

## ABSTRACT

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Let  $X_n$  be  $n \times N$  with i.i.d. complex entries having unit variance (sum of variances of real and imaginary parts equals 1),  $\sigma > 0$  constant, and  $R_n$  an  $n \times N$  random matrix independent of  $X_n$ . Assume, almost surely, as  $n \rightarrow \infty$ , the empirical distribution function (e.d.f.) of the eigenvalues of  $\frac{1}{N}R_nR_n^*$  converges in distribution to a nonrandom probability distribution function (p.d.f.), and the ratio  $\frac{n}{N}$  tends to a positive number. Then it is shown that, almost surely, the e.d.f. of the eigenvalues of  $\frac{1}{N}(R_n + \sigma X_n)(R_n + \sigma X_n)^*$  converges in distribution to a nonrandom p.d.f. being characterized in terms of its Stieltjes transform, which satisfies a certain equation. It is also shown that, away from zero, the limiting distribution possesses a continuous density. The density is analytic where it is positive and, for the most relevant cases of  $a$  in the boundary of its support, exhibits behavior closely resembling that of  $\sqrt{|x - a|}$  for  $x$  near  $a$ . A procedure to determine its support is also analyzed.

# Existence and Analysis of the Limiting Spectral Distribution of Large Dimensional Information-Plus-Noise Type Matrices

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## BIOGRAPHY

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# 1. INTRODUCTION

For any square matrix  $A$  with only real eigenvalues, let  $F^A$  denote the empirical distribution function (e.d.f.) of the eigenvalues of  $A$  (that is,  $F^A(x)$  is the proportion of eigenvalues of  $A \leq x$ ). The focus of this paper is on the limiting e.d.f. of the eigenvalues of matrices of the form  $C_n = \frac{1}{N}(R_n + \sigma X_n)(R_n + \sigma X_n)^*$  where  $X_n$  is  $n \times N$  containing i.i.d. complex entries with unit variance (sum of variances of real and imaginary parts equals 1),  $\sigma > 0$  is constant,  $R_n$  is an  $n \times N$  random matrix independent of  $X_n$ , and  $n$  and  $N$  both converge to infinity but their ratio  $\frac{n}{N}$  converges to a positive quantity  $c$ , and  $F^{\frac{1}{N}R_n R_n^*}$  converges, almost surely, in distribution to a nonrandom probability distribution function (p.d.f.)  $H$ . The aim of this paper is twofold. First, we will show that, almost surely,  $F^{C_n}$  converges in distribution to a nonrandom p.d.f.  $F$ , and second, we will derive analytic properties of  $F$ .

The matrix  $C_n$  can be thought of as the sample correlation matrix of  $N$  samples of the form  $R_{\cdot i} + \sigma X_{\cdot i}$ , where the  $n \times 1$  vectors  $R_{\cdot i}$  are stationary ergodic with correlation matrix  $S_n \equiv ER_{\cdot 1} R_{\cdot 1}^*$  and the  $X_{\cdot i}$ 's represent components of additive noise (variance  $\sigma^2$  unknown) that corrupt the  $R_{\cdot i}$ 's. If the noise is centered ( $EX_{11} = 0$ ), and  $N$  is sufficiently large, then  $C_n$  provides a reasonable estimate of  $S_n + \sigma^2 I$  ( $I$  denoting the identity matrix), which would reveal  $S_n$ , if  $S_n$  were known to be singular. However, if  $n$  is large, then the number of samples needed to provide an adequate approximation of  $S_n + \sigma^2 I$  is unattainable. The assumption  $\frac{n}{N} \rightarrow c > 0$  models the situation of sample size and vector dimension being on the same order of magnitude.

An area in which our results have significance is that of the detection problem in array signal processing, that is, the problem of observing data collected at  $n$  sensors

which receive signals transmitted from an unknown number of sources in a noise-filled environment, and using this data to determine the number of sources. The importance of such results to array signal processing is discussed in Silverstein and Combettes [11], however, in a less general setting. In that paper certain internal independence assumptions are imposed upon the signal matrix  $R_n$ , specifically, independence across samples is assumed. In this paper we require only that, almost surely, the e.d.f. of the eigenvalues of  $\frac{1}{N}R_nR_n^*$  converges in distribution to some nonrandom p.d.f.  $H$ , thus allowing the detection problem to be studied under more general settings. Further details on the detection problem are presented later in chapter 8 along with a discussion of the applicability of results in the theory of large dimensional random matrices.

The main tool we use is the Stieltjes transform. For any p.d.f.  $G$ , the Stieltjes transform of  $G$  is defined as the analytic function

$$m_G(z) = \int \frac{1}{\lambda - z} dG(\lambda), \quad z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : \text{Im } z > 0\},$$

and  $G$  can be retrieved by the inversion formula

$$G\{[a, b]\} = \frac{1}{\pi} \lim_{\eta \rightarrow 0^+} \int_a^b \text{Im } m_G(\xi + i\eta) d\xi,$$

where  $a, b$  are continuity points of  $G$ . Due to the inversion formula, convergence of a tight sequence of p.d.f.'s is guaranteed by showing convergence of the corresponding Stieltjes transforms on a countable subset of  $\mathbb{C}^+$  possessing at least one accumulation point in  $\mathbb{C}^+$ .

A property of Stieltjes transforms that will be needed later is that if  $G$  is any p.d.f. with nonnegative support, then

$$\text{Im } z m_G(z) = \int \frac{\lambda z_2}{|\lambda - z|^2} dG(\lambda) \geq 0, \tag{S.1}$$

for any  $z = z_1 + iz_2 \in \mathbb{C}^+$ .

For  $p \times p$  matrix  $A$  with real eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_p$  the Stieltjes transform of  $F^A$ ,

$$m_{F^A}(z) = \frac{1}{p} \sum_{i=1}^p \frac{1}{\lambda_i - z} = \frac{1}{p} \text{tr}(A - zI)^{-1},$$

involves the resolvent of  $A$  and is well-suited for our analysis ( $\text{tr}$  denoting trace).

As mentioned earlier, we will provide results on both the existence and analysis of the limiting e.d.f. of  $C_n$ . We deal with the existence issue by proving the following theorem.

**THEOREM 1.1.** Assume on a common probability space:

(a) For  $n=1,2,\dots$ ,  $X_n = (X_{ij}^n)$ ,  $n \times N$ ,  $X_{ij}^n \in \mathbb{C}^+$ , i.d. for all  $n, i, j$ , independent across  $i, j$  for each  $n$ ,  $E|X_{11}^1 - EX_{11}^1|^2 = 1$

(b)  $R_n$  is  $n \times N$  independent of  $X_n$  with  $F^{\frac{1}{N}R_n R_n^*} \xrightarrow{\mathcal{D}} H$ , a.s., ( $\mathcal{D}$  denoting weak convergence) where  $H$  is a nonrandom p.d.f.

(c)  $N = N(n)$  and  $c_n = \frac{n}{N} \rightarrow c > 0$  as  $n \rightarrow \infty$

(d)  $C_n = \frac{1}{N}(R_n + \sigma X_n)(R_n + \sigma X_n)^*$  where  $\sigma > 0$ .

Then  $F^{C_n} \xrightarrow{\mathcal{D}} F$ , a.s., where  $F$  is a nonrandom p.d.f. whose Stieltjes transform  $m = m(z)$  satisfies

$$m = \int \frac{dH(t)}{\frac{t}{1+\sigma^2 cm} - (1 + \sigma^2 cm)z + \sigma^2(1 - c)} \quad (1.1)$$

for any  $z \in \mathbb{C}^+$ .

From (S.1) we have for any  $t \geq 0$  and any  $z \in \mathbb{C}^+$

$$\mathcal{I}m \left( \frac{t}{1 + \sigma^2 cm} - (1 + \sigma^2 cm)z + \sigma^2(1 - c) \right) \leq -\mathcal{I}m z < 0.$$

Therefore for any  $z \in \mathbb{C}^+$  the integral in (1.1) is well-defined.

The proof of Theorem 1.1 is covered in chapters 2, 3, and 4. Chapter 2 mirrors Silverstein and Bai [9] in that justification is presented for restricting the assumptions on the matrices  $R_n$  and  $X_n$ . Chapter 3 contains the bulk of the proof, and chapter 4 is devoted to showing that solutions,  $m$ , to equation (1.1) (with  $z \in \mathbb{C}^+$ ) are unique if  $\mathcal{I}m m > 0$  and  $\mathcal{I}m (1 + \sigma^2 cm)z > 0$  (specifically, if  $m$  is the Stieltjes transform of a

p.d.f. with nonnegative support). In chapters 5, 6, 7, and 8 we analyze the limiting spectral distribution  $F$ . Namely, in chapter 5 we show that  $F$  has a density away from zero which is analytic where it is positive. Chapter 6 provides a procedure for determining the support of  $F$ , and chapter 7 contains an analysis of the behavior of the density near certain points on the boundary of its support. In particular, it is shown that near these boundary points the density is similar to a square root function. Finally, in chapter 8 we provide an example with specific choices for  $H$ ,  $c$ , and  $\sigma$  and a discussion of the detection problem in array signal processing. For the example given, the graph of the density is shown along with a histogram and scatterplot of eigenvalues resulting from a simulation of the matrix  $C_n$ . The Appendix contains the proof of a lemma from chapter 2.

Let  $\mathbf{C}_n = \frac{1}{N}(R_n + \sigma X_n)^*(R_n + \sigma X_n)$ . The spectra of  $C_n$  and  $\mathbf{C}_n$  differ by  $|n - N|$  zero eigenvalues and are related by

$$F^{\mathbf{C}_n} = \left(1 - \frac{n}{N}\right) \mathbf{1}_{[0, \infty)} + \frac{n}{N} F^{C_n} \quad (1.2)$$

( $\mathbf{1}_B$  denoting the indicator function over the set  $B$ ). Because of this, information on the limit of  $F^{\mathbf{C}_n}$  can be inferred from knowledge of  $F$ .

Notice that the eigenvalues of  $C_n$  are directly related to those of the  $N \times n$  matrix  $\frac{1}{n}(R_n^* + \sigma X_n^*)(R_n + \sigma X_n) = \frac{N}{n} \mathbf{C}_n$ . With this fact it is straightforward to show that if  $m$  satisfies (1.1) when  $c \leq 1$ , then  $m$  will also satisfy (1.1) when  $c > 1$ . We therefore assume, without loss of generality, that  $0 < c \leq 1$ .

Let  $m_n(z) = m_{FC_n}(z)$ . Defining  $\mathbf{m}_n(z) = m_{F\mathbf{C}_n}(z)$  we get from (1.2)

$$\mathbf{m}_n = -\frac{1 - c_n}{z} + c_n m_n, \quad (1.3)$$

which will be used later for notational convenience.

## 2. TRUNCATION AND CENTRALIZATION

The first step in proving Theorem 1.1 is similar to that of Silverstein and Bai [9], in that, we truncate and centralize twice with regard to  $X_n$ , and, as in Silverstein [8], we truncate  $R_n$ . The reason for these truncations and centralizations is to justify our later replacing the matrices  $X_n$  and  $R_n$  with ones more suitable for analysis. We compare the e.d.f.'s of these matrices by the following metric presented in Silverstein and Bai [9]. Let  $\{f_i\}$  be an enumeration of all continuous functions that take a constant  $\frac{1}{m}$  value ( $m$  a positive integer) on  $[a, b]$ , where  $a, b$  are rational, 0 on  $(-\infty, a - \frac{1}{m}] \cup [b + \frac{1}{m}, \infty)$ , and linear on each of  $[a - \frac{1}{m}, a]$ ,  $[b, b + \frac{1}{m}]$ . For probability measures  $F, G$  on  $\mathbb{R}$  the metric

$$D(F, G) \equiv \sum_{i=1}^{\infty} \left| \int f_i dF - \int f_i dG \right| 2^{-i}$$

induces the topology of weak convergence, and, as noted in Silverstein and Bai [9], for sequences  $\{F_n\}, \{G_n\}$  of probability measures on  $\mathbb{R}$ , we have

$$\lim_{n \rightarrow \infty} \|F_n - G_n\| = 0 \Rightarrow \lim_{n \rightarrow \infty} D(F_n, G_n) = 0 \quad (2.1)$$

where  $\|\cdot\|$  denotes the sup-norm on bounded functions from  $\mathbb{R}$  to  $\mathbb{R}$ .

Note that for  $x, y \in \mathbb{R}$ ,  $|f_i(x) - f_i(y)| \leq |x - y|$ . Then, restating from Silverstein and Bai [9], we have for e.d.f.'s  $F, G$  on the (respective) sets  $\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\}$ ,

$$D^2(F, G) \leq \left( \frac{1}{n} \sum_{j=1}^n |x_j - y_j| \right)^2 \leq \frac{1}{n} \sum_{j=1}^n (x_j - y_j)^2. \quad (2.2)$$

Before continuing, some needed results are presented.

For  $q \in \mathbb{C}^n$  and  $n \times N$  matrix  $A$ ,  $\|q\|$  will denote the Euclidean norm, and  $\|A\|$  the induced spectral norm on matrices, that is, the largest singular value of  $A$ . We

also use the notation  $F_{sing}^A$  to denote the e.d.f. of the square root of the eigenvalues of  $AA^*$ , which are the  $n$  largest singular values of  $A$ . The constants, denoted by  $K$ , appearing henceforth in some of the expressions are nonrandom and may take on different values from one appearance to the next.

LEMMA 2.1 [Corollary 7.3.8 of Horn and Johnson [3]]. For  $r \times s$  matrices  $A$  and  $B$  with respective singular values  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_q$ ,  $\tau_1 \geq \tau_2 \geq \dots \geq \tau_q$ , where  $q = \min\{r, s\}$ , we have

$$\left( \sum_{i=1}^q (\sigma_i - \tau_i)^2 \right)^{\frac{1}{2}} \leq \|A - B\|_2,$$

where  $\|\cdot\|_2$  is the Frobenius matrix norm.

LEMMA 2.2 [Lemma 2.5 of Silverstein and Bai [9]]. For  $n \times N$  matrices  $Q, \bar{Q}$ ,

$$\|F^{QQ^*} - F^{\bar{Q}\bar{Q}^*}\| \leq \frac{2}{n} \text{rank}(Q - \bar{Q}).$$

The following are well-known properties of matrices.

MATRIX PROPERTIES.

(MP1) For  $n \times n$   $A, B$ ,

$$|\text{tr}AB| \leq (\text{tr}AA^* \text{tr}BB^*)^{\frac{1}{2}} \leq n\|A\|\|B\|.$$

(MP2) For rectangular  $A$ ,  $\text{rank}(A) \leq$  the number of nonzero entries of  $A$ .

*Proof of Theorem 1.1.* Following Silverstein and Bai [9] we use the convention of occasionally suppressing the variables' dependence on  $n$ . All convergence statements are as  $n \rightarrow \infty$ . Let  $\hat{X}_{ij} = X_{ij} \mathbf{1}_{(|X_{ij}| < \sqrt{n})}$  and  $\hat{C}_n = \left( \frac{1}{\sqrt{N}}R + \sigma \hat{X} \right) \left( \frac{1}{\sqrt{N}}R + \sigma \hat{X} \right)^*$ , where  $\hat{X} = \left( \frac{1}{\sqrt{N}} \hat{X}_{ij} \right)$ . It is shown in the Appendix that

$$\|F^{C_n} - F^{\hat{C}_n}\| \xrightarrow{a.s.} 0. \tag{2.3}$$

Let  $\tilde{C}_n = \left(\frac{1}{\sqrt{N}}R + \sigma\tilde{X}\right) \left(\frac{1}{\sqrt{N}}R + \sigma\tilde{X}\right)^*$ , where  $\tilde{X} = \left(\frac{1}{\sqrt{N}}\tilde{X}_{ij}\right) = \left(\frac{1}{\sqrt{N}}\hat{X}_{ij} - E\frac{1}{\sqrt{N}}\hat{X}_{ij}\right)$ . Since  $\text{rank}(E\hat{X}) \leq 1$ , we have from Lemma 2.2

$$\|F^{\tilde{C}_n} - F^{\hat{C}_n}\| \rightarrow 0. \quad (2.4)$$

Write  $\frac{1}{\sqrt{N}}R$  in its singular value decomposition  $\frac{1}{\sqrt{N}}R = U\Lambda V$ . Let  $R_\alpha = U\Lambda_\alpha V$  where  $\Lambda_\alpha$  is the matrix  $\Lambda$  with each singular value  $s$  replaced by  $s\mathbf{1}_{(s \leq \alpha)}$ , for  $\alpha > 0$ .

Let  $Q$  be any  $n \times N$  matrix. If  $\alpha^2$  is a continuity point of  $H$ , we have by Lemma 2.2 and assumptions (b), (c)

$$\begin{aligned} \|F^{(\frac{1}{\sqrt{N}}R+Q)(\frac{1}{\sqrt{N}}R+Q)^*} - F^{(R_\alpha+Q)(R_\alpha+Q)^*}\| &\leq \frac{2}{n} \text{rank}\left(\frac{1}{\sqrt{N}}R - R_\alpha\right) \\ &= \frac{2}{n} \sum_{i=1}^n \mathbf{1}_{(s_i > \alpha)} = \frac{2}{n} \sum_{i=1}^n \mathbf{1}_{(\lambda_i > \alpha^2)} \xrightarrow{a.s.} 2H\{(\alpha^2, \infty)\}, \end{aligned}$$

where the  $s_i$ 's are the  $n$  largest singular values of  $\frac{1}{\sqrt{N}}R$  and the  $\lambda_i$ 's are the eigenvalues of  $\frac{1}{N}RR^*$ , i.e.,  $\lambda_i = s_i^2$ . Let  $\alpha \equiv \alpha_n = \ln(n)$ . It follows that as  $n \rightarrow \infty$

$$\|F^{(\frac{1}{\sqrt{N}}R+Q)(\frac{1}{\sqrt{N}}R+Q)^*} - F^{(R_\alpha+Q)(R_\alpha+Q)^*}\| \xrightarrow{a.s.} 0. \quad (2.5)$$

Let  $\bar{X}_{ij} = \tilde{X}_{ij}\mathbf{1}_{(|X_{ij}| < \ln(n))} - E\tilde{X}_{ij}\mathbf{1}_{(|X_{ij}| < \ln(n))}$ ,  $\bar{X} = \left(\frac{1}{\sqrt{N}}\bar{X}_{ij}\right)$ ,  $\overline{\bar{X}}_{ij} = \tilde{X}_{ij} - \bar{X}_{ij}$ , and  $\overline{\bar{X}} = \left(\frac{1}{\sqrt{N}}\overline{\bar{X}}_{ij}\right)$ . Let  $\tilde{s}_1 \geq \tilde{s}_2 \geq \dots \geq \tilde{s}_n$  and  $\bar{s}_1 \geq \bar{s}_2 \geq \dots \geq \bar{s}_n$  be the  $n$  largest singular values of  $R_\alpha + \sigma\tilde{X}$  and  $R_\alpha + \sigma\bar{X}$ , respectively. Then using (2.2), (MP1), and Lemma 2.1 we get

$$\begin{aligned} D^2\left(F_{\text{sing}}^{R_\alpha + \sigma\tilde{X}}, F_{\text{sing}}^{R_\alpha + \sigma\bar{X}}\right) &\leq \frac{1}{n} \sum_{j=1}^n (\tilde{s}_j - \bar{s}_j)^2 \leq \frac{1}{n} \sigma^2 \|\overline{\bar{X}}\|_2^2 \\ &= \frac{1}{n} \sigma^2 \text{tr} \overline{\bar{X}}^* \overline{\bar{X}} \leq \sigma^2 \left(\frac{1}{n} \text{tr} \left(\overline{\bar{X}}^* \overline{\bar{X}}\right)^2\right)^{\frac{1}{2}} \xrightarrow{a.s.} 0 \end{aligned}$$

by (3.6) of Silverstein and Bai [9]. It follows that

$$D\left(F^{(R_\alpha + \sigma\tilde{X})(R_\alpha + \sigma\tilde{X})^*}, F^{(R_\alpha + \sigma\bar{X})(R_\alpha + \sigma\bar{X})^*}\right) \xrightarrow{a.s.} 0. \quad (2.6)$$

Therefore, by (2.1), (2.3), (2.4), (2.5), and (2.6), in order to show that  $F^{C_n} \xrightarrow{\mathcal{D}} F$ , it is sufficient to show that for any  $z \in \mathbb{C}^+$ ,

$$m_{F^{(R_\alpha + \sigma \bar{X})} (R_\alpha + \sigma \bar{X})^*}(z) \xrightarrow{a.s.} m_F(z).$$

We may therefore add to the conditions of Theorem 1.1 the following:

- (1)  $|X_{11}| \leq \ln(n)$ ,
- (2)  $EX_{11} = 0, E|X_{11}|^2 = 1$ ,
- (3)  $\|\frac{1}{N}RR^*\| \leq \ln(n)$ .

### 3. EXISTENCE OF THE LIMITING DISTRIBUTION

Fix  $z = z_1 + iz_2 \in \mathbb{C}^+$ . The next four results are used to complete the proof of Theorem 1.1.

LEMMA 3.1. For  $n \times n$   $A$  and  $n \times 1$  vectors  $q, v$  where  $A$  and  $A + vv^*$  are invertible, we have

$$q^* (A + vv^*)^{-1} = q^* A^{-1} - \frac{q^* A^{-1} v}{1 + v^* A^{-1} v} v^* A^{-1}.$$

Notice if  $q = v$  then

$$v^* (A + vv^*)^{-1} = \frac{1}{1 + v^* A^{-1} v} v^* A^{-1}.$$

*Proof.* Let  $q^* (A + vv^*)^{-1} \equiv r^*$  so that  $q^* = r^* A + r^* vv^*$ . Multiplying by  $A^{-1}$  on the right we get

$$q^* A^{-1} = r^* + r^* vv^* A^{-1}, \tag{3.1.1}$$

and then multiplying by  $v$  on the right we get

$$q^* A^{-1} v = r^* v + r^* vv^* A^{-1} v = r^* v (1 + v^* A^{-1} v).$$

Since  $q$  is arbitrary we must have  $1 + v^* A^{-1} v \neq 0$ . Then

$$r^* v = \frac{q^* A^{-1} v}{1 + v^* A^{-1} v},$$

and hence by (3.1.1) we have

$$r^* = q^* A^{-1} - \frac{q^* A^{-1} v}{1 + v^* A^{-1} v} v^* A^{-1},$$

and the proof is complete.

LEMMA 3.2 [Lemma 3.1 of Silverstein and Bai [9]]. Let  $C = (c_{ij})$ ,  $c_{ij} \in \mathbb{C}$ , be an  $n \times n$  matrix with  $\|C\| \leq 1$ , and  $Y = (X_1, \dots, X_n)^T$ ,  $X_i \in \mathbb{C}$ , where the  $X_i$ 's are i.i.d. satisfying conditions (1) and (2). Then

$$E|Y^*CY - \text{tr}C|^6 \leq Kn^3(\ln(n))^{12},$$

where the constant  $K$  does not depend on  $n$ ,  $C$ , or on the distribution of  $X_1$ .

LEMMA 3.3 [Lemma 2.6 of Silverstein and Bai [9]]. Let  $z = z_1 + iz_2 \in \mathbb{C}^+$  with  $A$  and  $B$   $n \times n$ ,  $B$  Hermitian and  $r \in \mathbb{C}^n$ . Then

$$\begin{aligned} & \left| \text{tr} \left( (B - zI)^{-1} - (B + rr^* - zI)^{-1} \right) A \right| \\ &= \left| \frac{r^*(B - zI)^{-1}A(B - zI)^{-1}r}{1 + r^*(B - zI)^{-1}r} \right| \leq \frac{\|A\|}{z_2}. \end{aligned}$$

LEMMA 3.4 [Lemma 2.3 of Silverstein and Bai [9]]. Let  $x, y$  be nonnegative numbers. For rectangular matrices  $A, B$  of the same size,

$$F_{sing}^{A+B} \{(x + y, \infty)\} \leq F_{sing}^A \{(x, \infty)\} + F_{sing}^B \{(y, \infty)\}.$$

Using Lemma 3.2 we get  $\frac{1}{n} \text{tr} \frac{1}{N} XX^* \xrightarrow{a.s.} 1$  which yields the almost sure tightness of  $\{F^{\frac{1}{N}XX^*}\}$ . This together with Lemma 3.4 and assumption (b) gives us  $\{F^{C_n}\}$  being almost surely tight, and therefore the quantity

$$\delta \equiv \inf_n \mathcal{I}m(m_{F^{C_n}}(z)) \geq \inf_n \int \frac{z_2 dF^{C_n}(\lambda)}{2(\lambda^2 + z_1^2) + z_2^2}$$

is positive almost surely.

For  $j = 1, 2, \dots, N$  let  $x_j (= x_j^n)$  and  $r_j (= r_j^n)$  denote the  $j$ th column of  $X$  and  $R$  respectively and define  $y_j = \frac{1}{\sqrt{N}}(r_j + \sigma x_j)$  so that  $C_n = \sum_{j=1}^N y_j y_j^*$ .

Note that  $RR^* = \sum_{j=1}^N r_j r_j^*$ , and since for each  $j = 1, 2, \dots, N$  the matrix  $RR^* - r_j r_j^* = \sum_{i \neq j} r_i r_i^*$  is positive semidefinite, then using condition (3) from the previous chapter we get

$$\|r_j\|^2 = \|r_j r_j^*\| \leq \|RR^*\| \leq N \ln(n). \quad (3.1)$$

Define  $D = C_n - zI$ ,  $B = A_n - zI$ , where

$$A_n \equiv \left( \frac{1}{1 + \sigma^2 c_n m_n} \right) \frac{1}{N} RR^* - \sigma^2 z \mathbf{m}_n I,$$

and for  $j = 1, 2, \dots, N$  let  $C_{(j)} = C_n - y_j y_j^*$  and  $D_j = D - y_j y_j^* (= C_{(j)} - zI)$ . Write

$$D + zI = \sum_{j=1}^N y_j y_j^*.$$

Multiplying by  $D^{-1}$  on the right on both sides and using Lemma 3.1 we get

$$I + zD^{-1} = \sum_{j=1}^N \frac{1}{1 + y_j^* D_j^{-1} y_j} y_j y_j^* D_j^{-1}.$$

Taking the trace on both sides and dividing by  $N$  we have

$$c_n + z c_n m_n = \frac{1}{N} \sum_{j=1}^N \frac{y_j^* D_j^{-1} y_j}{1 + y_j^* D_j^{-1} y_j} = 1 - \frac{1}{N} \sum_{j=1}^N \frac{1}{1 + y_j^* D_j^{-1} y_j}.$$

From our definition (1.3) of  $\mathbf{m}_n$ , we see that

$$\mathbf{m}_n = -\frac{1}{N} \sum_{j=1}^N \frac{1}{z(1 + y_j^* D_j^{-1} y_j)}. \quad (3.2)$$

Following the steps leading up to (2.3) of Silverstein [8] we get

$$\frac{1}{|z(1 + y_j^* D_j^{-1} y_j)|} \leq \frac{1}{z_2}. \quad (3.3)$$

For  $j = 1, 2, \dots, N$ , we make the following scalar definitions:

$$\begin{aligned} \rho_j &= \frac{1}{N} r_j^* D_j^{-1} r_j, & \omega_j &= \frac{1}{N} \sigma^2 x_j^* D_j^{-1} x_j, \\ \beta_j &= \frac{1}{N} r_j^* D_j^{-1} \sigma x_j, & \gamma_j &= \frac{1}{N} \sigma x_j^* D_j^{-1} r_j, \\ \hat{\rho}_j &= \frac{1}{N} r_j^* D_j^{-1} B^{-1} r_j, & \hat{\omega}_j &= \frac{1}{N} \sigma^2 x_j^* D_j^{-1} B^{-1} x_j, \\ \hat{\beta}_j &= \frac{1}{N} r_j^* D_j^{-1} B^{-1} \sigma x_j, & \hat{\gamma}_j &= \frac{1}{N} \sigma x_j^* D_j^{-1} B^{-1} r_j. \end{aligned}$$

We begin the next stage of the proof by factoring the difference of inverses and expanding the middle factor to get

$$\begin{aligned}
B^{-1} - D^{-1} &= B^{-1}(D - B)D^{-1} = B^{-1}(C_n - A_n)D^{-1} \\
&= B^{-1}\left(\frac{\sigma^2 c_n m_n}{1 + \sigma^2 c_n m_n} \frac{1}{N} RR^* + \frac{1}{N} \sigma X R^* + \frac{1}{N} R \sigma X^* + \frac{1}{N} \sigma^2 X X^* + \sigma^2 z \mathbf{m}_n I\right) D^{-1} \\
&= \sum_{j=1}^N B^{-1} \left[ \frac{\sigma^2 c_n m_n}{1 + \sigma^2 c_n m_n} \frac{1}{N} r_j r_j^* + \frac{1}{N} \sigma x_j r_j^* + \frac{1}{N} r_j \sigma x_j^* + \frac{1}{N} \sigma^2 x_j x_j^* + \frac{1}{N} \sigma^2 z \mathbf{m}_n I \right] D^{-1} \\
&= \sum_{j=1}^N \left[ \frac{\sigma^2 c_n m_n}{1 + \sigma^2 c_n m_n} B^{-1} \frac{1}{N} r_j r_j^* D^{-1} + B^{-1} \frac{1}{N} \sigma x_j r_j^* D^{-1} + B^{-1} \frac{1}{N} r_j \sigma x_j^* D^{-1} \right. \\
&\quad \left. + B^{-1} \frac{1}{N} \sigma^2 x_j x_j^* D^{-1} + \frac{1}{N} \sigma^2 z \mathbf{m}_n B^{-1} D^{-1} \right].
\end{aligned}$$

While using (3.2), we take the trace of both sides and divide by  $n$  to get

$$\begin{aligned}
\frac{1}{n} \text{tr}(A_n - zI)^{-1} - m_n &= \frac{1}{n} \sum_{j=1}^N \left[ \frac{\sigma^2 c_n m_n}{1 + \sigma^2 c_n m_n} \frac{1}{N} r_j^* D^{-1} B^{-1} r_j + \frac{1}{N} r_j^* D^{-1} B^{-1} \sigma x_j \right. \\
&\quad \left. + \frac{1}{N} \sigma x_j^* D^{-1} B^{-1} r_j + \frac{1}{N} \sigma^2 x_j^* D^{-1} B^{-1} x_j \right. \\
&\quad \left. - \frac{1}{1 + \frac{1}{N} (r_j + \sigma x_j)^* D_j^{-1} (r_j + \sigma x_j)} \frac{1}{N} \sigma^2 \text{tr} D^{-1} B^{-1} \right] \\
&\equiv \frac{1}{n} \sum_{j=1}^N \left[ W_1^{n,j} + W_2^{n,j} + W_3^{n,j} + W_4^{n,j} - W_5^{n,j} \right]. \tag{3.4}
\end{aligned}$$

Let  $\alpha^{n,j} = 1 + \frac{1}{N} (r_j + \sigma x_j)^* D_j^{-1} (r_j + \sigma x_j) = 1 + \rho_j + \beta_j + \gamma_j + \omega_j$ .

Since  $D^{-1} = (D_j + \frac{1}{\sqrt{N}}(r_j + \sigma x_j)\frac{1}{\sqrt{N}}(r_j + \sigma x_j)^*)^{-1}$  we can use Lemma 3.1 to get

$$\begin{aligned}
W_1^{n,j} &= \frac{1}{\alpha^{n,j}} \left( \frac{\sigma^2 c_n m_n}{1 + \sigma^2 c_n m_n} \right) \left[ \alpha^{n,j} \frac{1}{N} r_j^* D_j^{-1} B^{-1} r_j \right. \\
&\quad \left. - \frac{1}{N} (r_j^* D_j^{-1} (r_j + \sigma x_j)) \frac{1}{N} (r_j + \sigma x_j)^* D_j^{-1} B^{-1} r_j \right] \\
&= \frac{1}{\alpha^{n,j}} \left( \frac{\sigma^2 c_n m_n}{1 + \sigma^2 c_n m_n} \right) \left[ (1 + \gamma_j + \omega_j) \hat{\rho}_j - (\rho_j + \beta_j) \hat{\gamma}_j \right], \\
W_2^{n,j} &= \frac{1}{\alpha^{n,j}} \left[ \alpha^{n,j} \frac{1}{N} r_j^* D_j^{-1} B^{-1} \sigma x_j \right. \\
&\quad \left. - \frac{1}{N} (r_j^* D_j^{-1} (r_j + \sigma x_j)) \frac{1}{N} (r_j + \sigma x_j)^* D_j^{-1} B^{-1} \sigma x_j \right] \\
&= \frac{1}{\alpha^{n,j}} \left[ (1 + \gamma_j + \omega_j) \hat{\beta}_j - (\rho_j + \beta_j) \hat{\omega}_j \right], \\
W_3^{n,j} &= \frac{1}{\alpha^{n,j}} \left[ \alpha^{n,j} \frac{1}{N} \sigma x_j^* D_j^{-1} B^{-1} r_j \right. \\
&\quad \left. - \frac{1}{N} (\sigma x_j^* D_j^{-1} (r_j + \sigma x_j)) \frac{1}{N} (r_j + \sigma x_j)^* D_j^{-1} B^{-1} r_j \right] \\
&= \frac{1}{\alpha^{n,j}} \left[ (1 + \rho_j + \beta_j) \hat{\gamma}_j - (\gamma_j + \omega_j) \hat{\rho}_j \right], \\
W_4^{n,j} &= \frac{1}{\alpha^{n,j}} \left[ \alpha^{n,j} \frac{1}{N} \sigma x_j^* D_j^{-1} B^{-1} \sigma x_j \right. \\
&\quad \left. - \frac{1}{N} (\sigma x_j^* D_j^{-1} (r_j + \sigma x_j)) \frac{1}{N} (r_j + \sigma x_j)^* D_j^{-1} B^{-1} \sigma x_j \right] \\
&= \frac{1}{\alpha^{n,j}} \left[ (1 + \rho_j + \beta_j) \hat{\omega}_j - (\gamma_j + \omega_j) \hat{\beta}_j \right], \quad \text{and} \\
W_5^{n,j} &= \frac{1}{\alpha^{n,j}} \frac{1}{N} \sigma^2 \text{tr} D^{-1} B^{-1}.
\end{aligned}$$

Therefore, after simplification, we have

$$\begin{aligned}
(3.4) &= \frac{1}{n} \sum_{j=1}^N \frac{1}{\alpha^{n,j}} \left[ \frac{1}{1 + \sigma^2 c_n m_n} (\sigma^2 c_n m_n - \omega_j - \gamma_j) \hat{\rho}_j + \hat{\beta}_j \right. \\
&\quad \left. + \frac{1}{1 + \sigma^2 c_n m_n} (\rho_j + \beta_j + 1 + \sigma^2 c_n m_n) \hat{\gamma}_j + \hat{\omega}_j - \frac{1}{N} \sigma^2 \text{tr} D^{-1} B^{-1} \right]
\end{aligned}$$

$$\equiv \frac{1}{n} \sum_{j=1}^N \frac{1}{\alpha^{n,j}} d^{n,j}.$$

For  $j = 1, 2, \dots, N$  we make the following definitions

$$m_{(j)} \equiv m_{F^{c_{(j)}}}(z), \quad \mathbf{m}_{(j)} \equiv -\frac{1-c_n}{z} + c_n m_{(j)},$$

$$B_j \equiv \left( \frac{1}{1 + \sigma^2 c_n m_{(j)}} \right) \frac{1}{N} R R^* - \sigma^2 z \mathbf{m}_{(j)}.$$

As noted below (2.5) of Silverstein [8],  $\mathbf{m}_{(j)}$  is the Stieltjes transform of a p.d.f. (on  $[0, \infty)$ ).

The following expressions hold for any  $j = 1, 2, \dots, N$  and any  $n$ .

From (3.3) we get

$$\frac{1}{|\alpha^{n,j}|} \leq \frac{|z|}{z_2}, \quad (3.5)$$

and since for any Hermitian matrix  $A$ ,  $\|(A - zI)^{-1}\| \leq \frac{1}{z_2}$ , we have

$$\|D_j^{-1}\| \leq \frac{1}{z_2}. \quad (3.6)$$

By (S.1) we get

$$\frac{1}{|1 + \sigma^2 c_n m_n|} \leq \frac{|z|}{z_2 + \sigma^2 c_n \mathcal{I}m z m_n} \leq \frac{|z|}{z_2} \quad (3.7)$$

and similarly

$$\frac{1}{|1 + \sigma^2 c_n m_{(j)}|} \leq \frac{|z|}{z_2}. \quad (3.8)$$

From Lemma 3.3 we have

$$\max_{j \leq N} |m_n - m_{(j)}| \leq \frac{1}{n z_2}. \quad (3.9)$$

Suppose  $\lambda$  is an eigenvalue of  $\frac{1}{N} R R^*$  and  $\lambda^B = \frac{1}{1 + \sigma^2 c_n m_n} \lambda - \sigma^2 z \mathbf{m}_n - z$  is the corresponding eigenvalue of  $B$ . Then (S.1) gives

$$|\lambda^B| \geq |\mathcal{I}m \lambda^B| = \left| \frac{\sigma^2 c_n \mathcal{I}m m_n}{|1 + \sigma^2 c_n m_n|^2} \lambda + \sigma^2 \mathcal{I}m z \mathbf{m}_n + z_2 \right| \geq z_2.$$

Therefore

$$\|B^{-1}\| = \frac{1}{|\lambda_{min}^B|} \leq \frac{1}{z_2}, \quad (3.10)$$

and similarly

$$\|B_j^{-1}\| \leq \frac{1}{z_2}. \quad (3.11)$$

Using (3.7), (3.8), (3.9), (3.10), (3.11), and condition (3) we get

$$\begin{aligned} \|B_j^{-1} - B^{-1}\| &= \|B_j^{-1}(B - B_j)B^{-1}\| \leq \frac{1}{z_2^2} \|B - B_j\| \\ &= \frac{\sigma^2 c_n |m_{(j)} - m_n|}{z_2^2} \left\| \frac{1}{(1 + \sigma^2 c_n m_n)(1 + \sigma^2 c_n m_{(j)})} \frac{1}{N} RR^* + zI \right\| \\ &\leq \frac{\sigma^2 c_n}{nz_2^3} \left( \frac{1}{|1 + \sigma^2 c_n m_n| |1 + \sigma^2 c_n m_{(j)}|} \left\| \frac{1}{N} RR^* \right\| + |z| \right) \\ &\leq \frac{\sigma^2 c_n}{nz_2^3} \left( \frac{|z|^2}{z_2^2} \ln(n) + |z| \right) \leq K \frac{\ln(n)}{n}. \end{aligned} \quad (3.12)$$

A simple application of Lemma 3.2 gives

$$E\|x_j\|^{12} \leq Kn^6(\ln(n))^{12}. \quad (3.13)$$

The combination of (3.1), (3.6), (3.10), and the Cauchy-Schwarz inequality yields

$$|\hat{\rho}_j| \leq K \ln(n) \quad \text{and} \quad |\rho_j| \leq K \ln(n).$$

The Cauchy-Schwarz inequality along with Lemma 3.2, Lemma 3.3, (3.6), (3.10), (3.11), (3.12), (3.13), and (MP1) gives

$$\begin{aligned} E|\hat{\omega}_j - \frac{1}{N} \sigma^2 \text{tr} D^{-1} B^{-1}|^6 &= \frac{\sigma^{12}}{N^6} E|x_j^* D_j^{-1} B^{-1} x_j - \text{tr} D^{-1} B^{-1}|^6 \\ &\leq \frac{K}{N^6} E|x_j^* D_j^{-1} (B^{-1} - B_j^{-1}) x_j|^6 + \frac{K}{N^6} E|x_j^* D_j^{-1} B_j^{-1} x_j - \text{tr} D_j^{-1} B_j^{-1}|^6 \\ &\quad + \frac{K}{N^6} E|\text{tr} D_j^{-1} (B_j^{-1} - B^{-1})|^6 + \frac{K}{N^6} E|\text{tr} (D_j^{-1} - D^{-1}) B^{-1}|^6 \\ &\leq K \frac{(\ln(n))^6}{N^{12}} E\|x_j\|^{12} + K \frac{(\ln(n))^{12}}{N^3} + K \frac{(\ln(n))^6}{N^6} + \frac{K}{N^6} \\ &\leq K \frac{(\ln(n))^{18}}{N^3}. \end{aligned}$$

Using (3.6), Lemma 3.2, and Lemma 3.3 we get

$$E|\omega_j - \sigma^2 c_n m_n|^6 = \frac{\sigma^{12}}{N^6} E|x_j D_j^{-1} x_j - \text{tr} D^{-1}|^6$$

$$\begin{aligned}
&\leq \frac{K}{N^6} (E|x_j^* D_j^{-1} x_j - \text{tr} D_j^{-1}|^6 + E|\text{tr}(D_j^{-1} - D^{-1})|^6) \\
&\leq \frac{K}{N^3} (\ln(n))^{12} + \frac{K}{N^6} \\
&\leq K \frac{(\ln(n))^{12}}{N^3}.
\end{aligned}$$

From (3.1), (3.6), (3.11), (3.12), (3.13) Lemma 3.2, and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
E|\hat{\gamma}_j|^{12} &\leq \frac{K}{N^{12}} E|x_j^* D_j^{-1} (B^{-1} - B_j^{-1}) r_j|^{12} + \frac{K}{N^{12}} E||x_j^* D_j^{-1} B_j^{-1} r_j|^2|^6 \\
&\leq \frac{K}{N^{12}} E\|x_j\|^{12} \|r_j\|^{12} \|D_j^{-1}\|^{12} \|B^{-1} - B_j^{-1}\|^{12} \\
&\quad + \frac{K}{N^{12}} E|x_j^* D_j^{-1} B_j^{-1} r_j r_j^* B_j^{-1*} D_j^{-1*} x_j|^6 \\
&\leq K \frac{(\ln(n))^{18}}{N^{18}} E\|x_j\|^{12} + \frac{K}{N^{12}} E|x_j^* D_j^{-1} B_j^{-1} r_j r_j^* B_j^{-1*} D_j^{-1*} x_j \\
&\quad - \text{tr} D_j^{-1} B_j^{-1} r_j r_j^* B_j^{-1*} D_j^{-1*}|^6 \\
&\quad + \frac{K}{N^{12}} E|r_j^* B_j^{-1*} D_j^{-1*} D_j^{-1} B_j^{-1} r_j|^6 \\
&\leq K \frac{(\ln(n))^{30}}{N^{12}} + \frac{K}{N^9} (\ln(n))^{12} E\|D_j^{-1} B_j^{-1} r_j r_j^* B_j^{-1*} D_j^{-1*}\|^6 \\
&\quad + \frac{K}{N^{12}} E\|B_j^{-1*} D_j^{-1*} D_j^{-1} B_j^{-1}\|^6 \|r_j\|^{12} \\
&\leq K \frac{(\ln(n))^{30}}{N^{12}} + K \frac{(\ln(n))^{18}}{N^3} + K \frac{(\ln(n))^6}{N^6} \\
&\leq K \frac{(\ln(n))^{30}}{N^3},
\end{aligned}$$

and similarly

$$E|\hat{\beta}_j|^{12} \leq K \frac{(\ln(n))^{30}}{N^3}.$$

Using (3.1), (3.6), Lemma 3.2, and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
E|\gamma_j|^{12} &= \frac{\sigma^{12}}{N^{12}} E||x_j^* D_j^{-1} r_j|^2|^6 = \frac{\sigma^{12}}{N^{12}} E|x_j^* D_j^{-1} r_j r_j^* D_j^{-1*} x_j|^6 \\
&\leq \frac{K}{N^{12}} E|x_j^* D_j^{-1} r_j r_j^* D_j^{-1*} x_j - \text{tr} D_j^{-1} r_j r_j^* D_j^{-1*}|^6 \\
&\quad + \frac{K}{N^{12}} E|r_j^* D_j^{-1*} D_j^{-1} r_j|^6 \\
&\leq \frac{K}{N^9} (\ln(n))^{12} E\|D_j^{-1} r_j r_j^* D_j^{-1*}\|^6 + \frac{K}{N^{12}} E\|D_j^{-1}\|^{12} \|r_j\|^{12}
\end{aligned}$$

$$\begin{aligned} &\leq K \frac{(\ln(n))^{18}}{N^3} + K \frac{(\ln(n))^6}{N^6} \\ &\leq K \frac{(\ln(n))^{18}}{N^3}, \end{aligned}$$

and similarly

$$E|\beta_j|^{12} \leq K \frac{(\ln(n))^{18}}{N^3}.$$

From the Cauchy-Schwarz inequality and the above bounds we get

$$E|\beta_j \hat{\gamma}_j|^6 \leq K \frac{(\ln(n))^{24}}{N^3}.$$

Therefore, using (3.7) with the above, we have as  $n \rightarrow \infty$

$$\begin{aligned} \max_{j \leq N} \max \left\{ \left| \frac{(\sigma^2 c_n m_n - \omega_j) \hat{\rho}_j}{1 + \sigma^2 c_n m_n} \right|, \left| \frac{\gamma_j \hat{\rho}_j}{1 + \sigma^2 c_n m_n} \right|, |\hat{\beta}_j|, |\hat{\gamma}_j|, \left| \frac{\rho_j \hat{\gamma}_j}{1 + \sigma^2 c_n m_n} \right|, \right. \\ \left. \left| \frac{\beta_j \hat{\gamma}_j}{1 + \sigma^2 c_n m_n} \right|, \left| \hat{\omega}_j - \frac{\sigma^2}{N} \text{tr} D^{-1} B^{-1} \right| \right\} \xrightarrow{a.s.} 0. \end{aligned} \quad (3.14)$$

We now concentrate on a realization for which (3.14) holds,  $\{F^{C_n}\}$  is tight, and  $F^{\frac{1}{N}RR^*}$  converges in distribution to  $H$ . From (3.14) we get  $\max_{j \leq N} |d^{n,j}| \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore, using (3.5),

$$\frac{1}{n} \text{tr}(A_n - zI)^{-1} - m_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Consider a subsequence  $\{n_i\}$  on which  $m_{n_i}$  ( bounded in absolute value by  $\frac{1}{z_2}$ ) converges to a number  $m$ . We have  $\mathcal{I}m m \geq \delta > 0$ . Let  $b = 1 + \sigma^2 c m$  and  $b_n = 1 + \sigma^2 c_n m_n$ . Since  $\mathcal{I}m z m_{n_i} \geq 0$  then  $\mathcal{I}m z m \geq 0$ . From this we find for all  $t \geq 0$

$$\mathcal{I}m \left( \frac{t}{b} - bz + \sigma^2(1 - c) \right) \leq -z_2 < 0,$$

and similarly

$$\mathcal{I}m \left( \frac{t}{b_{n_i}} - b_{n_i} z + \sigma^2(1 - c_{n_i}) \right) \leq -z_2 < 0.$$

Then

$$\left| \frac{1}{\frac{t}{b_{n_i}} - b_{n_i} z + \sigma^2(1 - c_{n_i})} - \frac{1}{\frac{t}{b} - bz + \sigma^2(1 - c)} \right|$$

$$\begin{aligned}
&= \left| \frac{\frac{t}{bb_{n_i}}(b_{n_i} - b) + z(b_{n_i} - b) + \sigma^2(c_{n_i} - c)}{\left(\frac{t}{b_{n_i}} - b_{n_i}z + \sigma^2(1 - c_{n_i})\right) \left(\frac{t}{b} - bz + \sigma^2(1 - c)\right)} \right| \\
&\leq \frac{|z||b_{n_i} - b| + \sigma^2|c_{n_i} - c|}{z_2^2} + \frac{|b_{n_i} - b|}{z_2|b_{n_i}|} \left| \frac{\frac{t}{b}}{\frac{t}{b} - bz + \sigma^2(1 - c)} \right| \\
&= \frac{|z||b_{n_i} - b| + \sigma^2|c_{n_i} - c|}{z_2^2} + \frac{|b_{n_i} - b|}{z_2|b_{n_i}|} \left| 1 - \frac{-bz + \sigma^2(1 - c)}{\frac{t}{b} - bz + \sigma^2(1 - c)} \right| \\
&\leq \frac{|z||b_{n_i} - b| + \sigma^2|c_{n_i} - c|}{z_2^2} + \frac{|b_{n_i} - b|}{z_2|b_{n_i}|} \left( 1 + \frac{|bz| + \sigma^2|1 - c|}{\left|\frac{t}{b} - bz + \sigma^2(1 - c)\right|} \right) \\
&\leq \frac{|z||b_{n_i} - b| + \sigma^2|c_{n_i} - c|}{z_2^2} + \frac{|b_{n_i} - b|}{z_2\sigma^2c_{n_i}\delta} \left( 1 + \frac{|z| \left( 1 + \frac{\sigma^2c}{z_2} \right) + \sigma^2|1 - c|}{z_2} \right),
\end{aligned}$$

which converges to zero uniformly in  $t$ . Therefore as  $n_i \rightarrow \infty$

$$\begin{aligned}
\frac{1}{n_i} \text{tr}(A_{n_i} - zI)^{-1} &= \int \frac{1}{\frac{t}{b_{n_i}} - b_{n_i}z + \sigma^2(1 - c_{n_i})} dF_{\frac{1}{N_i}RR^*}(t) \\
&\longrightarrow \int \frac{1}{\frac{t}{b} - bz + \sigma^2(1 - c)} dH(t).
\end{aligned}$$

Thus  $m$  satisfies (1.1).

Now, using the result from the next chapter we have that  $m$  is unique. Therefore, with probability one,  $F^{C_n}$  converges in distribution to the p.d.f.  $F$  having Stieltjes transform defined by (1.1), and the proof of Theorem 1.1 is complete.

## 4. UNIQUE SOLUTION TO (1.1)

We now prove that a certain type of solution to equation (1.1) is unique.

**THEOREM 4.1.** Let  $z = z_1 + iz_2 \in \mathbb{C}^+$ ,  $m = m_1 + im_2 \in \mathbb{C}^+$ , and  $\mathbf{m} = \mathbf{m}_1 + i\mathbf{m}_2 \in \mathbb{C}^+$  with  $\mathcal{I}m(1 + \sigma^2 cm)z > 0$ , and  $\mathcal{I}m(1 + \sigma^2 c\mathbf{m})z > 0$ . If both  $m$  and  $\mathbf{m}$  satisfy (1.1), then  $m = \mathbf{m}$ .

*Proof.* Define  $b \equiv 1 + \sigma^2 cm = b_1 + ib_2$  and  $\mathbf{b} \equiv 1 + \sigma^2 c\mathbf{m} = \mathbf{b}_1 + i\mathbf{b}_2$  and suppose that both  $m$  and  $\mathbf{m}$  satisfy (1.1). We have  $m - \mathbf{m} = (m - \mathbf{m})\alpha$ , where

$$\alpha = \sigma^2 c \int \frac{\frac{t}{b\mathbf{b}} + z}{\left(\frac{t}{b} - bz + \sigma^2(1-c)\right)\left(\frac{t}{\mathbf{b}} - \mathbf{b}z + \sigma^2(1-c)\right)} dH(t).$$

Using the triangle and Cauchy-Schwarz inequalities we get

$$\begin{aligned} |\alpha| &\leq \sigma^2 c \int \frac{\frac{t}{|b|\mathbf{b}} dH(t)}{\left|\frac{t}{b} - bz + \sigma^2(1-c)\right| \left|\frac{t}{\mathbf{b}} - \mathbf{b}z + \sigma^2(1-c)\right|} \\ &\quad + \sigma^2 c |z| \int \frac{dH(t)}{\left|\frac{t}{b} - bz + \sigma^2(1-c)\right| \left|\frac{t}{\mathbf{b}} - \mathbf{b}z + \sigma^2(1-c)\right|} \\ &\leq \left( \int \frac{\sigma^2 c \frac{t}{|b|^2} dH(t)}{\left|\frac{t}{b} - bz + \sigma^2(1-c)\right|^2} \right)^{\frac{1}{2}} \left( \int \frac{\sigma^2 c \frac{t}{|\mathbf{b}|^2} dH(t)}{\left|\frac{t}{\mathbf{b}} - \mathbf{b}z + \sigma^2(1-c)\right|^2} \right)^{\frac{1}{2}} \\ &\quad + |z| \left( \int \frac{\sigma^2 c dH(t)}{\left|\frac{t}{b} - bz + \sigma^2(1-c)\right|^2} \right)^{\frac{1}{2}} \left( \int \frac{\sigma^2 c dH(t)}{\left|\frac{t}{\mathbf{b}} - \mathbf{b}z + \sigma^2(1-c)\right|^2} \right)^{\frac{1}{2}} \\ &\equiv (g(b))^{\frac{1}{2}} (g(\mathbf{b}))^{\frac{1}{2}} + |z| (G(b))^{\frac{1}{2}} (G(\mathbf{b}))^{\frac{1}{2}}. \end{aligned} \tag{4.1}$$

Note that  $g(b), g(\mathbf{b}) \geq 0$  and  $G(b), G(\mathbf{b}) > 0$ .

The following statements are valid for both  $b$  and  $\mathbf{b}$ .

From (1.1) we get

$$b_1 = 1 + b_1 g(b) + (\sigma^2(1 - c) - \mathcal{R}e bz)G(b) \quad (4.2)$$

$$b_2 = b_2 g(b) + (\mathcal{I}m bz)G(b). \quad (4.3)$$

Writing  $\mathcal{I}m bz = b_1 z_2 + b_2 z_1$ , (4.3) implies

$$b_1 = b_2 \frac{1 - g(b) - z_1 G(b)}{z_2 G(b)}. \quad (4.4)$$

Since (4.2) can be written as

$$b_1(1 - g(b) + z_1 G(b)) = (1 + \sigma^2(1 - c)G(b)) + b_2 z_2 G(b)$$

we replace  $b_1$  using (4.4) and get

$$b_2((1 - g(b))^2 - |z|^2 G^2(b)) = (1 + \sigma^2(1 - c)G(b))z_2 G(b) > 0$$

(recall  $c \leq 1$ ).

Therefore,

$$(1 - g(b))^2 - |z|^2 G^2(b) > 0. \quad (4.5)$$

Since  $G(b)$ ,  $b_2$ , and  $\mathcal{I}m bz$  are positive, we get from (4.3) that  $g(b) < 1$ , and hence (4.5) implies

$$0 < |z|G(b) < 1 - g(b). \quad (4.6)$$

We now have

$$g(b) < 1 - |z|G(b) \quad \text{and} \quad g(\mathbf{b}) < 1 - |z|G(\mathbf{b}). \quad (4.7)$$

For real numbers  $x$  and  $y$  with  $x, y \in [0, 1]$  it is easy to show that

$$(1 - x)^{\frac{1}{2}}(1 - y)^{\frac{1}{2}} \leq 1 - (xy)^{\frac{1}{2}} \quad (4.8)$$

with equality holding if and only if  $x = y$ .

To complete the theorem's proof we use (4.1), (4.7), and (4.8) to get

$$\begin{aligned} |\alpha| &\leq (g(b))^{\frac{1}{2}}(g(\mathbf{b}))^{\frac{1}{2}} + |z|(G(b)G(\mathbf{b}))^{\frac{1}{2}} \\ &< (1 - |z|G(b))^{\frac{1}{2}}(1 - |z|G(\mathbf{b}))^{\frac{1}{2}} + |z|(G(b)G(\mathbf{b}))^{\frac{1}{2}} \\ &\leq 1 - (|z|G(b))^{\frac{1}{2}}(|z|G(\mathbf{b}))^{\frac{1}{2}} + |z|(G(b)G(\mathbf{b}))^{\frac{1}{2}} = 1. \end{aligned}$$

Therefore  $|\alpha| < 1$ , and hence  $m = \mathbf{m}$ .

## 5. EXISTENCE OF A DENSITY

In this chapter we establish the following result.

**THEOREM 5.1.** For all  $x \in \mathbb{R} - \{0\}$ ,  $\lim_{z \in \mathbb{C}^+ \rightarrow x} m_F(z) \equiv \underline{m}(x)$  exists. The function  $\underline{m}$  is continuous on  $\mathbb{R} - \{0\}$ , and  $F$  has a continuous derivative  $f$  on  $\mathbb{R} - \{0\}$  given by  $f(x) = \frac{1}{\pi} \mathcal{I}m \underline{m}(x)$ . Furthermore, if  $\mathcal{I}m \underline{m}(x) > 0$  ( $f(x) > 0$ ) for  $x \neq 0$ , then  $\underline{m}(x)$  is a solution to (1.1) for  $z = x$ , and the density  $f$  is analytic about  $x$ .

We note that it is shown in Silverstein [8] that, almost surely,  $F^{\frac{1}{N} \sigma^2 X_n X_n^*}$  converges in distribution to a nonrandom p.d.f.  $F_*$  whose Stieltjes transform  $m_* = m_{F_*}(z)$ , for  $z \in \mathbb{C}^+$ , satisfies the equation

$$m_* = \frac{1}{\sigma^2(1 - c - czm_*) - z} = \frac{1}{-(1 + \sigma^2 cm_*)z + \sigma^2(1 - c)},$$

which is equation (1.1) with  $H = \mathbf{1}_{[0, \infty)}$  ( $\mathbf{1}_B$  denoting the indicator function over the set  $B$ ). Therefore, by uniqueness of solution (Theorem 4.1), we have  $m_* = m_F$  (for  $H = \mathbf{1}_{[0, \infty)}$ ), and hence  $F = F_*$ . This function has an explicit expression (Marčenko and Pastur [5]), satisfying all properties to be investigated in this paper. Therefore for the remainder of this paper we may assume  $H \neq \mathbf{1}_{[0, \infty)}$ .

We see from equation (1.1) that if  $c \downarrow 0$ , we get for any  $z \in \mathbb{C}^+$

$$m_F(z) \longrightarrow \int \frac{dH(t)}{(t + \sigma^2) - z},$$

which is the Stieltjes transform of the p.d.f. of a random variable  $Y + \sigma^2$ , where  $Y$  has distribution  $H$ . In terms of the aforementioned application to array signal processing, the condition  $c \downarrow 0$  corresponds to the situation when the number of samples,  $N$ , is significantly larger than the number of sensors,  $n$  which is equivalent to  $n$  being

fixed and  $N$  approaching infinity. Assuming  $X_n$  to be centered and the samples independent, we can conclude by the strong law of large numbers that  $C_n \rightarrow S_n + \sigma^2 I$ , almost surely; however, if we assume for each  $N$  there is a new set of  $N$  samples, we may only use the weak law of large numbers to get convergence in probability. Both cases coincide with our result on  $m_F$  as  $c \downarrow 0$ .

Two theorems from Silverstein and Choi [10] that will be needed are the following.

**THEOREM 5.2** [Theorem 2.1 of Silverstein and Choi [10]]. Let  $F$  be a p.d.f. and  $x \in \mathbb{R}$ . Suppose  $\mathcal{I}m m_F(x) \equiv \lim_{z \in \mathbb{C}^+ \rightarrow x} \mathcal{I}m m_F(z)$  exists. Then  $F$  is differentiable at  $x$ , and its derivative is  $\frac{1}{\pi} \mathcal{I}m m_F(x)$ .

**THEOREM 5.3** [Theorem 2.2 of Silverstein and Choi [10]]. Let  $X$  be an open and bounded subset of  $\mathbb{R}^n$ , let  $Y$  be an open and bounded subset of  $\mathbb{R}^m$ , and let  $f : \overline{X} \rightarrow Y$  be a function, continuous on  $X$ . If, for all  $x_0 \in \partial X$ ,  $\lim_{x \in X \rightarrow x_0} f(x) = f(x_0)$ , then  $f$  is continuous on all of  $\overline{X}$ .

As indicated in Theorem 5.1, once existence of  $\underline{m}$  is verified, we immediately have continuity of  $\underline{m}$  and existence of the density  $f$  by Theorems 5.2 and 5.3. To prove the existence of  $\underline{m}$  and the analyticity of  $f$ , we rely on a series of lemmas which will be stated and proved throughout this chapter.

For notational convenience we will often write equation (1.1) in terms of the variable  $b = 1 + \sigma^2 cm$  in which case we have the equation

$$b = 1 + \sigma^2 c \int \frac{dH(t)}{\frac{t}{b} - bz + \sigma^2(1-c)}. \quad (1.1')$$

Therefore, when we say that such a  $b$  satisfies (1.1'), the meaning is understood to be that the corresponding variable  $m$  satisfies (1.1). At times we will also write  $b_F = 1 + \sigma^2 cm_F$  to make a reference to the Stieltjes transform  $m_F$ .

Let  $z = z_1 + iz_2 \in \mathbb{C}^+$ , and let  $m = m_F(z)$  and  $b = b_1 + ib_2 = 1 + \sigma^2 cm$ . Recall from the previous chapter the functions  $g(b)$  and  $G(b)$  given by

$$g(b) = \int \frac{\sigma^2 c \frac{t}{|b|^2} dH(t)}{\left| \frac{t}{b} - bz + \sigma^2(1-c) \right|^2}$$

$$G(b) = \int \frac{\sigma^2 c dH(t)}{\left| \frac{t}{b} - bz + \sigma^2(1-c) \right|^2}.$$

Note that  $G(b) > 0$ , and since  $H \neq \mathbf{1}_{[0, \infty)}$ , we have  $g(b) > 0$ . It is clear that (4.2) through (4.6) hold true for  $b$  since it is a solution to (1.1') with  $\mathcal{I}m bz > 0$ . These equations will be used often in establishing our results for this chapter.

We now prove the following lemma.

**LEMMA 5.1.** Let  $z = z_1 + iz_2 \in \mathbb{C}^+$ ,  $m = m_F(z)$ , and  $b = b_1 + ib_2 = 1 + \sigma^2 cm$ . Then we have the following three results:

- (a)  $b_1 > 0$ ,
- (b)  $|m| < \left( \frac{1}{\sigma^2 c |z|} \right)^{\frac{1}{2}}$ ,
- (c) If  $\lim_{z_n \rightarrow x} b \equiv \underline{b} = \underline{b}_1 + i\underline{b}_2$  exists for  $\{z_n\} \subset \mathbb{C}^+$  and  $x \in \mathbb{R} - \{0\}$ , then  $\underline{b}_1 > 0$ .

*Proof.* For simplicity of notation we suppress the subscript  $n$  in the proof of (c).

First, to prove (a), suppose  $1 - g(b) - z_1 G(b) \leq 0$ . Since  $g(b) < 1$  we get

$$0 < (1 - g(b))^2 \leq z_1^2 G^2(b) < |z|^2 G^2(b),$$

a contradiction of (4.6). Therefore  $1 - g(b) - z_1 G(b) > 0$ , and since  $b_2 > 0$ ,  $z_2 > 0$ , and  $G(b) > 0$  we have  $b_1 > 0$  by (4.4).

To prove (b) we first note that since  $0 < g(b) < 1$ , (4.6) gives

$$0 < G(b) < \frac{1}{|z|}. \tag{5.1}$$

Then using the Cauchy-Schwarz inequality we get

$$|m| \leq \int \frac{dH(t)}{\left| \frac{t}{b} - bz + \sigma^2(1-c) \right|} \leq \left( \int \frac{dH(t)}{\left| \frac{t}{b} - bz + \sigma^2(1-c) \right|^2} \right)^{\frac{1}{2}}$$

$$= \left( \frac{G(b)}{\sigma^2 c} \right)^{\frac{1}{2}} < \left( \frac{1}{\sigma^2 c |z|} \right)^{\frac{1}{2}}$$

Finally, for part (c) we note that part (b) gives  $|b| < \infty$ . Solving (4.2) and (4.3) for  $G(b)$  we find

$$\begin{aligned} G(b) &= \frac{b_2}{\mathcal{I}m(b^2 z - b\sigma^2(1-c))} = \frac{1}{\mathcal{R}e bz + b_1 \frac{\mathcal{I}m bz}{b_2} - \sigma^2(1-c)} \\ &= \frac{1}{2b_1 z_1 - b_2 z_2 + b_1^2 \frac{z_2}{b_2} - \sigma^2(1-c)}. \end{aligned} \quad (5.2)$$

Since  $F$  is proper we have  $\frac{z_2}{b_2} = \left( \sigma^2 c \int \frac{dF(\lambda)}{|\lambda - z|^2} \right)^{-1}$  is bounded as  $z \rightarrow x$ . Then if  $b_1 = 0$  and  $c < 1$  we get

$$\lim_{z \rightarrow x} G(b) \equiv G = \frac{1}{-\sigma^2(1-c)} < 0,$$

a contradiction since (5.1) gives

$$0 \leq G \leq \frac{1}{|x|}. \quad (5.3)$$

If  $c = 1$ , then, as  $z \rightarrow x$ ,  $G(b)$  goes unbounded, again contradicting (5.3). Therefore,  $b_1 > 0$  and the proof is complete.

In the next lemma we show that  $m_F(z)$  has a unique limit as  $z \rightarrow x \in \mathbb{R} - \{0\}$ .

**LEMMA 5.2.** Let  $\{z_n\}, \{\hat{z}_n\} \subset \mathbb{C}^+$  with  $z_n$  and  $\hat{z}_n$  both converging to  $x \in \mathbb{R} - \{0\}$  as  $n \rightarrow \infty$ . If  $m = m_F(z_n) \rightarrow \underline{m}$  and  $\hat{m} = m_F(\hat{z}_n) \rightarrow \underline{\hat{m}}$  as  $n \rightarrow \infty$ , then  $\underline{m} = \underline{\hat{m}}$ .

*Proof.* The result is obvious for  $x < 0$  since  $m_F$  is analytic outside the support of  $F$ . Therefore, we assume  $x > 0$ . We let  $b = b_1 + ib_2 = 1 + \sigma^2 cm$  and  $\hat{b} = \hat{b}_1 + i\hat{b}_2 = 1 + \sigma^2 c\hat{m}$  and define the functions  $g(\hat{b})$  and  $G(\hat{b})$  in the same way that  $g(b)$  and  $G(b)$  are defined with the exception that  $b$  and  $z$  are replaced by  $\hat{b}$  and  $\hat{z}$ , respectively.

To prevent the confusion of multiple subscripts, we will suppress the dependence on  $n$  of the sequence terms  $z_n, \hat{z}_n$  and write  $z_n = z = z_1 + iz_2$  and  $\hat{z}_n = \hat{z} = \hat{z}_1 + i\hat{z}_2$ .

We now take the difference  $m - \hat{m} = \frac{(z - \hat{z})\beta_n}{1 - \alpha_n}$  where

$$\alpha_n = \sigma^2 c \int \frac{\frac{t}{b\hat{b}} + z}{\left(\frac{t}{b} - bz + \sigma^2(1-c)\right)\left(\frac{t}{\hat{b}} - \hat{b}\hat{z} + \sigma^2(1-c)\right)} dH(t)$$

and

$$\beta_n = \int \frac{\hat{b}dH(t)}{\left(\frac{t}{b} - bz + \sigma^2(1-c)\right)\left(\frac{t}{\hat{b}} - \hat{b}\hat{z} + \sigma^2(1-c)\right)}.$$

Using the Cauchy-Schwarz inequality, (5.1), and Lemma 5.1 (b), we get for all  $n$

$$|\beta_n| \leq \frac{|\hat{b}|(G(b)G(\hat{b}))^{\frac{1}{2}}}{\sigma^2 c} \leq \frac{1 + \sigma^2 c |\hat{m}|}{\sigma^2 c (|z||\hat{z}|)^{\frac{1}{2}}} \leq \frac{1 + \left(\frac{\sigma^2 c}{|\hat{z}|}\right)^{\frac{1}{2}}}{\sigma^2 c (|z||\hat{z}|)^{\frac{1}{2}}} \leq K < \infty.$$

Therefore  $|m - \hat{m}| \leq \frac{K|z - \hat{z}|}{|1 - |\alpha_n||}$ , and consequently we need only show that  $|\alpha_n|$  stays uniformly away from 1.

Following the procedure from the previous chapter, we use the triangle and Cauchy-Schwarz inequalities to get

$$|\alpha_n| \leq (g(b))^{\frac{1}{2}}(g(\hat{b}))^{\frac{1}{2}} + |z|(G(b)G(\hat{b}))^{\frac{1}{2}}. \quad (5.4)$$

Therefore, since  $g(b)$ ,  $g(\hat{b})$ ,  $G(b)$ , and  $G(\hat{b})$  are bounded, we can choose a subsequence  $\{n_j\}$  for which  $\alpha_{n_j}$ ,  $g(b)$ ,  $g(\hat{b})$ ,  $G(b)$ , and  $G(\hat{b})$  converge, and we define their respective limits as  $\alpha$ ,  $g$ ,  $\hat{g}$ ,  $G$ , and  $\hat{G}$ .

From (5.4) and (4.6) we get for all  $j$

$$|\alpha_{n_j}| < (1 - |z|G(b))^{\frac{1}{2}}(1 - |\hat{z}|G(\hat{b}))^{\frac{1}{2}} + |z|(G(b)G(\hat{b}))^{\frac{1}{2}}, \quad (5.5)$$

and then taking the limit we have

$$|\alpha| \leq (1 - xG)^{\frac{1}{2}}(1 - x\hat{G})^{\frac{1}{2}} + x(G\hat{G})^{\frac{1}{2}}. \quad (5.6)$$

If  $G \neq \hat{G}$ , then applying (4.8) to (5.7) we get the strict inequality

$$|\alpha| < 1 - (xGx\hat{G})^{\frac{1}{2}} + x(G\hat{G})^{\frac{1}{2}} = 1$$

as desired.

For the rest of the proof, we assume that  $G = \hat{G}$ .

From (5.2) we see that  $G(b)$  converges if and only if  $\frac{z_2}{b_2}$  converges. Therefore,  $\frac{z_2}{b_2}$  and similarly  $\frac{\hat{z}_2}{\hat{b}_2}$  must converge, and we call their respective limits  $y$  and  $\hat{y}$ .

Solving (4.3) for  $g(b)$  gives

$$g(b) = 1 - \frac{\mathcal{I}m bz}{b_2} G(b) = 1 - \left( b_1 \frac{z_2}{b_2} + z_1 \right) G(b).$$

We solve for  $g(\hat{b})$  the same way and substitute the results into (5.4) to get

$$|\alpha_{n_j}| \leq \left( 1 - \left( b_1 \frac{z_2}{b_2} + z_1 \right) G(b) \right)^{\frac{1}{2}} \left( 1 - \left( \hat{b}_1 \frac{\hat{z}_2}{\hat{b}_2} + \hat{z}_1 \right) G(\hat{b}) \right)^{\frac{1}{2}} + |z| (G(b)G(\hat{b}))^{\frac{1}{2}}.$$

Defining  $\underline{b} = \underline{b}_1 + i\underline{b}_2 = 1 + \sigma^2 \underline{c} \underline{m}$  we take the limit and use (4.8) to get

$$\begin{aligned} |\alpha| &\leq (1 - (\underline{b}_1 y + x)G)^{\frac{1}{2}} (1 - (\hat{\underline{b}}_1 \hat{y} + x)G)^{\frac{1}{2}} + xG \\ &\leq 1 - \left( (\underline{b}_1 y + x)^{\frac{1}{2}} (\hat{\underline{b}}_1 \hat{y} + x)^{\frac{1}{2}} - x \right) G. \end{aligned} \quad (5.7)$$

By Lemma 5.1 (c), we have  $\underline{b}_1 > 0$  and  $\hat{\underline{b}}_1 > 0$ . Therefore if either  $y > 0$  or  $\hat{y} > 0$ , we have  $(\underline{b}_1 y + x)^{\frac{1}{2}} (\hat{\underline{b}}_1 \hat{y} + x)^{\frac{1}{2}} > x$ , and hence  $|\alpha| < 1$  by (5.7).

Suppose  $y = \hat{y} = 0$ . Then (5.2) gives

$$\frac{1}{2\underline{b}_1 x - \sigma^2(1 - c)} = G = \hat{G} = \frac{1}{2\hat{\underline{b}}_1 x - \sigma^2(1 - c)},$$

and hence  $\underline{b}_1 = \hat{\underline{b}}_1$ .

If  $\underline{b}_2 = \hat{\underline{b}}_2 = 0$ , we are done. Suppose that either  $\underline{b}_2 > 0$  or  $\hat{\underline{b}}_2 > 0$ . Define

$$k_{n_j}(t) \equiv \left| \frac{t}{b\hat{b}} \right| + |z| - \left| \frac{t}{b\hat{b}} + z \right| \quad \text{for } t \geq 0.$$

Since  $\mathcal{I}m \frac{1}{b\hat{b}} = \frac{-(b_1 \hat{b}_2 + b_2 \hat{b}_1)}{|b\hat{b}|^2} \rightarrow \frac{-(\underline{b}_1 \hat{\underline{b}}_2 + \underline{b}_2 \hat{\underline{b}}_1)}{|\underline{b}\hat{\underline{b}}|^2} < 0$  as  $j \rightarrow \infty$ , then  $z$  and  $\frac{1}{b\hat{b}}$  are noncolinear for  $j$  large. Therefore, since  $k_{n_j}(t)$  is the residual of the triangle inequality, we have for large  $j$ ,  $k_{n_j}(t) \geq 0$  for  $t \geq 0$  with  $k_{n_j}(t) = 0$  if and only if  $t = 0$ .

Define

$$\gamma_{n_j} \equiv \sigma^2 c \int \frac{k_{n_j}(t) dH(t)}{\left| \frac{t}{b} - bz + \sigma^2(1 - c) \right| \left| \frac{t}{\hat{b}} - \hat{b}z + \sigma^2(1 - c) \right|}.$$

Since  $k_{n_j}(t) \leq \left| \frac{t}{bb} \right| + |z|$ , we have

$$\begin{aligned} \gamma_{n_j} &\leq \sigma^2 c \int \frac{\left| \frac{t}{bb} \right| + |z|}{\left| \frac{t}{b} - bz + \sigma^2(1-c) \right| \left| \frac{t}{\hat{b}} - \hat{b}\hat{z} + \sigma^2(1-c) \right|} dH(t) \\ &\leq (g(b))^{\frac{1}{2}} (g(\hat{b}))^{\frac{1}{2}} + |z| (G(b)G(\hat{b}))^{\frac{1}{2}} \leq 1 \end{aligned}$$

for all  $j$ . Therefore by Fatou's lemma we get

$$\gamma \equiv \sigma^2 c \int \frac{\lim_{j \rightarrow \infty} k_{n_j}(t) dH(t)}{\left| \frac{t}{\underline{b}} - \underline{b}x + \sigma^2(1-c) \right| \left| \frac{t}{\underline{\hat{b}}} - \underline{\hat{b}}x + \sigma^2(1-c) \right|} \leq \liminf_{j \rightarrow \infty} \gamma_{n_j} \leq 1.$$

Since  $H$  is proper,  $H \neq \mathbf{1}_{[0, \infty)}$ , and  $\underline{b}, \underline{\hat{b}}$  are finite we get  $\gamma > 0$ .

Going back to the definition of  $\alpha$  we follow similar steps as before to derive

$$\begin{aligned} |\alpha_{n_j}| &\leq \sigma^2 c \int \frac{\left| \frac{t}{bb} \right| + |z| - k_{n_j}(t)}{\left| \frac{t}{b} - bz + \sigma^2(1-c) \right| \left| \frac{t}{\hat{b}} - \hat{b}\hat{z} + \sigma^2(1-c) \right|} dH(t) \\ &\leq (g(b))^{\frac{1}{2}} (g(\hat{b}))^{\frac{1}{2}} + |z| (G(b)G(\hat{b}))^{\frac{1}{2}} - \gamma_{n_j} \\ &< (1 - |z|G(b))^{\frac{1}{2}} (1 - |\hat{z}|G(\hat{b}))^{\frac{1}{2}} + |z| (G(b)G(\hat{b}))^{\frac{1}{2}} - \gamma_{n_j}. \end{aligned}$$

Then

$$\begin{aligned} |\alpha| &\leq \liminf_{j \rightarrow \infty} \left( (1 - |z|G(b))^{\frac{1}{2}} (1 - |\hat{z}|G(\hat{b}))^{\frac{1}{2}} + |z| (G(b)G(\hat{b}))^{\frac{1}{2}} - \gamma_{n_j} \right) \\ &\leq \liminf_{j \rightarrow \infty} \left( (1 - |z|G(b))^{\frac{1}{2}} (1 - |\hat{z}|G(\hat{b}))^{\frac{1}{2}} + |z| (G(b)G(\hat{b}))^{\frac{1}{2}} \right) - \liminf_{j \rightarrow \infty} \gamma_{n_j} \\ &= 1 - \liminf_{j \rightarrow \infty} \gamma_{n_j} \\ &\leq 1 - \gamma < 1. \end{aligned}$$

Therefore in every case we have  $\underline{m} = \hat{\underline{m}}$ , and hence the proof is complete.

By Theorems 5.2 and 5.3 and Lemmas 5.1 (b) and 5.2 we now have the existence and continuity of both  $\underline{m}$  and  $f$  on  $\mathbb{R} - \{0\}$ . Moreover, when  $f(x) > 0$  we have

$$\mathcal{I}m \left( \frac{t}{\underline{b}(x)} - \underline{b}(x)x - \sigma^2(1-c) \right) \leq -\mathcal{I}m \underline{b}(x)x < 0$$

for all  $t \geq 0$ , and therefore, by dominated convergence,  $\underline{m}(x)$  satisfies (1.1) for  $z = x$ . Therefore, the only part of Theorem 5.1 that remains to be shown is the analyticity of  $f$ .

We now complete the proof of Theorem 5.1 with the following lemma.

LEMMA 5.3. If  $x_0 \in (0, \infty)$  and  $f(x_0) > 0$ , then  $f$  is analytic near  $x_0$ .

*Proof.* Let  $\underline{b} = \underline{b}_1 + i\underline{b}_2 = 1 + \sigma^2 c \underline{m}(x_0)$ . For  $z \in \mathbb{C}^+$  and any  $m \in \mathbb{C}^+$  satisfying (1.1), we get

$$\frac{m}{1 + \sigma^2 cm} = \int \frac{dH(t)}{t - (b^2 z - b\sigma^2(1 - c))} = m_H(b^2 z - b\sigma^2(1 - c)), \quad (5.8)$$

where  $m_H(\cdot)$  denotes the Stieltjes transform of  $H$  and  $b = 1 + \sigma^2 cm$ . Let  $\underline{w} \equiv \underline{b}^2 x_0 - \underline{b}\sigma^2(1 - c)$ . Since the denominator in (5.2) is bounded, we can tighten inequality (5.3) to get

$$0 < G \leq \frac{1}{x_0}. \quad (5.9)$$

From (5.2) we get  $\mathcal{I}m \underline{w} = \frac{\underline{b}_2}{G}$ , and since  $\underline{b}_2 = \pi \sigma^2 c f(x_0) > 0$  we have  $\mathcal{I}m \underline{w} > 0$ , and hence  $m_H$  is analytic near  $\underline{w}$ .

First, suppose that  $m'_H(\underline{w}) \neq 0$ . Then in a neighborhood of  $\underline{w}$ , the analytic inverse  $m_H^{-1}$  exists. It is clear that for  $z$  near  $x_0$  and  $b$  near  $\underline{b}$ , we have  $w \equiv b^2 z - b\sigma^2(1 - c)$  near  $\underline{w}$ . Therefore, if  $b$  is near  $\underline{b}$  and  $b$  satisfies equation (1.1') for  $z$  near  $x_0$ , then we have

$$\frac{1}{\sigma^2 c} \left(1 - \frac{1}{b}\right) = m_H(b^2 z - b\sigma^2(1 - c)) = m_H(w), \quad (5.10)$$

and hence

$$z = \frac{1}{b^2} m_H^{-1} \left( \frac{1}{\sigma^2 c} \left(1 - \frac{1}{b}\right) \right) + \frac{1}{b} \sigma^2 (1 - c). \quad (5.11)$$

Let  $z(b)$  be the right hand side of (5.11). In a neighborhood of  $\underline{b}$ ,  $z(b)$  is clearly analytic, and we will show that it is also one-to-one.

For complex numbers  $b, \hat{b}$  near  $\underline{b}$  and  $z$  close enough to  $x_0$  so that  $\mathcal{I}m bz > 0$  and  $\mathcal{I}m \hat{b}z > 0$ , define the function

$$\alpha = \sigma^2 c \int \frac{\frac{t}{\hat{b}\hat{b}} + z}{\left(\frac{t}{\hat{b}} - bz + \sigma^2(1-c)\right)\left(\frac{t}{\hat{b}} - \hat{b}z + \sigma^2(1-c)\right)} dH(t).$$

Note that for  $t \geq 0$ ,

$$\mathcal{I}m \left( \frac{t}{\hat{b}} - bz + \sigma^2(1-c) \right) = -\frac{t\mathcal{I}m b}{|\hat{b}|^2} - \mathcal{I}m bz \leq -\mathcal{I}m bz < 0,$$

and similarly for  $\hat{b}$ . Therefore the integrand of  $\alpha$  is bounded since for any  $t \geq 0$

$$\begin{aligned} \frac{\left| \frac{t}{\hat{b}\hat{b}} + z \right|}{\left| \frac{t}{\hat{b}} - bz + \sigma^2(1-c) \right| \left| \frac{t}{\hat{b}} - \hat{b}z + \sigma^2(1-c) \right|} &\leq \frac{\frac{t}{|\hat{b}\hat{b}|} + |z|}{\left| \frac{t}{\hat{b}} - bz + \sigma^2(1-c) \right| \mathcal{I}m \hat{b}z} \\ &\leq \frac{1}{|\hat{b}|\mathcal{I}m \hat{b}z} \left| \frac{\frac{t}{\hat{b}}}{\frac{t}{\hat{b}} - bz + \sigma^2(1-c)} \right| + \frac{|z|}{(\mathcal{I}m bz)(\mathcal{I}m \hat{b}z)} \\ &= \frac{1}{|\hat{b}|\mathcal{I}m \hat{b}z} \left| 1 - \frac{-bz + \sigma^2(1-c)}{\frac{t}{\hat{b}} - bz + \sigma^2(1-c)} \right| + \frac{|z|}{(\mathcal{I}m bz)(\mathcal{I}m \hat{b}z)} \\ &\leq \frac{1}{|\hat{b}|\mathcal{I}m \hat{b}z} \left( 1 + \frac{|b||z| + \sigma^2(1-c)}{\left| \frac{t}{\hat{b}} - bz + \sigma^2(1-c) \right|} \right) + \frac{|z|}{(\mathcal{I}m bz)(\mathcal{I}m \hat{b}z)} \\ &\leq \frac{1}{|\hat{b}|\mathcal{I}m \hat{b}z} \left( 1 + \frac{|b||z| + \sigma^2(1-c)}{\mathcal{I}m bz} \right) + \frac{|z|}{(\mathcal{I}m bz)(\mathcal{I}m \hat{b}z)} < K, \end{aligned}$$

and hence  $\alpha$  is well-defined and, in fact, continuous in the variables  $b, \hat{b}$ , and  $z$ . Define  $\underline{\alpha}$  to be the value of  $\alpha$  when  $b = \hat{b} = \underline{b}$  and  $z = x_0$ , that is,

$$\underline{\alpha} = \sigma^2 c \int \frac{\frac{t}{\underline{b}^2} + x_0}{\left(\frac{t}{\underline{b}} - \underline{b}x_0 + \sigma^2(1-c)\right)^2} dH(t).$$

Define

$$\underline{k}(t) \equiv \left| \frac{t}{\underline{b}^2} + x_0 \right| - \left| \frac{t}{\underline{b}} - \underline{b}x_0 + \sigma^2(1-c) \right|$$

and

$$\underline{\gamma} \equiv \sigma^2 c \int \frac{\underline{k}(t)}{\left| \frac{t}{\underline{b}} - \underline{b}x_0 + \sigma^2(1-c) \right|^2} dH(t).$$

Now,  $\mathcal{I}m \frac{1}{\underline{b}^2} = -\frac{\hat{b}_1 \hat{b}_2}{|\underline{b}|^2} < 0$ , and therefore  $\frac{1}{\underline{b}^2}$  and  $x_0$  are noncolinear. Since  $\underline{k}$  is the residual of the triangle inequality, we have  $\underline{k}(t) \geq 0$  for  $t \geq 0$  with  $\underline{k}(t) = 0$  if and

only if  $t = 0$ . Therefore since  $H \neq \mathbf{1}_{[0, \infty)}$ , we have  $\underline{\gamma} > 0$ , and since (4.3) gives  $1 - g = x_0 G$ , we get, as in the proof of Lemma 5.2,

$$\begin{aligned} |\underline{\alpha}| &\leq \sigma^2 c \int \frac{\left| \frac{t}{\underline{b}^2} + x_0 \right|}{\left| \frac{t}{\underline{b}} - \underline{b}x_0 + \sigma^2(1 - c) \right|^2} dH(t) = \sigma^2 c \int \frac{\frac{t}{|\underline{b}|^2} + x_0 - \underline{k}(t)}{\left| \frac{t}{\underline{b}} - \underline{b}x_0 + \sigma^2(1 - c) \right|^2} dH(t) \\ &= g + x_0 G - \underline{\gamma} = 1 - \underline{\gamma} < 1. \end{aligned}$$

Suppose we have  $b$  and  $\hat{b}$  both satisfying (1.1') for the same  $z$ , where  $b, \hat{b}$  are close to  $\underline{b}$  and  $z$  is close enough to  $x_0$  so that  $\mathcal{I}m bz > 0$  and  $\mathcal{I}m \hat{b}z > 0$ . Then we can write  $b - \hat{b} = (b - \hat{b})\alpha$ , and by continuity of  $\alpha$  and the fact that  $|\underline{\alpha}| < 1$ , we must have  $|\alpha| < 1$  for all of these  $b, \hat{b}$  and  $z$  sufficiently close to  $\underline{b}$  and  $x_0$ , respectively. Therefore,  $b = \hat{b}$ . Then the function  $z(b)$  is one-to-one near  $\underline{b}$  and hence has an analytic inverse  $b(z)$  for  $z$  near  $x_0$ . By Theorem 4.1 we must have  $b(z) = 1 + \sigma^2 c m_F(z)$  for  $z \in \mathbb{C}^+$ , and hence  $m_F$  extends analytically onto an interval about  $x_0$ . Therefore we get

$$\underline{m}(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for  $x$  near  $x_0$  and some  $a_n \in \mathbb{C}$ , and hence

$$f(x) = \frac{1}{\pi} \sum_{n=0}^{\infty} \mathcal{I}m a_n (x - x_0)^n. \quad (5.12)$$

Now, suppose  $m'_H(\underline{w}) = 0$ . We form the function  $u$  of the two complex variables  $b, z$  by

$$u(b, z) = m_H(b^2 z - b\sigma^2(1 - c)) - \frac{1}{\sigma^2 c} \left( 1 - \frac{1}{b} \right)$$

which is analytic near  $(\underline{b}, x_0) \in \mathbb{C}^2$ . Then we have  $u(\underline{b}, x_0) = 0$ . Taking the derivative with respect to  $b$  we get

$$\frac{\partial u}{\partial b}(\underline{b}, x_0) = m'_H(\underline{w})(2\underline{b}x_0 - \sigma^2(1 - c)) - \frac{1}{\sigma^2 c \underline{b}^2} = -\frac{1}{\sigma^2 c \underline{b}^2} \neq 0.$$

By the implicit function theorem (Krantz [4] p.54) there is a unique analytic solution  $b(z)$  in a neighborhood of  $x_0$  such that  $b(x_0) = \underline{b}$ . Since  $m_F$  is an analytic solution to (1.1) in  $\mathbb{C}^+$ , we must have  $b(z) = 1 + \sigma^2 c m_F(z)$  by uniqueness of  $b(z)$ , and hence  $m_F$

extends analytically to an interval about  $x_0$ , and again we have (5.12). Therefore,  $f(x)$  is analytic where it is positive, and the proof is complete.

## 6. THE SUPPORT OF $F$

Let  $S_F$  and  $S_H$  denote the support of  $F$  and  $H$ , respectively. Clearly, by definition of  $F$  and  $H$ , we have  $S_F \subset [0, \infty)$  and  $S_H \subset [0, \infty)$ . We begin our analysis of  $S_F$  with the following result.

**THEOREM 6.1.**  $F$  has no mass at 0.

*Proof.* The method we will use to prove the lemma was previously used in Silverstein and Choi [10].

For any p.d.f.  $G$  we have

$$\lim_{y \downarrow 0} iym_G(iy) = -G\{0\} + \lim_{y \downarrow 0} \int_{(0, \infty)} \frac{iy}{\lambda - iy} dG(\lambda) = -G\{0\},$$

by dominated convergence, and therefore, if  $G\{0\} > 0$ , we must deduce that  $|m_G(iy)| \rightarrow \infty$  as  $y \downarrow 0$ .

Suppose  $F\{0\} > 0$ . From (1.1) we have

$$iym(iy) = \int \frac{iy}{\frac{t}{1+\sigma^2 cm(iy)} - (1 + \sigma^2 cm(iy))iy + \sigma^2(1-c)} dH(t).$$

Since  $F\{0\} > 0$  we find, for any  $t \geq 0$ , as  $y \downarrow 0$

$$\frac{iy}{\frac{t}{1+\sigma^2 cm(iy)} - (1 + \sigma^2 cm(iy))iy + \sigma^2(1-c)} \rightarrow \frac{0}{\sigma^2 cF\{0\} + \sigma^2(1-c)} = 0,$$

and since

$$\left| \frac{iy}{\frac{t}{1+\sigma^2 cm(iy)} - (1 + \sigma^2 cm(iy))iy + \sigma^2(1-c)} \right| \leq \frac{y}{\mathcal{I}m iyb(iy)} = \frac{1}{b_1(iy)} < \infty,$$

by Lemma 5.1 (a), we have, by dominated convergence,  $\lim_{y \downarrow 0} iym(iy) = 0$ , a contradiction. Therefore,  $F\{0\} = 0$ .

The fact that  $F\{0\} = 0$  gives no information on whether or not  $0 \in S_F$ . Simulations have shown that either case can occur, depending on  $H$  and the values of  $c$  and  $\sigma$ .

A method to identify the support of  $F$  is presented next.

First, we give a lemma that will be used in the proof of Theorem 6.3.

**LEMMA 6.1.** If  $b, \mathbf{b} \in \mathbb{R}$  are positive and both satisfy (1.1') for  $z = x \in \mathbb{R}$ ,  $x < 0$ , then  $b = \mathbf{b}$ .

*Proof.* First, note that for  $t \geq 0$ ,  $b > 0$ , and  $x < 0$  we have

$$\frac{1}{|\frac{t}{b} - bx + \sigma^2(1-c)|} = \frac{1}{\frac{t}{b} + b|x| + \sigma^2(1-c)} \leq \frac{1}{b|x|} < \infty,$$

and therefore the integral in (1.1') is well-defined for both  $b$  and  $\mathbf{b}$ . We write  $b - \mathbf{b} = (b - \mathbf{b})\alpha$ , where

$$\alpha = \sigma^2 c \int \frac{\frac{t}{b\mathbf{b}} + x}{(\frac{t}{b} - bx + \sigma^2(1-c))(\frac{t}{\mathbf{b}} - \mathbf{b}x + \sigma^2(1-c))} dH(t).$$

Again, following the procedure from chapter 4, we use the Cauchy-Schwarz and triangle inequalities to obtain

$$|\alpha| \leq (g(b))^{\frac{1}{2}}(g(\mathbf{b}))^{\frac{1}{2}} + |x|(G(b)G(\mathbf{b}))^{\frac{1}{2}}. \quad (6.1)$$

From (4.2) we get  $b(1 - g(b) + xG(b)) = 1 + \sigma^2(1-c)G(b) > 0$ , and since  $b > 0$  we have

$$g(b) < 1 + xG(b) = 1 - |x|G(b) \quad (6.2)$$

and similarly for  $\mathbf{b}$ . Substituting this into (6.1) and using (4.8) we find

$$\begin{aligned} |\alpha| &< (1 - |x|G(b))^{\frac{1}{2}}(1 - |x|G(\mathbf{b}))^{\frac{1}{2}} + |x|(G(b)G(\mathbf{b}))^{\frac{1}{2}} \\ &\leq 1 - (|x|G(b)|x|G(\mathbf{b}))^{\frac{1}{2}} + |x|(G(b)G(\mathbf{b}))^{\frac{1}{2}} = 1. \end{aligned}$$

Therefore,  $b = \mathbf{b}$ , and the proof is complete.

Suppose we have  $x \in \mathbb{R} - \{0\}$ . If  $x \in S_F^c$ , we have that  $\underline{m}(x)$  is real, continuous, and increasing, and therefore so is  $\underline{m}^{-1}(x)$ . Let  $b(z) = b_1(z) + ib_2(z) = 1 + \sigma^2 c m_F(z)$ .

Since  $b(z)$  is a well-defined, analytic function for  $z$  in a neighborhood of  $x$ , we have that the function  $w(z) \equiv b^2(z)z - b(z)\sigma^2(1-c)$  is also well-defined and analytic in such a neighborhood. In the next theorem, we will show that  $w(x) \in S_H^c$ , and therefore we may write the inverse of  $\underline{m}$ , expressed in terms of  $b \in \mathbb{R}$ , as

$$x(b) = \frac{1}{b^2} m_H^{-1} \left( \frac{1}{\sigma^2 c} \left( 1 - \frac{1}{b} \right) \right) + \frac{1}{b} \sigma^2 (1-c).$$

**THEOREM 6.2.** If  $x \in S_F^c$ , then  $w(x) \in S_H^c$ .

*Proof.* Let  $(l_1, l_2) \subset [L_1, L_2] \subset S_H^c$  and choose  $x_0 \in (l_1, l_2)$ . Since  $x_0 \in S_F^c$ ,  $m_F(z)$  is analytic in a neighborhood  $V$  of  $x_0$  with  $V \cap \mathbb{R} \subset (l_1, l_2)$ , and therefore  $w(z)$  is also analytic in  $V$ . Note that  $w(z) = z + 2\sigma^2 c m_F(z) z + (\sigma^2 c)^2 m_F^2(z) z - \sigma^2(1-c) - \sigma^2 c \sigma^2(1-c) m_F(z)$ . Let  $\zeta = u + iv$ , where  $u \in \mathbb{R}$  is fixed and  $v \rightarrow \infty$ . Since  $m_F(\zeta) \rightarrow 0$ ,  $m_F(\zeta)\zeta$  is bounded, and  $m_F^2(\zeta)\zeta \rightarrow 0$ , we have  $w(\zeta) \rightarrow \infty$ , and hence  $w(z)$  is nonconstant. Therefore, by the open mapping theorem,  $w(V)$  is an open set.

For  $z \in \mathbb{C}^+$  we have  $b(z) \in \mathbb{C}^+$ , and therefore  $w(z) \in \mathbb{C}^+$  by (5.2). Therefore, by (1.1) and Lemma 5.1 (a), we get for any  $z \in \mathbb{C}^+$

$$\frac{m_F(z)}{1 + \sigma^2 c m_F(z)} = m_H(w(z)) = \int \frac{dH(t)}{t - w(z)},$$

which gives

$$\mathcal{I}m m_H(w(z)) = \mathcal{I}m w(z) \int \frac{dH(t)}{|t - w(z)|^2} = \frac{\mathcal{I}m m_F(z)}{|1 + \sigma^2 c m_F(z)|^2}. \quad (6.3)$$

Let  $w_0 \in w(V) \cap \mathbb{R}$  be arbitrary. Take a sequence  $\{w_j\} \subset w(V) \cap \mathbb{C}^+$  such that  $w_j \rightarrow w_0$ . There exists a sequence  $\{z_j\} \subset V$  for which  $w_j = w(z_j)$  for each  $j$ . For any  $z \in V$  we have  $\overline{b(\bar{z})} = b(z)$ , and consequently,  $\overline{w(\bar{z})} = w(z)$ . Therefore,  $\{z_j\} \subset \mathbb{C}^+$ . Since the  $z_j$ 's are bounded, there exists a subsequence  $\{z_{j_k}\} \subset \{z_j\}$  that converges to some  $z_0 \in \overline{V}$ . If  $z_0 \in \mathbb{C}^+$ , then  $G(b(z_0)) > 0$  and  $b_2(z_0) > 0$ , and therefore (4.4) gives  $\mathcal{I}m w(z_0) = \mathcal{I}m w_0 > 0$ , a contradiction. Then we must have  $z_0 \in \mathbb{R}$ , and hence  $z_0 \in S_F^c$ . Therefore  $\mathcal{I}m m_F(z_0) = 0$ . If  $z_0 \neq 0$ , Lemma 5.1 (c) gives  $b(z_0) > 0$ . If  $z_0 = 0$  we may assume, without loss of generality, that  $0 \in [L_1, L_2]$ , and therefore we

have

$$b(z_0) = 1 + \sigma^2 c \int \frac{dF(\lambda)}{\lambda - z_0} = 1 + \sigma^2 c \int_{(L_2, \infty)} \frac{dF(\lambda)}{\lambda} > 1.$$

Therefore, in either case,  $b(z_0) > 0$ , and (6.3) gives

$$\lim_{k \rightarrow \infty} \mathcal{I}m m_H(w(z_{jk})) = \frac{\mathcal{I}m m_F(z_0)}{|b(z_0)|^2} = 0.$$

Hence by Theorem 5.2,  $H$  is differentiable at  $w_0$  and its derivative is 0. Since  $w_0$  is arbitrary in  $w(V) \cap \mathbb{R}$ , we have  $w(V) \cap \mathbb{R} \subset S_H^c$ , and therefore  $w(x_0) \in S_H^c$ , and since  $x_0$  is arbitrary in  $S_F^c$ , the proof is complete.

So far we have shown that if we have an  $x$  outside the support of  $F$ , the corresponding  $w(x)$  is outside the support of  $H$ , and we have an expression for the inverse of  $\underline{m}$ . Therefore, if we graph the inverse  $x(b)$  and identify an interval of points in  $S_F^c$  on the vertical axis,  $x(b)$  will be increasing on that interval, but does the presence of an interval on the vertical axis for which  $x(b)$  is increasing always yield an interval in  $S_F^c$ ? The answer is yes, if  $b > 0$ , as Theorem 6.3 will show. To prove this semi-converse we proceed as follows.

Suppose we have  $(l_1, l_2) \subset [L_1, L_2] \subset S_H^c$ . Then  $m_H(\cdot)$  is increasing on  $(l_1, l_2)$  and maps  $(l_1, l_2)$  onto some interval  $(d_1, d_2)$ . Now,  $\frac{1}{\sigma^2 c} \left(1 - \frac{1}{b}\right)$  is an increasing function of  $b$  from  $(0, \infty)$  onto  $(-\infty, \frac{1}{\sigma^2 c})$ . Since  $b \leq 0$  does not correspond to our Stieltjes transform by Lemma 5.1 (a),(c), we may assume  $(l_1, l_2)$  is chosen so that  $(d_1, d_2) \subset (-\infty, \frac{1}{\sigma^2 c})$ . Therefore there is an interval  $(k_1, k_2) \subset (0, \infty)$  such that the mapping

$$b \mapsto \frac{1}{\sigma^2 c} \left(1 - \frac{1}{b}\right) \tag{6.4}$$

is a one-to-one correspondence from  $(k_1, k_2)$  to  $(d_1, d_2)$ . Therefore  $m_H^{-1} \left( \frac{1}{\sigma^2 c} \left(1 - \frac{1}{b}\right) \right)$  is well-defined from  $(k_1, k_2)$  to  $(l_1, l_2)$ , and hence we define

$$x(b) = \frac{1}{b^2} m_H^{-1} \left( \frac{1}{\sigma^2 c} \left(1 - \frac{1}{b}\right) \right) + \frac{1}{b} \sigma^2 (1 - c) \tag{6.5}$$

for  $b \in (k_1, k_2)$ . The next theorem will show that at a point  $b \in (k_1, k_2)$  for which  $x'(b) > 0$ , we have  $x(b) \in S_F^c$ , and  $b = 1 + \sigma^2 c m_F(x(b))$ .

**THEOREM 6.3.** Let  $b \in (k_1, k_2)$  and  $x'(b) > 0$ . Then  $x(b) \in S_F^c$  and  $b = 1 + \sigma^2 c m_F(x(b))$ .

*Proof.* Let  $(\underline{k}_1, \underline{k}_2) \subset (k_1, k_2)$  be an interval on which  $x'(b) > 0$ . Fix  $\mathbf{b} \in (\underline{k}_1, \underline{k}_2)$ . If  $x(\mathbf{b}) < 0$ , we immediately have  $x(\mathbf{b}) \in S_F^c$ , and by Lemma 6.1 we must have  $\mathbf{b} = 1 + \sigma^2 c m_F(x(\mathbf{b}))$ . Therefore we assume  $x(\mathbf{b}) \geq 0$ . Let  $D$  be an open set in  $\mathbb{C}$  such that  $D \cap \mathbb{R} = (\underline{k}_1, \underline{k}_2)$ . Since  $x$  is analytic on  $(\underline{k}_1, \underline{k}_2)$ , we may write  $x(b)$  in a power series expansion centered at  $\mathbf{b}$ , and therefore, for  $b \in D$ , the function

$$z(b) \equiv x(\mathbf{b}) + \sum_{j=1}^{\infty} \frac{x^{(j)}(\mathbf{b})}{j!} (b - \mathbf{b})^j \quad (6.6)$$

is the analytic extension of  $x$  onto  $D$ . Using (6.6) we write  $z(b) = x(\mathbf{b}) + x'(\mathbf{b})(b - \mathbf{b}) + \theta(b)$  where  $\theta(b) = o(b - \mathbf{b})$ . Since  $x'(\mathbf{b}) > 0$ , it is clear that we may choose  $\hat{b} = \hat{b}_1 + i\hat{b}_2 \in D \cap \mathbb{C}^+$  sufficiently close to  $\mathbf{b}$  to ensure that  $z(\hat{b}) \in \mathbb{C}^+$ , and since  $\hat{b}z(\hat{b}) = \hat{b}x(\mathbf{b}) + x'(\mathbf{b})\hat{b}(\hat{b} - \mathbf{b}) + \hat{b}\theta(\hat{b})$  and  $x(\mathbf{b}) \geq 0$  we have  $\mathcal{I}m \hat{b}z(\hat{b}) = \hat{b}_2(x(\mathbf{b}) + x'(\mathbf{b})(2\hat{b}_1 - \mathbf{b})) + \mathcal{I}m \hat{b}\theta(\hat{b}) > 0$  for  $\hat{b} \in D \cap \mathbb{C}^+$  close enough to  $\mathbf{b}$ . Therefore we have  $\mathcal{I}m z(\hat{b}) > 0$ ,  $\mathcal{I}m \hat{b}z(\hat{b}) > 0$ , and

$$z(\hat{b}) = \frac{1}{\hat{b}^2} m_H^{-1} \left( \frac{1}{\sigma^2 c} \left( 1 - \frac{1}{\hat{b}} \right) \right) + \frac{1}{\hat{b}} \sigma^2 (1 - c). \quad (6.7)$$

Hence, by Theorem 4.1,  $\hat{b}$  is the unique solution to (1.1') for  $z = z(\hat{b})$ , that is,  $\hat{b} = 1 + \sigma^2 c m_F(z(\hat{b}))$ . Therefore,  $b_F = 1 + \sigma^2 c m_F$  extends analytically onto a neighborhood  $B$  of  $\mathbf{b}$  and its inverse is given by (6.7).

Choose a sequence  $\{z_j\} \subset z(B) \cap \mathbb{C}^+$  such that  $z_j \rightarrow z(\mathbf{b})$  ( $= x(\mathbf{b})$ ). Then we have  $b_F(z_j) = 1 + \sigma^2 c m_F(z_j) \rightarrow b_F(z(\mathbf{b})) = \mathbf{b}$ , and consequently  $\mathcal{I}m m_F(z_j) \rightarrow 0$  as  $j \rightarrow \infty$ . By Theorem 5.2,  $F$  is differentiable at  $x(\mathbf{b})$ , and its derivative is 0. Since  $\mathbf{b} \in (\underline{k}_1, \underline{k}_2)$  is arbitrary we have  $F'(x) = 0$  for all  $x = x(b) \in (x(\underline{k}_1), x(\underline{k}_2))$ , and therefore these  $x$ 's are outside  $S_F$ . Moreover,  $m_F$  is analytic in  $\mathbb{C}^+ \cup (x(\underline{k}_1), x(\underline{k}_2))$ ,

and therefore  $b = 1 + \sigma^2 c m_F(x(b))$  for any  $b \in (k_1, k_2)$  for which  $x'(b) > 0$ , and this completes the proof.

As a result of Theorems 6.2 and 6.3, we now have a method whereby we may graphically identify the support of  $F$ . The first step of the procedure is to choose an open interval  $I_H \subset S_H^c$  such that  $\bar{I}_H$  is not in  $S_H^c$ , that is,  $I_H$  is not a subset of a larger interval in  $S_H^c$ . On  $I_H$ ,  $m_H$  is increasing and maps to an interval  $(d_1, d_2)$ . Since the function (6.4) maps positive values of  $b$  onto  $(-\infty, \frac{1}{\sigma^2 c})$ , we take only those intervals  $I_H$  for which  $(\underline{d}_1, \underline{d}_2) \equiv (d_1, d_2) \cap (-\infty, \frac{1}{\sigma^2 c})$  is nonempty, and disregard any  $I_H$  for which this intersection is empty. Let  $(k_1, k_2)$  be the pre-image of  $(d_1, d_2)$  under the mapping given in (6.4). Therefore,  $m_H^{-1} \left( \frac{1}{\sigma^2 c} \left( 1 - \frac{1}{b} \right) \right)$  is well-defined from  $(k_1, k_2)$  to  $\tilde{I}_H \equiv \{t \in I_H : m_H(t) \in (\underline{d}_1, \underline{d}_2)\}$ , and hence we may graph the function  $x(b)$  given by (6.5) for  $b \in (k_1, k_2)$ . We then identify all intervals on the vertical axis where the graph of  $x$  is increasing. By Theorem 6.3, we know that these intervals are outside  $S_F$ , and therefore we remove them from  $\mathbb{R}$ , and  $S_F$  must be contained in what is left. We continue in this manner for every interval  $I_H \subset S_H^c$ . Let  $D$  be the resulting set. Since, by Theorem 6.2, every  $x \in S_F^c$  corresponds (via  $w(x)$ ) to a point in  $S_H^c$ , we must have  $D = S_F$ . Also, for each interval  $I_F \subset S_F^c$ , there is only one interval  $I_H \subset S_H^c$  for which our procedure produces  $I_F$ . In other words, the intervals outside  $S_F$  that are being removed from  $\mathbb{R}$  in the above procedure will not overlap each other. To see this, we note that (6.5) gives

$$m_H^{-1} \left( \frac{1}{\sigma^2 c} \left( 1 - \frac{1}{b} \right) \right) = b^2 x(b) - b\sigma^2(1 - c). \quad (6.8)$$

By Theorem 6.3, for each  $x \in I_F$ , there is a unique  $b$ , namely  $b = 1 + \sigma^2 c m_F(x)$ , such that  $x = x(b)$ . Therefore,  $I_F$  uniquely determines the range of the left-hand side of (6.8), which is an interval in  $S_H^c$ . Consequently, once we eliminate an interval from being in  $S_F$ , we will never again encounter any portion of this interval in subsequent steps of the procedure.

## 7. BEHAVIOR NEAR A BOUNDARY POINT

We now focus on the behavior of the density  $f$  near boundary points of  $S_F$ . Let  $a$  be a left end-point of  $S_F$ , and let  $\epsilon > 0$  be sufficiently small so that  $(a - \epsilon, a) \subset S_F^c$ . Therefore, by the previous chapter, there exists an interval  $(l_1, l_2) \subset S_H^c$  from which we can construct a well-defined, analytic function  $x(b)$  given by the representation in (6.5), for  $b$  in some interval  $(k_1, k_2) \subset (0, \infty)$ , such that  $(a - \epsilon, a)$  is in the range of  $x(b)$  and  $x'(b)$  is positive over these range values. We now assume that  $[a - \epsilon, a]$  is in the range of  $x(b)$ , and, in particular, we define  $b_*, b_a \in (k_1, k_2)$  so that  $x(b_*) = a - \epsilon$  and  $x(b_a) = a$ . Therefore  $b_* < b_a$ , and  $x(b)$  is defined on both sides of  $b_a$ .

Note that our assumption may not occur for certain choices of  $H$ . It may be the case that  $\lim_{b \uparrow b_a} x(b)$  exists, but  $x(b)$  is not defined at  $b_a$ , which can possibly occur if  $b_a^2 a - b_a \sigma^2 (1 - c) \in \partial S_H$  and  $m'_H(w)$  exists as  $w \rightarrow b_a^2 a - b_a \sigma^2 (1 - c)$ . However, our assumption is valid, for example, when  $H$  is discrete, since  $m'_H$  will not exist on  $\partial S_H$ . This constitutes the most relevant cases of application of our model. A non-discrete  $H$  would only be considered if it approximates the population eigenvalues in an analytically tractable manner.

Since  $x(b)$  is analytic with  $x'(b) > 0$  for all  $b \in (b_*, b_a)$  and  $a$  is a left end-point of  $S_F$ , we must have  $x'(b_a) = 0$ , and the next theorem will imply that  $b_a$  is a relative maximum of  $x$ .

**THEOREM 7.1.** For some  $\delta > 0$  for which  $b_a + \delta < k_2$  we have  $x'(b) < 0$  for all  $b \in (b_a, b_a + \delta)$ .

*Proof.* Suppose  $x^{(j)}(b_a)$  is the first non-vanishing derivative of  $x(b)$  at  $b_a$ . Then for all  $b$  in some interval  $(b_a, b_a + \delta) \subset (b_a, k_2)$ ,  $x^{(j)}(b)$  is of one sign, and therefore each of the

first  $j - 1$  derivatives do not change sign in this interval. If  $x'(b) > 0$  on  $(b_a, b_a + \delta)$  then we would have  $(x(b_a), x(b_a + \delta)) = (a, x(b_a + \delta)) \subset S_F^c$ , and consequently,  $a$  would be an isolated point in  $S_F$ , an impossibility since  $F$  has a continuous density on  $\mathbb{R} - \{0\}$ . Therefore we must have  $x'(b) < 0$  for all  $b \in (b_a, b_a + \delta)$ , and the proof is complete.

Let  $k^* \in (b_a, k_2)$  be such that  $x'(b) < 0$  for all  $b \in (b_a, k^*)$ . Define the interval  $(\underline{l}_1, \underline{l}_2) \subset (l_1, l_2)$  to be the image of  $(b_*, k^*)$  under the mapping  $m_H^{-1} \left( \frac{1}{\sigma^2 c} \left( 1 - \frac{1}{b} \right) \right)$ . For  $z \in \mathbb{C}^+$  let  $b_F(z) \equiv 1 + \sigma^2 c m_F(z)$ . Write  $\lim_{z \in \mathbb{C}^+ \rightarrow x} b_F(z) \equiv \underline{b}(x) = \underline{b}_1(x) + i \underline{b}_2(x)$  for  $x \in \mathbb{R} - \{0\}$ . We have  $(\underline{b}_1(a), \underline{b}_2(a)) = (b_a, 0)$ . Choose  $\delta$  sufficiently small so that for  $x \in (a, a + \delta)$  we have  $\underline{b}_1(x) \in (b_*, k^*)$  and  $\underline{b}_1^2(x)x - \underline{b}_1(x)\sigma^2(1 - c) \in (\underline{l}_1, \underline{l}_2)$ .

We argue that  $f(x) = \frac{1}{\sigma^2 c \pi} \underline{b}_2(x) > 0$  for all  $x \in (a, a + \delta)$ . Suppose  $x_0 \in (a, a + \delta)$  is such that  $\underline{b}_2(x_0) = 0$ . Letting  $\hat{b} = \underline{b}_1(x_0)$ , we have  $x(\hat{b}) = x_0$ . It is obvious that  $\hat{b} \neq b_a$ , and if  $\hat{b} \in (b_*, b_a)$ , then  $x_0 < a$ , a contradiction. Therefore  $\hat{b} \in (b_a, k^*)$ , and hence,  $x'(\hat{b}) < 0$ . For any  $b \in (b_a, k^*)$  we have from (1.1')

$$b = 1 + \sigma^2 c b m_H(b^2 x(b) - b \sigma^2 (1 - c)),$$

and therefore differentiating implicitly with respect to  $b$  we get

$$x'(b) = \frac{1 - \sigma^2 c b^2 m'_H(b^2 x(b) - b \sigma^2 (1 - c))(2b x(b) - \sigma^2 (1 - c))}{\sigma^2 c b^4 m'_H(b^2 x(b) - b \sigma^2 (1 - c))} < 0. \quad (7.1)$$

Since  $b$  is real, we have

$$\begin{aligned} \sigma^2 c b^2 m'_H(b^2 x(b) - b \sigma^2 (1 - c)) &= \sigma^2 c b^2 \int \frac{dH(t)}{(t - (b^2 x(b) - b \sigma^2 (1 - c)))^2} \\ &= \sigma^2 c \int \frac{dH(t)}{\left(\frac{t}{b} - b x(b) + \sigma^2 (1 - c)\right)^2} = G(b), \end{aligned}$$

and therefore (7.1) yields

$$x'(b) = \frac{1 - G(b)(2b x(b) - \sigma^2 (1 - c))}{b^2 G(b)} < 0. \quad (7.2)$$

Let  $z = z_1 + iz_2 \in \mathbb{C}^+$  and  $b(z) = b_1(z) + ib_2(z) \equiv b_F(z)$ . From (5.2) we get

$$\frac{z_2}{b_2(z)} = \frac{1 - G(b(z))(2b_1(z)z_1 - \sigma^2(1 - c)) + b_2(z)z_2G(b(z))}{b_1^2(z)G(b(z))} > 0. \quad (7.3)$$

Letting  $z \rightarrow x_0$  we have  $b(z) \rightarrow \hat{b}$  and therefore (7.3) gives

$$\frac{1 - G(\hat{b})(2\hat{b}x_0 - \sigma^2(1 - c))}{\hat{b}^2G(\hat{b})} \geq 0,$$

a contradiction of (7.2). Therefore,  $\underline{b}_2(x_0) > 0$ , and hence  $f(x) > 0$  for all  $x \in (a, a + \delta)$ .

**THEOREM 7.2.**  $x''(b_a) < 0$ .

*Proof.* Since  $x'(b_a) = 0$ , we have, by Theorem 7.1, that  $b_a$  is a relative maximum of  $x$ . Therefore,  $x''(b_a) \leq 0$ . Since the first non-vanishing derivative of a function at a relative extreme must be of even order, we will assume  $x''(b_a) = 0$  and  $x'''(b_a) = 0$ , and proceed to show a contradiction.

Let  $w \equiv b^2x(b) - b\sigma^2(1 - c)$ ,  $w_a \equiv b_a^2a - b_a\sigma^2(1 - c)$ ,  $d \equiv 2bx(b) - \sigma^2(1 - c)$ ,  $d_a \equiv 2b_aa - \sigma^2(1 - c)$ , and define

$$A_j = \int \frac{dH(t)}{(t - w_a)^j} \quad \text{for } j = 2, 3, 4$$

so that  $m'_H(w_a) = A_2$ ,  $m''_H(w_a) = 2A_3$ , and  $m'''_H(w_a) = 6A_4$ . Writing (1.1') as

$$\frac{1}{\sigma^2c} \left(1 - \frac{1}{b}\right) = m_H(w),$$

and differentiating implicitly with respect to  $b$  three times results in the following three equations

$$\begin{aligned} \frac{1}{\sigma^2cb^2} &= m'_H(w)(d + b^2x'(b)) \\ \frac{-2}{\sigma^2cb^3} &= m''_H(w)(d + b^2x'(b))^2 + m'_H(w)(2x(b) + 4bx'(b) + b^2x''(b)) \\ \frac{6}{\sigma^2cb^4} &= m'''_H(w)(d + b^2x'(b))^3 + 3m''_H(w)(d + b^2x'(b))(2x(b) + 4bx'(b) + b^2x''(b)) \\ &\quad + m'_H(w)(6x'(b) + 6bx''(b) + b^2x'''(b)). \end{aligned}$$

Now, we evaluate these equations at the point  $b_a$  and use the assumption that the first three derivatives of  $x$  are zero to get the following three equations in terms of the  $A_j$ 's

$$\begin{aligned}d_a A_2 &= \frac{1}{\sigma^2 c b_a^2} \\d_a^2 A_3 + a A_2 &= -\frac{1}{\sigma^2 c b_a^3} \\d_a^3 A_4 + 2a d_a A_3 &= \frac{1}{\sigma^2 c b_a^4}.\end{aligned}$$

Note that the first equation implies  $d_a > 0$ . Solving for  $A_3$  and  $A_4$  we get

$$A_3 = -\frac{1}{\sigma^2 c} \left( \frac{a}{d_a^3 b_a^2} + \frac{1}{d_a^2 b_a^3} \right)$$

and

$$A_4 = \frac{1}{\sigma^2 c} \left( \frac{2a^2}{d_a^5 b_a^2} + \frac{2a}{d_a^4 b_a^3} + \frac{1}{d_a^3 b_a^4} \right).$$

Writing  $w_a = d_a b_a - b_a^2 a$  and  $A_3 = \int \frac{t}{(t-w_a)^4} dH(t) - w_a A_4$  we get

$$0 < \int \frac{t}{(t-w_a)^4} dH(t) = w_a A_4 + A_3 = \frac{-2a^3 b_a^4}{\sigma^2 c d_a^5 b_a^4},$$

a contradiction since  $a$  and  $d_a$  are both positive. Therefore,  $x''(b_a) < 0$ .

We now show that the density  $f$  resembles a square root function in a neighborhood to the right of  $a$ .

Since  $m'_H(w_a) \neq 0$ , there exists a neighborhood  $W \subset \mathbb{C}$  of  $w_a$  on which  $m_H$  is one-to-one, and hence has an analytic inverse. Let  $B \subset \mathbb{C}$  and  $U \subset \mathbb{C}$  be neighborhoods of  $b_a$  and  $a$ , respectively. Define

$$W_0 \equiv \{w \in \mathbb{C} : w = b^2 z - b\sigma^2(1-c) \text{ for } b \in B \text{ and } z \in U\}.$$

Choose  $B$  and  $U$  sufficiently small so that  $B \cap \mathbb{R} \subset (k_1, k_2)$ ,  $W_0 \subset W$  and  $W_0 \cap \mathbb{R} \subset (l_1, l_2)$ . Then  $m_H^{-1} \left( \frac{1}{\sigma^2 c} \left( 1 - \frac{1}{b} \right) \right)$  is an analytic mapping from  $B$  to  $W_0$ . For  $b \in B$  define

$$z(b) = \frac{1}{b^2} m_H^{-1} \left( \frac{1}{\sigma^2 c} \left( 1 - \frac{1}{b} \right) \right) + \frac{1}{b} \sigma^2 (1-c).$$

Therefore, if  $b \in B \cap \mathbb{R}$ , we have  $z(b) = x(b)$ , and hence  $z'(b_a) = x'(b_a) = 0$  and  $z''(b_a) = x''(b_a) < 0$ . By Theorem 10.32 of Rudin [6], there is a neighborhood  $V \subset B$  of  $b_a$  and a function  $\phi$ , analytic in  $V$ , such that

$$z(b) - a = (\phi(b))^2 \quad \text{for all } b \in V, \quad (7.4)$$

$\phi'$  has no zero in  $V$ , and  $\phi$  is an invertible mapping of  $V$  onto a disc centered at the origin. We then have  $\phi(b_a) = 0$ ,  $\phi'(b_a) \neq 0$ , and computing the first two derivatives on both sides of (7.4) we get

$$z'(b) = 2\phi(b)\phi'(b)$$

and

$$z''(b) = 2[\phi'(b)]^2 + 2\phi(b)\phi''(b).$$

Therefore

$$0 > z''(b_a) = 2[\phi'(b_a)]^2, \quad (7.5)$$

and hence  $\phi'(b_a)$  must be purely imaginary. Write  $\frac{1}{\phi'(b_a)} = i\alpha$ , where  $\alpha \in \mathbb{R}$  is nonzero.

Let  $\delta > 0$  be small enough so that  $f$  is positive over  $(a, a+\delta)$  and  $(a, a+\delta) \subset U \cap \mathbb{R}$ . Fix  $x \in (a, a+\delta)$ . Since  $\underline{m}(x)$  satisfies (1.1) for  $z = x$  we immediately have

$$x - a = [\phi(\underline{b}(x))]^2.$$

Since  $x > a$ , we may take the square root of both sides to get

$$\phi(\underline{b}(x)) = \sqrt{x - a},$$

where we assume that  $\phi(\underline{b}(x))$  is the positive root. Let  $\Gamma$  be the inverse of  $\phi$  on  $V$ .

Then  $\Gamma(0) = b_a$  and  $\Gamma'(0) = \frac{1}{\phi'(b_a)} = i\alpha$ , and expanding  $\Gamma$  about 0 we have

$$\begin{aligned} \underline{b}(x) &= \Gamma(\sqrt{x - a}) = \Gamma(0) + \Gamma'(0)\sqrt{x - a} + (\text{higher order terms}) \\ &= b_a + i\alpha\sqrt{x - a} + (\text{higher order terms}). \end{aligned}$$

Therefore

$$\underline{b}_2(x) = \sqrt{x - a}(\alpha + (\text{higher order terms})),$$

and hence, for  $x \in (a, a + \delta)$ , we have expressed  $f(x) = \frac{1}{\sigma^2 c \pi} \underline{b}_2(x)$  as an analytic function of  $\sqrt{x - a}$ . This is a stronger result than what was proven for the density in Silverstein and Choi [10], although the same method used here may be applied to that case and yield the same strong result.

The  $a$  we used was a left end-point of  $S_F$ . If  $a$  were a right end-point of  $S_F$ , the analysis would differ only slightly from what we have done here. In that case we assume that  $b_a$  is a relative minimum of  $x(b)$ , and therefore (7.5) becomes

$$0 < z''(b_a) = 2[\phi'(b_a)]^2,$$

giving that  $\phi'(b_a)$  is nonzero and real. Write  $\frac{1}{\phi'(b_a)} = \alpha$ . Let  $\delta > 0$  be small enough so that  $f$  is positive over  $(a - \delta, a)$  and  $(a - \delta, a) \subset U \cap \mathbb{R}$ . Fixing  $x \in (a - \delta, a)$ , we again have

$$x - a = [\phi(\underline{b}(x))]^2,$$

and hence, since  $x < a$ , we get

$$\phi(\underline{b}(x)) = i\sqrt{|x - a|},$$

where the square root is assumed to be positive. Again letting  $\Gamma$  be the inverse of  $\phi$  on  $V$  we have  $\Gamma(0) = b_a$  and  $\Gamma'(0) = \alpha$ . Expanding  $\Gamma$  about 0 we get

$$\begin{aligned} \underline{b}(x) &= \Gamma(i\sqrt{|x - a|}) = \Gamma(0) + i\Gamma'(0)\sqrt{|x - a|} + (\text{higher order terms}) \\ &= b_a + i\alpha\sqrt{|x - a|} + (\text{higher order terms}), \end{aligned}$$

and therefore

$$\underline{b}_2(x) = \sqrt{|x - a|}(\alpha + (\text{higher order terms})).$$

## 8. AN EXAMPLE AND APPLICATION

In this chapter we graphically analyze the limiting density and the procedure for finding  $S_F$  for a particular example of  $F$ . We compare the results of a simulation to our density graph, and use the comparisons to analyze the problem of signal detection in array signal processing.

As noted earlier,  $F$  is nonrandom and only depends on the distribution  $H$  and the constants  $c$  and  $\sigma$ . We construct our example by letting  $c = .1$  and  $\sigma = 1$  and taking  $H$  to be discrete with mass .2, .4, and .4 at the respective values 0, 3, and 10.

In chapter 6 we described a method by which  $S_F$  may be obtained. From each interval  $I_H \subset S_H^c$  we construct a well-defined function  $x$  given (in terms of  $b = 1 + \sigma^2 cm$ ) by

$$x(b) = \frac{1}{b^2} m_H^{-1} \left( \frac{1}{\sigma^2 c} \left( 1 - \frac{1}{b} \right) \right) + \frac{1}{b} \sigma^2 (1 - c) \quad (8.1)$$

for  $b$  in some interval  $(k_1, k_2) \subset (0, \infty)$  prescribed by  $I_H$ . We graph this function and remove the intervals along the vertical axis where the graph is increasing. We repeat this procedure for each interval  $I_H \subset S_H^c$ , and the set of points that have not been removed from the vertical axis will be  $S_F$ .

For our example,  $S_H^c$  is composed of the four intervals  $I_{(i)} = (-\infty, 0)$ ,  $I_{(ii)} = (0, 3)$ ,  $I_{(iii)} = (3, 10)$ , and  $I_{(iv)} = (10, \infty)$ , and therefore we have four functions given by (8.1). The graphs of these four functions (given as  $x(m)$ ), obtained using Newton's method, are shown in Figure 1 (a). The thick lines on the vertical axis represent  $S_F$ . As noted in chapter 6, we see that the intervals on the vertical axis where the graphs are increasing do not overlap each other from one function to the next.

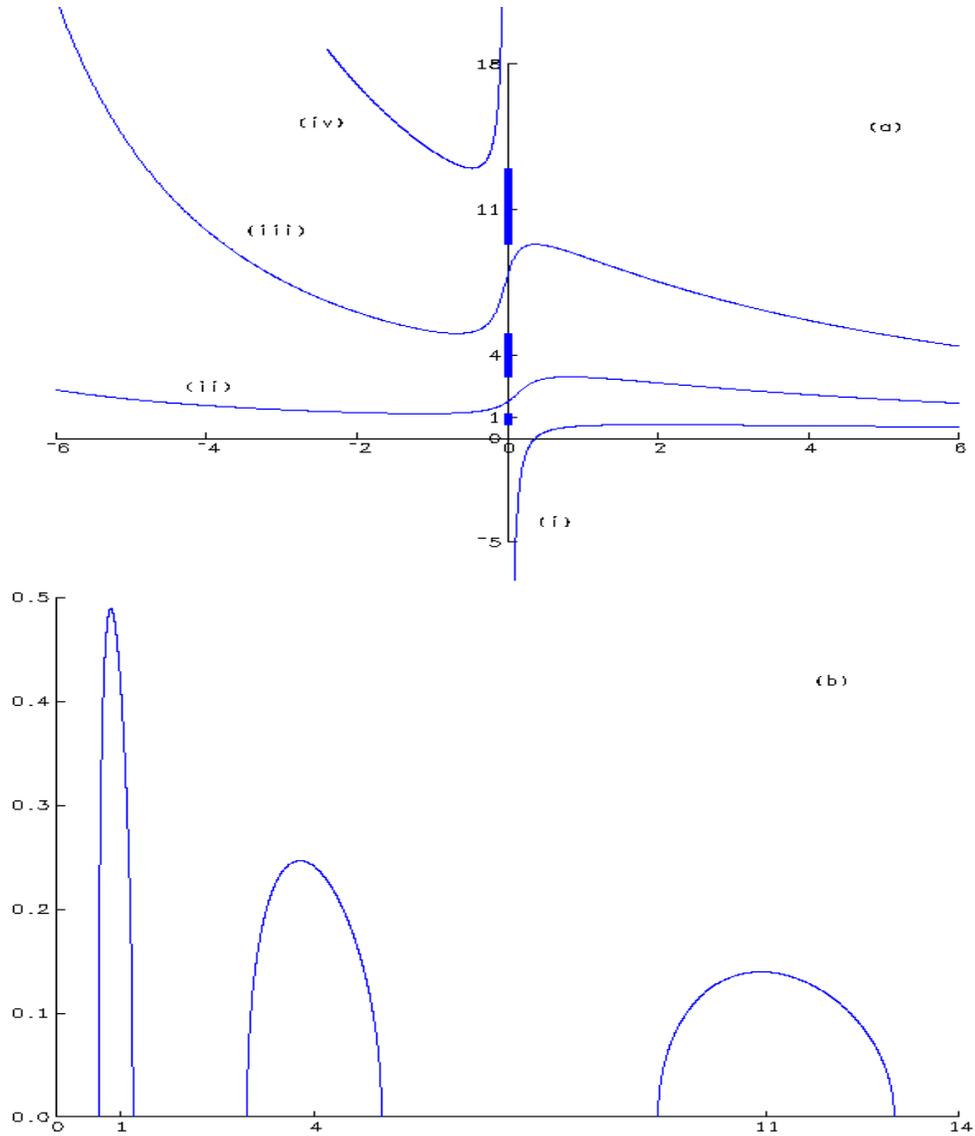


FIG1.

Once we have obtained  $S_F$  it is a simple matter of applying Newton's method to equation (1.1) with  $z = x$  and  $m = \underline{m}(x)$  to numerically obtain the density  $f(x) = \frac{1}{\pi} \text{Im} \underline{m}(x)$  for each  $x \in S_F$ . Figure 1 (b) shows the graph of the limiting density  $f$ . Note that when positive,  $f$  is a smooth function, and, at the boundary of its support,  $f$  goes down vertically to the  $x$ -axis, thus behaving in a similar fashion to a square root.

Recall that when  $c \downarrow 0$ ,  $F$  will converge to the distribution of a random variable  $Y + \sigma^2$ , where  $Y$  has distribution  $H$ . For our example, as  $c \downarrow 0$ ,  $F$  will converge to the discrete distribution having mass .2, .4, and .4 at the respective points 1, 4, and 11. It is evident that our choice of  $c = .1$  is small enough to see the mass beginning to accumulate around 1, 4, and 11.

In Figure 2 we have overlaid the density graph with a histogram and a scatterplot of the eigenvalues of a simulation of the matrix  $C_n$ . We choose  $n = 200$  so that, since  $c = .1$ ,  $N = 2000$ . We construct  $R_n$  in a deterministic manner so that the e.d.f. of the eigenvalues of  $\frac{1}{N}R_nR_n^*$  is exactly  $H$ , and let the entries of  $X_n$  be i.i.d. standardized Gaussian. We see that the histogram of the eigenvalues of  $C_n$  follows the shape of the density and the scatterplot, with each eigenvalue marked by the symbol ‘o’, stays close to  $S_F$ . The eigenvalues exhibit a clear separation into three distinct groups clustering near the points 1, 4, and 11. In fact, the distribution of the eigenvalues among the three groups is, from left to right, .2, .4, and .4. That is, of the 200 eigenvalues, 40 are in the first group, 80 are in the second group, and 80 are in the last group.

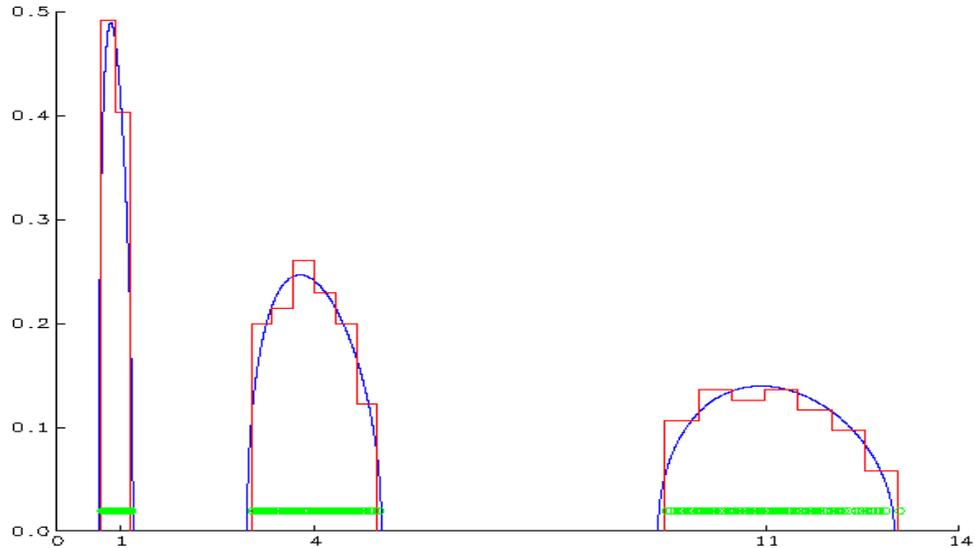


FIG2.

We use this example to illustrate the connection to the detection problem in array signal processing, where an array of  $n$  sensors receives signals transmitted by an unknown number  $q < n$  of sources with unknown locations in a noise-filled environment. The goal is mainly to identify the number of sources (signal detection) and their direction of arrival (DOA). The model is given by an  $n \times N$  matrix  $Y_n = R_n + \sigma X_n$  in which the columns represent  $N$  “snapshots” (samples) of the received signals. The matrix  $R_n$  represents the pure signal information and contains values detailing sensor orientation, the signal values at the source, and components such as steering vectors which provide information on the unknown direction of arrival of the signals. The signals are commonly assumed to be stationary ergodic processes. The matrix  $X_n$  represents additive noise (variance  $\sigma^2$  unknown) that contaminates the signal during transmission and processing. The entries of  $X_n$  are assumed to be i.i.d. standardized random variables. If the population matrix  $S_n + \sigma^2 I$  ( $S_n \equiv E \frac{1}{N} R_n R_n^*$ ) were known, or at least adequately approximated, then using the MUSIC (multiple signal classification) algorithm, as presented in Schmidt [7], one could determine the number of sources and, depending on the accuracy of the approximation, their direction of arrival. The sample covariance matrix  $C_n = \frac{1}{N} Y_n Y_n^*$  is used to estimate the population matrix; however, as stated in the introduction, if the number of sensors,  $n$ , is large then it may not be possible to collect enough samples to adequately estimate it. In this case, limiting results on the eigenvalues of  $C_n$  can aid in the detection problem: determining the number of sources. As noted in Schmidt [7], if  $q < n$  then  $S_n$  is singular with  $n - q$  zero eigenvalues. Therefore the  $n - q$  smallest eigenvalues of the matrix  $S_n + \sigma^2 I$  are equal to  $\sigma^2$ . These are called the “noise” eigenvalues, and the  $q$  larger eigenvalues are called the “signal” eigenvalues. Therefore, obtaining the value of  $q$ , the number of sources, can be accomplished by determining the multiplicity of the noise eigenvalues. From this it is clear that limiting results on the eigenvalues of the sample covariance matrix  $C_n$  can play an important role in signal detection. Indeed, if it can be shown that, for large  $n$ , the eigenvalues of  $C_n$  display this “splitting”

into groups of smaller and larger eigenvalues with the correct number of eigenvalues in each group corresponding to the noise and signal eigenvalues, then determining the number of sources can be accomplished with fewer samples than needed to approximate the population matrix itself. It will only require enough samples for the eigenvalues of  $C_n$  to separate into distinct, separate clusters.

Results of this type were proven for a different class of matrices in Bai and Silverstein [1], [2] with the first paper showing that, for  $n$  large, no eigenvalues appear where they should not, i.e., outside the support of the limiting distribution, and the second paper showing that, for  $n$  large, each interval of the support contains the correct number of eigenvalues. As yet, there are no such results proven for our limiting distribution, but from simulations it appears that similar results hold true for our case as well.

In the simulation above the number of sensors is 200, sample size is 2000, the (unknown) number of sources is 160, and  $\sigma^2 = 1$ . Since  $R_n$  was artificially constructed so that  $\frac{1}{N}R_nR_n^*$  has only two distinct nonzero eigenvalues, our example is limited in its comparison to an actual signal detection problem. Even so, this example is useful for illustrative purposes. The scatterplot shows a clear separation of the 40 noise eigenvalues from the 160 signal eigenvalues. The value  $c = .1$  is certainly small enough to see the separation of the support of  $F$  into disjoint intervals. In fact, by analyzing the density for different values of  $c$ , we discover that the separation of the smaller eigenvalues from the larger ones occurs when  $c$  is approximately .37555. Therefore, for a particular value of  $n$ , it would take less than  $3n$  samples for separation of the support to occur. This number is substantially smaller than the number of samples required to adequately approximate the population matrix  $S_n + \sigma^2 I$  using conventional multivariate inference methodology.

Further research is needed to make rigorous the mathematical arguments for exact eigenvalue separation in our information-plus-noise model.

## APPENDIX

## APPENDIX

Here we prove (2.3) which states that

$$\|F^{C_n} - F^{\hat{C}_n}\| \xrightarrow{a.s.} 0 \quad \text{as } n \rightarrow \infty.$$

First, we define  $p_n \equiv P(|X_{11}| \geq \sqrt{n})$  and note that since  $E|X_{11}|^2 < \infty$  we have  $p_n = \frac{o(1)}{n}$ .

Now, to prove (2.3) we will need the following theorem.

**THEOREM A.1.** Let  $X_1, X_2, \dots, X_n$  be i.i.d. Bernoulli with  $p \equiv P(X_1 = 1) < \frac{1}{2}$ . Then for any  $\epsilon > 0$  such that  $p + \epsilon \leq \frac{1}{2}$  we have

$$P\left(\frac{1}{n} \sum_{i=1}^n X_i - p \geq \epsilon\right) \leq e^{-\frac{n\epsilon^2}{2(p+\epsilon)}}.$$

*Proof.* For  $t > 0$

$$\begin{aligned} P\left(\frac{1}{n} \sum_{i=1}^n X_i - p \geq \epsilon\right) &\leq e^{-tn(p+\epsilon)} E\left[e^{t \sum_{i=1}^n X_i}\right] \\ &= \left(pe^{t(1-(p+\epsilon))} + (1-p)e^{-t(p+\epsilon)}\right)^n. \end{aligned}$$

Minimizing over  $t$  we get

$$P\left(\frac{1}{n} \sum_{i=1}^n X_i - p \geq \epsilon\right) \leq \left[\left(\frac{1-p}{1-(p+\epsilon)}\right)^{1-(p+\epsilon)} \left(\frac{p}{p+\epsilon}\right)^{p+\epsilon}\right]^n \equiv e^{n\psi(p, \epsilon)},$$

where  $\psi$  is defined by

$$\psi(p, \epsilon) = (1-(p+\epsilon)) \ln\left(1 + \frac{\epsilon}{1-(p+\epsilon)}\right) + (p+\epsilon) \ln\left(1 - \frac{\epsilon}{p+\epsilon}\right).$$

Now, using the Taylor series

$$\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \quad \text{for } |x| < 1,$$

we get

$$\psi(p, \epsilon) = - \sum_{k=2}^{\infty} \frac{\epsilon^k}{k} \left( \frac{1}{(p + \epsilon)^{k-1}} + \frac{(-1)^k}{(1 - (p + \epsilon))^{k-1}} \right).$$

Since  $p + \epsilon \leq \frac{1}{2}$ , the terms in the sum are all nonnegative, and therefore, dropping all but the first term, we get

$$\psi(p, \epsilon) \leq - \frac{\epsilon^2}{2} \left( \frac{1}{p + \epsilon} + \frac{1}{1 - (p + \epsilon)} \right) < - \frac{\epsilon^2}{2(p + \epsilon)}$$

and the theorem is proven.

We now prove (2.3) by first noting that for  $\epsilon > 0$  we get from Lemma 2.2 and (MP2)

$$\begin{aligned} P\left(\|F^{C_n} - F^{\hat{C}_n}\| \geq \epsilon\right) &\leq P\left(\frac{2}{n} \sum_{i,j} \mathbf{1}_{(|X_{ij}| \geq \sqrt{n})} \geq \epsilon\right) \\ &= P\left(\frac{1}{Nn} \sum_{i,j} \mathbf{1}_{(|X_{ij}| \geq \sqrt{n})} - p_n \geq \frac{\epsilon}{2n} - p_n\right). \end{aligned}$$

Since  $p_n = \frac{o(1)}{n}$ , for any  $\epsilon \in (0, \frac{1}{2})$  we can apply Theorem A.1 to get for all  $n$  large

$$P\left(\|F^{C_n} - F^{\hat{C}_n}\| \geq \epsilon\right) \leq e^{-\frac{n\epsilon}{16}}$$

when  $p_n < \frac{\epsilon}{4n}$ . Therefore  $P\left(\|F^{C_n} - F^{\hat{C}_n}\| \geq \epsilon\right)$  is summable, and hence  $\|F^{C_n} - F^{\hat{C}_n}\| \xrightarrow{a.s.} 0$  as  $n \rightarrow \infty$  which proves (2.3).

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