Abstract

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Let A be an algebraic system with product a*b between elements a and b in A. It is of interest to compare the solvable length t with other invariants, for instance size, order, or dimension of A. Thus we ask, for a given t what is the smallest n such that there is an A of length t and invariant n. It is this problem that we consider for associative algebras, matrix groups, and Lie algebras. We consider A in each case to be subsets of (strictly) upper triangular n by n matrices. Then the invariant is n. We do these for the associative (Lie) algebras of all strictly upper triangular n by n matrices and for the full n by n upper triangular unipotent groups. The answer for n is the same in all cases. Then we restrict the problem to a fixed number of generators. In particular, using only 3 generators and we get the same results for matrix groups and Lie algebras as for the earlier problem. For associative algebras with 1 generator we also get the same result as the general associative algebra case. Finally we consider Lie algebras with 2 generators and here n is larger than in the general case. We also consider the problem of finding the dimension in the associative algebra, the general, and 3 generator Lie algebra cases.

ON THE SOLVABLE LENGTH OF ASSOCIATIVE ALGEBRAS, MATRIX GROUPS, AND LIE ALGEBRAS

by
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DEDICATION

This dissertation is dedicated to my mother, Albina, not only my mentor, and best friend, but the person I admire the most, my father, Leon, who has taught me that I can accomplish anything as long as I set goals and work hard, and my brother, Lee Michael, who has always challenged my mind and analytical thinking skills. Because of all your love, patience, understanding, and support, I was able to achieve my goal of obtaining a PhD in Mathematics.

BIOGRAPHY

I was born in Buffalo, New York on October 3, 1977. I received my BA in Mathematics and Computer Science Summa Cum Laude at the State University of New York College at Geneseo in 1999. After graduation, I moved down to a warmer place Cary, North Carolina, where I took a job as a Software Engineer at IBM. In 2000, I started part-time graduate school at North Carolina State University and received my MS in Computational Mathematics in May of 2003. After completion of my PhD in Applied Mathematics, I will continue to work with IBM and plan to teach a class per semester at a local college.

On a more personal note, I currently live with my boyfriend, Jimmy, and my two dogs Bucca and Bailey in a cute little house in Cary. In my spare time, I enjoy traveling, boating, socializing with my friends, and mentoring young women.

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CHAPTER 1: Introduction

Let A be an algebraic system with product a*b between elements a and b in A. Define $A^1 = A*A$ and $A^{k+1} = A^k*A^k$. A is called solvable of length t if there exists a t such that $A^t = 0$ and A^{t-1} is nonzero. It is of interest to compare the solvable length t with other invariants, for instance size, order, or dimension of A. Thus we ask, for a given t what is the smallest n such that there is an A of length t and invariant n. It is this problem that we consider for associative algebras, matrix groups, and Lie algebras. We consider A in each case to be subsets of (strictly) upper triangular n by n matrices. Then the invariant is n. We do these for the associative (Lie) algebras of all strictly upper triangular n by n matrices and for the full n by n upper triangular unipotent groups. The answer for n is the same in all cases. Then we restrict the problem to a fixed number of generators. In particular, using only 3 generators and we get the same results for matrix groups and Lie algebras as for the earlier problem. For associative algebras with 1 generator we also get the same result as the general associative algebra case. Finally we consider Lie algebras with 2 generators and here n is larger than in the general case. We also consider the problem of finding the dimension in the associative algebra, the general, and 3 generator Lie algebra cases. We begin with further discussion of the problem and it's history.

In 1913 [1] Burnside proposed the question, "What is the least among the orders of p-groups with a given soluble length?" Recall that p-groups are defined as the groups of prime power order. G has solvable length t if $G^t = 0$ and G^{t-1} is non-zero. In this discussion, t will denote the derived length of G. Burnside started his work by showing that there were groups of order p^3 and p^6 with corresponding solvable lengths 2 and 3.

He then made a start on his famous question with finding the order of G to be $|G| \geq p^{3(t-1)}$ with solvable length t, but stated that it seems probable that for greater values of t the actual lower limit for the order exceeds $p^{3(t-1)}$ and in [2] he improves this order to $|G| \geq p^{(t+1)(t+2)/2}$. Continuing with Burnside's question, early results in group theory are shown in a classic 1933 paper by Philip Hall [6]. Hall found the derived length of p-groups to be bounded between $p^{2^{t-1}+t-1} \leq |G| \leq p^{2^{(t-2)}*(2^{t-1}-1)}$.

In [7] Itˆo refined the upper bound of Hall's and found the order of $|G| \leq p^{3*2^{t-1}}$. Recently in [4], Evans-Riley, Newman & Schneider showed that for every integer $t \geq 3$ and every prime $p \geq 5$ there is a group with solvable length t and order p^{2^t-2}, $|G| \leq p^{2^t-2}$, and Schneider in [9] found that $|G| \geq p^{2^{t-1}+3t-10}$ and Mann in [8] showed $|G| \geq p^{2^{t-1}+2t-4}$ with solvable length t. Schneider's result is better for larger t and Mann's is better for smaller t.

There are many recent results in groups and we would like to see what is true in Lie algebras. There are many similarities between groups and Lie algebras, in particular between groups of order p^n where p is prime, and nilpotent Lie algebras of dimension n. Both have a derived series and a lower central series which end at the identity. Also, in both cases, the derived series is contained in the lower central series ($L^w \subseteq L_{2^w}$).

The following examples help to explain the derived series and central series, where we let L be a Lie algebra. Let $L_2 = [L, L]$, $L_3 = [L, L_2] \ldots L_{n+1} = [L, L_n]$. L is called a *nilpotent* if there exists an n such that $L_n = 0$. L has *class* n if $L_{n+1} = 0$ and $L_n \neq 0$.

Example 1.1

$L = (x, y, z)$ where $[x, y] = z$ and $[x, z] = [y, z] = 0$. $L_2 = (z)$ and $L_3 = 0$ leaving L to have a class of 2. □

Example 1.2

Let M = 5x5 strictly upper triangular matrix $M = \begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

then $M_2 = \begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $M_3 = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $M_5 = 0$.

Leaving M to have a class of 4. □

Define $L^1 = [L, L]$, $L^2 = [L^1, L^1]$, … $L^{n+1} = [L^n, L^n]$. L is *solvable* if there exists a t such that $L^t = 0$. L has derived *length* t if $L^t = 0$ and $L^{t-1} \neq 0$.

Example 1.3

Using example 1.1 above we find $L^1 = (z) = L_2$ and $L^2 = 0$. L has a length of 2. □

Example 1.4

Using example 1.2 we find that $M^1 = \begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = M_2$ and $M^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = M_4$ and

$M^3 = 0$. M has a length of 3. □

Example 1.5

A is abelian and $A^1 = A_2 = 0$. A has a length of 1 and a class of 1. □

Note: $L^1 = L_2$, $L^2 \subseteq L_4$, $L^3 \subseteq L_8$ … $L^n \subseteq L_{2^n}$.

There are some more similarities between groups and Lie algebras. The center of a group G is $Z(G) = \{x \backslash xy = yx \ \forall y \in G\}$, and the center of a Lie algebra L is $Z(L) = \{x \backslash [x, y] = 0 \ \forall y \in L\}$. For groups we use the computation of $\langle x, y\rangle = xyx^{-1}y^{-1}$ and for Lie algebra we use the commutator multiplication where $[A, B] = AB - BA$. In the case of strictly upper triangular nxn matrices we have $Z(L) = \{E_{1,n}\}$. Burnsides theorem applied to groups and Lie algebras would result with: if G (or L) is abelian, then $G^1 = e$ ($L^1 = 0$), and if $|G| = p^2$ (dimension of L = 2), then $G^1=e$ ($L^1 = 0$). So if G (or L) is to have length 2 then $|G| \geq 3$ (dimension of $L \geq 3$). (Such an L is $L = \langle x, y, z\rangle$ where $[x, y] = z$ and $[x, z] =[y, z] =0$. Then $L^1 = \langle z\rangle$ and $L^2 = 0$ leaving the length of L to be 2). In [6] it is shown that for G to have length t and $|G| = p^n$, then $n \geq 2^{t-1} + t -1$ leaving the following table:

Table 1.6 Using Hall's bound $2^{t-1} + t$ for length t

t	n
2	3
3	6
4	11
5	20

The last entry of table 1.6 says for $G^4 \neq e$, $G^5 = e$ then $|G| \geq p^{20}$. The Lie algebra analogue, if it is true, would be $L^4 \neq 0$ and $L^5 = 0$ then dimension of $L \geq 20$. In fact, it might be that in the group case $2^{t-1} + t-1$ may not be big enough. This may also be the case for the Lie algebras.

We will apply the ideas of the group results and investigate the bounds using strictly upper n x n triangular matrices in the case of associative algebras, matrix groups, and Lie algebras. The aim is to find the smallest n for each derived length in these cases.

In a related direction, we would like to find the least size n of the nxn matrices which have a derived length t. For a given n, we use n-1 generators. These generator elements come from diagonal d_1, the diagonal directly above the main diagonal. For derived length t we find $n = 2^{t-1}+1$. We consider the case of having less than n-1 generators. If we fix the number of generators, it is conceivable that for a given t the size of n might need to increase. I have considered the following cases: 1 and n-1 generators for associative algebras; 3 and n-1 generators for matrix groups; 3 and n-1 generators for Lie algebras.

The final problem in this collection is what can be done with just two generators for the Lie algebras. There are group results due to Glasby [5] in this direction. He studied unipotent groups of nxn upper triangular matrices for this problem. He finds the size of n to be greater then $(21/32)*2^t$ with a given derived length t for the 2 generated case. We conclude with finding the matching results for Lie algebras of nxn strictly upper triangular matrices.

CHAPTER 2: Associative Algebra General Case

We are looking at the strictly upper triangular nxn matrices, N. The derived series of N is defined as $N^1 = N*N$ and $N^{k+1} = N^k*N^k$ that we find using associative multiplication. Evidently, N^{t-1} is generated by the set of all products of 2^{t-1} generator elements from N. These generator elements will be chosen from diagonal d_1, the diagonal directly above the main diagonal. Letting $E_{i,j}$ be the usual matrix unit with 1 in the (i,j) position and 0 elsewhere, the generators are $E_{12}, E_{23}, \ldots, E_{n-1,n}$. The N^* are the matrices we find starting with the first diagonal, d_1. Hence, for N to have length t, N must have 2^{t-1} generator elements whose product is non–zero while each product of 2^t elements is zero.

To make this clearer let T be the associative algebra of all strictly upper triangular nxn matrices. Also let d_1 represent the diagonal directly above the main diagonal, d_j represents the diagonal j steps above the main diagonal, and d_j' represent the basis elements of the diagonal j steps above the main diagonal. Clearly, $T = d_1 \oplus d_2 \oplus \ldots \oplus d_{n-1}$. Using matrix units, it is clear that the derived series is

$T^1 = d_2 \oplus d_3 \oplus \ldots \oplus d_{n-1}$

$T^2 = d_{2^2} \oplus d_5 \oplus \ldots \oplus d_{n-1}$

$\vdots$

$T^k = d_{2^k} \oplus d_{2^k+1} \oplus \ldots \oplus d_{n-1}$

Notice that $T^{t-1} \neq 0$ if and only if $2^{t-1} \leq n-1$ or $n \geq 2^{t-1}+1$ (diagonal $d_{2^{t-1}}$ has to exist). If $n \geq 2^{t-1}+1$ and $n < 2^t+1$ then T has length t. If $n = 2^{t-1}+1$, then n is the least possible n for T to have derived length t. In this case the dimension, the number, m, of matrix units in

T is $\frac{n(n-1)}{2}$. Since we are dealing with strictly upper triangular matrices we compute this dimension by starting with the size of the nxn matrix, n^2, minus the number of elements on the main diagonal, n, and then since we are only dealing with the upper half of the matrix we divide that number by 2. Therefore, we have $\frac{n^2-n}{2}=\frac{n(n-1)}{2}$. Letting $n = 2^{t-1}+1$, we find the dimension of T is $\frac{(2^{t-1}+1)2^{t-1}}{2}=(2^{t-1}+1)2^{t-2}$. These results can be incorporated into Lemma 2.1.

Lemma 2.1 For T to have length t, the smallest matrix size is $n = 2^{t-1}+1$. In this construction the dimension of T = $(2^{t-1}+1)2^{t-2}$, and T has 2^{t-1} generators.

We obtain the table:

Table 2.2: Associative algebra general case results

Length (t)	**Minimum Matrix size of T (n)**	**Dimension of T (m)**	**Number of Generators (Dimension of d_1)**
2	3	3	2
3	5	10	4
4	9	36	8
5	17	136	16
t	$2^{t-1}+1$	$(2^{t-1}+1)2^{t-2}$	2^{t-1}

Since T is a special case of general nilpotent algebras, we can also state Lemma 2.8.

Lemma 2.3 The minimum dimension for a nilpotent associative algebra to have derived length t is less than or equal to $(2^{t-1}+1)2^{t-2}$.

We can do much better than that. The problems that arise are:

1.) Reduce the upper bound found in Lemma 2.1.

2.) For a fixed number of generators and a fixed length, find a good upper bound for the dimension.

3.) Consider question 2 in the context of a subalgebra of T and find the minimum size of the matrices.

We will consider these questions simultaneously in the next section.

CHAPTER 3: Associative Algebras with One Generator

In the beginning problem, we started with a basis of the first diagonal as the generators and there were 2^{t-1} generators. Now we look at the case were the generator is a combination of the basis for the first diagonal, $F_{12} = E_{12} + E_{23} + \ldots + E_{n-1,n}$. Using associative matrix multiplication with strictly upper triangular nxn matrices, we find the derived series of the matrix and answer the three main questions at the end of Chapter 2. Set $F_{1,j+1} = E_{1,j+1} + E_{2,j+2} + \ldots + E_{n-j,n}$ where $F_{1,j+1}$ is the sum of the elementary matrices whose non–zero entries are on the diagonal d_j. To make this clear we start with an example.

Example 3.1

Let n = 5 and A be the subalgebra of T generated by the basis of d_1.

We have the 5x5 matrix starting with $F_{12} = E_{12} + E_{23} + E_{34} + E_{45}$

$$F_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now we compute the derived series of the matrix. The elements on d_2 are $F_{12} * F_{12} = (E_{12} + E_{23} + E_{34} + E_{45}) * (E_{12} + E_{23} + E_{34} + E_{45}) = F_{13} = E_{13} + E_{24} + E_{35}$. Then the elements on d_4, $F_{13} * F_{13} = F_{15} = E_{15}$. Note we also compute the elements of d_4 by multiplying $F_{12} * F_{14}$. Now we compute the rest of the products. We compute elements of d_3 by multiplying elements of d_1 by the elements of d_2, $F_{12} * F_{13} = F_{14} = E_{14} + E_{25}$. All other products between the other F's are 0. Now we have the derived series where $A = \langle F_{12} \rangle \oplus \langle F_{13} \rangle \oplus \langle F_{14} \rangle \oplus \langle F_{15} \rangle$, $A^1 = \langle F_{13} \rangle \oplus \langle F_{14} \rangle \oplus \langle F_{15} \rangle$, $A^2 = \langle F_{15} \rangle$, and $A^3 = 0$. Thus A has the length 3 and $\langle F_{12} \rangle \oplus \langle F_{13} \rangle \oplus \langle F_{14} \rangle \oplus \langle F_{15} \rangle$ is a basis for A and has dimension m = 4. This is the least possible dimension a nilpotent algebra of length 3 can have since such an algebra must have a product of 4 elements that are not 0. □

Example 3.2

Let n = 9 and A be the subalgebra of T generated by F_{12}.

We have the 9x9 matrix starting with $F_{12} = E_{12} + E_{23} + E_{34} + E_{45} + E_{56} + E_{67} + E_{78} + E_{89}$.

$$F_{12} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

Now we compute the derived series of the matrix. We start with computing the elements on d_2 by multiplying $F_{12} * F_{12} = F_{13} = E_{13} + E_{24} + E_{35} + E_{46} + E_{57} + E_{68} + E_{79}$. Then use the elements of d_2 to compute the elements of d_4, $F_{13} * F_{13} = F_{15} = E_{15} + E_{26} + E_{37} + E_{48} + E_{59}$. Then use the elements of d_4 to compute the elements of d_8, $F_{15} * F_{15} = F_{19} = E_{19}$. Also $F_{12} * F_{13} = F_{14} = E_{14} + E_{25} + E_{36} + E_{47} + E_{58} + E_{69} \in d_3$. By multiplying $F_{12} * F_{15}$ or $F_{13} * F_{14}$, we get $F_{16} = E_{16} + E_{27} + E_{38} + E_{49} \in d_5$. By multiplying $F_{12} * F_{16}$, $F_{13} * F_{15}$, or $F_{14} * F_{14}$, we get $F_{17} = E_{17} + E_{28} + E_{39} \in d_6$. By multiplying $F_{12} * F_{17}$, $F_{13} * F_{16}$, or $F_{14} * F_{15}$, we get $F_{18} = E_{18} + E_{29} \in d_7$. All other products between the other F's are 0. Note that $F_{1,s+1} \in d_s$.

Now we have that $A = \langle F_{12}\rangle \oplus \langle F_{13}\rangle \oplus \ldots \oplus \langle F_{19}\rangle$, $A^1 = \langle F_{13}\rangle \oplus \langle F_{14}\rangle \oplus \ldots \oplus \langle F_{19}\rangle$, $A^2 = \langle F_{15}\rangle \oplus \langle F_{16}\rangle \oplus \ldots \oplus \langle F_{19}\rangle$, $A^3 = \langle F_{19}\rangle$, $A^4 = 0$. Thus A has the length 4 and $\langle F_{12}\rangle \oplus \langle F_{13}\rangle \oplus \ldots \oplus \langle F_{19}\rangle$ is the basis for A and has dimension m = 8. For length 4, this example gives the best possible results for each of the three problems we are considering. □

In the general case we claim that $F_{1,s+1} \in d_s$ and that $F_{1,p+1} * F_{1,q+1} = F_{1,p+q+1}$. The first of these is by definition and for the second we note that

$$E_{i,i+p} * E_{j,j+q} = \begin{cases} 0 & \text{if } i+p \neq j \\ E_{i,i+p+q} & \text{if } i+p = j \end{cases}$$

Of course $F_{1,s+1} \in d_s$, where $s = p+q$. Hence $F_{1,p+1} * F_{1,q+1} = (E_{1,p+1} + E_{2,p+2} + \ldots. + E_{n-p,n}) * (E_{1,q+1} + E_{2,q+2} + \ldots. + E_{n-q,n}) = E_{1,p+q+1} + E_{2,p+q+2} + \ldots + E_{n-(p+q),n} = F_{1,p+q+1} = F_{1,s+1} \in d_{p+q} = d_s$. Therefore, $F_{12}, F_{13}, \ldots, F_{1,n-1}$ are a basis for A and the dimension of A = n–1.

Also

$A = \langle F_{12} \rangle \oplus \langle F_{13} \rangle \oplus \ldots \oplus \langle F_{1,n} \rangle$

$A^1 = \langle F_{13} \rangle \oplus \langle F_{14} \rangle \oplus \ldots \oplus \langle F_{1,n} \rangle$

$A^2 = \langle F_{15} \rangle \oplus \langle F_{16} \rangle \oplus \ldots \oplus \langle F_{1,n} \rangle$

$\vdots$

$A^k = \langle F_{1,2^k+1} \rangle \oplus \langle F_{1,2^k+2} \rangle \oplus \ldots \oplus \langle F_{1,n} \rangle$

Theorem 3.3 Let A be the subalgebra of the algebra T of strictly upper triangular nxn matrices that is generated by F_{12}. Then the dimension of A is n–1.

Theorem 3.4 There is a subalgebra A of T having length t and dimension 2^{t-1} if A has the size $n = 2^{t-1} + 1$. A is generated by one element, F_{12}. This result is the best possible in that there is not a subalgebra of smaller dimension that has length t.

From Theorem 3.4 we get this Corollary:

Corollary 3.5 There is a nilpotent associative algebra having length t and dimension 2^{t-1} and this algebra is generated by one element. This result is the best possible in that there is not a subalgebra of smaller dimension that has length t.

Proof. Let A be a subalgebra of T generated by one element, F_{12}, where T has a size n. As usual $F_{1,p+1} * F_{1,q+1} = F_{1,p+q+1} = F_{1,s+1} \in d_s$ if $p+q+1 \le n$ and it is 0 otherwise. Hence $A = \langle F_{12}\rangle \oplus \langle F_{13}\rangle \oplus \ldots \oplus \langle F_{1,n}\rangle$ and dimension of A = n–1. Furthermore,

$A^1 = \langle F_{13}\rangle \oplus \langle F_{14}\rangle \oplus \ldots \oplus \langle F_{1,n}\rangle$

$A^2 = \langle F_{15}\rangle \oplus \langle F_{16}\rangle \oplus \ldots \oplus \langle F_{1,n}\rangle$

$\vdots$

$A^k = \langle F_{1,2^k+1}\rangle \oplus \langle F_{1,2^k+2}\rangle \oplus \ldots \oplus \langle F_{1,n}\rangle$

Hence $A^{t-1} \neq 0$ if and only if $2^{t-1} \le n-1$ or $n \ge 2^{t-1}+1$ (if diagonal $d_{2^{t-1}}$ exists). If $n \ge 2^{t-1}+1$ and $n < 2^t+1$ then A has length t. If $n = 2^{t-1}+1$, then n is the least possible n for A to have derived length t. Now we know that the dimension of A is n–1, therefore when we want the smallest n of length t we let $n = 2^{t-1}+1$ and the dimension of A is 2^{t-1}. □

Table 3.6: Associative algebra results with one generator

Length (t)	Minimum Matrix Size of A (n)	Dimension of A (m)
1	2	1
2	3	2
3	5	4
4	9	8
5	17	16
t	$n = 2^{t-1}+1$	2^{t-1}

From our table we see that in order to obtain an algebra of length 4 we need to have a 9x9 matrix to start with. Now we compare the results when we start with one generator to when we start with the n-1 elementary matrices directly above the main diagonal as generators (Lemma 2.1).

Table 3.7: Comparison chart of associative algebras general case (T) and associative algebra with one generator (A):

Length (t)	**Minimum Matrix Size of A and T (n)**	**Dimension of T**	**Dimension of A**
1	2	1	1
2	3	3	2
3	5	10	4
4	9	36	8
5	17	136	16
t	$n = 2^{t-1} + 1$	$(2^{t-1} + 1)2^{t-2}$	2^{t-1}

CHAPTER 4: Matrix Group General Case

We now consider the matrix group versions of the associative algebra results. In this investigation, we work exclusively with subgroups M of the group of unipotent nxn upper triangular matrices, U. Then $U = I \oplus d_1 \oplus d_2 \oplus \ldots \oplus d_{n-1}$, where the sum is a vector space direct sum and the d_i are the upper diagonals of U as in the last section. We will consider the problem: For a given length, find the minimum n necessary.

Again using matrix units we compute the derived length where $[A,B] = ABA^{-1}B^{-1}$ is the commutator. Then $U^1 = <[A,B]; A,B \in U>$ and $U^d = <[A,B]; A,B \in U^{d-1}>$. Then

$U^1 = I \oplus d_2 \oplus d_3 \oplus \ldots \oplus d_{n-1}$

$U^2 = I \oplus d_{2^2} \oplus d_5 \oplus \ldots \oplus d_{n-1}$

$\vdots$

$U^k = I \oplus d_{2^k} \oplus d_{2^{k+1}} \oplus \ldots \oplus d_{n-1}$

Notice that $U^{t-1} \neq 0$ if and only if $2^{t-1} \leq n-1$ or $n \geq 2^{t-1}+1$ (diagonal $d_{2^{t-1}}$ has to exist).

The least n for U to have length t, is $n = 2^{t-1}+1$ and the dimension of $U = (2^{t-1}+1)2^{t-2}$, just as in the associative case. Hence $I + E_{12}, I + E_{23}, \ldots, I + E_{n-1,n}$ are generators of U, hence U has 2^{t-1} generators.

Lemma 4.1 For U to have length t, the smallest $n = 2^{t-1}+1$, and U has 2^{t-1} generators.

CHAPTER 5: Matrix Groups with 3 Generators

Now we consider a subgroup of U where there are three generators: F_{12}, F_{23}, and F_{34}. These generators are combinations of the elements on the first diagonal added to the identity matrix. We use the following terms:

$$F_{ij} = I + \sum_{k=0}^{j+3k\leq n} E_{i+3k,j+3k} \text{ where } i = 1,2,3 \text{ and } j > i.$$

Using multiplication with unipotent strictly upper triangular nxn matrices, we can find the derived length of the group. To make this clear we start with an example.

Example 5.1

Let n = 9 and let F_{12}, F_{23}, and F_{34} be generators of M in T. Thus $M \supseteq \{F_{12}, F_{23}, F_{34}\}$ where $F_{12} = I + E_{12} + E_{45} + E_{78}$, $F_{23} = I + E_{23} + E_{56} + E_{89}$, and $F_{34} = I + E_{34} + E_{67}$.

$$F_{12} = \begin{pmatrix} 1&1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \quad F_{23} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \quad F_{34} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

Now we compute the derived length of the group. Compute the derived series starting with M to get M^1 by the following multiplications:

$[F_{12},F_{23}] = (F_{12} * F_{23})*(F_{12}^{-1} * F_{23}^{-1}) =$

$[(I + E_{12} + E_{45} + E_{78}) * (I + E_{23} + E_{56} + E_{89})] * [(I - E_{12} - E_{45} - E_{78}) * (I - E_{23} - E_{56} - E_{89})]=$

$(I+E_{23}+E_{56}+E_{89}+E_{12}+E_{13}+E_{45}+E_{46}+E_{78}+E_{79})*(I-E_{23}-E_{56}-E_{89}-E_{12}+E_{13}-E_{45}+E_{46}-E_{78}+E_{79}) =$

$$I + E_{13} + E_{46} + E_{79} = F_{13}$$

$[F_{12},F_{34,}] = (F_{12} * F_{34}) * (F_{12}^{-1} * F_{34}^{-1}) =$

$[(I + E_{12} + E_{45} + E_{78}) * (I + E_{34} + E_{67})] * [(I - E_{12} - E_{45} - E_{78}) * (I - E_{34} - E_{67})] =$

$(I + E_{12} + E_{45} + E_{78} + E_{34} + E_{67}) * (I – E_{12} – E_{45} – E_{78} – E_{34}– E_{67}) =$

$$I - E_{35} - E_{68} = (F_{35})^{-1}$$

$[F_{23},F_{34}] = (F_{23} * F_{34}) * (F_{23}^{-1} * F_{34}^{-1}) =$

$[(I + E_{23} + E_{56} + E_{89}) * (I + E_{34} + E_{67})] * [(I – E_{23} – E_{56} – E_{89}) * (I – E_{34} – E_{67})] =$

$(I + E_{34} + E_{67} + E_{23} + E_{24} + E_{56} + E_{57} + E_{89}) * (I – E_{34} – E_{67} – E_{23} + E_{24} – E_{56} + E_{57} – E_{89}) =$

$$I + E_{24} + E_{57} = F_{24}$$

$M^1 \supseteq \{F_{13}, F_{24}, F_{35}\}$ and we continue to find $M^2 \supseteq \{F_{15}, F_{26}, F_{37}\}$ where $F_{15} = I + E_{15} + E_{48}$, $F_{26} = I + E_{26} + E_{59}$, and $F_{37} = I + E_{37}$; and $M^3 \supseteq \{ F_{19}\}$ where $F_{19} = I + E_{19}$; and $M^4 = 0$ since $M \subseteq U$ and $U^4 = 0$. Hence M has length 4. □

Lemma 5.2 The multiplication of $[F_{i,p+i},F_{j,p+j}]$ where i, j = 1,2,3 and $i \neq j$ and p is not congruent to 0 mod 3 yields three possible results:

Case 1: I if j is not congruent to p+i mod 3 and i is not congruent to p+j mod 3

Case 2: $F_{i,2p+i}$ if $j \equiv p+i$ mod 3 and i is not congruent to p+j mod 3

Case 3: $(F_{j,2p+j})^{-1}$ if $i \equiv p+j$ mod 3 and j is not congruent to p+i mod 3

Proof. We will be computing $[F_{i,p+i},F_{j,p+j}]$ where i=1,2; j = 2,3; i<j and we assume that p is not congruent to 0 mod 3. It will be useful to let $H_{a,b} = F_{a,b} – I$. Then

$[F_{i,p+i},F_{j,p+j}] =$

$[(I + H_{i,p+i}) * (I + H_{j,p+j})] * [(I - H_{i,p+i}) * (I - H_{j,p+j})] =$

$[I + H_{j,p+j} + H_{i,p+i} + H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0)]*[I - H_{j,p+j} - H_{i,p+i} + H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0)] =$

$I * I +I * (-H_{j,p+j}) + I * (-H_{i,p+i}) + I * H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0) + H_{j,p+j} * I + H_{j,p+j} * (-H_{j,p+j}) + H_{j,p+j} * (-H_{i,p+i}) + H_{j,p+j} * [H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0)] + H_{i,p+i} * I + H_{i,p+i} * (-H_{j,p+j}) + H_{i,p+i} * (-H_{i,p+i}) + H_{i,p+i} * [H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0)] + H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0) * I +$

$[H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0)] * (-H_{j,p+j}) + [H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0)] * (-H_{i,p+i}) + [H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0)] * [H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0)]$. We go through each of these multiplications:

$I * I = \mathbf{I}$

$I * (-H_{j,p+j}) = -H_{j,p+j}$ this will cancel with its positive opposite

$I * (-H_{i,p+i}) = -H_{i,p+i}$ this will cancel with its positive opposite

$I * H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0) = \mathbf{H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0)}$

$H_{j,p+j} * I = H_{j,p+j}$ this will cancel with its negative opposite

$H_{j,p+j} * (-H_{j,p+j}) = 0$ since p+j is not congruent to j mod 3

$H_{j,p+j} * (-H_{i,p+i}) = \mathbf{-H_{j,2p+j}\,\delta\,(p+j - i \bmod 3,0)}$

$H_{j,p+j} * [H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0)] = 0$, for if not,

$p+j \equiv i \bmod 3$ and $p+i \equiv j \bmod 3 \rightarrow$

$p \equiv i - j \bmod 3$ and $p \equiv -i + j \bmod 3 \rightarrow$

$p \equiv i - j \bmod 3$ and $-p \equiv i - j \bmod 3 \rightarrow$

$p \equiv -p \bmod 3 \rightarrow$

$2p \equiv 0 \bmod 3$

which is a contradiction to our original assumption.

$H_{i,p+i} * I = H_{i,p+i}$ this will cancel with its negative opposite

$H_{i,p+i} * (-H_{j,p+j}) = \mathbf{-H_{i,2p+i}\,\delta\,(p+i - j \bmod 3,0)}$

$H_{i,p+i} * (-H_{i,p+i}) = 0$ since p+i is not congruent to i mod 3

$H_{i,p+i} * [H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0)] = 0$, since if $p+i \equiv i \bmod 3$ then $p \equiv 0 \bmod 3$ which is a contradiction to our original assumption.

$H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0) * I = \mathbf{H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0)}$

$[H_{i,2p+i}\,\delta\,(p+i-j \bmod 3,0)] * (-H_{j,p+j}) = 0$, since if $2p+i \equiv j \bmod 3$ then $p+p+i-j \equiv 0 \bmod 3$, but p is not congruent to 0 mod 3 and $p+i-j \equiv 0$ mod 3 which is impossible.

$[H_{i,2p+i}\delta(p+i\text{-}j \bmod 3,0)] * (-H_{i,p+i}) = 0$, since if $2p+i \equiv i$ mod 3 then $2p \equiv 0$ mod 3 which is a contradiction to our original assumption.

$[H_{i,2p+i}\delta(p+i\text{-}j \bmod 3,0)] * [H_{i,2p+i}\delta(p+i\text{-}j \bmod 3,0)] = 0$, since if $2p+i \equiv i$ mod 3 then $p \equiv 0$ mod 3 which is a contradiction to our original assumption.

Leaving: $I + 2H_{i,2p+i}\delta(p+i\text{-}j \bmod 3,0) - H_{j,2p+j}\delta(p+j\text{-}i \bmod 3,0) - H_{i,2p+i}\delta(p+i\text{-}j \bmod 3,0)$

$= I + H_{i,2p+i}\,\delta(p+i - j \bmod 3,0) - H_{j,2p+j}\,\delta(p+j - i \bmod 3,0) =$

Case 1: **I** if j is not congruent to p+i mod 3 and i is not congruent to p+j mod 3

Case 2: $I + H_{i,2p+i} = \mathbf{F_{i,2p+i}}$ if $j \equiv p+i$ mod 3 and i is not congruent to p+j mod 3

Case 3: $I - H_{j,2p+j} = \mathbf{(F_{j,2p+j})^{-1}}$ if $i \equiv p+j$ mod 3 and j is not congruent to p+i mod 3

Note that $i \equiv p+j$ mod 3 and $j \equiv p+i$ mod 3 cannot hold simultaneously, if they did then

$p+j \equiv i$ mod 3 and $p+i \equiv j$ mod 3 $\rightarrow$

$p \equiv i - j$ mod 3 and $p \equiv -i + j$ mod 3 $\rightarrow$

$p \equiv i - j$ mod 3 and $-p \equiv i - j$ mod 3 $\rightarrow$

$p \equiv -p$ mod 3 $\rightarrow$

$2p \equiv 0$ mod 3

which is a contradiction to our original assumption. □

Lemma 5.3 $F_{1,2^k+1}, F_{2,2^k+2}, F_{3,2^k+3} \in M^k$.

Proof. Base case: Let k = 1. We start with $F_{12}, F_{23}, F_{34} \in M$. We find with the following multiplications of $[F_{1,2}, F_{2,3}]$, $[F_{1,2}\,F_{3,4}]$, and $[F_{2,3}, F_{3,4}]$ and with lemma 5.2 we get $F_{1,3}$, $(F_{3,5})^{-1}$, and $F_{2,4}$ respectively, where $F_{1,3}, F_{2,4}, F_{3,5}$ are elements of M^1.

Induction Hypothesis: let k = r where $F_{1,2^r+1}, F_{2,2^r+2}, F_{3,2^r+3} \in M^r$

To show that when k = r+1 that $F_{1,2^{r+1}+1}$, $F_{2,2^{r+1}+2}$, $F_{3,2^{r+1}+3} \in M^{r+1}$, we start with M^{r+1} = $[M^r, M^r]$ where $F_{1,2^r+1}$, $F_{2,2^r+2}$, $F_{3,2^r+3} \in M^r$. We do the calculations

$[F_{1,2^r+1}, F_{2,2^r+2}]$

$[F_{1,2^r+1}, F_{3,2^r+3}]$

$[F_{2,2^r+2}, F_{3,2^r+3}]$

using lemma 5.2 to get $F_{1,2*2^r+1}$, $(F_{3,2*2^r+3})^{-1}$, and $F_{2,2*2^r+2}$, giving us $F_{1,2^{r+1}+1}$, $F_{3,2^{r+1}+3}$, and $F_{2,2^{r+1}+2} \in M^{r+1}$. □

Theorem 5.4 M has length t if M has the matrix size $n = 2^{t-1} + 1$. M is generated by three elements; F_{12}, F_{23}, F_{34}. This result is the best possible.

Proof. Let M be the matrix group of the nxn unipotent strictly upper triangular matrices U generated by the three elements, F_{12}, F_{23}, and F_{34}. M has a size n and is a unipotent strictly upper triangular matrix group. If $n \geq 2^{t-1}+1$ then $M^{t-1} \neq 0$ (because $F_{1,2^{t-1}+1}$ is the smallest element of M^{t-1}). If $n \geq 2^{t-1}+1$ and $n < 2^t+1$, then M has length t. If $n=2^{t-1}+1$, then n is the least possible n for M to have derived length t. This number matches the least possible n for U to have derived length t. Hence using three or more generators, $n = 2^{t-1}+1$ is the smallest n for a unipotent group to have derived length t. □

Table 5.5: Matrix Group results with three generators

Length (t)	**Minimum Matrix Size of M (n)**
1	2
2	3
3	5
4	9
5	17
t	$n = 2^{t-1} + 1$

CHAPTER 6: Lie Algebra General Case

We now consider the Lie algebra versions of the associative algebra results. We have four problems to consider:

1.) Find a good upper bound for the dimension of nilpotent Lie algebras with derived length t. In other words, for a given t, find as small an m as possible such that there is an algebra of dimension, m, with derived length, t.

2.) Answer the same question when restricted to a fixed number of generators.

3.) Answer the same question when the Lie algebras are subalgebras of strictly upper triangular matrices of size n (denoted again by T). Here the multiplication is commutator multiplication.

4.) For a given length, find the minimum n necessary.

In this investigation, we work exclusively with subalgebras of T. As before $T = d_1 \oplus d_2 \oplus \ldots \oplus d_{n-1}$ where the d_i are the upper diagonals of T. Again using matrix units, we compute the derived series of T:

$T^1 = d_2 \oplus d_3 \oplus \ldots \oplus d_{n-1}$

$T^2 = d_{2^2} \oplus d_5 \oplus \ldots \oplus d_{n-1}$

$\vdots$

$T^k = d_{2^k} \oplus d_{2^{k+1}} \oplus \ldots \oplus d_{n-1}$

Notice that $T^{t-1} \neq 0$ if and only if $2^{t-1} \leq n-1$ or $n \geq 2^{t-1}+1$ (diagonal $d_{2^{t-1}}$ has to exist).

The least n for T to have length t, is $n = 2^{t-1}+1$ and the dimension of $T = (2^{t-1}+1)2^{t-2}$, just as in the associative case. Again $E_{12}, E_{23}, \ldots, E_{n-1,n}$ are the generators of T, hence T has 2^{t-1} generators.

Lemma 6.1 For T to have length t, the smallest matrix size $n = 2^{t-1}+1$, the dimension of T $= (2^{t-1} + 1)2^{t-2}$, and T has 2^{t-1} generators.

CHAPTER 7: Lie Algebras with 3 Generators

In lemma 6.1 there are 2^{t-1} generators. Now we look at the case where there are three generators: F_{12}, F_{23}, and F_{34}. These generators are combinations of the elements on the first diagonal. We define:

$$F_{ij} = \sum_{k=0}^{j+3k\leq n} E_{i+3k,j+3k} \text{ where } i = 1,2,3 \text{ and } j > i.$$

Let L be generated by F_{12}, F_{23}, and F_{34}.

Using commutator multiplication, we find the derived series of the algebra. To make this clear we start with an example.

Example 7.1

Let n = 9 and let F_{12}, F_{23}, and F_{34} be generators of L in T. Let $d_j' = L \cap d_j$. $d_1' = \langle F_{12}, F_{23}, F_{34}\rangle$ where $F_{12} = E_{12} + E_{45} + E_{78}$, $F_{23} = E_{23} + E_{56} + E_{89}$; and $F_{34} = E_{34} + E_{67}$.

$$F_{12} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{pmatrix} \quad F_{23} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0 \end{pmatrix} \quad F_{34} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

Now we obtain the derived series of the algebra. We start by finding the elements of d_2'.

$[F_{12},F_{23}] = (F_{12} * F_{23}) – (F_{23} * F_{12}) =$

$[(E_{12} + E_{45} + E_{78}) * (E_{23} + E_{56} + E_{89})] – [(E_{23} + E_{56} + E_{89}) * (E_{12} + E_{45} + E_{78})] =$

$E_{13} + E_{46} + E_{79} = F_{13} \in L^1$

$[F_{12},F_{34}] = (F_{12} * F_{34}) – (F_{34} * F_{12}) =$

$[(E_{12} + E_{45} + E_{78}) * (E_{34} + E_{67})] – [(E_{34} + E_{67}) * (E_{12} + E_{45} + E_{78})] =$

$– E_{35} – E_{68} = –F_{35} \in L^1$

$[F_{23},F_{34}] = (F_{23} * F_{34}) – (F_{34} * F_{23}) =$

$$[(E_{23} + E_{56} + E_{89}) * (E_{34} + E_{67})] – [(E_{34} + E_{67}) * (E_{23} + E_{56} + E_{89})] =$$

$$E_{24} + E_{57} = F_{24} \in L^1$$

Resulting with $d_2' = <F_{13,} F_{24}, F_{35}>$. Then use these elements to find $F_{15}, F_{26}, F_{37} \in L^2$ where $d_4' = <F_{15}, F_{26}, F_{37}>$ and $F_{15} = E_{15} + E_{48}$, $F_{26} = E_{26} + E_{59}$, and $F_{37} = E_{37}$. Then use these elements to find $F_{19} \in L^3$ where $d_8' = < F_{19}>$ and $F_{19} = E_{19}$. Clearly $L^4 = [L^3, L^3] = 0$ and L has derived length 4. We have computed the linearly independent elements d_1', d_2', d_4', and d_8'. We find the other diagonals, starting with d_3':

$[F_{12},F_{13}] = (F_{12} * F_{13}) – (F_{13} * F_{12}) =$

$$[(E_{12} + E_{45} + E_{78}) * (E_{13} + E_{46} + E_{79})] – [(E_{13} + E_{46} + E_{79}) * (E_{12} + E_{45} + E_{78})] =$$

$$0$$

$[F_{12},F_{24}] = (F_{12} * F_{24}) – (F_{24} * F_{12}) =$

$$[(E_{12} + E_{45} + E_{78}) * (E_{24} + E_{57})] – [(E_{24} + E_{57}) * (E_{12} + E_{45} + E_{78})] =$$

$$E_{14} + E_{47} – E_{25} – E_{58} = F_{14} – F_{25}$$

$[F_{12},F_{35}] = (F_{12} * F_{35}) – (F_{35} * F_{12}) =$

$$[(E_{12} + E_{45} + E_{78}) * (E_{35} + E_{68})] – [(E_{35} + E_{68}) * (E_{12} + E_{45} + E_{78})] =$$

$$0$$

$[F_{23},F_{13}] = (F_{23} * F_{13}) – (F_{13} * F_{23})=$

$$[(E_{23} + E_{56} + E_{89}) * (E_{13} + E_{46} + E_{79})] – [(E_{13} + E_{46} + E_{79}) * (E_{23} + E_{56} + E_{89})] =$$

$$0$$

$[F_{23},F_{24}] = (F_{23} * F_{24}) – (F_{24} * F_{23}) =$

$$[(E_{23} + E_{56} + E_{89}) * (E_{24} + E_{57})] – [(E_{24} + E_{57}) * (E_{23} + E_{56} + E_{89})] =$$

$$0$$

$[F_{23},F_{35}] = (F_{23} * F_{35}) – (F_{35} * F_{23}) =$

$$[(E_{23} + E_{56} + E_{89}) * (E_{35} + E_{68})] – [(E_{35} + E_{68}) * (E_{23} + E_{56} + E_{89})] =$$

$$E_{25} + E_{58} – E_{36} – E_{69} = F_{25} – F_{36}$$

$[F_{34},F_{13}] = (F_{34} * F_{13}) – (F_{13} * F_{34}) =$

$$[(E_{34} + E_{67}) * (E_{13} + E_{46} + E_{79})] – [(E_{13} + E_{46} + E_{79}) * (E_{34} + E_{67})] =$$

$$E_{36} + E_{69} – E_{14} – E_{47} = F_{36} – F_{14}$$

$[F_{34},F_{24}] = (F_{34} * F_{24}) – (F_{24} * F_{34}) =$

$$[(E_{34} + E_{67}) * (E_{24} + E_{57})] – [(E_{24} + E_{57}) * (E_{34} + E_{67})] =$$

$$0$$

$[F_{34},F_{35}] = (F_{34} * F_{35}) – (F_{35} * F_{34}) =$

$$[(E_{34} + E_{67}) * (E_{35} + E_{68}) – [(E_{35} + E_{68}) * (E_{34} + E_{67})] =$$

$$0$$

leaving $d_3' = \langle[d_1',d_2']\rangle = \langle F_{14} – F_{25}, F_{25} – F_{36}, F_{36} – F_{14}\rangle$ where $F_{14} – F_{25} = E_{14} + E_{47} – E_{25} – E_{58}$, $F_{25} – F_{36} = E_{25} + E_{58} – E_{36} – E_{69}$, and $F_{36} – F_{14} = E_{36} + E_{69} – E_{14} – E_{47}$. Notice that $F_{36} – F_{14} = – (F_{25} – F_{36}) – (F_{14} – F_{25})$, therefore, $d_3' = \langle F_{14} – F_{25}, F_{25} – F_{36}\rangle$. Continuing this computation yields $d_5' = \langle[d_1',d_4'], [d_2',d_3']\rangle = \langle F_{16}, F_{27}, F_{38}\rangle$ where $F_{16} = E_{16} + E_{49}$, $F_{27} = E_{27}$, and $F_{38} = E_{38}$. $d_6' = \langle[d_1',d_5'], [d_2',d_4'], [d_3',d_3']\rangle = \langle F_{17} – F_{28}, F_{28} – F_{39}, F_{39} – F_{17}\rangle$ where $F_{17} – F_{28} = E_{17} – E_{28}$, $F_{28} – F_{39} = E_{28} – E_{39}$, and $F_{39} – F_{17} = E_{39} – E_{17}$, and since two of the elements in d_6 are a combination of the other element, $d_6' = \langle F_{17} – F_{28}, F_{28} – F_{39}\rangle$. $d_7' = \langle[d_1',d_6'], [d_2',d_5'], [d_3',d_4']\rangle = \langle F_{18}, F_{29}\rangle$ where $F_{18} = E_{18}$ and $F_{29} = E_{29}$. Now we have the derived series of L where $L = d_1' \oplus d_2' \oplus \ldots \oplus d_8'$, $L^1 = d_2' \oplus d_3' \oplus \ldots \oplus d_8'$, $L^2 = d_4' \oplus d_5' \oplus \ldots \oplus d_8'$, $L^3 = d_8'$, $L^4 = 0$. Thus L has the length 4 and $d_1' \oplus d_2' \oplus \ldots \oplus d_8'$ contains a basis for L, as demonstrated, and L has dimension 3+3+2+3+3+2+2+1 = 19. □

With Lie Algebras we are dealing with commutator multiplication where $[F_{i,p+i}, F_{j,q+j}] = F_{i,p+i} * F_{j,q+j} - F_{j,q+j} * F_{i,p+i}$. Let s = p+q.

Recall that:

$$F_{i,p+i} * F_{j,q+j} = \begin{cases} 0 & \text{, if i+p is not congruent to j mod 3} \\ F_{i,p+q+i} = F_{i,s+i} \text{ where } i = 1,2,\text{and } 3 & \text{, if i+p is congruent to j mod 3} \end{cases}$$

and

$$F_{j,q+j} * F_{i,p+i} = \begin{cases} 0 & \text{, if q+j is not congruent to i mod 3} \\ F_{j,q+p+j} = F_{j,s+j} \text{ where } j = 1,2,\text{and } 3 & \text{, if q+j is congruent to i mod 3} \end{cases}$$

Let M1 represent $F_{i,p+i} * F_{j,q+j}$ and let M2 represent $F_{j,q+j} * F_{i,p+i}$. In order for M1 ≠ 0, p+i is congruent to j mod 3 or 3 divides p+i–j. In order for M2 ≠ 0, q+j is congruent to i mod 3 or 3 divides q+j–i.

Lemma 7.2 If both M1 ≠ 0 and M2 ≠ 0 then s is a multiple of 3 where d_s is the diagonal s steps above the main diagonal.

Proof. If both M1 ≠ 0 and M2 ≠ 0 then 3 divides p+i–j and q+j–i. Since q = s–p, q+j–i = s–p+j–i = s–(p+i–j). Hence, s is a multiple of 3. □

Lemma 7.3 If s is a multiple of 3 then either both M1 ≠ 0 and M2 ≠ 0 or both M1 = 0 and M2 = 0.

Proof. Let s be a multiple of 3. Assume that M1 ≠ 0 (3 divides p+i–j) and M2 = 0 (3 does not divide q+j–i) and s = p–q. Then 3 divides p+i–j = s–q+i–j = s–(q+j–i). Since 3 divides s, this shows that 3 divides q+j–i which is a contradiction. Similarly, we get the same results if we start with M2 $\neq$ 0 and M1 = 0. □

Corollary 7.4 If s is not a multiple of 3, then one of the following holds where M1 represent $(F_{i,p+i} * F_{j,q+j})$ and M2 represent $(F_{j,q+j} * F_{i,p+i})$: M1 $\neq$ 0 & M2 = 0, M1 = 0 & M2 $\neq$ 0, or M1 = 0 & M2 = 0.

Lemma 7.5 $F_{1,2^s+1}, F_{2,2^s+2}, F_{3,2^s+3} \in L^s$ (provided the second subscript $\leq$ n).

Note that $d_2^{s}{}' = < F_{1,2^s+1}, F_{2,2^s+2}, F_{3,2^s+3}>$, therefore, we know that the diagonal we are dealing with is never a multiple of 3.

Proof. Base Case: Let s = 1. We start with L, where $d_1' = <F_{12}, F_{23}, F_{34}>$ and $F_{12} = E_{12} + E_{45} + \ldots + E_{3k+1,3k+2}$, $F_{23} = E_{23} + E_{56} + \ldots + E_{3k+2,3k+3}$, and $F_{34} = E_{34} + E_{67} + \ldots + E_{3k+3,3k+4}$ where k starts at zero, and the terms are within the matrix size.

Then

$[F_{12},F_{23}] = (F_{12} * F_{23}) - (F_{23} * F_{12}) = F_{13}$

$[F_{12},F_{34}] = (F_{12} * F_{34}) - (F_{34} * F_{12}) = -F_{35}$

$[F_{23},F_{34}] = (F_{23} * F_{34}) - (F_{34} * F_{23}) = F_{24}$

Hence $d_2' = <F_{13}, F_{24}, F_{35}>$ and $F_{1,2^1+1}, F_{2,2^1+2}, F_{3,2^1+3} \in L^1$.

Induction Hypothesis: let s = k where $F_{1,2^k+1}, F_{2,2^k+2}, F_{3,2^k+3} \in L^k$. We show that $F_{1,2^{k+1}+1}$, $F_{2,2^{k+1}+2}, F_{3,2^{k+1}+3} \in L^{k+1}$. We compute

$[F_{1,2^k+1}, F_{2,2^k+2}]$,

$[F_{1,2^k+1}, F_{3,2^k+3}]$, and

$[F_{2,2^k+2}, F_{3,2^k+3}]$.

In $[F_{i,2^k+i}, F_{j,2^k+j}] = (F_{i,2^k+i} * F_{j,2^k+j}) - (F_{j,2^k+j} * F_{i,2^k+i})$ where i = 1,2; j = 2,3; i < j, $(F_{i,2^k+i} * F_{j,2^k+j})$ will be nonzero if 2^k+i is congruent to j mod 3 and $(F_{j,2^k+j} * F_{i,2^k+i})$ will be nonzero if 2^k+j is congruent to i mod 3, but at no time will these results both be nonzero (corollary 7.4). One of them will be non-zero since either i-j is congruent to 2^k mod 3 or j-i is congruent to 2^k mod 3.

Thus either $(F_{i,2^k+i} * F_{j,2^k+j}) = F_{i,2^k+2^k+i} = F_{i,2^{k+1}+i}$ or $(F_{j,2^k+j} * F_{i,2^k+i}) = F_{j,2^k+2^k+j} = F_{j,2^{k+1}+j}$. Therefore $d_2^{(k+1)'} = \langle F_{1,2^{k+1}+1}, F_{2,2^{k+1}+2}, F_{3,2^{k+1}+3}\rangle$ and $F_{1,2^{k+1}+1}, F_{2,2^{k+1}+2}, F_{3,2^{k+1}+3} \in L^{k+1}$. □

Theorem 7.6 L has length t if L has the matrix size $n = 2^{t-1} + 1$. L is generated by three elements; F_{12}, F_{23}, F_{34}. This result is the best possible for three generators.

Proof. Let L be the Lie subalgebra of the nxn strictly upper triangular matrices T generated by three elements, F_{12}, F_{23}, F_{34}. If $n \geq 2^{t-1}+1$ then $L^{t-1} \neq 0$ (because $F_{1,2^{t-1}+1}$ is in L^{t-1}). If $n \geq 2^{t-1}+1$ and $n < 2^t+1$ then L has length t. If $n = 2^{t-1}+1$, then n is the least possible n for L to have derived length t. This number matches the least possible n for T to have derived length t and thus $L \subseteq T$. Hence for any number of generators ≥ 3, $n = 2^{t-1}+1$ is the smallest n to obtain derived length t. □

Table 7.7: Lie Algebra results with three generators

Length (t)	Minimum Matrix Size of L (n)
1	2
2	3
3	5
4	9
5	17
t	$n = 2^{t-1}+1$

We now look at the dimension of L.

Lemma 7.8 $L = d_1' \oplus d_2' \oplus \ldots \oplus d_{n-1}'$ where $d_i' = L \cap d_i$. The dimension of $d_s' = 3$ if s is not congruent to 0 mod 3 and $s \neq n-2$ or $n-1$. The dimension of $d_s' = 2$ if s is congruent to 0 mod 3 and $s \neq n-2$ or $n-1$. The dimension of $d_{n-2}' = 2$ and dimension of $d_{n-1}' = 1$.

Proof. We will first look at the dimension of d_s' where s is not congruent to 0 mod 3, hence s is not a multiple of 3. d_s' is computed from $[d_p', d_q']$ where $s = p+q$, and we calculate d_s' with $[F_{i,p+i}, F_{j,q+j}] = (F_{i,p+i} * F_{j,q+j}) - (F_{j,q+j} * F_{i,p+i}) = F_{i,p+q+i} = F_{i,s+i}$ or $-F_{j,p+q+j} = F_{i,s+i}$ where i=1, 2, and 3, and $d_s' = \langle F_{1,s+1}, F_{2,s+2}, F_{3,s+3} \rangle$. Therefore, the dimension of d_s' is 3 when s is not congruent to 0 mod 3 and $s \neq n-2$ or $n-1$.

Now we look at the dimension of d_s' where s is congruent to 0 mod 3, hence s is a multiple of 3. We compute d_s' with the calculation $[F_{i,p+i}, F_{j,q+j}] = (F_{i,p+i} * F_{j,q+j}) - (F_{j,q+j} * F_{i,p+i}) = (F_{i,p+q+i} - F_{j,p+q+j}) = (F_{i,s+i} - F_{j,s+j})\delta_{p+i,j}$. If i = 1, p = 1, j = 2, q = s-1 this equals $F_{1,s+1} - F_{2,s+2}$. If i = 2, p = 1, j = 3, q = s-1 this equals $F_{2,s+2} - F_{3,s+3}$. If i = 1, p =2, j = 3, q = s-2 this equals $F_{1,s+1} - F_{3,s+3}$. Where $d_s' = \langle F_{1,s+1} - F_{2,s+2}, F_{2,s+2} - F_{3,s+3}, F_{1,s+1} - F_{3,s+3} \rangle$. Only two of these are linearly independent hence $d_s' = \langle F_{1,s+1} - F_{2,s+2}, F_{2,s+2} - F_{3,s+3} \rangle$ and the dimension of $d_s' = 2$ when s is congruent to 0 mod 3 and $s \neq n-2$ or $n-1$.

The case of d_{n-2}' and d_{n-1}' are easily checked to be $d_{n-2}' = \langle F_{1,n-1}, F_{2,n} \rangle$ where $F_{1,n-1} = E_{1,n-1}$ and $F_{2,n} = E_{2,n}$ and $d_{n-1}' = \langle F_{1,n} \rangle$ where $F_{1,n} = E_{1,n}$ remembering that in the last diagonal there is only one element in the matrix and in the second to last diagonal there are 2 elements in the matrix. Hence, the results hold. □

Lemma 7.9 The dimension of $L = \begin{cases} (k-1)*8+3 \text{ if } n=3k \\ (k-1)*8+6 \text{ if } n=3k+1 \\ (k-1)*8+9 \text{ if } n=3k+2 \end{cases}$

Proof. $L = d_1' \oplus d_2' \oplus \ldots \oplus d_{n-1}'$, Dimension of L = 3+3+2+3+3+2…+2+1 where there are n–1 terms in the sum. First consider the case when we have a matrix of size n=3k we know the matrix size is a multiple of 3 and that there are n–1 diagonals above the main diagonal, breaking down to k–1 sets of consecutive diagonals that have dimension 8 = (3+3+2), d_{n-2}' with dimension 2, and d_{n-1}' with dimension 1. Making the total dimension of L = (k–1)*8+3. In the case where n = 3k+1 break down to k–1 sets of diagonals that have dimension 8, d_{n-3}' where n–3 is not a multiple of 3 because 3k+1–3 mod 3 is not 0 leaving d_{n-3}' with dimension 3, d_{n-2}' with dimension 2, and d_{n-1}' with dimension 1. Making the total dimension of L = (k–1)*8+6. The last case where n = 3k+2 breaks down to k–1 sets of diagonals that have dimension 8, d_{n-4}' where n–4 is not a multiple of 3 because 3k+2–4 mod 3 is not 0 leaves d_{n-4}' with dimension 3, d_{n-3}' where n–3 is not a multiple of 3 because 3k+2–3 mod 3 is not 0 leaves d_{n-3}' with dimension 3, d_{n-2}' with dimension 2, and d_{n-1}' with dimension 1. Making the total dimension of L= (k–1)*8+9.□

We can use the 9x9 Lie Algebra, Example 7.1, to check this lemma. In the 9x9 example we showed that the dimension of L = 3+3+2+3+3+2+2+1 = 19. Using lemma 7.9 when n = 9 we see we have the case n = 3k where k = 3 and find the dimension to be the same as in our example (k-1) * 8 + 3 = (3–1) * 8 + 3 = 19.

We now find the dimension of L in Theorem 7.6. Recall that for a given length t, Theorem 7.6 gives that the minimum matrix size needed for L to have length t is n = $2^{t-1}+1$.

Proposition 7.10 If t is even, then n = 3k ($n \equiv 0 \bmod 3$), and if t is odd then n = 3k+2 ($n \equiv 2 \bmod 3$).

Proof. Induct on d.

If t = 2 then matrix size is $2^{t-1}+1 = 3$ where n = 3k with k = 1. Note that 3 mod 3 = 0.

If t = 3 then matrix size is $2^{t-1}+1 = 5$ where k = 1 and we use n = 3k+2. Note that 5 mod 3 = 2. We see that $2^{t-1+2}+1 \equiv 3*2^{t-1}+2^{t-1}+1 \equiv 2^{t-1}+1$ mod 3. Hence all even (odd) t have their corresponding n's congruent mod 3. □

Proposition 7.10 is easier to understand by looking at some examples. For the even case we start with t = 2. Then $2^{2-1+2}+1 \equiv 3*2^{2-1}+2^{2-1}+1 \equiv 2^{2-1}+1$ mod 3 = 3 mod 3 = 0. To check this we take t = 4, where $2^{t-1}+1 = 9$ and 9 mod 3 is 0. Notice that when t = 2 and n = 3, $n \equiv 0$ mod 3; and when t = 4 and n = 9, $n \equiv 0$ mod 3 showing that the even t have an $n \equiv 0$ mod 3. We can also show an odd example where we start with t = 3. Then $2^{3-1+2}+1 \equiv 3*2^{3-1}+2^{3-1}+1 \equiv 2^{3-1}+1$ mod 3 = 5 mod 3 = 2. To check this we take t = 5, where $2^{t-1}+1 = 17$ and 17 mod 3 is 2. Notice that when t = 3 and n = 5, $n \equiv 2$ mod 3; and when t = 5 and n = 17, $n \equiv 2$ mod 3 showing that the odd t have an $n \equiv 2$ mod 3.

Theorem 7.11 There is a three generator Lie Algebra (L) of derived length t and dimension $\frac{2^{t+2}-7}{3}$ if t is even and $\frac{2^{t+2}-5}{3}$ if t is odd. These algebras are subalgebras of T of matrix size $2^{t-1}+1$.

Proof. For any t, n = $2^{t-1}+1$ is the smallest matrix size which will support algebras L of length t as seen earlier. If t is even, $n \equiv 0$ mod 3 and n = 3k and the dimension of L is 8(k–1) + 3. Using both $n = 2^{t-1}+1$ and n = 3k we get the dimension of L is 8(k–1) + 3 = $8(\frac{n}{3}-1) + 3 = 8(\frac{2^{t-1}+1}{3}-1)+3 = \frac{2^{t+2}-7}{3}$. If t is odd, $n \equiv 2$ mod 3 and n = 3k + 2 and the dimension of L is 8(k–1) + 9. Using both $n = 2^{t-1}+1$ and n = 3k + 2 we get the dimension of L = 8(k–1) + 9 = $8(\frac{n-2}{3}-1)+9 = 8(\frac{2^{t-1}+1-2}{3}-1)+9 = \frac{2^{t+2}-5}{3}$. □

Table 7.12: Lie algebra results including dimension with three generators

Length (t)	**Minimum Matrix Size of L**	**Dimension of L (t is even)**	**Dimension of L (t is odd)**
1	2		1
2	3	3	
3	5		9
4	9	19	
5	17		41
t	$n = 2^{t-1} + 1$	$\frac{2^{t+2} - 7}{3}$	$\frac{2^{t+2} - 5}{3}$

Table 7.13: Comparison chart with strictly upper triangular matrices general case (T), Lie algebra with three generators (L), and associative algebra with one generator (A)

Length (t)	**Minimum Matrix Size(T,A,L)**	**Dimension of T**	**Dimension of L**	**Dimension of A**
1	2	1	1	1
2	3	3	3	2
3	5	10	9	4
4	9	36	19	8
5	17	136	41	16
t	$n = 2^{t-1} + 1$	$(2^{t-1} + 1)2^{t-2}$	t even $\frac{2^{t+2} - 7}{3}$ t odd $\frac{2^{t+2} - 5}{3}$	2^{t-1}

Remark for t, we have found the smallest n such that T has a subalgebra of length t. The result is sharp. We have also found the dimension of these algebras. It is conceivable that the dimension could be lower, we do not know if our bound is sharp. It is also possible that there are nilpotent N, not just subalgebras of T, for which the dimension bound could be lowered.

CHAPTER 8: Lie Algebras with 2 Generators

For a given t, we have found the smallest n such that there is a 3 generated subalgebra of the strictly upper triangular nxn matrices which has derived length t. We now consider the same problem for the 2 generated subalgebras.

We start by finding a lower bound for n.

Theorem 8.1 If $n \leq \frac{5}{8}2^t$ then T does not have a 2 generated subalgebra with derived length t.

Proof. Let L be a 2 generated subalgebra of T. Let L = <A, B>. In order for the derived length of L to be t, $L^t = 0$ and $L^{t-1} \neq 0$.

When t = 2 we find that $n \leq \frac{5}{8}2^2 = \frac{5}{2}$, leaving n < 3. When n = 2 we have $L^{t-1} = L^1 = \langle [A, B] \rangle = L_2 = 0$. Hence, T does not have a 2 generated subalgebra with derived length t = 2 when $n \leq \frac{5}{8}2^t$.

When t = 3 we find that $n \leq 5$. When L = <A, B>, $L_2 = [A, B] + L_3$. Then:

$$
\begin{aligned}
&L^{t-1} = L^2 = [L^1, L^1] = [L_2, L_2] = \\
&[[A, B] + L_3, [A, B] + L_3] = \\
&[L_3, [A, B] + L_3] = \\
&[L_3, L_2] \subseteq L_5 = 0 \text{ since } n \leq 5.
\end{aligned}
$$

Hence T does not have a 2 generated subalgebra with derived length t = 3 when $n \leq \frac{5}{8}2^t$.

Using induction, we assume $L^{t-1} \subseteq L_{5 * 2^{t-3}}$ and compute $L^t = [L^{t-1}, L^{t-1}]$

$\subseteq [L_{5 * 2^{t-3}}, L_{5 * 2^{t-3}}] \subseteq L_{2(5 * 2^{t-3})} = L_{5 * 2^{t-2}}$. Now $L^{t-1} \subseteq L_{5 * 2^{t-3}}$ for all t. If $n \le \frac{5}{8}2^t$, then

$L_{5 * 2^{t-3}} = 0$ leaving $L^{t-1} = 0$. Hence, T does not have a 2 generated subalgebra of derived

length t when $n \le \frac{5}{8}2^t$. Therefore, in order for a 2 generated subalgebra to have derived

length t, $n > \frac{5}{8}2^t$, leaving us with a lower bound on n. □

Let L be a Lie algebra generated by x and y and let $z \in L$ such that z can be expressed as a product of the generators. Of all expressions of z as a product of generators, let w(z) be the number of generators in the shortest such expression of z. Then w(z) will be called the width of z.

Let F be a free Lie algebra generated by a and b. Let $F \supset F^1 \supset F^2 \supset \ldots$ be the derived series of F. The terms [a, b], [[a, b], a], [[a, b], b] are all in F^1 with [a, b] having the shortest width, w([a, b]) = 2. The pairwise products of these elements are in F^2 with the shortest width of any term being 5. These products are [[a, b], [[a, b], a]]; [[a, b], [[a, b], b]]; and [[[a, b], a], [[a, b], b]]. The products of these elements have widths 10, 11 and 11 and are the shortest elements in F^3. Taking products again, we obtain products of width 21, 21, and 22. The new products are in F^4 and the shortest width is $21 = 21 * 2^{4-4}$. The shortest width in F^5 is at least $42 = 21 * 2^{5-4}$ which would be obtained by multiplying two elements of width 21 from F^4. This process continues and we find that the shortest width in F^s is at least $21 * 2^{s-4}$.

Theorem 8.2 Let F be a free Lie algebra with 2 generators. Then the term with the shortest width in F^s is $[21 * 2^{s-4}]$ ([] represents the greatest integer function).

Since any 2 generated Lie algebra L is the homomorphic image of F, the shortest width element in L^n is at least as wide as the shortest width element in F^n. The approach we take is that for each derived length t, we construct a 2 generated Lie algebra of strictly upper triangular matrices that meets the bound. The generator matrices A and B will be sums of $E_{i,i+1}$ of the appropriate size with each $E_{i,i+1}$ appearing in A or B but not both A and B. We let $A \in T_{1,n}(\alpha_1,\alpha_2,...\alpha_{n-1})$ and $B \in T_{1,n}(\beta_1,\beta_2,...\beta_{n-1})$ where α_i and β_j are regarded as variables, and $T_{1,n}$ denotes the nxn matrices with non-zero elements only on the first diagonal along the main diagonal.

The following examples consider the cases for small values of t. Computations have been aided by the use of Maple (see Appendix A). As usual, t will be the solvable length, L is a subalgebra of $T = T_n$ where n is the size of the matrices and j stands for the width of the smallest non-zero term in L^{t-1}. We always need n = j + 1.

Example 8.3

Let t = 2. Using s = 1 in Theorem 8.2 we find that j = 2 and n ≥ 3. Let A and B be matrices of size n = 3 with non-zero entries only on the super diagonal and zero's elsewhere. Clearly for j = 2 we use [A, B] or [B, A]. Let $c_2(x, y) = [y, x]$ and consider $c_2(A, B) = [B, A]$ where

$$A = \begin{pmatrix} 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & \beta_1 & 0 \\ 0 & 0 & \beta_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Expanding using Maple, we find $c_2(A, B)_{1,3} = \beta_1 \alpha_2 - \alpha_1 \beta_2$. Let $m_2=\beta_1\alpha_2$ and let $\beta_1=\alpha_2=1$ and $\alpha_1=\beta_2=0$, we find $[c_2(A, B)]_{1,3} = 1$. Hence, for n = 3 the Lie algebra generated by A and B has t = 2. From the result on free Lie algebras, n = 3 is the best possible result. That is, n = 3 is the smallest size matrix that contains a two generated Lie algebra of solvable length 2.

Note: that the number of alphas, 1, in m_2 equals the number of A's in $c_2(A, B)$ and the number of betas, 1, in m_2 equals the number of B's in $c_2(A, B)$. □

Example 8.4

Let t = 3. Using s = 2 in Theorem 8.2 we find that j = 5 and n ≥ 6. Let A and B be matrices of size n = 6. Let $c_5(x, y) = [[[y, x], x], [y, x]]$ and consider $c_5(A, B) = [[[B, A], A], [B, A]]$ where A and B have non-zero entries only on the super diagonal and zero's elsewhere and the elements of this diagonal for A are $\alpha_1, \ldots \alpha_5$ and for B are $\beta_1 \ldots \beta_5$.

Expanding using Maple, we find $c_5(A, B)_{1,6} = 3\alpha_3\beta_1\alpha_2\beta_4\alpha_5 - 2\alpha_3\beta_1\alpha_2\alpha_4\beta_5 - 4\alpha_3\alpha_1\beta_2\beta_4\alpha_5 + 3\alpha_3\alpha_1\beta_2\alpha_4\beta_5 + \alpha_1\alpha_2\beta_3\beta_4\alpha_5 - \alpha_1\alpha_2\beta_3\alpha_4\beta_5 - \beta_1\alpha_2\alpha_5\beta_3\alpha_4 + \alpha_1\beta_2\alpha_5\beta_3\alpha_4$. Let $m_5 = \alpha_1\alpha_2\beta_3\beta_4\alpha_5$ and let $\alpha_1 = \alpha_2 = \beta_3 = \beta_4 = \alpha_5 = 1$ and the remaining α's and β's be zero, we find $[c_5(A, B)]_{1,6} = 1$. Hence, for n = 6 the Lie algebra generated by A and B has t = 3. From the result on free Lie algebras, n = 6 is the best possible result. That is, n = 6 is the smallest size matrix that contains a two generated Lie algebra of solvable length 3.

Note: that the number of alphas, 3, in m_5 equals the number of A's in $c_5(A, B)$ and the number of betas, 2, in m_5 equals the number of B's in $c_5(A, B)$. □

For the next case we also will use $c_5'(x, y) = [[[y, x], y], [y, x]]$ and consider $c_5'(A, B) = [[[B, A], B], [B, A]]$ where A and B have non-zero entries only on the super diagonal and zero's elsewhere and the elements of this diagonal for A are $\alpha_1, \ldots \alpha_5$ and for B are $\beta_1 \ldots \beta_5$.

Expanding using Maple, we find $c_5'(A, B)_{1,6} = 3\beta_3\beta_1\alpha_2\beta_4\alpha_5 - 4\beta_3\beta_1\alpha_2\alpha_4\beta_5 - 2\beta_3\alpha_1\beta_2\beta_4\alpha_5 + 3\beta_3\alpha_1\beta_2\alpha_4\beta_5 - \beta_1\beta_2\alpha_3\beta_4\alpha_5 + \beta_1\beta_2\alpha_3\alpha_4\beta_5 + \beta_1\alpha_2\beta_5\alpha_3\beta_4 - \alpha_1\beta_2\beta_5\alpha_3\beta_4$. Let

$m_5' = \beta_1\beta_2\alpha_3\alpha_4\beta_5$ and let $\beta_1 = \beta_2 = \alpha_3 = \alpha_4 = \beta_5 = 1$ and the remaining α's and β's be zero, we find $[c_5'(A, B)]_{1,6} = 1$.

Note: that the number of alphas, 2, in m_5' equals the number of A's in $c_5'(A, B)$ and the number of betas, 3, in m_5' equals the number of B's in $c_5'(A, B)$.

Example 8.5

Let $t = 4$. Using $s = 3$ in Theorem 8.2 we find that $j = 10$ and $n \geq 11$. Let A and B be matrices of size $n = 11$. Let $c_{10}(x, y) = [c_5'(x, y), c_5(x, y)] = [[[[y, x], y], [y, x]], [[[y, x], x], [y, x]]]$ and consider $c_{10}(A, B) = [c_5'(A, B), c_5(A, B)] = [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]$ where A and B have non-zero entries only on the super diagonal and zero's elsewhere and the elements of this diagonal for A are $\alpha_1, \ldots \alpha_{10}$ and for B are $\beta_1 \ldots \beta_{10}$.

Expanding using Maple, we find $[c_{10}(A,B)]_{1,11}$ contains $\beta_1\beta_2\alpha_3\alpha_4\beta_5\alpha_6\alpha_7\beta_8\beta_9\alpha_{10}$ and many other terms, all different from this one. Let $m_{10} = \beta_1\beta_2\alpha_3\alpha_4\beta_5\alpha_6\alpha_7\beta_8\beta_9\alpha_{10}$ and let $\beta_1 = \beta_2 = \alpha_3 = \alpha_4 = \beta_5 = \alpha_6 = \alpha_7 = \beta_8 = \beta_9 = \alpha_{10} = 1$ and the remaining α's and β's be zero, we find $[c_{10}(A,B)]_{1,11} = 1$. Hence, for $n = 11$ the Lie algebra generated by A and B has $t = 4$. From the result on free Lie algebras, $n = 11$ is the best possible result. That is, $n = 11$ is the smallest size matrix that contains a two generated Lie algebra of solvable length 4.

Note: that the number of alphas, 5, in m_{10} equals the number of A's in $c_{10}(A, B)$ and the number of betas, 5, in m_{10} equals the number of B's in $c_{10}(A, B)$. □

Example 8.6

Let $t = 5$. Using $s = 4$ in Theorem 8.2 we find that $j = 21$ and $n \geq 22$. Let A and B be matrices of size $n = 22$ with non-zero entries only on the supper diagonal and zero's elsewhere. Let

$\text{term1}(x, y) = [[[[y, x], x], x], [y, x]]$

term2(x, y) = [[[[y, x], x], y], [y, x]]

term3(x, y) = [[[[y, x], y], y], [y, x]]

and

c_{21}(x, y) = [[term1(x, y), c_5(x, y)],c_{10}(x, y)]

c_{21}' (x, y) = [term2(x, y), c_5(x, y)],c_{10}(x, y)]

c_{21}'' (x, y) = [term3(x, y), c_5(x, y)],c_{10}(x, y)]

and consider c_{21}(A, B)=[[term1(A,B), c_5(A,B)],c_{10}(A, B)] =

[[[[[[B, A], A], A], [B, A]], [[[B, A], A], [B, A]]], [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]].

We first look at term1(A,B) = [[[[B, A], A], A], [B, A]] where A and B are 7x7 matrices of the usual super diagonal form, we expand using Maple and we find $[\text{term1}(A,B)]_{1,7}$ contains $\alpha_1\alpha_2\alpha_3\beta_4\beta_5\alpha_6$ and many other terms, all different from this one. We consider c_5(A,B) = [[[B, A], A], [B, A]] where A and B are 6x6 matrices of the usual super diagonal form, we expand using Maple and we find $[c_5(A,B)]_{1,6}$ contains $\alpha_1\alpha_2\beta_3\beta_4\alpha_5$ and many other terms, all different from this one (see example 8.4). We consider c_{10}(A,B)=[[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]] where A and B are 11x11 matrices of the usual super diagonal form, we expand using Maple and we find $[c_{10}(A,B)]_{1,11}$ contains $\beta_1\beta_2\alpha_3\alpha_4\beta_5\alpha_6\alpha_7\beta_8\beta_9\alpha_{10}$ and many other terms, all different from this one (see example 8.5).

To build A and B of size 22, we are guided by the above discussion. In the upper 7x7 block we put the A and B dictated from calculation using term 1; that is, we let $\alpha_1=\alpha_2=\alpha_3=\beta_4=\beta_5=\alpha_6=1$ and the rest of the super diagonal terms be 0. For the next 5 terms we use the computations for c_5; namely $\alpha_1=\alpha_2=\beta_3=\beta_4=\alpha_5=1$, but add 6 to the subscript to put them in the correct part of A and B. Hence $\alpha_7=\alpha_8=\beta_9=\beta_{10}=\alpha_{11}=1$ and the rest are 0. For the last part we use the calculations from c_{10} and add 11 to the subscripts giving $\beta_{12}=\beta_{13}=\alpha_{14}=\alpha_{15}=\beta_{16}=\alpha_{17}=\alpha_{18}=\beta_{19}=\beta_{20}=\alpha_{21}=1$ and the rest are 0. Substituting A and B into term1(x,y), $c_5(x,y)$ and $c_{10}(x,y)$ yields $\pm$ 1 in postions (1,7), (7,12), and (12,22) in the respective matrices. Then $c_{21}(A,B)$ has ± 1 (actually -1) in the (1,22) position and zeros elsewhere. This can also be described as in the next paragraph.

To build the matrices of size 22, term1(A,B) has $\alpha_1\alpha_2\alpha_3\beta_4\beta_5\alpha_6$ in the (1,7) position, $c_5(A,B)$ has $\alpha_7\alpha_8\beta_9\beta_{10}\alpha_{11}$ in the (7,12) position, and $c_{10}(A,B)$ has $\beta_{12}\beta_{13}\alpha_{14}\alpha_{15}\beta_{16}\alpha_{17}\alpha_{18}\beta_{19}\beta_{20}\alpha_{21}$ in the (12,22) position. The large A and B matrices have been built in 3 steps. In the first 6 supper diagonal positions we put the α's and β's found in term1(A,B). In the next 5 positions we put the term found in $c_5(A,B)$ where the subscripts have been increased by 6. In the last 10 positions we use the term found in $c_{10}(A,B)$ with the subscripts increased by 11. Then $[c_{21}(A,B)]_{1,22}$ has the product of these terms as one of many summands. Let $m_{21}=\alpha_1\alpha_2\alpha_3\beta_4\beta_5\alpha_6\alpha_7\alpha_8\beta_9\beta_{10}\alpha_{11}\beta_{12}\beta_{13}\alpha_{14}\alpha_{15}\beta_{16}\alpha_{17}\alpha_{18}\beta_{19}\beta_{20}\alpha_{21}$ and let these values for α's and β's be 1 and the remaining α's and β's be zero, we find $[c_{21}(A,B)]_{1,22} = -1$. Hence, for n = 22 the Lie algebra generated by A and B has t = 5. From the result on free Lie algebras, n = 22 is the best possible result. That is, n = 22 is the smallest size matrix that contains a two generated Lie algebra of solvable length 5.

Note: that the number of alphas, 12, in m_{21} equals the number of A's in $c_{21}(A, B)$ and the number of betas, 9, in m_{21} equals the number of B's in $c_{21}(A, B)$. □

Similar computations are carried out for $c_{21}'(A,B)$ and $c_{21}''(A,B)$ with the results again being that $[c_{21}'(A,B)]_{1,22} = 1$ and $[c_{21}''(A,B)]_{1,22} = 1$ for the right choices of α's and β's. Note that the number of A's in $c_{21}'(A,B)$ is 11 and the number of A's in $c_{21}''(A,B)$ is

10, thus the number of A's in $c_{21}(A,B)$, $c_{21}'(A,B)$, and $c_{21}''(A,B)$ is congruent to 0, 1, and 2 mod 3 respectively. □

Theorem 8.7 T has a 2 generated subalgebra of derived length t if n = $[21*2^{t-5}] + 1$ and n is the smallest possible size for T.

Proof. The cases $t \leq 5$ has been shown in the previous examples. Theorem 8.2 shows that for any t the shortest width of an element in F^s is $[21*2^{s-4}]$. For such an element to be non-zero $n \geq [21*2^{s-4}] + 1$. Since s = t-1, $n \geq [21*2^{t-5}] + 1$ is the smallest possible n. It remains to show that we can find an example that shows that $n = [21*2^{t-5}] + 1$.

To construct these examples we need some preliminary lemmas. Let A and B be enlarged (2n-1 x 2n-1) matrices each of which is the sum of elementary matrices $E_{i,i+1}$ such that each $E_{i,i+1}$ is a summand in A or B but not both. Let m_{n-1} denote an element in the free Lie Algebra on two generators, α and β, such that m_{n-1} is a single term with n-1 factors. We compute the $c_{n-1}(A,B)$ elements using [A,B] = AB-BA and when expanded, each associative product consists of n-1 terms of the $E_{i,i+1}$. In order for one of these products to be $E_{1,n}$, the terms must be $E_{1,2}\, E_{2,3} \ldots E_{n-1,n}$ and similarly for $-E_{1,n}$. Any terms containing $E_{i,i+1}$ where $i \geq n$ will not contribute to the scalar in the (1,n) position.

The following descriptions of A and B are needed for the next two lemmas: Let A_1 and B_1 be the upper nxn blocks of matrix A and B respectively, and let A_2 and B_2 be the lower nxn blocks of matrix A and B respectively.

Note: A and B are (2n-1 x 2n-1) size matrices, so the blocks of A_1 and A_2 overlap in row n and column n. For example, if we look at the case where n = 2 we have

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ where } A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Lemma 8.8 $c_{n-1}(A,B)$ has the same element in the (1,n) position as does $c_{n-1}(A_1,B_1)$.

Lemma 8.9 $c_{n-1}(A,B)$ has the same element in the (n, 2n-1) position as does $c_{n-1}(A_2,B_2)$.

Suppose that $c_{n-1}(A,B)$ has A as a factor r times. In order that the coefficient of $E_{1,n}$ is not 0 when we expand $c_{n-1}(A,B)$ the $E_{i,i+1}$, $i<n$ must be distributed such that r come from A and the remainder from B. If $r \neq |\{ E_{i,i+1} / E_{i,i+1} \text{ from A}, i<n\}|$ then the coefficient of $E_{1,n}$ is 0 in the $c_{n-1}(A,B)$. In our example 8.3 we had the matrix calculation of [B,A] where A and B had size n=3, and the $c_2(A,B)$ yielding $E_{1,3}$ was $m_2 = \beta_1\alpha_2$ where the number of α's equaled the number of A's in the calculation and the number of β's equaled the number of B's in the calculation. In particular, if m_{n-1}' and m_{n-1}'' have a different number of α's as factors, then the coefficient of $E_{1,n}$ must be 0 in one of them (at least). Similar remarks hold for the coefficient of $E_{n,2n-1}$.

Lemma 8.10 If $E_{1,n}$ has a non-zero coefficient in $c_{n-1}(A,B)$, then $|\{E_{i,i+1} / i<n, E_{i,i+1} \text{ in A}\}|$ = the number of times α is a factor of m_{n-1}.

Similar if $E_{1,n}$ has a non-zero coefficient in $c_{n-1}(A,B)$, then $|\{E_{i,i+1} / i<n, E_{i,i+1} \text{ in B}\}|$ = the number of times β is a factor of m_{n-1}.

Lemma 8.11 If $E_{n,2n-1}$ has a non-zero coefficient in $c_{n-1}(A,B)$, then $|\{E_{i,i+1} / i<n, E_{i,i+1} \text{ in A}\}|$ = the number of times α is a factor of m_{n-1}.

Similar if $E_{n,2n-1}$ has a non-zero coefficient in $c_{n-1}(A,B)$, then $|\{E_{i,i+1} / i<n, E_{i,i+1} \text{ in B}\}|$ = the number of times β is a factor of m_{n-1}.

Lemma 8.12 If $E_{1,n}$ has a non-zero coefficient in $c_{n-1}(A,B)$ then $E_{1,n}$ has a zero coefficient in $c_{n-1}'(A,B)$ if the number of times α is a factor of m_{n-1} and m_{n-1}' are different. The same remark holds for $E_{n,2n-1}$.

In example 8.6, when t = 5

$c_{21}(A, B) =$ [[term1, $c_5(A, B)$],$c_{10}(A, B)$] =

[[[[[[B, A], A], A], [B, A]], [[[B, A], A], [B, A]]], [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]]

c_{21}' (A, B) = [term2, $c_5(A, B)$],$c_{10}(A, B)$] =

[[[[[[B, A], A], B], [B, A]], [[[B, A], A], [B, A]]], [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]]

$c_{21}''(A, B)$ = [term3, $c_5(A, B)$],$c_{10}(A, B)$] =

[[[[[[B, A], B], B], [B, A]], [[[B, A], A], [B, A]]], [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]] .

where each matrix has the size 22. We chose the α's and β's that yield the $E_{1,22} = \pm 1$.

They are:

$m_{21} = \alpha_1\alpha_2\alpha_3\beta_4\beta_5\alpha_6\alpha_7\alpha_8\beta_9\beta_{10}\alpha_{11}\beta_{12}\beta_{13}\alpha_{14}\alpha_{15}\beta_{16}\alpha_{17}\alpha_{18}\beta_{19}\beta_{20}\alpha_{21}$,

where the number of α's = 12;

$m_{21}' = \alpha_1\alpha_2\beta_3\beta_4\beta_5\alpha_6\alpha_7\alpha_8\beta_9\beta_{10}\alpha_{11}\beta_{12}\beta_{13}\alpha_{14}\alpha_{15}\beta_{16}\alpha_{17}\alpha_{18}\beta_{19}\beta_{20}\alpha_{21}$,

where the number of α's = 11; and

$m_{21}'' =\beta_1\beta_2\beta_3\alpha_4\alpha_5\beta_6\alpha_7\alpha_8\beta_9\beta_{10}\alpha_{11}\beta_{12}\beta_{13}\alpha_{14}\alpha_{15}\beta_{16}\alpha_{17}\alpha_{18}\beta_{19}\beta_{20}\alpha_{21}$,

where the number of α's = 10.

Individually for each case m_{21}, m_{21}' and m_{21}'' the alphas show which entries above the main diagonal of matrix A are 1's with the rest of the entries being 0 and the betas show which entries above the main diagonal of matrix B are 1's with the rest of the entries 0 in order for each case to yield a coefficient of 1 or –1 for $E_{1,22}$. We note that if we choose the α's and β's from m_{21} we will get $E_{1,22}$ to have coefficient 1 but m_{21}' and m_{21}'' will both have coefficient 0 for $E_{1,22}$.

Suppose that in our original nxn size matrices, A and B are determined from m_{21}, A' and B' are determined from m_{21}', and A'' and B'' are determined from m_{21}''. When we consider the case when t = 6 we expand A and B to 2n-1 x 2n-1 matrices. We use A_i and

B_i to represent nxn blocks of the 2n-1 x 2n-1 matrices A and B respectively. We let A_1 be the nxn matrix A determined from m_{21} and B_1 be the nxn matrix B determined from m_{21}, A_2 be the nxn matrix A' determined from m_{21}' and B_2 be the nxn matrix B' determined from m_{21}', and A_3 be the nxn matrix A'' determined from m_{21}'' and B_3 be the nxn matrix B'' determined from m_{21}''.

The elementary matrices (A's and B's) for t = 5 with size n = 22 are enlarged as to size 2n–1 = 43 whereas the enlarging of $E_{i,i+1}$ becomes $E_{n+i,n+i+1}$. Formally, let $W_i = \{j$ / $E_{j,j+1}$ is a summand for A_i } with i=1, 2, 3 and $X_i = \{j$ / $E_{j,j+1}$ is a summand for B_i } with i=1, 2, 3. In the following the elementary matrices are 2n–1 by 2n–1.

Let $A = \sum_{i\in W_1} E_{i,i+1} + \sum_{i\in W_2} E_{n-1+i,n+i}$, $B = \sum_{i\in X_1} E_{i,i+1} + \sum_{i\in X_2} E_{n-1+i,n+i}$, and $c_{42}(x,y) =$

$[c_{21}(x,y), c_{21}'(x,y)]$. Then $[c_{42}(A,B)]_{1,43} = \pm E_{1,43}$, and the number of alphas in m_{42} is 23, where m_{42} represent the α's and β's chosen so that $[c_{42}(A,B)]_{1,43} = \pm 1$.

Using A and B we are assured a ± 1 in the $E_{1,43}$ position for the following reason: The coefficient of $E_{1,22}$ is 1 in $c_{21}(A, B)$ (from the t = 5 example 8.6), and the coefficient of $E_{1,22}$, by lemma 8.12, is 0 in $c_{21}'(A,B)$. The coefficient of $E_{22,43}$ is 1 in $c_{21}'(A,B)$, and the coefficient of $E_{22,43}$, by lemma 8.12, is 0 in $c_{21}(A, B)$. This leaves a 1 in the $E_{1,43}$ position.

Let $A' = \sum_{i\in W_3} E_{i,i+1} + \sum_{i\in W_1} E_{n-1+i,n+i}$, $B' = \sum_{i\in X_3} E_{i,i+1} + \sum_{i\in X_1} E_{n-1+i,n+i}$, and $c_{42}'(x,y) =$

$[c_{21}''(x,y), c_{21}(x,y)]$. Then $[c_{42}'(A', B')]_{1,43} = \pm E_{1,43}$, and the number of alphas in m_{42}' is 22, where m_{42} represent the α's and β's chosen so that $[c_{42}'(A',B')]_{1,43} = \pm 1$.

Using A' and B' we are assured a ± 1 in the $E_{1,43}$ position for the following reason: The coefficient of $E_{1,22}$ is 1 in $c_{21}''(A, B)$, and the coefficient of $E_{1,22}$, by lemma 8.12, is

0 in $c_{21}(A,B)$. The coefficient of $E_{22,43}$ is 1 in $c_{21}(A,B)$, and the coefficient of $E_{22,43}$, by lemma 8.12, is 0 in $c_{21}''(A, B)$. This leaves a 1 in the $E_{1,43}$ position.

Let $A'' = \sum_{i\in W_2} E_{i,i+1} + \sum_{i\in W_3} E_{n-1+i,n+i}$, $B'' = \sum_{i\in X_2} E_{i,i+1} + \sum_{i\in X_3} E_{n-1+i,n+i}$, and $c_{42}''(x,y) = [c_{21}'(x,y), c_{21}''(x,y)]$. Then $[c_{42}''(A'', B'')]_{1,43} = \pm E_{1,43}$, and the number of alphas in m_{42}'' is 21, where m_{42}'' represents the alpha's and beta's to obtain $[c_{42}''(A'', B'')]_{1,43} = \pm 1$.

Using A'' and B'' we are assured a ± 1 in the $E_{1,43}$ position for the following reason: The coefficient of $E_{1,22}$ is 1 in $c_{21}'(A, B)$, and the coefficient of $E_{1,22}$, by lemma 8.12, is 0 in $c_{21}''(A,B)$. The coefficient of $E_{22,43}$ is 1 in $c_{21}''(A,B)$, and the coefficient of $E_{22,43}$, by lemma 8.12, is 0 in $c_{21}'(A, B)$. This leaves a 1 in the $E_{1,43}$ position.

With these results, we look at the number of alphas occurring in m_{42}'', m_{42}', and m_{42} where the number of alphas is congruent to 0, 1, and 2 modulo 3 respectively. The same remark held for m_{21}, m_{21}', and m_{21}'', in the later case t = 5. Thus a recursive process follows in which going from t to t+1 when $t \geq 5$, we obtain $n = 21(2^{t-5}) + 1$, the smallest n such that L has a derived length of t. □

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APPENDIX

APPENDIX

We are always trying to get the smallest n such that G has length t. Set j = j(t) where j represents the number of terms needed for the length to be t.

We start out with t = 2 and we want to find the smallest amount of terms so that G^1 exists and G^2 = 0. We note that the smallest about of terms is j = 2 where we can use either [A, B] or [B, A] yielding the smallest n, n = 3, for the length t = 2.

We will use the c2(A,B) = [B, A] and look at the [1, 3] position to make sure it exists.

```
> restart:with(linalg):
```

```
Warning, new definition for norm
```

```
Warning, new definition for trace
```

```
> A:=matrix(3,3,0):
> for i from 1 by 1 to 2 do A[i,i+1] :=alpha(i):  od:
> print(A);
```

$$\begin{bmatrix} 0 & \alpha(1) & 0 \\ 0 & 0 & \alpha(2) \\ 0 & 0 & 0 \end{bmatrix}$$

```
> B:=matrix(3,3,0):
> for i from 1 by 1 to 2 do B[i,i+1] :=beta(i):  od:
> print(B);
```

$$\begin{bmatrix} 0 & \beta(1) & 0 \\ 0 & 0 & \beta(2) \\ 0 & 0 & 0 \end{bmatrix}$$

```
> BA:=evalm((B&*A)- (A&*B)):
```

get [B,A] position *1,3*

```
> BA13:=simplify(BA[1,3]);
```

$$BA13 := \beta(1)\,\alpha(2) - \alpha(1)\,\beta(2)$$

Let m2 = $\beta(1)\,\alpha(2)$ where m2 is an element of [B, A] sub in values of 1's and 0's as appropriate and check to make sure position 1,3 of [B,A] exits

```
> beta(1):=1:alpha(2):=1:alpha(1):=0:beta(2):=0:
> print(BA13);
```

$$1$$

This shows that we have found the smallest n, n = 3, where G has length 2. G^1 exists and G^2 = 0 and j = 2. Notice we choice [B, A] but [A, B] would have yield the same results.

Note that the number of alphas in m2, 1, equals the number of A's in c2(A, B) = [B, A] and the number of betas in m2, 1, equals the number of B's in c2(A, B)

We now go on to when t =3 and we have to find the least amount of terms needed to make G^2 exist and G^3 = 0. We know that G^1 has two 2 term combos but we cannot use them to get a 4 term combo because the multiplications will cancel themselves out so we need to use a combo of a two term element of G^1 and a 3 term element of G^1 to find the smallest number of terms. The smallest number of terms j = 5 is found using either the

2 term combo [A, B] or [B, A] and the 3 term combos [[A, B], B],[[B, A], B],[[A,B], A], or [[B, A], A]. Note that [[A, B], B] = [[B, A], B] and [[A, B], A] = [[B, A], A] and [A, B] = [B, A] so we choose the 5 term combos to be [[[B, A], A], [B, A]] and [[[B, A], B], [B, A]] yielding j = 5 and the smallest n, n=6, for the length t = 3.

We will use the 5 term combo of [[[B, A], A],[B, A]] and call it c5(A,B)

We are interested in looking at the [1, 6] position to make sure it exists. We will get c5[A, B] = [[[B,A], A], [B, A]] and check the [1, 6] position.

```
> restart:with(linalg):
```

```
Warning, new definition for norm
```

```
Warning, new definition for trace
```

```
> A:=matrix(6,6,0):
> for i from 1 by 1 to 5 do A[i,i+1] :=alpha(i):  od:
> print(A);
```

$$\begin{bmatrix} 0 & \alpha(1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha(2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha(3) & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha(4) & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha(5) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> B:=matrix(6,6,0):
> for i from 1 by 1 to 5 do B[i,i+1] :=beta(i):  od:
> print(B);
```

$$\begin{bmatrix} 0 & \beta(1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta(2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta(3) & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta(4) & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta(5) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> BA:=evalm((B&*A)- (A&*B)): BAA:=evalm((BA&*A)- (A&*BA)):
BAABA:=evalm((BAA&*BA)- (BA&*BAA)):
```

look at position 1,6 of c5[A,B] = [[[B, A], A], [B, A]]

```
> BAABA16:=simplify(BAABA[1,6]);
```

$$
\begin{aligned}
BAABA16 := {} & 3\,\alpha(3)\,\beta(1)\,\alpha(2)\,\beta(4)\,\alpha(5) - 2\,\alpha(3)\,\beta(1)\,\alpha(2)\,\alpha(4)\,\beta(5) \\
& - 4\,\alpha(3)\,\alpha(1)\,\beta(2)\,\beta(4)\,\alpha(5) + 3\,\alpha(3)\,\alpha(1)\,\beta(2)\,\alpha(4)\,\beta(5) \\
& + \alpha(1)\,\alpha(2)\,\beta(3)\,\beta(4)\,\alpha(5) - \alpha(1)\,\alpha(2)\,\beta(3)\,\alpha(4)\,\beta(5) - \beta(1)\,\alpha(2)\,\alpha(5)\,\beta(3)\,\alpha(4) \\
& + \alpha(1)\,\beta(2)\,\alpha(5)\,\beta(3)\,\alpha(4)
\end{aligned}
$$

let m5' = $\alpha(1)\,\alpha(2)\,\beta(3)\,\beta(4)\,\alpha(5)$

where m5' is an element of c5(A,B) = [[[B, A], A], [B, A]]. We choose our alphas and betas based on m5'

```
> alpha(1):=1:alpha(2):=1:beta(3):=1:beta(4):=1:alpha(5):=1:
> beta(1):=0:beta(2):=0:alpha(3):=0:alpha(4):=0:beta(5):=0:
```

sub in the values to get the [1,6] entry of [[[B, A], A], [B, A]]

```
> print(BAABA16);
```

$$1$$

This shows that we have found the smallest n, n = 6, where G has length 3. G^2 exists and G^3 = 0 and j = 5.

Note that the number of alphas in m5', 3, equals the number of A's in c5(A, B) = [[[B, A], A], [B, A]] and the number of betas in m5', 2, equals the number of B's in c5(A, B)

To go to the next case of t = 4 we need to first do the other case for when t = 3 and we have 5 terms using [[[B, A], B], [B, A]]. We get m5 which is an element of [[[B, A], B], [B, A]]

```
> restart;with(linalg):
```

```
Warning, new definition for norm
```

```
Warning, new definition for trace
```

```
> A:=matrix(6,6,0):
> for i from 1 by 1 to 5 do A[i,i+1] :=alpha(i):  od:
> print(A);
```

$$\begin{bmatrix} 0 & \alpha(1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha(2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha(3) & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha(4) & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha(5) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> B:=matrix(6,6,0):
> for i from 1 by 1 to 5 do B[i,i+1] :=beta(i):  od:
> print(B);
```

$$\begin{bmatrix} 0 & \beta(1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta(2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta(3) & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta(4) & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta(5) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> BA:=evalm((B&*A)- (A&*B)): BAB:=evalm((BA&*B)- (B&*BA)):
BABBA:=evalm((BAB&*BA)- (BA&*BAB)):
```

get the [1,6] position of [[[B, A], A], [B, A]]

```
> BABBA16:=simplify(BABBA[1,6]);
```

$$\begin{aligned} BABBA16 := {} & 3\,\beta(3)\,\beta(1)\,\alpha(2)\,\beta(4)\,\alpha(5) - 4\,\beta(3)\,\beta(1)\,\alpha(2)\,\alpha(4)\,\beta(5) \\ & - 2\,\beta(3)\,\alpha(1)\,\beta(2)\,\beta(4)\,\alpha(5) + 3\,\beta(3)\,\alpha(1)\,\beta(2)\,\alpha(4)\,\beta(5) \\ & - \beta(1)\,\beta(2)\,\alpha(3)\,\beta(4)\,\alpha(5) + \beta(1)\,\beta(2)\,\alpha(3)\,\alpha(4)\,\beta(5) + \beta(1)\,\alpha(2)\,\beta(5)\,\alpha(3)\,\beta(4) \\ & - \alpha(1)\,\beta(2)\,\beta(5)\,\alpha(3)\,\beta(4) \end{aligned}$$

choose m5 $= \beta(1)\,\beta(2)\,\alpha(3)\,\alpha(4)\,\beta(5)$ where m5 is an element of [[[B, A], B], [B, A]]

```
> beta(1):=1:beta(2):=1:alpha(3):=1:alpha(4):=1:beta(5):=1:
> alpha(1):=0:alpha(2):=0:beta(3):=0:beta(4):=0:alpha(5):=0:
```

sub in the values to get the [1,6] entry of [[[B, A], B], [B, A]]

```
> print(BABBA16);
```

Note that the number of alphas in m5, 2, equals the number of A's in [[[B, A], B], [B, A]] and the number of betas in m5, 3, equals the number of B's in [[[B, A], B], [B, A]] also note that the number of alphas and betas are different for the 2 different cases for t = 3.

Now we will do the case where t = 4. We would like to find the smallest n such that G^3 exists and G^4 = 0. Since G^2 had two terms of 5 that are different we know that G^3 smallest length of terms will be 10. We will use the two cases m5 and m5' (both with 5 terms) and both from G^2 to get the 10 term case, m10. For j = 10, the number of terms, we know that n = 11 and we will show that [1, 11] position exists and that n = 11 is the smallest matrix such that t = 4. We are interested in the [1,11] position and will use c10(A,B) = [[[[B, A], B], [B, A]],c5(A,B)] = [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]] to show this.

```
> restart;with(linalg):

Warning, new definition for norm

Warning, new definition for trace

> A:=matrix(11,11,0):

> for i from 1 by 1 to 10 do A[i,i+1] :=alpha(i):  od:

> print(A);
```

$$\begin{bmatrix} 0 & \alpha(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha(2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha(3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha(4) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha(5) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha(6) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha(7) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha(8) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha(9) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha(10) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> B:=matrix(11,11,0):

> for i from 1 by 1 to 10 do B[i,i+1] :=beta(i):  od:

> print(B);
```

$$
\begin{bmatrix}
0 & \beta(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \beta(2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta(3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \beta(4) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \beta(5) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \beta(6) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta(7) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta(8) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta(9) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta(10) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

```
> BA:=evalm((B&*A)- (A&*B)): BAA:=evalm((BA&*A)- (A&*BA)):
BAB:=evalm((BA&*B)- (B&*BA)):
> BAABA:=evalm((BAA&*BA)- (BA&*BAA)):
> BABBA:=evalm((BAB&*BA)- (BA&*BAB)):
> c10AB:=evalm((BABBA&*BAABA)- (BAABA&*BABBA)):
```

Get the [1,11] position of [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]

```
> c10AB111:=simplify(c10AB[1,11]);
```

$$
\begin{aligned}
c10AB111 := & -3\,\alpha(3)\,\beta(1)\,\alpha(2)\,\beta(4)\,\alpha(5)\,\beta(6)\,\beta(7)\,\alpha(8)\,\alpha(9)\,\beta(10) \\
& -3\,\alpha(3)\,\beta(1)\,\alpha(2)\,\beta(4)\,\alpha(5)\,\beta(6)\,\alpha(7)\,\beta(10)\,\alpha(8)\,\beta(9) \\
& +3\,\alpha(3)\,\beta(1)\,\alpha(2)\,\beta(4)\,\alpha(5)\,\alpha(6)\,\beta(7)\,\beta(10)\,\alpha(8)\,\beta(9) \\
& +6\,\alpha(3)\,\beta(1)\,\alpha(2)\,\alpha(4)\,\beta(5)\,\beta(8)\,\beta(6)\,\alpha(7)\,\beta(9)\,\alpha(10) \\
& -8\,\alpha(3)\,\beta(1)\,\alpha(2)\,\alpha(4)\,\beta(5)\,\beta(8)\,\beta(6)\,\alpha(7)\,\alpha(9)\,\beta(10) \\
& -4\,\alpha(3)\,\beta(1)\,\alpha(2)\,\alpha(4)\,\beta(5)\,\beta(8)\,\alpha(6)\,\beta(7)\,\beta(9)\,\alpha(10) \\
& +6\,\alpha(3)\,\beta(1)\,\alpha(2)\,\alpha(4)\,\beta(5)\,\beta(8)\,\alpha(6)\,\beta(7)\,\alpha(9)\,\beta(10) \\
& -2\,\alpha(3)\,\beta(1)\,\alpha(2)\,\alpha(4)\,\beta(5)\,\beta(6)\,\beta(7)\,\alpha(8)\,\beta(9)\,\alpha(10) \\
& +2\,\alpha(3)\,\beta(1)\,\alpha(2)\,\alpha(4)\,\beta(5)\,\beta(6)\,\beta(7)\,\alpha(8)\,\alpha(9)\,\beta(10) \\
& +2\,\alpha(3)\,\beta(1)\,\alpha(2)\,\alpha(4)\,\beta(5)\,\beta(6)\,\alpha(7)\,\beta(10)\,\alpha(8)\,\beta(9) \\
& -9\,\alpha(3)\,\beta(1)\,\alpha(2)\,\beta(4)\,\alpha(5)\,\beta(8)\,\beta(6)\,\alpha(7)\,\beta(9)\,\alpha(10) \\
& +12\,\alpha(3)\,\beta(1)\,\alpha(2)\,\beta(4)\,\alpha(5)\,\beta(8)\,\beta(6)\,\alpha(7)\,\alpha(9)\,\beta(10) \\
& +6\,\alpha(3)\,\beta(1)\,\alpha(2)\,\beta(4)\,\alpha(5)\,\beta(8)\,\alpha(6)\,\beta(7)\,\beta(9)\,\alpha(10) \\
& +3\,\beta(1)\,\alpha(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\beta(8)\,\alpha(6)\,\beta(7)\,\alpha(9)\,\beta(10) \\
& -3\,\alpha(1)\,\alpha(2)\,\beta(3)\,\beta(4)\,\alpha(5)\,\beta(8)\,\beta(6)\,\alpha(7)\,\beta(9)\,\alpha(10) \\
& +4\,\alpha(1)\,\alpha(2)\,\beta(3)\,\beta(4)\,\alpha(5)\,\beta(8)\,\beta(6)\,\alpha(7)\,\alpha(9)\,\beta(10) \\
& +2\,\alpha(1)\,\alpha(2)\,\beta(3)\,\beta(4)\,\alpha(5)\,\beta(8)\,\alpha(6)\,\beta(7)\,\beta(9)\,\alpha(10) \\
& -3\,\alpha(1)\,\alpha(2)\,\beta(3)\,\beta(4)\,\alpha(5)\,\beta(8)\,\alpha(6)\,\beta(7)\,\alpha(9)\,\beta(10) \\
& +\alpha(1)\,\alpha(2)\,\beta(3)\,\beta(4)\,\alpha(5)\,\beta(6)\,\beta(7)\,\alpha(8)\,\beta(9)\,\alpha(10) \\
& -\alpha(1)\,\alpha(2)\,\beta(3)\,\beta(4)\,\alpha(5)\,\beta(6)\,\beta(7)\,\alpha(8)\,\alpha(9)\,\beta(10) \\
& -\alpha(1)\,\alpha(2)\,\beta(3)\,\beta(4)\,\alpha(5)\,\beta(6)\,\alpha(7)\,\beta(10)\,\alpha(8)\,\beta(9) \\
& +3\,\alpha(3)\,\alpha(1)\,\beta(2)\,\alpha(4)\,\beta(5)\,\alpha(6)\,\beta(7)\,\beta(10)\,\alpha(8)\,\beta(9) \\
& +\alpha(1)\,\alpha(2)\,\beta(3)\,\beta(4)\,\alpha(5)\,\alpha(6)\,\beta(7)\,\beta(10)\,\alpha(8)\,\beta(9)
\end{aligned}
$$

$$- \beta(1)\,\alpha(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\beta(6)\,\beta(7)\,\alpha(8)\,\beta(9)\,\alpha(10)$$
$$- 9\,\alpha(3)\,\beta(1)\,\alpha(2)\,\beta(4)\,\alpha(5)\,\beta(8)\,\alpha(6)\,\beta(7)\,\alpha(9)\,\beta(10)$$
$$- \alpha(1)\,\beta(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\beta(6)\,\beta(7)\,\alpha(8)\,\alpha(9)\,\beta(10)$$
$$- \alpha(1)\,\beta(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\beta(6)\,\alpha(7)\,\beta(10)\,\alpha(8)\,\beta(9)$$
$$+ \alpha(1)\,\beta(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\alpha(6)\,\beta(7)\,\beta(10)\,\alpha(8)\,\beta(9)$$
$$+ 3\,\alpha(1)\,\alpha(2)\,\beta(3)\,\alpha(4)\,\beta(5)\,\beta(8)\,\beta(6)\,\alpha(7)\,\beta(9)\,\alpha(10)$$
$$- 4\,\alpha(1)\,\alpha(2)\,\beta(3)\,\alpha(4)\,\beta(5)\,\beta(8)\,\beta(6)\,\alpha(7)\,\alpha(9)\,\beta(10)$$
$$- 2\,\alpha(1)\,\alpha(2)\,\beta(3)\,\alpha(4)\,\beta(5)\,\beta(8)\,\alpha(6)\,\beta(7)\,\beta(9)\,\alpha(10)$$
$$+ 3\,\alpha(1)\,\alpha(2)\,\beta(3)\,\alpha(4)\,\beta(5)\,\beta(8)\,\alpha(6)\,\beta(7)\,\alpha(9)\,\beta(10)$$
$$- \alpha(1)\,\alpha(2)\,\beta(3)\,\alpha(4)\,\beta(5)\,\beta(6)\,\beta(7)\,\alpha(8)\,\beta(9)\,\alpha(10)$$
$$+ \alpha(1)\,\alpha(2)\,\beta(3)\,\alpha(4)\,\beta(5)\,\beta(6)\,\beta(7)\,\alpha(8)\,\alpha(9)\,\beta(10)$$
$$+ \alpha(1)\,\alpha(2)\,\beta(3)\,\alpha(4)\,\beta(5)\,\beta(6)\,\alpha(7)\,\beta(10)\,\alpha(8)\,\beta(9)$$
$$- \alpha(1)\,\alpha(2)\,\beta(3)\,\alpha(4)\,\beta(5)\,\alpha(6)\,\beta(7)\,\beta(10)\,\alpha(8)\,\beta(9)$$
$$+ 3\,\beta(1)\,\alpha(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\beta(8)\,\beta(6)\,\alpha(7)\,\beta(9)\,\alpha(10)$$
$$- 4\,\beta(1)\,\alpha(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\beta(8)\,\beta(6)\,\alpha(7)\,\alpha(9)\,\beta(10)$$
$$- 3\,\alpha(1)\,\beta(2)\,\beta(5)\,\alpha(3)\,\beta(4)\,\alpha(8)\,\beta(6)\,\alpha(7)\,\beta(9)\,\alpha(10)$$
$$+ 2\,\alpha(1)\,\beta(2)\,\beta(5)\,\alpha(3)\,\beta(4)\,\alpha(8)\,\beta(6)\,\alpha(7)\,\alpha(9)\,\beta(10)$$
$$+ \beta(1)\,\alpha(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\beta(6)\,\beta(7)\,\alpha(8)\,\alpha(9)\,\beta(10)$$
$$+ \beta(1)\,\alpha(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\beta(6)\,\alpha(7)\,\beta(10)\,\alpha(8)\,\beta(9)$$
$$- \beta(1)\,\alpha(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\alpha(6)\,\beta(7)\,\beta(10)\,\alpha(8)\,\beta(9)$$
$$- 3\,\alpha(1)\,\beta(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\beta(8)\,\beta(6)\,\alpha(7)\,\beta(9)\,\alpha(10)$$
$$+ 4\,\alpha(1)\,\beta(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\beta(8)\,\beta(6)\,\alpha(7)\,\alpha(9)\,\beta(10)$$
$$+ 2\,\alpha(1)\,\beta(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\beta(8)\,\alpha(6)\,\beta(7)\,\beta(9)\,\alpha(10)$$
$$- 3\,\alpha(1)\,\beta(2)\,\alpha(5)\,\beta(3)\,\alpha(4)\,\beta(8)\,\alpha(6)\,\beta(7)\,\alpha(9)\,\beta(10)$$
$$+ 9\,\beta(3)\,\alpha(1)\,\beta(2)\,\alpha(4)\,\beta(5)\,\alpha(8)\,\alpha(6)\,\beta(7)\,\alpha(9)\,\beta(10)$$

$$- 3\beta(1)\beta(2)\alpha(3)\beta(4)\alpha(5)\alpha(8)\alpha(6)\beta(7)\alpha(9)\beta(10)$$

$$- \beta(1)\beta(2)\alpha(3)\alpha(4)\beta(5)\alpha(6)\alpha(7)\beta(8)\alpha(9)\beta(10)$$

$$- \beta(1)\beta(2)\alpha(3)\alpha(4)\beta(5)\beta(6)\alpha(7)\alpha(10)\beta(8)\alpha(9)$$

$$+ \beta(1)\beta(2)\alpha(3)\alpha(4)\beta(5)\alpha(6)\beta(7)\alpha(10)\beta(8)\alpha(9)$$

$$+ 3\beta(1)\alpha(2)\beta(5)\alpha(3)\beta(4)\alpha(8)\beta(6)\alpha(7)\beta(9)\alpha(10)$$

$$- 2\beta(1)\alpha(2)\beta(5)\alpha(3)\beta(4)\alpha(8)\beta(6)\alpha(7)\alpha(9)\beta(10)$$

$$- 4\beta(1)\alpha(2)\beta(5)\alpha(3)\beta(4)\alpha(8)\alpha(6)\beta(7)\beta(9)\alpha(10)$$

$$+ 3\beta(1)\alpha(2)\beta(5)\alpha(3)\beta(4)\alpha(8)\alpha(6)\beta(7)\alpha(9)\beta(10)$$

$$+ \beta(1)\alpha(2)\beta(5)\alpha(3)\beta(4)\alpha(6)\alpha(7)\beta(8)\beta(9)\alpha(10)$$

$$- \beta(1)\alpha(2)\beta(5)\alpha(3)\beta(4)\alpha(6)\alpha(7)\beta(8)\alpha(9)\beta(10)$$

$$- \beta(1)\alpha(2)\beta(5)\alpha(3)\beta(4)\beta(6)\alpha(7)\alpha(10)\beta(8)\alpha(9)$$

$$+ 8\beta(3)\alpha(1)\beta(2)\beta(4)\alpha(5)\alpha(8)\alpha(6)\beta(7)\beta(9)\alpha(10)$$

$$+ \beta(1)\alpha(2)\beta(5)\alpha(3)\beta(4)\alpha(6)\beta(7)\alpha(10)\beta(8)\alpha(9)$$

$$+ \alpha(1)\beta(2)\alpha(5)\beta(3)\alpha(4)\beta(6)\beta(7)\alpha(8)\beta(9)\alpha(10)$$

$$- 2\beta(1)\alpha(2)\alpha(5)\beta(3)\alpha(4)\beta(8)\alpha(6)\beta(7)\beta(9)\alpha(10)$$

$$+ 3\alpha(3)\beta(1)\alpha(2)\beta(4)\alpha(5)\beta(6)\beta(7)\alpha(8)\beta(9)\alpha(10)$$

$$+ 2\beta(1)\beta(2)\alpha(3)\beta(4)\alpha(5)\alpha(8)\beta(6)\alpha(7)\alpha(9)\beta(10)$$

$$+ 4\beta(1)\beta(2)\alpha(3)\beta(4)\alpha(5)\alpha(8)\alpha(6)\beta(7)\beta(9)\alpha(10)$$

$$- 6\beta(3)\alpha(1)\beta(2)\beta(4)\alpha(5)\alpha(8)\alpha(6)\beta(7)\alpha(9)\beta(10)$$

$$- 2\beta(3)\alpha(1)\beta(2)\beta(4)\alpha(5)\alpha(6)\alpha(7)\beta(8)\beta(9)\alpha(10)$$

$$+ 2\beta(3)\alpha(1)\beta(2)\beta(4)\alpha(5)\alpha(6)\alpha(7)\beta(8)\alpha(9)\beta(10)$$

$$+ 2\beta(3)\alpha(1)\beta(2)\beta(4)\alpha(5)\beta(6)\alpha(7)\alpha(10)\beta(8)\alpha(9)$$

$$- 2\beta(3)\alpha(1)\beta(2)\beta(4)\alpha(5)\alpha(6)\beta(7)\alpha(10)\beta(8)\alpha(9)$$

$$+ 9\beta(3)\alpha(1)\beta(2)\alpha(4)\beta(5)\alpha(8)\beta(6)\alpha(7)\beta(9)\alpha(10)$$

$$- 6\beta(3)\alpha(1)\beta(2)\alpha(4)\beta(5)\alpha(8)\beta(6)\alpha(7)\alpha(9)\beta(10)$$

$$-12\beta(3)\alpha(1)\beta(2)\alpha(4)\beta(5)\alpha(8)\alpha(6)\beta(7)\beta(9)\alpha(10)$$

$$+3\beta(3)\beta(1)\alpha(2)\beta(4)\alpha(5)\alpha(6)\alpha(7)\beta(8)\beta(9)\alpha(10)$$

$$-3\beta(3)\beta(1)\alpha(2)\beta(4)\alpha(5)\alpha(6)\alpha(7)\beta(8)\alpha(9)\beta(10)$$

$$-3\beta(3)\beta(1)\alpha(2)\beta(4)\alpha(5)\beta(6)\alpha(7)\alpha(10)\beta(8)\alpha(9)$$

$$+3\beta(3)\beta(1)\alpha(2)\beta(4)\alpha(5)\alpha(6)\beta(7)\alpha(10)\beta(8)\alpha(9)$$

$$-12\beta(3)\beta(1)\alpha(2)\alpha(4)\beta(5)\alpha(8)\beta(6)\alpha(7)\beta(9)\alpha(10)$$

$$+8\beta(3)\beta(1)\alpha(2)\alpha(4)\beta(5)\alpha(8)\beta(6)\alpha(7)\alpha(9)\beta(10)$$

$$+16\beta(3)\beta(1)\alpha(2)\alpha(4)\beta(5)\alpha(8)\alpha(6)\beta(7)\beta(9)\alpha(10)$$

$$-12\beta(3)\beta(1)\alpha(2)\alpha(4)\beta(5)\alpha(8)\alpha(6)\beta(7)\alpha(9)\beta(10)$$

$$-4\beta(3)\beta(1)\alpha(2)\alpha(4)\beta(5)\alpha(6)\alpha(7)\beta(8)\beta(9)\alpha(10)$$

$$+4\beta(3)\beta(1)\alpha(2)\alpha(4)\beta(5)\alpha(6)\alpha(7)\beta(8)\alpha(9)\beta(10)$$

$$+4\beta(3)\beta(1)\alpha(2)\alpha(4)\beta(5)\beta(6)\alpha(7)\alpha(10)\beta(8)\alpha(9)$$

$$-4\beta(3)\beta(1)\alpha(2)\alpha(4)\beta(5)\alpha(6)\beta(7)\alpha(10)\beta(8)\alpha(9)$$

$$-6\beta(3)\alpha(1)\beta(2)\beta(4)\alpha(5)\alpha(8)\beta(6)\alpha(7)\beta(9)\alpha(10)$$

$$+4\beta(3)\alpha(1)\beta(2)\beta(4)\alpha(5)\alpha(8)\beta(6)\alpha(7)\alpha(9)\beta(10)$$

$$+9\beta(3)\beta(1)\alpha(2)\beta(4)\alpha(5)\alpha(8)\beta(6)\alpha(7)\beta(9)\alpha(10)$$

$$-6\beta(3)\beta(1)\alpha(2)\beta(4)\alpha(5)\alpha(8)\beta(6)\alpha(7)\alpha(9)\beta(10)$$

$$-12\beta(3)\beta(1)\alpha(2)\beta(4)\alpha(5)\alpha(8)\alpha(6)\beta(7)\beta(9)\alpha(10)$$

$$+9\beta(3)\beta(1)\alpha(2)\beta(4)\alpha(5)\alpha(8)\alpha(6)\beta(7)\alpha(9)\beta(10)$$

$$+3\alpha(3)\alpha(1)\beta(2)\alpha(4)\beta(5)\beta(6)\beta(7)\alpha(8)\beta(9)\alpha(10)$$

$$-3\alpha(3)\alpha(1)\beta(2)\alpha(4)\beta(5)\beta(6)\beta(7)\alpha(8)\alpha(9)\beta(10)$$

$$-3\alpha(3)\alpha(1)\beta(2)\alpha(4)\beta(5)\beta(6)\alpha(7)\beta(10)\alpha(8)\beta(9)$$

$$-\beta(1)\beta(2)\alpha(3)\beta(4)\alpha(5)\alpha(6)\alpha(7)\beta(8)\beta(9)\alpha(10)$$

$$+\beta(1)\beta(2)\alpha(3)\beta(4)\alpha(5)\alpha(6)\alpha(7)\beta(8)\alpha(9)\beta(10)$$

$$+\beta(1)\beta(2)\alpha(3)\beta(4)\alpha(5)\beta(6)\alpha(7)\alpha(10)\beta(8)\alpha(9)$$

$$- \beta(1)\,\beta(2)\,\alpha(3)\,\beta(4)\,\alpha(5)\,\alpha(6)\,\beta(7)\,\alpha(10)\,\beta(8)\,\alpha(9)$$

$$+ 3\,\beta(1)\,\beta(2)\,\alpha(3)\,\alpha(4)\,\beta(5)\,\alpha(8)\,\beta(6)\,\alpha(7)\,\beta(9)\,\alpha(10)$$

$$- 2\,\beta(1)\,\beta(2)\,\alpha(3)\,\alpha(4)\,\beta(5)\,\alpha(8)\,\beta(6)\,\alpha(7)\,\alpha(9)\,\beta(10)$$

$$- 4\,\beta(1)\,\beta(2)\,\alpha(3)\,\alpha(4)\,\beta(5)\,\alpha(8)\,\alpha(6)\,\beta(7)\,\beta(9)\,\alpha(10)$$

$$+ 3\,\beta(1)\,\beta(2)\,\alpha(3)\,\alpha(4)\,\beta(5)\,\alpha(8)\,\alpha(6)\,\beta(7)\,\alpha(9)\,\beta(10)$$

$$+ \beta(1)\,\beta(2)\,\alpha(3)\,\alpha(4)\,\beta(5)\,\alpha(6)\,\alpha(7)\,\beta(8)\,\beta(9)\,\alpha(10)$$

$$+ 3\,\beta(3)\,\alpha(1)\,\beta(2)\,\alpha(4)\,\beta(5)\,\alpha(6)\,\alpha(7)\,\beta(8)\,\beta(9)\,\alpha(10)$$

$$- 3\,\beta(3)\,\alpha(1)\,\beta(2)\,\alpha(4)\,\beta(5)\,\alpha(6)\,\alpha(7)\,\beta(8)\,\alpha(9)\,\beta(10)$$

$$- 3\,\beta(3)\,\alpha(1)\,\beta(2)\,\alpha(4)\,\beta(5)\,\beta(6)\,\alpha(7)\,\alpha(10)\,\beta(8)\,\alpha(9)$$

$$+ 3\,\beta(3)\,\alpha(1)\,\beta(2)\,\alpha(4)\,\beta(5)\,\alpha(6)\,\beta(7)\,\alpha(10)\,\beta(8)\,\alpha(9)$$

$$- 3\,\beta(1)\,\beta(2)\,\alpha(3)\,\beta(4)\,\alpha(5)\,\alpha(8)\,\beta(6)\,\alpha(7)\,\beta(9)\,\alpha(10)$$

$$+ 4\,\alpha(1)\,\beta(2)\,\beta(5)\,\alpha(3)\,\beta(4)\,\alpha(8)\,\alpha(6)\,\beta(7)\,\beta(9)\,\alpha(10)$$

$$- 2\,\alpha(3)\,\beta(1)\,\alpha(2)\,\alpha(4)\,\beta(5)\,\alpha(6)\,\beta(7)\,\beta(10)\,\alpha(8)\,\beta(9)$$

$$+ 12\,\alpha(3)\,\alpha(1)\,\beta(2)\,\beta(4)\,\alpha(5)\,\beta(8)\,\beta(6)\,\alpha(7)\,\beta(9)\,\alpha(10)$$

$$- 16\,\alpha(3)\,\alpha(1)\,\beta(2)\,\beta(4)\,\alpha(5)\,\beta(8)\,\beta(6)\,\alpha(7)\,\alpha(9)\,\beta(10)$$

$$- 8\,\alpha(3)\,\alpha(1)\,\beta(2)\,\beta(4)\,\alpha(5)\,\beta(8)\,\alpha(6)\,\beta(7)\,\beta(9)\,\alpha(10)$$

$$+ 12\,\alpha(3)\,\alpha(1)\,\beta(2)\,\beta(4)\,\alpha(5)\,\beta(8)\,\alpha(6)\,\beta(7)\,\alpha(9)\,\beta(10)$$

$$- 4\,\alpha(3)\,\alpha(1)\,\beta(2)\,\beta(4)\,\alpha(5)\,\beta(6)\,\beta(7)\,\alpha(8)\,\beta(9)\,\alpha(10)$$

$$+ 4\,\alpha(3)\,\alpha(1)\,\beta(2)\,\beta(4)\,\alpha(5)\,\beta(6)\,\beta(7)\,\alpha(8)\,\alpha(9)\,\beta(10)$$

$$+ 4\,\alpha(3)\,\alpha(1)\,\beta(2)\,\beta(4)\,\alpha(5)\,\beta(6)\,\alpha(7)\,\beta(10)\,\alpha(8)\,\beta(9)$$

$$- 4\,\alpha(3)\,\alpha(1)\,\beta(2)\,\beta(4)\,\alpha(5)\,\alpha(6)\,\beta(7)\,\beta(10)\,\alpha(8)\,\beta(9)$$

$$- 9\,\alpha(3)\,\alpha(1)\,\beta(2)\,\alpha(4)\,\beta(5)\,\beta(8)\,\beta(6)\,\alpha(7)\,\beta(9)\,\alpha(10)$$

$$+ 12\,\alpha(3)\,\alpha(1)\,\beta(2)\,\alpha(4)\,\beta(5)\,\beta(8)\,\beta(6)\,\alpha(7)\,\alpha(9)\,\beta(10)$$

$$+ 6\,\alpha(3)\,\alpha(1)\,\beta(2)\,\alpha(4)\,\beta(5)\,\beta(8)\,\alpha(6)\,\beta(7)\,\beta(9)\,\alpha(10)$$

$$- 9\,\alpha(3)\,\alpha(1)\,\beta(2)\,\alpha(4)\,\beta(5)\,\beta(8)\,\alpha(6)\,\beta(7)\,\alpha(9)\,\beta(10)$$

$$-3\,\alpha(1)\,\beta(2)\,\beta(5)\,\alpha(3)\,\beta(4)\,\alpha(8)\,\alpha(6)\,\beta(7)\,\alpha(9)\,\beta(10)$$
$$-\alpha(1)\,\beta(2)\,\beta(5)\,\alpha(3)\,\beta(4)\,\alpha(6)\,\alpha(7)\,\beta(8)\,\beta(9)\,\alpha(10)$$
$$+\alpha(1)\,\beta(2)\,\beta(5)\,\alpha(3)\,\beta(4)\,\alpha(6)\,\alpha(7)\,\beta(8)\,\alpha(9)\,\beta(10)$$
$$+\alpha(1)\,\beta(2)\,\beta(5)\,\alpha(3)\,\beta(4)\,\beta(6)\,\alpha(7)\,\alpha(10)\,\beta(8)\,\alpha(9)$$
$$-\alpha(1)\,\beta(2)\,\beta(5)\,\alpha(3)\,\beta(4)\,\alpha(6)\,\beta(7)\,\alpha(10)\,\beta(8)\,\alpha(9)$$

we choose m10 based on m5 and m5'

m10 = m5(move 5 positions up) m5' =

$\beta(1)\,\beta(2)\,\alpha(3)\,\alpha(4)\,\beta(5)\,\alpha(6)\,\alpha(7)\,\beta(8)\,\beta(9)\,\alpha(10)$ which is and element of c10(A, B)

choosing the alphas and betas based on m10

```
> beta(1):=1:beta(2):=1:alpha(3):=1:alpha(4):=1:beta(5):=1:
alpha(6):=1:alpha(7):=1:alpha(10):=1:beta(8):=1:beta(9):=1:
> alpha(1):=0:alpha(2):=0:beta(3):=0:beta(4):=0:alpha(5):=0:
beta(6):=0:beta(7):=0:alpha(8):=0:alpha(9):=0:beta(10):=0:
```

sub in the entries to get the [1,11] position of c10(A, B) = [[[[B, A], B], [B, A]],[[[B, A], A],[B, A]]]

```
> print(c10AB111);
```

$$1$$

This shows that we have found the smallest n, n = 11, where G has length 4. G^3 exists and G^4 = 0 and j = 10.

Note that the number of alphas in m10, 5, equals the number of A's in c10(A, B) = [[[[B, A], B], [B, A]],[[[B, A], A],[B, A]]] and the number of betas in m10, 5, equals the number of B's in c10(A, B).

We now go on to when t = 5 and we have to find the least amount of terms needed to make G^4 exist and G^5 = 0. We know that G^3 has only 1 term with length 10 so we cannot use only this to get a combo of length 20 because the multiplications will cancel themselves out. We need to use a combo with length 11 of G^3 and then the 10 term element from G^3 to find the smallest number of terms. The smallest number of terms j = 21 is found using the 10 term combo c10[A,B] = [[[[B, A], B], [B, A]],c5(A,B)] = [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]] and then a 11 term combo. For the 11 term combo we will use the 5 term combo c5(A,B)= [[[B, A], A], [B, A]] and find a 6 term combo. We find a 6 term combo by using a combo of 4 and a combo of 2. There are three 4 term combos [[[B, A], A], A], [[[B, A], A], B], and [[[B, A], B], B] (note that [[[B, A], A], A] = [[[A, B], B], B] and [[[B, A], A], B] = [[[A, B], B], A] and [[[B, A], B], B] = [[[A, B], A], A]). For the 2 term combo we use [B, A] (note that [A, B] = [B, A]). The different 6 term combos (using the 3 different 4 terms and the one 2 term) yield term1 = [[[[B, A], A], A], [B, A]], term2 = [[[[B, A], A], B], [B, A]], and term3 = [[[[B, A], B], B], [B, A]]. We will use the following combination (a 6 term 5 term and then 10 term = 21) to yield the smallest n when t = 5,

c21(A, B) = [[term1, c5(A, B)],c10(A, B)] = [[[[[[B, A], A], A], [B, A]], [[[B, A], A], [B, A]]], [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]]

Note the other 21 combos are

c'21(A, B) = [term2, c5(A, B)],c10(A, B)] = [[[[[[B, A], A], B], [B, A]], [[[B, A], A], [B, A]]], [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]]

and

c"21(A, B) = [term3, c5(A, B)],c10(A, B)] = [[[[[[B, A], B], B], [B, A]], [[[B, A], A], [B, A]]], [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]]

```
> restart;with(linalg):

Warning, new definition for norm

Warning, new definition for trace

> A:=matrix(22,22,0):

> for i from 1 by 1 to 21 do A[i,i+1] :=alpha(i):  od:

> print(A);
```

$$
\begin{bmatrix}
0, \alpha(1), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, \alpha(2), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, \alpha(3), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, \alpha(4), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, \alpha(5), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, \alpha(6), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, \alpha(7), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, \alpha(8), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha(9), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha(10), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha(11), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha(12), 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha(13), 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha(14), 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha(15), 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha(16), 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha(17), 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha(18), 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha(19), 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha(20), 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha(21) \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
\end{bmatrix}
$$

```
> B:=matrix(22,22,0):

> for i from 1 by 1 to 21 do B[i,i+1] :=beta(i):  od:

> print(B);
```

$$
\begin{bmatrix}
0, \beta(1), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, \beta(2), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, \beta(3), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, \beta(4), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, \beta(5), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, \beta(6), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, \beta(7), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, \beta(8), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, \beta(9), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \beta(10), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \beta(11), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \beta(12), 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \beta(13), 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \beta(14), 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \beta(15), 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \beta(16), 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \beta(17), 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \beta(18), 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \beta(19), 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \beta(20), 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \beta(21) \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
\end{bmatrix}
$$

c21(A, B) = [[term1, c5(A, B)],c10(A, B)] = [[[[[[B, A], A], A], [B, A]], [[[B, A], A], [B, A]]], [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]]

find term1 first

```
> BA:=evalm((B&*A)- (A&*B)): BAA:=evalm((BA&*A)- (A&*BA)):
BAAA:=evalm((BAA&*A)- (A&*BAA)): BAAABA:=evalm((BAAA&*BA)-
(BA&*BAAA)):
```

```
> term1:=BAAABA:
```

find c5(A,B)= [[[B, A], A], [B, A]]

```
> c5:=evalm((BAA&*BA)- (BA&*BAA)):
```

find c10[A,B] = [[[[B, A], B], [B, A]],c5(A,B)] = [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]

```
> BAB:=evalm((BA&*B)- (B&*BA)):
> BABBA:=evalm((BAB&*BA)- (BA&*BAB)):
> c10:=evalm((BABBA&*c5)- (c5&*BABBA)):
```

do mult1 = term1 * c5(A, B)

```
> mult1:=evalm((term1&*c5)- (c5&*term1)):
```

then do (term1*c5(A,B)) * c10(A, B)

```
> c21:=evalm((mult1&*c10)- (c10&*mult1)):
```

we choose m21 based on term1 (move 6 positions up) m5' and then (move 11 positions up) m10

term 1 yields $\alpha(1)$ $\alpha(2)$ $\alpha(3)\,\beta(4)\,\beta(5)\,\alpha(6)$

6 positions up from m5' yields $\alpha(7)\,\alpha(8)\,\beta(9)\,\beta(10)\,\alpha(11)$

11 positions up from m10 yields

$\beta(12)\beta(13)\alpha(14)\alpha(15)\beta(16)\alpha(17)\alpha(18)\beta(19)\beta(20)\alpha(21)$

Yielding m21 =

$\alpha(1)\alpha(2)\alpha(3)\beta(4)\beta(5)\alpha(6)\alpha(7)\alpha(8)\beta(9)\beta(10)\alpha(11)\beta(12)\beta(13)\alpha(14)\alpha(15)$

$\beta(16)\alpha(17)\alpha(18)\beta(19)\beta(20)\alpha(21)$

```
> E122:=simplify(c21[1,22]):
```

choosing the alphas and betas based on m21

```
> alpha(1):=1:alpha(2):=1:alpha(3):=1:beta(4):=1:beta(5):=1:
alpha(6):=1:alpha(7):=1:alpha(8):=1:beta(9):=1:beta(10):=1:
alpha(11):=1:beta(12):=1:beta(13):=1:alpha(14):=1:alpha(15):=1:
beta(16):=1:alpha(17):=1:alpha(18):=1:beta(19):=1:beta(20):=1:
alpha(21):=1:
```

```
> beta(1):=0:beta(2):=0:beta(3):=0:alpha(4):=0:alpha(5):=0:
beta(6):=0:beta(7):=0:beta(8):=0:alpha(9):=0:alpha(10):=0:
beta(11):=0:alpha(12):=0:alpha(13):=0:beta(14):=0:beta(15):=0:
alpha(16):=0:beta(17):=0:beta(18):=0:alpha(19):=0:alpha(20):=0:
beta(21):=0:
```

Sub in the entries to find the (1,22) position of c21(A, B) = [[term1, c5(A, B)],c10(A, B)] = [[[[[B, A], A], A], [B, A]], [[[B, A], A], [B, A]]], [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]]

```
> print(E122);
```

$$-1$$

This shows that we have found the smallest n, n = 22, where G has length 5. G^4 exists and G^5 = 0 and j = 21.

Note that the number of A's in c21(A, B) = [[term1, c5(A, B)],c10(A, B)] = [[[[[B, A], A], A], [B, A]], [[[B, A], A], [B, A]]], [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]] equals the number of alphas in m21

m21 =

$\alpha(1)\alpha(2)\alpha(3)\beta(4)\beta(5)\alpha(6)\alpha(7)\alpha(8)\beta(9)\beta(10)\alpha(11)\beta(12)\beta(13)\alpha(14)\alpha(15)$

$\beta(16)\alpha(17)\alpha(18)\beta(19)\beta(20)\alpha(21)$

and the number of B's in c21 equals the number of betas in m21.

Note the number of alphas.

For the term m21 which is an element of c21(A, B) = [[term1, c5(A, B)],c10(A, B)] = [[[[[B, A], A], A], [B, A]], [[[B, A], A], [B, A]]], [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]] we have 12 alphas.

For the term m21' which is an element of c'21(A, B) = [term2, c5(A, B)],c10(A, B)] = [[[[[B, A], A], B], [B, A]], [[[B, A], A], [B, A]]], [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]] we have 11 alphas.

For the term m21" which is an element of c"21(A, B) = [term3, c5(A, B)],c10(A, B)] = [[[[[B, A], B], B], [B, A]], [[[B, A], A], [B, A]]], [[[[B, A], B], [B, A]], [[[B, A], A], [B, A]]]] we have 10 alphas.

Notice that the number of alphas in m21, m21", and m21' is congruent to 0, 1, 2 module 3 respectively.