ABSTRACT

LEVY, LOUIS AGNEW. Multipliers for the Lower Central Series of Strictly Upper Triangular Matrices. (Under the direction of Professor E. L. Stitzinger).

Lie algebra multipliers and their properties is a recent area of study. A multiplier is the Lie algebra analogue of the Schur multiplier from group theory. By definition a multiplier is central, so we only need to find its dimension in order to characterize it. We will investigate how to find the dimensions of the multipliers for the lower central series of strictly upper triangular matrices. The closed form result is a set of six polynomial answers in two variables: the size of the matrix and the position in the series.

Multipliers for the Lower Central Series of Strictly Upper Triangular Matrices

by
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DEDICATION

This dissertation is dedicated to the memory of my grandmothers

Rena Dweck Levy ז"ל

and

Mabel Bedwell Agnew ז"ל

both of whom I lost while in graduate school.

BIOGRAPHY

Louis A. Levy was born in August 1980, in Washington, DC and moved with his family to North Carolina two years later. He received his Bachelors degree in Computer Engineering from the University of Maryland, College Park in 2003. Before starting his graduate work he worked as a radar engineer in suburban Washington DC. He earned a Masters degree in Pure Mathematics in 2006 from North Carolina State University. He is a member of several honor societies including Phi Kappa Phi.

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Chapter 1

Introduction

Much has been shown in [1], [2], [3], [4], [5], [6], and [7] about Lie algebras and their multipliers. We adopt the name multiplier to describe the Lie algebra analogue of the Schur multiplier from group theory. Schur did much of his work in the early 20^{th} century. To learn more about the Schur multiplier, please see [8] for a very thorough collection of Schur's contributions to group theory. Here we will study multipliers for the lower central series of strictly upper triangular matrices.

We begin with a few definitions. Suppose L is a finite dimensional Lie algebra over a field with characteristic not equal to two.

Definition A pair of Lie algebras (C, M) is called a defining pair for L if

1. $L \cong C/M$
2. $M \subset Z(C) \cap C^2$.

It is mentioned in [1] that if $\dim L = n$ then $\dim M \leq \frac{1}{2}n(n-1)$. If M is maximal, then $\dim M = \frac{1}{2}n(n-1) \Leftrightarrow L$ is abelian. Therefore L finite dimensional implies M finite dimensional, which together give C finite dimensional. In [3] we see that $\dim(C) \leq \frac{1}{2}n(n+1)$.

Definition If (C, M) is a defining pair for L, then a C of maximal dimension is called a cover for L. Likewise an M of maximal dimension is called a multiplier.

It is shown in [3] that for Lie algebras all covers are isomorphic. The multiplier is unique since it is abelian, hence we denote it by $M(L)$ and only need to find its dimension in order to characterize it.

Definition If $\dim L = n$, then define $t(L)$ to be as in [1], that is $t(L) = \frac{1}{2}n(n-1) - \dim M(L)$.

Notice $t(L) = 0 \Leftrightarrow L$ is abelian. Moneyhun, Batten, Hardy, and Stitzinger have completely classified Lie algebras where $t(L) \leq 8$ in [1], [5], and [6]. Here the goal is to find a formula for $\dim M(L)$ which will therefore also give a formula for $t(L)$ for the lower central series of strictly upper triangular matrices. For this family of Lie algebras we can achieve $t(L)$ arbitrarily large.

Chapter 2

Strictly Upper Triangular Matrices

Consider the strictly upper triangular matrices of size $n \times n$, which we will denote as St_n. Again we assume the entries in these matrices are from a field whose characteristic is not two. The cover and multiplier for St_n were found in [3]. The goal now is to find the dimension of the multiplier for this Lie algebra if in addition to being strictly upper, these matrices now have an additional k diagonals above the main diagonal which are all zero. We would like to accomplish this for an arbitrary nonnegative integer k.

Let (C, M) be a defining pair for St_n, therefore $C/M \cong St_n$ and $M \subset Z(C) \cap C^2$. We will use E_{ab} to denote the usual matrix units that form a basis for St_n. Since $C/M \cong St_n$, each E_{ab} in St_n also represents a coset in C/M. Let $u : St_n \to C$ denote the transversal map from St_n back into C. Thus $u(E_{ab})$ is some element in C, that belongs to the E_{ab} coset in C/M. Define F_{ab} to be such that $u(E_{ab}) = F_{ab} \in C$. Also let $J(x, y, z) = 0$ denote the Jacobi identity. We can completely describe the bracket operation as

$$[F_{st}, F_{ab}] = \begin{cases} F_{sb} + y(s,t,a,b) & \text{if } t = a \\ y(s,t,a,b) & \text{if } t \neq a \end{cases}$$

where $y(s,t,a,b) \in M$. For brevity we often denote $y(s,t,a,b)$ as y_{stab}. Due to the anti-symmetry of the bracket (alternating property) we establish that either $s < a$ or $s = a$ and $t < b$. (Note: this is consistent with the convention established in [3]).

For the usual St_n, when we do not require any entries above the main diagonal to be zero, this corresponds to our variable k being set to zero. As a consequence of being strictly upper triangular, the matrix units E_{ab} must be such that $a < b$, or $a + 1 \leq b$.

If $k = 1$ (*i.e.* one diagonal above the main diagonal must be zero), then this forces

$a+2 \leq b$. If $k=2$, then the extra diagonal of zeros forces $a+3 \leq b$. In general if k diagonals above the main diagonal are all zero then this requires $a+(k+1) \leq b$.

Following the model of [3], first make a change in the choice of F_{rt}. Set

$$G_{rt} = \begin{cases} F_{rt} & \text{if } t-r < 2(k+1) \\ F_{rt} + y_{r,t-(k+1),t-(k+1),t} = [F_{r,t-(k+1)}, F_{t-(k+1),t}] & \text{otherwise} \end{cases}$$

As stated in [3], "Thus $\{G(r,t)\}$ and $\{F(r,t)\}$ are complete sets of images of matrix units[.] Since the $y's$ are central, $G(r,t)$ and $F(r,t)$ induce the same multiplication in C." After much calculation we will remove all immediate (pre-existing) dependencies among the $y's$ as well as all trivial $y's$. With what remains we can conclude as [3] states, "These $y's$ are completely arbitrary. We can assume them to be a set of linearly independent vectors and no contradiction arises. In this case they would be a basis for a multiplier M." In other words $M = M(L)$. Therefore we wish to count the number of remaining $y's$ that occur for any values of n and k. Often times the absorbtion into G_{rt} or use of the Jacobi identity can reduce the number of $y's$ allowed to contribute to the multiplier's dimension.

The initial result we get is that $\dim M(L)$ is equal to

$$\sum_{i=1}^{k} i \cdot (n-(2(k+1)+i)) + \sum_{j=1}^{k}(k-j+1) \cdot (n-(2(k+1)+k+j)) +$$

$$2(n-2(k+1)) + \sum_{j=0}^{k}\sum_{i=0}^{k} \frac{(n-(2(k+1)+i+j+1)) \times (n-(2(k+1)+i+j))}{2} +$$

$$\sum_{i=1}^{k} \frac{3i(i+1)}{2} \cdot (n-((k+1)+i)) + \sum_{i=1}^{k}\left((k+1)^2 - \frac{i(i+1)}{2}\right) \cdot (n-((k+1)+k+i)) +$$

$$\sum_{i=1}^{k} \frac{(k-(i-1))(k-(i-1)+1)}{2} \cdot (n-((k+1)+2k+i))$$

Notice that every sum involves some quantity being subtracted from n. This quantity corresponds to a distance between positions within the matrices, hence this number cannot exceed n. While developing this formula, we make the assumption that n is sufficiently large to prevent any distances from exceeding n. When $n-x$ appears in a sum, such an expression arises to count multiplier elements produced by matrices with at least $x+1$ rows (and columns). As such if $n - x \leq 0$ occurs, then the matrices are too small to produce these multiplier elements. This means that any time $n - x \leq 0$ shows up, it should be replaced by zero. Also notice that if a particular i or j causes $n - x \leq 0$, then further incrementing of i or j will cause the same problem. Therefore sums should be terminated early at the first occurence of the appearance of $n - x \leq 0$. Notice in the previous formula for $\dim M(L)$ that every sum will run through its entirety when $n \geq 4k + 3$, so $4k + 3$ is the sufficiently large value of n we originally assumed. This will become more clear as we develop the above formula.

Considering that specific values of n and k when $n < 4k+3$ will force modifications to the formula, we will break this result into several cases based on how n and k interact with each other. Once these cases are established we will be able to convert all results into a more elegant polynomial form.

The complexity of the above formula comes from there being many different potential relationships among the values of $s, t, a,$ and b when computing $[G_{st}, G_{ab}]$, especially as k gets larger and widens the necessary gap between both (s, t) and (a, b). This general result will be easier to derive if we first investigate the patterns that show up when we look at specific values of k. As such we begin with two examples; we will derive this formula as it would simplify for the cases of $k = 1$ and 2. The logic for these examples is very similar to the approach used to prove this formula in the general case for any value of k.

Chapter 3

Example $k = 1$

Reminders: As before, let F_{ab} denote the image of the matrix unit E_{ab} under the transversal map. In addition to being strictly upper triangular, since $k = 1$ we have one additional diagonal above the main diagonal which is all zero. Since this implies that F_{ab} must be such that $a + (k+1) \leq b$, here we have $a + 2 \leq b$. Also the change from F_{rt} to G_{rt} simplifies to

$$G_{rt} = \begin{cases} F_{rt} & \text{if } t - r < 4 \\ F_{rt} + y_{r,t-2,t-2,t} = [F_{r,t-2}, F_{t-2,t}] & \text{otherwise} \end{cases}$$

We divide our investigation of multiplier elements into two sections: elements produced by $[G_{rs}, G_{st}]$ and elements produced by $[G_{st}, G_{ab}]$, where $t \neq a$.

3.1 $[G_{rs}, G_{st}]$

Consider first the case where $[G_{rs}, G_{st}] = [F_{rs}, F_{st}] = F_{rt} + y_{rsst}$. Suppose that $t \geq r + 7$. If $s = t - 2$ then $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = F_{rt} + y_{r,t-2,t-2,t} = G_{rt}$. If $s \neq t - 2$, then we wish to show $y_{rsst} = y_{r,t-2,t-2,t}$ because this will give $[G_{rs}, G_{st}] = G_{rt}$. Since F_{rs} is defined, we know $r + 2 \leq s$. Suppose first that $s = r + 2$ or $s = r + 3$, then $F_{s,t-2}$ is defined since $s + 2 \leq r + 5 \leq t - 2$. Therefore $J(F_{rs}, F_{s,t-2}, F_{t-2,t}) = 0 \;\Rightarrow\; y_{rsst} = y_{r,t-2,t-2,t}$. On the other hand, when $s > r + 3$ let $c = r + 2$. Notice that F_{cs} is defined so $J(F_{rc}, F_{cs}, F_{st}) = 0 \;\Rightarrow\; y_{rsst} = y_{rcct}$ and $J(F_{rc}, F_{c,t-2}, F_{t-2,t}) = 0 \;\Rightarrow\; y_{rcct} = y_{r,t-2,t-2,t}$, which together give $y_{rsst} = y_{r,t-2,t-2,t}$. Thus $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = F_{rt} + y_{r,t-2,t-2,t} = G_{rt}$ for any valid

s when $t \geq r+7$.

On the other extreme suppose $t = r+4$, (the minimum distance from r to t is 4). In this case the only choice for s is $s = r+2 = t-2$. Therefore $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = F_{rt} + y_{r,t-2,t-2,t} = G_{rt}$.

It remains to consider $t = r+5$ and $t = r+6$. We wish to equate y_{rsst} and $y_{r,t-2,t-2,t}$ whenever possible so that $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = F_{rt} + y_{r,t-2,t-2,t} = G_{rt}$.

Suppose $t = r+5$. If $s = r+3$ then $s = r+3 = t-2$ so $y_{rsst} = y_{r,t-2,t-2,t}$ is trivially satisfied. If $s = r+2$, there are no Jacobi identities available to equate y_{rsst} and $y_{r,t-2,t-2,t}$. Therefore $[G_{r,r+2}, G_{r+2,r+5}] = F_{r,r+5} + y_{r,r+2,r+2,r+5}$ where $y_{r,r+2,r+2,r+5}$ is some non-trivial element in the multiplier.

Suppose $t = r+6$. If $s = r+2$ then $J(F_{rs}, F_{s,t-2}, F_{t-2}, t) = 0$ gives $y_{rsst} = y_{r,t-2,t-2,t}$ and if $s = r+4$ then $s = t-2$ so $y_{rsst} = y_{r,t-2,t-2,t}$ is trivially satisfied. However if $s = r+3$, there are no Jacobi identities available to equate y_{rsst} and $y_{r,t-2,t-2,t}$. Therefore $[G_{r,r+3}, G_{r+3,r+6}] = F_{r,r+6} + y_{r,r+3,r+3,r+6}$ where $y_{r,r+3,r+3,r+6}$ is some non-trivial element in the multiplier.

In conclusion, the only non-trivial multiplier elements arising from $[G_{rs}, G_{st}]$ are of the form $y_{r,r+2,r+2,r+5}$ and $y_{r,r+3,r+3,r+6}$. Please note that whenever $F_{rt} + y_{rsst} \neq G_{rt}$ we can make a change in the choice of $y's$ to convert all $F's$ to $G's$. For instance let $\widehat{y_{rsst}} = y_{rsst} - y_{r,t-2,t-2,t}$ in which case $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = G_{rt} + \widehat{y_{rsst}}$. Regardless of whether we use F_{rt} or G_{rt} we get the same number of $y_{rsst}'s$ and $\widehat{y_{rsst}}'s$. Therefore it is acceptable to count the occurences of either y_{rsst} or $\widehat{y_{rsst}}$ to find the contribution to the multiplier. For convenience, we will count the $y_{rsst}'s$.

3.2 $[G_{st}, G_{ab}]$, where $t \neq a$

Consider the second case where $[G_{st}, G_{ab}] = [F_{st}, F_{ab}] = y_{stab}$ since $t \neq a$. Due to the antisymmetry of the bracket (alternating property), it is sufficient to consider only the cases when $s < a$ or $s = a$ and $t < b$. This in mind, we do not need to consider $s = b$ since it violates $s < a$ and the alternating property would put it back into the $t = a$ case.

Thus $[G_{st}, G_{ab}] = [F_{st}, F_{ab}] = y_{stab}$ where y_{stab} is some element in the multiplier, $M(L)$. Since no F is produced by the bracket operation and hence $[G_{st}, G_{ab}] = [F_{st}, F_{ab}] \in M(L)$, we will work with the $F's$ rather than the $G's$ as both produce the same elements in $M(L)$. We wish to find all the scenarios (relationships between the values of s, t, a, and

b) where $y_{stab} = 0$ versus $y_{stab} \neq 0$. We begin by imposing an upper bound on the distance between a and b as well as s and t for which y_{stab} may be non-zero.

Lemma 3.2.1 *If $b \geq a+5$ or $t \geq s+5$ then $y_{stab} = 0$. So $y_{stab} \neq 0$ only has the potential to occur if both $b \leq a+4$ and $t \leq s+4$.*

Proof Suppose $b \geq a+5$. If $t \neq a+2$ then let $c = a+2$. If $t = a+2$ then choose $c = a+3$. The number line is shown in Figure 3.1. By construction $c \neq t, a, b$. Notice $s \leq a < c \Rightarrow c \neq s$. This gives $c \neq s, t, a, b$ and therefore $J(F_{st}, F_{ac}, F_{cb}) = 0 \quad \Rightarrow \quad y_{stab} = 0$.

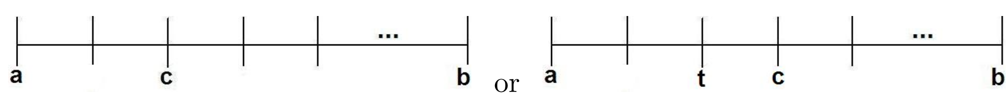

Figure 3.1: $k = 1 : b \geq a+5$

Similarly suppose $t \geq s+5$. If $a \neq s+2$ then let $c = s+2$. If $a = s+2$ then choose $c = s+3$. The number line is shown in Figure 3.2. By construction $c \neq s, t, a$, but we want to show $c \neq b$ also.

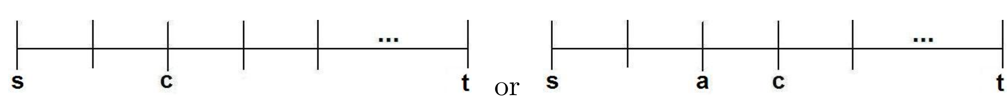

Figure 3.2: $k = 1 : t \geq s+5$

Notice when $s = a \quad \Rightarrow \quad t < b \quad \Rightarrow \quad c = s+2 < s+5 \leq t < b$. When $s < a$ then $a+2 \leq b \quad \Rightarrow \quad s+2 < a+2 \leq b \quad$ (*i.e.* $s+3 \leq b$). Therefore either $b > s+3 \geq c$ or $b = s+3 \quad \Rightarrow \quad a = s+1 \quad \Rightarrow \quad a \neq s+2$ so $c = s+2 \neq b$. Therefore in all cases $c \neq b$. This gives $c \neq s, t, a, b$ and therefore $J(F_{sc}, F_{ct}, F_{ab}) = 0 \quad \Rightarrow \quad y_{stab} = 0$. ∎

Lemma 3.2.2 *If $b = a+4$ then $y_{stab} = 0 \Leftrightarrow t \neq a+2$ or $s < a$ $(s \neq a)$. Similarly if $t = s+4$ then $y_{stab} = 0 \Leftrightarrow a \neq s+2$ or $b \neq t$.*

Proof ($\Leftarrow$) Suppose $b = a+4$. If $t \neq a+2$ then let $c = a+2$, so $c \neq t, a, b$ and furthermore $s \leq a < a+2 = c \Rightarrow c \neq s$. Therefore $c \neq s, t, a, b$ and $J(F_{st}, F_{ac}, F_{cb}) = 0 \Rightarrow y_{stab} = 0$. The number line is shown in Figure 3.3. On the other hand if $t = a+2$ and $s < a$ then let $c = t-1$, which can be seen in Figure 3.4. In this case $J(F_{st}, F_{tb}, F_{at}) = 0 \quad \Rightarrow \quad y_{stab} = -y_{sbat}$ and $J(F_{sc}, F_{cb}, F_{at}) = 0 \quad \Rightarrow \quad y_{sbat} = 0$ which together give $y_{stab} = 0$.

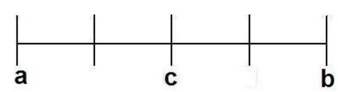

Figure 3.3: $k = 1 : b = a+4, t \neq a+2$

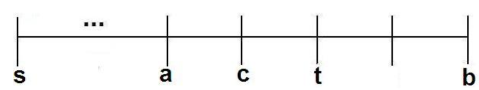

Figure 3.4: $k = 1 : b = a+4, t = a+2$

($\Rightarrow$) Suppose $b = a+4$, $t = a+2$, and $s = a$, as in Figure 3.5.

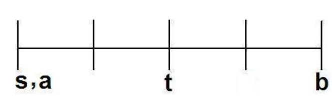

Figure 3.5: $k = 1 : b = a+4, t = a+2 = s+2$

There is no value of c such that F_{ac} and F_{cb} are both defined while $c \neq t$. As such, there are no Jacobi identities available to zero out y_{stab}, thus $y_{stab} \neq 0$.

($\Leftarrow$) Suppose $t = s+4$. If $a \neq s+2$ then let $c = s+2$, so $c \neq s, t, a$, but we want to show $c \neq b$ also. Notice when $s = a$ then $t < b$ so $s < c < t < b \Rightarrow c \neq b$. If $s < a$ then $a+2 \leq b \quad \Rightarrow \quad c = s+2 < a+2 \leq b \quad \Rightarrow \quad c \neq b$. Therefore $c \neq s, t, a, b$ and $J(F_{sc}, F_{ct}, F_{ab}) = 0 \Rightarrow y_{stab} = 0$. This can be seen on the number line in Figure 3.6.

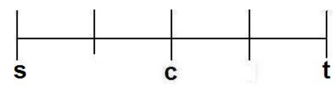

Figure 3.6: $k = 1 : t = s + 4, a \neq s + 2$

On the other hand if $a = s + 2$ and $b > t$ then let $c = a + 1$, as in Figure 3.7.

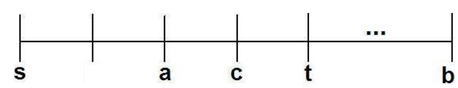

Figure 3.7: $k = 1 : t = s + 4, a = s + 2$

In this case $J(F_{sa}, F_{at}, F_{ab}) = 0 \;\Rightarrow\; y_{stab} = y_{sbat}$ and $J(F_{sc}, F_{cb}, F_{at}) = 0 \;\Rightarrow\; y_{sbat} = 0$ which together give $y_{stab} = 0$. If $a = s+2$ and $b < t$ then $s < s+2 = a < a+2 \leq b < t \;\Rightarrow$ $s + 4 < t$ and hence by Lemma 3.2.1, $y_{stab} = 0$.

($\Rightarrow$) Suppose $t = s + 4$, $a = s + 2$, and $b = t$, as in Figure 3.8.

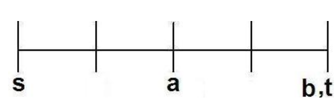

Figure 3.8: $k = 1 : b = t = s + 4, a = s + 2$

There is no value of c such that F_{sc} and F_{ct} are both defined while $c \neq a$. As such, there are no Jacobi identities available to zero out y_{stab}, thus $y_{stab} \neq 0$. ∎

Now that we have an upper bound on the distance from a to b and s to t for which non-trivial y_{stab} values may be produced, we continue our search by separating the variable relationships into three cases. Either (1) $s = a$, (2) $a > t$, or (3) $s < a < t$.

<u>**Case 1:** $s = a$</u>

For a fixed value of s, suppose $s = a$. Lemmas 3.2.1 and 3.2.2 discuss $b \geq s + 4$, so consider $b < s + 4$. Let $t_{min} = s + 2$, thus denoting the minimum possible value of t. Since $s = a \;\Rightarrow\; t < b$ this gives $t_{min} < b < s + 4 = t_{min} + 2$. In other words $b = t_{min} + j$, where $j = 1$.

Lemma 3.2.3 *When $b = t_{min} + 1$, we get* 1 *new non-zero value for y_{stab}.*

Proof Since $b = t_{min} + 1$ we have $b = s + 3 = a + 3$. Additionally $t < b \;\Rightarrow\; t < s + 3$. Therefore $\nexists c$ such that F_{ac} and F_{cb} are both defined, similarly $\nexists c$ such that F_{sc} and F_{ct} are both defined. Therefore $y_{stab} \neq 0$.

Also $b = t_{min} + 1$, $t < b \;\Rightarrow\; t = t_{min} \;\Rightarrow\; t$ may take on only 1 value for this fixed b, hence we get 1 distinct new non-zero value for y_{stab}. Putting this together we have $b = t_{min} + 1 = s + 3 \;\Rightarrow\; t = s + 2 \;\Rightarrow\; y(s, s + 2, s, s + 3) \neq 0$. ∎

<u>**Case 2:** $a > t$</u>

Lemma 3.2.4 *If $a > t$ then $y_{stab} \neq 0$ for all t and b such that both $t < s + 4$ and $b < a + 4$. Otherwise $y_{stab} = 0$ when $a > t$. That is $y(s, t, a, a + 2), y(s, t, a, a + 3) \neq 0$ when $t = s + 2, s + 3$.*

Proof If $t \geq s + 5$ or $b \geq a + 5$ then Lemma 3.2.1 $\;\Rightarrow\; y_{stab} = 0$. If $t = s + 4$ or $b = a + 4$ then Lemma 3.2.2 $\;\Rightarrow\; y_{stab} = 0$ since $a > t \;\Rightarrow\; t \neq a + 2$ and $a \neq s + 2$.

If $t < s + 4$ and $b < a + 4$ then there is no value of c, such that F_{sc} and F_{ct} are both defined for $s < c < t$. Similarly there is no value of c, such that F_{ac} and F_{cb} are both defined for $a < c < b$. Therefore the idea in Lemma 3.2.1, of using $J(F_{sc}, F_{ct}, F_{ab}) = 0$ or $J(F_{st}, F_{ac}, F_{cb}) = 0$ will not work here. Also, placing a c such that $s < t < c < a < b$ will not provide any helpful Jacobi identities, no matter how large the gap between t and a. Thus y_{stab} will always be non-zero in this case. ∎

Case 3: $s < a < t$

If $b \geq a+5$ or $t \geq s+5$ then Lemma 3.2.1 implies $y_{stab} = 0$. If $b = a+4$ then Lemma 3.2.2 implies $y_{stab} = 0$ since $s < a$. If $t = s+4$ and $a \neq s+2$ then Lemma 3.2.2 gives $y_{stab} = 0$. If $t = s+4$ and $a = s+2 \;\Rightarrow\; b \geq a+2 = s+4 = t$. When $b > t$ then Lemma 3.2.2 gives $y_{stab} = 0$. When $b = t$ then Lemma 3.2.2 gives $y_{stab} \neq 0$.

Thus it is only left to consider $t < s+4$ and $b < a+4$. No such c exists, nor do suitable Jacobi identities exist to zero out y_{stab} when $a+2 \leq b \leq a+3$ and $s+2 \leq t \leq s+3$. Therefore $y_{stab} \neq 0$ when $t = s+2$ or $s+3$ and $b = a+2$ or $a+3$.

Collecting all this information, we get $y_{stab} \neq 0$ when:

1. Lemma 3.2.2 result: $b = a+4$, $t = a+2$, and $s = a$.
2. Lemma 3.2.2 result: $t = s+4$, $a = s+2$ and $b = t$.
3. Lemma 3.2.3 result: $s = a$, $b = t_{min} + 1 = s+3$, and $t = t_{min} = s+2$.
4. Lemma 3.2.4 result: $a > t$, $t < s+4$, and $b < a+4$.
5. $s < a < t$, $t = s+2, s+3$, and $b = a+2, a+3$.

3.3 Counting the multiplier elements

We are interested in counting all the cases when $y_{stab} \neq 0$. There are two types of elements: (1) $y(s, s+x_1, s+x_2, s+x_3)$ where x_1, x_2, x_3 are all fixed and (2) $y(s, s+x_1, a, a+x_2)$ where $a > s + x_1$ and x_1, x_2 are both fixed. Remember that the numbers $s + x_i$ and $a + x_2$ were subscripts of the $G's$, and initially subscripts of the standard matrix units. Hence these numbers represent positions in the original $n \times n$ matrices. If ever these numbers exceed n, then the original matrix units were not available to work with, and so the corresponding multiplier element(s) cannot be produced.

Type 1: $y(s, s + x_1, s + x_2, s + x_3)$

Without loss of generality suppose $x_3 = \max\{x_1, x_2, x_3\}$. As before, assume the $F's$ we are working with are images of $n \times n$ matrix units. The minimum value of s is 1 and the maximum value of $s + x_3$ is n, thus the maximum value of s is $n - x_3$. Therefore $y(s, s + x_1, s + x_2, s + x_3)$ may assume $n - x_3$ different values since $s \in \{1, 2, \ldots, n - x_3\}$. If $n - x_3 \leq 0$ there is no contribution to the multiplier, $M(L)$, since the necessary matrix positions exceed the given number of positions n.

Notice also that the elements y_{rsst} from the $[G_{rs}, G_{st}]$ case also fall into this category since both s and t can be described by their distances from r.

Type 2: $y(s, s + x_1, a, a + x_2)$ where $a > s + x_1$

The minimum value of s is 1, so the minimum value of a is $2 + x_1$. The maximum value of $a + x_2$ is n, so the maximum value of a is $n - x_2$. Therefore a may range in value from $2 + x_1$ up to $n - x_2$, giving $(n - x_2) - (2 + x_1) + 1 = n - x_2 - x_1 - 1$ different values for a when $s = 1$. If $s = 2$, the minimum value of a now increases by 1 to be $3 + x_1$, giving one less possible value of a. If $s = 3$, the minimum value of a increases by 1 again, giving one less possible value of a again. This pattern continues until a can assume only one value. Thus as s increases, the number of possible values of a go from $n - x_2 - x_1 - 1$ down to 1, giving

$$\sum_{i=1}^{n-x_2-x_1-1} i \qquad \text{possible values of } y(s, s + x_1, a, a + x_2)$$

$$\sum_{i=1}^{n-x_2-x_1-1} i = \frac{(n-(x_2+x_1+1))(n-(x_2+x_1))}{2}$$

As in Type 1, this type demands a distance of $x_2 + x_1 + 1$ between matrix positions, hence if $n \leq x_2 + x_1 + 1$ there is no contribution to the multiplier, $M(L)$, since the original matrix units are not available to work with.

Using these two counting techniques, Table 3.1 lists all non-trivial y_{rsst} and y_{stab} possibilities and the number of times they occur. Adding all of these together gives $\dim M(L) = (n-6)+2(n-5)+5(n-4)+3(n-3)+(n-5)(n-6)+\frac{(n-6)(n-7)+(n-4)(n-5)}{2}$. Also, as stated above if any $n-x$ term is negative, simply omit it from the formula because the matrices will not be large enough to produce the corresponding matrix units and the resulting multiplier elements they would have produced.

Table 3.1: Counting multiplier elements for $k = 1$

y_{rsst} or y_{stab} non-trivial	Number of occurences
$y_{r,r+2,r+2,r+5}$	$n-5$
$y_{r,r+3,r+3,r+6}$	$n-6$
$y_{s,s+2,s,s+4}$	$n-4$
$y_{s,s+4,s+2,s+4}$	$n-4$
$y_{s,s+2,s,s+3}$	$n-3$
$y_{s,s+2,a,a+2}$	$\frac{(n-4)(n-5)}{2}$
$y_{s,s+2,a,a+3}$	$\frac{(n-5)(n-6)}{2}$
$y_{s,s+3,a,a+2}$	$\frac{(n-5)(n-6)}{2}$
$y_{s,s+3,a,a+3}$	$\frac{(n-6)(n-7)}{2}$
$y_{s,s+2,s+1,s+3}$	$n-3$
$y_{s,s+2,s+1,s+4}$	$n-4$
$y_{s,s+3,s+1,s+3}$	$n-3$
$y_{s,s+3,s+1,s+4}$	$n-4$
$y_{s,s+3,s+2,s+4}$	$n-4$
$y_{s,s+3,s+2,s+5}$	$n-5$

Chapter 4

Example $k = 2$

Reminders: As before, let F_{ab} denote the image of the matrix unit E_{ab} under the transversal map. In addition to being strictly upper triangular, since $k = 2$ we have two additional diagonals above the main diagonal which are all zero. Since this implies that F_{ab} must be such that $a + (k+1) \leq b$, here we have $a + 3 \leq b$. Also the change from F_{rt} to G_{rt} simplifies to

$$G_{rt} = \begin{cases} F_{rt} & \text{if } t - r < 6 \\ F_{rt} + y_{r,t-3,t-3,t} = [F_{r,t-3}, F_{t-3,t}] & \text{otherwise} \end{cases}$$

As in the $k = 1$ example we divide our investigation of multiplier elements into two sections: elements produced by $[G_{rs}, G_{st}]$ and elements produced by $[G_{st}, G_{ab}]$, where $t \neq a$.

4.1 $[G_{rs}, G_{st}]$

Consider first the case where $[G_{rs}, G_{st}] = [F_{rs}, F_{st}] = F_{rt} + y_{rsst}$. Suppose that $t \geq r + 11$. If $s = t - 3$ then $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = F_{rt} + y_{r,t-3,t-3,t} = G_{rt}$. If $s \neq t - 3$, then we wish to show $y_{rsst} = y_{r,t-3,t-3,t}$ because this will give $[G_{rs}, G_{st}] = G_{rt}$. Since F_{rs} is defined, we know $r + 3 \leq s$. Suppose first that $s = r + 3$, $s = r + 4$, or $s = r + 5$, then $F_{s,t-3}$ is defined since $s + 3 \leq r + 8 \leq t - 3$. Therefore $J(F_{rs}, F_{s,t-3}, F_{t-3,t}) = 0 \;\Rightarrow\; y_{rsst} = y_{r,t-3,t-3,t}$. On the other hand, when $s > r + 5$ let $c = r + 3$. Notice that F_{cs} is defined so $J(F_{rc}, F_{cs}, F_{st}) = 0 \;\Rightarrow\; y_{rsst} = y_{rcct}$ and $J(F_{rc}, F_{c,t-3}, F_{t-3,t}) = 0 \;\Rightarrow\; y_{rcct} = y_{r,t-3,t-3,t}$, which together give $y_{rsst} = y_{r,t-3,t-3,t}$. Thus $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = F_{rt} + y_{r,t-3,t-3,t} = G_{rt}$ for any valid s when $t \geq r + 11$.

On the other extreme suppose $t = r + 6$, (the minimum distance from r to t is 6). In this case the only choice for s is $s = t - 3$. Therefore $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = F_{rt} + y_{r,t-3,t-3,t} = G_{rt}$.

It remains to consider $r + 7 \leq t \leq r + 10$. We wish to equate y_{rsst} and $y_{r,t-3,t-3,t}$ whenever possible so that $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = F_{rt} + y_{r,t-3,t-3,t} = G_{rt}$.

Suppose $t = r + 7$, then $r + 3 \leq s \leq r + 4$. If $s = r + 4$ then $s = r + 4 = t - 3$ so $y_{rsst} = y_{r,t-3,t-3,t}$ is trivially satisfied. If $s = r + 3$, there are no Jacobi identities available to equate y_{rsst} and $y_{r,t-3,t-3,t}$. Therefore $[G_{r,r+3}, G_{r+3,r+7}] = F_{r,r+7} + y_{r,r+3,r+3,r+7}$ where $y_{r,r+3,r+3,r+7}$ is some non-trivial element in the multiplier.

Suppose $t = r + 8$, then $r + 3 \leq s \leq r + 5$. If $s = r + 5$ then $s = r + 5 = t - 3$ so $y_{rsst} = y_{r,t-3,t-3,t}$ is trivially satisfied. If $s = r+3$ or $s = r+4$, there are no Jacobi identities available to equate y_{rsst} and $y_{r,t-3,t-3,t}$. Therefore $[G_{r,r+3}, G_{r+3,r+8}] = F_{r,r+8} + y_{r,r+3,r+3,r+8}$ and $[G_{r,r+4}, G_{r+4,r+8}] = F_{r,r+8} + y_{r,r+4,r+4,r+8}$ where $y_{r,r+3,r+3,r+8}$ and $y_{r,r+4,r+4,r+8}$ are some non-trivial elements in the multiplier.

Suppose $t = r + 9$, then $r + 3 \leq s \leq r + 6$. If $s = r + 6$ then $s = r + 6 = t - 3$ so $y_{rsst} = y_{r,t-3,t-3,t}$ is trivially satisfied. If $s = r + 3$ then $J(F_{rs}, F_{s,t-3}, F_{t-3,t}) = 0 \quad \Rightarrow \quad y_{rsst} = y_{r,t-3,t-3,t}$. If $s = r + 4$ or $s = r + 5$, there are no Jacobi identities available to equate y_{rsst} and $y_{r,t-3,t-3,t}$. Therefore $[G_{r,r+4}, G_{r+4,r+9}] = F_{r,r+9} + y_{r,r+4,r+4,r+9}$ and $[G_{r,r+5}, G_{r+5,r+9}] = F_{r,r+9} + y_{r,r+5,r+5,r+9}$ where $y_{r,r+4,r+4,r+9}$ and $y_{r,r+5,r+5,r+9}$ are some non-trivial elements in the multiplier.

Suppose $t = r + 10$, then $r + 3 \leq s \leq r + 7$. If $s = r + 7$ then $s = r + 7 = t - 3$ so $y_{rsst} = y_{r,t-3,t-3,t}$ is trivially satisfied. If $s = r + 3$ or $s = r + 4$ then $J(F_{rs}, F_{s,t-3}, F_{t-3,t}) = 0 \quad \Rightarrow \quad y_{rsst} = y_{r,t-3,t-3,t}$. If $s = r + 6$ then let $c = r + 3$, in which case $J(F_{rc}, F_{cs}, F_{st}) = 0 \quad \Rightarrow \quad y_{rsst} = y_{rcct}$, that is $y_{r,r+6,r+6,t} = y_{r,r+3,r+3,t}$, but we just saw from $J(F_{r,r+3}, F_{r+3,t-3}, F_{t-3,t}) = 0$ that $y_{r,r+3,r+3,t} = y_{r,t-3,t-3,t}$, which means $y_{r,r+6,r+6,t} = y_{r,t-3,t-3,t}$. If $s = r+5$, there are no Jacobi identities available to equate y_{rsst} and $y_{r,t-3,t-3,t}$. Therefore $y_{rsst} = y_{r,t-3,t-3,t}$ for $s = r + 3, r + 4, r + 6, r + 7$, but $[G_{r,r+5}, G_{r+5,r+10}] = F_{r,r+10} + y_{r,r+5,r+5,r+10}$ where $y_{r,r+5,r+5,r+10}$ is some non-trivial element in the multiplier.

In conclusion, the only non-trivial multiplier elements arising from $[G_{rs}, G_{st}]$ are of the form $y_{r,r+3,r+3,r+7}$, $y_{r,r+3,r+3,r+8}$, $y_{r,r+4,r+4,r+8}$, $y_{r,r+4,r+4,r+9}$, $y_{r,r+5,r+5,r+9}$, $y_{r,r+5,r+5,r+10}$.

As in the $k = 1$ example if we define $\widehat{y_{rsst}}$ such that $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = G_{rt} + \widehat{y_{rsst}}$, we get the same number of $y_{rsst}'s$ and $\widehat{y_{rsst}}'s$, so it is sufficient to count the occurences of y_{rsst}.

4.2 $[G_{st}, G_{ab}]$, where $t \neq a$

Consider the second case where $[G_{st}, G_{ab}] = [F_{st}, F_{ab}] = y_{stab}$ since $t \neq a$. Due to the antisymmetry of the bracket (alternating property), it is sufficient to consider only the cases when $s < a$ or $s = a$ and $t < b$. This in mind, we do not need to consider $s = b$ since it violates $s < a$ and the alternating property would put it back into the $t = a$ case.

Thus $[G_{st}, G_{ab}] = [F_{st}, F_{ab}] = y_{stab}$ where y_{stab} is some element in the multiplier, $M(L)$. Since no F is produced by the bracket operation and hence $[G_{st}, G_{ab}] = [F_{st}, F_{ab}] \in M(L)$, we will work with the $F's$ rather than the $G's$ as both produce the same elements in $M(L)$ when $t \neq a$. We wish to find all the scenarios (relationships between the values of $s, t, a,$ and b) where $y_{stab} = 0$ versus $y_{stab} \neq 0$. As in the previous example we begin by imposing an upper bound on the distance between a and b as well as s and t for which y_{stab} may be non-zero.

Lemma 4.2.1 *If $b \geq a + 7$ or $t \geq s + 7$ then $y_{stab} = 0$. So $y_{stab} \neq 0$ only has the potential to occur if both $b \leq a + 6$ and $t \leq s + 6$.*

Proof Suppose $b \geq a + 7$. If $t \neq a + 3$ then let $c = a + 3$. If $t = a + 3$ then choose $c = a + 4$. The number line is shown in Figure 4.1. By construction $c \neq t, a, b$. Notice $s \leq a < c \Rightarrow c \neq s$. This gives $c \neq s, t, a, b$ and therefore $J(F_{st}, F_{ac}, F_{cb}) = 0 \;\; \Rightarrow \;\; y_{stab} = 0$.

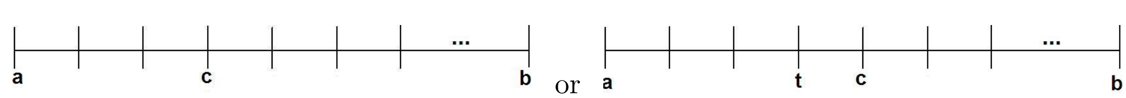

Figure 4.1: $k = 2 : b \geq a + 7$

Similarly suppose $t \geq s + 7$. If $a \neq s + 3$ then let $c = s + 3$. If $a = s + 3$ then choose $c = s + 4$. The number line is shown in Figure 4.2. By construction $c \neq s, t, a$, but we want to show $c \neq b$ also.

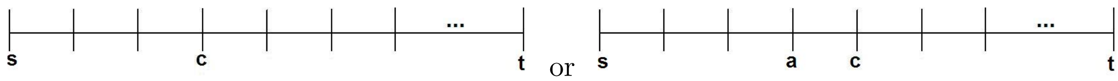

Figure 4.2: $k = 2 : t \geq s + 7$

Notice when $s = a \Rightarrow t < b \Rightarrow c = s + 3 < s + 7 \leq t < b$ and when $s < a$ then $a + 3 \leq b \Rightarrow s + 3 < a + 3 \leq b$ (*i.e.* $s + 4 \leq b$). Therefore either $b > s + 4 \geq c$ or $b = s + 4 \Rightarrow a = s + 1 \Rightarrow a \neq s + 3$ so $c = s + 3 \neq b$. Therefore in all cases $c \neq b$. This gives $c \neq s, t, a, b$ and therefore $J(F_{sc}, F_{ct}, F_{ab}) = 0 \Rightarrow y_{stab} = 0$. ∎

Lemma 4.2.2 *If $b = a + 6$ then $y_{stab} = 0 \Leftrightarrow t \neq a + 3$ or $s < a$ $(s \neq a)$. Similarly if $t = s + 6$ then $y_{stab} = 0 \Leftrightarrow a \neq s + 3$ or $b \neq t$.*

Proof ($\Leftarrow$) Suppose $b = a+6$. If $t \neq a+3$ then let $c = a+3$, so $c \neq t, a, b$ and furthermore $s \leq a < a + 3 = c \Rightarrow c \neq s$. Therefore $c \neq s, t, a, b$ and $J(F_{st}, F_{ac}, F_{cb}) = 0 \Rightarrow y_{stab} = 0$. The number line is shown in Figure 4.3.

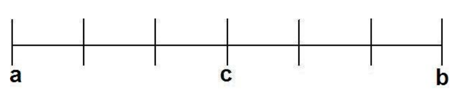

Figure 4.3: $k = 2 : b = a + 6, t \neq a + 3$

On the other hand if $t = a + 3$ and $s < a$ then let $c = t - 1$, as in Figure 4.4.

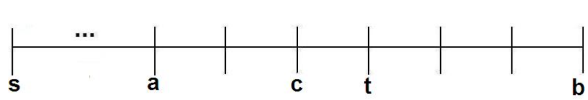

Figure 4.4: $k = 2 : b = a + 6, t = a + 3$

In this case $J(F_{st}, F_{tb}, F_{at}) = 0 \Rightarrow y_{stab} = -y_{sbat}$ and $J(F_{sc}, F_{cb}, F_{at}) = 0 \Rightarrow y_{sbat} = 0$ which together give $y_{stab} = 0$.

(⇒) Suppose $b = a + 6$, $t = a + 3$, and $s = a$, as in Figure 4.5.

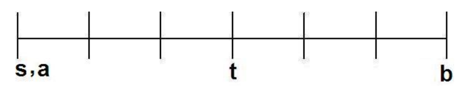

Figure 4.5: $k = 2 : b = a + 6, t = a + 3 = s + 3$

There is no value of c such that F_{ac} and F_{cb} are both defined while $c \neq t$. As such, there are no Jacobi identities available to zero out y_{stab}, thus $y_{stab} \neq 0$.

(⇐) Suppose $t = s + 6$. If $a \neq s + 3$ then let $c = s + 3$, so $c \neq s, t, a$, but we want to show $c \neq b$ also. Notice when $s = a$ then $t < b$ so $s < c < t < b \Rightarrow c \neq b$. If $s < a$ then $a + 3 \leq b \Rightarrow c = s + 3 < a + 3 \leq b \Rightarrow c \neq b$. Therefore $c \neq s, t, a, b$ and $J(F_{sc}, F_{ct}, F_{ab}) = 0 \Rightarrow y_{stab} = 0$. This can be seen on the number line in Figure 4.6.

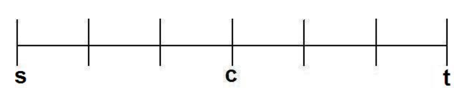

Figure 4.6: $k = 2 : t = s + 6, a \neq s + 3$

On the other hand if $a = s + 3$ and $b > t$ then let $c = a + 1$, as in Figure 4.7.

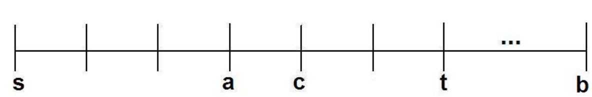

Figure 4.7: $k = 2 : t = s + 6, a = s + 3$

In this case $J(F_{sa}, F_{at}, F_{ab}) = 0 \Rightarrow y_{stab} = y_{sbat}$ and $J(F_{sc}, F_{cb}, F_{at}) = 0 \Rightarrow y_{sbat} = 0$ which together give $y_{stab} = 0$. If $a = s + 3$ and $b < t$ then $s < s + 3 = a < a + 3 \leq b < t \Rightarrow s + 6 < t$ and hence by Lemma 4.2.1, $y_{stab} = 0$.

($\Rightarrow$) Suppose $t = s + 6$, $a = s + 3$, and $b = t$, as in Figure 4.8.

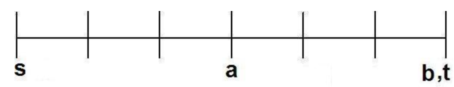

Figure 4.8: $k = 2 : b = t = s + 6, a = s + 3$

There is no value of c such that F_{sc} and F_{ct} are both defined while $c \neq a$. As such, there are no Jacobi identities available to zero out y_{stab}, thus $y_{stab} \neq 0$. ∎

Now that we have an upper bound on the distance from a to b and s to t for which non-trivial y_{stab} values may be produced, we continue our search by separating the variable relationships into three cases. Either (1) $s = a$, (2) $a > t$, or (3) $s < a < t$.

Case 1: $s = a$

For a fixed value of s, suppose $s = a$. Lemmas 4.2.1 and 4.2.2 discuss $b \geq s+6$, so consider $b < s + 6$. Let $t_{min} = s + 3$, thus denoting the minimum possible value of t. Since $s = a \Rightarrow t < b$ this gives $t_{min} < b < s + 6 = t_{min} + 3$. In other words $b = t_{min} + j$, where $j \in \{1, 2\}$.

Lemma 4.2.3 *When $b = t_{min} + j$ where $j \in \{1, 2\}$ we get j new non-zero values for y_{stab}.*

Proof For either value of j we have $b < s + 6 = a + 6$. Additionally $t < b \Rightarrow t < s + 6$. Therefore $\nexists c$ such that F_{ac} and F_{cb} are both defined, similarly $\nexists c$ such that F_{sc} and F_{ct} are both defined. Therefore $y_{stab} \neq 0$.

For $b = t_{min} + j$, $t < b \Rightarrow t \in \{t_{min}, \ldots, t_{min} + j - 1\} \Rightarrow t$ may take on j different values for a fixed b, hence we get j distinct new non-zero values for y_{stab} as t fluctuates. Putting this together we have

$j = 1 \Rightarrow b = t_{min} + 1 = s + 4 \Rightarrow t = t_{min} = s + 3 \Rightarrow y(s, s + 3, s, s + 4) \neq 0.$

$j = 2 \Rightarrow b = t_{min} + 2 = s+5 \Rightarrow t = t_{min}, t_{min} + 1 = s+3, s+4 \Rightarrow y(s, s+3, s, s+5)$ and $y(s, s + 4, s, s + 5) \neq 0.$ ∎

<u>**Case 2:** $a > t$</u>

Lemma 4.2.4 *If $a > t$ then $y_{stab} \neq 0$ for all t and b such that both $t < s+6$ and $b < a+6$. Otherwise $y_{stab} = 0$ when $a > t$. That is $y(s,t,a,a+3), y(s,t,a,a+4), y(s,t,a,a+5) \neq 0$ when $t = s+3, s+4, s+5$.*

Proof If $t \geq s+7$ or $b \geq a+7$ then Lemma 4.2.1 $\Rightarrow$ $y_{stab} = 0$. If $t = s+6$ or $b = a+6$ then Lemma 4.2.2 $\Rightarrow$ $y_{stab} = 0$ since $a > t$ $\Rightarrow$ $t \neq a+3$ and $a \neq s+3$.

If $t < s+6$ and $b < a+6$ then there is no value of c, such that F_{sc} and F_{ct} are both defined for $s < c < t$. Similarly there is no value of c, such that F_{ac} and F_{cb} are both defined for $a < c < b$. Therefore the idea in Lemma 4.2.1, of using $J(F_{sc}, F_{ct}, F_{ab}) = 0$ or $J(F_{st}, F_{ac}, F_{cb}) = 0$ will not work here. Also, placing a c such that $s < t < c < a < b$ will not provide any helpful Jacobi identities, no matter how large the gap between t and a. Thus y_{stab} will always be non-zero in this case. ∎

<u>**Case 3:** $s < a < t$</u>

If $b \geq a+7$ or $t \geq s+7$ then Lemma 4.2.1 implies $y_{stab} = 0$. If $b = a+6$ then Lemma 4.2.2 implies $y_{stab} = 0$ since $s < a$. If $t = s+6$ and $a \neq s+3$ then Lemma 4.2.2 gives $y_{stab} = 0$. If $t = s+6$ and $a = s+3$ $\Rightarrow$ $b \geq a+3 = s+6 = t$. When $b > t$ then Lemma 4.2.2 gives $y_{stab} = 0$. When $b = t$ then Lemma 4.2.2 gives $y_{stab} \neq 0$.

Thus it is only left to consider $t < s+6$ and $b < a+6$. No such c exists, nor do suitable Jacobi identities exist to zero out y_{stab} when $a+3 \leq b \leq a+5$ and $s+3 \leq t \leq s+5$. Therefore $y_{stab} \neq 0$ when $t = s+3, s+4$, or $s+5$ and $b = a+3, a+4$, or $a+5$.

Collecting all this information, we get $y_{stab} \neq 0$ when:

1. Lemma 4.2.2 result: $b = a + 6$, $t = a + 3$, and $s = a$.
2. Lemma 4.2.2 result: $t = s + 6$, $a = s + 3$ and $b = t$.
3. Lemma 4.2.3 result: $s = a$, $b = t_{min} + 1, t_{min} + 2 = s + 4, s + 5$, and $t < b$.
4. Lemma 4.2.4 result: $a > t$, $t < s + 6$, and $b < a + 6$.
5. $s < a < t$, $t = s + 3, s + 4, s + 5$, and $b = a + 3, a + 4, a + 5$.

4.3 Counting the multiplier elements

We are interested in counting all the cases when y_{rsst} and y_{stab} cannot be eliminated. Notice that just as in the $k = 1$ example there are two types of elements: (1) $y(s, s + x_1, s + x_2, s + x_3)$ where x_1, x_2, x_3 are all fixed and (2) $y(s, s + x_1, a, a + x_2)$ where $a > s + x_1$ and x_1, x_2 are both fixed. Recall that in the first, $y(s, s + x_1, s + x_2, s + x_3)$ may assume $n - w$ different values (provided that $w = max\{x_1, x_2, x_3\}$) and in the second, $y(s, s + x_1, a, a + x_2)$ may assume $\frac{1}{2}(n - (x_2 + x_1 + 1))(n - (x_2 + x_1))$ different values.

Using these two counting techniques, Tables 4.1 and 4.2 list all non-trivial y_{rsst} and y_{stab} possibilities and the number of times they occur. Adding all of these together gives $\dim M(L) = 3(n-4) + 9(n-5) + 10(n-6) + 7(n-7) + 5(n-8) + 3(n-9) + (n-10) + \frac{(n-6)(n-7)}{2} + (n-7)(n-8) + \frac{3}{2}(n-8)(n-9) + (n-9)(n-10) + \frac{(n-10)(n-11)}{2}$. Also, as stated before if any $n - x$ term is not positive, simply omit it from the formula because the original matrices will not be large enough to produce the corresponding matrix units and the resulting multiplier elements they would have produced.

Table 4.1: Counting multiplier elements for $k = 2$, the $[G_{rs}, G_{st}]$ case

y_{rsst} non-trivial	Number of occurences
$y_{r,r+3,r+3,r+7}$	$n-7$
$y_{r,r+3,r+3,r+8}$	$n-8$
$y_{r,r+4,r+4,r+8}$	$n-8$
$y_{r,r+4,r+4,r+9}$	$n-9$
$y_{r,r+5,r+5,r+9}$	$n-9$
$y_{r,r+5,r+5,r+10}$	$n-10$

Table 4.2: Counting multiplier elements for $k = 2$, the $[G_{st}, G_{ab}]$ case

y_{stab} non-trivial	Number of occurences
$y_{s,s+3,s,s+6}$	$n-6$
$y_{s,s+6,s+3,s+6}$	$n-6$
$y_{s,s+3,s,s+4}$	$n-4$
$y_{s,s+3,s,s+5}$	$n-5$
$y_{s,s+4,s,s+5}$	$n-5$
$y_{s,s+3,a,a+3}$	$\frac{(n-6)(n-7)}{2}$
$y_{s,s+3,a,a+4}$	$\frac{(n-7)(n-8)}{2}$
$y_{s,s+3,a,a+5}$	$\frac{(n-8)(n-9)}{2}$
$y_{s,s+4,a,a+3}$	$\frac{(n-7)(n-8)}{2}$
$y_{s,s+4,a,a+4}$	$\frac{(n-8)(n-9)}{2}$
$y_{s,s+4,a,a+5}$	$\frac{(n-9)(n-10)}{2}$
$y_{s,s+5,a,a+3}$	$\frac{(n-8)(n-9)}{2}$
$y_{s,s+5,a,a+4}$	$\frac{(n-9)(n-10)}{2}$
$y_{s,s+5,a,a+5}$	$\frac{(n-10)(n-11)}{2}$
$y_{s,s+3,s+1,s+4}$	$n-4$
$y_{s,s+3,s+1,s+5}$	$n-5$
$y_{s,s+3,s+1,s+6}$	$n-6$
$y_{s,s+3,s+2,s+5}$	$n-5$
$y_{s,s+3,s+2,s+6}$	$n-6$
$y_{s,s+3,s+2,s+7}$	$n-7$
$y_{s,s+4,s+1,s+4}$	$n-4$
$y_{s,s+4,s+1,s+5}$	$n-5$
$y_{s,s+4,s+1,s+6}$	$n-6$
$y_{s,s+4,s+2,s+5}$	$n-5$
$y_{s,s+4,s+2,s+6}$	$n-6$
$y_{s,s+4,s+2,s+7}$	$n-7$
$y_{s,s+4,s+3,s+6}$	$n-6$
$y_{s,s+4,s+3,s+7}$	$n-7$
$y_{s,s+4,s+3,s+8}$	$n-8$
$y_{s,s+5,s+1,s+4}$	$n-5$
$y_{s,s+5,s+1,s+5}$	$n-5$
$y_{s,s+5,s+1,s+6}$	$n-6$
$y_{s,s+5,s+2,s+5}$	$n-5$
$y_{s,s+5,s+2,s+6}$	$n-6$
$y_{s,s+5,s+2,s+7}$	$n-7$
$y_{s,s+5,s+3,s+6}$	$n-6$
$y_{s,s+5,s+3,s+7}$	$n-7$
$y_{s,s+5,s+3,s+8}$	$n-8$
$y_{s,s+5,s+4,s+7}$	$n-7$
$y_{s,s+5,s+4,s+8}$	$n-8$
$y_{s,s+5,s+4,s+9}$	$n-9$

Chapter 5

General Case

Recall that we let F_{ab} denote the image of the matrix unit E_{ab} under the transversal map. Having E_{ab} strictly upper triangular requires $a+1 \le b$ and having k superdiagonals of zeros implies $a+(k+1) \le b$. (Note that the strictly upper triangular matrices correspond to $k=0$, one superdiagonal of zeros corresponds to $k=1$, etc.) Accordingly we completely describe the bracket as

$$[F_{st}, F_{ab}] = \begin{cases} F_{sb} + y(s,t,a,b) & \text{if } t = a \\ y(s,t,a,b) & \text{if } t \neq a \end{cases}$$

where $y(s,t,a,b) \in M(L)$ and we often abbreviate $y(s,t,a,b)$ with y_{stab}. Also recall that as in the model of [3], we first make a change in the choice of F_{rt}. Set

$$G_{rt} = \begin{cases} F_{rt} & \text{if } t-r < 2(k+1) \\ F_{rt} + y_{r,t-(k+1),t-(k+1),t} = [F_{r,t-(k+1)}, F_{t-(k+1),t}] & \text{otherwise} \end{cases}$$

We want to see how many different $y's$ exist for all possible brackets assuming the original matrices are of size $n \times n$ with k superdiagonals of zeros. Presently assume n is sufficiently large to define all desired computations; later we will relax this restriction.

As in the cases $k=1$ and $k=2$, we divide our investigation of multiplier elements into two sections: elements produced by $[G_{rs}, G_{st}]$ and elements produced by $[G_{st}, G_{ab}]$ where $t \neq a$. In the latter we establish $s < a$ or $s = a$ and $t < b$.

5.1 $[G_{rs}, G_{st}]$

Consider $[G_{rs}, G_{st}] = [F_{rs}, F_{st}] = F_{rt} + y_{rsst} = G_{rt} + \widehat{y_{rsst}}$ where $\widehat{y_{rsst}} = y_{rsst} - y_{r,t-(k+1),t-(k+1),t}$. As in the examples, we will count the number of occurences of y_{rsst} since this will be the same as the number of occurences of $\widehat{y_{rsst}}$ provided that we account for the case when y_{rsst} is absorbed into G_{rt} and $\widehat{y_{rsst}} = 0$, that is $[G_{rs}, G_{st}] = G_{rt}$ and there is no contribution to the multiplier. Notice that the requirements of $s - r, t - s \geq k + 1$ give $t - r \geq 2(k+1)$ or $r + 2(k+1) \leq t$. From the example $k = 1$ we noticed that given a fixed r, we get all $y's$ trivial when $t = r + 2(k+1) = r + 4$ and $t \geq r + 3(k+1) + k = r + 7$. However for the cases in between when $t = r + 5, r + 6$ we get 1 non-trivial y in each case. Similarly when $k = 2$ (and r fixed) we noticed we get all $y's$ trivial when $t = r + 2(k+1) = r + 6$ and $t \geq r+3(k+1)+k = r+11$. However for the cases in between when $t = r+7, r+8, r+9, r+10$, the number of non-trivial $y's$ we attain are $1, 2, 2, 1$ respectively. Fortunately this pattern will continue. For a general k (and r fixed) we get all $y's = 0$ when $t = r + 2(k+1)$ and $t \geq r+3(k+1)+k$. However as t traverses the $2k$ values between $r+2(k+1)$ and $r+3(k+1)+k$ (*i.e.* $t = r+2(k+1)+1, r+2(k+1)+2, \ldots, r+3(k+1)+(k-1)$) the number of non-trivial $y's$ we attain are $1, 2, 3, \ldots, k-3, k-2, k-1, k, k, k-1, k-2, k-3, \ldots, 3, 2, 1$ respectively. The theorem below shows this, and also takes into account what happens as r changes. Additionally, case 1 shows the first set of k ascending numbers $1, 2, 3, \ldots, k-3, k-2, k-1, k$ and case 2 shows the second set of k descending numbers $k, k-1, k-2, k-3, \ldots, 3, 2, 1$.

Theorem 5.1.1 *As r, s, and t range over all values for which the bracket is defined, the number of $y's$ produced from $[G_{rs}, G_{st}]$ is* $\sum_{i=1}^{k} i \cdot (n - (2(k+1) + i)) + \sum_{j=1}^{k} (k - j + 1) \cdot (n - (2(k+1) + k + j))$.

Proof In general if $\exists c$ such that F_{rc} and F_{cs} are both defined (*i.e.* $r + (k+1) \leq c$ and $c + (k+1) \leq s$) then the Jacobi identity $J(F_{rc}, F_{cs}, F_{st}) = 0 \Rightarrow y_{rsst} = y_{rcct}$. Similarly if $\exists c$ such that F_{sc} and F_{ct} are both defined (i.e. $s + (k+1) \leq c$ and $c + (k+1) \leq t$) then the Jacobi identity $J(F_{rs}, F_{sc}, F_{ct}) = 0 \Rightarrow y_{rsst} = y_{rcct}$.

<u>**Case 1:** $t = r + 2(k+1) + i$ where $1 \leq i \leq k$</u>

We do not need to consider $i = 0$ because $t = r + 2(k+1) \Rightarrow s = r + (k+1) = t - (k+1) \Rightarrow [G_{rs}, G_{st}] = G_{rt}$. For any value of i in this range, the distance from r to t is $t - r = 2(k+1) + i \leq 2(k+1) + k$. In order to define G_{ab} there must be at least

k integers between a and b, or $a + (k + 1) \leq b$. Given a distance from r to t of at most $2(k + 1) + k$ does not provide sufficient room to define either of the Jacobi identities above (*i.e.* $J(F_{rc}, F_{cs}, F_{st}) = 0$ or $J(F_{rs}, F_{sc}, F_{ct}) = 0$). While a number c may still be defined to produce other Jacobi identities, none of these will involve y_{rsst} and hence none of the $y's$ resulting from the bracket operation $[G_{rs}, G_{st}]$ may be equated. Thus for each value of i (from 1 to k) and a fixed r there will be i such distinct values for s and hence i distinct $y's$ produced, namely y_{rsst} where s is any integer from $r + (k + 1)$ up to $r + (k + 1) + i - 1$. Notice we have excluded the last possible case when $s = r + (k + 1) + i$ because this also means $s = t - (k + 1)$, in which case $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = G_{rt}$, thus y_{rsst} gets absorbed into the G_{rt} and is not counted. That is

if $i = 1$ then $s = r + (k + 1)$ or $s = r + (k + 1) + 1$

if $i = 2$ then $s = r + (k + 1)$ or $s = r + (k + 1) + 1$ or $s = r + (k + 1) + 2$

if $i = 3$ then $s = r + (k + 1)$ or $s = r + (k + 1) + 1$ or $s = r + (k + 1) + 2$ or $s = r + (k + 1) + 3$

etc.

In all of which we do not count the last case $s = t - (k + 1) = r + (k + 1) + i$ where y_{rsst} is absorbed into G_{rt}.

Now for a fixed number i, it is necessary to see how many $y's$ may be produced as r varies. Notice that $1 \leq r < r + 2(k + 1) + i = t \leq n$. Thus r is at least 1 and t is at most n. In order words $1 \leq r \leq n - (2(k + 1) + i)$. Therefore r may assume $n - (2(k + 1) + i)$ different values, each producing a different set of i distinct values for y_{rsst} as s varies. So as r varies and i remains fixed, there are $i \cdot (n - (2(k + 1) + i))$ different $y's$.

In total this produces $\displaystyle\sum_{i=1}^{k} i \cdot (n - (2(k + 1) + i))$ distinct values for y_{rsst} as r, s and t vary.

Case 2: $t = r + 2(k + 1) + i$ where $k + 1 \leq i$

For convenience let $j = i - k$ so that $t = r + 2(k + 1) + k + j$ where $j \geq 1$. As stated in the previous case G_{ab} may only be defined if there are k integers between a and b, or $a + (k + 1) \leq b$. Now the luxury exists to pick a c and a d so that G_{rc}, G_{cd}, and G_{dt} are all defined. For the rest of this case, we will allow c to vary and force d to be $d = c + (k + 1)$ (thus depending on the choice of c). Thus we are using d in lieu of s when discussing new values of s in y_{rsst} and c in lieu of s when discussing old values of s in y_{rsst} from case 1. Enforcing the relationship between c and d, we are interested to count the number of $c's$ in existence that will allow G_{rc}, G_{cd}, and G_{dt} to all be defined. We know that the minimum value of c is $r+(k+1)$ in order to define G_{rc}, but we need an upper bound also. Using the fact that we need $d \leq t-(k+1)$ in order to define G_{dt} means

$$d \leq t-(k+1) \quad \Rightarrow \quad c+(k+1) \leq t-(k+1) \quad \Rightarrow \quad c \leq t-2(k+1) \qquad (*)$$

However $t = r+2(k+1)+k+j$
$\Rightarrow \quad c \leq (r+2(k+1)+k+j)-2(k+1) = r+k+j = r+(k+1)+(j-1)$
$\Rightarrow c \leq r+(k+1)+(j-1)$.

Therefore
if $j=1$ then $c=r+(k+1)$
if $j=2$ then $c=r+(k+1)$ or $c=r+(k+1)+1$
if $j=3$ then $c=r+(k+1)$ or $c=r+(k+1)+1$ or $c=r+(k+1)+2$
if $j=4$ then $c=r+(k+1)$ or $c=r+(k+1)+1$ or $c=r+(k+1)+2$ or $c=r+(k+1)+3$
etc.

So in general there are j different choices for the variable c which will allow G_{rc}, G_{cd}, and G_{dt} to all be defined.

Notice that the largest distance between r and t from case 1 occured when what we are now calling j would be equal to zero (*i.e.* $i=k$). We found that there were k different possible values of y_{rsst} that could arise from $[G_{rs}, G_{st}]$ for a fixed r, because s can assume $k+1$ different values (the largest of which does not get counted since $y_{r,t-(k+1),t-(k+1),t}$ gets absorbed into G_{rt}).

Case 2 now considers $j \geq 1$. There is the potential for j new values of y_{rsst} as there are j new values of s for a fixed r (reminder: we call these y_{rddt}) because there are j additional integers between r and t and hence s may assume j more values in the expression $y_{rsst} = y_{rddt}$ which arise from the j additional brackets $[G_{rs}, G_{st}] = [G_{rd}, G_{dt}]$. Even though the potential is there, none of these $y's$ are new. The Jacobi identity (1) $J(F_{rc}, F_{cd}, F_{dt}) = 0 \Rightarrow y_{rcct} = y_{rddt}$. This means that each of the j new $y_{rddt}{}'s$ can be equated to its corresponding y_{rcct} from the case 1 scenario. Even more fortunate is that c is always at least $k+1$ integers away from $t-(k+1)$, as shown earlier in $(*)$. Hence the Jacobi identity (2) $J(F_{rc}, F_{c,t-(k+1)}, F_{t-(k+1),t}) = 0$ gives $y_{rcct} = y_{r,t-(k+1),t-(k+1),t}$ for all j possible values for c. Keep in mind that $c \leq t-2(k+1)$, however if $c = t-2(k+1)$ then $d = t-(k+1)$ which would mean these two Jacobi identities (1) and (2) are the exact same. So rather, we are only interested in the $j-1$ different c values when $c < t-2(k+1)$

and therefore $\Rightarrow y_{rddt} = y_{rcct} = y_{r,t-(k+1),t-(k+1),t}$ where $c < c + (k+1) = d < t - (k+1)$ and hence these $j-1$ values of y_{rcct} where c is as in case 1 can all be equated to $y_{r,t-(k+1),t-(k+1),t}$, which are all absored into G_{rt}. This eliminates these $j-1$ $y_{rcct}'s$ from our original count of k when j was zero.

Therefore for a fixed r there are $k-(j-1) = k-j+1$ values of y when $1 \leq j \leq k$, and if $j > k$, all the $y's$ are equal to each other and to $y_{r,t-(k+1),t-(k+1),t}$ hence producing no non-trivial values for y_{rsst}.

Now for a fixed number j (between 1 and k), it is necessary to see how many $y's$ can be produced as r varies. Notice that $1 \leq r < r + 2(k+1) + k + j = t \leq n$. Thus r is at least 1 and t is at most n. In other words $1 \leq r \leq n - (2(k+1)+k+j)$. Therefore r may assume $n - (2(k+1)+k+j)$ different values, each producing a set of $k-j+1$ distinct values for y_{rsst}. So as r varies and j remains fixed, there are
$(k-j+1)\cdot(n-(2(k+1)+k+j))$ different $y's$.

In total this produces $\sum_{j=1}^{k}(k-j+1)\cdot(n-(2(k+1)+k+j))$ distinct values for y_{rsst} as r and j vary, keeping in mind that if $j > k$ we get zero $y's$, hence it is sufficient to sum up to $j = k$.

In total this produces $\sum_{i=1}^{k} i\cdot(n-(2(k+1)+i)) + \sum_{j=1}^{k}(k-j+1)\cdot(n-(2(k+1)+k+j))$ different y_{rsst} values. $\blacksquare$

Notice every part of both sums involve a term of the form $n-x$. As in the examples, x denotes a distance between subscripts of the $G's$ and hence a distance between matrix positions in the original $n \times n$ matrices. Therefore whenever $n - x \leq 0$ occurs, it should be replaced with a zero since the necessary matrix units were not available to produce the corresponding multiplier elements.

5.2 $[G_{st}, G_{ab}]$, where $t \neq a$

Recall that $[G_{st}, G_{ab}] = [F_{st}, F_{ab}] = y_{stab}$ where y_{stab} is some element in the multiplier, $M(L)$, since $t \neq a$ and the $y's$ are central. As in the cases of $k = 1, 2$, since no F results from the bracket operation, we will work with the $F's$ rather than the $G's$ as both produce the same elements in $M(L)$ when $t \neq a$. Again we wish to find all the relationships between the values of s, t, a, and b where $y_{stab} = 0$ versus $y_{stab} \neq 0$. As in the examples we begin by imposing an upper bound on the distance between a and b as well as s and t for which y_{stab} may be non-zero.

Theorem 5.2.1 *If $b \geq a + 2(k+1) + 1$ or $t \geq s + 2(k+1) + 1$ then $y_{stab} = 0$. So $y_{stab} \neq 0$ only has the potential to occur if both $b \leq a + 2(k+1)$ and $t \leq s + 2(k+1)$.*

Proof Suppose $b \geq a + 2(k+1) + 1$. If $t \neq a + (k+1)$ then let $c = a + (k+1)$. If $t = a + (k+1)$ then choose $c = a + (k+1) + 1$. This can be seen in Figure 5.1. Where $b_1 = a + 2(k+1) + 1$, so $b \geq b_1$. By construction $c \neq t, a, b$. Notice $s \leq a < c \Rightarrow c \neq s$. This gives $c \neq s, t, a, b$ and therefore $J(F_{st}, F_{ac}, F_{cb}) = 0 \quad \Rightarrow \quad y_{stab} = 0$.

Similarly suppose $t \geq s + 2(k+1) + 1$. If $a \neq s + (k+1)$ then let $c = s + (k+1)$. If $a = s + (k+1)$ then choose $c = s + (k+1) + 1$. This can be seen in Figure 5.2. Where $t_1 = s + 2(k+1) + 1$, so $t \geq t_1$. By construction $c \neq s, t, a$, but we want to show $c \neq b$ also.

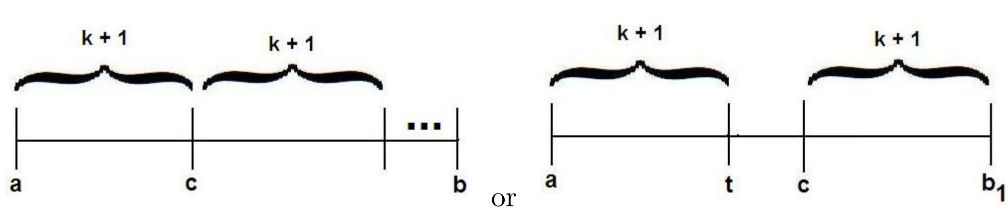

Figure 5.1: $b \geq a + 2(k+1) + 1$

Notice when $s = a \quad \Rightarrow \quad t < b \quad \Rightarrow \quad c = s + (k+1) < s + 2(k+1) + 1 \leq t < b$ so $c \neq b$. When $s < a$ then $a + (k+1) \leq b \quad \Rightarrow \quad s + (k+1) < a + (k+1) \leq b \quad$ (*i.e.* $b \geq s + (k+1) + 1$). Therefore either $b > s + (k+1) + 1 \geq c$ or $b = s + (k+1) + 1 \quad \Rightarrow$

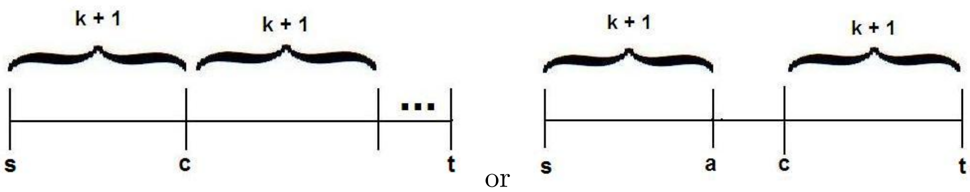

Figure 5.2: $t \geq s + 2(k+1) + 1$

$a = s+1$ (since $a + (k+1) \leq b$ and $s < a$) $\Rightarrow$ $a \neq s + (k+1)$ so $c = s + (k+1) \neq b$. Therefore in all cases $c \neq b$. (This assumed $k \neq 0$. If $k = 0$ then the result still holds according to [3].) This gives $c \neq s, t, a, b$ and therefore $J(F_{sc}, F_{ct}, F_{ab}) = 0 \Rightarrow y_{stab} = 0$. $\blacksquare$

Theorem 5.2.2 *If $b = a + 2(k+1)$ then $y_{stab} = 0 \Leftrightarrow t \neq a + (k+1)$ or $s < a$ $(s \neq a)$. Similarly if $t = s + 2(k+1)$ then $y_{stab} = 0 \Leftrightarrow a \neq s + (k+1)$ or $b \neq t$.*

Proof ($\Leftarrow$) Suppose $b = a + 2(k+1)$. If $t \neq a + (k+1)$ then let $c = a + (k+1)$, so $c \neq t, a, b$ and furthermore $s \leq a < a + (k+1) = c \Rightarrow c \neq s$. Therefore $c \neq s, t, a, b$ and $J(F_{st}, F_{ac}, F_{cb}) = 0 \Rightarrow y_{stab} = 0$. Please see Figure 5.3. On the other hand if $t = a + (k+1)$ and $s < a$ then let $c = t - 1$, which is shown in Figure 5.4. In this case $J(F_{st}, F_{tb}, F_{at}) = 0 \Rightarrow y_{stab} = -y_{sbat}$ and $J(F_{sc}, F_{cb}, F_{at}) = 0 \Rightarrow y_{sbat} = 0$ which together give $y_{stab} = 0$.

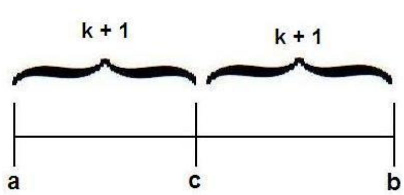

Figure 5.3: $b = a + 2(k+1), t \neq a + (k+1)$

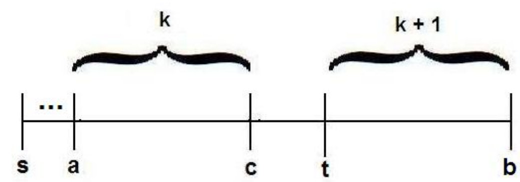

Figure 5.4: $b = a + 2(k+1), t = a + (k+1)$

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($\Rightarrow$) Suppose $b = a + 2(k+1)$, $t = a + (k+1)$, and $s = a$, as in Figure 5.5.

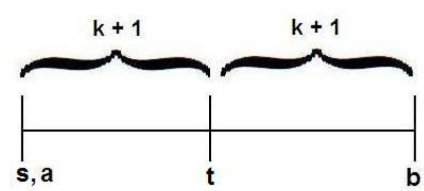

Figure 5.5: $b = a + 2(k+1), t = a + (k+1) = s + (k+1)$

There is no value of c such that F_{ac} and F_{cb} are both defined while $c \neq t$, nor is there a c such that both F_{sc} and F_{ct} can be defined. As such, there are no Jacobi identities available to zero out y_{stab}, thus $y_{stab} \neq 0$.

($\Leftarrow$) Suppose $t = s + 2(k+1)$. If $a \neq s + (k+1)$ then let $c = s + (k+1)$, so $c \neq s, t, a$, but we want to show $c \neq b$ also. Notice when $s = a$ then $t < b$ so $s < c < t < b \Rightarrow c \neq b$. If $s < a$ then $a + (k+1) \leq b \;\Rightarrow\; c = s + (k+1) < a + (k+1) \leq b \;\Rightarrow\; c \neq b$. Therefore $c \neq s, t, a, b$ and $J(F_{sc}, F_{ct}, F_{ab}) = 0 \Rightarrow y_{stab} = 0$. Please see Figure 5.6. On the other hand if $a = s + (k+1)$ and $b > t$ then let $c = a + 1$, which can be seen in Figure 5.7. In this case $J(F_{sa}, F_{at}, F_{ab}) = 0 \;\Rightarrow\; y_{stab} = y_{sbat}$ and $J(F_{sc}, F_{cb}, F_{at}) = 0 \;\Rightarrow\; y_{sbat} = 0$ which together give $y_{stab} = 0$.

If $a = s + (k+1)$ and $b < t$ then $s < s + (k+1) = a < a + (k+1) \leq b < t \;\Rightarrow$ $s + 2(k+1) + 1 \leq t$ and hence by Theorem 5.2.1, $y_{stab} = 0$.

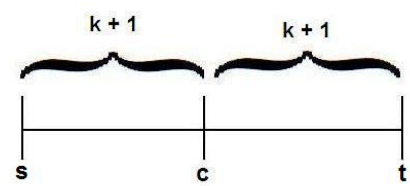

Figure 5.6: $t = s + 2(k+1), a \neq s + (k+1)$

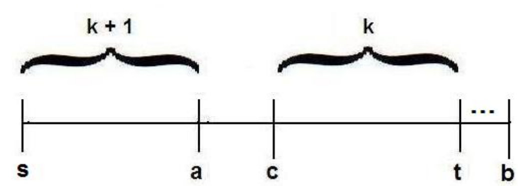

Figure 5.7: $t = s + 2(k+1), a = s + (k+1)$

(⇒) Suppose $t = s + 2(k+1)$, $a = s + (k+1)$, and $b = t$, as shown in Figure 5.8.

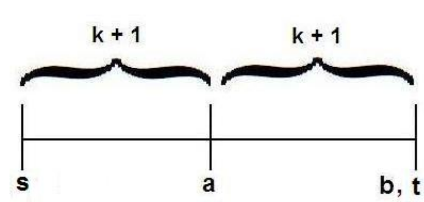

Figure 5.8: $b = t = s + 2(k+1), a = s + (k+1)$

There is no value of c such that F_{sc} and F_{ct} are both defined while $c \neq a$, nor is there a c such that both F_{ac} and F_{cb} can be defined. As such, there are no Jacobi identities available to zero out y_{stab}, thus $y_{stab} \neq 0$. ∎

As in the examples, $k = 1, 2$, now that we have an upper bound on the distance from a to b and s to t for which non-trivial y_{stab} values may be produced we continue our search by separating the variable relationships into three cases. Either (1) $s = a$, (2) $a > t$, or (3) $s < a < t$.

<u>**Case 1:** $s = a$</u>

For a fixed value of s, suppose $s = a$. Theorems 5.2.1 and 5.2.2 discuss $b \geq s + 2(k+1)$, so consider $b < s + 2(k+1)$. Let $t_{min} = s + (k+1)$, denoting the minimum possible value of t. Since $s = a \Rightarrow t < b$ this gives $t_{min} < b < s+2(k+1) = t_{min}+(k+1)$. In other words $b = t_{min} + j$, where $j \in \{1, 2, \ldots, k\}$.

Theorem 5.2.3 *When $b = t_{min} + j$ where $j \in \{1, 2, \ldots, k\}$ we get j new non-zero values for y_{stab}.*

Proof For any value of j we get $b < s + 2(k+1) = a + 2(k+1)$. Additionally $t < b \Rightarrow t < s + 2(k+1)$. Therefore $\nexists c$ such that F_{ac} and F_{cb} are both defined, similarly $\nexists c$ such that F_{sc} and F_{ct} are both defined. Therefore $y_{stab} \neq 0$.

For $b = t_{min} + j$, $t < b \Rightarrow t \in \{t_{min}, t_{min} + 1, \ldots, t_{min} + j - 1\} \Rightarrow t$ may take on j different values for a fixed b, hence we get j distinct new non-zero values for y_{stab} as t fluctuates. So the $y's$ are $y(s, t_{min}, s, t_{min} + j), y(s, t_{min} + 1, s, t_{min} + j), \ldots, y(s, t_{min} + j - 1, s, t_{min} + j)$. ∎

<u>**Case 2:** $a > t$</u>

Theorem 5.2.4 *If $a > t$ then $y_{stab} \neq 0$ for all t and b such that both $t < s + 2(k+1)$ and $b < a + 2(k+1)$. Otherwise $y_{stab} = 0$ when $a > t$.*

Proof If $t \geq s + 2(k+1) + 1$ or $b \geq a + 2(k+1) + 1$ then Theorem 5.2.1 $\Rightarrow$ $y_{stab} = 0$. If $t = s + 2(k+1)$ or $b = a + 2(k+1)$ then Theorem 5.2.2 $\Rightarrow$ $y_{stab} = 0$ since $a > t \Rightarrow t \neq a + (k+1)$ and $a \neq s + (k+1)$.

If $t < s + 2(k+1)$ and $b < a + 2(k+1)$ then there is no value of c, such that F_{sc} and F_{ct} are both defined for $s < c < t$. Similarly there is no value of c, such that F_{ac} and F_{cb} are both defined for $a < c < b$. Therefore the idea in Theorem 5.2.1, of using $J(F_{sc}, F_{ct}, F_{ab}) = 0$ or $J(F_{st}, F_{ac}, F_{cb}) = 0$ will not work here. Also, placing a c such that $s < t < c < a < b$ will not provide any helpful Jacobi identities, no matter how large the gap between t and a. Thus y_{stab} will always be non-zero in this case. ∎

<u>**Case 3:** $s < a < t$</u>

If $b \geq a + 2(k+1) + 1$ or $t \geq s + 2(k+1) + 1$ then Theorem 5.2.1 implies $y_{stab} = 0$. If $b = a + 2(k+1)$ then Theorem 5.2.2 implies $y_{stab} = 0$ since $s < a$. If $t = s + 2(k+1)$ and $a \neq s + (k+1)$ then Theorem 5.2.2 gives $y_{stab} = 0$. If $t = s + 2(k+1)$ and $a = s + (k+1) \Rightarrow b \geq a + (k+1) = s + 2(k+1) = t$. When $b > t$ then Theorem 5.2.2 gives $y_{stab} = 0$. When $b = t$ then Theorem 5.2.2 gives $y_{stab} \neq 0$.

Thus it is only left to consider $t < s + 2(k+1)$ and $b < a + 2(k+1)$. No such c exists nor do any suitable Jacobi identities exist to zero out y_{stab} when $a + (k+1) \leq b \leq a + 2(k+1) - 1$ and $s + (k+1) \leq t \leq s + 2(k+1) - 1$. Therefore $y_{stab} \neq 0$ when b and t lie within these intervals.

<u>Collecting all this information, we get $y_{stab} \neq 0$ when:</u>

1. Theorem 5.2.2 result:

   $b = a + 2(k+1)$, $t = a + (k+1)$, and $s = a$.

2. Theorem 5.2.2 result:

   $t = s + 2(k+1)$, $a = s + (k+1)$ and $b = t$.

3. Theorem 5.2.3 result:

   $s = a$ and $b = t_{min} + j$ where $j \in \{1, 2, \ldots, k\}$.

4. Theorem 5.2.4 result:

   $a > t$, $t < s + 2(k+1)$, and $b < a + 2(k+1)$.

5. $s < a < t$, $t < s + 2(k+1)$, and $b < a + 2(k+1)$.

5.3 Counting elements produced by $[G_{st}, G_{ab}]$, where $t \neq a$

We are interested to count all such cases when $y_{stab} \neq 0$. As in the $k = 1, 2$ examples there are two types of elements: (1) $y(s, s + x_1, s + x_2, s + x_3)$ which produce $n - w$ multiplier elements when $w = max\{x_1, x_2, x_3\}$ and (2) $y(s, s + x_1, a, a + x_2)$ where

$a > s + x_1$ which produce $\frac{1}{2}(n-(x_2+x_1+1))(n-(x_2+x_1))$ multiplier elements. We now use this to count the number of non-trivial values for y_{stab} when $t \neq a$.

1. <u>Theorem 5.2.2 result: $b = a + 2(k+1)$, $t = a + (k+1)$, and $s = a$</u>

   $y_{stab} = y(s, s+(k+1), s, s+2(k+1))$ which assumes $n - 2(k+1)$ values as s varies.

2. <u>Theorem 5.2.2 result: $t = s + 2(k+1)$, $a = s + (k+1)$ and $b = t$</u>

   $y_{stab} = y(s, s+2(k+1), s+(k+1), s+2(k+1))$ which assumes $n - 2(k+1)$ values as s varies.

3. <u>Theorem 5.2.3 result: $s = a$ and $b = t_{min} + j$ where $j \in \{1, 2, \ldots, k\}$</u>

   Recall: $t_{min} = s + (k+1)$ and $s = a \Rightarrow t < b$. So $y_{stab} = y(s, t_{min}+i, s, t_{min}+j)$ where $i < j$. It is acceptable for $t = t_{min}$ which gives $i \in \{0, 1, 2, \ldots, j-1\}$ for $j \in \{1, 2, \ldots, k\}$. Therefore $t_{min}+j = max\{s, t_{min}+i, s, t_{min}+j\}$ and since $t_{min}+j = s+(k+1)+j$ this means that y_{stab} will assume $n-((k+1)+j)$ values as s varies, for a fixed i and j. Also notice that for a fixed j, i may assume j different values (namely $i \in \{0, 1, 2, \ldots, j-1\}$), thus we get $n-((k+1)+j)$ values of y_{stab} for each i. In total this gives $j \times (n-((k+1)+j))$ values for y_{stab} as s and i vary but j remains fixed.

   Since we get $j \times (n-((k+1)+j))$ values of y_{stab} for a fixed j, and j can range from 1 up to k, in total this produces $\sum_{j=1}^{k} j \times (n-((k+1)+j))$ non-trivial values for y_{stab} in this situation.

4. <u>Theorem 5.2.4 result: $a > t$, $t < s + 2(k+1)$, and $b < a + 2(k+1)$</u>

   Let $b_{min} = a + (k+1) =$ the minimum possible value for b. So $y_{stab} = y(s, t_{min} + i, a, b_{min} + j) = y(s, s+(k+1)+i, a, a+(k+1)+j)$ where $i, j \in \{0, 1, 2, \ldots, k\}$.

If we let $x_1 = (k+1)+i$ and $x_2 = (k+1)+j$, then $y_{stab} = y(s, s+x_1, a, a+x_2)$ and we know from our earlier discussion that this may assume $\frac{(n-(x_2+x_1+1))(n-(x_2+x_1))}{2}$ different values for a fixed x and y.

Also $\frac{(n-(x_2+x_1+1))\times(n-(x_2+x_1))}{2} = \frac{(n-(2(k+1)+i+j+1))\times(n-(2(k+1)+i+j))}{2}$. This fraction corresponds to the number of non-trivial y_{stab} values that arise for fixed i and j. Since i and j can both range in value from 0 up to k, in total this produces

$$\sum_{j=0}^{k}\sum_{i=0}^{k}\frac{(n-(2(k+1)+i+j+1))\times(n-(2(k+1)+i+j))}{2}$$

non-trivial values for y_{stab} in this situation.

5. <u>$s < a < t$, $t < s+2(k+1)$, and $b < a+2(k+1)$</u>

This is the most difficult case to count. Even though y_{stab} looks like $y(s, s+x_1, s+x_2, s+x_3)$, sometimes $t < b$ and other times $b \leq t$. The former is more numerous, but both will occur. The larger the value of k, the more common the latter becomes. In fact the latter is slightly easier to count, so we will deal with this first. In order to explain this it will be useful to mention both $t_{min} = s+(k+1)$ and $t_{max} = s+2(k+1)-1$. As the name implies, t_{max} is the largest possible value of t that can occur. In order to more easily see what is happening, Figure 5.9 illustrates this situation when $k = 6$. The numbers across the top refer to the distance on the number line beyond s. Since $s < a < t$ we note that a may take any value $s < a < t_{max}$. Each a_i corresponds to placing a, such that $a_i = s+i$. We continue to increase i until we reach $t_{max}-1$. Since $a+(k+1) \leq b \leq a+2(k+1)-1$ we get $(a+2(k+1)-1)-(a+(k+1))+1 = k+1$ values of b for each a.

In the figure, the $k+1$ possible values of b that correspond to a_i are labeled as b_i. To reduce clutter each a_i and its corresponding b_i's are written one line lower than a_{i-1} and its corresponding b_{i-1}'s in the chart.

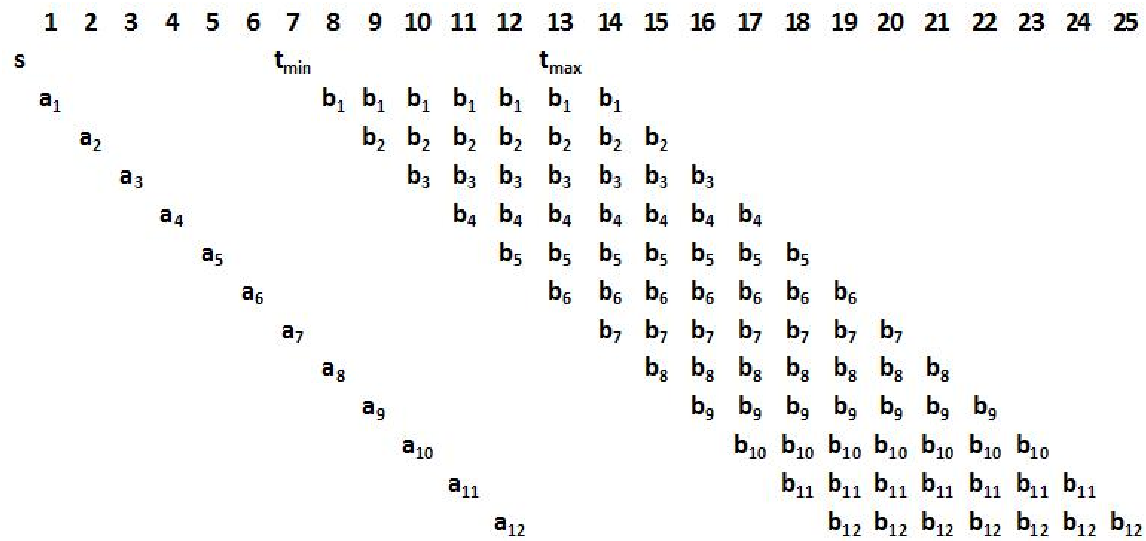

Figure 5.9: Counting Example, $k = 6$ and $s < a < t$

<u>(i) $b \leq t$</u>

We will now count the $b \leq t$ possibility. Since a and b have the same relationship as s and t, that being $a+(k+1) \leq b \leq a+2(k+1)-1$ and $s+(k+1) \leq t \leq s+2(k+1)-1$, and we have $s < a < t$, this means that $\forall b$, $t_{min} < b$ and the smallest possible value b may assume is $t_{min} + 1$.

If $t = t_{min}$ then $b \leq t$ cannot occur.

If $t = t_{min} + 1$ then $b \leq t$ only occurs when $b = t = t_{min} + 1$. The restriction of $a + (k + 1) \leq b \leq a + 2(k + 1) - 1$ then forces $a = s + 1$. Thus $b \leq t$ can only happen one way.

If $t = t_{min} + 2$ then $b \leq t$ can occur when $b \in \{t_{min} + 1, t_{min} + 2\}$. As noted in the previous case, $b = t_{min} + 1$ corresponds to one possibility (which must be counted again because t has taken on a different value), but $b = t_{min} + 2$ corresponds to two possibilities as a is either $a = s + 1$ or $a = s + 2$. So there are $1 + 2$ possibilities in this case.

If $t = t_{min} + 3$ then $b \leq t$ can occur when $b \in \{t_{min} + 1, t_{min} + 2, t_{min} + 3\}$. If $b = t_{min} + 3$ then $a + (k + 1) \leq b \leq a + 2(k + 1) - 1$ gives $a \in \{s + 1, s + 2, s + 3\}$, giving three new possibilities. If $b \neq t_{min} + 3$ then revert to the previous case. In total this implies there are $1+2+3$ possibilities in this case.

So in general if $t = t_{min} + i$ where $1 \leq i \leq k$, then requiring $a + (k+1) \leq b \leq a+2(k+1)-1$ gives $a \in \{s+1, s+2, s+3, \ldots, s+i\}$ when $b = t_{min}+i$. However when $b \neq t_{min}+i$ we revert to the previous case. In total this therefore gives $1+2+3+\ldots+i = \sum_{j=1}^{i} j = \frac{i(i+1)}{2}$ occurences of $b \leq t$.

Note that $t_{max} = s+2(k+1)-1 = t_{min}+k$, so this continues until $t = t_{max} = t_{min}+k$.

If $b \leq t$, then as shown earlier if $t = s+x$ there are $n-x$ possible values of y_{stab} as s varies. If $t = t_{min}+i$ then $t = s+(k+1)+i$, giving $n-((k+1)+i)$ non-trivial values of y_{stab} for a fixed i. Since $1 \leq i \leq k$ we get a total of $\sum_{i=1}^{k} \frac{i(i+1)}{2} \cdot (n-((k+1)+i))$ values of y_{stab} here.

(ii) $t < b$

Thus now it only remains to count the non-trivial occurences of y_{stab} for $t < b$ within this case (*i.e.* $s < a < t < b$). These actually come in 3 sets of k (three different patterns), and will produce three summations each going from 1 up to k.

As previously mentioned, the smallest possible value of b is $t_{min}+1 = s+(k+1)+1$. The largest possible b occurs when both a and the distance from a to b are as large as possible. That is, the largest b is $b = a+2(k+1)-1$ and $a = t_{max}-1$. Since $t_{max} = s+2(k+1)-1$, we get the largest b to be $b = (t_{max}-1)+2(k+1)-1 = ((s+2(k+1)-1)-1)+2(k+1)-1 = s+4(k+1)-3$, which brings the total number of potential b's to $(s+4(k+1)-3)-(s+(k+1)+1)+1 = 3(k+1)-3-1+1 = 3k$. We will now consider the potential values of b, k at a time.

Suppose $b \in \{t_{min}+1, t_{min}+2, ..., t_{min}+k\}$.

If $b = t_{min}+1$, then $t < b$ implies $t = t_{min}$ and $a+(k+1) \leq b \leq a+2(k+1)-1$ together with $s < a$ imply $a = s+1$. Thus, $t < b$ can only happen one way. For the example of $k = 6$, we are considering that b is in the column under the number 8.

If $b = t_{min}+2$, then $t < b$ implies $t \in \{t_{min}, t_{min}+1\}$ and $a+(k+1) \leq b \leq a+2(k+1)-1$

together with $s < a$ imply $a \in \{s+1, s+2\}$. Since a and t may each assume 2 values, there are 2^2 arrangements of placing a and t, and thus 2^2 possibilities in this case. For the example in $k = 6$, we are considering that b is in the column under the 9. Since there are 2 b's in this column, they are generated from the 2 a's: a_1, a_2. Thus a can be in column 1 or 2, and t can be in column 7 or 8.

If $b = t_{min} + 3$, then $t < b$ implies $t \in \{t_{min}, t_{min} + 1, t_{min} + 2\}$ and $a + (k+1) \leq b \leq a + 2(k+1) - 1$ together with $s < a$ imply $a \in \{s+1, s+2, s+3\}$. Since a and t may each assume 3 values, there are 3^2 arrangements of placing a and t, and thus 3^2 possibilities in this case. For the example in $k = 6$, we are considering that b is in the column under the 10. Since there are 3 b's in this column, they are generated from the 3 a's: a_1, a_2, a_3. Thus a can be in column 1, 2, or 3, and t can be in column 7, 8, or 9.

This pattern continues.

If $b = t_{min} + k = t_{max}$, then $t < b$ implies $t \in \{t_{min}, t_{min} + 1, t_{min} + 2, \ldots, t_{min} + k - 1\}$ and $a + (k+1) \leq b \leq a + 2(k+1) - 1$ together with $s < a$ imply $a \in \{s+1, s+2, s+3, \ldots, s+k\}$. Since the largest a is $a = s + k < s + (k+1) = t_{min}$, there is no need to reduce the number of t's to enforce the restriction of $s < a < t$. Since a and t may each assume k values, there are k^2 arrangements of placing a and t, and thus k^2 possibilities in this case. For the example in $k = 6$, we are considering that b is in the column under the 13. Since there are $k = 6$ b's in this column, they are generated from the $k = 6$ a's: $a_1, a_2, \ldots, a_6$. Thus a can be in column $1, 2, \ldots, 6$ and t can be in column $7, 8, \ldots, 12$.

So in general if $b = t_{min} + i$ where $1 \leq i \leq k$ there are i^2 non-trivial occurences of y_{stab} for a fixed s when $t < b$. Since $t < b$, then as shown earlier if $b = s + x_3$ there are $n - x_3$ possible values of y_{stab} as s varies. If $b = t_{min} + i$ then $b = s + (k+1) + i$, giving $n - ((k+1) + i)$ non-trivial values of y_{stab} as s fluctuates and i remains fixed. Since $1 \leq i \leq k$ we get a total of $\displaystyle\sum_{i=1}^{k} i^2 \cdot (n - ((k+1) + i))$ values of y_{stab} here as s and i both change.

Now as b exceeds $t_{min} + k$, a has the potential of being t_{min} or larger, which will reduce the number of acceptable values for t since $s < a < t$ is in effect.

Suppose that $b \in \{t_{min} + k + 1, t_{min} + k + 2, ..., t_{min} + 2k\}$.

In such case $t_{max} < b$, so the requirement of $t < b$ is automatic.

If $b = t_{min} + k + 1$, as with the smaller values of b, the criterion that $a + (k+1) \leq b \leq a + 2(k+1) - 1$ together with $s < a$ imply $a \in \{s+1, s+2, s+3, \ldots, s+k, s+(k+1)\}$. Notice that $t_{min} = s + (k+1)$, so extra caution is in order to ensure $s < a < t$ is maintained. If $a \neq s + (k+1)$ then any of the $k+1$ possible values of t meet the $s < a < t$ criterion. (Note: $k+1$ possible values comes from $t_{min} \leq t \leq t_{max}$, or $s + (k+1) \leq t \leq s + 2(k+1) - 1$, in total giving $t_{max} - t_{min} + 1 = k + 1$ possible values of t). On the other hand if $a = s + (k+1) = t_{min}$, then t can be any value except t_{min}, giving k values (one fewer) of t now that follow $s < a < t$. In other words we can say that each of the $k+1$ values of a can match up with any of the $k+1$ values of t, with this one exception thus yielding $(k+1)^2 - 1$ possibilities in this case. For the example in $k = 6$, we are considering that b is in column 14, so a is in any column 1 to 7 ($k+1$ columns).

If $b = t_{min} + k + 2 = s + 2(k+1) + 1$, then $a + (k+1) \leq b \leq a + 2(k+1) - 1$ implies $a \in \{s+2, s+3, s+4, \ldots, s+k, s+(k+1), s+k+2\}$. If a is any value $s+2, \ldots, s+k$ then $a < t_{min}$, hence there are $k+1$ possible values of t for each of these a's. As in the last case if $a = s + (k+1) = t_{min}$, we lose the possibility of $t = t_{min}$, likewise if $a = s + k + 2 = t_{min} + 1$ we lose the possibilities of $t = t_{min}, t_{min} + 1$ due to $s < a < t$. Hence each possible a with each possible t produces $(k+1)^2$ combinations with $1 + 2$ exceptions. Therefore we get $(k+1)^2 - (1+2)$ possibilities in this case. For the example in $k = 6$, we are considering that b is in column 15, so a is any column 2 through 8 ($k+1$ columns).

This pattern continues.

If $b = t_{min} + k + k$ then $a + (k+1) \leq b \leq a + 2(k+1) - 1$ implies $a \in \{s+k, \ldots, s+k+k\}$. If $a = s + k < t_{min}$ then a can match up with any of the $k+1$ possible values of t. If $a = s + k + 1 = t_{min}$ we lose one potential t, that being $t = t_{min}$. If $a = s + k + 2$ we lose $t = t_{min}, t_{min} + 1$. Continuing this until $a = s + k + k$, we lose $t = t_{min}, \ldots, t_{min} + k - 1$.

So in total we get the $(k+1)^2$ original possibilities, but lose $1+2+3+\ldots+k = \frac{k(k+1)}{2}$ combinations, thus giving $(k+1)^2 - \frac{k(k+1)}{2}$ possibilities in this case. For the example in $k = 6$, we are considering that b is in column 19, so a is any column 6 to 12 ($k+1$ columns).

So in general if $b = t_{min}+k+i$, where $1 \leq i \leq k$ then $a+(k+1) \leq b \leq a+2(k+1)-1$ implies $a \in \{s+i, \ldots, s+k+i\}$. As before each a such that $a < t_{min} = s+(k+1)$ can match up with any of the $k+1$ possible values of t. For the largest value of a, $a = s+k+i$ we lose i potential values of t ($t = s+k+1, \ldots, s+k+i$), for the second largest value of a, $a = s+k+(i-1)$ we lose $i-1$ potential values of t ($t = s+k+1, \ldots, s+k+(i-1)$), and so on until we lose only one potential value of t. In total we lose $1+2+\ldots+i = \frac{i(i+1)}{2}$ values of t, bringing the total number of possibilities to $(k+1)^2 - \frac{i(i+1)}{2}$. Since $b = t_{min}+k+i = s+(k+1)+k+i$ we get $n-((k+1)+k+i)$ non-trivial values of y_{stab} for a fixed i as s varies. Since $1 \leq i \leq k$ we get a total of $\displaystyle\sum_{i=1}^{k}\left((k+1)^2 - \frac{i(i+1)}{2}\right)\cdot(n-((k+1)+k+i))$ values of y_{stab} here.

<u>Suppose that $b \in \{t_{min}+2k+1, t_{min}+2k+2, ..., t_{min}+3k\}$.</u>

In such case $t_{max} < b$, so the requirement of $t < b$ is automatic.

If $b = t_{min}+2k+1$ then $a+(k+1) \leq b \leq a+2(k+1)-1$ together with $s < a < t_{max}$ imply $a \in \{s+k+1, \ldots, s+2k\}$. Notice now that the upper bound of $a < t_{max}$ is being utilized. Given $t_{max} = t_{min}+k = s+(k+1)+k = s+2k+1$ together with $a < t_{max}$ yield $s+2k$ is the largest possible a. Thus for $b = t_{min}+2k+1$ we get k possible values of a, rather than $k+1$. This can be seen in the $k = 6$ example. If b is in column 20, there are only 6 b's (k of them) in this column rather than 7 (or $k+1$) like there are in column 19. The largest possible a, $a = s+2k = t_{max}-1$, yields only one possible value of t, namely $t = t_{max}$. The second largest possible value of a, $a = s+2k-1 = t_{max}-2$, yields two possible values of t, that is $t = t_{max}, t_{max}-1$. This continues until the minimum value of a, $a = s+k+1 = t_{max}-k$ when t may take on any of k possible values, $t = t_{max}, t_{max}-1, \ldots, t_{max}-k+1$. So as a moves from largest to smallest, the potential number of t's go from 1 to k, giving $1+2+\ldots+k = \frac{k(k+1)}{2}$ possibilities in this case.

If $b = t_{min}+2k+2$ then $a+(k+1) \leq b \leq a+2(k+1)-1$ together with $s < a < t_{max}$

imply $a \in \{s+k+2, \ldots, s+2k\}$. As in the previous case, the largest possible a, $a = s+2k$ corresponds to 1 possible values of t. The second largest a corresponds to 2 possible values of t. This continues until the minimum value of a, when $a = s+k+2 = t_{max} - (k-1)$ which produces $k-1$ possible values of t. So as a traverses this list, the number of t's goes from 1 to $k-1$, giving $1+2+\ldots+(k-1) = \frac{(k-1)k}{2}$ possibilities in this case.

This pattern continues.

If $b = t_{min} + 3k$ then $a + (k+1) \leq b \leq a + 2(k+1) - 1$ together with $s < a < t_{max}$ imply $a = s + 2k = t_{max} - 1$. This also forces $t = t_{max}$, giving only one possible non-trivial value to y_{stab}. In the example of $k = 6$, we are considering that b is in column 25, which forces a and t to be in columns 12 and 13 respectively.

So in general if $b = t_{min}+2k+i$, where $1 \leq i \leq k$ then $a+(k+1) \leq b \leq a+2(k+1)-1$ implies $a \in \{s+k+i, \ldots, s+2k\}$ There are $(s+2k)-(s+k+i)+1 = k-i+1 = k-(i-1)$ possible values for a. The minimum a gives $k-(i-1)$ possible $t's$ and the maximum a gives one t, meaning in total we get $1+2+\ldots+(k-(i-1)) = \frac{(k-(i-1))(k-(i-1)+1)}{2}$ possibilities as a and t move (for a fixed i). Since $b = t_{min}+2k+i = s+(k+1)+2k+i$ we get $n-((k+1)+2k+i)$ non-trivial values of y_{stab} for a fixed i as s varies. Since $1 \leq i \leq k$ we get a total of $\displaystyle\sum_{i=1}^{k} \frac{(k-(i-1))(k-(i-1)+1)}{2} \cdot (n-((k+1)+2k+i))$ values of y_{stab} here.

This exhausts all possibilities, as we have considered all $3k$ possible values of b. As stated earlier, the largest possible value of b is $b = s+4(k+1)-3$. Notice that $b = s+4(k+1)-3 = s+(k+1)+3(k+1)-3 = t_{min}+3(k+1)-3 = t_{min}+3k$, which was the final b considered in this case.

In summary when we put all of this together including both $[G_{rs}, G_{st}]$ and $[G_{st}, G_{ab}]$ we get the total number of non-trivial y_{rsst}, y_{stab} values as all subscripts vary to be
$\dim M(L) =$

$$\sum_{i=1}^{k} i \cdot (n-(2(k+1)+i))+$$

$$\sum_{j=1}^{k}(k-j+1)\cdot(n-(2(k+1)+k+j))+$$

$$2(n-2(k+1))+$$

$$\sum_{j=1}^{k} j \times (n-((k+1)+j))+$$

$$\sum_{j=0}^{k}\sum_{i=0}^{k}\frac{(n-(2(k+1)+i+j+1))\times(n-(2(k+1)+i+j))}{2}+$$

$$\sum_{i=1}^{k}\frac{i(i+1)}{2}\cdot(n-((k+1)+i))+$$

$$\sum_{i=1}^{k} i^2\cdot(n-((k+1)+i))+$$

$$\sum_{i=1}^{k}\left((k+1)^2-\frac{i(i+1)}{2}\right)\cdot(n-((k+1)+k+i))+$$

$$\sum_{i=1}^{k}\frac{(k-(i-1))(k-(i-1)+1)}{2}\cdot(n-((k+1)+2k+i))$$

Notice the $4^{th}, 6^{th}$, and 7^{th} line all sum from 1 to k and subtract the same quantity from n. Thus we may simplify this formula to be $\dim M(L) =$

$$\sum_{i=1}^{k} i\cdot(n-(2(k+1)+i))+\ \sum_{j=1}^{k}(k-j+1)\cdot(n-(2(k+1)+k+j))+$$

$$2(n-2(k+1))+\ \sum_{j=0}^{k}\sum_{i=0}^{k}\frac{(n-(2(k+1)+i+j+1))\times(n-(2(k+1)+i+j))}{2}+$$

$$\sum_{i=1}^{k}\frac{3i(i+1)}{2}\cdot(n-((k+1)+i))+\ \sum_{i=1}^{k}\left((k+1)^2-\frac{i(i+1)}{2}\right)\cdot(n-((k+1)+k+i))+$$

$$\sum_{i=1}^{k}\frac{(k-(i-1))(k-(i-1)+1)}{2}\cdot(n-((k+1)+2k+i))$$

Notice that every term above involves $n - w$ for some w. Just as in the examples of $k = 1$ and $k = 2$, this number w represents either the distance from r to t in the element y_{rsst} or the distance from either s to t or s to b in the element y_{stab}. As a reminder, each y_{rsst} and y_{stab} are produced from $[G_{rs}, G_{st}]$ and $[G_{st}, G_{ab}]$ respectively. Also each G_{ij} is a convenient element change from F_{ij}, the image of the matrix unit E_{ij} under the transversal map. Therefore whenever $n \leq w$ $(n - w \leq 0)$ occurs, n is too small to produce the distance of w between positions in the matrices creating the multiplier elements being counted. As such any negative $n - w$ should be replaced with zero.

In the case discussed in [3], the strictly upper triangular matrices were examined without requiring that there be diagonals of zeros above the main diagonal. As such, that case corresponds to $k = 0$ here. In this formula any sum going from $i = 1$ to k is eliminated in the case of $k = 0$. Only the double sum and the $2(n - 2(k + 1))$ survive. Thus when $k = 0$ this formula simplifies to

$$2(n - 2(0+1)) + \frac{(n - (2(0+1) + 0 + 0 + 1)) \times (n - (2(0+1) + 0 + 0))}{2} =$$

$2(n-2) + \dfrac{(n-3)(n-2)}{2}$, which is consistent with the formula for $\dim M(L)$, found in [3].

Chapter 6

Sample $\dim M(L)$ and $t(L)$ computations

In the event $k \geq 1$, even if $k = 1$ and n is some fixed constant, computing this sum is quite an arduous task. To avoid these tedious calculations, we have coded the formula into Matlab which will calculate everything and give $\dim L, \dim M(L)$, and $t(L)$. The user need only enter the values of n and k. Each sum is written as a for-loop. In the event some i or j causes a term in the sum that looks like $n - x$ to be nonpositive, notice that further incrementing i or j will also cause this problem. Hence if a particular value of i or j causes $n - x \leq 0$ then that particular sum should be terminated at that point (hence the "break" statements), as n is not large enough to produce the remaining non-zero y_{stab} values. Also notice that the last two sums each pick up where the previous leaves off, hence if the second or third to last sums terminate early then all remaining sums should not run at all. To account for this, the code also includes a flag to prevent superfluous sums from running. Notice that in order for everything to run to its final step (and get a y_{stab} of every possible type) that n must be at least $4k + 3$, however the code will work for any n and k. The Matlab code is available in section 8.1.

Here are a few examples runs:

1. If $k = 1$ then we would like n to be at least $4k + 3$, so everything runs. Let $n = 8$. The program produces the output

   ```
   >> multiplier
   ```

```
If a strictly upper triangular matrix is of size nxn and k denotes
the number of diagonals above the main diagonal which are zero.
Note that k must be at least zero.
Enter a value for k = 1
You may enter any positive integer for n, however
to get elements of every type n must be at least
     7

Enter a value for n = 8
dim M(L) =
    56
dim L =
    21

t(L) =
   154
```

2. Again, the program will work for any n, even if not all possible y_{stab} values are produced. Let $k = 2$ and $n = 7$.

```
>> multiplier
If a strictly upper triangular matrix is of size nxn and k denotes
the number of diagonals above the main diagonal which are zero.
Note that k must be at least zero.
Enter a value for k = 2
You may enter any positive integer for n, however
to get elements of every type n must be at least
    11

Enter a value for n = 7
dim M(L) =
    37
dim L =
    10

t(L) =
     8
```

3. In the event $k = 2$ and $n = 4$ the restriction of $a + (k + 1) \leq b$ (or $a + 3 \leq b$) means the Lie algebra L has only one independent element: E_{14}, which the transversal map takes to F_{14} in C. Therefore L is abelian and $[F_{14}, F_{14}] = 0$ is the only possible bracket involving $F's$. So $\dim M(L) = 0$.

```
>> multiplier
If a strictly upper triangular matrix is of size nxn and k denotes
the number of diagonals above the main diagonal which are zero.
Note that k must be at least zero.
Enter a value for k = 2
You may enter any positive integer for n, however
to get elements of every type n must be at least
    11

Enter a value for n = 4
dim M(L) =
     0
dim L =
     1

t(L) =
     0

L is abelian
```

4. A slightly less extreme abelian case, suppose $k = 1$ and $n = 4$. Since now $a+2 \le b$, the Lie algebra L corresponds to 3 elements in C: F_{13}, F_{14}, F_{24}. Considering all possible brackets gives

   $[F_{13}, F_{14}] = y_{1314}$

   $[F_{13}, F_{24}] = y_{1324}$

   $[F_{14}, F_{24}] = y_{1424}$

   No bracket produces an F, thus each bracket only produces an element in the multiplier. Also any possible Jacobi identity will trivially give zero without any information about the y's emerging. Hence $\dim M(L) = 3$.

```
>> multiplier
If a strictly upper triangular matrix is of size nxn and k denotes
the number of diagonals above the main diagonal which are zero.
Note that k must be at least zero.
Enter a value for k = 1
You may enter any positive integer for n, however
to get elements of every type n must be at least
     7
```

```
Enter a value for n = 4
dim M(L) =
     3
dim L =
     3

t(L) =
     0

L is abelian
```

5. If $k = 1$ and $n = 5$ then C contains $F_{13}, F_{14}, F_{15}, F_{24}, F_{25}, F_{35}$. The possible brackets are

$[F_{13}, F_{14}] = y_{1314}$

$[F_{13}, F_{15}] = y_{1315}$

$[F_{13}, F_{24}] = y_{1324}$

$[F_{13}, F_{25}] = y_{1325}$

$[F_{13}, F_{35}] = F_{15} + y_{1335} = G_{15}$

$[F_{14}, F_{15}] = y_{1415} = 0$

$[F_{14}, F_{24}] = y_{1424}$

$[F_{14}, F_{25}] = y_{1425}$

$[F_{14}, F_{35}] = y_{1435}$

$[F_{15}, F_{24}] = y_{1524} = 0$

$[F_{15}, F_{25}] = y_{1525} = 0$

$[F_{15}, F_{35}] = y_{1535}$

$[F_{24}, F_{25}] = y_{2425}$

$[F_{24}, F_{35}] = y_{2435}$

$[F_{25}, F_{35}] = y_{2535}$

We get $y_{1415} = 0$ from $J(F_{13}, F_{35}, F_{14}) = 0$, $y_{1524} = 0$ from $J(F_{13}, F_{35}, F_{24}) = 0$, and $y_{1525} = 0$ from $J(F_{13}, F_{35}, F_{25}) = 0$.

Thus counting the y's shows $\dim M(L) = 11$. Therefore $t(L) = \frac{1}{2}(\dim L)(\dim L - 1) - 11 = \frac{1}{2}(6)(5) - 11 = 4$.

```
>> multiplier
If a strictly upper triangular matrix is of size nxn and k denotes
the number of diagonals above the main diagonal which are zero.
Note that k must be at least zero.
Enter a value for k = 1
You may enter any positive integer for n, however
to get elements of every type n must be at least
     7

Enter a value for n = 5
dim M(L) =
    11
dim L =
     6

t(L) =
     4
```

6. Consider now the case where $k = 1$ and $n = 6$. Computing all possible bracket operations produces Table 6.1. We place a $*$ wherever we would have $[x, x]$ or violate $s < a$ or $s = a, t < b$. All the zeros come from some Jacobi identity.

Table 6.1: Bracket Operation $[F_{st}, F_{ab}]$, when $k = 1$ and $n = 6$

	F_{13}	F_{14}	F_{15}	F_{16}	F_{24}	F_{25}	F_{26}	F_{35}	F_{36}	F_{46}
F_{13}	$*$	y_{1314}	y_{1315}	0	y_{1324}	y_{1325}	0	G_{15}	$G_{16} + y_{1336}$	y_{1346}
F_{14}	$*$	$*$	0	0	y_{1424}	y_{1425}	0	y_{1435}	y_{1436}	G_{16}
F_{15}	$*$	$*$	$*$	0	0	0	0	y_{1535}	0	0
F_{16}	$*$	$*$	$*$	$*$	0	0	0	0	0	0
F_{24}	$*$	$*$	$*$	$*$	$*$	y_{2425}	y_{2426}	y_{2435}	y_{2436}	G_{26}
F_{25}	$*$	$*$	$*$	$*$	$*$	$*$	0	y_{2535}	y_{2536}	y_{2546}
F_{26}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	0	y_{2646}
F_{35}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	y_{3536}	y_{3546}
F_{36}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	y_{3646}
F_{46}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

Counting the y's reveals that $\dim M(L) = 22$ and so $t(L) = \frac{1}{2}(10)(9) - 22 = 23$.

```
>> multiplier
If a strictly upper triangular matrix is of size nxn and k denotes
the number of diagonals above the main diagonal which are zero.
Note that k must be at least zero.
Enter a value for k = 1
You may enter any positive integer for n, however
to get elements of every type n must be at least
     7

Enter a value for n = 6
dim M(L) =
    22

dim L =
    10

t(L) =
    23
```

Calculating the values of $\dim M(L)$ and $t(L)$ by hand can be very tedious, especially as n and k grow large. To further illustrate the usefulness of the Matlab program, we have used it to compute $\dim M(L)$ and $t(L)$ in the case where $n = 100$. (Please see Tables 6.2 and 6.3.) We let k take on all values 0 through 48. We omit the cases where $k \geq 49$ because L is abelian. When L is abelian, $\dim M(L) = \frac{1}{2}(\dim L)(\dim L - 1)$ and $t(L) = 0$.

Table 6.2: $\dim M(L)$ and $t(L)$ for $n = 100$ and $0 \leq k \leq 24$

k	$\dim L$	$\dim M(L)$	$t(L)$
0	4,950	4,949	12,243,826
1	4,851	18,916	11,744,759
2	4,753	41,237	11,251,891
3	4,656	71,075	10,765,765
4	4,560	107,620	10,286,900
5	4,465	150,089	9,815,791
6	4,371	197,726	9,352,909
7	4,271	249,802	8,898,701
8	4,171	305,615	8,453,590
9	4,071	364,490	8,017,975
10	3,971	425,779	7,592,231
11	3,871	488,861	7,176,709
12	3,771	553,142	6,771,736
13	3,671	618,055	6,377,615
14	3,571	683,060	5,994,625
15	3,471	747,644	5,623,021
16	3,371	811,321	5,263,034
17	3,271	873,632	4,914,871
18	3,171	934,145	4,578,715
19	3,071	992,455	4,254,725
20	2,971	1,048,184	3,943,036
21	2,871	1,100,981	3,643,759
22	2,771	1,150,522	3,356,981
23	2,671	1,196,510	3,082,765
24	2,571	1,238,675	2,821,150

Table 6.3: $\dim M(L)$ and $t(L)$ for $n = 100$ and $25 \leq k \leq 48$

k	$\dim L$	$\dim M(L)$	$t(L)$
25	2,471	1,276,774	2,572,151
26	2,371	1,310,591	2,335,759
27	2,271	1,339,937	2,111,941
28	2,171	1,364,650	1,900,640
29	2,071	1,384,595	1,701,775
30	1,971	1,399,664	1,515,241
31	1,871	1,409,776	1,340,909
32	1,771	1,414,877	1,178,626
33	1,671	1,414,939	1,028,216
34	1,571	1,409,955	889,485
35	1,471	1,399,951	762,209
36	1,371	1,384,981	646,139
37	1,271	1,365,126	541,002
38	1,171	1,340,494	446,501
39	1,071	1,311,220	362,315
40	971	1,277,466	288,099
41	871	1,239,421	223,484
42	771	1,197,301	168,077
43	671	1,151,349	121,461
44	571	1,101,835	83,195
45	471	1,049,056	52,814
46	371	993,336	29,829
47	271	935,026	13,727
48	171	874,504	3,971

Chapter 7

Polynomial Closed Form

The current formula for $\dim M(L)$ is a collection of sums, and works well for implementation in Matlab when the user inputs specific values of n and k. Considering that the open form requires continuous adjustment based on the choices of n and k and that calculation by computer is convenient but by hand is extremely long, we welcome an alternative. A closed form, or a polynomial in n and k, would be nice to have as well. In order to achieve a closed form, some manipulation is necessary. Remember that in each sum, the quantity subtracted from n denotes the distance between positions within an $n \times n$ matrix. Such a distance is required so that the respective multiplier elements, y_{stab}, may be produced (*e.g.* if the distance from s to b is being subtracted from n then the original matrices must have at least $b - s + 1$ rows and columns for this multiplier element to exist). The initial calculation of $\dim M(L)$ went under the assumption that n is sufficiently large to produce every type of multiplier element (so that $n - x$ will always be positive, though nonnegative would work too). In the event that $n - x \leq 0$ appears in a sum, then the corresponding F_{st} and F_{ab} that would have produced this multiplier element do not both exist. As such, this portion of the summation should be replaced with zero. Also notice that in each sum as the index of summation increases, so does the quantity subtracted from n, therefore when $n \leq x$ occurs, the sum should be terminated early (as later elements in the sum will cause the same problem). Fortunately the Matlab code accounts for this, as it is decorated with "if" statements that constantly run checks. In order to get a closed form, the potential for the sums to terminate early must be accounted for earlier with a general n and k rather than specific values (like Matlab uses).

Calculation of this closed form will be much more convenient if all sums are indexed by the quantity subtracted from n. Doing this will cause the upper bound of a sum to be n when it must terminate early. Also, the presence of the double sum poses some difficulty. First we will rewrite the double sum more conveniently, and second we will reindex everything.

7.1 Double sum to single sum conversion

Consider $\displaystyle\sum_{j=0}^{k}\sum_{i=0}^{k}\frac{(n-(2(k+1)+i+j+1))\times(n-(2(k+1)+i+j))}{2}$

To assist in converting this into single sums, let $\alpha = i+j+1$. Since $0 \leq i, j \leq k$ we observe that α may take on any values between 1 and $2k+1$. If for instance $\alpha = 1$ then $i = j = 0$. If $\alpha = 2$ then either $i = 1, j = 0$ or $i = 0, j = 1$. Table 7.1 lists all relationships between i, j, and α.

Table 7.1: Double to single sum reindexing

α	number of times $\alpha = i+j+1$ is in the double sum	(i, j) pairs that produce α
1	1	(0,0)
2	2	(1,0), (0,1)
3	3	(2,0), (1,1), (0,2)
4	4	(3,0), (2,1), (1,2), (0,3)
$\vdots$	$\vdots$	$\vdots$
k	k	$(k-1,0)$, $(k-2,1)$,…,$(0,k-1)$
$k+1$	$k+1$	$(k,0)$, $(k-1,1)$,…,$(0,k)$
$k+2$	k	$(k,1)$, $(k-1,2)$,…,$(1,k)$
$k+3$	$k-1$	$(k,2)$, $(k-1,3)$,…,$(2,k)$
$k+4$	$k-2$	$(k,3)$, $(k-1,4)$,…,$(3,k)$
$\vdots$	$\vdots$	$\vdots$
$2k+1$	1	(k,k)

Case 1: $1 \leq \alpha \leq k$

$(n-(2(k+1)+i+j+1))\times(n-(2(k+1)+i+j)) = (n-(2(k+1)+\alpha))\times(n-(2(k+1)+\alpha-1))$

appears in the double sum α times.

Case 2: $k+1 \leq \alpha \leq 2k+1$

Let $\beta = \alpha - (k+1)$, so $0 \leq \beta \leq k$ and the second half of Table 7.1 becomes Table 7.2.

Table 7.2: Double to single sum reindexing, descending occurences

α	β	number of times $\alpha = i+j+1$ is in the double sum	(i,j) pairs that produce α
$k+1$	0	$k+1$	$(k,0)$, $(k-1,1)$,...,$(0,k)$
$k+2$	1	k	$(k,1)$, $(k-1,2)$,...,$(1,k)$
$k+3$	2	$k-1$	$(k,2)$, $(k-1,3)$,...,$(2,k)$
$k+4$	3	$k-2$	$(k,3)$, $(k-1,4)$,...,$(3,k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2k+1$	k	1	(k,k)

Notice $(n-(2(k+1)+i+j+1)) \times (n-(2(k+1)+i+j)) = (n-(3(k+1)+\beta)) \times (n-(3(k+1)+\beta-1))$ appears in the double sum $(k+1)-\beta$ times.

Therefore putting all this together gives

$$\sum_{j=0}^{k}\sum_{i=0}^{k}\frac{(n-(2(k+1)+i+j+1)) \times (n-(2(k+1)+i+j))}{2} = \sum_{\alpha=1}^{k}\frac{\alpha}{2}\cdot(n-(2(k+1)+\alpha))\times$$

$$(n-(2(k+1)+\alpha-1)) + \sum_{\beta=0}^{k}\frac{(k+1)-\beta}{2}\cdot(n-(3(k+1)+\beta)) \times (n-(3(k+1)+\beta-1))$$

Therefore for sufficiently large n, $\dim M(L) =$

$$\sum_{i=1}^{k} i\cdot(n-(2(k+1)+i)) +$$

$$\sum_{i=1}^{k}(k-i+1)\cdot(n-(2(k+1)+k+i)) +$$

$2(n-2(k+1))+$

$$\sum_{\alpha=1}^{k} \frac{\alpha}{2} \cdot (n-(2(k+1)+\alpha)) \times (n-(2(k+1)+\alpha-1))+$$

$$\sum_{\beta=0}^{k} \frac{(k+1)-\beta}{2} \cdot (n-(3(k+1)+\beta)) \times (n-(3(k+1)+\beta-1))+$$

$$\sum_{i=1}^{k} \frac{3i(i+1)}{2} \cdot (n-((k+1)+i))+$$

$$\sum_{i=1}^{k} \left((k+1)^2 - \frac{i(i+1)}{2}\right) \cdot (n-((k+1)+k+i))+$$

$$\sum_{i=1}^{k} \frac{(k-(i-1))(k-(i-1)+1)}{2} \cdot (n-((k+1)+2k+i))$$

with the exception of the newest result, we let all sums be indexed over i.

7.2 Reindexing all sums

1. Consider $\sum_{i=1}^{k} i \cdot (n-(2(k+1)+i))$

   Let $j = 2(k+1)+i$, which makes the sum

   $$\sum_{j=2k+3}^{3k+2} (j-2k-2) \cdot (n-j)$$

2. Consider $\sum_{i=1}^{k}(k-i+1) \cdot (n-(2(k+1)+k+i))$

   Let $j = 2(k+1)+k+i$, which makes the sum

   $$\sum_{j=3k+3}^{4k+2} (4k+3-j) \cdot (n-j)$$

3. Consider $\sum_{\alpha=1}^{k} \frac{\alpha}{2} \cdot (n-(2(k+1)+\alpha)) \times (n-(2(k+1)+\alpha-1))$

   Let $j = 2(k+1)+\alpha$, which makes the sum

   $$\sum_{j=2k+3}^{3k+2} \frac{j-2k-2}{2} \cdot (n-j)\,(n-j+1)$$

4. Consider $\displaystyle\sum_{\beta=0}^{k} \frac{(k+1)-\beta}{2} \cdot (n-(3(k+1)+\beta)) \times (n-(3(k+1)+\beta-1))$

   Let $j = 3(k+1)+\beta$, which makes the sum

   $$\sum_{j=3k+3}^{4k+3} \frac{4k+4-j}{2} \cdot (n-j)\,(n-j+1)$$

5. Consider $\displaystyle\sum_{i=1}^{k} \frac{3i(i+1)}{2} \cdot (n-((k+1)+i))$

   Let $j = (k+1)+i$, which makes the sum

   $$\sum_{j=k+2}^{2k+1} \frac{3}{2} \cdot (j-k-1)(j-k)\,(n-j)$$

6. Consider $\displaystyle\sum_{i=1}^{k} \left((k+1)^2 - \frac{i(i+1)}{2}\right) \cdot (n-((k+1)+k+i))$

   Let $j = (k+1)+k+i = 2k+1+i$, which makes the sum

   $$\sum_{j=2k+2}^{3k+1} \left((k+1)^2 - \frac{(j-2k-1)(j-2k)}{2}\right) \cdot (n-j)$$

7. Consider $\displaystyle\sum_{i=1}^{k} \frac{(k-(i-1))(k-(i-1)+1)}{2} \cdot (n-((k+1)+2k+i))$

   Let $j = (k+1)+2k+i = 3k+1+i$, which makes the sum

   $$\sum_{j=3k+2}^{4k+1} \frac{1}{2} \cdot (4k+2-j)(4k+3-j)(n-j)$$

Therefore for sufficiently large n, $\dim M(L) =$

$$\sum_{j=k+2}^{2k+1} \frac{3}{2} \cdot (j-k-1)(j-k)\,(n-j) +$$

$2(n-(2k+2))+$

$$\sum_{j=2k+2}^{3k+1} \left((k+1)^2 - \frac{(j-2k-1)(j-2k)}{2}\right) \cdot (n-j)+$$

$$\sum_{j=2k+3}^{3k+2} (j-2k-2) \cdot (n-j) +$$

$$\sum_{j=2k+3}^{3k+2} \frac{j-2k-2}{2} \cdot (n-j)\,(n-j+1)+$$

$$\sum_{j=3k+2}^{4k+1} \frac{1}{2} \cdot (4k+2-j)(4k+3-j)(n-j)+$$

$$\sum_{j=3k+3}^{4k+2} (4k+3-j) \cdot (n-j) +$$

$$\sum_{j=3k+3}^{4k+3} \frac{4k+4-j}{2} \cdot (n-j)\,(n-j+1)$$

7.3 Observations

1. If $n \geq 4k+3$ then n is sufficiently large and every sum will run through its entirety. It remains to consider what happens when $n < 4k+3$ and not all pieces above are needed.

2. For a fixed k, n must be at least $k+1$ (otherwise the matrices do not have k "super-diagonals"). The above confirms this, as n must be at least $k+2$ for any of the sums to run at all. If $n = k+1$ then $L = 0$, so no calculation is necessary.

3. E_{ab} (and hence F_{ab}) is defined $\Leftrightarrow a + (k+1) \leq b$. Also

   $$[F_{st}, F_{ab}] = \begin{cases} F_{sb} + y(s,t,a,b) & \text{if } t = a \\ y(s,t,a,b) & \text{if } t \neq a \end{cases}$$

   In order for F_{rs} and F_{st} to both be defined, $r + (k+1) \leq s$ and $s + (k+1) \leq t$, so if $r = 1$, then $s \geq k+2$ and $t \geq 2k+3$. Therefore if $n < 2k+3$

   $\Rightarrow [F_{rs}, F_{st}]$ is never defined,

   $\Rightarrow [F_{st}, F_{ab}] = y(s,t,a,b)$, $\forall s,t,a,b$.

   $\Rightarrow [E_{st}, E_{ab}] = 0$, $\forall s,t,a,b$.

   $\Rightarrow L$ is abelian.

   $\Rightarrow \dim M(L) = \frac{1}{2}(\dim L)(\dim L - 1)$ and $t(L) = 0$.

   Notice $\dim L = \displaystyle\sum_{i=1}^{n-(k+1)} i = \frac{1}{2}(n-(k+1))(n-(k+1)+1)$

So, L is abelian $\Rightarrow \dim M(L) = \frac{1}{8}(n-k-1)(n-k)(n-k+1)(n-k-2)$.

4. If L is abelian ($n < 2k+3$) then only the first sum can run, and it must terminate at $\min\{n, 2k+1\}$. This sum will simplify to $\frac{1}{2}(\dim L)(\dim L - 1)$.

5. To accurately describe $\dim M(L)$ for any n and k, the sum $\sum_{j=f(k)}^{g(k)} (\ldots)$ should be replaced with zero if $n < f(k)$ and $\sum_{j=f(k)}^{\min\{g(k),n\}} (\ldots)$ if $n \geq f(k)$.

7.4 Polynomials: $\dim M(L)$ and $t(L)$

The final point in the previous section breaks the calculation of $\dim M(L)$ into several cases. We will develop polynomials for all nonnegative k values. We will return to $k = 0, 1$ soon, but first we focus on a fixed $k \geq 2$ where we establish the following cases.

1. Case 1: $k + 2 \leq n < 2k + 3$

   L is abelian, hence $\dim M(L) = \frac{1}{2}(\dim L)(\dim L - 1) = \frac{1}{8}(n-k-1)(n-k)(n-k+1)(n-k-2)$

   $t(L) = 0$

2. Case 2: $n = 2k+3, \ldots, 3k+1$

$$\dim M(L) = \left(\sum_{j=k+2}^{2k+1} \frac{3}{2} \cdot (j-k-1)(j-k)(n-j) \right) + 2(n-(2k+2)) +$$

$$\sum_{j=2k+2}^{n} \left((k+1)^2 - \frac{(j-2k-1)(j-2k)}{2} \right) \cdot (n-j) + \sum_{j=2k+3}^{n} (j-2k-2) \cdot (n-j) +$$

$$\sum_{j=2k+3}^{n} \frac{j-2k-2}{2} \cdot (n-j)\,(n-j+1)$$

3. Case 3: $n = 3k+2$

$$\dim M(L) = \left(\sum_{j=k+2}^{2k+1} \frac{3}{2} \cdot (j-k-1)(j-k)(n-j) \right) + 2(n-(2k+2)) +$$

$$\sum_{j=2k+2}^{3k+1} \left((k+1)^2 - \frac{(j-2k-1)(j-2k)}{2} \right) \cdot (n-j) + \sum_{j=2k+3}^{3k+2} (j-2k-2) \cdot (n-j) +$$

$$\sum_{j=2k+3}^{3k+2} \frac{j-2k-2}{2} \cdot (n-j)\,(n-j+1)$$

4. Case 4: $n = 3k+3, \ldots, 4k+1$

$$\dim M(L) = \left(\sum_{j=k+2}^{2k+1} \frac{3}{2} \cdot (j-k-1)(j-k)(n-j) \right) + 2(n-(2k+2)) +$$

$$\sum_{j=2k+2}^{3k+1} \left((k+1)^2 - \frac{(j-2k-1)(j-2k)}{2} \right) \cdot (n-j) + \sum_{j=2k+3}^{3k+2} (j-2k-2) \cdot (n-j) +$$

$$\sum_{j=2k+3}^{3k+2} \frac{j-2k-2}{2} \cdot (n-j)\,(n-j+1) + \sum_{j=3k+2}^{n} \frac{1}{2} \cdot (4k+2-j)(4k+3-j)(n-j) +$$

$$\sum_{j=3k+3}^{n} (4k+3-j) \cdot (n-j) + \sum_{j=3k+3}^{n} \frac{4k+4-j}{2} \cdot (n-j)\,(n-j+1)$$

5. Case 5: $n = 4k+2$

$$\dim M(L) = \left(\sum_{j=k+2}^{2k+1} \frac{3}{2} \cdot (j-k-1)(j-k)(n-j) \right) + 2(n-(2k+2)) +$$

$$\sum_{j=2k+2}^{3k+1} \left((k+1)^2 - \frac{(j-2k-1)(j-2k)}{2} \right) \cdot (n-j) + \sum_{j=2k+3}^{3k+2} (j-2k-2) \cdot (n-j) +$$

$$\sum_{j=2k+3}^{3k+2} \frac{j-2k-2}{2} \cdot (n-j)\,(n-j+1) + \sum_{j=3k+2}^{4k+1} \frac{1}{2} \cdot (4k+2-j)(4k+3-j)(n-j) +$$

$$\sum_{j=3k+3}^{n} (4k+3-j) \cdot (n-j) + \sum_{j=3k+3}^{n} \frac{4k+4-j}{2} \cdot (n-j)\,(n-j+1)$$

6. Case 6: $n \geq 4k+3$

   No sums terminate early, hence the original calculation of $\dim M(L)$ may be used.

Now we use the help of Maple to expand the sums into polynomials. (The Maple code is available in section 8.2.)

1. Case 1: $k+2 \leq n < 2k+3$

   L is abelian, hence $\dim M(L) = \frac{1}{2}(\dim L)(\dim L - 1) = \frac{1}{8}(n-k-1)(n-k)(n-k+1)(n-k-2)$

   $t(L) = 0$

2. Case 2: $n = 2k+3, \ldots, 3k+1$

   $\dim M(L) = -4 - \frac{3}{2}nk^3 + 2n - \frac{13}{4}k - 2nk - 4nk^2 + \frac{15}{4}k^3 + \frac{27}{8}k^2 + \frac{1}{2}n^2k^2 + n^2k + \frac{9}{8}k^4$

   $t(L) = 4 + nk^3 - \frac{7}{4}n + 3k + \frac{9}{4}nk - \frac{1}{8}n^2 + \frac{13}{4}nk^2 - \frac{1}{4}n^3 - \frac{7}{2}k^3 - \frac{7}{2}k^2 + \frac{1}{4}n^2k^2$
   $-\frac{1}{2}n^3k - \frac{1}{4}n^2k - k^4 + \frac{1}{8}n^4$

3. Case 3: $n = 3k+2$

   $\dim M(L) = -4 + 2n - \frac{27}{4}k + \frac{5}{4}nk + \frac{5}{4}nk^2 - \frac{21}{4}k^3 - \frac{55}{8}k^2 + \frac{1}{4}n^2k^2 + \frac{1}{4}n^2k - \frac{9}{8}k^4$
   $= \frac{11}{4}k + \frac{27}{8}k^2 + \frac{15}{4}k^3 + \frac{9}{8}k^4$

   $t(L) = 4 - \frac{1}{2}nk^3 - \frac{7}{4}n + \frac{13}{2}k - nk - \frac{1}{8}n^2 - 2nk^2 - \frac{1}{4}n^3 + \frac{11}{2}k^3 + \frac{27}{4}k^2 + \frac{1}{2}n^2k^2$
   $-\frac{1}{2}n^3k + \frac{1}{2}n^2k + \frac{5}{4}k^4 + \frac{1}{8}n^4$

   $= \frac{7}{8}k^4 + \frac{9}{4}k^3 + \frac{17}{8}k^2 - \frac{5}{4}k$

4. Case 4: $n = 3k+3, \ldots, 4k+1$

   $\dim M(L) = -1 - \frac{3}{2}nk^3 - \frac{1}{2}n + \frac{17}{4}k - 5nk + \frac{1}{2}n^2 - 4nk^2 + \frac{15}{4}k^3 + \frac{63}{8}k^2 + \frac{1}{2}n^2k^2 + n^2k + \frac{9}{8}k^4$

$t(L) = 1 + nk^3 + \frac{3}{4}n - \frac{9}{2}k + \frac{21}{4}nk - \frac{5}{8}n^2 + \frac{13}{4}nk^2 - \frac{1}{4}n^3 - \frac{7}{2}k^3 - 8k^2$
$+\frac{1}{4}n^2k^2 - \frac{1}{2}n^3k - \frac{1}{4}n^2k - k^4 + \frac{1}{8}n^4$

5. Case 5: $n = 4k + 2$

   $\dim M(L) = -2 + \frac{55}{6}nk^3 + \frac{19}{12}n - \frac{49}{12}k + \frac{20}{3}nk - \frac{23}{24}n^2 + 16nk^2 + \frac{5}{12}n^3 - \frac{275}{12}k^3 - \frac{371}{24}k^2$

   $-\frac{7}{2}n^2k^2 + \frac{2}{3}n^3k - 4n^2k - \frac{229}{24}k^4 - \frac{1}{24}n^4$
   $= \frac{17}{4}k + \frac{47}{8}k^2 + \frac{35}{4}k^3 + \frac{25}{8}k^4$

   $t(L) = 2 - \frac{29}{3}nk^3 - \frac{4}{3}n + \frac{23}{6}k - \frac{77}{12}nk + \frac{5}{6}n^2 - \frac{67}{4}nk^2 - \frac{2}{3}n^3 + \frac{139}{6}k^3 + \frac{46}{3}k^2$

   $+\frac{17}{4}n^2k^2 - \frac{7}{6}n^3k + \frac{19}{4}n^2k + \frac{29}{3}k^4 + \frac{1}{6}n^4$
   $= -2k + \frac{13}{2}k^2 + \frac{23}{2}k^3 + 7k^4$

6. Case 6: $n \geq 4k + 3$

   $\dim M(L) = -1 - \frac{3}{2}nk^3 - \frac{1}{2}n + \frac{17}{4}k - 5nk + \frac{1}{2}n^2 - 4nk^2 + \frac{15}{4}k^3 + \frac{63}{8}k^2 + \frac{1}{2}n^2k^2 + n^2k + \frac{9}{8}k^4$

   $t(L) = 1 + nk^3 + \frac{3}{4}n - \frac{9}{2}k + \frac{21}{4}nk - \frac{5}{8}n^2 + \frac{13}{4}nk^2 - \frac{1}{4}n^3 - \frac{7}{2}k^3 - 8k^2 + \frac{1}{4}n^2k^2$

   $-\frac{1}{2}n^3k - \frac{1}{4}n^2k - k^4 + \frac{1}{8}n^4$

When $k = 1$ only cases 1, 3, 5, 6 are applicable ($n < 5, n = 5, n = 6, n \geq 7$ respectively), and the above polynomials simplify to our original formula for $k = 1$ that we developed in the first example before we found a general formula for any k. When $k = 0$ only cases 1 and 6 are applicable ($n < 3$ and $n \geq 3$ respectively). These two polynomials simplify to the results found for $k = 0$ in [3]. Suppose we fix a value for k. As n grows large we remain in case 6 and the formula for $t(L)$ becomes a degree four polynomial in n. Since the leading term is positive we can attain $t(L)$ arbitrarily large. For example if $k = 1$, then case 6 simplifies to say that when $n \geq 7$ the formula for $t(L)$ will be $t(L) = \frac{1}{8}n^4 - \frac{3}{4}n^3 - \frac{5}{8}n^2 + \frac{41}{4}n - 16$, which grows arbitrarily large since it behaves like $\frac{1}{8}n^4$ for large values of n.

Chapter 8

Computer Code

8.1 Matlab code

```
function multiplier;

disp('If a strictly upper triangular matrix is of size nxn and k denotes');
disp('the number of diagonals above the main diagonal which are zero.');
disp('Note that k must be at least zero.');
k = input('Enter a value for k = ');
minimum_value = 4*k+3;
disp('You may enter any positive integer for n, however');
disp('to get elements of every type, n must be at least ')
disp(minimum_value);
n=1;

n = input('Enter a value for n = ');

if k > n-1         %invalid because an nxn has n-1 superdiagonals
    disp('Invalid values for n and k')
    return
end

total = 0;

flag = 1;

for i=1:k                                   %inner match (t=a)
    if n-(2*(k+1)+i) < 0
        flag = 0;
        break                               %stop if n is too small
```

```
    end
    total=total+i*(n-(2*(k+1)+i));
end

if flag                                 %only continue counting inner match
    for j=1:k                           %if n is large enough
        if n-(2*(k+1)+k+j) < 0
            break                       %stop if n is too small
        end
        total=total+(k-j+1)*(n-(2*(k+1)+k+j));
    end
end

flag = 1;

if n-2*(k+1) >= 0
    total = total+2*(n-2*(k+1));
end

for i=1:k
    if n-((k+1)+i) < 0                  %stop counting if n is too small
        flag = 0;
        break
    end
    total=total+(3/2)*(i^2+i)*(n-((k+1)+i));
end

%only continue counting s<a<t<b if n is large enough

if flag
    for i=1:k
        if n-((k+1)+k+i) < 0
            flag = 0;
            break                       %stop counting when n is too small
        end
        total=total+(((k+1)^2)-i*(i+1)/2)*(n-((k+1)+k+i));
    end
end

%only continue counting s<a<t<b if n is large enough
if flag
    for i=1:k
        if n-((k+1)+2*k+i) < 0
            break
```

```
        end
        total=total+(1/2)*(k-(i-1))*(k-(i-1)+1)*(n-((k+1)+2*k+i));
    end
end

for j=0:k
    for i=0:k
        if n-(2*(k+1)+i+j+1) >= 0
            total=total+(1/2)*(n-(2*(k+1)+i+j+1))*(n-(2*(k+1)+i+j));
        end
    end
end

disp('dim M(L) =');
disp(total);

if k+1 < n
    L = (1/2)*(n-(k+1))*(n-k);
else
    L = 0;
end

disp('dim L =');
disp(L);
disp('t(L) =');
disp((1/2)*L*(L-1)-total);

if (1/2)*L*(L-1)-total == 0
    disp('L is abelian');
end
```

8.2 Maple code

$L := \frac{1}{2} \cdot (n-k-1) \cdot (n-k)$:

$upperBound := \frac{1}{2} \cdot L \cdot (L-1)$:

$\#Case\ 1,\ n < 2k+3$

$\#\ L\ is\ abelian,\ dimM(L) = upperBound\ and\ t(L) = 0$

$\#\ Case\ 2,\ n = 2k+3, \ldots, 3k+1$

$x_1 := sum\left(\frac{3}{2} \cdot (j-k-1) \cdot (j-k) \cdot (n-j), j = (k+2)..(2 \cdot k+1)\right)$
$+2 \cdot (n-(2 \cdot k+2))$:

$x_2 := sum\left(\left((k+1)^2 - \frac{1}{2} \cdot (j-2 \cdot k-1) \cdot (j-2 \cdot k)\right) \cdot (n-j), j = (2 \cdot k+2)..n\right)$:

$x_3 := sum\left((j-2 \cdot k-2) \cdot (n-j), j = (2 \cdot k+3)..n\right)$:

$x_4 := sum\left(\frac{1}{2} \cdot (j-2 \cdot k-2) \cdot (n-j) \cdot (n-j+1), j = (2 \cdot k+3)..n\right)$:

$ML := x_1 + x_2 + x_3 + x_4$:

$simplify(ML);$

$-4 - \frac{3}{2}nk^3 + 2n - \frac{13}{4}k - 2nk - 4nk^2 + \frac{15}{4}k^3 + \frac{27}{8}k^2 + \frac{1}{2}n^2k^2 + n^2k + \frac{9}{8}k^4$

$simplify(upperBound - ML);$

$4 + nk^3 - \frac{7}{4}n + 3k + \frac{9}{4}nk - \frac{1}{8}n^2 + \frac{13}{4}nk^2 - \frac{1}{4}n^3 - \frac{7}{2}k^3 - \frac{7}{2}k^2 + \frac{1}{4}n^2k^2$

$-\frac{1}{2}n^3k - \frac{1}{4}n^2k - k^4 + \frac{1}{8}n^4$

$\#\ Case\ 3,\ n = 3k+2$

$x_2 := sum(\left((k+1)^2 - \frac{1}{2} \cdot (j-2 \cdot k-1) \cdot (j-2 \cdot k)\right) \cdot (n-j),$
$j = (2 \cdot k+2)..(3 \cdot k+1))$:

$x_3 := sum\left((j-2 \cdot k-2) \cdot (n-j), j = (2 \cdot k+3)..(3 \cdot k+2)\right)$:

$x_4 := sum\left(\frac{1}{2} \cdot (j-2 \cdot k-2) \cdot (n-j) \cdot (n-j+1), j = (2 \cdot k+3)..(3 \cdot k+2)\right)$:

$ML := x_1 + x_2 + x_3 + x_4$:

$simplify(ML);$

$-4+2n-\frac{27}{4}k+\frac{5}{4}nk+\frac{5}{4}nk^2-\frac{21}{4}k^3-\frac{55}{8}k^2+\frac{1}{4}n^2k^2+\frac{1}{4}n^2k-\frac{9}{8}k^4$

$simplify(upperBound-ML);$

$4-\frac{1}{2}nk^3-\frac{7}{4}n+\frac{13}{2}k-nk-\frac{1}{8}n^2-2nk^2-\frac{1}{4}n^3+\frac{11}{2}k^3+\frac{27}{4}k^2+\frac{1}{2}n^2k^2$
$-\frac{1}{2}n^3k+\frac{1}{2}n^2k+\frac{5}{4}k^4+\frac{1}{8}n^4$

$\#\ Case\ 4,\ n=3k+3,\ldots,4k+1$

$x_5 := sum\left(\frac{1}{2}\cdot(4\cdot k+2-j)\cdot(4\cdot k+3-j)\cdot(n-j), j=(3\cdot k+2)..n\right):$

$x_6 := sum\left((4\cdot k+3-j)\cdot(n-j), j=(3\cdot k+3)..n\right):$

$x_7 := sum\left(\frac{1}{2}\cdot(4\cdot k+4-j)\cdot(n-j)\cdot(n-j+1), j=(3\cdot k+3)..n\right):$

$ML := x_1+x_2+x_3+x_4+x_5+x_6+x_7:$

$simplify(ML);$

$-1-\frac{3}{2}nk^3-\frac{1}{2}n+\frac{17}{4}k-5nk+\frac{1}{2}n^2-4nk^2+\frac{15}{4}k^3+\frac{63}{8}k^2+\frac{1}{2}n^2k^2+n^2k+\frac{9}{8}k^4$

$simplify(upperBound-ML);$

$1+nk^3+\frac{3}{4}n-\frac{9}{2}k+\frac{21}{4}nk-\frac{5}{8}n^2+\frac{13}{4}nk^2-\frac{1}{4}n^3-\frac{7}{2}k^3-8k^2$
$+\frac{1}{4}n^2k^2-\frac{1}{2}n^3k-\frac{1}{4}n^2k-k^4+\frac{1}{8}n^4$

$\#\ Case\ 5,\ n=4k+2$

$x_5 := sum\left(\frac{1}{2}\cdot(4\cdot k+2-j)\cdot(4\cdot k+3-j)\cdot(n-j), j=(3\cdot k+2)..(4\cdot k+1)\right):$

$x_6 := sum\left((4\cdot k+3-j)\cdot(n-j), j=(3\cdot k+3)..n\right):$

$x_7 := sum\left(\frac{1}{2}\cdot(4\cdot k+4-j)\cdot(n-j)\cdot(n-j+1), j=(3\cdot k+3)..n\right):$

$ML := x_1+x_2+x_3+x_4+x_5+x_6+x_7:$

$simplify(ML);$

$-2+\frac{55}{6}nk^3+\frac{19}{12}n-\frac{49}{12}k+\frac{20}{3}nk-\frac{23}{24}n^2+16nk^2+\frac{5}{12}n^3-\frac{275}{12}k^3-\frac{371}{24}k^2$
$-\frac{7}{2}n^2k^2+\frac{2}{3}n^3k-4n^2k-\frac{229}{24}k^4-\frac{1}{24}n^4$

$simplify(upperBound - ML);$

$2 - \frac{29}{3}nk^3 - \frac{4}{3}n + \frac{23}{6}k - \frac{77}{12}nk + \frac{5}{6}n^2 - \frac{67}{4}nk^2 - \frac{2}{3}n^3 + \frac{139}{6}k^3 + \frac{46}{3}k^2$

$+\frac{17}{4}n^2k^2 - \frac{7}{6}n^3k + \frac{19}{4}n^2k + \frac{29}{3}k^4 + \frac{1}{6}n^4$

$\# \ Case\ 6,\ n \geq 4k + 3$

$x_6 := sum\left((4 \cdot k + 3 - j) \cdot (n - j), j = (3 \cdot k + 3)..(4 \cdot k + 2)\right) :$

$x_7 := sum\left(\frac{1}{2} \cdot (4 \cdot k + 4 - j) \cdot (n - j) \cdot (n - j + 1), j = (3 \cdot k + 3)..(4 \cdot k + 3)\right) :$

$ML := x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 :$

$simplify(ML);$

$-1 - \frac{3}{2}nk^3 - \frac{1}{2}n + \frac{17}{4}k - 5nk + \frac{1}{2}n^2 - 4nk^2 + \frac{15}{4}k^3 + \frac{63}{8}k^2 + \frac{1}{2}n^2k^2 + n^2k + \frac{9}{8}k^4$

$simplify(upperBound - ML);$

$1 + nk^3 + \frac{3}{4}n - \frac{9}{2}k + \frac{21}{4}nk - \frac{5}{8}n^2 + \frac{13}{4}nk^2 - \frac{1}{4}n^3 - \frac{7}{2}k^3 - 8k^2 + \frac{1}{4}n^2k^2$

$-\frac{1}{2}n^3k - \frac{1}{4}n^2k - k^4 + \frac{1}{8}n^4$

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