ABSTRACT

BROWN, JONATHAN D. N-symplectic Quantization. (Under the direction of Dr. Larry Norris.)

A quantization scheme based on n-symplectic geometry is defined. Using this new definition a generalized Van Hove prequantization is given for the frame bundle of $\mathbb{R}^n$, $L\mathbb{R}^n$. The full set of operators of the generalized Van Hove prequantization is full rank irreducible and the components of these tensor valued operators are essentially self adjoint. However, this prequantization is reducible when it is restricted to the Heisenberg algebra. Several full quantizations are also given for $L\mathbb{R}^n$ proving there is no Groenwold Van Hove type obstruction for quantizing $L\mathbb{R}^n$. Using the covering theory of n-symplectic geometry we analyze why this quantization fails under symplectic quantization. Throughout the paper, emphasis is placed on comparison to the symplectic theory.

N-symplectic Quantization

by

Jonathan D. Brown

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fullfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2008

APPROVED BY:

Dr. Arkady Kheyfets	Dr. Irina Kogan
Dr. Larry Norris Chair of Advisory Committee	Dr. Ronald Fulp

BIOGRAPHY

Jonathan Dale Brown is the son of Royce and Brenda Brown of Blairsville Georgia. The author graduated from Union County High School in 1998, Young Harris College in 2000, and North Georgia College and State University in 2003. After obtaining his undergraduate degree in mathematics with a minor in physics, the author attended graduate school at North Carolina State University. He obtained his masters of science in mathematics in 2005.

ACKNOWLEDGMENTS

The author would like to thank Dr. Larry Norris for his guidance and influence. He would also like to thank Dr. Ron Fulp for his careful critique of chapter four. His suggestions led to many improvements to the manuscript. The author would also like to thank Denise Seabrooks for all her help in negotiating the paperwork of being a graduate student. Finally, I would like to thank my wife Jessica and my daughter Delaney whose encouragement and understanding helped the author through many long Saturdays working on this material.

TABLE OF CONTENTS

1 Introduction **1**

2 Preliminary Material **5**
- 2.1 The Canonical n-Symplectic Manifold LM 5
- 2.2 Poisson Bracket on LM 7
- 2.3 Momentum Mappings 10
- 2.4 Hilbert Spaces 11
- 2.5 Definition of Symplectic Quantization 13
- 2.6 Definition of n-Symplectic Quantization 15

3 Generalization of the Van Hove Prequantization **16**
- 3.1 Vector Fields on AM 16
- 3.2 Irreducibility and Reducibility of the Generalized Van Hove Prequantization 18
  - 3.2.1 Notation 18
  - 3.2.2 The Transformation W 18
  - 3.2.3 Function of Operators 19
  - 3.2.4 Choosing a Basis 20
  - 3.2.5 Reduction of $\mathcal{L}$ 22
  - 3.2.6 The Irreducibility of $\mathcal{U}_\Gamma^{(\alpha)}$ 30
- 3.3 The Operators of the Generalized Van Hove Prequantization are Essentially Self Adjoint 32
  - 3.3.1 The Group Γ 32
  - 3.3.2 Infinitesimal Generators 33
  - 3.3.3 Operator $\mathcal{U}_\gamma^{(\alpha)}$ 34
  - 3.3.4 Essentially Self Adjoint on $L^2(L\mathbb{R}^n, \mathbb{C})$ 36

4 Quantization of $L\mathbb{R}^n$ **38**
- 4.1 The Basic Set for $L\mathbb{R}^n$ 38
- 4.2 Comparison of the Poisson Algebra of Polynomials for $T^*\mathbb{R}^n$ vs $L\mathbb{R}^n$ 40
  - 4.2.1 The Polynomial Algebra $P(\mathsf{b}_{2n})$ 40
  - 4.2.2 The Polynomial Algebra $P(\mathsf{b}_L)$ 40
- 4.3 Go theorem for $L(\mathbb{R}^n)$ 42
  - 4.3.1 Ideal Quantization 42
  - 4.3.2 Another Full Polynomial Quantization 43
  - 4.3.3 Subalgebras of $P(\mathsf{b}_L)$ 44
- 4.4 Groenwold Van Hove Obstruction for $T^*\mathbb{R}^n$ 45
  - 4.4.1 n-Symplectic Case 46
- 4.5 No-Go theorems 47
- 4.6 Map From LM to T^*M 51

5 Conclusion **55**

Bibliography **59**

Chapter 1

Introduction

The purpose of this thesis is to develop a definition of quantization based on n-symplectic geometry rather than the traditional symplectic geometry. The major result of this dissertation is that not only can we develop a quantization scheme based on n-symplectic geometry but this quantization scheme also removes some, if not all, of the obstructions to quantization encountered in the symplectic case. To appreciate the results we first must understand the problems of quantization. To do this we will address two questions. First, "Why change from symplectic geometry?" and second, "Why use n-symplectic geometry?"

Quantization is the relation between classical mechanics and quantum mechanics. A classical configuration space is often a symplectic manifold and classical observables are function on that manifold. A quantum configuration space is a Hilbert space and quantum observables are symmetric operators on that Hilbert space. The classical system has a natural Lie algebra structure on the observables induced by the Poisson bracket, while the quantum observables form a Lie algebra with the commutator bracket. A quantization is basically a Dirac map Q between these two spaces that satisfies additional requirements, the most important being an irreducibility requirement for the "important" observables. A Dirac map is a linear map between two Lie algebras such that the map L satisfies the Dirac condition [Lx,Ly]=L{x,y}, L(1)=1.

"Why change from symplectic geometry" The process of quantization based on symplectic geometry is plagued by difficulties. For example, if one wishes to *try* to quantize a simple system of n moving particles one runs into a famous obstruction theorem. The classical space for n moving particles is the cotangent bundle $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$. To be physically acceptable this quantization must be unitarily equivalent to the Schrodinger representation

$Q(q^i) \to q^i$ and $Q(p_j) \to -i\hbar\partial/\partial q^j$. This quantization must also be a Dirac map. There have been many attempts made to find such a map. When Dirac described his canonical quantization rule in [1] he made an interesting remark. Dirac acknowledged that his "Poisson bracket→commutator" rule holds for all quantizable functions "or at any rate the simpler ones of them." His hesitation was well deserved. In [2] Groenwold proved that it is *impossible* to quantize the set of all polynomials, in q^i and p_j, of $\mathbb{R}^{2n}$ consistent with the Schrodinger representation. Explicitly, taking n=1 for simplicity, he proved that when one quantizes both sides of the equation $\{q^3, p^3\} = 3\{q^2p, p^2q\}$ the quantizations are not equal, $Q(\{q^3, p^3\}) \neq 3Q(\{q^2p, p^2q\})$. In [3], Van Hove later refined Groenwold's work by proving a stronger result. The result being that there exists no quantization of $C^\infty(\mathbb{R}^{2n})$. It is important to note that the Groenwold-Van Hove theorem only applies to the pairs $(P[q^i, p_j], \mathbb{R}^{2n})$ and $(C^\infty(\mathbb{R}^{2n}), \mathbb{R}^{2n})$. Furthermore the theorem only holds if $Q(q^i) = q^i$ and $Q(p_j) = -i\hbar\partial/\partial q^j$. The Stone von Neumann theorem made the theorems of Groenwold and Van Hove much stronger. The Stone von Neumann theorem states that any pair of operators that are irreducible on $L^2(\mathbb{R}^n)$ and that satisfy the Heisenberg commutation relations are unitarily equivalent to the Schrodinger representation, modulo technical difficulties [4]. This theorem improved the Groenwold-Van Hove theorem to the following.

Theorem 1.1 *There exists no quantization of* $(P[q^i, p_j], \mathbb{R}^{2n})$*, respectively* $(C^\infty(\mathbb{R}^{2n}), \mathbb{R}^{2n})$ *such that the operators* $Q(q^i)$ *and* $Q(p_j)$ *act irreducibly.*

A strong theorem indeed. This theorem shows that the Lie algebra of $C^\infty(\mathbb{R}^{2n})$ with the Poisson bracket, which is explicitly related to the symplectic structure of the cotangent bundle of $\mathbb{R}^n$, is incompatible with the Lie algebra of symmetric operators with the commutator. Other papers [5] and [6] have also emphasized this fact. The symplectic space $\mathbb{R}^{2n}$ is not the only symplectic space that exhibits obstruction to quantization. The symplectic manifolds S^2, T^*S^1, also exhibit similar obstructions to quantizations[5]. The author believes the inherent incompatibility of the Lie algebra structures necessitates a change of setting for the classical systems. A natural choice for the replacement of symplectic geometry is n-symplectic geometry. n-Symplectic geometry is one of three generalizations of symplectic geometry that originated in the late 80's and early 90's. See [7] for a review of all three theories side by side. Playing the role of the cotangent bundle T^*M, the frame bundle of a manifold LM is the canonical n-symplectic manifold. The n-symplectic potential is the soldering 1-form, which is $\mathbb{R}^n$ valued. The tensor valued nature of the n-symplectic form drives the main differences between symplectic and n-symplectic geometry. The symplectic geometry of polynomials can be completely recovered using n-symplectic geometry[8]. In this way, n-symplectic geometry is a covering theory for symplectic geometry. This fact makes n-symplectic geometry a natural choice to replace symplectic geometry.

In chapter one we review the preliminary ideas needed to study a quantization based on n-symplectic geometry. In sections one and two we review symplectic geometry and n-symplectic geometry to highlight the similarities of the two structures. In section three we review the definition of quantization based on symplectic geometry.

In chapter two we give the definition of an n-symplectic quantization. This definition is similar to the definition of symplectic quantization given in chapter one. The main difference is that the domain of the Dirac map is now functions on an n-symplectic manifold.

In chapter three we generalize a lesser known result of Van Hove and give the first full prequantization of $L\mathbb{R}^n$. We show that the prequantization is full rank irreducible and the components are essentially self adjoint but the operators becomes reducible when restricted to a certain subset of the domain. For completeness, a recent paper [9], by Tuynman proves the Van Hove prequantization is irreducible for all symplectic manifolds. I suspect the results of this dissertation could be extended to prove the prequantization in chapter three works for the frame bundle of any manifold.

Finally, in chapter four we give our main result. We show there exists a full polynomial quantization of $L\mathbb{R}^n$. Therefore there is no Groenwold Van Hove theorem for n-symplectic quantization. We investigate the difference that allows a quantization to exist for $L\mathbb{R}^n$ but not for $T^*\mathbb{R}^n$. As a result of these observations we give some restricted no-go theorems. As a final remark we use the covering properties of LM over T^*M to study why this quantization on $L\mathbb{R}^n$ does not give rise to a quantization on $T^*\mathbb{R}^n$.

For convenience, we list some notation used throughout the thesis.

- $L^2(M)$ denotes square integrable functions of a manifold M.

- LM is the bundle of linear frames of a manifold M.

- $\otimes_s$ denotes symmetric tensor product.

- $\otimes_s^p \mathbb{R}^n$ denotes repeated symmetric tensor product of $\mathbb{R}^n$, $\otimes_s^p \mathbb{R}^n = \mathbb{R}^n \underbrace{\otimes_s \ldots \otimes_s}_{p\ times} \mathbb{R}^n$

- We will use the multi-index notation $f^{I_p} = f^{i_1 \ldots i_p}$ for functions and $\hat{r}_{I_p} = \hat{r}_{i_1} \otimes_s \ldots \otimes_s \hat{r}_{i_p}$ for vectors.

- A multi-index on $\hat{\pi}_k$, $\hat{q}^i_j$, or $\hat{r}_k$ denotes repeated symmetric tensor product over multiple indicies, $\hat{q}^{I_a}_{J_b} = \hat{q}^{i_1}_{j_1} \otimes_s \ldots \otimes_s \hat{q}^{i_a}_{j_b}$.

- Parenthesis around indicies means symmetrize the indicies.

- Denote the set of vector fields on a manifold M by $\mathcal{X}(M)$.

Chapter 2

Preliminary Material

2.1 The Canonical n-Symplectic Manifold LM

In symplectic geometry the canonical symplectic manifold is $P = T^*M$, the cotangent bundle of a manifold M. The symplectic structure, in local symplectic coordinates (q^i, p_j), is given by the differential of the canonical one form $\tilde{\theta} = p_j dq^j$. To each observable $f \in C^\infty(T^*M)$ one assigns a Hamiltonian vector field by the structure equation

$$df = X_f \lrcorner\, d\tilde{\theta}$$

Each symplectic coordinate is C^∞ and hence is an allowable observable for T^*M. The corresponding Hamiltonian vector fields are

$$X_{q^i} = -\frac{\partial}{\partial p_i}, \qquad X_{p_j} = \frac{\partial}{\partial q^j}$$

Definition 2.1 *An n-symplectic manifold is a manifold P together with an $\mathbb{R}^n$ valued non-degenerate two form $\omega = \omega^i \hat{r}_i$. Here $\{\hat{r}_i\}$ is the standard basis of $\mathbb{R}^n$.*

An equivalent definition for a polysymplectic manifold is given by Gunther [10]. For n-symplectic geometry the canonical n-symplectic manifold is $P = LM$, the linear frame bundle of an n dimensional manifold M. Define coordinates on LM in the standard way. Let $(\tilde{q}^i, U)$ be a chart on M and $\pi : LM \to M$ the standard projection to M. For a point $(m, e_i) \in \pi^{-1}(U) \subset LM$ define coordinates (q^i, π^i_j) by $q^i(m, e_i) = \tilde{q}^i(m)$ and $\pi^i_j(m, e_k) = e^i(\partial/\partial\tilde{q}^j|_m)$. The n-symplectic structure is given by the differential of the $\mathbb{R}^n$ valued soldering one form $\theta = \theta^i \hat{r}_i$ defined by $\theta(m, e_i)(X) = e^i(d\pi(X))\hat{r}_i$. In local coordinates it has the form $\theta = \theta^i \hat{r}_i = \pi^i_j dq^j \hat{r}_i$. Here the similarity with symplectic geometry starts to differ. The set of observables are $\otimes^p\mathbb{R}^n$ valued. Moreover, the observables are no longer all of $C^\infty(LM)$ but rather are polynomials in the momenta π^i_j with coefficients in $C^\infty(M)$. The observables naturally split into symmetric tensor valued Hamiltonian functions, SHF, and totally antisymmetric tensor valued Hamiltonian functions, AHF [11]. For the remainder of this paper we will only consider SHF leaving the antisymmetric case to a future work. On LM, all $\otimes^p_s\mathbb{R}^n$ valued functions, for which there exists a $\otimes^{p-1}_s\mathbb{R}^n$ valued Hamiltonian vector field, are denoted SHF^p. Following [11] we assign to each $\hat{f} \in SHF^p$ an equivalence class $[X_{\hat{f}}]$ of $\otimes^{p-1}_s\mathbb{R}^n$ valued Hamiltonian vector fields by the structure equation

$$d\hat{f}^I_p = -p!X^{(I_{p-1}}_{\hat{f}} \lrcorner\, d\theta^{i_p)}$$

The equivalence class of $\otimes^{p-1}_s\mathbb{R}^n$ valued vector fields is denoted by $[X_{\hat{f}} = X^{I_{p-1}}\hat{r}_{I_{p-1}}]$. In local (q^i, π^j_k) coordinates, the vector fields can be written for $\hat{f} \in SHF^p$

$$X^{I_{p-1}}_{\hat{f}} = \frac{1}{p!}\frac{\partial \hat{f}^{I_{p-1}b}}{\partial \pi^b_a}\frac{\partial}{\partial q^a} - \frac{1}{p!}\frac{\partial \hat{f}^{I_{p-1}a}}{\partial q^b} + T^{I_{p-1}a}_b\frac{\partial}{\partial \pi^a_b} \quad \text{where } T^{(I_{p-1}a)}_b = 0. \tag{2.1}$$

The equivalence classes of Hamiltonian vector fields generated by SHF form a Lie Algebra relative to the bracket defined in [11] as follows. For $\hat{f} \in SHF^p$ and $\hat{g} \in SHF^q$, define the bracket of their corresponding Hamiltonian vector fields by

$$[[X_{\hat{f}}], [X_{\hat{g}}]] = [[X^{I_{p-1}}_{\hat{f}}\hat{r}_{I_{p-1}}], [X^{J_{q-1}}_{\hat{g}}\hat{r}_{J_{q-1}}]] \overset{def}{=} [X^{I_{p-1}}_{\hat{f}}, X^{J_{q-1}}_{\hat{g}}]\hat{r}_{I_{p-1}} \otimes_s \hat{r}_{J_{q-1}}$$

The bracket on the right hand side is the ordinary Lie bracket of vector fields, and $X^{I_{p-1}}_{\hat{f}}$ and $X^{J_{q-1}}_{\hat{g}}$ are arbitrary representatives of the equivalence classes $[X^{I_{p-1}}_{\hat{f}}]$ and $[X^{J_{q-1}}_{\hat{g}}]$. The symmetrization on the upper indices in the bracket destroys the non uniqueness making the bracket independent of choice of representative. These vector fields also preserve the n-symplectic form.

Lemma 2.1 *Let $\hat{g} \in SHF^q$ and $[X^{J_{q-1}}_{\hat{g}}]$ the corresponding Hamiltonian vector field. This vector field preserves the n-symplectic form $d\theta$ in the sense that,*

$$L_{X^{(J_{q-1}}_{\hat{g}}} d\theta^{i)} = 0$$

Proof

The Lie derivative of forms satisfies the familiar relation

$$L_X\omega = X \lrcorner\, d\omega + d(X \lrcorner\, \omega)$$

Therefore we have

$$L_{X_{\hat{g}}^{(J_{q-1}}} d\theta^{i)} = X_{\hat{g}}^{(J_{q-1}} \lrcorner \; d(d\theta^{i)}) + d(X_{\hat{g}}^{(J_{q-1}} \lrcorner \; d\theta^{i)}) = 0$$

The last relation being zero since $X_{\hat{g}}^{(J_{q-1}} \lrcorner \; d\theta^{i)} = -\frac{1}{q!} dg^{J_q}$ and $d^2 = 0$. $\square$

In contradistinction to the situation on T^*M, the local coordinates of LM are no longer observables. Each observable must be $\otimes_s^p \mathbb{R}^n$ valued. However, the local coordinates define some basic observables.

$$q^i \quad \rightarrow \quad \hat{q}^i_j \stackrel{def}{=} q^i \hat{r}_j \tag{2.2}$$

$$\pi^a_k \quad \rightarrow \quad \hat{\pi}_k \stackrel{def}{=} \pi^a_k \hat{r}_a \tag{2.3}$$

The corresponding Hamiltonian vector fields are

$$X_{\hat{q}^i_j} = -\frac{\partial}{\partial \pi^j_i}, \qquad X_{\hat{\pi}_k} = \frac{\partial}{\partial q^k}$$

These are not the only observables we can construct. We can create many observables from the coordinates, one for each SHF^p. For example, $q^i \hat{r}_j$, $q^i \hat{r}_j \otimes_s \hat{r}_k$, $q^i \hat{r}_j \otimes_s \hat{r}_k \otimes_s \hat{r}_l$, etc. are all different observables created from the coordinate q^i.

2.2 Poisson Bracket on LM

The Poisson bracket on LM plays a fundamental role in our discussion. In this section we review the Poisson bracket for LM. We define the Poisson bracket of two symmetric Hamiltonian functions as follows:

Definition 2.2 *Let $\hat{f} \in SHF^p$ and $\hat{g} \in SHF^q$ then $\{\hat{f}, \hat{g}\} \in SHF^{p+q-1}$ where $\{.,.\}$ is defined by*

$$\{\hat{f}, \hat{g}\} = p! X_{\hat{f}}^{I_{p-1}} (\hat{g}^{J_q}) \hat{r}_{I_{p-1}} \otimes_s \hat{r}_{J_q}$$

Here $X_{\hat{f}}$ is any representative of the equivalence class of symmetric Hamiltonian vector fields of $\hat{f}$.

In [11] it is shown that this bracket is independent of choice of representative and hence well defined. The bracket is anti-symmetric and satisfies the Jacobi identity. Also, it is fundamental to a later discussion to note

$$\{SHF^p, SHF^q\} \subset SHF^{p+q-1} \tag{2.4}$$

The Poisson bracket in n-symplectic geometry is linked to the bracket of Hamiltonian vector fields. We extend the result in [11].

Theorem 2.3 *Let $\hat{f} \in SHF^p$ and $\hat{g} \in SHF^q$ then the Hamiltonian vector fields satisfy the relation*

$$CX_{\{\hat{f},\hat{g}\}} = [X_{\hat{f}}, X_{\hat{g}}]$$

The square bracket is the one defined in section 1 and $C = \frac{(p+q-1)!}{p!q!}$.

Proof

Let $\hat{f} \in SHF^p$ and $\hat{g} \in SHF^q$. Using $L_X\omega = X\lrcorner\, d\omega + d(X\lrcorner\, \omega)$ we get

$$[X_{\hat{f}}^{(I_{p-1}}, X_{\hat{g}}^{J_{q-1}}]\lrcorner\, d\theta^{i)} = L_{X_{\hat{f}}^{(I_{p-1}}}(X_{\hat{g}}^{J_{q-1}}\lrcorner\, d\theta^{i)}) - X_{\hat{f}}^{(I_{p-1}}\lrcorner\, L_{X_{\hat{g}}^{J_{q-1}}} d\theta^{i)}$$

By lemma 2.1, $L_{X_{\hat{g}}^{J_{q-1}}} d\theta^{i)} = 0$. Our equation becomes

$$\begin{aligned}
[X_{\hat{f}}^{(I_{p-1}}, X_{\hat{g}}^{J_{q-1}}]\lrcorner\, d\theta^{i)} &= L_{X_{\hat{f}}^{(I_{p-1}}}(X_{\hat{g}}^{J_{q-1}}\lrcorner\, d\theta^{i)}) \\
&= X_{\hat{f}}^{(I_{p-1}}\lrcorner\, d(X_{\hat{g}}^{J_{q-1}}\lrcorner\, d\theta^{i)}) + d(X_{\hat{f}}^{(I_{p-1}}\lrcorner\, X_{\hat{g}}^{J_{q-1}}\lrcorner\, d\theta^{i)}) \\
&= 0 + d(X_{\hat{f}}^{(I_{p-1}}\lrcorner\, X_{\hat{g}}^{J_{q-1}}\lrcorner\, d\theta^{i)})
\end{aligned}$$

The last line follows from the structure equation $-q!X_{\hat{g}}^{(J_{q-1}}\lrcorner\, d\theta^{i)} = dg^{J_q}$.

$$\begin{aligned}
[X_{\hat{f}}^{(I_{p-1}}, X_{\hat{g}}^{J_{q-1}}]\lrcorner\, d\theta^{i)} &= d(X_{\hat{f}}^{(I_{p-1}}\lrcorner\, X_{\hat{g}}^{J_{q-1}}\lrcorner\, d\theta^{i)}) \\
&= d(X_{\hat{f}}^{(I_{p-1}}\lrcorner\, \frac{1}{-q!} d\hat{g}^{J_q i)}) \\
&= \frac{1}{-q!} d(X_{\hat{f}}^{(I_{p-1}}(\hat{g}^{J_q i)}) \\
&= \frac{1}{-q!p!} d(\{\hat{f}, \hat{g}\}^{I_{p-1}J_q}) \\
&= \frac{(p+q-1)!}{p!q!} X_{\{\hat{f},\hat{g}\}}^{I_{p-1}J_{q-1}}\lrcorner\, d\theta^{i)} \qquad \square
\end{aligned}$$

The n-symplectic Poisson bracket and the bracket of equivalence classes of Hamiltonian vector fields are independent of choice of equivalence class. Hence we will no longer emphasize the the equivalence class and simply refer to a representative $X_{\hat{f}}$.

2.3 Momentum Mappings

The symplectic structure of T^*M supports an object called a momentum mapping. A generalization of this mapping to n-symplectic geometry will be useful for calculating basic sets which will be defined in a later section. The definition given in Foundations of Mechanics edition two [12] is the following.

Definition 2.4 *Let $\Phi : G \times T^*M \to T^*M$ be a symplectic action of a Lie group G on $(T^*M, \omega = dq^i \wedge dp_i)$. The mapping $J : T^*M \to \mathcal{G}^*$ is a momentum mapping if for each $\xi \in \mathcal{G}$*

$$d\hat{J}(\xi) = -\xi_{T^*M} \lrcorner\, \omega$$

where ξ_{T^*M} is the infinitesimal generator of the action of G on T^*M generated by ξ, and $\hat{J}(\xi) : T^*M \to \mathbb{R}$ is defined by

$$\hat{J}(\xi)(u) =< J(u), \xi >$$

In the previous definition $\mathcal{G}$ is the Lie algebra for the Lie group G and a symplectic action of a Lie group is one that preserves the symplectic form. Following [8] there is a momentum map for n-symplectic geometry.

Definition 2.5 *Let $\Phi : G \times LM \to LM$ be an n-symplectic action of a Lie group G on $(LM, d\theta)$. The mapping $J : LM \to \mathcal{G}^* \otimes \mathbb{R}^n$ is a momentum mapping if for each $\xi \in \mathcal{G}$*

$$d\hat{J}(\xi) = -\xi_{LM} \lrcorner\, d\theta$$

where ξ_{LM} is the infinitesimal generator of the action of G on LM generated by ξ, and $\hat{J}(\xi) : LM \to \mathbb{R}^n$ is defined by

$$\hat{J}(\xi)(u) =< J(u), \xi >$$

The inner product $< .,. >$ is the natural extension of the one on $\mathcal{G} \times \mathcal{G}^*$.

$$< \xi, \xi^* \otimes \hat{r}_j >=< \xi, \xi^* > \otimes \hat{r}_j$$

Similar to symplectic momentum maps, if $\{\xi_i\}$ is a basis of $\mathcal{G}$ let $\{J_i\}$ be the $\otimes^p \mathbb{R}^n$ valued Hamiltonian functions for $(\xi_i)_{LM}$. Define $\hat{J}$ by $\hat{J}(\xi_i) = J_i$. This gives a n-symplectic momentum map J with components J_i.

2.4 Hilbert Spaces

The quantization we will define is a general Hilbert space based quantization. In this section we will define the Hilbert spaces needed to discuss the properties of the operators of these quantizations. Before we define the new Hilbert spaces we mention the standard Hilbert spaces for some quantizations of $T^*\mathbb{R}^n$. For the metaplectic quantization [5] the Hilbert space is the set of all measurable complex valued square integrable functions of $\mathbb{R}^n$. We commonly denote the Hilbert space of measurable complex valued square integrable functions of M by $L^2(M, \mathbb{C})$. For the Van Hove prequantization [3] the Hilbert space is essentially $L^2(\mathbb{R}^{2n}, \mathbb{C})$. The measure in both cases is the one induced by the canonical volume form.

To describe the appropriate Hilbert spaces for $L\mathbb{R}^n$ we first need to describe an integral for LM. The volume for LM is given by $dV = \Delta(\omega) \wedge (\theta)^n$, where ω is a torsion free connection on LM and θ is the soldering one-form. Let ω^i_j be the associated one-forms to ω. Define $\Delta(\omega) \overset{def}{=} \omega^1_1 \wedge \omega^2_1 \wedge \cdots \wedge \omega^{n-1}_n \wedge \omega^n_n$ and $(\theta)^n \overset{def}{=} n!\theta^1 \wedge \cdots \wedge \theta^n$. This definition is independent of choice of connection[13]. For the simple frame bundle $L\mathbb{R}^n$ a judicious choice of connection gives a more familiar volume $dV = dq^1 dq^2 \cdots dq^n d\pi^1_1 d\pi^1_2 \cdot d\pi^n_n$. We have suppressed the wedge products in the previous and following equation. We also define a volume for the affine frame bundle $A\mathbb{R}^n$, $dW = dV dy^1 \cdots dy^n$. For the given volumes on $L\mathbb{R}^n$ and $A\mathbb{R}^n$ we make the following definitions.

Definition 2.6 $L^2(L\mathbb{R}^n, \mathbb{C})$ *is the Hilbert space of measurable square integrable functions from* $L\mathbb{R}^n$ *to* $\mathbb{C}$. *Let* $\phi, \psi \in L^2(L\mathbb{R}^n, \mathbb{C})$ *then the inner product is defined by*

$$< \phi, \psi >= \int \phi \bar{\psi} dV$$

Definition 2.7 $L^2(A\mathbb{R}^n, \mathbb{C})$ *is the Hilbert space of measurable square integrable functions from* $A\mathbb{R}^n$ *to* $\mathbb{C}$. *Let* $\phi, \psi \in L^2(A\mathbb{R}^n, \mathbb{C})$ *then the inner product is defined by*

$$< \phi, \psi >= \int \phi \bar{\psi} dW$$

Definition 2.8 *The Hilbert space* $\mathcal{H}^p = \{\psi^{I_p} z_{I_p} | \psi^{I_p} \in L^2(L\mathbb{R}^n, \mathbb{C})\}$. *Here* z_i *is the standard basis for* $\mathbb{C}^n$. *Let* $\phi, \psi \in \mathcal{H}^p$ *then the inner product is defined by*

$$< \phi, \psi >=< \phi^{I_p} z_{I_p}, \psi^{I_p} z_{I_p} >= \sum_{I_p} < \phi^{I_p}, \psi^{I_p} >$$

Definition 2.9 *The Hilbert space $\mathcal{H}$ is the completion of the direct sum $\tilde{\mathcal{H}}$ of $\mathcal{H}^p$ for all p.*

$$\tilde{\mathcal{H}} = \bigoplus_{p=1}^{\infty} \mathcal{H}^p$$

The inner product is the standard inner product for a direct sum.

$$< \oplus_{p=1}^{\infty} \psi^{I_p}, \oplus_{q=1}^{\infty} \psi^{I_q} >= \sum_{p=1}^{\infty} < \phi^{I_p}, \psi^{I_p} >$$

For each $\psi \in \mathcal{H}$ there will be only finitely many non-zero terms, so this inner product is well defined.

2.5 Definition of Symplectic Quantization

We use the definition of a symplectic quantization given in [5] restricted to the specific symplectic manifold T^*M. A detailed explanation of the motivation behind each condition is given in [5]. We choose this definition as it is independent of quantization method. Let M be an n dimensional manifold. Then $(T^*M, d\theta)$ is a symplectic manifold.

Definition 2.10 *A prequantization of the cotangent bundle T^*M is a linear map Q together with a Lie subalgebra $\mathcal{O}$ of $C^\infty(T^*M)$ where Q takes observables $f \in \mathcal{O}$ to symmetric operators on a dense domain D of a Hilbert space $\mathcal{H}$ such that the following hold:*
(1) $Q(\{f,g\}) = \frac{i}{\hbar}[Q(f), Q(g)]$
(2) If $1 \in \mathcal{O}$ then $Q(1) = Identity$.
(3) If the Hamiltonian vector field X_f of f is complete, then $Q(f)$ is essentially self-adjoint on D.

Here $\{.,.\}$ denotes the symplectic Poisson bracket [14].

Definition 2.11 *A basic set of observables b is a Lie subalgebra of $C^\infty(T^*M)$ such that:*
(4) b is finitely generated,
(5) the Hamiltonian vector fields X_f, $f \in \mathsf{b}$ are complete,
(6) b is transitive and separating, and
(7) b is a minimal Lie algebra satisfying these requirements.

The conditions required for a basic set are modeled on the properties of the components of a symplectic momentum map for an elementary system from geometric quantization [15]. A set of functions, $\mathcal{F}$, on a symplectic manifold, M, is transitive if $\{X_f | f \in \mathcal{F}\}$ span TM. We say a set of functions separates points if for $x \neq y \in M$ there exists an $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

Definition 2.12 *A quantization of T^*M is a prequantization $(\mathcal{O}, Q)$ such that for the basic set b:*
(8) $Q(\mathsf{b})$ acts irreducibly on $\mathcal{H}$,
(9) $Q|_{\mathsf{b}}$ is faithful, and
(10) D contains a dense set of separately analytic vectors for $Q(\mathsf{b})$.
To clarify condition 10, a vector $\phi \in D$ is analytic for an operator X on $\mathcal{H}$ given the series

$$\sum_{k=0}^{\infty} \frac{||X^k||\phi}{k!} t^k$$

is defined and converges for some $t > 0$. We say a vector $\phi \in D$ is separately analytic for a set of operators $\{X_1, .., X_k\}$ defined on a common invariant dense domain D if ϕ is analytic for each X_j. A vector is separately analytic for a Lie algebra if it is separately analytic for the set of generators for that Lie algebra.

A quantization is said to be a full quantization if $\mathcal{O} = C^\infty(T^*M)$.

Definition 2.13 $P(\mathsf{b})$ *is the polynomial algebra for a basic set* b.

A quantization is said to be a full polynomial quantization if $\mathcal{O} = P(\mathsf{b})$.

2.6 Definition of n-Symplectic Quantization

The definition of n-symplectic quantization is modeled on symplectic quantization. Let M be an n dimensional manifold. Then $(LM, d\theta)$ is an n-symplectic manifold.

Definition 2.14 *A prequantization of the frame bundle LM is a linear map Q together with a Lie subalgebra $\mathcal{O}$ of SHF where Q takes observables $f \in \mathcal{O}$ to symmetric operators on a dense domain D of a Hilbert space $\mathcal{H}$ such that the following hold:*
(1) $Q(\{f, g\}) = \frac{i}{\hbar}[Q(f), Q(g)]$
(2) If $\hat{r}_i \in \mathcal{O}$ then $Q(\hat{r}_i) = c_i$. The constants c_i are complex numbers.
(3) If the Hamiltonian vector field X_f of f is complete, then $Q(f)$ is essentially self-adjoint on D.

Here $\{.,.\}$ denotes the n-symplectic Poisson bracket [**?**].

Definition 2.15 *A basic set of observables b is a Lie subalgebra of SHF such that:*
(4) b is finitely generated,
(5) the Hamiltonian vector fields X_f, $f \in \mathsf{b}$ are complete,
(6) b is transitive and separating, and
(7) b is a minimal Lie algebra satisfying these requirements.

A set of functions, $\mathcal{F}$, on an n-symplectic manifold, LM, is transitive if $\{X_f | f \in \mathcal{F}\}$ span LM. We say a set of functions separates points if for $x \neq y \in LM$ there exists an $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

Definition 2.16 *A quantization of LM is a prequantization $(\mathcal{O}, Q)$ such that for the basic set b:*
(8) $Q(\mathsf{b})$ acts irreducibly on $\mathcal{H}$,
(9) $Q|_{\mathsf{b}}$ is faithful, and
(10) D contains a dense set of separately analytic vectors for $Q(\mathsf{b})$.
Separately analytic has the same meaning here as it does for the symplectic quantization since it is defined in terms of operators.

A quantization is said to be a full quantization if $\mathcal{O} = SHF$.

Definition 2.17 *$P(\mathsf{b})$ is the polynomial algebra for a basic set b.*

A quantization is said to be a full polynomial quantization if $\mathcal{O} = P(\mathsf{b})$.

Chapter 3

Generalization of the Van Hove Prequantization

In 1951 Leon Van Hove proved in his thesis [3] that the "Van Hove Prequantization," which is the prequantization of geometric quantization, is irreducible on the set of all operators generated by square integrable functions on $\mathbb{R}^{2n}$ and reducible on the smaller sub set of operators generated by $\mathcal{L} \cong \mathbb{R}[p,q]^2$. He also proved that these operators are essentially self adjoint on the Hilbert space of square integrable functions on $\mathbb{R}^{2n}$. In the following sections we prove a similar theorem for a generalized Van Hove n-symplectic prequantization of the frame bundle of $\mathbb{R}^n$.

3.1 Vector Fields on AM

Consider the affine frame bundle AM with base space LM, the linear frame bundle of M. For the manifold $M = \mathbb{R}^n$, $L\mathbb{R}^n$ and hence $A\mathbb{R}^n$ have globally defined coordinates (q^i, π^i_j) and (q^i, π^i_j, y^i) respectively. On the bundle $\beta : A\mathbb{R}^n \to L\mathbb{R}^n$, there exists the $\mathbb{R}^n$ valued one-form

$$\sigma = \beta^*\theta + d\lambda$$

In the above equation $\lambda = y^i\hat{r}_i$ and $\theta = (\pi^i_j dq^j)\hat{r}_i$ and σ is a connection on $\beta : A\mathbb{R}^n \to L\mathbb{R}^n$. The set of all real vector fields on $A\mathbb{R}^n$ which preserve the connection σ, i.e. that satisfy the equations

$$\mathcal{L}_{\xi^{(I_{p-1}}}\sigma^{i_p)} = 0$$

are generated by functions of the base bundle $\hat{f} \in SHF^p$ and have the specific form [8]

$$\xi_{\hat{f}} = X^{\#}_{\hat{f}} + \frac{1}{p!}\eta_{\hat{f}}$$

The symbol $X^{\#}_{\hat{f}}$ is the horizontal lift of the Hamiltonian vector field for $\hat{f}$ to $A\mathbb{R}^n$ and $\eta_{\hat{f}}$ is a vertical vector field involving the partial with respect to y^i. See [8] for details as the next equation is more important here. In global (q^i, π^i_j, y^i) coordinates, the vector fields can be written for $\hat{f} = \hat{f}^{I_p}\hat{r}_{I_p} \in SHF^p$,

$$\xi^{I_{p-1}}_{\hat{f}} = \frac{1}{p!}\frac{\partial \hat{f}^{I_{p-1}b}}{\partial \pi^b_a}\frac{\partial}{\partial q^a} - \frac{1}{p!}\frac{\partial \hat{f}^{I_{p-1}a}}{\partial q^b}\frac{\partial}{\partial \pi^a_b} + \frac{1}{p!}[\frac{\partial \hat{f}^{I_{p-1}b}}{\partial \pi^b_a}\pi^c_a - \hat{f}^{I_{p-1}c}]\frac{\partial}{\partial y^c} \qquad (3.1)$$

If we let $\xi_{\hat{f}}$ act on a specific element of $L^2(A\mathbb{R}^n, \mathbb{C})$ $\psi = e^{i\alpha_j y^j}\phi(\hat{q}, \hat{\pi})$ we have

$$\xi^{I_{p-1}}_{\hat{f}}\psi = \left[\frac{1}{p!}\frac{\partial \hat{f}^{I_{p-1}b}}{\partial \pi^b_a}\frac{\partial}{\partial q^a} - \frac{1}{p!}\frac{\partial \hat{f}^{I_{p-1}a}}{\partial q^b}\frac{\partial}{\partial \pi^a_b} - i\alpha_c\frac{1}{p!}[\frac{\partial \hat{f}^{I_{p-1}b}}{\partial \pi^b_a}\pi^c_a - \hat{f}^{I_{p-1}c}]\right]\psi \qquad (3.2)$$

Denote the operator acting on ψ on the right hand side of this equation by $\xi^{(\alpha)}_{\hat{f}}$.

3.2 Irreducibility and Reducibility of the Generalized Van Hove Prequantization

3.2.1 Notation

It is convenient to define some notation that is unique to this chapter.

- $\mathcal{L}$ is a sub Lie algebra of SHF. It is the infinite Lie algebra on $L\mathbb{R}^n$ generated by polynomials up to polynomial degree two in $\hat{\pi}$ and $\hat{q}$.

- Let $\mathcal{F}$ be the set of all $\hat{f} \in SHF$ such that the n-symplectic Hamiltonian vector field $X_{\hat{f}}$ is complete.
- In this section we will be working exclusively with $\xi_{\hat{f}}^{(\alpha)}$ defined in (3.2). Hence, we will simply write $\xi_{\hat{f}}$ for convenience.
- A delta used with only 1 upper or 1 lower index should be interpreted as follows; for $h^{M-\delta_k}$, $M - \delta_k = m^1 - \delta_k^1,\ m^2 - \delta_k^2, \cdots, m^n - \delta_k^n$

- The symbol $\not\sum$ means "no sum".

3.2.2 The Transformation W

In this section we define an important unitary operator on $L^2(L\mathbb{R}^n, \mathbb{C})$. Fix an $\alpha = \alpha_i \hat{r}^i \in \mathbb{R}^{n*}$ such that $\alpha_i \neq 0$ for each i. Define the inverse of α, by $\beta^a = \frac{1}{|\alpha|^2}\delta^{ab}\alpha_b$. For $\phi(\pi, q) \in L^2(L\mathbb{R}^n, \mathbb{C})$ define the transformation $W : L^2(L\mathbb{R}^n, \mathbb{C}) \to L^2(L\mathbb{R}^n, \mathbb{C})$ by

$$W\phi(\pi_j^i, q^k) = \left(\frac{1}{2\pi}\right)^{n^2/2} \int e^{iw_b^a \pi_a^b} \phi(w_j^i, q^k - \beta^l w_l^k) dw \tag{3.3}$$

where the convention $\int dw = \int_\infty^\infty \int_\infty^\infty \cdot\cdot \int_\infty^\infty dw_1^1 dw_2^1 \cdot\cdot dw_n^n$ has been used for simplification. Note that W is a Fourier transform of $\psi(\pi) = \phi(\pi, q - \alpha\pi)$; therefore by the Plancherel theorem W is unitary. Also since W is a Fourier transform it has an inverse W^{-1}. Now consider the following operators on $L^2(L\mathbb{R}^n, \mathbb{C})$:

$$\begin{aligned}
A^a_b &= -i\beta^a \frac{\partial}{\partial q^b} \\
B^{ja}_b &= \frac{1}{n}\delta^a_b q^j \\
\bar{A}^a_b &= -i\frac{\partial}{\partial \pi^b_a} \\
B^i_j &= \pi^i_j.
\end{aligned} \tag{3.4}$$

Given these definitions, one can check the following relations.

$$\begin{aligned}
P^a_b &\equiv WA^a_bW^{-1} = -i\beta^a\frac{\partial}{\partial q^b} = A^a_b \\
Q^{ja}_b &\equiv WB^{ja}_bW^{-1} = \frac{1}{n}[\delta^a_b q^j + i\delta^a_b\beta^c\frac{\partial}{\partial \pi^c_j}] \\
\bar{P}^a_b &\equiv W\bar{A}^a_bW^{-1} = -\pi^a_b - i\beta^a\frac{\partial}{\partial q^b} \\
\bar{Q}^a_b &\equiv WB^a_bW^{-1} = -i\frac{\partial}{\partial \pi^b_a} = \bar{A}^a_b
\end{aligned} \tag{3.5}$$

It is important to note the following;

$$\begin{aligned}
P^a_b &= -i\beta^a\xi_{\hat{\pi}_b} \\
Q^{ja}_b &= -i\beta^a\xi_{\hat{q}^j_b}
\end{aligned} \tag{3.6}$$

where β^a is the inverse of α_a defined by $\beta^a = \frac{1}{|\alpha|^2}\delta^{ab}\alpha_b$. Therefore the operators P, Q satisfy various commutation relations induced by their relations to ξ, e.g. $[P^a_b, P^k_l] = -\beta^c\beta^k[\xi(\hat{\pi}_b), \xi(\hat{\pi}_l)] = 0$.

3.2.3 Function of Operators

Now we want to represent $\xi_{\hat{f}}$ as a function of the operators $P, Q, \bar{P}, \bar{Q}$ defined above. First notice that we have the following relations

$$\begin{aligned}
q^j &= Q^{ja}_a + \beta^c\bar{Q}^j_c \\
\pi^i_j &= P^i_j - \bar{P}^i_j.
\end{aligned} \tag{3.7}$$

These are clear from equations (3.5). When we do not want to stress the indices, we will denote each of the previous bases symbolically by $q = Q + \beta \cdot \bar{Q}$ and $\pi = P - \bar{P}$. Now recall from (3.2), $\hat{f} \in SHF^{p+1}$

$$\xi^{I_p}_{\hat{f}}\psi = \left[\frac{1}{(p+1)!}\frac{\partial \hat{f}^{I_p b}}{\partial \pi^b_a}\frac{\partial}{\partial q^a} - \frac{1}{(p+1)!}\frac{\partial \hat{f}^{I_p a}}{\partial q^b}\frac{\partial}{\partial \pi^a_b} - i\alpha_c \frac{1}{(p+1)!}[\frac{\partial \hat{f}^{I_p b}}{\partial \pi^b_a}\pi^c_a - \hat{f}^{I_p c}]\right]\psi$$

Collecting terms on $\frac{\partial \hat{f}}{\partial \pi}$ and $\frac{\partial \hat{f}}{\partial q}$ we get

$$\xi^{I_p}_{\hat{f}}\psi = \frac{1}{(p+1)!}[i\alpha_c \hat{f}^{I_p c} + \frac{\partial \hat{f}^{I_p b}}{\partial \pi^b_a}(-i\alpha_c \pi^c_a + \frac{\partial}{\partial q^a}) - \frac{\partial \hat{f}^{I_p c}}{\partial q^b}\frac{\partial}{\partial \pi^c_b}]\psi. \tag{3.8}$$

Finally, using (3.5,3.7) and this last equation we can write

$$\begin{aligned}
\xi^{I_p}_{\hat{f}}\psi &= \frac{1}{(p+1)!}[i\alpha_c \hat{f}^{I_p c}(P - \bar{P}, Q + \beta \cdot \bar{Q}) + \frac{\partial \hat{f}^{I_p} b}{\partial \pi^b_a}(P - \bar{P}, Q + \beta \cdot \bar{Q}) i\alpha_c \bar{P}^c_a \\
&- \frac{\partial \hat{f}^{I_p c}}{\partial q^b}(P - \bar{P}, Q + \beta \cdot \bar{Q}) i\bar{Q}^b_c]\psi.
\end{aligned} \tag{3.9}$$

Therefore we have proved the following result:

Theorem 3.1 *For $\hat{f} \in \mathcal{F}$ and for $\alpha_i \neq 0$ one has the equality*

$$\xi_{\hat{f}}\psi = \frac{1}{(p+1)!} i[\alpha_j \hat{f}(P - \bar{P}, Q + \beta \cdot \bar{Q}) + \frac{\partial \hat{f}}{\partial \pi}(P - \bar{P}, Q + \beta \cdot \bar{Q})\alpha_j \bar{P} - \frac{\partial \hat{f}}{\partial q}(P - \bar{P}, Q + \beta \cdot \bar{Q})\bar{Q}]\psi$$

This theorem allows us to write any Hamiltonian vector field as a function of these special operators $P, Q, \bar{P}, \bar{Q}$.

3.2.4 Choosing a Basis

Our proof is a generalization of a proof by Van Hove[3]. For the proof of reducibility of $\mathcal{U}^{(\alpha)}_{\mathcal{L}} = \{\xi_{\hat{f}} | \hat{f} \in \mathcal{L}\}$ and subsequent proof of full rank irreducibility of $\mathcal{U}^{(\alpha)}_{\Gamma} = \{\xi_{\hat{f}} | \hat{f} \in SHF\}$ we need to construct a certain basis of the Hilbert space $\mathcal{H}$. Recall that the Hermite functions of one real variable have the form

$$h_m(x) = e^{x^2/2}\frac{d^m}{dx^m}e^{-x^2}, \quad m = (0, 1, 2..).$$

They enjoy the following well-known properties

$$\begin{aligned}
xh_m(x) &= mh_{m-1}(x) + \frac{1}{2}h_{m+1}(x), \\
\frac{d}{dx}h_m(x) &= mh_{m-1}(x) - \frac{1}{2}h_{m+1}(x), \\
\frac{1}{2}(x^2 - \frac{d^2}{dx^2})h_m(x) &= (m + \frac{1}{2})h_m(x).
\end{aligned} \tag{3.10}$$

The last property we need is that the Hermite functions form an orthogonal basis for $L^2(\mathbb{R}, \mathbb{C})$. Now we want to extend these in a natural way to $L^2(L\mathbb{R}^n, \mathbb{C})$. Define

$$h_K^M = h_{m_1}(q^1) \cdot \cdot h_{m_n}(q^n) h_{k_1^1}(\pi_1^1) h_{k_1^2}(\pi_1^2) \cdot \cdot h_{k_n^n}(\pi_n^n) \tag{3.11}$$

These new functions form an orthogonal basis in the Hilbert space $L^2(L\mathbb{R}^n, \mathbb{C})$ since products of 1-variable functions are dense in functions of more than one variable. Now consider the transformation of these functions by W. Define

$$\tilde{h}_K^M = W h_K^M. \tag{3.12}$$

These new functions also form an orthogonal basis for $L^2(L\mathbb{R}^n, \mathbb{C})$ since W is unitary. This is the basis of the proof we have in mind. Notice that this basis depends on choice of α since W depends on α. Finally a basis for $\mathcal{H}$ can be constructed from the following objects. For each I_p let ${}^{I_p}\tilde{h}\hat{r}_{I_p}$ ($\sum$ I_p) be an $\otimes_s^p \mathbb{R}^n$ valued Hermite function. Denote the set of these objects

$$\mathcal{B}^p = \{{}^{I_p}\tilde{h}\hat{r}_{I_p} \quad (\quad \sum \; I_p)\} \tag{3.13}$$

These form an orthogonal basis for $\mathcal{H}^p$. Since $\mathcal{H}$ is the completion of the direct sum of all $\mathcal{H}^p$ then the union of all $\mathcal{B}^p$ forms an orthogonal basis for $\mathcal{H}$. Denote this set

$$\mathcal{B} = \bigcup_{p=1}^{\infty} \mathcal{B}^p \tag{3.14}$$

Now we will look at the operation of some combinations of the operators P, Q, etc. on this basis. Since the components of each basis element is a Hermite function it is enough to show the action on an arbitrary Hermite function.

$$\begin{aligned}\frac{1}{2}((\alpha_a P_k^a)^2 + (Q_b^{kb})^2)(\tilde{h}_N^M) &= (m_k + \frac{1}{2})(\tilde{h}_K^M)\\ \frac{1}{2}((\bar{P}_l^k)^2 + (\bar{Q}_l^k)^2)(\tilde{h}_N^M) &= (n_l^k + \frac{1}{2})(\tilde{h}_K^M)\end{aligned} \tag{3.15}$$

$$\begin{aligned}i\alpha_a P_b^a(\tilde{h}_K^M) &= [m_b(\tilde{h}_K^{M-\delta_b}) - \frac{1}{2}(\tilde{h}_K^{M+\delta_b})]\\ i\alpha_a \bar{P}_b^a(\tilde{h}_K^M) &= \alpha_a[k_b^a(\tilde{h}_{K-\delta_b\delta^a}^M) - \frac{1}{2}(\tilde{h}_{K+\delta_b\delta^a}^M)]\end{aligned} \tag{3.16}$$

From equations (3.6) and equation (3.15) we immediately have

$$\frac{1}{i^2 2}\sum_a \beta^a\beta^a(\alpha_a\alpha_a[\xi(\hat{\pi}_b)]^2 + [\xi(\hat{q}_a^b)]^2)(\tilde{h}_K^M) = (m_b + \frac{1}{2})(\tilde{h}_K^M). \tag{3.17}$$

Direct applications of Theorem 3.1 give

$$\xi_{\hat{q}_b^j \hat{q}_c^i} = \frac{i}{2}\alpha_b[\sum_a Q_a^{ja}Q_a^{ia} - \sum_s \beta^s\beta^s\bar{Q}_s^j\bar{Q}_s^i]\hat{r}_c \tag{3.18}$$

$$\xi_{\hat{\pi}_k\hat{\pi}_l} = \frac{i}{2}\alpha_b[P_k^a P_l^b - \bar{P}_k^a\bar{P}_l^b]\hat{r}_a \tag{3.19}$$

$$\xi_{\hat{q}_j^i\hat{\pi}_k} = \frac{i}{2}\delta_j^a[\alpha_c Q_b^{ib}P_k^c + \bar{P}_k^l\bar{Q}_l^i]\hat{r}_a \tag{3.20}$$

$$\xi_{q^i\hat{\pi}_k} = \frac{i}{2}[\alpha_c Q_b^{ib}P_k^c + \bar{P}_k^l\bar{Q}_l^i] \tag{3.21}$$

3.2.5 Reduction of $\mathcal{L}$

First notice that our set of operators $\mathcal{U}_\Gamma^{(\alpha)}$ is highly reducible in the traditional sense. For example let H_k be the set of all Hermite polynomials of rank k. Then $\mathcal{U}_\Gamma^{(\alpha)}$ preserves the subset $\bigoplus_{m=k}^{\infty}$ for each k. However if we force our subsets to be full rank we eliminate this possibility.

Definition 3.2 *A subspace of $\mathcal{H}$ is full rank if it contains at least one non zero element of each tensor rank. A set of operators X is full rank irreducible if $X(A) \in A$ where A a full rank subset then $A = \mathcal{H}$.*

In this section we will give the proof of reducibility for $\mathcal{U}_\mathcal{L}^{(\alpha)}$. Again, we follow Van Hove's guidance and apply his method of proof to our unique spaces. We start by assuming there exists a decomposition of $\mathcal{H}$ into a direct sum of full rank linearly closed manifolds invariant under $\mathcal{U}_{\mathcal{L}}^{(\alpha)}$. The structure of our Hilbert space, namely our special basis, forces conditions, written as a series of lemmas, on these manifolds. Finally, we show there exists a unique decomposition.

Suppose we can decompose $\mathcal{H}$ into a direct sum of two full rank linearly closed manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ invariant under transformations by $\mathcal{U}_{\mathcal{L}}^{(\alpha)}$. Let E_i represent the orthogonal projection to each space $\mathcal{M}_i$. Then for each $\psi \in \mathcal{H}$, $\psi = E_1\psi + E_2\psi$. Furthermore:

Lemma 3.1 *Every element satisfying equations (3.15) is of the form $\lambda\tilde{h}$ where λ is some complex scalar and $\tilde{h}$ is a generalized Hermite function .*

Proof

Suppose $\psi = \psi^{I_p}\hat{r}_{I_p} \in \mathcal{H}^p$ such that ψ^{I_p} satisfies (3.15). If $\psi^{I_p} = 0$ we are done. Suppose $\psi^{I_p} \neq 0$. Since $\mathcal{B}^p$ forms an orthogonal basis for $\mathcal{H}^p$ we can write $\psi^{I_p} = \sum_{M,K} a_{M(I_p)}^{K(I_p)}\, {}^{I_p}h_{K(I_p)}^{M(I_p)}$ where M and K are functions of the multi-index I_p. The sum is over all possible multi-indices M and K, and $a_M^K =< \psi^{I_p}, {}^{I_p}h_K^M >$. For this proof denote $\frac{1}{2}((\alpha_a P_k^a)^2 + (Q_b^{kb})^2) = Op_k$. Since ψ^{I_p} satisfies (3.15) we know

$$Op_k(\psi^{I_p}) = (m'_k(I_p) + 1/2)\psi^{I_p} \quad \forall k$$

In the equation directly above, $m'_k(I_p)$ is a fixed function of I_p, for each I_p. Expanding $\psi^{I_p} = \sum_{M,K} a_{M(I_p)}^{K(I_p)}\, {}^{I_p}h_{K(I_p)}^{M(I_p)}$ and substituting we get

$$Op_k(\sum_{M,K} a_{M(I_p)}^{K(I_p)}\, {}^{I_p}h_{K(I_p)}^{M(I_p)}) = (m'_k(I_p) + 1/2) \sum_{M,K} a_{M(I_p)}^{K(I_p)}\, {}^{I_p}h_{K(I_p)}^{M(I_p)} \qquad (3.22)$$

To analyze the components, bracket $Op_k(\psi^{I_p})$ with the basis element, ${}^{I_p}h_{\tilde{K}(I_p)}^{\tilde{M}(I_p)} \in \mathcal{B}^p$. Since Op_k is an essentially self adjoint operator, we have the following

$$< Op_k(\psi^{I_p}), \quad {}^{I_p}h_{\tilde{K}(I_p)}^{\tilde{M}(I_p)} >=< \psi^{I_p}, \quad Op_k({}^{I_p}h_{\tilde{K}(I_p)}^{\tilde{M}(I_p)}) >$$

Substituting from (3.22) above and using (3.15) to expand $Op_k({}^{I_p}h_{\tilde{K}(I_p)}^{\tilde{M}(I_p)})$ the equation becomes

$$
\begin{aligned}
&< (m'_k(I_p)+1/2)\sum_{M,K} a^{K(I_p)}_{M(I_p)}\,{}^{I_p}h^{M(I_p)}_{K(I_P)},\quad {}^{I_p}h^{\tilde{M}(I_p)}_{\tilde{K}(I_p)} > \\
= \quad &< \sum_{M,K} a^{K(I_p)}_{M(I_p)}\,{}^{I_p}h^{M(I_p)}_{K(I_P)},\quad (\tilde{m}(I_p)+1/2)({}^{I_p}h^{\tilde{M}(I_p)}_{\tilde{K}(I_p)}) >
\end{aligned}
$$

Evaluating both of the brackets yields

$$
(m'_k(I_p)+1/2)a^{\tilde{K}(I_p)}_{\tilde{M}(I_p)} = (\tilde{m}_k(I_p)+1/2)a^{\tilde{K}(I_p)}_{\tilde{M}(I_p)}
$$

This statement implies: either

$$
m'_k(I_p) = \tilde{m}_k(I_p) \qquad \forall k
$$

or

$$
a^{\tilde{K}(I_p)}_{\tilde{M}(I_p)} = 0
$$

Since $\psi^{I_p} \neq 0$ and the basis element, ${}^{I_p}h^{\tilde{M}(I_p)}_{\tilde{K}(I_p)}$, was arbitrarily chosen, without loss of generality we can assume $a^{\tilde{K}(I_p)}_{\tilde{M}(I_p)} \neq 0$. Hence, we have $m'_k(I_p) = \tilde{m}_k(I_p) \quad \forall k$. Since this is true for all k,

$$
\tilde{M}(I_p) = M'(I_p)
$$

Therefore, for every basis element ${}^{I_p}h^{\tilde{M}(I_p)}_{\tilde{K}(I_p)}$ such that $a^{\tilde{K}}_{\tilde{M}} =< \psi^{I_p}, {}^{I_p}h^{\tilde{M}(I_p)}_{\tilde{K}(I_p)} > \neq 0$ we have ${}^{I_p}h^{\tilde{M}(I_p)}_{\tilde{K}(I_p)} = {}^{I_p}h^{M'(I_p)}_{\tilde{K}(I_p)}$. Hence, we can write $\psi^{I_p} = \sum_K b^{K(I_p)} \quad {}^{I_p}h^{M'(I_p)}_{K(I_p)}$. By a similar argument, one can show there is only one $K'(I_p)$, therefore ${}^{I_p}\psi = B^{I_p}h^{M'(I_p)}_{K'(I_p)}$.$\Box$

Lemma 3.2 *The subspace $\mathcal{B}_k$ defined by $E_k\mathcal{B} = \mathcal{B}_k$ forms a basis of $\mathcal{M}_k$ (k=1,2).*

Proof

Consider the basis element ${}^{I_p}\tilde{h}\hat{r}_{I_p}$. This basis element can be written ${}^{I_p}\tilde{h} = E_1\,{}^{I_p}\tilde{h} + E_2\,{}^{I_p}\tilde{h}$ and satisfies equations (3.15). Since the operators P,Q etc. are linear $E_i\,{}^{I_p}\tilde{h}$ satisfies (3.15) for each i. Therefore $E_i\,{}^{I_p}\tilde{h} = a_i^{I_p}\tilde{h}$ for some scalar a_i. This implies that $a_1 + a_2 = 1$. Finally, $\mathcal{M}_1$ and $\mathcal{M}_2$ are orthogonal so one of the scalars a_1, a_2 must be zero. $\Box$

Lemma 3.3 *Let ${}^{I_p}\tilde{h}^M_K\hat{r}_{I_p} \in \mathcal{B}_1 \quad \sum I_p$ and M, K fixed multi-indices. Then ${}^{I_p}\tilde{h}^{M'}_K\hat{r}_{I_p} \in \mathcal{B}_1\ \forall M' \quad \sum I_p$ and $f^{I_c\,I_p}\tilde{h}^{M'}_K\hat{r}_{I_p} \otimes_s \hat{r}_{I_c} \in \mathcal{B}_1\ \forall c, M', \quad \sum I_p, \quad \sum I_c$ where f^{I_c} is a complex scalar for all $i_1, \cdots, i_c$. Similarly for $\mathcal{B}_2$*

Proof

To prove this lemma we will need a couple of propositions.

Suppose ${}^{I_p}\tilde{h}^M_K\hat{r}_{I_p} \in \mathcal{B}_1 \subset \mathcal{M}_1 \quad \sum I_p$ and M, K fixed multi-indices. Then by equation (3.15),

$$\xi_{\hat{\pi}_b}({}^{I_p}\tilde{h}^M_K) = \alpha_a P^a_b({}^{I_p}\tilde{h}^M_K) = [m_b({}^{I_p}\tilde{h}^{M-\delta_b}_K) - \frac{1}{2}({}^{I_p}\tilde{h}^{M+\delta_b}_K)] \tag{3.23}$$

The right hand side of this equation is also in $\mathcal{M}_1$ since $\mathcal{U}^{(\alpha)}_{\hat{\pi}_b}\mathcal{M}_1 \subset \mathcal{M}_1$.

Proposition 3.1 *If ${}^{I_p}\tilde{h}^M_K\hat{r}_{I_p} \in \mathcal{B}_1 \quad \sum I_p$, and M, K fixed multi-indices, then ${}^{I_p}\tilde{h}^{M-\delta_b}_K$, ${}^{I_p}\tilde{h}^{M+\delta_b}_K \in \mathcal{B}_1$. Similarly for $\mathcal{B}_2$.*

Proof

Suppose not, then ${}^{I_p}\tilde{h}^{M-\delta_b}_K$, ${}^{I_p}\tilde{h}^{M+\delta_b}_K \notin \mathcal{B}_1$. Since the right hand side of (3.23) belongs to $\mathcal{M}_1$ we can expand in terms of elements of $\mathcal{B}_1$.

$$\alpha_a P^a_b({}^{I_p}\tilde{h}^M_K) = [m_b({}^{I_p}\tilde{h}^{M-\delta_b}_K) - \frac{1}{2}({}^{I_p}\tilde{h}^{M+\delta_b}_K)] = \sum_{M',K'} a^{K'}_{M'}\,{}^{I_p}\tilde{h}^{M'}_{K'}$$

where the sum is over all M', K' such that ${}^{I_p}\tilde{h}^{M'}_{K'} \in \mathcal{B}_1$. Now bracket both sides with an element of $\mathcal{B}_1$, ${}^{I_p}\tilde{h}^{M'}_{K'}$. The LHS will always be zero since by supposition ${}^{I_p}\tilde{h}^{M-\delta_b}_K$, ${}^{I_p}\tilde{h}^{M+\delta_b}_K \notin \mathcal{B}_1$. We have the condition,

$$0 = a^{K'}_{M'} \quad \forall M', K'$$

This implies all components are zero and hence $m_b({}^{I_p}\tilde{h}^{M-\delta_b}_K) - \frac{1}{2}({}^{I_p}\tilde{h}^{M+\delta_b}_K) = 0$. This is a contradiction since $m_b({}^{I_p}\tilde{h}^{M-\delta_b}_K) - \frac{1}{2}({}^{I_p}\tilde{h}^{M+\delta_b}_K)$ can never be zero. Therefore, one or the other of ${}^{I_p}\tilde{h}^{M-\delta_b}_K$, ${}^{I_p}\tilde{h}^{M+\delta_b}_K$ is in $\mathcal{B}_1$. Since $\mathcal{M}_1$ is a linearly closed manifold, the other must belong to $\mathcal{B}_1$ as well.□

Consider the symmetric Hamiltonian function $f^{I_c}\hat{\pi}_b\hat{r}_{I_c} \in \mathcal{L}$ ($\sum I_c$), where f^{I_c} is a complex scalar for all $i_1, \cdots, i_c$, and the operator generated by this function $\xi_{f^{I_c}\hat{\pi}_b}\hat{r}_{I_c} \in \mathcal{U}^{(\alpha)}_{\mathcal{L}}$. Let this new operator act on ${}^{I_p}\tilde{h}^M_K\hat{r}_{I_p} \in \mathcal{B}_1, \quad \sum I_p$.

$$\xi_{f^{I_c}\hat{\pi}_b}\hat{r}_{I_c}({}^{I_p}\tilde{h}^M_K\hat{r}_{I_p}) = \frac{1}{(c+1)!}f^{I_c}[m_b({}^{I_p}\tilde{h}^{M-\delta_b}_K) - \frac{1}{2}({}^{I_p}\tilde{h}^{M+\delta_b}_K)]\hat{r}_{I_c}\otimes_s\hat{r}_{I_p} \qquad \sum I_p$$

Again, since $\mathcal{M}_1$ is closed the right hand side also belongs to $\mathcal{M}_1$.

Proposition 3.2 *If* ${}^{I_p}\tilde{h}^M_K\hat{r}_{I_p}\in\mathcal{B}_1 \quad \sum I_p$ *and* M*,* K *fixed multi-indices, then*

$$f^{I_cI_p}\tilde{h}^{M-\delta_b}_K,\ f^{I_cI_p}\tilde{h}^{M+\delta_b}_K \in \mathcal{B}_1\subset\mathcal{M}_1 \qquad \sum I_c$$

Similarly for $\mathcal{B}_2$*.*

A similar argument to the proof of proposition 3.1 will prove this one as well. □

In other words, closure under the operators $\mathcal{U}^{(\alpha)}_{\mathcal{L}}$ depends *only* on the lower(π) indices. We now investigate the specific requirements on those indicies.

Lemma 3.4 *If* ${}^{I_p}\tilde{h}^M_K\hat{r}_{I_p}\in\mathcal{B}_1 \quad \sum I_p$ *and* M*,* K *fixed multi-indices, then*

$${}^{I_p}\tilde{h}^M_{K-\delta_k\delta^b+\delta_j\delta^a}\hat{r}_{I_p},\ {}^{I_p}\tilde{h}^M_{K-\delta_k\delta^b-\delta_j\delta^a}\hat{r}_{I_p},\ {}^{I_p}\tilde{h}^M_{K+\delta_k\delta^b+\delta_j\delta^a}\hat{r}_{I_p}\in\mathcal{B}_1\subset\mathcal{M}_1$$

Similarly for $\mathcal{B}_2$*.*

Proof

Consider the operator $\xi_{\hat{\pi}_j\hat{\pi}_k}\in\mathcal{U}^{(\alpha)}_{\mathcal{L}}$ and let it act on the basis element ${}^{I_p}\tilde{h}^M_K\hat{r}_{I_p}\in\mathcal{B}_1, \quad \sum I_p$. Recall from (3.19) that

$$\xi_{\hat{\pi}_j\hat{\pi}_k} = \frac{i}{2}\alpha_b[P^a_jP^b_k - \bar{P}^a_j\bar{P}^b_k]\hat{r}_a$$

Then using equations (3.16) we see

$$\begin{aligned}
\xi_{\hat{\pi}_j\hat{\pi}_k}({}^{I_p}\tilde{h}^M_K\hat{r}_{I_p}) &= \frac{i}{2}\alpha_b[P^a_jP^b_k - \bar{P}^a_j\bar{P}^b_k]\hat{r}_a({}^{I_p}\tilde{h}^M_K\hat{r}_{I_p})\\
&= \frac{i}{2}\alpha_b[P^a_jP^b_k - \bar{P}^a_j\bar{P}^b_k]({}^{I_p}\tilde{h}^M_K)\hat{r}_a\otimes_s\hat{r}_{I_p}\\
&= [\frac{-i}{2}\beta^a\{m_k[(m_j-\delta^j_k){}^{I_p}\tilde{h}^{M-\delta_k-\delta_j}_K - \frac{1}{2}{}^{I_p}\tilde{h}^{M-\delta_k+\delta_j}_K]\\
&- \frac{1}{2}[(m_j+\delta^j_k){}^{I_p}\tilde{h}^{M+\delta_k-\delta_j}_K - \frac{1}{2}{}^{I_p}\tilde{h}^{M+\delta_k+\delta_j}_K]\}\\
&+ \frac{i}{2}\alpha_b\{k^b_k[(k^a_j-\delta^b_a\delta^j_k){}^{I_p}\tilde{h}^M_{K-\delta_k\delta^b-\delta_j\delta^a} - \frac{1}{2}{}^{I_p}\tilde{h}^M_{K-\delta_k\delta^b+\delta_j\delta^a}]\\
&+ \frac{1}{2}[(k^a_j+\delta^b_a\delta^j_k){}^{I_p}\tilde{h}^M_{K+\delta_k\delta^b-\delta_j\delta^a}\\
&- \frac{1}{2}{}^{I_p}\tilde{h}^M_{K+\delta_k\delta^b+\delta_j\delta^a}]\}]\hat{r}_a\otimes_s\hat{r}_{I_p}
\end{aligned} \tag{3.24}$$

All the terms which have the form $f^{aI_p}\tilde{h}^{\bar{M}}_{K}\hat{r}_a \otimes_s \hat{r}_{I_p}$ belong to $\mathcal{B}_1$ by lemma (3.2). All the other terms which vary in the lower multi-index have the form $f^{aI_p}\tilde{h}^{M}_{\bar{K}}\hat{r}_a \otimes_s \hat{r}_{I_p}$. These belong to $\mathcal{B}_1$ by using a linear combination arguement similar to the proof of (3.1).□
Notice that the parity, either even or odd, of the sum of the lower multi-indices, K, does not change after evaluation by $\xi_{\hat{\pi}_j\hat{\pi}_k}$. Now we come to the one of the main theorems.

Theorem 3.3 *The only possible choices for $\mathcal{M}_1$ and $\mathcal{M}_2$ are the manifolds $\mathcal{M}_+^{(\alpha)}$ and $\mathcal{M}_-^{(\alpha)}$, where $\mathcal{M}_+^{(\alpha)}$ has the basis $\mathcal{B}_+^{(\alpha)}=\{{}^{I_p}\tilde{h}^{\bar{M}}_{K}\hat{r}_{I_p} | \sum K$ is even, M anything$\}$ and $\mathcal{M}_-^{(\alpha)}$ has the basis $\mathcal{B}_-^{(\alpha)}=\{{}^{I_p}\tilde{h}^{\bar{M}}_{K}\hat{r}_{I_p} | \sum K$ is odd, M anything$\}$.*

Proof
Lemmas (3.3) and (3.4) show that $\mathcal{M}_+^{(\alpha)}$ and $\mathcal{M}_-^{(\alpha)}$ are possible choices, i.e. the operators used in the proofs of these lemmas don't change the parity of the basis elements. All that is left to show is uniqueness. If we can show one is strictly even then the orthogonality (and direct sum property) of these manifolds proves the other is strictly odd. Suppose ${}^{I_p}\tilde{h}^{M}_{K}\hat{r}_{I_p} \in \mathcal{B}_1 \quad \sum I_p$ such that $\sum K$ is even and ${}^{I_a}\tilde{h}^{M'}_{K'}\hat{r}_{I_a} \in \mathcal{B}_2 \quad \sum I_a$ such that $\sum K'$ is also even. Without loss of generality assume $p > a$. From lemma (3.4) it is clear that we can change the lower multi-index, K, of ${}^{I_p}\tilde{h}^{M}_{K}\hat{r}_{I_p}$ by a multiple of two, and thus not change the parity of the sum of the lower index, and stay in $\mathcal{B}_1$. We can do this multiple times, adding 1 to k_1 and -1 to k_4 or adding +2 to k_3 etc., to get

$$f^{I_cI_p}\tilde{h}^{M}_{K'}\hat{r}_{I_p} \otimes_s \hat{r}_{I_c} \in \mathcal{B}_1 \quad (\ \sum I_p)\ (\ \sum I_c) \tag{3.25}$$

Note two things. We have changed the lower index to K' but the operators used to do this have increased the tensor rank by some constant c, $f^{I_c} \in \mathbb{C}$. However, from our starting assumption, ${}^{I_a}\tilde{h}^{M'}_{K'}\hat{r}_{I_a} \in \mathcal{B}_2$. This implies, by lemma (3.3),

$$\tilde{f}^{I_{c+(p-a)}I_a}\tilde{h}^{M'}_{K'}\hat{r}_{I_a} \otimes_s \hat{r}_{I_{c+(p-a)}} \in \mathcal{B}_2 \quad (\ \sum I_a, I_{c+(p-a)})$$

Let $\tilde{f}^{I_{c+(p-a)}} = f^c\delta^{I_{p-a}}_{I_{p-a}}$, then we have

$$\begin{aligned}
& \tilde{f}^{I_{c+(p-a)}I_a}\tilde{h}^{M'}_{K'}\hat{r}_{I_a} \otimes_s \hat{r}_{I_{c+(p-a)}} \\
= \ & f^c\delta^{I_{p-a}I_a}_{I_{p-a}}\tilde{h}^{M'}_{K'}\hat{r}_{I_a} \otimes_s \hat{r}_{I_{c+(p-a)}} \\
= \ & f^{cI_{(p-a)+a}}\tilde{h}^{M'}_{K'}\hat{r}_{I_c} \otimes_s \hat{r}_{I_{(p-a)+a}} \text{ by rearranging the symmetric indices} \\
= \ & f^{cI_p}\tilde{h}^{M'}_{K'}\hat{r}_{I_c} \otimes_s \hat{r}_{I_p} \in \mathcal{B}_2 \quad (\ \sum I_c, I_p)
\end{aligned} \tag{3.26}$$

Equations (3.25, 3.26) give a contradiction since they violate the first part of lemma (3.3). Therefore, both of the basis elements given at the beginning of the proof must be in either $\mathcal{B}_1$ or $\mathcal{B}_2$. □

It will make the following proofs easier if we investigate $\mathcal{M}_+^{(\alpha)}$ and $\mathcal{M}_-^{(\alpha)}$ a little further. If a basis element ${}^{I_p}\tilde{h}_M^K$ is even or odd then the corresponding Hermite function ${}^{I_p}h_M^K = W^{-1}{}^{I_p}\tilde{h}_M^K$ is even in the π variables. In other words, $\mathcal{M}_+^{(\alpha)}$ and $\mathcal{M}_-^{(\alpha)}$ are determined only by the π slots of a Hilbert space element $\psi(\pi, q) \in \mathcal{H}$. From these two observations we see

$$W^{-1}\mathcal{M}_+^{(\alpha)} = \{\psi \in \mathcal{H}| \;\; \psi(-\pi_1^1, \cdots, -\pi_n^n, q^1, \cdots, q^n) = \psi(\pi_1^1, \cdots, \pi_n^n, q^1, \cdots, q^n)\}$$

$$W^{-1}\mathcal{M}_-^{(\alpha)} = \{\psi \in \mathcal{H}| \;\; \psi(-\pi_1^1, \cdots, -\pi_n^n, q^1, \cdots, q^n) = -\psi(\pi_1^1, \cdots, \pi_n^n, q^1, \cdots, q^n)$$

It is left to show that $\mathcal{M}_+^{(\alpha)}$ and $\mathcal{M}_-^{(\alpha)}$ are indeed full rank invariant linearly closed manifolds that orthogonally decompose $\mathcal{H}$.

Theorem 3.4 *$\mathcal{M}_+^{(\alpha)}$ and $\mathcal{M}_-^{(\alpha)}$, as defined above, are the unique linearly closed manifolds that orthogonally decompose $\mathcal{H}$, $\mathcal{H} = \mathcal{M}_+^{(\alpha)} \oplus \mathcal{M}_-^{(\alpha)}$ such that $\mathcal{M}_+^{(\alpha)}$ and $\mathcal{M}_-^{(\alpha)}$are invariant subspaces of $\mathcal{U}_{\mathcal{L}}^{(\alpha)}$.*

Proof

The spaces $\mathcal{M}_+^{(\alpha)}$ and $\mathcal{M}_-^{(\alpha)}$ clearly span $\mathcal{H}^n$. The spaces $\mathcal{M}_+^{(\alpha)}$ and $\mathcal{M}_-^{(\alpha)}$ are orthogonal since $\mathcal{B}_+^{(\alpha)}$ and $\mathcal{B}_-^{(\alpha)}$ are orthogonal. These spaces, $\mathcal{M}_+^{(\alpha)}$, $\mathcal{M}_-^{(\alpha)}$, are vector spaces and hence are linearly closed. Finally, we will show that $\mathcal{M}_+^{(\alpha)}$ and $\mathcal{M}_-^{(\alpha)}$ are invariant subspaces. Let ${}^{I_p}\tilde{h}_K^M \hat{r}_{I_p} \in \mathcal{B}_+^{(\alpha)} \;\; \sum I_p$. By definition of $\mathcal{B}_+^{(\alpha)}$ $\sum K$ is even. The subspace $\mathcal{L}$ is generated by five operators, $\{\xi_{\hat{\pi}_j\hat{\pi}_k}, \xi_{\hat{\pi}_k}, \xi_{\hat{q}_j^i\hat{\pi}_k}, \xi_{\hat{q}_j^i\hat{q}_k^a}, \xi_{\hat{q}_j^i}\}$. We will investigate each of these in turn.

In the proof of lemma 3.4, we found

$$\begin{aligned}
\xi_{\hat{\pi}_j\hat{\pi}_k}({}^{I_p}\tilde{h}_K^M \hat{r}_{I_p}) \;\; &= \;\; [\frac{-i}{2}\beta^a\{m_k[(m_j - \delta_k^j)^{I_p}\tilde{h}_K^{M-\delta_k-\delta_j} - \frac{1}{2}{}^{I_p}\tilde{h}_K^{M-\delta_k+\delta_j}] \\
&- \;\; \frac{1}{2}[(m_j + \delta_k^j)^{I_p}\tilde{h}_K^{M+\delta_k-\delta_j} - \frac{1}{2}{}^{I_p}\tilde{h}_K^{M+\delta_k+\delta_j}]\} \\
&+ \;\; \frac{i}{2}\alpha_b\{k_k^b[(k_j^a - \delta_a^b\delta_k^j)^{I_p}\tilde{h}^M_{K-\delta_k\delta^b-\delta_j\delta^a} - \frac{1}{2}{}^{I_p}\tilde{h}^M_{K-\delta_k\delta^b+\delta_j\delta^a}] \\
&+ \;\; \frac{1}{2}[(k_j^a + \delta_a^b\delta_k^j)^{I_p}\tilde{h}^M_{K+\delta_k\delta^b-\delta_j\delta^a} - \frac{1}{2}{}^{I_p}\tilde{h}^M_{K+\delta_k\delta^b+\delta_j\delta^a}]\}]\hat{r}_a \otimes_s \hat{r}_{I_p}
\end{aligned}$$

It is clear that this operation does not change the parity of ${}^{I_p}\tilde{h}_K^M \hat{r}_{I_p}$, since the lower indicies always change by two. We see from 3.15 that

$$\xi_{\hat{\pi}_b}({}^{I_p}\tilde{h}_K^M) = \alpha_a P_b^a({}^{I_p}\tilde{h}_K^M) = [m_b({}^{I_p}\tilde{h}_K^{M-\delta_b}) - \frac{1}{2}({}^{I_p}\tilde{h}_K^{M+\delta_b})]$$

This operator does not change the parity of ${}^{I_p}\tilde{h}_K^M \hat{r}_{I_p}$ as the lower index remains unchanged. For the last three operators it is useful to notice

$$\begin{aligned} W^{-1}Q_b^{ja} &= B_b^{ja}W^{-1} \qquad (3.27)\\ &= \frac{1}{n}\delta_b^a q^j W^{-1} \\ W^{-1}\bar{Q}_b^a &= B_b^a W^{-1} \qquad (3.28)\\ &= \pi_b^a W^{-1} \end{aligned}$$

Also recall that W^{-1} is linear, and

$$\bar{Q}_b^a \bar{Q}_k^c = W B_b^a W^{-1} W B_k^c W^{-1} = W B_b^a B_k^c W^{-1}$$

Similar relations hold for Q, P, and $\bar{P}$. Now from (3.18) we have

$$\xi_{(\hat{q}_b^j)^2} = \frac{i}{2}\alpha_b[\sum_a Q_a^{ja} Q_a^{ja} - \sum_s \beta^s \beta^s \bar{Q}_s^j \bar{Q}_s^j]\hat{r}_b$$

From above, acting by W^{-1} gives

$$W^{-1}\xi_{(\hat{q}_b^j)^2}\psi = \frac{i}{2}\alpha_b[q^j q^j - \sum_s \beta^s \beta^s \pi_s^j \pi_s^j](W^{-1}\psi)\hat{r}_b$$

The right hand side will have the same parity as ψ since the only added terms are q^2 and π^2, neither of which alter the parity in the π variables. Now consider the operator $\xi_{\hat{q}_j^i}$. From (3.6) we have

$$\xi_{\hat{q}_j^i}\psi = i\alpha_a Q_j^{ia}\psi$$

Let W^{-1} act on both sides to get

$$W^{-1}\xi_{\hat{q}_j^i}\psi = i\alpha_a \frac{1}{n}\delta_j^a q^i W^{-1}\psi$$

Clearly this does not alter the parity of the π variables. Finally, from (3.20) we have

$$\xi_{\hat{q}_j^i \hat{\pi}_k} = \frac{i}{2}\delta_j^a[\alpha_c Q_b^{ib} P_k^c + \bar{P}_k^l \bar{Q}_l^i]\hat{r}_a$$

Again we let W^{-1} act on both sides to get

$$W^{-1}\xi_{\hat{q}^i_j \hat{\pi}_k}\psi = \frac{1}{2}\delta^a_j[q^i\frac{\partial}{\partial q^k}\psi - (\delta^i_k\psi + \pi^i_l\frac{\partial}{\partial \pi^k_l}\psi)]\hat{r}_a$$

The first term of the right hand side only involves q's so it does not change the parity of the π variables. Likewise, the second term does not change the parity since it is just the identity. It is clear that the third term will not change the parity of polynomials or $e^{(\pi)^2}$. Since a Hermite function is a product of these two types of functions, the third term will not change the parity of a basis element. Since none of the generators change the parity of the basis element that one starts with, $\mathcal{M}^{(\alpha)}_+$ and $\mathcal{M}^{(\alpha)}_-$ are invariant subspaces of $\mathcal{U}^{(\alpha)}_L$.□

We get a main result of this paper as a corollary to Theorem 3.4.

Corollary 3.5 *The subspace of operators* $\mathcal{U}^{(\alpha)}_{\mathcal{L}}$ *is (full rank) reducible.*

3.2.6 The Irreducibility of $\mathcal{U}^{(\alpha)}_\Gamma$

In this section we will prove the full rank irreducibility of $\mathcal{U}^{(\alpha)}_\Gamma$. The hard work of the previous sections reduces the proof of this statement to essentially giving a counter example. We begin with the statement of the theorem.

Theorem 3.6 *The space of operators* $\mathcal{U}^{(\alpha)}_\Gamma$ *is full rank irreducible.*

Proof

Any full rank subspace of $\mathcal{H}$ that is invariant under transformations by $\mathcal{U}^{(\alpha)}$ must be invariant under transformations by $\mathcal{U}^{(\alpha)}_L$ since $\mathcal{L} \subset \Gamma$. By Theorem 3.4, the only possible choices for full rank invariant subspaces are $\mathcal{M}^{(\alpha)}_+$ and $\mathcal{M}^{(\alpha)}_-$. Consider the operator $\xi_{\hat{f}}$ generated by the function $\hat{f} = f^i\hat{r}_i$, where $f^1 = e^{-(q^1)^2}$ and $f^A = 0,\ \forall A \neq 1$. By theorem 3.1,

$$\xi_{f^i\hat{r}_i}\psi = i[\alpha_1 + 2Q^1\bar{Q}^1_1 + 2\beta^c\bar{Q}^1_c\bar{Q}^1_1]e^{-(q^1)^2}\psi$$

Taking the transform of both sides by W^{-1} and using the identity $W^{-1}(\phi e^{-(q^1)^2}) = e^{-(q^1+\beta^a\pi^1_a)^2}W^{-1}(\phi)$ we have

$$W^{-1}\xi_{f^i\hat{r}_i}\psi = i[W^{-1}\alpha_1 + 2q^1\pi^1_1 + 2\beta^c\pi^1_c\pi^1_1]e^{-(q^1+\beta^c\pi^1_c)^2}W^{-1}\psi \tag{3.29}$$

If $\mathcal{M}^{(\alpha)}_+$ was invariant under this transformation, then for $\psi \in \mathcal{M}^{(\alpha)}_+$ this would imply $\xi_{f^i\hat{r}_i}\psi \in \mathcal{M}^{(\alpha)}_+$. However, the above computation (3.29) shows this is not true for this operator. This operator takes even functions of π to odd functions. Thus by theorem 3.4, $\mathcal{U}_\Gamma^{(\alpha)}$ is irreducible.$\Box$

3.3 The Operators of the Generalized Van Hove Prequantization are Essentially Self Adjoint

3.3.1 The Group Γ

In the following section, we will show that a subset of the vector fields, $\xi_{\hat{f}}^{(\alpha)}$, described in(**??**) are in one to one correspondence to infinitesimal generators of a certain one parameter subgroup of transformations. In this section, we describe the group Γ and some of it's properties. The definition given is generalization of the definition of Γ in [3]. The main differences being a different space $A\mathbb{R}^n$ verses (q, p, s) and an $\mathbb{R}^n$ valued one form rather than an $\mathbb{R}$ valued one form.

Definition 3.7 *Denote by Γ the set of all bijective, C^∞ transformations from $A\mathbb{R}^n$ to $A\mathbb{R}^n$ that leave σ invariant.*

We can write the invariance of σ in several ways. First, let $\gamma \in \Gamma$. The invariance of σ gives

$$\gamma^* \sigma = \sigma \tag{3.30}$$

This follows immediately from $d_p\gamma(X) = X_{\gamma p}$, for $p \in A\mathbb{R}^n$ and $X \in \mathcal{X}(A\mathbb{R}^n)$.

Secondly, let (q^i, π^i_j, y^j) be global coordinates on $A\mathbb{R}^n$, then σ has the global coordinate representation

$$\sigma^i = dy^i + \pi^i_j dq^j$$

Let $\gamma \in \Gamma$ with $\gamma(q^i, \pi^i_j, y^i) = (q'^i, \pi'^i_j, y'^i)$. The invariance of σ gives the equality

$$dy^i + \pi^i_j dq^j = dy'^i + \pi'^i_j dq'^j \tag{3.31}$$

From (3.31) we see that $d\sigma$ is invariant as well as exterior products $(d\sigma)^n$ and $\sigma \wedge (d\sigma)^n$. We get the following relations from the invariance of $d\sigma$, $(d\sigma)^n$, and $\sigma \wedge (d\sigma)^n$; respectively

$$\begin{aligned} d\pi^i_j dq^j &= d\pi'^i_j dq'^j \qquad (3.32) \\ dq^1 \cdots dq^n d\pi^1_1 d\pi^1_2 \cdots d\pi^n_n &= dq'^1 \cdots dq'^n d\pi'^1_1 d\pi'^1_2 \cdots d\pi'^n_n \\ dy^1 \cdots dy^n dq^1 \cdots dq^n d\pi^1_1 d\pi^1_2 \cdots d\pi^n_n &= dy'^1 \cdots dy'^n dq'^1 \cdots dq'^n d\pi'^1_1 d\pi'^1_2 \cdots d\pi'^n_n \end{aligned}$$

We want to use the equations above to show transformations by Γ have a restricted form.

Proposition 3.3 *Let γ be a transformation in Γ. If $\gamma(q^i, \pi^i_j, y^i) = (q'^i, \pi'^i_j, y'^i)$ then*

$$\begin{aligned} q'^i &= q'^i(q^i, \pi^i_j) \\ \pi'^i_j &= \pi'^i_j(q^i, \pi^i_j) \\ y'^i &= y^i + \rho^i(q^i, \pi^i_j) \end{aligned}$$

Here ρ is an $\mathbb{R}^n$ valued C^∞ function of $\hat{q}$ and $\hat{\pi}$.

Proof

Let γ be a transformation in Γ such that $\gamma(q^i, \pi^i_j, y^i) = (q'^i, \pi'^i_j, y'^i)$. Expanding the differentials dq'^i and $d\pi'^i_j$ and using the equality (3.32) we have

$$\left(\frac{\partial q'^i}{\partial q^j}\frac{\partial \pi'^k_i}{\partial \pi^k_j} - \frac{\partial q'^i}{\partial \pi^k_j}\frac{\partial \pi'^k_i}{\partial q^j}\right) = 1 \tag{3.33}$$

$$\left(\frac{\partial q'^i}{\partial q^a}\frac{\partial \pi'^k_i}{\partial y^b} - \frac{\partial q'^i}{\partial y^b}\frac{\partial \pi'^k_i}{\partial q^a}\right) = 0 \tag{3.34}$$

$$\left(\frac{\partial q'^i}{\partial \pi^a_b}\frac{\partial \pi'^k_i}{\partial y^m} - \frac{\partial q'^i}{\partial y^m}\frac{\partial \pi'^k_i}{\partial \pi^a_b}\right) = 0 \tag{3.35}$$

Equations (3.33,3.34) give

$$\frac{\partial \pi'^i_j}{\partial y^a} = \frac{\partial q'^i}{\partial y^a} = 0$$

Now for the y'^i. Consider the vector field $X = \frac{\partial}{\partial y^i} = \frac{\partial y'^a}{\partial y^i}\frac{\partial}{\partial y'^a}$. Compute $\sigma(X)$ using both sides of (3.31) to get

$$\frac{\partial y'^a}{\partial y^i} = \delta^a_i$$

This shows $y'^i = y^i + \rho^i(q^i, \pi^i_j)$.$\square$

3.3.2 Infinitesimal Generators

Definition 3.8 *Let $\mathcal{F} \subset SHF$ be the set of all $\hat{f} \in SHF$ such that the Hamiltonian vector field of $\hat{f}$, $X_{\hat{f}}$, is complete.*

Theorem 3.9 *There is an bijective correspondence between C^∞ infinitesimal transformations of Γ and the observables $\mathcal{F}$.*

Proof

We know that all vector fields on AM that satisfy the relation $L_{X(^I\sigma^j)} = 0$ have a special form (3.1), and are generated by SHF. We show that $L_{\xi_{LM}}\sigma = 0$ for all $\xi \in l(\Gamma)$, the Lie algebra of Γ. We know (3.30),

$$\gamma^*\sigma = \sigma \quad \forall\gamma \in \Gamma$$

Therefore $\forall\xi \in l(\Gamma)$,

$$exp(t\xi)^*\sigma = \sigma$$

Recall $exp(t\xi)$ is the flow of ξ_{LM}. Hence we have

$$\begin{aligned} L_{\xi_{LM}}\sigma &= \frac{d}{dt}exp(t\xi)^*\sigma \\ &= \frac{d}{dt}\sigma = 0 \end{aligned} \tag{3.36}$$

Therefore the infinitesimal transformation of Γ must have the form $\xi_{\hat{f}}$ for some $\hat{f}$. The relation $\xi_{\hat{f}} \to \hat{f}$ is one to one. If one tries to generate a group element γ given a function $\hat{f}$ one sees that not all $\hat{f}$ work. Writing out the equations one would need to integrate it is not hard to see that the $\hat{f}$ that generate a $\gamma_{\hat{f}} \in \Gamma$ are the ones such that the Hamiltonian vector fields $X_{\hat{f}}$, are complete. $\square$

3.3.3 Operator $\mathcal{U}_\gamma^{(\alpha)}$

In this section we will define an important unitary operator $\mathcal{U}_\gamma^{(\alpha)}$. We will show this operator gives a copy of $L^2(L\mathbb{R}^n, \mathbb{C})$ in $L^2(A\mathbb{R}^n, \mathbb{C})$ for each α. Finally we will use this unitary operator to prove our prequantization is essentially self adjoint. To do so, write points of $A\mathbb{R}^n$ as (ω, y^i) where $\omega = (\hat{q}, \hat{\pi})$. For all transformations $\gamma \in \Gamma$, denote $\gamma(\omega, y^i) = (\omega', y^{i\prime})$ where $\omega' = \gamma(\omega)$ and $y^{i\prime} = y^i + \rho^i_\gamma(\omega)$.

Definition 3.10 *For each $\gamma \in \Gamma$ and $\alpha = (\alpha_i) \in \mathbb{R}^n$ such that $\alpha_i \neq 0$ let $\mathcal{U}_\gamma^{(\alpha)} : L^2(L\mathbb{R}^n, \mathbb{C}) \to L^2(A\mathbb{R}^n, \mathbb{C})$ be given by*

$$\mathcal{U}_\gamma^{(\alpha)}\psi(\omega) = e^{i\alpha_j\rho^j_\gamma(\gamma^{-1}\omega)}\psi(\gamma^{-1}\omega)$$

Proposition 3.4 *The operator $\mathcal{U}_\gamma^{(\alpha)}$ is unitary for all $\gamma \in \Gamma$ and for all $\alpha = (\alpha_i) \in \mathbb{R}^n$ such that $\alpha_i \neq 0$.*

Proof

Let $\psi,\ \phi \in \mathcal{H}$.

$$\begin{aligned}
< \mathcal{U}^{(\alpha)}_{\gamma}\psi(\omega), \mathcal{U}^{(\alpha)}_{\gamma}\phi(\omega) > &= \int_{A\mathbb{R}^n} \mathcal{U}^{(\alpha)}_{\gamma}\psi(\omega)\overline{\mathcal{U}^{(\alpha)}_{\gamma}\phi(\omega)}dW \\
&= \int_{A\mathbb{R}^n} e^{i\alpha_j\rho^j_\gamma(\gamma^{-1}\omega)}\psi(\gamma^{-1}\omega)e^{-i\alpha_j\rho^j_\gamma(\gamma^{-1}\omega)}\overline{\phi(\gamma^{-1}\omega)}dW \\
&= \int_{A\mathbb{R}^n} \psi(\gamma^{-1}\omega)\overline{\phi(\gamma^{-1}\omega)}dW \\
&= \int_{A\mathbb{R}^n} \psi(\omega)\overline{\phi(\omega)}d\gamma W \\
&= \int_{A\mathbb{R}^n} \psi(\omega)\overline{\phi(\omega)}dW \\
&= < \psi(\omega), \phi(\omega) > \quad \square
\end{aligned}$$

Given

$$\rho^i_{\gamma_1\gamma_2}(\omega) = \rho^i_{\gamma_1}(\omega) + \rho^i_{\gamma_2}(\gamma_1\omega)$$

we have

$$\mathcal{U}^{(\alpha)}_{\gamma_1\gamma_2} = \mathcal{U}^{(\alpha)}_{\gamma_1}\mathcal{U}^{(\alpha)}_{\gamma_2}$$

Finally, let γ_0 be the identity transformation in Γ. Using the definition of U we see

$$\mathcal{U}^{(\alpha)}_{\gamma_0}\phi(\omega) = e^{i\alpha_j\rho^j_{\gamma_0}(\gamma_0^{-1}\omega)}\phi(\gamma_0^{-1}\omega) = e^{i0}\phi(\omega) = \phi(\omega)$$

By a theorem of Stone(B,C page 8 [3]) the above implies there exists a unique self adjoint operator, call it $H^\alpha[\hat{f}]$ such that

$$\mathcal{U}^{(\alpha)}_{\gamma_t^{\hat{f}}} = e^{itH^\alpha[\hat{f}]} \tag{3.37}$$

In the above equation $\gamma_t^{\hat{f}}$ is the one parameter group generated by the infinitesimal transformation $\xi_{\hat{f}}$. The domain of $H^\alpha[\hat{f}]$ is the set of functions of $L^2(L\mathbb{R}^n, \mathbb{C})$ such that $\lim_{t\to 0} t^{-1}\left[\mathcal{U}^{(\alpha)}_{\gamma_t^{\hat{f}}}\phi - \phi\right]$ exists. We denote this domain $D^\alpha[\hat{f}]$.

Using equation (3.37) and expanding the exponential one easily sees

$$lim_{t\to 0}\frac{1}{t}\left[\mathcal{U}^{(\alpha)}_{\gamma_t^{\hat{f}}}\phi - \phi\right] = iH^\alpha[\hat{f}]\phi \quad \forall \phi \in D^\alpha[\hat{f}] \tag{3.38}$$

We want to construct a formula for $H^\alpha[\hat{f}]$. With this in mind consider the infinitesimal generator $\xi_{\hat{f}}$ and let it act on $\psi \in \mathcal{H}$. By definition

$$\xi_{\hat{f}}\psi(\omega, y^i) = lim_{t\to 0}\frac{1}{t}\left[\psi(\gamma_t^{\hat{f}}(\omega), y^i_{\gamma_t^{\hat{f}}}) - \psi(\omega, y^i)\right]$$

This last equation can be rewritten

$$\xi_{\hat{f}}\psi(\omega, y^i) = lim_{-t\to 0}\frac{1}{-t}\left[\psi(\gamma_{-t}^{\hat{f}}(\omega), y^i_{\gamma_{-t}^{\hat{f}}}) - \psi(\omega, y^i)\right] \tag{3.39}$$

Since $y^i_\gamma = y^i + \rho^i_\gamma(\omega)$ the inverse is $y^i_{\gamma^{-1}} = y^i - \rho^i_\gamma(\gamma^{-1}\omega)$. Also we have the relation $\gamma^{\hat{f}}_{-t} = (\gamma^{\hat{f}}_t)^{-1}$. Now take a specific function $\psi(\omega, y^i) = e^{i\alpha_j y^i}\phi(\omega)$ for $\phi \in D^\alpha[\hat{f}]$ and use equation (3.39).

$$\begin{aligned}
\xi_{\hat{f}}e^{i\alpha_j y^i}\phi(\omega) &= lim_{t\to 0}\frac{1}{t}\left[e^{i\alpha_j y^i_{\gamma^{\hat{f}}_{-t}}}\phi(\gamma^{\hat{f}}_{-t}(\omega)) - e^{i\alpha_j y^i}\phi(\omega)\right] \\
&= lim_{t\to 0}\frac{1}{t}\left[e^{i\alpha_j(y^j - \rho^j_\gamma(\gamma_{-t}\omega)}\phi(\gamma^{\hat{f}}_{-t}(\omega)) - e^{i\alpha_j y^i}\phi(\omega)\right] \\
&= lim_{t\to 0}\frac{1}{t}\left[e^{i\alpha_j y^j}e^{-i\alpha_j\rho^j_\gamma(\gamma_{-t}\omega)}\phi(\gamma^{\hat{f}}_{-t}(\omega)) - e^{i\alpha_j y^i}\phi(\omega)\right] \\
&= e^{i\alpha_j y^j}lim_{t\to 0}\frac{1}{t}\left[\mathcal{U}^{(\alpha)}_{\gamma^{\hat{f}}_t}\phi(\omega) - \phi(\omega)\right]
\end{aligned} \tag{3.40}$$

Hence we have

$$\xi_{\hat{f}}e^{i\alpha_j y^i}\phi(\omega) = e^{i\alpha_j y^j}lim_{t\to 0}\frac{1}{t}\left[\mathcal{U}^{(\alpha)}_{\gamma^{\hat{f}}_t}\phi(\omega) - \phi(\omega)\right]$$

Using (3.37) and (3.2) this equation becomes

$$H^\alpha[\hat{f}] = -i\xi^{(\alpha)}_{\hat{f}} \quad \forall\phi \in D^\alpha[\hat{f}] \tag{3.41}$$

Thus,

Theorem 3.11 *The components of the vector fields $H^\alpha[\hat{f}] = -i\xi^{(\alpha)}_{\hat{f}}$ are self adjoint on $D^\alpha[\hat{f}]$.*

3.3.4 Essentially Self Adjoint on $L^2(L\mathbb{R}^n, \mathbb{C})$

Let $\mathcal{D}$ be the set of all C^∞ functions in $L^2(L\mathbb{R}^n, \mathbb{C})$ vanishing outside of compact sets.

Theorem 3.12 *For all $\hat{f} \in SHF$ and all $\alpha^i \in \mathbb{R} - \{0\}$, the restriction of $H^\alpha[\hat{f}]$ to the domain $\mathcal{D}$ is essentially self adjoint.*

Proof

Notice that $\mathcal{D}$ is a subset of $D^\alpha[\hat{f}]$. Furthermore, $H^\alpha[\hat{f}]$ is an automorphism of $\mathcal{D}$. Given these two facts and the lemma in Foundations of Mechanics 2nd edition page 141 [12] gives the operator $H^\alpha[\hat{f}]$ restricted to $\mathcal{D}$ is essentially self adjoint on $\mathcal{D}$. $\square$

Chapter 4

Quantization of $L\mathbb{R}^n$

4.1 The Basic Set for $L\mathbb{R}^n$

Before we compute the basic set for $L\mathbb{R}^n$ we review the basic set for the cotangent bundle of $\mathbb{R}^n$. In [5] the basic set for $T^*\mathbb{R}^n$ is the Heisenberg algebra $b_{2n} \subseteq C^\infty(\mathbb{R}^n)$

$$\mathsf{b}_{2n} = span\{q^i, p_j, 1\} \cong h(2n) \tag{4.1}$$

This basic set consists of the span of the components of the momentum map created by the action of Heisenberg group $H(2n)$ acting on $\mathbb{R}^{2n}$[14]. This is the basic set needed for the Schrödinger representation.

Recall a technique mentioned earlier. If $\{\xi_i\}$ is a basis of $\mathcal{G}$ let $\{J_i\}$ be the $\otimes^p\mathbb{R}^n$ valued Hamiltonian functions for $(\xi_i)_{LM}$. Define $\hat{J}$ by $\hat{J}(\xi_i) = J_i$. This gives a n-symplectic momentum map J with components J_i. This is the procedure we outline below. Before we begin recall also that $L\mathbb{R}^n \cong \mathbb{R}^n \times Gl(n)$ and we have a global chart from $L\mathbb{R}^n \to \mathbb{R}^n \times Gl(n)$ given by the coordinates (q^i, π^i_j). To compute the n-symplectic momentum map needed to construct the basic set for the frame bundle $L\mathbb{R}^n$ we define a new group.

Definition 4.1 *Define the group*

$$H(L\mathbb{R}^n) = \mathcal{X}(L\mathbb{R}^n) \times^n S^1$$

with product

$$(u_1, z^1, \ldots, z^n) \cdot (u_2, w^1, \ldots, w^n) = (u_1 + u_2, z^1 w^1 exp(\frac{i}{2}A^1), \ldots, z^n w^n exp(\frac{i}{2}A^n))$$

In the above equation $A^k = d\theta^k(u_1, u_2)$ evaluated at the identity $(0, I)$.

For every element $(u, \vec{z})$ we have the inverse $(-u, \vec{z}^{-1})$. Notice the similarity to the Heisenberg group $H(2n)$. Given this similarity and following Guillemen and Sternberg [14] (section 15) we may identify the Lie algebra of this group $h(L\mathbb{R}^n)$ with $\mathcal{X}(L\mathbb{R}^n) \times^n \mathbb{R} = \mathcal{X}(L\mathbb{R}^n) \times \mathbb{R}^n$. The bracket for the Lie algebra being

$$[(u_1, \vec{v}), (u_2, \vec{w})] = (0, d\theta^1(u_1, u_2), \ldots, d\theta^n(u_1, u_2)) = (0, d\theta^i(u_1, u_2)\hat{r}_i)$$

Now we are ready to compute the components of the n-symplectic momentum map for $H(L\mathbb{R}^n)$ acting on $L\mathbb{R}^n$. Identify $L\mathbb{R}^n$ with a subset of $h(L\mathbb{R}^n)$ by $q^i\hat{r}_j \to \hat{q}^i_j \to (X_{\hat{q}^i_j}, 0)$, $\pi^j_k\hat{r}_j \to \hat{\pi}_k \to (X_{\hat{\pi}_k}, 0)$. Also identify the identity $\hat{r}_k \to (0, \hat{r}_k)$ and let $H(L\mathbb{R}^n)$ act on $h(L\mathbb{R}^n)$ via the adjoint action. For the elements $\xi = \xi^c_r\hat{q}^r_c + \xi^l\hat{\pi}_l,\ m = m^b_a\hat{q}^a_b + m^k\hat{\pi}_k \in h(L\mathbb{R}^n)$

$$\xi_{L\mathbb{R}^n}(m) = [\xi, m] = (0, d\theta^i(\xi, m)\hat{r}_i) = d\theta^i(\xi, m)\hat{r}_i = (\xi^b_a m^a - m^b_l\xi^l)\hat{r}_b \tag{4.2}$$

Given the above identification, a basis for $h(L\mathbb{R}^n)$ is $\{\hat{q}^i_j, \hat{\pi}_k, \hat{r}_k\} \to \{X_{\hat{q}^i_j}, X_{\hat{\pi}_k}, \hat{r}_k\}$. Using (4.2) we compute the infinitesimal generators of the basis, $(\hat{q}^i_j)_{L\mathbb{R}^n} = \frac{\partial}{\partial\hat{\pi}^j_i}$, $(\hat{\pi}_k)_{L\mathbb{R}^n} = \frac{\partial}{\partial\hat{q}^k}$, and $(\hat{r}_k)_{L\mathbb{R}^n} = 0$. The components of the momentum map are the Hamiltonian functions for these infinitesimal generators:

$$J^a_b = \hat{q}^a_b, J_k = \hat{\pi}_k, J_j = \hat{r_j}$$

We choose our basic set for $L\mathbb{R}^n$ to be the span of the components of this n-symplectic momentum map.

$$\mathsf{b}_L = span\{\hat{q}^i_j, \hat{\pi}_k, \hat{r}_j\} \cong h(L\mathbb{R}^n) \tag{4.3}$$

This is the analogue of the Heisenberg algebra for $L\mathbb{R}^n$. The Poisson brackets are

$$\{\hat{q}^i_j, \hat{\pi}_k\} = \delta^i_k\hat{r}_j$$

From the bracket we see b_L is a subalgebra of SHF. The Hamiltonian vector fields for the subalgebra b_L are $\{-\partial/\partial\pi^i_j, \partial/\partial q^k, 0\}$ respectively. The integral curves of these vector fields are linear and hence defined for all time. The set b_L is finitely generated and since q^i and π^i_j are global coordinates on $L\mathbb{R}^n$ they separate points. Likewise their Hamiltonian vector fields span $T(L\mathbb{R}^n)$. Thus b_L is indeed a basic set for $L\mathbb{R}^n$. The fact that we get the "hatted" versions of the coordinates instead of the coordinates themselves is a consequence of the fact that all observables on $L\mathbb{R}^n$ must be $\mathbb{R}^n$ valued.

4.2 Comparison of the Poisson Algebra of Polynomials for $T^*\mathbb{R}^n$ vs $L\mathbb{R}^n$

The existence of a "no-go" theorem for $T^*\mathbb{R}^n$ and the absence thereof for $L\mathbb{R}^n$ stems from the difference in their Poisson algebras, as we will see in a later section. In this section we explicitly show the differences.

4.2.1 The Polynomial Algebra $P(\mathsf{b}_{2n})$

Recall the basic set $\mathsf{b}_{2n} = span\{q^i, p_j, 1\} \subseteq C^\infty(\mathbb{R}^n)$ for $T^*\mathbb{R}^n$ from equation (4.1). Then $P(\mathsf{b}_{2n})$ is just polynomials of the variables (q^i, p_j), and the bracket is the standard Poisson bracket for $T^*\mathbb{R}^n$.

Computing a few important Poisson brackets we see

$$\begin{aligned}
\{q^i, p_j\} &= \delta^i_j \\
\{(q^i)^a, (q^j)^b\} &= 0 \\
\{(p_i)^a, (p_j)^b\} &= 0 \\
\{(q^i)^2, (p_j)^2\} &= 4\delta^i_j q^i p_j \\
\{(q^i)^3, (p_j)^3\} &= 9\delta^i_j (q^i)^2 (p_j)^2 \\
\{(q^i)^2 p_b, q^a (p_j)^2\} &= 3\delta^a_b (q^i)^2 (p_j)^2
\end{aligned} \tag{4.4}$$

The notations $(q^i)^a$ and $(p_j)^a$ denote an a-fold product in the underlying commutative algebra $P(b_{2n})$. The last two relations are the Poisson relations that lead to the Groenwold obstruction. Notable subalgebras of $P(\mathsf{b}_{2n})$ are polynomials of degree two or less and the affine subalgebra, which is the subalgebra of all polynomials linear in p_j. The Poisson algebra $P(\mathsf{b}_{2n})$ has no non-trivial ideals and satisfies

$$[P(\mathsf{b}_{2n}), P(\mathsf{b}_{2n})] = P(\mathsf{b}_{2n})$$

4.2.2 The Polynomial Algebra $P(\mathsf{b}_L)$

Recall $\mathsf{b}_L = span\{\hat{q}^i_j, \hat{\pi}_k, \hat{r}_k\}$ from equation (4.3) and consider the Poisson algebra $P(\mathsf{b}_L)$ with bracket defined in section 2.2. Elements of $P(\mathsf{b}_L)$ are polynomials of $(\hat{q}^i_j, \hat{\pi}_k, \hat{r}_k)$ and hence are $\otimes^m_s \mathbb{R}^n$ valued functions on $L\mathbb{R}^n$, m being the degree of the polynomial.

A typical monomial looks like $\hat{q}^{I_n}_{J_n}\hat{\pi}_{K_m}\hat{r}_{M_l} \in SHF^{n+m+l}$. Here the multiplication is the symmetric tensor product, $\hat{q}^i_j\hat{\pi}_k \equiv \hat{q}^i_j \otimes_s \hat{\pi}_k$.

Using the n-symplectic Poisson bracket defined in section 2.2 we compute some relevant brackets.

$$\begin{aligned}
\{\hat{q}^i_k, \hat{\pi}_j\} &= \delta^i_j \hat{r}_k \\
\{(\hat{q}^i_k)^a, (\hat{q}^j_l)^b\} &= 0 \\
\{(\hat{\pi}_i)^a, (\hat{\pi}_j)^b\} &= 0 \\
\{(\hat{q}^i_k)^2, (\hat{\pi}_j)^2\} &= 4\delta^i_j \hat{q}^i_k \hat{\pi}_j \hat{r}_k \\
\{(\hat{q}^i_k)^3, (\hat{\pi}_j)^3\} &= 9\delta^i_j (\hat{q}^i_k)^2 (\hat{\pi}_j)^2 \hat{r}_k \\
\{(\hat{q}^i_k)^2\hat{\pi}_b, \hat{q}^a_c(\hat{\pi}_j)^2\} &= 3\delta^a_b (\hat{q}^i_k)^2 (\hat{\pi}_j)^2 \hat{r}_c
\end{aligned} \tag{4.5}$$

Notice the similarity to the symplectic Poisson brackets above. The main difference is the $\otimes^p_s\mathbb{R}^n$ valued rank. We also have for any two polynomials $\hat{f}, \hat{g} \in P(\mathsf{b}_L)$

$$\{\hat{f} \otimes^{I_k}_s \hat{r}_{I_k}, \hat{g} \otimes^{J_l}_s \hat{r}_{J_l}\} = \{\hat{f}, \hat{g}\} \otimes^{M_{k+l}}_s \hat{r}_{M_{k+l}}$$

The multi-index M_{k+l} is (I_k, J_l). Now we define some important subalgebras.

Definition 4.2 *The set of all polynomials of degree 2 or less in* $\hat{q}^i_j$, $\hat{\pi}_k$ *with no restriction on the degree of* $\hat{r}_k$ *is denoted* P^2_L.

Definition 4.3 *The set of all polynomials linear in* $\hat{\pi}_k$ *is denoted* C_L.

Notice that the last part of definition 4.2 is necessary since the linear space of all polynomials of degree no greater than 2 no longer closes for $P(\mathsf{b}_L)$.

Next we define an important set within $P(\mathsf{b}_L)$.

Definition 4.4 *The set of all polynomials of* $\hat{q}^i_j$ *and* $\hat{\pi}_k$ *is denoted* $P(\hat{q}, \hat{\pi})$

This is just a subspace of $P(\mathsf{b}_L)$ and not a subalgebra, unlike $T^*\mathbb{R}^n$ where $P(q,p) = P(\mathsf{b}_{2n})$. Specifically it follows from the equations (4.5) that

$$\{P(\hat{q}, \hat{\pi}), P(\hat{q}, \hat{\pi})\} = P(\hat{q}, \hat{\pi})\hat{r}$$

However, we can partition $P(\mathsf{b}_L)$ using $P(\hat{q}, \hat{\pi})$.

$$P(\mathsf{b}_L) = \bigoplus_{k=0}^{\infty} P(\hat{q}, \hat{\pi})\hat{r}_{I_k}$$

This direct sum gives us an easy way to find ideals of $P(\mathsf{b}_L)$.

Definition 4.5 *Define subalgebras P^m of $P(\mathsf{b}_L)$ by $P^m = \bigoplus_{k=m}^{\infty} P(\hat{q}, \hat{\pi})\hat{r}_{I_k}$*

These subalgebras are nested

$$P^1 \supset P^2 \supset P^3 \supset \dots$$

Furthermore, for each m, P^m is an *ideal* of $P(\mathsf{b}_L)$ since

$$\{P^m, P^s\} \subset P^{m+s-1} \tag{4.6}$$

The previous equation is the most important feature of $P(\mathsf{b}_L)$. This equation states that *one cannot decrease rank in $P(\mathsf{b}_L)$ by taking Poisson brackets.*

4.3 Go theorem for $L(\mathbb{R}^n)$

Theorem 4.6 *There exists a full polynomial quantization of the polynomial algebra $P(\mathsf{b}_L)$ for the space $L\mathbb{R}^n$.*

We prove this existence theorem by giving two examples!

4.3.1 Ideal Quantization

Recall from section 4.2 equation (4.6) that

$$\{P^m, P^s\} \subset P^{m+s-1}$$

Also recall that $P^m = \bigoplus_{k=m}^{\infty} P(\hat{q}, \hat{\pi})\hat{r}_{I_k}$ is a Lie ideal for each m. Therefore we can write

$$P(\mathsf{b}_L) = \mathsf{b}_L + P^1$$

In the above equation, $+$ represents semi-direct sum with bracket given by

$$\{(\xi_1, \eta_1), (\xi_2, \eta_2)\} = (\{\xi_1, \xi_2\}, \{\xi_1, \eta_2\} - \{\xi_2, \eta_1\} + \{\eta_1, \eta_2\}\}$$

Thus we can obtain a full quantization of $P(\mathsf{b}_L)$ by quantizing b_L and setting $Q(P^1) = 0$. This is the approach taken by Gotay [5] when he exhibited a quantization of $T^*\mathbb{R}_+$. Quantize b_L as follows:

$$\begin{aligned} Q(\hat{q}^i_j) &= \alpha_j q^i && (4.7) \\ Q(\hat{\pi}_k) &= -i\hbar \frac{\partial}{\partial q^k} && (4.8) \\ Q(\hat{r}_k) &= \alpha_k && (4.9) \end{aligned}$$

For the last equation the α_k are constants such that $\alpha_k \neq 0$ for all k. It is trivial to check this map satisfies the definition of a prequantization. The Hilbert space is $L^2(\mathbb{R}^n, \mathbb{C})$ and the domain is the Schwartz space of all C^∞ rapidly decreasing functions. When restricted to b_L the quantization map is faithful and is equivalent to the n-symplectic Schrodinger representation. Irreducibility of $Q(\mathsf{b}_L)$ follows from the fact that the Schrodinger representation $\{q^i, \frac{\partial}{\partial q^k}\}$ is irreducible on $L^2(\mathbb{R}^n, \mathbb{C})$. Likewise the operators $Q|_{\mathsf{b}_L}$ are essentially self adjoint on $\mathcal{D}$. Finally, it is known that the Hermite polynomials form a dense set in $\mathcal{D}$ of separately analytic vectors for the Schrodinger representation.

Corollary 4.7 *There is no Groenwold van Hove type obstruction for quantizing $L\mathbb{R}^n$.*

By Groenwold van Hove type obstruction we mean an obstruction to quantization that:

- arises as a consequence of the irreducibility condition and the Poission bracket goes commutator condition and

- requires a restriction of the quantization to a subalgebra of observables to correct.

4.3.2 Another Full Polynomial Quantization

Let $A = (A^i_j)$ be a constant $n \times n$ Hermitian matrix. Another full quantization is given by the map $Q(P^2) = 0$ and

$$\begin{array}{ll}
Q(\hat{r}_k) = \alpha_k & Q(\hat{q}^{I_m}_{J_m}\hat{\pi}_{K_t}\hat{r}_s) = 0 \\
Q(\hat{q}^i_j) = \alpha_j q^i & Q(\hat{\pi}_k) = -i\hbar\frac{\partial}{\partial q^k} \\
Q(\hat{q}^{I_m}_{J_m}) = m\sum_{p=1}^m A^{i_1 i_2 \cdots \hat{i}_p \cdots i_m}_{J_{m-1}}\alpha_{j_m} q^{i_p} & Q(\hat{\pi}_{K_m}) = \alpha_{M_c}\pi^{M_c}_{K_m}\ \forall m \neq 1 \\
Q(\hat{q}^{I_m}_{J_m}\hat{r}_k) = A^{I_m}_{J_m}\alpha_k & Q(\hat{\pi}_{K_m}\hat{r}_s) = 0 \\
Q(\hat{q}^{I_m}_{J_m}\hat{\pi}_k) = -i\hbar A^{I_m}_{J_m}\frac{\partial}{\partial q^k} & Q(\hat{q}^{I_m}_{J_m}\hat{\pi}_{K_t}) = A^{I_m}_{J_m}\alpha_{M_c}\pi^{M_c}_{K_t} \quad \forall t \neq 1
\end{array} \tag{4.10}$$

In the above we have used the notation $A^{I_m}_{J_m} = A^{i_1}_{j_1}\cdots A^{i_m}_{j_m}$. This quantization is also easy to check since the only significant brackets have the form

$$\{\hat{q}^{I_t}_{J_t}, \hat{q}^{L_m}_{R_m}\hat{\pi}_k\} = \sum_{p=1}^{t} -t\delta^{i_p}_k \hat{q}^{i_1 i_2 \cdots \hat{i}_p \cdots i_t}_{J_{t-1}} \hat{q}^{L_M}_{R_m}\hat{r}_{j_t}$$

Computing the commutator of the operators we have

$$\frac{i}{\hbar}[Q(\hat{q}^{I_t}_{J_t}), Q(\hat{q}^{L_m}_{R_m}\hat{\pi}_k)] = \frac{i}{\hbar}[t\sum_{p=1}^{t} A^{i_1 i_2\cdots\hat{i}_p\cdots i_t}_{J_{t-1}}\alpha_{j_t}q^{i_p}, -i\hbar A^{I_m}_{J_m}\frac{\partial}{\partial q^k}]$$

$$= -t\sum_{p=1}^{t}\delta^{i_p}_k A^{i_1 i_2\cdots\hat{i}_p\cdots i_t}_{J_{t-1}} A^{L_m}_{R_m}\alpha_{j_t} = -t\sum_{p=1}^{t}\delta^{i_p}_k Q(q^{i_1 i_2\cdots\hat{i}_p\cdots i_t}_{J_{t-1}}\hat{q}^{L_M}_{R_m}\hat{r}_{j_t})$$

Notice that when restricted to the basic set this quantization is the same as the previous one. Thus the map Q satisfies the definition of a quantization given in section **??**.

4.3.3 Subalgebras of $P(\mathsf{b}_L)$

We can also quantize the two subalgebras mentioned in section 4.2. First consider the subalgebra P^2_L. Define a map

$$\begin{array}{ll} Q(\hat{r}_k) = \alpha_k & Q(\hat{q}^i_j\hat{\pi}_k) = -i\hbar\left(\alpha_j q^i\frac{\partial}{\partial q^k} + \delta^i_k\frac{1}{2}\right) \\ Q(\hat{q}^i_j) = \alpha_j q^i & Q(\hat{\pi}_k) = -i\hbar\frac{\partial}{\partial q^k} \\ Q(\hat{q}^i_j\hat{q}^a_b) = \alpha_j\alpha_b q^i q^a & Q(\hat{\pi}_k\hat{\pi}_b) = -\hbar^2\frac{\partial^2}{\partial q^k\partial q^b} \end{array} \tag{4.11}$$

Extend this map to all of P^2_L by linearity and the generalized von Neumann rule

$$Q(\hat{f}\hat{r}_k) = Q(\hat{f})\alpha_k \quad \forall\hat{f}\in P^2_L$$

In fact this particular choice of $Q(\hat{q}^i_j\hat{\pi}_k)$ forces the generalized von Neumann rule above to apply. This quantization of the subalgebra P^2_L is the direct analogue of the extended metaplectic quantization**??** of polynomials of degree 2 or less on $T^*\mathbb{R}^n$, and hence these operators are essentially self adjoint. Unfortunately, like its counterpart in symplectic geometry, this quantization cannot be extended to all of $P(\mathsf{b}_L)$. See section 4.5 for a proof. We can also quantize the subalgebra C_L. Recall

$$C_L = \{(\hat{f}^{I_k i}\pi^j_i + \hat{g}^{I_k j})\hat{r}_{I_k j}\},$$

where $\hat{f}^{I_k i}$ and $\hat{g}^{I_k j}$ are polynomials of $\hat{q}$ only. For each ν the map

$$Q_\nu(\hat{f}^{I_k i}\pi^j_i + \hat{g}^{I_k j})\hat{r}_{I_k j}) = -i\hbar\alpha_{I_k}((\hat{f}^{I_k i}\frac{\partial}{\partial q^i} + [\frac{1}{2} + i\nu]\frac{\partial\hat{f}^{I_k i}}{q^i}) + \alpha_j\hat{g}^{I_k j})$$

is a quantization of C_L. In the above equation $\alpha_{I_k} = \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}$ For $\nu = 0$ this is the n-symplectic version of the coordinate representation of all polynomials linear in p_i on $T^*\mathbb{R}^n$. Unfortunately, this quantization of C_L can not be extended in $P(\mathsf{b}_L)$ either.

4.4 Groenwold Van Hove Obstruction for $T^*\mathbb{R}^n$

To better understand these results a careful study of the proof of the Groenwold Van Hove obstruction for $T^*\mathbb{R}^n$ is needed. Complete proofs are given in [5],[2],[3], and [12] to name a few. Below we give a part of the proof given in [5]. We will show a contradiction. Suppose Q is a symplectic quantization of $(P(\mathsf{b}_{2n}), \mathsf{b}_{2n})$. The first step is to determine $Q((q^i)^2)$ and $Q((p_i)^2)$. Let $\Delta(i)$=$Q((q^i)^2)$-$(Q(q^i))^2$. Now we will compute the brackets

$$[\Delta(i), Q(q^k)]$$

$$[\Delta(i), Q(p_k)]$$

From the definition of a quantization property (1) we have

$$[Q(q^k), Q(p_j)] = i\hbar\delta^k_j$$

Therefore,

$$\begin{aligned} [\Delta(i), Q(q^k)] &= [Q((q^i)^2), Q(q^k)] - [(Q(q^i))^2, Q(q^k)] \quad (4.12)\\ &= 0 - 0 = 0 \quad (4.13) \end{aligned}$$

We also have

$$\begin{aligned} [\Delta(i), Q(p_k)] &= [Q((q^i)^2), Q(p_k)] - [(Q(q^i))^2, Q(p_k)]\\ &= 2i\hbar\delta^i_k Q(q^i) - [(Q(q^i))^2, Q(p_k)]\\ &= 2i\hbar\delta^i_k Q(q^i) - \{(Q(q^i))^2 Q(p_k) - Q(p_k)(Q(q^i))^2\}\\ &= 2i\hbar\delta^i_k Q(q^i) - (Q(q^i))(Q(p_k)Q(q^i) + i\hbar\delta^i_k)\\ &- (Q(q^i)Q(p_k) - i\hbar\delta^i_k)Q(q^i)\}\\ &= 0 \quad (4.14) \end{aligned}$$

The algebraic irreducibility ([5] prop. 5) of $Q(p_k)$ and $Q(q^i)$ implies

$$Q((q^i)^2) = (Q(q^i))^2 + EI$$

A similar argument gives

$$Q((p_i)^2) = (Q(p_i))^2 + FI$$

Computing more brackets, [5] one sees for the constants E and F that $E = F = 0$ or

$$Q((q^i)^2) = (Q(q^i))^2$$

$$Q((p_i)^2) = (Q(p_i))^2$$

These relations lead to higher Von Neumann rules and eventually to an obstruction.

4.4.1 n-Symplectic Case

Now compute the same relations for an n-symplectic quantization of $(P(\mathsf{b}_L), \mathsf{b}_L)$. We start our computation with the same goal: to determine $Q((\hat{q}^i_j)^2)$. Let $\Delta(i,j)$=$Q((\hat{q}^i_j)^2)$-$(Q(\hat{q}^i_j))^2$. Compute the brackets

$$[\Delta(i,j), Q(\hat{q}^k_l)]$$

$$[\Delta(i,j), Q(\hat{\pi}_k)]$$

From the definition of an n-symplectic quantization property (1) we have for some constant c_l

$$[Q(\hat{q}^k_l), Q(\hat{\pi}_j)] = i\hbar\delta^k_j c_l \tag{4.15}$$

and moreover,

$$\begin{aligned} [\Delta(i,j), Q(\hat{q}^k_l)] &= [Q((\hat{q}^i_j)^2), Q(\hat{q}^k_l)] - [(Q(\hat{q}^i_j))^2, Q(\hat{q}^k_l)] & (4.16) \\ &= 0 - 0 = 0 & (4.17) \end{aligned}$$

We also have, using the definition of an n-symplectic quantization property (1) and equation (4.15),

$$
\begin{aligned}
[\Delta(i,j), Q(\hat{\pi}_k)] &= [Q((\hat{q}^i_j)^2), Q(\hat{\pi}_k)] - [(Q(\hat{q}^i_j))^2, Q(\hat{\pi}_k)] \\
&= 2i\hbar\delta^i_k Q(\hat{q}^i_j \hat{r}_j) - [(Q(\hat{q}^i_j))^2, Q(\hat{\pi}_k)] \\
&= 2i\hbar\delta^i_k Q(\hat{q}^i_j \hat{r}_j) - \{(Q(\hat{q}^i_j))^2 Q(\hat{\pi}_k) - Q(\hat{\pi}_k)(Q(\hat{q}^i_j))^2\} \\
&= 2i\hbar\delta^i_k Q(\hat{q}^i_j \hat{r}_j) - (Q(\hat{q}^i_j))(Q(\hat{\pi}_k)Q(\hat{q}^i_j) + i\hbar\delta^i_k c_j) \\
&- (Q(\hat{q}^i_j)Q(\hat{\pi}_k) - i\hbar\delta^i_k c_j)Q(\hat{q}^i_j)\} \\
&= 2i\hbar\delta^i_k Q(\hat{q}^i_j \hat{r}_j) - 2i\hbar\delta^i_k Q(\hat{q}^i_j) c_j \qquad (4.18)
\end{aligned}
$$

Hence, $[\Delta(i,j), Q(\hat{\pi}_k)]$ is *not necessarily zero for all quantizations* Q! Compare this with the computation leading to equation (4.14). The difference being that $\{P(\hat{q},\hat{\pi}), P(\hat{q},\hat{\pi})\} = P(\hat{q},\hat{\pi})\hat{r}$ for the n-symplectic case. The irreducibility of $Q(\hat{\pi}_k)$ and $Q(\hat{q}^i_j)$ implies nothing unless we *choose* the generalized Von Nuemann relation

$$Q(\hat{q}^i_j \hat{r}_j) = Q(\hat{q}^i_j) c_j$$

An identical argument yields the same result for $(\hat{\pi}_k)^2$, namely

$$Q(\hat{\pi}_k \hat{r}_j) = Q(\hat{\pi}_k) c_j$$

These results and other similar observations lead to the restricted no-go theorems of section 4.5.

4.5 No-Go theorems

The examples from section 4.3 prove there is no global "no-go" theorem for quantizing $L\mathbb{R}^n$. However, some simple conditions lead to "no-go" theorems. Explicitly,

Theorem 4.8 *There is no full quantization of $L\mathbb{R}^n$ such that $Q(\mathsf{b}_L)$ acts irreducibly and $Q(\hat{f}\hat{r}_i) = Q(\hat{f})Q(\hat{r}_i) \;\; \forall \hat{f} \in SHF$.*

Proof

This is the exact condition needed to make n-symplectic quantization behave like a symplectic quantization. Suppose Q is a quantization of $(P(\mathsf{b}_L), \mathsf{b}_L)$. Let $\Delta(i,j)$=$Q((\hat{q}^i_j)^2)$-$(Q(\hat{q}^i_j))^2$. Compute the brackets

$$[\Delta(i,j), Q(\hat{q}^k_l)]$$

$$[\Delta(i,j), Q(\hat{\pi}_k)]$$

From property (1) and (2) of an n-symplectic quantization we have for some constants c_l

$$[Q(\hat{q}^k_l), Q(\hat{\pi}_j)] = i\hbar\delta^k_j c_l \tag{4.19}$$

Since $Q(\hat{f}\hat{r}_i) = Q(\hat{f})Q(\hat{r}_i)$ we have

$$Q(\hat{\pi}_k\hat{r}_j) = Q(\hat{\pi}_k)c_j$$

$$Q(\hat{q}^i_j\hat{r}_j) = Q(\hat{q}^i_j)c_j$$

Therefore,

$$[\Delta(i,j), Q(\hat{q}^k_l)] = 0$$

$$[\Delta(i,j), Q(\hat{\pi}_k)] = 2i\hbar\delta^i_k Q(\hat{q}^i_j\hat{r}_j) - 2i\hbar\delta^i_k Q(\hat{q}^i_j)c_j = 0$$

By the algebraic irreducibility of $Q(\hat{q}^k_l)$ and $Q(\hat{\pi}_k)$, we have $\Delta(i,j) = EI$, where E is some constant matrix. Hence,

$$Q((\hat{q}^i_j)^2) = Q(\hat{q}^i_j)^2 + EI \tag{4.20}$$

A similar argument for $Q((\hat{\pi}_i)^2)$ gives

$$Q((\hat{\pi}_i)^2) = Q(\hat{\pi}_i)^2 + FI \tag{4.21}$$

From equation (4.5) we have $\{(\hat{q}^i_j)^2, (\hat{\pi}_k)^2\} = 4\delta^i_k\hat{q}^i_j\hat{\pi}_k\hat{r}_j$. To get the quantization of $Q(\delta^i_k\hat{q}^i_j\hat{\pi}_k)$ we will quantize both sides of the previous equation. To illustrate the type of computation we put the full calculation here. Using relations (4.19), (4.20), and (4.21) to simplify, we see

$$\begin{aligned}
Q(\{(\hat{q}^i_j)^2,(\hat{\pi}_k)^2\}) &= [Q((\hat{q}^i_j)^2),Q((\hat{\pi}_k)^2)] \\
&= [Q(\hat{q}^i_j)^2+E,Q(\hat{\pi}_k)^2+F] \\
&= Q(\hat{q})^2Q(\hat{\pi})^2-Q(\hat{\pi})^2Q(\hat{q})^2 \\
&= Q(\hat{q})Q(\hat{q})Q(\hat{\pi})Q(\hat{\pi})-Q(\hat{\pi})^2Q(\hat{q})^2 \\
&= Q(\hat{q})[Q(\hat{\pi})Q(\hat{q})+i\hbar\delta^i_kc_j]Q(\hat{\pi})-Q(\hat{\pi})^2Q(\hat{q})^2 \\
&= Q(\hat{q})Q(\hat{\pi})Q(\hat{q})Q(\hat{\pi})+i\hbar\delta^i_kc_jQ(\hat{q})Q(\hat{\pi})-Q(\hat{\pi})^2Q(\hat{q})^2 \\
&= [Q(\hat{\pi})Q(\hat{q})+i\hbar\delta^i_kc_j]Q(\hat{q})Q(\hat{\pi})+i\hbar Q(\hat{q})Q(\hat{\pi})-Q(\hat{\pi})^2Q(\hat{q})^2 \\
&= Q(\hat{\pi})Q(\hat{q})Q(\hat{q})Q(\hat{\pi})+2i\hbar Q(\hat{q})Q(\hat{\pi})-Q(\hat{\pi})^2Q(\hat{q})^2 \\
&= Q(\hat{\pi})Q(\hat{q})[Q(\hat{\pi})Q(\hat{q})+i\hbar\delta^i_kc_j]+2i\hbar\delta^i_kc_jQ(\hat{q})Q(\hat{\pi})-Q(\hat{\pi})^2Q(\hat{q})^2 \\
&= Q(\hat{\pi})Q(\hat{q})Q(\hat{\pi})Q(\hat{q})+i\hbar\delta^i_kc_jQ(\hat{\pi})Q(\hat{q}) \\
&+ 2i\hbar\delta^i_kc_jQ(\hat{q})Q(\hat{\pi})-Q(\hat{\pi})^2Q(\hat{q})^2 \\
&= Q(\hat{\pi})[Q(\hat{\pi})Q(\hat{q})+i\hbar\delta^i_kc_j]Q(\hat{q})+i\hbar\delta^i_kc_jQ(\hat{\pi})Q(\hat{q}) \\
&+ 2i\hbar\delta^i_kc_jQ(\hat{q})Q(\hat{\pi})-Q(\hat{\pi})^2Q(\hat{q})^2 \\
&= 2i\hbar\delta^i_kc_jQ(\hat{\pi})Q(\hat{q})+2i\hbar\delta^i_kc_jQ(\hat{q})Q(\hat{\pi})
\end{aligned} \tag{4.22}$$

Therefore we have the quantization relation

$$Q(\{(\hat{q}^i_j)^2,(\hat{\pi}_k)^2\})=Q(\delta^i_k\hat{q}^i_j\hat{\pi}_k\hat{r}_j)=Q(\delta^i_k\hat{q}^i_j\hat{\pi}_k)Q(\hat{r}_j)=\frac{1}{2}\delta^i_k(Q(\hat{q}^i_j)Q(\hat{\pi}_k)+Q(\hat{\pi}_k)Q(\hat{q}^i_j))c_j$$

Quantizing the Poisson bracket relations $\{(\hat{q}^i_j)^2,\hat{q}^i_j\hat{\pi}_k\}=-2\delta^i_k(\hat{q}^i_j)^2\hat{r}_j$ and $\{(\hat{\pi}_k)^2,\hat{q}^i_j\hat{\pi}_k\}=2\delta^i_k(\hat{\pi}_k)^2\hat{r}_j$ and simplifying, we see that $E=F=0$. Using similar techniques one can show

$$Q((\hat{q}^i_j)^n)=Q(\hat{q}^i_j)^n$$

$$Q((\hat{\pi}_k)^n)=Q(\hat{\pi}_k)^n$$

$$Q((\hat{q}^i_j)^2\hat{\pi}_k)=\frac{1}{2}[Q(\hat{q}^i_j)^2Q(\hat{\pi}_k)+Q(\hat{\pi}_k)Q(\hat{q}^i_j)^2]$$

$$Q((\hat{\pi}_k)^2\hat{q}^i_j)=\frac{1}{2}[Q(\hat{\pi}_k)^2Q(\hat{q}^i_j)+Q(\hat{q}^i_j)Q(\hat{\pi}_k)^2]$$

Now consider the Poisson bracket relation $\{(\hat{q}^i_j)^3,(\hat{\pi}_k)^3\}=3\{(\hat{q}^i_j)^2\hat{\pi}_k,\hat{q}^i_j(\hat{\pi}_k)^2\}$. Quantizing both sides and using the above to simplify, we have the desired contradiction. Specifically, quantizing $\{(\hat{q}^i_j)^3,(\hat{\pi}_k)^3\}$ gives

$$9Q(\hat{q}^i_j)^2 Q(\hat{\pi}_k)^2 - 18i\hbar Q(\hat{q}^i_j)Q(\hat{\pi}_k) - 6\hbar^2 I \tag{4.23}$$

Quantizing $3\{(\hat{q}^i_j)^2\hat{\pi}_k, \hat{q}^i_j(\hat{\pi}_k)^2\}$ gives

$$9Q(\hat{q}^i_j)^2 Q(\hat{\pi}_k)^2 - 18i\hbar Q(\hat{q}^i_j)Q(\hat{\pi}_k) - 3\hbar^2 I \tag{4.24}$$

□

An important corollary to the previous theorem is the following:

Corollary 4.9 *There exists no full quantization of $L(\mathbb{R}^n)$ such that Q acts like an anti-commutator, $Q(\hat{f}\hat{g}) = \frac{1}{2}(Q(\hat{f})Q(\hat{g}) + Q(\hat{g})Q(\hat{f}))$.*

Proof

Let $\hat{g} = \hat{r}_k$, and then use Theorem 4.8. □

Another interesting theorem is given by the following observation. Suppose that $Q(\mathsf{b}_L\hat{r}_{K_p}) = Q(\mathsf{b}_L)Q(\hat{r}_{K_p}) = Q(\mathsf{b}_L)c_{K_p}$ for all p. Given the quantization of the basic set $Q(\mathsf{b}_L)$ acts irreducibly, then $Q(\mathsf{b}_L\hat{r}_{K_p}) = Q(\mathsf{b}_L)c_{K_p}$ acts irreducibly for all p.

Theorem 4.10 *There is no full quantization of $L(\mathbb{R}^n)$ such that $Q(\mathsf{b}_L\hat{r}_{K_p})$ acts irreducibly for any fixed p and $Q(\hat{f}\hat{r}_i) = Q(\hat{f})Q(\hat{r}_i)\ \ \forall \hat{f} \in \bigoplus_{m=p}^{\infty} P(\hat{q},\hat{\pi})\hat{r}_{K_m}$.*

Proof

The proof of this theorem is identical to the main theorem of this section. Replace every polynomial $\hat{f}$ with $\hat{f}\hat{r}_{K_p}$ and all the computations follow through. The contradiction arises from the classical equation $\{(\hat{q}^i_j)^3\hat{r}_{K_p}, (\hat{\pi}_k)^3\hat{r}_{K_s}\} = 3\{(\hat{q}^i_j)^2\hat{\pi}_k\hat{r}_{K_m}, \hat{q}^i_j(\hat{\pi}_k)^2\hat{r}_{K_t}\}$ where $p+s = t+m$. Notice that the proof of the main theorem only uses $P(\mathsf{b}_L)$ and not all of SHF. So the requirement of the generalized von Neumann rule only for $\bigoplus_{m=p}^{\infty} P(\hat{q},\hat{\pi})\hat{r}_{K_m}$ is sufficient. □

As a corollary we get the results stated in section 4.3.

Corollary 4.11 *The extended metaplectic n-symplectic quantization cannot be extended past P^2_L in $P(\mathsf{b}_L)$.*

Proof

Let the quantization of P^2_L be the extended metaplectic n-symplectic quantization given in section 4.3.3. Extend P^2_L by any monomial $\hat{q}^{I_l}_{J_l}\hat{\pi}_{K_m}\hat{r}_{N_c}$. The subalgebra P^2_L is "maximal" in the sense that adding $\hat{q}^{I_l}_{J_l}\hat{\pi}_{K_m}\hat{r}_{N_c}$ will generate every monomial of P_c, but no lower. Hence the smallest subalgebra containing P^2_L and $\hat{q}^{I_l}_{J_l}\hat{\pi}_{K_m}\hat{r}_{N_c}$ is $\{P^2_L,\} \cup P_c$. The quantization of $\hat{q}^i_j\hat{\pi}_k$ forces the generalized von Nuemann rule $Q(\hat{f}\hat{r}_i) = Q(\hat{f})Q(\hat{r}_i) \quad \forall \hat{f} \in P_c$. Finally the irreducibility of $Q(\mathsf{b}_L)$ implies the irreducibility of $Q(\mathsf{b}_L)c_{K_c}$. $\square$

Corollary 4.12 *The n-symplectic version of the coordinate representation cannot be extended past C_L in $P(\mathsf{b}_L)$.*

The proof is the same as the last corollary with the following observations. First the n-symplectic version of the coordinate representation obeys the generalized von Neumann relation $Q(\hat{f}\hat{r}_i) = Q(\hat{f})Q(\hat{r}_i) \quad \forall \hat{f} \in C_L$. The subalgebra of C_L is again weakly maximal. Hence, adding any monomial $\hat{q}^{I_l}_{J_l}\hat{\pi}_{K_m}\hat{r}_{N_c}$ forces one to consider $\{C_L,\} \cup P_c$.

4.6 Map From LM to T^*M

In [8] it is shown that for all $\alpha \in \mathbb{R}^n$ such that $\alpha_i \neq 0 \quad \forall i$ there is a map $\alpha : ST^p \to C^\infty(T^*M)$ which recovers symplectic geometry. The space ST^p is the set of all symmetric Hamiltonian observables with homogeneous degree p in $\hat{\pi}_k$. There is also an induced map $\psi_\alpha : AM \to L^\times$, where AM is the affine frame bundle of a manifold M and $L^\times$ is the $\mathbb{C}^\times$ bundle of geometric quantization. The author has extended the map α to $SHF^p \to C^\infty(T^*M)$ for each p.

Definition 4.13 *For the symmetric Hamiltonian observable $\hat{f} = \hat{f}^{I_p}\hat{r}_{I_p} \in SHF^p$ define the map $\alpha : SHF^p \to C^\infty(T^*M)$ by*

$$\alpha(\hat{f}) = \alpha_{I_p}\hat{f}^{I_p}$$

For the purpose of this paper we can use a simpler version of ψ_α which we will also denote ψ_α.

Definition 4.14 *Let $u = (p, e_i)$ be a point in LM. Define the map $\psi_\alpha : LM \to T^*(M)$ by*

$$\psi_\alpha(u) = \psi_\alpha(p, e_i) = (p, \alpha_i e^i)$$

In this definition $\{e^i\}$ is the dual basis to $\{e_i\}$. Let $\tilde{f}$ denote the observable on T^*M obtained by $\alpha(\hat{f})$. Before we give the next theorem recall that a Hamiltonian vector field on $L\mathbb{R}^n$ is

$\otimes_s^p \mathbb{R}^n$ valued $X_{\hat{f}} = X_{\hat{f}}^{I_p} \hat{r}_{I_p}$. Notice that for $p = 0$ the Hamiltonian vector field is $\mathbb{R}$ valued. Hence for all $p \geq 0$, $X^{I_p} \alpha_{I_p}$ is an $\mathbb{R}^n$ valued Hamiltonian vector field of $L\mathbb{R}^n$.

Theorem 4.15 *Let $\hat{f} \in SHF^p$, then*

$$d\psi_\alpha(X_{\hat{f}}^{I_p} \alpha_{I_p}) = X_{\alpha(\hat{f})} = X_{\tilde{f}} \tag{4.25}$$

In the above theorem $X_{\tilde{f}}$ is the symplectic Hamiltonian vector field on T^*M generated by $\tilde{f}$, and $X_{\hat{f}}$ is the n-symplectic Hamiltonian vector field on LM generated by $\hat{f}$.

Proof

Using definition 4.14 we first compute

$$d\psi_\alpha\left(\frac{\partial}{\partial \pi_j^i}\right) = \alpha_i \frac{\partial}{\partial p_j}$$
$$d\psi_\alpha\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial q^i}$$

Let $u \in LM$ and let $w = \psi_\alpha(u)$. For $\hat{f}^{I_{p+1}} \in SHF^{p+1}$ the Hamiltonian vector field $X_{\hat{f}}$ is given by (3.1). Let $w = \psi_\alpha(u)$ with $u \in L\mathbb{R}^n$. Computing we find:

$$\begin{aligned}
p! d\psi_\alpha(X_{\hat{f}}^{I_{p-1}} \alpha_{I_{p-1}})(u) &= p! d\psi_\alpha\left(\frac{1}{p!} \frac{\partial \hat{f}^{I_{p-1}b}}{\partial \pi_a^b} \frac{\partial}{\partial q^a} - \frac{1}{p!} \frac{\partial \hat{f}^{I_{p-1}a}}{\partial q^b} \frac{\partial}{\partial \pi_b^a} \alpha_{I_{p-1}}\right)(u) \\
&= \frac{\partial \hat{f}^{I_{p-1}b}}{\partial \pi_a^b} \alpha_{I_{p-1}} \frac{\partial}{\partial q^a}(w) - \frac{1}{p} \frac{\partial \hat{f}^{I_p a}}{\partial q^b} \alpha_{I_{p-1}} \alpha_a \frac{\partial}{\partial p_b}(w) \\
&= \frac{\partial \tilde{f}}{\partial p_a} \frac{\partial}{\partial q^a}(w) - \frac{\partial \tilde{f}}{\partial q^b} \frac{\partial}{\partial p_b}(w) \\
&= X_{\tilde{f}}
\end{aligned} \tag{4.26}$$

The third line depends on the relation $\frac{\partial}{\partial \pi_b^a} = \alpha_a \frac{\partial}{\partial p_b}$. Hence $d\psi_\alpha$ has the required property. $\Box$This map preserves the Poisson bracket in each respective space. Denote the set of equivalence classes of n-symplectic Hamiltonian vector fields on LM by $HV(LM)$. Denote the set of symplectic Hamiltonian vector fields on T^*M by $HV(T^*M)$.

Definition 4.16 *Let $X = X^{I_p} \hat{r}_{I_p} \in HV(LM)$ then define the map $T : HV(LM) \to HV(T^*M)$ as follows*

$$T(X) = T(X^{I_p} \hat{r}_{I_p}) = (p+1)! d\psi_\alpha(X^{I_p} \alpha_{I_p})$$

From above we have

$$T(X_{\hat{f}}) = X_{\alpha \hat{f}} = X_{\tilde{f}}$$

Theorem 4.17 *The map $T : HV(LM) \to HV(T^*M)$ is a Lie algebra homomorphism.*

Proof

From section 2.1 we have

$$[X_{\hat{f}}, X_{\hat{g}}] = CX_{\{\hat{f},\hat{g}\}}$$

From standard symplectic geometry on T^*M we have

$$[X_{\tilde{f}}, X_{\tilde{g}}] = X_{\{\tilde{f},\tilde{g}\}}$$

Using these relations one has

$$\begin{aligned} T[X_{\hat{f}}, X_{\hat{g}}] &= T(CX_{\{\hat{f},\hat{g}\}}) \\ &= CX_{\alpha\{\hat{f},\hat{g}\}} \\ &= CX_{\{\tilde{f},\tilde{g}\}} \\ &= C[X_{\tilde{f}}, X_{\tilde{g}}] \qquad \square \end{aligned} \tag{4.27}$$

The map T can be extended to a map $T : op(\mathcal{H}_{L\mathbb{R}^n}) \to op(\mathcal{H}_{T^*\mathbb{R}^n})$ for operators of a specific form. First some notation. Denote a differential operator of $op(\mathcal{H}_{L\mathbb{R}^n})$ by

$$F = F^{I_k A_m J_r}_{B_m}(q^i, \pi^a_b)\partial_{q^{I_k}} \circ \partial_{\pi^{A_m}_{B_m}} \hat{r}_{J_r}$$

This equation uses the notation $\partial_{q^{I_k}} \stackrel{def}{=} \frac{\partial}{\partial q^{i_1}} \circ \cdots \circ \frac{\partial}{\partial q^{i_k}}$ and $\partial_{\pi^{A_m}_{B_m}} \stackrel{def}{=} \frac{\partial}{\partial \pi^{a_1}_{b_1}} \circ \cdots \circ \frac{\partial}{\partial \pi^{a_m}_{b_m}}$. Note also that if all differential indicies are zero, F is a multiplication operator $F = F^{J_r}\hat{r}_{J_r}$. Extend T in a natural way by the following definition.

Definition 4.18 *Let $F = F^{I_k A_m J_r}_{B_m}(q^i, \pi^a_b)\partial_{q^{I_k}}\partial_{\pi^{A_m}_{B_m}}\hat{r}_{J_r} \in op(\mathcal{H}_{L\mathbb{R}^n})$ and define the map $T : op(\mathcal{H}_{L\mathbb{R}^n}) \to op(\mathcal{H}_{T^*\mathbb{R}^n})$ by*

$$T(F) = T(F^{I_k A_m J_r}_{B_m}(q^i, \pi^a_b)\partial_{q^{I_k}}\partial_{\pi^{A_m}_{B_m}}\hat{r}_{J_r}) = \alpha_{J_r} F^{I_k A_m J_r}_{B_m}(q^i, \pi^a_b) d\psi_\alpha(\partial_{q^{I_k}}) d\psi_\alpha(\partial_{\pi^{A_m}_{B_m}})$$

For a multiplication operator $F = F^{J_r}\hat{r}_{J_r}$

$$T(F) = \alpha(F)$$

In this definition we have used the notation $d\psi_\alpha(\partial_{q^{I_k}}) \stackrel{def}{=} d\psi_\alpha\left(\frac{\partial}{\partial q^{i_1}}\right)\circ\cdots\circ d\psi_\alpha\left(\frac{\partial}{\partial q^{i_k}}\right)$ and $d\psi_\alpha(\partial_{\pi^{A_m}_{B_m}}) \stackrel{def}{=} d\psi_\alpha\left(\frac{\partial}{\partial \pi^{a_1}_{b_1}}\right)\circ\cdots\circ d\psi_\alpha\left(\frac{\partial}{\partial \pi^{a_m}_{b_m}}\right)$. This map clearly maps into $op(\mathcal{H}_{T^*\mathbb{R}^n})$.

Definition 4.19 *Let $Q_{T^*\mathbb{R}^n}$ be the map induced on $T^*\mathbb{R}^n$ given a quantization Q on LM.*

$$Q_{T^*\mathbb{R}^n}(\tilde{f}) = T(Q(\hat{f}))$$

From the definition of a quantization

$$Q(\{\hat{f},\hat{g}\}) = [Q(\hat{f}),Q(\hat{g})]$$

$$\begin{aligned} T[Q(\hat{f}),Q(\hat{g})] &= T(Q(\{\hat{f},\hat{g}\})) \\ &= Q_{T^*\mathbb{R}^n}(\alpha\{\hat{f},\hat{g}\}) \\ &= Q_{T^*\mathbb{R}^n}(\{\tilde{f},\tilde{g}\}) \end{aligned} \tag{4.28}$$

The map α can be thought of as a map $P(\mathsf{b}_L)\to P(\mathsf{b}_{2n})$ and $T: op(\mathcal{H}_{L\mathbb{R}^n})\to op(\mathcal{H}_{T^*\mathbb{R}^n})$. However the map $Q_{T^*\mathbb{R}^n}$ does not give a quantization on $T^*\mathbb{R}^n$. Using the maps above we can map the quantization of $L\mathbb{R}^n$ to $T^*\mathbb{R}^n$ and see where it breaks down. In n-symplectic geometry the observables $\hat{q}^i_j$ and $\hat{q}^i_j\hat{r}_k$ are distinctly different. Quantizing each as in the full polynomial quantization of section 8.2 we get

$$Q(\hat{q}^i_j) = \alpha_j q^i$$

$$Q(\hat{q}^i_j\hat{r}_k) = A^i_j\alpha_k$$

Using the above maps we see that $\alpha(\hat{q}^i_j) = q^i\alpha_j$ and $\alpha(\hat{q}^i_j\hat{r}_k) = \alpha_j\alpha_k q^i$ on $T^*\mathbb{R}^n$. Using definition 4.18 we have

$$Q_{T^*\mathbb{R}^n}(\alpha_j q^i) = Q_{T^*\mathbb{R}^n}(\alpha(\hat{q}^i_j)) = T(Q(\hat{q}^i_j)) = T(\alpha_j q^i) = \alpha_j q^i$$

Also, we have

$$Q_{T^*\mathbb{R}^n}(\alpha_k\alpha_j q^i) = Q_{T^*\mathbb{R}^n}(\alpha(\hat{q}^i_j\hat{r}_k)) = T(Q(\hat{q}^i_j\hat{r}_k)) = T(A^i_j\alpha_k) = A^i_j\alpha_k$$

Hence, on the cotangent bundle, $Q_{T^*\mathbb{R}^n}$ is not a linear map since $Q_{T^*\mathbb{R}^n}(\alpha_k\alpha_j q^i) \neq \alpha_k Q_{T^*\mathbb{R}^n}(\alpha_j q^i)$. In fact, the map α *removes the Lie ideals in the Poisson algebra*!

Chapter 5

Conclusion

To avoid the obstructions to quantizing the canonical symplectic manifold $T^*\mathbb{R}^n$ something must change. The most obvious candidates for modification are either Dirac's "Poisson bracket $\rightarrow$ commutator" quantization rule or the underlying setting of quantization. The symplectic geometry of polynomial observables on $T^*\mathbb{R}^n$ is induced from the n-symplectic geometry of $L\mathbb{R}^n$. Hence, n-symplectic geometry is the natural choice of a larger geometry in which to base quantization. As shown in chapter 4, the n-symplectic geometry of $L\mathbb{R}^n$ allows for the existence of ideals in the Poisson algebra of polynomials. On the other hand there are no such ideals in the Poisson algebra of polynomial observables on T^*M. The existence of ideals in the Poisson algebra of polynomial observables on $L\mathbb{R}^n$ allows the frame bundle of $\mathbb{R}^n$ to support full polynomial quantizations.The author believes this is a necessary and sufficient condition. With that conjecture in mind it is important to notice that ideals exist in the Poisson algebra of polynomial observables of LM for all M.

The change from symplectic geometry to n-symplectic geometry is better than weakening Dirac's "Poisson bracket $\rightarrow$ commutator" quantization rule because of an important observation in [5]. There is an apparent link between obstructions to Hilbert space based quantization and the absence of a strict deformation quantization. In response to these observations Gotay makes the following comment,

> It is generally believed that the existence of Groenwold-Van Hove obstructions necessitates a weakening of the Poisson Bracket $\rightarrow$ commutator rule(by insisting that it hold only to order $\hbar$), but these observations indicate that this may not suffice to remove the obstructions.

The existence of full polynomial quantizations for $L\mathbb{R}^n$ and none for $T^*\mathbb{R}^n$ is just one more reason n-symplectic geometry is a rich subject with great physical potential. The theory of n-symplectic quantization is also fertile ground for new research. I would like to emphasize some important topics that need further consideration.

- Momentum maps play an important role in quantization. They are the foundation that the basic sets are built on. The momentum mapping for n-symplectic geometry is only roughly understood. A full treatment of momentum maps would be useful. The definition of basic set for n-symplectic geometry given here is based on the properties of a symplectic momentum map. A better understanding of the n-symplectic momentum mapping could lead to a better definition of an n-symplectic basic set.

- The only space considered in this paper is $L\mathbb{R}^n$. There are other manifolds that cannot be quantized namely S^2 and T^*S^1. The n-symplectic quantization of these manifolds has yet to be studied. The natural n-symplectic analogue of T^*S^1, or T^*S^n in general, is the frame bundle of S^n, LS^n. The n-symplectic analogue of S^2 is more subtle. A general n-symplectic manifold, not necessarily a frame bundle, may be required.

- The quantization given here is just a specific example of a quantization for a specific n-symplectic manifold. We have made no attempt to develop a quantization method to instantly quantize every n-symplectic manifold. However, this would be a desirable result.

- There exists a map from the set of polynomials of degree 3 or less to e.s.a. operators on the Hilbert space, $\mathcal{L}$. This map satisfies all the requirements of a quantization except the space is not a subalgebra of $P(\mathsf{b}_L)$. We note that this map is not possible in symplectic geometry because of the Greonwold obstruction for cubic polynomials.

The specific map is:

$$
\begin{aligned}
Q(\hat{q}^i_j \otimes^k_s \hat{r}_b) &= (-1)^k \alpha_j q^i \otimes^k_s \hat{r}_b \\
Q(\hat{\pi}_k \otimes^k_s \hat{r}_b) &= (-1)^k \frac{\partial}{\partial q^k} \otimes^k_s \hat{r}_b \\
Q(\hat{r}_k \otimes^k_s \hat{r}_b) &= \alpha_k \otimes^k_s \hat{r}_b \\
Q((\hat{q}^i_j)^2 \otimes^k_s \hat{r}_b) &= \alpha_j q^i \hat{q}^i_j \otimes^k_s \hat{r}_b \\
Q((\hat{\pi}_k)^2 \otimes^k_s \hat{r}_b) &= -\beta \frac{\partial^2}{\partial q^k \partial q^k} \otimes^{k+1}_s \hat{r}_b \\
Q(\hat{q}^i_j \hat{\pi}_k) &= \hat{q}^i_j \frac{\partial}{\partial q^k} + \delta^i_k 3\hat{\pi} \frac{\partial}{\partial \pi} \\
Q(\hat{q}^i_j \hat{\pi}_k \otimes^k_s \hat{r}_b) &= (-1)^k (\hat{q}^i_j \frac{\partial}{\partial q^k} + \delta^i_k \frac{1}{2}) \otimes^k_s \hat{r}_b \\
Q((\hat{q}^i_j)^3 \otimes^k_s \hat{r}_b) &= (\alpha_j q^i (\hat{q}^i_j)^2 - 3\alpha \hat{\pi}\hat{r}) \otimes^k_s \hat{r}_b \\
Q((\hat{\pi}_k)^3 \otimes^k_s \hat{r}_b) &= \beta^2 \hat{r}^2 (\frac{\partial^3}{\partial q^k \partial q^k \partial q^k} + \hat{r}^2 \beta^2 \frac{\partial}{\partial \pi}) \otimes^{k+1}_s \hat{r}_b \\
Q(\hat{q}^i_j)^2 \hat{\pi}_k) &= (\hat{q}^i_j)^2 \frac{\partial}{\partial q} + \hat{q}\hat{r} \\
Q(\hat{\pi}_k)^2 \hat{q}^i_j) &= \hat{q}^i_j \hat{r} \frac{\partial}{\partial q} \frac{\partial}{\partial q} + \frac{\partial}{\partial q} \hat{r}\hat{r}
\end{aligned}
\tag{5.1}
$$

What makes this map remarkable is the the relation

$$[Q((\hat{q}^i_j)^3), Q((\hat{\pi}_k)^3)] = 3[Q(\hat{q}^i_j)^2 \hat{\pi}_k), Q(\hat{\pi}_k)^2 \hat{q}^i_j)]$$

This is the relation that leads to the Groenwold Van Hove obstruction for $\mathbb{R}^{2n}$ (see equations (4.23,4.24)). The author believes that this map cannot be extended to a full polynomial quantization but has not proved this statement at this time. Since there is no Groenwold Van Hove obstruction for n-symplectic geometry, it is interesting to search for a full quantization that includes the metaplectic quantization or some close variant thereof.

- We have only studied a quantization for the symmetric observables of $L\mathbb{R}^n$. There is another class of observables, the totally antisymmetric Hamiltonian functions AHF. In [11] it is shown that these functions form a graded Poisson algebra and the antisymmetric Hamiltonian vector fields form a graded Lie algebra. A quantization for these observables would be very interesting. For example, consider $L\mathbb{R}^3$ with AHF. The antisymmetric observables are again polynomials in the π. By the properties of the wedge product these polynomials terminate after degree three. Therefore the map given in (5.1) is a full polynomial quantization of $L\mathbb{R}^3$! Given the graded nature of AHF this could also have implications for supersymmetry. Obviously, this is an area that needs investigation.

- Finally, we have made no attempt to analyze these results physically.

Bibliography

[1] P. Dirac. *Quantum Mechanics, 4th ed.* Oxford University Press, 1958.

[2] H. J. Groenewold. On the principles of elementary quantum mechanics. *Physics*, 12:405–458, 1946.

[3] Leon Van Hove. *On Certain Unitary Representations of an Infinite Group of Transformations*. World Scientific, 2001.

[4] J. Rosenberg. A selective history of the stone-von neumann theorem. In *Operator algebras, quantization, and noncommutative geometry*, pages 123–158. American Mathematical Society, 2004.

[5] Mark J. Gotay. Obstructions to quantization. In *Mechanics:From Theory to Computation*. Springer, 2000.

[6] A. Joseph. Derivations of lie brackets and canonical quantization. *Commun. Math. Phys.*, 17:210–232, 1970.

[7] M. Leon et. al. Geometric structures in field theory. *AriXiv Mathematical Physics e-prints*, eprint math-ph/0208036, 2002.

[8] L.K. Norris. Syplectic geometry on t^*m derived from n-symplectic geometry on lm. *Geometry and Physics*, 13:51–78, 1994.

[9] G.M. Tuynman. Prequantization is irreducible. *Indag. Math.*, 9:607–618, 1998.

[10] Ch. Gunther. The polysymplectic hamiltonian formalism in field theory and calculus of variations i: The local case. *J. Differential Geometry*, 25:25–53, 1987.

[11] L.K. Norris. Generalized symplectic geometry on the frame bundle of a manifold. *Symposiuim in Pure Mathematics*, 54:435–465, 1993.

[12] R. Abraham and J. E. Marsden. *Foundations of Mechanics, 2nd ed.* The Benjamin/Cummings Publishing Company, Inc., 1978.

[13] Daniel Cartin. Some aspects of geometric pre-quantization on the affine frame bundle. Master's thesis, North Carolina State University, 1995.

[14] V. Guillemin and S. Sternberg. *Symplectic techniques in physics.* Cambridge University Press, 1984.

[15] N. M. J. Woodhouse. *Geometric Quantiation, 2nd ed.* Oxford University Press, Inc., 1991.