Abstract

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L_∞ structures are a natural generalization of Lie algebras, which need satisfy the standard graded Jacobi identity only up to homotopy. They have also been a subject of recent interest in physics, where they occur in closed string theory and in gauge theory. This dissertation classifies **all** possible L_∞ structures which can be constructed on a $\mathbb{Z}$-graded (characteristic 0) vector space of dimension three or less. It also includes necessary and sufficient conditions under which a space with an L_3 structure is a differential graded Lie algebra. Additionally, it is shown that some of these differential graded Lie algebras possess a nontrivial L_n structure for higher n.

L_∞ Structures on Spaces of Low Dimension

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Biography

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1. Introduction.

An L_∞ structure is a natural generalization of a graded Lie algebra. However, whereas a Lie algebra has just a skew bilinear bracket $[\ ,\ ] : V \otimes V \to V$ of degree 0, an L_∞ structure can include an infinite number of skew multilinear "brackets" $l_n : V^{\otimes n} \to V$ of degree $2 - n$. Additionally, a Lie algebra need only satisfy a standard Jacobi identity, while an L_∞ algebra must satisfy an infinite number of generalized Jacobi identities.

L_∞ algebras, also known as "strongly homotopy Lie algebras", first arose in the study of deformation theory [**15**], and the maps l_n in an L_∞ algebra have natural mathematical interpretations. If we look at the first few generalized Jacobi identities, this idea should become clear. The nth generalized Jacobi identity is given by

$$\mathcal{J}_n := \sum_{p=1}^{n} (-1)^{p(n-p)} l_{n-p+1} \circ l_p = 0.$$

We will now expand this simple formula for some small values of n, and watch some familiar mathematical constructions appear, repeating formulations which were developed earlier in [**10**] and [**11**].

(1) The first generalized Jacobi identity $\mathcal{J}_1$ simply requires that $l_1 \circ l_1 = 0$. In other words, the degree one operator l_1 is a differential.

(2) When $n = 2$, we have $-l_2 \circ l_1 + l_1 \circ l_2 = 0$. In other words, l_2 is a chain map. Additionally, the identity $\mathcal{J}_2 = 0$ can be viewed as a graded Leibniz formula, as explained in Remark (20). The degree zero map l_2 also has a natural interpretation as a Lie bracket, which will become clear when we look at $\mathcal{J}_3$.

(3) The third generalized Jacobi identity says that $l_3 \circ l_1 + l_2 \circ l_2 + l_1 \circ l_3 = 0$. If $l_3 \equiv 0$, then this reduces to the simpler equation $l_2 \circ l_2 = 0$. This last

equation is actually the classical graded Jacobi identity, as will be expained in more detail in Remark (14). If $l_3 \neq 0$, the situation becomes much more interesting. A closer look at the formula $l_3 \circ l_1 + l_2 \circ l_2 + l_1 \circ l_3 = 0$ reveals that the degree -1 map l_3 can actually be viewed as a cochain homotopy. It is this point of view that gives rise to the term "strongly homotopy Lie algebra". Even if a bracket l_2 fails to satisfy the usual Jacobi identity $l_2 \circ l_2 = 0$, appropriate maps l_1 and l_3 would allow $l_2 \circ l_2$ to be *homotopic* to zero. For a simple illustration of this, refer to Example (72).

(4) If we expand $\mathcal{J}_4 = 0$, we can interpret the degree -2 map l_4 as a cochain homotopy of a higher dimension. Furthermore, one can continue this process indefinitely, constructing higher and higher homotopies.

L_∞ structures have been a subject of recent interest in physics. In the gauge theory model of Berends, Burgers, and van Dam [**1**], the interactions between massless particles of high spin can be reformulated by giving the graded vector space of gauge parameters, together with the fields, the structure of an L_∞ algebra [**7, 4**]. L_∞ structures also occur in closed string theory [**17, 18**], and in the study of D-Brane superpotentials [**12**].

There is also some related research which classifies L_∞ structures on $\mathbb{Z}_2$-graded vector spaces [**5, 6**]. That work also includes some interesting results about the possible deformations from one $\mathbb{Z}_2$-graded L_∞ structure to another.

After giving some important definitions and background information in the next chapter, we will then proceed to identify **all** possible L_∞ structures which can be constructed on a $\mathbb{Z}$-graded vector space of dimension three or less. Although it is possible to completely classify the Lie algebras which are concentrated in grade zero in just a few pages [**9**], we will see that it it is quite a bit more complicated to characterize arbitrarily graded low-dimensional L_∞ structures. It is also worth noting that although semisimple Lie algebras have long been well understood, nilpotent Lie algebras have only been classified up to dimension seven [**16**].

In the process of characterizing the low-dimensional L_∞ structures, we will encounter many interesting cases, such as

- Structures which are not differential graded Lie algebras, but which do satisfy the Jacobi identity up to homotopy, and which can in some cases be extended to satisfy an infinite number of generalized Jacobi identities.
- Nontrivial operators which can be built on top of a classical Lie algebra, yielding higher levels of structure.
- L_n structures on which it is possible to impose a variety of different higher structures.
- L_n structures on which it is impossible to impose any higher L_{n+1} structure.

The interested reader might also want to refer to the additional catalog of examples of L_n and L_∞ structures in an appendix at the end of this dissertation.

2. Basic Definitions and Notation.

Throughout this paper, it is assumed that $V = \oplus V_m$ is a $\mathbb{Z}$-graded vector space over a fixed field of characteristic zero. Since we are working with graded vector spaces, the Koszul sign convention will be employed: whenever two objects of degrees p and q are commuted, a factor of $(-1)^{pq}$ is introduced. For a permutation σ acting on a string of symbols, let $\varepsilon(\sigma)$ denote the total effect of these signs. We then define $\chi(\sigma) = (-1)^{\sigma}\varepsilon(\sigma)$, where $(-1)^{\sigma}$ is the sign of the permutation σ.

Remark 1. $\chi(\sigma)$ has the following properties, which can be easily verified:

(1) If a permutation σ just reorders a string of consecutive elements of odd degree, then $\chi(\sigma) = 1$.
(2) If a permutation σ just reorders a string of consecutive elements of even degree, then $\chi(\sigma) = (-1)^{\sigma}$.
(3) If a permutation σ just transposes two (arbitrarily positioned) elements of even degree, then $\chi(\sigma) = -1$.
(4) If a permutation σ moves an element of even degree past a string of k elements of arbitrary degree, then $\chi(\sigma) = (-1)^k$.
(5) If a permutation σ moves an element of odd degree past a string of k elements of even degree, then $\chi(\sigma) = (-1)^k$.
(6) If a permutation σ moves a string of j even elements past a string of k elements of arbitrary degree, then $\chi(\sigma) = (-1)^{jk}$.

Definition 2. A $\mathbb{Z}$-graded vector space $V = \oplus_{k\in\mathbb{Z}} V_k$ is the direct sum of an indexed family of subspaces V_k.

Remark 3. We will usually list only the nonempty components of a graded vector space V (e.g. $V = V_{-10} \oplus V_{-3} \oplus V_3$).

Definition 4. A graded Lie algebra is a graded vector space V together with a graded Lie bracket $[\,,]: V_p \otimes V_q \to V_{p+q}$ such that

(1) $[x, y] = -(-1)^{|x|\,|y|}[y, x]$.

(2) $[[x, y], z] - (-1)^{|y||z|}[[x, z], y] + (-1)^{|x|(|y|+|z|)}[[y, z], x] = 0$.

Remark 5. It is an elementary exercise to verify that property (2) of a graded Lie algebra is equivalent to a classical graded Jacobi identity [**14**], usually written

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|y||z|}[z, [x, y]] = 0.$$

When we generalize the generalized Jacobi identity to allow the n-brackets which form an L_∞ structure, it naturally takes a form which is directly analogous to formula (2) of Definition (4), where the inside brackets are moved to the front.

Definition 6. A $(p, n-p)$ *unshuffle* is a permutation $\sigma \in S_n$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \text{ and } \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(n).$$

Definition 7. A map $l_n : V^{\otimes n} \to V$ has *degree* k if $l_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n) \subset V_N$, where $N = k + \sum_{i=1}^{n} |x_i|$, where $|x_i|$ is the degree of x_i.

Definition 8. A linear map $l_n : V^{\otimes n} \to V$ is *skew symmetric* if

$$l_n(x_{\pi(1)} \otimes \cdots \otimes x_{\pi(n)}) = \chi(\pi) l_n(x_1 \otimes \cdots \otimes x_n) \;\forall\; \pi \in S_n.$$

Definition 9. A skew symmetric linear map $l : V^{\otimes n} \to V$ can be extended to a map $l : V^{\otimes n+k} \to V^{\otimes 1+k}$ by the rule,

$$l(x_1 \otimes \cdots \otimes x_{n+k}) = \sum_{\substack{\sigma \text{ is an } (n,k) \\ \text{unshuffle}}} \chi(\sigma) l(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}) \otimes x_{\sigma(n+1)} \otimes \cdots \otimes x_{\sigma(n+k)}.$$

Remark 10. The extension of a skew linear map need not be skew. For example: $l_1(v \otimes u \otimes w) \neq -(-1)^{uv} l_1(u \otimes v \otimes w)$. However, it can be shown that the composition $l_{n-p+1} \circ l_p : V^{\otimes n} \to V$ is a skew symmetric linear map.

Definition 11. Let $\mathcal{J}_n : V^{\otimes n} \to V$ be defined by

$$\mathcal{J}_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = \sum_{p=1}^{n} (-1)^{p(n-p)} l_{n-p+1} \circ l_p(x_1 \otimes x_2 \otimes \cdots \otimes x_n).$$

(using the notation of extended maps defined above).

Remark 12. For ease of computation, $\mathcal{J}_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n)$ can be rewritten as

$$\sum_{p=1}^{n} (-1)^{p(n-p)} \sum_{\substack{\sigma \text{ is } (p,n-p) \\ \text{unshuffle}}} \chi(\sigma) l_{n-p+1}(l_p(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p)}) \otimes x_{\sigma(p+1)} \otimes \cdots \otimes x_{\sigma(n)}).$$

Definition 13. The *nth generalized Jacobi identity* is the equation

$$\mathcal{J}_n = 0.$$

Remark 14. If we denote $l_2(x \otimes y) = [x, y]$ and consider the special case when $n = 3$ and $l_3 = 0$, the generalized Jacobi identity becomes the more familiar graded Jacobi identity, since the equation

$$\sum_{\substack{\sigma \text{ is } (2,1) \\ \text{unshuffle}}} \chi(\sigma) l_2(l_2(x_{\sigma(1)} \otimes x_{\sigma(2)}) \otimes x_{\sigma(3)}) = 0$$

can be rewritten in bracket notation as

$$\sum_{\substack{\sigma \text{ is } (2,1) \\ \text{unshuffle}}} \chi(\sigma)[[x_{\sigma(1)}, x_{\sigma(2)})], x_{\sigma(3)}] = 0,$$

which then expands to the usual graded Jacobi identity,

$$[[x_1, x_2], x_3] - (-1)^{|x_2||x_3|}[[x_1, x_3], x_2] + (-1)^{|x_1|(|x_2|+|x_3|)}[[x_2, x_3], x_1] = 0.$$

Remark 15. Note that $l_{n-p+1} \circ l_p : V^{\otimes n} \to V$ is a skew linear map of degree $3-n$. Therefore, $\mathcal{J}_n$ is also a skew linear map of degree $3 - n$. In particular, whenever an element of even degree is repeated, the Jacobi identity is automatically satisfied.

DEFINITION 16. An L_m *structure* is a graded vector space V endowed with a collection of linear maps $\{l_k : V^{\otimes k} \to V,\ 1 \le k \le m\}$ with $deg(l_k) = 2 - k$ which are skew symmetric and satisfy each Jacobi identity $\mathcal{J}_n = 0$, $1 \le n \le m$.

REMARK 17. This is the "cochain version" of the definition of an L_m structure, having $deg(l_k) = 2 - k$. The definition for chain complexes is the same, but with $deg(l_k) = k - 2$. We will be working exclusively with the cochain version in this paper.

DEFINITION 18. An L_∞ *structure* is a graded vector space V endowed with a collection of linear maps $\{l_k : V^{\otimes k} \to V,\ k \in \mathbb{N}\}$ such that $\mathcal{J}_n = 0\ \forall\ n \in \mathbb{N}$.

DEFINITION 19. A *differential graded (d.g.) Lie algebra* is an L_2 structure in which $l_2 \circ l_2 = 0$.

REMARK 20. In an L_2 structure, the identity $\mathcal{J}_2 = 0$ has a natural interpretation as a graded Leibniz formula. In a structure in which $l_2 \circ l_2 = 0$, it is appropriate to think of l_2 as a Lie bracket $[\,,]$. If we also think of the degree one map l_1 as a differential ∂, then $\mathcal{J}_2 = 0$ expands to $-\partial([x, y]) + [\partial(x), y] - (-1)^{|x|\,|y|}[\partial(y), x]$, which is equivalent to a standard graded Leibniz formula[**8, 13**]: $\partial([x, y]) = [\partial(x), y] + (-1)^x[x, \partial(y)]$. Thus the previous definition is exactly equivalent to the usual definition of a differential graded Lie algebra.

3. Preliminaries.

At the beginning of this chapter, we will consider L_∞ structures on a space V which contains only vectors of the *same* parity, meaning that either all of the vectors are elements of even-graded components, or that all of the vectors are elements of odd-graded components. On such a space of dimension three or less, we will show that either all l_n are forced to be zero, or l_2 is the only nonzero operator, and V with the l_2 bracket is a classical graded Lie algebra. After proving some general lemmas, we will then identify the possible L_∞ structures on $\mathbb{Z}$-graded vector spaces of dimension two or less.

3.1. Spaces in which all vectors have the same parity

It is interesting to note that if V consists of just one component (of arbitrary dimension), one can only construct a nontrivial L_∞ structure if the nonempty component is of degree 0. Furthermore, the only possible L_∞ structures on such a space of grade zero are just the classical Lie algebras, extended by $l_n \equiv 0 \ \forall \ n \neq 2$.

Lemma 21. *Suppose $V = V_\alpha$ is a graded vector space of arbitrary dimension, in which all vectors are concentrated in a single component of grade α. Then*

- *If $\alpha \neq 0$, then $l_n \equiv 0 \ \forall \ n \in \mathbb{N}$.*
- *If $\alpha = 0$, then every L_∞ structure on V consists of an l_2 bracket which satisfies the classical Jacobi identity ($l_2 \circ l_2 = 0$), with $l_n \equiv 0 \ \forall \ n \neq 2$.*

Proof. $l_n(V_\alpha^n) \subset V_{\alpha n+2-n} = V_{n(\alpha-1)+2}$. Since l_n can only be nonzero if it maps into V_α, we require $n(\alpha - 1) + 2 = \alpha$. Note that this equation is not satisfied by $\alpha = 1$!

When $\alpha \neq 1$, we have $n = \frac{\alpha-2}{\alpha-1} = 1 - \frac{1}{\alpha-1}$. This is only an integer if $\alpha \in \{0, 2\}$. Since $\alpha = 2$ gives $n = 0$ which is invalid, we see that l_n can only be nonzero if $\alpha = 0$. □

The next lemma shows that if a vector space contains only odd elements, then all operators l_n are forced to be trivial. It is true for odd graded vector spaces of arbitrary dimension.

LEMMA 22. *If V contains only basis elements of odd degree, then $l_n = 0 \;\forall\; n \in \mathbb{N}$.*

PROOF. Suppose that $v_1, \ldots v_n$ are arbitrary vectors of odd degrees $\alpha_1 \ldots \alpha_n$.

$$l_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) \subset V_{2-n+\alpha_1+\cdots+\alpha_n}.$$

If n is odd, $\sum_{i=1}^{n} \alpha_i$ is odd, which makes $2 - n + \alpha_1 + \cdots + \alpha_n$ even. If n is even, $\sum_{i=1}^{n} \alpha_i$ is even, which also makes $2 - n + \alpha_1 + \cdots + \alpha_n$ even. Since the output must go into an evenly graded component, and all even components are zero, $l_n = 0 \;\forall\; n$. □

LEMMA 23. *Suppose that V contains exactly N basis elements, all of which are of* even *degree. Then the following results are true.*

(1) *$l_n \equiv 0$ for all odd n.*
(2) *$l_n \equiv 0 \;\forall\; n > N$.*
(3) *$\mathcal{J}_n \equiv 0$ for all even n.*

PROOF. (1) When n is odd, l_n must map into a component of odd degree, and these are all zero.

(2) $l_n \equiv 0 \;\forall\; n > N$ because of skewness, since some even element would be repeated as in input to l_n.

(3) The degree $3 - n$ map $\mathcal{J}_n$ maps even input into an odd component whenever n is even.

□

From the simple yet useful facts listed above, the following lemma follows.

Lemma 24. *If dim$(V) \leq 3$, and V contains only even vectors, then the possible L_∞ structures on V are precisely the classical graded Lie algebras, with all operators besides l_2 being trivial.*

Remark 25. If V is a four-dimensional space containing only even vectors, then the possible L_∞ structures on V are like those in the preceding lemma, except that l_4 can be defined arbitrarily. (since $\mathcal{J}_4 \equiv 0$ in this case).

3.2. Some basic lemmas

Before considering vector spaces of lower dimension, we will first prove a few necessary lemmas, which will also be useful in later sections. The next two lemmas identify which operators l_n on a given vector space $V = V_\alpha \oplus V_\beta$ (where α and β have opposite parity) can be nonzero. We will see that the only such spaces on which l_n can be defined to be nonzero for all n are $V = V_0 \oplus V_1$ and $V = V_1 \oplus V_2$. Lemma (27), however, will show that there exist many degenerate cases with very sparse nonzero structures.

Lemma 26. *Suppose $V = V_\alpha \oplus V_1$, where α is even.*

- *If $V = V_0 \oplus V_1$, the only possible nonzero l_n are* $\left\{ \begin{array}{r} l_n(V_0 \otimes V_1^{\otimes n-1}) \subset V_1 \ \forall\ n. \\ l_n(V_0^{\otimes 2} \otimes V_1^{\otimes n-2}) \subset V_0 \ \forall\ n. \end{array} \right\}$.
- *If $V = V_1 \oplus V_2$, the only possible nonzero l_n is $l_n(V_1^{\otimes n}) \subset V_2 \ \forall\ n$.*
- *If $V = V_1 \oplus V_\alpha$ where α is an even number other than 0 or 2, then all operators l_n are trivial.*

Proof. $l_n(V_\alpha^{\otimes k} \otimes V_1^{\otimes n-k}) \subset V_{\alpha k+n-k+2-n} = V_{k(\alpha-1)+2}$, which is of the same parity as k.

If k is even, this can only be nonzero if $k\alpha - k + 2 = \alpha$, which implies that $\alpha = \frac{k-2}{k-1} = 1 - \frac{1}{k-1}$. This can only be an integer if $k \in \{0, 2\}$. If $k = 0$, this forces

$\alpha = 2$. Thus when $\alpha = 2$, we can define $l_n(V_1^{\otimes n})$ to be nonzero for all n. If $k = 2$, this forces $\alpha = 0$. Thus when $\alpha = 0$, we can define $l_n(V_0^{\otimes 2} \otimes V_1^{\otimes n})$ to be nonzero for all n.

Similarly, if k is odd, $l_n(V_\alpha^{\otimes k} \otimes V_1^{\otimes n-k})$ can only be nonzero if $k\alpha - k + 2 = 1$, which implies that $\alpha = \frac{k-1}{k} = 1 - \frac{1}{k}$. This can only be an integer if $k \in \{0, 1\}$. Since k is odd, we can only have nonzero $l_n(V_\alpha^{\otimes k} \otimes V_1^{\otimes n-k})$ when $k = 1$. If $k = 1$, this forces $\alpha = 0$. Thus when $\alpha = 0$, we can define $l_n(V_0 \otimes V_1^{\otimes n-1})$ to be nonzero for all n. $\square$

LEMMA 27. *Suppose $V = V_\alpha \oplus V_\beta$, where α is even and β is an odd number other than 1. Then for each value of k, $l_n(V_\alpha^{\otimes k} \otimes V_\beta^{\otimes n-k})$ can be nonzero for at most one value of n. We list the special cases which are of interest in this paper below:*

$$\left\{ \begin{aligned} l_n(V_\beta^{\otimes n}) \subset V_\alpha \text{ when } n = \tfrac{\alpha-2}{\beta-1} \in \mathbb{N}. \\ l_n(V_\alpha \otimes V_\beta^{\otimes n-1}) \subset V_\beta \text{ when } n = 2 - \tfrac{\alpha}{\beta-1} \in \mathbb{N}. \\ l_n(V_\alpha^{\otimes 2} \otimes V_\beta^{\otimes n-2}) \subset V_\alpha \text{ when } n = 2 - \tfrac{\alpha}{\beta-1} \in \mathbb{N}. \end{aligned} \right\}$$

PROOF. $l_n(V_\alpha^{\otimes k} \otimes V_\beta^{\otimes n-k}) \subset V_{\alpha k + \beta(n-k) + 2 - n} = V_{n(\beta-1) + (\alpha-\beta)k + 2}$, which is of the same parity as k.

If k is even, this can only be nonzero if $n(\beta - 1) + (\alpha - \beta)k + 2 = \alpha$, which implies that $n = \frac{\beta k - \alpha k + \alpha - 2}{\beta - 1} = k + \frac{k - \alpha k + \alpha - 2}{\beta - 1}$. In particular, when $k = 0$, $n = \frac{\alpha-2}{\beta-1}$, and when $k = 2$, $n = 2 - \frac{\alpha}{\beta-1}$.

Similarly, if k is odd, $l_n(V_\alpha^{\otimes k} \otimes V_\beta^{\otimes n-k}) \neq 0 \implies n(\beta - 1) + (\alpha - \beta)k + 2 = \beta$, which implies that $n = \frac{\beta - 2 - \alpha k + \beta k}{\beta - 1} = 1 + k + \frac{(1-\alpha)k - 1}{\beta - 1}$. In particular, when $k = 1$, $n = 2 - \frac{\alpha}{\beta-1}$, $\square$

We will now calculate some generalized Jacobi identities on spaces with exactly two nonzero components of opposite parity. Note that the following lemma is true regardless of the dimension of each component.

LEMMA 28. *Suppose $V = V_\alpha \oplus V_\beta$ where α is even and β is odd. Then*

- *If* $\beta = -1$, $\begin{cases} \mathcal{J}_n(V_{-1}^{\otimes n}) = 0 \ \forall \ n \neq 2. \\ \mathcal{J}_n(V_\alpha \otimes V_{-1}^{\otimes n-1}) = 0 \ \forall \ n \neq 2. \\ \mathcal{J}_n(V_\alpha^{\otimes 2} \otimes V_{-1}^{\otimes n-2}) = 0 \ \forall \ n \neq 3 + \alpha. \end{cases}$

- *If* $\beta = 1$, $\begin{cases} \textit{On } V_1 \oplus V_\alpha \textit{ where } \alpha \neq 0, \ \mathcal{J}_n \equiv 0 \ \forall \ n \in \mathbb{N}. \\ \textit{On } V_0 \oplus V_1, \ \mathcal{J}_n \equiv 0 \Longleftrightarrow \mathcal{J}_n(V_0^{\otimes 2} \otimes V_1^{\otimes n-2}) \equiv 0. \\ \mathcal{J}_n(V_0^{\otimes 2} \otimes V_1^{\otimes n-2}) \textit{ can be nonzero } \forall \ n \in \mathbb{N}. \end{cases}$

- *If* $\beta \neq \pm 1$, $\begin{cases} \mathcal{J}_n \equiv 0 \Longleftrightarrow \mathcal{J}_n(V_\alpha^{\otimes 2} \otimes V_\beta^{\otimes n-2}) \equiv 0. \\ \mathcal{J}_n(V_\alpha^{\otimes 2} \otimes V_\beta^{\otimes n-2}) = 0 \ \forall \ n \neq 3 - \frac{2\alpha}{\beta - 1}. \end{cases}$

Proof. $\mathcal{J}_n(V_\beta^{\otimes n}) \subset V_{\beta n + 3 - n} = V_{n(\beta-1)+3}$, which is odd. $\mathcal{J}_n(V_\beta^{\otimes n})$ can only be nonzero if $n(\beta - 1) + 3 = \beta$. Note that this equation cannot be satisfied by $\beta = 1$! If $\beta \neq 1$, $n(\beta - 1) + 3 = \beta \Longrightarrow n = \frac{\beta - 3}{\beta - 1} = 1 - \frac{2}{\beta - 1}$. This can only be an integer if $\beta \in \{-1, 0, 2, 3\}$. Since β is odd, we need $\beta \in \{-1, 3\}$. But if $\beta = 3$, this makes $n = 0$, which is not valid. When $\beta = -1$, this forces $n = 2$. Thus

$$\beta \neq -1 \Longrightarrow \mathcal{J}_n(V_\beta^{\otimes n}) = 0 \ \forall \ n \in \mathbb{N} \quad \text{and} \quad \beta = -1 \Longrightarrow \mathcal{J}_n(V_{-1}^{\otimes n}) = 0 \ \forall \ n \neq 2.$$

$\mathcal{J}_n(V_\alpha \otimes V_\beta^{\otimes n-1}) \subset V_{\alpha + \beta(n-1) + 3 - n} = V_{n(\beta-1)+\alpha-\beta+3}$, which is even. Therefore, $\mathcal{J}_n(V_\alpha \otimes V_\beta^{\otimes n-1})$ can only be nonzero if $n(\beta - 1) + \alpha - \beta + 3 = \alpha$, which implies that $n(\beta - 1) + 3 = \beta$, which is just like the equation in the previous paragraph. Thus

$$\beta \neq -1 \Longrightarrow \mathcal{J}_n(V_\alpha \otimes V_\beta^{\otimes n-1}) = 0 \ \forall \ n \in \mathbb{N} \text{ and } \beta = -1 \Longrightarrow \mathcal{J}_n(V_\alpha \otimes V_\beta^{\otimes n-1}) = 0 \ \forall \ n \neq 2.$$

$\mathcal{J}_n(V_\alpha^{\otimes 2} \otimes V_\beta^{\otimes n-2}) \subset V_{2\alpha + \beta(n-2) + 3 - n} = V_{n(\beta-1)+2(\alpha-\beta)+3}$, which is odd. Thus $\mathcal{J}_n(V_\alpha^{\otimes 2} \otimes V_\beta^{\otimes n-2})$ can only be nonzero if $n(\beta - 1) + 2(\alpha - \beta) + 3 = \beta$, or equivalently, if $3\beta - 3 - 2\alpha = n(\beta - 1)$. If $\beta = 1$, this forces $\alpha = 0$. If $\beta \neq 1$, this implies that $n = 3 - \frac{2\alpha}{\beta - 1}$. $\square$

Finally, we'll prove one other little lemma which will come in handy later.

LEMMA 29. *Suppose V is a graded vector space with only one odd component V_γ, and suppose that $l_n(V_\gamma^{\otimes n}) = 0\ \forall\ n \in \mathbb{N}$. Then if V_α is any other (even) component, $\mathcal{J}_n(V_\alpha \otimes V_\gamma^{\otimes n-1}) \equiv 0\ \forall\ n \in \mathbb{N}$.*

PROOF. Since $l_p(V_\gamma^{\otimes p}) = 0$, we have $l_{n-p+1}(l_p(V_\gamma^{\otimes p}) \otimes V_\alpha \otimes V_\gamma^{\otimes n-p-1}) = 0$. Now note that $l_p(V_\alpha \otimes V_\gamma^{\otimes p-1}) \subset V_{\alpha+(p-1)\gamma+2-p} = V_{p(\gamma-1)+\alpha-\gamma+2}$, which is an odd component. Since the only odd component is V_γ, we have $l_{n-p+1}(l_p(V_\alpha \otimes V_\gamma^{\otimes p-1}) \otimes V_\gamma^{\otimes n-p}) = 0$.

$\square$

3.3. L_∞ STRUCTURES ON TWO-DIMENSIONAL SPACES

We will now identify all possible L_∞ structures on spaces of dimension two or less. Since our previous results have already characterized the possible structures on spaces where all elements have the same parity, we now only need to consider a two-dimensional space with elements of opposite parity.

Consider $V = V_\alpha \oplus V_\beta$ where α is even and β is odd, and each component is one-dimensional. From Lemmas (26) and (27), we can list all possible nontrivial operators l_n which can be defined on such a space:

$$\begin{cases} l_n(V_0 \otimes V_1^{\otimes n-1}) \subset V_1 \text{ can be nonzero on } V = V_0 \oplus V_1 \text{ for all } n \in \mathbb{N}. \\ l_n(V_1^{\otimes n}) \subset V_2 \text{ can be nonzero on } V = V_1 \oplus V_2. \text{ for all } n \in \mathbb{N}. \\ l_n(V_\beta^{\otimes n}) \subset V_\alpha \text{ can be nonzero on } V = V_\alpha \oplus V_\beta \text{ only when } n = \frac{\alpha-2}{\beta-1} \in \mathbb{N}. \\ l_n(V_\alpha \otimes V_\beta^{\otimes n-1}) \subset V_\beta \text{ can be nonzero on } V = V_\alpha \oplus V_\beta \text{ only when } n = 2 - \frac{\alpha}{\beta-1} \in \mathbb{N}. \end{cases}$$

By Lemma (28), such operators trivially form an L_∞ structure whenever $\beta \neq -1$, since $\mathcal{J}_n \equiv 0$ on such a space whenever $\beta \neq -1$. Lemma (28) also tells us that $\mathcal{J}_n \equiv 0$ on $V = V_\alpha \oplus V_{-1}$ for all $n \neq 2$. The next lemma finds necessary and sufficient conditions for $\mathcal{J}_2 \equiv 0$ on $V = V_\alpha \oplus V_{-1}$.

LEMMA 30. *Let $V = V_\alpha \oplus V_{-1}$, where α is even. Then*

- *If $\alpha \notin \{0, -2\}$, then $l_1 \equiv 0$ and $l_2 \equiv 0$ (forming a trivial L_2 structure).*
- *If $\alpha = 0$, then the possible L_2 structures on $V = V_\alpha \oplus V_{-1}$ are*

$$\left\{\begin{array}{l} l_1(v_{-1}) = a\, v_0 \\ \\ l_2 \equiv 0 \end{array}\right\} \quad \textit{and} \quad \left\{\begin{array}{l} l_1 \equiv 0 \\ \\ l_2(v_0 \otimes v_{-1}) = b\, v_{-1} \end{array}\right\}.$$

- *If $\alpha = -2$, then the possible L_2 structures on $V = V_\alpha \oplus V_{-1}$ are*

$$\left\{\begin{array}{l} l_1(v_{-2}) = a\, v_{-1} \\ \\ l_2 \equiv 0 \end{array}\right\} \quad \textit{and} \quad \left\{\begin{array}{l} l_1 \equiv 0 \\ \\ l_2(v_{-1} \otimes v_{-1}) = b\, v_{-2} \end{array}\right\}.$$

Proof. We will start by using the above table to determine the values of α which allow l_1 or l_2 to be nonzero (since $\mathcal{J}_2$ involves only these operators). If $l_1(V_{-1}) \neq 0$, $1 = \frac{\alpha-2}{-1-1} \implies \alpha = 0$. If $l_2(V_{-1} \otimes V_{-1}) \neq 0$, $2 = \frac{\alpha-2}{-1-1} \implies \alpha = -2$. If $l_1(V_\alpha) \neq 0$, $1 = 2 - \frac{\alpha}{-1-1} \implies \alpha = -2$. If $l_1(V_\alpha \otimes V_{-1}) \neq 0$, $2 = 2 - \frac{\alpha}{-1-1} \implies \alpha = 0$. Thus when $\alpha \notin \{0, -2\}$, $\mathcal{J}_2 \equiv 0$ on $V = V_\alpha \oplus V_{-1}$.

When $\alpha = 0$, l_1 and l_2 can be defined in general by $l_1(v_{-1}) = a\, v_0$ (zero otherwise), and $l_2(v_0 \otimes v_{-1}) = b\, v_{-1}$ (zero otherwise).

$$\begin{aligned} \mathcal{J}_2(v_0 \otimes v_{-1}) &= -l_2(l_1(v_0) \otimes v_{-1}) + l_2(l_1(v_{-1}) \otimes v_0) + l_1(l_2(v_0 \otimes v_{-1})) \\ &= 0 + l_2(a\, v_0 \otimes v_0) + l_1(b\, v_{-1}) = a\, b\, v_0. \end{aligned}$$

$$\mathcal{J}_2(v_{-1} \otimes v_{-1}) = -2\, l_2(l_1(v_{-1}) \otimes v_{-1}) + l_1(l_2(v_{-1} \otimes v_{-1})) = -2\, l_2(a\, v_0 \otimes v_{-1}) = -2\, a\, b\, v_{-1}.$$

Thus we see that either a or b must be zero, which forces one of l_1, l_2 to be zero. It is also easy to see that we have an L_2 structure on V when one of l_1, l_2 is zero, so we have necessary and sufficient conditions for an L_2 structure on $V = V_0 \oplus V_{-1}$.

When $\alpha = -2$, l_1 and l_2 can be defined in general by $l_1(v_{-2}) = a\, v_{-1}$ (zero otherwise), and $l_2(v_{-1} \otimes v_{-1}) = b\, v_{-2}$ (zero otherwise).

$$\mathcal{J}_2(v_{-2} \otimes v_{-1}) = -l_2(l_1(v_{-2}) \otimes v_{-1}) + l_2(l_1(v_{-1}) \otimes v_{-2}) + l_1(l_2(v_{-2} \otimes v_{-1}) = 0.$$

$$\mathcal{J}_2(v_{-1} \otimes v_{-1}) = -2\, l_2(l_1(v_{-1}) \otimes v_{-1}) + l_1(l_2(v_{-1} \otimes v_{-1})) = 0 + l_1(b\, v_{-2}) = a\, b\, v_{-1}.$$

Again, either a or b must be zero, which forces one of l_1, l_2 to be zero. $\square$

We can now classify all possible L_∞ structures which are possible on a two dimensional graded vector space. It is interesting to note that $V_0 \oplus V_1$ and $V_1 \oplus V_2$ are the only two-dimensional spaces on which it is possible to form an L_∞ structure with all operators $l_n \neq 0$. Indeed, they are the only two-dimensional spaces on which it is possible to define an L_∞ structure with more than one nonzero operation.

THEOREM 31. *We identify all nonzero L_∞ structures which can be built on a two dimensional $\mathbb{Z}$-graded vector space:*

(1) *On $V_0 \oplus V_1$, all L_∞ structures are given by $l_n(v_0 \otimes v_1^{\otimes n-1}) = c_n v_1$ (c_n arbitrary).*

(2) *On $V_1 \oplus V_2$, all L_∞ structures are given by $l_n(v_1^{\otimes n}) = c_n v_2$ (c_n arbitrary).*

(3) *On $V_{-1} \oplus V_0$, all L_∞ structures can be divided into two types:*
   - $l_1(v_{-1}) = c\, v_0$ *and* $l_n \equiv 0 \ \forall\ n \neq 1$.
   - $l_2(v_0 \otimes v_{-1}) = c\, v_{-1}$ *and* $l_n \equiv 0 \ \forall\ n \neq 2$.

(4) *On $V_{-2} \oplus V_{-1}$, all L_∞ structures can be divided into two types:*
   - $l_1(v_{-2}) = c\, v_{-1}$ *and* $l_n \equiv 0 \ \forall\ n \neq 1$.
   - $l_2(v_{-1} \otimes v_{-1}) = c\, v_{-2}$ *and* $l_n \equiv 0 \ \forall\ n \neq 2$.

(5) *Other structures of the form $V_\alpha \oplus V_\beta$ where α is even and $\beta \neq 1$ can contain at most one nonzero operator, which must be one of the following:*
   - $l_n(v_\beta^{\otimes n}) = c\, v_\alpha$ *when* $n = \frac{\alpha-2}{\beta-1} \in \mathbb{N}$.
   - $l_n(v_\alpha \otimes v_\beta^{\otimes n-1}) = c\, v_\beta$ *when* $n = 2 - \frac{\alpha}{\beta-1} \in \mathbb{N}$.

(6) *On $V = V_\alpha \oplus V_\beta$, where both α and β are even, all L_∞ structures are given by l_2 which is a classical Lie bracket, with $l_n \equiv 0 \ \forall\ n \neq 2$.*

PROOF. Statements (1) and (2) follow from Lemmas (26), (27), and (28). Statements (3) and (4) follow from Lemma (30). Statement (5) follows from Lemma (27). Statement (6) follows from Lemma (24). □

REMARK 32. One can easily verify that all of the above structures are differential graded Lie algebras.

4. Two Components of Opposite Parity.

In this chapter, will find all possible L_∞ structures on a three-dimensional graded vector space $V = V_\alpha \oplus V_\beta$, where α is even and β is odd. We will see that it is only possible to construct an L_∞ structure with an infinite number of nonzero l_n when $\beta = 1$, although there are many degenerate cases of L_∞ structures with only one or two nonzero operators. In particular, Theorem (34) will show that $V_0 \oplus V_1$ can have an infinite number of nonzero l_n, and that there are nontrivial constraints involved in constructing an L_∞ structure on $V_0 \oplus V_1$.

4.1. Two even vectors and one odd vector (of degree one)

In this section, we identify all possible L_n and L_∞ structures on $V = V_\alpha \oplus V_1$, where α is even and $V_\alpha =< v, x >$ and $V_1 =< w >$. The results can be summarized as follows:

(1) If $V = V_1 \oplus V_2$, then any collection of operators $l_n(w^{\otimes n}) = a_n v + b_n x$ trivially forms an L_∞ structure on V.

(2) If $V = V_0 \oplus V_1$, we will show that there are two types of L_∞ structure.
   - $l_n(V_0 \otimes V_1^{n-1}) \equiv 0 \; \forall n \in \mathbb{N}$ but $l_n(v \otimes x \otimes w^{\otimes n-2}) \in V_0$ can be arbitrarily assigned for each $n \in \mathbb{N}$.
   - $l_k(V_0 \otimes V_1^{n-1}) \neq 0$ for some k, but $l_n(v \otimes x \otimes w^{\otimes n-2}) \in V_0$ must be assigned according to a recursion formula.

(3) On $V = V_\alpha \oplus V_1$, where $\alpha \notin \{0, 2\}$, all l_n must be zero, creating only trivial L_∞ structures.

According to Lemma (28), if $\beta = 1$ and $\alpha \neq 0$, then $\mathcal{J}_n \equiv 0 \; \forall \; n \in \mathbb{N}$. Thus any collection of skew linear operators l_n of degree $2-n$ form an L_∞ structure on $V_1 \oplus V_\alpha$ where $\alpha \neq 0$. With most choices of α, all operators l_n must be trivial anyway because of the grading structure itself. Specifically, Lemma (26) tells us that if $V = V_1 \oplus V_\alpha$ where α is an even number other than 0 or 2, then all l_n are forced to be zero.

However, if $V = V_1 \oplus V_2$, where $V_1 =< w >$ and $V_2 =< v, x >$, one can define an infinite number of nonzero operators on V by $l_n(w^{\otimes n}) = a_n v + b_n x$. According to Lemma (28), these operators will then form an L_∞ structure on $V = V_1 \oplus V_2$ for any choice of constants a_n, b_n.

On $V = V_0 \oplus V_1$, however, Lemma (26) allows many more possibilities for nonzero operators. In particular, Lemma (33) and Theorem (34) will show that nontrivial constraints result when $l_k(V_0 \otimes V_1^{n-1}) \neq 0$ for some k.

Lemma 33. *Suppose* $V = V_0 \oplus V_1$ *where* $V_0 =< v, x >$ *and* $V_1 =< w >$ *has operators defined by*

$$\left\{ \begin{array}{c} l_n(v \otimes w^{\otimes n-1}) = (n-1)! \; a_n w. \\ l_n(x \otimes w^{\otimes n-1}) = (n-1)! \; b_n w. \\ l_n(v \otimes x \otimes w^{\otimes n-2}) = (n-2)! \; c_n v + (n-2)! \; d_n x. \end{array} \right\}$$

Further, suppose that $a_n, b_n = 0 \; \forall \; n < k$ *but* $b_k \neq 0$, *and denote* $m = n - k + 1$. *Then* $\mathcal{J}_n(v \otimes x \otimes w^{\otimes n-2}) = 0$ *if and only if* $n < k$ *or*

$$d_m = (-1)^m (m-k) a_m - \frac{c_m a_k}{b_k} + \frac{\sum_{p=1}^{m-1} (-1)^{p(m+k)} [c_p a_{m+k-p} + d_p b_{m+k-p} + (-1)^p (m+k-2p) a_p b_{m+k-p}]}{-(-1)^{m(m+k)} b_k}.$$

Proof. $l_{n-p+1} \circ l_p(v \otimes x \otimes w^{\otimes n-2})$ is equal to

$\binom{n-2}{p-2} l_{n-p+1}(l_p(v \otimes x \otimes w^{\otimes p-2}) \otimes w^{\otimes n-p}) + (-1)^{p-1}\binom{n-2}{p-1} l_{n-p+1}(l_p(v \otimes w^{\otimes p-1}) \otimes x \otimes w^{\otimes n-p-1})$

$+ (-1)^p \binom{n-2}{p-1} l_{n-p+1}(l_p(x \otimes w^{\otimes p-1}) \otimes v \otimes w^{\otimes n-p-1}) + \binom{n-2}{p} l_{n-p+1}(l_p(w^{\otimes p}) \otimes v \otimes x \otimes w^{\otimes n-p-2})$

$= \frac{(n-2)!}{(n-p)!} l_{n-p+1}(c_p v \otimes w^{\otimes n-p}) + \frac{(n-2)!}{(n-p)!} l_{n-p+1}(d_p x \otimes w^{\otimes n-p})$

$\quad + (-1)^p \frac{(n-2)!}{(n-p-1)!} l_{n-p+1}(a_p x \otimes w^{\otimes n-p}) - (-1)^p \frac{(n-2)!}{(n-p-1)!} l_{n-p+1}(b_p v \otimes w^{\otimes n-p}) + 0$

$= (n-2)! \left[c_p a_{n-p+1} + d_p b_{n-p+1} + (-1)^p (n-p) a_p b_{n-p+1} - (-1)^p (n-p) b_p a_{n-p+1}\right] w.$

Since $\sum_{p=1}^{n} (-1)^{p(n-p)} (-1)^p (n-p) b_p a_{n-p+1} = \sum_{q=1}^{n} (-1)^{q(n-q)} (-1)^q (q-1) b_{n-q+1} a_q$, we have $\mathcal{J}_n(v \otimes x \otimes w^{\otimes n-2}) = 0$ if and only if

$$0 = \sum_{p=1}^{n} (-1)^{p(n-p)} \left[c_p a_{n-p+1} + d_p b_{n-p+1} + (-1)^p \left((n-p) - (p-1)\right) a_p b_{n-p+1}\right].$$

Since $a_n, b_n = 0 \ \forall \ n < k$, $\mathcal{J}_n(v \otimes x \otimes w^{\otimes n-2}) = 0$ if and only if

$$0 = \sum_{p=1}^{n-k+1} (-1)^{p(n-p)} \left[c_p a_{n-p+1} + d_p b_{n-p+1} + (-1)^p (n-2p+1)) a_p b_{n-p+1}\right].$$

Note that when $n < k$, $\mathcal{J}_n \equiv 0$. When $n \geq k$, though, the sum is nontrivial, and we can solve for d_{n-k+1}. Before pulling terms out of the sum, though, we can substitute $m = n - k + 1$ to get a nicer formula in the end:

$$0 = \sum_{p=1}^{m} (-1)^{p(m+k)} \left[c_p a_{m+k-p} + d_p b_{m+k-p} + (-1)^p (m+k-2p) a_p b_{m+k-p}\right].$$

Now, we can remove the terms with $p = m$ from the sum and solve for d_m:

$$d_m = (-1)^m (m-k) a_m - \tfrac{c_m a_k}{b_k} + \frac{\sum_{p=1}^{m-1} (-1)^{p(m+k)} \left[c_p a_{m+k-p} + d_p b_{m+k-p} + (-1)^p (m+k-2p) a_p b_{m+k-p}\right]}{-(-1)^{m(m+k)} b_k}.$$

□

Theorem 34. *Suppose that $V = V_0 \oplus V_1$, where $V_0 =< v, x >$ and $V_1 =< w >$. Then the possible L_n structures on V can be characterized as follows.*

(1) If $l_p(V_0 \otimes V_1^{\otimes p-1}) \equiv 0 \ \forall \ p \leq n$, then

$$\{ \ l_p(v \otimes x \otimes w^{\otimes p-2}) = c_p v + d_p x \ \forall \ p \leq n \ \},$$

where c_n and d_n are arbitrarily chosen constants, form an L_∞ structure on V.

(2) Otherwise, we can suppose without loss of generality that $\exists k \leq n$ such that $a_p = b_p = 0 \ \forall \ p < k$ but $b_k \neq 0$ (since if not, the basis vectors can simply be reordered). Then the operators

$$\left\{ \begin{array}{c} l_p(v \otimes w^{\otimes p-1}) = (p-1)! \ a_p w \\ l_p(x \otimes w^{\otimes p-1}) = (p-1)! \ b_p w \\ l_p(v \otimes x \otimes w^{\otimes p-2}) = (p-2)! \ c_p v + (p-2)! \ d_p x \end{array} \right\}$$

form an L_n structure on $V = V_0 \oplus V_1$ if and only if

$$\begin{aligned} d_m = (-1)^m (m-k) a_m - \tfrac{c_m a_k}{b_k} \\ + \frac{\sum_{p=1}^{m-1} (-1)^{p(m+k)} [c_p a_{m+k-p} + d_p b_{m+k-p} + (-1)^p (m+k-2p) a_p b_{m+k-p}]}{-(-1)^{m(m+k)} b_k} \end{aligned}$$

for all $m \leq n-k+1$ (where a_n, b_n, and c_n are arbitrarily chosen constants).

Proof. By Lemma (26), the most general l_n on $V = V_0 \oplus V_1$ are given by

$$\left\{ \begin{array}{c} l_n(v \otimes w^{\otimes n-1}) = (n-1)! \ a_n w \\ l_n(x \otimes w^{\otimes n-1}) = (n-1)! \ b_n w \\ l_n(v \otimes x \otimes w^{\otimes n-2}) = (n-2)! \ c_n v + (n-2)! \ d_n x \end{array} \right\}.$$

Furthermore, Lemma (28) tells us that on $V = V_0 \oplus V_1$,

$$\mathcal{J}_n \equiv 0 \Longleftrightarrow \mathcal{J}_n(v \otimes x \otimes w^{\otimes n-2}) = 0.$$

If $a_p = b_p = 0 \ \forall \ p \leq n$, then the only possible nonzero l_p requires both v and x as input. This in turn forces $\mathcal{J}_m(v \otimes x \otimes w^{\otimes m-2}) = 0 \ \forall \ m \leq n$, which verifies that there is an L_n structure on V.

Otherwise, we can suppose without loss of generality that $\exists k \leq n$ such that $a_p = b_p = 0 \ \forall \ p < k$ but $b_k \neq 0$ (since if not, the basis vectors can simply be reordered). Then Lemma (33) completes the proof. $\square$

Corollary 35. *Suppose that $V = V_0 \oplus V_1$, where $V_0 =< v, x >$ and $V_1 =< w >$. Suppose $V = V_0 \oplus V_1$ where $V_0 =< v, x >$ and $V_1 =< w >$. There are two types of L_∞ structure on V:*

- *$l_n(v \otimes x \otimes w^{\otimes n-2}) = c_n v + d_n x$ where c_n, d_n can be arbitrarily chosen, but $l_n(V_0 \otimes V_1^{\otimes n-1}) = 0 \ \forall \ n$, or*
- *$\exists \ k$ such that $b_k \neq 0$ and $a_n, b_n = 0 \ \forall \ n < k$ (up to a reordering of the basis vectors of V_0), and*

$$\left\{ \begin{array}{c} l_n(v \otimes w^{\otimes n-1}) = (n-1)! \ a_n w. \\ l_n(x \otimes w^{\otimes n-1}) = (n-1)! \ b_n w. \\ l_n(v \otimes x \otimes w^{\otimes n-2}) = (n-2)! \ c_n v + (n-2)! \ d_n x. \end{array} \right\}$$

where a_n, b_n, c_n are arbitrary, but

$$d_m = (-1)^m (m-k) a_m - \tfrac{c_m a_k}{b_k} + \frac{\sum_{p=1}^{m-1} (-1)^{p(m+k)} [c_p a_{m+k-p} + d_p b_{m+k-p} + (-1)^p (m+k-2p) a_p b_{m+k-p}]}{-(-1)^{m(m+k)} b_k}.$$

Remark 36. Using the notation of Theorem (34), an L_2 structure on $V = V_0 \oplus V_1$ is a differential graded Lie algebra if and only if $a_2 c_2 + b_2 d_2 = 0$.

4.2. Two even vectors and one odd vector ($\beta \neq 1$)

In this section, we will consider $V_\alpha \oplus V_\beta$, where α is even and $\beta \neq 1$ is odd. and $V_\alpha =< v, x >$ and $V_\beta =< w >$. According to Lemma (27), such a space can only have $l_n \neq 0$ for (at most) a few particular values of n. Specifically,

$$\left\{ \begin{array}{r} l_n(V_\beta^{\otimes n}) \subset V_\alpha \text{ when } n = \frac{\alpha-2}{\beta-1} \in \mathbb{N}. \\ l_n(V_\alpha \otimes V_\beta^{\otimes n-1}) \subset V_\beta \text{ when } n = 2 - \frac{\alpha}{\beta-1} \in \mathbb{N}. \\ l_n(V_\alpha^{\otimes 2} \otimes V_\beta^{\otimes n-2}) \subset V_\alpha \text{ when } n = 2 - \frac{\alpha}{\beta-1} \in \mathbb{N}. \end{array} \right\}$$

Before finding the possible L_∞ structures on such a space, we will first prove a little technical lemma:

LEMMA 37. *Suppose $V_\alpha \oplus V_\beta$, where $\beta \neq 1$ is odd and $\alpha = (\beta - 1)(2 - k)$ where $k \in \mathbb{N}$, and $V_\alpha =< v, x >$ and $V_\beta =< w >$, and suppose that $l_n(V_\beta^{\otimes n}) = 0 \ \forall\ n \in \mathbb{N}$. Then the possible L_∞ structures on V are*

$$l_k(v \otimes w^{\otimes k-1}) = a\, w, \qquad l_k(x \otimes w^{\otimes k-1}) = b\, w, \qquad l_k(v \otimes x \otimes w^{\otimes k-2}) = c\, v + d\, x.$$

where $a\, c + b\, d = 0$.

PROOF. Since $l_n(V_\beta^{\otimes n}) = 0 \ \forall\ n \in \mathbb{N}$, $\mathcal{J}_n(V_\beta^{\otimes n}) = 0 \ \forall\ n \in \mathbb{N}$, and Lemma (29) also forces $\mathcal{J}_n(V_\alpha \otimes V_\beta^{\otimes n-1}) = 0 \ \forall\ n \in \mathbb{N}$. Thus we need only prove $\mathcal{J}_n(v \otimes x \otimes w^{\otimes n-2}) = 0$. Since $k = 2 - \frac{\alpha}{\beta - 1} \in \mathbb{N}$, Lemma (27) says that the most general l_n are given by

$$l_k(v \otimes w^{\otimes k-1}) = a\, w, \qquad l_k(x \otimes w^{\otimes k-1}) = b\, w, \qquad l_k(v \otimes x \otimes w^{\otimes k-2}) = c\, v + d\, x.$$

Since l_n is only nonzero when $n = k$, the composition $l_{n-p+1} \circ l_p$ can only be nonzero when $n - p + 1 = p = k$. By Lemma (28), $\mathcal{J}_n \neq 0 \implies n = 3 - \frac{2\alpha}{\beta - 1}$. Since $k = 2 - \frac{\alpha}{\beta - 1}$, $\mathcal{J}_n \neq 0 \implies n = 2k - 1$. When $n = 2k - 1$, $\mathcal{J}_n(v \otimes x \otimes w^{\otimes n-2})$ is equal to $(-1)^{k(n-k)} l_k \circ l_k(v \otimes x \otimes w^{\otimes n-2})$. But $l_k \circ l_k(v \otimes x \otimes w^{\otimes n-2})$ is equal to

$$\begin{aligned}
&\tbinom{n-2}{k-2} l_k(l_k(v \otimes x \otimes w^{\otimes k-2}) \otimes w^{\otimes n-k}) + (-1)^{k-1} \tbinom{n-2}{k-1} l_k(l_k(v \otimes w^{\otimes k-1}) \otimes x \otimes w^{\otimes n-k-1}) \\
&\qquad + (-1)^k \tbinom{n-2}{k-1} l_k(l_k(x \otimes w^{\otimes k-1}) \otimes v \otimes w^{\otimes n-k-1}) \\
&= \tbinom{n-2}{k-2} l_k((cv + dx) \otimes w^{\otimes n-k}) + (-1)^k \tbinom{n-2}{k-1} \left[l_k(ax \otimes w^{\otimes n-k}) - l_k(bv \otimes w^{\otimes n-k}) \right].
\end{aligned}$$

Thus $\mathcal{J}_n = 0 \implies \binom{n-2}{k-2}(c\, a\, w + d\, b\, w) + (-1)^k \binom{n-2}{k-1} [a\, b\, w - b\, a\, w] = 0$, which implies that $a\, c + b\, d = 0$. $\square$

The next lemma characterizes the possible nonzero L_∞ structures on $V_\alpha \oplus V_\beta$ when α is even, $\beta \neq \pm 1$ is odd, $V_\alpha =< v, x >$, and $V_\beta =< w >$. This is a very degenerate case, since an L_∞ structure on such a space can consist of at most one nonzero operator (for some specific k).

LEMMA 38. *Suppose $V = V_\alpha \oplus V_\beta$ where $\beta \neq \pm 1$. The possible L_∞ structures on V are as follows:*

- *If $k = \frac{\alpha-2}{\beta-1} \in \mathbb{N}$, $l_k(w^{\otimes k}) = a\,v + b\,x$ forms the only possible L_∞ structure.*
- *If $k = 2 - \frac{\alpha}{\beta-1} \in \mathbb{N}$, then the most general skew maps of degree $2-n$ are*

$$l_k(v \otimes w^{\otimes k-1}) = a\,w, \qquad l_k(x \otimes w^{\otimes k-1}) = b\,w, \qquad l_k(v \otimes x \otimes w^{\otimes k-2}) = c\,v + d\,x.$$

These form an L_∞ structure on V if and only if $ac + bd = 0$.

PROOF. By Lemma (28), we know that $\mathcal{J}_n \equiv 0 \iff \mathcal{J}_n(V_\alpha^{\otimes 2} \otimes V_\beta^{\otimes n-2}) \equiv 0$, and that $\mathcal{J}_n(V_\alpha^{\otimes 2} \otimes V_\beta^{\otimes n-2}) = 0\ \forall\ n \neq 3 - \frac{2\alpha}{\beta-1}$. There are also strong constraints on the possible operators l_n, which we will now consider.

By Lemma (27), we can only define nonzero l_n on V when $\frac{\alpha-2}{\beta-1} \in \mathbb{N}$ or $2 - \frac{\alpha}{\beta-1} \in \mathbb{N}$. If both conditions were true, this would imply that $\frac{2}{\beta-1} \in \mathbb{Z}$, which implies that $\beta \in \{-1, 0, 2, 3\}$. Since $\beta \neq \pm 1$ is odd, this would force $\beta = 3$. If $\beta = 3$ and $\frac{\alpha-2}{\beta-1} \in \mathbb{N}$ and $2 - \frac{\alpha}{\beta-1} \in \mathbb{N}$, then $\frac{\alpha}{2} - 1 \in \mathbb{N}$ and $2 - \frac{\alpha}{2} \in \mathbb{N}$, which is impossible!

Suppose that $k = \frac{\alpha-2}{\beta-1} \in \mathbb{N}$. Then we can define $l_k(w^{\otimes k}) = a\,v + b\,x$, but this is the only possible nonzero l_n on the space! It is easy to see, though, that this does in fact form a little L_∞ structure on V.

If $k = 2 - \frac{\alpha}{\beta-1} \in \mathbb{N}$, then Lemma (37). gives the possible L_∞ structures on V. $\square$

The next lemma shows that on $V = V_0 \oplus V_{-1}$, we have an L_∞ structure if and only if we have an L_2 structure. The constraints on l_1 and l_2 in order to form an L_2 structure on $V = V_0 \oplus V_{-1}$ are detailed in the next lemma.

LEMMA 39. *On $V = V_0 \oplus V_{-1}$, where $V_0 = < v, x >$ and $V_{-1} = < w >$, the possible L_∞ structures on V are as follows.*

$$(1) \quad \left\{ \begin{array}{lll} l_1 \equiv 0, & l_2(v \otimes w) = c\,w, & l_n \equiv 0\ \forall\ n > 2, \\ & l_2(x \otimes w) = d\,w, & \\ & l_2(v \otimes x) = k(d\,v - c\,x) & \text{for some } k \in \mathbb{R}. \end{array} \right\}$$

$$(2)\ \left\{\begin{array}{lll} l_1(w)=b\,x, & l_2(v\otimes w)=c\,w, & l_n\equiv 0\ \forall\ n>2, \\ & l_2(v\otimes x)=c\,x. & \end{array}\right\}$$

$$(3)\ \left\{\begin{array}{lll} l_1(w)=a\,v & l_2(x\otimes w)=d\,w, & l_n\equiv 0\ \forall\ n>2, \\ & l_2(v\otimes x)=-e\,v. & \end{array}\right\}$$

$$(4)\ \left\{\begin{array}{lll} l_1(w)=a\,v+b\,x, & l_2(v\otimes w)=b\,k\,w, & l_n\equiv 0\ \forall\ n>2, \\ & l_2(x\otimes w)=-a\,k\,w, & \\ & l_2(v\otimes x)=k(a\,v+b\,x)\ for\ some\ k\in\mathbb{R}. & \end{array}\right\}$$

Proof. According to Lemma (27), the most general l_n on $V=V_0\oplus V_{-1}$ are

$$l_1(w)=a\,v+b\,x,\qquad l_2(v\otimes w)=c\,w,\qquad l_2(x\otimes w)=d\,w,\qquad l_2(v\otimes x)=e\,v+f\,x.$$

By Lemma (28), $\mathcal{J}_n=0\ \forall\ n\notin\{2,3\}$, but we need to check $\mathcal{J}_2(w\otimes w)$, $\mathcal{J}_2(v\otimes w)$, $\mathcal{J}_2(x\otimes w)$, and $\mathcal{J}_3(v\otimes w\otimes x)$.

$$\mathcal{J}_2(w\otimes w)=-2l_2(l_1(w)\otimes w)+l_1(l_2(w\otimes w)=-2l_2(av+bx)w)=-2(ac+bd)w.$$

$$\mathcal{J}_2(v\otimes w)=-l_2(l_1(v)\otimes w)+l_2(l_1(w)\otimes v)+l_1(l_2(v\otimes w)=l_2((av+bx)\otimes v)+l_1(cw)$$

$$=-b\,(ev+fx)+c\,(av+bx)=(-be+ac)v+b\,(-f+c)x.$$

$$\mathcal{J}_2(x\otimes w)=-l_2(l_1(x)\otimes w)+l_2(l_1(w)\otimes x)+l_1(l_2(x\otimes w)=l_2((av+bx)\otimes x)+l_1(dw)$$

$$=a\,(ev+fx)+d\,(av+bx)=a\,(d+e)v+(af+bd)x.$$

$$\mathcal{J}_3(v\otimes x\otimes w)=l_2\circ l_2(v\otimes x\otimes w)=l_2(l_2(v\otimes x)\otimes w)-l_2(l_2(v\otimes w)\otimes x)+l_2(l_2(x\otimes w)\otimes v)$$

$$=l_2((ev+fx)\otimes w)-l_2(cw\otimes x)+l_2(dw\otimes v)$$

$$=(ec+fd+cd-dc)w=(ec+fd)w.$$

Thus we have an L_∞ structure on V if and only if all of the following are true:

$$ac+bd=0,\quad be=ac,\quad b(c-f)=0,\quad a(d+e)=0,\quad af+bd=0,\quad ec+fd=0.$$

This implies that there are just four cases which give an L_∞ structure:

(1) If $a=b=0$ and c,d are given, $e=kd$ and $f=-kc$ for some $k\in\mathbb{R}$.

(2) If $a = 0$ and $b \neq 0$, then $d = e = 0$ and $c = f$.

(3) If $b = 0$ and $a \neq 0$, then $c = f = 0$ and $d = -e$.

(4) If $b \neq 0$ and $a \neq 0$, then $c = f$ and $d = -e$ and $ac + bd = 0$.

$\square$

On $V = V_{-2} \oplus V_{-1}$, we also have an L_∞ structure if and only if we have an L_2 structure. However, this is a very degenerate case in which there is at most one nonzero operator, as the next lemma shows.

Lemma 40. *Suppose $V = V_{-2} \oplus V_{-1}$, and $V_{-2} =< v, x >$ and $V_{-1} =< w >$. The possible L_∞ structures on V are as follows:*

$$(1) \left\{ \begin{array}{ll} l_1(v) = a\, w, & l_n = 0 \ \forall\ n \neq 1, \\ l_1(x) = b\, w. & \end{array} \right\}$$

$$(2) \left\{ \begin{array}{ll} l_2(w \otimes w) = c\, v + d\, x, & l_n = 0 \ \forall\ n \neq 2. \end{array} \right\}$$

Proof. According to Lemma (27), the most general l_n on $V = V_{-2} \oplus V_{-1}$ are

$$l_1(v) = a\, w \qquad l_1(x) = b\, w \qquad l_2(w \otimes w) = c\, v + d\, x.$$

By Lemma (28), $\mathcal{J}_n = 0 \ \forall\ n \neq 2$, but we still need to check $\mathcal{J}_2(w \otimes w)$, $\mathcal{J}_2(v \otimes w)$, and $\mathcal{J}_2(x \otimes w)$.

$$\mathcal{J}_2(w \otimes w) = -2l_2(l_1(w) \otimes w) + l_1(l_2(w \otimes w) = 0 + l_1(cv + dx) = (ac + bd)w.$$

$$\begin{aligned} \mathcal{J}_2(v \otimes w) &= -l_2(l_1(v) \otimes w) + l_2(l_1(w) \otimes v) + l_1(l_2(v \otimes w) = -l_2(aw \otimes w) \\ &= -acv - adx. \end{aligned}$$

$$\begin{aligned} \mathcal{J}_2(x \otimes w) &= -l_2(l_1(x) \otimes w) + l_2(l_1(w) \otimes x) + l_1(l_2(x \otimes w) = -l_2(bw \otimes w) \\ &= -bcv - bdx. \end{aligned}$$

Thus we have an L_∞ structure on V if and only if all of the following are true:

$$ac = 0, \qquad ad = 0, \qquad bc = 0, \qquad bd = 0.$$

If $a = 0$ and $b = 0$, we have $l_1 \equiv 0$, but $l_2(w \otimes w)$ can be freely defined. If $a \neq 0$ or $b \neq 0$, then $c = 0$ and $d = 0$, forcing $l_2(w \otimes w)$=0. $\square$

The following theorem summarizes all nonzero L_∞ structures which can be built on $V = V_\alpha \oplus V_{-1}$, where $\alpha > 0$ is even and $V_\alpha =< v, x >$ and $V_{-1} =< w >$. Note that they are all degenerate cases, with at most two nonzero l_n.

THEOREM 41. *We identify all possible L_∞ structures which can be built on a space $V = V_\alpha \oplus V_{-1}$, where $\alpha > 0$ is even and $V_\alpha =< v, x >$ and $V_{-1} =< w >$:*

- *On $V = V_0 \oplus V_{-1}$, there are four possible types of L_∞ structure.*

$$(1) \left\{ \begin{array}{lll} l_1 \equiv 0, & l_2(v \otimes w) = c\, w, & l_n \equiv 0 \ \forall\ n > 2, \\ & l_2(x \otimes w) = d\, w, & \\ & l_2(v \otimes x) = k(d\, v - c\, x) & \textit{for some } k \in \mathbb{R}. \end{array} \right\}$$

$$(2) \left\{ \begin{array}{lll} l_1(w) = b\, x, & l_2(v \otimes w) = c\, w, & l_n \equiv 0 \ \forall\ n > 2, \\ & l_2(x \otimes w) = 0, & \\ & l_2(v \otimes x) = c\, x. & \end{array} \right\}$$

$$(3) \left\{ \begin{array}{lll} l_1(w) = a\, v & l_2(x \otimes w) = d\, w, & l_n \equiv 0 \ \forall\ n > 2, \\ & l_2(v \otimes w) = 0, & \\ & l_2(v \otimes x) = -e\, v. & \end{array} \right\}$$

$$(4) \left\{ \begin{array}{lll} l_1(w) = a\, v + b\, x, & l_2(v \otimes w) = b\, k\, w, & l_n \equiv 0 \ \forall\ n > 2, \\ & l_2(x \otimes w) = -a\, k\, w, & \\ & l_2(v \otimes x) = k(a\, v + b\, x) & for\ some\ k \in \mathbb{R}. \end{array} \right\}$$

- *On $V = V_{-2} \oplus V_{-1}$, there are two possible types of L_∞ structure:*

$$(1) \left\{ \begin{array}{ll} l_1(v) = a\, w, & l_n = 0 \ \forall\ n \neq 1, \\ l_1(x) = b\, w. & \end{array} \right\}$$

$$(2) \left\{ \begin{array}{ll} l_2(w \otimes w) = c\, v + d\, x, & l_n = 0 \ \forall\ n \neq 2. \end{array} \right\}$$

- *If $\alpha = 2k - 4$ where $k - 3 \in \mathbb{N}$, then the possible L_∞ structures on V are $l_k(v \otimes w^{\otimes k-1}) = a\, w, \quad l_k(x \otimes w^{\otimes k-1}) = b\, w, \quad l_k(v \otimes x \otimes w^{\otimes k-2}) = c\, v + d\, x$ where $a\, c + b\, d = 0$.*

- *If $\alpha < -2$, then $V_\alpha \oplus V_{-1}$ can just have an L_∞ structure with a single non-trivial operator: $l_m(V_{-1}^{\otimes m}) = av + bx$, where $m = 1 - \frac{\alpha}{2}$.*

Proof. According to Lemma (27), l_n can only be nonzero on $V_\alpha \oplus V_{-1}$ in very particular cases:

$$\left\{ \begin{array}{r} l_m(V_{-1}^{\otimes m}) \subset V_\alpha \text{ when } m = 1 - \frac{\alpha}{2} \in \mathbb{N}. \\ l_k(V_\alpha \otimes V_{-1}^{\otimes k-1}) \subset V_{-1} \text{ when } k = 2 + \frac{\alpha}{2} \in \mathbb{N}. \\ l_k(V_\alpha^{\otimes 2} \otimes V_{-1}^{\otimes k-2}) \subset V_\alpha \text{ when } k = 2 + \frac{\alpha}{2} \in \mathbb{N}. \end{array} \right\}$$

If $m = 1 - \frac{\alpha}{2} \in \mathbb{N}$ and $k = 2 + \frac{\alpha}{2} \in \mathbb{N}$, then $\alpha \in \{0, -2\}$. The possible structures on $V_0 \oplus V_{-1}$ follow from Lemma (39), and the possible structures on $V_{-1} \oplus V_{-2}$ follow from Lemma (40).

If $k = 2 + \frac{\alpha}{2} \in \mathbb{N}$, and $\alpha \notin \{0, -2\}$, then $m = 0$, forcing $l_n(V_{-1}^{\otimes n}) = 0 \; \forall \; n \in \mathbb{N}$. The possible L_∞ structures on V then follow from Lemma (37).

If $m = 1 - \frac{\alpha}{2} \in \mathbb{N}$ and and $\alpha \notin \{0, -2\}$, then $k = 0$. which implies that the only possible nonzero l_n is $l_m(V_{-1}^{\otimes m}) \subset V_\alpha$. It is clear that $\mathcal{J}_n$ must always be zero in this case. $\square$

4.3. Two odd vectors and one even vector

We now consider $V = V_\alpha \oplus V_\beta$, where α is even and $V_\alpha =< v >$ and $V_1 =< u, w >$. Lemma (28) tells us that if $\beta \neq -1$, any skew linear operators l_n on V gives an L_∞ structure. Furthermore, when $\beta = -1$, one need only check $\mathcal{J}_2$ to verify an L_∞ structure on $V = V_\alpha \oplus V_{-1}$.

According to Lemma (27), the only possible nonzero operators on $V_\alpha \oplus V_{-1}$ are

$$\left\{ \begin{array}{r} l_n(V_{-1}^{\otimes n}) \subset V_\alpha \text{ when } n = 1 - \frac{\alpha}{2} \in \mathbb{N}. \\ l_n(V_\alpha \otimes V_{-1}^{\otimes n-1}) \subset V_{-1} \text{ when } n = 2 + \frac{\alpha}{2} \in \mathbb{N}. \end{array} \right\}$$

We only need to check $\mathcal{J}_2$ in spaces with nonzero l_1 or l_2, and these operators can only be nonzero when $\alpha \in \{0, -2\}$.

- On $V_0 \oplus V_{-1}$, $l_1(V_{-1}) \subset V_0$ and $l_2(V_0 \otimes V_{-1}) \subset V_{-1}$.
- On $V_{-2} \oplus V_{-1}$, $l_1(V_{-2}) \subset V_{-1}$ and $l_2(V_{-1} \otimes V_{-1}) \subset V_{-2}$.

LEMMA 42. *Suppose that $V = V_0 \oplus V_{-1}$, where $V_0 =< v >$ and $V_{-2} =< u, x >$. Then every L_∞ structure on V is given by*

$$\left\{ \begin{array}{lll} l_1(u) = av, & l_2(v \otimes u) = cu + dw, & l_n \equiv 0 \ \forall \ n > 2, \\ l_1(w) = bv, & l_2(v \otimes w) = eu + fw & \end{array} \right\},$$

where $ae + bf = 0$ and $ae + bc = 0$ and $af + bd = 0$.

PROOF. The most general nonzero l_n on V are given by

$$l_1(u) = av, \qquad l_1(w) = bv, \qquad l_2(v \otimes u) = cu + dw, \qquad l_2(v \otimes w) = eu + fw.$$

$$\mathcal{J}_2(v \otimes u) = -l_2(l_1(v) \otimes u) + l_2(l_1(u) \otimes v) + l_1(l_2(v \otimes u)) = -l_1(cu + dw) = (ae + bf)v.$$
$$\mathcal{J}_2(v \otimes w) = -l_2(l_1(v) \otimes w) + l_2(l_1(w) \otimes v) + l_1(l_2(v \otimes w)) = l_1(eu + fw) = (ae + bf)v.$$
$$\begin{aligned}\mathcal{J}_2(u \otimes w) &= -l_2(l_1(u) \otimes w) + l_2(l_1(w) \otimes u) + l_1(l_2(u \otimes w)) = -l_2(av \otimes w) + l_2(bv \otimes u) \\ &= a(eu + fw) + b(cu + dw) = (ae + bc)u + (af + bd)w.\end{aligned}$$

Thus we have an L_∞ structure on V if and only if

$$ae + bf = 0, \qquad ae + bc = 0, \qquad af + bd = 0.$$

□

LEMMA 43. *Suppose that $V = V_{-2} \oplus V_{-1}$, where $V_{-2} =< v >$ and $V_{-1} =< u, x >$. Then every L_∞ structure on V is given by*

$$l_1(v) = au + bw, \quad l_2(u \otimes u) = cv, \quad l_2(w \otimes w) = dv, \quad l_2(u \otimes w) = ev,$$

where $ae = 0$ and $be = 0$ and $ac = 0$ and $bd = 0$.

Proof. The most general nonzero l_n on V are given by

$$l_1(v) = au + bw, \quad l_2(u \otimes u) = cv, \quad l_2(w \otimes w) = dv, \quad l_2(u \otimes w) = ev.$$

$$\begin{aligned}
\mathcal{J}_2(v \otimes u) &= -l_2(l_1(v) \otimes u) + l_2(l_1(u) \otimes v) + l_1(l_2(v \otimes u)) \\
&= -l_2((au + bw) \otimes u) = acv + bev = (ac + be)v. \\
\mathcal{J}_2(v \otimes w) &= -l_2(l_1(v) \otimes w) + l_2(l_1(w) \otimes v) + l_1(l_2(v \otimes w)) \\
&= -l_2((au + bw) \otimes w) = aev + bdv = (ae + bd)v. \\
\mathcal{J}_2(u \otimes w) &= -l_2(l_1(u) \otimes w) + l_2(l_1(w) \otimes u) + l_1(l_2(u \otimes w)) = l_1(ev) = aeu + bew.
\end{aligned}$$

Thus we have an L_∞ structure on V if and only if

$$ae = 0, \qquad be = 0, \qquad ac = 0, \qquad bd = 0.$$

$\square$

5. Three Components (Two Even and One Odd).

In this chapter, we will identify all possible L_∞ structures on $V = V_\alpha \oplus V_\beta \oplus V_\gamma$ where α, β are even, γ is odd, and each component is one-dimensional. We will first consider the degenerate cases which have very sparse structures, and conclude that the most interesting space with two even one-dimensional components and one odd one-dimensional component is $V = V_0 \oplus V_1 \oplus V_2$. This case will then be considered in full detail at the end of the chapter, in Section (5.4).

5.1. Spaces in which the odd component is not of degree ± 1

Here, we will identify all possible L_∞ structures on $V = V_\alpha \oplus V_\beta \oplus V_\gamma$, where α, β are even and $\gamma \neq \pm 1$ is odd, and each component of V is one-dimensional. We'll start with some little technical lemmas. The following lemma is true whenever $\gamma \neq 1$. The case when $\gamma = -1$ will be considered in a separate section because it allows some additional special cases, as suggested by Lemma (45).

Lemma 44. *Suppose $V = V_\alpha \oplus V_\beta \oplus V_\gamma$, where α and β are even, $\gamma \neq 1$ is odd, and each component is one-dimensional. Let $k = 2 - \frac{\alpha}{\gamma-1}$, $m = 2 - \frac{\beta}{\gamma-1}$, $q = \frac{\alpha-2}{\gamma-1}$, and $t = \frac{\beta-2}{\gamma-1}$. Then the most general possible operators on V are the following:*

$$\left\{\begin{array}{lll} l_k(v_\alpha \otimes v_\gamma^{\otimes k-1}) = a\, v_\gamma & \text{and} & l_k(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes k-2}) = b\, v_\beta \text{ when } k \in \mathbb{N}. \\ l_m(v_\beta \otimes v_\gamma^{\otimes m-1}) = c\, v_\gamma & \text{and} & l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = d\, v_\alpha \text{ when } m \in \mathbb{N}. \\ l_q(v_\gamma^{\otimes q}) = e\, v_\alpha \text{ when } q \in \mathbb{N}. & & \\ l_t(v_\gamma^{\otimes t}) = f\, v_\beta \text{ when } t \in \mathbb{N}. & & \end{array}\right\}$$

PROOF. $l_n(v_\alpha \otimes v_\gamma^{\otimes n-1}) \in V_{\alpha+(n-1)\gamma+2-n} = V_{n(\gamma-1)+\alpha-\gamma+2}$, which is odd. Thus $l_n(v_\alpha \otimes v_\gamma^{\otimes n-1})$ can only be nonzero if $n(\gamma-1)+\alpha-\gamma+2=\gamma$, which implies that $n = \frac{2\gamma-2-\alpha}{\gamma-1} = 2 - \frac{\alpha}{\gamma-1}$.

$l_n(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes n-2}) \in V_{\alpha+\beta+(n-2)\gamma+2-n} = V_{n(\gamma-1)+\alpha+\beta-2\gamma+2}$, which is even. Thus $l_n(v_\alpha \otimes v_\gamma^{\otimes n-1})$ can only be nonzero if $n(\gamma-1)+\alpha+\beta-2\gamma+2 \in \{\alpha, \beta\}$. But $n(\gamma-1)+\alpha+\beta-2\gamma+2 = \alpha \Longrightarrow n = 2 - \frac{\alpha}{\gamma-1}$. Similarily, it can only map to V_β if $n = 2 - \frac{\beta}{\gamma-1}$. The rest of the proof follows the same pattern! $\square$

LEMMA 45. *Suppose $2 - \frac{\alpha}{\gamma-1} \in \mathbb{N}$ and $\frac{\alpha-2}{\gamma-1} \in \mathbb{N}$, where α, β, and γ are integers, and γ is odd. Then $\gamma = -1$ and $\alpha \in \{0, -2\}$.*

PROOF. Suppose $2 - \frac{\alpha}{\gamma-1} \in \mathbb{N}$ and $\frac{\alpha-2}{\gamma-1} \in \mathbb{N}$. Then $\frac{2}{\gamma-1} \in \mathbb{Z}$, which implies that $\gamma \in \{-1, 0, 2, 3\}$. Since γ is odd, $\gamma \in \{-1, 3\}$. If $\gamma = 3$, then $2 - \frac{\alpha}{3-1} \geq 1$ and $\frac{\alpha-2}{3-1} \geq 1$, which implies that $1 \geq \frac{\alpha}{2}$ and $\frac{\alpha}{2} \geq 2$, which is impossible.

If $\gamma = -1$, then $2 - \frac{\alpha}{-1-1} \geq 1$ and $\frac{\alpha-2}{-1-1} \geq 1$, implying that $\alpha \geq -2$ and $\alpha \leq 0$. $\square$

LEMMA 46. *Let $= V_\alpha \oplus V_\beta \oplus V_\gamma$, where $\gamma \neq 1$ is odd, and each component of V is one-dimensional. Further, suppose that $\alpha = (2-k)(\gamma-1)$ for some $k \in \mathbb{N}$, and $\beta = (2-m)(\gamma-1)$ for some $m \in \mathbb{N}$, and suppose that $l_n(V_\beta^{\otimes n}) = 0 \ \forall\ n \in \mathbb{N}$. Then the possible L_∞ structures on V are as follows (listing only the nonzero l_n).*

(1) $\left\{ l_m(v_\beta \otimes v_{-1}^{\otimes m-1}) = c\, v_\gamma, \qquad l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = d\, v_\alpha. \right\}$

(2) $\left\{ l_k(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes k-2}) = b\, v_\beta, \qquad l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = d\, v_\alpha. \right\}$

(3) $$\left\{ \begin{array}{ll} l_k(v_\alpha \otimes v_\gamma^{\otimes m-1}) = a\, v_\gamma, & l_k(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = b\, v_\beta \\ l_m(v_\beta \otimes v_\gamma^{\otimes m-1}) = c\, v_\gamma, & l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = d\, v_\alpha, \\ \textit{where } d = \frac{(-1)^k(k-1)b\,c + (m+k-2)a\,c}{(-1)^{m+1}(m-1)a}. & \end{array} \right\}$$

PROOF. Since $k = 2 - \frac{\alpha}{\gamma-1}$ and $m = 2 - \frac{\beta}{\gamma-1}$, where $\gamma \neq 1$, and $l_n(V_\beta^{\otimes n}) = 0\ \forall\ n$, we know from Lemma (44) that the most general l_n on V are the following:

$$\left\{ \begin{array}{ll} l_k(v_\alpha \otimes v_\gamma^{\otimes k-1}) = a\, v_\gamma. & l_k(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes k-2}) = b\, v_\beta. \\ l_m(v_\beta \otimes v_\gamma^{\otimes m-1}) = c\, v_\gamma. & l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = d\, v_\alpha. \end{array} \right\}$$

Since $l_n(V_\gamma^{\otimes n}) = 0$, $\mathcal{J}_n(V_\gamma^{\otimes n}) = 0 \; \forall \; n \in \mathbb{N}$. By Lemma (29), $\mathcal{J}_n(V_\alpha \otimes V_\gamma^{\otimes n-1}) = 0$ and $\mathcal{J}_n(V_\beta \otimes V_\gamma^{\otimes n-1}) = 0$ for all n. Thus it suffices to prove $\mathcal{J}_n(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes n-2}) = 0$. Furthermore, we need only check this when $n = m + k - 1$, since it will clearly be zero for all other values of n in this space. $\mathcal{J}_{m+k-1}(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m+k-3})$ is equal to

$$
\begin{aligned}
&(-1)^{k(m-1)+k-1}\tbinom{m+k-3}{k-1} l_m(l_k(v_\alpha \otimes v_\gamma^{\otimes k-1}) \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) \\
&+ (-1)^{k(m-1)}\tbinom{m+k-3}{k-2} l_m(l_k(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes k-2}) \otimes v_\gamma^{\otimes m-1}) \\
&+ (-1)^{m(k-1)+m-1}\tbinom{m+k-3}{m-1} l_k(l_m(v_\beta \otimes v_\gamma^{\otimes m-1}) \otimes v_\alpha \otimes v_\gamma^{\otimes k-2}) \\
&+ (-1)^{m(k-1)}\tbinom{m+k-3}{m-2} l_k(l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) \otimes v_\gamma^{\otimes k-1}) \\
= &(-1)^{km+1}\tfrac{(m+k-3)!}{(k-1)!(m-2)!} l_m(a v_\gamma \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) + (-1)^{km+k}\tfrac{(m+k-3)!}{(k-2)!(m-1)!} l_m(b v_\beta \otimes v_\gamma^{\otimes m-1}) \\
&+ (-1)^{km+1}\tfrac{(m+k-3)!}{(m-1)!(k-2)!} l_k(c v_\gamma \otimes v_\alpha \otimes v_\gamma^{\otimes k-2}) + (-1)^{km+m}\tfrac{(m+k-3)!}{(m-2)!(k-1)!} l_k(d v_\alpha \otimes v_\gamma^{\otimes k-1}).
\end{aligned}
$$

Thus $\mathcal{J}_{m+k-1}(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m+k-3}) = 0$ if and only if

$$(m-1)a\,c + (-1)^k(k-1)b\,c + (k-1)c\,a + (-1)^m(m-1)d\,a = 0.$$

If $a = 0$, then we need $b = 0$ or $c = 0$ to have an L_∞ structure. If $a \neq 0$, then we have an L_∞ structure if and only if $d = \frac{(m+k-2)ac+(-1)^k(k-1)bc}{(-1)^{m+1}(m-1)a}$. $\square$

Lemma 47. *Let $= V_\alpha \oplus V_\beta \oplus V_\gamma$, where $\gamma \neq 1$ is odd, and each component of V is one-dimensional. Further, suppose that $\beta = (2-m)(\gamma-1)$ for some $m \in \mathbb{N}$, and $\alpha = 2 + (q)(\gamma - 1)$ for some $q \in \mathbb{N}$, and suppose that $\frac{\beta-2}{\gamma-1} \notin \mathbb{N}$ and $2 - \frac{\alpha}{\gamma-1} \notin \mathbb{N}$. Then the only possible L_∞ structures on V are as follows.*

(1) $\{l_m(v_\beta \otimes v_\gamma^{\otimes m-1}) = 0, \quad l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = d\,v_\alpha, \quad l_q(v_\gamma^{\otimes q}) = e\,v_\alpha.\}$

(2) $\{l_m(v_\beta \otimes v_\gamma^{\otimes m-1}) = c\,v_\gamma, \quad l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = d\,v_\alpha, \quad l_q(v_\gamma^{\otimes q}) = 0. \}$

Proof. By Lemma (44), the most general possible l_n on this space are

$$l_m(v_\beta \otimes v_\gamma^{\otimes m-1}) = c\,v_\gamma, \qquad l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = d\,v_\alpha, \qquad l_q(v_\gamma^{\otimes q}) = e\,v_\alpha.$$

A moment's reflection will show that $\mathcal{J}_n(v_\gamma^{\otimes n})$, $\mathcal{J}_n(v_\alpha \otimes v_\gamma^{\otimes n-1})$, and $\mathcal{J}_n(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes n-2})$ are always zero. Furthermore, $\mathcal{J}_n(v_\beta \otimes v_\gamma^{\otimes n-1})$ can only be nonzero when $n = m+q-1$.

$$\mathcal{J}_{m+q-1}(v_\beta \otimes v_\gamma^{\otimes m+q-2}) = \binom{m+q-2}{m-1} l_q(l_m(v_\beta \otimes v_\gamma^{\otimes m-1}) \otimes v_\gamma^{\otimes q-1}) = l_q(c\, v_\gamma^{\otimes q}) = c\, e\, v_\alpha.$$

Thus we have an L_∞ structure on V if and only if $c\,e = 0$. $\square$

LEMMA 48. *Let $= V_\alpha \oplus V_\beta \oplus V_\gamma$, where $\gamma \neq \pm 1$ is odd, and each component of V is one-dimensional. Further, suppose that $\alpha = (2-k)(\gamma-1)$ for some $k \in \mathbb{N}$, and $\beta = (2-m)(\gamma-1)$ for some $m \in \mathbb{N}$. Then the possible L_∞ structures on V are as follows (listing only the nonzero l_n, to save space).*

(1) $\left\{ l_m(v_\beta \otimes v_{-1}^{\otimes m-1}) = c\, v_\gamma, \qquad l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = d\, v_\alpha. \right\}$

(2) $\left\{ l_k(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes k-2}) = b\, v_\beta, \qquad l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = d\, v_\alpha. \right\}$

(3) $$\left\{ \begin{array}{ll} l_k(v_\alpha \otimes v_\gamma^{\otimes m-1}) = a\, v_\gamma, & l_k(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = b\, v_\beta, \\ l_m(v_\beta \otimes v_\gamma^{\otimes m-1}) = c\, v_\gamma, & l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = d\, v_\alpha, \\ \textit{where } d = \frac{(-1)^k(k-1)b\,c + (m+k-2)a\,c}{(-1)^{m+1}(m-1)a}. & \end{array} \right\}$$

PROOF. Since we have $k = 2 - \frac{\alpha}{\gamma-1}$ and $m = 2 - \frac{\beta}{\gamma-1}$, where $\gamma \neq -1$, we know from Lemma (45) that $q = \frac{\alpha-2}{\gamma-1} \notin \mathbb{N}$ and $t = \frac{\beta-2}{\gamma-1} \notin \mathbb{N}$. The rest of the proof follows from Lemma (46). $\square$

LEMMA 49. *Let $= V_\alpha \oplus V_\beta \oplus V_\gamma$, where $\gamma \neq 1$ is odd, and each component of V is one-dimensional. Further, suppose that $\beta = (2-m)(\gamma-1)$ for some $m \in \mathbb{N}$, and $\alpha = 2 + (q)(\gamma-1)$ for some $q \in \mathbb{N}$. Then the only possible L_∞ structures on V are as follows.*

(1) $\left\{ l_m(v_\beta \otimes v_\gamma^{\otimes m-1}) = 0, \quad l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = d\, v_\alpha, \quad l_q(v_\gamma^{\otimes q}) = e\, v_\alpha. \right\}$

(2) $\left\{ l_m(v_\beta \otimes v_\gamma^{\otimes m-1}) = c\, v_\gamma, \quad l_m(v_\alpha \otimes v_\beta \otimes v_\gamma^{\otimes m-2}) = d\, v_\alpha, \quad l_q(v_\gamma^{\otimes q}) = 0. \right\}$

PROOF. Since we have $m = 2 - \frac{\beta}{\gamma-1}$ and $q = \frac{\alpha-2}{\gamma-1}$, where $\gamma \neq 1$, we know from Lemma (45) that $q = \frac{\alpha-2}{\gamma-1} \notin \mathbb{N}$ and $k = 2 - \frac{\alpha}{\gamma-1} \notin \mathbb{N}$. The rest of the proof follows from Lemma (47). $\square$

LEMMA 50. *Let* $= V_\alpha \oplus V_\beta \oplus V_\gamma$*, where* $\gamma \neq \pm 1$ *is odd, and each component of* V *is one-dimensional. Further, suppose that* $\alpha = (1-q)(\gamma-1)$ *for some* $q \in \mathbb{N}$*, and* $\beta = (1-t)(\gamma-1)$ *for some* $t \in \mathbb{N}$*. Then the only possible nonzero* l_n *on* V *are*

$$\left\{\ l_q(v_\gamma^{\otimes q}) = e\, v_\alpha, \qquad\qquad l_t(v_\gamma^{\otimes t}) = f\, v_\beta.\ \right\}$$

These operators form an L_∞ *structure on* V*, regardless of the values of* e *and* f*.*

5.2. $V = V_\alpha \oplus V_\beta \oplus V_{-1}$ (WHERE α, β ARE EVEN)

It requires a bit of extra effort to classify the L_∞ structures on $V = V_\alpha \oplus V_\beta \oplus V_{-1}$, even though there are very few of them. This is because there are more combinations of operators to check, as implied by Lemma (45). The following lemmas, which fill the remainder of this section, identify all possible L_∞ structures on $V = V_\alpha \oplus V_\beta \oplus V_{-1}$.

LEMMA 51. *Suppose* $V = V_\alpha \oplus V_\beta \oplus V_{-1}$*, where* α *and* β *are even. Let* $k = 2 + \frac{\alpha}{2}$*,* $m = 2 + \frac{\beta}{2}$*,* $q = 1 - \frac{\alpha}{2}$*, and* $t = 1 - \frac{\beta}{2}$*. Then the most general possible skew linear operators of degree* $2-n$ *on* V *are the following:*

$$\left\{\begin{array}{rlrl} l_k(v_\alpha \otimes v_{-1}^{\otimes k-1}) = a\, v_{-1} & \text{if } k \in \mathbb{N}. & l_m(v_\beta \otimes v_{-1}^{\otimes m-1}) = c\, v_{-1} & \text{if } m \in \mathbb{N}. \\ l_k(v_\alpha \otimes v_\beta \otimes v_{-1}^{\otimes k-2}) = b\, v_\beta & \text{if } k \in \mathbb{N}. & l_m(v_\alpha \otimes v_\beta \otimes v_{-1}^{\otimes m-2}) = d\, v_\alpha & \text{if } m \in \mathbb{N}. \\ l_q(v_{-1}^{\otimes q}) = e\, v_\alpha & \text{if } q \in \mathbb{N}. & l_t(v_{-1}^{\otimes t}) = f\, v_\beta & \text{if } t \in \mathbb{N}. \end{array}\right\}$$

PROOF. Special case of Lemma (44). $\square$

LEMMA 52. *Suppose* $V = V_\alpha \oplus V_\beta \oplus V_{-1}$*, where* α *and* β *are even and positive. Let* $k = 2 + \frac{\alpha}{2}$ *and* $m = 2 + \frac{\beta}{2}$*. Then the possible* L_∞ *structures on* V *are as follows (listing only the nonzero* l_n*, to save space).*

(1) $\{\ l_m(v_\beta \otimes v_{-1}^{\otimes m-1}) = c\, v_{-1}, \qquad l_m(v_\alpha \otimes v_\beta \otimes v_{-1}^{\otimes m-2}) = d\, v_\alpha\ \}$.

(2) $\{\ l_k(v_\alpha \otimes v_\beta \otimes v_{-1}^{\otimes k-2}) = b\, v_\beta, \qquad l_m(v_\alpha \otimes v_\beta \otimes v_{-1}^{\otimes m-2}) = d\, v_\alpha\ \}$.

$$(3)\ \left\{\begin{array}{ll} l_k(v_\alpha\otimes v_{-1}^{\otimes m-1}) = a\,v_{-1}, & l_k(v_\alpha\otimes v_\beta\otimes v_{-1}^{\otimes m-2}) = b\,v_\beta, \\ l_m(v_\beta\otimes v_{-1}^{\otimes m-1}) = c\,v_{-1}, & l_m(v_\alpha\otimes v_\beta\otimes v_{-1}^{\otimes m-2}) = d\,v_\alpha, \\ \textit{where } d = \frac{(-1)^k(k-1)b\,c+(m+k-2)a\,c}{(-1)^{m+1}(m-1)a}. & \end{array}\right\}$$

Proof. The result follows from Lemma (46). □

Lemma 53. *Suppose $V = V_\alpha \oplus V_\beta \oplus V_{-1}$, where α and β are even and $\alpha < -2$ and $\beta > 0$. Then the only possible L_∞ structures on V are as follows.*

(1) $\{\ l_m(v_\beta\otimes v_\gamma^{\otimes m-1}) = 0, \quad l_m(v_\alpha\otimes v_\beta\otimes v_\gamma^{\otimes m-2}) = d\,v_\alpha, \quad l_q(v_\gamma^{\otimes q}) = e\,v_\alpha.\ \}$

(2) $\{\ l_m(v_\beta\otimes v_\gamma^{\otimes m-1}) = c\,v_\gamma, \quad l_m(v_\alpha\otimes v_\beta\otimes v_\gamma^{\otimes m-2}) = d\,v_\alpha, \quad l_q(v_\gamma^{\otimes q}) = 0.\ \}$

Proof. Since $\alpha < -2$, $k = 2+\frac{\alpha}{2} \notin \mathbb{N}$, and $q = 1-\frac{\alpha}{2} \in \mathbb{N}$. Since $\beta > 0$, $m = 2+\frac{\beta}{2} \in \mathbb{N}$, and $t = 1-\frac{\beta}{2} \notin \mathbb{N}$. The rest of the proof follows from Lemma(47). □

Lemma 54. *Suppose $V = V_\alpha \oplus V_\beta \oplus V_{-1}$, where $\alpha, \beta < -1$ are even, and let $q = 1-\frac{\alpha}{2}$ and $t = 1-\frac{\beta}{2}$. Then the only possible nonzero l_n on V are*

$$\{\ l_q(v_{-1}^{\otimes q}) = e\,v_\alpha, \qquad l_t(v_{-1}^{\otimes t}) = f\,v_\beta\ \}.$$

These operators form an L_∞ structure on V, regardless of the values of e and f.

Lemma 55. *Suppose $V = V_\alpha \oplus V_0 \oplus V_{-1}$, where each component is one-dimensional, and $\alpha > 0$ is even. Let $k = 2+\frac{\alpha}{2}$. Then the possible L_∞ structures on V are as follows (listing only the nonzero operators, to save space).*

(1) $\{\ l_2(v_\alpha\otimes v_0) = d\,v_\alpha, \quad l_k(v_\alpha\otimes v_0\otimes v_{-1}^{\otimes k-2}) = b\,v_0.\ \}$

(2) $\{\ l_k(v_\alpha\otimes v_{-1}^{\otimes k-1}) = a\,v_{-1}, \quad l_k(v_\alpha\otimes v_0\otimes v_{-1}^{\otimes k-2}) = b\,v_0.\ \}$

(3) $\{\ l_1(v_{-1}) = f\,v_0, \quad l_2(v_\alpha\otimes v_0) = d\,v_\alpha.\ \}$

(4) $\{\ l_1(v_{-1}) = f v_0, \quad l_k(v_\alpha\otimes v_0\otimes v_{-1}^{\otimes k-2}) = b\,v_0, \quad l_k(v_\alpha\otimes v_{-1}^{\otimes k-1}) = [(-1)^{k+1}k\,b]v_{-1}.\}$

$$(5)\ \left\{\begin{array}{l} l_2(v_0\otimes v_{-1}) = cv_{-1}, \quad l_2(v_\alpha\otimes v_0) = dv_\alpha, \quad l_k(v_\alpha\otimes v_{-1}^{\otimes k-1}) = av_{-1}, \\ l_k(v_\alpha\otimes v_0\otimes v_{-1}^{\otimes k-2}) = \left(\dfrac{-ad-kac}{(-1)^k(k-1)c}\right) v_0. \end{array}\right\}$$

PROOF. Since $\alpha > 0$, $k = 2 + \frac{\alpha}{2} \in \mathbb{N}$, and $q = 1 - \frac{\alpha}{2} \notin \mathbb{N}$. Therefore, by Lemma (51), the most general possible l_n on V when $\alpha > 0$ are

$$\left\{ \begin{array}{cc} l_1(v_{-1}) = f\, v_0. & l_k(v_\alpha \otimes v_{-1}^{\otimes k-1}) = a\, v_{-1}. \\ l_2(v_0 \otimes v_{-1}) = c\, v_{-1}. & l_k(v_\alpha \otimes v_0 \otimes v_{-1}^{\otimes k-2}) = b\, v_0. \\ l_2(v_\alpha \otimes v_0) = d\, v_\alpha. & \end{array} \right\}$$

Given these possible l_n, $l_{n-p+1} \circ l_p(v_{-1}^{\otimes n})$ can only be nonzero when $p = 1$, so

$$\mathcal{J}_n(v_{-1}^{\otimes n}) = (-1)^{(1)(n-1)} \binom{n}{1} l_n(l_1(v_{-1}) \otimes v_{-1}^{\otimes n-1}) = (-1)^{n-1}(n) l_n(f v_0 \otimes v_{-1}^{\otimes n-1}).$$

But with the l_n that we have, this can only be nonzero when $n = 2$:

$$\mathcal{J}_2(v_{-1}^{\otimes 2}) = -2 l_2(f v_0 \otimes v_{-1}) = -2\, c\, f\, v_{-1}.$$

Similarly, $l_{n-p+1}(l_p(v_\alpha \otimes v_{-1}^{\otimes p-1}) \otimes v_{-1}^{\otimes n-p})$ can only be nonzero when $p = k$, and $l_{n-p+1}(l_p(v_{-1}^{\otimes p}) \otimes v_\alpha \otimes v_{-1}^{\otimes n-p-1})$ can only be nonzero when $p = 1$. Thus

$$\mathcal{J}_n(v_\alpha \otimes v_{-1}^{\otimes n-1}) = (-1)^{k(n-k)} \binom{n-1}{k-1} l_{n-k+1}(a\, v_{-1}^{n-k+1}) + (-1)^{1(n-1)}(n) l_n(f\, v_0 \otimes v_\alpha \otimes v_{-1}^{\otimes n-2}).$$

Examining the summands again, we see that they can only be nonzero when $n = k$. Completing the calculation, we get

$$\mathcal{J}_k(v_\alpha \otimes v_{-1}^{\otimes k-1}) = \binom{k-1}{k-1} a\, f\, v_0 + (-1)^k (k) b\, f\, v_0 = f\, [\, a + (-1)^k k\, b\,]\, v_0.$$

Now, note that $l_{n-p+1}(l_p(v_0 \otimes v_{-1}^{\otimes p-1}) \otimes v_{-1}^{\otimes n-p})$ can only be nonzero when $p = 2$, and $l_{n-p+1}(l_p(v_{-1}^{\otimes p}) \otimes v_0 \otimes v_{-1}^{\otimes n-p-1}) = 0\ \forall p$ (since v_0 would be repeated). Thus $\mathcal{J}_n(v_0 \otimes v_{-1}^{\otimes n-1})$ is equal to $(-1)^{2(n-2)} \binom{n-1}{1} l_{n-1}(c\, v_{-1}^{n-1})$, which can only be nonzero when $n = 2$. Thus

$$\mathcal{J}_2(v_\alpha \otimes v_{-1}) = c\, f\, v_0.$$

$l_{n-p+1}(l_p(v_\alpha \otimes v_0 \otimes v_{-1}^{\otimes p-2}) \otimes v_{-1}^{\otimes n-p})$ can only be nonzero if ($p = k$ and $n-k+1 = 2$), or ($p = 2$ and $n - 1 = k$). $l_{n-p+1}(l_p(v_\alpha \otimes v_{-1}^{\otimes p-1}) \otimes v_0 \otimes v_{-1}^{\otimes n-p-1})$ can only be nonzero if $p = k$ and $n - p + 1 = 2$, and $l_{n-p+1}(l_p(v_0 \otimes v_{-1}^{\otimes p-1}) \otimes v_\alpha \otimes v_{-1}^{\otimes n-p-1})$ can only be nonzero when $p = 2$ and $n - p + 1 = k$. Thus we only need to check when $n = k + 1$:

$$
\begin{aligned}
\mathcal{J}_{k+1}(v_\alpha \otimes v_0 \otimes v_{-1}^{\otimes k-1}) &= (-1)^{k(1)}\tbinom{k-1}{k-2} l_2(b\, v_0 \otimes v_{-1}) + (-1)^{2(k-1)} l_k(d\, v_\alpha \otimes v_{-1}^{\otimes k-1}) \\
&\quad + (-1)^{k(1)}(-1)^{k-1}\tbinom{k-1}{k-1} l_2(a\, v_{-1} \otimes v_0) \\
&\quad + (-1)^{2(k-1)}(-1)\tbinom{k-1}{1} l_k(c\, v_{-1} \otimes v_\alpha \otimes v_{-1}^{\otimes k-2}) \\
&= [(-1)^k (k-1) b\, c + a\, d + a\, c + (k-1) a\, c] v_{-1}.
\end{aligned}
$$

Therefore, we have an L_∞ structure if and only if

$$c\, f = 0 \qquad \textit{and} \qquad f\,[\,a + (-1)^k k\, b\,] = 0 \qquad \textit{and} \qquad (-1)^k (k-1) b\, c + a\, d + k\, a\, c = 0.$$

If $c = f = 0$, we need $a = 0$ or $d = 0$. If $c = 0$ and $f \neq 0$, we need ($a = 0$ and $b = 0$) or ($d = 0$ and $a = (-1)^{k+1} k\, b$). If $f = 0$ and $c \neq 0$, we need $b = \frac{-a\,d - k\,a\,c}{(-1)^k (k-1) c}$. $\square$

Lemma 56. *Suppose $V = V_\alpha \oplus V_0 \oplus V_{-1}$, where each component is one-dimensional, and $\alpha < -2$ is even. Let $q = 1 - \frac{\alpha}{2}$. Then the possible L_∞ structures on V are as follows (listing only the nonzero operators, to save space).*

- $\{\ l_1(v_{-1}) = f\, v_0, \qquad l_q(v_{-1}^{\otimes q}) = e\, v_\alpha. \ \}$
- $\{\ l_2(v_\alpha \otimes v_0) = d\, v_\alpha, \qquad l_q(v_{-1}^{\otimes q}) = e\, v_\alpha \ \}$
- $\{\ l_2(v_0 \otimes v_{-1}) = c\, v_{-1}, \qquad l_2(v_\alpha \otimes v_0) = d\, v_\alpha. \ \}$

Proof. Since $\alpha < -2$, $k = 2 + \frac{\alpha}{2} \notin \mathbb{N}$, and $q = 1 - \frac{\alpha}{2} \in \mathbb{N}$. Therefore, by Lemma (51), the most general possible operators on V when $\alpha < -2$ are

$$l_1(v_{-1}) = f\, v_0, \qquad l_2(v_0 \otimes v_{-1}) = c\, v_{-1}, \qquad l_2(v_\alpha \otimes v_0) = d\, v_\alpha, \qquad l_q(v_{-1}^{\otimes q}) = e\, v_\alpha.$$

$l_{n-p+1}(l_p(v_{-1}^{\otimes p}) \otimes v_{-1}^{\otimes n-p})$ can only be nonzero if $p = 1$ or $p = q$. If $p = 1$, we have $l_n(f\, v_0 \otimes v_{-1}^{\otimes n-1})$, which can only be nonzero if $n = 2$. If $p = q$, we have $l_{n-q+1}(e\, v_\alpha \otimes v_{-1}^{\otimes n-q})$, which is zero. Thus the only constraint of this type is

$$\mathcal{J}_2(v_{-1} \otimes v_{-1}) = -2 l_2(l_1(v_{-1}) \otimes v_{-1}) = -2 l_2(f\, v_0 \otimes v_{-1}) = -2c\, f\, v_{-1}.$$

Note that $l_{n-p+1}(l_p(v_\alpha \otimes v_{-1}^{\otimes p-1}) \otimes v_{-1}^{\otimes n-p}) = 0\ \forall p$. Also, we can only have nonzero $l_{n-p+1}(l_p(v_{-1}^{\otimes p}) \otimes v_\alpha \otimes v_{-1}^{\otimes n-p-1})$ when $p = 1$, and $l_n(f\, v_0 \otimes v_\alpha \otimes v_{-1}^{\otimes n-2})$ can only be nonzero when $n = 2$. Thus we have only one constraint of this type:

$$\mathcal{J}_2(v_\alpha \otimes v_{-1}) = l_2(f\, v_0 \otimes v_\alpha) = -d\, f\, v_\alpha.$$

We can only have nonzero $l_{n-p+1}(l_p(v_0 \otimes v_{-1}^{\otimes p-1}) \otimes v_{-1}^{\otimes n-p})$ when $p = 2$, but $l_{n-1}(c\, v_{-1}^{\otimes n-1})$ can be nonzero when $n - 1 = 1$ or $n - 1 = q$. Additionally, we can only have nonzero $l_{n-p+1}(l_p(v_{-1}^{\otimes p-1}) \otimes v_0 \otimes v_{-1}^{\otimes n-p})$ when $p = q$ and $n - p + 1 = 2$. Therefore, we have constraints of this type when $n = 2$ and when $n = q + 1$:

$$\mathcal{J}_2(v_0 \otimes v_{-1}) = l_1(l_2(v_0 \otimes v_{-1})) = l_1(c\, v_{-1}) = c\, f\, v_0.$$

$$\mathcal{J}_{q+1}(v_0 \otimes v_{-1}^{\otimes q}) = (-1)^{2(q-1)} \binom{q+1}{1} l_q(c\, v_{-1}^{\otimes q}) = (q + 1)c\, e\, v_\alpha.$$

We have $l_{n-p+1}(l_p(v_\alpha \otimes v_{-1}^{\otimes p-1}) \otimes v_{-1}^{\otimes n-p}) = 0\ \forall p$. Also, $l_{n-p+1}(l_p(v_{-1}^{\otimes p}) \otimes v_\alpha \otimes v_{-1}^{\otimes n-p-1})$ can only be nonzero when $p = 1$ and $n - p + 1 = 2$. Thus our constraint here is

$$\mathcal{J}_2(v_\alpha \otimes v_{-1}) = l_2(l_1(v_{-1}) \otimes v_\alpha) = l_2(f\, v_0 \otimes v_\alpha) = -d\, f\, v_\alpha.$$

Finally, $\mathcal{J}_n(v_\alpha \otimes v_0 \otimes v_{-1}^{\otimes n-2}) = 0\ \forall n$. Therefore, for an L_∞ structure on V, we require that

$$c\, f = 0 \qquad \textit{and} \qquad d\, f = 0 \qquad \textit{and} \qquad ce = 0.$$

$\square$

Lemma 57. *Let $V = V_\alpha \oplus V_{-2} \oplus V_{-1}$, where each component is one-dimensional, and $\alpha > 0$ is even. Let $k = 2 + \frac{\alpha}{2}$. Then the possible L_∞ structures on V are as follows (listing only the nonzero operators, to save space).*

- $\{\ l_k(v_\alpha \otimes v_{-1}^{\otimes k-1}) = a\, v_{-1}, \qquad l_k(v_\alpha \otimes v_{-2} \otimes v_{-1}^{\otimes k-2}) = b\, v_{-2}. \ \}$
- $\left\{ \begin{array}{ll} l_k(v_\alpha \otimes v_{-2} \otimes v_{-1}^{\otimes k-2}) = b\, v_{-2}, & l_k(v_\alpha \otimes v_{-1}^{\otimes k-1}) = (-1)^k \binom{k-1}{2} b v_{-1}, \\ l_2(v_{-1}^{\otimes 2}) = f\, v_{-2}. & \end{array} \right\}$
- $\left\{ \begin{array}{ll} l_k(v_\alpha \otimes v_{-1}^{\otimes k-1}) = a\, v_{-1}, & l_k(v_\alpha \otimes v_{-2} \otimes v_{-1}^{\otimes k-2}) = (-1)^k (2 - k) a\, v_{-2}, \\ l_1(v_{-2}) = c\, v_{-1}. & \end{array} \right\}$

PROOF. Since $\alpha > 0$, $k = 2 + \frac{\alpha}{2} \in \mathbb{N}$, and $q = 1 - \frac{\alpha}{2} \notin \mathbb{N}$. Therefore, by Lemma (51), the most general possible operators on V when $\alpha > 0$ are

$$\begin{Bmatrix} l_1(v_{-2}) = c\, v_{-1}. & l_k(v_\alpha \otimes v_{-1}^{\otimes k-1}) = a\, v_{-1}. \\ \\ l_2(v_{-1}^{\otimes 2}) = f\, v_{-2}. & l_k(v_\alpha \otimes v_{-2} \otimes v_{-1}^{\otimes k-2}) = b\, v_{-2}. \end{Bmatrix}$$

Since $l_{n-p+1}(l_p(v_{-1}^{\otimes p}) \otimes v_{-1}^{\otimes n-p})$ can only be nonzero if $p = 2$ and $n - p + 1 = 1$,

$$\mathcal{J}_2(v_{-1}^{\otimes n}) = l_1(l_2(v_1^{\otimes 2})) = l_1(f\, v_{-2}) = c\, f\, v_{-1}.$$

Note that $l_{n-p+1}(l_p(v_{-2} \otimes v_{-1}^{\otimes p-1}) \otimes v_{-1}^{\otimes n-p})$ can only be nonzero when $p = 1$, and $n - p + 1 = 2$, and $l_{n-p+1}(l_p(v_{-1}^{\otimes p}) \otimes v_{-2} \otimes v_{-1}^{\otimes n-p-1}) = 0\ \forall p$. Thus

$$\mathcal{J}_2(v_{-2} \otimes v_{-1}) = -l_2(l_1(v_{-2}) \otimes v_{-1}) = c\, f\, v_{-2}.$$

Since $l_{n-p+1}(l_p(v_\alpha \otimes v_{-1}^{\otimes p-1}) \otimes v_{-1}^{\otimes n-p})$ can only be nonzero if $p = k$ and $n - p + 1 = 2$, and since we can only have nonzero $l_{n-p+1}(l_p(v_{-1}^{\otimes p}) \otimes v_\alpha \otimes v_{-1}^{\otimes n-p-1})$ when $p = 2$ and $n - p + 1 = k$, we get a constraint when $n = k + 1$:

$$\begin{aligned} \mathcal{J}_{k+1}(v_\alpha \otimes v_{-1}^{\otimes k}) &= (-1)^{k(1)} l_2(a\, v_{-1} \otimes v_{-1}) + (-1)^{2(k-1)} \tbinom{k-1}{2} l_k(f\, v_{-2} \otimes v_\alpha \otimes v_{-1}^{\otimes k-2}) \\ &= (-1)^k a\, f\, v_{-2} - \tbinom{k-1}{2} b\, f\, v_{-2} = f\, [(-1)^k a\ - \tbinom{k-1}{2} b\,] v_{-2}. \end{aligned}$$

Finally, $l_{n-p+1}(l_p(v_\alpha \otimes v_{-2} \otimes v_{-1}^{\otimes p-2}) \otimes v_{-1}^{\otimes n-p})$ can only be nonzero when $p = k$ and $n - p + 1 = 1$, and $l_{n-p+1}(l_p(v_\alpha \otimes v_{-1}^{\otimes p-1}) \otimes v_{-2} \otimes v_{-1}^{\otimes n-p-1}) = 0\ \forall p$. Also, $l_{n-p+1}(l_p(v_{-2} \otimes v_{-1}^{\otimes p-1}) \otimes v_\alpha \otimes v_{-1}^{\otimes n-p-1}) = 0$ can only be nonzero when $p = 1$ and $n - p + 1 = k$, and $l_{n-p+1}(l_p(v_{-1}^{\otimes p}) \otimes v_\alpha \otimes v_{-2} \otimes v_{-1}^{\otimes n-p-2}) = 0\ \forall p$. Therefore, we get a constraint when $n = k$:

$$\begin{aligned} \mathcal{J}_k(v_\alpha \otimes v_{-2} \otimes v_{-1}^{\otimes k-2}) &= l_1(b\, v_{-2}) + (-1)^{1(k-1)} \tbinom{k-2}{1} l_k(c\, v_{-1} \otimes v_\alpha \otimes v_{-1}^{\otimes k-2})) \\ &= b\, c\, v_{-1} + (-1)^k (k-2) a\, c\, v_{-1} = c\, [b\ + (-1)^k (k-2) a\,] v_{-1}. \end{aligned}$$

Therefore, for an L_∞ structure, we require

$$c\, f = 0 \qquad \textit{and} \qquad f\, [(-1)^k a\ - \tbinom{k-1}{2} b\,] = 0 \qquad \textit{and} \qquad c\, [b\ + (-1)^k (k-2) a\,] = 0.$$

$\square$

LEMMA 58. *Let $V = V_\alpha \oplus V_{-2} \oplus V_{-1}$, where each component is one-dimensional, and $\alpha < -2$ is even. Let $q = 1 - \frac{\alpha}{2}$. Then the possible L_∞ structures on V are as follows (listing only the nonzero operators, to save space).*

- $\{\ l_q(v_{-1}^{\otimes q}) = e\, v_\alpha, \qquad l_2(v_{-1}^{\otimes 2}) = f\, v_{-2}.\ \}$
- $\{\ l_1(v_{-2}) = c\, v_{-1}.\ \}$

PROOF. Since $\alpha < -2$, $k = 2 + \frac{\alpha}{2} \notin \mathbb{N}$, and $q = 1 - \frac{\alpha}{2} \in \mathbb{N}$. Therefore, by Lemma (51), the most general possible l_n on $V = V_\alpha \oplus V_{-2} \oplus V_{-1}$ when $\alpha < -2$ are

$$l_1(v_{-2}) = c\, v_{-1}, \qquad l_q(v_{-1}^{\otimes q}) = e\, v_\alpha, \qquad l_2(v_{-1}^{\otimes 2}) = f\, v_{-2}.$$

Since $l_{n-p+1}(l_p(v_{-1}^{\otimes p}) \otimes v_{-1}^{\otimes n-p})$ can only be nonzero if $p = 2$ and $n - p + 1 = 1$,

$$\mathcal{J}_2(v_{-1}^{\otimes n}) = l_1(l_2(v_1^{\otimes 2})) = l_1(f\, v_{-2}) = c\, f\, v_{-1}.$$

Note that $l_{n-p+1}(l_p(v_{-2} \otimes v_{-1}^{\otimes p-1}) \otimes v_{-1}^{\otimes n-p})$ can only be nonzero when $p = 1$, and either $n - p + 1 = 2$ or $n - p + 1 = q$, and $l_{n-p+1}(l_p(v_{-1}^{\otimes p}) \otimes v_{-2} \otimes v_{-1}^{\otimes n-p-1})) = 0\ \forall p$. Thus we get constraints when $n = 2$ and $n = q$:

$$\mathcal{J}_2(v_{-2} \otimes v_{-1}) = -l_2(l_1(v_{-2}) \otimes v_{-1}) = -l_2(c\, v_{-1}^{\otimes 2}) = c\, f\, v_{-2}.$$

$$\mathcal{J}_q(v_{-2} \otimes v_{-1}^{\otimes q-1}) = l_q(c\, v_{-1}^{\otimes q}) = c\, e\, v_{-2}.$$

Finally, $\mathcal{J}_n(v_\alpha \otimes v_{-1}^{\otimes n-1}) = 0\ \forall\ n$, and $\mathcal{J}_n(v_\alpha \otimes v_\beta \otimes v_{-1}^{\otimes n-1}) = 0\ \forall\ n$, since no nonzero l_p takes v_α as input. Therefore, we have an L_∞ structure on $V = V_\alpha \oplus V_{-2} \oplus V_{-1}$ if and only if $ce = 0$ and $cf = 0$. $\square$

LEMMA 59. *Let $V = V_{-2} \oplus V_{-1} \oplus V_0$, where each component is one-dimensional. Then the possible L_∞ structures on V are as follows (listing only the nonzero operators, to save space).*

- $\{\ l_1(v_{-2}) = a\, v_{-1}, \qquad l_2(v_0 \otimes v_{-1}) = c\, v_{-1}, \qquad l_2(v_{-2} \otimes v_0) = d\, v_{-2}.\ \}$
- $\{\ l_1(v_{-1}) = f\, v_0.\ \}$

PROOF. Let $k = 1$, $m = 2$, and $t = 1$. The most general possible l_n on V are

$$l_1(v_{-2}) = a\,v_{-1}, \qquad l_1(v_{-1}) = f\,v_0, \qquad l_2(v_0 \otimes v_{-1}) = c\,v_{-1}, \qquad l_2(v_{-2} \otimes v_0) = d\,v_{-2}.$$

$\mathcal{J}_1(v_{-2}) = a\,f\,v_0$. $\mathcal{J}_2(v_{-2} \otimes v_0) = -l_2(a\,v_{-1} \otimes v_0) + l_1(d\,v_{-2}) = 0$. $\mathcal{J}_2(v_{-2} \otimes v_{-1}) = l_2(f\,v_0 \otimes v_{-2}) = -f\,d\,v_{-2}$. $\mathcal{J}_2(v_0 \otimes v_{-1}) = l_1(c\,v_{-1}) = c\,f\,v_{-1}$. $\mathcal{J}_3(v_{-2} \otimes v_{-1} \otimes v_0) = 0$. Thus we need $af = 0$ and $df = 0$ and $cf = 0$. □

5.3. $V = V_\alpha \oplus V_\beta \oplus V_1$ (WHERE α, β ARE EVEN)

The next lemma shows that if $V = V_\alpha \oplus V_\beta \oplus V_1$ (where α, β are even), but V is not $V_0 \oplus V_1 \oplus V_2$, then any arbitrary collection of skew linear operators l_n of degree $2 - n$ forms an L_∞ structure on V.

LEMMA 60. *Suppose that $V = V_\alpha \oplus V_\beta \oplus V_1$ where α and β are distinct even numbers such that $\{\alpha, \beta\} \neq \{0, 2\}$, and any set of skew linear operators l_n of degree $2 - n$ are defined on V. Then these operators form an L_∞ structure on V.*

PROOF. First, suppose that $\alpha, \beta \neq 0$. Since the only odd component is of degree 1, this forces $l_n(v_\alpha \otimes v_1^{\otimes n-1}) \in V_{\alpha+1} = 0$ and $l_n(v_\beta \otimes v_1^{\otimes n-1}) \in V_{\beta+1} = 0$. Similarly, we have $l_n(v_\alpha \otimes v_\beta \otimes v_1^{\otimes n-1}) \in V_{\alpha+1} = 0$ (since $\alpha + \beta$ is even, but cannot be equal to either α or β). Given all of these constraints together, we see that l_n is forced to be zero whenever it receives at least one even vector as input, which in turn forces $\mathcal{J}_n \equiv 0\ \forall\ n \geq 2$. One can easily verify that our assumptions about grading also force $l_1 \circ l_1 = 0$, which proves that $\mathcal{J}_n \equiv 0\ \forall\ n \in \mathbb{N}$ when neither of the even components is of grade zero.

Now suppose that $\beta = 0$, and $\alpha \neq 2$. Then $\mathcal{J}_n(v_1^{\otimes n}) \in V_3 = 0$, since the only odd component is of degree one. Similarly, $\mathcal{J}_n(v_\alpha \otimes v_\beta \otimes v_1^{\otimes n-2}) \in V_{\alpha+1} = 0$. Also, $\mathcal{J}_n(v_\beta \otimes v_1^{\otimes n-1}) \in V_2 = 0$ (by assumption). Finally, $\mathcal{J}_n(v_\alpha \otimes v_1^{\otimes n-1}) \in V_{\alpha+2}$, which can only be nonzero if $\alpha = -2$. Since the above calculations include all inputs to the $\mathcal{J}_n$ which do not repeat an even vector, we see that $\mathcal{J}_n \equiv 0$ whenever $\alpha \neq -2$.

However, when $\alpha = -2$, $\mathcal{J}_n$ is also zero. This is because of the way that the grading restricts l_n:

- $l_n(v_1^{\otimes n}) \in V_2$, which is zero by assumption.
- $l_n(v_{-2} \otimes v_1^{\otimes n-1}) \in V_{-1}$, which is zero by assumption.

This forces $l_{n-p+1} \circ l_p(v_{-2} \otimes v_1^{\otimes n-1})$ to be zero, which forces $\mathcal{J}_n(v_\alpha \otimes v_1^{\otimes n-1})$ to be zero. $\square$

5.4. Proof of the case in which $\mathbf{V} = \mathbf{V_0} \oplus \mathbf{V_1} \oplus \mathbf{V_2}$

In this section, we will show that there are exactly two types of L_∞ structures on $V = V_0 \oplus V_1 \oplus V_2$. If we denote $V_0 =< v >$, $V_1 =< w >$, and $V_2 =< x >$, then the possible L_∞ structures are as follows:

- A structure with maps $\left\{ \begin{array}{rl} l_n(v \otimes w^{\otimes n-1}) = a_n w & \forall\, n \geq 1 \\ l_n(v \otimes w^{\otimes n-2} \otimes x) = c_n x & \forall\, n \geq 2 \end{array} \right\}$, in which a_n and c_n are arbitrary chosen constants, but all other maps are zero. It will be shown that any structure of this type which is L_3 is also a differential graded Lie algebra.
- A structure with maps $\left\{ \begin{array}{rl} l_n(v \otimes w^{\otimes n-1}) = a_n w & \forall\, n \geq 1 \\ l_n(w^{\otimes n}) = b_n x & \forall\, n \geq 1 \\ l_n(v \otimes w^{\otimes n-2} \otimes x) = c_n x & \forall\, n \geq 2 \end{array} \right\}$, in which some b_n is nonzero and the remaining b_n and all of the c_n are arbitrarily chosen, but each constant a_n is uniquely determined by the recursion formula,

$$a_n = \binom{n-1}{k} c_n + \frac{\sum_{p=2}^{n-1} (-1)^{p(n+k)} b_{n+k-p} \left[\binom{n+k-2}{p-1} a_p - \binom{n+k-2}{p-2} c_p \right]}{(-1)^{n(n+k)+1} \binom{n+k-2}{n-1} b_k}$$

(where k is the least value such that b_k is nonzero). It will be proven that an L_3 structure of this type is a differential graded Lie algebra if and only if $b_1 = 0$ or $b_2 = 0$ or $a_2 = 0$.

We will also provide necessary and sufficient conditions under which skew linear maps form an L_m structure on $V = V_0 \oplus V_1 \oplus V_2$, as well as necessary and sufficient conditions for an L_3 structure on this space to be a differential graded Lie algebra. To accomplish these goals, we first need to establish some notation.

Let $V = V_0 \oplus V_1 \oplus V_2$, where $V_0 =< v >$, $V_1 =< w >$, and $V_2 =< x >$. Define skew linear maps $l_n : V^{\otimes n} \to V$ of degree $2 - n$ by

$$\tag{1} \begin{cases} l_n(v \otimes w^{\otimes n-1}) = a_n w & \forall\ n \geq 1 \\ l_n(w^{\otimes n}) = b_n x & \forall\ n \geq 1 \\ l_n(v \otimes w^{\otimes n-2} \otimes x) = c_n x & \forall\ n \geq 2 \\ l_n(w^{\otimes n-1} \otimes x) = 0 & \forall\ n \\ l_n(v^{\otimes i} \otimes w^{\otimes n-i-j} \otimes x^{\otimes j}) = 0 & \forall\ i,\ j > 1 \end{cases}$$

where a_n, b_n, c_n are constants. Since c_n is defined above only for $n \geq 2$, we can define $c_1 = 0$ for the sake of convenience.

Remark 61. The above list (1) includes all possible skew linear maps of degree $2 - n$, by the following argument:

- $l_n(v^{\otimes i} \otimes w^{\otimes n-i-j} \otimes x^{\otimes j}) = 0 \quad \forall\ i,\ j > 1$ because a repeated component of even degree makes a skew map zero.
- $l_n(w^{\otimes n-1} \otimes x) \subset V_{n-1+2+2-n} = V_3$ which is zero.

The action of l_k on the remaining generators is determined by its skewness.

In order to determine when these maps define an L_m structure on V, we need to know when $\mathcal{J}_n = 0\ \forall\ n \leq m$. The following two lemmas establish the conditions which force $\mathcal{J}_n = 0$:

Lemma 62. *Given* $V = V_0 \oplus V_1 \oplus V_2$ *with the skew linear maps defined in* (1), $\mathcal{J}_n = 0$ *if and only if* $\mathcal{J}_n(v \otimes w^{\otimes n-1}) = 0$.

Proof. Since $deg(\mathcal{J}_n) = 3 - n$,

$$\mathcal{J}_n(V_0^{\otimes i} \otimes V_1^{\otimes n-i-j} \otimes V_2^{\otimes j}) \subset V_{n-i-j+2j+3-n} = V_{-i+j+3}.$$

Thus $\mathcal{J}_n(V_0^{\otimes i} \otimes V_1^{\otimes n-i-j} \otimes V_2^{\otimes j})$ can only be nonzero if $j - i + 3 \in \{0, 1, 2\}$ (since these are the only nonzero components of V). Equivalently, $\mathcal{J}_n(V_0^{\otimes i} \otimes V_1^{\otimes n-i-j} \otimes V_2^{\otimes j})$ can only be nonzero if $j - i \in \{-3, -2, -1\}$. But $\mathcal{J}_n(V_0^{\otimes i} \otimes V_1^{\otimes n-i-j} \otimes V_2^{\otimes j}) = 0$ whenever $i > 1$ (since $\mathcal{J}_n$ is skew and a parameter of even degree would be repeated). Thus $\mathcal{J}_n(V_0^{\otimes i} \otimes V_1^{\otimes n-i-j} \otimes V_2^{\otimes j})$ can only be nonzero when $i = 1$ and $j = 0$. □

Lemma 63. *Given* $V = V_0 \oplus V_1 \oplus V_2$ *with the maps defined in* (1),

$$\mathcal{J}_n(v \otimes w^{\otimes n-1}) = \sum_{p=1}^{n} (-1)^{p(n-p)}(-1)^n b_p \left[(-1)\binom{n-1}{n-p} a_{n-p+1} + \binom{n-1}{p} c_{n-p+1}\right] x.$$

Proof. Before calculating $l_{n-p+1} \circ l_p(v \otimes w^{\otimes n-1})$, it is helpful to consider the ways in which a $(p, n-p)$ unshuffle of $v \otimes w^{\otimes n-1}$ could possibly rearrange the terms. Here, there are only two possibilities, since ordering guarantees that the v must be placed in either position 1 or position $p + 1$.

There are $\binom{n-1}{p-1}$ unshuffles which place v in position 1. In each case, $\chi(\sigma) = +1$, since each of these permutations can be accomplished solely by commuting elements of odd degree.

Similarly, there are $\binom{n-1}{p}$ unshuffles which place v in position $p+1$. In each case, $\chi(\sigma) = (-1)^p$. This is because we can reorder the consecutive w terms (which does not change the sign), and then simply shift v past p w terms.

$$
\begin{aligned}
l_{n-p+1} \circ l_p(v \otimes w^{\otimes n-1}) &= \tbinom{n-1}{p-1} l_{n-p+1}(l_p(v \otimes w^{\otimes p-1}) \otimes w^{\otimes n-p}) \\
&\quad + \tbinom{n-1}{p}(-1)^p l_{n-p+1}(l_p(w^{\otimes p}) \otimes v \otimes w^{\otimes n-p-1}) \\
&= \tbinom{n-1}{p-1} l_{n-p+1}(a_p w \otimes w^{\otimes n-p}) \\
&\quad + \tbinom{n-1}{p}(-1)^p(-1)^{n-p} l_{n-p+1}(b_p v \otimes w^{\otimes n-p-1} \otimes x) \\
&= \tbinom{n-1}{p-1} a_p b_{n-p+1} x + (-1)^n \tbinom{n-1}{p} b_p c_{n-p+1} x.
\end{aligned}
$$

Since $\mathcal{J}_n = \sum_{p=1}^{n} (-1)^{p(n-p)} l_{n-p+1} \circ l_p$, we have

$$
\mathcal{J}_n(v \otimes w^{\otimes n-1}) = \sum_{p=1}^{n} (-1)^{p(n-p)} \left[\tbinom{n-1}{p-1} a_p b_{n-p+1} x + (-1)^n \tbinom{n-1}{p} b_p c_{n-p+1} x \right].
$$

Thus $\mathcal{J}_n(v \otimes w^{\otimes n-1})$ is equal to

$$
\sum_{p=1}^{n} (-1)^{p(n-p)} \tbinom{n-1}{p-1} a_p b_{n-p+1} x + \sum_{p=1}^{n} (-1)^{p(n-p)} (-1)^n \tbinom{n-1}{p} b_p c_{n-p+1} x.
$$

Substituting $q = n - p + 1$ in the first sum and re-indexing, this is equivalent to

$$
\sum_{q=1}^{n} (-1)^{(n-q+1)(q-1)} \tbinom{n-1}{n-q} a_{n-q+1} b_q x + \sum_{p=1}^{n} (-1)^{p(n-p)} (-1)^n \tbinom{n-1}{p} b_p c_{n-p+1} x.
$$

Recombining the two sums, this is equivalent to

$$
\sum_{p=1}^{n} (-1)^{p(n-p)} (-1)^n b_p \left[(-1) \tbinom{n-1}{n-p} a_{n-p+1} + \tbinom{n-1}{p} c_{n-p+1} \right] x.
$$

□

Remark 64. Thus in $V = V_0 \oplus V_1 \oplus V_2$,

$$
\mathcal{J}_n = 0 \Longleftrightarrow 0 = \sum_{p=1}^{n} (-1)^{p(n-p)} (-1)^n b_p \left[(-1) \tbinom{n-1}{n-p} a_{n-p+1} + \tbinom{n-1}{p} c_{n-p+1} \right].
$$

In particular, v is L_1 (i.e. a differential graded vector space) if and only if $0 = a_1 b_1$.

Lemma 65. *Suppose $V = V_0 \oplus V_1 \oplus V_2$ with the skew linear maps defined in* (1) *has an L_∞ structure and $a_1 \neq 0$. Then $b_n = 0 \ \forall \ n \in \mathbb{N}$.*

PROOF. Suppose $a_1 \neq 0$. Then $b_1 = 0$ (since the first Jacobi identity requires that $0 = a_1 b_1$). Now we do an induction proof: Suppose $b_k = 0 \ \forall\ k < n$. The nth Jacobi identity requires that

$$0 = \sum_{p=1}^{n} (-1)^{p(n-p)} (-1)^n b_p \left[(-1) \binom{n-1}{n-p} a_{n-p+1} + \binom{n-1}{p} c_{n-p+1} \right].$$

Since we're supposing that $b_k = 0 \ \forall\ k < n$, this collapses to $0 = (-1)^n b_n (n-1) a_1$. But we also supposed that $a_1 \neq 0$, which forces $b_n = 0$. $\square$

If some constant b_k is nonzero, then the following lemma proves that the constraint $\mathcal{J}_n = 0$ is equivalent to an explicit formula for a_{n-k+1}:

LEMMA 66. *Given $V = V_0 \oplus V_1 \oplus V_2$ with the maps defined in* (1)*, suppose that $1 \leq k < n$ & $b_p = 0 \ \forall\ p < k$ & $b_k \neq 0$. Then $\mathcal{J}_n = 0$ if and only if*

$$a_{n-k+1} = \binom{n-k}{k} c_{n-k+1} + \frac{\sum_{q=2}^{n-k} (-1)^{q(n-q)} b_{n-q+1} \left[\binom{n-1}{q-1} a_q - \binom{n-1}{q-2} c_q \right]}{(-1)^{k(n-k)+n} \binom{n-1}{n-k} b_k}.$$

PROOF. We know that

$$\mathcal{J}_n = 0 \Longleftrightarrow 0 = \sum_{p=1}^{n} (-1)^{p(n-p)} (-1)^n b_p \left[(-1) \binom{n-1}{n-p} a_{n-p+1} + \binom{n-1}{p} c_{n-p+1} \right].$$

Since $b_k \neq 0$ by assumption, $a_1 = 0$ by the preceding lemma. Thus when $p = n$, $a_{n-p+1} = 0$. Also, when $p = n$, $\binom{n-1}{p} = 0$. Thus we can change the upper limit of the summation to $n-1$. Furthermore, since $b_p = 0 \ \forall\ p < k$, the lower limit can be changed to k. Thus $\mathcal{J}_n = 0$ if and only if

$$0 = \sum_{p=k}^{n-1} (-1)^{p(n-p)} b_p \left[(-1) \binom{n-1}{n-p} a_{n-p+1} + \binom{n-1}{p} c_{n-p+1} \right].$$

Removing the term with $p = k$ from the summation, we get an equivalent equation:

$$(-1)^{k(n-k)} b_k \binom{n-1}{n-k} a_{n-k+1} = (-1)^{k(n-k)} b_k \binom{n-1}{k} c_{n-k+1}$$
$$+ \sum_{p=k+1}^{n-1} (-1)^{p(n-p)} b_p \left[(-1) \binom{n-1}{n-p} a_{n-p+1} + \binom{n-1}{p} c_{n-p+1} \right].$$

Since we supposed that $1 \leq k < n$, $\binom{n-1}{n-k} \neq 0$. Also $b_k \neq 0$ by supposition. Thus the previous equation is equivalent to the formula,

$$a_{n-k+1} = \frac{\binom{n-1}{k} c_{n-k+1}}{\binom{n-1}{n-k}} + \frac{\sum_{p=k+1}^{n-1} (-1)^{p(n-p)} b_p \left[(-1) \binom{n-1}{n-p} a_{n-p+1} + \binom{n-1}{p} c_{n-p+1} \right]}{(-1)^{k(n-k)} b_k \binom{n-1}{n-k}}.$$

Substituting $q = n - p + 1$, we get

$$a_{n-k+1} = \frac{\binom{n-1}{k} c_{n-k+1}}{\binom{n-1}{n-k}} + \frac{\sum_{q=2}^{n-k} (-1)^{(n-q+1)(q-1)} b_{n-q+1} \left[(-1) \binom{n-1}{q-1} a_q + \binom{n-1}{n-q+1} c_q \right]}{(-1)^{k(n-k)} b_k \binom{n-1}{n-k}}.$$

Simplifying, we get

$$a_{n-k+1} = \binom{n-k}{k} c_{n-k+1} + \frac{\sum_{q=2}^{n-k} (-1)^{q(n-q)} b_{n-q+1} \left[\binom{n-1}{q-1} a_q - \binom{n-1}{q-2} c_q \right]}{(-1)^{k(n-k)+n} \binom{n-1}{n-k} b_k}.$$

□

LEMMA 67. *If $b_p = 0 \ \forall \ p < n$ then $V = V_0 \oplus V_1 \oplus V_2$, with the skew maps defined in (1) is L_n.*

PROOF. Suppose $1 \leq m \leq n$. We know that

$$\mathcal{J}_m = 0 \Longleftrightarrow 0 = \sum_{p=1}^{m} (-1)^{p(m-p)} (-1)^m b_p \left[(-1) \binom{m-1}{m-p} a_{m-p+1} + \binom{m-1}{p} c_{m-p+1} \right].$$

If $m < n$, we have $b_p = 0 \ \forall \ p \leq m$, and $\mathcal{J}_m = 0$. So we only need to show that $\mathcal{J}_n = 0$.

$$\mathcal{J}_n = \sum_{p=n}^{n} (-1)^{p(n-p)} (-1)^n b_p \left[(-1) \binom{n-1}{n-p} a_{n-p+1} + \binom{n-1}{p} c_{n-p+1} \right]$$
$$= (-1)^n b_n \left[(-1) \binom{n-1}{o} a_1 + \binom{n-1}{n} c_1 \right] = (-1)^{n+1} (n-1) b_n a_1.$$

If $b_n = 0$, then this is zero. If $b_n \neq 0$, we know from Lemma (65) that $a_1 = 0$, which makes the sum zero in all cases. Thus $\mathcal{J}_n = 0 \ \forall \ m \leq n$. Therefore, V is L_n. □

Now the following theorem provides necessary and sufficient conditions which identify all possible L_n structures on $V = V_0 \oplus V_1 \oplus V_2$. This also leads to a corollary which specifies all possible L_∞ structures on this space.

THEOREM 68. *Suppose that $V = V_0 \oplus V_1 \oplus V_2$ has skew maps defined by*

$$\left\{ \begin{aligned} l_n(v \otimes w^{\otimes n-1}) &= a_n w \quad \forall\ n \geq 1 \\ l_n(w^{\otimes n}) &= b_n x \quad \forall\ n \geq 1 \\ l_n(v \otimes w^{\otimes n-2} \otimes x) &= c_n x \quad \forall\ n \geq 2 \end{aligned} \right\}.$$

V is L_n if and only if

- $b_p = 0\ \forall\ p < n$ *or*
- $\exists\ k$ *such that* $1 \leq k < n$ & $b_p = 0\ \forall\ p < k$ & $b_k \neq 0$ & $\forall\ 1 \leq m \leq n-k+1$,

$$a_m = \binom{m-1}{k} c_m + \frac{\sum_{p=2}^{m-1} (-1)^{p(m+k)} b_{m+k-p} \left[\binom{m+k-2}{p-1} a_p - \binom{m+k-2}{p-2} c_p \right]}{(-1)^{m(m+k)+1} \binom{m+k-2}{m-1} b_k}.$$

PROOF. If $b_p = 0\ \forall\ p < n$ then we know from Lemma (67) that V is L_n. So suppose otherwise.

Then $\exists\ k$ such that $1 \leq k < n$ & $b_p = 0\ \forall\ p < k$ & $b_k \neq 0$. Using Lemma (67) again, we know that V is L_k. Thus $\mathcal{J}_q = 0\ \forall\ q \leq k$. Thus V is L_n if and only if $\mathcal{J}_q = 0\ \forall\ k < q \leq n$, which by Lemma (66) is true if and only if

$$a_{q-k+1} = \binom{q-k}{k} c_{q-k+1} + \frac{\sum_{p=2}^{q-k} (-1)^{p(q-p)} b_{q-p+1} \left[\binom{q-1}{p-1} a_p - \binom{q-1}{p-2} c_p \right]}{(-1)^{k(q-k)+q} \binom{q-1}{q-k} b_k}$$

$\forall\ k < q \leq n$. If we substitute $m = q-k+1$, this formula becomes

$$a_m = \binom{m-1}{k} c_m + \frac{\sum_{p=2}^{m-1} (-1)^{p(m+k-1-p)} b_{m+k-p} \left[\binom{m+k-2}{p-1} a_p - \binom{m+k-2}{p-2} c_p \right]}{(-1)^{k(m-1)+m+k-1} \binom{m+k-2}{m-1} b_k}.$$

Note that $k < q \leq n \iff 1 < q-k+1 \leq n-k+1$. Thus V is L_n if and only if

$$a_m = \binom{m-1}{k} c_m + \frac{\sum_{p=2}^{m-1} (-1)^{p(m+k)} b_{m+k-p} \left[\binom{m+k-2}{p-1} a_p - \binom{m+k-2}{p-2} c_p \right]}{(-1)^{m(m+k)+1} \binom{m+k-2}{m-1} b_k}$$

$\forall\ 1 < m \leq n-k+1$. $\square$

Corollary 69. *Suppose that* $V = V_0 \oplus V_1 \oplus V_2$, *with the skew maps defined in* (1) *is* L_n. *Then* V *is* L_{n+1} *if and only if*

- $b_p = 0 \; \forall \; p < n+1$ *or*
- $\exists \; k$ *such that* $1 \le k < n+1$ & $b_p = 0 \; \forall \; p < k$ & $b_k \neq 0$ *and*

$$a_{n-k+2} = \binom{n-k+1}{k} c_{n-k+2} + \frac{\sum_{p=2}^{n-k+1} (-1)^{p(n+2)} b_{n+2-p} \left[\binom{n}{p-1} a_p - \binom{n}{p-2} c_p \right]}{(-1)^{n(n-k)+1} \binom{n}{k-1} b_k}.$$

Corollary 70. $V = V_0 \oplus V_1 \oplus V_2$ *with the skew maps defined in* (1) *is* L_∞ *if and only if*

- $b_p = 0 \; \forall \; p \in \mathbb{N}$ *or*
- $\exists \; k \in \mathbb{N}$ *such that* $b_p = 0 \; \forall \; p < k \quad \& \quad b_k \neq 0 \quad \& \quad \forall \; m \in \mathbb{N}$,

$$a_m = \binom{m-1}{k} c_m + \frac{\sum_{p=2}^{m-1} (-1)^{p(m+k)} b_{m+k-p} \left[\binom{m+k-2}{p-1} a_p - \binom{m+k-2}{p-2} c_p \right]}{(-1)^{m(m+k)+1} \binom{m+k-2}{m-1} b_k}.$$

The following result gives necessary and sufficient conditions under which an L_3 structure on $V = V_0 \oplus V_1 \oplus V_2$ is a differential graded Lie algebra.

Theorem 71. *Suppose that* $V = V_0 \oplus V_1 \oplus V_2$ *with the maps defined in* (1) *is an* L_3 *structure. Then* V *is a differential graded Lie algebra if and only if* $b_1 = 0$ *or* $b_2 = 0$ *or* $a_2 = 0$.

Proof. Recall that a differential graded (d.g.) Lie algebra is an L_2 structure in which $l_2 \circ l_2 \equiv 0$. In this space, $l_2 \circ l_2 \equiv 0 \Longleftrightarrow l_2 \circ l_2(v \otimes w^{\otimes 2}) = 0$, by the argument used in Lemma (62).

$$\begin{aligned} & l_2 \circ l_2(v \otimes w^{\otimes 2}) \\ &\quad = l_2(l_2(v \otimes w) \otimes w) - (-1)^1 l_2(l_2(v \otimes w) \otimes w) + (-1)^0 l_2(l_2(w \otimes w) \otimes v) \\ &\quad\quad = 2 l_2(a_2 w \otimes w) + l_2(b_2 x \otimes v) = 2a_2 b_2 x - b_2 c_2 x = (2a_2 - c_2) b_2 x. \end{aligned}$$

Thus V is a d.g. Lie algebra if and only if $(2a_2 - c_2)b_2 = 0$. Note that when $b_2 = 0$, this condition makes V a d.g. Lie algebra.

To prove the rest, we need to use the fact that V is L_3. Recall that in this space,

$$\mathcal{J}_n = 0 \Longleftrightarrow 0 = \sum_{p=1}^{n} (-1)^{p(n-p)}(-1)^n b_p \left[(-1)\binom{n-1}{n-p} a_{n-p+1} + \binom{n-1}{p} c_{n-p+1}\right].$$

Therefore, V is L_3 if and only if the following 3 constraints hold:

$$0 = a_1 b_1 \qquad \text{(since } \mathcal{J}_1 = 0\text{).}$$

$$0 = b_1 a_2 - b_1 c_2 - b_2 a_1 \qquad \text{(since } \mathcal{J}_2 = 0\text{).}$$

$$0 = b_1(a_3 - 2c_3) + b_2(2a_2 - c_2) + b_3 a_1 \qquad \text{(since } \mathcal{J}_3 = 0\text{).}$$

Suppose $b_2 \neq 0$ and $b_1 = 0$. By plugging into the second Jacobi constraint, we get $a_1 = 0$ (since $b_2 \neq 0$). Plugging $a_1 = 0$ and $b_1 = 0$ into the third constraint, we get $b_2(2a_2 - c_2) = 0$, which makes V is a d.g. Lie algebra.

So suppose $b_2 \neq 0$ and $b_1 \neq 0$ and $a_2 = 0$. From the first constraint, $a_1 = 0$. Plugging $a_1 = a_2 = 0$ into the second Jacobi constraint, we get $b_1 c_2 = 0$. Thus $c_2 = 0$ (since $b_1 \neq 0$). Since $a_2 = 0$ and $c_2 = 0$, $(2a_2 - c_2)b_2 = 0$, and V is a d.g. Lie algebra.

Finally, suppose $b_2 \neq 0$ and $b_1 \neq 0$ and $a_2 \neq 0$. From the first constraint, $a_1 = 0$. Plugging $a_1 = 0$ into the second Jacobi constraint, we get $0 = b_1(a_2 - c_2)$. Thus $c_2 = a_2$ (since $b_1 \neq 0$). Then $(2a_2 - c_2)b_2 = (2a_2 - a_2)b_2 = a_2 b_2 \neq 0$ (since we assumed that both a_2 and b_2 are nonzero). Thus V is not a d.g. Lie algebra in this case. □

Example 72. Let $V = V_0 \oplus V_1 \oplus V_2$ where

$$\begin{array}{lll} l_1(v) = 0 & l_2(v \otimes w) = w & l_3(v \otimes w \otimes w) = -w \\ l_1(w) = x & l_2(w \otimes w) = x & l_3(w \otimes w \otimes w) = 0 \\ & l_2(v \otimes x) = x & l_3(v \otimes w \otimes x) = 0 \end{array}$$

In the notation of (1), $a_1 = b_3 = c3 = 0$, $b_1 = a_2 = b_2 = c_2 = 1$, and $a_3 = -1$. This example is L_3 since a_1, a_2, and a_3 satisfy the formula in Theorem (68). However, it is **not** a differential graded Lie algebra since

$$l_2 \circ l_2(v \otimes w \otimes w) = l_2(l_2(v \otimes w) \otimes w) - (-1)^1 l_2(l_2(v \otimes w) \otimes w) + (-1)^0 l_2(w \otimes w) \otimes v)$$
$$= l_2(w \otimes w) + l_2(w \otimes w) + l_2(x \otimes v) = x + x - x = x.$$

Suppose that we extend this example by defining l_4:

$$l_4(v \otimes w^{\otimes 3}) = a_4 w \qquad l_4(w^{\otimes 4}) = b_4 x \qquad l_4(v \otimes w^{\otimes 2} \otimes x) = c_4 x$$

Then by Corollary (69), this structure is L_4 if and only if $a_4 = 3c_4 - 1$. □

Example 73. Let $V = V_0 \oplus V_1 \oplus V_2$ where

$$V_0 =< v > \qquad V_1 =< w > \qquad V_2 =< x >$$

Define skew linear maps $l_n : V^{\otimes n} \to V$ of degree $2 - n$ by

$$l_n(v \otimes w^{\otimes n-1}) = 0 \qquad l_n(w^{\otimes n}) = x \qquad l_n(v \otimes w^{\otimes n-2} \otimes x) = 0 \qquad \forall\, n \in \mathbb{N}$$

In the notation of (1), $a_n = c_n = 0$ and $b_n = 1\ \forall\, n \in \mathbb{N}$, and it is easy to see that the conditions of corollary (70) are satisfied. Therefore, this is an L_∞ structure on V. Furthermore, it is a differential graded Lie algebra since $a_2 = 0$. Note that in this simple example, we have a differential graded Lie algebra in which the higher maps (l_n where $n > 2$) are also nonzero. □

6. Three Components (Two Odd and One Even).

In this chapter, we will identify all possible non-trivial L_∞ structures on a graded vector space $V = V_\alpha \oplus V_\beta \oplus V_\gamma = <v_\alpha> \oplus <v_\beta> \oplus <v_\gamma>$, where α is even and β, γ are odd, and each component is one-dimensional. Since we have two components of odd degree, we can assume that $\gamma \neq 1$ without loss of generality.

At first, it might seem like the nonzero skew linear operators of degree $2-n$ on V would be difficult to define, since we now have two odd basis vectors which can be repeated. However, the next three lemmas show that for each value of n, there are at most three types of inputs to l_n which could map to nonzero components.

Lemma 74. *Suppose $l_n(V_\alpha \otimes V_\beta^{\otimes k} \otimes V_\gamma^{\otimes n-k-1})$ and $l_n(V_\alpha \otimes V_\beta^{\otimes m} \otimes V_\gamma^{\otimes n-m-1})$ map to the same component of V, where $0 \leq k, m < n$. Then $k = m$.*

Proof. If $l_n(V_\alpha \otimes V_\beta^{\otimes k} \otimes V_\gamma^{\otimes n-k-1})$ and $l_n(V_\alpha \otimes V_\beta^{\otimes m} \otimes V_\gamma^{\otimes n-m-1})$ map to the same component, then $\alpha + k\beta + (n-k-1)\gamma + 2 - n = \alpha + m\beta + (n-m-1)\gamma + 2 - n$, which implies that $k\beta - k\gamma = m\beta - m\gamma$. Thus $(k-m)(\beta - \gamma) = 0$. Since $\beta \neq \gamma$, this forces $k = m$. $\square$

Lemma 75. *Suppose $l_n(V_\beta^{\otimes k} \otimes V_\gamma^{\otimes n-k})$ and $l_n(V_\beta^{\otimes m} \otimes V_\gamma^{\otimes n-m})$ map to the same component of V, where $0 \leq k, m < n$. Then $k = m$.*

Proof. Similar to the proof of Lemma (74). $\square$

Lemma 76. *Suppose $0 < q < n$. Then $l_n(V_\alpha \otimes V_\beta^{\otimes q} \otimes V_\gamma^{\otimes n-q-1}) \subset V_\beta$ if and only if $l_n(V_\alpha \otimes V_\beta^{\otimes q-1} \otimes V_\gamma^{\otimes n-q}) \subset V_\gamma$.*

PROOF.

$$\begin{aligned}
l_n(V_\alpha \otimes V_\beta^{\otimes q} \otimes V_\gamma^{\otimes n-q-1}) \subset V_\beta &\iff \alpha + q\beta + (n-q-1)\gamma + 2 - n = \beta \\
&\iff \alpha + (q-1)\beta + \beta + (n-q)\gamma - \gamma + 2 - n = \beta \\
&\iff \alpha + (q-1)\beta + (n-q)\gamma + 2 - n = \gamma \\
&\iff l_n(V_\alpha \otimes V_\beta^{\otimes q-1} \otimes V_\gamma^{\otimes n-q}) \subset V_\gamma.
\end{aligned}$$

□

LEMMA 77. *Suppose $V = V_\alpha \oplus V_\beta \oplus V_\gamma$, where α is even and β, γ are odd. Then the most general nonzero skew linear operators of degree $2-n$ on V are*

- $l_n(v_\beta^{\otimes k} \otimes v_\gamma^{\otimes n-k}) = a_n v_\alpha$ *when* $n = k + \frac{\alpha + k(1-\beta) - 2}{\gamma - 1} \in \mathbb{N}$.
- $l_n(v_\alpha \otimes v_\beta^{\otimes k} \otimes v_\gamma^{\otimes n-k-1}) = b_n v_\beta$ *when* $n = k + 1 + \frac{(k-1)(1-\beta) - \alpha}{\gamma - 1} \in \mathbb{N}$.
- $l_n(v_\alpha \otimes v_\beta^{\otimes k} \otimes v_\gamma^{\otimes n-k-1}) = c_n v_\gamma$ *when* $n = k + 2 + \frac{k(1-\beta) - \alpha}{\gamma - 1} \in \mathbb{N}$.

PROOF. $l_n(V_\beta^{\otimes k} \otimes V_\gamma^{\otimes n-k}) \subset V_{k\beta+(n-k)\gamma+2-n} = V_{n(\gamma-1)+k(\beta-\gamma)+2}$, which is even. $n(\gamma-1) + k(\beta-\gamma) + 2 = \alpha \implies n = \frac{\alpha - k\beta + k\gamma - 2}{\gamma - 1} \implies n = k + \frac{\alpha + k(1-\beta) - 2}{\gamma - 1}$.

$l_n(V_\alpha \otimes V_\beta^{\otimes k} \otimes V_\gamma^{\otimes n-k-1}) \subset V_{\alpha + k\beta + (n-k-1)\gamma + 2 - n} = V_{n(\gamma-1)+k(\beta-\gamma)+\alpha-\gamma+2}$, which is odd. $l_n(V_\alpha \otimes V_\beta^{\otimes k} \otimes V_\gamma^{\otimes n-k-1}) \subset V_\beta$ if and only if $n(\gamma-1)+k(\beta-\gamma)+\alpha-\gamma+2 = \beta$, which is true if and only if $n = \frac{\beta - k\beta + k\gamma - \alpha + \gamma - 2}{\gamma - 1} = k + 1 + \frac{\beta - k\beta - \alpha + k - 1}{\gamma - 1} = k + 1 + \frac{(k-1)(1-\beta) - \alpha}{\gamma - 1}$. Similarly, $l_n(V_\alpha \otimes V_\beta^{\otimes k} \otimes V_\gamma^{\otimes n-k-1}) \subset V_\gamma$ if and only if $n(\gamma-1)+k(\beta-\gamma)+\alpha-\gamma+2 = \gamma$, which is true if and only if $n = \frac{2\gamma - k\beta + k\gamma - \alpha - 2}{\gamma - 1} = k + 2 + \frac{k(1-\beta) - \alpha}{\gamma - 1}$. □

REMARK 78. The operators defined in Lemma (77) are well-defined for every n which satisfies the criteria stated, since Lemmas (74) and (75) guarantee that a given operator l_n cannot map to the same component for multiple values of k. However, Lemma (77) does not necessarily define a_n, b_n, c_n for all $n \in \mathbb{N}$, since the necessary criteria might not be satisfied for certain values of n. We can remedy that by defining the unassigned constants to be zero in such cases. Specifically,

- If $n \neq k + \frac{\alpha + k(1-\beta) - 2}{\gamma - 1}$ $\forall\ 0 \leq k \leq n$, then define $a_n = 0$.

- If $n \neq k+1+\frac{(k-1)(1-\beta)-\alpha}{\gamma-1}\ \forall\ 0 \leq k < n$, then define $b_n = 0$.
- If $n \neq k+2+\frac{k(1-\beta)-\alpha}{\gamma-1}\ \forall\ 0 \leq k < n$, then define $c_n = 0$.

THEOREM 79. *Suppose $V = V_\alpha \oplus V_\beta \oplus V_\gamma$, where α is even and β, γ are odd, and each component is one-dimensional. Then the possible L_∞ structures on V consist of*

$$\left\{ \begin{array}{c} l_n(v_\beta^{\otimes k} \otimes v_\gamma^{\otimes n-k}) = a_n v_\alpha \quad \textit{when} \quad n = k + \frac{\alpha+k(1-\beta)-2}{\gamma-1} \in \mathbb{N}. \\ l_n(v_\alpha \otimes v_\beta^{\otimes k} \otimes v_\gamma^{\otimes n-k-1}) = b_n v_\beta \quad \textit{when} \quad n = k+1+\frac{(k-1)(1-\beta)-\alpha}{\gamma-1} \in \mathbb{N}. \\ l_n(v_\alpha \otimes v_\beta^{\otimes k} \otimes v_\gamma^{\otimes n-k-1}) = c_n v_\gamma \quad \textit{when} \quad n = k+2+\frac{k(1-\beta)-\alpha}{\gamma-1} \in \mathbb{N}. \end{array} \right\},$$

where either $a_n = 0\ \forall\ n \in \mathbb{N}$ or $b_n = c_n = 0\ \forall\ n \in \mathbb{N}$.

PROOF. If $a_n = 0\ \forall\ n$, then the only possible nonzero l_n require v_α as input. Since these operators do not return v_α, the composition $l_{n-p+1} \circ l_p$ would then require two v_α. But if the even vector v_α is repeated, it also forces the composition to zero because of skewness.

To prove the remainder of the theorem, suppose that $a_k \neq 0$ but $a_p = 0\ \forall\ p < k$. First, we observe that by Lemma (77), if $a_p \neq 0$, then $\exists\ i$ such that $0 \leq i \leq p$ and $p = i + \frac{\alpha+i(1-\beta)-2}{\gamma-1}$, and if $b_q \neq 0$, then $\exists\ j$ such that $0 \leq j < q$ and $q = j+1+\frac{(j-1)(1-\beta)-\alpha}{\gamma-1}$.

Now suppose that $b_1 \neq 0$. Since $a_k \neq 0$, there exists i such that $k = i + \frac{\alpha+i(1-\beta)-2}{\gamma-1}$. Since $a_p = 0\ \forall\ p < k$,

$$\mathcal{J}_k(v_\beta^{\otimes i} \otimes v_\gamma^{\otimes k-i}) = \binom{k}{i}\binom{k-i}{k-i} l_1(l_k(v_\beta^{\otimes i} \otimes v_\gamma^{\otimes k-i})) = \binom{k}{i} l_1(a_k(v_\alpha) = \binom{k}{i} b_1 a_k v_\beta.$$

Since $a_k \neq 0$, we have $b_1 = 0$. A similar calculation shows that $c_1 = 0$. Now we can do a little induction proof.

Suppose that $b_q = c_q = 0\ \forall\ q < m$, but $b_m \neq 0$. Then $\exists\ j$ such that $0 \leq j < m$ and $m = j+1+\frac{(j-1)(1-\beta)-\alpha}{\gamma-1}$. Thus $\mathcal{J}_{m+k-1}(v_\beta^{\otimes i+j} \otimes v_\gamma^{\otimes m+k-1-i-j})$ is equal to

$$\sum_{p=1}^{m+k-1} (-1)^{p(m+k-1-p)} \sum_{r=1}^{p} \binom{i+j}{r}\binom{m+k-1-i-j}{p-r} l_{m+k-p}(l_p(v_\beta^{\otimes r} \otimes v_\gamma^{\otimes p-r}) \otimes v_\beta^{\otimes j} \otimes v_\gamma^{\otimes m+k-1-j-p}).$$

Note that if $l_p(v_\beta^{\otimes r} \otimes v_\gamma^{\otimes p-r}) \neq 0$, then $l_p(v_\beta^{\otimes r} \otimes v_\gamma^{\otimes p-r}) = a_p v_\alpha$. Since $a_p = 0 \ \forall \ p < k$, we only need sum for $p \geq k$. Furthermore, the induction hypothesis states that $l_{m+k-p}(l_p(v_\beta^{\otimes r} \otimes v_\gamma^{\otimes p-r}) \otimes v_\beta^{\otimes j} \otimes v_\gamma^{\otimes m+k-1-j-p}) = 0$ when $m+k-p < m$, or equivalently, when $k < p$. Since this just leaves $p = k$, $\mathcal{J}_{m+k-1}(v_\beta^{\otimes i+j} \otimes v_\gamma^{\otimes m+k-1-i-j})$ is equal to

$$(-1)^{k(m-1)} \sum_{r=1}^{k} \tbinom{i+j}{r} \tbinom{m+k-1-i-j}{k-r} l_m(l_k(v_\beta^{\otimes r} \otimes v_\gamma^{\otimes k-r}) \otimes v_\beta^{\otimes j} \otimes v_\gamma^{\otimes m-1-j}),$$

which is equal to

$$(-1)^{k(m-1)} \sum_{r=1}^{k} \tbinom{i+j}{r} \tbinom{m+k-1-i-j}{k-r} l_m(a_k v_\alpha \otimes v_\beta^{\otimes j} \otimes v_\gamma^{\otimes m-1-j}),$$

which is equal to $(-1)^{k(m-1)} \sum_{r=1}^{k} \tbinom{i+j}{r} \tbinom{m+k-1-i-j}{k-r} b_m a_k v_\beta$ (since i was originally chosen so that l_m would map the input to V_β). Since $a_k = 0$, this forces $b_m = 0$. A similar calculation forces $c_m = 0$ as well. $\square$

Here is an example of a three-graded space which can have an infinite number of nonzero l_n, which are defined for only certain values of n.

Example 80. Let $\alpha = -2$, $\beta = 3$, and $\gamma = -3$. Given $m \in \mathbb{N}$, we can define skew linear operators of degree $2 - n$ by

- $l_{3m+1}(v_3^{\otimes 2m} \otimes v_{-3}^{\otimes m+1}) = a_{3m+1} v_{-2}$.
- $l_{3m}(v_{-2} \otimes v_3^{\otimes 2m} \otimes v_{-3}^{\otimes m-1}) = b_{3m} v_3$.
- $l_{3m}(v_{-2} \otimes v_3^{\otimes 2m-1} \otimes v_{-3}^{\otimes m}) = c_{3m} v_{-3}$.

In order for these l_n to form an L_∞ structure, though, we need either $a_n = 0 \ \forall \ n \in \mathbb{N}$ or $b_n = c_n = 0 \ \forall \ n \in \mathbb{N}$.

6.1. $\mathbf{V = V_{-1} \oplus V_0 \oplus V_1}$ where each is 1-dimensional

In this section, we will show that there are exactly two types of L_∞ structures on $V = V_{-1} \oplus V_0 \oplus V_1$. If we denote $V_{-1} =< u >$, $V_0 =< v >$, and $V_1 =< w >$, then the possible L_∞ structures are as follows:

- A structure with maps $\left\{ \begin{array}{r} l_n(u \otimes v \otimes w^{\otimes n-2}) = a_n u \quad \forall\, n \geq 2 \\ \\ l_n(v \otimes w^{\otimes n-1}) = c_n w \quad \forall\, n \geq 1 \end{array} \right\}$, in which a_n and c_n are arbitrarily chosen constants, but all other maps are zero.
- A structure with maps $\left\{ \ l_n(u \otimes w^{\otimes n-1}) = b_n v \quad \forall\, n \geq 1 \ \right\}$, in which b_n are arbitrarily chosen constants, but all other maps are zero.

It can also be shown that every L_3 structure on $V = V_{-1} \oplus V_0 \oplus V_1$ is a differential graded Lie algebra (even when l_3 is nonzero). To accomplish these goals, we first need to establish some notation.

Let $V = V_{-1} \oplus V_0 \oplus V_1$, where $V_{-1} =< u >$, $V_0 =< v >$, and $V_1 =< w >$. Define skew linear maps $l_n : V^{\otimes n} \to V$ of degree $2 - n$ by

$$
(2) \qquad \left\{ \begin{array}{rl} l_n(u \otimes v \otimes w^{\otimes n-2}) = a_n u & \quad \forall\, n \geq 2 \\ \\ l_n(u \otimes w^{\otimes n-1}) = b_n v & \quad \forall\, n \geq 1 \\ \\ l_n(v \otimes w^{\otimes n-1}) = c_n w & \quad \forall\, n \geq 1 \\ \\ l_n(w^{\otimes n}) = 0 & \quad \forall\, n \\ \\ l_n(u^{\otimes i} \otimes v^{\otimes j} \otimes w^{\otimes n-i-j}) = 0 & \quad \forall\, i, j > 1 \end{array} \right.
$$

where a_n, b_n, c_n are constants. Since a_n is defined above only for $n \geq 2$, we can define $a_1 = 0$ for the sake of convenience.

Remark 81. The above list includes all possible skew linear maps of degree $2-n$, by an argument similar to the one in the preceding section.

In order to determine when these maps define an L_m structure on V, we need to know when $\mathcal{J}_n = 0 \ \forall\, n \leq m$. The following three lemmas establish when $\mathcal{J}_n = 0$.

Lemma 82. *Given* $V = V_{-1} \oplus V_0 \oplus V_1$ *with the skew maps defined in* (2), $\mathcal{J}_n = 0$ *if and only if*

$$
\mathcal{J}_n(u^{\otimes 2} \otimes w^{\otimes n-2}) = 0 \ \ \& \ \ \mathcal{J}_n(u \otimes v \otimes w^{\otimes n-2}) = 0 \ \ \& \ \ \mathcal{J}_n(u \otimes w^{\otimes n-1}) = 0.
$$

Proof. Similar to Lemma (62). ☐

Lemma 83. *Given $V = V_{-1} \oplus V_0 \oplus V_1$ with the skew maps defined in* (2),

$$\mathcal{J}_n(u \otimes w^{\otimes n-1}) = \sum_{p=1}^{n} (-1)^{p(n-p)} \binom{n-1}{p-1} b_p c_{n-p+1} w.$$

Proof. Similar to Lemma (63). ☐

Lemma 84. *Given $V = V_{-1} \oplus V_0 \oplus V_1$ with the skew maps defined in* (2),

$$\mathcal{J}_n(u^{\otimes 2} \otimes w^{\otimes n-2}) = \sum_{p=1}^{n-1} (-1)^{p(n-p)} (-2) \binom{n-2}{p-1} b_p a_{n-p+1} u.$$

Proof. Before calculating $l_{n-p+1} \circ l_p(u^{\otimes 2} \otimes w^{\otimes n-2})$, it is helpful to consider the ways in which a $(p, n-p)$ unshuffle of $u^{\otimes 2} \otimes w^{\otimes n-2}$ could possibly rearrange the two u terms.

- If both u are on the left, $l_{n-p+1}(l_p(u^{\otimes 2} \otimes w^{\otimes p-2}) \otimes w^{\otimes n-p}) = l_{n-p+1}(0) = 0$.
- If both u are on the right, $l_{n-p+1}(l_p(w^{\otimes p}) \otimes u^{\otimes 2} \otimes w^{\otimes n-p-2}) = l_{n-p+1}(0) = 0$.

Thus we only need to consider unshuffles σ which put one u on each side. Because of ordering, these are the unshuffles such that either ($\sigma(1) = 1$ & $\sigma(p+1) = 2$) or ($\sigma(1) = 2$ & $\sigma(p+1) = 1$).

There are $\binom{n-2}{p-1}$ unshuffles such that ($\sigma(1) = 1$ & $\sigma(p+1) = 2$). In each case, $\chi(\sigma) = +1$, since each of these permutations can be accomplished solely by commuting elements of odd degree. Similarly, there are $\binom{n-2}{p-1}$ unshuffles such that ($\sigma(1) = 2$ & $\sigma(p+1) = 1$), each with $\chi(\sigma) = +1$. Thus

$$\begin{aligned}
l_{n-p+1} \circ l_p(u^{\otimes 2} \otimes w^{\otimes n-2}) &= 2\tbinom{n-2}{p-1}(+1) l_{n-p+1}(l_p(u^{\otimes} w^{\otimes p-1}) \otimes u \otimes w^{\otimes n-p+1}) \\
&= 2\tbinom{n-2}{p-1} l_{n-p+1}(b_p v \otimes u \otimes w^{\otimes n-p+1}) \\
&= 2\tbinom{n-2}{p-1} b_p(-1) l_{n-p+1}(u \otimes v \otimes w^{\otimes n-p+1}) \\
&= (-2)\tbinom{n-2}{p-1} b_p a_{n-p+1} u.
\end{aligned}$$

Thus

$$\mathcal{J}_n(u^{\otimes 2} \otimes w^{\otimes n-2}) = \sum_{p=1}^{n} (-1)^{p(n-p)}(-2)\tbinom{n-2}{p-1} b_p a_{n-p+1} u.$$

Since $a_1 = 0$, we get

$$\mathcal{J}_n(u^{\otimes 2} \otimes w^{\otimes n-2}) = \sum_{p=1}^{n-1} (-1)^{p(n-p)}(-2)\tbinom{n-2}{p-1} b_p a_{n-p+1} u.$$

□

LEMMA 85. *Given $V = V_{-1} \oplus V_0 \oplus V_1$ with the skew maps defined in* (2)*,*

$$\mathcal{J}_n(u \otimes v \otimes w^{\otimes n-2}) = \sum_{p=1}^{n} (-1)^{p(n-p)} b_{n-p+1} \left[\binom{n-2}{p-2} a_p - \binom{n-2}{p-1} c_p \right] v.$$

PROOF. Denote $x_1 \otimes \cdots \otimes x_n = u \otimes v \otimes w^{\otimes n-2}$. Consider the different ways in which a $(p, n-p)$ unshuffle σ could possibly arrange the u and v terms:

- If both u and v are on the left, $l_{n-p+1}(l_p(u \otimes v \otimes w^{\otimes p-2}) \otimes w^{\otimes n-p})$ is equal to $l_{n-p+1}(a_p u \otimes w^{\otimes n-p}) = a_p b_{n-p+1} v$. There are $\binom{n-2}{p-2}$ such unshuffles, with $\chi(\sigma) = +1$ in each case.
- If u is on the left and v is on the right, $l_{n-p+1}(l_p(u \otimes w^{\otimes p-1}) \otimes v \otimes w^{\otimes n-p-1}) = l_{n-p+1}(b_p v^{\otimes 2} \otimes w^{\otimes n-p-1}) = 0$
- If v is on the left and u is on the right, $l_{n-p+1}(l_p(v \otimes w^{\otimes p-1}) \otimes u \otimes w^{\otimes n-p+1}) = l_{n-p+1}(c_p w \otimes u \otimes w^{\otimes n-p+1}) = c_p b_{n-p+1} v$. There are $\binom{n-2}{p-1}$ of these, with $\chi(\sigma) = -1$.
- If both are on the right, $l_p(w^p) = 0$.

Thus

$$\mathcal{J}_n(u \otimes v \otimes w^{\otimes n-2}) = \sum_{p=1}^{n} (-1)^{p(n-p)} \left[\tbinom{n-2}{p-2} a_p b_{n-p+1} v - \tbinom{n-2}{p-1} c_p b_{n-p+1} v \right].$$

□

Thus $\mathcal{J}_n = 0$ if and only if the following 3 constraints hold:

$$0 = \sum_{p=1}^{n} (-1)^{p(n-p)} \binom{n-1}{p-1} b_p c_{n-p+1} \qquad \text{(from } u \otimes w^{\otimes n-1}\text{)}.$$

$$0 = \sum_{p=1}^{n-1} (-1)^{p(n-p)} \binom{n-2}{p-1} b_p a_{n-p+1} \qquad \text{(from } u^{\otimes 2} \otimes w^{\otimes n-2}\text{)}.$$

$$0 = \sum_{p=1}^{n} (-1)^{p(n-p)} b_{n-p+1} \left[\binom{n-2}{p-2} a_p - \binom{n-2}{p-1} c_p \right] \qquad \text{(from } u \otimes v \otimes w^{\otimes n-2}\text{)}.$$

The following lemma shows that if any constant b_k is nonzero, then this forces many other constants a_q, c_q to be zero in any L_m structure on $V = V_{-1} \oplus V_0 \oplus V_1$.

Lemma 86. *Suppose $1 \leq k \leq m$ and $b_k \neq 0$ and $b_p = 0\ \forall\ p < k$. Then $V = V_{-1} \oplus V_0 \oplus V_1$ with the skew linear maps defined in* (2) *is L_m if and only if $a_q = c_q = 0\ \forall\ q \leq m - k + 1$.*

Proof. Since $b_p = 0\ \forall\ p < k$, $\mathcal{J}_n = 0$ if and only if

$$0 = \sum_{p=k}^{n} (-1)^{p(n-p)} \tbinom{n-1}{p-1} b_p c_{n-p+1} \ \&\ 0 = \sum_{p=k}^{n-1} (-1)^{p(n-p)} \tbinom{n-2}{p-1} b_p a_{n-p+1}$$

$$\&\ 0 = \sum_{p=1}^{n-k+1} (-1)^{p(n-p)} b_{n-p+1} \left[\tbinom{n-2}{p-2} a_p - \tbinom{n-2}{p-1} c_p \right].$$

Substituting $q = n - p + 1$, this is equivalent to

$$0 = \sum_{q=1}^{n-k+1} (-1)^{(n+1)(q-1)} \tbinom{n-1}{n-q} b_{n-q+1} c_q \quad \& \quad 0 = \sum_{q=2}^{n-k+1} (-1)^{(n+1)(q-1)} \tbinom{n-2}{n-q} b_{n-q+1} a_q$$

$$\&\ 0 = \sum_{p=1}^{n-k+1} (-1)^{p(n-p)} b_{n-p+1} \left[\tbinom{n-2}{p-2} a_p - \tbinom{n-2}{p-1} c_p \right].$$

Suppose $a_q = c_q = 0\ \forall\ q \leq m - k + 1$ and $n \leq m$. Then $\mathcal{J}_n = 0\ \forall\ n \leq m$ (since all of the above sums are clearly zero in that case). Thus V is L_m.

Now suppose that V is L_m. Since we assumed $1 \leq k \leq m$, V is $\mathcal{J}_k$. When we plug $n = k$ into

$$0 = \sum_{q=1}^{n-k+1} (-1)^{(n+1)(q-1)} \tbinom{n-1}{n-q} b_{n-q+1} c_q,$$

the expression simplifies to $0 = b_k c_1$. Since $b_k \neq 0$ by assumption, $c_1 = 0$. Also, we know $a_1 = 0$ because of the way that the skew maps were defined.

Now we can do induction: Suppose $2 \leq i \leq m-k+1$ & $a_q = c_q = 0\ \forall\ q < i$. We will now show that $a_i = c_i = 0$, which will complete the proof! Since $i \leq m-k+1$, $i+k-1 \leq m$. Thus $\mathcal{J}_{i+k-1} = 0$. Thus we can plug $n = i+k-1$ into

$$0 = \sum_{q=1}^{n-k+1} (-1)^{(n+1)(q-1)} \binom{n-1}{n-q} b_{n-q+1} c_q \quad \& \quad 0 = \sum_{q=2}^{n-k+1} (-1)^{(n+1)(q-1)} \binom{n-2}{n-q} b_{n-q+1} a_q.$$

Since we assumed that $a_q = c_q = 0\ \forall\ q < i$, this simplifies to

$$0 = (-1)^{(k)(i-1)} \binom{i+k-2}{k-1} b_k c_i \quad \& \quad 0 = (-1)^{(k)(i-1)} \binom{i+k-3}{k-1} b_k a_i.$$

But the binomial expressions are nonzero since $i \geq 2$, and $b_k \neq 0$ by assumption. Thus $c_i = 0$ and $a_i = 0$. $\square$

Now we get to the main results, which follow easily from the lemmas. Theorems (87) and (88) give necessary and sufficient conditions under which skew linear maps form an L_m structure on $V = V_{-1} \oplus V_0 \oplus V_1$, and Corollary (89) provides necessary and sufficient conditions for an L_∞ structure.

THEOREM 87. *Suppose* $V = V_{-1} \oplus V_0 \oplus V_1$ *has skew linear maps defined by* $\left\{ \begin{array}{rl} l_n(u \otimes v \otimes w^{\otimes n-2}) = a_n u & \forall\ n \geq 2 \\ l_n(u \otimes w^{\otimes n-1}) = b_n v & \forall\ n \geq 1 \\ l_n(v \otimes w^{\otimes n-1}) = c_n w & \forall\ n \geq 1 \end{array} \right\}$. V *is* L_m *if and only if*

- $b_p = 0\ \forall\ p \leq m$ *or*
- $\exists\ k \in \mathbb{N}$ *such that* $1 \leq k \leq m$ *and* $b_p = 0\ \forall\ p < k$ *and* $b_k \neq 0$ *and* $a_q = c_q = 0\ \forall\ q \leq m-k+1$.

PROOF. $\mathcal{J}_n = 0$ if and only if

$$0 = \sum_{p=1}^{n} (-1)^{p(n-p)} \binom{n-1}{p-1} b_p c_{n-p+1} \;\; \& \;\; 0 = \sum_{p=1}^{n-1} (-1)^{p(n-p)} \binom{n-2}{p-1} b_p a_{n-p+1}$$

$$\& \;\; 0 = \sum_{p=1}^{n} (-1)^{p(n-p)} b_{n-p+1} \left[\binom{n-2}{p-2} a_p - \binom{n-2}{p-1} c_p \right].$$

If $b_p = 0 \; \forall \; p \leq m$, then $\mathcal{J}_n = 0 \; \forall \; p \leq m$ (since all of the above sums are clearly zero). Thus V is L_m.

Now suppose that V is L_m & $\exists \; b_{\overline{k}} \neq 0$ such that $\overline{k} \leq m$. Then $\exists \; k$ such that $1 \leq k \leq m$ & $b_p = 0 \; \forall \; p < k$ & $b_k \neq 0$. Thus by the preceding lemma, V is L_m if and only if $a_q = c_q = 0 \; \forall \; q \leq m - k + 1$. □

THEOREM 88. *Suppose $V = V_{-1} \oplus V_0 \oplus V_1$ is an L_∞ structure with the skew linear maps defined in (2). If $\exists \, m \in \mathbb{N}$ such that $b_m \neq 0$, then $a_n = c_n = 0 \; \forall \; n$.*

PROOF. A simple induction argument. □

COROLLARY 89. *Given the skew linear maps defined in (2), every L_∞ structure on $V = V_{-1} \oplus V_0 \oplus V_1$ has one of the following forms:*

- *$b_n = 0 \; \forall \; n$ but the constants a_n, c_n are arbitrary*
- *$b_m \neq 0$ for some m but all constants a_n, c_n are zero*

The following result shows that every L_3 structure on $V = V_{-1} \oplus V_0 \oplus V_1$ is a differential graded Lie algebra.

THEOREM 90. *Suppose $V = V_{-1} \oplus V_0 \oplus V_1$ is an L_3 structure with the skew linear maps defined in (2). Then this structure is a differential graded Lie algebra.*

PROOF. Recall that a differential graded (d.g.) Lie algebra is an L_2 structure in which $l_2 \circ l_2 \equiv 0$. In the space $V_{-1} \oplus V_0 \oplus V_1$, we have $l_2 \circ l_2 = 0$ if and only if

$l_2 \circ l_2(u^{\otimes 2} \otimes w) = l_2 \circ l_2(u \otimes v \otimes w) = l_2 \circ l_2(u \otimes w^{\otimes 2}) = 0.$

$$l_2 \circ l_2(u^{\otimes 2} \otimes w)$$

$$= l_2(l_2(u \otimes u) \otimes w) - (-1)^1 l_2(l_2(u \otimes w) \otimes u) \; + (-1)^0 l_2(l_2(u \otimes w) \otimes u)$$

$$= l_2(0) + l_2(b_2 v \otimes u) + l_2(b_2 v \otimes u) = 0 - 2l_2(b_2 u \otimes v) = -2b_2 a_2 u.$$

$$l_2 \circ l_2(u \otimes v \otimes w)$$

$$= l_2(l_2(u \otimes v) \otimes w) - (-1)^0 l_2(l_2(u \otimes w) \otimes v) + (-1)^1 l_2(l_2(v \otimes w) \otimes u)$$

$$= l_2(a_2 u \otimes w) - l_2(b_2 v \otimes v) - l_2(c_2 w \otimes u) = a_2 b_2 v - 0 + c_2 b_2 v.$$

$$l_2 \circ l_2(u \otimes w^{\otimes 2})$$

$$= l_2(l_2(u \otimes w) \otimes w) - (-1)^1 l_2(l_2(u \otimes w) \otimes w) + (-1)^0 l_2(l_2(w \otimes w) \otimes u)$$

$$= 2l_2(b_2 v \otimes w) + 0 = 2b_2 c_2 w.$$

Thus V is a d.g. Lie algebra if and only if $b_2 a_2 = 0$ and $b_2 c_2 = 0$. If $b_2 = 0$, then clearly V is d.g. Lie.

If $b_2 \neq 0$, then $\exists\, k$ such that $k \leq 2$ and $b_k \neq 0$ and $b_p = 0 \;\forall\, p < k$. Because of this and the fact that V is L_3, Theorem (87) requires that $a_q = c_q = 0 \;\forall\, q \leq 3 - k + 1$. Since $k \leq 2$, this forces $a_2 = c_2 = 0$, which again makes V d.g. Lie. $\square$

7. Appendix.

Here is an additional catalog of some other interesting examples of L_n and L_∞ structures on $V = V_0 \oplus V_1$. This is an informal collection, and all of the calculations here are done by brute force. Nevertheless, it is useful to know a variety of small examples when exploring more elaborate theories. Most of the toy structures in this appendix were constructed as a response to specific questions from other mathematicians about whether such a structure was possible. It is in this spirit that I hope that this little bestiary will be enjoyed.

Example 91. Here, we give a simple L_∞ Structure in which V_0 is a nonabelian Lie algebra and V_1 is a Lie module. Let $V_0 =< v_1, v_2 >$, and $V_1 =< w_i >$ be a vector space of arbitrary dimension. Define skew linear operators on $V = V_0 \oplus V_1$ by

$$\left\{ \begin{array}{r} l_n(v_i \otimes w_{j_1} \otimes \cdots \otimes w_{j_{n-1}}) = \sum_k A^k w_k \\ l_n(v_{i_1} \otimes v_{i_2} \otimes w_{j_1} \otimes \cdots \otimes w_{j_{n-2}}) = v_{i_1} - v_{i_2} \end{array} \right\}.$$

In this example, $l_{n-p+1} \circ l_p \equiv 0$, which follows from the following calculations:

$$\begin{aligned} & l_{n-p+1} \circ l_p(v_{i_1} \otimes v_{i_2} \otimes w_{j_1} \otimes \cdots \otimes w_{j_{n-2}}) \\ & \quad = \tbinom{n-2}{p-2} l_{n-p+1}(l_p(v_{i_1} \otimes v_{i_2} \otimes w\text{-}terms) \otimes w\text{-}terms) \\ & \qquad + (-1)^{p-1} \tbinom{n-2}{p-1} l_{n-p+1}(l_p(v_{i_1} \otimes w\text{-}terms) \otimes v_{i_2} \otimes w\text{-}terms) \\ & \qquad + (-1)^{p} \tbinom{n-2}{p-1} l_{n-p+1}(l_p(v_{i_2} \otimes w\text{-}terms) \otimes v_{i_1} \otimes w\text{-}terms) \\ & \qquad + \tbinom{n-2}{p-1} l_{n-p+1}(l_p(w\text{-}terms) \otimes v_{i_1} \otimes v_{i_2} \otimes w\text{-}terms), \end{aligned}$$

which is equal to

$$
\begin{aligned}
&\tbinom{n-2}{p-2} l_{n-p+1}((v_{i_1} - v_{i_2}) \otimes w\textit{-terms}) \\
&\qquad + (-1)^{p-1} \tbinom{n-2}{p-1} l_{n-p+1}(\sum_k A^k w_k \otimes v_{i_2} \otimes w\textit{-terms}) \\
&\qquad\qquad + (-1)^{p} \tbinom{n-2}{p-1} l_{n-p+1}(\sum_k A^k w_k \otimes v_{i_1} \otimes w\textit{-terms}) \\
&= \tbinom{n-2}{p-2} l_{n-p+1}(v_{i_1} \otimes w\textit{-terms}) - \tbinom{n-2}{p-2} l_{n-p+1}(v_{i_2} \otimes w\textit{-terms}) \\
&\qquad + (-1)^{p} \tbinom{n-2}{p-1} \sum_k A^k l_{n-p+1}(v_{i_2} \otimes w\textit{-terms}) \\
&\qquad\qquad - (-1)^{p} \tbinom{n-2}{p-1} \sum_k A^k l_{n-p+1}(v_{i_1} \otimes w\textit{-terms}) \\
&= \ \tbinom{n-2}{p-2} \sum_k A^k w_k - \tbinom{n-2}{p-2} \sum_k A^k w_k \\
&\qquad + (-1)^{p} \tbinom{n-2}{p-1} \sum_k A^k \sum_m A^m w_m - (-1)^{p} \tbinom{n-2}{p-1} \sum_k A^k \sum_m A^m w_m \\
&= 0.
\end{aligned}
$$

Similarly, $l_{n-p+1} \circ l_p(v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes w_{j_1} \otimes \cdots \otimes w_{j_{n-3}}) = 0$. Thus $l_{n-p+1} \circ l_p$ is zero for all $n \in \mathbb{N}$. Therefore, we have an L_∞ structure on V. In particular, since $l_2 \circ l_2 = 0$, V_0 is a Lie algebra and V_1 is a Lie module.

Remark 92. This is one of the first L_∞ structures that I found on $V_0 \oplus V_1$. It occurred to me later that the choice of A^k has no real effect on the problem... The key point that really makes this example work is that we picked one particular element $w = \sum_k A^k w_k$ in V_1, and then consistently assigned $l_n(v \otimes w\textit{-terms}) = w$. So the choice of A^k really just amounts to a choice of basis for V_1.

Remark 93. If instead we had defined $l_n(v_{i_1} \otimes v_{i_2} \otimes w_{j_1} \otimes \cdots \otimes w_{j_{n-2}}) = 0$, then this would have been an example of an L_∞ structure in which V_0 is an *abelian* Lie algebra.

Example 94. Now, we will construct an L_∞ Structure in which V_0 is an abelian Lie algebra, but V_1 is not a Lie module. In this example, we show how the operators l_n can be recursively defined in order to get the desired result. It is amusing to note that if we are given any abelian Lie algebra V_0 and any V_1 with $dim(V_1) \geq dim(V_0)$, then the following construction creates a nontrivial L_∞ structure on $V_0 \oplus V_1$. Let $V_0 =< v_i >$ be a vector space of arbitrary dimension, and let $V_1 =< w_i >$ be a vector space with $dim(V_1) \geq dim(V_0)$. Define skew linear operators on $V = V_0 \oplus V_1$ by

$$\left\{ \begin{array}{llll} l_1(v_i) = w_i, & l_2(v_i \otimes v_j) = 0, & l_n(v_i \otimes v_j \otimes w\text{-}terms) = 0, & \forall\ n \geq 3, \\ & l_2(v_i \otimes w_j) = w_i + w_j, & l_n(v_i \otimes w\text{-}terms) = C_n w_i, & \forall\ n \geq 3. \end{array} \right\}$$

The C_n will be will be determined recursively by the calculations below, but first we'll verify that the structure is L_2:

$$\begin{aligned} l_1 \circ l_2(v_i \otimes v_j) - l_2 \circ l_1(v_i \otimes v_j) &= l_1(0) - (l_2(l_1(v_i) \otimes v_j) - l_2(l_1(v_j) \otimes v_i)) \\ &= 0 - l_2(w_i \otimes v_j) + l_2(w_j \otimes v_i) \\ &= w_j + w_i - (w_i + w_j) = 0. \end{aligned}$$

Now we'll see what C_1 needs to be in order to make the structure L_3:

$$\begin{aligned} l_1 \circ l_3(v_i \otimes v_j \otimes w_k) &= 0. \\ l_2 \circ l_2(v_i \otimes v_j \otimes w_k) &= l_2(l_2(v_i \otimes v_j) \otimes w_k) - l_2(l_2(v_i \otimes w_k) \otimes v_j) + l_2(l_2(v_j \otimes w_k) \otimes v_i) \\ &= 0 - l_2((w_i + w_k) \otimes v_j) + l_2(w_j + w_k) \otimes v_i) \\ &= (w_j + w_i) + (w_j + w_k) - (w_i + w_j) - (w_i + w_k) \\ &= -w_i + w_j. \\ l_3 \circ l_1(v_i \otimes v_j \otimes w_k) &= l_3(w_i \otimes v_j \otimes w_k) - l_3(w_j \otimes v_i \otimes w_k) \\ &= -C_3 w_j + C_3 w_i. \end{aligned}$$

Thus $\mathcal{J}_3(v_i \otimes v_j \otimes w_k) = 0 \Longleftrightarrow -w_i + w_j - C_3 w_j + C_3 w_i = 0 \Longleftrightarrow C_3 = 1$. It is also easy to see that $\mathcal{J}_3(v_i \otimes v_j \otimes v_k) = 0$. Thus we have an L_3 structure when $C_3 = 1$. Now consider the case when $n \geq 4$.

$$\begin{aligned}
&l_n \circ l_1(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) \\
&\qquad = l_n(w_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) - l_n(w_j \otimes v_i \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) \\
&\qquad = -C_n w_j + C_n w_i = C_n(w_i - w_j).
\end{aligned}$$

$$\begin{aligned}
l_{n-1} \circ l_2(v_i \otimes v_j \otimes w_{k_1} \otimes \ldots \otimes w_{k_{n-2}}) &= \sum_a (-1) l_{n-1}(l_2(v_i \otimes w_{k_a}) \otimes v_j \otimes w\text{-}terms) \\
&\quad + \sum_a (+1) l_{n-1}(l_2(v_j \otimes w_{k_a}) \otimes v_i \otimes w\text{-}terms) \\
&= \sum_a (-1) l_{n-1}(w_i + w_{k_a}) \otimes v_j \otimes w\text{-}terms) \\
&\quad + \sum_a (+1) l_{n-1}(w_j + w_{k_a}) \otimes v_i \otimes w\text{-}terms) \\
&= 2(n-2)C_{n-1} w_j - 2(n-2) C_{n-1} w_i \\
&= -2(n-2) C_{n-1}(w_i - w_j).
\end{aligned}$$

$$\begin{aligned}
l_2 \circ l_{n-1}(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) &= (-1)^{n-2} l_2(l_{n-1}(v_i \otimes w\text{-}terms)) \otimes v_j) \\
&\quad - (-1)^{n-2} l_2(l_{n-1}(v_j \otimes w\text{-}terms)) \otimes v_i) \\
&= (-1)^{n-2} l_2(C_{n-1} w_i \otimes v_j) \\
&\quad - (-1)^{n-2} l_2(C_{n-1} w_j \otimes v_i) \\
&= (-1)^{n-1} C_{n-1}(w_j + w_i) \\
&\quad - (-1)^{n-1} C_{n-1}(w_i + w_j) = 0.
\end{aligned}$$

$$l_1 \circ l_n(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) = 0.$$

In the case where $3 \leq p \leq n-2$,

$$l_{n-p+1} \circ l_p(v_i \otimes v_j \otimes w_{k_1} \otimes \ldots \otimes w_{k_{n-2}})$$

$$= (-1)^{p-1}\tbinom{n-2}{p-1} l_{n-p+1}(l_p(v_i \otimes w\text{-}terms) \otimes v_j \otimes w\text{-}terms)$$

$$- (-1)^{p-1}\tbinom{n-2}{p-1} l_{n-p+1}(l_p(v_j \otimes w\text{-}terms) \otimes v_i \otimes w\text{-}terms)$$

$$= (-1)^{p-1}\tbinom{n-2}{p-1} l_{n-p+1}(C_p w_i \otimes v_j \otimes w\text{-}terms)$$

$$- (-1)^{p-1}\tbinom{n-2}{p-1} l_{n-p+1}(C_p w_j \otimes v_i \otimes w\text{-}terms)$$

$$= (-1)^{p}\tbinom{n-2}{p-1} C_{n-p+1} C_p w_j - (-1)^{p}\tbinom{n-2}{p-1} C_{n-p+1} C_p w_i$$

$$= (-1)^{p+1}\tbinom{n-2}{p-1} C_{n-p+1} C_p (w_i - w_j).$$

Thus $\mathcal{J}_n(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}})$ is equal to

$$= \sum_{p=1}^{n} (-1)^{p(n-p)} l_{n-p+1} \circ l_p(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}})$$

$$(-1)^{1(n-1)} l_n \circ l_1(v_i \otimes v_j \otimes w_{k_1} \otimes \ldots \otimes w_{k_{n-2}})$$

$$+ (-1)^{2(n-2)} l_{n-1} \circ l_2(v_i \otimes v_j \otimes w_{k_1} \otimes \ldots \otimes w_{k_{n-2}})$$

$$+ \sum_{p=3}^{n-2} (-1)^{p(n-p)} l_{n-p+1} \circ l_p(v_i \otimes v_j \otimes w_{k_1} \otimes \ldots \otimes w_{k_{n-2}})$$

$$+ (-1)^{(n-1)(1)} l_2 \circ l_{n-1}(v_i \otimes v_j \otimes w_{k_1} \otimes \ldots \otimes w_{k_{n-2}})$$

$$+ (-1)^{n(0)} l_1 \circ l_n(v_i \otimes v_j \otimes w_{k_1} \otimes \ldots \otimes w_{k_{n-2}})$$

$$= (-1)^{n-1} C_n (w_i - w_j) - 2(n-2) C_{n-1}(w_i - w_j)$$

$$+ \sum_{p=3}^{n-2} (-1)^{p(n-p)} (-1)^{p+1}\tbinom{n-2}{p-1} C_{n-p+1} C_p (w_i - w_j) + 0 + 0.$$

Thus $\mathcal{J}_n(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) = 0$ if and only if

$$(-1)^{n-1} C_n - 2(n-2) C_{n-1} + \sum_{p=3}^{n-2} (-1)^{pn+1}\tbinom{n-2}{p-1} C_{n-p+1} C_p = 0.$$

But then it is possible to solve for each C_n in terms of prior constants:

$$C_n = (-1)^n \left[-2(n-2)C_{n-1} + \sum_{p=3}^{n-2} (-1)^{pn+1} \binom{n-2}{p-1} C_{n-p+1} C_p \right].$$

Thus $\mathcal{J}_n(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) = 0$ if and only if each constant C_n is defined according to the above formula (where $C_1 = 1$). As far as the other generators are concerned, it is simple to show that $\mathcal{J}_n(v_i \otimes v_j \otimes v_k \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-3}}) = 0$. Thus if we define the C_n as specified, all generalized Jacobi identities will be satisfied, making this an L_∞ structure.

Example 95. Here, we have an L_∞ structure in which V_0 is not a Lie algebra. Let $V_0 =< v_1, v_2, v_3 >$ and $V_1 =< w >$. Define skew operators on $V = V_0 \oplus V_1$ by

$$\left\{ \begin{array}{llll} l_1(v_1) = w & l_2(v_1 \otimes v_2) = v_1, & l_3(v_1 \otimes v_3 \otimes w) = v_1, & l_4(v_2 \otimes v_3 \otimes w \otimes w) = 2v_1, \\ & l_2(v_1 \otimes v_3) = v_2, & l_3(v_2 \otimes v_3 \otimes w) = -v_2, & \\ & l_2(v_2 \otimes w) = -w. & & \end{array} \right\}$$

Define $l_n \equiv 0 \ \forall \ n > 4$.

$$l_1 \circ l_2(v_1 \otimes v_2) - l_2(l_1(v_1) \otimes v_2) + l_2(l_1(v_2) \otimes v_1) = l_1(v_1) - l_2(w \otimes v_2) = w - w = 0.$$

$$l_1 \circ l_2(v_1 \otimes v_3) - l_2(l_1(v_1) \otimes v_3) + l_2(l_1(v_3) \otimes v_1) = l_1(v_2) - 0 = 0.$$

$$l_1 \circ l_2(v_2 \otimes v_3) - l_2(l_1(v_2) \otimes v_3) + l_2(l_1(v_3) \otimes v_2) = 0 - 0 + 0.$$

Thus we have an L_2 structure. The following calculations prove that it's an L_3 structure as well. Note, however, that since $l_2 \circ l_2(v_1 \otimes v_2 \otimes v_3) \neq 0$, V_0 is *not* a Lie algebra in the usual sense.

$$l_1 \circ l_3(v_1 \otimes v_2 \otimes v_3) = 0.$$

$$\begin{aligned} l_2 \circ l_2(v_1 \otimes v_2 \otimes v_3) &= l_2(l_2(v_1 \otimes v_2) \otimes v_3) - l_2(l_2(v_1 \otimes v_3) \otimes v_2) + l_2(l_2(v_2 \otimes v_3) \otimes v_1) \\ &= l_2(v_1 \otimes v_3) - l_2(v_2 \otimes v_2) + 0 = v_2 - 0 = v_2. \end{aligned}$$

$$l_3 \circ l_1(v_1 \otimes v_2 \otimes v_3) = l_3(w \otimes v_2 \otimes v_3) = -v_2.$$

To finish showing that it's an L_3 structure, we can just show that $\mathcal{J}_n(v_i \otimes v_j \otimes w) = 0$. Without loss of generality, we can assume $i < j$ in the following calculation:

$$
\begin{aligned}
l_1 \circ l_3(v_i \otimes v_j \otimes w) &= l_1(\delta_{i1}\delta_{j3}v_1 - \delta_{i2}\delta_{j3}v_2) = \delta_{i1}\delta_{j3}w. \\
l_2 \circ l_2(v_i \otimes v_j \otimes w) &= l_2(l_2(v_i \otimes v_j) \otimes w) - l_2(l_2(v_i \otimes w) \otimes v_j) + l_2(l_2(v_j \otimes w) \otimes v_i) \\
&= l_2((\delta_{i1}\delta_{j2}v_1 + \delta_{i1}\delta_{j3}v_2) \otimes w) + l_2(\delta_{i2}w \otimes v_j) - l_2(\delta_{j2}w \otimes v_i) \\
&= -\delta_{i1}\delta_{j3}w + \delta_{i2}\delta_{j2}w - \delta_{j2}\delta_{i2}w = -\delta_{i1}\delta_{j3}w. \\
l_3 \circ l_1(v_i \otimes v_j \otimes w) &= l_3(\delta_{i1}w \otimes v_j \otimes w) = 0.
\end{aligned}
$$

To show that this is an L_∞ structure, we'll first need to show that it's L_4, then L_5, then L_6, then L_7. This is rather tedious, but I suppose that it needs to be done. A lot of annoying calculation is necessary here, since examples in which V_0 is not a Lie algebra in the usual sense seem to lack the sort of pleasing symmetry which makes things work well. Anyway, we can start by verifying that it's an L_4 structure:

$$
\begin{aligned}
l_1 \circ l_4(v_1 \otimes v_2 \otimes v_3 \otimes w) &= 0. \\
l_2 \circ l_3(v_1 \otimes v_2 \otimes v_3 \otimes w) &= l_2(l_3(v_1 \otimes v_3 \otimes w) \otimes v_2) - l_2(l_3(v_2 \otimes v_3 \otimes w) \otimes v_1) \\
&= l_2(v_1 \otimes v_2) + l_2(v_2 \otimes v_1) = 0. \\
l_3 \circ l_2(v_1 \otimes v_2 \otimes v_3 \otimes w) &= l_3(l_2(v_1 \otimes v_2) \otimes v_3 \otimes w) - l_3(l_2(v_1 \otimes v_3) \otimes v_2 \otimes w) \\
&\qquad + l_3(l_2(v_2 \otimes w) \otimes v_1 \otimes v_3) \\
&= l_3(v_1 \otimes v_3 \otimes w) - l_3(v_2 \otimes v_2 \otimes w) - l_3(w \otimes v_1 \otimes v_3) \\
&= v_1 - 0 - v_1 = 0. \\
l_4 \circ l_1(v_1 \otimes v_2 \otimes v_3 \otimes w) &= l_4(w \otimes v_2 \otimes w \otimes w) = 0.
\end{aligned}
$$

$$
l_1 \circ l_4(v_i \otimes v_j \otimes w \otimes w) = l_1(\delta_{i2}\delta_{j3}2v_1) = 2\delta_{i2}\delta_{j3}w.
$$

$$l_2 \circ l_3(v_i \otimes v_j \otimes w \otimes w) = 2l_2(l_3(v_i \otimes v_j \otimes w) \otimes w) = 2l_2(\delta_{i1}\delta_{j3}v_1 - \delta_{i2}\delta_{j3}v_2) \otimes w)$$
$$= 2\delta_{i2}\delta_{j3}w.$$
$$l_3 \circ l_2(v_i \otimes v_j \otimes w \otimes w) = l_3(l_2(v_i \otimes v_j) \otimes w \otimes w) - 2l_3(l_2(v_i \otimes w) \otimes v_j \otimes w)$$
$$+ 2l_3(l_2(v_j \otimes w) \otimes v_i \otimes w)$$
$$= l_3((\delta_{i1}\delta_{j2}v_1 + \delta_{i1}\delta_{j3}v_2) \otimes w \otimes w) + 2l_3(\delta_{i2}w \otimes v_j \otimes w)$$
$$- 2l_3(\delta_{j2}w \otimes v_i \otimes w) = 0.$$
$$l_4 \circ l_1(v_i \otimes v_j \otimes w \otimes w) = l_4(l_1(v_i) \otimes v_j \otimes w \otimes w) - l_4(l_1(v_j) \otimes v_i \otimes w \otimes w)$$
$$= l_4(\delta_{i1}w \otimes v_j \otimes w \otimes w) = 0.$$

Now we show that it's an L_5 structure:

$$l_1 \circ l_5(v_1 \otimes v_2 \otimes v_3 \otimes w^{\otimes 2}) = 0.$$
$$l_2 \circ l_4(v_1 \otimes v_2 \otimes v_3 \otimes w^{\otimes 2}) = 0.$$
$$l_3 \circ l_3(v_1 \otimes v_2 \otimes v_3 \otimes w^{\otimes 2}) = 2l_3(l_3(v_1 \otimes v_3 \otimes w) \otimes v_2 \otimes w) - 2l_3(l_3(v_2 \otimes v_3 \otimes w) \otimes v_1 \otimes w)$$
$$= 2l_3(v_1 \otimes v_2 \otimes w) + 2l_3(v_2 \otimes v_1 \otimes w) = 0.$$
$$l_4 \circ l_2(v_1 \otimes v_2 \otimes v_3 \otimes w^{\otimes 2}) = l_4(l_2(v_1 \otimes v_2) \otimes v_3 \otimes w^{\otimes 2}) - l_4(l_2(v_1 \otimes v_3) \otimes v_2 \otimes w^{\otimes 2})$$
$$- 2l_4(l_2(v_2 \otimes w) \otimes v_1 \otimes v_3 \otimes w)$$
$$= l_4(v_1 \otimes v_3 \otimes w^{\otimes 2}) - l_4(v_2 \otimes v_2 \otimes w^{\otimes 2}) + 2l_4(w \otimes v_1 \otimes v_3 \otimes w) = 0.$$
$$l_5 \circ l_1(v_1 \otimes v_2 \otimes v_3 \otimes w^{\otimes 2}) = 0.$$

$$l_1 \circ l_5(v_i \otimes v_j \otimes w^{\otimes 3}) = 0.$$
$$l_2 \circ l_4(v_i \otimes v_j \otimes w^{\otimes 3}) = 3l_2(l_4(v_i \otimes v_j \otimes w^{\otimes 2}) \otimes w) = 3l_2(\delta_{i2}\delta_{j3}2v_1 \otimes w) = 0.$$
$$l_3 \circ l_3(v_i \otimes v_j \otimes w^{\otimes 3}) = 3l_3(l_3(v_i \otimes v_j \otimes w) \otimes w^{\otimes 2}) = 3l_3((\delta_{i1}\delta_{j3}v_1 - \delta_{i2}\delta_{j3}v_2) \otimes w^{\otimes 2}) = 0.$$

$$l_4 \circ l_2(v_i \otimes v_j \otimes w^{\otimes 3}) = l_4(l_2(v_i \otimes v_j) \otimes w^{\otimes 3}) - 3\, l_4(l_2(v_i \otimes w) \otimes v_j \otimes w^{\otimes 2}) + 3\, l_4(l_2(v_j \otimes w) \otimes v_i \otimes w^{\otimes 2})$$

$$= l_4((\delta_{i1}\delta_{j2}v_1 + \delta_{i1}\delta_{j3}v_2) \otimes w^{\otimes 2}) + 3\, l_4(\delta_{i2} w \otimes v_j \otimes w^{\otimes 2}) - 3\, l_4(\delta_{j2} w \otimes v_i \otimes w^{\otimes 2}) = 0.$$

$$l_5 \circ l_1(v_i \otimes v_j \otimes w^{\otimes 3}) = 0.$$

To show that it's an L_6 structure, we only need to check $l_3 \circ l_4$ and $l_4 \circ l_3$, since all of the other compositons are definitely zero.

$$l_3 \circ l_4(v_1 \otimes v_2 \otimes v_3 \otimes w^{\otimes 3}) = 3\, l_3(l_4(v_2 \otimes v_3 \otimes w^{\otimes 2}) \otimes v_1 \otimes w) = 3\, l_3(2v_1 \otimes v_1 \otimes w) = 0.$$

$$l_4 \circ l_3(v_1 \otimes v_2 \otimes v_3 \otimes w^{\otimes 3}) = 3\, l_4(l_3(v_1 \otimes v_3 \otimes w) \otimes v_2 \otimes w^{\otimes 2}) - 3\, l_4(l_3(v_2 \otimes v_3 \otimes w) \otimes v_1 \otimes w^{\otimes 2})$$

$$= 3\, l_4(v_1 \otimes v_2 \otimes w^{\otimes 2}) + 3\, l_4(v_2 \otimes v_1 \otimes w^{\otimes 2}) = 0.$$

$$l_3 \circ l_4(v_i \otimes v_j \otimes w^{\otimes 4}) = \binom{4}{2} l_3(l_4(v_i \otimes v_j \otimes w^{\otimes 2}) \otimes w^{\otimes 2}) = 6\, l_3(\delta_{i2}\delta_{j3} 2v_1 \otimes w^{\otimes 2}) = 0.$$

$$l_4 \circ l_3(v_i \otimes v_j \otimes w^{\otimes 4}) = 4 l_4(l_3(v_i \otimes v_j \otimes w) \otimes w^{\otimes 3}) = 0.$$

Finally, to show that it's an L_7 structure, we only need to check $l_4 \circ l_4$.

$$l_4 \circ l_4(v_1 \otimes v_2 \otimes v_3 \otimes w^{\otimes 4}) = \binom{4}{2} l_4(l_4(v_2 \otimes v_3 \otimes w^{\otimes 2}) \otimes v_1 \otimes w^{\otimes 2})$$

$$= 6\, l_4(2v_1 \otimes v_1 \otimes w^{\otimes 2}) = 0.$$

$$l_4 \circ l_4(v_i \otimes v_j \otimes w^{\otimes 5}) = \binom{5}{2} l_4(l_4(v_i \otimes v_j \otimes w^{\otimes 2}) \otimes w^{\otimes 3})$$

$$= \binom{5}{2} l_4(\delta_{i2}\delta_{j3} 2v_1 \otimes w^{\otimes 3}) = 0.$$

For $n \geq 8$, $l_{n-p+1} \circ l_p \equiv 0$. Thus we have an L_∞ structure!

Example 96. Here, we have a small L_3 structure in which V_0 is not a Lie algebra. Let $V_0 =< v_1, v_2, v_3 >$ and $V_1 =< w_1, w_2, w_3 >$. Define skew linear operators on

$V = V_0 \oplus V_1$ by

$$\left\{\begin{array}{llll} l_1(v_1) = w_1, & l_2(v_1 \otimes v_2) = v_1, & l_3(v_1 \otimes v_3 \otimes w_2) & = v_2. \\ l_1(v_2) = w_2, & l_2(v_1 \otimes v_3) = v_2, & l_3(v_2 \otimes w_1 \otimes w_2) & = w_1. \\ l_1(v_3) = w_3, & l_2(v_1 \otimes w_2) = w_1, & l_3(v_2 \otimes w_1 \otimes w_3) & = w_2. \\ & l_2(v_1 \otimes w_3) = w_2, & l_3(v_3 \otimes w_1 \otimes w_2) & = w_2. \end{array}\right\}$$

As usual, we'll do some preliminary calculations to show that this structure is L_2:

$$\begin{aligned} l_1 \circ l_2(v_1 \otimes v_2) - l_2 \circ l_1(v_1 \otimes v_2) &= l_1(v_1) - (l_2(l_1(v_1) \otimes v_2) - l_2(l_1(v_2) \otimes v_1)) \\ &= w_1 - l_2(w_1 \otimes v_2) + l_2(w_2 \otimes v_1) = w_1 + 0 - w_1 = 0. \\ l_1 \circ l_2(v_1 \otimes v_3) - l_2 \circ l_1(v_1 \otimes v_3) &= l_1(v_2) - (l_2(l_1(v_1) \otimes v_3) - l_2(l_1(v_3) \otimes v_1)) \\ &= w_2 - l_2(w_1 \otimes v_3) + l_2(w_3 \otimes v_1) = w_2 + 0 - w_2 = 0. \\ l_1 \circ l_2(v_2 \otimes v_3) - l_2 \circ l_1(v_2 \otimes v_3) &= 0 - (l_2(l_1(v_2) \otimes v_3) - l_2(l_1(v_3) \otimes v_2)) \\ &= -l_2(w_2 \otimes v_3) + l_2(w_3 \otimes v_2) = 0 - 0 = 0. \end{aligned}$$

The next three sets of calculations show that the structure is L_3. Note that since $l_2 \circ l_2(v_1 \otimes v_2 \otimes v_3) = v_2$, V_0 does not satisify the standard Jacobi identity. Thus V_0 is not a Lie algebra.

$$\begin{aligned} l_1 \circ l_3(v_1 \otimes v_2 \otimes v_3) &= 0. \\ l_2 \circ l_2(v_1 \otimes v_2 \otimes v_3) &= l_2(l_2(v_1 \otimes v_2) \otimes v_3) - l_2(l_2(v_1 \otimes v_3) \otimes v_2) + l_2(l_2(v_2 \otimes v_3) \otimes v_1) \\ &= l_2(v_1 \otimes v_3) - l_2(v_2 \otimes v_2) + 0 = v_2. \\ l_3 \circ l_1(v_1 \otimes v_2 \otimes v_3) &= l_3(w_1 \otimes v_2 \otimes v_3) - l_3(w_2 \otimes v_1 \otimes v_3) + l_3(w_3 \otimes v_1 \otimes v_2) \\ &= 0 - l_3(v_1 \otimes v_3 \otimes w_2) + 0 = -v_2. \end{aligned}$$

$$l_1 \circ l_3(v_1 \otimes v_2 \otimes w_i) = 0.$$

$$\begin{aligned}
l_2 \circ l_2(v_1 \otimes v_2 \otimes w_i) &= l_2(l_2(v_1 \otimes v_2) \otimes w_i) - l_2(l_2(v_1 \otimes w_i) \otimes v_2) + l_2(l_2(v_2 \otimes w_i) \otimes v_1) \\
&= l_2(v_1 \otimes w_i) - l_2(\delta_{i2}w_1 + \delta_{i3}w_2) \otimes v_2) + 0 \\
&= (\delta_{i2}w_1 + \delta_{i3}w_2) - 0 = \delta_{i2}w_1 + \delta_{i3}w_2.
\end{aligned}$$

$$\begin{aligned}
l_3 \circ l_1(v_1 \otimes v_2 \otimes w_i) &= l_3(w_1 \otimes v_2 \otimes w_i) - l_3(w_2 \otimes v_1 \otimes w_i) \\
&= -l_3(v_2 \otimes w_1 \otimes w_i) + l_3(v_1 \otimes w_2 \otimes w_i) \\
&= -(\delta_{i2}w_1 + \delta_{i3}w_2) + 0.
\end{aligned}$$

$$l_1 \circ l_3(v_1 \otimes v_3 \otimes w_i) = l_1(\delta_{i2}v_2) = \delta_{i2}w_2$$

$$\begin{aligned}
l_2 \circ l_2(v_1 \otimes v_3 \otimes w_i) &= l_2(l_2(v_1 \otimes v_3) \otimes w_i) - l_2(l_2(v_1 \otimes w_i) \otimes v_3) + l_2(l_2(v_3 \otimes w_i) \otimes v_1) \\
&= l_2(v_2 \otimes w_i) - l_2(\delta_{i2}w_1 + \delta_{i3}w_2) \otimes v_3) + 0 \\
&= 0 + \delta_{i2}l_2(v_3 \otimes w_1) + \delta_{i3}l_2(v_3 \otimes w_2) = 0 + 0.
\end{aligned}$$

$$\begin{aligned}
l_3 \circ l_1(v_1 \otimes v_3 \otimes w_i) &= l_3(w_1 \otimes v_3 \otimes w_i) - l_3(w_3 \otimes v_1 \otimes w_i) \\
&= -l_3(v_3 \otimes w_1 \otimes w_i) + l_3(v_1 \otimes w_3 \otimes w_i) \\
&= -\delta_{i2}w_2 + 0.
\end{aligned}$$

$$l_1 \circ l_3(v_2 \otimes v_3 \otimes w_i) = 0.$$

$$\begin{aligned}
l_2 \circ l_2(v_2 \otimes v_3 \otimes w_i) &= l_2(l_2(v_2 \otimes v_3) \otimes w_i) - l_2(l_2(v_2 \otimes w_i) \otimes v_3) + l_2(l_2(v_3 \otimes w_i) \otimes v_2) \\
&= 0 + 0 + 0.
\end{aligned}$$

$$\begin{aligned}
l_3 \circ l_1(v_2 \otimes v_3 \otimes w_i) &= l_3(w_2 \otimes v_3 \otimes w_i) - l_3(w_3 \otimes v_2 \otimes w_i) \\
&= -l_3(v_3 \otimes w_i \otimes w_2) + l_3(v_2 \otimes w_i \otimes w_3) \\
&= -\delta_{i1}w_2 + \delta_{i1}w_2 = 0.
\end{aligned}$$

Example 97. Here, we will examine a small L_3 structure in which V_0 is an abelian Lie algebra, but V_1 is not a Lie module. Let $V_0 =< v_1, v_2 >$ and $V_1 =< w_1, w_2, w_3 >$, and define skew linear operators on $V = V_0 \oplus V_1$ by

$$\left\{\begin{array}{llll} l_1(v_2) & = w_2, & l_2(v_2 \otimes w_1) = w_1, & l_3(v_1 \otimes w_2 \otimes w_3) & = -w_1, \\ & & l_2(v_1 \otimes w_3) = w_1. & & \end{array}\right\}$$

Now we'll do some quick calculations to verify that it's an L_3 structure:

$$l_1 \circ l_2(v_1 \otimes v_2) - l_2 \circ l_1(v_1 \otimes v_2) = l_1(0) - (l_2(l_1(v_1) \otimes v_2) - l_2(l_1(v_2) \otimes v_1))$$

$$= 0 - l_2(0) + l_2(w_2 \otimes v_1) = 0.$$

$$l_1 \circ l_3(v_1 \otimes v_2 \otimes w_i) = 0.$$

$$l_2 \circ l_2(v_1 \otimes v_2 \otimes w_i) = l_2(l_2(v_1 \otimes v_2) \otimes w_i) - l_2(l_2(v_1 \otimes w_i) \otimes v_2) + l_2(l_2(v_2 \otimes w_i) \otimes v_1)$$

$$= 0 - l_2(\delta_{i3} w_1 \otimes v_2) + l_2(\delta_{i1} w_1 \otimes v_1)$$

$$= \delta_{i3} l_2(v_2 \otimes w_1) + 0 = \delta_{i3} w_1.$$

$$l_3 \circ l_1(v_1 \otimes v_2 \otimes w_i) = l_3(w_1 \otimes v_2 \otimes w_i) - l_3(w_2 \otimes v_1 \otimes w_i) = -l_3(w_2 \otimes v_1 \otimes w_i)$$

$$= l_3(v_1 \otimes w_2 \otimes w_i) = -\delta_{i3} w_1.$$

Thus we see that this is an L_3 structure in which V_0 is an abelian Lie algebra, but V_1 is *not* a Lie module, since $l_2 \circ l_2(v_1 \otimes v_2 \otimes w_3) = w_1 \neq 0$.

Example 98. Now, we will look at a tiny L_3 structure in which V_0 is a nonabelian Lie algebra, but V_1 is not a Lie module. Let $V_0 =< v_1, v_2 >$ and $V_1 =< w >$. Define skew linear operators on $V = V_0 \oplus V_1$ by

$$\left\{\begin{array}{llll} l_1(v_1) = w, & l_2(v_1 \otimes v_2) & = v_1, & l_3(v_2 \otimes w \otimes w) & = w. \\ l_1(v_2) = w, & l_2(v_1 \otimes w) & = w. & & \end{array}\right\}$$

This would seem to be the smallest possible example of this type, since when V_0 is one-dimensional, the "module action" condition is always (trivially) satisfied!

$$l_1 \circ l_2(v_1 \otimes v_2) - l_2 \circ l_1(v_1 \otimes v_2) = l_1(v_1) - (l_2(l_1(v_1) \otimes v_2) - l_2(l_1(v_2) \otimes v_1))$$

$$= w - l_2(w \otimes v_2) + l_2(w \otimes v_1) = w + 0 - w = 0.$$

$l_1 \circ l_3(v_1 \otimes v_2 \otimes w) = 0.$

$$l_2 \circ l_2(v_1 \otimes v_2 \otimes w) = l_2(l_2(v_1 \otimes v_2) \otimes w) - l_2(l_2(v_1 \otimes w) \otimes v_2) + l_2(l_2(v_2 \otimes w) \otimes v_1)$$
$$= l_2(v_1 \otimes w) - l_2(w \otimes v_2) + 0 = w - 0 = w.$$

$$l_3 \circ l_1(v_1 \otimes v_2 \otimes w) = l_3(w \otimes v_2 \otimes w) - l_3(w \otimes v_1 \otimes w) = -l_3(v_2 \otimes w \otimes w) + 0 = -w.$$

Thus we have an L_3 structure. Note that the preceding calculations demonstrate that V_1 is *not* a Lie module over V_0, since $l_2 \circ l_2(v_1 \otimes v_2 \otimes w) \neq 0$.

REMARK 99. It is possible to extend the above example to an L_∞ structure by defining $l_n(v_2 \otimes w^{\otimes n-1}) = C_n w$, where each constant C_n is defined recursively as a function of its predecessor. Simply let $C_3 = 1$, and then assign $C_n = (-1)^{n-1}(n-3)C_{n-1}$. The details of the proof are left as an exercise for any interested reader.

REMARK 100. This example was used in the paper in [**4**] to explore the the gauge theory model of Berends, Burgers, and van Dam in a very specific context.

Example 101. Finally, we will give an extremely simple example of a case in which it is impossible to extend a given L_∞ structure in order to create an L_{n+1} structure (even if all of the higher operators are defined to be zero). Let $V_0 =< v_1, v_2, v_3 >$ and $V_1 =< w >$. Define skew operators on $V = V_0 \oplus V_1$ by

$$\left\{\begin{array}{ll} l_1(v_1) = 0, & l_2(v_1 \otimes v_2) \quad = v_2, \\ l_1(v_2) = 0, & l_2(v_1 \otimes v_3) \quad = v_1, \\ l_1(v_3) = 0, & l_2(v_2 \otimes v_3) \quad = 0. \end{array}\right\}$$

Note that since $l_1 \equiv 0$, this structure is trivially L_2.

$$l_2 \circ l_2(v_1 \otimes v_2 \otimes v_3) = l_2(l_2(v_1 \otimes v_2) \otimes v_3) - l_2(l_2(v_1 \otimes v_3) \otimes v_2) + l_2(l_2(v_2 \otimes v_3) \otimes v_1)$$
$$= l_2(v_2 \otimes v_3) - l_2(v_1 \otimes v_2) + 0 = 0 - v_2 = -v_2.$$

Thus we see that V_0 is not a Lie algebra in the usual sense. Since we have a trivial differential in this example, $l_1 \circ l_3 \equiv 0$ and $l_3 \circ l_1 \equiv 0$. Therefore, there is no possible way to define an operator l_3 which would make this an L_3 structure.

Bibliography

[1] F.A. Berends, G.H.J. Burgers, H. van Dam. *On the Theoretical Problems in Constructing Interactions Involving Higher Spin Massless Particles*. Nucl. Phys. B 1985, *260*, 295–322.

[2] A. Cattaneo, G. Felder. *A path-integral approach to the Kontsevich quantization formula*. Comm. Math. Phys. 2000, *212*, (3), 591–611.

[3] M. Daily. *L_∞ Structures on Spaces with Three One-Dimensional Components*. Comm. in Alg 2004, *32* (5), 2041–2059.

[4] M. Daily, T. Lada. *A Finite Dimensional L_∞ Algebra Example in Gauge Theory*. To appear in *Homology, Homotopy, and Applications*.

[5] A. Fialowski, M. Penkava. *Strongly Homotopy Lie Algebras of One Even and Two Odd Dimension*. Preprint math.QA/0308016, 2003.

[6] A. Fialowski, M. Penkava. *Classification and Versal Deformations of L_∞ Algebras on a $2|1$-Dimensional Space*. Preprint math.QA/0401025, 2004.

[7] R. Fulp, T. Lada, J. Stasheff. *Sh-Lie Algebras Induced by Gauge Transformations*. Commun. Math Phys. 2002, *231*, 25–43.

[8] M. Gerstenhaber. *The cohomology structure of an associative ring*. Ann. Math. 1962, *78*, 267–288.

[9] N. Jacobson. *Lie Algebras*. John Wiley & Sons, 1962, Dover 1979.

[10] T. Lada, M. Markl. *Strongly Homotopy Lie Algebras*. Comm. in Algebra 1995, *23* (6), 2147–2161.

[11] T. Lada, J. Stasheff. *Introduction to sh-Lie Algebras for Physicists*. Intern'l J. Theor. Phys. 1993, *32* (7), 1087–1103.

[12] C. Lazaroiu. *String Field Theory and Brane Superpotentials*. JHEP 2001, *10* (018).

[13] S. Mac Lane.. *Homology*. Springer-Verlag, 1963.

[14] J. Milnor, J. Moore. *On the Structure of Hopf Algebras*. Ann. of Math. 1965, *81* (2), 211–264.

[15] M. Schlessinger, J. Stasheff. *The Lie Algebra Structure of Tangent Cohomology and Deformation Theory*. J. of Pure and Appl. Alg. 1985, *38*, 479–496.

[16] C. Seeley. 7 *dimensional nilpotent Lie algebras*. Trans. Amer. Math. Soc. 1993 *335* (2),

[17] E. Witten, B. Zwiebach. *Algebraic Structures and Differential Geometry in Two-Dimensional String Theory*. Nucl.Phys. B 1992, *390*, 55–112.

[18] B. Zwiebach, Closed string field theory: *Quantum action and the Batalin-Vilkovisky master equation*. Nucl.Phys.B 1993, *390* (1993), 33–152.