Abstract

Farrah Jackson. *Characterization of Involutions of* $\mathrm{SP}(2n,k)$ (Under the direction of Aloysius Helminck). In this thesis, we discuss the relationship between involutions of the two matrix groups $\mathrm{SL}(2n,k)$ and $\mathrm{SP}(2n,k)$. Involutions determine symmetric spaces hence a complete classification of involutions of both $\mathrm{SL}(n,k)$ and $\mathrm{SP}(2n,k)$ will in turn classify the symmetric spaces coming from these involutions. We begin by giving a complete classification of involutions of the group $\mathrm{SL}(n,k)$ over the algebraically closed fields, the real numbers, the rational numbers, and the finite fields. As a method of classifying a particular type of involution of $\mathrm{SL}(n,k)$ we focus on how they may be obtained from a non-degenerate symmetric or skew-symmetric bilinear form. With the classification of involutions of $\mathrm{SL}(n,k)$ in hand we focus our attention on the subgroup $\mathrm{SP}(2n,k)$ of $\mathrm{SL}(2n,k)$. We first show that all involutions of $\mathrm{SP}(2n,k)$ are the restriction of an involution of $\mathrm{SL}(2n,k)$ to $\mathrm{SP}(2n,k)$. We determine that an automorphism $\theta = \mathrm{Inn}_A$ leaves $\mathrm{SP}(2n,k)$ invariant if and only if $A = pM$ for some $p \in \bar{k}$ and $M \in \mathrm{SP}(2n,k)$. Next we give specific criteria to characterize which involutions of $\mathrm{SL}(2n,k)$ remain involutions when restricted to $\mathrm{SP}(2n,k)$. Lastly, we determine that if two involutions of $\mathrm{SP}(2n,k)$ are isomorphic under $\mathrm{SP}(2n,k)$ then they are isomorphic under $\mathrm{SL}(2n,k)$.

CHARACTERIZATION OF INVOLUTIONS OF SP$(2N, K)$

BY

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Dedication

In loving memory of my father, Henry A. Jackson.

Biography

Farrah Monique Jackson was born in Washington, D.C. on June 3, 1977 to Henry and Doretha Jackson. She has one brother Henry Jackson, Jr. and one sister Terri Jackson. Farrah attended Suitland High School in Forestville, Maryland where she graduated Salutatorian of her 1995 Class. Farrah went on to attend North Carolina Agricultural and Technical State University where she received her Bachelor of Science in Mathematics Education. In 1999, Farrah began her studies at North Carolina State University in Raleigh, NC. She later received her Master of Science Degree in Mathematics from North Carolina State in 2001. Farrah continued her PhD studies at North Carolina State and taught mathematics courses at both North Carolina State University and Meredith College. Farrah enjoys skiing, bowling and swimming.

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Chapter 1

Introduction

1.1 Motivation

This thesis involves symmetric spaces and how they are obtained from Lie groups, algebraic groups. Symmetric spaces describe various symmetries in nature. One way to visualize a symmetric space is by viewing the group of symmetries or motions acting on the space. For example, you can think of the symmetric space as the plane and the group as the set of translations in the plane. The study of symmetric spaces combines group theory, geometry, field theory, Lie theory and linear algebra. Symmetric spaces have been studied for over 100 years and have played an important role in mathematical physics, representation theory and differential geometry. Although initially symmetric spaces were only studied over the real numbers, more recently the area has expanded to include the study over various fields. In fact, in the last 15 to 20 years symmetric spaces have become of importance in many other areas of mathematics and have been studied by many mathematicians.

1.2 Overview

We are interested in classifying symmetric spaces. Let $\bar{G}$ be a reductive linear algebraic group over a field k and G its set of k-rational points. An automorphism θ is an involution if $\theta \neq \mathrm{Id}$ and $\theta^2 = \mathrm{Id}$. Hence, given an involution θ on our group G, the symmetric space

X is defined as G/H where $G^\theta = H$ is the fixed-point group of the involution θ. As you can see the sheer definition of a symmetric space relies heavily on the fixed point group of the involution. Because of this reliance we focus exclusively on classifying the involutions which define the symmetric space up to isomorphy as a method of classifying the symmetric space itself. We begin by considering involutions on the group $\mathrm{GL}(n,k)$. In this case we are able to see that one method of classifying such involutions can be realized by looking at their relationship with bilinear forms. We first observe that given a bilinear form $\beta(x,y) = x^T My$ and a matrix A the adjoint of A, denoted A', is the matrix satisfying

$$\beta(Ax,y) = \beta(x,A'y) \quad \text{for all} \quad x,y \in V.$$

In particular we see that $A' = M^{-1}A^TM$. More importantly if β represents a non-degenerate symmetric or skew-symmetric bilinear form (i.e. $\det(M) \neq 0$ and $M = M^T$ or $M = -M^T$) then we are able to define an involution $\theta_M(X) = (X')^{-1}$ based on that bilinear form. This invokes interest on the connection between isomorphy classes of such involutions and congruence classes of symmetric and skew-symmetric bilinear forms. It is at this point that we formally give this relationship via

The Classification Theorem: If θ_{M_1} and θ_{M_2} are involutions on $\mathrm{GL}(n,k)$ which come from a symmetric or skew-symmetric bilinear form as stated before, then

$$M_1 \cong^s M_2 \quad \text{over} \quad k \iff \theta_{M_1} \approx \theta_{M_2}.$$

(where semi-congruence is simply congruence up to a scalar α in the algebraic closure of the field).

It is then shown that these results carry over to the subgroups $\mathrm{SL}(n,k)$ and $\mathrm{SP}(n,k)$ of $\mathrm{GL}(n,k)$. Finally we provide a classification of the involutions on $\mathrm{SL}(n,k)$ up to isomorphy and give a characterization of the involutions of $\mathrm{SP}(2n,k)$. A detailed summary of all the results are provided in the next section.

1.3 Summary of Results

We begin Chapter 2 with the definition of symmetric spaces an the concept of isomorphic involutions. We give a detailed analysis of the congruence classes of symmetric and skew-symmetric bilinear forms which will prove extremely useful in later chapters. The most significant result in this chapter is the Classification Theorem which links the congruence classes (truly semi-congruence classes) of symmetric and skew-symmetric bilinear forms to the isomorphism classes of involutions.

In Chapter 3 we turn our attention to the subgroup $\mathrm{SL}(2,k)$ of $\mathrm{GL}(2,k)$. For $\mathrm{SL}(2,k)$ we observe that all involutions are of type inner. In addition, we give the result that all involutions of $\mathrm{SL}(2,k)$ come from a symmetric bilinear form, however no involution of $\mathrm{SL}(2,k)$ is obtained from the skew-symmetric bilinear form. We go on to give a classification of the involutions over the algebraically closed fields, the real numbers, the rational numbers, and the finite fields.

We move on to the subgroup $\mathrm{SL}(n,k)$ of $\mathrm{GL}(n,k)$ in Chapter 4. We first observe that unlike $\mathrm{SL}(2,k)$, there are two types of involutions on $\mathrm{SL}(n,k)$, both inner and outer. We show that all the outer involutions of $\mathrm{SL}(n,k)$ come from bilinear forms. Because of this we are able to use the Classification Theorem to determine the isomorphy classes of involutions on $\mathrm{SL}(n,k)$ over the algebraically closed fields, the real numbers, the rational numbers, and the finite fields. We then focus on the inner involutions of $\mathrm{SL}(n,k)$ and state that they do not come from bilinear forms. Since the Classification Theorem is not applicable in this case we give a new set of criteria for the classification of inner involutions and provide such a classification.

We begin Chapter 5 by discussing the skew-symmetric bilinear forms in detail. Here we are able to see that there is in fact only one skew-symmetric bilinear form up to a change of basis. We then review the orthogonal group $\mathrm{O}(m, k, \beta)$ discussed in Chapter 2 and conclude that $\mathrm{SP}(m, k) = \mathrm{O}(m, k, \beta)$ where β represents the sole skew-symmetric bilinear form. In addition, m must be even so we switch to the notation $\mathrm{SP}(2n, k)$.

Since we are truly interested in characterizing involutions of $\mathrm{SP}(2n, k)$ we show that every involution of $\mathrm{SP}(2n, k)$ is the restriction of an involution of $\mathrm{SL}(2n, k)$ to $\mathrm{SP}(2n, k)$. Hence to classify the involutions on $\mathrm{SP}(2n, k)$ we investigate the involutions of $\mathrm{SL}(2n, k)$.

We start off by looking at involutions when $n = 1$, that is on $\mathrm{SP}(2, k)$. In this situation we show that $\mathrm{SP}(2, k) = \mathrm{SL}(2, k)$. Thererfore we are able to conclude that the isomorphism classes of involutions of $\mathrm{SP}(2, k)$ are given by the isomorphism classes of involutions of $\mathrm{SL}(2, k)$ which leave $\mathrm{SP}(2, k)$ invariant.

Next we look at the more complicated situation of involutins of $G = \mathrm{SP}(2n, k)$ with $n > 1$. We begin by focusing on the fact that if k is algebraically closed then all the automorphisms of G are of type inner. We use this fact to give the result that an automorpism $\mathrm{Inn}_A|_G = \mathrm{Id}$ for some $A \in \mathrm{SL}(2n, \bar{k})$ if and only if $A = pI$ for some $p \in \bar{k}$. Although it will initially appear that we have slightly deviated from our objective the above result will play an integral part in our next theorem, The Characterization Theorem.

Continuing to focus on $\mathrm{SP}(2n, k)$ we now state what is called the Characterization Theorem. The first part of this theorem answers the question, which inner automorphisms Inn_A with $A \in \mathrm{GL}(2n, \bar{k})$ keep $\bar{G} = \mathrm{SP}(2n, \bar{k})$ invariant. What we determine is that Inn_A keeps $\bar{G}$ invariant if and only if $A = pM$ with $p \in \bar{k}$ and $M \in \bar{G}$. Now part 2 of the Characterization Theorem states that the inner automorphisms Inn_B with $B \in \bar{G}$ keeps G invariant if and only if $B = qN$ for some $q \in \bar{k}$ and $N \in G$. Hence we are able to conclude that any automorphism θ, and more importantly any involution θ, which leaves $\mathrm{SP}(2n, k)$ invariant has the property that $\theta = \mathrm{Inn}_A$ where $A = pM$ for some $p \in \bar{k}$ and $M \in \mathrm{SP}(2n, k)$. Since the proof of the second part of the Characterization Theorem is quite complex we first demonstrate the procedure of the proof via an example for $2n = 6$, followed by a complete proof in the general case.

We then turn our attention to the outer involutions of $\mathrm{SL}(2n,k)$. We first redefine the outer involutions of $\mathrm{SL}(2n,k)$ by redefining the fixed outer automorphism used in Chapter 2. By redefining the fixed outer automorphism we are able to give the result that the outer involutions of $\mathrm{SL}(2n,k)$ become inner involutions when restricted to $\mathrm{SP}(2n,k)$. This will follow directly from the fact that $\mathrm{SP}(2n,k)$ is the fixed point group of our new fixed outer automorphism.

Since we focus on involutions of $\mathrm{SL}(2n,k)$ restricted to $\mathrm{SP}(2n,k)$ we now begin to characterize which involutions of $\mathrm{SL}(2n,k)$ remain involutions when restricted to $\mathrm{SP}(2n,k)$. We give the result that an involution τ of $\mathrm{SL}(2n,k)$ keeps G invariant if and only if $\tau\phi = \phi\tau$ where $\phi = \mathrm{Inn}_J\,\theta$ where $\theta(X) = (X^T)^{-1}$. Moreover, we determine specific criteria in order for an involutions of $\mathrm{SL}(2n,k)$ to remain an involution when restricted to $\mathrm{SP}(2n,k)$. With the aforementioned criteria in hand we are able to state exactly which involutions of $\mathrm{SL}(2n,k)$ will not remain involutions when restricted to $\mathrm{SP}(2n,k)$.

Finally, we give the result that the isomorphy class of outer involutions of $\mathrm{SL}(2n,k)$ coming from the skew-symmetric matrix $M = J_{2n}$ does not exist when restricted to $\mathrm{SP}(2n,k)$. In addition, we show that if two involutions τ_1 and τ_2 on G come from the restriction of outer involutions of $\mathrm{SL}(2n,k)$ then if $\tau_1 \approx \tau_2$ over G then the outer involutions of $\mathrm{SL}(2n,k)$ from which they came must be isomorphic over $\mathrm{SL}(2n,k)$.

Chapter 2

Symmetric Spaces and Bilinear Forms

2.1 Notation

Throughout this thesis our terminology and notation for reductive groups will come from the books of Borel [Bor91], Humphreys [Hum72] and Springer [Spr81]. We will also use information provided in the papers of Borel and Tits [BT65], [BT72]. All algebraic groups and algebraic varieties are taken over an arbitrary field k with the characteristic k not equal to 2 and all algebraic groups considered are linear algebraic groups. In addition, throughout this thesis some standard notation is used. With an attempt to limit the introduction of new notation we provide the following list.

k $-$ denotes a field of characteristic not equal to 2

k_1 $-$ an extension field of k

$\bar{k}$ $-$ the algebraic closure of k

$V = k^n$- a finite dimensional vector space over the field k

$\bar{V} = \bar{k}^n$

k^* $-$ the product group of all nonzero elements of k

$(k^*)^2 = \{a^2 \mid a \in k^*\}$

$\mathrm{M}_n(k) = \mathrm{M}(n,k)$ =\{$n \times n$ matrices with entries in k\}

$\mathrm{GL}(V) = \mathrm{GL}(n,k) = \mathrm{GL}_n(k) = \{A \in M_n(k) \mid \det(A) \neq 0\}$

$\mathrm{SL}(V) = \mathrm{SL}(n,k) = \mathrm{SL}_n(k) = \{A \in M_n(k) \mid \det(A) = 1\}$

Id - denotes the identity automorphism

$\mathrm{Aut}(G)$ - the set of all automorphisms on G

2.2 Symmetric Spaces

In this section we begin by giving the necessary background material required to give a complete definition of a symmetric space and conclude with said definition. We note here that an alternative definition of a symmetric space will be given later on in this chapter.

Throughout this thesis we will define $\bar{G}$ to be a reductive linear algebraic group over a field k. G will denote its set of k-rational points. We will assume G is a subgroup of $\mathrm{GL}(n.k)$.

2.2.1 Definition of Symmetric Space

Definition 1. Let θ be an automorphism. The ***order of*** $\boldsymbol{\theta}$, denoted $\mathrm{ord}(\theta)$, is defined to be the smallest integer y such that $\theta^y = \mathrm{Id}$.

Definition 2. Let $\theta \in \mathrm{Aut}(G)$ then $\boldsymbol{\theta}$ is an ***involution*** of G if $\theta \neq \mathrm{Id}$ and $\mathrm{ord}(\theta) = 2$, (i.e. $\theta^2 = \mathrm{Id}$).

Definition 3. Given an involution θ on our group G, the ***symmetric space*** $\boldsymbol{X}$ is defined as G/H where $G^\theta = H$ is the fixed-point group of the involution θ. One can also characterize this symmetric space as the subset $X = \{x\theta(x)^{-1} \mid x \in G\}$ of G. Then $X \approx G/H$.

Remark 1. Two symmetric spaces X_1 and X_2 are isomorphic if and only if their corresponding fixed-point groups H_1 and H_2 are isomorphic.

The above remark will allow us to give criteria for the classification of symmetric spaces. In essence we will classify the fixed-point group of an involution and then use the above remark to give a characterization of the symmetric space.

2.3 Isomorphic Involutions

As stated in the previous section determining the fixed point group of an involution will prove to be extremely significant in classifying symmetric spaces. That being said, it will be necessary for us to define what is meant by isomorphic involutions. Some notation is first needed.

For $A \in \mathrm{GL}(n.k)$

(1) Inn_A denote the ***inner automorphisms*** defined by

$$\mathrm{Inn}_A(X) = A^{-1}XA \ \ \forall X \in \mathrm{GL}(n,k).$$

(2) $\mathrm{Inn}_k(G) = \{\mathrm{Inn}_A \mid A \in G\}$

(3) $\mathrm{Inn}(G) = \{\mathrm{Inn}_A \mid A \in \bar{G}, \ \ \mathrm{Inn}_A(G) \subseteq G\}$.

Definition 4. (1) Let $\theta, \tau \in \mathrm{Aut}(G)$. We say that θ and τ are $\mathrm{Inn}_k(G)$-*isomorphic* denoted, $\theta \approx^{Inn_k} \tau$, if there exist a $\phi \in \mathrm{Inn}_k(G)$ such that $\tau = \phi^{-1}\theta\phi$.

(2) Let $\theta, \tau \in \mathrm{Aut}(G)$. We say that θ and τ are $\mathrm{Inn}(G)$-*isomorphic*, denoted, $\theta \approx^{Inn} \tau$, if there exists a $\phi \in \mathrm{Inn}(G)$ such that $\tau = \phi^{-1}\theta\phi$.

(3) We say that θ and τ are $\mathrm{Aut}(G)$-*isomorphic* denoted, $\theta \approx^{Aut} \tau$, if there exists a $\phi \in \mathrm{Aut}(G)$ such that $\tau = \phi^{-1}\theta\phi$.

When trying to determine the isomorphy classes of an automorphism one must be specific as to which type of isomorphism they are referring. It should be clear that the set of $\mathrm{Inn}_k(G)$-isomorphism classes are contained in the $\mathrm{Inn}(G)$-isomorphism classes, that is $\theta \approx^{\mathrm{Inn}_k} \tau \implies \theta \approx^{\mathrm{Inn}} \tau$. In addition, it is also easily seen that the set of $\mathrm{Inn}(G)$-isomorphism classes are contained in the $\mathrm{Aut}(G)$-isomorphism classes, that is $\theta \approx^{\mathrm{Inn}} \tau \implies \theta \approx^{\mathrm{Aut}} \tau$. Throughout this thesis we will only investigate $\mathrm{Inn}(G)$-isomorphism classes hence, $\theta \approx \tau$ will always indicate an $\mathrm{Inn}(G)$-isomorphism. The cases of classifying automorphisms, specifically involutions, using the $\mathrm{Inn}_k(G)$-isomorphic or $\mathrm{Aut}(G)$-isomorphic criteria still remain open problems.

2.4 Bilinear Forms

The existence of this section may initially seem unusual. As you will see in the next section there is a clear and precise relationship between bilinear forms and involutions. In fact, the wealth of knowledge already known about Bilinear Algebra will be the cornerstone in our ability to classify involuitons. Before we proceed we first must give some background information on bilinear forms.

2.4.1 Definition

Definition 5 ([Art91])**.** Let $V = k^n$ be a vector space, where k is any field such that $char(k) \neq 2$. A ***bilinear form on V*** is a function of two variables on V with values in the field k, $\beta : V \times V \to k$, satisfying the bilinear axioms below.

(1) $\beta(v_1 + v_2, w) = \beta(v_1, w) + \beta(v_2, w)$

(2) $\beta(v, w_1 + w_2) = \beta(v, w_1) + \beta(v, w_2)$

(3) $\beta(cv, w) = c\beta(v, w)$

(4) $\beta(v, cw) = c\beta(v, w)$

for all $v, w, v_i, w_i \in V$ and $c \in k$

Definition 6. The ***matrix associated with the form β*** on a vector space V is the matrix $M = (m_{i,j})$ with $m_{i,j} = \beta(e_i, e_j)$, where $\{e_i\}$ is any ordered basis for V.

Hence our bilinear form β can be viewed as

$$\beta(x, y) = x^T My \quad \forall x, y \in V$$

Note: The matrix of the form β is dependent upon the choice of basis for V.

In Section 2.5 we will see that the relationship between involutions and bilinear forms center specifically around symmetric and skew-symmetric non-degenerate bilinear forms. Therefore we define such terms and their relationship with matrices of the form.

Theorem 2.1 ([Art91]). *The properties of the bilinear form β carry over to the matrix of the form M. Using this fact we list 3 well know properties*

(1) β is a symmetric bilinear form $\iff$ M is a symmetric matrix.
($\beta(x, y) = \beta(y, x)\ \ \forall x, y \in V \iff M = M^T$)

(2) β is a skew-symmetric bilinear form $\iff$ M is a skew-symmetric matrix
($\beta(x, y) = -\beta(y, x)\ \ \forall x, y \in V) \iff M = -M^T$)

(3) β is non-degenerate (i.e. $\beta(x, y) = 0\ \ \forall y \in V$ only if $x = 0$) $\iff$ nullspace$(M) = 0$.
(β is non-degenerate $\iff \det(M) \neq 0 \iff$ M is invertible)

In this thesis all of the bilinear forms that we will use will be non-degenerate. Hence all the matrices representing these bilinear forms will be invertible.

2.4.2 Congruence

The idea of determining when two involutions are isomorphic will rely on the congruence classes of matrices of symmetric and skew-symmetric bilinear forms. Throughout this thesis we will consistently refer to the theorems in this subsection which characterize such congruence classes.

Definition 7. Let M_1 and M_2 be the matrices of two bilinear forms on $V = k^n$. Then M_1 is ***congruent*** to M_2, denoted $M_1 \cong M_2$, over the field k if there exists a matrix $Q \in \mathrm{GL}(n,k)$ such that $M_2 = Q^T M_1 Q$.

Theorem 2.2. *Two matrices M_1 and M_2 represent the same bilinear form with respect to different bases if and only if $M_1 \cong M_2$.*

Remark 2. If M_1 represents the matrix of the form β with respect to the basis $\mathfrak{B}_{\mathcal{N}}$ and M_2 represents the matrix of the form β with respect to the basis $\mathfrak{B}_{\mathcal{O}}$, then the matrix Q that gives the congruence relation is the change of basis matrix from $\mathfrak{B}_{\mathcal{O}}$ to $\mathfrak{B}_{\mathcal{N}}$.

Lemma 1 ([Szy97])**.** *If M is the matrix of a non-degenerate symmetric bilinear form, then:*

(1) M is congruent to a diagonal matrix with non-zero diagonal entries.

(2) Any rearrangement of the diagonal matrix M results in another matrix in the same congruence class, congruent via an orthogonal permutation matrix P.

In addition, we state the following theorem about congruence classes of symmetric and skew-symmetric matrices.

Theorem 2.3 ([Sch85])**.** *(1) Symmetric matrices are congruent to diagonal matrices whose entries are representatives of the square-class group $k^*/(k^*)^2$.*

(2) Skew-Symmetric matrices are congruent to the $2m \times 2m$ matrix J_{2m}, where $n = 2m$ and

$$J_{2m} = \begin{pmatrix} 0 & I_{m\times m} \\ -I_{m\times m} & 0 \end{pmatrix}.$$

2.5 Induced Involutions of Bilinear Forms

In this section we discuss the connection between bilinear forms and involutions. Specifically, we will determine the method of constructing an involution from a given symmetric or skew-symmetric bilinear form.

2.5.1 Definition of the Adjoint

Definition 8. Let M be the matrix of a non-degenerate bilinear form β over a vector space $V = k^n$. Given $A \in \mathrm{GL}(n,k)$ we define the ***adjoint of*** A ***with respect to*** β, denoted A', as the matrix that satisfies the relation $\beta(Ax, y) = \beta(x, A'y)$. When the bilinear form we are using is clear we will simply refer to A' as the adjoint of A.

Given a bilinear form $\beta(x, y) = x^T My$ we note that:

$$\begin{aligned}
\beta(Ax, y) &= \beta(x, A'y) \\
(Ax)^T My &= x^T M(A'y) \\
x^T A^T My &= x^T MA'y \\
A^T M &= MA' \\
M^{-1} A^T M &= A'
\end{aligned}$$

Hence, using the definition of the adjoint we are able to see that

$$A' = M^{-1} A^T M.$$

2.5.2 Constructing Involutions via the Adjoint

Definition 9. Given a non-degenerate bilinear form on V with matrix M, we define $\theta = \theta_M$ by $\theta_M(A) = (A')^{-1}$.

Using the fact that $A' = M^{-1} A^T M$ we have

$$\theta_M(A) = M^{-1}(A^T)^{-1} M$$

It turns out that the automorphism $\theta = \theta_M$ given in Definition 9 is an involution if the matrix M is symmetric or skew-symmetric. Before we formally state this result we observe the following properties of the adjoint of a symmetric or skew-symmetric bilinear form.

Theorem 2.4. *Let A' be the adjoint of a symmetric or skew-symmetric matrix. Then*

(1) $(A')' = A$

(2) $(A')^{-1} = (A^{-1})'$

(3) $(AB)' = B'A'$

Proof. (1)

$$\begin{aligned}(A')' &= (M^{-1}A^TM)' \\ &= M^{-1}(M^{-1}A^TM)^T)M \\ &= M^{-1}M^TA(M^{-1})^TM\end{aligned}$$

If M is symmetric then $M = M^T$ and $M^{-1}M^TA(M^{-1})^TM = A$
Similarly, if M is skew-symmetric then $M = -M^T$ and again $M^{-1}M^TA(M^{-1})^TM = A$. In either case, $(A')' = A$.

(2)

$$\begin{aligned}(A')^{-1} &= (M^{-1}A^TM)^{-1} \\ &= M^{-1}(A^T)^{-1}M \\ &= M^{-1}(A^{-1})^TM \\ &= (A^{-1})'\end{aligned}$$

The 2nd and 3rd steps of the above equality are true since we know the transpose and inverse operations commute.

(3)

$$\begin{aligned}
(AB)' &= M^{-1}(AB)^T M \\
&= M^{-1}B^T A^T M \\
&= M^{-1}B^T(MM^{-1})A^T M \\
&= (M^{-1}B^T M)(M^{-1}A^T M) \\
&= B'A'
\end{aligned}$$

□

Proposition 1. *If β is either symmetric or skew-symmetric then $\theta = \theta_M$ is an involution.*

Proof. From Theorem 2.4 it is easy to see that θ is indeed an involution since

$$\begin{aligned}
\theta(AB) &= ((AB)')^{-1} \\
&= (B'A')^{-1} \\
&= (A')^{-1}(B')^{-1} \\
&= \theta(A)\theta(B)
\end{aligned}$$

and

$$\theta^2(A) = (((A')^{-1})')^{-1} = (A')' = A.$$

□

Utilizing the method described above, given an non-degenerate symmetric or skew-symmetric bilinear form on a vector space V, one can always obtain an involution of a matrix group over a field k. Because of this relationship a natural question arises, can all involutions be obtained in this manner, (i.e. Do all involutions come from the adjoint of a bilinear form?). As you will see in the next few chapters the answer to this question is primarily based on the matrix group and vector space in question. Another question which arises is, does there exist any correlation between congruence classes of symmetric and skew-symmetric bilinear forms and isomorphism classes of their induced involutions? The next section will explore this question in detail.

2.6 Congruent Bilinear Forms versus Isomorphic Involutions

Given the construction of an involution via the adjoint of a symmetric or skew-symmetric bilinear form it is natural to explore the relationship between the two ideas. In this section we will see that congruence of bilinear forms does not exactly correspond to isomorphic involutions however, by slightly altering the idea of congruence a nice correspondence is obtained.

2.6.1 Semi-Congruence

Theorem 2.5. *Let M_1 and M_2 be two matrices of symmetric or skew-symmetric bilinear forms β_1 and β_2 over $V = k^n$ respectively. Let θ_{M_1} and θ_{M_2} be their corresponding involutions on $\mathrm{GL}(n,k)$. If $M_1 \cong M_2$, then $\theta_{M_1} \approx \theta_{M_2}$.*

Proof. Suppose $M_1 \cong M_2$ over k. Then there exists a $Q \in \mathrm{GL}(n,k)$ such that $M_2 = Q^T M_1 Q$. Now using the fact that $\theta_{M_i} = M_i^{-1}(A^T)^{-1}M_i$ for $i = 1,2$ we have that $\forall A \in \mathrm{GL}(n,k)$,

$$\begin{aligned}
\theta_{M_2}(A) = \mathrm{Inn}_{M_2}(A^T)^{-1} &= \mathrm{Inn}_{Q^T M_1 Q}(A^T)^{-1} \\
&= (Q^T M_1 Q)^{-1}(A^T)^{-1}(Q^T M_1 Q) \\
&= Q^{-1}(M_1^{-1}((Q^T)^{-1}(A^T)^{-1}Q^T)M_1)Q \\
&= Q^{-1}\,\mathrm{Inn}_{M_1}((\mathrm{Inn}_{Q^{-1}}(A))^T)^{-1}Q \\
&= Q^{-1}\theta_{M_1}(\mathrm{Inn}_{Q^{-1}}(A))Q \\
&= \mathrm{Inn}_Q\,\theta_1\,\mathrm{Inn}_{Q^{-1}}(A), \Longrightarrow \theta_{M_2} = \mathrm{Inn}_Q\,\theta_{M_1}\,\mathrm{Inn}_{(Q)^{-1}}.
\end{aligned}$$

This means $\theta_{M_2} = (\phi)^{-1}\theta_{M_1}\phi$ with $\phi = \mathrm{Inn}_{Q^{-1}}$, and $\theta_{M_2} \approx \theta_{M_1}$ over $\mathrm{GL}(n, k)$. $\square$

A correspondence between congruent symmetric and skew-symmetric bilinear forms and isomorphisms of the induced involution begins to appear. That is congruence of bilinear forms implies an isomorphism of their induced involutions. In the reverse direction we get an ”almost” one-to-one correspondence between congruence classes of bilinear forms and isomorphy classes of involutions.

Theorem 2.6. *Suppose θ_{M_1} and θ_{M_2} are involutions on $\mathrm{GL}(n, k)$ which come from symmetric or skew-symmetric bilinear forms over $V = k^n$ with associated matrices M_1 and M_2 respectively. If $\theta_{M_1} \approx \theta_{M_2}$ then $M_2 = \alpha Q^T M_1 Q$ for some matrix $Q \in \mathrm{GL}(n, k)$ and some scalar $\alpha \in \bar{k}$.*

Proof. Suppose there exists a $\phi \in \mathrm{Inn}(\mathrm{GL}(n, k))$ such that $\theta_2 = \phi^{-1}\theta_1\phi$. Let $P \in \mathrm{GL}(n, \bar{k})$ such that $\phi = \mathrm{Inn}_P$. Then for all $A \in \mathrm{GL}(n, k)$,

$$\begin{aligned}
\theta_{M_2}(A) &= \phi^{-1}\theta_{M_1} \\
\phi(A) &= \mathrm{Inn}_{P^{-1}}\,\mathrm{Inn}_{M_1}((\mathrm{Inn}_P(A))^T)^{-1} \\
&= PM_1^{-1}(P^T)(A^T)^{-1}(P^T)^{-1}M_1P^{-1} \\
&= M_2^{-1}(A^T)^{-1}M_2
\end{aligned}$$

which imples

$$M_2P(M_1)^{-1}P^T(A^T)^{-1}(P^T)^{-1}M_1P^{-1}M_2^{-1} = (A^T)^{-1}$$

This holds for all A^T, so it holds for all A. Which means

$$((P^T)^{-1}M_1P^{-1}M_2^{-1})^{-1}A((P^T)^{-1}M_1P^{-1}M_2^{-1}) = A.$$

So $\mathrm{Inn}_{(P^T)^{-1}M_1P^{-1}M_2^{-1}} = \mathrm{Id}$, $(P^T)^{-1}M_1P^{-1}M_2^{-1} = \gamma I_{nxn}$ for some $\gamma \in k^*$, and $M_2 = 1/\gamma(P^T)^{-1}M_1P^{-1}$. The result follows by substituting $Q = P^{-1}$, $\alpha = 1/\gamma$. $\square$

The relationship $M_2 = \alpha Q^T M_1 Q$ for some matrix $Q \in \mathrm{GL}(n,k)$ and some scalar $\alpha \in \bar{k}$ is very similar to the definition of congruent matrices with the exception of the scalar α. Hence we define this relationship as semi-congruence in an effort to obtain the equivalence of Theorem 2.5 and Theorem 2.6.

Definition 10. Two bilinear forms on $V = k^n$ with associated matrices M_1 and M_2 are ***semi-congruent over k***, denoted $M_1 \cong^s M_2$, if there exists a $Q \in \mathrm{GL}(n,k)$ and an $\alpha \in \bar{k}$ such that $M_2 = \alpha Q^T M_1 Q$.

It is clear from the above definition of semi-congruence that if two matrices are congruent then they must be semi-congruent. We are now able to rewrite Theorem 2.6.

Theorem 2.7. *Suppose θ_1 and θ_2 are involutions on $\mathrm{GL}(n,k)$ which come from symmetric or skew-symmetric bilinear forms over $V = k^n$ with associated matrices M_1 and M_2 respectively. Then $\theta_{M_1} \approx \theta_{M_2} \Rightarrow M_1 \cong^s M_2$.*

2.6.2 Classification Theorem

The two Theorems discussed in Section 2.6.1 can be combined to create precise relationship between associated matrices of symmetric and skew-symmetric bilinear forms and their corresponding involutions which is given below.

Theorem 2.8 (Classification Theorem). *If θ_{M_1} and θ_{M_2} are involutions on $\mathrm{GL}(n,k)$ which come from a symmetric or skew-symmetric bilinear form as stated before, then*

$$M_1 \cong^s M_2 \quad \text{over} \quad k \iff \theta_{M_1} \approx \theta_{M_2}.$$

Proof. $\Longleftarrow$ The proof of the above theorem in this direction follows directly from Theorem 2.7.

$\Longrightarrow$ Suppose $M_1 \cong^s M_2$, then this is equivalent to $M_1 \cong \alpha M_2$. Now Theorem 2.5 says that $\theta_{M_1} \approx \theta_{\alpha M_2}$. Now

$$\begin{aligned}
\theta_{\alpha M_2} &= \mathrm{Inn}_{\alpha M_2}(A^T)^{-1} \\
&= (\alpha M_2)^{-1}(A^T)^{-1}(\alpha M_2) \\
&= (\alpha^{-1}\alpha)M_2^{-1}(A^T)^{-1}M_2 = M_2^{-1}(A^T)^{-1}M_2 \\
&= \mathrm{Inn}_{M_2}(A^T)^{-1} \\
&= \theta_{M_2}
\end{aligned}$$

Hence, $\theta_{\alpha M_2} = \theta_{M_2}$ which means $\theta_{M_1} \approx \theta_{M_2}$ □

Remark 3. There is an important concept which is addressed in the proof of the above Lemma which will be commented upon throughout this thesis. This is the fact that the scalar α does not affect the automorphism, ie. $\mathrm{Inn}_{\alpha M_2} = \mathrm{Inn}_{M_2}$.

2.6.3 Alternative Definition of Symmetric Spaces

Since we have been able to create a correspondence between associated matrices of symmetric and skew-symmetric bilinear forms and involutions which are defined by these forms one must question how this relates to symmetric spaces. By redefining our notion of symmetric spaces this will become clear.

Definition 11. Given a non-degenerate symmetric or skew-symmetric bilinear form β on $V = k^n$, the ***orthogonal group*** $\mathbf{O}(\boldsymbol{n}, \boldsymbol{k}, \boldsymbol{\beta})$ is defined as

$$\mathrm{O}(n, k, \beta) = \{A \in \mathrm{GL}(n, k) \mid \beta(Ax, Ay) = \beta(x, y)\}.$$

$$= \{A \in GL(n, k) \mid AA^{'} = \mathrm{I}\}$$

Recall: When we first defined a symmetric space X we said that $X = G/H$ where $H = G^{\theta}$, the fixed point group of the involution θ. In Section 2.5.2, we defined involutions which come from a bilinear form via the adjoint as $\theta(A) = (A')^{-1}$. Let's now observe the fixed point group H of such an involution

$$\begin{aligned} H &= \{A \mid \theta(A) = A\} \\ &= \{A \mid (A')^{-1} = A\} \\ &= \{A \mid AA' = \mathrm{I}\} \\ &= \mathrm{O}(n, k, \beta) \end{aligned}$$

Therefore, if an involution comes from a symmetric or skew-symmetric bilinear form then the fixed point group of the involution is precisely the orthogonal group of the form. With this relationship we can redefine our symmetric space.

Definition 12 (***Symmetric Space***)**.** Let $X = \{AA' \mid A \in \mathrm{GL}(n,k)\}$. Then $X = \{A(\theta(A))^{-1} \mid A \in \mathrm{GL}(n,k)\} \simeq \mathrm{GL}(n,k)/\mathrm{O}(n,k,\beta)$.

If G is a subgroup of $\mathrm{GL}(n,k)$, invariant under taking the adjoint, then $\hat{X} = X \cap G \simeq G/(G \cap \mathrm{O}(n,k,\beta))$ is exactly the symmetric space defined in the previous definition.

Recall: Two symmetric spaces X_1 and X_2 are isomorphic if and only if their corresponding fixed-point groups H_1 and H_2 are isomorphic.

Theorem 2.9 ([HW93])**.** *Two fixed point groups $H_{\beta_1} = \mathrm{O}(n,k,\beta_1)$ and $H_{\beta_2} = \mathrm{O}(n,k,\beta_2)$ are isomorphic if and only if $\theta_1 \approx \theta_2$.*

Therefore, if an involution comes from a non-degenerate symmetric or skew-symmetric bilinear form by using the Classification Theorem we can determine the isomorphy classes of involutions which will in turn give us a classification of the related symmetric space via its fixed point group.

2.7 Results for $\mathrm{SL}(n,k)$

Although all the results in the previous section have been given based on the group $\mathrm{GL}(n,k)$ from this point on we will focus strictly on subgroups of $\mathrm{GL}(n,k)$. In particular we will begin with the subgroup $\mathrm{SL}(n,k)$ of $\mathrm{GL}(n,k)$. In this section we will show that all of the results given up to this point for $\mathrm{GL}(n,k)$ in fact hold for $\mathrm{SL}(n,k)$.

Lemma 2. *An automorphism θ coming from a symmetric or skew-symmetric bilinear form is an involution on $\mathrm{GL}(n,k)$ $\Longleftrightarrow$ θ is an involution on $\mathrm{SL}(n,k)$.*

Proof. $\Longrightarrow$ Let θ be an involution on $\mathrm{GL}(n,k)$ which comes from a bilinear form with associated matrix M. Then $\theta(A) = M^{-1}(A^T)^{-1}M$. Since $\mathrm{SL}(n,k)$ is a subgroup of $\mathrm{GL}(n,k)$ we can restrict θ to $\mathrm{SL}(n,k)$. W now only need to check if θ keeps $\mathrm{SL}(n,k)$ invariant. Let $A \in \mathrm{SL}(n,k)$ then $\det(\theta(A)) = \det(M^{-1}(A^T)^{-1}M) = \det(A^{-1}) = 1/\det(A) = 1$ since $A \in \mathrm{SL}(n,k)$. Hence θ is an involution on $\mathrm{SL}(n,k)$
$\Longleftarrow$ This direction is trivial since if θ is an involution on $\mathrm{SL}(n,k)$ it is clear that if lifted to act on $\mathrm{GL}(n,k)$, θ will still remain an involution. □

Since the Classification Theorem focuses on the isomorphism classes of the involutions it is necessary to explore the relationship between these classes of $\mathrm{GL}(n,k)$ versus $\mathrm{SL}(n,k)$. The following theorem addresses this issue.

Theorem 2.10. *Two involutions θ_1 and θ_2 are isomorphic over $\mathrm{GL}(n,k)$ if and only if they are isomorphic over $\mathrm{SL}(n,k)$.*

Proof. $\Longleftarrow$: This direction is clear.

$\Longrightarrow$: Suppose θ_1 and θ_2 are isomorphic over $\mathrm{GL}(n,k)$ via some $P \in \mathrm{GL}(n,k)$. Then we can replace P with $\hat{P} = \left(1/\sqrt[n]{\det(P)}\right) P$ and still retain $\mathrm{Inn}-G$ isomorphism via $\mathrm{Inn}_{\hat{P}}$. Now

$$\det(\hat{P}) = \left(1/\sqrt[n]{\det(P)}\right)^n (\det(P)) = 1$$

Since $P \in \mathrm{SL}(n,k)$ and thus $\det(P) = 1$. Therefore we may conclude that $\hat{P} \in \mathrm{SL}(n,k)$ and hence $\theta_1 \approx \theta_2$ over $\mathrm{SL}(n,k)$. □

Chapter 3

Involutions on $G = \mathrm{SL}(2, k)$

3.1 Introduction

This chapter focuses on involutions of the subgroup $\mathrm{SL}(2, k)$ of $\mathrm{GL}(2, k)$. As we will see in the later chapters the classification of the involutions on $\mathrm{SL}(2, k)$ will play a vital role in classifying involutions on $\mathrm{SP}(2, k)$, therefore the development of this classification will be explored in detail.

In this chapter $G = \mathrm{SL}(2, k)$, $\bar{G} = \mathrm{SL}(2, \bar{k})$, $G_1 = \mathrm{SL}(2, k_1)$ and all bilinear forms are taken over either the vector space $V = k^2$ or $\bar{V} = \bar{k}^2$.

3.2 Involutions and Bilinear Forms

This subsection gives the framework needed in order to determine the isomorphy classes of involutions on $\mathrm{SL}(2, k)$.

Lemma 3 ([Bor91])**.** *If k is an algebraically closed field,* $\mathrm{Aut}(G) = \mathrm{Inn}(G)$.

Remark 4. If $\theta \in \mathrm{Aut}(G)$ and k is not algebraically closed then there always exists an extension field k_1 and a $\tau \in \mathrm{Inn}(\mathrm{SL}(2, k_1))$ such that $\tau|_G = \theta$.

Lemma 4 ([HW02]). *The inner automorphism $\mathrm{Inn}_A \in \mathrm{Inn}\, G_1$ keep G invariant if and only if $A = \alpha B$ for some α in k_1 and some B in $\mathrm{GL}(2,k)$.*

We first recall that adding a scalar to a conjugation does not alter the automorphism, that is $\mathrm{Inn}_{\alpha B} = \mathrm{Inn}_B$ as discussed in Remark 3 of Chapter 2. Because of this fact the above Lemma proves to be extremely useful. We now have that every automorphism, and hence every involution, of $\mathrm{SL}(2,k)$ can be written as Inn_B where $B \in \mathrm{GL}(2,k)$. Adding the additional criteria that our automorphism is an involution gives us the following result.

Theorem 3.1 ([HW02]). *All involutions on $\mathrm{SL}(2,k)$ have the form $\theta = \mathrm{Inn}_B$, where B has the form $\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$ and $b \in k$.*

We are now able to link the form of involuitons of $\mathrm{SL}(2,k)$ given in Theorem 3.1 to involutions derived from bilinear forms as discussed in Chapter 2.

Theorem 3.2. *All involutions on $\mathrm{SL}(2,k)$ have the form θ_M where M is the matrix of a symmetric bilinear form and θ_M its corresponding involution.*

Proof. Consider an involution ϕ of $\mathrm{SL}(2,k)$. Then by Theorem 3.1 $\phi = \mathrm{Inn}_B$ where $B = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$. Let M' be the matrix of a non-degenerate symmetric bilinear form on $V = k^2$. By Lemma 1 we know that M' is congruent to a diagonal matrix, that is M' can be viewed as, $M' = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$. Normalizing the $(1,1)$ entry of M' does not change the involution since

$$\begin{aligned}
\theta_{M'}(A) &= (M')^{-1}(A^T)^{-1}M' \\
&= \mathrm{Inn}_{M'}(A^T)^{-1} \\
&= \mathrm{Inn}_{\frac{1}{m_1}M'}(A^T)^{-1} \\
&= \theta_{\frac{1}{m_1}M'}(A)
\end{aligned}$$

Hence we can let our symmetric bilinear form be represented by the matrix $M = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$, where $m = -b$. Let $Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then $YM = \begin{pmatrix} 0 & -m \\ 1 & 0 \end{pmatrix}$. Now $\phi(A) = \mathrm{Inn}_B(A) = \mathrm{Inn}_{YM}(A) = \mathrm{Inn}_M \mathrm{Inn}_Y(A) = \mathrm{Inn}_M(A^T)^{-1} = \theta_M(A)$. Hence Inn_B comes from a bilinear form and all involutions of $\mathrm{SL}(2,k)$ have the form θ_M where M is the matrix of a symmetric bilinear form. □

Since the above theorem tells us that all involutions of $\mathrm{SL}(2,k)$ come from bilinear forms we are able to invoke The Classification Theorem 2.8 in order to determine the isomorphism classes of involutions of $\mathrm{SL}(2,k)$.

Theorem 3.3. *Let M_1 and M_2 be associated matrices of symmetric bilinear forms on $V = k^2$. Then $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & m_1 \end{pmatrix} \cong^s M_2 = \begin{pmatrix} 1 & 0 \\ 0 & m_2 \end{pmatrix} \iff m_1 = \alpha^2 m_2$ for some $\alpha \in k^*$ (i.e. the semi-congruence classes of symmetric bilinear forms on $V = k^n$ are determined by the square class group $k^*/(k^*)^2$.)*

Proof. $\Longrightarrow$: Suppose M_1 and M_2 are semi-congruent. Then we know there exist a $Q \in \mathrm{GL}(2,k)$ and a scalar $c \in k^*$ such that $M_1 = cQ^T M_2 Q$. By taking the determinant of both sides of we see that $m_1 = c^2(\det Q)^2 m_2 = (c\det Q)^2 m_2$. By letting $\alpha = c\det Q$ we obtain the desired result that $m_1 = \alpha^2 * m_2$ for some $\alpha \in k$.

$\Longleftarrow$: Let M_1 and M_2 be defined as above. Suppose $m_1 = \alpha^2 * m_2$. Then $M_1 = Q^T M_2 Q$ where $Q = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$. This of course implies that $M_1 \cong M_2$. Since congruence implies semi-congruence we have that $M_1 \cong^s M_2$.

□

Theorem 3.4. *The number of isomorphism classes of involutions of G is equal to $|k^*/(k^*)^2|$, and each representative has the form $\theta = \mathrm{Inn}_B$, where $B = \begin{pmatrix} 0 & -m \\ 1 & 0 \end{pmatrix}$ and m is a representative of $k^*/(k^*)^2$.*

3.3 Skew-Symmetric Bilinear Form on $V = k^2$

Before we give a complete classification for the involutions on $\mathrm{SL}(2,k)$ we note one important fact. The Classification Theorem clearly deals with involutions which come from both symmetric and skew-symmetric bilinear forms on V via the adjoint. However, in the previous section there is no mention of any involutions coming from a skew-symmetric bilinear form on V. This case has not accidently been overlooked. For $V = k^2$ there are no involutions which come from skew-symmetric bilinear forms. The reason for this is given in this section.

Lemma 5. *(1) All skew-symmetric bilinear forms on k^2 are semi-congruent to the form represented by the matrix*

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(2) The only induced automorphism from the skew-symmetric form on k^2 is therefore $\theta(A) = \mathrm{Inn}_M(A^T)^{-1}$ which is not an involution.

Proof. (1) We know from Theorem 2.1 that a skew-symmetric form is represented by a skew-symmetric matrix M with $M = -M^T$. Since M is a 2×2 matrix M has the form

$$\begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}$$

which is semi-congruent to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(2) From part 1 we know that the only induced automorphism from the skew-symmetric form on k^2 is $\theta(A) = \mathrm{Inn}_M(A^T)^{-1}$ where $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. However if $A \in \mathrm{SL}(2,k)$ then it is simple to see that $\theta(A) = \mathrm{Inn}_M(A^T)^{-1} = A$. That is $\theta(A) = \mathrm{Inn}_M(A^T)^{-1}$ is the identity automorphism and is therefore not an involution. □

Hence we can conclude that there are no involutions induced by the skew-symmetric form on $\mathrm{SL}(2, k)$.

3.4 Isomorphy Classes of Involutions of $\mathrm{SL}(2, k)$

In this section we invoke The Classification Theorem 2.8 in order to give a complete classification of the involutions of $\mathrm{SL}(2, k)$. However, because of the sole nature of the Classification Theorem it is necessary that we also identify the semi-congruence classes of the symmetric bilinear forms. Theorem 3.3 told us that the semi-congruence classes of symmetric bilinear forms are determined by the square class group $|(k^*)/(k^*)^2|$. When we give the characterization of such classes we will rely on this fact. In addition, we are able to use Theorem 3.4 which tells us that each involution of $\mathrm{SL}(2, k)$ has the form Inn_B where $B = \begin{pmatrix} 0 & -m \\ 1 & 0 \end{pmatrix}$ and $m \in (k^*)/(k^*)^2$ to give a complete classification of the isomorphy classes of involutions for various fields as seen below.

(1) $k = \bar{k}$: If k is algebraically closed then $|(k^*)/(k^*)^2| = 1$ hence there is only 1 semi-congruence class of matrices.
All involutions are isomorphic to $\theta = \mathrm{Inn}_{B_1}$, where $B_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with $\theta(A) = \mathrm{Inn}_B(A) = (A^T)^{-1}$.

(2) $k = \mathbb{R}$: In this case we know that $|(k^*)/(k^*)^2| = 2$ with representatives given by 1 and -1. There are 3 congruence classes of symmetric bilinear forms over the real numbers represented by $M_1 = \mathrm{Id}$, $M_2 = -\mathrm{Id}$ and $M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. However, when we look at the semi-congruence classes over $\mathbb{R}$ we see that M_1 and M_2 represent the same semi-congruence class.
There are 2 isomorphism classes of involutions (1) $\theta = \mathrm{Inn}_{B_1}$ as defined above in

the case where $k = \bar{k}$ and (2) $\theta_2 = \mathrm{Inn}_{B_2}$, where $B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The action of θ_2 is: $\theta_2(A) = \begin{pmatrix} a_{2,2} & a_{2,1} \\ a_{1,2} & a_{1,1} \end{pmatrix}$.

(3) $k = \mathbb{Q}$: Unfortunately $|(k^*)/(k^*)^2| = \infty$ and there are infinitely many congruence classes over $\mathbb{Q}$ therefore there are infinitely many semi-congruence classes over $\mathbb{Q}$. In addition, we can only conclude that there are infinitely many isomorphism classes in this case.

(4) $k = \mathbb{F}_p \quad (p \neq 2)$: In this case $|(k^*)/(k^*)^2| = 2$. This can be seen as follows. Let $\phi : \mathbb{F}_p \to \mathbb{F}_p$ be the map defined by $\phi(x) = x^2$. Then $\phi(\mathbb{F}_p^*) = \mathbb{F}_p^{*2}$ is a normal subgroup of $\mathbb{F}_p^*$ and $|\mathbb{F}_p^*/\mathbb{F}_p^{*2}| = |\mathrm{Ker}(\phi)| = 2$. Hence, the 2 semi-congruence classes are represented by $M = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$, where m is either 1 or S_p =the "smallest" non-square element of $\mathbb{F}_p$.
There are two isomorphism classes of involutios given by (1) $\theta = \mathrm{Inn}_{B_1}$ as defined in the algebraically closed case and (2) $\theta_3 = \mathrm{Inn}_{B_3}$ where $B_3 = \begin{pmatrix} 0 & -S_p \\ 1 & 0 \end{pmatrix}$.

3.5 Resulting Symmetric Space

Since we began this thesis with an interest in viewing symmetric spaces it is natural to question what such a space looks like. The simplest illustration of the resulting symmetric space can be viewed when $G = \mathrm{SL}(2,\mathbb{R})$. Recall that a symmetric space is defined as $X = G/H$ where H is the fixed point group of the involution. When $k = \mathbb{R}$ we know from Section 3.4 that there are two isomorphy classes of involutions and hence we are able to view their two corresponding symmetric spaces.

(1) As the previous section indicates the action of the first involution $\theta_1 = \mathrm{Inn}_B$, where $B_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is

$$\theta_1(A) = (A^T)^{-1}$$

The fixed point group of this involution is given by

$$\begin{aligned} H_1 = G^{\theta_1} &= \{A \in G \mid (A^T)^{-1} = A\} \\ &= \{A \in G \mid (A^T) = A^{-1}\} \\ &= \mathrm{SO}(2,\mathbb{R}) \end{aligned}$$

Hence the fixed point group of θ_1 is the well known 2-Dimensional Special Orthogonal group of length preserving rotations. In addition, if $A \in \mathrm{SL}(2,k)$ and $A^T = A^{-1}$ then $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, where $\alpha^2 + \beta^2 = 1$. That is H_1 is isomorphic to the unit circle. Moreover, the resulting symmetric space is

$$X_1 = G/H_1 = G/\,\mathrm{SO}(2,\mathbb{R}) \cong \{AA^T \mid A \in \mathrm{SL}(2,\mathbb{R})$$

which is precisely the set of positive definite symmetric matrices.

(2) For the second symmetric space we consider the other involution of $\mathrm{SL}(2,\mathbb{R})$, $\theta_2 = \mathrm{Inn}_{B_2}$ where $B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which has the given action $\theta_2(A) = \begin{pmatrix} a_{2,2} & a_{2,1} \\ a_{1,2} & a_{1,1} \end{pmatrix}$. (i.e. switches the diagonal and anti-diagonal entries of A). We examine the fixed point group of θ_2 and see that

$$H_2 = G^{\theta_2} = \{A \in G \mid A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \text{ and } \alpha^2 - \beta^2 = 1\}$$

Hence, H_2 is isomorphic to a hyperbola. The resulting symmetric space is simply $X_2 = G/H_2$.

3.6 Table of Involutions on $\mathrm{SL}(2,k)$

This section gives a table which summarizes the information given in the two previous sections. The table includes the Semi-Congruence classes, the isomorphism Classes of

Involutions as well as the action of the involutions over the algebraically closed field, the real numbers, the rational numbers and the finite fields. An important thing to note when looking at the table is the connection between the semi-congruence classses of symmetric matrices and the the isomorphy classes of their induced involutions. This information is given below.

(1) The semi-congruence classes are given by $M = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$, where $m \in (k^*)/(k^*)^2$.

(2) The involutions of $\mathrm{SL}(2,k)$ can be viewed as $\theta = \mathrm{Inn}_M(A^T)^{-1}$ where $M = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$.

(3) The Involutions of $\mathrm{SL}(2,k)$ can also be viewed as $\theta = \mathrm{Inn}_B(A)$ where $B = \begin{pmatrix} 0 & -m \\ 1 & 0 \end{pmatrix}$

Table 3.1: Involutions of $\mathrm{SL}(2, k)$

Field	Semi-Congruence Class M	Involution $\theta = \mathrm{Inn}_B$	Action of the Involution on G
$k = \bar{k}$	$M = I_{2\times 2}$	$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\theta(A) = (A^T)^{-1}$
$k = \mathbb{R}$	$M_1 = I_{2\times 2}$	$B_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\theta_1(A) = (A^T)^{-1}$
	$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\theta_2(A) = \begin{pmatrix} a_{2,2} & a_{2,1} \\ a_{1,2} & a_{1,1} \end{pmatrix}$
$k = \mathbb{Q}$	$M_i = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_i \end{pmatrix}$ ($\alpha_{i_1} \neq \alpha_{i_2}(q^2)$ for any $i = i_1, i_2 \in \mathbb{N}$ $q \in \mathbb{Q}^*$)	$B_i = \begin{pmatrix} 0 & -\alpha_i \\ 1 & 0 \end{pmatrix}$	$\theta_i(A) = \begin{pmatrix} a_{2,2} & -\alpha_i a_{2,1} \\ -a_{1,2}/\alpha_i & a_{1,1} \end{pmatrix}$
$k = \mathbb{F}_p$ $p \neq 2$	$M_1 = I_{2\times 2}$	$B_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\theta_1(A) = (A^T)^{-1}$
	$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & S_p \end{pmatrix}$	$B_2 = \begin{pmatrix} 0 & -S_p \\ 1 & 0 \end{pmatrix}$	$\theta_2(A) = \begin{pmatrix} a_{2,2} & -S_p a_{2,1} \\ -a_{1,2}(S_p)^{-1} & a_{1,1} \end{pmatrix}$

Chapter 4

Involutions on $G = \mathrm{SL}(n, k), n > 2$

4.1 Introduction

As was the case for $\mathrm{SL}(2, k)$ the classification for involutions of $\mathrm{SL}(n, k)$, $n > 2$ plays a role in characterizing involutions of $\mathrm{SP}(2n, k)$ and therefore must be studied in detail. In the following chapter a complete classification of involutions on $\mathrm{SL}(n, k)$ will be given.

In this chapter, $G = \mathrm{SL}(n, k)$, $\bar{G} = \mathrm{SL}(n, \bar{k})$ and $n > 2$. All bilinear forms are taken over the vector space $V = k^n$ or $\bar{V} = \bar{k}^n$.

4.2 Outer Involutions of $G = \mathrm{SL}(n, k)$

In the previous chapter we discovered that all involutions of $\mathrm{SL}(2, k)$ are of type inner and were determined by a symmetric bilinear form via the adjoint. When we look at involutions of $\mathrm{SL}(n, k)$ where $n > 2$ this result does not carry over. The first distinction will appear in the fact that all the automorphisms, and thus involutions, of $\mathrm{SL}(n, k)$ for $n > 2$ are not inner. Secondly, we will see that the involutions of $\mathrm{SL}(n, k), n > 2$ that are not of type inner will be the ones which come from a symmetric or skew-symmetric bilinear form. We begin by defining a non-inner automorphism of $\mathrm{SL}(n, k), n > 2$.

Definition 13. Any automorphism θ of G such that $\theta \neq \mathrm{Inn}_M$ for any matrix $M \in G$ is an ***outer automorphism*** of G.

Lemma 6 ([Bor91])**.** *(1) If k is algebraically closed, then $|\operatorname{Aut}(G)/\operatorname{Inn}(G)| = 2$.*

(2) Any outer automorphism θ of G can be written as $\operatorname{Inn}_B\phi$, where ϕ is a fixed outer automorphism.

As previously stated, it will turn out that the outer involutions of $\mathrm{SL}(n,k), n > 2$ will come from a symmetric or skew-symmetric bilinear form via the adjoint. Choosing our fixed outer autormorphism ϕ discussed in Lemma 6 as $\phi(A) = (A^T)^{-1}$, enables us to begin to see this connection.

Let $M \in G$, then using ϕ as defined above we see that

$$\operatorname{Inn}_M(\phi(A)) = \operatorname{Inn}_M(A^T)^{-1} = M^{-1}(A^T)^{-1}M = \theta_M(A).$$

Although this appears to be the method we used to define an involution via the adjoint of a bilinear form we can not yet draw that conclusion. The reason we must hesitate is because Proposition 1 said that $\theta = \theta_M$ is an involution when M represents a symmetric or skew-symmetric bilinear form. The next theorem however will give us this result when we add the criteria that the outer automorphism $\operatorname{Inn}_M\phi$ be an involution.

Lemma 7. *(1) $\operatorname{Inn}_M\phi$ is an involution $\iff$ $\phi(M)M \in Z(G)$.*

(2) $\phi(M)M \in Z(G) \iff M$ is symmetric or skew-symmetric.

(3) $\operatorname{Inn}_M\phi$ is an involution $\iff$ M is symmetric or skew-symmetric, and M is only skew-symmetric if n is even.

Proof. (1) We know that $\operatorname{Inn}_M\phi$ is an involution $\iff$ $(\operatorname{Inn}_M\phi)^2 = \mathrm{Id}$ on G. Then the following are each equivalent:

(a) $\operatorname{Inn}_M\phi\operatorname{Inn}_M\phi(X) = X$ for all $X \in G$

(b) $\operatorname{Inn}_M\phi(M^{-1}\phi(X)M) = \operatorname{Inn}_M\phi(M^{-1})\phi(\phi(X))\phi(M) = X$

(c) $M^{-1}\phi(M^{-1})X\phi(M)M = X$

(d) $X\phi(M)M = \phi(M)MX$ for all $X \in G$

(e) $\phi(M)M \in Z(G)$

(2) $\Longleftarrow$: If M is symmetric, then $M = M^T$ and $\phi(M)M = (M^T)^{-1}M = M^{-1}M = I_{n\times n}$, which is clearly in $Z(G)$. If M is skew-symmetric, then $M = -M^T$ and $\phi(M)M = (M^T)^{-1}M = (-M)^{-1}M = -I_{n\times n} \in Z(G)$.

$\Longrightarrow$: If $\phi(M)M = (M^T)^{-1}M \in Z(G)$ then

$$(M^T)^{-1}MX = X(M^T)^{-1}M \quad \forall X \in G.$$

So $(M^T)^{-1}MXM^{-1}M^T = X \quad \forall X \in G$ and $\mathrm{Inn}_{M^{-1}M^T} = \mathrm{Id}$. Therefore $M^{-1}M^T = \alpha I_{n\times n}$ for some $\alpha \in k$, and $M^T = \alpha M$. Taking determinants of both sides we see that $\alpha^n = 1$. If n is odd, then $\alpha = 1$ and $M = M^T$. If n is even, then $\alpha = 1$ or -1, and $M = M^T$ or $M = -M^T$.

The third statement follows immediately from Proposition 1. $\square$

Remark 5. This lemma tells us that all outer involutions on G are of the form θ_M where M represents a symmetric or skew-symmetric bilinear form. Hence we have the following application of the Classification Theorem 2.8.

Theorem 4.1 (Outer Classification Theorem)**.** *Let θ_{M_1} and θ_{M_2} be outer involutions on* $\mathrm{SL}(n,k)$*, then they come from bilinear symmetric or skew-symmetric forms represented by M_1 and M_2. Then*

$$\mathrm{Inn}_{M_1}\phi \approx \mathrm{Inn}_{M_2}\phi \iff M_1 \cong^s M_2$$

Proof. The proof of the above theorem follows directly from The Classification Theorem 2.8 by simply observing that $\theta_{M_1} = \mathrm{Inn}_{M_1}\phi$ and $\theta_{M_2} = \mathrm{Inn}_{M_2}\phi$. $\square$

We are now able to see that in order to classify the outer involutions of $\mathrm{SL}(n,k)$ we once again have been reduced to determining the semi-congruence classes of the symmetric and skew-symmetric bilinear forms. From Theorem 2.3 we know this reduces to focusing on diagonal matrices with entries in $k^*/(k^*)^2$ and the skew-symmetric matrix J.

4.3 Classification of Outer Involutions of $\mathrm{SL}(n, k)$

In this section we give a complete classification of both the semi-congruence classes of the matrices of symmetric bilinear forms as well as the isomorphy classes of outer involutions of $\mathrm{SL}(n,k)$ for various fields k. It is important to note that over all of the fields below there will only be one semi-congruence class of skew symmetric bilinear forms which is represented by the matrix $M = J_{2m}$ which only occurs when $n = 2m$ is even.

Remark 6. In each case of the following cases the outer involutions are given by

$$\theta_M = \mathrm{Inn}_M\,\phi,\ \text{where } \phi(A) = (A^T)^{-1}$$

The symmetric/skew-symmetric matrices M are given below.

(1) $k = \bar{k}$: For k an algebraically closed field there is only one congruence class of symmetric bilinear forms and thus only one semi-congruence class.

(a) **n odd**: There is one isomorphism class of involutions represented by $M = \mathrm{Id}$.

(b) **n even**: There are two isomorphism classes of involutions represented by $M = \mathrm{Id}$ and $M = J_{2m}$.

(2) $k = \mathbb{R}$: Since $|k^*/(k^*)^2| = 2$ with representatives 1 and -1, we know from Theorem 2.3 that the congruence classes of symmetric forms are given by $M = I_{n-i,i} \quad i = 0, 1, \ldots, n$ where $I_{n-i,i} = \begin{pmatrix} I_{n-i\times n-i} & 0 \\ 0 & -I_{i\times i} \end{pmatrix}$. That is there are $n+1$ congruence classes of symmetric forms. However since $I_{n-i,i} \cong^s -I_{i,n-i}$ we have a reduced number of semi-congruence classes which depends on whether n is even or odd.

(a) **n odd**: There are $\frac{n+1}{2}$ isomorphism classes of involutions, represented by $M = I_{n-i,i} \quad i = 0, 1, \ldots, \frac{n-1}{2}$.

(b) **n even**: There are $\frac{n}{2} + 2$ classes of involutions, represented by $M = I_{n-i,i} \quad i = 0, 1, \ldots, \frac{n}{2}$ or J_{2m}.

(3) $k = \mathbb{Q}$: As in the case when $n = 2$, there are infinitely many congruence classes and therefore infinitely many semi-congruence classes of symmetric forms. Therefore, we can only conclude that there are infinitely many isomorphism classes of involutions.

Remark 7. Before we give the isomorphism classes of involutions over the finite fields we state the following well known result.

Lemma 8 ([Sch85]). *(1)* $|\mathbb{F}_p^*/(\mathbb{F}_p^*)^2| = 2$ *when* $p \neq 2$.

(2) Every element of $k = \mathbb{F}_p$ *can be written as the sum of* 2 *squares in* $\mathbb{F}_p$.

(3) For $\alpha \in \mathbb{F}_p$ *we have* $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \cong I_{2\times 2}$.

Therefore we can get either all 1's on the diagonal or all 1's with the exception of S_p, the ”smallest” representative of a non-square element, left over in the (n,n) entry.

With the following Lemma in hand we can now discuss involutions over the finite fields.

(1) $\boldsymbol{k = \mathbb{F}_p}$: Over the finite fields there are 2 congruence classes of symmetric bilinear forms and 2 semi-congruence classes.

(a) ***n* odd**: There are two isomorphy classes of involutions represented by $M = Id$ and $M = M_{n,S_p} = \begin{pmatrix} I_{(n-1)x(n-1)} & 0 \\ 0 & S_p \end{pmatrix}$.

(b) ***n* even**: We have 3 isomorphism classes of involutions, the 2 from above plus $M = J_{2m}$.

Remark 8. Table 4.1 includes a complete summary of the Outer Involutions of $\mathrm{SL}(n,k)$ and can be found at the end of Chapter 4.

4.4 Inner Involutions

The inner involutions of G do not come from bilinear forms. We will need the classification of these involutions when we focus on the subgroup $\mathrm{SP}(2n,k)$ of $\mathrm{SL}(2n,k)$ therefore it is necessary to give an overview of these results. We begin by noting that for any automorphism θ of inner type, there exist a $n \times n$ matrix $Y \in \mathrm{GL}(n, \bar{k})$, such that $\theta = \mathrm{Inn}_Y|_G$.

Lemma 9. *Let $Y \in \mathrm{GL}(n, \bar{k})$. If $\mathrm{Inn}_Y|_G = \mathrm{Id}$, then $Y = pI$ for some $p \in \bar{k}$, i.e. $\mathrm{Inn}_Y = \mathrm{Id}$ over $\mathrm{GL}(\bar{V})$.*

Proof. Since $\mathrm{Inn}_Y|_G = \mathrm{Id}$, we have for all $A \in \mathrm{SL}(n,k)$, $\mathrm{Inn}_Y(A) = Y^{-1}AY = A$, i.e. $YA = AY$. Since A is arbitrary it follows that $Y = pI$ for some $p \in \bar{k}$. Furthermore $\mathrm{Inn}_Y = \mathrm{Inn}_{pI} = \mathrm{Inn}_I = \mathrm{Id}$. □

Lemma 10 ([HWD04])**.** *For any inner automorphism $\theta \in \mathrm{Inn}(G)$, suppose $Y \in \mathrm{GL}(n, \bar{k})$. Then $\theta = \mathrm{Inn}_Y \in \mathrm{Inn}(\bar{G})$ keeps G invariant if and only if $Y = pB$, for some $p \in \bar{k}$ and $B \in \mathrm{GL}(n,k)$. In other words, there is a matrix $B \in \mathrm{GL}(n,k)$ such that $\theta = \mathrm{Inn}_B|_G$.*

Remark 9. Let $Y \in \mathrm{GL}(n,k)$. If we add the additional criteria that our automorphism is an involution then we get the following result. If Inn_Y is an involution, then $\mathrm{Inn}_{Y^2} = \mathrm{Id}$ and $Y^2 = cI_{n\times n}$.

Lemma 11. *Let $Y \in \mathrm{GL}(n,k)$ with $Y^2 = pI$. Then*

(1) If $p = c^2 \in (k^*)^2$, then Y is conjugate to $cI_{n-i,i} = c\begin{pmatrix} I_{n-i} & 0 \\ 0 & -I_i \end{pmatrix}$ $(i = 0, 1, \ldots n)$.

(2) If p is not in $(k^*)^2$, then n is even and Y is conjugate to $L_{n,p} = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ p & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & p & 0 \end{pmatrix}$.

Proof. If there is a $c \in k$ such that $p = c^2$, then the characteristic polynomial of Y is $(x-c)^{n-i}(x+c)^i$, and the minimal polynomial is a factor of $(x+c)(x-c)$. So Y is conjugate to $cI_{n-i,i}$ for some $i = 0, 1, \ldots, n$.

If p is not in k^{*2}, then the minimal polynomial is $(x^2 - p)$, which does not factor over k, therefore the characteristic polynomial is a power of the minimal polynomial. Hence n, which is the degree of the characteristic polynomial, is even. Furthermore, Y is conjugate to $L_{\frac{n}{2},p}$ since they have the same minimal and characteristic polynomials. □

We must now determine which of the above matrices gives us conjugate involutions.

Lemma 12 ([HWD04])**.** *The matrices* $I_{n-i,i}$ *and* $cI_{n-j,j}$ *are conjugate for some* $c \in k$ *if and only if one of the following is true:*

(1) $c = 1$ *and* $i = j$.

(2) $c = -1$ *and* $j = n - i$.

Lemma 13 ([HWD04])**.** *Let* $p, q \in \bar{k}^*/(\bar{k}^*)^2$. *The matrices* $L_{n,p}$ *is conjugate to* $cL_{m,q}$ *for some* $c \in k$ *if and only if* $\frac{p}{q} \in (k^*)^2$.

Theorem 4.2 ([HWD04])**.** *Suppose the involution* $\theta \in \mathrm{Aut}(G)$ *is of inner type. Then up to isomorphism* θ *is one of the following:*

(1) $\mathrm{Inn}_Y|_G$, *where* $Y = I_{n-i,i} \in \mathrm{GL}(n,k)$ *where* $i \in \left\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\right\}$.

(2) $\mathrm{Inn}_Y|_G$, *where* $Y = L_{\frac{n}{2},p} \in \mathrm{GL}(n,k)$ *where* $p \in k^*/k^{*2}$, $p \not\equiv 1 \mod k^{*2}$.

Note that (2) *can only occur when n is even.*

Corollary 1. *The number of involutions of inner type of* $\mathrm{SL}(n,k)$ $(n > 2)$ *up to isomorphism is equal to* $\frac{n-1}{2}$ *if n is odd and* $\frac{n}{2} + ||(k^*)/((k^*)^2)|| - 1$ *if n is even.*

Remark 10. (1) Table 4.1 contains a list of all the outer involutions of $\mathrm{SL}(n,k)$.

(2) Table 4.2 contains a list of all the inner involutions of $\mathrm{SL}(n,k)$.

(3) Table 4.3 provides a summary of both the inner and outer involutions of $\mathrm{SL}(n,k)$.

Table 4.1: Outer Involutions of $\mathrm{SL}(n,k)$

Field	Semi-Congruence Class M	Involution Class on G
$k = \bar{k}$		
n odd	$M = I_{n\times n}$	$\theta(A) = (A^T)^{-1}$
n even	$M = I_{n\times n}$ $M = J_{2m}$ $(n = 2m)$	$\theta_1(A) = (A^T)^{-1}$ $\theta_2(A) = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$
$k = \mathbb{R}$		
n odd	$M_i = I_{n-i,i}$ $i = 0, 1, 2, \ldots, \frac{n-1}{2}$	$\theta_i(A) = \mathrm{Inn}_{M_i}(A^T)^{-1}$
n even	$M_i = I_{n-i,i}$ $i = 0, 1, 2, \ldots, \frac{n}{2}$ $M_{\frac{n}{2}+2} = J_{2m}$	$\theta_i(A) = \mathrm{Inn}_{M_i}(A^T)^{-1}$ $\theta_{M_{\frac{n}{2}+2}}(A) = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$
$k = \mathbb{F}_p$ $p \neq 2$		
n odd	$M_1 = I_{n\times n}$ $M_2 = \begin{pmatrix} I_{(n-1)\times(n-1)} & 0 \\ 0 & S_p \end{pmatrix}$	$\theta_1(A) = (A^T)^{-1}$ $\theta_2(A) = \mathrm{Inn}_{M_2}(A^T)^{-1}$
n even	$M_1 = I_{n\times n}$ $M_2 = \begin{pmatrix} I_{(n-1)\times(n-1)} & 0 \\ 0 & S_p \end{pmatrix}$ $M_3 = J_{2m}$	$\theta_1(A) = (A^T)^{-1}$ $\theta_2(A) = \mathrm{Inn}_{M_2}(A^T)^{-1}$ $\theta_3(A) = \mathrm{Inn}_{M_3}(A^T)^{-1}$

Table 4.2: Inner Involutions of $\mathrm{SL}(n,k)$

Field	Number of Inner Involutions	Representative Matrix Y such that $\theta = \mathrm{Inn}_Y$
n odd, $k =$ any field	$\frac{n-1}{2}$	$Y = I_{n-i,i} \quad i = 1,2,\ldots,\frac{n-1}{2}$
n even		
$k = \bar{k}$	$\frac{n}{2}$	$Y = I_{n-i,i} \quad i = 1,2,\ldots,\frac{n}{2}$
$k = \mathbb{R}$	$\frac{n}{2}+1$	$Y = I_{n-i,i} \quad i = 1,2,\ldots,\frac{n}{2}$ $Y = L_{n,-1}$
$k = \mathbb{Q}$	∞	$Y = I_{n-i,i} \quad i = 1,2,\ldots,\frac{n}{2}$ $Y = L_{n,\alpha} \quad \alpha \neq 1 \mod (\mathbb{Q}^*)^2$
$k = \mathbb{F}_p \quad p \neq 2$	$\frac{n}{2}+1$	$Y = I_{n-i,i} \quad i = 1,2,\ldots,\frac{n}{2}$ $Y = L_{n,S_p}$

Table 4.3: Number of Involution Classes of $\mathrm{SL}(n,k)$

Field	$\dim(V)$	Number of Outer Involutions	Number of Inner Involutions	Total Number of Involutions
$k = \bar{k}$	n odd	1	$\frac{n-1}{2}$	$\frac{n+1}{2}$
	n even	2	$\frac{n}{2}$	$\frac{n}{2}+2$
$k = \mathbb{R}$	n odd	$\frac{n+1}{2}$	$\frac{n-1}{2}$	n
	n even	$\frac{n}{2}+2$	$\frac{n}{2}+1$	$n+3$
$k = \mathbb{Q}$	n odd	∞	$\frac{n-1}{2}$	∞
	n even	∞	∞	∞
$k = \mathbb{F}_p$ $p \neq 2$	n odd	2	$\frac{n-1}{2}$	$\frac{n+3}{2}$
	n even	3	$\frac{n}{2}+1$	$\frac{n}{2}+4$

Chapter 5

Involutions of $G = \mathrm{SP}(2n, k)$

5.1 Introduction

We begin this chapter be giving an introduction to the symplectic group $\mathrm{SP}(2n, k)$. We give a detailed investigation of the skew-symmetric bilinear form and provide a formal definition of $\mathrm{SP}(2n, k)$. Next we state the Characterization Theorem which will give us an identifiable form for automorphisms of $\mathrm{SP}(2n, k)$. We conclude with a characterization of involutions of $\mathrm{SP}(2n, k)$.

5.2 The Skew-Symmetric Form

Since throughout this thesis we have concentrated solely on fields of characteristic not equal to 2, we have two equivalent definitions for a skew-symmetric bilinear form.

Definition 14. A bilinear form $\beta : V \times V \to k$ on a vector space $V = k^n$ is skew-symmetric if

$$\beta(x, x) = 0 \text{ for all } x \in V.$$

Equivalently, $\beta : V \times V \to k$ is skew-symmetric if

$$\beta(x, y) = -\beta(y, x) \text{ for all } x, y \in V.$$

To derive the second definition of a skew-symmetric bilinear form from the first one consider the following,

$$\beta(x+y, x+y) = \beta(x,x) + \beta(x,y) + \beta(y,x) + \beta(y,y)$$

and use the fact that $\beta(x+y, x+y) = \beta(x,x) = \beta(y,y) = 0$.
Conversely, if the second definition holds, that is $\beta(x,y) = -\beta(y,x)$ then by setting $x = y$ we see that $\beta(x,x) = -\beta(x,x)$ which implies that $2\beta(x,x) = 0$ which means $\beta(x,x) = 0$. We note here that the following case of course does not hold if the characteristic of k is 2.

Theorem 2.1 told us that a bilinear form is skew-symmetric if and only if the matrix A of the form has the property that $A = -A^T$. In addition, unlike a symmetric matrix B, where B must be congruent to some diagonal matrix, all skew-symmetric matrices are congruent to J_m where m is even. This is seen in the following theorem.

Theorem 5.1 ([Sch85])**.** *Let A be a nonsingular skew-symmetric $m \times m$ matrix. Then m is even and there is a matrix $Q \in \mathrm{GL}(m,k)$ such that $Q^T AQ = J_m$.*

Remark 11.

(1) Since all skew-symmetric matrices are congruent to J_m there is only one skew-symmetric bilinear form, up to a change of basis, represented by $\beta(x,y) = x^T Jy$.

(2) If m is odd and A represents a skew-symmetric bilinear form β then we know $A = -A^T$. This means that $\det(A) = (-1)^m \det(A^T)$, which implies $\det(A) = 0$ and hence, A must be singular. Since all our matrices are non-singular we know that m will never be odd.

5.3 Definition of $\mathbf{SP(2n, k)}$

In Definition 11 we defined the orthogonal group $\mathrm{O}(m,k,\beta)$ as the following.
Given a non-degenerate symmetric or skew-symmetric bilinear form β on $V = k^m$

$$\begin{aligned}\mathrm{O}(m,k,\beta) &= \{A \in \mathrm{GL}(m,k) \mid \beta(Ax, Ay) = \beta(x,y)\} \\ &= \{A \in |GL(m,k)|AA' = \mathrm{I}\}.\end{aligned}$$

Similarly, one can also define the special orthogonal group $\mathrm{SO}(m, k, \beta)$ as

$$\begin{aligned} \mathrm{SO}(m, k, \beta) &= \{A \in \mathrm{SL}(m, k) \mid \beta(Ax, Ay) = \beta(x, y)\} \\ &= \{A \in \mathrm{SL}(m, k) | AA^{'} = \mathrm{I}\}. \end{aligned}$$

Now by considering the sole skew-symmetric bilinear form given by $\beta(x, y) = x^T Jy$ we are able to define the symplectic group $\mathrm{SP}(m, k)$ as

$$\begin{aligned} \mathrm{SP}(m, k) &= \mathrm{O}(m, k, \beta) \\ &\{A \in \mathrm{GL}(m, k) | \beta(Ax, Ay) = \beta(x, y)\} \\ &= \{A \in \mathrm{GL}(m, k) | AA^{'} = \mathrm{I}\} \\ &= \{A \in \mathrm{GL}(m, k) | AJ^{-1}A^T J = \mathrm{I}\} \\ &= \{A \in \mathrm{GL}(m, k) | A^T JA = J\}. \end{aligned}$$

Since we know m must be even from this point on we will use the notation $\mathrm{SP}(2n, k)$ to represent our symplectic group. The following theorem gives us more insight into $\mathrm{SP}(2n, k)$.

Theorem 5.2 ([Sch85])**.** *All elements of* $\mathrm{SP}(2n, k)$ *have determinant* 1.

We originally defined $\mathrm{SP}(2n, k)$ as $\mathrm{O}(2n, k, \beta)$, where β represents the skew-symmetric bilinear form, however, Theorem 5.2 tells that for any $A \in \mathrm{SP}(2n, k)$ $\det(A) = 1$. That being said we may view $\mathrm{SP}(2n, k) = \mathrm{SO}(2n, k, \beta)$. Hence $\mathrm{SP}(2n, k)$ is a subgroup of $\mathrm{SL}(2n, k)$.

$\mathrm{SP}(2n, k)$ is generated by the matrices

$$\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}, \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}, \begin{pmatrix} I_n & 0 \\ B & I_n \end{pmatrix}$$

where A runs through $\mathrm{SL}(n, k)$ and B runs through all the $n \times n$ symmetric matrices over k, (i.e. $B = B^T$).

We can also view $\mathrm{SP}(2n, k)$ as the space generated by the matrices.

$$\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}, \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where A and B are as defined above. [Ome78]

The reason for this is that

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$$

and

$$\begin{pmatrix} I & 0 \\ B & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^{-1} \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Throughout this chapter, $G = \mathrm{SP}(2n,k)$ and $\bar{G} = \mathrm{SP}(n,\bar{k})$.

5.4 Involutions of $G = \mathrm{SP}(2,k)$

In Chapter 3 we discussed the results for $\mathrm{SL}(2,k)$ in detail. The reason for this is demonstrated by the next theorem.

Theorem 5.3. $\mathrm{SP}(2,k) = \mathrm{SL}(2,k)$.

Proof. If $A \in \mathrm{SP}(2,k)$ then by definition $A \in \mathrm{SL}(2,k)$ thus, $\mathrm{SP}(2,k) \subset \mathrm{SL}(2,k)$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,k)$. Consider

$$\begin{aligned} A^T J A &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} 0 & ad-bc \\ -ad+bc & 0 \end{pmatrix}. \end{aligned}$$

Since $A \in \mathrm{SL}(2, k)$, det(A)=1, ie. $ad - bc = 1$ and $-ad + bc = -1$. Therefore

$$\begin{pmatrix} 0 & ad - bc \\ -ad + bc & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and thus $A^T JA = J$ so $\mathrm{SL}(2, k) \subset \mathrm{SP}(2, k)$. Hence $\mathrm{SL}(2, k) = \mathrm{SP}(2, k)$ $\square$

The above theorem tells us that the isomorphism classes of involutions of $\mathrm{SP}(2, k)$ are precisely the isomorphism classes of $\mathrm{SL}(2, k)$ which leave $\mathrm{SP}(2, k)$ invariant.

5.5 Automorphisms of $\mathrm{SP}(2n, k)$

Lemma 14 ([Bor91])**.** *(1) If k is an algebraically closed field, then* $\mathrm{Aut}(G) = \mathrm{Inn}(G)$.

(2) For any $\theta \in \mathrm{Aut}(G)$ there is a matrix $A \in \mathrm{SL}(2n, \bar{k})$ such that $\theta = \mathrm{Inn}_A |_G$.

i.e. All automorphisms of $\mathrm{SP}(2n, \bar{k})$ are of type inner.

5.5.1 $\mathrm{Inn}_A = \mathrm{Id}$

Our true desire is to determine what happens when we require our inner automorphism $\theta = \mathrm{Inn}_A$ to hold $\bar{G}$ and G invariant. Before we explore this concept in detail we discuss a topic which will prove extremely useful in proving numerous results.

Theorem 5.4. *If $\mathrm{Inn}_A |_G = \mathrm{Id}$ for some $A \in \mathrm{GL}(2n, \bar{k})$ then $A = pI$ for some $p \in \bar{k}$.*

Proof. Suppose $\mathrm{Inn}_A |_G = \mathrm{Id}$ for some $A \in \mathrm{GL}(2n, \bar{k})$. Then for all $X \in G$ we have $\mathrm{Inn}_A(X) = A^{-1}XA = X$ which means that $AX = XA$ for all $X \in G$. Let

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

and consider the matrix

$$W_1 = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}.$$

Since $W_1 \in G$, $AW_1 = W_1 A$ which implies

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & A_1 + A_2 \\ A_3 & A_3 + A_4 \end{pmatrix} = \begin{pmatrix} A_1 + A_3 & A_2 + A_4 \\ A_3 & A_4 \end{pmatrix}.$$

Hence, $A_3 = 0$ and $A_1 = A_4$. With this information in hand we are now able to rewrite A as $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_1 \end{pmatrix}$. We now consider the matrix $W_2 = \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix}$. Now W_2 is also in G and thus $AW_2 = W_2 A$ and thus

$$\begin{pmatrix} A_1 & A_2 \\ 0 & A_1 \end{pmatrix}\begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ 0 & A_1 \end{pmatrix}$$

$$\begin{pmatrix} A_1 + A_2 & A_2 \\ A_1 & A_1 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ A_1 & A_2 + A_1 \end{pmatrix}.$$

Which implies that $A_2 = 0$ and thus $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}$.

Let

$$\bar{X}_k = \begin{pmatrix} X_k & 0 \\ 0 & X_k \end{pmatrix}$$

where

$$X_k := \begin{pmatrix} I_{n-k-1} & \dots & 0 \\ \vdots & -1 & \vdots \\ 0 & \dots & I_k \end{pmatrix}$$

and $k = 0, 1, ..., n-1$. Then $\bar{X}_k \in G$ and hence we may utilize the fact that $A\bar{X}_k = \bar{X}_k A$, to conclude that

$$\begin{pmatrix} A_1X_k & 0 \\ 0 & A_1X_k \end{pmatrix} = \begin{pmatrix} X_kA_1 & 0 \\ 0 & X_kA_1 \end{pmatrix}.$$

From the above equality we see that $A_1X_k = X_kA_1$. Define $A_1 = (a_{i,j})$ for $i, j = 1, 2, .., n$. Then $A_1X_k = X_kA_1$ implies

$$\begin{pmatrix} a_{11} & a_{12} & \dots & -a_{1,n-k} & \dots & a_{1,n} \\ a_{21} & a_{22} & \dots & -a_{2,n-k} & \dots & a_{2,n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ a_{n-k,1} & a_{n-k,2} & \dots & -a_{n-k,n-k} & \dots & a_{n-k,n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ a_{n,1} & a_{n,2} & \dots & -a_{n,n-k} & \dots & a_{n,n} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1,n-k} & \dots & a_{1,n} \\ a_{21} & a_{22} & \dots & a_{2,n-k} & \dots & a_{2,n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ -a_{n-k,1} & -a_{n-k,2} & \dots & -a_{n-k,n-k} & \dots & -a_{n-k,n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n-k} & \dots & a_{n,n} \end{pmatrix}.$$

Hence, it follows that $a_{n-k,j} = a_{j,n-k} = 0$ for $j \neq n-k$ and $k = 0, 1..., n-1, \quad j = 1, 2, .., n$. Therefore we now obtain the fact that A is a diagonal matrix say,

$$A = \begin{pmatrix} A_d & 0 \\ 0 & A_d \end{pmatrix} \text{ with } A_d = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & a_{n,n} \end{pmatrix}.$$

Let

$$\bar{Y}_l = \begin{pmatrix} Y_l & 0 \\ 0 & Y_l \end{pmatrix} \text{ where } \quad Y_l = \begin{pmatrix} I_l & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & & & I_{n-l-2\times n-l-2} \\ 0 & 0 & & & \end{pmatrix}$$

and $l = 0, 1, \ldots, n-2$. Then $\bar{Y}_l \in \mathrm{SP}(2n,k)$ and again $A\bar{Y}_l = \bar{Y}_l A$ which implies $A_d Y_l = Y_l A_d$. Therefore, we obtain the following equality

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & a_{ll} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{l+1,l+1} & 0 & 0 & \\ 0 & 0 & 0 & 0 & a_{l+2,l+2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{l+3,l+3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{n,n} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & a_{ll} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{l+2,l+2} & 0 & 0 & \\ 0 & 0 & 0 & 0 & a_{l+1,l+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{l+3,l+3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{n,n} \end{pmatrix}$$

Hence $a_{l+1,l+1} = a_{l+2,l+2}$ for $l = 0, 1, \ldots, n-2$. That is $A = p\,\mathrm{Id}$ for some $p \in \bar{k}$. $\square$

5.5.2 Characterization Theorem

With Theorem 5.4 in hand we now return to the question of which inner automorphisms Inn_A with $A \in \mathrm{GL}(2n, \bar{k})$ keep $\bar{G} = \mathrm{SP}(2n, \bar{k})$ and $G = \mathrm{SP}(2n, k)$ invariant. The following theorem provides us with the answers.

Theorem 5.5. *Suppose* $A \in \mathrm{GL}(2n, \bar{k})$, $\bar{G} = \mathrm{SP}(2n, \bar{k})$ *and* $G = \mathrm{SP}(2n, k)$.

(1) The inner automorphism Inn_A *keeps* $\bar{G}$ *invariant if and only if* $A = pM$ *for some* $p \in \bar{k}$ *and* $M \in \bar{G}$.

(2) If $A \in \bar{G}$, *then* Inn_A *keeps* G *invariant if and only if* $A = pM$ *for some* $p \in \bar{k}$ *and* $M \in G$.

The proof of Theorem 5.5 (1) will go through smoothly however the second claim will require significantly more work. Because of the additional work required to prove Theorem 5.5 (2) we begin by giving an outline of the procedure used in the proof followed by an example for the specific case when $2n = 6$.

Outline of Proof

After proving Theorem 5.5 (1), which will follow through with ease, we now are able to use the fact that Inn_A keeps $\bar{G}$ invariant if and only if $A = pM$ for some $p \in \bar{k}$ and $M \in \bar{G}$. In addition, we also use the fact that $\mathrm{Inn}_M = \mathrm{Inn}_{pM}$ (see Remark 3, Chapter 2) to prove Theorem 5.5 (2). Assuming that $A = pM$ for some $p \in \bar{k}$ and $M \in G$ and deriving that Inn_A keeps G invariant is relatively simple. However, the forward direction must be broken down into several steps. We begin by assuming that $A \in \bar{G}$ and Inn_A keeps G invariant. Our first step is to show that $a_{ri}a_{rj} + a_{si}a_{sj}$ are in our base field k of G for all $i, j, r, s = 1, 2, \ldots, 2n$ with $r \neq s$. The proof of this is broken into 3 cases depending on whether (1) $r, s \leq n$ (2) $r, s > n$, (3) $r \leq n$ and $s > n$. We then are able to use the fact that $a_{ri}a_{rj} + a_{si}a_{sj} \in k$ to show that $a_{ri}a_{rj} \in k$. The proof of this portion is broken into two cases for $i > n$ and $i \leq n$. Lastly, we will show that $a_{ri}a_{sj} \in k$ which will give us our desired result. The proof of this last fact will be by far the most complex portion and will involve 3 cases each with 3 subcases.

5.5.3 Notation

The following is a list of notation which will be used in the proof of Theorem 5.5.

Let $X_{r,s}$ be the $n \times n$ diagonal matrix with a -1 in the (r, r) and (s, s) entries and 1's everywhere else.

Let X_r be the $n \times n$ diagonal matrix with a -1 in the (r, r) position and 1's everywhere else.

Let $E_{r,s}$ be the $n \times n$ matrix with a 1 in the (r, s) entry and $0's$ everywhere else.

Let T_c be the $c \times c$ antidiagonal matrix with 1's on the antidiagonal and 0's everywhere else.

Let I_c be the $c \times c$ identity matrix. If the size of the identity matrix is understood from the context then I may be used to represent I_c.

$$J_{2n} = J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{5.1}$$

$$Y_{r,s} = \begin{pmatrix} T_{r+s-1} & 0 \\ 0 & I_{n-(r+s-1)} \end{pmatrix} \tag{5.2}$$

$$Z_{r,s} = \begin{pmatrix} Y_{r,s} & 0 \\ E_{r,s} & Y_{r,s} \end{pmatrix} \tag{5.3}$$

$$\bar{Z}_{r,s} = \begin{pmatrix} -Y_{r,s} & 0 \\ E_{r,s} & -Y_{r,s} \end{pmatrix} \tag{5.4}$$

$$Z'_{r,s} = \begin{pmatrix} Y_{r-n,s-n} & E_{r-n,s-n} \\ 0 & Y_{r-n,s-n} \end{pmatrix} \tag{5.5}$$

$$\bar{Z}'_{r,s} = \begin{pmatrix} Y_{r-n,s-n} & E_{r-n,s-n} \\ 0 & Y_{r-n,s-n} \end{pmatrix} \tag{5.6}$$

$$M_{r,s} = \begin{pmatrix} E_{s-n,r} & Y_{s-n,r} \\ -Y_{s-n,r} & 0 \end{pmatrix} \tag{5.7}$$

$$\bar{M}_{r,s} = \begin{pmatrix} E_{s-n,r} & -Y_{s-n,r} \\ Y_{s-n,r} & 0 \end{pmatrix} \tag{5.8}$$

$$U_{r,s} = \begin{pmatrix} I_{-n+(r+s-1)} & 0 \\ 0 & T_{2n-(r+s-1)} \end{pmatrix} \tag{5.9}$$

$$V_{r,s} = \begin{pmatrix} U_{r,s} & 0 \\ E_{r,s} & U_{r,s} \end{pmatrix} \tag{5.10}$$

$$\bar{V}_{r,s} = \begin{pmatrix} -U_{r,s} & 0 \\ E_{r,s} & -U_{r,s} \end{pmatrix} \tag{5.11}$$

$$\tag{5.12}$$

$$V'_{r,s} = \begin{pmatrix} U_{r-n,s-n} & E_{r-n,s-n} \\ 0 & U_{r-n,s-n} \end{pmatrix} \tag{5.13}$$

$$\bar{V}'_{r,s} = \begin{pmatrix} -U_{r-n,s-n} & E_{r-n,s-n} \\ 0 & -U_{r-n,s-n} \end{pmatrix} \tag{5.14}$$

$$N_{r,s} = \begin{pmatrix} E_{s-n,r} & U_{s-n,r} \\ -U_{s-n,r} & 0 \end{pmatrix} \tag{5.15}$$

$$\bar{N}_{r,s} = \begin{pmatrix} E_{s-n,r} & -U_{s-n,r} \\ U_{s-n,r} & 0 \end{pmatrix} \tag{5.16}$$

$$W_{r,s} = \begin{pmatrix} T_n & 0 \\ E_{r,s} & T_n \end{pmatrix} \tag{5.17}$$

$$\bar{W}_{r,s} = \begin{pmatrix} -T_n & 0 \\ E_{r,s} & -T_n \end{pmatrix} \tag{5.18}$$

$$W'_{r,s} = \begin{pmatrix} T_n & E_{r-n,s-n} \\ 0 & T_n \end{pmatrix} \tag{5.19}$$

$$\bar{W}'_{r,s} = \begin{pmatrix} -T_n & E_{r-n,s-n} \\ 0 & -T_n \end{pmatrix} \tag{5.20}$$

$$F_{r,s} = \begin{pmatrix} E_{s-n,r} & T_n \\ T_n & 0 \end{pmatrix} \tag{5.21}$$

$$\bar{F}_{r,s} = \begin{pmatrix} E_{s-n,r} & -T_n \\ -T_n & 0 \end{pmatrix} \tag{5.22}$$

5.5.4 Proof of Characterization Theorem Part II, $2n = 6$

Remark 12. In the proof of Theorem 5.5 (2) we will make constant use of Theorem 5.5 (1), that is an inner automorphism Inn_A keeps $\bar{G}$ invariant if and only if $A = pM$ for some $p \in \bar{k}$ and $M \in \bar{G}$. We will begin by assuming that $A \in \bar{G}$ which means that $A^T JA = J$ or more importantly that $A^{-1} = J^{-1}A^T J$. By viewing $A^{-1} = J^{-1}A^T J$ we are able to define our inner automorphisms as

$$\mathrm{Inn}_A(X) = A^{-1}XA = J^{-1}A^T JXA.$$

So every time we apply our inner automorphism to G we will be using the fact that $\mathrm{Inn}_A(X) = J^{-1}A^T JXA$.

Theorem 5.6. *Suppose $A \in \mathrm{GL}(6, \bar{k})$, $\bar{G} = \mathrm{SP}(6, \bar{k})$ and $G = \mathrm{SP}(6, k)$.*

If $A \in \bar{G}$, then Inn_A keeps G invariant if and only if $A = pM$ for some $p \in \bar{k}$ and $M \in G$.

Proof. Suppose $A = (a_{ij}) \in \bar{G}$ and Inn_A keeps G invariant. We will first show that $a_{ri}a_{rj} + a_{si}a_{sj} \in k$ for $r \neq s$.

Case I: Suppose $r, s \leq 3$. Without loss of generality let $r < s$.
We look at the specific case wher $r = 1$ and $s = 2$ to illustrate this situation.
To isolate the element $a_{1i}a_{1j} + a_{2i}a_{2,j}$ we need to concentrate on the lower left hand block.
We begin by noting that the matrices

$$J,\ \begin{pmatrix} I & I \\ 0 & I \end{pmatrix},\ \text{ and } \begin{pmatrix} I & 0 \\ X_{12} & I \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

all lie in G. More importantly, since Inn_A keeps G invariant the sum $\mathrm{Inn}_A(J) - \mathrm{Inn}_A\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} + \mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_{12} & I \end{pmatrix}$ is in G and thus has entries in k. Therefore, since

$$\mathrm{Inn}_A(J) - \mathrm{Inn}_A\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} + \mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_{12} & I \end{pmatrix} =$$

$$\begin{pmatrix} 2a_{14}a_{1j} + 2a_{24}a_{2j} \\ 2a_{15}a_{1j} + 2a_{25}a_{2j} \\ 2a_{16}a_{1j} + 2a_{26}a_{2j} \\ -2a_{11}a_{1j} - 2a_{21}a_{2j} \\ -2a_{12}a_{1j} - 2a_{22}a_{2j} \\ -2a_{13}a_{1j} - 2a_{23}a_{2j} \end{pmatrix}$$

where j corresponds to the column we obtain our desired result that $a_{1,i}a_{1,j} + a_{2,i}a_{2,j} \in k$. We note here that in the proof of the general case, this case actually splits into two subcases. This is hinted upon in this example since you see that for $i \leq 3$ the entries are multiplied by a 2 while for $i > 3$ the entries are multiplied by a -2. Although this fact is handled in detail in the general proof we simply mention it during this illustration.

Note: To obtain the results that $a_{1i}a_{1j} + a_{3i}a_{3j}$ and $a_{2i}a_{2j} + a_{3i}a_{3j}$ are in k, follow the exact procedure above but replace X_{12} with X_{13} and X_{23} respectively.

Case II: Suppose $r, s > 3$. Without loss of generality we assume $r < s$.
Let's consider the case where $r = 4$ and $s = 5$.
Similar to Case I we are able to determine that in order to isolate the $a_{4i}a_{4j} + a_{5i}a_{5j}$ element we need to focus on the lower left hand block. We first observe that each of the following matrices are in G

$$J, \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \text{ and } \begin{pmatrix} I & X_{4-3,5-3} \\ 0 & I \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Again utilizing the fact that Inn_A holds G invariant we know the sum

$$\mathrm{Inn}_A(J) + \mathrm{Inn}_A \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} - \mathrm{Inn}_A \begin{pmatrix} I & X_{4-3,5-3} \\ 0 & I \end{pmatrix} =$$

$$\begin{pmatrix} 2a_{44}a_{4j} + 2a_{54}a_{5j} \\ 2a_{45}a_{4j} + 2a_{55}a_{5j} \\ 2a_{46}a_{4j} + 2a_{56}a_{5j} \\ -2a_{41}a_{4j} - 2a_{51}a_{5j} \\ -2a_{42}a_{4j} - 2a_{52}a_{5j} \\ -2a_{43}a_{4j} - 2a_{53}a_{5j} \end{pmatrix}$$

where j corresponds to the column, lies in G with entries in the base field k. Hence we may conclude that $a_{4i}a_{4j} + a_{5i}a_{5j} \in k$.

Note: To conclude that the elements $a_{4i}a_{4j} + a_{6i}a_{6j}$ and $a_{5i}a_{5j} + a_{6i}a_{6j}$ both lie in k one must simply follow the above calculation substituting $X_{4-3,6-3}$ and $X_{5-3,6-3}$ in for $X_{4-3,5-3}$ respectively.

Case III: Suppose $r \leq 3$ and $s > 3$.

In this case we look at the situation where $r = 3$ and $s = 4$.

Since $r \leq 3$ and $s > 3$ in this case we must focus on both the lower left hand block to obtain entries involving $r = 3$ and the upper right hand block to obtain entries involving $s = 4$.

We begin by observing that the matrix J and the matrices

$$\begin{pmatrix} I & X_{4-3} \\ I & O \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} I & 0 \\ X_3 & I \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

all lie in G. We again focus on the sum

$$\mathrm{Inn}_A(J) + \mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_3 & I \end{pmatrix} - \mathrm{Inn}_A\begin{pmatrix} I & X_{4-3} \\ I & O \end{pmatrix} =$$

$$\begin{pmatrix} 2a_{34}a_{3j} + 2a_{44}a_{4j} \\ 2a_{35}a_{3j} + 2a_{45}a_{4j} \\ 2a_{36}a_{3j} + 2a_{46}a_{4j} \\ -2a_{31}a_{3j} - 2a_{41}a_{4j} \\ -2a_{32}a_{3j} - 2a_{42}a_{4j} \\ -2a_{33}a_{3j} - 2a_{43}a_{4j} \end{pmatrix}$$

where j corresponds to the column, which must lie in G since Inn_A keeps G invariant. Therefore, we are able to conclude that $a_{3i}a_{3j} + a_{4i}a_{4j}$ must lie in k.

Note: Below I list the matrices that you need to replace X_{4-3} and X_3 respectively, in the above procedure to obtain the desired element.

(1) $a_{3i}a_{3j} + a_{5i}a_{5j}$ use X_{5-3} and X_3

(2) $a_{3i}a_{3j} + a_{6i}a_{6j}$ use X_{6-3} and X_3

(3) $a_{1i}a_{1j} + a_{4i}a_{4j}$ use X_{4-3} and X_1

(4) $a_{1i}a_{1j} + a_{5i}a_{5j}$ use X_{5-3} and X_1

(5) $a_{1i}a_{1j} + a_{6i}a_{6j}$ use X_{6-3} and X_1

(6) $a_{2i}a_{2j} + a_{4i}a_{4j}$ use X_{4-3} and X_2

(7) $a_{2i}a_{2j} + a_{5i}a_{5j}$ use X_{5-3} and X_2

(8) $a_{2i}a_{2j} + a_{6i}a_{6j}$ use X_{6-3} and X_2

By combining Cases I, II, and III we can conclude that $a_{ri}a_{rj} + a_{si}a_{sj} \in k$ for all $i, j = 1, 2, \ldots 6$ and $r \neq s$.

We are now able to use the fact that $a_{r,i}a_{r,j} + a_{s,i}a_{s,j} \in k$ for all $i, j = 1, 2, \ldots 6$ and $r \neq s$ to show that $a_{r,i}a_{r,j} \in k$ for all $i, j = 1, 2, \ldots, 6$. However, we must show this in two cases. We will first show that $a_{r,l}a_{r,j} \in k$ for all $l \leq 3$ and then show that $a_{r,l}a_{r,j} \in k$ for all $l > 3$. Without loss of generality it shall suffice to show $a_{1,l}a_{1,j} \in k$ for all l.

Case I: Assume $i > 3$

The (i, j) entry of $\mathrm{Inn}_A(J)$ is given by

$$a_{1,i-3}a_{1,j} - a_{2,i-3}a_{2,j} - a_{3,i-3}a_{3,j} - a_{4,i-3}a_{4,j} - a_{5,i-3}a_{5,j} - a_{6,i-3}a_{6,j}$$

which must lie in k since Inn_A holds G invariant. Moreover, -1 times the (i, j) entry of $\mathrm{Inn}_A(J)$ lies in k. From our previous argument we know that $a_{ri}a_{rj} + a_{si}a_{sj} \in k$ for all $r, s, i, j = 1, 2, \ldots 6$ with $r \neq s$. Hence the equality

$$a_{1,i-3}a_{1,j} =$$

$(a_{1,i-3}a_{1,j} - a_{2,i-3}a_{2,j} - a_{3,i-3}a_{3,j} - a_{4,i-3}a_{4,j} - a_{5,i-3}a_{5,j} - a_{6,i-3}a_{6,j}) + (-1/2)(a_{2,i-3}a_{2,j} - a_{3,i-3}a_{3,j}) + (-1/2)(a_{3,i-3}a_{3,j} - a_{4,i-3}a_{4,j}) + (-1/2)(a_{4,i-3}a_{4,j} - a_{5,i-3}a_{5,j}) + (-1/2)(a_{5,i-3}a_{5,j} - a_{6,i-3}a_{6,j}) + (-1/2)(a_{6,i-3}a_{6,j} + a_{2,i-3}a_{2,j})$

must lie in k, i.e. $a_{1,i-3}a_{1,j} \in k$. Since we assumed $i > 3$ we can conclude that $a_{1l}a_{1,j} \in k$ for $l \leq 3$.

Case II: Assume $i \leq 3$.
In this situation the (i, j) entry of $\mathrm{Inn}_A(J)$ is given by

$$a_{1,i+3}a_{1,j} - a_{2,i+3}a_{2,j} - a_{3,i+3}a_{3,j} - a_{4,i+3}a_{4,j} - a_{5,i+3}a_{5,j} - a_{6,i+3}a_{6,j}$$

which of course lies in k. In addition, -1 times this entry also lies in k. Again using the fact that $a_{ri}a_{rj} + a_{si}a_{sj} \in k$ for all $r, s, i, j = 1, 2, ...6$ with $r \neq s$ we observe the equality

$$a_{1,i+3}a_{1,j} =$$

$(a_{1,i+3}a_{1,j} - a_{2,i+3}a_{2,j} - a_{3,i+3}a_{3,j} - a_{4,i+3}a_{4,j} - a_{5,i+3}a_{5,j} - a_{6,i+3}a_{6,j}) + (-1/2)(a_{2,i+3}a_{2,j} - a_{3,i+3}a_{3,j}) + (-1/2)(a_{3,i+3}a_{3,j} - a_{4,i+3}a_{4,j}) + (-1/2)(a_{4,i+3}a_{4,j} - a_{5,i+3}a_{5,j}) + (-1/2)(a_{5,i+3}a_{5,j} - a_{6,i+3}a_{6,j}) + (-1/2)(a_{6,i+3}a_{6,j} + a_{2,i+3}a_{2,j})$.
The above equality enables us to conclude that the element $a_{1,i+3}a_{1,j}$ must lie in k. Since in this case we assumed that $i \leq 3$ we obtain the fact that $a_{1l}a_{1,j} \in k$ for $l > 3$.

Combining Cases I and II we see that $a_{1i}a_{ij} \in k$ for $i, j = 1, 2, ..., 6$.
Using a similar argument it is easily verified that $a_{ri}a_{rj} \in k$ for $r = 2, 3, .., 6$.

I will finally show that $a_{ri}a_{sj} \in k$

In each of the following cases the methodology involved in proving the case is the same. In every case we will use our assumption that Inn_A keeps G invariant and thus if we choose any general matrix $B_1 \in G$ then $\mathrm{Inn}_A(B_1) \in G$. Next, we use the fact that if B_1 and B_2 are both in G then the sum $\mathrm{Inn}_A(B_1) + \mathrm{Inn}_A(B_2) \in G$ to obtain our desired result. We will explain this in detail in case I subcase 1 only.

CASE I: Suppose $r, s \leq 3$. Without loss of generality we will assume that $r < s$.
The lower left block controls the $a_{ri}a_{sj}$ elements for r and s less than 3. Hence in each of the following subcases we put a 1 in the (r, s) entry of this block to isolate that element. The diagonal blocks must also have a 1 in the (r, s) entry but require a different form depending on the relationship between r and s. The following subcases illustrates this fact.

Subcase 1: Suppose $r+s<4$.

We will look at the specific case where $r=1$ and $s=2$.

Here we first choose $Y_{1,2} = \begin{pmatrix} T_2 & 0 \\ 0 & I_1 \end{pmatrix}$ and

$$Z_{1,2} = \begin{pmatrix} Y_{1,2} & 0 \\ E_{1,2} & Y_{1,2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now $Z_{1,2} \in G$ and since Inn_A keeps G invariant $\mathrm{Inn}_A(Z_{1,2}) \in G$. In addition, the matrix

$$\bar{Z}_{1,2} = \begin{pmatrix} -Y_{1,2} & 0 \\ E_{1,2} & -Y_{1,2} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

lies in G and thus $\mathrm{Inn}_A(\bar{Z}_{1,2}) \in G$. The sum $\mathrm{Inn}_A(Z_{1,2}) + \mathrm{Inn}_A(\bar{Z}_{1,2}) \in G$ which means that its entries must lie in the base field k. We now observe the matrix

$$\mathrm{Inn}_A(Z_{1,2}) + \mathrm{Inn}_A(\bar{Z}_{1,2}) =$$

$$\begin{pmatrix} -2a_{2,1}a_{1,4} & -2a_{2,2}a_{1,4} & -2a_{2,3}a_{1,4} & -2a_{2,4}a_{1,4} & -2a_{2,5}a_{1,4} & -2a_{2,6}a_{1,4} \\ -2a_{2,1}a_{1,5} & -2a_{2,2}a_{1,5} & -2a_{2,3}a_{1,5} & -2a_{2,4}a_{1,5} & -2a_{2,5}a_{1,5} & -2a_{2,6}a_{1,5} \\ -2a_{2,1}a_{1,6} & -2a_{2,2}a_{1,6} & -2a_{2,3}a_{1,6} & -2a_{2,4}a_{1,6} & -2a_{2,5}a_{1,6} & -2a_{2,6}a_{1,6} \\ 2a_{2,1}a_{1,1} & 2a_{2,2}a_{1,1} & 2a_{2,3}a_{1,1} & 2a_{2,4}a_{1,1} & 2a_{2,5}a_{1,1} & 2a_{2,6}a_{1,1} \\ 2a_{2,1}a_{1,2} & 2a_{2,2}a_{1,2} & 2a_{2,3}a_{1,2} & 2a_{2,4}a_{1,2} & 2a_{2,5}a_{1,2} & 2a_{2,1}a_{1,2} \\ 2a_{2,1}a_{1,4} & 2a_{2,2}a_{1,3} & 2a_{2,3}a_{1,3} & 2a_{2,4}a_{1,3} & 2a_{2,5}a_{1,3} & 2a_{2,1}a_{1,3} \end{pmatrix}.$$

Which tells us that the element $a_{2,i}a_{1,j} \in k$ for i, $j = 1,2,\ldots 6$. Since $a_{1,i}a_{2,j} = a_{2,j}a_{1,i}$ we may also conclude that $a_{1,j}a_{2,i} \in k$.

It is important to note that in the general proof of Theorem 5.5 (2) the entries of $\mathrm{Inn}_A(Z_{r,s})$, $\mathrm{Inn}_A(\bar{Z}_{r,s})$ and $\mathrm{Inn}_A(Zr,s) + \mathrm{Inn}_A(\bar{Z}_{r,s})$ are broken into two cases depending on whether $i \leq n$ or $i > n$. The reason for this can be seen in the example above for the sum $\mathrm{Inn}_A(Z1,2) + \mathrm{Inn}_A(\bar{Z}_{1,2})$. For $i \leq 3$ we get -2 being multiplied by each entry and for $i > 3$ we simply get a 2 multiplied by each entry. Although this difference is addressed in detail in the general proof we simply illustrate the procedure here.

Subcase 2: Suppose $r+s > 4$.
We look at the case where $r = 2$ and $s = 3$.

We first consider $U_{2,3} = \begin{pmatrix} I_1 & 0 \\ 0 & T_2 \end{pmatrix}$. We then choose

$$V_{2,3} = \begin{pmatrix} U_{2,3} & 0 \\ E_{2,3} & U_{2,3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\bar{V}_{2,3} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Since $V_{2,3}$ and $\bar{V}_{23}$ both lie in G the sum

$$\mathrm{Inn}_A(V_{2,3}) + \mathrm{Inn}_A(\bar{V}_{2,3}) =$$

$$\begin{pmatrix} -2a_{3,1}a_{2,4} & -2a_{3,2}a_{2,4} & -2a_{3,3}a_{2,4} & -2a_{3,4}a_{2,4} & -2a_{3,5}a_{2,4} & -2a_{3,6}a_{2,4} \\ -2a_{3,1}a_{2,5} & -2a_{3,2}a_{2,5} & -2a_{3,3}a_{2,5} & -2a_{3,4}a_{2,5} & -2a_{3,5}a_{2,5} & -2a_{3,6}a_{2,5} \\ -2a_{3,1}a_{2,6} & -2a_{3,2}a_{2,6} & -2a_{3,3}a_{2,6} & -2a_{3,4}a_{2,6} & -2a_{3,5}a_{2,6} & -2a_{3,6}a_{2,6} \\ 2a_{3,1}a_{2,1} & 2a_{3,2}a_{2,1} & 2a_{3,3}a_{2,1} & 2a_{3,4}a_{2,1} & 2a_{3,5}a_{2,1} & 2a_{3,6}a_{2,1} \\ 2a_{3,1}a_{2,2} & 2a_{3,2}a_{2,2} & 2a_{3,3}a_{2,2} & 2a_{3,4}a_{2,2} & 2a_{3,5}a_{2,2} & 2a_{3,6}a_{2,2} \\ 2a_{3,1}a_{2,3} & 2a_{3,2}a_{2,3} & 2a_{3,3}a_{2,3} & 2a_{3,4}a_{2,3} & 2a_{3,5}a_{2,3} & 2a_{3,6}a_{2,3} \end{pmatrix}$$

has entries in k. We specifically are able to conclude that $a_{3,i}a_{2,j} = a_{2,j}a_{3,i} \in k$ as desired.

Subcase 3: Suppose $r + s = 4$.

We will look at the case when $r = 1$ and $s = 3$.

For this case we choose

$$W_{1,3} = \begin{pmatrix} T_3 & 0 \\ E_{1,3} & T_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$\bar{W}_{1,3} = \begin{pmatrix} -T_3 & 0 \\ E_{1,3} & -T_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

We then consider the sum

$$\mathrm{Inn}_A(W_{1,3}) + \mathrm{Inn}_A(\bar{W}_{1,3}) =$$

$$\begin{pmatrix} -2a_{3,1}a_{1,4} & -2a_{3,2}a_{1,4} & -2a_{3,3}a_{1,4} & -2a_{3,4}a_{1,4} & -2a_{3,5}a_{1,4} & -2a_{3,6}a_{1,4} \\ -2a_{3,1}a_{1,5} & -2a_{3,2}a_{1,5} & -2a_{3,3}a_{1,5} & -2a_{3,4}a_{1,5} & -2a_{3,5}a_{1,5} & -2a_{3,6}a_{1,5} \\ -2a_{3,1}a_{1,6} & -2a_{3,2}a_{1,6} & -2a_{3,3}a_{1,6} & -2a_{3,4}a_{1,6} & -2a_{3,5}a_{1,6} & -2a_{3,6}a_{1,6} \\ 2a_{3,1}a_{1,1} & 2a_{3,2}a_{1,1} & 2a_{3,3}a_{1,1} & 2a_{3,4}a_{1,1} & 2a_{3,5}a_{1,1} & 2a_{3,6}a_{1,1} \\ 2a_{3,1}a_{1,2} & 2a_{3,2}a_{1,2} & 2a_{3,3}a_{1,2} & 2a_{3,4}a_{1,2} & 2a_{3,5}a_{1,2} & 2a_{3,6}a_{1,2} \\ 2a_{3,1}a_{1,3} & 2a_{3,2}a_{1,3} & 2a_{3,3}a_{1,3} & 2a_{3,4}a_{1,3} & 2a_{1,5}a_{1,3} & 2a_{3,6}a_{1,3} \end{pmatrix}$$

and again observe that it lies in G and thus must have entries in k. Thus we may conclude that $a_{3,i}a_{1,j} = a_{1,j}a_{3,i} \in k$ as desired.

CASE II: Suppose $r, s > 3$. Without loss of generality assume $r < s$.
In this case the upper right hand block controls the elements $a_{ri}a_{sj}$. However, since both r and s are greater than 3 we must place a 1 in the $(r-3, s-3)$ entry of the upper right block to isolate the element. As in Case I, the entries on the diagonal blocks vary based on the relationship between r and s and are given by the following subcases.

Subcase 1: Suppose $r + s - 6 < 4$
Let's consider the specific case where $r = 4$ and $s = 5$ for this situation.
We begin by choosing

$$Z'_{4,5} = \begin{pmatrix} Y_{4-3,5-3} & E_{4-3,5-3} \\ 0 & Y_{4-3,5-3} \end{pmatrix} = \begin{pmatrix} Y_{1,2} & E_{1,2} \\ 0 & Y_{1,2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\bar{Z}'_{4,5} = \begin{pmatrix} -Y_{1,2} & E_{1,2} \\ 0 & -Y_{1,2} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

both of which lie in G. Next we observe that

$$\mathrm{Inn}_A(Z'_{4,5}) + \mathrm{Inn}_A(\bar{Z}'_{4,5}) =$$

$$\begin{pmatrix} 2a_{5,1}a_{4,4} & 2a_{5,2}a_{4,4} & 2a_{5,3}a_{4,4} & 2a_{5,4}a_{4,4} & 2a_{5,5}a_{4,4} & 2a_{5,6}a_{4,4} \\ 2a_{5,1}a_{4,5} & 2a_{5,2}a_{4,5} & 2a_{5,3}a_{4,5} & 2a_{5,4}a_{4,5} & 2a_{5,5}a_{4,5} & 2a_{5,6}a_{4,5} \\ 2a_{5,1}a_{4,6} & 2a_{5,2}a_{4,6} & 2a_{5,3}a_{4,6} & 2a_{5,4}a_{4,6} & 2a_{5,5}a_{4,6} & 2a_{5,6}a_{4,6} \\ -2a_{5,1}a_{4,1} & -2a_{5,2}a_{4,1} & -2a_{5,3}a_{4,1} & -2a_{5,4}a_{4,1} & -2a_{5,5}a_{4,1} & -2a_{5,6}a_{4,1} \\ -2a_{5,1}a_{4,2} & -2a_{5,2}a_{4,2} & -2a_{5,3}a_{4,2} & -2a_{5,4}a_{4,2} & -2a_{5,5}a_{4,2} & -2a_{5,6}a_{4,2} \\ -2a_{5,1}a_{4,3} & -2a_{5,2}a_{4,3} & -2a_{5,3}a_{4,3} & -2a_{5,4}a_{4,3} & -2a_{5,5}a_{4,3} & -2a_{5,6}a_{4,3} \end{pmatrix}$$

is in G which of course means it has entries in k. Therefore, $a_{5,i}a_{4,j} = a_{4,j}a_{5,i} \in k$.

Subcase 2: Suppose $r + s - 6 > 4$

In this case we will look at the situation where $r = 5$ and $s = 6$.

We begin by choosing

$$V'_{5,6} = \begin{pmatrix} U_{5-3.6-3} & E_{5-3.6-3} \\ 0 & U_{5-3.6-3} \end{pmatrix} = \begin{pmatrix} U_{2,3} & E_{2,3} \\ 0 & U_{2,3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\bar{V'}_{5,6} = \begin{pmatrix} -U_{2,3} & E_{2,3} \\ 0 & -U_{2,3} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

We then consider the sum

$$\mathrm{Inn}_A(V'_{4,5}) + \mathrm{Inn}_A(\bar{V'}_{4,5}) =$$

$$\begin{pmatrix} 2a_{6,1}a_{5,4} & 2a_{6,2}a_{5,4} & 2a_{6,3}a_{5,4} & 2a_{6,4}a_{5,4} & 2a_{6,5}a_{5,4} & 2a_{6,6}a_{5,4} \\ 2a_{6,1}a_{5,5} & 2a_{6,2}a_{5,5} & 2a_{6,3}a_{5,5} & 2a_{6,4}a_{5,5} & 2a_{6,5}a_{5,5} & 2a_{6,6}a_{5,5} \\ 2a_{6,1}a_{5,6} & 2a_{6,2}a_{5,6} & 2a_{6,3}a_{5,6} & 2a_{6,4}a_{5,6} & 2a_{6,5}a_{5,6} & 2a_{6,6}a_{5,6} \\ -2a_{6,1}a_{5,1} & -2a_{6,2}a_{5,1} & -2a_{6,3}a_{5,1} & -2a_{6,4}a_{5,1} & -2a_{6,5}a_{5,1} & -2a_{6,6}a_{5,1} \\ -2a_{6,1}a_{5,2} & -2a_{6,2}a_{5,2} & -2a_{6,3}a_{5,2} & -2a_{6,4}a_{5,2} & -2a_{6,5}a_{5,2} & -2a_{6,6}a_{5,2} \\ -2a_{6,1}a_{5,3} & -2a_{6,2}a_{5,3} & -2a_{6,3}a_{5,3} & -2a_{6,4}a_{5,3} & -2a_{6,5}a_{5,3} & -2a_{6,6}a_{5,3} \end{pmatrix}$$

which in in G. Hence, we can conclude that $a_{6,i}a_{5,j} = a_{5,j}a_{6,i} \in k$.

Subcase 3: Suppose $r + s - 6 = 4$.

Let's consider $r = 4$ and $s = 6$.

We first look at the two matrices in G given by

$$W'_{4,6} = \begin{pmatrix} T_3 & E_{4-3,6-3} \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} T_3 & E_{1.3} \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ and}$$

$$\bar{W'}_{4,6} = \begin{pmatrix} -T_3 & E_{1,3} \\ 0 & -T_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

We then observe the sum

$$\mathrm{Inn}_A(W'_{4,6}) + \mathrm{Inn}_A(\bar{W'}_{4,6}) =$$

$$\begin{pmatrix} 2a_{6,1}a_{4,4} & 2a_{6,2}a_{4,4} & 2a_{6,3}a_{4,4} & 2a_{6,4}a_{4,4} & 2a_{6,5}a_{4,4} & 2a_{6,6}a_{4,4} \\ 2a_{6,1}a_{4,5} & 2a_{6,2}a_{4,5} & 2a_{6,3}a_{4,5} & 2a_{6,4}a_{4,5} & 2a_{6,5}a_{4,5} & 2a_{6,6}a_{4,5} \\ 2a_{6,1}a_{4,6} & 2a_{6,2}a_{4,6} & 2a_{6,3}a_{4,6} & 2a_{6,4}a_{4,6} & 2a_{6,5}a_{4,6} & 2a_{6,6}a_{4,6} \\ -2a_{6,1}a_{4,1} & -2a_{6,2}a_{4,1} & -2a_{6,3}a_{4,1} & -2a_{6,4}a_{4,1} & -2a_{6,5}a_{4,1} & -2a_{6,6}a_{4,1} \\ -2a_{6,1}a_{4,2} & -2a_{6,2}a_{4,2} & -2a_{6,3}a_{4,2} & -2a_{6,4}a_{4,2} & -2a_{6,5}a_{4,2} & -2a_{6,6}a_{4,2} \\ -2a_{6,1}a_{4,3} & -2a_{6,2}a_{4,3} & -2a_{6,3}a_{4,3} & -2a_{6,4}a_{4,3} & -2a_{6,5}a_{4,3} & -2a_{6,6}a_{4,3} \end{pmatrix}$$

which has entries in k. Hence, we are able to see that the entries $a_{6,i}a_{4,j} = a_{4,j}a_{6,i} \in k$ as desired.

CASE III: Suppose $r \leq 3$ and $s > 3$.

Case III differs slightly from Case I and Case II in that the upper left block controls the element $a_{ri}a_{sj}$, a block on the diagonal. Since in this case $r \leq 3$ and $s > 3$ we put a 1 in the $(s-3, r)$ entry of the upper left hand block. In this case the off diagonal blocks vary depending on the relationship between r and s and divide into the following subcases.

Subcase 1: Suppose $r + s < 7$.

In this situation we will look at the specific case where $r = 1$ and $s = 5$.

We begin by considering the two matrices

$$M_{1,5} = \begin{pmatrix} E_{5-3,1} & Y_{5-3,1} \\ -Y_{5-3,1} & 0 \end{pmatrix} = \begin{pmatrix} E_{2,1} & Y_{2,1} \\ -Y_{2,1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\bar{M}_{1,5} = \begin{pmatrix} E_{2,1} & -Y_{2,1} \\ Y_{2,1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We then observe the sum

$$\mathrm{Inn}_A(M_{1,5}) + \mathrm{Inn}_A(\bar{M}_{1,5}) =$$

$$\begin{pmatrix} 2a_{5,4}a_{1,1} & 2a_{5,4}a_{1,2} & 2a_{5,4}a_{1,3} & 2a_{5,4}a_{1,4} & 2a_{5,4}a_{1,5} & 2a_{5,4}a_{1,6} \\ 2a_{5,5}a_{1,1} & 2a_{5,5}a_{1,2} & 2a_{5,5}a_{1,3} & 2a_{5,5}a_{1,4} & 2a_{5,5}a_{1,5} & 2a_{5,5}a_{1,6} \\ 2a_{5,6}a_{1,1} & 2a_{5,6}a_{1,2} & 2a_{5,6}a_{1,3} & 2a_{5,6}a_{1,4} & 2a_{5,6}a_{1,5} & 2a_{5,6}a_{1,6} \\ -2a_{5,1}a_{1,1} & -2a_{5,1}a_{1,2} & -2a_{5,1}a_{1,3} & -2a_{5,1}a_{1,4} & -2a_{5,1}a_{1,5} & -2a_{5,1}a_{1,6} \\ -2a_{5,2}a_{1,1} & -2a_{5,2}a_{1,2} & -2a_{5,2}a_{1,3} & -2a_{5,2}a_{1,4} & -2a_{5,2}a_{1,5} & -2a_{5,2}a_{1,6} \\ -2a_{5,3}a_{1,1} & -2a_{5,3}a_{1,2} & -2a_{5,3}a_{1,3} & -2a_{5,3}a_{1,4} & -2a_{5,3}a_{1,5} & -2a_{5,3}a_{1,6} \end{pmatrix}$$

which we know must be in G. We finally are able to conclude that $a_{5,i}a_{1,j} = a_{1,j}a_{5,i}$ both lie in k.

Note: To obtain $a_{1,i}a_{4,j} = a_{4,j}a_{1,i} \in k$ simply choose $M_{1,4} = \begin{pmatrix} E_{1,1} & Y_{1,1} \\ -Y_{1,1} & 0 \end{pmatrix}$ and $\bar{M}_{1,4}$ accordingly and follow the procedure above. To obtain $a_{2,i}a_{4,j} = a_{4,j}a_{2,i} \in k$ choose

$M_{2,4} = \begin{pmatrix} E_{1,2} & Y_{1,2} \\ -Y_{1,2} & 0 \end{pmatrix}$ and $\bar{M}_{2,4}$ and again follow the procedure above.

Subcase 2 :Suppose $r+s>7$.

Specifically, let $r=2$ and $s=6$. Then by selecting

$$N_{2,6} = \begin{pmatrix} E_{6-3,2} & U_{6-3,2} \\ -U_{6-3,2} & 0 \end{pmatrix} = \begin{pmatrix} E_{3,2} & U_{3,2} \\ -U_{3,2} & 0 \end{pmatrix}$$

and

$$\bar{N}_{2,6} = \begin{pmatrix} E_{3,2} & -U_{3,2} \\ U_{3,2} & 0 \end{pmatrix}$$

both of which are in G, we are able to obtain the fact that

$$\mathrm{Inn}_A(\bar{N}_{3,2}) + \mathrm{Inn}_A(N_{3,2}) =$$

$$\begin{pmatrix} 2a_{6,4}a_{2,1} & 2a_{6,4}a_{2,2} & 2a_{6,4}a_{2,3} & 2a_{6,4}a_{2,4} & 2a_{6,4}a_{2,5} & 2a_{6,4}a_{2,6} \\ 2a_{6,5}a_{2,1} & 2a_{6,5}a_{2,2} & 2a_{6,5}a_{2,3} & 2a_{6,5}a_{2,4} & 2a_{6,5}a_{2,5} & 2a_{6,5}a_{2,6} \\ 2a_{6,6}a_{2,1} & 2a_{6,6}a_{2,2} & 2a_{6,6}a_{2,3} & 2a_{6,6}a_{2,4} & 2a_{6,6}a_{2,5} & 2a_{6,6}a_{2,6} \\ -2a_{6,1}a_{2,1} & -2a_{6,1}a_{2,2} & -2a_{6,1}a_{2,3} & -2a_{6,1}a_{2,4} & -2a_{6,1}a_{2,5} & -2a_{6,1}a_{2,6} \\ -2a_{6,2}a_{2,1} & -2a_{6,2}a_{2,2} & -2a_{6,2}a_{2,3} & -2a_{6,2}a_{2,4} & -2a_{6,2}a_{2,5} & -2a_{6,2}a_{2,6} \\ -2a_{6,3}a_{2,1} & -2a_{6,3}a_{2,2} & -2a_{6,3}a_{2,3} & -2a_{6,3}a_{2,4} & -2a_{6,3}a_{2,5} & -2a_{6,3}a_{2,6} \end{pmatrix}$$

must lie in G. More importantly its entries $a_{6,i}a_{2,j} = a_{2,j}a_{6,i} \in k$.

Note: To conclude that $a_{3,i}a_{5,j} = a_{5,j}a_{3,i}$ and $a_{3,i}a_{6,j} = a_{6,j}a_{3,i} \in k$ simply follow the procedure above chooseing $N_{3,5} = \begin{pmatrix} E_{2,3} & U_{2,3} \\ -U_{2,3} & 0 \end{pmatrix}$ and $N_{3,6} = \begin{pmatrix} E_{3,3} & U_{3,3} \\ -U_{3,3} & 0 \end{pmatrix}$ respectively.

Subcase 3: Suppose $r+s=7$.

Let's look at the situation when $r=2$ and $s=5$. Here we choose

$$F_{2,5} = \begin{pmatrix} E_{5-3,2} & T_3 \\ T_3 & 0 \end{pmatrix} = \begin{pmatrix} E_{2,2} & T_3 \\ T_3 & 0 \end{pmatrix} \text{ and } \bar{F}_{2,5} = \begin{pmatrix} E_{2,2} & -T_3 \\ -T_3 & 0 \end{pmatrix}$$

As in the other cases we look specifically at the entries of $\mathrm{Inn}_A(S_{2,5}) + \mathrm{Inn}_A(\bar{S}_{2,5})$ which must lie in G. Now

$$\mathrm{Inn}_A(F_{2,5}) + \mathrm{Inn}_A(\bar{F}_{2,5}) =$$

$$\begin{pmatrix} 2a_{5,4}a_{2,1} & 2a_{5,4}a_{2,2} & 2a_{5,4}a_{2,3} & 2a_{5,4}a_{2,4} & 2a_{5,4}a_{2,5} & 2a_{5,4}a_{2,6} \\ 2a_{5,5}a_{2,1} & 2a_{5,5}a_{2,2} & 2a_{5,5}a_{2,3} & 2a_{5,5}a_{2,4} & 2a_{5,5}a_{2,5} & 2a_{5,5}a_{2,6} \\ 2a_{5,6}a_{2,1} & 2a_{5,6}a_{2,2} & 2a_{5,6}a_{2,3} & 2a_{5,6}a_{2,4} & 2a_{5,6}a_{2,5} & 2a_{5,6}a_{2,6} \\ -2a_{5,1}a_{2,1} & -2a_{5,1}a_{2,2} & -2a_{5,1}a_{2,3} & -2a_{5,1}a_{2,4} & -2a_{5,1}a_{2,5} & -2a_{5,1}a_{2,6} \\ -2a_{5,2}a_{2,1} & -2a_{5,2}a_{2,2} & -2a_{5,2}a_{2,3} & -2a_{5,2}a_{2,4} & -2a_{5,2}a_{2,5} & -2a_{5,2}a_{2,6} \\ -2a_{5,3}a_{2,1} & -2a_{5,3}a_{2,2} & -2a_{5,3}a_{2,3} & -2a_{5,3}a_{2,4} & -2a_{5,3}a_{2,5} & -2a_{5,3}a_{2,6} \end{pmatrix}$$

Therefore we may conclude that $a_{5,i}a_{2,j} = a_{2,j}a_{5,i} \in k$ as desired.

Note: To obtain the two cases of $a_{1,i}a_{6,j} = a_{6,j}a_{1,i}$ and $a_{3,i}a_{4,j} = a_{4,j}a_{3,i}$ both residing in k follow the same procedure above but replacing $E_{2,2}$ with $E_{3,1}$ and $E_{1,3}$ respectively.

Cases I, II and III show that $a_{r,i}a_{s,j} \in k$ and hence $A = pM$ for some $p \in \bar{k}$ and $M \in \mathrm{SP}(6, k)$. □

5.5.5 General Proof of Characterization Theorem

Theorem 5.7. *Suppose $A \in \mathrm{GL}(2n, \bar{k})$, $\bar{G} = \mathrm{SP}(2n, \bar{k})$ and $G = \mathrm{SP}(2n, k)$.*

(1) The inner automorphism Inn_A keeps $\bar{G}$ invariant if and only if $A = pM$ for some $p \in \bar{k}$ and $M \in \bar{G}$.

(2) If $A \in \bar{G}$, then Inn_A keeps G invariant if and only if $A = pM$ for some $p \in \bar{k}$ and $M \in G$.

Proof. (1) $\Longleftarrow$ Suppose $A = pM$ for some $p \in \bar{k}$ and $M \in \bar{G}$. Let $X \in \bar{G}$, then

$$\mathrm{Inn}_A(X) = \mathrm{Inn}_{pM}(X) = (pM)^{-1}X(PM) = M^{-1}XM$$

Since $M, M^{-1}, X \in \bar{G}$, $M^{-1}XM \in \bar{G}$ and thus Inn_A keeps $\bar{G}$ invariant.

$\Longrightarrow$ Suppose Inn_A keeps $\bar{G}$ invariant. Then for any $X \in \bar{G}$,

$B = \mathrm{Inn}_A(X) = A^{-1}XA \in \bar{G}$. Since $B \in \bar{G}$, by definition $B^TJB = J$ which implies that $B = J^{-1}(B^T)^{-1}J$. In addition, since $B = A^{-1}XA$, we have that $(B^T)^{-1} = A^T(X^T)^{-1}(A^T)^{-1}$. Thus the following is true

$$A^{-1}XA = B$$

implies

$$A^{-1}XA = J^{-1}(B^T)^{-1}J$$

which implies

$$A^{-1}XA = J^{-1}(A^T(X^T)^{-1}(A^T)^{-1})J$$

hence

$$X = AJ^{-1}A^T(X^T)^{-1}(A^T)^{-1}JA^{-1}.$$

Now since $X \in \bar{G}$, we know $(X^T)^{-1} = JXJ^{-1}$ which means

$$X = AJ^{-1}A^T(JXJ^{-1})(A^T)^{-1}JA^{-1}$$

that is

$$X = (AJ^{-1}A^TJ)X(AJ^{-1}A^TJ)^{-1}$$

$$\text{i.e. } \mathrm{Inn}_{AJ^{-1}A^TJ}(X) = X.$$

Therefore by Lemma 5.4 $AJ^{-1}A^TJ = q\,\mathrm{Id}$ for some $q \in \bar{k}^*$ which implies $q^{-1}AJ^{-1}A^TJ = \mathrm{Id}$. Let $p \in \bar{k}^*$ such that $p^2 = q^{-1}$. Then for $M = pA$ we have

$$MJ^{-1}M^TJ = pAJ^{-1}pA^TJ = p^2AJ^{-1}A^TJ = q^{-1}AJ^{-1}A^TJ = \mathrm{I}.$$

Therefore, $MJ^{-1}M^TJ = \mathrm{Id}$ which implies $M^TJM = J$ ie. $M \in \bar{G}$.

(2) $\Longleftarrow$ Suppose $A = pM$ for some $p \in \bar{k}$ and $M \in G$. Let $X \in G$, then

$$\mathrm{Inn}_A(X) = \mathrm{Inn}_{pM}(X) = p^{-1}M^{-1}XpM = M^{-1}XM$$

Since $M^{-1}, X, M \in G$ we know $\mathrm{Inn}_A(X) = M^{-1}XM \in G$ and thus Inn_A keeps G invariant.

$\Longrightarrow$ Suppose $A = (a_{ij}) \in \bar{G}$ and Inn_A keeps G invariant.

We will first show that $a_{ri}a_{rj} + a_{si}a_{sj} \in G$.

CASE 1: Suppose $r, s \leq n$.

Subcase a: Suppose $i \leq n$.

The (i, j) entry of $\mathrm{Inn}_A(J)$ is given by

$$a_{1,n+i}a_{1,j} + a_{2,n+i}a_{2,j} + \ldots + a_{2n,n+i}a_{2n,j} \in k$$

since $J \in G$ and Inn_A keeps G invariant. By the same argument the (i, j) entry of $\mathrm{Inn}_A\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ given by
$a_{1j}a_{n+1,n+i} + a_{2j}a_{n+2,n+i} + \ldots + a_{nj}a_{2n,n+i} + a_{n+1,n+i}a_{n+1,j} + a_{n+2,n+i}a_{n+2,j} + \ldots + a_{2n,n+i}a_{2n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \ldots - a_{n,n+i}a_{2n,j} \in k$

Hence the (i, j) position of $\mathrm{Inn}_A(J) - \mathrm{Inn}_A\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ given by
$-a_{1j}a_{n+1,n+i} - a_{2j}a_{n+2,n+i} - \ldots - a_{nj}a_{2n,n+i} + a_{1,n+i}a_{1,j} + a_{2,n+i}a_{2,j} + \ldots + a_{n,n+i}a_{n,j} + a_{1,n+i}a_{n+1,j} + a_{2,n+i}a_{n+2,j} + \ldots + a_{n,n+i}a_{2n,j} \in k$.

We know the matrix $\begin{pmatrix} I & 0 \\ X_{rs} & I \end{pmatrix}$ is in G and hence the (i, j) entry of $\mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_{rs} & I \end{pmatrix}$ given by
$a_{1j}a_{n+1,n+i} + a_{2j}a_{n+2,n+i} + \ldots + a_{nj}a_{2n,n+i} - a_{1,n+i}a_{1,j} - a_{2,n+i}a_{2,j} - (-a_{r,n+i}a_{r,j}) - a_{r+1,n+i}a_{r+1,j} - \ldots - (-a_{s,n+i}a_{s,j}) - a_{s+1,n+i}a_{s+1,j} - \ldots - a_{n,n+i}a_{n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \ldots - a_{2n,n+i}a_{2n,j} \in k$

Now, the (i, j) entry of $\mathrm{Inn}_A(J) - \mathrm{Inn}_A\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} + \mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_{rs} & I \end{pmatrix}$ is given by $2a_{r,n+i}a_{rj} +$

$2a_{s,n+i}a_{s,j}$ and hence $a_{rl}a_{rj} + a_{sl}a_{sj} \in k$ for all $l > n$ and $j = 1, 2, \ldots, 2n$.

Subcase b: Suppose $i > n$

For $i > n$ the (i, j) entry of $\mathrm{Inn}_A(J)$ yields $-a_{1,i-n}a_{1,j} - a_{2,i-n}a_{2,j} - \ldots - a_{2n,i-n}a_{2n,j}$ and the (i, j) position of $\mathrm{Inn}_A\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ for $i > n$ is $-a_{n+1,i-n}a_{1j} - a_{n+2,i-n}a_{2,j} - \ldots - a_{2n,i-n}a_{n,j} - a_{n+1,i-n}a_{n+1,j} - a_{n+2,i-n}a_{n+2,j} - \ldots - a_{2n,i-n}a_{2n,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + \ldots + a_{n,i-n}a_{2n,j}$.

Hence the (i, j) entry of $\mathrm{Inn}_A(J) - \mathrm{Inn}_A\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ is given by $a_{n+1,i-n}a_{1j} + a_{n+2,i-n}a_{2,j} + \ldots + a_{2n,i-n}a_{n,j} - a_{1,i-n}a_{1,j} - a_{2,i-n}a_{2,j} - \ldots - a_{n,i-n}a_{n,j} - a_{1,i-n}an+1, j - a_{2,i-n}a_{n+2,j} - \ldots - a_{n,i-n}a_{2n,j}$.

For $i > n$ the (i, j) entry of $\mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_{rs} & I \end{pmatrix}$ is

$-a_{n+1,i-n}a_{1,j} - a_{n+2,i-n}a_{2j} - \ldots - a_{2n,i-n}a_{n,j} + a_{1,i-n}a_{1,j} + a_{2,i-n}a_{2,j} + \ldots + a_{n,i-n}a_{n,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + \ldots + a_{n,i-n}a_{2n,j}$.

Therefore the (i, j) entry of $\mathrm{Inn}_A(J) - \mathrm{Inn}_A\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} + \mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_{rs} & I \end{pmatrix}$ yields $-2a_{r,i-n}a_{rj} - 2a_{s,i-n}a_{s,j}$ and since $i > n$ we have that $a_{rl}a_{rj} + a_{sl}a_{sj} \in k$ for all $l \leq n$ and $j = 1, 2, \ldots, 2n$. Combining subcases a and b we have that $a_{rl}a_{rj} + a_{sl}a_{sj} \in k$ whenever $r, s \leq n$.

CASE 2: Suppose $r, s > n$. Without loss of generality assume $r < s$.

Subcase a: Suppose $i \leq n$. Now the matrix $\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$ is in G and since Inn_A keeps G

invariant the (i, j) entry of $\mathrm{Inn}_A\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$ given by

$a_{1,j}a_{n+1,n+i} + a_{2,j}a_{n+2,n+i} + \ldots + a_{n,j}a_{2n,n+i} - a_{1,n+i}a_{1,j} - a_{2,n+i}a_{2,j} - \ldots - a_{n,n+i}a_{n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \ldots - a_{n,n+i}a_{2n,j} \in k$

Now the (i, j) entry of $\mathrm{Inn}_A(J)$ was given in case 1 subcase a, therefore the (i, j) entry of $\mathrm{Inn}_A(J) + \mathrm{Inn}_A\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$ is

$a_{1,j}a_{n+1,n+i} + a_{2,j}a_{n+2,n+i} + \ldots + a_{n,j}a_{2n,n+i} + a_{n+1,n+i}a_{n+1,j} + a_{n+2,n+i}a_{n+2,j} + \ldots + a_{2n,n+i}a_{2n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \ldots - a_{n,n+i}a_{2n,j}$

which must lie in k. We know the matrix $\begin{pmatrix} I & X_{r-n,s-n} \\ 0 & I \end{pmatrix} \in G$ and thus the automorphism $\mathrm{Inn}_A\begin{pmatrix} I & X_{r-n,s-n} \\ 0 & I \end{pmatrix} \in G$ and its (i, j) entry given by

$a_{1,j}a_{n+1,n+i} + a_{2j}a_{n+2,n+i} + \ldots + a_{n,j}a_{2n,n+i} + a_{n+1,n+i}a_{n+1,j} + a_{n+2,n+i}a_{n+2,j} + \ldots + (-a_{r,n+i}a_{r,j}) + a_{r+1,n+i}a_{r+1,j} + \ldots + (-a_{s,n+i}a_{s,j}) + a_{s+1,n+i}a_{s+1,j} + \ldots + a_{2n,n+i}a_{2n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \ldots - a_nn + ia_{2n,j} \in k$.

Finally we observe that the (i, j) entry of $\mathrm{Inn}_A(J) + \mathrm{Inn}_A\begin{pmatrix} I & 0 \\ I & I \end{pmatrix} -$

$\mathrm{Inn}_A\begin{pmatrix} I & X_{r-n,s-n} \\ 0 & I \end{pmatrix}$ is given by $2a_{r,n+i}a_{rj} + 2a_{s,n+i}a_{s,j}$ and hence $a_{rl}a_{rj} + a_{sl}a_{sj} \in k$ for all $l > n$ and $j = 1, 2, \ldots, 2n$.

Subcase b: Suppose $i > n$. The (i, j) entry of $\mathrm{Inn}_A\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$ is in k and is given by

$-a_{n+i,i-n}a_{1,j} - a_{n+2,i-n}a_{2,j} - \ldots - a_{2n,i-n}a_{n,j} + a_{1,i-n}a_{1,j} + a_{2,i-n}a_{2,j} + \ldots + a_{n,i-n}a_{n,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + \ldots + a_{n,i-n}a_{2n,j}$.

Hence the (i, j) position of $\mathrm{Inn}_A(J) + \mathrm{Inn}_A\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$ is

$-a_{n+i,i-n}a_{1,j} - a_{n+2,i-n}a_{2,j} - \ldots - a_{2n,i-n}a_{n,j} - a_{n+1,i-n}a_{n+1,j} - a_{n+2,i-n}a_{n+2,j} - \ldots - a_{2n,i-n}a_{2,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + \ldots + a_{n,i-n}a_{2n,j}$

must reside in k. For $i > n$ the (i, j) entry of $\mathrm{Inn}_A\begin{pmatrix} I & X_{r-n,s-n} \\ 0 & I \end{pmatrix}$ is given by $-a_{n+1,i-n}a_{1,j} - a_{n+2,i-n}a_{2,j} - \ldots - a_{2n,i-n}a_{n,j} - a_{n+1,i-n}a_{n+1,j} - a_{n+2,i-n}a_{n+2,j} - \ldots - (-a_{r,i-n}a_{rj}) - a_{r+1,i-n}a_{r+1,j} - \ldots - (-a_{s,i-n}a_{s,j}) - a_{s+1,i-n}a_{s+1,j} - \ldots - a_{2n,i-n}a_{2n,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + \ldots + a_{n,i-n}a_{2n,j}$.

Therefore by considering the (i, j) entry of $\mathrm{Inn}_A(J) + \mathrm{Inn}_A\begin{pmatrix} I & 0 \\ I & I \end{pmatrix} -$ $\mathrm{Inn}_A\begin{pmatrix} I & X_{r-n,s-n} \\ 0 & I \end{pmatrix}$ we see that $-2a_{r,i-n}a_{rj} - 2a_{s,i-n}a_{s,j}$ must be in k. Since we assumed $i > n$ we have that $a_{rl}a_{rj} + a_{sl}a_{sj} \in k$ for all $l \leq n$ and $j = 1, 2, \ldots, 2n$. By combining subcases a and b we obtain $a_{rl}a_{rj} + a_{sl}a_{sj} \in k$ whenever $r, s > n$.

CASE 3: Suppose $r \leq n$ and $s > n$.

Subcase a: Suppose $i \leq n$. The matrix $\begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix} \in G$ and therefore $\mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix} \in G$. Specifically, the (i, j) entry of $\mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix}$ given by $a_{1,j}a_{n+1,n+i} + a_{2,j}a_{n+2,n+i} + \ldots + a_{n,j}a_{2n,n+i} - a_{1,n+i}a_{1,j} - a_{2,n+i}a_{2,j} - \ldots - (-a_{r,n+i}a_{r,j}) - a_{r+1,n+i}a_{r+1,j} - \ldots - a_{n,n+i}a_{n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \ldots - a_{n,n+i}a_{2n,j}$

lies in k. Now the (i, j) entry of $\mathrm{Inn}_A(J) + \mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix}$, which must be in k, is $a_{1,j}a_{n+1,n+i} + a_{2,j}a_{n+2,n+i} + \ldots + a_{n,j}a_{2n,n+i} + 2a_{r,n+i}a_{r,j} + a_{n+1,n+i}a_{n+1,j} + a_{n+2,n+i}a_{n+2,j} + \ldots + a_{2n,n+i}a_{2n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \ldots - a_{n,n+i}a_{2n,j}$.

If we now consider the automorphism Inn_A on the matrix $\begin{pmatrix} I & X_{s-n} \\ 0 & I \end{pmatrix} \in G$ then we see that the (i, j) entry of $\mathrm{Inn}_A\begin{pmatrix} I & X_{s-n} \\ 0 & I \end{pmatrix}$ is given by $a_{1,j}a_{n+1,n+i} + a_{2,j}a_{n+2,n+i} + \ldots + a_{n,j}a_{2n,n+i} + a_{n+1,n+i}a_{n+1,j} + a_{n+2,n+i}a_{n+2,j} + \ldots + (-a_{s,n+i}a_{s,j}) + \ldots + a_{2n,n+i}a_{2n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \ldots - a_{n,n+i}a_{2n,j}$.

Hence, the (i, j) entry of $\mathrm{Inn}_A(J) + \mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix} - \mathrm{Inn}_A\begin{pmatrix} I & X_{s-n} \\ 0 & I \end{pmatrix}$ gives us $2a_{r,n+i}a_{r,j} + 2a_{s,n+i}a_{s,j}$ and more importantly since we assumed $i \leq n$ we have that $a_{r,l}a_{r,j} + a_{s,l}a_{s,j} \in k$ for all $l > n$ and $j = 1, 2, ..., 2n$.

Subcase b: Suppose $i > n$. For $i > n$ the (i, j) entry of $\mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix}$ yields $-a_{n+1,i-n}a_{i,j} - a_{n+2,i-n}a_{2,j} - ... - a_{2n,i-n}a_{n,j} + a_{1,i-n}a_{1,j} + a_{2,i-n}a_{2,j} + ...+) - a_{r,i-n}a_{r,j} + ... + a_{n,i-n}a_{n,j} + a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + ... + a_{n,i-n}a_{2n,j}$.
Therefore the (i, j) entry of $\mathrm{Inn}_A(J) + \mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix}$ is
$-a_{n+1,i-n}a_{i,j} - a_{n+2,i-n}a_{2,j} - ... - a_{2n,i-n}a_{n,j} - 2a_{r,i-n}a_{r,j} - a_{n+1,i-n}a_{n+1,j} - a_{n+2,i-n}a_{n+2,j} - ... - a_{2n,i-n}a_{2n,j}a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + ... + a_{n,i-n}a_{2n,j.}$
Lastly we consider the (i, j) entry of $\mathrm{Inn}_A\begin{pmatrix} I & X_{s-n} \\ 0 & I \end{pmatrix}$ which is given by
$-a_{n+1,i-n}a_{i,j} - a_{n+2,i-n}a_{2,j} - ... - a_{2n,i-n}a_{n,j} - a_{n+1,i-n}a_{n+1,j} - a_{n+2,i-n}a_{n+2,j} - ... - (-a_{s,i-n}a_{s,j}) - ... - a_{2n,i-n}a_{2n,j}a_{1,i-n}a_{n+1,j} + a_{2,i-n}a_{n+2,j} + ... + a_{n,i-n}a_{2n,j}$.
So the (i, j) entry of $\mathrm{Inn}_A(J) + \mathrm{Inn}_A\begin{pmatrix} I & 0 \\ X_r & I \end{pmatrix} - \mathrm{Inn}_A\begin{pmatrix} I & X_{s-n} \\ 0 & I \end{pmatrix}$ gives us $-2a_{r,i-n}a_{r,j} - 2a_{s,i-n}a_{s,j}$ and since $i > n$ we have that $a_{r,l}a_{ar,j} + a_{s,l}a_{s,j} \in k$ for all $l \leq n$ and $j = 1, 2, ..., 2n$.
Combining subcases a and b we have that $a_{rl}a_{rj} + a_{sl}a_{sj} \in k$ whenever $r \leq n$ and $s > n$

In conclusion, by combining Cases 1,2,and 3 we can conclude that $a_{r,i}a_{r,j} + a_{s,i}a_{s,j} \in k$ for all $i, j = 1, 2, ...2n$ and $r \neq s$.

We are now able to use the fact that $a_{r,i}a_{r,j} + a_{s,i}a_{s,j} \in k$ for all $i, j = 1, 2, ...2n$ and $r \neq s$ to show that $a_{r,i}a_{r,j} \in k$ for all $i, j = 1, 2, ..., 2n$. However, we must show this in two cases. We will first show that $a_{r,l}a_{r,j} \in k$ for all $l \leq n$ and then show that

$a_{r,l}a_{r,j} \in k$ for all $l > n$. Without loss of generality it shall suffice to show $a_{1,l}a_{1,j} \in k$ for all l.

CASE 1: Assume $i > n$. The (i, j) entry or $\mathrm{Inn}_A(J)$ is given by $-a_{1,i-n}a_{1,j} - a_{2,i-n}a_{2,j} - \ldots - a_{2n,i-n}a_{2n,j}$ which is in k and implies that $a_{1,i-n}a_{1,j} + a_{2,i-n}a_{2,j} + \ldots + a_{2n,i-n}a_{2n,j} \in k$. From our previous argument we know that $a_{r,i}a_{r,j} + a_{s,i}a_{s,j} \in k$ for all $i, j = 1, 2, \ldots, 2n$, so obviously $a_{r,i}a_{r,j} + a_{s,i}a_{s,j} \in k$ for $i > n$. Making use of that fact the equality given by

$$a_{1,i-n}a_{1,j} =$$

$$(a_{1,i-n}a_{1,j} + a_{2,i-n}a_{2,j} + \ldots + a_{2n,i-n}a_{2n,j}) - (1/2)(a_{2,i-n}a_{2,j} + a_{3,i-n}a_{3,j}) - (1/2)(a_{3,i-n}a_{3,j} + a_{4,i-n}a_{4,j}) - (1/2)(a_{4,i-n}a_{4,j} + a_{5,i-n}a_{5,j}) - \ldots - (1/2)(a_{2n,i-n}a_{2n,j} + a_{2,i-n}a_{2,j})$$

must be in k, ie. $a_{1,i-n}a_{1,j} \in k$. Since we assumed that $i > n$ we have that $a_{1,l}a_{1,j} \in k$ for $l \leq n$. Furthermore, we can conclude that $a_{r,l}a_{r,j} \in k$ for $l \leq n$

CASE 2: Assume $i \leq n$. Then the (i, j) entry of $\mathrm{Inn}_A(J)$, which is in k, is given by $a_{1,i+n}a_{1,j} + a_{2,i+n}a_{2,j} + \ldots + a_{2n,i+n}a_{2n,j}$. We again make use of the fact that $a_{r,i}a_{r,j} + a_{s,i}a_{s,j} \in k$ for $i = 1, 2, \ldots, 2n$, and have an equality similar to the one in case 1 ($i - n$ is simply replaced by $i + n$)

$$a_{1,i+n}a_{1,j} =$$

$$(a_{1,i+n}a_{1,j} + a_{2,i+n}a_{2,j} + \ldots + a_{2n,i+n}a_{2n,j}) - (1/2)(a_{2,i+n}a_{2,j} + a_{3,i+n}a_{3,j}) - (1/2)(a_{3,i+n}a_{3,j} + a_{4,i+n}a_{4,j}) - (1/2)(a_{4,i+n}a_{4,j} + a_{5,i+n}a_{5,j}) - \ldots - (1/2)(a_{2n,i+n}a_{2n,j} + a_{2,i+n}a_{2,j})$$

which again must be in k. Since we assumed $i \leq n$ we have that $a_{1,l}a_{1,j} \in k$ for $l > n$ and furthermore, $a_{r,l}a_{r,j} \in k$ for $l > n$. Combining cases 1 and 2 shows that $a_{r,l}a_{r,j} \in k$ for $i, j = 1, 2, \ldots, 2n$.

I will finally show that $a_{ri}a_{sj} \in k$ for $r \neq s$

CASE I: Suppose $r, s \leq n$. Without loss of generality we will assume that $r < s$.

(1) **Subcase 1**: Suppose $r + s < n + 1$.
Let $Y_{r,s} = \begin{pmatrix} T_{s+r-1} & 0 \\ 0 & I_{n-(s+r-1)} \end{pmatrix}$ and

$$Z_{r,s} = \begin{pmatrix} Y_{r,s} & 0 \\ E_{r,s} & Y_{r,s} \end{pmatrix}$$

Now $Z_{r,s} \in G$ and hence Inn_A must keep $Z_{r,s}$ invariant and thus all the entries of $\mathrm{Inn}_A(Z_{r,s})$ must lie in$k$.

(a) Assume $i \leq n$. Then the (i,j) entry of $\mathrm{Inn}_A(Z_{r,s})$ is given by
$-a_{r,n+i}a_{s,j} + a_{n+s+r-1,n+i}a_{1,j} + a_{n+s+r-2,n+i}a_{2,j} + \ldots +$
$a_{n+2,n+i}a_{s+r-2,j} + a_{n+1,n+i}a_{s+r-1,j} - a_{1,n+i}a_{n+s+r-1,j} - a_{2,n+i}a_{n+s+r-2,j} -$
$\ldots - a_{s+r-2,n+i}a_{n+2,j} - a_{s+r-1,n+i}a_{n+1,j} + a_{n+r+s,n+i}a_{r+s,j} + a_{n+r+s+1,n+i}a_{r+s+1,j} +$
$\ldots + a_{2n,n+i}a_{n,j} - a_{r+s,n+i}a_{n+r+s,j} - a_{r+s+1,n+i}a_{n+r+s+1,j} - \ldots - a_{n,n+i}a_{2n,j}$.
Let $\bar{Z}_{r,s} \begin{pmatrix} -Y_{r,s} & 0 \\ E_{r,s} & -Y_{r,s} \end{pmatrix}$. Now $\bar{Z}_{r,s} \in G$ and thus $\mathrm{Inn}_A(\bar{Z}_{r,s}) \in G$. In fact, the (i,j) entry of $\mathrm{Inn}_A(\bar{Z}_{r,s})$ is the negative of the (i,j) entry of $\mathrm{Inn}_A\, Z_{r,s}$ with the exception of $-a_{r,n+i}a_{s,j}$ which remains negative. Therefore, $\mathrm{Inn}_A(Z_{r,s}) + \mathrm{Inn}_A(\bar{Z}_{r,s})$ has an (i,j) entry of $-2a_{r,n+i}a_{s,j}$. Since both $\mathrm{Inn}_A(Z_{r,s})$ and $\mathrm{Inn}_A(\bar{Z}_{r,s})$ are both in G their sum is in G and hence $-2a_{r,n+i}a_{s,j} \in k$. Since we assumed $i \leq n$ we have $a_{r,l}a_{s,j} \in k$ for $l > n$.

(b) Assume $i > n$. Then the (i,j) entry of $\mathrm{Inn}_A(Z_{r,s})$ is given by
$a_{r,i-n}a_{s,j} - a_{n+s+r-1,i-n}a_{1,j} - a_{n+s+r-2,i-n}a_{2,j} - \ldots - a_{n+2,i-n}a_{s+r-2,j} -$
$a_{n+1,i-n}a_{s+r-1,j} + a_{1,i-n}a_{n+s+r-1,j} - a_{2,i-n}a_{n+s+r-2,j} + \ldots + a_{s+r-2,i-n}a_{n+2,j} +$
$a_{s+r-1,i-n}a_{n+1,j} + a_{n+r+s,i-n}a_{r+s,j} - a_{n+r+s+1,i-n}a_{r+s+1,j} - \ldots - a_{2n,i-n}a_{n,j} +$

$a_{r+s,i-n}a_{n+r+s,j} + a_{r+s+1,i-n}a_{n+r+s+1,j} + \ldots + a_{n,i-n}a_{2n,j}$

Note that the (i, j) entry of $\mathrm{Inn}_A(Z_{r,s})$ for $i > n$ is the negative of the (i, j) entry of $\mathrm{Inn}_A(Z_{r,s})$ for $i \leq n$ with the simple change that $n+i$ becomes $i-n$. Again we have that the (i, j) entry of $\mathrm{Inn}_A(\bar{Z}_{r,s})$ is the negative of the (i, j) entry of $\mathrm{Inn}_A(Z_{r,s})$ with the exception of $a_{r,i-n}a_{s,j}$ which remains positive. Hence as in the previous case the (i, j) entry of $\mathrm{Inn}_A Z_{r,s} + \mathrm{Inn}_A(\bar{Z}_{r,s})$, gives us $2a_{r,i-n}a_{s,j} \in k$. Since we assumed that $i > n$ we can conclude that $a_{r,l}a_{s,j} \in k$ for $l < n$.

Combining a and b we have that $a_{r,i}a_{s,j} \in k$ for $r + s < n + 1$.

(2) **Subcase 2**: Suppose $r + s > n + 1$.

Let

$$U_{r,s} = \begin{pmatrix} I_{-n+(r+s-1)} & 0 \\ 0 & (T_{2n-(r+s-1)} \end{pmatrix}$$

and

$$V_{r,s} = \begin{pmatrix} U_{r,s} & 0 \\ E_{r,s} & U_{r,s} \end{pmatrix}.$$

$V_{r,s} \in G$ so $\mathrm{Inn}_A(V_{r,s}) \in G$ since Inn_A keeps G invariant.

(a) Suppose $i \leq n$. Then the (i, j) entry of $\mathrm{Inn}_A(V_{r,s})$ is given by

$-a_{r,n+i}a_{s,j} + a_{2n,n+i}a_{s+r-n,j} + a_{2n-1,n+i}a_{s+r-n+1,j} + \ldots + a_{s+r,n+i}a_{n,j} - a_{2n,j}a_{s+r-n,n+i} - a_{2n-1,j}a_{s+r-n+1,n+i} - \ldots - a_{s+r,j}a_{n,n+i} + a_{n+1,n+i}a_{1,j} + a_{n+2,n+i}a_{2,j} + \ldots + a_{r+s-1,n+i}a_{(r+s-1)-n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \ldots - a_{(r+s-1)-n,n+i}a_{r+s-1,j}$.

Let $\bar{V}_{r,s} = \begin{pmatrix} -U_{r,s} & 0 \\ E_{r,s} & -U_{r,s} \end{pmatrix}$. Now $\bar{V}_{r,s} \in G$ which implies that $\mathrm{Inn}_A(\bar{V}_{r,s}) \in G$. The (i, j) entry of $\mathrm{Inn}_A(\bar{V}_{r,s})$ is the negative of the (i,j) entry of $\mathrm{Inn}_A(V_{r,s})$ with the exception of the term $-a_{r,n+i}a_{s,j}$ which remains negative. Hence the (i, j) entry of $\mathrm{Inn}_A(V_{r,s}) + \mathrm{Inn}_A(\bar{V}_{r,s})$, $-2a_{r,n+i}a_{s,j}$ is in k. Since we assumed $i \leq n$ we have $a_{r,l}a_{s,j} \in k$ for $l > n$.

(b) Assume $i > n$. As in the previous case, the (i, j) entry of $\mathrm{Inn}_A(V_{r,s})$ for $i > n$ is the negative of the (i, j) entry of $\mathrm{Inn}_A(V_{r,s})$ for $i \le n$ with the simple change that $n+i$ becomes $i-n$. Again we have that the (i, j) entry of $\mathrm{Inn}_A(\bar{V}_{r,s})$ is the negative of the (i, j) entry of $\mathrm{Inn}_A(V_{r,s})$ with the exception of $a_{r,i-n}a_{s,j}$ which remains positive. Hence as in the previous case the (i, j) entry of $\mathrm{Inn}_A Z_{r,s} + \mathrm{Inn}_A(\bar{Z}_{r,s})$, gives us $2a_{r,i-n}a_{s,j}$. Since we assumed that $i > n$ we can conclude that $a_{r,l}a_{s,j} \in k$ for $l < n$.

Combining a and b we have that $a_{r,i}a_{s,j} \in k$ for $r + s < n + 1$

(3) **Subcase 3:** Suppose $r + s = n + 1$. Here we choose $W_{r,s} = \begin{pmatrix} T_n & 0 \\ E_{r,s} & T_n \end{pmatrix}$. Now $W_{r,s} \in G$ and hence, $\mathrm{Inn}_A(W_{r,s}) \in G$ since Inn_A keeps G invariant.

(a) Suppose $i \le n$ Then the (i, j) entry of $\mathrm{Inn}_A(W_{r,s})$ is given by
$-a_{r,n+i}a_{s,j} + a_{2n,n+i}a_{1,j} + a_{2n-1,n+i}a_{2,j} + \ldots + a_{n+1,n+i}a_{n,j} - a_{2n,j}a_{1,n+i} + a_{2n-1,j}a_{2,n+i} + \ldots + a_{n+1,j}a_{n,n+i}$.
Let $\bar{W}_{r,s} = \begin{pmatrix} -T_n & 0 \\ E_{r,s} & -T_n \end{pmatrix}$. $\bar{W}_{r,s} \in G$ which means that $\mathrm{Inn}_A(\bar{W}_{r,s}) \in G$. The (i, j) entry of $\mathrm{Inn}_A(\bar{W}_{r,s})$ is the negative of the (i, j) entry of $\mathrm{Inn}_A(W_{r,s})$ with the exception that the term $-a_{r,n+i}a_{s,j}$ which remains negative. Using the fact that $\mathrm{Inn}_A(W_{r,s}) + \mathrm{Inn}_A(\bar{W}_{r,s}) \in G$ we have that the term $-2a_{r,n+i}a_{s,j} \in k$. However, since we assumed that $i \le n$ we have that $a_{r,l}a_{s,j} \in k$ for $l > n$.

(b) The case where $i > n$ follows exactly as above by simply changing the signs of each term and replacing $n + i$ by $i - n$.

Combining Subcases 1,2, and 3 gives us $a_{r,i}a_{s,j} \in k$ for $r, s > n$.

CASE II: Suppose $r, s > n$. Without loss of generality assume $r < s$.

(1) **Subcase 1:** Suppose $r + s - 2n < n + 1$.

Let $Z'_{r,s} = \begin{pmatrix} Y_{r-n,s-n} & E_{r-n,s-n} \\ 0 & Y_{r-n,s-n} \end{pmatrix}$

(a) Suppose $i \leq n$, Since $Z'_{r,s} \in G$, $\mathrm{Inn}_A(Z'_{r,s})$ must lie in G and hence its (i, j) entry of

$a_{r,n+i}a_{s,j} + a_{n+s+r-1,n+i}a_{1,j} + a_{n+s+r-2,n+i}a_{2,j} + \ldots + a_{n+2,n+i}a_{s+r-2,j} + a_{n+1,n+i}a_{s+r-1,j} - a_{1,n+i}a_{n+s+r-1,j} - a_{2,n+i}a_{n+s+r-2,j} - \ldots - a_{s+r-2,n+i}a_{n+2,j} - a_{s+r-1,n+i}a_{n+1,j} + a_{n+r+s,n+i}a_{r+s,j} + a_{n+r+s+1,n+i}a_{r+s+1,j} + \ldots + a_{2n,n+i}a_{n,j} - a_{r+s,n+i}a_{n+r+s,j} - a_{r+s+1,n+i}a_{n+r+s+1,j} - \ldots - a_{n,n+i}a_{2n,j}$

is in k. Note that the (i, j) entry of $\mathrm{Inn}_A(Z'_{r,s})$ is precisely the (i, j) entry of $\mathrm{Inn}_A(Z_{r,s})$ given in part I with the exception of the first term. Let $\bar{Z}'_{r,s} = \begin{pmatrix} -Y_{r-n,s-n} & E_{r-n,s-n} \\ 0 & -Y_{r-n,s-n} \end{pmatrix}$. The (i, j) entry of $\mathrm{Inn}_A(\bar{Z}'_{r,s})$ is the negative of the (i, j) entry of $\mathrm{Inn}_A(Z'_{r,s})$ excluding the term $a_{r,n+i}a_{s,j}$ which remains positive. Hence the (i, j) entry of $\mathrm{Inn}_A(Z'_{r,s}) + \mathrm{Inn}_A(\bar{Z}'_{r,s}), givenby 2a_{r,n+i}a_{s,j}$ lies in k. Since we assumed $i \leq n$ we have $a_{r,l}a_{s,j} \in k$ for $l > n$.

(b) As in the previous cases, for $i > n$ the proof follows exactly as above by simply changing the signs of each term and replacing $n + i$ by $i - n$. You will get that the (i, j) entry of $\mathrm{Inn}_A(Z'_{r,s}) + \mathrm{Inn}_A(\bar{Z}'_{r,s})$ yields $-2a_{r,i-n}a_{s,j} \in k$. Or more specifically, $a_{r,l}a_{s,j} \in k$ for $l \leq n$.

Combining a and b gives $a_{r,i}a_{s,j} \in k$ for $r + s - 2n < n + 1$

(2) **Subcase 2:** Suppose $r + s - 2n > n + 1$

(a) Let $V'_{r,s} = \begin{pmatrix} U_{r-n,s-n} & E_{r-n,s-n} \\ 0 & U_{r-n,s-n} \end{pmatrix}$. Now $\mathrm{Inn}_A(V'_{r,s})$ must lie in G and hence its (i, j) entry of

$a_{r,n+i}a_{s,j} + a_{2n,n+i}a_{s+r-n,j} + a_{2n-1,n+i}a_{s+r-n+1,j} + \ldots + a_{s+r,n+i}a_{n,j} - a_{2n,j}a_{s+r-n,n+i} - a_{2n-1,j}a_{s+r-n+1,n+i} - \ldots - a_{s+r,j}a_{n,n+i} + a_{n+1,n+i}a_{1,j} + a_{n+2,n+i}a_{2,j} + \ldots + a_{r+s-1,n+i}a_{(r+s-1)-n,j} - a_{1,n+i}a_{n+1,j} - a_{2,n+i}a_{n+2,j} - \ldots - a_{(r+s-1)-n,n+i}a_{r+s-1,j}$

must lie in k. If we define $\bar{V'}_{r,s} = \begin{pmatrix} -U_{r,s} & E_{r-n,s-n} \\ 0 & -U_{r,s} \end{pmatrix}$, which is in G, then we see that the (i, j) entry of $\mathrm{Inn}_A(\bar{V'}_{r,s})$ is the negative of the (i, j) entry of $\mathrm{Inn}_A(V'_{r,s})$ excluding the term $a_{r,n+i}a_{s,j}$ which remains positive. Hence the (i, j) entry of $\mathrm{Inn}_A(V'_{r,s}) + \mathrm{Inn}_A(\bar{V'}_{r,s})$, $2a_{r,n+i}a_{s,j}$ is in$k$. Since we assumed $i \le n$ we have $a_{r,l}a_{s,j} \in k$ for $l > n$.

(b) Again as in the previous cases, for $i > n$ the proof follows exactly as above by simply changing the signs of each term and replacing $n+i$ by $i-n$. You will get that the (i, j) entry of $\mathrm{Inn}_A(V'_{r,s}) + \mathrm{Inn}_A(\bar{V'}_{r,s})$ yields that the term $-2a_{r,i-n}a_{s,j}$ is in k. Or more specifically, $a_{r,l}a_{s,j} \in k$ for $l \le n$.

Combining a and b gives $a_{r,i}a_{s,j} \in k$ for $r+s-2n > n+1$

(3) **Subcase 3**: Suppose $r+s-2n = n+1$. Let $W'_{r,s} = \begin{pmatrix} T_n & E_{r-n,s-n} \\ 0 & T_n \end{pmatrix}$. Now $W'_{r,s} \in G$ and thus $\mathrm{Inn}_A(W'_{r,s}) \in G$.

(a) Suppose $i \le n$, then the (i, j) entry of $\mathrm{Inn}_A(W'_{r,s})$ is given by $a_{r,n+i}a_{s,j} + a_{2n,n+i}a_{1,j} + a_{2n-1,n+i}a_{2,j} + \ldots + a_{n+1,n+i}a_{n,j} - a_{2n,j}a_{1,n+i} + a_{2n-1,j}a_{2,n+i} + \ldots + a_{n+1,j}a_{n,n+i}$.

If we let $\bar{W'}_{r,s} = \begin{pmatrix} -T_n & E_{r-n,s-n} \\ 0 & -T_n \end{pmatrix}$, then we see that the (i, j) entry is simply the negative of the (i, j) entry of $\mathrm{Inn}_A(W'_{r,s})$ excluding the term $a_{r,n+i}a_{s,j}$ which remains positive. Hence the (i, j) entry of $\mathrm{Inn}_A(W'_{r,s}) + \mathrm{Inn}_A(\bar{W'}_{r,s})$, $2a_{r,n+i}a_{s,j}$ is in k. Since we assumed $i \le n$ we have $a_{r,l}a_{s,j} \in k$ for $l > n$.

(b) Again as in the previous cases, for $i > n$ the proof follows exactly as above by simply changing the signs of each term and replacing $n+i$ by $i-n$. You will get that the (i,j) entry of $\mathrm{Inn}_A(W'_{r,s}) + \mathrm{Inn}_A(\bar{W}'_{r,s})$ yields that the term $-2a_{r,i-n}a_{s,j}$ is in k. Or more specifically, $a_{r,l}a_{s,j} \in k$ for $l \leq n$.

Combining a and b gives $a_{r,i}a_{s,j} \in k$ for $r+s-2n > n+1$

CASE III: Suppose $r \leq n$ and $s > n$

(1) **Subcase 1:** Suppose $r+s < 2n+1$.

Let $M_{r,s} = \begin{pmatrix} E_{s-n,r} & Y_{s-n,r} \\ -Y_{s-n,r} & 0 \end{pmatrix}$. Now $M_{r,s} \in G$ and thus $\mathrm{Inn}_A(M_{r,s}) \in G$ by assumption.

(a) Suppose $i \leq n$. Now the (i,j) entry of $\mathrm{Inn}_A(M_{r,s})$ is given by $a_{r,j}a_{s,n+i} + a_{s+r,n+i}a_{s+r,j} + a_{s+r+1,n+i}a_{s+r+1,j} + \ldots + a_{2n,n+i}a_{2n,j} + a_{s+r-n,n+i}a_{s+r-n,j} + a_{s+r-n+1,n+i}a_{s+r-n+1,j} + \ldots + a_{n,n+i}a_{n,j} + a_{s+r-1,n+i}a_{n+1,j} + a_{s+r-2,n+i}a_{n+2,j} + \ldots + a_{n+1,n+i}a_{s+r-1,j} + a_{s-n+r-1,n+i}a_{1,j} + a_{s-n+r-2,n+i}a_{2,j} + \ldots + a_{1,n+i}a_{s-n+r-1,j}$.

Now define $\bar{M}_{r,s} = \begin{pmatrix} E_{s-n,r} & -Y_{s-n,r} \\ Y_{s-n,r} & 0 \end{pmatrix}$, then $\bar{M}_{r,s} \in G$ and therefore $\mathrm{Inn}_A(\bar{M}_{r,s}) \in G$. In addition the (i,j) entry of $\bar{M}_{r,s}$ is the negative of the (i,j) entry of $\mathrm{Inn}_A(M_{r,s})$ with the exception of the term $a_{r,j}a_{s,i+n}$ which remains positive. The sum $\mathrm{Inn}_A(M_{r,s}) + \mathrm{Inn}_A(\bar{M}_{r,s}) \in G$ and thus its (i,j) entry of $2a_{r,j}a_{s,i+n} \in k$. Since we assumed that $i \leq n$ this gives us $a_{r,j}a_{s,l} \in k$ for $l > n$.

(b) As in the previous cases, for $i > n$ the proof follows exactly as above by simply changing the signs of each term and replacing $n+i$ by $i-n$. You will get that the (i,j) entry of $\mathrm{Inn}_A(M_{r,s}) + \mathrm{Inn}_A(\bar{M}_{r,s})$ yields that the term $-2a_{r,j}a_{s,i-n}$ is in k. Or more specifically, $a_{r,j}a_{s,l} \in k$ for $l \leq n$.

Combining a and b gives $a_{r,j}a_{s,l} \in k$ for $r + s < 2n + 1$.

(2) **Subcase 2:** Suppose $r + s > 2n + 1$.
With $N_{r,s} = \begin{pmatrix} E_{s-n,r} & U_{s-n,r} \\ -U_{s-n,r} & 0 \end{pmatrix}$ it is seen that $N_{r,s} \in G$ and thus by assumption $\mathrm{Inn}_A(N_{r,s}) \in G$

(a) Suppose $i \leq n$ then the (i, j) entry of $\mathrm{Inn}_A(N_{r,s})$ is given by
$a_{r,j}a_{s,n+i} + a_{2n,n+i}a_{s+r-n,j} + a_{2n-1,n+i}a_{s+r-n+1,j} + \ldots + a_{s+r-n,n+i}a_{2n,j} + a_{n,n+i}a_{s+r-2n,j} + a_{n-1,n+i}a_{s+r-2n+1,j} + \ldots + a_{s+r-2n,n+i}a_{n,j} + a_{n+1,n+i}a_{n+1,j} + a_{n+2,n+i}a_{n+2,j} + \ldots + a_{-n+r+s-1,n+i}a_{-n+r+s-1,j} + a_{1,n+i}a_{1,j} + a_{2,n+i}a_{2,j} + \ldots + a_{-2n+r+s-1,n+i}a_{-2n+r+s-1,j}$
Define $\bar{N}_{r,s} = \begin{pmatrix} E_{s-n,r} & -U_{s-n,r} \\ U_{s-n,r} & 0 \end{pmatrix}$. We again can make use of the fact that $\bar{N}_{r,s} \in G$ implies that $\mathrm{Inn}_A(\bar{N}_{r,s}) \in G$. Now the (i, j) entry of $\mathrm{Inn}_A(\bar{N}_{r,s})$ is the negative of the (i, j) entry of $\mathrm{Inn}_A(N_{r,s})$ with the exception of the the term $a_{r,j}a_{s,n+i}$ which remains positive. Hence the (i, j) entry of $\mathrm{Inn}_A(\bar{N}_{r,s}) + \mathrm{Inn}_A(N_{r,s})$ given by $2a_{r,j}a_{s,n+i}$ must lie in k. Furthermore, since we assumed that $i \leq n$ we can conclude that $a_{r,j}a_{s,l} \in k$ for $l > n$.

(b) As in the previous cases, if $i > n$ the proof follows exactly as above by simply changing the signs of each term and replacing $n + i$ by $i - n$. You will get that the (i, j) entry of $\mathrm{Inn}_A(N_{r,s}) + \mathrm{Inn}_A(\bar{N}_{r,s})$ yields that the term $-2a_{r,j}a_{s,i-n}$ is in k. Or more specifically, $a_{r,j}a_{s,l} \in k$ for $l \leq n$.

Combining a and b gives $a_{r,j}a_{s,l} \in k$ for $r + s < 2n + 1$

(3) **Subcase 3:** Suppose $r+s=2n+1$. Let $F_{r,s} = \begin{pmatrix} E_{s-n,r} & T_n \\ T_n & 0 \end{pmatrix}$.

Now $F_{r,s} \in G$ and therefore $\mathrm{Inn}_A(F_{r,s}) \in G$ since Inn_A keeps G invariant.

(a) Suppose $i \leq n$. Then the (i, j) entry of $\mathrm{Inn}_A(F_{r,s})$ is given by

$a_{r,j}a_{s,n+i} + a_{1,n+i}a_{1,j} + a_{2,n+i}a_{2,j} + a_{2,n+i}a_{3,j} + \ldots + a_{2n,n+i}a_{2n,j}$

Let $\bar{F}_{r,s} = \begin{pmatrix} E_{s-n,r} & -T_n \\ -T_n & 0 \end{pmatrix}$. Then since $\bar{F}_{r,s} \in G$ we have that $\mathrm{Inn}_A(\bar{F}_{r,s}) \in G$. More importantly, the (i, j) entry of $\mathrm{Inn}_A(\bar{F}_{r,s})$ is the negative of the (i, j) entry of $\mathrm{Inn}_A(F_{r,s})$ with the exception that the term $a_{r,j}a_{s,i+n}$ remains positive. Again using the fact that $\mathrm{Inn}_A(\bar{F}_{r,s}) + \mathrm{Inn}_A(F_{r,s}) \in G$ we have that its (i, j) entry of $2a_{r,j}a_{s,n+i} \in k$. Since we assumed $i \leq n$ we have that $a_{r,j}a_{s,l} \in k$ for $l > n$.

(b) As in the previous cases, if $i > n$ the proof follows exactly as above by simply changing the signs of each term and replacing $n+i$ by $i-n$. You will get that the (i, j) entry of $\mathrm{Inn}_A(\bar{F}_{r,s}) + \mathrm{Inn}_A(F_{r,s}) \in G$ yields that the term $-2a_{r,j}a_{s,i-n}$ is in k. Or more specifically, $a_{r,j}a_{s,l} \in k$ for $l \leq n$

Combining subcases a and b gives us $a_{r,j}a_{s,l} \in k$ for $r+s=2n+1$.

Combining cases 1,2, and 3 gives us $a_{r,i}a_{s,j} \in k$ for $r \leq n$ and $s > n$.

Cases I, II and III show that $a_{r,i}a_{s,j} \in k$ and hence $A = pM$. □

5.6 Involutions of $\mathrm{SL}(2n,k)$ on $\mathrm{SP}(2n,k)$

It is easy to see that every automorphism, hence every involution, of $\mathrm{SP}(2n,k)$ is the restriction of an automorphism of $\mathrm{SL}(2n,k)$. Therefore, to characterize the involutions of $\mathrm{SP}(2n,k)$ we need to look at the involutions of $\mathrm{SL}(2n,k)$ restricted to $\mathrm{SP}(2n,k)$.

5.6.1 Outer Involutions of $\mathrm{SL}(2n, k)$ on $\mathrm{SP}(2n, k)$

We begin by investigating what happens when we restrict an outer involution of $\mathrm{SL}(2n, k)$ to $\mathrm{SP}(2n, k)$. A few lemmas will prove to be useful in obtaining this result.

Lemma 15. *The outer involutions of* $\mathrm{SL}(2n, k)$ *coming from a symmetric or skew-symmetric matrix* M*, defined as* $\tau = \mathrm{Inn}_M \theta$ *where* $\theta(A) = (A^T)^{-1}$ *can be viewed as* $\tau = \mathrm{Inn}_{J^{-1}M} \phi$ *where* ϕ *is the fixed outer automorphism given by* $\phi(A) = J^{-1}(A^T)^{-1}J = \mathrm{Inn}_J \theta(A)$.

Proof. Consider the outer involution $\tau = \mathrm{Inn}_M \theta$ on $\mathrm{SL}(2n, k)$ coming from the symmetric or skew-symmetric bilinear form with matrix M. Then

$$\tau(A) = \mathrm{Inn}_M \theta = \mathrm{Inn}_M (A^T)^{-1} = M^{-1}(A^T)^{-1}M$$

Let $\phi(A) = J^{-1}(A^T)^{-1}J$ then

$$\begin{aligned}
\tau(A) &= M^{-1}(A^T)^{-1}M \\
&= M^{-1}JJ^{-1}(A^T)^{-1}JJ^{-1}M \\
&= (J^{-1}M)^{-1}(J^{-1}(A^T)^{-1}J)J^{-1}M \\
&= \mathrm{Inn}_{J^{-1}M}((J^{-1}(A^T)^{-1}J) \\
&= \mathrm{Inn}_{J^{-1}M} \phi(A)
\end{aligned}$$

Hence the outer involutions of $\mathrm{SL}(2n, k)$ can be viewed as $\tau = \mathrm{Inn}_{J^{-1}M} \phi$. □

Lemma 16. *The fixed outer automorphism* $\phi = \mathrm{Inn}_J \theta$ *where* $\theta(A) = (A^T)^{-1}$ *is the identity when restricted to* G*. ie.* $\phi|_G = \mathrm{Inn}_J \theta|_G = \mathrm{Id}$.

Proof. Let $\phi = \mathrm{Inn}_J \theta$ where $\theta(A) = (A^T)^{-1}$. Suppose $A \in G$, then

$$\begin{aligned}
\phi(A) &= \mathrm{Inn}_J \phi(A) \\
&= \mathrm{Inn}_J (A^T)^{-1} \\
&= J^{-1}(A^T)^{-1}J
\end{aligned}$$

Now since $A \in G$, $A^T JA = J$ or more importantly $(A^T)^{-1} = JAJ^{-1}$ hence,

$$\phi(A) = J^{-1}(A^T)^{-1}J = J^{-1}(JAJ^{-1})J = J.$$

Therefore, $\phi|_G = \mathrm{Inn}_J\,\theta|_G = \mathrm{Id}$. □

The above Lemma is true since G is the fixed point group of $\phi = \mathrm{Inn}_J\,\theta$.

Lemma 17. *The outer involutions of* $\mathrm{SL}(2n, k)$ *become inner involutions when restricted to* $\mathrm{SP}(2n, k)$.

Proof. Let τ be an outer involution of $\mathrm{SL}(2n, k)$. Then from Lemma 15 we can let $\tau = \mathrm{Inn}_{J^{-1}M}\,\phi$ where, $\phi = \mathrm{Inn}_J\,\theta$, $\theta(A) = (A^T)^{-1}$ and M is a symmetric or skew-symmetric matrix. Now from Lemma 16 we know that $\phi|_G = \mathrm{Inn}_J\,\theta|_G = \mathrm{Id}$, hence

$$\tau|_G = \mathrm{Inn}_{J^{-1}M}\,\phi|_G = \mathrm{Inn}_{J^{-1}M}$$

Therefore, the outer involutions of $\mathrm{SL}(2n, k)$ become inner involutions when restricted to $\mathrm{SP}(2n, k)$. □

5.6.2 Involutions of $\mathrm{SL}(2n, k)$ Which Leave $\mathrm{SP}(2n, k)$ Invariant

We now turn our attention to trying to characterize which involutions of $\mathrm{SL}(2n, k)$ leave $\mathrm{SP}(2n, k)$ invariant. That is which involutions of $\mathrm{SL}(2n, k)$ will remain involutions when restricted to $\mathrm{SP}(2n, k)$. The following Lemma helps us with this endeavor.

Theorem 5.8. *Let ϕ be the involution of* $\mathrm{SL}(2n, k)$ *coming from the skew-symmetric bilinear form with matrix representation J. Then its corresponding fixed point group is* $G = \mathrm{SP}(2n, k)$. *Now since ϕ is coming from the skew-symmetric matrix J, ϕ must be of type outer and thus $\phi = \mathrm{Inn}_J\,\theta$ where $\theta(X) = (X^T)^{-1}$. The involution τ of* $\mathrm{SL}(2n, k)$ *keeps G invariant if and only if $\tau\phi = \phi\tau$ on* $\mathrm{SL}(2n, k)$.

Proof. $\Longleftarrow$ Suppose $\tau\phi = \phi\tau$ on $\mathrm{SL}(2n, k)$ and let $X \in G$. Since G is the fixed point group of ϕ, $\tau\phi(X) = \tau(X)$. By assumption $\tau\phi = \phi\tau$ so $\phi\tau(X) = \tau\phi(X) = \tau(X)$, ie. $\phi\tau(X) = \tau(X)$, which states that $\tau(X)$ is in the fixed point group of ϕ, ie. $\tau(X) \in G$ Hence, τ keeps G invariant.

$\Longrightarrow$ Suppose the involution τ of $\mathrm{SL}(2n, k)$ keeps G invariant. We know that every automorphism of G, and thus every involution of G can be written as $\tau = \mathrm{Inn}_A$ where $A \in SL(2n, \bar{k})$. In addition, since by assumption τ keeps G invariant, Theorem 5.5 tells us that $\tau = \mathrm{Inn}_A = \mathrm{Inn}_{pM}$ where $p \in \bar{k}$ and$M \in G$. Now

$$\begin{aligned}
\tau\phi(X) &= \mathrm{Inn}_A\,\phi(X) \\
&= \mathrm{Inn}_{pM}\,\phi(X) \\
&= \mathrm{Inn}_{pM}(J^{-1}(X^T)^{-1}J) \\
&= pM^{-1}J^{-1}(X^T)^{-1}JpM
\end{aligned}$$

Since $M \in G$ we have that $M^{-1} = J^{-1}M^TJ$ and $M = J^{-1}(M^T)^{-1}J$ so

$$\begin{aligned}
\tau\phi(X) &= M^{-1}J^{-1}(X^T)^{-1}JM \\
&= (J^{-1}M^TJ)J^{-1}(X^T)^{-1}JJ^{-1}(M^T)^{-1}J) \\
&= J^{-1}M^T(X^T)^{-1}(M^T)^{-1}J
\end{aligned}$$

On the other hand

$$\begin{aligned}
\phi\tau(X) &= \phi(\mathrm{Inn}_A(X)) \\
&= \phi(\mathrm{Inn}_{pM}(X)) \\
&= \phi(\mathrm{Inn}_M(X)) \\
&= \phi(M^{-1}XM) \\
&= J^{-1}((M^{-1}XM)^T)^{-1}J \\
&= J^{-1}M^T(X^T)^{-1}(M^T)^{-1}J
\end{aligned}$$

Thus, $\tau\phi = \phi\tau$ on $\mathrm{SL}(2n, k)$. $\square$

Corollary 2. *An involution* Inn_A *of* $\mathrm{SL}(2n,k)$ *leaves* $\mathrm{SP}(2n,k)$ *invariant if and only if* $J(A^T)^{-1}J^{-1} = cA$ *where* $c \in k^*$.

Proof. Suppose $\tau = \mathrm{Inn}_A$ is an involution of $\mathrm{SL}(2n,k)$ which leaves $\mathrm{SP}(2n,k)$ invariant. Theorem 5.8 tells us that $\phi\tau = \tau\phi$, where $\phi = \mathrm{Inn}_J\theta$ and $\theta(X) = (X^T)^{-1}$. Now

$$\begin{aligned}
\phi\tau(X) &= \mathrm{Inn}_J\theta\,\mathrm{Inn}_A(X)\\
&= \mathrm{Inn}_J\theta(A^{-1}XA)\\
&= \mathrm{Inn}_J((A^{-1}XA)^T)^{-1})\\
&= \mathrm{Inn}_J(A^T(X^T)^{-1}(A^T)^{-1})\\
&= J^{-1}A^T(X^T)^{-1}(A^T)^{-1}J
\end{aligned}$$

and

$$\tau\phi(X) = \mathrm{Inn}_A\,\mathrm{Inn}_J\theta(X) = \mathrm{Inn}_{JA}\theta(X) = A^{-1}J^{-1}(X^T)^{-1}JA$$

Hence we have the following equivalent statements

$$\begin{aligned}
J^{-1}A^T(X^T)^{-1}(A^T)^{-1}J &= A^{-1}J^{-1}(X^T)^{-1}JA\\
J^TA^{-1}X^{-1}A(J^{-1})^T &= A^TJ^TX^{-1}(J^{-1})^T(A^{-1})^T\\
J^TA^{-1}XA(J^T)^{-1} &= A^TJ^TX(J^{-1})^T(A^{-1})^T\\
(J^T)^{-1}(A^T)^{-1}J^TA^{-1}XA(J^T)^{-1}A^TJ^T &= X\\
\mathrm{Inn}_{A(J^T)^{-1}A^TJ^T}(X) &= X
\end{aligned}$$

By Theorem 5.4 we know that this means $A(J^T)^{-1}A^TJ^T = p\,\mathrm{I}$ for some $p \in \bar{k}^*$. With minimal rearrangement and utilizing the fact that $J^T = J^{-1}$ one can easily see that this is equivalent to $J(A^T)^{-1}J^{-1} = cA$ where $c = 1/p \in \bar{k}^*$. Now since J, $(A^T)^{-1}$ and J^{-1} all reside in $\mathrm{SP}(2n,k)$ we have that $cA \in \mathrm{SP}(2n,k)$ which means that $c \in k^*$. The above steps may be reversed to obtain the other direction. □

Corollary 3. *An outer involution* $\mathrm{Inn}_{J^{-1}M}\phi$ *of* $\mathrm{SL}(2n,k)$ *coming from a symmetric or skew-symmetric bilinear form leaves* $\mathrm{SP}(2n,k)$ *invariant if and only if* $J(M^T)^{-1}J^{-1} = cM$ *for some* $c \in k^*$.

Proof. From Lemma 17 we know that the outer involution $\mathrm{Inn}_{J^{-1}M}\phi$ becomes the inner involution $\mathrm{Inn}_{J^{-1}M}$ when restricted to $\mathrm{SP}(2n,k)$. Hence, the proof will follow directly from Corollary 2 by simply replacing A with $J^{-1}M$. $\square$

We now have specific criteria to determine whether an involution of $\mathrm{SL}(2n,k)$ remains an involution when restricted to $\mathrm{SP}(2n,k)$. To actually obtain which involutions leave $\mathrm{SP}(2n,k)$ invariant we simply need to consider whether the matrices of the form $A = I_{s,t}$ and $A = L_{n,p}$ obey the criteria given in Corollary 2.

Before we analyze this situation we introduce the following notation.

$$I_{s,t,m} = \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} \text{ where } m \text{ represents the size}$$

$$I^{u,v,n}_{s,t,2n} = \begin{pmatrix} I_u & 0 & 0 & 0 \\ 0 & -I_v & 0 & 0 \\ 0 & 0 & I_u & 0 \\ 0 & 0 & 0 & -I_v \end{pmatrix}, \quad \text{where } s = 2u,\, t = 2v,\, u + v = n$$

Let's consider the inner involutions of $\mathrm{SL}(2n,k)$ which have the form Inn_A, where $A = I_{s,t,2n} = \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}$ and $s < t$. We first rewrite A as

$$A = \begin{pmatrix} I_s & 0 & 0 \\ 0 & -I_{n-s} & 0 \\ 0 & 0 & -I_n \end{pmatrix}$$

where $t = 2n - s$. If Inn_A is an involution on $\mathrm{SP}(2n,k)$ then Corollary 2 tells us that

$J(A^T)^{-1}J^{-1} = cA$ so let's consider the right hand side of this equality.

$$\begin{aligned}
J(A^T)^{-1}J^{-1} &= JAJ^{-1} \\
&= \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} I_s & 0 & 0 \\ 0 & -I_{n-s} & 0 \\ 0 & 0 & -I_n \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \\
&= \begin{pmatrix} -I_n & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & I_{n-s} \end{pmatrix}
\end{aligned}$$

which is not equal to cA for any c.

However, when s is even we may reorder the basis elements to obtain an equivalent form for $I_{s,t,2n}$, hence we have the following lemma.

Lemma 18. *If s is even then the involution* Inn_A *where* $A = I_{s,t,2n}$ *of* $\mathrm{SL}(2n, k)$ *leaves* $\mathrm{SP}(2n, k)$ *invariant.*

Proof. Suppose s is even and the involution Inn_A with $A = I_{s,t,2n} = \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}$ of $\mathrm{SL}(2n, k)$ leaves $\mathrm{SP}(2n, k)$. With a reordering of the basis we may rewrite A as

$$A = I_{s,t,2n}^{u,v,n} = \begin{pmatrix} I_u & 0 & 0 & 0 \\ 0 & -I_v & 0 & 0 \\ 0 & 0 & I_u & 0 \\ 0 & 0 & 0 & -I_v \end{pmatrix} = \begin{pmatrix} I_{u,v,n} & 0 \\ 0 & I_{u,v,n} \end{pmatrix}.$$

Hence for Inn_A to be an involution of $\mathrm{SP}(2n, k)$ Corollary 2 tells us that we need only check to see if $J(A^T)^{-1}J^{-1} = cA$. We observe that

$$\begin{aligned} J(A^T)^{-1}J^{-1} &= JAJ^{-1} \\ &= \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\begin{pmatrix} I_{u,v,n} & 0 \\ 0 & I_{u,v,n} \end{pmatrix}\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \\ &= \begin{pmatrix} I_{u,v,n} & 0 \\ 0 & I_{u,v,n} \end{pmatrix} \\ &= A. \end{aligned}$$

Therefore, by choosing $c = 1$ we can conclude that Inn_A with $A = I_{s,t,2n}$ is an involution on $\mathrm{SP}(2n,k)$. □

We now turn our attention to involutions of $\mathrm{SL}(2n,k)$ with the form Inn_A with

$$A = L_{n,p} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ p & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & p & 0 \end{pmatrix}$$

If we try to apply the Criteria of Corollary 2 directly to A then we would see that $J^{-1}(A^T)^{-1}J^{-1} \neq cA$ for any c. However, by rewriting our matrix $A = L_{n,p}$ we are able to obtain the following result.

Lemma 19. *The involutions of* $\mathrm{SL}(2n,k)$ *given by* Inn_A *where* $A = L_{n,p}$ *keep* $\mathrm{SP}(2n,k)$ *invariant and remain involutions when restricted to* $\mathrm{SP}(2n,k)$.

Proof. In order for the involution Inn_A where $A = L_{n,p} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ p & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & p & 0 \end{pmatrix}$ to leave $\mathrm{SP}(2n,k)$ Corollary 2 tells us that $J(A^T)^{-1}J^{-1} = cA$. Before considering this equality we note that with a reordering of the basis elements we may rewrite $A = L_{n,p}$ as

$$A = \begin{pmatrix} 0 & I_n \\ pI_n & 0 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} J(A^T)^{-1}J^{-1} &= \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\begin{pmatrix} 0 & I_n \\ p^{-1}I_n & 0 \end{pmatrix}\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -p^{-1}I_n \\ -I_n & 0 \end{pmatrix}. \end{aligned}$$

Hence by choosing $c = -p$ we see that $J(A^T)^{-1}J^{-1} = cA$ which tells us that Inn_A is an involution on $\mathrm{SP}(2n,k)$. □

5.7 Isomorphy Classes of Involutions of $\mathrm{SP}(2n,k)$

In this section we begin to investigate how the isomorphy classes of involutions of $\mathrm{SL}(2n,k)$ react when restricted to $\mathrm{SP}(2n,k)$. We begin by recalling that Corollary 3 gave us criteria to determine whether an outer involution of $\mathrm{SL}(2n,k)$ left $\mathrm{SP}(2n,k)$ invariant. However the following result let's us know that one of the involutions does not remain an involution when restricted to $\mathrm{SP}(2n,k)$.

Lemma 20. *The isomorphy class of outer involution of* $\mathrm{SL}(2n,k)$ *which comes from the skew-symmetric matrix $M = J_{2n}$ does not exist on* $\mathrm{SP}(2n,k)$.

Proof. We begin by recalling that Lemma 15 enables us to view the outer involution of $\mathrm{SL}(2n,k)$ coming from the skew-symmetric matrix $M = J_{2n}$ as the involution $\theta = \mathrm{Inn}_{J^{-1}J}\,\phi = \mathrm{Inn}_{\mathrm{Id}}\,\phi$, where $\phi = \mathrm{Inn}_J\,\theta$ and $\theta(A) = (A^T)^{-1}$. In addition, Lemma 16 tells us that that ϕ is the identity when restricted to G. Hence, $\theta = \mathrm{Inn}_{\mathrm{Id}}\,\phi = \mathrm{Inn}_{\mathrm{Id}}$ when restricted to $\mathrm{SP}(2n,k)$ which is not an involution. Thus, we see that there will always be one less isomorphy class of outer involutions on $\mathrm{SP}(2n,k)$. □

Lemma 21. *If two involutions* ϕ_1 *and* ϕ_2 *of* $\mathrm{SP}(2n, k)$ *are isomorphic over* $\mathrm{SP}(2n, k)$ *then they are isomorphic over* $\mathrm{SL}(2n, k)$.

Proof. The proof of this theorem is trivial. Assume two involutions ϕ_1 and ϕ_2 of $\mathrm{SP}(2n, k)$ are isomorphic over $\mathrm{SP}(2n, k)$ via some automorphism $\psi = \mathrm{Inn}_A$ with $A \in \mathrm{SP}(2n, k)$. It is clear that $A \in \mathrm{SL}(2n, k)$ and thus ϕ_1 and ϕ_2 are isomorphic over $\mathrm{SL}(2n, k)$ via $\psi = \mathrm{Inn}_A$ too. $\square$

Lemma 22. *Let* τ_1 *and* τ_2 *be two involutions on* $\mathrm{SP}(2n, k)$ *which come from the restriction of outer involutions of* $\mathrm{SL}(2n, k)$. *If* $\tau_1 \approx \tau_2$ *over* $\mathrm{SP}(2n, k)$ *then the outer involutions of* $\mathrm{SL}(2n, k)$ *from which they came are isomorphic over* $\mathrm{SL}(2n, k)$.

Proof. Let τ_1 and τ_2 be two involutions on $\mathrm{SP}(2n, k)$ which come from outer involutions of $\mathrm{SL}(2n, k)$, say $\tau_1 = \mathrm{Inn}_{J^{-1}M_1}\phi|_G = \mathrm{Inn}_{J^{-1}M_1}$ and $\tau_2 = \mathrm{Inn}_{J^{-1}M_2}\phi|_G = \mathrm{Inn}_{J^{-1}M_2}$, where M_1 and M_2 are symmetric and $\phi(A) = J^{-1}(A^T)^{-1}J$. Suppose $\tau_1 \approx \tau_2$ over $\mathrm{SP}(2n, k)$, i.e. $\mathrm{Inn}_{J^{-1}M_1} \approx \mathrm{Inn}_{J^{-1}M_2}$ over G. Then there exists an inner automorphism Inn_Q with $Q \in \mathrm{SP}(2n, k)$ such that

$$\begin{aligned}\mathrm{Inn}_{J^{-1}M_2} &= \mathrm{Inn}_{Q^{-1}}\mathrm{Inn}_{J^{-1}M_1}\mathrm{Inn}_Q \\ &= \mathrm{Inn}_{QJ^{-1}M_1Q^{-1}}\end{aligned}$$

Therfore, by Theorem 5.4 we have $\mathrm{Inn}_{(J^{-1}M_2)Q(J^{-1}M_1)^{-1}Q^{-1}} = \alpha\,\mathrm{I}$ for some $\alpha \in \bar{k}^*$. This means that

$$\begin{aligned}(J^{-1}M_2)Q(J^{-1}M_1)^{-1}Q^{-1} &= \alpha\,\mathrm{Id} \\ J^{-1}M_2 &= \alpha Q(J^{-1}M_1)^{-1}Q^{-1} \\ M_2 &= \alpha JQ(J^{-1}M_1)^{-1}Q^{-1}\end{aligned}$$

Now since $Q \in \mathrm{SP}(2n, k)$, $Q = J^{-1}(Q^T)^{-1}J$, we see that

$$\begin{aligned} M_2 &= \alpha J Q (J^{-1}M_1)^{-1}Q^{-1} \\ &= \alpha J(J^{-1}(Q^T)^{-1}J)(J^{-1}M_1)^{-1}Q^{-1} \\ &= \alpha (Q^T)^{-1}M_1 Q^{-1}. \end{aligned}$$

Which means that $M_1 \cong^s M_2$ over k. By Theorem 4.1 their corresponding outer involutions are isomorphic, i.e. $\tau_1 = \mathrm{Inn}_{J^{-1}M_1}\phi \approx 0\tau_2 = \mathrm{Inn}_{J^{-1}M_2}\phi$. □

The converse of Lemma 21 and Lemma 22 are not true in general. That is two involutions ϕ_1 and ϕ_2 of $\mathrm{SP}(2n, k)$ being isomorphic over $\mathrm{SL}(2n, k)$ does not imply that they remain isomorphic when restricted to $\mathrm{SP}(2n, k)$.

5.8 Future Goals

Isomorphy classes of involutions of $\mathrm{SP}(2n, k)$ have been classified over algebraically closed fields and the real numbers (see [Hel88]). In the future we wish to classify the isomorphy classes of involutions of $\mathrm{SP}(2n, k)$ over finite fields, the p-adic numbers, and number fields. We have already determined which involutions of $\mathrm{SL}(n, k)$ leave $\mathrm{SP}(2n, k)$ invariant and actually remain involutions when restricted to $\mathrm{SP}(2n, k)$. Therefore, to give a complete classification of the involutions of $\mathrm{SP}(2n, k)$ the next step is to determine how many $\mathrm{SP}(2n, k)$-isomorphism classes each $\mathrm{SL}(2n, k)$-isomorphism class splits.

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