

Abstract

WILLIAMS, VICKY. Root Multiplicities of the indefinite Kac-Moody algebras $HC_n^{(1)}$.

(Under the direction of Kailash C. Misra)

Victor Kac and Robert Moody independently introduced Kac-Moody algebras around 1968. These Lie algebras have numerous application in physics and mathematics and thus have been the subject of much study over the last three decades. Kac-Moody algebras are classified as finite, affine, or indefinite type. A basic problem concerning these algebras is finding their root multiplicities. The root multiplicities of finite and affine type Kac-Moody algebras are well known. However, determining the root multiplicities of indefinite type Kac-Moody algebras is an open problem. In this thesis we determine the multiplicities of some roots of the indefinite type Kac-Moody algebras $HC_n^{(1)}$.

A well known construction allows us to view $HC_n^{(1)}$ as the minimal graded Lie algebra with local part $V \oplus \mathfrak{g}_0 \oplus V'$, where \mathfrak{g}_0 is the affine Kac-Moody algebra $C_n^{(1)}$ and V, V' are suitable \mathfrak{g}_0 -modules. From this viewpoint root spaces of $HC_n^{(1)}$ become

weight spaces of certain $C_n^{(1)}$ -modules. Using a multiplicity formula due to Kang we reduce our problem to finding weight multiplicities in certain irreducible highest weight $C_n^{(1)}$ -modules. We then use crystal basis theory for the affine Kac-Moody algebras $C_n^{(1)}$ to find these weight multiplicities.

With this strategy we calculate the multiplicities of some roots of $HC_n^{(1)}$. In particular, we determine the multiplicities of the level two roots $-2\alpha_{-1} - k\delta$ of $HC_2^{(1)}$ for $1 \leq k \leq 10$. We also show that the multiplicities of the roots of $HC_n^{(1)}$ of the form $-l\alpha_{-1} - k\delta$ are n for $l = k$ and 0 for $l > k$. In the process, we observe that Frenkel's conjectured bound for root multiplicities does not hold for the indefinite Kac-Moody algebras $HC_n^{(1)}$.

ROOT MULTIPLICITIES OF THE INDEFINITE KAC-MOODY ALGEBRAS $HC_n^{(1)}$

BY

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A DISSERTATION SUBMITTED TO THE GRADUATE FACULTY OF

NORTH CAROLINA STATE UNIVERSITY

IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

MATHEMATICS

RALEIGH, NORTH CAROLINA

JUNE 2003

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Biography

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Acknowledgments

Most importantly, I must thank my advisor, Dr. Kailash Misra. Without his help and never-ending patience, I would have accomplished nothing. I would also like to thank the additional members of my committee: Dr. Ernest Stitzinger, Dr. Naihuan Zing, and Dr. Jacqueline Hughes-Oliver. In addition, I wish to express my gratitude to my parents, Richard and Virginia, who have always let me make my own decisions and have, along with my wonderful brother, Wade, supported those decisions wholeheartedly. Finally, I want to thank my friends: Kathryn, who was there when I first considered graduate work and who helped me choose the right school; Karen Yokley who has protected me from my scatter brained behaviors over the past three years (my keys also thank her); Karen Bliss, who has listened to me complain and worry and scream and cry and ... always finding a way to make me laugh in the end; and Katie whom I have looked to each time I thought I could not deal with the obstacles before me and whose thoughtfulness has carried me through those obstacles. “Yo Katie - I did it!”

Table of Contents

List of Tables	vi
List of Figures	viii
1 Introduction	1
2 Kac-Moody algebras	6
3 Construction of $HC_n^{(1)}$	19
4 Multiplicity formula	32
5 Crystal basis theory	40
5.1 Motivation	41
5.2 Basic Concepts	42
5.3 Path Realizations	48
6 Root multiplicities of $HC_n^{(1)}$	61
6.1 Level One Root Multiplicities	61

6.2	Level Two Root Multiplicities	81
6.3	General Observations	104
	References	108
	A MATLAB Code	113
A.1	phi.m	113
A.2	epsilon.m	114
A.3	tenzorrule.m	115
A.4	identify_action.m	116
A.5	one_move.m	117
A.6	next_move.m	119
A.7	crystal_path.m	121
A.8	mult.m	122

List of Tables

6.1	$\underline{x}, \underline{y} \neq \underline{0}$ in \mathcal{B}_1 such that $(x_i + y_i - \bar{x}_i - \bar{y}_i) = 0$	67
6.2	The paths $(\cdots \otimes \underline{0} \otimes \underline{0} \otimes \underline{y} \otimes \underline{x})$ in $\mathcal{P}(\Lambda_0)$ of weight $\Lambda_0 - 2\delta$	67
6.3	Paths $p = (\cdots \underline{0} \otimes \underline{y} \otimes \underline{0})$ in $\mathcal{P}(\Lambda_0)$ for which $wt(p) = \Lambda_0 - 3\delta$	71
6.4	Paths $p = (\cdots \underline{0} \otimes \underline{y} \otimes \underline{x})$ ($\underline{x} \neq \underline{0}$) for which $wt(p) = \Lambda_0 - 3\delta$	71
6.5	Some paths in category A	74
6.6	Paths from Category A for which $wt(p) = \Lambda_0 - 3\delta$	74
6.7	Some paths in Category B	75
6.8	Some paths in Category C	76
6.9	Paths from Category C of weight $\Lambda_0 - 3\delta$	77
6.10	Some paths from Category D	78
6.11	Some paths from Category D	78
6.12	Paths from Category D with weight $\Lambda_0 - 3\delta$	79
6.13	Multiplicities of roots $-\alpha_{-1} - k\delta$ in $HC_2^{(1)}$	81
6.14	$\omega\rho - \rho$ for $\omega \in W(S)$	82
6.15	$\lambda_{i,j}^{(t)}$ and $\eta_{i,j}^{(t)}$ such that $\lambda_{i,j}^{(t)} + \eta_{i,j}^{(t)} = 2\Lambda_0 - 3\delta$	86

6.16	paths, $(\cdots \otimes \underline{0} \otimes \underline{0} \otimes \underline{y} \otimes \underline{x})$, with weights $\eta_{i,j}^{(3)}$	89
6.17	number of paths with weight $\eta_{i,j}^{(t)}$	89
6.18	Paths in Case I, Category A with \underline{x}' and \underline{y}' of type one which are of weight $2\Lambda_0 - 2\delta$	98
6.19	Paths from Case I, Category A(i) of weight $2\Lambda_0 - 2\delta$	100
6.20	Paths from Case I, Category A(ii) of weight $2\Lambda_0 - 2\delta$	100
6.21	Number of paths of weight $2\Lambda_0 - 2\delta$ for cases one through nine	101
6.22	The multiplicity of $-2\alpha_{-1} - k\delta$ as a root of $HC_2^{(1)}$	104
6.23	The multiplicity of $-2\alpha_{-1} - k\delta$ as a root of $HC_2^{(1)}$	107

List of Figures

5.1	Crystal Graph $V \otimes V$	47
5.2	A partial path realization of the $C_2^{(1)}$ -module $L(\Lambda_0)$	57
6.1	A partial path realization of the module $V(\Lambda_0)$ as a $U'_q(C_2^{(1)})$ -module. (Zero arrows are not shown after the first.)	63

Chapter 1

Introduction

In 1968, Victor Kac [14] and Robert Moody [33] independently introduced a new class of Lie algebras, which are known today as Kac-Moody algebras. For the past thirty years, Kac-Moody algebras have been a rich area of study due to their numerous applications to other areas of mathematics and physics. Kac-Moody algebras are classified as finite, affine, or indefinite type. Root multiplicities in Kac-Moody algebras of finite and affine type are well known. However, we know root multiplicities of indefinite type Kac-Moody algebras in only a few cases (for example see, [6, 1, 2, 3, 15, 20, 19, 21, 24, 11, 10, 24, 25]). In this thesis we determine some root multiplicities of the indefinite type Kac-Moody algebras $HC_n^{(1)}$.

A Kac-Moody algebra \mathfrak{g} associated with generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ (I an index set) affords a triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$, where \mathfrak{g}_- is the direct sum of negative root spaces, \mathfrak{h} is the Cartan subalgebra of \mathfrak{g} , and \mathfrak{g}_+ is the

direct sum of positive root spaces. The multiplicity of any root α of \mathfrak{g} is equal to that of $-\alpha$. Thus we may choose to study either the negative or the positive roots; we choose to study the multiplicities of negative roots of $HC_n^{(1)}$. We will review a few of the basic concepts of Kac-Moody theory in chapter 2. For more details see [16].

The indefinite type Kac-Moody algebra $\mathfrak{g} = HC_n^{(1)}$ is the Kac-Moody algebra associated with the indecomposable generalized Cartan matrix (for definition see chapter 2)

$$A = (a_{ij})_{i,j \in I} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 & 0 \\ & & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix},$$

where the index set $I = \{-1, 0, \dots, n\}$. Let $S = \{0, 1, \dots, n\}$, and let \mathfrak{g}_0 be the Kac-Moody algebra associated with the indecomposable generalized Cartan matrix $A = (a_{ij})_{i,j \in S}$. Then, \mathfrak{g}_0 is the affine Kac-Moody algebra $C_n^{(1)}$. The Lie algebra \mathfrak{g} may be constructed as the minimal \mathbb{Z} -graded Lie algebra with local part $V \otimes \mathfrak{g}_0 \otimes V^*$, where V is the basic \mathfrak{g}_0 -module, $V(\Lambda_0)$, and V^* its restricted dual [6, 14, 1]. This

construction allows us to view the root spaces of $HC_n^{(1)}$ as weight spaces of appropriate $C_n^{(1)}$ -modules. Thus, we may use the representation theory of the affine Kac-Moody algebras $C_n^{(1)}$ to study root multiplicities of the indefinite Kac-Moody algebras $HC_n^{(1)}$. We give the explicit construction of $HC_n^{(1)}$ in chapter 3.

For an affine algebra, \mathfrak{g}_0 , the level of a weight λ is defined as $\lambda(K)$ where K is the cononical central element of \mathfrak{g}_0 . We define the level of the root α of \mathfrak{g} to be the level of the weight α of \mathfrak{g}_0 . Root multiplicities of all level two roots of $HA_1^{(1)}$ were computed in [6] and for all level two roots of E_{10} in [15]. Generalizing the results in [4], using homology theory for Lie algebras Kang obtained two multiplicity formulas - one closed form, the other recursive - for roots of a class of Kac-Moody algebras which includes $HC_n^{(1)}$ [18, 20, 19, 21]. Using this formula, Kang [20, 19] was able to extend the results of [6]. Some multiplicities were found for roots of the algebras $HA_n^{(1)}$ in [24, 10], for roots of the algebras $IA_n(a, b)$ in [1, 2, 3], and for roots of the algebras $HG_2^{(1)}$, $HD_4^{(1)}$ in [11]. In each of these cases, a multiplicity formula analogous to Kang's was used. We review Kang's multiplicity formula in chapter 4.

In [24], Kang and Melville chose \mathfrak{g}_0 to be $A_n^{(1)}$ and used Crystal Basis theory to compute weight multiplicities for the appropriate $A_n^{(1)}$ -modules. We use a similar method in this thesis. The quantized universal enveloping algebras (also known as quantum groups), $U_q(\mathfrak{g})$, associated with the symmetrizable Kac-Moody algebras \mathfrak{g} were introduced independently by Drinfel and Jimbo [5, 13]. These algebras are related to solvable lattice models in statistical mechanics. In this context, the inde-

terminate q of the quantized universal enveloping algebra corresponds to the temperature parameter. Thus, it makes sense to expect nice behavior of the representation space for these algebras at $q = 0$. A crystal basis of an integrable $U_q(\mathfrak{g})$ -module, M , can be roughly thought of as a basis for M at $q = 0$. The notion of crystal bases was first introduced by Kashiwara [27, 26]. Kashiwara showed the existence and uniqueness of crystal basis for integrable representations of symmetrizable Kac-Moody algebras. Kashiwara's crystal bases are the same as the canonical bases at $q = 0$ introduced by Lusztig [31, 32]. In 1988, Lusztig proved that the weight multiplicities of $U_q(\mathfrak{g})$ -modules are the same as those of \mathfrak{g} -modules for any symmetrizable Kac-Moody algebra \mathfrak{g} [30]. We use the concrete realizations of the crystal bases for integrable highest weight $C_n^{(1)}$ -modules given in [22]. These realization are in terms of certain combinatorial objects called “paths” which arise naturally in statistical mechanics. A path can be thought of as an element of the semi-infinite produce $(\cdots \otimes B \otimes B \otimes B)$ where B denotes the so called “perfect crystal” (see [23, 9]). We use the path realizations to calculate weight multiplicities of certain highest weight $C_n^{(1)}$ -modules. We briefly review the combinatorial aspects of crystal basis theory in chapter 5.

In chapter 6, we determine multiplicities of some roots of $HC_n^{(1)}$. More specifically, we find the multiplicity of the level two root $-2\alpha_{-1} - 3\delta$ of $HC_n^{(1)}$ for any n and we determine the multiplicities of the roots $-2\alpha_{-1} - k\delta$ of $HC_2^{(1)}$ for $1 \leq k \leq 10$. We also show that for $HC_n^{(1)}$ roots of the form $-k\alpha_{-1} - k\delta$ are of multiplicity n and show

$-\alpha_{-1} - k\delta$ is not a root of $HC_n^{(1)}$ for $l > k$. In [7] Frenkel conjectured a bound for root multiplicities of hyperbolic Lie algebras. Kac, Moody, and Wakimoto showed that Frenkel's conjectured bound does not hold in E_{10} [15]. We observe that this bound also fails for $HC_n^{(1)}$.

Chapter 2

Kac-Moody algebras

We begin this chapter by defining the basic objects of our study: Lie algebras over the field of complex numbers and their representations. We then move to discuss the specific type of Lie algebras we will investigate, Kac-Moody algebras. For more details see [12] and [16].

Definition 2.1. *A Lie algebra is a vector space \mathfrak{g} over \mathbb{C} with an anti-symmetric, bilinear operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ (called the bracket) with the following property (called the Jacobi identity):*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Example 2.1. *The Lie algebra of trace zero 2×2 complex matrices with bracket*

$[A, B] = AB - BA$ is known as $sl(2, \mathbb{C})$. This Lie algebra has basis

$$\left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

and relations

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$$

A Lie algebra is a non-associative algebra. However, each Lie algebra has a unique universal enveloping algebra which is associative.

Definition 2.2. Let \mathfrak{g} be a Lie algebra. A universal enveloping algebra of \mathfrak{g} is a pair $(U(\mathfrak{g}), j)$ where $U(\mathfrak{g})$ is an associative algebra and $j : \mathfrak{g} \rightarrow U(\mathfrak{g})$ is a Lie algebra homomorphism satisfying the following universal property. If (A, ϕ) is another such pair then there exists a unique associative algebra homomorphism $\psi : U(\mathfrak{g}) \rightarrow A$ such that $\phi = \psi \circ j$.

The Poincare-Birkhoff-Witt theorem stated below gives a linear basis for $U(\mathfrak{g})$.

Theorem 2.1. Let \mathfrak{g} be a Lie algebra with ordered basis $\{x_\alpha | \alpha \in \Omega\}$ and let $(U(\mathfrak{g}), j)$ be the universal enveloping algebra of \mathfrak{g} . Then, $\{x_{\alpha_1} x_{\alpha_2} \dots x_{\alpha_n} | n \geq 0, \alpha_1 \leq \alpha_2 \dots \leq \alpha_n\}$ form a basis for $U(\mathfrak{g})$.

We wish to investigate the representation theory of Lie algebras.

Definition 2.3. Let \mathfrak{g} be a Lie algebra over \mathbb{C} and let V be a vector space over \mathbb{C} .

1. A representation of \mathfrak{g} on V is a homomorphism $\phi : \mathfrak{g} \rightarrow gl(V)$.
2. V is a \mathfrak{g} module if there is a bilinear map from $\mathfrak{g} \times V$ into V given by $(g, v) \rightarrow g \cdot v$ such that $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ for all $x, y \in \mathfrak{g}$ and $v \in V$.

Note, if ϕ is a representation of \mathfrak{g} on V then V is a \mathfrak{g} -module with module action $g \cdot v = \phi(g)v$ and conversely. For example, the adjoint map $ad : \mathfrak{g} \rightarrow gl(\mathfrak{g})$ given by $ad(x) = ad_x$ for any $x \in \mathfrak{g}$, where $ad_x(g) = [x, g]$ for all $g \in \mathfrak{g}$, is a representation of \mathfrak{g} on \mathfrak{g} . In other words, \mathfrak{g} is a \mathfrak{g} -module under the adjoint action.

A representation of a Lie algebra \mathfrak{g} naturally extends to a representation of $U(\mathfrak{g})$. Conversely, a representation of $U(\mathfrak{g})$ is also a representation of \mathfrak{g} . Therefore, the representation theory of a Lie algebra and that of its universal enveloping algebra are essentially equivalent.

We will study the representation theory of a certain class of Lie algebras known as Kac-Moody algebras. The remainder of this chapter briefly introduces the concepts of Kac-Moody theory. For a much more thorough treatment of the topic see [16].

A matrix $A = (a_{i,j})_{i,j \in I}$ is called a *generalized Cartan matrix* if it satisfies the following conditions

$$a_{ii} = 2 \text{ for all } i \in I$$

$$a_{ij} \in \mathbb{Z}_{\leq 0} \text{ for } i \neq j \in I$$

$$a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \text{ for all } i, j \in I,$$

where I denotes the index set. A generalized Cartan matrix, A , is indecomposable if, after any reordering of the indices, A can never be written in the form $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$. Any indecomposable generalized Cartan matrix, $A = (a_{i,j})_{i,j \in I = \{1, \dots, n\}}$, is of one and only one of the following types.

1. Finite type: There exists an n -dimensional column vector, θ , of positive integers such that all the coordinates of $A\theta$ are positive. In such a case, A is positive definite.
2. Affine type: There exists an n -dimensional column vector, θ , of positive integers such that $A\theta=0$. In such a case, A is positive-semidefinite of corank 1.
3. Indefinite type: There exists an n -dimensional column vector, θ , of positive integers such that all the coordinates of $A\theta$ are negative.

Let $A = (a_{i,j})_{i,j \in I = \{1, \dots, n\}}$ be a generalized Cartan matrix of rank l . A realization of A is a triple, $(\mathfrak{h}, \Pi, \check{\Pi})$, where \mathfrak{h} is a complex vector space, $\Pi = \{\alpha_i | i \in I\} \subset \mathfrak{h}^*$ and $\check{\Pi} = \{\check{\alpha}_i | i \in I\} = \{h_i | i \in I\} \subset \mathfrak{h}$ such that

Π and $\check{\Pi}$ are both linearly independent sets ,

$$\alpha_j(h_i) = \langle \alpha_j, h_i \rangle = a_{ij} \ (i, j \in I) \ ,$$

$$n - l = \dim(\mathfrak{h}) - n$$

The set Π is called the *root basis* and elements of Π are known as *simple roots*.

Similarly, $\check{\Pi}$ is the *co-root basis* and its elements are the *simple co-roots*. Later, we will refer to the following lattices (the root lattice and the positive root lattice, respectively).

$$Q = \sum_{i \in I} \mathbb{Z}\alpha_i \quad Q_+ = \sum_{i \in I} \mathbb{Z}_+\alpha_i$$

We introduce a partial ordering, \geq , on \mathfrak{h}^* : for $\alpha, \beta \in \mathfrak{h}^*$, $\alpha \geq \beta$ if and only if $\alpha - \beta \in Q_+$.

With these few preliminary definitions, we are now in a position to define a Kac-Moody algebra.

Definition 2.4. *Let A be a generalized Cartan matrix, and let $(\mathfrak{h}, \Pi, \check{\Pi})$ be a realization of A . Let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(A)$ be the Lie algebra on generators e_i, f_i ($i \in I$), and \mathfrak{h} with relations*

$$\begin{aligned} [e_i, f_j] &= \delta_{i,j} h_i \quad (i, j \in I) \\ [h, h'] &= 0 \quad (h, h' \in \mathfrak{h}) \\ [h, e_i] &= \langle \alpha_i, h \rangle e_i \quad (i \in I, h \in \mathfrak{h}) \\ [h, f_i] &= -\langle \alpha_i, h \rangle f_i \quad (i \in I; h \in \mathfrak{h}) \end{aligned} \tag{2.1}$$

Let ξ be the unique maximal ideal of $\tilde{\mathfrak{g}}$ intersecting \mathfrak{h} trivially. The Lie algebra $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\xi$ is the Kac-Moody algebra with generalized Cartan Matrix A .

The $e_i, f_i \in \mathfrak{g}(A)$ ($i \in I$) are known as the *Chevalley generators*, and the subalgebra \mathfrak{h} of $\mathfrak{g}(A)$ is known as the *Cartan subalgebra*. We define \mathfrak{g}_+ as the subalgebra

of \mathfrak{g} generated by the elements e_i . Similarly, \mathfrak{g}_- is the subalgebra of \mathfrak{g} generated by the f_i . For each i Let $\mathfrak{g}_{(i)} = \mathbb{C}e_i + \mathbb{C}h_i + \mathbb{C}f_i$. Then $\mathfrak{g}_{(i)}$ is isomorphic to $sl(2, \mathbb{C})$, with basis $\{e_i, f_i, h_i\}$ and relations,

$$[e_i, f_i] = h_i \quad [h_i, e_i] = 2e_i \quad [h_i, f_i] = -2f_i.$$

A matrix $A_{n \times n}$ is *symmetrizable* if there exists an invertible diagonal matrix $D = \text{diag}(s_1, \dots, s_n)$ and a symmetric matrix B such that $DB = A$. For a symmetrizable generalized Cartan matrix, $A = (a_{ij})_{i,j \in I}$ with $I = \{1, \dots, n\}$, define the bilinear form $(\cdot|\cdot)$ on $\mathfrak{h} = \text{span}\{h_1, \dots, h_n, d_1, \dots, d_{n-\text{rank}(A)}\}$, the Cartan subalgebra of $\mathfrak{g}(A)$, as follows.

$$\begin{aligned} (h_i|h) &= s_i \alpha_i(h) \quad \text{for } i = 1, \dots, n \text{ and } h \in \mathfrak{h} \\ (d_i|d_j) &= 0 \quad \text{for } i, j = 1, \dots, n - \text{rank}(A) \\ (h_i|d_j) &= \alpha_i(d_j) \quad \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, n - \text{rank}(A). \end{aligned} \tag{2.2}$$

So defined, $(\cdot|\cdot)$ is an invariant, symmetric, non-degenerate, bilinear form. Furthermore, $(\cdot|\cdot)$ can be uniquely extend to an invariant, symmetric, non-degenerate, bilinear form on all of \mathfrak{g} . Let $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$ be the linear map given by $\nu(h)(h') = (h|h')$. Since, $(\cdot|\cdot)$ is non-degenerate, ν is a vector space isomorphism. Thus, there is an invariant, symmetric, non-degenerate, bilinear form on \mathfrak{h}^* induced by $(\cdot|\cdot)$. We use the same notation for the bilinear form $(\cdot|\cdot)$ on \mathfrak{h} and the induced bilinear form on \mathfrak{h}^* . Notice,

$(\alpha_i | \alpha_j) = b_{i,j}$ for $i, j \in I$.

Every Kac Moody algebra, $\mathfrak{g}(A)$, has an associated group of reflections known as the Weyl group. For each $i \in I$, define the *simple reflection* r_i on \mathfrak{h}^* by $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$. The *Weyl group*, W , is the subgroup of $\text{End}(\mathfrak{h}^*)$ generated by all simple reflections. Any element $\omega \in W$ may be expressed as a product of simple reflections, i.e. $\omega = \prod_{k=1}^t r_{i_k}$. If t is minimal amongst all such expressions we say ω is a *reduced expression* and call t the *length of ω* (denoted $l(\omega)$).

For each $\alpha \in \mathfrak{h}^*$, we define the α -root space of \mathfrak{g} as follows.

Definition 2.5. Let \mathfrak{g} be a Kac-Moody algebra with Cartan subalgebra \mathfrak{h} . For $\alpha \in \mathfrak{h}^*$, define

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, we say α is a root of \mathfrak{g} , \mathfrak{g}_α is the α root space, and $\dim(\mathfrak{g}_\alpha)$ is the multiplicity of the root α .

Note that $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$ and $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$. The Kac-Moody algebra \mathfrak{g} has the root space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha.$$

All roots are either positive (i.e. $\alpha \in Q_+$) or negative (i.e. $\alpha \in Q_- = -Q_+$). Let Δ , Δ_+ , and Δ_- represent the set of roots, positive roots, and negative roots respectively. Then, $\mathfrak{g}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}$ (resp. $\mathfrak{g}_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}$) and we have the following triangular

decomposition

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+.$$

The *Chevalley involution*, $\zeta : \mathfrak{g} \rightarrow \mathfrak{g}$ given by

$$\zeta(e_i) = -f_i, \zeta(f_i) = -e_i, \zeta(h) = -h \quad \text{for } h \in \mathfrak{h}$$

is an automorphism. Since $\zeta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, we conclude $\text{mult}(\alpha) = \text{mult}(-\alpha)$ for all $\alpha \in \Delta$.

If α is a root of a Kac-Moody algebra, $\mathfrak{g}(A)$, of finite type the following statements are true: there exists an $\omega \in W$ such that $\omega(\alpha) = \alpha_i$ for some $i \in I$, $\text{mult}(\alpha) = 1$, and $k\alpha$ is a root if and only if $k = \pm 1$. These properties do not hold for all roots of Kac-Moody algebras of affine or indefinite types. Let α be a root of a Kac-Moody algebra, \mathfrak{g} . If α is ω -conjugate to a simple root we say α is a real root. Otherwise, we say α is an imaginary root. The properties of imaginary roots differ drastically from those of their real counterparts. For example, if α is an imaginary root of an affine Lie algebra \mathfrak{g} then $k\alpha$ is also an imaginary root of \mathfrak{g} for all $k \in \mathbb{Z}$. In addition, imaginary roots have multiplicities greater than one.

The λ -weight space of the \mathfrak{g} -module V is defined in the same manner as the α -root space of the Lie algebra \mathfrak{g} .

Definition 2.6. For any $\lambda \in \mathfrak{h}^*$ the λ weight space, V_λ , is defined as

$$V_\lambda = \{v \in V \mid h \cdot v = \lambda(h) \cdot v \text{ for all } h \in \mathfrak{h}\}.$$

If $V_\lambda \neq 0$, we call λ a weight of V and $\dim(V_\lambda)$ the weight multiplicity of λ in V (denoted $\text{mult}_V(\lambda)$).

A module, V , is called a *weight module* if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$. When all weight spaces of a weight module V are finite dimensional, we define the *character* of V as follows.

$$\text{ch}V = \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda) e^\lambda.$$

Let $\text{wt}(V)$ be the set of all weights of the module V . The *category* \mathcal{O} has as objects weight modules, V , over \mathfrak{g} with finite dimensional weight spaces for which there exists a finite number of elements $\lambda_1, \dots, \lambda_s \in \mathfrak{h}^*$ such that

$$\text{wt}(V) \subseteq \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda_1\} \cup \dots \cup \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda_s\}$$

and as morphisms \mathfrak{g} -module homomorphisms.

Important examples of \mathfrak{g} -modules in the category \mathcal{O} are highest weight modules.

Definition 2.7. Let V be a weight module of $\mathfrak{g}(A)$. V is a *highest weight module* if

there exists a $\lambda \in \mathfrak{h}^*$ and a $v_\lambda \in V$, $v_\lambda \neq 0$, such that

$$e_i \cdot v_\lambda = 0 \quad \text{for all } i \in I$$

$$h \cdot v_\lambda = \lambda(h)v_\lambda \quad \text{for all } h \in \mathfrak{h}$$

$$V = U(\mathfrak{g})v_\lambda$$

In such a case, we call v_λ a highest weight vector and λ the highest weight of V .

Example 2.2. Let $\mathfrak{g} = sl(2, \mathbb{C})$ the Lie algebra described in example 2.1. Let $V(\lambda)$ be a finite dimensional irreducible highest weight module of $sl(2, \mathbb{C})$ with highest weight $\lambda \in \mathbb{C}$, and highest weight vector v . Then, $\lambda \in \mathbb{Z}$ and $V(\lambda)$ is $(\lambda + 1)$ dimensional with basis $\{v_k\}_{k=0..\lambda}$, where

$$v_k = \frac{f^k(v)}{k!}.$$

Again, let \mathfrak{g} be any Kac-Moody algebra. Let V be a highest weight \mathfrak{g} -module with highest weight λ . Then $V = \bigoplus_{\mu \leq \lambda} V_\mu$ where each V_μ is finite dimensional and $\dim(V_\lambda) = 1$. For every λ in \mathfrak{h}^* there exists a unique (up to isomorphism) type of highest weight module called a Verma module. A \mathfrak{g} -module $M(\lambda)$ with highest weight λ is a *Verma module* if every other \mathfrak{g} -module with highest weight λ is a quotient of $M(\lambda)$. Not only is $M(\lambda)$ unique but it also contains a unique proper maximal submodule, $M'(\lambda)$. Let $L(\lambda) = M(\lambda)/M'(\lambda)$. Then $L(\lambda)$ is an irreducible

\mathfrak{g} -module with highest weight λ . In fact, every irreducible \mathfrak{g} -module in the category \mathcal{O} is isomorphic to $L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.

We say V is an integrable $\mathfrak{g}(A)$ -module if all e_i and f_i ($i \in I$) are locally nilpotent on V . A weight, λ , of V is called integral if $\lambda(h_i) \in \mathbb{Z}$ for $i \in I$. The set of all integral weights is known as the weight lattice, denoted by P . Define the following subsets of P , the set of all dominant integral weights and the set of all regular dominant integral weights respectively.

$$P_+ = \{\lambda \in P \mid \lambda(h_i) \geq 0 \text{ for all } i \in I\}$$

$$P_{++} = \{\lambda \in P \mid (\lambda, h_i) > 0 \text{ for all } i \in I\}$$

The Category \mathcal{O}_{int} consists of integrable \mathfrak{g} -modules in the category \mathcal{O} such that all weights of V are integral weights. Every \mathfrak{g} -module in the category \mathcal{O}_{int} is completely reducible and every irreducible \mathfrak{g} -module in \mathcal{O}_{int} is isomorphic to a highest weight module $L(\lambda)$ with $\lambda \in P_+$. We end our discussion of the category \mathcal{O}_{int} with the following information concerning weight multiplicities and the action of the Weyl group for modules in this category. Let $V \in \mathcal{O}_{int}$, $\lambda \in wt(V)$, and $\omega \in W$. Then,

$$\dim(V_\lambda) = \dim(V_{\omega\lambda}).$$

We will also deal with the quantized universal enveloping algebra of a Kac-Moody algebra, \mathfrak{g} , which we will denote $U_q(\mathfrak{g})$. Before defining the quantized universal en-

veloping algebra, $U_q(\mathfrak{g})$, we introduce the following notations: $t_i = q^{(\alpha_i|\alpha_i)h_i} = q_i^{h_i}$, $b = 1 - \alpha_j(h_i)$, $[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}$, $e_i^{(k)} = \frac{e_i^k}{[k]_i!}$, $f_i^{(k)} = \frac{f_i^k}{[k]_i!}$, and $[k]_i! = \prod_{m=1}^k [m]_i$.

Definition 2.8. $U_q(\mathfrak{g})$ is the quantized universal enveloping algebra over $\mathbb{C}(q)$ associated with P , generated by the set $\{e_i, f_i, q^h | i \in I, h \in P^*\}$, and satisfying the following relations.

$$\begin{aligned} q^0 &= 1, q^h q^{h'} = q^{h+h'} \\ q^h e_i q^{-h} &= q^{\alpha_i(h)} e_i \\ q^h f_i q^{-h} &= q^{-\alpha_i(h)} f_i \\ [e_i, f_j] &= \delta_{ij} \left(\frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \right) \\ \sum_{k=0}^b (-1)^k e_i^{(k)} e_j e_i^{(b-k)} &= 0 \quad \text{for } i \neq j \\ \sum_{k=0}^b (-1)^k f_i^{(k)} f_j f_i^{(b-k)} &= 0 \quad \text{for } i \neq j. \end{aligned}$$

Note,

$$\lim_{q \rightarrow 1} U_q(\mathfrak{g}) = U(\mathfrak{g}).$$

Example 2.3. We have considered the Lie algebra $sl(2, \mathbb{C})$; now we will look at the quantized universal enveloping algebra of $sl(2, \mathbb{C})$, $U_q(sl(2, \mathbb{C}))$. $U_q(sl(2, \mathbb{C}))$ is the associative algebra over $\mathbb{C}(q)$ with generators $\{e, f, t^{\pm 1}\}$ and relations

$$tet^{-1} = q^2 e, \quad tft^{-1} = q^{-2} f, \quad [e, f] = \frac{t - t^{-1}}{q - q^{-1}}.$$

We have seen that $\mathfrak{g}_{(i)} = \mathbb{C}e_i + \mathbb{C}h_i + \mathbb{C}f_i$ is isomorphic to $sl(2, \mathbb{C})$. It follows that the quantized universal enveloping algebra $U_q(\mathfrak{g}_{(i)})$ is isomorphic to $U_q(sl(2, \mathbb{C}))$.

The representation theory of quantized universal enveloping algebras is essentially parallel to that of Kac-Moody algebras. For more details see [9, 28]. We mention the following fact which will be important for us in subsequent chapters.

Theorem 2.2 (see, for example, [9]). *If $\lambda \in P_+$ and $L^q(\lambda)$ is the irreducible highest weight $U_q(\mathfrak{g})$ -module with highest weight λ , then $L^1(\lambda)$ is isomorphic to the irreducible highest weight module $L(\lambda)$ over $U(\mathfrak{g})$ with highest weight λ . Hence, the character of $L^q(\lambda)$ over $U_q(\mathfrak{g})$ is the same as the character of $L(\lambda)$ over $U(\mathfrak{g})$.*

Chapter 3

Construction of $HC_n^{(1)}$

In the following section, we review the construction described in [6, 14, 1] and apply this construction to the indefinite Kac-Moody algebra $HC_n^{(1)}$. Let $I = \{-1, 0, 1, \dots, n\}$ then $HC_n^{(1)}$ is the Kac-Moody algebra associated with the generalized Cartan matrix

$$A = (a_{ij})_{i,j \in I} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 & 0 \\ & & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix} \quad (3.1)$$

We begin by considering the Lie algebra $\mathfrak{g}_0 = C_n^{(1)}$. We construct two \mathfrak{g}_0 -modules, V and V^* , and a \mathfrak{g}_0 -module homomorphism $\psi : V^* \otimes V \rightarrow \mathfrak{g}_0$. With these four ingredients, we build a graded Lie algebra, \mathfrak{g} , which intersects its local part $V \oplus \mathfrak{g}_0 \oplus V^*$ trivially. Finally, we show $\mathfrak{g} \cong HC_n^{(1)}$.

The algebra $\mathfrak{g}_0 = C_n^{(1)}$ is the affine Kac-Moody algebra $\mathfrak{g}_0(A_0)$ associated with the generalized Cartan matrix $A_0 = (a_{ij})_{i,j \in S}$ where $S = \{0, 1, \dots, n\}$. A_0 has the realization $\{\mathfrak{h}_0, \Pi = \{\alpha_0, \alpha_1, \dots, \alpha_n\}, \check{\Pi} = \{h_0, h_1, \dots, h_n\}\}$. The center of \mathfrak{g}_0 is one dimensional and is spanned by $K = h_0 + h_1 + \dots + h_n \in \mathfrak{h}_0$. Let $d \in \mathfrak{h}$ be such that

$$\langle \alpha_0, d \rangle = 1, \text{ and } \langle \alpha_i, d \rangle = 0 \text{ for } i = 1, \dots, n$$

Then, $\{d, h_0, h_1, \dots, h_n\}$ form a basis for \mathfrak{h}_0 . The algebra \mathfrak{g}_0 is generated by the elements $\{e_i\}_{i=0}^n$, $\{f_i\}_{i=0}^n$, and the Cartan subalgebra, \mathfrak{h}_0 .

Let $\Lambda_0 \in \mathfrak{h}^*$ be such that

$$\langle \Lambda_0, d \rangle = 0, \text{ and } \langle \Lambda_0, h_i \rangle = \delta_{0,i} \text{ for } i = 0, 1, \dots, n \quad (3.2)$$

and let $V = V(\Lambda_0)$ be the irreducible highest weight module over \mathfrak{g}_0 with highest weight Λ_0 . Let v_0 represent the highest weight vector of weight Λ_0 in $V(\Lambda_0)$. Then

we have the following relations for $i = 0, 1, \dots, n$.

$$\begin{aligned} h_i \cdot v_0 &= \delta_{0,i} \cdot v_0 \\ d \cdot v_0 &= 0 \\ e_i \cdot v_0 &= 0 \end{aligned} \tag{3.3}$$

Using the module action given below, the restricted dual $V^* = V(\Lambda_0)^*$ is an irreducible lowest weight \mathfrak{g}_0 -module with lowest weight Λ_0 .

$$\langle g \cdot v^*, v \rangle = -\langle v^*, g \cdot v \rangle \quad \forall v^* \in V(\Lambda_0)^*, \quad v \in V(\Lambda_0), \quad g \in \mathfrak{g}_0. \tag{3.4}$$

Let v_0^* represent the lowest weight vector of $V(\Lambda_0)^*$. The following relations hold for $i = 0, 1, \dots, n$.

$$\begin{aligned} h_i \cdot v_0^* &= -\delta_{0,i} \cdot v_0^* \\ d \cdot v_0^* &= 0 \\ f_i \cdot v_0^* &= 0 \end{aligned} \tag{3.5}$$

The last ingredient we need for our construction is a homomorphism from $V^* \otimes V$ to \mathfrak{g}_0 . This homomorphism makes use of the standard invariant, nondegenerate, symmetric, bilinear form, $(\cdot | \cdot)_{\mathfrak{h}_0}$, given in (2.2). For reference, we describe $(\cdot | \cdot)_{\mathfrak{h}_0}$ explicitly below.

$$\begin{aligned}
 (h_i|h_j) &= a_j a_{ij} \quad (i, j = 0, 1, \dots, n) \\
 a_0 &= 1, a_1 = 2, \dots, a_{n-1} = 2, a_n = 1 \\
 (h_i|d) &= 0 \quad (i = 1, \dots, n) \\
 (h_0|d) &= 1 \\
 (d|d) &= 0.
 \end{aligned} \tag{3.6}$$

As mentioned previously, the standard bilinear form on \mathfrak{h}_0 may be uniquely extended to an invariant, nondegenerate, symmetric, bilinear form on all of \mathfrak{g}_0 , which we will denote $(\cdot|\cdot)_{\mathfrak{g}_0}$.

We define the map $\psi : V^* \otimes V \rightarrow \mathfrak{g}_0$ by

$$\psi(v^* \otimes v) = - \sum_{i \in \mathcal{I}} \langle v^*, x_i \cdot v \rangle x_i - 2 \langle v^*, v \rangle K, \tag{3.7}$$

where $\{x_i | i \in \mathcal{I}\}$ is a basis for \mathfrak{g}_0 which is orthonormal with respect to the bilinear form $(\cdot|\cdot)_{\mathfrak{g}_0}$. We claim ψ is a \mathfrak{g}_0 -module homomorphism. In proving our claim, we make use of the following fact.

$$\text{If } [x_m, x_n] = \sum_{t \in \mathcal{I}} c_{m,n}^t x_t \text{ then } c_{m,n}^k = c_{n,k}^m. \tag{3.8}$$

To see this consider,

$$\begin{aligned}
 ([x_m, x_n] | x_k) &= \left(\sum_{t \in I} c_{m,n}^t x_t \mid x_k \right) \\
 &= \sum_{t \in \mathcal{I}} c_{m,n}^t (x_t | x_k) \\
 &= c_{m,n}^k
 \end{aligned}$$

and

$$\begin{aligned}
 (x_m | [x_n, x_k]) &= \left(x_m \mid \sum_{t \in \mathcal{I}} c_{n,k}^t x_t \right) \\
 &= \sum_{t \in \mathcal{I}} c_{n,k}^t (x_m | x_t) \\
 &= c_{n,k}^m.
 \end{aligned}$$

The invariance of the form $(\cdot | \cdot)_{\mathfrak{g}_0}$ implies $([x_m, x_n] | x_k) = (x_m | [x_n, x_k])$. So that, $c_{m,n}^k$ must equal $c_{n,k}^m$.

Recall, $\{x_i\}_{i \in \mathcal{I}}$ is an orthonormal basis for \mathfrak{g}_0 . Then, $[x_m, x_n] = \sum_{t \in \mathcal{I}} c_{m,n}^t x_t$ for some $c_{m,n}^t$ in \mathbb{C} . For any $x_j \in \{x_i\}_{i \in \mathcal{I}}$,

$$\begin{aligned}
 \psi(x_j \cdot (v^* \otimes v)) &= \psi(x_j \cdot v^* \otimes v) + \psi(v^* \otimes x_j \cdot v) \\
 &= - \sum_{i \in \mathcal{I}} \langle x_j \cdot v^*, x_i \cdot v \rangle x_i - 2 \langle x_j \cdot v^*, v \rangle K \\
 &\quad - \sum_{i \in \mathcal{I}} \langle v^*, x_i \cdot x_j \cdot v \rangle x_i - 2 \langle v^*, x_j \cdot v \rangle K
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{i \in \mathcal{I}} \langle v^*, x_j \cdot x_i \cdot v \rangle x_i + 2 \langle v^*, x_j \cdot v \rangle K \\
&\quad - \sum_{i \in \mathcal{I}} \langle v^*, x_i \cdot x_j \cdot v \rangle x_i - 2 \langle v^*, x_j \cdot v \rangle K \quad (\text{by (3.4)}) \\
&= \sum_{i \in \mathcal{I}} \langle v^*, x_j \cdot x_i \cdot v - x_i \cdot x_j \cdot v \rangle x_i \\
&= \sum_{i \in \mathcal{I}} \langle v^*, [x_j, x_i] \cdot v \rangle x_i \\
&= \sum_{i \in \mathcal{I}} \left\langle v^*, \sum_{k \in \mathcal{I}} c_{j,i}^k x_k \cdot v \right\rangle x_i \\
&= \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{I}} \langle v^*, x_k \cdot v \rangle c_{j,i}^k x_i \\
&= \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{I}} \langle v^*, x_i \cdot v \rangle c_{j,k}^i x_k \\
&= \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{I}} \langle v^*, x_i \cdot v \rangle c_{i,j}^k x_k \quad (\text{by (3.8)})
\end{aligned}$$

while

$$\begin{aligned}
x_j \cdot \psi(v^* \otimes v) &= - \sum_{i \in \mathcal{I}} \langle v^*, x_i \cdot v \rangle x_j \cdot x_i - 2 \langle v^*, v \rangle x_j \cdot K \\
&= - \sum_{i \in \mathcal{I}} \langle v^*, x_i \cdot v \rangle [x_j, x_i], \text{ since } [x_j, K] = 0 \\
&= \sum_{i \in \mathcal{I}} \langle v^*, x_i \cdot v \rangle [x_i, x_j] \\
&= \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{I}} \langle v^*, x_i \cdot v \rangle c_{i,j}^k x_k.
\end{aligned}$$

Thus, $\psi(x_j \cdot (v^* \otimes v)) = x_j \cdot \psi(v^* \otimes v)$ for all x_j , making ψ a \mathfrak{g}_0 -module homomorphism.

Now, we begin our construction of \mathfrak{g} . The space $V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V(\Lambda_0)^*$ has a local Lie algebra structure with the bracket defined below.

$$[v^*, v] = \psi(v^* \otimes v) \quad (3.9)$$

$$[g, v] = g \cdot v \quad (3.10)$$

$$[g, v^*] = g \cdot v^* \quad (3.11)$$

for all $v \in V(\Lambda_0)$, $v^* \in V(\Lambda_0)^*$, and $g \in \mathfrak{g}_0$. For $j \geq 1$ define $\hat{\mathfrak{g}}_{-j}$ to be the space spanned by all products of j vectors from $V(\Lambda_0)$, and define $\hat{\mathfrak{g}}_j$ to be the space spanned by all products of j vectors from $V(\Lambda_0)^*$. Then,

$$\hat{\mathfrak{g}}_{-1} = V(\Lambda_0)$$

$$\hat{\mathfrak{g}}_0 = \mathfrak{g}_0$$

$$\hat{\mathfrak{g}}_1 = V(\Lambda_0)^*.$$

and $\hat{\mathfrak{g}} = \bigoplus_{j \in \mathbb{Z}} \hat{\mathfrak{g}}_j$ is the maximal graded Lie algebra with local part

$V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V(\Lambda_0)^*$. Notice, $\hat{\mathfrak{g}}_- = \bigoplus_{j \geq 1} \hat{\mathfrak{g}}_{-j}$ and $\hat{\mathfrak{g}}_+ = \bigoplus_{j \geq 1} \hat{\mathfrak{g}}_{+j}$ are the free lie algebras generated by $V(\Lambda_0)$ and $V(\Lambda_0)^*$ respectively. For all $k > 1$ define the subspaces

$$J_k = \{ x \in \hat{\mathfrak{g}}_k \mid [y_1, [\dots [y_{k-1}, x]] \dots] = 0 \quad \forall y_1, \dots, y_{k-1} \in V(\Lambda_0) \}$$

$$J_{-k} = \{ x \in \hat{\mathfrak{g}}_{-k} \mid [y_1, [\dots [y_{k-1}, x]] \dots] = 0 \quad \forall y_1, \dots, y_{k-1} \in V(\Lambda_0)^* \}$$

$$\begin{aligned} J_{\pm} &= \sum_{k>1} J_{\pm k} \\ J &= J_- \oplus J_+. \end{aligned}$$

J_{\pm} are ideals of $\hat{\mathfrak{g}}$ and J is the largest graded ideal of $\hat{\mathfrak{g}}$ which intersects the local part of \hat{g} trivially.

Finally, we define

$$\begin{aligned} \mathfrak{g} &= \hat{\mathfrak{g}}/J \\ &= \left(\bigoplus_{i>1} \mathfrak{g}_{-i} \right) \oplus V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V(\Lambda_0)^* \oplus \left(\bigoplus_{i>1} \mathfrak{g}_i \right) \end{aligned} \quad (3.12)$$

where $\mathfrak{g}_{\pm i} = \hat{\mathfrak{g}}_{\pm i}/J_{\pm i}$ for $i > 1$

which is the minimal graded Lie algebra with local part $V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V(\Lambda_0)^*$. Proposition (3.1) below shows that \mathfrak{g} is isomorphic to $HC_n^{(1)}$. In our construction, each subspace \mathfrak{g}_{-j} (respectively \mathfrak{g}_{+j}) is a direct sum of irreducible highest weight (respectively lowest weight) modules over \mathfrak{g}_0 . This fact allows us to view the root spaces of $HC_n^{(1)} \cong \mathfrak{g}$ as weight spaces of \mathfrak{g}_0 -modules. Therefore, we may use the representation theory of the affine Kac-Moody algebra $\mathfrak{g}_0 = C_n^{(1)}$ to find the root multiplicities of the indefinite Kac-Moody algebra $HC_n^{(1)}$.

Proposition 3.1. *Consider the map ϕ from $HC_n^{(1)}$ to \mathfrak{g} given by*

$$\begin{aligned} E_{-1} &\rightarrow v_0^* & F_{-1} &\rightarrow v_0 & H_{-1} &\rightarrow -2K - d \\ E_i &\rightarrow e_i & F_i &\rightarrow f_i & H_i &\rightarrow h_i \end{aligned} \quad (i = 0, 1, \dots, n)$$

where $\{E_i\}_{i=-1}^n, \{F_i\}_{i=-1}^n$, and $\mathfrak{h} = \text{span} \{H_i\}_{i=-1}^n$ are the Chevalley generators and Cartan subalgebra of $HC_n^{(1)}$ respectively. The map $\phi : HC_n^{(1)} \rightarrow \mathfrak{g}$ is an isomorphism.

Proof:

Recall, $HC_n^{(1)} = \mathfrak{g}(A)$ where A is given in (3.1). $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\xi$, where ξ is the unique maximal ideal of $\tilde{\mathfrak{g}}(A)$ which intersects \mathfrak{h} trivially, and $\tilde{\mathfrak{g}}(A)$ is the Lie algebra on generators $\{E_i, F_i\}_{i=-1}^n$ and \mathfrak{h} with defining relations given in (2.1). Since J is the maximal graded ideal of $\hat{\mathfrak{g}}$ which intersects the local part of $\hat{\mathfrak{g}}$ trivially and ϕ is clearly linear and bijective, we need only show the following.

$$[\phi(E_i), \phi(F_i)] = \delta_{ij} \phi(H_i) \quad (i, j = -1, 0, 1, \dots, n)$$

$$[\phi(H), \phi(H')] = 0 \quad (H, H' \in \mathfrak{h})$$

$$[\phi(H), \phi(E_i)] = \langle \alpha_i, H \rangle \phi(E_i) \quad (i = -1, 0, 1, \dots, n; H \in \mathfrak{h})$$

$$[\phi(H), \phi(F_i)] = -\langle \alpha_i, H \rangle \phi(F_i) \quad (i = -1, 0, 1, \dots, n; H \in \mathfrak{h}).$$

Notice for $i, j = 0, 1, \dots, n$ the above statements follow directly from the following two facts:

1. $\phi(E_i) = e_i \quad \phi(F_i) = f_i \quad \phi(H_i) = h_i \quad (i = 0, 1, \dots, n).$
2. The generalized Cartan matrix for $\mathfrak{g}_0 = C_n^{(1)}$ is identical to the generalized Cartan matrix for $HC_n^{(1)}$ with column (-1) and row (-1) removed.

We have reduced our task to showing the following relations.

$$[\phi(E_{-1}), \phi(F_{-1})] = \phi(H_{-1}) \quad (3.13)$$

$$[\phi(E_{-1}), \phi(F_i)] = 0 \quad (i = 0, 1, \dots, n) \quad (3.14)$$

$$[\phi(E_i), \phi(F_{-1})] = 0 \quad (i = 0, 1, \dots, n) \quad (3.15)$$

$$[\phi(H), \phi(H')] = 0 \quad (H, H' \in \mathfrak{h}) \quad (3.16)$$

$$[\phi(H), \phi(E_{-1})] = \langle \alpha_{-1}, H \rangle \phi(E_{-1}) \quad (H \in \mathfrak{h}) \quad (3.17)$$

$$[\phi(H), \phi(F_{-1})] = \langle \alpha_{-1}, H \rangle \phi(F_{-1}) \quad (H \in \mathfrak{h}) \quad (3.18)$$

To show (3.13):

$\phi(E_{-1}) = v_0^*$ and $\phi(F_{-1}) = v_0$. Therefore we must use the homomorphism given in (3.7) to compute their bracket. Let $y_{-1} = \frac{1}{\sqrt{2}}(K + d)$ and $y_0 = \frac{1}{\sqrt{-2}}(K - d)$. Also, define $\{y_i\}_{i=1}^n$ as the orthonormal basis for the space $\mathcal{H} = \text{span}\{h_1, \dots, h_n\}$. We know such a basis exists because the bilinear form given in (3.6), restricted to \mathcal{H} , is an inner product. Then, the set $\{y_i\}_{i=-1,0,1,\dots,n}$ forms an orthonormal basis for \mathfrak{h}_0 . Let $\{x_i | i \in \mathcal{I}\}$ be the orthonormal basis for g_0 ordered in such a way that $x_i = y_i$ for $i = -1, 0, 1, \dots, n$. Then we have

$$\begin{aligned}
[\phi(E_{-1}), \phi(F_{-1})] &= \psi(v_0^* \otimes v_0) \\
&= \sum_{i \in \mathcal{I}} \langle v_0^*, x_i \cdot v_0 \rangle x_i - 2 \langle v^*, v \rangle K \\
&= - \left\langle v_0^*, \frac{1}{\sqrt{2}}(K+d) \cdot v_0 \right\rangle \left(\frac{1}{\sqrt{2}}(K+d) \right) \\
&\quad - \left\langle v_0^*, \frac{1}{\sqrt{-2}}(K-d) \cdot v_0 \right\rangle \left(\frac{1}{\sqrt{-2}}(K-d) \right) \\
&\quad - \sum_{i=1}^n \langle v_0^*, x_i \cdot v_0 \rangle x_i \\
&\quad - 2 \langle v_0^*, v_0 \rangle K
\end{aligned}$$

Let $i \in \{1, 2, \dots, n\}$, and suppose $\langle v_0^*, x_i \cdot v_0 \rangle \neq 0$. Then $x_i \cdot v_0$ must be some non-zero multiple of v_0 . Thus $x_i \in \text{span}\{h_i\}_{i=1,2,\dots,n}$. But, (3.3) shows that $h_i \cdot v_0 = 0$ for $i = 1, 2, \dots, n$, a contradiction. Therefore $\langle v_0^*, x_i \cdot v_0 \rangle = 0$ for $i = 1, 2, \dots, n$. Thus,

$$\begin{aligned}
[\phi(E_{-1}), \phi(F_{-1})] &= - \left\langle v_0^*, \frac{1}{\sqrt{2}}(K+d) \cdot v_0 \right\rangle \left(\frac{1}{\sqrt{2}}(K+d) \right) \\
&\quad - \left\langle v_0^*, \frac{1}{\sqrt{-2}}(K-d) \cdot v_0 \right\rangle \left(\frac{1}{\sqrt{-2}}(K-d) \right) \\
&\quad - 2 \langle v_0^*, v_0 \rangle K \\
&= -\frac{1}{2}(K+d) - \frac{1}{-2}(K-d) - 2K \quad (\text{here we use (3.3)}) \\
&= -2K - d \\
&= \phi(H_{-1})
\end{aligned}$$

To show (3.14) and (3.15):

Recall, v_0^* is the lowest weight vector of $V(\Lambda_0)^*$, and thus we have the relations give in (3.5). Therefore,

$$\begin{aligned} [\phi(E_i), \phi(F_{-1})] &= [v_0^*, f_i] \quad (i = 0, 1, \dots, n) \\ &= -f_i \cdot v_0 \\ &= 0. \end{aligned}$$

This shows (3.14). Similarly (3.15) follows from (3.3).

To show (3.16) :

Let, H and H' be in \mathfrak{h} . Notice, $\phi(H_{-1}) = -2h_0 - 2h_1 - \dots - 2h_{n-1} - 2h_n - d \in \mathfrak{h}_0$. Clearly $\phi(H_i) \in \mathfrak{h}_0$ for $i = 0, 1, \dots, n$. Since H and H' are both linear combinations of $\{H_{-1}, H_0, H_1, \dots, H_n\}$, $\phi(H)$ and $\phi(H')$ must both be elements of \mathfrak{h}_0 . Therefore,

$$[\phi(H), \phi(H')] = 0$$

To show (3.17) and (3.18):

Let $H = c_{-1}H_{-1} + c_0H_0 + c_1H_1 + \dots + c_nH_n \in \mathfrak{h}$ where $c_{-1}, c_0, c_1, \dots, c_n \in \mathbb{C}$. Then,

$$\begin{aligned} \langle \alpha_{-1}, H \rangle \phi(E_{-1}) &= \langle \alpha_{-1}, H \rangle v_0^* \\ &= (2c_{-1} - c_0)v_0^* \end{aligned}$$

and

$$\begin{aligned}
[\phi(H), \phi(E_{-1})] &= [\phi(H), v_0^*] \\
&= [c_{-1}(-2h_0 - 2h_1 - \cdots - 2h_n - d), v_0^*] + [c_0 h_0, v_0^*] + \\
&\quad c_1 [h_1, v_0^*] + \cdots + c_n [h_n, v_0^*] \\
&= 2c_{-1}v_0^* - c_0v_0^* \quad (\text{here we use (3.5)}) \\
&= (2c_{-1} - c_0)v_0^*
\end{aligned}$$

This shows (3.17). The proof of (3.18) parallels the proof of (3.17), replacing $\phi(E_{-1}) = v_0^*$ with $\phi(F_{-1}) = v_0$, and replacing the use of (3.5) with the use of (3.3).

We have now shown all relations (3.13) through (3.18) hold, proving our assertion.

We will now identify E_i, F_i, H_i with e_i, f_i, h_i respectively.

□

Chapter 4

Multiplicity formula

The purpose of the current chapter is to recall a multiplicity formula developed by Kang which we will need later in determining root multiplicities of $HC_n^{(1)}$. For more details about this material see [18, 20, 19, 21]. We will make use of ideas from homological algebra and will begin by recalling some pertinent definitions and theorems in this subject.

Definition 4.1. *A chain complex of \mathfrak{g} -modules is a family $\{C_n\}_{n \in \mathbb{Z}}$ of \mathfrak{g} -modules together with \mathfrak{g} -module maps $d_n : C_n \rightarrow C_{n-1}$ such that $d_n \circ d_{n+1} \equiv 0$. We call the maps d_n differentials. A chain complex, \mathcal{C} , of \mathfrak{g} -modules is admissible if $\cup \{C_n\}_{n \in \mathbb{Z}}$ is itself a \mathfrak{g} -module.*

Definition 4.2. *Let $\{C_n\}_{n \in \mathbb{Z}}$ be a chain complex of \mathfrak{g} -modules with differentials d_n .*

The n^{th} homology module of $\{C_n\}_{n \in \mathbb{Z}}$ is given by.

$$H_n(C) = \ker(d_n) / \text{im}(d_{n+1})$$

Theorem 4.1. (*Euler-Poincare Principle*) Let $\{C_n\}_{n \in \mathbb{Z}}$ be an admissible chain complex of \mathfrak{g} -modules. Then

$$\sum_{k \in \mathbb{Z}_+} (-1)^k \text{ch}(C_k) = \sum_{k \in \mathbb{Z}_+} (-1)^k \text{ch}(H_k(C))$$

Before we move on, let us fix our notation.

$I = \{-1, \dots, n\}$: the index set of the simple roots for $\mathfrak{g} = HC_n^{(1)}$

Δ : the set of roots of \mathfrak{g}

Δ_{\pm} : the set of positive (resp. negative) roots of \mathfrak{g}

W : the Weyl group of \mathfrak{g}

P : the weight lattice of \mathfrak{g}

$\rho \in P$: the element of P satisfying $\rho(h_i) = 1$ for $i = -1, \dots, n$

$L(\lambda)$: the highest weight \mathfrak{g} -module with highest weight λ

$S = \{0, \dots, n\}$: the index set of simple roots of $\mathfrak{g}_0 = C_n^{(1)}$

Δ_S : the set of roots of \mathfrak{g}_0

$\Delta_{S,\pm}$: the set of positive (resp. negative) roots of \mathfrak{g}_0

W_S : the Weyl group of \mathfrak{g}_0

P_S : the weight lattice of \mathfrak{g}_0

$L_0(\lambda)$: the highest weight \mathfrak{g}_0 -module with highest weight λ

$\Delta_{\pm}(S)$: $\Delta_{\pm} \setminus \Delta_{S,\pm}$

$W(S)$: $\{w \in W \mid w\Delta_- \cap \Delta_+ \subseteq \Delta_+(S)\}$

Theorem 4.2. (*Kostant's Formula for Kac-Moody algebras*) [8, 29] Let $\lambda \in P$. Then

$$\cdots \rightarrow \bigwedge^j \mathfrak{g}_- \otimes L(\lambda) \xrightarrow{d_j} \bigwedge^j \mathfrak{g}_- \otimes L(\lambda) \rightarrow \cdots \rightarrow \bigwedge^0 \mathfrak{g}_- \otimes L(\lambda) \xrightarrow{d_0} L(\lambda) \rightarrow 0 \rightarrow \cdots$$

with,

$$\begin{aligned} d_j((c_1 \wedge \cdots \wedge c_j) \otimes v) &= \sum_{i=1}^j (-1)^i (c_1 \wedge \cdots \wedge \hat{c}_i \wedge \cdots \wedge c_j) \otimes c_i v \\ &\quad + \sum_{r < t} (-1)^{r+t} [c_r, c_t] \wedge c_1 \wedge \cdots \wedge \hat{c}_r \wedge \cdots \wedge \hat{c}_t \wedge \cdots \wedge c_j \otimes v \\ &\text{for } k \geq 1, \text{ and } d_k = 0 \text{ for } k \leq 0. \end{aligned}$$

is a \mathfrak{g}_0 -module complex. In addition, the homology modules, $H_k(\mathfrak{g}, L(\lambda)) = \ker(d_n) /$

$im(d_{n+1})$, of this complex are \mathfrak{g}_0 -modules and

$$H_k(\mathfrak{g}_-, L(\lambda)) \cong \sum_{\substack{w \in W(S) \\ l(w)=k}} L_0(w(\lambda + \rho) - \rho)$$

Let \mathbb{C} be the trivial \mathfrak{g} -module. We know by theorem 4.2 that

$$\cdots \bigwedge^j \mathfrak{g}_- \xrightarrow{d_j} \bigwedge^{j-1} \mathfrak{g}_- \rightarrow \cdots \rightarrow \bigwedge^1 \mathfrak{g}_- \xrightarrow{d_1} \bigwedge^0 \mathfrak{g}_- \xrightarrow{d_0} \mathbb{C} \rightarrow 0 \rightarrow \cdots \quad (4.1)$$

where

$$\begin{aligned} d_j(c_1 \wedge \cdots \wedge c_j) &= \sum_{r < t} (-1)^{r+t} [c_r, c_t] \wedge c_1 \wedge \cdots \wedge \hat{c}_r \wedge \cdots \wedge \hat{c}_t \wedge \cdots \wedge c_k \\ j &\geq 2, \text{ and } d_j = 0 \text{ for } j \leq 1 \end{aligned}$$

is a chain complex. Applying the Euler-Poincare principle to this complex yields the following equality.

$$\sum_{k=0}^{\infty} (-1)^k \text{ch} \bigwedge^k (\mathfrak{g}_-) = \sum_{k=0}^{\infty} (-1)^k \text{ch} H_k(\mathfrak{g}_-) \quad (4.2)$$

First, let us consider the left hand side of this equality. Making use of the Weyl-

Kac denominator identity, we see

$$LHS = \sum_{k=0}^{\infty} (-1)^k \text{ch} \bigwedge^k (\mathfrak{g}_-) = \prod_{\alpha \in \Delta_-(S)} (1 - e(\alpha))^{\dim \mathfrak{g}_\alpha}$$

Now consider the right hand side of equality (4.2).

$$\begin{aligned} \text{RHS} &= \sum_{k=0}^{\infty} (-1)^k \text{ch} H_k(\mathfrak{g}_-) \\ &= (-1)^0 \text{ch} H_0(\mathfrak{g}_-) + \sum_{k=1}^{\infty} (-1)^k \text{ch} H_k(\mathfrak{g}_-) \\ &= 1 - \sum_{k=1}^{\infty} (-1)^{k+1} \text{ch} H_k(\mathfrak{g}_-) \\ &= 1 - \sum_{k=1}^{\infty} (-1)^{k+1} \sum_{\substack{w \in W(S) \\ l(w)=k}} \text{ch} L_0(w\rho - \rho) \quad (\text{Kostant's formula, 4.2}) \\ &= 1 - \sum_{\substack{w \in W(S) \\ l(w) \geq 1}} (-1)^{l(w)+1} \text{ch} L_0(w\rho - \rho) \\ &= 1 - \sum_{\substack{w \in W(S) \\ l(w) \geq 1}} (-1)^{l(w)+1} \sum_{\tau \in \mathfrak{h}_0^*} \dim L_0(w\rho - \rho)_\tau e(\tau) \\ &= 1 - \sum_{\tau \in \mathfrak{h}_0^*} \sum_{\substack{w \in W(S) \\ l(w) \geq 1}} (-1)^{l(w)+1} \dim L_0(w\rho - \rho)_\tau e(\tau) \end{aligned}$$

Let

$$K_\tau = \sum_{\substack{w \in W(S) \\ l(w) \geq 1}} (-1)^{l(w)+1} \dim L_0(w\rho - \rho)_\tau \quad (4.3)$$

Then,

$$RHS = 1 - \sum_{\tau \in \mathfrak{h}_0^*} K_\tau e(\tau)$$

Equating the right and left hand sides of equality 4.2 we arrive at

$$\prod_{\alpha \in \Delta_-(S)} (1 - e(\alpha))^{\dim \mathfrak{g}_\alpha} = 1 - \sum_{\tau \in \mathfrak{h}_0^*} K_\tau e(\tau) \quad (4.4)$$

Taking the log of both sides of 4.4 we obtain

$$\sum_{\alpha \in \Delta_-(S)} \dim \mathfrak{g}_\alpha \log(1 - e(\alpha)) = \log \left(1 - \sum_{\tau \in \mathfrak{h}^*} K_\tau e(\tau) \right) \quad (4.5)$$

Using the formal power series expansion $\log(1 - t) = - \sum_{k=1}^{\infty} \frac{t^k}{k}$ the left hand side of 4.5

becomes

$$\begin{aligned} LHS &= - \sum_{\alpha \in \Delta_-(S)} \dim \mathfrak{g}_\alpha \sum_{k=1}^{\infty} \frac{1}{k} e(k\alpha) \\ &= - \sum_{\alpha \in \Delta_-(S)} \sum_{k=1}^{\infty} \frac{1}{k} \dim \mathfrak{g}_\alpha e(k\alpha) \end{aligned}$$

and the right hand side of 4.5 becomes

$$RHS = - \sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{\tau \in \mathfrak{h}_0^*} K_\tau e(\tau) \right)^k$$

Let $\{\tau_i | i = 1, 2, \dots, \}$ be an enumeration of the elements of \mathfrak{h}_0^* .

$$\begin{aligned}
 RHS &= - \sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{i=1}^{\infty} K_{\tau_i} e(\tau_i) \right)^k \\
 &= - \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{(n)=(n_i) \\ \sum n_i = k}} \frac{(\sum n_i)!}{\prod (n_i!)} \prod K_{\tau_i}^{n_i} e(\sum n_i \tau_i) \quad (\text{multinomial expansion}) \\
 &= - \sum_{\tau \in \mathfrak{h}_0^*} \left(\sum_{\substack{(n)=(n_i) \\ \sum n_i \tau_i = \tau}} \frac{(\sum n_i - 1)!}{\prod (n_i!)} \prod K_{\tau_i}^{n_i} \right) e(\tau)
 \end{aligned}$$

Let $B(\tau) = \sum_{\substack{(n)=(n_i) \\ \sum n_i \tau_i = \tau}} \frac{(\sum n_i - 1)!}{\prod (n_i!)} \prod K_{\tau_i}^{n_i}$. Then,

$$RHS = - \sum_{\tau \in \mathfrak{h}_0^*} B(\tau) e(\tau)$$

Equating the right and left hand sides of 4.5 we see

$$\sum_{\tau \in \mathfrak{h}_0^*} B(\tau) e(\tau) = \sum_{\alpha \in \Delta_-(S)} \sum_{k=1}^{\infty} \frac{1}{k} \dim \mathfrak{g}_{\alpha} e(k\alpha)$$

Therefore,

$$\begin{aligned}
 B(\tau) &= \sum_{\substack{\tau = k\alpha \\ \alpha \in \Delta_-(S)}} \frac{1}{k} \dim \mathfrak{g}_{\alpha} \\
 &= \sum_{\substack{\tau = k\alpha \\ \alpha \in \Delta_-(S)}} \left(\frac{\alpha}{\tau} \right) \dim \mathfrak{g}_{\alpha}
 \end{aligned}$$

Using Mobius inversion, we see for $\alpha \in \Delta_-(S)$,

$$\dim \mathfrak{g}_\alpha = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B(\tau)$$

We summarize these results in the following theorem.

Theorem 4.3. *Let $\mathfrak{g} = HC_n^{(1)}$ and $\mathfrak{g}_0 = C_n^{(1)}$. Let \mathfrak{h} (resp. \mathfrak{h}_0) be the Cartan subalgebra of \mathfrak{g} (resp. \mathfrak{g}_0), and $L(\lambda)$ (resp. $L_0(\lambda)$) be the highest weight \mathfrak{g} -module (resp. \mathfrak{g}_0 -module) with highest weight $\lambda \in \mathfrak{h}$ (resp. $\lambda \in \mathfrak{h}_0$). Let $\alpha \in \Delta_-(S)$, then*

$$\dim(\mathfrak{g}_\alpha) = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B(\tau)$$

where,

$$\begin{aligned} \mu &= \text{the classical Mobius function} \\ B(\tau) &= \sum_{(n_i \tau_i) \in T(\tau)} \frac{(\sum n_i - 1)!}{\prod (n_i!)} \prod K_{\tau_i}^{n_i} \\ T(\tau) &= \left\{ (n_i \tau_i) \mid n_i \in \mathbb{Z}_{\geq 0}, \sum n_i \tau_i = \tau \right\} \\ K_{\tau_i} &= \sum_{\substack{\omega \in W(S) \\ l(\omega) \geq 1}} (-1)^{l(\omega)+1} \dim L_0(\omega \rho - \rho)_{\tau_i} \end{aligned}$$

Chapter 5

Crystal basis theory

This chapter reviews the ideas of Crystal Basis Theory for $U_q(\mathfrak{g})$ modules. A crystal base of a $U_q(\mathfrak{g})$ -module, V , may be roughly thought of as a base for the module at $q = 0$. As mentioned previously (see theorem 2.2), the character of $L^q(\lambda)$, the highest weight $U_q(\mathfrak{g})$ -module of highest weight λ , is equal to the character of $L(\lambda)$, the highest weight \mathfrak{g} -module with highest weight λ . Thus, by using crystal basis theory to calculate weight multiplicities in highest weight $U_q\left(C_n^{(1)}\right)$ -modules we will determine weight multiplicities in highest weight $C_n^{(1)}$ -modules. These weight multiplicities are the last piece of information we need in order to determine root multiplicities of $HC_n^{(1)}$. We begin our discussion of Crystal Basis theory with a motivating example. Next, we review the notions of Crystal Basis theory more thoroughly. Finally, we review path realizations for crystal bases of highest weight modules over $U_q(\mathfrak{g})$, where \mathfrak{g} is an affine Kac-Moody algebra. For more details concerning Crystal Basis theory

see [9].

5.1 Motivation

Consider the three dimensional simple Lie algebra $U_q(sl(2, \mathbb{C}))$ given in example (2.3).

Let $V = \text{span} \{v_1, v_2\}$ be the two dimensional irreducible representation of $U_q(sl(2, \mathbb{C}))$ defined by:

$$\begin{aligned} ev_1 &= 0, & fv_1 &= v_2, & tv_1 &= qv_1, & t^{-1}v_1 &= q^{-1}v_1 \\ ev_2 &= v_1, & fv_2 &= 0, & tv_2 &= q^{-1}v_2, & t^{-1}v_2 &= qv_2. \end{aligned} \tag{5.1}$$

Then the tensor product $V \otimes V$ has the $U_q(sl(2, \mathbb{C}))$ -module structure given below.

$$\begin{aligned} t(u_1 \otimes u_2) &= tu_1 \otimes tu_2, & t^{-1}(u_1 \otimes u_2) &= t^{-1}u_1 \otimes t^{-1}u_2 \\ e(u_1 \otimes u_2) &= eu_1 \otimes t^{-1}u_1 + u_1 \otimes eu_2 \\ f(u_1 \otimes u_2) &= fu_1 \otimes u_2 + tu_1 \otimes fu_2 \end{aligned} \tag{5.2}$$

$V \otimes V$ is completely reducible and by the Clebsch-Gordan formula:

$$V \otimes V \cong V(2) \oplus V(0)$$

where $V(2)$ is the three dimensional module with basis

$$\{v_1 \otimes v_1, f(v_1 \otimes v_1) = v_2 \otimes v_1 + qv_1 \otimes v_2, f^{(2)}(v_1 \otimes v_1) = v_2 \otimes v_2\}$$

and $V(0)$ is the one dimensional module with basis

$$\{v_1 \otimes v_2 - qv_2 \otimes v_1\}.$$

Here $f^{(2)}(v_1 \otimes v_1) = \frac{1}{[2]}f^2(v_1 \otimes v_1)$, and $[2] = \frac{q^2 - q^{-2}}{q - q^{-1}}$. When $q = 0$, the basis for $V(2)$ and $V(0)$ can be parameterized by pure tensors $\{v_1 \otimes v_1, v_2 \otimes v_1, v_2 \otimes v_2\}$ and $\{v_1 \otimes v_2\}$ respectively. This indicates that it is natural to expect simple structure in the limiting case $q = 0$. As will be seen in example 5.2,

$$\{v_1 \otimes v_1, v_2 \otimes v_1, v_2 \otimes v_2\} \text{ and } \{v_1 \otimes v_2\} \quad (5.3)$$

form the crystal base for $V(2)$ and $V(0)$ respectively.

5.2 Basic Concepts

We begin our discussion of crystal bases by giving some definitions. Let, $F = \mathbb{C}(q)$, the field of rational functions in q with coefficients in \mathbb{C} . Define the subring, A , of F as follows,

$$A = \left\{ \frac{f(q)}{g(q)} \middle| f, g \in \mathbb{C}[q] \text{ and } g(0) \neq 0 \right\}.$$

The evaluation map $\xi : A \bmod qA \rightarrow \mathbb{C}$ given by $\xi(f(q) + q \cdot g(q)) = f(0) + 0 \cdot g(0) = f(0)$ is an isomorphism.

Definition 5.1. *Let V be a vector space over F . A local base at $q = 0$ of V is a pair*

$(\mathcal{L}, \mathcal{B})$ where

1. \mathcal{L} is a free A -module such that V is generated by \mathcal{L} as a vector space over F .
(i.e. $V \cong F \otimes_A \mathcal{L}$).
2. \mathcal{B} is a base of the vector space $\mathcal{L} \bmod q\mathcal{L}$ over F

Let $\mathfrak{g} = \mathfrak{g}(A)$ be a symmetrizable Kac-Moody Lie algebra with realization $\{\mathfrak{h}, \Pi = \{\alpha_i | i \in I\} \subset \mathfrak{h}^*, \check{\Pi} = \{h_i | i \in I\} \subset \mathfrak{h}\}$, let P be the weight lattice of \mathfrak{g} , let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of \mathfrak{g} , and let $U_q(\mathfrak{g}_{(i)})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the set $\{e_i, f_i, t_i^\pm\}$. Recall, for each $i \in I$, $U_q(\mathfrak{g}_{(i)})$ is isomorphic to $U_q(sl(2, \mathbb{C}))$.

Let M be an integrable $U_q(\mathfrak{g})$ -module. By $U_q(sl(2, \mathbb{C}))$ representation theory, for each $i \in I$, any element $u \in M_\lambda$ can be uniquely written as $u = \sum_{k \geq 0} f_i^{(k)} u_k$, where $u_k \in \ker(e_i) \cap M_{\lambda + k\alpha_i}$. The endomorphism \tilde{e}_i and \tilde{f}_i on M given below are known as the *Kashiwara operators*.

$$\begin{aligned} \tilde{e}_i(u) &= \sum_{k \geq 0} f_i^{(k-1)} u_k \\ \tilde{f}_i(u) &= \sum_{k \geq 0} f_i^{(k+1)} u_k, \end{aligned} \tag{5.4}$$

for $u \in M_\lambda$.

Definition 5.2. A crystal base of an integrable $U_q(\mathfrak{g})$ -module, M , is a pair $(\mathcal{L}, \mathcal{B})$

such that:

$(\mathcal{L}, \mathcal{B})$ is a local base at $q = 0$

$$\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda, \text{ where } \mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$$

$$\mathcal{B} = \bigcup_{\lambda \in P} \mathcal{B}_\lambda, \text{ where } \mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}_\lambda / q\mathcal{L}_\lambda)$$

$$\tilde{e}_i \mathcal{L} \subset \mathcal{L}, \tilde{f}_i \mathcal{L} \subset \mathcal{L}$$

$$\tilde{e}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}, \tilde{f}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$$

for $b, b' \in \mathcal{B}$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$.

The existence and uniqueness of crystal bases for integrable $U_q(\mathfrak{g})$ -modules follows from the following theorem due to Kashiwara.

Theorem 5.1. Let $\mathcal{L}(\lambda) = \sum_{\substack{l \geq 0 \\ i_1, \dots, i_l \in I}} A \tilde{f}_{i_1} \dots \tilde{f}_{i_l} u_\lambda$, and let $\mathcal{B}(\lambda) = \left\{ \tilde{f}_{i_1} \dots \tilde{f}_{i_l} u_\lambda \bmod q\mathcal{L}(\lambda) \mid l \geq 0, i_1 \dots i_l \in I \right\} \setminus \{0\}$. Then, $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is a crystal base of $L(\lambda)$, the irreducible $U_q(\mathfrak{g})$ -module with highest weight λ .

Each crystal base is uniquely associated with a crystal graph, as described in the definition below.

Definition 5.3. Let M be an integrable $U_q(\mathfrak{g})$ -module with crystal basis $(\mathcal{L}, \mathcal{B})$. The crystal graph of M has \mathcal{B} as its set of vertices. For each $i \in I$, we join $b \in \mathcal{B}$ to $b' \in \mathcal{B}$ with an i -colored arrow $(b \xrightarrow{i} b')$ if and only if $b' = \tilde{f}_i b$ (iff $b = \tilde{e}_i b'$).

Example 5.1. Let $V = \text{span}\{v_1, v_2 = f v_1\}$ be the irreducible two dimensional $U_q(\mathfrak{sl}(2, \mathbb{C}))$ -module given in (5.1). V is a highest weight module of highest weight

1 and highest weight vector v_1 . Let,

$$\mathcal{L} = Av_1 \oplus Afv_1$$

$$\mathcal{B} = \{\bar{v}_1, f\bar{v}_1\}$$

where $\bar{x} = x + q\mathcal{L}$ for all $x \in \mathcal{L}$

Then, $(\mathcal{L}, \mathcal{B})$ forms a crystal basis for V , with crystal graph

$$\bar{v}_1 \rightarrow f\bar{v}_1$$

At this time, we take a moment to introduce some definitions and discuss some general properties relating to crystal graphs. Let \mathcal{B} be a crystal graph of an integrable $U_q(\mathfrak{g})$ -module, $M = \bigoplus_{\lambda \in P} (M_\lambda)$. For $b \in \mathcal{B}_\lambda$ we say b is of weight λ and write $wt(b) = \lambda$. Define the maps ε_i and φ_i as follows.

$$\varepsilon_i(b) = \max\{n \neq 0, \tilde{e}_i^{(n)}b \neq 0\} \tag{5.5}$$

$$\varphi_i(b) = \max\{n \neq 0, \tilde{f}_i^{(n)}b \neq 0\} \tag{5.6}$$

$\varepsilon_i(b)$ gives the number of i -colored arrows coming into the vertex b , $\varphi_i(b)$ gives the number of i -colored arrows coming out of the vertex b and $\varphi_i(b) + \varepsilon_i(b)$ gives the length of the i -string through b . Notice, $\varphi_i(b) - \varepsilon_i(b) = \langle h_i, wt(b) \rangle$.

One of the most useful combinatorial features of the crystal bases is their stability under the tensor product.

Theorem 5.2. (*Tensor Product Rule*) *Let M_j be an integrable $U_q(\mathfrak{g})$ -module and let $(\mathcal{L}, \mathcal{B})$ be a crystal basis of M_j for $j = 1, 2$. Let, $\mathcal{L} = \mathcal{L}_1 \otimes_A \mathcal{L}_2$ and $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$. Then,*

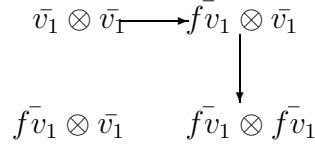
$(\mathcal{L}, \mathcal{B})$ is a crystal basis of $M_1 \otimes_F M_2$ \tilde{e}_i and

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

Example 5.2. *Let $\mathfrak{g} = U_q(\mathfrak{sl}(2, \mathbb{C}))$ and let V be the two dimensional irreducible module we considered in example 5.1. Applying the tensor product rule to the crystal graph of V given in 5.1 we obtain the crystal graph of $V \otimes V$, shown in figure 5.1. Looking at the crystal graph we see, as we expect, $V \otimes V \cong V(2) \otimes V(0)$, where $V(2)$ has crystal basis $\{\bar{v}_1 \otimes \bar{v}_1, \bar{f}v_1 \otimes \bar{v}_1, \bar{f}v_1 \otimes \bar{f}v_1\}$, and $V(0)$ has crystal basis $\{\bar{v}_1 \otimes \bar{f}v_1\}$, which coincide with the bases given in (5.3).*

There is a combinatorial rule using theorem 5.2 to determine the action of the Kashiwara operators on multi-fold tensor products of crystal graphs (see [9]). Let


 Figure 5.1: Crystal Graph $V \otimes V$

M_j be an integrable $U_q(\mathfrak{g})$ -module with crystal bases $(\mathcal{L}_j, \mathcal{B}_j)$ for $j = 1, \dots, n$ and let $b = b_1 \otimes b_2 \otimes \dots \otimes b_n$ ($b_j \in \mathcal{B}_j$). Choose $i \in I$. Create the following sequence of minuses and pluses.

$$\left(\underbrace{- \dots -}_{\varepsilon_i(b_1)} \underbrace{+ \dots +}_{\varphi_i(b_1)}, \underbrace{- \dots -}_{\varepsilon_i(b_2)} \underbrace{+ \dots +}_{\varphi_i(b_2)}, \dots, \underbrace{- \dots -}_{\varepsilon_i(b_n)} \underbrace{+ \dots +}_{\varphi_i(b_n)} \right) \quad (5.7)$$

Cancel out all $(+-)$ -pairs in (5.7). (One may cancel across component boundaries.) The resulting sequence is called the i -signature of b . Let k be the number corresponding to the component in which the right most $-$ appears in the i -signature of b , and let p be the number corresponding to the component in which the left most $+$ appears in the i -signature of b . The Kashiwara operators act on b as follows.

$$\tilde{e}_i b = b_1 \otimes \dots \otimes \tilde{e}_i b_k \otimes \dots \otimes b_n \quad (5.8)$$

$$\tilde{f}_i b = b_1 \otimes \dots \otimes \tilde{f}_i b_p \otimes \dots \otimes b_n \quad (5.9)$$

5.3 Path Realizations

Let \mathfrak{g} be an affine Lie algebra over \mathbb{C} generated by: $\{e_i, f_i | i \in I = \{0, 1, \dots, n\}\} \cup \{h | h \in \mathfrak{h}\}$ where \mathfrak{h} is the Cartan subalgebra of \mathfrak{g} with dimension $n + 2$. Let $\Pi = \{\alpha_i | i \in I\} \subseteq \mathfrak{h}^*$ and $\check{\Pi} = \{h_i | i \in I\} \subseteq \mathfrak{h}$ be the set of simple roots and co-roots respectively. We fix the symmetric bilinear form $(\cdot | \cdot)$ on \mathfrak{h}^* so that $(\alpha_i | \alpha_i) \in \mathbb{Z}_{>0}$, for all $i \in I$. Let δ and K be the generators of the imaginary roots and center respectively. Set $\mathfrak{h}_{cl} = \bigoplus_{i=0}^n \mathbb{C}h_i \subseteq \mathfrak{h}$, and $\mathfrak{h}_{cl}^* = \left(\bigoplus_{i=0}^n \mathbb{C}h_i\right)^*$. Let $cl : \mathfrak{h}^* \rightarrow \mathfrak{h}_{cl}^*$ be the canonical morphism. Then we have an exact sequence:

$$0 \rightarrow \mathbb{C}\delta \rightarrow \mathfrak{h}^* \xrightarrow{cl} \mathfrak{h}_{cl}^* \rightarrow 0$$

Choose and fix a map $af : \mathfrak{h}_{cl}^* \rightarrow \mathfrak{h}^*$ satisfying: $cl \circ af = id$, and $af \circ cl(\alpha_i) = \alpha_i$, $i \neq 0$. Let $\Lambda_i, i \in I$, be the elements of $af(\mathfrak{h}_{cl}^*) \subset \mathfrak{h}^*$ such that $\Lambda_i(h_j) = \delta_{i,j}$. Set $P = \sum_{i=0}^n \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ and $P_{cl} = cl(P) \subset \mathfrak{h}_{cl}^*$, the affine weight lattice and classical weight lattice respectively. Let P^* and P_{cl}^* denote the respective dual lattices. Finally, let $U_q(\mathfrak{g})$ and $U_q(\mathfrak{g}_{(i)})$ be as defined in section 2. We also define the quantized universal enveloping algebra $U'_q(\mathfrak{g})$ to be the universal enveloping algebra over $\mathbb{C}(q)$ associated with P_{cl} generated by $\{e_i, f_i, q^h | i \in I, h \in P_{cl}^*\}$ and satisfying relations (2.8).

Abstracting the properties of crystal bases we define a crystal.

Definition 5.4. *Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of a Kac-Moody algebra with realization $(\mathfrak{h}, \Pi, \check{\Pi})$ and weight lattice, P . A $U_q(\mathfrak{g})$ -crystal asso-*

ciated with \mathfrak{g} is P -weighted set, \mathcal{B} , along with maps

$$\varepsilon_i : \mathcal{B} \rightarrow \mathbb{Z} \sqcup \{-\infty\}$$

$$\varphi_i : \mathcal{B} \rightarrow \mathbb{Z} \sqcup \{-\infty\}$$

$$\tilde{e}_i : \mathcal{B} \rightarrow \mathcal{B} \sqcup \{0\}$$

$$\tilde{f}_i : \mathcal{B} \rightarrow \mathcal{B} \sqcup \{0\}$$

$$wt : \mathcal{B} \rightarrow P$$

which satisfies the following properties.

$$\varphi_i(b) - \varepsilon_i(b) = \langle wt(b), h_i \rangle \quad \text{for all } i \in I, b \in \mathcal{B}$$

$$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1 \quad \text{for } b \in \mathcal{B} \text{ such that } \tilde{e}_i b \neq 0$$

$$\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1 \quad \text{for } b \in \mathcal{B} \text{ such that } \tilde{e}_i b \neq 0$$

$$wt(\tilde{e}_i b) = wt(b) + \alpha_i \quad \text{for } b \in \mathcal{B} \text{ such that } \tilde{e}_i b \neq 0$$

$$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1 \quad \text{for } b \in \mathcal{B} \text{ such that } \tilde{f}_i b \neq 0$$

$$\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1 \quad \text{for } b \in \mathcal{B} \text{ such that } \tilde{f}_i b \neq 0$$

$$wt(\tilde{f}_i b) = wt(b) - \alpha_i \quad \text{for } b \in \mathcal{B} \text{ such that } \tilde{f}_i b \neq 0$$

$$b_2 = \tilde{f}_i b_1 \Leftrightarrow b_1 = \tilde{e}_i b_2 \quad \text{for } b_1, b_2 \in \mathcal{B}$$

$$\varphi_i(b) = -\infty \Rightarrow \tilde{e}_i b = \tilde{f}_i b = 0 \quad \text{for } b \in \mathcal{B}$$

For two crystals \mathcal{B}_1 and \mathcal{B}_2 we define the tensor product $\mathcal{B}_1 \otimes \mathcal{B}_2$ as follows. The underlying set is $\mathcal{B}_1 \times \mathcal{B}_2$. We write $b_1 \otimes b_2$ for (b_1, b_2) and understand $b_1 \otimes 0 = 0 = 0 \otimes b_2$. The action of \tilde{e}_i and \tilde{f}_i are analogous to the action given in (5.2). Note that if \mathcal{B}_1 and \mathcal{B}_2 are P -weighted crystals then so is $\mathcal{B}_1 \otimes \mathcal{B}_2$ and for $b_1 \otimes b_2 \in \mathcal{B}_1 \otimes \mathcal{B}_2$ $wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2)$.

Set $P_{cl}^+ = \{\lambda \in P_{cl} | \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}$ and let $(P_{cl}^+)_l = \{\lambda \in P_{cl}^+ | \lambda(K) = l\}$ for all $l \in \mathbb{Z}$. Let \mathcal{B} be a classical crystal. For $b \in \mathcal{B}$, set $\varepsilon(b) = \sum_{i=0}^n \varepsilon_i(b) \Lambda_i$, and $\varphi(b) = \sum_{i=0}^n \varphi_i(b) \Lambda_i$. Note that $wt(b) = \varphi(b) - \varepsilon(b)$.

Definition 5.5. *A perfect crystal of level l , \mathcal{B}_l , is a finite $U'_q(\mathfrak{g})$ -crystal which satisfies the following properties.*

There exists a finite-dimensional $U'_q(\mathfrak{g})$ -module with a crystal basis whose crystal graph is isomorphic to \mathcal{B}_l .

$\mathcal{B}_l \otimes \mathcal{B}_l$ is connected.

There exists a classical weight $\lambda_0 \in P_{cl}$, with $\dim(\mathcal{B}_{\lambda_0}) = 1$, such that $wt(\mathcal{B}) \subset$

$$\lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i.$$

For any $b \in \mathcal{B}$, $\langle K, \varepsilon(b) \rangle \geq l$.

For each $\lambda \in P_{cl}$ of level l , there exists unique vectors $b^\lambda \in \mathcal{B}$ and $b_\lambda \in \mathcal{B}$ such that

$$\varepsilon(b^\lambda) = \lambda \text{ and } \varphi(b_\lambda) = \lambda.$$

Example 5.3. [22] For a fixed positive integer l a $U'_q(C_n^{(1)})$ perfect crystal of level l may be defined as follows

$$\mathcal{B}_l = \left\{ b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \left| x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, s(b) = \sum_{i=1}^n x_i + \bar{x}_i \leq 2l, s(b) \text{ even} \right. \right\}$$

with crystal path structure

$$\tilde{f}_0(b) = \begin{cases} (x_1 + 2, x_2, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) & x_1 \geq \bar{x}_1 \\ (x_1 + 1, x_2, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1 - 1) & x_1 = \bar{x}_1 - 1 \\ (x_1, x_2, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1 - 2) & x_1 \leq \bar{x}_1 - 2 \end{cases} \quad (5.10)$$

$$\tilde{f}_i(b) = \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) & x_{i+1} \geq \bar{x}_{i+1} \\ (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \dots, \bar{x}_1) & x_{i+1} < \bar{x}_{i+1} \end{cases}$$

(for $1 \leq i \leq n - 1$)

$$\tilde{f}_n(b) = (x_1, \dots, x_n - 1, \bar{x}_n + 1, \dots, \bar{x}_1)$$

$$\varphi_0(b) = l - \frac{1}{2}s(b) + (\bar{x}_1 - x_1)_+ \quad (5.11)$$

$$\varepsilon_0(b) = l - \frac{1}{2}s(b) + (x_1 - \bar{x}_1)_+$$

$$\varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+$$

$$\varepsilon_i(b) = \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+$$

(for $1 \leq i \leq n - 1$)

$$\begin{aligned}
 \varphi_n(b) &= x_n \\
 \varepsilon_n(b) &= \bar{x}_n \\
 wt(b) &= \sum_{i=0}^n (\phi_i(b) - \varepsilon_i(b)) \Lambda_i \\
 &= \sum_{i=1}^n (x_i - \bar{x}_i) (\Lambda_i - \Lambda_{i-1})
 \end{aligned} \tag{5.12}$$

Let us refocus our attention on devising a method to realize crystal graphs for highest weight $U_q(\mathfrak{g})$ -modules. Throughout the sequel assume the following.

\mathfrak{g} is an affine algebra.

l is an element of \mathbb{Z}_+

\mathcal{B}_l is a perfect crystal of level l .

λ is an element of $(P_{cl}^+)_l$

b_λ is the unique vector in \mathcal{B} such that $\varphi(b_\lambda) = \lambda$

$L(\lambda)$ is the highest weight $U_q(\mathfrak{g})$ -module of highest weight λ

$\mathcal{B}(\lambda)$ is the crystal graph of $L(\lambda)$

The following isomorphism theorem will be our main tool in developing a path realization for $\mathcal{B}(\lambda)$ (see, for example, [9]) .

Theorem 5.3. *The map*

$$\Psi : \mathcal{B}(\lambda) \rightarrow \mathcal{B}(\varepsilon(b_\lambda)) \otimes \mathcal{B}$$

$$\text{given by} \quad u_\lambda \rightarrow u_{\varepsilon(b_\lambda)} \otimes b_\lambda$$

where u_λ , and $u_{\varepsilon(b_\lambda)}$ are the highest weight vectors of $\mathcal{B}(\lambda)$ and $\mathcal{B}(\varepsilon(b_\lambda))$ respectively, is a strict isomorphism of crystals.

Repeated application of Theorem 5.3 gives a sequence of isomorphisms.

$$\Psi_1 : \mathcal{B}(\lambda) \rightarrow \mathcal{B}(\lambda_1) \otimes \mathcal{B}$$

$$\text{given by} : u_\lambda \rightarrow u_{\lambda_1} \otimes b_\lambda$$

$$\Psi_2 : \mathcal{B}(\lambda) \rightarrow \mathcal{B}(\lambda_2) \otimes \mathcal{B} \otimes \mathcal{B}$$

$$\text{given by} \quad u_\lambda \rightarrow u_{\lambda_2} \otimes b_{\lambda_1} \otimes b_\lambda$$

$$\dots \quad \dots$$

$$\Phi_k : \mathcal{B}(\lambda) \rightarrow \mathcal{B}(\lambda_k) \otimes \mathcal{B}^{\otimes k}$$

$$\text{given by} \quad u_\lambda \rightarrow u_{\lambda_k} \otimes b_{k-1} \otimes \dots \otimes b_1 \otimes b_0$$

where, $\lambda_0 = \lambda$, $b_k = b_{\lambda_k}$, $\lambda_{k+1} = \varepsilon(b_k)$. This sequence of isomorphisms leads to two

other important sequences:

$$w_\lambda = (\lambda_k)_{k=0}^\infty = (\dots, \lambda_{k+1}, \lambda_k, \dots, \lambda_1, \lambda_0) \in (P_{cl}^+)_l^\infty \quad (5.13)$$

$$p_\lambda = (b_k)_{k=0}^\infty = (\dots, b_{k+1}, b_k, \dots, b_1, b_0) \in \mathcal{B}^{\otimes \infty} \quad (5.14)$$

The set $(P_{cl}^+)_l$ is finite, as is the set \mathcal{B} . Therefore, there must exist an $N > 0$ such that $\lambda_N = \lambda_0$. Since the maps φ and ε are bijective, the following set of equalities hold.

$$b_0 = \varphi^{-1}(\lambda_0) = \varphi^{-1}(\lambda_N) = b_N$$

$$\lambda_1 = \varepsilon(b_0) = \varepsilon(b_N) = \lambda_{N+1}$$

$$b_1 = \varphi^{-1}(\lambda_1) = \varphi^{-1}(\lambda_{N+1}) = b_{N+1}$$

$$\dots = \dots$$

$$\lambda_j = \lambda_{j+N}$$

$$b_j = b_{j+N}$$

$$\dots = \dots$$

$$\lambda_{N-1} = \lambda_{2N-1}$$

$$b_{N-1} = b_{2N-1}$$

Therefore, w_λ and b_λ defined in (5.13) and (5.14) respectively are periodic with the same period, $N > 0$.

Definition 5.6. The path $p_\lambda = (\cdots \otimes b_k \otimes b_1 \otimes b_0)$, with $\lambda_0 = \lambda$, $b_k = b_{\lambda_k}$, $\lambda_{k+1} = \varepsilon(b_k)$, is called the ground-state path of weight λ .

Example 5.4. The ground state path of weight Λ_0 in \mathcal{B}_1 , the $U'_q(C_n^{(1)})$ perfect crystal of level one mentioned in example 5.3 is given below.

$$p = (\cdots \otimes \underline{0} \otimes \underline{0}),$$

where $\underline{0} = \underbrace{(0, 0, \dots, 0)}_{2n \text{ times}}$.

Definition 5.7. A λ -path in \mathcal{B} is any path $p = (\cdots \otimes p_{k+1} \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0)$ whose elements agree with those of the ground state path of weight λ in \mathcal{B} past some finite index, k .

We collect all λ -paths in \mathcal{B} into a set which we label $\mathcal{P}(\lambda)$. Theorem 5.4 below, shows that $\mathcal{P}(\lambda)$ can be given a crystal-structure. Theorem 5.5, also given below, then shows that $\mathcal{P}(\lambda)$ is a realization of the crystal base $\mathcal{B}(\lambda)$ for the highest weight $U'_q(\mathfrak{g})$ -module $L(\lambda)$.

Theorem 5.4. Let,

$$p \in \mathcal{P}(\lambda),$$

b be the λ -ground state path, and

$N > 0$ be the smallest positive integer such that $p_k = b_k$ for all $k \geq N$.

The maps $wt_{cl} : \mathcal{P}(\lambda) \rightarrow P_{cl}$; $\tilde{e}_i, \tilde{f}_i : \mathcal{P}(\lambda) \rightarrow \mathcal{P}(\lambda) \sqcup \{0\}$; and $\varepsilon_i, \varphi_i : \mathcal{P}(\lambda) \rightarrow \mathbb{Z}$ ($i \in I$) given below define a crystal-structure on $\mathcal{P}(\lambda)$.

$$\begin{aligned}
wt_{cl}(p) &= \lambda_N + \sum_{k=0}^{N-1} wt_{cl} p_k \\
\tilde{e}_i(p) &= \cdots \otimes p_{N+1} \otimes \tilde{e}_i(p_N \otimes \cdots \otimes p_0) \\
\tilde{f}_i(p) &= \cdots \otimes p_{N+1} \otimes \tilde{f}_i(p_N \otimes \cdots \otimes p_0) \\
\varepsilon_i(p) &= [\varepsilon(p_{N-1} \otimes \cdots \otimes p_0) - \varphi_i(b_N)]_+ \\
\varphi_i(p) &= \varphi_i(p_{N-1} \otimes \cdots \otimes p_0) + [\varphi(b_N) - \varepsilon_i(p_{N-1} \otimes \cdots \otimes p_0)]_+
\end{aligned}$$

Theorem 5.5. *Let \mathfrak{g} be an affine algebra, $\lambda \in (P_{cl}^+)_I$, $L(\lambda)$ be a highest weight $U'_q(\mathfrak{g})$ -module of highest weight λ , and $\mathcal{B}(\lambda)$ be the crystal base of $L(\lambda)$. Then $\mathcal{B}(\lambda)$ and $\mathcal{P}(\lambda)$ are isomorphic as $U'_q(\mathfrak{g})$ crystals.*

Example 5.5. *Let $\mathfrak{g} = C_2^{(1)}$. Then the (partial) path realization for the crystal graph of $\mathcal{B}(\Lambda_0)$ is given in figure 5.5.*

Theorem 5.4 tells us how to calculate the classical weight of a path in $\mathcal{P}(\lambda)$. We will also wish to calculate the affine weights of such paths. Before we explain how to calculate such weights, we must introduce the following definition.

Definition 5.8. *Let V be a finite dimensional $U'_q(\mathfrak{g})$ -module, and let $(\mathcal{L}, \mathcal{B})$ be a crystal basis for V . An energy function on \mathcal{B} is a function $H : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathbb{Z}$ such that*

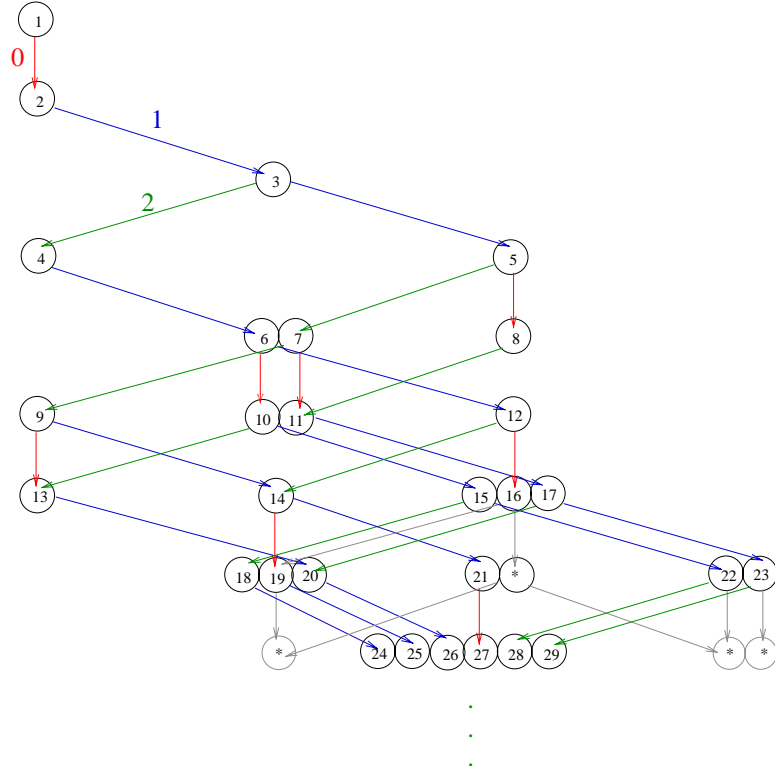


Figure 5.2: A partial path realization of the $C_2^{(1)}$ -module $L(\Lambda_0)$.

for all $i \in I$ and $b_1 \otimes b_2$ in $\mathcal{B} \otimes \mathcal{B}$ such that $\tilde{e}_i(b_1 \otimes b_2) \in \mathcal{B} \otimes \mathcal{B}$,

$$H(\tilde{e}_i(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) & \text{if } i \neq 0 \\ H(b_1 \otimes b_2) + 1 & \text{if } i = 0, \varphi_0(b_1) \geq \varepsilon_0(b_2) \\ H(b_1 \otimes b_2) - 1 & \text{if } i = 0, \varphi_0(b_1) < \varepsilon_0(b_2) \end{cases}$$

Example 5.6. [22] Let \mathcal{B}_l be the $U'_q(C_n^{(1)})$ perfect crystal of level l described in

example 5.3 . The function $H : \mathcal{B}_l \otimes \mathcal{B}_l \rightarrow \mathbb{Z}$ given below is an energy function of \mathcal{B}_l .

$$H(b \otimes b') = \max\{\theta_j(b \otimes b'), \theta'_j(b \otimes b'), \eta_j(b \otimes b'), \eta'_j(b \otimes b') | 1 \leq j \leq n\} \quad (5.15)$$

Where,

$$\begin{aligned} \theta_j(b \otimes b') &= \sum_{k=1}^{j-1} (\bar{x}_k - \bar{x}'_k) + \frac{1}{2}(s(b') - s(b))_+ \\ \theta'_j(b \otimes b') &= \sum_{k=1}^{j-1} (x'_k - x_k) + \frac{1}{2}(s(b) - s(b'))_+ \\ \eta_j(b \otimes b') &= \theta_j(b \otimes b') + \bar{x}_j - x_j \\ \eta'_j(b \otimes b') &= \theta'_j(b \otimes b') + x'_j - \bar{x}'_j. \end{aligned}$$

Theorem 5.6, stated below, uses energy functions to give a formula for calculating the affine weight of a path $p \in \mathcal{P}(\lambda)$.

Theorem 5.6. *Let $p = (\cdots \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0)$ be a λ path in $\mathcal{P}(\lambda)$, and let $b = (\cdots \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0)$ be the ground state path of weight λ . Then, the affine weight of p is given by the formula*

$$\begin{aligned} wt(p) &= \lambda + \sum_{k=0}^{\infty} (wt_{cl}(p_k) - wt_{cl}(b_k)) \\ &\quad - \left(\sum_{k=0}^{\infty} (k+1)(H(p_{k+1} \otimes p_k) - H(b_{k+1} \otimes b_k)) \right) \delta \end{aligned}$$

Example 5.7. Consider crystal graph of the $U'_q(C_2^{(1)})$ highest weight module $L(\Lambda_0)$ drawn in example 5.2. Let p be the path labeled as number three in the example. Thus,

$$p = \cdots \otimes (0, 0, 0, 0) \otimes (0, 0, 0, 0) \otimes (1, 1, 0, 0).$$

We will calculate the affine weight of the path p in two different ways. First, we will use the affine weight formula given in theorem 5.6 and second we will use the fact that $wt(\tilde{f}_i b) = wt(b) - \alpha_i$ for $b \in \mathcal{B}(\lambda)$ (see definition 5.4).

$$\begin{aligned}
wt(p) &= \Lambda_0 + (wt_{cl}(1, 1, 0, 0) - wt_{cl}(0, 0, 0, 0)) \\
&\quad - (H(1, 1, 0, 0) \otimes (0, 0, 0, 0)) - H((0, 0, 0, 0) \otimes (0, 0, 0, 0)) \delta \\
&= \Lambda_0 + wt((1, 1, 0, 0)) - H((1, 1, 0, 0) \otimes (0, 0, 0, 0)) \delta \\
&= \Lambda_0 - \Lambda_0 + \Lambda_2 - \delta \\
&= \Lambda_2 - \delta \\
wt(p) &= wt(\tilde{f}_1 \tilde{f}_0(\cdots (0, 0, 0, 0) \otimes (0, 0, 0, 0))) \\
&= \Lambda_0 - \alpha_0 - \alpha_1 \\
&= \Lambda_0 - (\delta + 2\Lambda_0 - 2\Lambda_1) - (-\Lambda_0 + 2\Lambda_1 - \Lambda_2) \\
&= \Lambda_2 - \delta.
\end{aligned}$$

In summary, using the notation we introduced at the beginning of this section,

$$\begin{aligned}
 \dim(L(\lambda))_\mu &= \dim L^q(\lambda)_\mu \\
 &= |\mathcal{B}(\lambda)_\mu| \\
 &= |p \in \mathcal{P}(\lambda) | wt(p) = \mu|
 \end{aligned}$$

where $wt(p)$ is the affine weight of the path p .

Chapter 6

Root multiplicities of $HC_n^{(1)}$

In this chapter, we determine root multiplicities of $HC_n^{(1)}$ for roots of the form $-l\alpha_{-1} - k\delta$. We use Kang's multiplicity formula (see theorem 4.3) and the path realizations of crystal bases for irreducible highest weight $C_n^{(1)}$ -modules to obtain these multiplicities.

6.1 Level One Root Multiplicities

We begin by considering multiplicities of level one roots in $\mathfrak{g} = HC_n^{(1)}$. If α is a root of level one, the construction given in chapter 3 implies

$$\dim \mathfrak{g}_\alpha = \dim V(\Lambda_0)_\alpha$$

where $V(\Lambda_0)$ is the highest weight $\mathfrak{g}_0 = C_n^{(1)}$ -module with highest weight Λ_0 . Therefore we may call upon the ideas of chapter 5 to calculate multiplicities of level one roots of \mathfrak{g} .

Example 6.1. *In this example we determine the multiplicity of the root $\alpha = -\alpha_{-1} - \delta$ of the Kac-Moody algebra $\mathfrak{g} = HC_2^{(1)}$. Using the techniques of chapter 5 we draw a portion of the path realization, $\mathcal{P}(\Lambda_0)$, of $V(\Lambda_0)$ as a $U'_q(\mathfrak{g}_0)$ -module which we show in figure 6.1. (We do not show any zero arrows after the first.) The paths in $\mathcal{P}(\Lambda_0)$ of weight α are the paths which follow exactly one 0-arrow, two 1-arrows, and one 2-arrow. We see these are the paths labeled eight and nine in the graph. Therefore, $\dim(\mathfrak{g})_\alpha = \dim V(\Lambda_0)_\alpha = 2$.*

Using the same techniques as in Example 6.1, we prove propositions 6.1, 6.2, and 6.3.

Proposition 6.1. *Let $\mathfrak{g} = HC_n^{(1)}$ for any $n \in \mathbb{Z}_{\geq 2}$, and let $\alpha = -\alpha_{-1} - \delta$. Then, $\dim \mathfrak{g}_\alpha = n$.*

Proof: Let $\mathcal{P}(\Lambda_0)$ be the path realization of $V(\Lambda_0)$ as a $U'_q(\mathfrak{g}_0)$ -module and let $p \in \mathcal{P}(\Lambda_0)$. Then $p = (\cdots \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0)$, where each b_i is an element of the level one $U'_q(C_n^{(1)})$ -perfect crystal \mathcal{B}_1 given in example 5.3. We wish to find the number of paths in $\mathcal{P}(\Lambda_0)$ of weight α . Thus, we are looking for the number of paths following exactly one 0-arrow, two 1-arrows, two 2-arrows, \dots , two $(n-1)$ -arrows, and one n -arrow. The ground state path in $\mathcal{P}(\Lambda_0)$ is $(\dots \otimes \underline{0} \otimes \underline{0})$. Referring to example

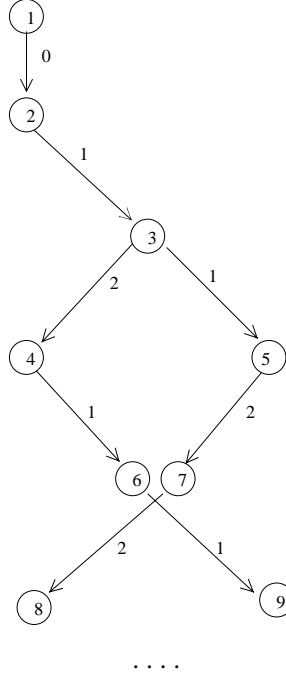


Figure 6.1: A partial path realization of the module $V(\Lambda_0)$ as a $U'_q(C_2^{(1)})$ -module. (Zero arrows are not shown after the first.)

5.3, we see $\tilde{f}_i(\underline{0})$ does not exist for any i not equal to zero. Thus any $p \in \mathcal{P}(\Lambda_0)$ with weight α must be of the form

$$p = (\cdots \otimes \underline{0} \otimes \underline{0} \otimes \underline{x}).$$

for some $\underline{x} = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \neq \underline{0} \in \mathcal{B}_1$.

Using theorem 5.6, we calculate the weight of such a path. In our calculation we use the classical weight formula given in example 5.3 and the energy function given

in example 5.6.

$$\begin{aligned}
 wt(p) &= \Lambda_0 + wt_{cl}(\underline{x}) - H(\underline{0} \otimes \underline{x})\delta \\
 &= \Lambda_0 + wt_{cl}(\underline{x}) - \delta \\
 &= \Lambda_0 + \sum_{i=1}^n (x_i - \bar{x}_i)(\Lambda_i - \Lambda_{i-1}) - \delta
 \end{aligned}$$

Next, we determine which $p \in \mathcal{P}(\Lambda_0)$ are of weight α .

$$\begin{aligned}
 \{p &\in \mathcal{P}(\Lambda_0) | wt(p) = \alpha\} \\
 &= \left\{ p = (\cdots \otimes \underline{0} \otimes \underline{0} \otimes \underline{x}) \mid \sum_{i=1}^n (x_i - \bar{x}_i)(\Lambda_i - \Lambda_{i-1}) = 0 \right\} \\
 &= \{p = (\cdots \otimes \underline{0} \otimes \underline{0} \otimes \underline{x}) \mid x_i - \bar{x}_i = 0 \text{ for } i=1 \text{ to } n \} \\
 &\quad (\text{since } \{\Lambda_i - \Lambda_{i-1}\}_{i=1}^n \text{ are linearly independent}) \\
 &= \{p = (\cdots \otimes \underline{0} \otimes \underline{0} \otimes \underline{x}) \mid \\
 &\quad x_i = \bar{x}_i = 1 \text{ for some } i = 1..n \text{ and } x_j = \bar{x}_j = 0 \text{ for all } j \neq i\} \\
 &\quad (\text{since } \underline{x} \neq \underline{0} \in \mathcal{B}_1 \Rightarrow s(\underline{x}) = 2)
 \end{aligned}$$

There are n such paths. Therefore,

$$\begin{aligned} \dim \mathfrak{g}_\alpha &= \#p \in \mathcal{P}(\Lambda_0) \text{ with weight } \alpha \\ &= n. \end{aligned}$$

□

Proposition 6.2. *Let $\mathfrak{g} = HC_n^{(1)}$ for any $n \in \mathbb{Z}_{\geq 2}$, and let $\alpha = -\alpha_{-1} - 2\delta$. Then, $\dim \mathfrak{g}_\alpha = n^2 + n$.*

Proof: $\dim \mathfrak{g}_\alpha$ is simply the number of paths in $\mathcal{P}(\Lambda_0)$ following exactly two 0-arrows, four 1-arrows, four 2-arrows, \dots , four $(n-1)$ -arrows, and two n -arrows. All such paths must be of the form $p = (\cdots \otimes \underline{0} \otimes \underline{0} \otimes \underline{y} \otimes \underline{x})$ for some $\underline{y} = (y_1, \dots, y_n, \bar{y}_n, \dots, \bar{y}_1)$ and $\underline{x} = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1)$ in \mathcal{B}_1 . Using theorem 5.6 we calculate the weight of such a path.

$$\begin{aligned} wt(p) &= \Lambda_0 + wt_d(\underline{x}) + wt_d(\underline{y}) - (H(\underline{y} \otimes \underline{x}) + 2H(\underline{0} \otimes \underline{y})) \delta \\ &= \Lambda_0 + \sum_{i=1}^n (x_i + y_i - \bar{x}_i - \bar{y}_i) (\Lambda_i - \Lambda_{i-1}) \\ &\quad - (H(\underline{y} \otimes \underline{x}) + 2H(\underline{0} \otimes \underline{y})) \delta \end{aligned}$$

Suppose $p = (\cdots \otimes \underline{0} \otimes \underline{0} \otimes \underline{y} \otimes \underline{x})$ is of weight $-\alpha_{-1} - 2\delta$ and $\underline{y} = \underline{0}$. Then,

$$\begin{aligned} \Lambda_0 - 2\delta &= wt(p) \\ &= \Lambda_0 + \sum_{i=1}^n (x_i + y_i - \bar{x}_i - \bar{y}_i) (\Lambda_i - \Lambda_{i-1}) \\ &\quad - (H(\underline{0} \otimes \underline{x})) \delta \end{aligned}$$

We know $\{\Lambda_i - \Lambda_{i-1}\}_{i=1}^n$ and $\{\delta\}$ are linearly independent. Therefore, $H(\underline{0} \otimes \underline{x})$ must equal 2. However, $H(\underline{0} \otimes \underline{a}) = 1$ for all $\underline{a} \neq \underline{0}$ and $H(\underline{0} \otimes \underline{0}) = 0$. Thus $\underline{y} \neq \underline{0}$ and

$$\begin{aligned} \Lambda_0 - 2\delta &= wt(p) \\ &= \Lambda_0 + \sum_{i=1}^n (x_i + y_i - \bar{x}_i - \bar{y}_i) (\Lambda_i - \Lambda_{i-1}) \\ &\quad - (H(\underline{y} \otimes \underline{x})) \delta - 2\delta. \end{aligned}$$

We conclude a path $p = (\cdots \otimes \underline{0} \otimes \underline{y} \otimes \underline{x})$ in $P(\Lambda_0)$ is of weight $\Lambda_0 - 2\delta$ if and only if

$$(x_i + y_i - \bar{x}_i - \bar{y}_i) = 0 \text{ for all } i = 1..n \text{ and} \quad (6.1)$$

$$H(\underline{y} \otimes \underline{x}) = 0. \quad (6.2)$$

Notice, (6.2) implies $\underline{x} \neq 0$ also. Table 6.1 lists all paths $p = (\cdots \otimes \underline{0} \otimes \underline{y} \otimes \underline{x})$ ($\underline{y}, \underline{x} \neq \underline{0}$) in $P(\Lambda_0)$ with property (6.1) along with $H(\underline{y} \otimes \underline{x})$ for the given \underline{y} and \underline{x} . In this table

and throughout the remainder of this chapter we assume $x_i = \bar{x}_i = y_i = \bar{y}_i = 0$ unless otherwise stated.

Label	\underline{x}	\underline{y}	restrictions	$H(\underline{y} \otimes \underline{x})$
Type 1	$\bar{x}_i = 2$	$y_i = 2$	none	0
Type 2	$\bar{x}_i=1$ $\bar{x}_j=1$	$y_i=1$ $y_j=1$	$i < j$	0
Type 3	$x_i=1$ $\bar{x}_j=1$	$y_i=1$ $y_j=1$	$i < j$	1
Type 4	$x_i=1$ $\bar{x}_j=1$	$y_i=1$ $y_j=1$	$i > j$	0
Type 5	$x_i=1$ $\bar{x}_i=1$	$y_j=1$ $\bar{y}_j=1$	$i \neq j$	1
Type 6	$x_i=1$ $\bar{x}_i=1$	$y_i=1$ $\bar{y}_i=1$	none	0
Type 7	$x_i = 2$	$\bar{y}_i = 2$	none	2
Type 8	$x_i=1$ $x_j=1$	$y_i=1$ $y_j=1$	$i < j$	2

Table 6.1: $\underline{x}, \underline{y} \neq \underline{0}$ in \mathcal{B}_1 such that $(x_i + y_i - \bar{x}_i - \bar{y}_i) = 0$

We collect those paths for which $H(\underline{y} \otimes \underline{x}) = 0$ in table 6.2. Notice, these are the only paths in $\mathcal{P}(\Lambda_0)$ of weight $\Lambda_0 - 2\delta$.

\underline{x}	\underline{y}	restrictions	count
$\bar{x}_i = 2$	$y_i = 2$	none	n
$\bar{x}_i=1$ $\bar{x}_j=1$	$y_i=1$ $y_j=1$	$i < j$	$\frac{(n)(n-1)}{2}$
$x_i=1$ $\bar{x}_j=1$	$y_i=1$ $y_j=1$	$i > j$	$\frac{(n)(n-1)}{2}$
$x_i=1$ $\bar{x}_i=1$	$y_i=1$ $y_i=1$	none	n

Table 6.2: The paths $(\cdots \otimes \underline{0} \otimes \underline{0} \otimes \underline{y} \otimes \underline{x})$ in $\mathcal{P}(\Lambda_0)$ of weight $\Lambda_0 - 2\delta$

Therefore,

$$\begin{aligned}
 \dim \mathfrak{g}_\alpha &= \dim V(\Lambda_0)_\alpha \\
 &= \#p \in \mathcal{P}(\Lambda_0) \text{ with weight } \alpha \\
 &= n + \frac{n(n-1)}{2} + \frac{n(n-1)}{2} + n \\
 &= n^2 + n.
 \end{aligned}$$

□

Proposition 6.3. *Let $\mathfrak{g} = HC_n^{(1)}$ for any $n \in \mathbb{Z}_{\geq 2}$, and let $\alpha = -\alpha_{-1} - 3\delta$. Then,*

$$\dim \mathfrak{g}_\alpha = \frac{5n^3 + 6n^2 + 7n}{3!}.$$

Proof: $\dim \mathfrak{g}_\alpha$ is simply the number of paths in $\mathcal{P}(\Lambda_0)$ following exactly three 0-arrows, six 1-arrows, six 2-arrows, \dots , six $(n-1)$ -arrows, and three n -arrows. All such paths are of the form

$$p = (\cdots \otimes \underline{0} \otimes \underline{0} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$$

for some

$$\underline{z} = (z_1, \dots, z_n, \bar{z}_n, \dots, \bar{z}_1),$$

$$\underline{y} = (y_1, \dots, y_n, \bar{y}_n, \dots, \bar{y}_1), \text{ and}$$

$$\underline{x} = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1)$$

in \mathcal{B}_1 . Using theorem 5.6, we calculate the weight of such a path.

$$\begin{aligned}
wt(p) &= \Lambda_0 + \sum_{i=1}^n (x_i + y_i + z_i - \bar{x}_i - \bar{y}_i - \bar{z}_i)(\Lambda_i - \Lambda_{i-1}) \\
&\quad - (H(\underline{y} \otimes \underline{x}) + 2H(\underline{z} \otimes \underline{y}) + 3H(\underline{0} \otimes \underline{z})) \delta
\end{aligned}$$

We divide our search into two cases; Case I: $\underline{z} = \underline{0}$ and Case II: $\underline{z} \neq \underline{0}$.

Case I: Suppose $wt(p) = \Lambda_0 - 3\delta$ and $\underline{z} = \underline{0}$. Then

$$\begin{aligned}
\Lambda_0 - 3\delta &= wt(p) \\
&= \Lambda_0 + \sum_{i=1}^n (x_i + y_i - \bar{x}_i - \bar{y}_i)(\Lambda_i - \Lambda_{i-1}) \\
&\quad - (H(\underline{y} \otimes \underline{x}) + 2H(\underline{0} \otimes \underline{y})) \delta
\end{aligned}$$

If $\underline{y} = 0$, then

$$\begin{aligned}
\Lambda_0 - 3\delta &= wt(p) \\
&= \Lambda_0 + \sum_{i=1}^n (x_i - \bar{x}_i)(\Lambda_i - \Lambda_{i-1}) - \delta,
\end{aligned}$$

but this is impossible. We conclude $\underline{y} \neq 0$ and

$$\begin{aligned} \Lambda_0 - 3\delta &= wt(p) \\ &= \Lambda_0 + \sum_{i=1}^n (x_i + y_i - \bar{x}_i - \bar{y}_i)(\Lambda_i - \Lambda_{i-1}) \\ &\quad - (H(\underline{y} \otimes \underline{x}) + 2)\delta. \end{aligned}$$

Thus, a path $p = (\cdots \otimes \underline{0} \otimes \underline{y} \otimes \underline{x})$ ($\underline{y} \neq \underline{0}$) in $\mathcal{P}(\Lambda_0)$ is of weight $\Lambda_0 - 3\delta$ if and only if

$$(x_i + y_i - \bar{x}_i - \bar{y}_i) = 0 \text{ for all } i \text{ and} \quad (6.3)$$

$$H(\underline{y} \otimes \underline{x}) = 1. \quad (6.4)$$

In listing these properties we have called upon the fact that $\{\Lambda_i - \Lambda_{i-1}\}_{i=1}^n$ and $\{\delta\}$ are linearly independent.

Property (6.4) would clearly be met if $\underline{x} = \underline{0}$. In this case, property (6.3) would reduce to the following statement.

$$y_i - \bar{y}_i = 0 \text{ for all } i$$

Table 6.3 lists all paths of the form $p = (\dots \underline{0} \otimes \underline{y} \otimes \underline{0})$ for which $wt(p) = \Lambda_0 - 3\delta$.

We are still considering the case $\underline{z} = \underline{0}$. We know $\underline{y} \neq \underline{0}$ and we have accounted

\underline{y}	restrictions	count
$\begin{smallmatrix} y_i=1 \\ \bar{y}_i=1 \end{smallmatrix}$	none	n

Table 6.3: Paths $p = (\dots \underline{0} \otimes \underline{y} \otimes \underline{0})$ in $\mathcal{P}(\Lambda_0)$ for which $wt(p) = \Lambda_0 - 3\delta$

for those paths for which $\underline{x} = \underline{0}$. The only paths left to consider in this case are those of the form $p = (\dots \otimes \underline{0} \otimes \underline{y} \otimes \underline{x})$ for which both \underline{y} and \underline{x} are not equal to $\underline{0}$. We have already listed all paths of this form with property (6.3) in table 6.1. We collect the paths with both properties (6.3) and (6.4) in table 6.4.

\underline{x}	\underline{y}	restrictions	count
$\begin{smallmatrix} x_i=1 \\ \bar{x}_j=1 \end{smallmatrix}$	$\begin{smallmatrix} y_j=1 \\ \bar{y}_i=1 \end{smallmatrix}$	$i < j$	$\frac{(n)(n-1)}{2}$
$\begin{smallmatrix} x_i=1 \\ \bar{x}_i=1 \end{smallmatrix}$	$\begin{smallmatrix} y_j=1 \\ \bar{y}_j=1 \end{smallmatrix}$	$i \neq j$	$(n)(n-1)$

Table 6.4: Paths $p = (\dots \underline{0} \otimes \underline{y} \otimes \underline{x})$ ($\underline{x} \neq \underline{0}$) for which $wt(p) = \Lambda_0 - 3\delta$

We have now exhausted the possibilities in Case I. Tables 6.3 and 6.4 lists all paths of the form $p = \dots (\dots \underline{0} \otimes \underline{y} \otimes \underline{x})$ for which $wt(p) = \Lambda_0 - 3\delta$.

Case II: Next we move to consider paths of the form $p = \dots (\dots \underline{0} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$

where $\underline{z} \neq \underline{0}$. We require

$$\begin{aligned}
 \Lambda_0 - 3\delta &= wt(p) \\
 &= \Lambda_0 + \sum_{i=1}^n (x_i + y_i + z_i - \bar{x}_i - \bar{y}_i - \bar{z}_i)(\Lambda_i - \Lambda_{i-1}) \\
 &\quad - (H(\underline{y} \otimes \underline{x}) + 2H(\underline{z} \otimes \underline{y}) + 3H(\underline{0} \otimes \underline{z})) \delta \\
 &= \Lambda_0 + \sum_{i=1}^n (x_i + y_i + z_i - \bar{x}_i - \bar{y}_i - \bar{z}_i)(\Lambda_i - \Lambda_{i-1}) \\
 &\quad - (H(\underline{y} \otimes \underline{x}) + 2H(\underline{z} \otimes \underline{y}) + 3) \delta
 \end{aligned}$$

If $\underline{y} = \underline{0}$ then

$$\begin{aligned}
 \Lambda_0 - 3\delta &= \Lambda_0 + \sum_{i=1}^n (x_i + y_i + z_i - \bar{x}_i - \bar{y}_i - \bar{z}_i)(\Lambda_i - \Lambda_{i-1}) \\
 &\quad - (H(\underline{y} \otimes \underline{x}) + 2 + 3) \delta.
 \end{aligned}$$

Thus, depending on the value of \underline{x} , either

$$\Lambda_0 - 3\delta = \Lambda_0 + \sum_{i=1}^n (x_i + y_i + z_i - \bar{x}_i - \bar{y}_i - \bar{z}_i)(\Lambda_i - \Lambda_{i-1}) - 6\delta$$

or

$$\Lambda_0 - 3\delta = \Lambda_0 + \sum_{i=1}^n (x_i + y_i + z_i - \bar{x}_i - \bar{y}_i - \bar{z}_i)(\Lambda_i - \Lambda_{i-1}) - 5\delta,$$

but this is not possible. Therefore, $\underline{y} \neq \underline{0}$. Now suppose $\underline{x} = \underline{0}$.

$$\begin{aligned}
 \Lambda_0 - 3\delta &= \Lambda_0 + \sum_{i=1}^n (x_i + y_i + z_i - \bar{x}_i - \bar{y}_i - \bar{z}_i)(\Lambda_i - \Lambda_{i-1}) \\
 &\quad - (H(\underline{y} \otimes \underline{x}) + 2H(\underline{z} \otimes \underline{y}) + 3) \delta \\
 &= \Lambda_0 + \sum_{i=1}^n (x_i + y_i + z_i - \bar{x}_i - \bar{y}_i - \bar{z}_i)(\Lambda_i - \Lambda_{i-1}) \\
 &\quad - (2H(\underline{z} \otimes \underline{y}) + 4) \delta
 \end{aligned}$$

meaning $2H(\underline{z} \otimes \underline{y}) + 4 = 3$. However, $H(\underline{z} \otimes \underline{y})$ is always a non-negative integer, a contradiction. Hence, $\underline{x} \neq \underline{0}$. We have now established that the only paths of weight $\Lambda_0 - 3\delta$ of the form $(\dots \underline{0} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$ with $\underline{z} \neq \underline{0}$ are paths for which \underline{y} and \underline{x} are not equal to $\underline{0}$ also. We divide such paths into the following categories.

Category A: $z_k = \bar{z}_k$ for some k ($k = 1, \dots, n$).

Category B: $x_k = \bar{x}_k$ and $z_q \neq \bar{z}_q$ for some k and any q ($k, q = 1, \dots, n$).

Category C: $y_k = \bar{y}_k$ and $z_q \neq \bar{z}_q$ and $x_s \neq \bar{x}_s$ for some k and any q, s ,
 $(k, q, s = 1, \dots, n)$.

Category D: $y_k \neq \bar{y}_k$, $z_q \neq \bar{z}_q$ and $x_s \neq \bar{x}_s$ for any k, q, s ($k, q, s = 1, \dots, n$).

As we have done in the previous cases, we use the affine weight formula along with the fact that $\{\Lambda_i - \Lambda_{i-1}\}_{i=1}^n$ and $\{\delta\}$ are linearly independent to formulate requirements for a path in a particular category to be of weight $\Lambda_0 - 3\delta$.

Category A: A path $p = (\dots \underline{0} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$ from category A is of weight $\Lambda_0 - 3\delta$ if and only if

$$x_i + y_i - \bar{x}_i - \bar{y}_i = 0 \text{ for all } i = 1, \dots, n, \quad (6.5)$$

$$H(\underline{y} \otimes \underline{x}) = 0, \text{ and} \quad (6.6)$$

$$H(\underline{z} \otimes \underline{y}) = 0. \quad (6.7)$$

Using table 6.2 we collect all paths $p = (\dots \underline{0} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$ in Category A with properties (6.5) and (6.6) and record our results in table 6.5. We also list $H(\underline{z} \otimes \underline{y})$ for each of these paths. Then, we collect those paths in category A of weight $\Lambda_0 - 3\delta$ in table 6.6.

\underline{x}	\underline{y}	restrictions	$H(\underline{z} \otimes \underline{y})$
$\bar{x}_i=1$ $\bar{x}_j=1$	$y_i=1$ $y_j=1$	$i < j$	$\begin{cases} 2 & 1 < j \leq k \\ 1 & \text{otherwise} \end{cases}$
$x_i=1$ $\bar{x}_j=1$	$\bar{y}_i=1$ $y_j=1$	$i > j$	1
$x_i=1$ $\bar{x}_i=1$	$\bar{y}_i=1$ $y_i=1$	none	$\begin{cases} 0 & i = k \\ 1 & \text{otherwise} \end{cases}$
$\bar{x}_i = 2$	$y_i = 2$	none	$\begin{cases} 2 & k \geq i \\ 1 & \text{otherwise} \end{cases}$

Table 6.5: Some paths in category A

\underline{x}	\underline{y}	\underline{z}	restrictions	count
$x_i=1$ $\bar{x}_i=1$	$y_i=1$ $\bar{y}_i=1$	$z_k=1$ $\bar{z}_k=1$	$i = k$	n

 Table 6.6: Paths from Category A for which $wt(p) = \Lambda_0 - 3\delta$

Category B: A path $p = (\dots \underline{0} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$ from Category B is of weight $\Lambda_0 - 3\delta$ if and only if

$$y_i + z_i - \bar{y}_i - \bar{z}_i = 0 \text{ for all } i = 1, \dots, n, \quad (6.8)$$

$$H(\underline{y} \otimes \underline{x}) = 0 \text{ and} \quad (6.9)$$

$$H(\underline{z} \otimes \underline{y}) = 0. \quad (6.10)$$

Using table 6.2 we collect all paths $p = (\dots \underline{0} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$ in Category B with properties (6.8) and (6.10) and record our results in table 6.7. We also list $H(\underline{y} \otimes \underline{x})$ for each path. We conclude that no paths in category B are of weight $\Lambda_0 - 3\delta$.

\underline{y}	\underline{z}	restrictions	$H(\underline{y} \otimes \underline{x})$
$\bar{y}_i = 2$	$z_i = 2$	none	$\begin{cases} 2 & i \leq k \\ 1 & \text{otherwise} \end{cases}$
$\bar{y}_i=1$ $\bar{y}_j=1$	$z_i=1$ $z_j=1$	$i < j$	$\begin{cases} 2 & i < j \leq k \\ 1 & \text{otherwise} \end{cases}$
$y_i=1$ $\bar{y}_j=1$	$\bar{z}_i=1$ $z_j=1$	$i > j$	1

Table 6.7: Some paths in Category B

Category C: A path $p = (\dots \underline{0} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$ from Category C is of weight

$\Lambda_0 - 3\delta$ if and only if

$$x_i + z_i - \bar{x}_i - \bar{z}_i = 0 \text{ for all } i = 1, \dots, n, \quad (6.11)$$

$$H(\underline{y} \otimes \underline{x}) = 0 \text{ and} \quad (6.12)$$

$$H(\underline{z} \otimes \underline{y}) = 0. \quad (6.13)$$

Using table 6.1 we collect all paths $p = (\dots \underline{z} \otimes \underline{y} \otimes \underline{x})$ in Category C with property (6.11) and record our results in table 6.1. We also list $H(\underline{y} \otimes \underline{x})$ in this table. Note, $H(\underline{z} \otimes \underline{y}) = H(\underline{y} \otimes \underline{z})$ for each path in the table. We list those paths from category C of weight $\Lambda_0 - 3\delta$ in table 6.9.

\underline{x}	\underline{z}	restrictions	$H(\underline{y} \otimes \underline{x})$
$\bar{x}_i = 2$	$z_i = 2$	none	$\begin{cases} 0 & i \geq k \\ 1 & \text{otherwise} \end{cases}$
$\bar{x}_i=1$ $\bar{x}_j=1$	$z_i=1$ $z_j=1$	$i < j$	$\begin{cases} 0 & i \leq k \\ 1 & \text{otherwise} \end{cases}$
$x_i=1$ $\bar{x}_j=1$	$\bar{z}_i=1$ $z_j=1$	$i < j$	1
$x_i=1$ $\bar{x}_j=1$	$\bar{z}_i=1$ $z_j=1$	$i > j$	$\begin{cases} 0 & j \leq k < i \\ 1 & \text{otherwise} \end{cases}$
$x_i = 2$	$\bar{z}_i = 2$	none	$\begin{cases} 2 & i \geq k \\ 1 & \text{otherwise} \end{cases}$
$x_i=1$ $x_j=1$	$\bar{z}_i=1$ $\bar{z}_j=1$	$i < j$	$\begin{cases} 2 & j \leq k \\ 1 & \text{otherwise} \end{cases}$

Table 6.8: Some paths in Category C

Category D: A path $p = (\dots \underline{0} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$ from Category D is of weight

\underline{x}	\underline{y}	\underline{z}	restrictions	count
$\bar{x}_i = 2$	$\frac{y_k=1}{\bar{y}_k=1}$	$z_i = 2$	$i \geq k$	$n + \frac{(n)(n-1)}{2}$
$\frac{\bar{x}_i=1}{\bar{x}_j=1}$	$\frac{y_k=1}{\bar{y}_k=1}$	$\frac{z_i=1}{z_j=1}$	$\frac{i < j}{i \leq k}$	$(n)(n-1) + \frac{(n)(n-1)(n-2)}{3}$
$\frac{x_i=1}{\bar{x}_j=1}$	$\frac{y_k=1}{\bar{y}_k=1}$	$\frac{\bar{z}_i=1}{z_j=1}$	$j \leq k < i$	$\frac{(n)(n-1)}{2} + \frac{(n)(n-1)(n-2)}{6}$

Table 6.9: Paths from Category C of weight $\Lambda_0 - 3\delta$

$\Lambda_0 - 3\delta$ if and only if

$$x_i + y_i + z_i - \bar{x}_i - \bar{y}_i - \bar{z}_i = 0 \text{ for all } i = 1, \dots, n, \quad (6.14)$$

$$H(\underline{y} \otimes \underline{x}) = 0 \text{ and} \quad (6.15)$$

$$H(\underline{z} \otimes \underline{y}) = 0. \quad (6.16)$$

We list all paths $p = (\dots \underline{0} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$ in Category D with property (6.14) in table 6.10. We include $H(\underline{y} \otimes \underline{x})$ in our list. Table 6.11 then lists those paths from table 6.10 with both properties (6.14) and (6.15). We include $H(\underline{z} \otimes \underline{y})$ in our list. Finally, we record all the paths from category D with weight $\Lambda_0 - 3\delta$ in table 6.12.

We are now finished considering Case II.

All that is left to do in calculating $\dim \mathfrak{g}_\alpha$ is to count the number of $\Lambda_0 - paths$ of weight $\Lambda_0 - 3\delta$. These paths are given in tables 6.3, 6.4, 6.6, 6.9, and 6.12.

\underline{x}	\underline{y}	\underline{z}	restrictions	$H(\underline{y} \otimes \underline{x})$
$x_i=1$ $\bar{x}_i=1$	$\bar{y}_j=1$ $\bar{y}_k=1$	$z_i=1$ $z_j=1$	$i \neq j \neq k$	$\begin{cases} 2 & i > k, i > j \\ 1 & \text{otherwise} \end{cases}$
$x_i=1$ $x_j=1$	$y_k=1$ $\bar{y}_i=1$	$\bar{z}_j=1$ $\bar{z}_k=1$	$i \neq j \neq k$	$\begin{cases} 2 & i < k, j < k \\ 1 & \text{otherwise} \end{cases}$
$x_j=1$ $\bar{x}_k=1$	$y_i=1$ $y_j=1$	$z_k=1$ $\bar{z}_i=1$	$i \neq j \neq k$	0
$\bar{x}_j=1$ $\bar{x}_k=1$	$y_k=1$ $\bar{y}_i=1$	$z_i=1$ $z_j=1$	$i \neq j \neq k$	$\begin{cases} 1 & j > k, i > k \\ 0 & \text{otherwise} \end{cases}$
$x_i=1$ $\bar{x}_i=1$	$y_j=1$ $\bar{y}_k=1$	$z_k=1$ $\bar{z}_i=1$	$i \neq j \neq k$	2
$x_k=1$ $\bar{x}_i=1$	$y_i=1$ $y_j=1$	$\bar{z}_j=1$ $\bar{z}_k=1$	$i \neq j \neq k$	$\begin{cases} 1 & j > k, i > k \\ 0 & \text{otherwise} \end{cases}$

Table 6.10: Some paths from Category D

\underline{x}	\underline{y}	\underline{z}	restrictions	$H(\underline{z} \otimes \underline{y})$
$\bar{x}_j=1$ $\bar{x}_k=1$	$y_i=1$ $y_j=1$	$z_k=1$ $\bar{z}_i=1$	$i \neq j \neq k$	$\begin{cases} 2 & i < k, j < k \\ 1 & \text{otherwise} \end{cases}$
$\bar{x}_j=1$ $\bar{x}_k=1$	$y_k=1$ $\bar{y}_i=1$	$z_i=1$ $z_j=1$	$i \neq j < k$ OR $j \neq k < i$	$\begin{cases} 1 & i > k > j \\ 0 & \text{otherwise} \end{cases}$
$x_k=1$ $\bar{x}_i=1$	$y_i=1$ $y_j=1$	$\bar{z}_j=1$ $\bar{z}_k=1$	$i \neq j < k$ OR $i < k < j$	2

Table 6.11: Some paths from Category D

Therefore,

$$\begin{aligned}
mult_{\mathfrak{g}_{-\alpha_{-1}-3\delta}} &= \dim V(\Lambda_0)_{\Lambda_0-3\delta} \\
&= |\{p \in \mathcal{P}(\Lambda_0) | wt(p) = \Lambda_0 - 3\delta\}| \\
&= n + \frac{(n)(n-1)}{2} + (n)(n-1) + n + n + \frac{(n)(n-1)}{2} \\
&\quad + (n)(n-1) + \frac{(n)(n-1)(n-2)}{3} + \frac{(n)(n-1)}{2} \\
&\quad + \frac{(n)(n-1)(n-2)}{6} + \frac{(n)(n-1)(n-2)}{3} \\
&= \frac{5n^3 + 6n^2 + 7n}{6}
\end{aligned}$$

\underline{x}	\underline{y}	\underline{z}	restrictions	count
$\bar{x}_j=1$ $\bar{x}_k=1$	$\bar{y}_k=1$ $\bar{y}_i=1$	$\bar{z}_i=1$ $\bar{z}_j=1$	$i < k, i \neq j \neq k$	$\frac{(n)(n-1)(n-2)}{3}$

Table 6.12: Paths from Category D with weight $\Lambda_0 - 3\delta$

□

Propositions 6.1, 6.2, and 6.3 give root multiplicities of roots, $-\alpha_{-1} - k\delta$, of $HC_n^{(1)}$ for fixed k and varying n . We would also like to investigate root multiplicities of roots, $-\alpha_{-1} - k\delta$, for fixed n and varying k . We will consider roots in $HC_2^{(1)}$ of the form $-\alpha_{-1} - k\delta$.

For any Kac-Moody Algebra of classical type, $X = A, B, C, D$, we have the following formula: (see [16] chapter 12).

$$\sum_{k=1}^{\infty} mult_{V(\Lambda)}(\lambda - k\delta)q^k = \phi(q)^{-n} q^{\frac{dim(X_n) - n(1+\check{h})}{24*(1+\check{h})}} b_{\lambda}^{\Lambda},$$

where, $V(\Lambda)$ is the highest weight $X_n^{(1)}$ -module with highest weight Λ , ϕ is the Euler-phi function, \check{h} is the dual Coxeter number associated with $X_n^{(1)}$, and b_{λ}^{Λ} is the branching function described in [16]. Let $\Lambda = \Lambda_0$, $\lambda = \Lambda_0$, and $X_n^{(1)} = C_2^{(1)}$. Then $dim(X_n) = 10$

, $\check{h} = 3$, and $b_\lambda^\Lambda = b_{\Lambda_0}^{\Lambda_0}$ is given in [17] to be

$$\begin{aligned} b_{\Lambda_0}^{\Lambda_0} &= \chi_{1,1}^{(1)} \\ &= q^{\frac{-1}{24}} \phi(q)^{-1} (f_{12,1} - f_{12,7}), \text{ where} \\ f_{a,b} &= \sum_{m \in \mathbb{Z}} q^{a(m + \frac{b}{2a})^2} \end{aligned}$$

Therefore, if $V(\Lambda_0)$ is the highest weight $C_n^{(1)}$ -module with highest weight Λ_0 we have the following formula,

$$\sum_{k=1}^{\infty} \text{mult}_{V(\Lambda_0)}(\Lambda_0 - k\delta) q^k = \sum_{m \in \mathbb{Z}} (q^{m(12m+1)} - q^{(2m+1)(4m+1)}) \phi^{-3}(q) \quad (6.17)$$

Notice (6.17) is a generating function for the multiplicity of $\Lambda_0 - k\delta$ as a weight in the highest weight $C_n^{(1)}$ -module, $V(\Lambda_0)$, which leads to proposition 6.4.

Proposition 6.4. *Let $\mathfrak{g} = HC_2^{(1)}$. Then the multiplicity of the root $-\alpha_{-1} - k\delta$ in \mathfrak{g} is given by the coefficient of q^k in the right hand side of expression (6.17).*

In table 6.13 we use proposition 6.4 to give the multiplicities of $-\alpha_{-1} - k\delta$ ($1 \leq k \leq 10$) as a root of $HC_2^{(1)}$.

k	$mult(-\alpha_{-1} - k\delta)$
1	2
2	6
3	13
4	29
5	57
6	112
7	205
8	372
9	647
10	1110

Table 6.13: Multiplicities of roots $-\alpha_{-1} - k\delta$ in $HC_2^{(1)}$

6.2 Level Two Root Multiplicities

In this section we wish to investigate multiplicities of roots, $\alpha = -2\alpha_{-1} - k\delta$, of $HC_n^{(1)}$.

In chapter 4, we gave a rather complicated formula for the multiplicity of a root α of $HC_n^{(1)}$. Recall,

$$\dim(\mathfrak{g}_\alpha) = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \left(\frac{\tau}{\alpha}\right) \mathcal{B}(\tau).$$

Now $\frac{\alpha}{2}$ is the only possible τ which divides $\alpha = -2\alpha_{-1} - k\delta$. Thus, we may simplify the multiplicity formula given above.

$$\dim \mathfrak{g}_{-2\alpha_{-1} - k\delta} = \mathcal{B}(-2\alpha_{-1} - k\delta) - \frac{\delta_{(0),(k \bmod 2)}}{2} \mathcal{B}\left(-\alpha_{-1} - \frac{k}{2}\delta\right),$$

where

$$\begin{aligned}\mathcal{B}(\tau) &= \sum_{(n_i \tau_i) \in T(\tau)} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod \kappa_{\tau_i}^{n_i} \\ \kappa_{\tau_i} &= \sum_{\omega \in W(S)} (-1)^{l(\omega)+1} \dim V(\omega\rho - \rho)_{\tau_i}.\end{aligned}$$

Next, we must calculate a sufficient number of $\omega \in W(S)$. We do so using the following well known lemma and record our results in table 6.14.

Lemma 6.1. *Suppose $\omega = \omega' r_j$ and $l(\omega) = l(\omega') + 1$. Then, $\omega \in W(S)$ if and only if $\omega' \in W(S)$ and $\omega'(\alpha_j) \in \Delta^+(S)$.*

length= l	$\omega \in W(S), \quad l(\omega) = l$	$\omega\rho - \rho, \alpha$ -basis	$\omega\rho - \rho, \Lambda$ -basis	level
1	r_{-1}	$-\alpha_{-1}$	Λ_0	1
2	$r_{-1}r_0$	$-2\alpha_{-1} - \alpha_0$	$2\Lambda_1 - \delta$	2

Table 6.14: $\omega\rho - \rho$ for $\omega \in W(S)$

Let,

$$\begin{aligned}M_1 &= \sum_{\substack{\lambda_i < \eta_i \\ \lambda_i + \eta_i = -2\alpha_{-1} - k\delta}} \dim V(\Lambda_0)_{\lambda_i} \dim V(\Lambda_0)_{\eta_i} \\ M_2 &= \dim V(2\Lambda_1 - \delta)_{-2\alpha_{-1} - k\delta}, \text{ and} \\ M_3 &= \dim V(\Lambda_0)_{-\alpha_{-1} - \frac{k}{2}\delta}.\end{aligned} \tag{6.18}$$

Then the multiplicity formula for a level two root $-2\alpha - k\delta$ of $HC_n^{(1)}$ becomes

$$\begin{aligned}
 \dim \mathfrak{g}_{-2\alpha_{-1}-k\delta} &= M_1 + \frac{\delta_{(0),(k \bmod 2)}}{2} M_3^2 - M_2 - \frac{\delta_{(0),(k \bmod 2)}}{2} M_3 \\
 &= M_1 - M_2 + \frac{\delta_{(0),(k \bmod 2)}}{2} M_3(M_3 - 1)
 \end{aligned} \tag{6.19}$$

Example 6.2. *In this example we calculate the multiplicity of the root $-2\alpha_{-1} - 2\delta$ of $HC_n^{(1)}$.*

We will rely upon the multiplicity formula given in (6.19). Let us begin by calculating M_1 . First, we list all partitions, (λ_i, η_i) of $-\alpha_{-1} - 2\delta$ into two distinct weights in $V(\Lambda_0)$.

$$\begin{aligned}
 \lambda_i &= \text{row } i \text{ of the matrix } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} \\
 \eta_i &= \text{row } i \text{ of the matrix } \begin{pmatrix} 1 & 2 & 4 & 2 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 3 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}
 \end{aligned}$$

Using the path realization, $\mathcal{P}(\Lambda_0)$, given in figure 6.1, we count the number of paths with weights λ_i , and η_i for each i and list our results in table 6.2. Summing the entries in the last column of table 6.2 leads us to conclude $M_1 = 10$.

i	# paths with weight λ_i	# paths with weight η_i	$\dim V(\Lambda_0)_{\lambda_i} \dim V(\Lambda_0)_{\eta_i}$
1	1	6	6
2	1	1	1
3	1	1	1
4	1	1	1
5	1	1	1

Next, we must calculate M_2 . Notice,

$$\dim V(2\Lambda_1 - \delta)_{-2\alpha_{-1} - 2\delta} = \dim V(2\Lambda_1)_{-2\alpha_{-1} - \delta}$$

We must find the number of paths in the path realization $\mathcal{P}(2\Lambda_1)$ following exactly one 0-arrow, four 1-arrows, and two 2-arrows. We perform this calculation using the MATLAB program, *mult.m* found in appendix A, and conclude $M_2 = 9$. Finally, we need to calculate M_3 . In example 6.1 we showed that

$$\dim V(-\alpha_{-1})_{-\alpha_{-1} - \delta} = 2$$

Therefore, $M_3 = 2$. We conclude

$$\dim \mathfrak{g}_\alpha = 10 - 9 + \frac{1}{2}(2)(2 - 1) = 2.$$

Using the same technique as in example 6.2 we will find the multiplicity of the root $-2\alpha_{-1} - 3\delta$ of $HC_n^{(1)}$, for any $n \geq 2$. (We will show in a later section that the multiplicity of the root $-2\alpha_{-1} - 2\delta$ of $HC_n^{(1)}$ is n .)

Referring to equation 6.19 we see

$$\dim \mathfrak{g}_{-2\alpha_{-1}-3\delta} = M_1 - M_2 \tag{6.20}$$

where

$$M_1 = \sum_{\substack{\lambda_i < \eta_i \\ \lambda_i + \eta_i = \alpha}} \dim V(-\alpha_{-1})_{\lambda_i} \dim V(-\alpha_{-1})_{\eta_i} \text{ and} \tag{6.21}$$

$$M_2 = \dim V(-2\alpha_{-1} - \alpha_0)_{-2\alpha_{-1}-3\delta} \tag{6.22}$$

Our problem now consists of calculating weight multiplicities in certain highest weight $C_n^{(1)}$ -modules. We make use of crystal basis theory to calculate these multiplicities.

Lemma 6.2. *Let $V(\Lambda_0)$ be the highest weight $C_n^{(1)}$ -module with highest weight Λ_0 .*

Then

$$\sum_{\substack{\lambda_i < \eta_i, \eta_i \neq \Lambda_0 \\ \lambda_i + \eta_i = -2\alpha_{-1} - 3\delta}} \dim V(\Lambda_0)_{\lambda_i} \cdot \dim V(\Lambda_0)_{\eta_i} = 5n^3 - 3n^2 + 2n$$

Proof:

In table 6.15 we list all possible possible λ_i and η_i such that $\lambda_i + \eta_i = 2\Lambda_0 - 3\delta$.

t	$\lambda_{i,j}^{(t)} = \Lambda_0 + \mu - \delta,$ μ given below	$\eta_{i,j}^{(t)} = \Lambda_0 + \nu - 2\delta$ ν given below	restrictions
1	$2(\Lambda_i - \Lambda_{i-1})$	$-2(\Lambda_i - \Lambda_{i-1})$	$i = 1..n$
2	$-2(\Lambda_i - \Lambda_{i-1})$	$2(\Lambda_i - \Lambda_{i-1})$	$i = 1..n$
3	$(\Lambda_i - \Lambda_{i-1}) + (\Lambda_j - \Lambda_{j-1})$	$-(\Lambda_i - \Lambda_{i-1}) - (\Lambda_j - \Lambda_{j-1})$	$i = 1..n$ $j = 1..n$ $i > j$
4	$-(\Lambda_i - \Lambda_{i-1}) - (\Lambda_j - \Lambda_{j-1})$	$(\Lambda_i - \Lambda_{i-1}) + (\Lambda_j - \Lambda_{j-1})$	$i = 1..n$ $j = 1..n$ $i > j$
5	$(\Lambda_i - \Lambda_{i-1}) - (\Lambda_j - \Lambda_{j-1})$	$-(\Lambda_i - \Lambda_{i-1}) + (\Lambda_j - \Lambda_{j-1})$	$i = 1..n$ $j = 1..n$ $i \neq j$
6	0	0	none

Table 6.15: $\lambda_{i,j}^{(t)}$ and $\eta_{i,j}^{(t)}$ such that $\lambda_{i,j}^{(t)} + \eta_{i,j}^{(t)} = 2\Lambda_0 - 3\delta$

For fixed i and j , $\dim V(\Lambda_0)_{\lambda_{i,j}} = 1$ for classes $t = 1 - 5$ and $\dim V(\Lambda_0)_{\lambda_{i,j}^{(6)}} = n$.

Thus,

$$\begin{aligned}
\sum_{\substack{\lambda_i < \eta_i, \eta_i \neq \Lambda_0 \\ \lambda_i + \eta_i = -2\alpha_{-1} - 3\delta}} \dim V(\Lambda_0)_{\lambda_i} \cdot \dim V(\Lambda_0)_{\eta_i} &= \sum_{\substack{(\lambda_{i,j}^{(t)}, \eta_{i,j}^{(t)}) \\ t=1..5}} \dim V(\Lambda_0)_{\lambda_{i,j}^{(t)}} \cdot \dim V(\Lambda_0)_{\eta_{i,j}^{(t)}} \\
&= \sum_{\substack{(\lambda_{i,j}^{(t)}, \eta_{i,j}^{(t)}) \\ t=1..5}} \dim V(\Lambda_0)_{\lambda_{i,j}^{(t)}} \cdot \dim V(\Lambda_0)_{\eta_{i,j}^{(t)}} \\
&\quad + \dim V(\Lambda_0)_{\Lambda_0 - \delta} \cdot \dim V(\Lambda_0)_{\Lambda_0 - 2\delta} \\
&= \sum_{t=1}^5 \dim V(\Lambda_0)_{\eta_{i,j}^{(t)}} + n \cdot \dim V(\Lambda_0)_{\Lambda_0 - 2\delta} \\
&= \sum_{t=1}^5 \dim V(\Lambda_0)_{\eta_{i,j}^{(t)}} + n \cdot (n^2 + n) \\
&\quad \text{(see proposition (6.2))}
\end{aligned}$$

Next, we find the number of Λ_0 -paths in $\mathcal{P}(\Lambda_0)$ with weights $\eta_{i,j}^{(t)}$ for $t = 1..5$.

Combining this information with the above equation, we will prove our lemma.

Consider $\eta_{i,j}^{(3)} = \Lambda_0 - (\Lambda_i - \Lambda_{i-1}) - (\Lambda_j - \Lambda_{j-1}) - 2\delta$. All paths of weights $\eta_{i,j}^{(3)}$ must follow exactly two zero arrows and thus be of the form $(\cdots \otimes \underline{0} \otimes \underline{y} \otimes \underline{x})$ for some $\underline{x}, \underline{y} \in B_1$.

$$\begin{aligned}
\eta_{i,j}^{(3)} &= \Lambda_0 - (\Lambda_i - \Lambda_{i-1}) - (\Lambda_j - \Lambda_{j-1}) - 2\delta \\
&= wt(\cdots \otimes \underline{0} \cdots \otimes \underline{0} \otimes \underline{y} \otimes \underline{x}) \\
&= \Lambda_0 + \sum_{k=1}^n (x_k + y_k - \bar{x}_k - \bar{y}_k)(\Lambda_k - \Lambda_{k-1}) \\
&\quad - [H(\underline{y} \otimes \underline{x}) + 2H(\underline{0} \otimes \underline{y})] \delta \\
&= \Lambda_0 + \sum_{k=1}^n (x_k + y_k - \bar{x}_k - \bar{y}_k)(\Lambda_k - \Lambda_{k-1}) - [2 + H(\underline{y} \otimes \underline{x})] \delta
\end{aligned}$$

So that,

$$\begin{aligned}
 0 &= (\Lambda_i - \Lambda_{i-1}) + (\Lambda_j - \Lambda_{j-1}) + \\
 &\quad \sum_{k=1}^n (x_k + y_k - \bar{x}_k - \bar{y}_k)(\Lambda_k - \Lambda_{k-1}) \\
 &\quad + [H(\underline{y} \otimes \underline{x})] \delta
 \end{aligned}$$

Therefore a path $p = (\dots \underline{0} \otimes \underline{y} \otimes \underline{x})$ is of weight $\eta_{ij}^{(3)}$ if and only if

$$(x_i + y_i - \bar{x}_i - \bar{y}_i) = -1, \quad (6.23)$$

$$(x_j + y_j - \bar{x}_j - \bar{y}_j) = -1, \quad (6.24)$$

$$(x_k + y_k - \bar{x}_k - \bar{y}_k) = 0 \text{ (for all } k \neq i, j \text{) and} \quad (6.25)$$

$$H(\underline{y} \otimes \underline{x}) = 0, \quad (6.26)$$

since $\{\Lambda_k - \Lambda_{k-1}\}_{k=1}^n$, and $\{\delta\}$ are linearly independent. Table 6.16 lists all paths $(\dots \otimes \underline{0} \otimes \underline{0} \otimes \underline{y} \otimes \underline{x})$ have properties (6.23) through (6.26).

Using similar techniques we find the number of paths $\eta_{i,j}^{(t)}$ for $t = 1 \dots 5$ and record our results in table 6.17.

$\underline{x}, \underline{y}$	restrictions	count
$y_j = 1 \quad \bar{y}_i = 1$ $\bar{x}_j = 2$	$i, j = 1..n$ $i > j$	$\frac{(n)(n-1)}{2}$
$y_j = 1 \quad \bar{y}_j = 1$ $\bar{x}_i = 1 \quad \bar{x}_j = 1$	$i, j = 1..n$ $i > j$	$\frac{(n)(n-1)}{2}$
$y_k = 1 \quad \bar{y}_i = 1$ $\bar{x}_j = 1 \quad \bar{x}_k = 1$	$i, j, k = 1..n$ $k > i > j$	$\frac{(n)(n-1)(n-2)}{6}$
$y_k = 1 \quad \bar{y}_k = 1$ $\bar{x}_i = 1 \quad \bar{x}_j = 1$	$i, j, k = 1..n$ $i > j, k > j$ $k \neq i$	$\frac{(n)(n-1)(n-2)}{3}$
$y_k = 1 \quad y_j = 1$ $\bar{x}_i = 1 \quad \bar{x}_k = 1$	$i, j, k = 1..n$ $i > j$ $k \neq i, k \neq j$	$\frac{(n)(n-1)(n-2)}{2}$

 Table 6.16: paths, $(\cdots \otimes \underline{0} \otimes \underline{0} \otimes \underline{y} \otimes \underline{x})$, with weights $\eta_{i,j}^{(3)}$

t	number of paths with weight $\eta_{i,j}^t$
1	n^2
2	n^2
3	$(n)(n-1)^2$
4	$(n)(n-1)^2$
5	$2(n^2)(n-1)$

 Table 6.17: number of paths with weight $\eta_{i,j}^{(t)}$

Therefore,

$$\begin{aligned}
 \sum_{\substack{\lambda_i < \eta_i, \eta_i \neq \Lambda_0 \\ \lambda_i + \eta_i = -2\alpha_{-1} - 3\delta}} \dim V(\Lambda_0)_{\lambda_i} \cdot \dim V(\Lambda_0)_{\eta_i} &= \sum_{t=1}^5 \dim V(\Lambda_0)_{\eta_{i,j}^{(t)}} + n(n^2 + n) \\
 &= n^2 + n^2 + (n)(n-1)^2 + (n)(n-1)^2 \\
 &\quad + 2(n)^2(n-1) + (n)(n^2 + n) \\
 &= 5n^3 - 3n^2 + 2n.
 \end{aligned}$$

□

Lemma 6.3. *Let $V(2\Lambda_1)$ be the highest weight $C_n^{(1)}$ -module with highest weight $2\Lambda_1$.*

Then we have the following formula.

$$\dim V(2\Lambda_1)_{2\Lambda_0 - 2\delta} = 5n^3 - 3n^2 + 2n.$$

Proof: Let $\mathcal{P}(2\Lambda_1)$ be the path realization of the highest weight $C_n^{(1)}$ -module $V(2\Lambda_1)$.

Let $p \in \mathcal{P}(2\Lambda_1)$ be of weight $2\Lambda_0 - 2\delta$. Notice, $2\Lambda_1 - (2\Lambda_0 - 2\delta) = 2\alpha_0 + 6\alpha_1 + \cdots + 6\alpha_{n-1} + 3\alpha_n$. Therefore p must follow two 0-arrows, six 1-arrows, \dots , six $(n-1)$ -

arrows, and three n -arrows. The ground state path of weight $2\Lambda_1$ is $(\dots \underline{2} \otimes \underline{2})$ where

$\underline{2} = (2, \underbrace{0, \dots, 0}_{2(n-1) \text{ times}}, 2)$. The only \tilde{f}_i for which $\tilde{f}_i(\underline{2})$ exists is \tilde{f}_1 . Thus,

$$p = (\dots \underline{2} \otimes \underline{s} \otimes \underline{r} \otimes \underline{w} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$$

for some $\underline{s}, \underline{r}, \underline{w}, \underline{z}, \underline{y}, \underline{x}$ in \mathcal{B}_2 , the perfect $C_n^{(1)}$ perfect crystal of level two given in

example 5.3. We require

$$\begin{aligned}
 2\Lambda_0 - 2\delta &= wt(p) \\
 &= 2\Lambda_1 - \sum_{i=1}^n (x_i + y_i + z_i + w_i + r_i + s_i \\
 &\quad - \bar{x}_i - \bar{y}_i - \bar{z}_i - \bar{w}_i - \bar{r}_i - \bar{s}_i)(\Lambda_i - \Lambda_{i-1}) \\
 &\quad - [H(\underline{y} \otimes \underline{x}) + 2H(\underline{z} \otimes \underline{y}) + 3H(\underline{w} \otimes \underline{z}) \\
 &\quad + 4H(\underline{r} \otimes \underline{w}) + 5H(\underline{s} \otimes \underline{r}) + 6H(\underline{2} \otimes \underline{s})] \delta
 \end{aligned}$$

We know $\{\Lambda_i - \Lambda_{i-1}\}_{i=1}^n$ and $\{\delta\}$ are linearly independent, and we know $H(\underline{a} \otimes \underline{b})$ is a non-negative integer for all $\underline{a}, \underline{b} \in \mathcal{B}_2$. Therefore,

$$H(\underline{w} \otimes \underline{z}) = H(\underline{r} \otimes \underline{z}) = H(\underline{s} \otimes \underline{r}) = H(\underline{2} \otimes \underline{s}) = 0. \quad (6.27)$$

First let us consider the implications of the statement $H(\underline{2} \otimes \underline{s}) = 0$. Referring to the definition of H given in example 5.6, we notice

$$\begin{aligned}
 0 &= H(\underline{2} \otimes \underline{s}) \\
 &\geq \theta'_1(\underline{2} \otimes \underline{s}) \\
 &= \frac{1}{2}(4 - s(\underline{s}))
 \end{aligned}$$

Thus, $s(\underline{s})$ must be greater than or equal to four. $\underline{s} \in \mathcal{B}_2$, meaning $s(\underline{s})$ must be less than or equal to four. Hence, $s(\underline{s})=4$.

Next notice

$$\begin{aligned} 0 &= H(\underline{2} \otimes \underline{s}) \\ &\geq \theta_2(\underline{2} \otimes \underline{s}) \\ &= 2 - \bar{s}_1 \end{aligned}$$

Thus $\bar{s}_1 \geq 2$. We conclude,

$$\text{if } H(\underline{2} \otimes \underline{s}) = 0 \text{ then } s(\underline{s}) = 4 \text{ and } \bar{s}_1 \geq 2. \quad (6.28)$$

Next let us consider the implications of the statement $H(\underline{2} \otimes \underline{s}) = 0$ and $H(\underline{s} \otimes \underline{r}) = 0$. By statement (6.28) we know that $s(\underline{s}) = 4$ and $\bar{s}_1 \geq 2$. Notice,

$$\begin{aligned} 0 &= H(\underline{s} \otimes \underline{r}) \\ &\geq \theta'_1(\underline{s} \otimes \underline{r}) \\ &= \frac{1}{2}(4 - s(\underline{r})) \end{aligned}$$

Thus, $s(\underline{r}) = 4$. Next consider

$$\begin{aligned} 0 &= H(\underline{s} \otimes \underline{r}) \\ &\geq \eta_1(\underline{s} \otimes \underline{r}) \\ &= 2 - s_1 \end{aligned}$$

Thus, $s_1 \geq 2$. Combining this fact with the restrictions given in (6.28), we see \underline{s} is completely determined. We conclude,

$$\text{if } H(\underline{2} \otimes \underline{s}) = 0 \text{ and } H(\underline{s} \otimes \underline{r}) = 0 \text{ then } \underline{s} = \underline{2} \quad (6.29)$$

Combining (6.29) and (6.27), we see

$$\underline{s} = \underline{r} = \underline{w} = \underline{2}, \quad s(\underline{z}) = 4, \quad \text{and } \bar{z}_1 \geq 2. \quad (6.30)$$

We now realize any path $p \in \mathcal{P}(2\Lambda_1)$ is of weight $2\Lambda_0 - 2\delta$ if and only if where $s(\underline{z}) = 4$, $\bar{z}_1 \geq 2$, and

$$\begin{aligned} 2\Lambda_0 - 2\delta &= 2\Lambda_1 + \sum_{i=1}^n (x_i + y_i + z_i - \bar{x}_i - \bar{y}_i - \bar{z}_i)(\Lambda_i - \Lambda_{i-1}) \\ &\quad - (H(\underline{y} \otimes \underline{x}) + 2H(\underline{z} \otimes \underline{y})) \delta \end{aligned} \quad (6.31)$$

Calling upon the fact that $\{\Lambda_i - \Lambda_{i-1}\}$ and $\{\delta\}$ are linearly independent, we restate (6.31) as follows

$$\bar{x}_1 + \bar{y}_1 + \bar{z}_1 - x_1 - y_1 - z_1 = 2 \quad (6.32)$$

$$\bar{x}_i + \bar{y}_i + \bar{z}_i - x_i - y_i - z_i = 0 \text{ for all } i = 2 \dots n \quad (6.33)$$

$$H(\underline{z} \otimes \underline{y}) = 1 \quad \text{and} \quad H(\underline{y} \otimes \underline{x}) = 0 \text{ or} \quad (6.34)$$

$$H(\underline{z} \otimes \underline{y}) = 0 \quad \text{and} \quad H(\underline{y} \otimes \underline{x}) = 2$$

We divide our search into the following cases:

Case 1: $s(\underline{z}) = 4, s(\underline{y}) = 4, s(\underline{x}) = 4$

Case 2: $s(\underline{z}) = 4, s(\underline{y}) = 4, s(\underline{x}) = 2$

Case 3: $s(\underline{z}) = 4, s(\underline{y}) = 4, s(\underline{x}) = 0$

Case 4: $s(\underline{z}) = 4, s(\underline{y}) = 2, s(\underline{x}) = 4$

Case 5: $s(\underline{z}) = 4, s(\underline{y}) = 2, s(\underline{x}) = 2$

Case 6: $s(\underline{z}) = 4, s(\underline{y}) = 2, s(\underline{x}) = 0$

Case 7: $s(\underline{z}) = 4, s(\underline{y}) = 0, s(\underline{x}) = 4$

Case 8: $s(\underline{z}) = 4, s(\underline{y}) = 0, s(\underline{x}) = 2$

Case 9: $s(\underline{z}) = 4, s(\underline{y}) = 0, s(\underline{x}) = 0$

In all cases, \underline{z} must meet the following requirement,

$$\underline{z} = \underline{q} + \underline{w} \text{ where } \bar{q}_1 = 2, \text{ and } s(\underline{q}) = s(\underline{w}) = 2 \quad (6.35)$$

We wish to count the number of paths from each case of weight $2\Lambda_0 - 2\delta$. That is we wish to find the number of paths in each case for which all requirements (6.35), (6.32), (6.33), and (6.34) hold.

No paths from case 4 are of weight $2\Lambda_0 - 2\delta$. To see this let p be a path in Case 4. Then $p = (\cdots \otimes \underline{2} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$, with $s(\underline{z}) = 4, s(\underline{y}) = 2$, and $s(\underline{x}) = 4$. Using

the energy function definition give in 5.6, we see $H(\underline{z} \otimes \underline{y}) \geq \theta'_1(\underline{z} \otimes \underline{y}) = 1$. Then requirement (6.34) forces $H(\underline{y} \otimes \underline{x}) = 0$. But $H(\underline{y} \otimes \underline{x}) \geq \theta_1(\underline{y} \otimes \underline{x}) = 1$. Therefore, no paths from Case 4 are of weight $2\Lambda_0 - 2\delta$. Similarly, Cases 6,7,8,and 9 do not lead to any paths of weight $2\Lambda_0 - 2\delta$. Let us consider the remaining cases.

Case 1: Let p be a path from case 1. Then, $p = (\cdots \otimes \underline{2} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$, with $s(\underline{z}) = 4$, $s(\underline{y}) = 4$, $s(\underline{x}) = 4$. A path p from case I of weight $2\Lambda_0 - 2\delta$ must be from one of the following two categories.

Case I, Category A:

$$\underline{z} = \underline{q} + \underline{y}'$$

$$\text{where } \bar{q}_1 = 2 \text{ and } s(\underline{q}) = s(\underline{y}') = 2$$

$$\underline{y} = \underline{x}' + \underline{y}''$$

$$\text{where } \underline{x}' \text{ is the corresponding entry to } \underline{y}' \text{ in table 6.1}$$

$$\underline{x} = \underline{x}'' + \underline{a}$$

$$\text{where } \underline{x}'' \text{ is the corresponding entry to } \underline{y}'' \text{ in table 6.1}$$

$$\text{and } a_k = \bar{a}_k = 1 \text{ with } s(\underline{a}) = 2$$

Case I, Category B:

$$\underline{z} = \underline{q} + \underline{y}'$$

$$\text{where } \bar{q}_1 = 2 \text{ and } s(\underline{q}) = s(\underline{y}') = 2$$

$$\underline{y} = \underline{y}'' + \underline{a}$$

$$\text{where } a_k = \bar{a}_k = 1 \text{ with } s(\underline{a}) = 2$$

$$\underline{x} = \underline{x}'' + \underline{x}'$$

$$\text{where } \underline{x}'' \text{ is the corresponding entry to } \underline{y}'' \text{ in table 6.1}$$

$$\text{and } \underline{x}' \text{ is the corresponding entry to } \underline{y}' \text{ in table 6.1}$$

First we will show categories A and B contain all the possible paths in case I of weight $2\Lambda_0 - 2\delta$. Suppose $p' = (\cdots \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$ is a path in Case I which is not in category A or in category B. Furthermore, suppose $z_1 = 0$. Then $\eta_1(\underline{z} \otimes \underline{y}) = 2$ which contradicts (6.34). Thus $z_1 = 1$ or $z_1 = 2$ (since $s(\underline{z})=4$). Suppose $z_1 = 2$. If $\bar{y}_1 \geq 2$ then p' will be in category A and if $\bar{x}_1 \geq 2$ then p' will be in category B. Thus (6.32) implies $\bar{y}_1 = 1, \bar{x}_1 = 1, y_1 = 0$, and $x_1 = 0$. Now, $\eta_1(\underline{y} \otimes \underline{x}) = \bar{y}_1 - y_1 = 1$ which means $H(\underline{y} \otimes \underline{x}) = 2$ and $H(\underline{z} \otimes \underline{y}) = 0$ (refer to 6.34). But, $\theta_2(\underline{z} \otimes \underline{y}) = \bar{z}_1 - \bar{y}_1 = 1$, a contradiction. Hence, $z_1 \neq 2$. Suppose $z_1 = 1$ and $z_k = 1$ (or $\bar{z}_k = 1$) for some $k = 2 \dots n$. Notice, $\eta_1(\underline{z} \otimes \underline{y}) = \bar{z}_1 - z_1 = 1$. Thus (6.34) implies $H(\underline{z} \otimes \underline{y}) = 1$ and $H(\underline{y} \otimes \underline{x}) = 0$. Also notice, $\theta_2(\underline{z} \otimes \underline{y}) = 2 - \bar{y}_1$, which implies \bar{y}_1 is greater than or equal

to one. Now, if $\bar{y}_k \geq 1$ (or $y_k \geq 1$) then p' will be a member of category A. Thus, (6.33) implies $\bar{x}_k \geq 1$ (or $x_k \geq 1$). Notice $\theta_2(\underline{y} \otimes \underline{x}) = \bar{y}_1 - \bar{x}_1$, meaning $\bar{x}_1 \geq \bar{y}_1 \geq 1$. We conclude p' is an element of category B. Hence, any path which falls under case I must be in either category A or category B.

Next, we determine which paths in category A and which new paths in category B are of weight $2\Lambda_0 - 2\delta$.

Case I Category A: Let us consider those paths $p = (\cdots \otimes \underline{2} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$ where

$\underline{z}, \underline{y}$, and \underline{x} are as described Case I, Category A above. Suppose that \underline{y}' and \underline{x}' are of type one in table 6.1. That is $y'_i = 2$ and $\bar{x}'_i = 2$ for some i . Notice, $\eta_1(\underline{z} \otimes \underline{y}) = \bar{z}_1 - z_1 = 2 - y'_1$. We know $H(\underline{z} \otimes \underline{y}) < 2$ and thus conclude $i = 1$. Therefore, if \underline{y}' and \underline{x}' are of type one we are looking for p of the form $p = (\cdots \underline{2} \otimes \underline{z} \otimes \underline{y} \otimes \underline{x})$ with

$$\underline{z} \quad : \quad \bar{z}_1 = z_1 = 2$$

$$\underline{y} \quad = \quad \underline{b} + \underline{y}''$$

$$\text{where } \bar{b}_1 = 2, s(\underline{b}) = 2. \tag{6.36}$$

$$\underline{x} \quad = \quad \underline{x}'' + \underline{a}$$

where \underline{x}'' is the corresponding entry to \underline{y}'' in table 6.1

and $a_k = \bar{a}_k = 1$ with $s(\underline{a}) = 2$

$H(\underline{z} \otimes \underline{y}) = 0$ for all such paths. Thus a path from Case I, category A with $\underline{x}', \underline{y}'$

$\underline{y}'', \underline{x}''$ of type	restrictions	count
one	$i \neq 1$	$(n)(n-1)$
two	$i \neq 1$	$\frac{(n)(n-1)(n-2)}{2}$
three	$\begin{cases} j = 1 & i \neq 1, i \leq k \text{ or} \\ j \neq 1 & j < i \end{cases}$	$(n-1) + \frac{(n-1)(n-2)}{2} + \frac{(n)(n-1)(n-2)}{2}$
four	$k < i < j$	$\frac{(n)(n-1)(n-2)}{6}$
five	$\begin{cases} i = k & i \neq j \text{ or} \\ i < k & i < j \end{cases}$ ($i \leq k$ to avoid double counting)	$(n)(n-1) + \frac{(n)(n-1)(n-2)}{3} + \frac{(n)(n-1)}{2}$
six	$i = j = k$ to avoid double counting	n
seven	never	0
eight	never	0

Table 6.18: Paths in Case I, Category A with \underline{x}' and \underline{y}' of type one which are of weight $2\Lambda_0 - 2\delta$

of type one is of weight $2\Lambda_0 - 2\delta$ if and only if

$$\underline{z}, \underline{y}, \text{ and } \underline{x} \text{ are as in (6.36)} \quad (6.37)$$

$$H(\underline{y} \otimes \underline{x}) = 2 \quad (6.38)$$

We list all paths from Case I, Category A with \underline{x}' and \underline{y}' of type one meeting both requirements (6.37) and (6.38) in table 6.18.

Next suppose that \underline{y}' and \underline{x}' are of type two in table 6.1. Furthermore, suppose $y'_1 \neq 1$. Then, $y'_1 = 0$ and $\eta_1(\underline{z} \otimes \underline{y}) = 2 - 0 = 2$. Yet, we know $H(\underline{z} \otimes \underline{y}) < 2$, a contradiction. Therefore, $y'_1 = 1$ and consequently $\bar{x}'_1 = 1$. Now, notice $H(\underline{z} \otimes \underline{y}) \geq \eta_1(\underline{z} \otimes \underline{y}) = 2 - 1 = 1$. We know $H(\underline{z} \otimes \underline{y})$ is equal to zero or one. Thus, $H(\underline{z} \otimes \underline{y}) = 1$ and $H(\underline{y} \otimes \underline{x}) = 0$. Finally, suppose $\bar{x}_1 = 0$. Then,

$\theta_2(\underline{y} \otimes \underline{x}) = \bar{y}_1 - 0 = \bar{y}_1$. However, this is impossible since $\bar{y}_1 \geq \bar{x}'_1 = 1$. Thus $\bar{x}_1 \geq 1$. Therefore, either

Case I, Category A(i):

$$\underline{z} \quad : \quad \bar{z}_1 = 2, z_1 = 1, z_k = 1 (k = 2, \dots, n)$$

$$\underline{y} \quad = \quad \underline{b} + \underline{y}''$$

$$\text{where } \bar{b}_1 = 1, \bar{b}_k = 1, s(\underline{b}) = 2$$

$$\underline{x} \quad = \quad \underline{x}'' + \underline{a}$$

$$\text{where } \underline{x}'' \text{ is the corresponding entry to } \underline{y}'' \text{ in table 6.1}$$

$$\text{and } a_1 = \bar{a}_1 = 1 \text{ with } s(\underline{a}) = 2$$

or

Case I, Category A(ii):

$$\underline{z} \quad : \quad \bar{z}_1 = 2, z_1 = 1, z_j = 1 (j = 2, \dots, n)$$

$$\underline{y} \quad = \quad \underline{b} + \underline{y}''$$

$$\text{where } \bar{b}_1 = 1, \bar{b}_j = 1, s(\underline{b}) = 2$$

$$\underline{x} \quad = \quad \underline{x}'' + \underline{a}$$

$$\text{where } \underline{x}'' \text{ is the corresponding entry to } \underline{y}'' \text{ in table 6.1 } (\bar{x}_1'' \neq 0).$$

$$\text{and } a_k = \bar{a}_k = 1 (k \neq 1) \text{ with } s(\underline{a}) = 2.$$

We list those paths from Case I, Category A(i) (resp. (ii)) for which $H(\underline{z} \otimes \underline{y}) = 1$ and $H(\underline{y} \otimes \underline{x}) = 0$ in table 6.19 (resp. 6.20). These paths are exactly the paths from Case I, Category A(i) (resp. (ii)) of weight $2\Lambda_0 - 2\delta$.

$\underline{y}'', \underline{x}''$ of type	restrictions	count
one	$\begin{matrix} i=1 \\ k=2 \dots n \end{matrix}$	$n - 1$
two	$\begin{matrix} i=1 \\ j,k=2 \dots n \end{matrix}$	$(n - 1)^2$
three	never	0
four	never	0
five	never	0
six	never	0
seven	never	0
eight	never	0

Table 6.19: Paths from Case I, Category A(i) of weight $2\Lambda_0 - 2\delta$

$\underline{y}'', \underline{x}''$ of type	restrictions	count
one $(\bar{x}_1 = 2, y_1 = 2)$	$\begin{matrix} j = 2 \dots n \\ k = 2 \dots n \end{matrix}$	$(n - 1)^2$
two $\begin{pmatrix} \bar{x}_1=1, \bar{x}_i=1 \\ y_1=1, y_i=1 \end{pmatrix}$	$\begin{matrix} i, j, k = 2 \dots n \\ \left\{ \begin{array}{l} i \leq j \text{ or} \\ k < j \text{ and } j < i \end{array} \right. \end{matrix}$	$(n - 1)^2 + (n - 1)^2(n - 2) + \frac{(n-1)(n-2)(n-3)}{6}$
three	never	0
four	never	0
five	never	0
six	never	0
seven	never	0
eight	never	0

Table 6.20: Paths from Case I, Category A(ii) of weight $2\Lambda_0 - 2\delta$

We have considered paths from Case I, Category A where $\underline{y}', \underline{x}'$ are of types one and two. If $\underline{y}', \underline{x}'$ are of types three through eight no paths from Case I, Category A will be of weight $2\Lambda_0 - 2\delta$. (This can be checked using the methods

we used in the previous cases). Thus, tables 6.18, 6.19, and 6.20 contain all paths from Case I, Category A of weight $2\Lambda_0 - 2\delta$. We find the total number of paths in Case I, Category A of weight $2\Lambda_0 - 2\delta$ by summing the last columns of these three tables. There are $\frac{13n^3-9n^2+2n}{6}$ paths from Case I, Category A of weight $2\Lambda_0 - 2\delta$.

Case I, Category B: Using similar methods it can be shown that there are

$\frac{4n^3-5n^2+n}{2}$ paths from Case I, Category B of weight $2\Lambda_0 - 2\delta$.

Therefore, $\frac{25n^3-24n^2+5n}{6}$ of the paths from Case I are of weight $2\Lambda_0 - 2\delta$. We have used a similar method to find the number of paths from each of the remaining cases of weight $2\Lambda_0 - 2\delta$. Due to the tedious nature of the calculations we will list only the results here.

Paths from Case	count
one	$\frac{25n^3-24n^2+5n}{6}$
two	$\frac{3n^2-3n}{2}$
three	n
four	0
five	$\frac{5n^3-3n^2+10n}{6}$
six	0
seven	0
eight	0
nine	0

Table 6.21: Number of paths of weight $2\Lambda_0 - 2\delta$ for cases one through nine

Summing the last column of table 6.21, we prove our lemma.

$$\begin{aligned}
\dim V(2\Lambda_1)_{2\Lambda_0-2\delta} &= \{ \#p \in \mathcal{P}(2\Lambda_1) \mid wt(p) = 2\Lambda_0 - 2\delta \} \\
&= \frac{25n^3 - 24n^2 + 5n}{6} + \frac{3n^2 - 3n}{2} \\
&\quad + n + \frac{5n^3 - 3n^2 + 10n}{6} \\
&= 5n^3 - 2n^2 + 3n
\end{aligned}$$

□

Proposition 6.5. *Let $\mathfrak{g} = HC_n^{(1)}$ for any $n \in \mathbb{Z}_{\geq 2}$ and let $\alpha = -2\alpha_{-1} - 3\delta$. Then,*

$$\dim \mathfrak{g}_\alpha = \frac{5n^3 + 6n^2 + 7n}{3!}$$

Proof: Referring to (6.20) we see that

$$\dim \mathfrak{g}_{-2\alpha_{-1}-3\delta} = M_1 - M_2,$$

where

$$M_1 = \sum_{\substack{\lambda_i < \eta_i \\ \lambda_i + \eta_i = \alpha}} \dim V(-\alpha_{-1})_{\lambda_i} \dim V(-\alpha_{-1})_{\eta_i} \text{ and} \quad (6.39)$$

$$M_2 = \dim V(-2\alpha_{-1} - \alpha_0)_{-2\alpha_{-1}-3\delta} \quad (6.40)$$

Notice,

$$M_1 = \text{mult}(\Lambda_0)_{\Lambda_0} \text{mult}(\Lambda_0)_{\Lambda_0 - 3\delta} + \sum_{\substack{\lambda_i < \eta_i, \eta_i \neq \Lambda_0 \\ \lambda_i + \eta_i = -2\alpha_{-1} - 3\delta}} \dim V(\Lambda_0)_{\lambda_i} \cdot \dim V(\Lambda_0)_{\eta_i} \quad (6.41)$$

Clearly, $\text{mult}(\Lambda_0)_{\Lambda_0} = 1$. Using proposition 6.2 and lemma 6.2, equation 6.41 reduces to

$$M_1 = \frac{5n^3 + 6n^2 + 7n}{6} + 5n^3 - 3n^2 + 2n. \quad (6.42)$$

Realizing $\dim(2\Lambda_1 - \delta)_{2\Lambda_0 - 3\delta} = \dim(2\Lambda_1)_{2\Lambda_0 - 2\delta}$, we may use lemma 6.3 to calculate M_2 .

$$M_2 = 5n^3 - 3n^2 + 2n. \quad (6.43)$$

Therefore,

$$\begin{aligned} \dim \mathfrak{g}_{-2\alpha_{-1} - 3\delta} &= M_1 - M_2 \\ &= \frac{5n^3 + 6n^2 + 7n}{6} + 5n^3 - 3n^2 + 2n \\ &\quad - (5n^3 - 3n^2 + 2n) \\ &= \frac{5n^3 + 6n^2 + 7n}{6} \end{aligned}$$

□

In section 6.1 we not only gave formulas for roots, $\alpha = -\alpha_{-1} - k\delta$, of $HC_n^{(1)}$ for fixed k and varying n , we also gave a formula for the multiplicity of α for fixed n

($n = 2$) and varying k . While we have not yet found a formula for the multiplicity of roots $\alpha = -2\alpha_{-1} - k\delta$ in $HC_2^{(1)}$ for varying k , we have calculated such multiplicities for k equal to two through ten. We record our results in table 6.22.

k	M_1	M_2	M_3	$mult(-2\alpha_{-1} - k\delta)$
2	10	2	9	2
3	45	0	32	13
4	139	6	97	57
5	469	0	264	205
6	1228	13	661	645
7	3396	0	1556	1840
8	7939	29	2477	4868
9	19570	0	7448	12122
10	42497	57	15386	28704

Table 6.22: The multiplicity of $-2\alpha_{-1} - k\delta$ as a root of $HC_2^{(1)}$

6.3 General Observations

In this section we state some general observations concerning root multiplicities of $HC_n^{(1)}$.

Proposition 6.6. *Let $\mathfrak{g} = HC_n^{(1)}$ and let $\alpha = -l\alpha_{-1} - k\delta$. Then*

$$\dim(\mathfrak{g})_\alpha = \begin{cases} n & \text{if } l = k \\ 0 & \text{if } l > k \end{cases}$$

Proof: Let $\mathfrak{g} = HC_n^{(1)}$ and let $k \in \mathbb{Z}$. Notice,

$$\begin{aligned}
 r_{-1}(-k\alpha_{-1} - k\delta) &= -k\alpha_{-1} - k\delta - (-k\alpha_{-1} - k\delta)(h_{-1})\alpha_{-1} \\
 &= -k\delta + k(-1 + \alpha_{-1}(h_{-1}) + \delta(h_{-1}))\alpha_{-1} \\
 &= -k\delta + k(-1 + 2 + (-1 + 0 + \cdots + 0))\alpha_{-1} \\
 &= -k\delta
 \end{aligned} \tag{6.44}$$

Therefore,

$$\begin{aligned}
 \dim(\mathfrak{g})_{-k\alpha_{-1}-k\delta} &= \dim(\mathfrak{g})_{r_{-1}(-k\alpha_{-1}-k\delta)} \\
 &= \dim(\mathfrak{g})_{-k\delta} \\
 &= \dim(\mathfrak{g})_{-\delta} \\
 &= \dim(\mathfrak{g})_{r_{-1}(-\alpha_{-1}-\delta)} \\
 &= \dim(\mathfrak{g})_{-\alpha_{-1}-\delta} \\
 &= n \text{ (see proposition 6.1)}
 \end{aligned}$$

Next, we show $\dim(\mathfrak{g})_{-l\alpha_{-1}-k\delta} = 0$ for $l > k$. Calculation (6.44) shows us $-k\delta$ is a root of \mathfrak{g} for all k . Consider the α_{-1} string through $-k\delta$. This string is of length,

$$-k\delta(h_{-1}) = k.$$

$\alpha_{-1} - k\delta$ is not a root of \mathfrak{g} , since the coefficients of the simple roots are of mixed signs.

Thus, the α_{-1} string through $-k\delta$ consists of roots of the form $\{-l\alpha_{-1} - k\delta | 0 < l \leq k\}$.

We conclude, $\dim(\mathfrak{g})_{-l\alpha_{-1} - k\delta} = 0$ for $l > k$.

□

In [7], Frankel made the following conjecture: For any hyperbolic Kac-Moody algebra \mathfrak{g} , with Cartan matrix A ,

$$\dim(\mathfrak{g}_\alpha) \leq p^{(\text{rank}(A)-2)} \left(1 - \frac{(\alpha|\alpha)}{2} \right), \quad (6.45)$$

Let $\mathfrak{g} = HC_n^{(1)}$ and $\alpha = -l\alpha_{-1} - k\delta$. (6.45) becomes

$$\begin{aligned} \dim(HC_n^{(1)})_{-l\alpha_{-1} - k\delta} &\leq p^{(n)} \left(1 - \frac{(-l\alpha_{-1} - k\delta | -l\alpha_{-1} - k\delta)}{2} \right) \\ &= p^{(n)} \left(1 - \frac{l^2(\alpha_{-1}|\alpha_{-1}) + 2kl(\alpha_{-1}|\delta) + k^2(\delta|\delta)}{2} \right) \\ &= p^{(n)} (1 - l(l - k)) \end{aligned}$$

Thus, Frenkel's conjecture says

$$\begin{aligned} \dim \mathfrak{g}_{-2\alpha_{-1} - 3\delta} &\leq p^{(n)}(3) \\ &= \frac{n^3 + 9n^2 + 8n}{6} \end{aligned}$$

However, we have shown that $\text{mult}(-2\alpha_{-1} - 3\delta) = \frac{5n^3 + 6n^2 + 7n}{6} > p^{(n)}(3)$, since $(5n^3 +$

$6n^2 = 7n) - (n^3 + 9n^2 + 8n) = (4n + 1)(n - 1) > 0$ for $n \geq 2$. Therefore, Frenkel's conjecture does not hold for $\mathfrak{g} = HC_n^{(1)}$ and $\alpha = -2\alpha_{-1} - 3\delta$ ($n \geq 2$). In fact, Frenkel's conjectured bound does not hold for any of the roots we have considered except for those of the form $-k\alpha_{-1} - k\delta$. In table 6.3 we re-state the multiplicities of the roots $\{-2\alpha_{-1} - k\delta | 2 \leq k \leq 10\}$ of $HC_n^{(1)}$ along with Frenkel's conjectured bound for each root.

k	$mult(-2\alpha_{-1} - k\delta)$	Frenkel's conjectured bound
2	2	2
3	13	10
4	57	36
5	205	110
6	645	300
7	1840	752
8	4868	1770
9	12122	3956
10	28704	8470

Table 6.23: The multiplicity of $-2\alpha_{-1} - k\delta$ as a root of $HC_2^{(1)}$

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Appendix A

MATLAB Code

In this appendix we include the MATLAB code we used to find M_2 in example 6.2.

We would like to point out that this code can be used to find the multiplicity of any weight $\alpha - a_0\alpha_0 + \cdots + -a_n\alpha_n$ in any highest weight $C_n^{(1)}$ -module of highest weight Λ , simply by making the appropriate changes to the final program, mult.m.

A.1 phi.m

function y= phi(index,x,lev); This function returns ϕ_i of x , where x is an element of the $C_n^{(1)}$ perfect crystal of level, lev.

```
function y= phi(index,x,lev);  
  
n=length(x)/2;  
  
A=[x(1:n);fliplr(x(n+1:2*n))];  
  
if index == 0
```

```

        y=lev -0.5*sum(x)+subplus(A(2,1)-A(1,1));

end;

if 0 < index & index < n

    y=A(1,index)  + subplus(A(2,index+1)-A(1,index+1));

end;

if index ==n

    y=A(1,n);

end;

```

A.2 epsilon.m

function y=epsilon(index,x,lev); This function works similarly to function phi, but for epsilon_index

```

function y=epsilon(index,x,lev)

n=length(x)/2;

A=[x(1:n);fliplr(x(n+1:2*n))];

%subplus(x) is a function which returns x if x>0 and 0 otherwise

if index == 0

    y=lev-0.5*sum(x)+subplus(A(1,1)-A(2,1));

end;

if 0 < index & index < n

    y= A(2,index) + subplus(A(1,index+1)-A(2,index+1));

```

```

end;

if index ==n

    y=A(2,n);

end;

```

A.3 tensorrule.m

function B=tensorrule(index, A,lev,n); This function can be used in conjunction with identify action to see which element of a multi-fold tensor,A, of elements of the $C_n^{(1)}$ perfect crystal of level n ftilde_index acts upon. The function tensorrule returns phi_index of the ith element in the sequence in B(i,2) and epsilon_index of the ith element in the tensor product in B(i,2).

```

function B=tensorrule(index, A,lev,n);

y=length(A);

B=zeros(y/(2*n),2);

for i=1:y/(2*n)

    B(i,1)=phi(index,A(2*n*(i-1)+1:(2*n*i)),lev);

    B(i,2)=epsilon(index, A(2*n*(i-1)+1:(2*n*i)),lev);

end;

```

A.4 identify_action.m

function action=identify_action(A); This function takes in ,A, the output from the function tensorrule.m Tensorrule acts on a multi-fold tensor (each element as a row) and returns a matrix A where A(i,2) is phi_index(the ith element in the multi-fold tensor) and A(i,1) is epsilon_index(the ith element in the multi-fold tensor). The function action then uses the multi-fold tensor product rule to determine which element of the multi-fold tensor ftilde_index will act upon and assigns the row number of this element to the variable action. If there should be no action, it returns the number 0.1.

```
function action=identify_action(A);

[a,b]=size(A);

j=1;

for j=1:a

    [x,y]=size(A);

    A=[A;0,0];

    A(:,2)=[0;A(1:x,2)];

    for i=1:x+1

        num=A(i,1)-A(i,2);

        if num >= 0

            A(i,1)=num;

            A(i,2)=0;
```

```

else

    A(i,1)=0;

    A(i,2)=-1*num;

end;

end;

j=j+1;

end;

A=A(:,1);

test=find(A>0);

[a,b]=size(test);

if b==0

    action=0.1;

else action=test(a,b);

end;

```

A.5 one_move.m

function [paths,arrows]=one_move(A, ground_state_path, prev_path_arrows, goalarrows, lev, n); This function starts at one path, A, of the path realization of the path $P(\lambda)$, the $C_n^{(1)}$ -highest weight module of highest weight λ . The function first checks to see if we need to go any further. That is, it reads in goalarrows, the number of arrows we wish to follow, and checks to see if prev_path_arrows, the number of arrows the

previous path follows is less than our goal. If so it lets the modified root vectors for the arrows we still must consider act on the path and returns `[results,path]` where `results` is a matrix containing the emanating paths from the path `A`, and `path` is a matrix which records which f-arrow leads to which emanating path. Here, `ground_state_path` is the ground state path of weight λ and `lev` is the level of the weight λ .

```
function [paths,arrows]=
one_move(A,ground_state_path,prev_path_arrows,goalarrows,lev,n);
arrows=zeros(0,n+1);
y=length(A);
b=length(ground_state_path);
paths=zeros(0,y+b);
for k=1:n+1
    c=zeros(1,n+1);
    c(1,k)=1;
    if prev_path_arrows(k)<goalarrows(k)
        B=A;
        num=identify_action(tensorrule(k-1,A,lev,n));
        if num ~= 0.1;
            B((2*n*(num-1)+1):(2*n*num))=ftilde(k-1,
            A(2*n*(num-1)+1:2*n*(num)));
        add=[B,ground_state_path];
```



```

    paths=[paths;add];

        arrows=[arrows;c];

    end;

end;

end;

```

A.6 next_move.m

function [paths,arrows]= next_move(oldpaths, oldarrows, ground_state_path, goalarrows, lev, n); This function takes in a row of paths, which we name oldpaths, in $P(\lambda)$, the path realization of the $C_n^{(1)}$ highest weight module with highest weight λ . Here ground_state_path is the ground state path of weight λ and lev is the level of λ , oldarrows is a matrix each row of which tells the number of arrows the path represented by the corresponding path follows, and goalarrows represents number of arrows the paths we are eventually looking for (see program mult.m) follow. The function returns the next row of the path realization, which we call paths, and a matrix whose rows represent the number of arrows the corresponding row in the matrix paths follows.

```

function [paths,arrows]=

next_move(oldpaths,oldarrows,ground_state_path,goalarrows,lev,n);

y=length(ground_state_path);

[x2,y2]=size(oldpaths);

```

```

newpaths=zeros(0,y2+y);

newarrows=zeros(0,n+1);

for i=1:x2;

    A=oldpaths(i,:);

    current_arrows=oldarrows(i,:);

    [paths,arrows]=

    one_move(A,ground_state_path,current_arrows,goalarrows,lev,n);

    [c,d]=size(arrows);

    mult=ones(c,1);

    addition=mult*current_arrows;

    newarrows=[newarrows;(addition+arrows)];

    newpaths=[newpaths;paths];

end;

% Next we remove unneeded repeating of ground state path

% at the end of the paths

test=[ground_state_path;newpaths(:,y2+1-2*n:y2+y-2*n)];

test=sum(abs(diff(test)));

if test==0

    newpaths=newpaths(:,1:y2);

end;

% some of our rows in newpaths may be repeats here we remove the

```

```

% repeated paths

[a,b]=size(newpaths);

if a>1

[sorted,index]=sortrows(newpaths);

D=sum(abs(diff(sorted)),2);

f=find(D~=0);

rows=[index(f);index(a)];

paths=[newpaths(rows,:)];

arrows=newarrows(rows,:);

else

paths=newpaths;

arrows=newarrows;

end;

```

A.7 crystal_path.m

function [final_paths]=crystal_path(ground_state_path,goalarrows,lev,n); This function finds all paths which follow goalarrows arrows in the path realization $P(\lambda)$ of the highest weight $C_n^{(1)}$ -module with highest weight λ . Here ground_state_path is the ground state path of weight lambda and lev is the level of lambda. Each path which follows exactly goalarrows arrows appears as one row of final_paths.

```
function [final_paths]=crystal_path(ground_state_path,goalarrows,lev,n);
```

```

prev_path_arrows=zeros(1,n+1);

[A,B]=one_move(ground_state_path,ground_state_path,prev_path_arrows,
               goalarrows,lev,n);

for i=1:sum(goalarrows)-1;

    [A,B]=next_move(A,B,ground_state_path,goalarrows,lev,n);

end;

final_paths=A;

```

A.8 mult.m

A program to find the multiplicity of the weight $-2\alpha_{-1} - \delta$ in $P(2\Lambda_1)$ note any such path must follow exactly one zero-arrow, four one-arrows, and two two-arrows.

```

root=[2,2,4,2];

ground_state_path=zeros(1,4);

ground_state_path(1)=2;

ground_state_path(4)=2;

arrows=[1,4,2]';

final_paths=crystal_path(ground_state_path,arrows,2,2);

[x,y]=size(final_paths);

mult=x

```