulating these conditions into SOS optimization problems. In the following subsections, we will develop a systematic design procedure for nonlinear output feedback $\mathcal{H}_\infty$ control and obtain computationally tractable solutions using SOS programming tools

when $U(x)$ and $V(x)$ are both of polynomial forms, i.e.

$$\begin{aligned} U(x) &= \frac{1}{2} M^{[n_p]}(x)^T P M^{[n_p]}(x) \\ V(x) &= \frac{1}{2} N^{[n_q]}(x)^T Q N^{[n_q]}(x) \end{aligned}$$

and $n_p \geq 1, n_q \geq 1$.

4.3 Iterative Algorithm to Solve the Nonlinear State Feedback $\mathcal{H}_\infty$ Condition

Suppose there is a quadratic Lyapunov function $U_0(x) = \frac{1}{2}x^T P_0 x > 0$ and an initial state feedback controller $u_0 = K_0(x)x$ for state feedback $\mathcal{H}_\infty$ control, then the closed-loop plant becomes

$$\begin{cases} \dot{x} &= [A(x) + B_2(x)K_0(x)]x + B_1(x)d \\ e &= [C_1(x) + D_{12}(x)K_0(x)]x \end{cases}$$

Due to the special form of quadratic Lyapunov function, it is sufficient to reformulate condition (4.2) as a SOS optimization problem:

$$\begin{aligned} \min \quad & \gamma_0 \\ \text{s.t.} \quad & -v^T \mathcal{S}_{FI0} v \in \Phi_{\text{SOS}}, \end{aligned} \tag{4.10}$$

where v is a free vector of compatible dimension and

$$\mathcal{S}_{FI0} := \begin{bmatrix} \begin{cases} \frac{1}{2}[A(x)R_0 + R_0A^T(x)] - \frac{1}{4}B_2(x)(D_{12}^T D_{12})^{-1}B_2^T(x) \\ -\frac{1}{2}B_2(x)(D_{12}^T D_{12})^{-1}D_{12}^T C_1(x)R_0 - \frac{1}{2}R_0C_1^T(x)D_{12}(D_{12}^T D_{12})^{-1}B_2^T(x) \\ [I - D_{12}(D_{12}^T D_{12})^{-1}D_{12}^T]^{\frac{1}{2}}C_1(x)R_0 \\ B_1^T(x) \end{cases} \\ \begin{matrix} \star & \star \\ -I & \star \\ 0 & -4\gamma_0^2 I \end{matrix} \end{bmatrix}$$

and $R_0 = P_0^{-1} > 0$. Consequently, condition (4.10) can be solved using SOS programming techniques to obtain a new control law $u_0(x)$ with optimized closed-loop $\mathcal{H}_\infty$ performance

γ_0 . To this end, a feasible state feedback control renders the closed-loop $\mathcal{H}_\infty$ norm less than γ_0 will be

$$u_0(x) = -(D_{12}^T D_{12})^{-1} [B_2^T(x) R_0^{-1} + D_{12}^T C_1(x)] x.$$

On the other hand, an equivalent SOS condition for state feedback $\mathcal{H}_\infty$ control will be

$$\begin{aligned} \min \quad & \gamma_i \\ \text{s.t.} \quad & U_i(x) \in \Phi_{\text{SOS}} \\ & -v^T \mathcal{S}_{FIi} v \in \Phi_{\text{SOS}}, \end{aligned} \tag{4.11}$$

where

$$\mathcal{S}_{FIi} := \begin{bmatrix} \left\{ \begin{array}{l} \frac{\partial U_i}{\partial x} [A(x)x + B_2(x)u_{i-1}(x)] \\ + [C_1(x)x + D_{12}u_{i-1}(x)]^T [C_1(x)x + D_{12}u_{i-1}(x)] \end{array} \right\} & \frac{\partial U_i}{\partial x} B_1(x) \\ B_1^T(x) \frac{\partial U_i^T}{\partial x} & -4\gamma_i^2 I \end{bmatrix}.$$

For any fixed $u_{i-1}(x)$, condition (4.11) is convex about variables $U_i(x)$ and γ_i . Using a polynomial representation of Lyapunov function $U_i(x)$, it is possible to solve for a $U_i(x)$ and γ_i through SOS programming and achieve a better closed-loop performance $\gamma_i < \gamma_{i-1}$. It is clear that optimization (4.11) is always feasible (at least $U_i(x) = U_{i-1}(x)$). Then computing a new controller u_i from

$$u_i(x) = -(D_{12}^T D_{12})^{-1} \left[\frac{1}{2} B_2^T(x) \frac{\partial U_i^T}{\partial x} + D_{12}^T C_1(x)x \right], \tag{4.12}$$

the next round of iteration could be started by resolving the SOS optimization (4.11) repeatedly.

Although we start from a quadratic Lyapunov function, the final Lyapunov function will be a complicated polynomial form of x . During the iteration, $u_i(x)$ could be specified as any type of polynomials of x . The resulting state feedback controller is given by eqn. (4.12) at exit. Finally, it is noted that the performance level γ_i will be improved gradually as i increases and finally converges to a sub-optimal solution.

4.4 Design Procedure for Nonlinear Output Feedback $\mathcal{H}_\infty$ Control

Summarizing previous discussions, we will have the following control design procedure to solve the nonlinear output feedback $\mathcal{H}_\infty$ synthesis condition for polynomial nonlinear

systems.

1. Through the iterative algorithm in section 4.3, the HJI inequality (4.2) will be solved to obtain a polynomial Lyapunov function $U(x) = \frac{1}{2}M^{[n_p]}(x)^T P M^{[n_p]}(x)$, $n_p \geq 1$ and a performance level $\gamma > 0$.

2. Calculate $F_0(x), F_1(x)$ by

$$F_0(x) = -(D_{12}^T D_{12})^{-1} \left[\frac{1}{2} B_2^T(x) \frac{\partial U^T}{\partial x} + D_{12}^T C_1(x)x \right]$$

$$F_1(x) := \tilde{F}_1(x)x = \frac{1}{2\gamma^2} B_1^T(x) \frac{\partial U^T}{\partial x}.$$

3. Reformulate conditions (4.3)-(4.5) as another SOS optimization problem:

$$\begin{aligned} \min \quad & \hat{\gamma} \\ \text{s.t.} \quad & V(x) \in \Phi_{\text{SOS}} \\ & -v_1^T \mathcal{S}_{FC} v_1 \in \Phi_{\text{SOS}} \\ & -v_2^T \left[\frac{\partial^2}{\partial x^2} (\mathcal{H}_{FC} - \gamma^{-2} \mathcal{H}_{FI}) \Big|_{x=0} \right] v_2 \in \Phi_{\text{SOS}} \\ & V(x) - \gamma^{-2} U(x) \in \Phi_{\text{SOS}}, \end{aligned} \tag{4.13}$$

where v_1, v_2 are free vectors of compatible dimensions and

$$\mathcal{S}_{FC} := \begin{bmatrix} \left\{ \begin{aligned} & -\gamma^{-2} \mathcal{H}_{FI}(U, \gamma, x) + \frac{\partial V}{\partial x} A(x)x - x^T C_2^T(x) (D_{21} D_{21}^T)^{-1} C_2(x)x \\ & -\frac{1}{2} x^T C_2^T(x) (D_{21} D_{21}^T)^{-1} D_{21} B_1^T(x) \frac{\partial V^T}{\partial x} - \frac{1}{2} \frac{\partial V}{\partial x} B_1(x) D_{21}^T (D_{21} D_{21}^T)^{-1} C_2(x)x \\ & \frac{1}{2} [I - D_{21}^T (D_{21} D_{21}^T)^{-1} D_{21}]^{\frac{1}{2}} B_1^T(x) \frac{\partial V^T}{\partial x} \\ & C_1(x)x \end{aligned} \right\} \\ \star & \quad \star \\ -I & \quad \star \\ 0 & \quad -\hat{\gamma}^2 I \end{bmatrix}.$$

Solve optimization problem (4.13) using SOS programming to obtain

$$V(x) = \frac{1}{2} N^{[n_q]}(x)^T Q N^{[n_q]}(x), \quad n_q \geq 1 \text{ and } \hat{\gamma} \leq \gamma.$$

4. Solve $L_0(x)$ from the equation

$$\frac{\partial W(x)}{\partial x} L_0(x) = -2 [C_2(x)x + D_{21} F_1(x)]^T (D_{21} D_{21}^T)^{-1} \tag{4.14}$$

as $L_0(x) = -2T^{-1}(x) \left[C_2(x) + D_{21}\tilde{F}_1(x) \right]^T (D_{21}D_{21}^T)^{-1}$, where $T(x)$ is a non-singular polynomial function satisfying $\frac{\partial W}{\partial x} = x^T T$.

5. Finally, we construct the output feedback controller in the form of (4.6). The closed-loop $\mathcal{H}_\infty$ norm will be bounded by γ .

In the proposed control design procedure, we first reformulate synthesis conditions (4.2) and (4.3)-(4.5) into the optimization problems (4.11) and (4.13). It is obvious that the resulting synthesis conditions are matrix inequalities with polynomial entries and linear about their decision variables. Both problems are then solved efficiently using SOS programming techniques with polynomial complexity. The Lyapunov function for the closed-loop system is given by $U(x) + W(x - x_c)$, which could be any higher order form polynomials. Note that $L_0(x)$ is not unique for a given $W(x)$. Since $W(x)$ has at least order 2, it is clear that $\frac{\partial W(x)}{\partial x}$ could be rewritten as $x^T T(x)$ for a non-singular polynomial matrix $T(x)$ of dimension $n \times n$. Therefore, one solution of $L_0(x)$ satisfying eqn. (4.14) will be

$$L_0(x) = -2T^{-1}(x) \left[C_2(x) + D_{21}\tilde{F}_1(x) \right]^T (D_{21}D_{21}^T)^{-1}.$$

Most of previous nonlinear control research considered global stabilization problem. Nevertheless, it is not always possible to find a global stabilizing controller. Moreover, in a restricted region, local controllers will often perform better than global controllers. To solve nonlinear $\mathcal{H}_\infty$ control problem locally, the SOS optimization problems (4.11) and (4.13) needs to be amended by adding state region constraints. This will help improve solvability of the synthesis conditions and lead to optimized solutions for nonlinear output feedback $\mathcal{H}_\infty$ control in a local region. These constraints also specify a state space region on which the synthesized nonlinear $\mathcal{H}_\infty$ control is valid, i.e., domain of validity.

4.5 Nonlinear $\mathcal{H}_\infty$ Control Designs for Axi-Symmetric Spacecraft

Spacecraft has inherent nonlinear characteristics such as cross-coupling gyroscopic terms that cannot be ignored. Moreover, its increasingly stringent operating requirements have demanded high performance flight control systems. For example, future commercial and military spacecraft are envisioned to have large angle maneuver capability and high

pointing accuracy. In addition, limited propellant on board necessitates optimal control strategy for long term spacecraft operations. These capabilities and operating requirements are imperative for the demanding tasks of future space missions, such as very large base interferometry, formation flying, and autonomous shuttle docking and servicing.

Stabilizing feedbacks of a spacecraft with or without disturbance inputs have been studied for a long time. Due to significant cross-coupling effect of spacecraft, linear control designs based on its linearized dynamics are often inadequate [17]. Wie and Barba [84] derived a number of simple nonlinear control schemes using quaternion and angular velocity feedback. Global stabilization strategy of spacecraft was also proposed in [83]. It is noted that global attitude stabilization is not achievable using smooth feedback. Tsiotras [76] extended these results and designed different linear/nonlinear controllers by proper selection of Lyapunov functions. These control laws could recover the optimal cost involving both attitude and velocity asymptotically but may lead to high-gain controllers. Generally speaking, the synthesis of optimal feedback control for a rigid body spacecraft remains a difficult problem and was mainly addressed in the context of time/fuel optimal maneuvers [29]. The main obstruction in obtaining optimal control laws is to efficiently solve the Hamilton-Jacobi-Isaacs (HJI) equation, especially when the cost includes a penalty term on the control effort.

Other than spacecraft regulation problems, another issue highly relevant to spacecraft operation is the tracking of prescribed attitude trajectories. Adaptive control and sliding mode control were proposed in [72] and [13] for spacecraft tracking problem. However, the full states have to be measured for these control algorithms and no robustness is guaranteed in the presence of unmodeled dynamics. On the other hand, a passivity-based control approach was proposed in [43] to achieve rigid body attitude regulation without velocity measure. The second approach that has also been used to avoid the velocity measurement is based on the use of nonlinear observers [10]. It is noted that the proposed output feedback controllers do not possess desired optimality and robustness properties.

In the following subsections, we will study both of the attitude regulation and trajectory tracking problems for an axi-symmetric rigid spacecraft with two controls.

4.5.1 Modeling of Axi-Symmetric Spacecraft

In this subsection, we will present an axi-symmetric rigid body spacecraft model [78] with two control torques acting perpendicular to the symmetry axis and to each other.

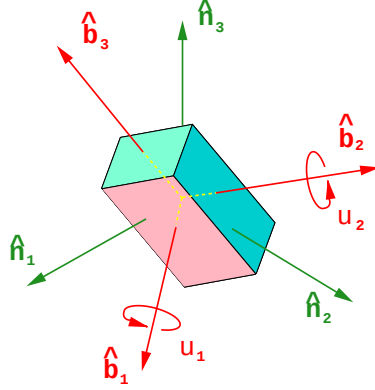


Figure 4.1: Axi-symmetric rigid body with two controls.

To describe spacecraft dynamics, two reference frames will be introduced (see Fig. 4.1 and [78, 70]). The first one is the body-fixed reference frame $\mathbf{b} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$, which is located at the center of mass and aligned along the principal axes of a rotating rigid body. In the body-fixed reference frame, denote the angular velocity vector and moment of inertia with respect to three body axes as $\omega_1, \omega_2, \omega_3$ and I_1, I_2, I_3 . We assume that $\hat{b}_3$ pointing along the symmetry axis and $I_1 = I_2$. Furthermore, there are two control torques T_1 and T_2 generated along the $\hat{b}_1$ and $\hat{b}_2$ axis. The two control torques span the two-dimensional plane orthogonal to the axis of symmetry. The dynamics equation of spacecraft with respect to this frame take the form of

$$I_1 \dot{\omega}_1 = (I_2 - I_3) \omega_2 \omega_3 + T_1 \quad (4.15)$$

$$I_2 \dot{\omega}_2 = (I_3 - I_1) \omega_3 \omega_1 + T_2 \quad (4.16)$$

$$I_3 \dot{\omega}_3 = (I_1 - I_2) \omega_1 \omega_2.$$

From the third equation, it is clear that $\dot{\omega}_3 = 0$. It is noted that once the symmetric axis is static, the spacecraft is not controllable using smooth control [38]. Therefore, the spacecraft will be assumed to rotate constantly about the symmetric axis ($\omega_3(t) = \omega_{30} \neq 0$) to render smooth nonlinear control laws. Besides the body-fixed reference frame, we also

introduce an inertial reference frame $\mathbf{n} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$. The two reference frames are related by a rotation matrix that belongs to the special orthogonal group $SO(3)$. Consequently, the attitude trajectory of the moving reference frame with respect to the inertial reference frame can be described by a curve traced by the corresponding rotation vector $R \in SO(3)$. $R(t)$ satisfies the following differential equation while moving along the trajectory

$$\dot{R} = S(\omega_1, \omega_2, \omega_3)R,$$

where $S(\omega_1, \omega_2, \omega_3)$ is a skew-symmetric matrix defined as

$$S(\omega_1, \omega_2, \omega_3) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}.$$

There are many ways to parameterize the rotation group, such as Euler angles, quaternion, Rodrigues parameters, etc. In this research, we will adopt a parameterization of $SO(3)$ from [78] to describe the relative orientation of two reference frames using two perpendicular rotations. Specifically, one rotation describes the location of the designated body axis (usually spin-axis of a spinning vehicle) in the inertial frame. Another one is an initial rotation about this axis. This formulation of rotation matrix is very useful to the axis-symmetric spacecraft control problem because its uniform revolution about the symmetric axis is not concerned. Moreover, the new parameterization will reduce the number of kinematic variables to 2. To this end, a stereographic projection is introduced to describe the location of inertial axis $\hat{n}_3$ in the body-fixed $\mathbf{b}$ frame by two variables w_1 and w_2

$$w_1 = \frac{p_2}{1 + p_3} \quad w_2 = \frac{-p_1}{1 + p_3},$$

where p_1, p_2 and p_3 are directional cosines of the axis $\hat{n}_3$ with respect to $\mathbf{b}$ frame, i.e., $\hat{n}_3 = p_1\hat{b}_1 + p_2\hat{b}_2 + p_3\hat{b}_3$ with $p_1^2 + p_2^2 + p_3^2 = 1$. The stereographic projection is well defined for all orientations except for an upside-down configuration $(p_1, p_2, p_3) = (0, 0, -1)$ of the spacecraft. It is easy to verify that its inverse map is given by

$$\begin{aligned} p_1 &= \frac{-2w_2}{1 + w_1^2 + w_2^2} \\ p_2 &= \frac{2w_1}{1 + w_1^2 + w_2^2} \\ p_3 &= \frac{1 - w_1^2 - w_2^2}{1 + w_1^2 + w_2^2} \end{aligned}$$

and can be used to determine p_1, p_2, p_3 for the given w_1, w_2 . Under this projection, w_1 and w_2 satisfy the differential equations

$$\dot{w}_1 = \omega_{30}w_2 + \omega_2w_1w_2 + \frac{\omega_1}{2}(1 + w_1^2 - w_2^2) \quad (4.17)$$

$$\dot{w}_2 = -\omega_{30}w_1 + \omega_1w_1w_2 + \frac{\omega_2}{2}(1 + w_2^2 - w_1^2) \quad (4.18)$$

Let

$$a = \frac{I_2 - I_3}{I_1}, \quad \text{assuming } -1 < a < 1 \text{ for physical consideration}$$

$$m = \omega_{30}$$

$$u_1 = \frac{T_1}{I_1} \quad u_2 = \frac{T_2}{I_2},$$

and define the states as $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T = \begin{bmatrix} \omega_1 & \omega_2 & w_1 & w_2 \end{bmatrix}^T$, a state-space model of the axi-symmetric spacecraft will be derived from equation (4.15)-(4.16) and (4.17)-(4.18)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & am & 0 & 0 \\ -am & 0 & 0 & 0 \\ \frac{1+x_3^2-x_4^2}{2} & x_3x_4 & 0 & m \\ x_3x_4 & \frac{1+x_4^2-x_3^2}{2} & -m & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \quad (4.19)$$

For the rest of this paper, we will assume that $a = 0.5, m = -0.5 \text{ rad/sec}$. It is noted that the spacecraft equation is given in cascade form. That is, the control input u drives the angular velocity equation and the angular velocity ω drives the kinematic equation. The kinematic equation can be accessed and manipulated only through the angular velocity vector ω . To facilitate the application of proposed nonlinear $\mathcal{H}_\infty$ control techniques to axi-symmetric spacecraft control, we will also augment the spacecraft model by some exogenous input. The external input used for attitude regulation problem will capture modeling uncertainty and actuator inaccuracy of the spacecraft. For spacecraft command tracking, external inputs will include modeling uncertainty, actuator and sensor inaccuracies, and command input.

4.5.2 Nonlinear $\mathcal{H}_\infty$ Control Design and Simulation

We are interested in solving the attitude regulation and the command tracking problems for the axi-symmetric spacecraft using computationally efficient nonlinear $\mathcal{H}_\infty$ control techniques. A key observation that makes rigid body control of spacecraft amenable

to an efficient (convex) design technique such as SOS programming is that spacecraft dynamic eqn. (4.19) represents its attitude dynamics by polynomial vector field. The problem of attitude stabilization of a symmetric spacecraft was treated in [38], using control torques about axes spanning a two-dimensional plane orthogonal to the axis of symmetry. It was also shown that non-spinning dynamics cannot be asymptotically stabilized using smooth feedback. [78] presented a new methodology for constructing feedback control laws using backstepping technique. Non-smooth feedback control designs for axi-symmetric spacecraft have also been addressed in [70].

Spacecraft Attitude Regulation

For attitude regulation, the spacecraft is subject to disturbances d_1, d_2 with respect to $\hat{b}_1, \hat{b}_2$ axes and expected to be stabilized on a two-dimensional plane instead of staying at an equilibrium point. When $\omega_{30} \neq 0$, the steady state motion of the axi-symmetric spacecraft is a uniform revolution about its symmetric axis [78]. The state space model of spacecraft including disturbance d and controlled output variable e is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & am & 0 & 0 \\ -am & 0 & 0 & 0 \\ \frac{1+x_3^2-x_4^2}{2} & x_3x_4 & 0 & m \\ x_3x_4 & \frac{1+x_4^2-x_3^2}{2} & -m & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (4.20)$$

$$e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.8 & 0 \\ 0 & 0.8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \quad (4.21)$$

Note that the spacecraft model (4.20)-(4.21) does not satisfy orthonormal and decoupling assumptions. We would like to design nonlinear state feedback $\mathcal{H}_\infty$ controller to stabilize the spacecraft in the existence of external input, such as modeling uncertainty and actuator inaccuracy. This corresponds to solving the first HJI inequality (4.2) in Theorem 3. For local control design, we will introduce state region constraints $R_1(x) = x_3^2 + x_4^2 - 0.5 < 0$ and $R_2(x) = x_1^2 + x_2^2 - 0.5 < 0$. These constraints specify a bounded operating range of spacecraft angular velocity and attitude in which the nonlinear $\mathcal{H}_\infty$ control is designed for. Consequently, the state feedback $\mathcal{H}_\infty$ control synthesis condition in optimization problem

(4.11) will be modified to

$$\begin{aligned}
& \min \gamma_i \tag{4.22} \\
& \text{s.t. } U_i(x) + \lambda_1(x)R_1(x) + \lambda_2(x)R_2(x) \in \Phi_{\text{SOS}} \\
& \quad v^T \left\{ -\mathcal{S}_{FIi} + \begin{bmatrix} [\lambda_3(x) + \alpha x_1^4 + \beta x_2^4]R_1(x) + \lambda_4(x)R_2(x) & \underline{\mathbf{0}} \\ & \underline{\mathbf{0}} \end{bmatrix} \right\} v \in \Phi_{\text{SOS}},
\end{aligned}$$

where the SOS multipliers $\lambda_i(x) = x^T \Lambda_i x$ with $\Lambda_i > 0, \alpha > 0, \beta > 0$.

Using the iterative algorithm outlined in previous section, we solve the modified SOS problem (4.22) by SOSTOOL [60] and obtain a higher order Lyapunov function $U(x)$ as

$$\begin{aligned}
& \gamma = 0.8943 \\
& U(x) = 46.2544x_1^2 + 8.4019x_2^2 + 152.8548x_3^2 + 60.7185x_4^2 + 4.6060x_1x_2 + 98.0724x_1x_3 \\
& \quad + 4.8102x_1x_4 + 4.7611x_2x_3 + 11.9792x_2x_4 + 66.7882x_3x_4 + 19.4649x_1^4 + 4.5473x_2^4 \\
& \quad = \frac{1}{2}M^{[2]}(x)^T \begin{bmatrix} 92.5088 & 4.606 & 98.0724 & 4.8102 & 0 & 0 \\ 4.606 & 16.8038 & 4.7611 & 11.9792 & 0 & 0 \\ 98.0724 & 4.7611 & 305.7097 & 66.7882 & 0 & 0 \\ 4.8102 & 11.9792 & 66.7882 & 121.437 & 0 & 0 \\ 0 & 0 & 0 & 0 & 38.9299 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9.0946 \end{bmatrix} M^{[2]}(x)
\end{aligned}$$

with $M^{[2]}(x) = [x_1 \ x_2 \ x_3 \ x_4 \ x_1^2 \ x_2^2]^T$. The corresponding nonlinear state feedback control law is

$$u = F(x) = \begin{bmatrix} -17.0455x_1 - 0.6534x_2 - 17.4743x_3 - 0.4122x_4 - 11.7462x_1^3 \\ -0.6534x_1 - 10.7249x_2 - 0.7326x_3 - 8.4464x_4 - 8.7985x_2^3 \end{bmatrix},$$

which renders the closed loop $\mathcal{H}_\infty$ norm less than $\gamma = 0.8943$. The domain of validity for this controller is the intersection of regions $R_1(x) < 0$ and $R_2(x) < 0$.

Under zero disturbance, we choose an initial point $x_0 = [0.45 \ 0.5 \ -0.6 \ 0.35]^T$ within the feasible state region. Fig. 4.2 provides the state trajectories and the control inputs of the closed-loop system starting from the given initial condition. It is observed that evolved state trajectories satisfy state region constraints. Moreover, the closed loop

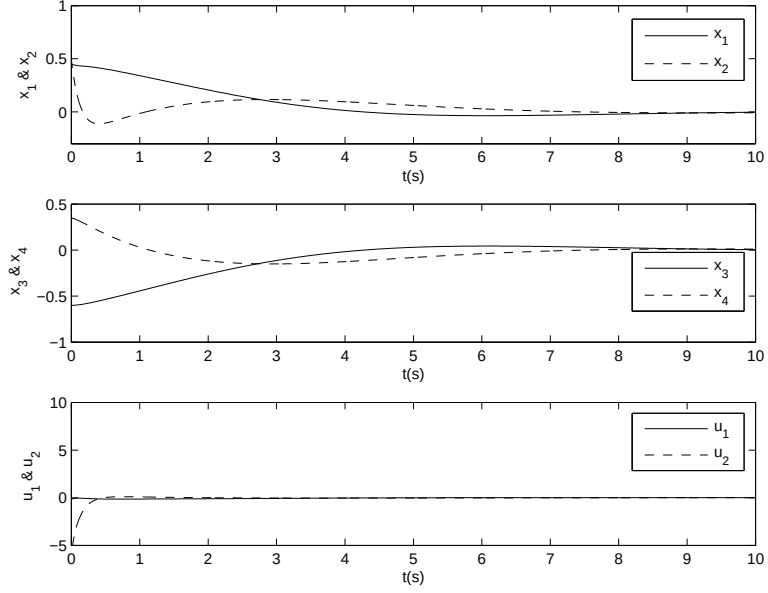


Figure 4.2: Spacecraft regulation using the nonlinear $\mathcal{H}_\infty$ control.

states converges to the origin, which implies the designed nonlinear control law stabilizes the spacecraft to a uniform revolution (circular attractor) rather than an isolated equilibrium.

As a comparison to nonlinear $\mathcal{H}_\infty$ control approach, we will also apply the backstepping control design developed in [78] to the axi-symmetric spacecraft and obtain a robust backstepping controller as

$$\begin{aligned}
 u_1 &= -amx_2 + k \left[-mx_4 - \left(\frac{1 + x_3^2 - x_4^2}{2} \right) x_1 - x_3x_4x_2 \right] - (\phi + C_d)(x_1 + kx_3) \\
 u_2 &= amx_1 + k \left[mx_3 - \left(\frac{1 + x_4^2 - x_3^2}{2} \right) x_2 - x_3x_4x_1 \right] - (\phi + C_d)(x_2 + kx_4).
 \end{aligned}$$

With $k = 20$, $\phi = 0.44$, $C_d = 0.15$, we are able to achieve the same convergence rate as the nonlinear $\mathcal{H}_\infty$ design. C_d is a disturbance bound satisfying

$$\|B_1(x)d\| \leq C_d \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + k \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \right\|. \quad (4.23)$$

The corresponding state trajectories and the control inputs of the backstepping control are shown in Fig. 4.3. It is observed that our proposed state feedback control has less state overshoots and requires smaller control efforts. This demonstrates the advantage of using

proposed $\mathcal{H}_\infty$ control to achieve better spacecraft controlled attitude regulation. The result is not surprising because the nonlinear $\mathcal{H}_\infty$ control is optimized over a local state region; while the backstepping control is only a feasible controller to stabilize the spacecraft.

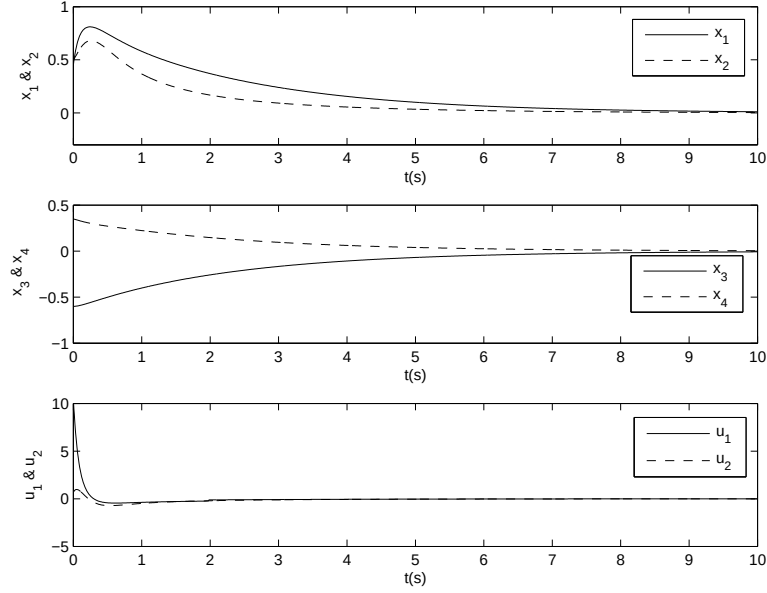


Figure 4.3: Spacecraft regulation using a robust backstepping control.

Moreover, we would like to compare the disturbance rejection property of both controllers. Starting from the same initial condition, we choose a disturbance profile satisfying the constraint (4.23) as

$$\begin{cases} d_1(t) &= 0.9x_1(t) + 4x_3(t) \\ d_2(t) &= 0.5x_2(t) + 4x_4(t) \end{cases}$$

Such a disturbance depends on the variation of the states and converges to zero asymptotically, as required by robust backstepping control design. Shown in Fig.4.4 is the comparison of states and inputs using two types of control designs. The three figures on the left hand side are the results using state feedback nonlinear $\mathcal{H}_\infty$ control. Parallel results from the backstepping control are provided on the right. As expected, backstepping control design guarantees the robust stability of the closed-loop system under external disturbances. The spacecraft eventually settles down to its equilibrium (uniform revolution about its symmetric axis). Using state feedback nonlinear $\mathcal{H}_\infty$ control, the closed-loop system is also stabilized

to the origin with better disturbance rejection performance. Note that the state region constraints are satisfied during the simulation. Moreover, the nonlinear $\mathcal{H}_\infty$ controller is able to accommodate persistent disturbances.

Spacecraft Command Tracking

To achieve accurate command tracking with output feedback, the control performance will be specified differently from the attitude regulation problem. Specifically, under the external disturbances such as modeling uncertainty, actuator and sensor inaccuracies, the spacecraft is desired to follow a command trajectory with tracking error as small as possible. The disturbance vector $\begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 \end{bmatrix}^T$ includes disturbances d_1, d_2 along $\hat{b}_1$ and $\hat{b}_2$ axes, output measurement noises d_3, d_4 , and the desired x_3 trajectory $x_{3d} = d_5$. The nonlinear control design objective is to minimize the effect of these disturbances on the tracking error $(x_3 - x_{3d})$ with reasonable control force.

We will introduce a new state variable $z = \frac{\tau}{s}(x_3 - x_{3d})$ to eliminate the tracking error asymptotically. z reflects the accumulated effect of tracking error over the past time and τ is a pre-specified integration coefficient. By incorporating the dynamics of z into the

original spacecraft model and redefining output variables, the augmented system becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & am & 0 & 0 & 0 \\ -am & 0 & 0 & 0 & 0 \\ \frac{1+x_3^2-x_4^2}{2} & x_3x_4 & 0 & m & 0 \\ x_3x_4 & \frac{1+x_4^2-x_3^2}{2} & -m & 0 & 0 \\ 0 & 0 & \tau & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ z \end{bmatrix} + \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\tau \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (4.24)$$

$$e = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ z \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (4.25)$$

$$y = \begin{bmatrix} 0 & 0 & 325.38 & 0 & 0 \\ 0 & 0 & 0 & 344.38 & 0 \\ 0 & 0 & 0 & 0 & 99.556 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ z \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix}. \quad (4.26)$$

The new state variable is $\bar{x} = [x_1 \ x_2 \ x_3 \ x_4 \ z]$. The output y include measured attitude w_1, w_2 and accumulated tracking error z . Note that the augmented plant fits into the plant structure of (4.1) and satisfies corresponding assumptions.

To solve the command tracking problem in output feedback form, we set $\tau = 0.1$ and choose $U(\bar{x})$ as a quadratic Lyapunov function and $V(\bar{x})$ as a higher order Lyapunov function. Similar to attitude regulation design, we will specify the feasible state region in terms of spacecraft angular velocity and attitude as the intersection of $R_1(x) = x_3^2 + x_4^2 - 0.5 < 0$ and $R_2(x) = x_1^2 + x_2^2 - 0.5 < 0$. The local state feedback synthesis condition in (4.11) will then be modified by adding state constraints. Moreover, the local output

feedback synthesis condition in (4.13) needs to be modified to

$$\begin{aligned}
& \min \hat{\gamma} \\
& \text{s.t. } V(\bar{x}) + \lambda_1(\tilde{x})R_1(x) + \lambda_2(\tilde{x})R_2(x) \in \Phi_{\text{SOS}} \\
& \quad v_1^T \left\{ -\mathcal{S}_{FC} + \begin{bmatrix} [\psi_1(\bar{x}) + \alpha x_1^4 + \beta x_2^4]R_1(x) + \psi_2(\bar{x})R_2(x) & \underline{\mathbf{0}} \\ & \underline{\mathbf{0}} \end{bmatrix} \right\} v_1 \in \Phi_{\text{SOS}} \\
& \quad v_2 \left[-\frac{\partial^2}{\partial x^2}(\mathcal{H}_{FC} - \gamma^{-2}\mathcal{H}_{FI}) \right] v_2 \in \Phi_{\text{SOS}} \\
& \quad V(\bar{x}) - \gamma^{-2}U(\bar{x}) + \lambda_3(\tilde{x})R_1(x) + \lambda_4(\tilde{x})R_2(x) \in \Phi_{\text{SOS}},
\end{aligned}$$

where $\tilde{x} = \begin{bmatrix} \bar{x}^T & x_1^2 & x_2^2 & x_1x_2 \end{bmatrix}^T$, $\lambda_i(\tilde{x}) = \tilde{x}^T \Lambda_i \tilde{x}$, $\psi_j(\bar{x}) = \bar{x}^T \Psi_j \bar{x}$, $\Lambda_i, \Psi_j > 0, \alpha, \beta > 0$.

Applying the control design procedure in previous section with SOS programming, we obtain

$$\gamma = 0.827$$

$$\begin{aligned}
U(\bar{x}) &= 0.277x_1^2 + 0.284x_2^2 + 1.008x_3^2 + 0.886x_4^2 + 31.248z^2 - 0.182x_1x_2 + 0.918x_1x_3 \\
&\quad - 0.582x_1x_4 + 4.593x_1z - 0.589x_2x_3 + 0.869x_2x_4 - 3.895x_2z - 1.481x_3x_4 \\
&\quad + 10.469x_3z - 8.950x_4z \\
&= \frac{1}{2}M^{[1]}(\bar{x})^T \begin{bmatrix} 0.554 & -0.182 & 0.918 & -0.582 & 4.593 \\ -0.182 & 0.568 & -0.589 & 0.869 & -3.895 \\ 0.918 & -0.589 & 2.016 & -1.481 & 10.469 \\ -0.582 & 0.869 & -1.481 & 1.772 & -8.95 \\ 4.593 & -3.895 & 10.469 & -8.95 & 62.496 \end{bmatrix} M^{[1]}(\bar{x})
\end{aligned}$$

listed as following:

$$\begin{aligned}
F_0 &= \begin{bmatrix} -2.7063x_1 + 0.8752x_2 - 4.4728x_3 + 2.8236x_4 - 22.3924e_1 \\ 0.8752x_1 - 2.7809x_2 + 2.8515x_3 - 4.2379x_4 + 18.9445e_1 \end{bmatrix} \\
F_1 &= \begin{bmatrix} 0.1982x_1 - 0.0641x_2 + 0.3275x_3 - 0.2068x_4 + 1.6397e_1 \\ -0.1282x_1 + 0.4073x_2 - 0.4176x_3 + 0.6206x_4 - 2.7744e_1 \\ 0 \\ 0 \\ -0.3279x_1 + 0.2774x_2 - 0.7479x_3 + 0.6388x_4 - 4.4820e_1 \end{bmatrix} \\
L_0 &= \frac{1}{\Delta} \times \begin{bmatrix} 8.1086 + 1.0280x_1x_2 + 22.2138x_2^2 & -2.4262 - 32.1055x_1x_2 - 0.4395x_2^2 \\ -3.7133 - 1.6557x_1^2 - 44.7680x_1x_2 & 17.3007 + 51.7120x_1^2 + 0.8856x_1x_2 \\ \left\{ \begin{array}{l} 0.6300 + 1.7353x_2^2 + 0.4273x_1^2 \\ -0.2952x_1^2x_2^2 - 0.0833x_1x_2 \end{array} \right\} & \left\{ \begin{array}{l} -0.1692 - 0.0219x_2^2 - 0.0667x_1^2 \\ -0.0036x_1^2x_2^2 - 1.9457x_1x_2 \end{array} \right\} \\ \left\{ \begin{array}{l} -0.1598 - 0.0206x_2^2 - 0.0630x_1^2 \\ -0.0034x_1^2x_2^2 - 1.8384x_1x_2 \end{array} \right\} & \left\{ \begin{array}{l} -0.1565 - 0.0126x_2^2 - 0.4010x_1^2 \\ +0.0064x_1^2x_2^2 - 0.3006x_1x_2 \end{array} \right\} \\ \left\{ \begin{array}{l} 0.1060 + 0.2137x_2^2 + 0.0204x_1^2 \\ -0.0054x_1^2x_2^2 + 0.3449x_1x_2 \end{array} \right\} & \left\{ \begin{array}{l} 0.8985 + 0.5446x_2^2 + 2.7100x_1^2 \\ -0.4101x_1^2x_2^2 - 0.2159x_1x_2 \end{array} \right\} \\ 6.1919 + 12.9614x_2^2 + 21.2844x_1x_2 & \\ -13.2727 - 26.1215x_1x_2 - 34.2826x_1^2 & \\ \left\{ \begin{array}{l} 0.4024 + 1.3092x_1x_2 + 0.8114x_2^2 \\ +0.0773x_1^2 - 0.0206x_1^2x_2^2 \end{array} \right\} & \\ \left\{ \begin{array}{l} 5.9156 - 8.5016x_1x_2 + 16.9043x_2^2 \\ +18.4738x_1^2 - 12.6486x_1^2x_2^2 \end{array} \right\} & \\ \left\{ \begin{array}{l} -0.5615 - 1.0784x_1x_2 - 0.0453x_2^2 \\ -1.4384x_1^2 + 0.0231x_1^2x_2^2 \end{array} \right\} & \end{bmatrix} \\
\Delta &= -5.9484 - 17.3403x_2^2 - 18.8181x_1^2 + 13.0681x_1^2x_2^2 + 9.1923x_1x_2
\end{aligned}$$

Note that the synthesized controller does not rely on the tracking trajectory explicitly, therefore it is capable to track difference command signals.

To examine tracking performance of the controlled spacecraft, we firstly set d_1 to d_4 to be zero and x_{3d} as a trapezoidal profile. Starting from zero initial condition $\bar{x} = 0$, we obtain the simulation results as shown in Fig. 4.5. It is observed that good tracking of

the given trajectory has been achieved. On the other hand, it is also possible to design a linear output feedback controller based on the linearized model of (4.24)-(4.26) so that the closed loop system has optimal $\mathcal{H}_\infty$ performance. However, the linear controller would not work well on the original nonlinear spacecraft for the given trajectory.

Using inverse map of stereographic projection, Fig. 4.6 shows the orientation of $\hat{n}_3$ axis in terms of cosines of three direction angles. It is observed that $\hat{n}_3$ axis initially coincides with the direction of $\hat{b}_3$ and is perpendicular to $\hat{b}_1 - \hat{b}_2$ plane. As w_1 tracking a command trajectory, $\hat{n}_3$ rotates in three directions. Each direction cosine follows a similar trapezoidal profile and finally goes back to its original position.

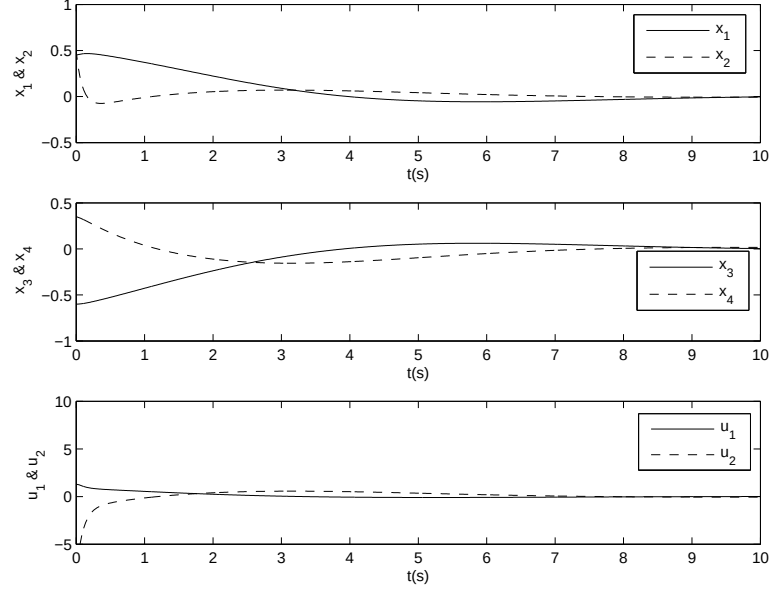
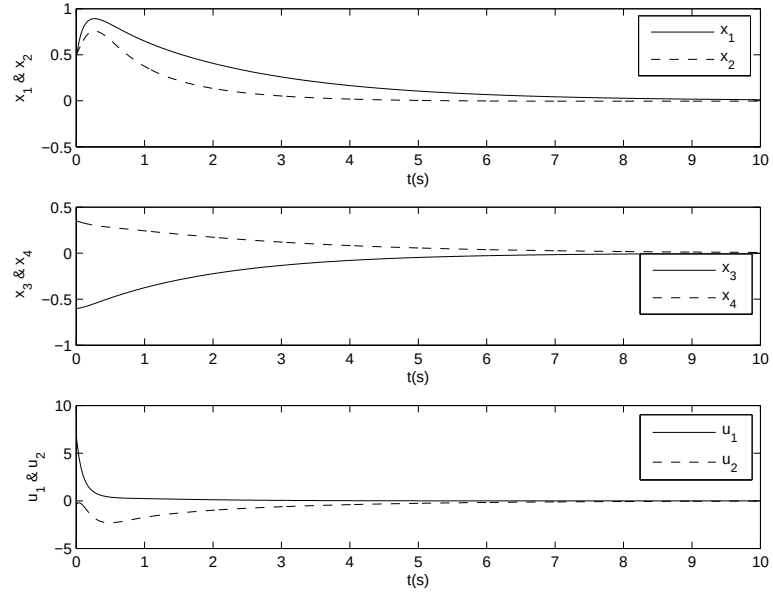
Secondly, we keep d_5 unchanged and set external disturbances $d_1 \sim d_4$ as random noises. Starting from the same initial point, we obtain the simulation results as shown in Fig. 4.7. It turns out that with small random noises, the trajectory of x_3 is still capable of tracking x_{3d} and the system behavior is similar to that of no noise situation.

The feasible region for the constructed nonlinear control law is generally a complicated state space domain in the four-dimensional space. To visualize it, we project the feasible region onto $x_1 - x_2$ and $x_3 - x_4$ planes separately. Shown in Fig. 4.8(a) is the cross-section of the resulting Lyapunov level set with $x_1 - x_2$ plane and the state region constraint on x_1, x_2 variables; while Fig. 4.8(b) describes the cross region where the nonlinear $\mathcal{H}_\infty$ controller is feasible in $x_3 - x_4$ plane. As can be seen, all specified constraints $R_1(x) < 0$ and $R_2(x) < 0$ are satisfied.

4.6 Summary

In this chapter, we have proposed a computational $\mathcal{H}_\infty$ control design procedure for a generalized nonlinear output feedback $\mathcal{H}_\infty$ condition and developed nonlinear $\mathcal{H}_\infty$ control laws for axi-symmetric spacecraft attitude regulation and trajectory tracking problems. Since the spacecraft attitude dynamics is in the form of polynomial nonlinear systems, we are able to use SOS-based solution approach to solve its nonlinear control design problems. Specifically, the difficult HJI inequalities associated with nonlinear $\mathcal{H}_\infty$ control synthesis are converted to polynomial matrix inequalities, and solved using SOS programming techniques in polynomial time. The proposed design approach covers a wider class of systems by relaxing the plant structure from standard form to a more generalized form, and extends

the Lyapunov function from trivial quadratic case to higher-order case, which were shown effective in achieving better controlled performance for axi-symmetric spacecraft with two control torques.

(a) nonlinear $\mathcal{H}_\infty$ control

(b) backstepping control

Figure 4.4: Comparison of two control designs for spacecraft disturbance rejection.

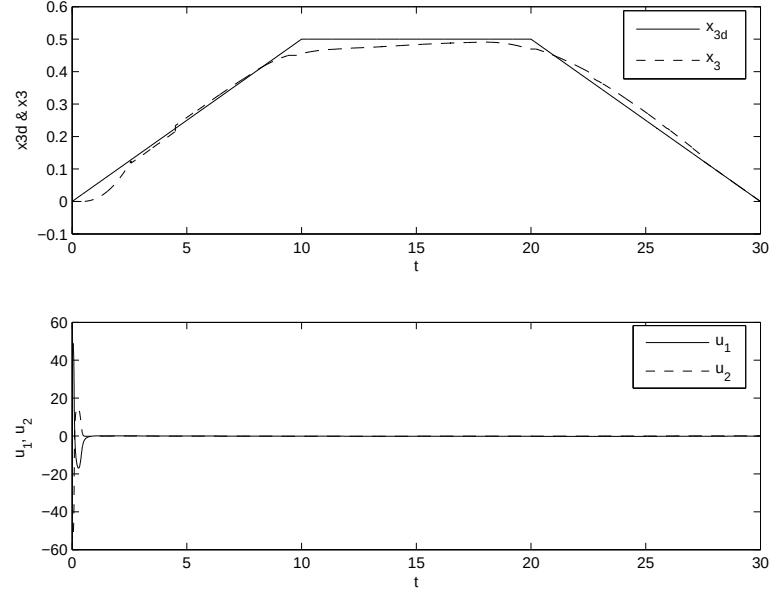


Figure 4.5: Spacecraft tracking performance using the nonlinear $\mathcal{H}_\infty$ control.

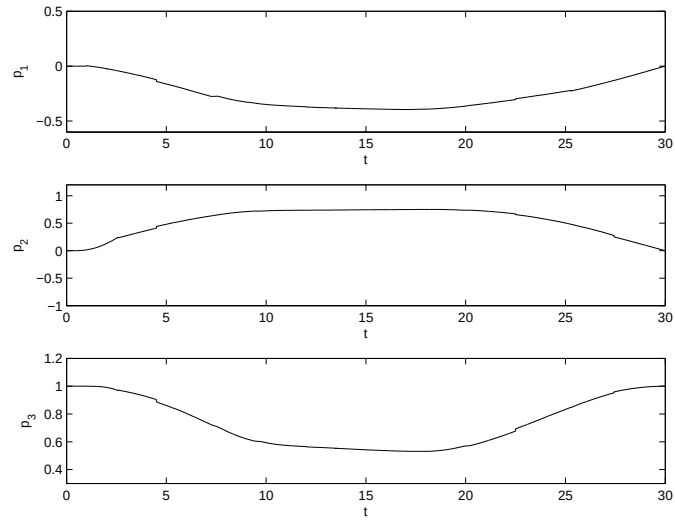


Figure 4.6: Orientation of $\hat{n}_3$ axis in terms of direction cosines.

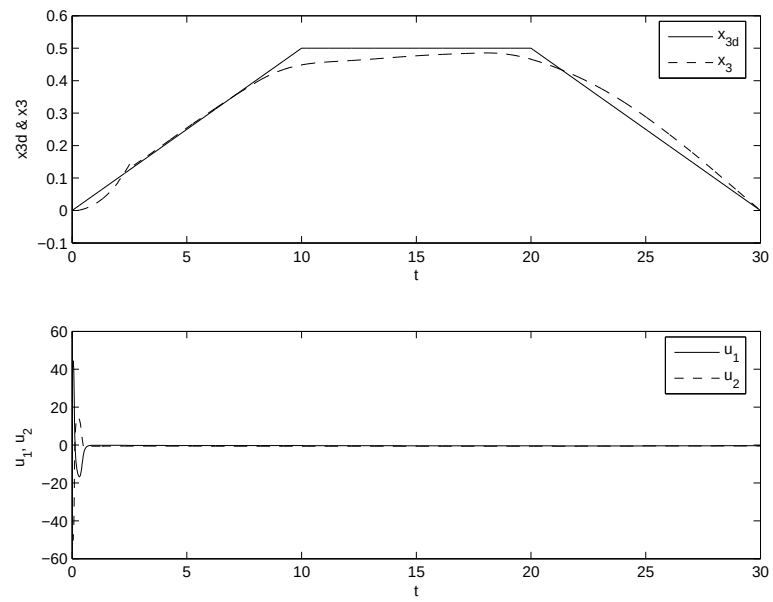
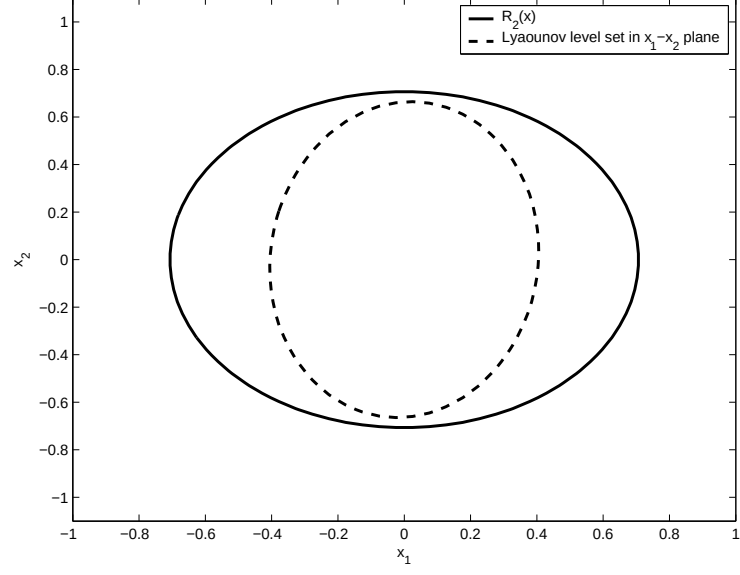
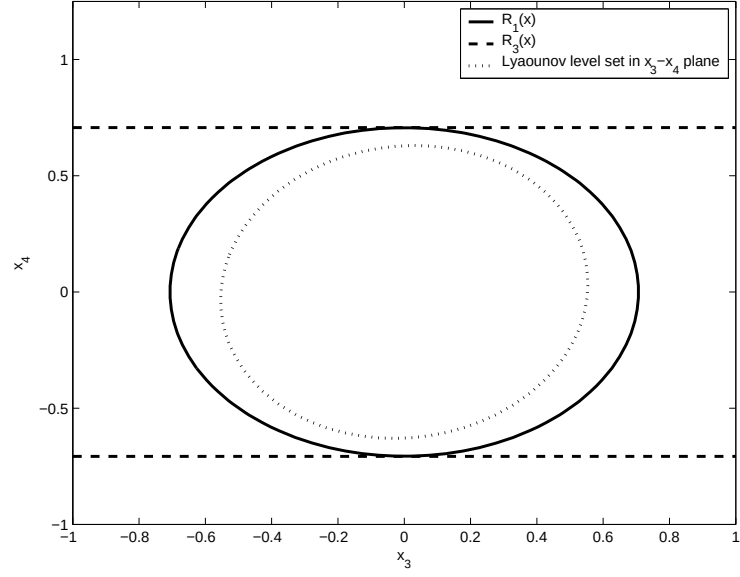


Figure 4.7: Spacecraft tracking performance using the nonlinear $\mathcal{H}_\infty$ control.

(a) in $x_1 - x_2$ plane(b) in $x_3 - x_4$ planeFigure 4.8: Feasible region for nonlinear output feedback $\mathcal{H}_\infty$ control.

Chapter 5

Lyapunov Redesign of Adaptive Controllers for Polynomial Nonlinear Systems

Interest in adaptive control of nonlinear systems was stimulated by the inability of nonlinear feedback control to handle the presence of unknown parameters. The first series of adaptive nonlinear control schemes were restricted to systems satisfying the matching condition [12], and later on relaxed to the extended matching condition [53]. For a period of time, the extended matching condition could not be removed by Lyapunov-based designs, which redirected researchers to estimation-based design. Nam and Arapostathis [52], Sastry and Isidori [68] combined feedback linearization with adaptation techniques from adaptive linear control. However, these schemes required the nonlinearities be restricted by linear growth conditions in order to achieve global stability. The only nonlinear estimation-based results which went beyond the linear growth constraints were obtained by using Lyapunov functions to characterize relationships between nonlinear growth constraints and controller stabilizing properties [57, 58], but still somehow restricted by matching conditions. The result presented in [31] finally broke the extended matching barrier, which was achieved with a new recursive design procedure called adaptive backstepping. Adaptive backstepping developed by [30] confluenced the adaptive estimation idea. On the other hand, nonlinear control ideas were expressed in works [75]. The properties of an early adaptive scheme [18]

also strongly influenced the adaptive backstepping, which preserved global stability under output feedback for a class of output nonlinearities [32].

The study of adaptive $\mathcal{H}_\infty$ control were conducted to design feedback controllers so that the external disturbances have the smallest impact on the controlled systems. Existing achievements include parameter-estimation based approaches by Krause et al. [36] and Basar et al. [4], a game-theoretical approach by Didinsky and Basar [14], a Lyapunov design approach by Yang et al. [89], a back-stepping approach by Pan and Basar [54], and a problem with finite parameter set by Rangan and Poolla [64].

In this chapter, we will consider a special class of disturbances - disturbances internally produced by the parameter estimation error. With the objectives of identifying the true parameter asymptotically and attenuating the impact of the parameter estimation error on the controlled output to a minimal level, it will be shown that the disturbances would finally converge to zero. In Section 5.1, the parametric uncertainties satisfying the matching condition will be considered. The construction of the adaptive control law and the parameter adaptation law are straightforward and determined simultaneously through the iterative synthesis conditions. However, for un-matched parametric uncertainty case, it becomes more complicated. Motivated by the results from [48], which presents the dissipation theory and vector projection methods for parameter-dependent nonlinear systems, we will develop parallel results for the polynomial nonlinear systems in Section 5.2.

5.1 Polynomial Nonlinear Systems with Matching Parametric Uncertainties

By transforming the system into its corresponding error dynamics, we will develop an adaptive control scheme in attenuating the effect of the unknown parameters on the controlled output, which is composed of tracking errors and control efforts. To achieve better controlled performance, the Lyapunov functions will be relaxed from quadratic to higher order and the resulting control gain is generalized from constant to parameter dependent. The synthesis conditions of adaptive control could be formulated as polynomial matrix inequalities and are thus solvable by recast the resulting conditions into a Sum of Squares (SOS) optimization problem, from which the adaptive control law as well as the parameter adaptation law are derived with zero tracking and parameter estimation errors. An example

will be provided to demonstrate effectiveness of the proposed adaptive control redesign approach.

5.1.1 Standard Adaptive Control Scheme for Polynomial Nonlinear Systems

Consider an uncertain polynomial nonlinear system with matching uncertainties as

$$\dot{x} = A(x)x + \phi^T(x)\theta + u \quad (5.1)$$

where the system state and control input $x, u \in \mathbf{R}^{n_x}$, $\phi^T(x)\theta$ describes the parametric uncertainties, which is composed of given basis functions $\phi^T(x) \in \mathbf{R}^\phi$ and unknown weights $\theta \in \mathbf{R}^\theta$. In addition, $A(x)$ and $\phi^T(x)$ are restricted as polynomial functions of state x . Let $x_d(t)$ be the command output trajectory and $z = x - x_d(t)$ be the tracking error. The desired trajectory $x_d(t)$ is assumed as continuous function of time t . The actual value of the unknown parameter θ is a constant. For standard adaptive control design, the goal is to make z as small as possible. Therefore, it is important to look at how the control input influences z :

$$\dot{z} = \dot{x} - \dot{x}_d(t) = A(x)x + \phi^T(x)\theta + u - \dot{x}_d(t)$$

To keep the transformed system in a clean form, we separate the polynomial and non-polynomial terms in the state equation. As a result, the transformed system is described by

$$\dot{z} = \tilde{A}(z, x_d)z + \tilde{\phi}^T(z, x_d)\theta + u + f(z, x_d, \dot{x}_d) \quad (5.2)$$

where $\tilde{A}(z, x_d)$ and $\tilde{\phi}^T(z, x_d)$ are polynomial functions of z and x_d . $f(z, x_d, \dot{x}_d)$ represents the nonlinear terms produced by the introduction of x_d through the transformation $x = z + x_d$.

With the design philosophy that assume the control law can be synthesized as if the system does not have any parameter uncertainties when the online parameter estimate $\hat{\theta}$ is used. Motivated by model reference adaptive control (MRAC) design, we propose the

adaptive control scheme as following

$$u = u_m + u_s \quad (5.3)$$

$$u_m = -(\tilde{A}(z, x_d)z + \tilde{\phi}^T(z, x_d)\hat{\theta} + f(z, x_d, \dot{x}_d))$$

$$u_s = -Kz, \quad K > 0$$

$$\dot{\hat{\theta}} = \Gamma \tilde{\phi}(z, x_d)z \quad (5.4)$$

with Lyapunov function

$$V(z, \tilde{\theta}) = z^T z + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}, \quad (5.5)$$

where $\tilde{\theta} = \theta - \hat{\theta}$ denotes the estimation error between $\hat{\theta}$ and its true value. Note that the control law is composed of two parts, the first part u_m is used to cancel out redundant nonlinearities and implement the unknown parameter θ by its estimation $\hat{\theta}$. $\hat{\theta}$ has its own dynamics, which is governed by the adaptation law. When $\hat{\theta}$ perfectly estimates θ , $\dot{z}$ is driven to zero with the control force u_m . To address non-zero initial tracking error ($z = 0$), we need to include another part of control law u_s .

It has been proven that with the adaptive control law (5.3) and the parameter adaptation law (5.4), all signals in the system are bounded and the tracking error asymptotically converges to zero [87]. However, the parameter estimation error can only be guaranteed as bounded unless the persistent exciting (PE) condition is satisfied. When the system is very sensitive to the variation of the parameter, or in the situation that the parameter is poorly estimated and performs bad so that it has very large discrepancy with its true value, the operational consistency of the whole system may be destroyed.

5.1.2 Adaptive Control Redesign for Performance Improvement

In this section, we will address the Lyapunov redesign of adaptive control problem for the purpose of improving system performance with respect to some optimality criteria. We will focus on how to minimize the effect of imperfect parameter estimation on the controlled output. To this end, the Lyapunov function will be not restricted to the simple quadratic form as in (5.5), but relaxed to higher order form, which helps provide more freedom to synthesize a better Lyapunov function. As can be seen later, the direct consequences of this newly developed Lyapunov redesign strategy is that the optimization

of the performance level, the synthesis of the controller gain, the selection of the adaptation law, the determination of the Lyapunov coefficients as well as the construction of the storage function are all integrated into a simple but systematic design procedure. We will firstly formulate the optimization problem, then an iterative algorithm will be proposed to recursively improve the performance level and extend the Lyapunov function.

Resort the system dynamics and specify the performance level as

$$\dot{z} = \tilde{\phi}^T(z, x_d)\tilde{\theta} + u_s \quad (5.6)$$

$$e = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix} u_s \quad (5.7)$$

where $Q \geq 0, R > 0$ are given weight penalties on z and u_s . As can be seen, $\tilde{\theta}$ is treated as a system disturbance. This disturbance is special since it implicitly depends on state z through its own dynamics. Our design objective is to synthesize a control law (5.3) together with an online parameter adaptation mechanism (5.4) so that the effect of $\tilde{\theta}$ is as small as possible on the controlled output e under a Lyapunov function $V(z, \tilde{\theta})$, i.e.

$$\begin{aligned} \min \quad & \gamma \\ \text{s.t} \quad & \dot{V}(z, \tilde{\theta}) + (z^T Q z + u_s^T R u_s) - \gamma^2 \tilde{\theta}^T \tilde{\theta} < 0 \end{aligned} \quad (5.8)$$

In the following, we will provide an adaptive control synthesis condition for the disturbance attenuation problem mentioned above.

Theorem 4 *For the system (5.6)-(5.7), if there exist a polynomial matrix function $V(z, \tilde{\theta}) = \mathcal{M}^T(z)P_1\mathcal{M}(z) + \tilde{\theta}^T P_2 \tilde{\theta}$, $P_1 > 0, P_2 > 0$, a matrix function $K(z)$ and a scalar $\gamma > 0$ so that following condition is satisfied*

$$\left[\begin{array}{c} \left\{ \begin{array}{l} -K^T(z)\frac{\partial \mathcal{M}(z)^T}{\partial z}P_1N(z) - N^T(z)P_1\frac{\partial \mathcal{M}(z)}{\partial z}K(z) \\ +Q + K^T(z)RK(z) \end{array} \right\} \quad \star \\ \tilde{\phi}(z, x_d)\frac{\partial \mathcal{M}(z)^T}{\partial z}P_1N(z) - P_2\Gamma\tilde{\phi}(z, x_d) \quad -\gamma^2 I \end{array} \right] < 0 \quad (5.9)$$

where $\mathcal{M}(z)$ is a pre-specified monomial vector of z and $N(z)$ is a matrix function of z so that $\mathcal{M}(z) = N(z)z$, then problem (5.8) is solvable by an adaptive control law in the form of (5.3) with controller gain $K(z)$ and an adaptation law in the form of

$$\dot{\tilde{\theta}} = \Gamma\tilde{\phi}(z, x_d)z.$$

Proof: For system (5.6)-(5.7) with Lyapunov function and adaptation law given in the theorem, calculating the analysis condition of problem (5.8) directly leads to

$$\begin{aligned}
& \dot{V}(z, \tilde{\theta}) + (z^T Q z + u_s^T R u_s) - \gamma^2 \tilde{\theta}^T \tilde{\theta} \\
&= -z^T \left[K^T \frac{\partial M(z)^T}{\partial z} P_1 N(z) + N^T(z) P_1 \frac{\partial M(z)}{\partial z} K(z) - Q - K^T(z) R K(z) \right] z \\
&+ \left[\tilde{\theta}^T \tilde{\phi}(z, x_d) \frac{\partial M(z)^T}{\partial z} P_1 N(z) z + z^T N^T(z) P_1 \frac{\partial M(z)}{\partial z} \tilde{\phi}^T(z, x_d) \tilde{\theta} \right] + \dot{\tilde{\theta}}^T P_2 \tilde{\theta} + \tilde{\theta}^T P_2 \dot{\tilde{\theta}} - \gamma^2 \tilde{\theta}^T \tilde{\theta} \\
&= -z^T \left[K^T \frac{\partial M(z)^T}{\partial z} P_1 N(z) + N^T(z) P_1 \frac{\partial M(z)}{\partial z} K(z) - Q - K^T(z) R K(z) \right] z \\
&+ \left[\tilde{\theta}^T \tilde{\phi}(z, x_d) \frac{\partial M(z)^T}{\partial z} P_1 N(z) z + z^T N^T(z) P_1 \frac{\partial M(z)}{\partial z} \tilde{\phi}^T(z, x_d) \tilde{\theta} \right] - z^T \tilde{\phi}^T(z, x_d) \Gamma^T P_2 \tilde{\theta} \\
&- \tilde{\theta}^T P_2 \Gamma \tilde{\phi}(z, x_d) z - \gamma^2 \tilde{\theta}^T \tilde{\theta} \\
&= \begin{bmatrix} z^T & \tilde{\theta}^T \end{bmatrix} \begin{bmatrix} \left\{ \begin{array}{c} -K^T(z) \frac{\partial M(z)^T}{\partial z} P_1 N(z) - N^T(z) P_1 \frac{\partial M(z)}{\partial z} K(z) \\ + Q + K^T(z) R K(z) \end{array} \right\} & \star \\ \tilde{\phi}(z, x_d) \frac{\partial M(z)^T}{\partial z} P_1 N(z) - P_2 \Gamma \tilde{\phi}(z, x_d) & -\gamma^2 I \end{bmatrix} \begin{bmatrix} z \\ \tilde{\theta} \end{bmatrix} \\
&< 0.
\end{aligned}$$

It is clear that its sufficient condition is (5.9).

Q.E.D.

Note that the solvability condition is bilinear about matrix functions $V(z, \tilde{\theta})$ and $K(z)$. In general, there does not exist a computationally efficient algorithm to solve the condition (5.9). For this reason, we will resort to an iterative computational scheme for its effective solution.

The iterative algorithm will start from the simplest case. Choosing Lyapunov function, control and adaptation laws as following

$$V(z, \tilde{\theta}) = z^T P_1 z + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}, \quad P_1 > 0, \Gamma^{-1} > 0 \quad (5.10)$$

$$u_s = -Kz, \quad K > 0 \quad (5.11)$$

$$\dot{\tilde{\theta}} = \Gamma \tilde{\phi}(z, x_d) z \quad (5.12)$$

Then the condition in problem (5.8) becomes

$$\begin{aligned}
& \dot{V}(z, \tilde{\theta}) + (z^T Q z + u_s^T R u_s) - \gamma^2 \tilde{\theta}^T \tilde{\theta} \\
&= \dot{z}^T P_1 z + z^T P_1 \dot{z} + \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} + \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} + (z^T Q z + u_s^T R u_s) - \gamma^2 \tilde{\theta}^T \tilde{\theta} \\
&= [\tilde{\phi}^T(z, x_d) \tilde{\theta} - K z]^T P_1 z + z^T P_1 [\tilde{\phi}^T(z, x_d) \tilde{\theta} - K z] - z^T \tilde{\phi}^T(z, x_d) \tilde{\theta} - \tilde{\theta}^T \tilde{\phi}(z, x_d) z \\
&\quad + (z^T Q z + u_s^T R u_s) - \gamma^2 \tilde{\theta}^T \tilde{\theta} \\
&= z^T (Q + K^T R K - K^T P_1 - P_1 K) z + z^T [P_1 \tilde{\phi}^T(z, x_d) - \tilde{\phi}^T(z, x_d)] \tilde{\theta} \\
&\quad + \tilde{\theta}^T [\tilde{\phi}(z, x_d) P_1 - \tilde{\phi}(z, x_d)] z - \gamma^2 \tilde{\theta}^T \tilde{\theta} \\
&< 0.
\end{aligned}$$

Using Schur complement, it is sufficient to solve the following condition

$$\mathbf{S}_{\text{itro}} := \begin{bmatrix} -M - M^T & \star & \star & \star \\ \tilde{\phi}(z, x_d) - \tilde{\phi}^T(z, x_d) P_1^{-1} & -\gamma_i^2 & \star & \star \\ Q^{\frac{1}{2}} P_1^{-1} & 0 & -I & \star \\ R^{\frac{1}{2}} M & 0 & 0 & -I \end{bmatrix} < 0 \quad (5.13)$$

where $M = K P_1^{-1} > 0$. This is a polynomial matrix inequality about variables M, P_1^{-1}, γ_i^2 and can be solved efficiently by SOS programming. Using SOSTOOL we will get K , which is the control gain of u_s . Denote this K as K_0 and it will be used in the following step.

Now for fixed K_0 , we generalize the Lyapunov function to the higher order form

$$V(z, \tilde{\theta}) = \mathcal{M}^T(z) P_1 \mathcal{M}(z) + \tilde{\theta}^T P_2 \tilde{\theta}, \quad (5.14)$$

where $P_1 > 0, P_2 > 0, \mathcal{M}(z)$ is a pre-specified monomial vector of z . Because the Lyapunov function is at least of 2nd order, any $\mathcal{M}(z)$ could be represented as $\mathcal{M}(z) = N(z)z$, where $N(z)$ is a matrix function of z . Clearly, when $N(z) = I$, $\mathcal{M}(z)$ is the monomial vector of a

quadratic Lyapunov function. Recalculating the condition in problem (5.8), we have

$$\begin{aligned}
& \dot{V}(z, \tilde{\theta}) + (z^T Q z + u_s^T R u_s) - \gamma^2 \tilde{\theta}^T \tilde{\theta} \\
&= -z^T [K_0^T \frac{\partial \mathcal{M}(z)}{\partial z} P_1 N(z) + N^T(z) P_1 \frac{\partial \mathcal{M}(z)}{\partial z} K_0 - Q - K_0^T R K_0] z \\
&\quad + z^T [N^T(z) P_1 \frac{\partial \mathcal{M}(z)}{\partial z} \tilde{\phi}^T(z, x_d) - \tilde{\phi}^T(z, x_d) \Gamma^T P_2] \tilde{\theta} \\
&\quad + \tilde{\theta}^T [\tilde{\phi}(z, x_d) \frac{\partial \mathcal{M}(z)}{\partial z} P_1 N(z) - P_2 \Gamma \tilde{\phi}(z, x_d)] z - \gamma^2 \tilde{\theta}^T \tilde{\theta} \\
&= \begin{bmatrix} z^T & \tilde{\theta}^T \end{bmatrix} \begin{bmatrix} \begin{Bmatrix} -K_0^T \frac{\partial \mathcal{M}(z)}{\partial z} P_1 N(z) \\ -N^T(z) P_1 \frac{\partial \mathcal{M}(z)}{\partial z} K_0 \\ +Q + K_0^T R K_0 \end{Bmatrix} & \star \\ \tilde{\phi}(z, x_d) \frac{\partial \mathcal{M}(z)}{\partial z} P_1 N(z) - P_2 \Gamma \tilde{\phi}(z, x_d) & -\gamma^2 I \end{bmatrix} \begin{bmatrix} z \\ \tilde{\theta} \end{bmatrix} < 0 \quad (5.15)
\end{aligned}$$

It is straightforward to solve this condition by transforming to its sufficient condition through Schur complement

$$\mathbf{S}_{\text{itr1}} := \begin{bmatrix} \begin{Bmatrix} -K_0^T \frac{\partial \mathcal{M}(z)}{\partial z} P_1 N(z) \\ -N^T(z) P_1 \frac{\partial \mathcal{M}(z)}{\partial z} K_0 \\ +Q + K_0^T R K_0 \end{Bmatrix} & \star \\ \tilde{\phi}(z, x_d) \frac{\partial \mathcal{M}(z)}{\partial z} P_1 N(z) - Y \tilde{\phi}(z, x_d) & -\gamma^2 I \end{bmatrix} < 0 \quad (5.16)$$

where $Y = P_2 \Gamma$. By selecting an adaptation gain Γ , one can get the Lyapunov matrix $P_2 = Y \Gamma^{-1}$. Actually, P_2 and Γ always show up as the multiplication form during the derivation, hence it is possible to simplify the synthesis of Y as the synthesis of Γ by absorbing P_2 into Γ so that the second part of the Lyapunov function keeps a simple form of $\tilde{\theta}^T \tilde{\theta}$.

Note that (5.16) is a z, x_d dependent polynomial matrix inequality with decision variables P_1, Y, γ^2 involved in. Sum of Squares (SOS) technique provides an efficient way to solve the polynomial matrix inequalities by recasting them as SOS optimization problems. Specifically, condition (5.16) can be reformulated as

$$\begin{aligned}
& \min \gamma \\
& \text{s.t. } \mathbf{S} := -v^T \mathbf{S}_{\text{itr1}} v \in \Phi_{\text{SOS}}
\end{aligned} \quad (5.17)$$

where v is a free vector of compatible dimension $\mathbf{R}^{n_v}$ and $\mathbf{S}$ is a polynomial.

Generally speaking, a global nonlinear controller will be obtained. Nevertheless, it is often too restrictive to synthesize a global stabilizing controller. Moreover, in a restricted region, local controllers will often perform better than global controllers. In such situations, it is suggested to add state region constraints into the original condition and search for a local solution.

Another interesting observation coming from the fundamental mechanism of the adaptive control is the form of Lyapunov function $V(z, \tilde{\theta}) = \mathcal{M}^T(z)P_1\mathcal{M}(z) + \tilde{\theta}^T P_2 \tilde{\theta}$, $P_1 > 0, P_2 > 0$. The Lyapunov function involves not only the state z , but also the disturbance $\tilde{\theta}$. This is motivated by the fact that the dynamics of $\tilde{\theta}$ is available and depends on z as well. Therefore, augmenting the Lyapunov function to include complete information of the system dynamics does not obstruct the identification of the derivative of the Lyapunov function.

Keep in mind that whenever we solve the synthesis condition, we are desired to minimize the γ value simultaneously. From this point of view, it is promising to obtain better γ value through the iteration because much more freedom is induced due to the extension on both of the Lyapunov function and the parameter adaptation.

Now we will state the iterative adaptive control algorithm for performance improvement using higher order Lyapunov function.

1. Let $i = 0$. Starting from a quadratic Lyapunov function (5.10) and adaptation law (5.12), solve the following condition to get the initial controller gain K_0 and performance level γ_0 :

$$\begin{aligned} \min \gamma_i & \\ \text{s.t.} \quad & -v^T \mathbf{S}_{\text{itr}0} v \in \Phi_{\text{SOS}} \end{aligned} \tag{5.18}$$

where $\mathbf{S}_{\text{itr}0}$ is defined in eqn. (5.13). Solve $M > 0, P_1 > 0, \gamma_i$ using SOS programming and the initial controller gain will be $K_i = MP_1$.

2. If problem (5.18) has a feasible solution, let $i = i + 1$. For given $\mathcal{M}(z)$ and fixed K_{i-1} , solve the following condition to get $Y_i, P_{1i} > 0$ and γ_i .

$$\begin{aligned} \min \gamma_i & \\ \text{s.t.} \quad & -v^T \mathbf{S}_{\text{itr}i} v \in \Phi_{\text{SOS}} \end{aligned} \tag{5.19}$$

where

$$\mathbf{S}_{\text{itri}} = \begin{bmatrix} \begin{Bmatrix} -K_{i-1}^T(z) \frac{\partial \mathcal{M}(z)^T}{\partial z} P_{1i} N(z) \\ -N^T(z) P_{1i} \frac{\partial \mathcal{M}(z)}{\partial z} K_{i-1}(z) \\ +Q + K_{i-1}^T(z) R K_{i-1}(z) \end{Bmatrix} & \star \\ \tilde{\phi}(z, x_d) \frac{\partial \mathcal{M}(z)^T}{\partial z} P_{1i} N(z) - Y_i \tilde{\phi}(z, x_d) & -\gamma_i^2 I \end{bmatrix}$$

If $|\gamma_i - \gamma_{i-1}| < \epsilon$ or the iteration number i is sufficiently large, let $\gamma = \gamma_i$ and STOP. Otherwise, go to step 4.

3. Let $i = i + 1$. For given $\mathcal{M}(z)$ and fixed $P_{1(i-1)}$, solve the following condition to get K_i, Y_i and γ_i .

$$\begin{aligned} \min \quad & \gamma_i \\ \text{s.t.} \quad & -v^T \tilde{\mathbf{S}}_{\text{itri}} v \in \Phi_{\text{SOS}} \end{aligned} \tag{5.20}$$

where

$$\tilde{\mathbf{S}}_{\text{itri}} = \begin{bmatrix} \begin{Bmatrix} -K_i^T(z) \frac{\partial \mathcal{M}(z)^T}{\partial z} P_{1(i-1)} N(z) \\ -N^T(z) P_{1(i-1)} \frac{\partial \mathcal{M}(z)}{\partial z} K_i(z) + Q \end{Bmatrix} & \star & \star \\ \tilde{\phi}(z, x_d) \frac{\partial \mathcal{M}(z)^T}{\partial z} P_{1(i-1)} N(z) - Y_i \tilde{\phi}(z, x_d) & -\gamma_i^2 I & \star \\ R^{\frac{1}{2}} K_i(z) & 0 & -I \end{bmatrix}$$

Note that in this step, K_i is not necessarily be a positive scalar, it could be a z dependent polynomial function with pre-specified polynomial form. If $|\gamma_i - \gamma_{i-1}| < \epsilon$ or the iteration number i is sufficiently large, let $\gamma = \gamma_i$ and STOP. Otherwise, go to step 3.

Although we start from a quadratic Lyapunov function, the final Lyapunov function will be a complicated polynomial form. Using a polynomial representation of Lyapunov function, it is possible to achieve a better closed-loop performance $\gamma_i < \gamma_{i-1}$. From step 2 to step 3, it is clear that (5.19) is always feasible (at least $\mathcal{M}(z) = z, \frac{\partial \mathcal{M}(z)}{\partial z} = I, N(z) = I, Y_i = I$ should work). When iterating between step 3 and step 4, the condition to be solved is always feasible (at least keep $K_{i-1} = K_i, Y_{i-1} = Y_i, P_{1(i-1)} = P_{1i}$ should work).

Finally, from derivation (5.15), we could claim that in each step, following condition is guaranteed

$$\dot{V}(z, \tilde{\theta}) + (z^T Q z + u_s^T R u_s) - \gamma^2 \tilde{\theta}^T \tilde{\theta} = \begin{bmatrix} z & \tilde{\theta} \end{bmatrix} \mathbf{S}_{\text{itri}} \begin{bmatrix} z \\ \tilde{\theta} \end{bmatrix} < 0$$

The feasibility of this condition indicates that the function of $z, \tilde{\theta}$ is nonnegative and the squares of $z(t), \tilde{\theta}(t)$ are integrable with respect to time, i.e. $z(t), \tilde{\theta}(t) \in \mathcal{L}_2$. Next by Barbalat's lemma, for any initial condition, equation (5.6) and (5.24) imply that $\dot{z}(t), \dot{\tilde{\theta}}(t) \in \mathcal{L}_\infty$, which in turn implies that $z(t), \tilde{\theta}(t) \rightarrow 0$ as $t \rightarrow \infty$.

5.1.3 Example

Consider the second order system with one unknown parameter θ

$$\dot{x} = A(x)x + \phi^T(x)\theta + u$$

where

$$A(x) = \begin{bmatrix} x_1^2 & x_1 x_2 \\ 2x_2 & x_2^2 \end{bmatrix}, \quad \phi^T(x) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

The uncertain system can be transformed to its corresponding error dynamics with

$$\begin{aligned} \tilde{A}(z, x_d) &= \begin{bmatrix} \begin{Bmatrix} (z_1 + x_{d1})^2 \\ + z_1 x_{d1} + 2x_{d1}^2 \end{Bmatrix} & \begin{Bmatrix} (z_1 + x_{d1})(z_2 + x_{d2}) \\ + z_1 x_{d2} + x_{d1} x_{d2} \end{Bmatrix} \\ 2(z_2 + x_{d2}) & \begin{Bmatrix} (z_2 + x_{d2})^2 \\ + z_2 x_{d2} + 2x_{d2}^2 \end{Bmatrix} \end{bmatrix} \\ \tilde{\phi}^T(z, x_d) &= \begin{bmatrix} z_1 + x_{d1} \\ 0 \end{bmatrix} \\ f(z, x_d, \dot{x}_d) &= \begin{bmatrix} x_{d1}^3 + z_1 x_{d2}^2 + x_{d1} x_{d2}^2 \\ 2z_2 x_{d1} + 2x_{d1} x_{d2} + x_{d2}^3 \end{bmatrix}. \end{aligned}$$

Given $\theta = 0.5, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$. We will apply the proposed iterative algorithm to solve the adaptive control problem.

1. Starting from a simple Lyapunov function (5.10) and an adaptation law (5.12), solve condition (5.18) to get initial performance $\gamma_0 = 1.1853$ and the control gain

$$K_0 = \begin{bmatrix} 0.5350 & 0 \\ 0 & 0.5350 \end{bmatrix}$$

During simulation, we set references trajectories as following

$$x_{1d}(t) = \begin{cases} 2 + 0.4t, & 0 \leq t \leq 7s, \\ 4.8, & 7s < t \leq 10s. \end{cases}$$

$$x_{2d}(t) = \begin{cases} -1 - 0.5t, & 0 \leq t \leq 3s, \\ -2.5, & 3s < t \leq 7s, \\ -2.5 + 0.5(t - 7), & 7s < t \leq 10s. \end{cases}$$

The starting point is $[x_{10}; x_{20}; \theta_0] = [0; 0; 0]$. The simulated performance of this initial adaptive controller is shown in Fig. 5.1.

2. Assume that the Lyapunov function has 4th order term of z . Specifically, $V = \mathcal{M}^T(z)P_1\mathcal{M}(z) + \tilde{\theta}^T P_2 \tilde{\theta}$ with $\mathcal{M}(z) = [z_1^2; z_1 z_2; z_2^2; z_1; z_2]$. For fixed K_0 , we solve condition (5.19) using SOSTOOL and obtain

$$\gamma_1 = 1.1095$$

$$Y_1 = 26.166$$

$$P_1 = \begin{bmatrix} 0.0810 & 0 & -0.0526 & -0.0933 & 0 \\ 0 & 0.2272 & 0 & 0 & -0.0039 \\ -0.0526 & 0 & 6.7129 & -0.0061 & 0 \\ -0.0933 & 0 & -0.0061 & 36.1680 & 0 \\ 0 & -0.0039 & 0 & 0 & 30.4730 \end{bmatrix}$$

3. Fix P_{11} and specify the control gain $K_2 = K(z) = K_{20} + K_{21}z_1 + K_{22}z_2$. Solving condition (5.20) to minimize γ , finally we have

$$\gamma = \gamma_2 = 1.0410$$

$$Y_2 = 36.405$$

$$K(z) = \begin{bmatrix} 14.511 + 0.1107z_1 & 0.0117z_2 \\ -0.0014z_2 & 14.019 + 0.0021z_1 \end{bmatrix}$$

Select $\Gamma = Y_2$, then $P_2 = 1$. Through three iterations, we are able to minimize the γ value by extending the Lyapunov function from 2nd order to 4th order and generalizing the control gain K from scalar form to parameter dependent form.

For comparison purpose, we will use the same command trajectories and initial conditions to conduct simulation. Shown in Figure 5.2(a) is the tracking performance of x_1 and x_2 . As can be seen, both of the states track the command signals quickly. Furthermore, $\hat{\theta}$ tracks its true value as well, as shown in Figure 5.2(b).

5.2 Polynomial Nonlinear Systems with Un-Matched Parametric Uncertainties

Consider the uncertain nonlinear system

$$\dot{x} = f(x, \theta) + B_2(x, \theta)u \quad (5.21)$$

where x, u are state and control input with dimensions n and n_u . θ is an r -dimensional vector of the constant parameters with $\theta \in \Theta \in \mathcal{R}^r$. The nonlinear functions $f(x, \theta)$ and $B_2(x, \theta)$ are in polynomial forms of state x . Also, we assume θ has the following parameter-dependence fashion

$$\begin{aligned} f(x, \theta) &= f_0(x) + \sum_{i=1}^r \theta_i f_i(x) \\ B_2(x, \theta) &= B_{20}(x) + \sum_{i=1}^r \theta_i B_{2i}(x) \end{aligned}$$

with $f_i(0) = 0$.

It is well known that when θ is known, there exist many ways such as backstepping design to construct the nonlinear stabilizing control law. However, when the system is not exactly known, the traditional nonlinear stabilizing control design techniques are not adequate to handle the parametric uncertainties. Specifically, assume θ is unknown and belongs to the compact set Θ , the estimation of the uncertain parameter $\hat{\theta}$ will produce the estimation error $\tilde{\theta} = \theta - \hat{\theta}$, which plays the role of an external disturbance on the open-loop system.

Our concern is how to develop the control design strategy to identify the true parameter asymptotically and minimize the effect of estimation error on the controlled system output. This belongs to an adaptive $\mathcal{H}_\infty$ control problem. Resorting equation (5.21) and defining the controlled output equation, the adaptive $\mathcal{H}_\infty$ control problem for the polynomial nonlinear system is in the following form

$$\begin{cases} \dot{x} &= A(x, \hat{\theta})x + B_1(x)\tilde{\theta} + B_2(x, \hat{\theta})u \\ e &= \begin{bmatrix} \lambda \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \alpha \end{bmatrix} u \end{cases} \quad (5.22)$$

where $A(x, \hat{\theta})x, B_1(x)\tilde{\theta}$ are determined by substituting θ in $f(x, \theta)$ with $\hat{\theta} + \tilde{\theta}$ and then splitting out $\hat{\theta}$ and $\tilde{\theta}$ terms. Since $\tilde{\theta}$ is produced by inexact estimation of θ , it is time-varying and will be treated as an disturbances on the controlled system. Furthermore, since θ is unknown, $B_2(x, \theta)$ could be replaced by its estimation $B_2(x, \hat{\theta})$. Our design objective is to develop state feedback nonlinear $\mathcal{H}_\infty$ control law and the adaptive control strategy so that $\hat{\theta}$ asymptotically catches up θ and the impact of $\tilde{\theta}$ on the closed-loop system is gradually reduced to zero.

5.2.1 Vector projection

Vector projection technique uses the technique from nonsmooth analysis and viability theory [3, 65]. It will play a very important role in the development of adaptation law for adaptive $\mathcal{H}_\infty$ control design.

Invariance and contingent cone

For a differential equation:

$$\dot{x} = f(x, t) \quad (5.23)$$

where $f : \mathbf{R}^n \times \mathbf{R}^+ \rightarrow \mathbf{R}^n$ is a continuous function. Suppose for all $x_0 \in \mathbf{R}^n$, the differential equation has a unique solution starting from $x(0) = x_0$. A set $\mathbf{X} \subset \mathbf{R}^n$ is an invariant set of (5.23), if its solution will stay in the set $\mathbf{X}$ for all $x_0 \in \mathbf{X}$ and all $t > 0$.

Given a compact set $\mathbf{\Omega} \subset \mathbf{R}^n$, the invariance of $\mathbf{\Omega}$ for the differential equation is characterized in terms of contingent cones [3]. The contingent cone to $\mathbf{\Omega}$ is well defined as

a set-valued map $T_{\Omega} : \Omega \rightarrow \mathbf{X}$:

$$T_{\Omega}(x) := \left\{ v \mid \liminf_{h \rightarrow 0^+} \frac{d_{\Omega}(x + hv)}{h} \leq 0. \right\}$$

where $d_{\Omega}(x) := \inf_{z \in \Omega} \|x - z\|$. For all $x \in \Omega$, the value $T_{\Omega}(x)$ is a closed cone.

Suppose Ω is convex and $0 \in INT(\Omega)$. The Minkowski function to Ω is defined as [44]

$$\Psi_{\Omega}(x) = \inf \{ \lambda \in \mathbf{R}^+ : x \in \lambda \Omega \};$$

Given $r \geq 0$, we define the set Ω_r as

$$\Omega_r := \{ x \in \mathbf{R}^n : \Phi_{\Omega}(x) \leq r \}.$$

Then $\Omega = \Omega_1$.

Lemma 1 *If Ω is convex, then*

$$T_{\Omega}(x) = \overline{\{ \psi : \exists t > 0 : x + t\psi \in \Omega \}}.$$

If $\Omega \subset \mathbf{R}^n$ is convex, then its normal cone is defined as

$$N_{\Omega}(x) = \{ z \in \mathbf{R}^n : \psi^T z \leq 0, \forall \psi \in T_{\Omega}(x) \}$$

Intuitively, the contingent cone on a compact set can be interpreted as the set of all vectors that could make the finite extension of a given vector, which belongs to the set, still in this set. If the vector is perpendicular to the contingent cone, then it belongs to the normal cone.

Lemma 2 *Given a convex compact set $\Omega \subset \mathbf{R}^n$, it is an invariant set of differential equation (5.23) if and only if for all $x \in \Omega$ and $t > 0$*

$$f(x, t) \in T_{\Omega}(x).$$

In summary, given a convex and compact set $\Omega \subset \mathbf{R}^n$, the solutions of the differential equation (5.23) are not necessarily always inside the set Ω . The invariance of this set can be characterized by its contingent cone. In adaptive control problems, we usually require some parameters, which are governed by differential equations, stay within the given

set during the adaptation with some properties satisfied. This objective will be achieved by the following vector projection technique.

Vector projection technique was introduced for constrained optimization [44] and was generalized to a non-smooth case [47, 48]. It has been widely used in the adaptive parameter estimation and adaptive control construction [23, 28, 51, 62]. The following theorem provides an adaptation law borrowed from [47], which is based on vector projection.

Theorem 5 *Consider the convex and compact set $\Omega \subset \mathbf{R}^n$ and its Minkowski function $\Psi_\Omega(x)$. Define the projection of a vector $\psi \in \mathbf{R}^n$ at a point $x \in \mathbf{R}^n$ on the contingent cone $T_\Omega(x)$ as following:*

$$\Pi_\Omega(x, \psi) = \begin{cases} \psi, & \text{if } x \in \text{INT}(\Omega) \text{ or } \psi \in T_{\Omega_r}(x); \\ 0, & \text{if } \psi \in N_{\Omega_r}(x); \\ \psi^T v v, & \text{otherwise, where } v = \text{argmax} \{ \psi^T v_0 | v_0 \in T_{\Omega_r}(x), \|v_0\| = 1 \}. \end{cases}$$

where $r = \Psi_\Omega(x)$. Then we have the following assertions:

- (1) $\Pi_\Omega(x, \psi) \in T_\Omega(x)$ for all $x \in \Omega$ and $\psi \in \mathbf{R}^n$.
- (2) $\|\Pi_\Omega(x, \psi)\| \leq \|\psi\|$ for all $x \in \Omega$ and $\psi \in \mathbf{R}^n$.
- (3) $(x - x^*)^T (\Pi_\Omega(x, \psi) - \psi) \leq 0$ for all $x^* \in \Omega, x \in \Omega$, and $\psi \in \mathbf{R}^n$.
- (4) Consider the system (5.23) under the above projection:

$$\dot{x} = \Pi_\Omega(x, f(x, t)),$$

then the set Ω is an invariant set for the projected system.

The proof of the theorem can be found in [47].

Theorem 5 provides a way to describe the projection of a vector at some point on the contingent cone. This can be classified as three cases: direct projection when ψ belongs to the contingent cone; none projection when ψ is perpendicular to the contingent cone; partly projection when some components of ψ belong to the contingent cone. This Theorem will be very useful in establishing the adaptive control mechanism in the next subsection. Furthermore, note that the allowable parameter sets are assumed as compact and convex, but not necessarily smooth, for instance, a cube in the parameter space. The vector projection to a more general setting such that the non-smooth parameter sets are allowed has been developed in [47].

5.2.2 Adaptive $\mathcal{H}_\infty$ control

Motivated by the vector projection technique introduced in subsection 5.2.1, we have following Theorem for the construction of adaptive control law.

Theorem 6 *Consider the polynomial nonlinear system (5.22). Suppose there exists a non-negative function $V(x) : \mathbf{R}^n \rightarrow \mathbf{R}^+$ and a controller $u(x, \hat{\theta})$ such that for each $\theta \in \Theta$, (5.26) is satisfied. Then the adaptive $\mathcal{H}_\infty$ control problem has a solution. And an adaptive control law is given by*

$$\dot{\hat{\theta}} = \Pi(\hat{\theta}, \Phi(x, u)), \quad (5.24)$$

where

$$\Phi(x, u) = \frac{1}{2\alpha} \begin{bmatrix} \frac{\partial V}{\partial x} B_{21}(x)u \\ \frac{\partial V}{\partial x} B_{22}(x)u \\ \vdots \\ \frac{\partial V}{\partial x} B_{2r}(x)u \end{bmatrix}, \quad (5.25)$$

Π is the vector projection with respect to the set Θ . In fact, $\hat{\theta} \rightarrow \theta$ as $t \rightarrow \infty$.

Proof: Suppose $V(x)$ is a positive definite function and independent of $\tilde{\theta}$, let $W(x, \tilde{\theta})$ be a positive definite function defined as

$$W(x, \tilde{\theta}) = V(x) + \alpha \tilde{\theta}^T \tilde{\theta},$$

where $\alpha > 0$ can be chosen such that

$$\max_{(\hat{\theta}, \theta) \in \Theta \times \Theta} \alpha (\hat{\theta} - \theta)^T (\hat{\theta} - \theta) \leq \epsilon,$$

because $\Theta \times \Theta$ is compact.

Taking W as a storage function candidate of the adaptive $\mathcal{H}_\infty$ control system and calculating the nonlinear $\mathcal{H}_\infty$ condition

$$\begin{aligned}
& \dot{W}(x, \tilde{\theta}) + e^T e - \gamma^2 \tilde{\theta}^T \tilde{\theta} \\
&= \frac{\partial V}{\partial x} [A(x, \hat{\theta})x + B_1(x)\tilde{\theta} + B_2(x, \hat{\theta})u] + \sum_{i=1}^r \frac{\partial V}{\partial x} (\theta_i - \hat{\theta}_i) B_{2i}(x)u + x^T \lambda^T \lambda x + u^T \alpha^T \alpha u \\
&\quad - \gamma^2 \tilde{\theta}^T \tilde{\theta} - 2\alpha \tilde{\theta} \dot{\hat{\theta}} \\
&= \frac{\partial V}{\partial x} A(x, \hat{\theta})x + [\frac{\partial V}{\partial x} B_1(x)\tilde{\theta} - \gamma^2 \tilde{\theta}^T \tilde{\theta}] + [\frac{\partial V}{\partial x} B_2(x, \hat{\theta})u + u^T \alpha^T \alpha u] + x^T \lambda^T \lambda x - 2\alpha \tilde{\theta} \dot{\hat{\theta}} \\
&\quad + \sum_{i=1}^r \frac{\partial V}{\partial x} (\theta_i - \hat{\theta}_i) B_{2i}(x)u \\
&= \frac{\partial V}{\partial x} A(x, \hat{\theta})x + [\frac{1}{4\gamma^2} \frac{\partial V}{\partial x} B_1(x)B_1(x)^T \frac{\partial V^T}{\partial x} + \frac{\partial V}{\partial x} B_1(x)\tilde{\theta} - \gamma^2 \tilde{\theta}^T \tilde{\theta}] - \frac{1}{4\gamma^2} \frac{\partial V}{\partial x} B_1(x)B_1(x)^T \frac{\partial V^T}{\partial x} \\
&\quad + [\frac{\partial V}{\partial x} B_2(x, \hat{\theta})u + u^T \alpha^T \alpha u] + x^T \lambda^T \lambda x - 2\alpha \tilde{\theta} \dot{\hat{\theta}} + \sum_{i=1}^r \frac{\partial V}{\partial x} (\theta_i - \hat{\theta}_i) B_{2i}(x)u \\
&= \frac{\partial V}{\partial x} A(x, \hat{\theta})x - \|\gamma \tilde{\theta} - \frac{1}{2\gamma} B_1(x)^T \frac{\partial V^T}{\partial x}\|^2 + \frac{1}{4\gamma^2} \frac{\partial V}{\partial x} B_1(x)B_1(x)^T \frac{\partial V^T}{\partial x} \\
&\quad + [\frac{\partial V}{\partial x} B_2(x, \hat{\theta})u + u^T \alpha^T \alpha u] + x^T \lambda^T \lambda x - 2\alpha \tilde{\theta} \dot{\hat{\theta}} + \sum_{i=1}^r \frac{\partial V}{\partial x} (\theta_i - \hat{\theta}_i) B_{2i}(x)u \\
&< \frac{\partial V}{\partial x} A(x, \hat{\theta})x + \frac{1}{4\gamma^2} \frac{\partial V}{\partial x} B_1(x)B_1(x)^T \frac{\partial V^T}{\partial x} + [\frac{\partial V}{\partial x} B_2(x, \hat{\theta})u + u^T \alpha^T \alpha u] \\
&\quad + x^T \lambda^T \lambda x - 2\alpha \tilde{\theta} \dot{\hat{\theta}} + \sum_{i=1}^r \frac{\partial V}{\partial x} (\theta_i - \hat{\theta}_i) B_{2i}(x)u \\
&= \frac{\partial V}{\partial x} A(x, \hat{\theta})x + \frac{1}{4\gamma^2} \frac{\partial V}{\partial x} B_1(x)B_1(x)^T \frac{\partial V^T}{\partial x} + [\frac{\partial V}{\partial x} B_2(x, \hat{\theta})u + u^T \alpha^T \alpha u] \\
&\quad + x^T \lambda^T \lambda x + 2\alpha(\hat{\theta} - \theta) \left(\dot{\hat{\theta}} - \begin{bmatrix} \frac{\partial V}{\partial x} B_{21}(x)u \\ \frac{\partial V}{\partial x} B_{22}(x)u \\ \vdots \\ \frac{\partial V}{\partial x} B_{2r}(x)u \end{bmatrix} \right),
\end{aligned}$$

If the following condition is satisfied

$$\begin{aligned}
& \frac{\partial V}{\partial x} A(x, \hat{\theta})x + \frac{1}{4\gamma^2} \frac{\partial V}{\partial x} B_1(x)B_1(x)^T \frac{\partial V^T}{\partial x} + [\frac{\partial V}{\partial x} B_2(x, \hat{\theta})u + u^T \alpha^T \alpha u] + x^T \lambda^T \lambda x < 0, \\
& \hspace{25em} (5.26)
\end{aligned}$$

we then have

$$\dot{W}(x, \tilde{\theta}) \leq -\|e\|^2 + \gamma^2 \|\tilde{\theta}\|^2 + 2\alpha(\hat{\theta} - \theta)(\dot{\hat{\theta}} - \Phi(x, u)).$$

Integrate both side of the above condition from 0 to T and notice $W(x(T), \hat{\theta}(T)) \geq 0$ and $x(0) = 0$, we have

$$\begin{aligned} \int_0^T \|e(t)\|^2 dt &\leq \int_0^T \gamma^2 \|\tilde{\theta}(t)\|^2 dt + \alpha(\hat{\theta}(0) - \theta)^T(\hat{\theta}(0) - \theta) + 2\alpha \int_0^T (\hat{\theta}(t) - \theta)^T(\dot{\hat{\theta}}(t) - \Phi(x, u)) dt \\ &\leq \int_0^T \gamma^2 \|\tilde{\theta}(t)\|^2 dt + 2\alpha \int_0^T (\hat{\theta}(t) - \theta)^T(\dot{\hat{\theta}}(t) - \Phi(x, u)) dt + \epsilon \end{aligned} \quad (5.27)$$

Therefore, if we can find a parameter adaptation law for $\hat{\theta}$ such that

$$\int_0^T (\hat{\theta}(t) - \theta)^T(\dot{\hat{\theta}}(t) - \Phi(x, u)) dt \leq 0 \quad (5.28)$$

and

$$\hat{\theta}(t) \in \Theta,$$

then the adaptive $\mathcal{H}_\infty$ control problem is solved with performance level γ .

From Theorem 5, we can conclude that the adaptation law (5.24) guarantees $\hat{\theta}(t) \in \Theta$ and

$$(\hat{\theta} - \theta)^T(\dot{\hat{\theta}} - \Phi(x, u)) = (\hat{\theta} - \theta)^T(\Pi(\hat{\theta}, \Phi(x, u)) - \Phi(x, u)) \leq 0.$$

Integrating from both sides from 0 to T directly leads to

$$\int_0^T (\hat{\theta}(t) - \theta)^T(\dot{\hat{\theta}}(t) - \Phi(x, u)) dt \leq 0.$$

Substituting this relation into condition (5.27), we have

$$\int_0^T \|z(t)\|^2 dt \leq \int_0^T \gamma^2 \|\tilde{\theta}(t)\|^2 dt + \epsilon. \quad (5.29)$$

Q.E.D.

In applications, the unknown parameter θ is usually bounded. Here we assume that allowable parameter sets Θ are compact and convex. During the adaptation, one of the requirements is that the parameter-adjustment mechanism of an adaptive control system keep the adjusted parameters θ in the parameter sets Θ so as not to invalidate the solvability conditions. As shown in Theorem 6, vector projection technique is used to achieve this goal.

Furthermore, from the proof of Theorem 6, we can conclude that the solution of adaptive $\mathcal{H}_\infty$ control problem is composed of two parts. The first part is the construction of the adaptation law which satisfies (5.28). We have proved that the adaptation law (5.24) is a qualified one. The second part is the synthesis condition (5.26).

Similar to nonlinear $\mathcal{H}_\infty$ control case, the synthesis condition (5.26) is a non-convex condition because of the bilinear term $\frac{\partial V}{\partial x} B_2(x, \hat{\theta})u$ is involved in. For this reason, we propose the following iterative algorithm. Firstly, starting from an initial controller $u_0(x, \hat{\theta})$, solve the following condition by specifying the form of $V_0(x)$,

$$\begin{aligned} \min \quad & \gamma_1 \\ \mathcal{S}_{itr1} = & \begin{bmatrix} \frac{\partial V_0}{\partial x} A(x, \hat{\theta})x + \frac{\partial V_0}{\partial x} B_2(x, \hat{\theta})u_0(x, \hat{\theta}) + x^T \lambda^T \lambda x + u_0(x, \hat{\theta})^T \alpha^T \alpha u_0(x, \hat{\theta}) & \frac{1}{2} \frac{\partial V_0}{\partial x} B_1(x) \\ * & -\gamma_1^2 I \end{bmatrix} < 0 \end{aligned} \quad (5.30)$$

Secondly, for fixed $V_0(x)$, solve the following condition by specifying the form of $u_1(x)$,

$$\begin{aligned} \min \quad & \gamma_2 \\ \mathcal{S}_{itr2} = & \begin{bmatrix} \frac{\partial V_0}{\partial x} A(x, \hat{\theta})x + \frac{\partial V_0}{\partial x} B_2(x, \hat{\theta})u_1(x, \hat{\theta}) + x^T \lambda^T \lambda x & \frac{1}{2} \frac{\partial V_0}{\partial x} B_1(x) & u_1(x, \hat{\theta})^T \alpha^T \\ * & -\gamma_2^2 I & 0 \\ * & * & -I \end{bmatrix} < 0 \end{aligned} \quad (5.31)$$

Repeat the above two steps by gradually augmenting the Lyapunov function and enriching the controller parametrization until γ is small enough.

Notice that both of condition (5.30) and (5.31) are polynomial matrix functions of x and $\hat{\theta}$. By recasting them as the SOS optimization problems, tractable solutions could be obtained via SOS programming. Generally, the synthesis condition can be reformulated as

$$\begin{aligned} \min \quad & \gamma_i \\ \mathbf{S}_i := & -v^T \mathcal{S}_{itri} v \quad is \quad SOS. \end{aligned} \quad (5.32)$$

where v is a free vector of compatible dimension $\mathbf{R}^{n_v}$ and $\mathbf{S}_i$ is a polynomial.

Finally, from above, we could claim that condition (5.26) is the polynomial function of x and $\hat{\theta}$. If it can be solved through the iterative algorithm, it must be able to be

formulated as the following form

$$\begin{bmatrix} x & \hat{\theta} \end{bmatrix} \Pi(x, \hat{\theta}) \begin{bmatrix} x \\ \hat{\theta} \end{bmatrix} < 0,$$

where $\Pi(x, \hat{\theta})$ is a polynomial matrix function of x and $\hat{\theta}$. The feasibility of this condition indicates that the squares of $x(t), \tilde{\theta}(t)$ are integrable, i.e. $x(t), \tilde{\theta}(t) \in \mathcal{L}_2$. Furthermore, for any initial condition, equations (5.22), (5.24) and (5.28) imply that $\dot{x}(t), \dot{\tilde{\theta}}(t) \in \mathcal{L}_\infty$. By Barbalat's lemma, we have $x(t), \tilde{\theta}(t) \rightarrow 0$ as $t \rightarrow \infty$.

5.2.3 Example

In this section, we will demonstrate the proposed nonlinear adaptive $\mathcal{H}_\infty$ control approach on an induction motor problem. Assuming linear magnetic circuits, the dynamics of a balanced non-saturated induction motor with one pole pair in a fixed reference frame attached to the stator are given by the well known fifth-order model [49]

$$\begin{aligned} \dot{\omega} &= \mu(\psi_a i_b - \psi_b i_a) - \frac{T_L}{J} \\ \dot{\psi}_a &= -\theta_2 \psi_a - \omega \psi_b + \theta_2 M i_a \\ \dot{\psi}_b &= -\theta_2 \psi_b + \omega \psi_a + \theta_2 M i_b \\ \dot{i}_a &= -\left(\frac{R_s}{\sigma} + \beta \theta_2 M\right) i_a + \frac{1}{\sigma} u_a + \beta \theta_2 \psi_a + \beta \omega \psi_b \\ \dot{i}_b &= -\left(\frac{R_s}{\sigma} + \beta \theta_2 M\right) i_b + \frac{1}{\sigma} u_b + \beta \theta_2 \psi_b - \beta \omega \psi_a \end{aligned}$$

where

ω is the rotor speed;

(ψ_a, ψ_b) are the rotor fluxes;

(i_a, i_b) are the stator currents;

(u_a, u_b) are the stator voltages in a fixed reference attached to the stator;

The model parameters are defined as

$J = 0.0075$: rotor moment of inertia

$(R_r, R_s) = (3.3, 5.3)$: rotor and stator windings resistances

$(L_r, L_s) = (0.375, 0.365)$: rotor and stator windings inductances

$M = 0.34$: mutual inductance

To simplify notations, we use the following reparametrization:

$$\theta_2 = \frac{R_r}{L_r}, \beta = \frac{M}{\sigma L_r}, \mu = \frac{M}{J L_r}, \sigma = L_s(1 - \frac{M^2}{L_s L_r}).$$

The load torque $T_L = T_{Ln} + \theta_1$ where $\theta_1 \in [-\theta_{1m}, \theta_{1m}]$ denotes the constant uncertain variation from the constant nominal value T_{Ln} . For simplicity, we will assume T_{Ln} as 0 in the following derivation. It is trivial to get the results for T_{Ln} nonzero case by imposing a constant displacement on θ_1 . Using $(\omega, \psi_a, \psi_b, i_a, i_b)$ to represent the state variables, the dynamics of the induction motor in the state space is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & -\mu x_3 & \mu x_2 \\ 0 & -\theta_2 & -x_1 & M\theta_2 & 0 \\ 0 & x_1 & -\theta_2 & 0 & M\theta_2 \\ \beta x_3 & \beta \theta_2 & 0 & -\frac{R_s}{\sigma} - \beta M\theta_2 & 0 \\ -\beta x_2 & 0 & 0 & 0 & -\frac{R_s}{\sigma} - \beta M\theta_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} -\frac{1}{J} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \theta_1 \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{\sigma} & 0 \\ 0 & \frac{1}{\sigma} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

Apparently, the origin is not an equilibrium point of the above system. For this reason, we will transform the system to an equivalent system where the origin is its equilibrium. Choosing the equilibrium point in the original system as $x_1 = -\frac{M\theta_2\theta_1}{\mu J}$, $x_2 = 0$, $x_3 = 1$, $x_4 = -\frac{\theta_1}{\mu J}$, $x_5 = \frac{1}{M}$. Define

$$\begin{aligned} \tilde{x}_1 &= x_1 + \frac{M\theta_2\theta_1}{\mu J} \\ \tilde{x}_2 &= x_2 \\ \tilde{x}_3 &= x_3 - 1 \\ \tilde{x}_4 &= x_4 + \frac{\theta_1}{\mu J} \\ \tilde{x}_5 &= x_5 - \frac{1}{M}. \end{aligned}$$

Then the dynamics of the transformed system is

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \\ \dot{\tilde{x}}_5 \end{bmatrix} &= \begin{bmatrix} 0 & \frac{\mu}{M} & \frac{\theta_1}{J} & -\mu - \mu\tilde{x}_3 & \mu\tilde{x}_2 \\ -1 - \tilde{x}_3 & -\theta_2 & \frac{M\theta_2\theta_1}{\mu J} & \theta_2 M & 0 \\ \tilde{x}_2 & -\frac{M\theta_2\theta_1}{\mu J} & -\theta_2 & 0 & \theta_2 M \\ \beta + \beta\tilde{x}_3 & \theta_2\beta & -\frac{M\beta\theta_2\theta_1}{\mu J} & -\frac{R_s}{\sigma} - \beta M\theta_2 & 0 \\ -\beta\tilde{x}_2 & \frac{M\beta\theta_2\theta_1}{\mu J} & 0 & 0 & -\frac{R_s}{\sigma} - \beta M\theta_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{\sigma} & 0 \\ 0 & \frac{1}{\sigma} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{R_s\theta_1}{\sigma\mu J} \\ -\frac{1}{M}(\frac{R_s}{\sigma} + \beta\theta_2 M) \end{bmatrix} \end{aligned}$$

Define $\eta = [\tilde{x}_1 \ \tilde{x}_2 \ \tilde{x}_3]^T$, $\xi = [\tilde{x}_4 \ \tilde{x}_5]^T$. Represent the transformed system as

$$\begin{cases} \dot{\eta} = f(\eta) + g(\eta)\xi \\ \dot{\xi} = f_a(\eta, \xi) + g_a(\eta, \xi)u + \delta \end{cases} \quad (5.33)$$

where

$$\begin{aligned} f(\eta) &= \begin{bmatrix} 0 & \frac{\mu}{M} & \frac{\theta_1}{J} \\ -1 - \tilde{x}_3 & -\theta_2 & \frac{M\theta_2\theta_1}{\mu J} \\ \tilde{x}_2 & -\frac{M\theta_2\theta_1}{\mu J} & -\theta_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} \\ g(\eta) &= \begin{bmatrix} -\mu - \mu\tilde{x}_3 & \mu\tilde{x}_2 \\ \theta_2 M & 0 \\ 0 & \theta_2 M \end{bmatrix} \\ f_a(\eta, \xi) &= \begin{bmatrix} \beta + \beta\tilde{x}_3 & \theta_2\beta & -\frac{M\beta\theta_2\theta_1}{\mu J} \\ -\beta\tilde{x}_2 & \frac{M\beta\theta_2\theta_1}{\mu J} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} + \begin{bmatrix} -\frac{R_s}{\sigma} - \beta M\theta_2 & 0 \\ 0 & -\frac{R_s}{\sigma} - \beta M\theta_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_4 \\ \tilde{x}_5 \end{bmatrix} \\ g_a(\eta, \xi) &= \begin{bmatrix} \frac{1}{\sigma} & 0 \\ 0 & \frac{1}{\sigma} \end{bmatrix} \\ \delta &= \begin{bmatrix} \frac{R_s\theta_1}{\sigma\mu J} \\ -\frac{1}{M}(\frac{R_s}{\sigma} + \beta\theta_2 M) \end{bmatrix} \end{aligned}$$

Denote $u_a = f_a(\eta, \xi) + g_a(\eta, \xi)u + \delta$, then

$$\begin{cases} \dot{\eta} &= f(\eta) + g(\eta)\xi \\ \dot{\xi} &= u_a \end{cases}$$

The backstepping design is as the following:

1. Consider $\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$, $\xi = \phi(\eta)$. Choose

$$\phi(\eta) = \begin{bmatrix} \phi_1(\eta) \\ \phi_2(\eta) \end{bmatrix} = \begin{bmatrix} 2\tilde{x}_1 + 0.5\tilde{x}_2 + 2\tilde{x}_1\tilde{x}_3 \\ 0.5\tilde{x}_3 - 2\tilde{x}_1\tilde{x}_2 \end{bmatrix}$$

$$W(\eta) = 10^{-13}(\tilde{x}_2^2 + \tilde{x}_3^2)$$

It can be proved that $\dot{V}(\eta) \leq -W(\eta)$ by selecting

$$V(\eta) = 0.12461 \times 10^{-11}\tilde{x}_1^2 + 0.12328 \times 10^{-9}\tilde{x}_2^2 + 0.20951 \times 10^{-9}\tilde{x}_3^2.$$

2. Define $z = \xi - \phi(\eta)$, $v = u_a - \dot{\phi}(\eta)$, we have

$$\begin{cases} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \\ \dot{z} &= v \end{cases}$$

Defining $V_c(\eta, \xi) = V(\eta) + \frac{1}{2}z^T M z$, $M > 0$. Calculating the derivative of V_c leads to

$$\dot{V}_c \leq -W(\eta) + \frac{\partial V}{\partial \eta} g(\eta)z + v^T M z$$

Then $\dot{V}_c \leq -W(\eta) - z^T K M z$ by choosing $v = -M^{-T} g^T(\eta) \frac{\partial V}{\partial \eta} - K z$, $K > 0$.

Set $M = \begin{bmatrix} 10^{-12} & 0 \\ 0 & 10^{-12} \end{bmatrix}$, $K = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, the resulting Lyapunov function is

$$\begin{aligned} V_c(\eta, \xi) &= 0.3246 \times 10^{-11}\tilde{x}_1^2 + 0.1234 \times 10^{-9}\tilde{x}_2^2 + 0.2096 \times 10^{-9}\tilde{x}_3^2 + 0.5000 \times 10^{-12}\tilde{x}_4^2 \\ &\quad - 0.2000 \times 10^{-11}\tilde{x}_4\tilde{x}_1 - 0.5000 \times 10^{-12}\tilde{x}_4\tilde{x}_2 - 0.2000 \times 10^{-11}\tilde{x}_4\tilde{x}_1\tilde{x}_3 \\ &\quad + 1.0000 \times 10^{-12}\tilde{x}_2\tilde{x}_1 + 0.4000 \times 10^{-11}\tilde{x}_1^2\tilde{x}_3 + 0.2000 \times 10^{-11}\tilde{x}_1^2\tilde{x}_3^2 \\ &\quad + 0.5000 \times 10^{-12}\tilde{x}_5^2 - 0.5000 \times 10^{-12}\tilde{x}_5\tilde{x}_3 + 0.2000 \times 10^{-11}\tilde{x}_5\tilde{x}_2\tilde{x}_1 \\ &\quad + 0.2000 \times 10^{-11}\tilde{x}_2^2\tilde{x}_1^2 \end{aligned}$$

and the backstepping controller is

$$u = g_a^{-1}(\eta, \xi)(u_a - f_a(\eta, \xi) - \delta)$$

$$= \begin{bmatrix} \left\{ \begin{aligned} &-0.7863\tilde{x}_2\tilde{x}_1 + 40.3437\tilde{x}_3\tilde{x}_2 - 0.7081\tilde{x}_1 + 32.1721\tilde{x}_2 + 38.2504\tilde{x}_3 - 5.7327\tilde{x}_4 \\ &-1.7066\tilde{x}_1\tilde{x}_3 + 13.7169\tilde{x}_2\tilde{x}_5 - 27.4337\tilde{x}_4\tilde{x}_3 + 31.7707\tilde{x}_3^2 - 13.7169\tilde{x}_4\tilde{x}_3^2 \\ &+ 0.1135\tilde{x}_2\tilde{x}_1^2 + 0.3395\tilde{x}_1\tilde{x}_5 - 12.2757 + 13.7169\tilde{x}_3\tilde{x}_2\tilde{x}_5 \end{aligned} \right\} \\ \left\{ \begin{aligned} &23.5669 + 1.7066\tilde{x}_2\tilde{x}_1 - 0.1929\tilde{x}_3 + 7.9842\tilde{x}_5 - 40.3437\tilde{x}_2^2 - 31.7707\tilde{x}_3\tilde{x}_2 \\ &+ 13.7169\tilde{x}_2\tilde{x}_4\tilde{x}_3 + 13.7169\tilde{x}_2\tilde{x}_4 - 13.7169\tilde{x}_2^2\tilde{x}_5 + 0.1135\tilde{x}_1^2 + 0.1135\tilde{x}_1^2\tilde{x}_3 \\ &- 0.7863\tilde{x}_1\tilde{x}_3 - 0.3395\tilde{x}_1\tilde{x}_4 - 6.4798\tilde{x}_2 \end{aligned} \right\} \end{bmatrix}$$

Figure 5.3 is the simulation result in the transformed system with initial condition $[\tilde{x}_1; \tilde{x}_2; \tilde{x}_3; \tilde{x}_4; \tilde{x}_5] = [0.5; 0.1; 0.1; 0.8; 0.7]$.

Backstepping is an effective way to achieve global stability with nonlinear control. However, when the system is not exactly known, it becomes an adaptive $\mathcal{H}_\infty$ control problem. For example, assume the constant load torque θ_1 is unknown and belongs to a set $\theta_1 \in \Theta, \Theta = [-\theta_{1m}, \theta_{1m}]$, then the estimation of the uncertain parameter $\hat{\theta}_1$ will produce the estimation error $\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1$, which plays the role of an external disturbance on the open-loop system. Resorting equation (5.33) leads to

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \\ \dot{\tilde{x}}_5 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\mu}{M} & \frac{\hat{\theta}_1}{J} & -\mu - \mu\tilde{x}_3 & \mu\tilde{x}_2 \\ -1 - \tilde{x}_3 & -\theta_2 & \frac{M\theta_2\hat{\theta}_1}{\mu J} & \theta_2 M & 0 \\ \tilde{x}_2 & -\frac{M\theta_2\hat{\theta}_1}{\mu J} & -\theta_2 & 0 & \theta_2 M \\ \beta + \beta\tilde{x}_3 & \theta_2\beta & -\frac{M\beta\theta_2\hat{\theta}_1}{\mu J} & -\frac{R_s}{\sigma} - \beta M\theta_2 & 0 \\ -\beta\tilde{x}_2 & \frac{M\beta\theta_2\hat{\theta}_1}{\mu J} & 0 & 0 & -\frac{R_s}{\sigma} - \beta M\theta_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{1}{J}\tilde{x}_3 \\ \frac{M\theta_2}{\mu J}\tilde{x}_3 \\ -\frac{M\theta_2}{\mu J}\tilde{x}_2 \\ -\frac{\beta M\theta_2}{\mu J}\tilde{x}_3 \\ \frac{\beta M\theta_2}{\mu J}\tilde{x}_2 \end{bmatrix} \tilde{\theta}_1 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \quad (5.34)$$

$$= A(\tilde{x}, \hat{\theta}_1)\tilde{x} + B_1(\tilde{x})\tilde{\theta}_1 + B_2\tilde{u} \quad (5.35)$$

where

$$\begin{aligned}\tilde{u} &= \left\{ \begin{bmatrix} \frac{1}{\sigma} & 0 \\ 0 & \frac{1}{\sigma} \end{bmatrix} u + \begin{bmatrix} \frac{R_s \hat{\theta}_1}{\sigma J \mu} \\ -\frac{1}{M} \left(\frac{R_s}{\sigma} + \beta M \theta_2 \right) \end{bmatrix} \right\} + \begin{bmatrix} \frac{R_s}{\sigma J \mu} \\ 0 \end{bmatrix} \tilde{\theta}_1 \\ &= \tilde{u}(\tilde{x}, \hat{\theta}_1) + \tilde{u}(\tilde{x}) \tilde{\theta}_1\end{aligned}$$

Note that comparing with backstepping design, the controller $\tilde{u}$ is divided into two parts because θ_1 is not available anymore. In the first part, $\hat{\theta}_1$ is used to estimate θ_1 during the controller implementation. The second part deals with the estimation error and it will be involved in the adaptation law construction. Our design objective is to develop state feedback adaptive $\mathcal{H}_\infty$ control strategy and the adaptation law so that $\hat{\theta}_1$ quickly catches up θ_1 and the impact of $\tilde{\theta}_1$ on the closed-loop system is finally minimized to zero.

For state feedback nonlinear $\mathcal{H}_\infty$ design, the controlled output equation is defined as following

$$\begin{aligned}e &= \begin{bmatrix} 0.0530 & 0 & 0 & 0 & 0 \\ 0 & 0.9156 & 0 & 0 & 0 \\ 0 & 0 & 0.8869 & 0 & 0 \\ 0 & 0 & 0 & 0.7936 & 0 \\ 0 & 0 & 0 & 0 & 0.8451 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.2487 & 0 \\ 0 & 0.2487 \end{bmatrix} \begin{bmatrix} \tilde{u}_1(\tilde{x}, \hat{\theta}_1) \\ \tilde{u}_2(\tilde{x}, \hat{\theta}_1) \end{bmatrix} \\ &= \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \tilde{u}(\tilde{x}, \hat{\theta}_1)\end{aligned}\tag{5.36}$$

Note that the control force is directly related with $\tilde{x}_4, \tilde{x}_5$ channel and it should be function of $\hat{\theta}_1$. Therefore, we start from the following simple formed controller:

$$\tilde{u}_0 = \begin{bmatrix} -\tilde{x}_4 \hat{\theta}_1^2 \\ -\tilde{x}_5 \hat{\theta}_1^2 \end{bmatrix}$$

1. Suppose the Lyapunov function V_0 is of quadratic form. Solve the synthesis condition

(5.30). We get V_0 and γ_1 .

$$V_0(\tilde{x}) = \mathcal{M}_0(\tilde{x})^T \Gamma_0 \mathcal{M}_0(\tilde{x})$$

$$= \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_5 \end{bmatrix} \begin{bmatrix} 0.0065 & 0.0120 & 0 & -0.0037 & 0 \\ 0.0120 & 2.5330 & 0 & 0.0578 & 0 \\ 0 & 0 & 3.9616 & 0 & 0.0960 \\ -0.0037 & 0.0578 & 0 & 0.0358 & 0 \\ 0 & 0 & 0.0960 & 0 & 0.0440 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \end{bmatrix}$$

$$\gamma_1 = 2.9120$$

2. Specify

$$\tilde{u}_1 = \begin{bmatrix} \tilde{u}_{10}\tilde{x}_1 + \tilde{u}_{11}\tilde{x}_4 + \tilde{u}_{12}\tilde{x}_5 + \tilde{u}_{13}\tilde{x}_4\hat{\theta}_1^2 \\ \tilde{u}_{20}\tilde{x}_1 + \tilde{u}_{21}\tilde{x}_4 + \tilde{u}_{22}\tilde{x}_5 + \tilde{u}_{23}\tilde{x}_5\hat{\theta}_1^2 \end{bmatrix},$$

solve the synthesis condition (5.31), we have

$$\tilde{u}_1 = \begin{bmatrix} 0.1158\tilde{x}_1 - 1.1854\tilde{x}_4 + 0.0057\tilde{x}_4\hat{\theta}_1^2 \\ -1.2553\tilde{x}_5 + 0.4050\tilde{x}_5\hat{\theta}_1^2 \end{bmatrix}$$

$$\gamma_2 = 1.7089$$

3. For fixed $\tilde{u}_1$, suppose $V_1(\tilde{x})$ is in the form of

$$V_1(\tilde{x}) = \mathcal{M}_1(\tilde{x})^T \Gamma_1 \mathcal{M}_1(\tilde{x}),$$

where

$$\mathcal{M}_1(\tilde{x})^T = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_5 & \tilde{x}_1\tilde{x}_3 & \tilde{x}_1\tilde{x}_2 \end{bmatrix}$$

Repeat step 1 to get Γ_1, γ_3 . The synthesis results are

$$\Gamma_1 = \begin{bmatrix} 0.0046 & 0.0166 & 0 & -0.0023 & 0 & -0.0003 & 0 \\ 0.0166 & 2.7063 & 0 & 0.0633 & 0 & 0.0004 & 0 \\ 0 & 0 & 4.2781 & 0 & 0.1145 & 0 & -0.0053 \\ -0.0023 & 0.0633 & 0 & 0.0183 & 0 & 0.0003 & 0 \\ 0 & 0 & 0.1145 & 0 & 0.0145 & 0 & -0.0007 \\ -0.0003 & 0.0004 & 0 & 0.0003 & 0 & 0.0007 & 0 \\ 0 & 0 & -0.0053 & 0 & -0.0007 & 0 & 0.0005 \end{bmatrix}$$

$$\gamma_3 = 0.6970$$

Paralleling with the theoretical part from subsection 5.2.2, to construct the adaptive control law, we define $\Phi(\tilde{x}, \tilde{u}(\tilde{x})) = \frac{1}{2\alpha} \frac{\partial V_1}{\partial \tilde{x}} B_2 \begin{bmatrix} \frac{R_s}{\sigma J \mu} \\ 0 \end{bmatrix}$. Take $\epsilon = 0.41$ in (5.29), then α can be chosen as 0.02. Thus, the adaptation law is

$$\dot{\hat{\theta}}_1 = \Pi(\hat{\theta}_1, \Phi(\tilde{x}, \tilde{u}(\tilde{x}))).$$

The state region constraints are

$$C_{r1} = 5.104 - \hat{\theta}_1^2 > 0,$$

$$C_{r2} = 10 - \tilde{x}_1^2 > 0,$$

$$C_{r3} = 0.5 - \tilde{x}_2^2 > 0,$$

$$C_{r4} = 0.5 - \tilde{x}_3^2 > 0,$$

$$C_{r5} = 7 - \tilde{x}_4^2 > 0,$$

$$C_{r6} = 7 - \tilde{x}_5^2 > 0.$$

Assume the actual value of θ_1 is 2.1. Initial condition is $\tilde{x}_0 = [\tilde{x}_{10}; \tilde{x}_{20}; \tilde{x}_{30}; \tilde{x}_{40}; \tilde{x}_{50}; \hat{\theta}_{10}] = [0; 0; 0; 0; 0; 1]$. or $x_0 = [x_{10}; x_{20}; x_{30}; x_{40}; x_{50}; \hat{\theta}_{10}] = [-6.9300; 0; 1; -2.3162; 2.9412; 1]$.

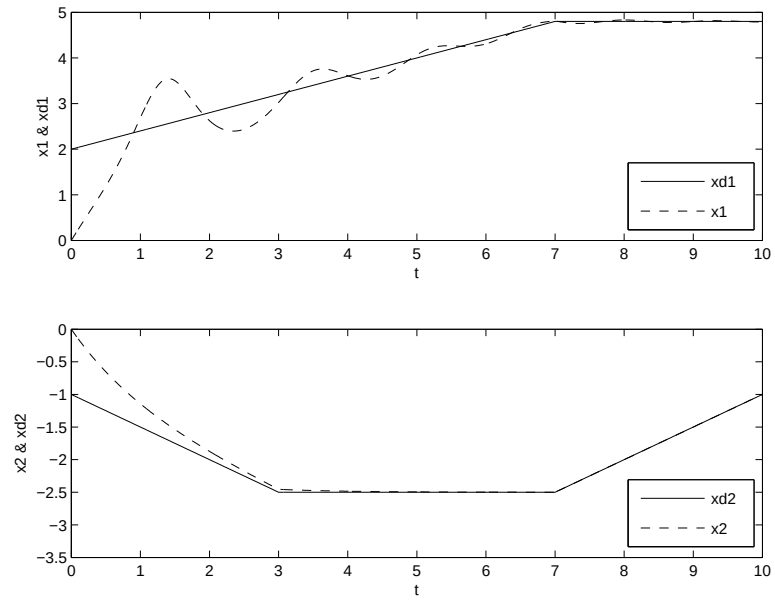
Figure 5.4 is the simulation result in the transformed system. Figure 5.5 is the result in the original system.

Lastly, to compare the performance of backstepping design and adaptive $\mathcal{H}_\infty$ control design, we choose a nonzero initial point inside the feasible region $\tilde{x}_0 = [\tilde{x}_{10}; \tilde{x}_{20}; \tilde{x}_{30}; \tilde{x}_{40}; \tilde{x}_{50}] = [2; 0.5; 0.5; 2.5; 2.5]$ and assume that the true value of θ_1 is 1.6. In backstepping design, θ_1

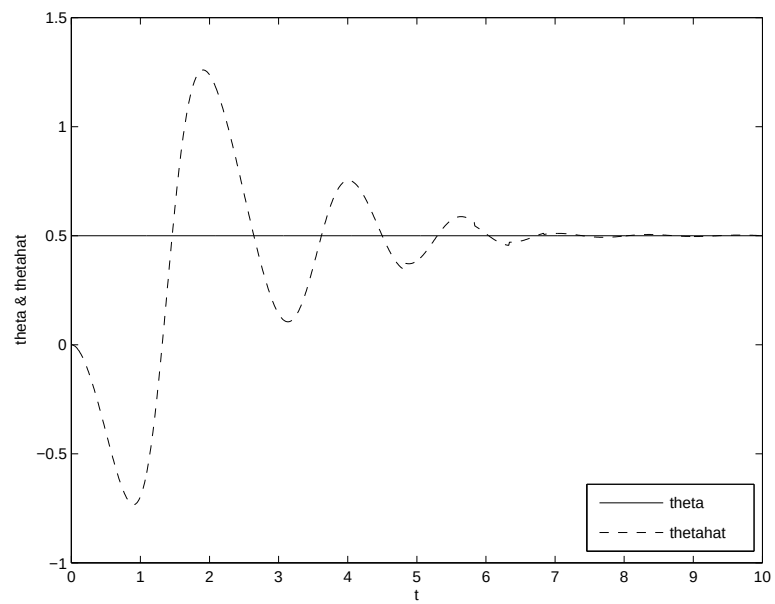
is implemented by its true value and Figure 5.6 shows that the trajectories of $\tilde{x}_1 \sim \tilde{x}_5$ are finally stabilized at the origin, as expected. In adaptive $\mathcal{H}_\infty$ control design, θ_1 is unknown and can only be implemented by its estimation $\hat{\theta}_1$, which starts from a guessed value 0.7. As shown in Figure 5.7, $\hat{\theta}_1$ quickly tracks its true value and the trajectories of $\tilde{x}_1 \sim \tilde{x}_5$ are driven to the origin much faster than those using backstepping design.

5.3 Summary

In this chapter, we developed an adaptive control approach to attenuate the effect of the unknown parameters on the controlled output for a class of polynomial nonlinear systems with matching and un-matching conditions. For the matching condition case, motivated by traditional adaptive control theory, the system is transformed into its error system and the adaptive control law as well as the parameter adaptation law are synthesized through an iterative algorithm. Meanwhile, the system trajectories track their command profiles and the estimated parameters finally converge to their true value. For the un-matching condition case, motivated by the dissipation theory and the vector projection technique, parallel results are established with exact parameter estimation. To improve adaptive controlled performance, the Lyapunov functions are relaxed from quadratic to higher order and the control gain is generalized from constant to parameter dependent. All of the synthesis conditions are formulated in the frame work of polynomial/constant linear matrix inequalities and solvable using available software packages.

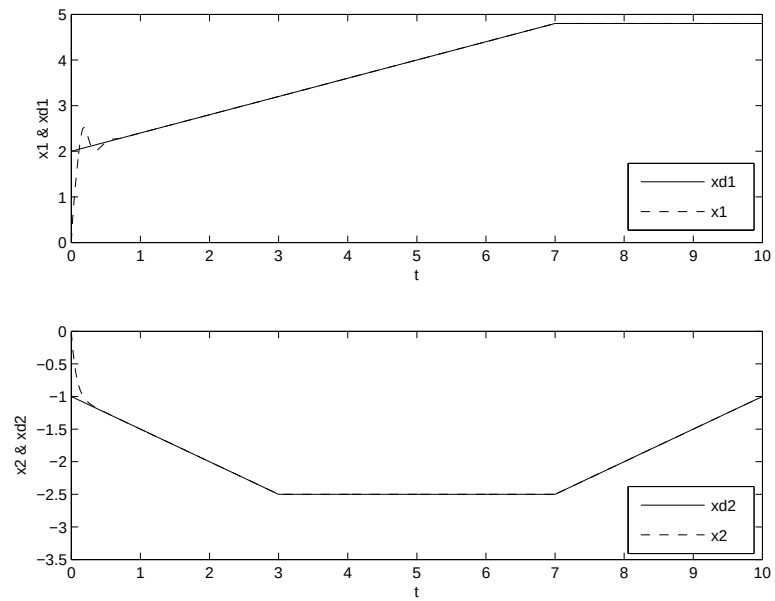


(a) Tracking performance of states

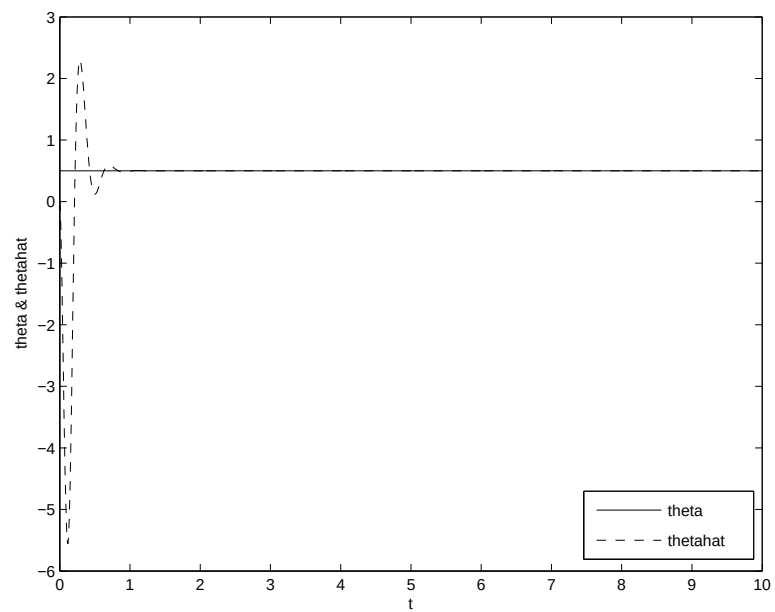


(b) Trajectory of unknown parameter

Figure 5.1: Trajectories of states and adaptive parameter w.r.t their reference values with an initial adaptive controller.



(a) Tracking performance of states



(b) Trajectory of unknown parameter

Figure 5.2: Trajectories of states and adaptive parameter w.r.t their reference values using iterative algorithm.

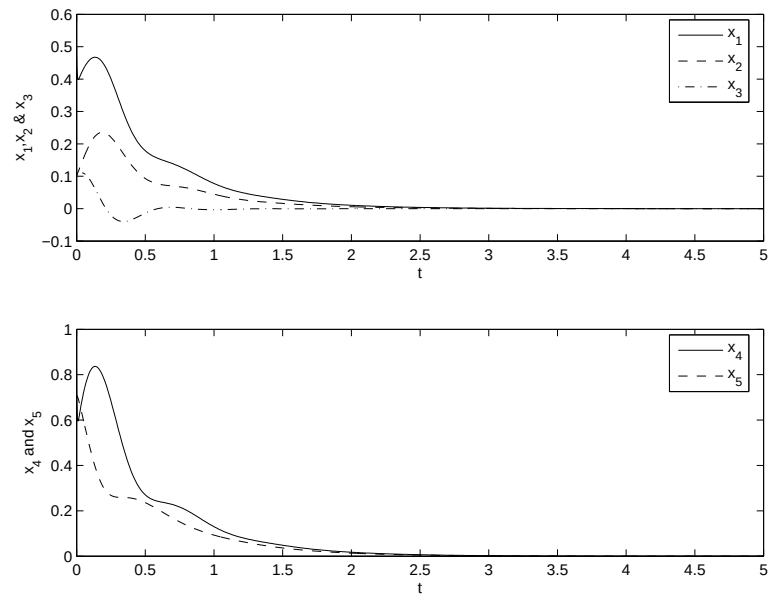
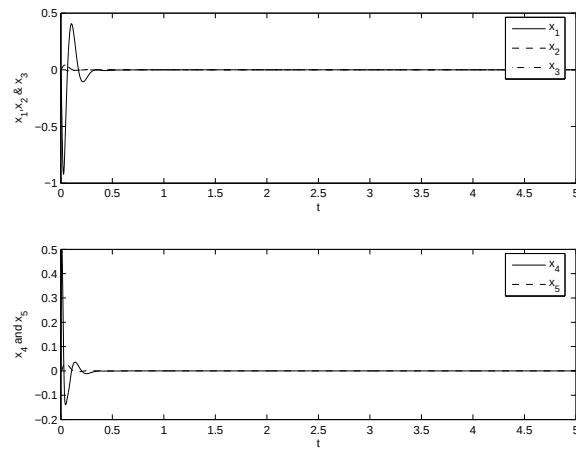
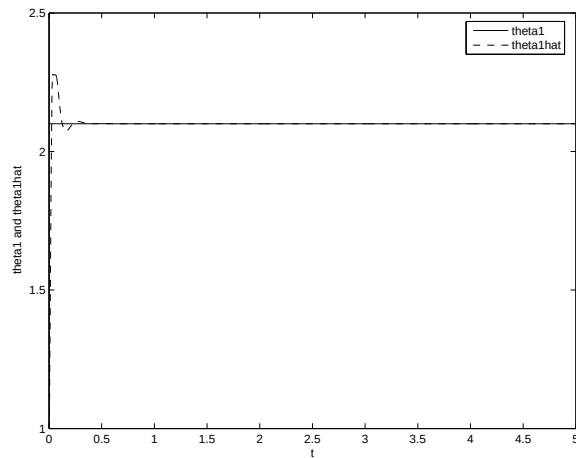
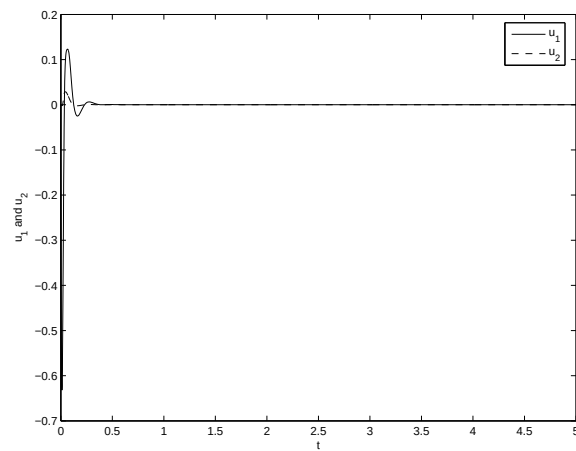


Figure 5.3: States in the transformed system using backstepping design.

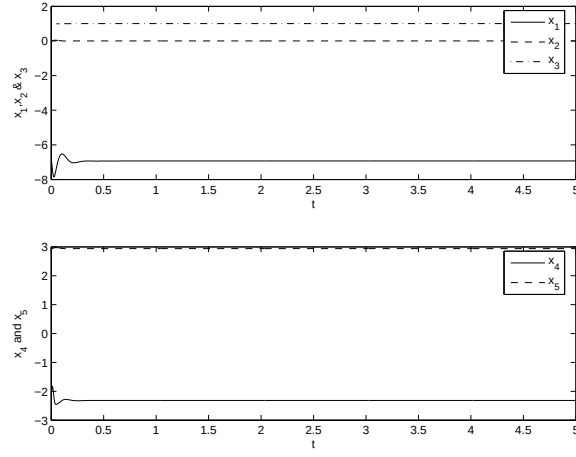


(a) States

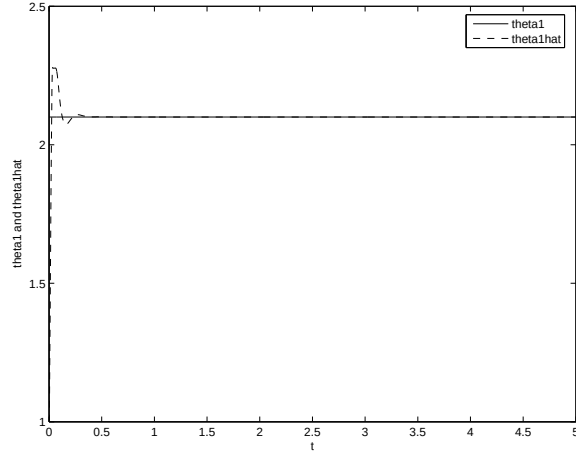
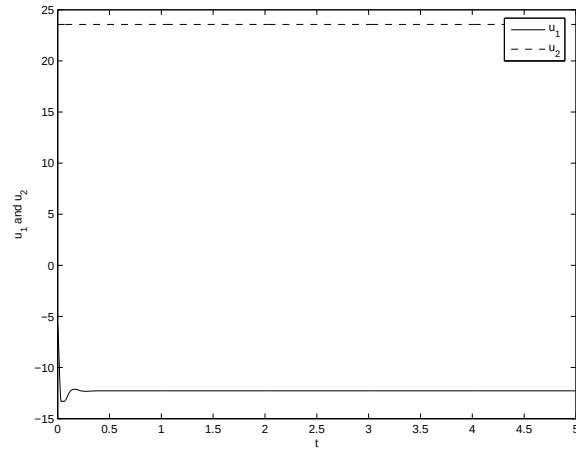
(b) Trajectory of unknown parameter θ_1 

(c) Control force

Figure 5.4: Simulation result in the transformed system.



(a) States

(b) Trajectory of unknown parameter θ_1 

(c) Control force

Figure 5.5: Simulation result in the original system.

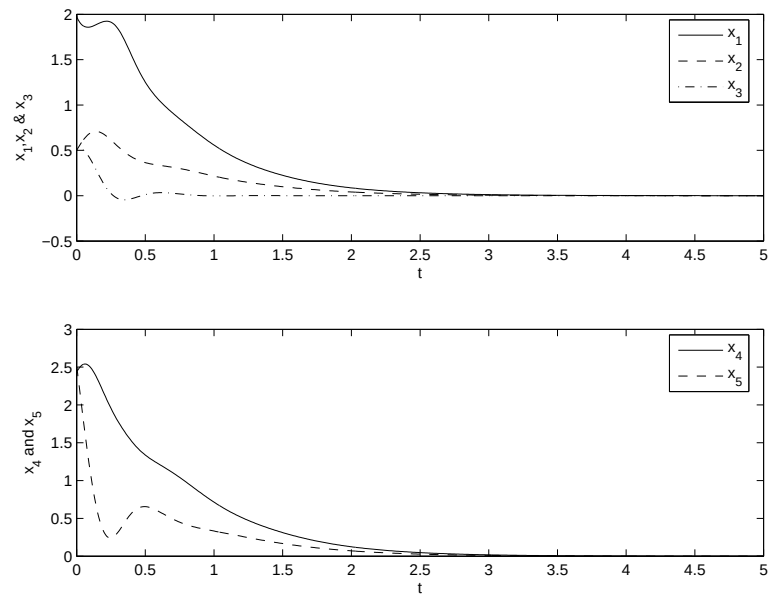
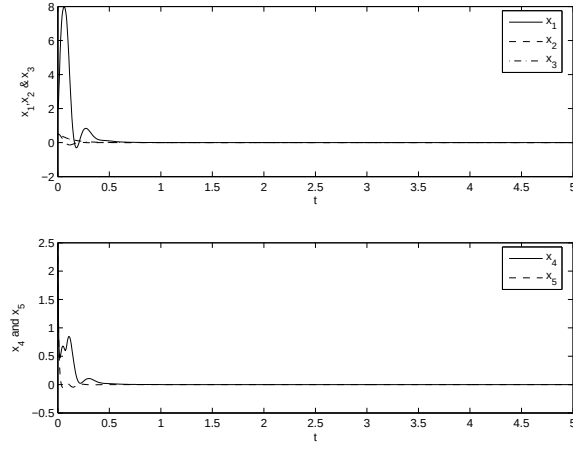
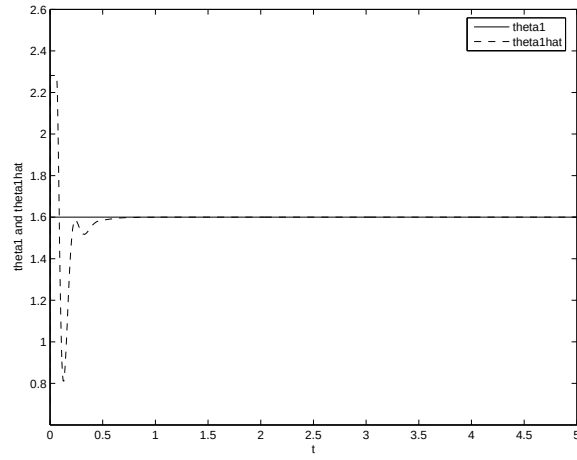


Figure 5.6: Simulation result using backstepping design.



(a) States

(b) Trajectory of unknown parameter θ_1 Figure 5.7: Simulation result using adaptive $\mathcal{H}_\infty$ control.

Chapter 6

Conclusion

6.1 Dissertation Conclusion and Contribution

In this dissertation, we investigated the stabilization, optimal and adaptive control designs of polynomial nonlinear systems. From both the theoretical and the computational aspects, we fully explored the newly developed nonlinear control design technique - Sum of Squares (SOS)-based nonlinear control design techniques. The main contribution of this research is highlighted as following.

- Firstly, we achieved the stability and performance optimization simultaneously through SOS programming and provided a coherent methodology of synthesizing Lyapunov functions and control laws for polynomial nonlinear systems [91, 95, 96, 92].
- In nonlinear $\mathcal{H}_\infty$ control design, we relaxed the orthonormal and decoupling assumptions on the plant structure and proposed a computationally efficient approach based on SOS programming to solve the generalized HJI inequalities [93, 94, 96].
- The $\mathcal{H}_\infty$ control design approach was successfully applied to an underactuated spacecraft precision control problem [93, 94]. Accurate control of axi-symmetric spacecraft has practical importance as it is required during deployment and station-keeping of spacecraft in orbit. The proposed output feedback designs were advantageous by enhancing control design flexibility and reducing on-board sensor hardware. Since the approach we developed is a general framework for the polynomial nonlinear systems,

its potential application is broad and can be applied to aircraft and robotics control problems.

- To overcome the limitations in existing adaptive control literature in terms of the robustness and restriction of the Lyapunov function form, we developed the adaptive $\mathcal{H}_\infty$ control strategy to maintain system robustness with respect to the parameter estimation error and iteratively generalized the Lyapunov function from simple quadratic to higher order [97]. The system was eventually guaranteed to achieve accurate estimation of the unknown parameters.
- Two types of non-quadratic Lyapunov functions were covered in the dissertation [91, 93, 94, 95, 96, 97]: higher order Lyapunov function and rational Lyapunov function. The parametric representation of Lyapunov functions in non-quadratic form were shown as providing an effective way to convert the original analysis conditions into linear matrix inequalities (LMI) with polynomial entries. And the testing of non-negativity of generalized Gram matrix from the resulting LMI was solvable using SOS programming techniques with polynomial-time complexity.
- To resolve the limited efficiency and solution tracking capability in current version of SOS software, we dedicated to develop some interface functions in conjunction with SOSTOOL and Matlab to improve their capabilities. This helped us identify the feasibility of SOS problem, track interim results and propose necessary monomial components for the success of SOS decomposition. These functions has been properly conjuncted with current SOS toolbox to form a more powerful software package, which is applicable for synthesizing general class of nonlinear control problems with polynomial vector field, especially those with large dimensions or higher-order nonlinearities.

6.2 Future Work

Besides the results presented in this dissertation, there are some remaining issues as well as promising directions that worth to be further studied in the future.

- In some practical problems, it is hard or even impossible to find feasible solutions by SOS programming. In these situations, we try to solve their sufficient conditions

by adding some state region constraints. This will help enrich the polynomial representation and improve the solvability of the synthesis conditions. Consequently, the composition of the state region constraints will greatly influence the solutions of the synthesis conditions. In this research, this has been done by checking the interim results, finding the necessary monomial form for the success of the decomposition, and manually adding state region constraints to fill up the necessary ones. For large dimensional systems or systems with higher-order nonlinearities, it will be a time consuming and complicated process. Therefore, it is desirable to continuously explore SOS toolbox and implement this mechanism automatically. Ideally, SOSTOOL would be powerful enough to identify potential monomial items, suggest the possible state region constraints, pick up the most suitable ones while keeping balance between the computational complexity and the solution optimality.

- In Chapter 4, we developed the output feedback $\mathcal{H}_\infty$ synthesis conditions for the generalized class of polynomial nonlinear systems without the orthonormal and decoupling constraints [93, 94, 96]. However, note that the $D_{11}(x)$ matrix is assumed to be zero. Therefore, the most complete and generalized results for the output feedback $\mathcal{H}_\infty$ control of polynomial nonlinear systems should be further explored in the future. Basically, none zero D_{11} term will result in HJI conditions with rational form. Equipped with current results of converting rational formed polynomial functions to equivalent polynomial functions. Both of the state and output feedback conditions are expected to be reformulated as semi-definite optimization conditions and locally solved by SOS programming. Since this will be a significant relaxations on the plant structure compared with current results in the literature, it will impact on the nonlinear $\mathcal{H}_\infty$ control design of a wider class of nonlinear systems.
- In Section 5.2, we developed the adaptive $\mathcal{H}_\infty$ control scheme based on the Lyapunov function $W(x, \tilde{\theta}) = V(x) + \alpha \tilde{\theta}^T \tilde{\theta}$, where $V(x)$ is independent of $\hat{\theta}$ [97]. It is possible to extend $V(x)$ to the form of $V(x, \hat{\theta})$ by including a new term $\frac{\partial V}{\partial \hat{\theta}} \dot{\hat{\theta}}$ into the derivation. Nevertheless, this term will lead to an extra part of the controller which is governed by the so called matching condition. This extension will apparently result in better performance level since $V(x)$ is a special case of $V(x, \hat{\theta})$.
- In Chapter 5, we consider a class of disturbances which are internally produced by the

parameter estimation error. In the future, these disturbances could be extended to a more general form by counting on both of the internal and external disturbances [47]. Also, considering that a specific parametric model does not exactly describe a physical system due to the neglected dynamics which are not captured by the uncertain real parameter. It is worthwhile to further involve norm bounded uncertainties into the model and come up with adaptive robust control strategies which combine both robust control and adaptive control schemes.

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