ABSTRACT

NEWTON, GREGORY ANSELM. Heisenberg Quantum Representation of the Bianchi I Cosmological Model. (Under the direction of Professor Arkady Kheyfets)

A Bianchi-I type cosmological model is considered. A Hamiltonian function for the Bianchi geometry is developed from differential geometric methods and general relativity. The Hamiltonian function allows the model to be analyzed via dynamical methods of classical mechanics. The system is then quantized by way of the the usual methods of transition from classical mechanics to quantum mechanics.

The motivation to express the system in the Heisenberg representation stems from the Heisenberg equations of motion being more closely parallel to the classical equations. This correspondence is largely due to dynamical time-dependent quantum mechanical operators consisting of derivative or matrix operators fashioned from classical observables that are considered to be time dependent in the classical theory. In the Heisenberg picture the time-dependent operators corresponding to classical observables act on a state vector in Hilbert space which is not time-dependent.

In contrast, the Schrodinger representation expresses evolution by means of application of non-dynamic non-time-dependent operators which act on a dynamic evolving time-dependent wave function. The correspondence is somewhat faulty because in the Schrodinger picture time-independent operators replace time-dependent classical observables.

It is important to consider Heisenberg evolution because the concepts of time in relativistic space-time geometric systems become difficult to analyze in a consistent fashion because of Lorentz transformations. An interpretation of a global time function with respect to local measurements of time should be consistently and logically related, and made to coincide with the fact that measurements of such observables as local times must be performed within the system itself. This is a unique problem in quantum cosmology because in usual quantum mechanical systems the observer is defined to be external to the system being analysed.

HEISENBERG QUANTUM REPRESENTATION OF THE BIANCHI-I COSMOLOGICAL MODEL

BY

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Dedication

This work is dedicated to those that I love, especially my daughter Josi and my mother Mary, but also all of my brothers and sisters.

Biography

The author of this work was born in Georgetown Virginia and attended elementary and high school in Louisville Kentucky. He then earned a Bachelor's of Science degree in physics at the University of Washington in Seattle, and a Master's of Science in Physics at the University of Louisville. He has a delightful eight year old daughter. He now lives in Raleigh North Carolina and is teaching mathematics at Randolph Community College in Asheboro, North Carolina. He became interested in cosmology when introduced to the Hubble redshift-distance relation as an undergraduate at the University of Washington.

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My advisor Professor Arkady Kheyfets has been more helpful than he will ever know, simply because he embodies a naturally and inherently helpful personality. He has been a very positive and helpful force in my life for a while now in ways that really count, and also I have observed him helping others in big ways and small, even those he doesn't even know. "Awesome" is the only way to describe his understanding of mathematics and physics. It has been also been a great pleasure to have met the other excellent professors that are assisting me by being on the committee, Professors David Brown, Hans Hallen, and Chung Ji. My own personal growth has been very much enhanced by way of knowing these admirable individuals, and I believe that I have changed for the better as a result. I would like also to emphasize that I am very deeply indebted to the help of Professor Micheal Paesler, Jenny Allen, Steven Jenkins, and Brenda Johnson. Of great assistance, as well as being a great friend since coming to North Carolina State is Mr. Samuel Acha. And I owe a debt of gratitude to good friends who are not associated with the university, but who have given help and encouragement.

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Chapter 1

Introduction

The quantum mechanical state of a system is often pictured as a ray in Hilbert space that has been normalized to unit length. The Hilbert state space is spanned by a set of basis vectors, each representing some particular base mode of the system. Operations (measurements) performed on the system will rotate the unit state vector with respect to the spanning basis into some particular mode corresponding to a measurement in the Schrodinger formulation of quantum mechanics. In order to analyse possible measurements on the system, the original Hilbert space basis must be transformed into a different set of basis functions whose eigenvalues are possible measurement values. Such differing base representations are realized as different phase plane coordinates, such as position representation or momentum representation.

In the Schrodinger picture, the state vector of the system is considered to have a direction in the Hilbert space as a function of time. That is, a time development prop-

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agation operator will rotate the direction of the state vector in Hilbert space away from the initial state direction. However, in the Heisenberg representation the basis vectors are functions of time, and their phases will advance at a rate proportional to the energy of that particular basis state. The picture is changed from that of an "active" one, where the state vector rotates as a function of time, to a "passive" one, where the state vector remains static but the coordinate system will rotate (phase at a rate proportional to the energy of the state) underneath it. The initial state vector in the Heisenberg representation is regarded as being fixed, but with basis vectors having temporal dependence. The Heisenberg equations of motion can be transferred more readily from classical equations of motion. It is the operators (such as the position operator) corresponding to the observables that change in time, due to the fact that operators are fashioned as linear combinations of the phasing Heisenberg basis vectors. For example, the quantum mechanical momentum operator corresponding to a free particle has a time dependence associated with it via the momentum base kets. The operator is formed from a linear combination of base kets, all which phase in time in proportion to the energy of the particle.

In standard quantum mechanics, usually a statistical wave function as a function of position coordinates is asked for, and usually on a flat Euclidean space or even possibly on a curved non-Euclidean space, but still a function of the coordinates of position within the geometry. What's more, measurements on the system are usually made from devices exterior to the system itself.

However, in considering the dynamics of a closed spatial geometry G, the approach is different than the approach to the usual quantum mechanical examples. In writing the equations for a dynamic geometry spatial geometry, $G(t)$ is the spatial system itself, and so it is difficult to consider a statistical wave function for G as a function of a coordinatized background geometry. And what's more, measurements on G must be obtained from within G because there is no points outside of G from which to perform measurements. More abstract degrees of freedom are available for some dynamic geometries, and analyses may be done in abstract symplectic spaces coordinatized by these degrees of freedom. Furthermore, it is convenient to incorporate abstract temporal parameters because the concept of time within generally curved compact dynamic geometries is highly dependent on localized Lorentz frames of reference and local curvature conditions, and not on G as an autonomous system. The state of a cosmological geometrodynamic hypersurface system Σ may be evolved within phase planes made up of these more fundamental or abstract degrees of freedom and their conjugate momenta degrees of freedom. The evolution may be propagated with respect to a more fundamental holistic temporal parameter, and the problem of local time within Σ may be left to local Lorentz covariance, and an absolute cosmological time measured by cosmological parameters perhaps such as overall curvature or degree of symmetry. Some very intriguing methods of representing local times and global times have been developed by Brown, York, Kuchar, Kheyfets, and others. [5, 22, 4, 28]

Quantum mechanical analyses may be done following guidelines inferred from better

known classical prototypes and their quantizations [11]. The Heisenberg representation is much more readily adapted from classical mechanics than the Schrodinger representation due to the similarity of the Heisenberg equations of motion to the classical equations of motion. The idea of a dynamical spatial geometry is very surprising and intriguing, not to mention somewhat problematic. However, the theory goes a long way and is consistent up to certain problems, mostly those concerning definitions of time.

Chapter 2

A Local Picture

Some tools and objects from differential geometry will enable development of results. It is known that, on the macroscopic scale, the universe exists within a four-dimensional manifold continuum called space-time which admits additional structure of a Lorentz pseudo-Reimannian metric function. That is, the metric function has a signature (-+++)=2 which admits a light cone at all non-singular four-points (ordered quadruplets).

However, when introducing Lagrangian and Hamiltonian methods into cosmology it is convenient to divide the spacetime 4-manifold up into a strata of spatial 3-manifolds which are foliated in time one on top of the other, parameterized by and thus labeled by some time-like parameter. This method is referred to as the Arnowitt-Deser-Misner approach to general relativity. The foliations are defined such that each 3-dimensional hypersurface contains all the spatial points (along with their topology and curvature) that are associated with a certain common time parameter in spacetime. That is, they

comprise a surface of simultaneity.

On the local spacelike scale, we may think of a local 3-d neighbourhood U which is small enough so that it may be considered a local flat part of a larger 3-region that is possibly curved. Let us call the curved 3-surface by the Greek letter Σ. U is the generator of and subset of a flat Euclidean 3-space which is the tangent space to Σ at a 3-point q. The tangent 3-space may be represented as $T_q\Sigma$, where $q \in \Sigma$ As discussed in the previous paragraph, a 3-manifold Σ may be parameterized as one of a series of stratifications labeled uniquely as $\Sigma(t)$, which fill out a 4-manifold spacetime which may be called M. $T_q\Sigma$ may be thought of as being (in general) "tilted" with respect to the arrow of time at that event point. This is because of the possible uneven propagation of time at varying points, i.e., the concept of "many fingered time". The construction of surfaces of simultaneity is sometimes called a "foliating" or "laminating" approach to filling out M, as opposed to a "threading" approach, which fills out M by a set of timelike 1-curves, each intersecting Σ at a different 3-point. Such one-dimensional curves can be parameterized by writing each timelike curve as $\gamma(q)$, where $q \in \Sigma$. These curves remain unique, but may meet and/or become undefined at borders of singularities. As long as the curves remain separate and do not cross, the coordinatization will be diffeomorphic.

It will be helpful to say a few words here about diagrams and figures. We have at our disposal three spatial dimensions available in which to picture configurations. Such spatial configurations changing in time can be thought of as being embedded in a four dimensional spacetime continuum. For pictorial purposes it is convenient to suppress one

spatial dimension in order to borrow it for use as an imitation of the temporal dimension. The three spatial dimensions in which we are used to envisioning things are only an introduction to geometry, topology, and mathematics in higher dimensions. Importantly however, there are just enough dimensions to observe, analyse, and classify some key concepts which become important in descriptions of higher dimensions. In three spatial dimensions it is possible to draw and envision the ideas of curvature, surfaces, vectors, embeddings, duality, compactness, and so forth.

2.0.1   Normal vectors and the spatial metric

Historically, normal vectors to surfaces and their derivatives have been employed extensively in differential geometry [43], and in general relativity this still holds true. It is convenient to construct a unit vector $\mathbf{n}$ [1] which is normal to a 3-surface Σ. (Please refer here to Figure 2.1.) However, a unit one-form is a more primitive object and could possibly be represented in a notation such as $n_\mu \sim (1, 0, 0, 0)$. Such an object may be thought of as a stack of copies of the flat local neighborhood 3-surface U, with one unit of time between the copies, or as the curl or gradient $\mathbf{d}t$ of the time function t. Lorentz geometry allows a metric function as additional structure in the manifold, and a unit timelike normal vector may be created from an inner product of the metric function and the unit normal one form. That is, the 1-form may be transformed into a contravariant vector by an inner product with the metric tensor. The representation of the vector will

---

[1]A boldface letter here represents a multidimensional tensor object with one or more indices. Suppressed is information about if the object is contravariant or covariant, vector-like or form-like. See [48] Chapters 1 and 2.

in general be more complicated than the unit normal 1-form due to the action of the

metric function. However, since **n** is timelike as well as unit length, it is always true that

$$n_a n^a = -1 \qquad\qquad (2.1)$$

In the threading approach, the contravariant normal vector n^μ may be thought of as

a tangent vector to a curve which is normal to Σ at a point q. In fact, the normal vector

is actually an equivalence class of all curves which are normal to Σ at q. (The unique

one of these curves which is a geodesic is the path in spacetime which test matter would

follow.) Such curves may be represented in a fashion $\phi(\tau)$, where ϕ represents the set of

points in spacetime making up the curve. τ is the affine parameter of the curve, which

may be thought of as either the proper time of an observer following the curve, or else

an affine function of the original parameter such as

$$\tau_1(\tau) = a_0 \tau + b_0. \qquad\qquad (2.2)$$

Note that an affine function composed of an affine function is still a member of the class

of affine functions by its definition.

The geodesic threading curves will enable one type of mapping of points from one

3-surface to the next. As long as the geodesics do not intersect or trade identities,

the mapping is called a diffeomorphic type of one-to-one mapping. If the topology of

the manifold is retained, the mapping is homeomorphic. Analysis of such geodesics are

important for determining singularities in the spacetime 4-manifold. Incorporated in such analyses are concepts such as limiting points, open and closed set theory, compactness, boundedness, etc. Analysis is continued up until a singular region is encountered, which is sometimes represented as a point or a region itself, although analysis is impossible there. Analysis may show that the singularity is not physical, only mathematical, and may be transformed away.

On extremely small scales of the order of the Planck Length, quantum analysis becomes important, the classical limit being the point sets in spacetime such as geodesic curves. See [48], chapter 43. A new physical world becomes evident at very small spatial and temporal scales, out of which many new theories are developing. Much about new physical theories corresponding to astrophysical phenomena and observations which couple to the quantum realm may be followed up through publications by Mottola et al. A very interesting theory concerning astrophysical compact objects and quantum mechanics melding with general relativity at extreme conditions may be investigated through the publications of this group of workers, as well as references given therein. [26, 25]

A very useful doubly covariant metric-like operator [2] may be formed from the metric and the unit normal vector, as;

$$h_{ab} = g_{ab} + n_a n_b. \tag{2.3}$$

---

[2]Here we are employing Wald's convention [46] for the early lower case Latin indices being covariant metacoordinate input slots. For the functional form of a tensor object in a given coordinate system, lower case Greek letters are usually employed for referral to all four spacetime indices, and lower case Latin letters such as i, j, k are usually employed for the three spatial coordinates in a 3-surface.

This important tensor is known as the induced 3-dimensional Reimannian metric tensor on the spatial 3-surface, or simply the induced 3-metric. One might suppose that adding the dyadic-like object $n_a n_b$ would add structure to the 4-metric g instead of subtracting out the timelike part, but the negative Lorentz signature in the timelike direction requires that a plus sign be used.

A quick calculation will show that the inner product of $\mathbf{h}$ and the normal vector $\mathbf{n}$ is indeed zero, showing orthogonality and normality of $\mathbf{n}$ to the 3-surface. Furthermore, $\mathbf{h}$ performs as the metric tensor in a 3-surface which is orthogonal to $\mathbf{n}$. This 3+1 arrangement is called the *synchronous system*, and is an important method for decomposing the Einstein field equations into a $3 + 1$ separable and tractable form.

Raising one index of h_{ab} with the full 4-metric g^{bc} will form the so-called *projection operator* h_a^c, which will operate such that an inner product of any vector object with h_a^c will eliminate the timelike component of the vector parallel to $\boldsymbol{n}$, and an inner product on each and every index of a tensor will eliminate all the tensor's components parallel to $\boldsymbol{n}$. The projection operator is idempotent in that subsequent inner products over the same index will not do any further subtractions to the tensor, the component parallel to $\mathbf{n}$ being already eliminated, i.e., $h_a^c h_c^d = h_a^d$. The representation of the projection operator is

$$h_a^b = \delta_a^b + n_a n^b \tag{2.4}$$

This operator has general utility in differential geometry in that any unit vector may be used instead of $\mathbf{n}$ (e.g. the 4-velocity $\mathbf{u}$),and the surface orthogonal to that arbitrary

Figure 2.1: A local picture of neighboring hypersurfaces, normal vector, normal vector times lapse function, timelike bridging vector, shift vector, and light cone.

vector may be considered as having a similar submetric and projection operator. However, a minus sign instead of a plus sign must be used if the metric is Reimannian as opposed to Lorentzian, i.e., because of eq. 2.1

2.0.2   Lapse Function and Shift Vector

If one is considering a hypersurface Σ and does a "wait" operation, then a 4-d region is swept out with two 3-surfaces as the boundary of the 4-region. Globally, there can be topological identifications and connectivity assigned to the boundary point sets. Locally however, the 4-region generated is sometimes called a "thin sandwich". It being thin enables a calculus type of analysis to first order. The "rigging vector" $\mathbf{t}$ shown in Figure 2.1 can be thought of as a local generator of a diffeomorphism in that it is able to map a point q in the first 3-surface to another point in the 4-manifold, but which could be

referred to as the same point q in the 3-surface at a later time. Obviously, a test mass following along **t** would be subject to the Lorentz transformation with respect to the light cone having the normal vector **n** as its axis, and the time lapse along **t** would be the proper time of the test mass.

The timelike distance along **n** from one 3-surface to the next is defined as the *lapse*, which is usually a parameter given to describe a certain physical configuration, and is given as the *lapse function N*. It may take on various mathematical representations depending on the geometry involved. The vector $N\boldsymbol{n}$ will take one from one 3-surface to the next with no Lorentz transformation with respect to the light cone (along the axis), while emitting normal from the first 3-surface. When an action integral is formed from the curvature invariants of the theory, the lapse function plays the role of a Lagrange multiplier on the Hamiltonian constraint function. This is possible because the Hamiltonian constraint is constrained to zero due to the Bianchi identities on the geometrical side of the Einstein field equations. However, this in turn creates a serious problem for quantum gravity because it eliminates the generator in the time direction (the "problem of time"). On the source side of the Einstein field equations, the same constraint is a conservation statement. Usually, some specific lapse function is initially written in an analysis as an ansatz, wherefrom other results follow. We will see that the lapse N enters into the metric tensor matrix, but in the time-time entry of the matrix.

To span a 4-space, three more functions are required, and these are given by the three components N^i of the spatial 3-vector **N**, the *shift* vector. As can be seen in Figure 2.1,

the spatial projection of **t** is given by $\boldsymbol{N}$. In fact the definitions of lapse and shift are given by Wald as [46]

$$
\begin{aligned}
N &= -t^a n_a \\
&= (n^a \nabla_a t)^{-1} \\
N_a &= h_{ab} t^b.
\end{aligned}
\tag{2.5}
$$

The first order diffeomorphisms generated by the fields of lapse and shift are arbitrary until provided by a model, and may include rotations, shears, scaling, etc, as long as the fields are continuous and provide one-to-one mappings.

Consider a differential quantity $\boldsymbol{N} dt$ which will be the timelike length along the normal vector $\boldsymbol{n}$ emitting from the lower hypersurface (labeled by time t) to the upper hypersurface of the subsequent foliation (labeled by time $t + dt$). This construction is known as a "thin sandwich", with differential thickness. Now consider a completely spatial contravariant vector-like object $\omega^i = dx^i + N^i dt$ which exists in the lower hypersurface of the thin sandwich. The shift vector $\boldsymbol{N}$ allows a recoordinatization from one hypersurface to the next. For example, let the actual changes in coordinates dx^i for all coordinates x^i be zero. Now, because $dx^i = 0$, this implies that the point with coordinate label $\boldsymbol{x}$ is then displaced on the next hypersurface by a spatial distance $\boldsymbol{N} dt$. Thus, the point $\boldsymbol{x}(t+dt)$ is not directly above the point $\boldsymbol{x}(t)$ along the normal direction. Therefore, if the shift vector $\boldsymbol{N}$ is zero, both points on the upper and lower hypersurfaces connected along the normal direction may be called $\boldsymbol{x}$ and movements to points with other coordi-

nate labels will be represented by the usual dx^i. The shift vector should be such that a

diffeomorphic mapping occurs.

Letting the lapse function times the normal vector $N dt$ be the "vertical" and $\omega^i =$

$dx^i + N^i dt$ be the "horizontal", the Lorentz Pythagorean theorem may be written as

$$ds^2 \quad = \quad -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{2.6}$$

Multiplying out this equation and collecting terms will yield the matrix representation

of the covariant metric array $g_{\mu\nu}$;

$$g_{\mu\nu} = \begin{bmatrix} -(N^2 - g_{ij}N^i N^j) & g_{1j}N^j & g_{2j}N^j & g_{3j}N^j \\ g_{1j}N^j & g_{11} & g_{12} & g_{13} \\ g_{2j}N^j & g_{21} & g_{22} & g_{23} \\ g_{3j}N^j & g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{2.7}$$

Note that the contravariant values for the shift vector are expressed within the covariant

metric tensor components. It is formulated in this fashion because it is the contravariant

shift components which are written down as an ansatz in accordance with specifying a

particular model. Solving $g_{\alpha\nu}g^{\nu\beta} = \delta_\alpha^\beta$ for the contravariant metric, which is the inverse

of the covariant metric gives

$$
g^{\mu\nu} = N^{-2}
\begin{bmatrix}
-1 & N^1 & N^2 & N^3 \\
N^1 & g_{11} - N^1 N^1 & g_{12} - N^1 N^2 & g_{13} - N^1 N^3 \\
N^2 & g_{21} - N^2 N^1 & g_{22} - N^2 N^2 & g_{13} - N^1 N^3 \\
N^3 & g_{31} - N^3 N^1 & g_{32} - N^3 N^2 & g_{33} - N^3 N^3
\end{bmatrix}
. \qquad (2.8)
$$

A covariant unit normal and its dual contravariant vector may be fashioned from the lapse and shift and the existence of the metric as

$$
\begin{aligned}
n_\mu &= (-N, 0, 0, 0) \\
n^\mu &= N^{-1}(1, -N^1, -N^2, -N^3) \qquad (2.9)
\end{aligned}
$$

(See [48], chapter 21.) The lapse function and shift vector functions are functions which are usually supplied in a particular analysis, i.e., they are functions supplied in order to characterize the type of model being analysed. For instance, $N = 1$ cosmology is a group of models that encompasses much of cosmological theory. It will be seen later on that the lapse and shift quantities serve as Lagrange multipliers in the action integral, and which scale constraint surfaces in the dynamic phase space.

Chapter 3

A Semi-Local Picture

In this chapter will be developed some geometric objects and structures from around the local neighbourhood of a point. Changes in geometrical constructions and fields with respect to small changes in position around the point will be considered. New objects and constructions may be derived from primary objects that reflect changes in position in space and spacetime. This implies the differentiation of some of the locally defined objects introduced in the previous chapter.

3.0.3 Extrinsic Curvature

Recall the normal vector, or rather, its more primitive covariant representation

$$\mathbf{n} = -N\mathbf{d}t = (-N, 0, 0, 0),$$ (3.1)

where boldface symbols are used to represent covariant 1-form objects. Since we are exploring the neighbourhood, consider now the covariant derivative of this normal form; $n_{\mu;\nu}$. However, we will only need the part of this tensor which is in the 3-surface, which is called the *extrinsic curvature tensor*, also called the *second fundamental form* K_{ij}. Because $boldsymbol K$ is 3 by 3, let us use only Latin indices as,

$$
\begin{aligned}
K_{ij} &= n_{i;j} \\
&= n_{i,j} - \Gamma^{\mu}_{ij} n_{\mu} \\
&= -N\Gamma^{0}_{ij},
\end{aligned}
\tag{3.2}
$$

where we have used the fact from definition 3.1 that $n_{\mu} = -N\delta_{0\mu}$. Notice that the curl of $\mathbf{n}$ is

$$
\begin{aligned}
\mathbf{dn} &= -\mathbf{d}N\mathbf{d}t \\
&= -N_{,i}\mathbf{d}t \wedge \boldsymbol{\omega}^{i} \\
&= -N_{,i}\mathbf{d}t \wedge (\mathbf{d}x^{i} + N^{i}\mathbf{d}t) \\
&= -N_{,i}\mathbf{d}t \wedge \mathbf{d}x^{i},
\end{aligned}
\tag{3.3}
$$

which shows the derivative of $\boldsymbol{n}$ having no antisymmetric spatial parts, i.e. terms such as in $\boldsymbol{d}x^{i} \wedge \boldsymbol{d}x^{j}$. This has the implication that the connection Γ^{0}_{ij} is symmetric with respect to the covariant spatial indices. Furthermore, $\mathbf{K}$ is contained in the hypersurface tangent space because of its obvious orthogonality to $\boldsymbol{d}t$. And because the derivative of $\boldsymbol{n}$ being

symmetric, the proportionality of $\boldsymbol{K}$ to the derivative of $\boldsymbol{n}$ implies that

$$K_{ij} = K_{ji}. \tag{3.4}$$

Notice further that the linear combination of spatial forms which expand the spatial derivative $N_{,i}$ of the lapse defines a preferred direction in 3-space. This is important in that it shows the spatial anisotropy in general of "many-fingered time" [48].

It is oftentimes convenient to have an expression for $\boldsymbol{K}$ in terms of the lapse and shift and hypersurface 3-metric. Recall that the lapse, shift, and 3-metric g_{ij} make up the spacetime 4-metric according to eqs. 2.7 and 2.8. Also recall the expression for the connection in terms of the first partial derivatives of the metric;

$$\Gamma_{\beta\mu\nu} = \frac{1}{2}[g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}]. \tag{3.5}$$

Inserting the metric in terms of lapse and shift into the expression for the connection and taking derivatives gives an expression for $\boldsymbol{K}$ (via equation 3.2) as

$$K_{ij} = \frac{1}{2N}[N_{i,j} + N_{j,i} - g_{ij,0} - 2\Gamma_{kij}N^k]. \tag{3.6}$$

Recalling that the covariant derivative of a tensor within the 3-surface is written with a stroke instead of semicolon, K is given as

$$K_{ij} = \frac{1}{2N}[N_{i|j} + N_{j|i} - \frac{\partial}{\partial t}g_{ij}]. \tag{3.7}$$

It is illustrative to construct the extrinsic curvature from a Lie derivative, which is a very

pictorial and geometrical construction. (See [6]) Taking the Lie derivative of the metric

with respect to the normal vector field gives

$$
\begin{aligned}
(\mathcal{L}_{\mathbf{n}}\mathbf{g})_{\mu\nu} &= g_{\mu\nu,\beta}n^{\beta} + g_{\beta\nu}n^{\beta}_{,\mu} + g_{\mu\beta}n^{\beta}_{,\nu} \\[1ex]
&= g_{\mu\nu,\beta}n^{\beta} + (g_{\beta\nu}n^{\beta})_{,\mu} - g_{\beta\nu,\mu}n^{\beta} + (g_{\mu\beta}n^{\beta})_{,\nu} - g_{\mu\beta,\nu}n^{\beta} \\[1ex]
&= n_{\mu,\nu} + n_{\nu,\mu} - n^{\beta}[g_{\beta\nu,\mu} + g_{\beta\mu,\nu} - g_{\mu\nu,\beta}] \\[1ex]
&= n_{\mu,\nu} + n_{\nu,\mu} - 2n^{\beta}\Gamma_{\beta\mu\nu} \\[1ex]
&= n_{\mu;\nu} + n_{\nu;\mu} \\[1ex]
&= 2n_{\mu;\nu}.
\end{aligned}
$$

$$(3.8)$$

Putting two projection operators, one on each index, of this equation shows that, accord-

ing to eq. 3.2;

$$K_{ij} = \frac{\mathcal{L}_{\mathbf{n}}g_{ij}}{2}. \tag{3.9}$$

The Lie derivative is illustrative in that the extrinsic curvature is essentially the

change of the metric tensor along the normal to the 3-surface, where the normal is the

direction of the axis of the local light cone. It is interesting that the change in the metric

tensor with respect to the timelike normal is determined by the change in the normal

vector itself. The extrinsic curvature may be decomposed into a group of three different

motions with respect to the normal vector;

$$K_{ij} = \frac{1}{3}\theta h_{ij} + \sigma_{ij} + \omega_{ij}. \tag{3.10}$$

The first term represents expansions or contractions, which may also be referred to as conformal or scale transformations. θ is the conformal factor, which multiplies the usual spatially induced metric tensor, and is defined by

$$\theta = K_i^i = K_{ij}h^{ij}. \tag{3.11}$$

The second term σ_{ij} is the term which represents the shear, and is defined by

$$\sigma_{ij} = K_{(ij)} - \frac{1}{3}\theta h_{ij}, \tag{3.12}$$

that is, the symmetric part of the extrinsic curvature tensor with the conformal transformation taken out. The third term is called the twisting or torsion term, and is defined by

$$\omega_{ij} = K_{[ij]}, \tag{3.13}$$

that is, the antisymmetric portion of the extrinsic curvature. However, the ADM approach is such that the coordinates are adapted to layered or stratified 3-surfaces that are non-intersecting, also termed synchronous. In order to have a well-defined normal vector everywhere on the hypersurfaces the torsion must be equal to zero, meaning in

turn that the antisymmetric part of the extrinsic curvature must vanish. That is,

$$\omega_{ij} = K_{[ij]} = 0, \tag{3.14}$$

The extrinsic curvature is the second fundamental form, or more specifically in this case, the second fundamental form of the spatial hypersurface with respect to its embedding in the 4-manifold. That K is proportional to the Christoffel connection symbol Γ in equation 3.2 implies that it may be employed as a matrix valued derivative operator, which usually enters as a second corrective term for curved spaces in some covariant differentiation.

Chapter 4

Extending the Scope

We would now like to extend the scope of the neighbourhood and allow some nice simplifications (symmetries) which will allow a tractable dynamic cosmological model.

4.0.4 Isometries

If one considers an oscillating soap bubble, it would have a general shape of a 2-sphere, but it could have waves traveling or standing on it such that many hills and valleys would exist on the bubble. The dynamics of the bubble could be analyzed by computing the curvature of the 2-surface at each point and writing some differential equations which would require initial position (shape) and velocity data. An arbitrary bubble would have many degrees of freedom and would require numerical analysis to predict the future shape and velocities. However, if the bubble admits some symmetries then the degrees of freedom would be reduced to a few variables. For example, the initial data could be

a cigar shape with velocity zero, the shape changing in time to an oblate spheroid. In a case like this the surface would have rotational symmetry about an axis, say the z axis. Furthermore, there would be a vector field $\partial/\partial\phi$ tangent to curves circling the z axis. Since the curvature of the surface is constant along this vector field, the metric tensor is constant as well. Therefore, the vector field $\partial/\partial\phi$ is called a *Killing vector field*, and is the *generator of an isometry*.

Isometries and symmetries are used in cosmology to be able to write the metric as a function of position, which globalizes the field equations, and gives many quantities an integrable form. After all, cosmology is the most global of sciences. The rationale for using the concept stems from the *cosmological principle*, which asserts that on the grand cosmological scale, the density of galaxies and matter appears as a fluid or dust. Even further, letting the limit of the density going to zero would render the galaxies as test matter in a dynamic gravitational field or geometry (a vacuum model). Einstein's equivalence principle was arrived at by by considering the acceleration of a particle in a gravitational field, and taking the limit of the acceleration as the mass goes to zero, leaving only a 1-dimensional timelike curve in spacetime.

If the Lie derivative of the metric tensor $\boldsymbol{g}$ is taken with respect to a vector field $\boldsymbol{\xi}$, and the result set equal to the zero tensor, an equation is left for $\boldsymbol{\xi}$ which is referred to as the *Killing equation* for $\boldsymbol{\xi}$. Just insert $\boldsymbol{\xi}$ into the above equation 3.8 instead of the normal vector $\boldsymbol{n}$, and a similar equation will result;

$$(\pounds_{\boldsymbol{\xi}}\boldsymbol{g})_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}. \tag{4.1}$$

Demanding that g does not change (Lie derivative is zero) along the vector field ξ gives

Killing's equation;

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} \overset{set}{=} 0. \tag{4.2}$$

Given g and taking its derivatives, one can solve this equation for all Killing vector fields

(KVF's) ξ that exist in a manifold, and can be assured that the metric will be constant

along these special vector fields. That is, the Killing vector fields (or the curves to which

they are tangent) describe the isometries of the manifold.

4.0.5 Isometry Group

In a 3-manifold it is known that there are at most $3(3 + 1)/2 = 6$ linearly independent

KVF's (See [46], appendix C). It is also easily proved that, for two linearly independent

KVF's ξ and η, that

$$\pounds_{[\xi,\eta]} g_{\mu\nu} = 0_{\mu\nu}. \tag{4.3}$$

Where the commutator of the two KVF's is defined as

$$
\begin{aligned}
[\xi, \eta]^\mu &= (\pounds_\xi \eta)^\mu \\
&= \eta^\mu_{,\delta} \xi^\delta - \xi^\mu_{,\delta} \eta^\delta. \tag{4.4}
\end{aligned}
$$

The fact that there are only at most 6 KVF's in the 3-manifold, and that the commutator

of two KVF's is also a KVF, it is recognized that the set of linearly independent KVF's

make up a closed set$\{\xi_s\}$ of number less than or equal to 6, and have an algebraic

relationship such that

$$[\boldsymbol{\xi}_r, \boldsymbol{\xi}_t] = C_{rt}^s \boldsymbol{\xi}_s. \tag{4.5}$$

The set of constants C_{rt}^s are known as the *structure constants* of the Lie algebra derived

from the Lie *invariance group* of the 3-manifold type, and are obviously antisymmetric in

two lower indices. Commutators in general also satisfy another algebraic identity which

completes the group algebra, usually referred to as the *Jacobi identities*;

$$[[\boldsymbol{\xi}_r, \boldsymbol{\xi}_s], \boldsymbol{\xi}_t] + [[\boldsymbol{\xi}_s, \boldsymbol{\xi}_t], \boldsymbol{\xi}_r] + [[\boldsymbol{\xi}_t, \boldsymbol{\xi}_r], \boldsymbol{\xi}_s] = 0, \tag{4.6}$$

which implies that, via eq. 4.5

$$C_{mn}^p C_{pq}^u + C_{nq}^p C_{pm}^u + C_{qm}^p C_{pn}^u = 0. \tag{4.7}$$

The set of structure constants are tensor class objects. The linearly independent

KVF's may be transformed locally into a set of unit vectors which span the 3-space and

have similar commutation relations. If the group dimension is larger than the dimension

of the space to be spanned, then the situation is termed *multiply transitive*. If the group

dimension equals the dimension of the space to be spanned, it is termed *simply transitive*.

The possible Lie symmetry groups for a 3-dimensional homogeneous manifold embed-

ded into 4-dimensional Lorentz signature spacetime have been worked out and tabulated,

and the structure constant sets solved for which correspond to several different possible

3-manifolds corresponding to their respective Lie group.

However, a convenient decomposition of the structure constants into irreducible terms has been worked out [18], and eventually allows for a diagonalization of the metric tensor itself. Because of the antisymmetry of the two lower indices, the structure constants may be dualized with the totally antisymmetric tensor ϵ into a doubly contravariant tensor density, and splits into a symmetric and an antisymmetric term, i.e., one writes

$$
\begin{aligned}
C^{ab} &= \frac{1}{2} C^a_{cd} \epsilon^{bcd} \\
&= C^{(ab)} + C^{[ab]} \\
&\stackrel{define}{=} n^{ab} + \epsilon^{abc} a_c,
\end{aligned}
\tag{4.8}
$$

where ϵ^{abc} is the totally antisymmetric tensor. The structure constants may be then written in terms of the irreducible terms as

$$
\begin{aligned}
C^a_{bc} &= C^{ad} \epsilon_{dbc} \\
&= n^{ad} \epsilon_{bcd} + a_f \delta^{fa}_{bc},
\end{aligned}
\tag{4.9}
$$

where n^{ad} is a symmetric matrix resulting from the split, and n hasn't any direct relation to the normal vector. The delta symbol δ^{fa}_{bc} is an object which is positive unity if the upper indices equal the lower ones, and negative unity if the lower indices equal the set of permuted upper ones. The a_f symbol determines if the Bianchi model is a member of class A or of class B depending on if it is zero or non-zero, and is a trace of the structure

constants, as;

$$a_b = \frac{1}{2}C^s_{bs}.$$

(4.10)

Furthermore, the following orthogonality conditions are identities;

$$0 = a_b n^{ba} = a_b C^{ab} = a_b C^b_{ac}.$$

(4.11)

The tables on the subsequent pages help to codify the several different Bianchi types based on the parameters which make them unique. The algebra of structure constants is listed, as are the Killing vectors and their dualized form objects. The invariant bases X_i in the tables are constructed to be invariant by way of the relations

$$[X_i, \xi_j] \quad = \quad 0$$

$$[X_i, X_j] \quad = \quad C^k_{ij}\xi_k$$

(4.12)

$$.$$

(4.13)

The structure constants are important in determining the curls of the base forms by way of the relations

$$d\boldsymbol{\omega}^i = \frac{1}{2}C^i_{jk}\boldsymbol{\omega}^i \wedge \boldsymbol{\omega}^j.$$

(4.14)

The Bianchi I and the Bianchi IX models are the most useful models. The former is a flat space that reduces to the Friedman model when the anisotropy decays to zero. The Bianchi IX model is a curved-in 3-sphere that reduces to the Robertson-Walker

model when the anisotropy decays to zero. Einstein favoured a 3-sphere model due to considerations of the Mach principle, and maintained that the cosmology should be self-enclosed for these reasons.

The transition from classical gravity and importance of Mach's principle is analysed with great and loving detail in a publication by J. Wheeler titled *The Geometric Steering Principle Reveals the Determiners of Inertia* [47]. Within this work is a tour-de-force of quantum gravity, and at the finish is a wealth of hundreds of references to many important publications, including those by J. York and Chouquet-Brouhat.

Table 4.1: Bianchi I Isometry group Relations

Structure Constants		
All $C^i_{jk} = 0$		
Killing Vectors ξ_i		
ξ_1	ξ_2	ξ_3
∂_1	∂_2	∂_3
Invariant Bases X_i		
X_1	X_2	X_3
∂_1	∂_2	∂_3
Bases Dual to X_i		
ω^1	ω^2	ω^3
dx^1	dx^2	dx^3
Curls of Duals $d\omega^i$		
dx^1	dx^2	dx^3
0	0	0

Table 4.2: Bianchi II Isometry group Relations

Structure Constants		
$C^1_{23} = 1 = -C^1_{32}$		
other $C^i_{jk} = 0$		
Killing Vectors ξ_i		
ξ_1	ξ_2	ξ_3
∂_2	∂_3	$\partial_1 + x^3\partial_2$
Invariant Bases X_i		
X_1	X_2	X_3
∂_2	$x^1\partial_2 + \partial_3$	∂_1
Bases Dual to X_i		
ω^1	ω^2	ω^3
$dx^2 - x^3 dx^3$	dx^3	dx^1
Curls of Duals $d\omega^i$		
dx^1	dx^2	dx^3
$\omega^2 \wedge \omega^3$	0	0

Table 4.3: Bianchi III Isometry group Relations

Structure Constants		
$C^1_{13} = 1 = -C^1_{31}$		
other $C^i_{jk} = 0$		
Killing Vectors ξ_i		
ξ_1	ξ_2	ξ_3
∂_2	∂_3	$\partial_1 + x^2\partial_2$
Invariant Bases X_i		
X_1	X_2	X_3
$e^{x^1}\partial_2$	∂_3	∂_1
Bases Dual to X_i		
ω^1	ω^2	ω^3
$e^{-x^1} dx^2$	dx^3	dx^1
Curls of Duals $d\omega^i$		
dx^1	dx^2	dx^3
$\omega^1 \wedge \omega^2$	0	0

Table 4.4: Bianchi IV Isometry group Relations

Structure Constants		
$C_{13}^1 = 1 = -C_{31}^1$		
$C_{23}^1 = 1 = -C_{32}^1$		
$C_{23}^2 = 1 = -C_{32}^1$		
other $C_{jk}^i = 0$		
Killing Vectors ξ_i		
ξ_1	ξ_2	ξ_3
∂_2	∂_3	$\partial_1 + (x^2 + x^3)\partial_2 + x^3\partial_3$
Invariant Bases X_i		
X_1	X_2	X_3
$e^{x^1}\partial_2$	$x^1 e^{x^1}\partial_2 + e^{x^1}\partial_3$	∂_1
Bases Dual to X_i		
ω^1	ω^2	ω^3
$e^{-x^1}dx^2 - x^1 e^{-x^1}dx^3$	$e^{-x^1}dx^3$	dx^1
Curls of Duals $d\omega^i$		
dx^1	dx^2	dx^3
$\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3$	$\omega^2 \wedge \omega^3$	0

At this point, a very important reduction in the degrees of freedom of the equations may be initiated. The matrix n^{ab} is symmetric, and therefore is hermitian. This hermitian matrix may be diagonalized by an adjoint transformation using orthonormal matrices of unit determinant, and still contain all vital information about the manifold. In effect, the diagonalization will project the matrix n^{ab}, which exists as a point (or vector) in the 6-dimensional Euclidean space of $gl(3, R)$ with the set of spanning basis unit vectors $\{e_b^a\}$, into a 3-dimensional subspace spanned independently by the 3 eigenvectors of the matrix. Therefore, this matrix may be written as

$$n^{ab} \sim \text{diag}(n^{(1)}, n^{(2)}, n^{(3)}). \qquad (4.15)$$

Table 4.5: Bianchi V Isometry group Relations

Structure Constants		
$C^1_{13} = 1 = -C^1_{31}$		
$C^2_{23} = 1 = -C^2_{32}$		
other $C^i_{jk} = 0$		
Killing Vectors ξ_i		
ξ_1	ξ_2	ξ_3
∂_2	∂_3	$\partial_1 + x^2\partial_2 + x^3\partial_3$
Invariant Bases X_i		
X_1	X_2	X_3
$e^{x^1}\partial_2$	$e^{x^1}\partial_3$	∂_1
Bases Dual to X_i		
ω^1	ω^2	ω^3
$e^{-x^1}dx^2$	$e^{-x^1}dx^3$	dx^1
Curls of Duals $d\omega^i$		
dx^1	dx^2	dx^3
$\omega^1 \wedge \omega^3$	$\omega^2 \wedge \omega^3$	0

Table 4.6: Bianchi VI Isometry group Relations

Structure Constants		
$C^1_{13} = 1 = -C^1_{31}$		
$C^2_{23} = h = -C^2_{32}, h \neq 0, 1$		
other $C^i_{jk} = 0$		
Killing Vectors ξ_i		
ξ_1	ξ_2	ξ_3
∂_2	∂_3	$\partial_1 + x^2\partial_2 + hx^3\partial_3$
Invariant Bases X_i		
X_1	X_2	X_3
$e^{x^1}\partial_2$	$e^{hx^1}\partial_3$	∂_1
Bases Dual to X_i		
ω^1	ω^2	ω^3
$e^{-x^1}dx^2$	$e^{-hx^1}dx^3$	dx^1
Curls of Duals $d\omega^i$		
dx^1	dx^2	dx^3
$\omega^1 \wedge \omega^3$	$h\omega^2 \wedge \omega^3$	0

Table 4.7: Bianchi VII Isometry group Relations

Structure Constants
$C^2_{13} = 1 = -C^2_{31}$
$C^1_{23} = -1 = -C^1_{32}$
$C^2_{23} = h = -C^2_{32}, (h^2 < 4)$
other $C^i_{jk} = 0$

Killing Vectors ξ_i		
ξ_1	ξ_2	ξ_3
∂_2	∂_3	$\partial_1 - x^3\partial_2 + (x^2 + hx^3)\partial_3$

Invariant Bases X_i		
X_1	X_2	X_3
$(A + BkB)\partial_2 - B\partial_3$	$B\partial_2 + (A - kB)\partial_3$	∂_1

Bases Dual to X_i		
ω^1	ω^2	ω^3
$(C - kD)dx^2 - Ddx^3$	$Ddx^2 + (C + kD)dx^3$	dx^1

Curls of Duals $d\omega^i$		
dx^1	dx^2	dx^3
$-\omega^2 \wedge \omega^3$	$\omega^1 \wedge \omega^3 + h\omega^2 \wedge \omega^3$	0

where $A = e^{kx^1}\cos(ax^1), \quad B = -\frac{1}{a}e^{kx^1}\sin(ax^1)$

$C = e^{-kx^1}\cos(ax^1), \quad D = -\frac{1}{a}e^{-kx^1}\sin(ax^1)$

$k = \frac{h}{2}, \quad a = (1 - k^2)^{\frac{1}{2}} = \frac{h}{2}(4 - h^2)^{\frac{1}{2}}$

Table 4.8: Bianchi VIII Isometry group Relations

Structure Constants
$C^1_{23} = -1 = -C^1_{32}$
$C^2_{31} = 1 = -C^2_{13}$
$C^3_{12} = 1 = -C^3_{21}, (h^2 < 4)$
other $C^i_{jk} = 0$

Killing Vectors ξ_i		
ξ_1	ξ_2	ξ_3
$\frac{1}{2}e^{-x^3}\partial_1 + \frac{1}{2}[e^{x^3} - (x^2)^2 e^{-x^3}]\partial_2 - x^2 e^{-x^3}\partial_3$	∂_3	$\frac{1}{2}e^{-x^3}\partial_1 - \frac{1}{2}[e^{x^3} + (x^2)^2 e^{-x^3}]\partial_2 - x^2 e^{-x^3}\partial_3$

Invariant Bases X_i		
X_1	X_2	X_3
$\frac{1}{2}[1 + (x^1)^2]\partial_1 + \frac{1}{2}[1 - 2x^1 x^2]\partial_2 - x^1\partial_3$	$-x^1\partial_1 + x^2\partial_2 + \partial_3$	$\frac{1}{2}[1 - (x^1)^2]\partial_1 + \frac{1}{2}[-1 + 2x^1 x^2]\partial_2 + x^1\partial_3$

Bases Dual to X_i		
ω^1	ω^2	ω^3
$dx^1 + [1 + (x^1)^2]dx^2 + [x^1 - x^2 - (x^1)^2 x^2]dx^3$	$2x^1 dx^2 + (1 - 2x^1 x^2)dx^3$	$dx^1 + [-1 + (x^1)^2]dx^2 + [x^1 + x^2 - (x^1)^2 x^2]dx^3$

Curls of Duals $d\omega^i$		
dx^1	dx^2	dx^3
$-\omega^2 \wedge \omega^3$	$\omega^3 \wedge \omega^1$	$\omega^1 \wedge \omega^2$

Table 4.9: Bianchi IX Isometry group Relations

Structure Constants		
	$C_{23}^1 = 1 = -C_{32}^1$	
	$C_{31}^2 = 1 = -C_{13}^2$	
	$C_{12}^3 = 1 = -C_{21}^3$	
	other $C_{jk}^i = 0$	
Killing Vectors ξ_i		
ξ_1	ξ_2	ξ_3
∂_2	$\cos(x^2)\partial_1 - (x^1)\sin x^2 \partial_2 + \frac{\sin(x^2)}{\sin(x^1)}\partial_3$	$-\sin(x^2)\partial_1 - (x^1)\cos x^2 \partial_2 + \frac{\cos(x^2)}{\sin(x^1)}\partial_3$
Invariant Bases X_i		
X_1	X_2	X_3
$-\sin(x^3)\partial_1 + \frac{\cos(x^3)}{\sin(x^1)}\partial_2 - (x^1)\cos x^3 \partial_3$	$\cos(x^3)\partial_1 + \frac{\sin(x^3)}{\sin(x^1)}\partial_2 - \sin(x^3)x^1 \partial_3$	∂_3
Bases Dual to X_i		
ω^1	ω^2	ω^3
$-\sin(x^1)dx^1 + \sin(x^1)\cos(x^3)dx^2$	$\cos(x^3)dx^1 + \sin(x^1)\sin(x^3)dx^2$	$\cos(x^1)dx^2 + dx^3$
Curls of Duals $d\omega^i$		
dx^1	dx^2	dx^3
$\omega^2 \wedge \omega^3$	$\omega^3 \wedge \omega^1$	$\omega^1 \wedge \omega^2$

Table 4.10: The potential terms in Hamiltonian cosmology for all diagonal Bianchi types
I through IX

Bianchi Type	Potential	Constraint
I	0	None
II	$1 + \frac{1}{3}e^{4(\beta_+ + \sqrt{3}\beta_-)}$	None
III	$1 + \frac{4}{3}e^{4\beta_+}$	$3p_+ - \sqrt{3}p_-$
IV	$1 + \frac{1}{2}e^{4\beta_+}\left(8 + \frac{2}{3}e^{4\sqrt{3}\beta_-}\right)$	$p_+ = 0$
V	$1 + 4e^{4\beta_+}$	$p_+ = 0$
VI	$1 + \frac{4}{3}(1 + h + h^2)e^{4\beta_+}$	$p_- = \sqrt{3}\left(\frac{h+1}{h-1}\right)$
VII	$1 + \frac{2}{3}\left[\cosh(4\sqrt{3}\beta_-) + 2h^2 - 1\right]e^{4\beta_+}$	None
VIII	$1 + \frac{2}{3}\left[\cosh(4\sqrt{3}\beta_-) - 1\right]e^{4\beta_+} + \frac{1}{3}e^{-8\beta_+} + \frac{4}{3}e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-)$	None
IX	$1 + \frac{2}{3}\left[\cosh(4\sqrt{3}\beta_-) - 1\right]e^{4\beta_+} + \frac{1}{3}e^{-8\beta_+} + \frac{4}{3}e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-)$	None

Note that the dualized covector $a_a\epsilon^{abc}$ is antisymmetric and will vanish upon performing

the diagonalization transformation which diagonalized the symmetric matrix $\boldsymbol{n}$. This

implies that the antisymmetric dual vector is is perpendicular to the surface spanned

by the eigenvectors of $\boldsymbol{n}$. This in turn implies that the covector a_b lies in the surface

spanned by the eigenvectors. One may choose $\boldsymbol{a}$ to lie along any of the three eigenvectors,

for instance parallel to $n^{(3)}$, i.e., $a_b = a\delta_a^3$. Furthermore, an invariant scalar h may be

introduced via the relation

$$a^2 = hn^{(1)}n^{(2)}. \tag{4.16}$$

This invariant also gives information about the type of manifold corresponding to the

different Bianchi types. The topology of the Bianchi manifolds is important for clas-

sification, and remains invariant upon diagonalization of the matrix $\boldsymbol{n}$, and also upon

normalization of the diagonal eigenvalues produced upon diagonalization. After this is

done, the Table of Bianchi Types listing the parameters of each Bianchi type may be

constructed (see [18]), and is shown above. We will be quantizing the Bianchi I model,

but it is important to see how this model compares to the other Bianchi types. The

parameters shown in the table create a lattice in the vector space having the matrices $\boldsymbol{n}$

as points. The lattice points may be viewed as a type of state space for cosmologies.

One of the most important of these types is Bianchi I (studied here in more detail),

which is a flat 3-space with the translational symmetry group T_3. The structure constants

of this model are all equal to zero, and therefore the curvature is zero. The other type

of major importance is Bianchi Type IX, which has the topology of a three sphere S^3,

and is the archetype for many well known cosmological models, including the Freidman-

Robinson-Walker types and the Einstein model. The structure constants for these models

are the totally antisymmetric tensor components ϵ_{ijk}. The curvature of these models is

non-zero, and is simply related to the scaling factor, or the "radius of the universe",

usually written as R (see [48]).

4.0.6   Diagonalizing the Metric Tensor

The 3x3 metric tensor of the 3-surface is symmetric, and hence hermitian, and therefore

diagonalizable. It is possible to diagonalize both the 3x3 symmetric irreducible part of the

structure constant tensor n^{ab} and the 3x3 metric tensor simultaneously (for details, see

e.g. [14], ch. 6). Thus, the 3-metric would have a basis of three eigenvectors associated

with it. The eigenvectors are not necessarily degenerate, and therefore the universe may

Table 4.11: Bianchi Homogeneous Cosmology Types and their Parameters

Bianchi Type	$n^{(1)}$	$n^{(2)}$	$n^{(3)}$	a	h
Class A:					
I	0	0	0	0	?
II	0	0	1	0	?
VI_0	1	-1	0	0	0
VII_0	1	1	0	0	0
VIII	1	1	-1	0	0
IX	1	1	1	0	0
Class B:					
V	0	0	0	1	?
IV	1	0	0	1	?
III	1	-1	0	1	-1
VI_{-1}	1	-1	0	1	-1
VI_{-a^2}	1	-1	0	a	$-a^2$
VI_{a^2}	1	1	0	a	a^2

have anisotropy. It is possible to have a universe with homogeneity as well as anisotropy. The universe may appear different by looking in different directions. However, this same anisotropy would appear in the same fashion throughout the universe, implying a homogeneous anisotropy.

Once considered diagonalized, the metric tensor may be put in a form which is admitted into the Lie group category with an associated Lie algebra. Consider the diagonal matrix

$$\boldsymbol{\beta} = \text{diag}(\beta^1, \beta^2, \beta^3). \tag{4.17}$$

Furthermore, consider a matrix fashioned from these parameters as exponents, as in

$$e^{\boldsymbol{\beta}} = \text{diag}(e^{\beta^1}, e^{\beta^2}, e^{\beta^3}). \tag{4.18}$$

The Lie algebra of a Lie group is the tangent space at the identity element of the Lie group. When the matrix β has all zero elements, the matrix e^{β} is the identity matrix. Then, let the matrix e^{β} be the "square root" of the 3-metric, in that

$$\boldsymbol{g} = e^{2\beta}. \tag{4.19}$$

It is convenient to rewrite the metric in a way where there is an overall multiplying factor which scales the entire Lie group manifold, and which multiplies a 3x3 matrix which contains the topological information, and has determinant always equal to unity. That is,

$$\boldsymbol{g} = e^{2\beta^0} e^{2\hat{\boldsymbol{\beta}}}. \tag{4.20}$$

Misner has developed such a method, and is based on a set of three basis matrices defined as

$$\{\boldsymbol{e_0}, \boldsymbol{e_+}, \boldsymbol{e_-}\} = \{\mathrm{diag}(1,1,1), \mathrm{diag}(1,1,-2), \mathrm{diag}(1,-1,0)\}. \tag{4.21}$$

This basis set is defined so as to be orthogonal with respect to the DeWitt inner product

$$< \boldsymbol{A}, \boldsymbol{B} >_{DW} = \mathrm{Tr}\boldsymbol{AB} - \mathrm{Tr}\boldsymbol{A}\mathrm{Tr}\boldsymbol{B}. \tag{4.22}$$

In this basis, the Lie group structure is apparent in the equation

$$\boldsymbol{\beta} = \beta^0 \boldsymbol{1} + \hat{\boldsymbol{\beta}}, \tag{4.23}$$

where the hatted values are given as the set

$$\{\beta^0, \hat{\beta}^1, \hat{\beta}^2, \hat{\beta}^3\} = \{(\frac{1}{3}\mathrm{Tr}\boldsymbol{\beta}), (\beta^+ + \sqrt{3}\beta^-), (\beta^+ - \sqrt{3}\beta^-), (-2\beta^+)\}, \qquad (4.24)$$

and such that

$$\hat{\boldsymbol{\beta}} = \beta^A \boldsymbol{e_A} = \beta^0 \boldsymbol{e_0} + \beta^+ \boldsymbol{e_+} + \beta^- \boldsymbol{e_-}. \qquad (4.25)$$

The Lie group structure is evident in this equation, as $\boldsymbol{e_0}$ is the identity element of the group algebra, the last two terms are the generators of the degrees of freedom away from the identity element, and β^0 is the overall scale of the Lie group manifold. β^0 may be thought of as a time dependent scale or conformal factor of the manifold. Qualitative dynamics of the cosmological models are oftentimes described on a 2-plane space with axes of β^+ and β^-, with contours of potential energy type functions given as functions of the β^0 parameter, or of a variant thereof. These variables represent the two true degrees of freedom of the gravitational field as a function of a temporal parameter.

In accordance with the definitions of Misner (see [41], p. 149), we will let $\beta_0 = -\Omega$, so that the complete 4-metric line element for the Bianchi models is then usually written accordingly as

$$ds^2 = -dt^2 + e^{-2\Omega}[e^{2(\beta^+ + \sqrt{3}\beta^-)}(dx^1)^2 + e^{2(\beta^+ - \sqrt{3}\beta^-)}(dx^2)^2 + e^{-4\beta^+}(dx^3)^2]. \qquad (4.26)$$

The diagonal subspace of such 3-metrics may be used to formulate a concise expression

for the 3-curvature as found in [18], which is

$$R = -\text{Tr}(\boldsymbol{m})^2 - \frac{1}{2}\text{Tr}^2\boldsymbol{m} - 6a^c a_c, \qquad (4.27)$$

where the covector a_c corresponds to the trace of the structure constant tensor as defined above in equation 4.10 and the matrix $\boldsymbol{m}$ is defined by

$$\boldsymbol{m} = \frac{1}{\sqrt{g}}\boldsymbol{ng}. \qquad (4.28)$$

Furthermore, by consulting the table of algebraic constants of the Bianchi homogeneous models as described above, it is easily seen that the scalar 3-curvature for the Bianchi I model is identically zero, due to the vanishing of the structure constant trace vector $\boldsymbol{n}$

Quantum mechanical gravitation and cosmology is a work in progress, with some theoretical problems left unsolved. In order to acquire more insight into what is happening, it is helpful to work with models that have been simplified to a certain extent. We will be looking at a Bianchi I model with the above metric line element. However, we will consider a subset of the models where it is taken from the start that $\beta^- = 0$ on the initial 3-surface. The equations of evolution ensure that it will remain zero on subsequent hypersurfaces. Therefore, the metric line element that we will use will appear as

$$ds^2 = -dt^2 + e^{-2\beta_0}[e^{2\beta}(dx^1)^2 + e^{2\beta}(dx^2)^2 + e^{-4\beta}(dx^3)^2], \qquad (4.29)$$

where $\beta = \beta^+$, and β_0 is the conformal degree of freedom. This symmetry results in a model that is referred to as being locally rotationally symmetric (LRS), or a Kantowski-Sachs geometry. It may be thought of as having one spatial direction (out of three) which is expanding or contracting at a rate which is different than the other two. The resulting group of motions is a symmetry transformation of ellipsoid (cigar shape) to oblate spheroid (pancake) transformations.

Chapter 5

Classical Mechanics Applied to Geometries

We have seen how the Bianchi classification scheme is organized. We would like now to apply classical mechanics to the model, in order to create a formalism upon which to apply quantum methods. It is quite intriguing and wondrous that spatial geometries can be dynamic and subject to the laws of mechanics. This is of course due to the fact that Einstein's equations can be solved in a 4-dimensional continuum. Even so, the ADM 3+1 split and Hilbert's method of determining the field equations through a Lagrangian and action integral are intriguing ways to view the same problem, not to mention allowing inroads to quantum mechanical solutions.

In 1917, Hilbert derived the Einstein field equations by applying the variational calculus to a Lagrangian with the scalar 4-curvature density as the integrand, in the same

fashion that the Maxwell equations are derived from variation of an action integral of an electromagnetic Lagrangian. The action integral is written in general as

$$I = \int \mathcal{L} \ dV, \tag{5.1}$$

where $\mathcal{L}$ is the Lagrangian density throughout a volume, and the integral is taken over the volume. For the case of a gravitational field, the Lagrangian density is the scalar 4-curvature multiplied times a 4-volume-correction factor (the square root of the determinant of the metric tensor) which is integrated over a spacetime 4-region. That is,

$$I = \int {}^4\!R\sqrt{{}^4g} \ d^4x. \tag{5.2}$$

The square root of the determinant of the 4-metric is necessary when integrating over the manifold because that is the factor which enables covariance of the volume differential. For instance, if the coordinates were changed from Cartesian to angular, the square root term would enable covariance. Of course, we would like this expression to be decomposed into the 3+1 spacetime split in order to formulate it in terms of the geometric objects we have been using, like the spatial hypersurface and its normal vector, and the extrinsic curvature. This split is worked out in detail in Wald [46], appendix E. The result for the action integral is

$$I = \frac{1}{16\pi} \int \sqrt{g}N[R + \text{Tr}(K^2) - (\text{Tr}K)^2]d^4x, \tag{5.3}$$

where g is the determinant of the 3-surface metric, R is the scalar spatial curvature of

the 3-surface. From the matrix representation of the metric tensor given in terms of the lapse and shift earlier, it is easy to see when the shift is zero that the 3+1 split of the determinant of the 4-metric is just $N^2 g$. That is,

$$\sqrt{^4g} = N\sqrt{g}. \tag{5.4}$$

Even though it is a simple matter to calculate the above relation when the shift is set to the zero vector, the relation is true in general, although the proof is more difficult. The overall multiplier of the integrand, the lapse function N, is a transformation between the spacetime temporal coordinate t and the symplectic time coordinate;

$$N = \dot{t}, \tag{5.5}$$

and enters from the square root of the determinant of the 4-metric.

The above Hamiltonian may be qualitatively seen in the following light. Recall the Lagrangian from mechanics

$$\mathcal{L} = \mathcal{T} - \mathcal{V}, \tag{5.6}$$

where $\mathcal{T}$ is the kinetic energy and $\mathcal{V}$ is the potential energy function. Now consider the configuration space of the metric components g_{ij}, which is spanned by the basis vector set $\{e_j^i\}$, and which has the DeWitt inner product as the metric on the symplectic-type mechanical space,

$$\mathcal{G}^{abcd} = \sqrt{g}(g^{a(c}g^{d)b} - g^{ab}g^{cd}), \tag{5.7}$$

along with its inverse metric

$$\mathcal{G}_{abcd}^{-1} = \sqrt{g}(g_{a(c}g_{d)b} - \frac{1}{2}g_{ab}g_{cd}).$$ (5.8)

We now recall equation 3.7 for the extrinsic curvature as two spatial derivative terms of the shift vector and one term a time derivative of the metric tensor. If we consider a synchronous system without diffeomorphic gauge transformations caused by a shift vector, i.e. $N_i = 0$ we will have that

$$K_{ij} = -\frac{\dot{g}_{ij}}{2N}.$$ (5.9)

Here is a good place to discuss again the difficult and complicated nature of time in the theory. The dot operator as employed above signifies a derivative with respect to the variable that parameterizes the hypersurfaces. Because it parameterizes the hypersurfaces, and because these hypersurfaces are defined as being dense and non-intersecting in the 4-manifold, the variable t may be employed as a coordinate in the 4-space. The difficulty is making a mapping function from t as a spacetime coordinate to t as a parameterization variable in the mathematical spaces of classical mechanics, and eventually to the quantum mechanical mathematical spaces. The mapping must be done so as to allow theory to match observations. An observer has an associated curve in 4-space that everywhere has a time-like tangent vector within the light cone at some 4-point event. The proper time of the observer associated with the curve must be related to the time

function that labels the hypersurfaces. The normal vector emits perpendicular to the hypersurface, and defines the axis of the light cone at that event point. The normal vector is also one of a whole equivalence class of vectors which are tangent to a geodesic curve which intersects all hypersurfaces in a normal fashion along its length. Normal curves of this type make up a special class of observers who would follow the path of a particle that was at rest on average with respect to all other particles in the universe. Another way to look at the path would be that of a test particle with a rest mass that originated in some initial singularity.

The dot operator considered here is the derivative with respect to the actual coordinate that labels the hypersurfaces. In a general spacetime without added symmetries, the hypersurfaces are tilted with respect to the direction of local time flow. However, when one thinks of the flow of time, one thinks of a set of curves, which is a very different picture than that of hypersurfaces of simultaneity labeled by some t. When symmetries are allowed, the situation becomes somewhat more tractable. The symmetry of homogeneity on the global scale (the cosmological principle) allows the whole global system to be considered as being parameterized by some fundamental fiducial time t. The time coordinate in the 4-space associated with the zeroth index of the four indices is what is inferred from the relation

$$K_{ij} = -N\Gamma_{ij}^0 = -\frac{1}{2N}\frac{\partial}{\partial t}g_{ij}, \qquad (5.10)$$

which is what we are using here as the variable with respect to which the time derivative is taken. We have taken the shift vector to be zero in the Bianchi I model which is under

consideration. This is another symmetry which makes the model less general and allows an overall global dynamical equation to be written. A vacuum model is made up basically of various modes of gravitational waves with spatial frequencies that fit cyclically within the global topological shape of the model. The first fundamental mode is that of a DC component where no oscillating standing modes are present. A general vacuum model would be made up of all wavelengths of gravitational waves and the issue of a direction of time and energy density would be a very involved one.

Once the choice of time variable has been made, and symmetries such as homogeneity and no shift are in effect, one may think of the time variable as that of the proper time of a test particle that follows a path which intersects each hypersurface at some spatial coordinate label (x, y, z). Then, this time may be re-introduced as the time variable in any space in which the mechanical dynamics develop, such as the phase space. The nature of time in the mechanical space is then considered usually as a Newtonian time, flowing evenly everywhere in the mechanical space, i.e., at every value of the dynamical coordinate and dynamical momenta. In this sense the parameterization of observables may be solved for, for example a momentum $p(t)$

We now regard the variables $\{g_{ij}, \dot{g}_{ij}\}$ as the tangent bundle of the configuration space, and we write for the quadratic inner product of these lifted coordinates

$$\mathcal{T} \;=\; \mathcal{G}^{abcd} \dot{g}_{ab} \dot{g}_{cd} \;=\; \sqrt{g} < K, K >_{DW} \;=\; \mathrm{Tr}(K^2) - (\mathrm{Tr}K)^2 \qquad (5.11)$$

as the kinetic energy of the Lagrangian. Thus we see from comparing this with the

integrand of equation 5.3 that the potential energy function is what is left over in the integrand, that is;

$$\mathcal{V} = \sqrt{g}R, \tag{5.12}$$

meaning that the scalar curvature of the 3-surface acts as the potential energy function in the symplectic phase space.

Next, one does a transformation from the tangent bundle to the cotangent bundle via the Legendre transformation

$$
\begin{aligned}
\pi^{ab} &= \frac{\partial \mathcal{L}}{\partial \dot{g}_{ab}} \\
&= \frac{\partial}{\partial \dot{g}_{ab}} \left[\frac{1}{(2N)^2} \mathcal{G}^{abcd} \dot{g}_{ab} \dot{g}_{cd} \right] \\
&= \frac{1}{(2N)^2} \mathcal{G}^{abcd} \dot{g}_{cd} \\
&= -\frac{1}{(2N)} \mathcal{G}^{abcd} K_{cd} \\
&= -\frac{1}{(2N)} \sqrt{g} (g^{a(c} g^{d)b} - g^{ab} g^{cd}) K_{cd} \\
&= \sqrt{g} (K^{ab} - g^{ab} \mathrm{Tr} \boldsymbol{K}). \tag{5.13}
\end{aligned}
$$

The transition from the Lagrangian to the Hamiltonian results in a transition from the tangent bundle to the cotangent bundle, represented by

$$\dot{g}_{ab} = \mathcal{G}^{-1}_{abcd} \pi^{cd}. \tag{5.14}$$

Then, one may write the Hamiltonian function in the cotangent bundle representation

as

$$
\begin{aligned}
\mathcal{H} &= \mathcal{T} + \mathcal{V} \\
&= \mathcal{G}^{-1}_{abcd} \pi^{ab} \pi^{cd} + \sqrt{g}R \\
&= \sqrt{g}\left(g_{a(c}g_{d)b} - \frac{1}{2}g_{ab}g_{cd}\right)\pi^{ab}\pi^{cd} + \sqrt{g}R \\
&= \sqrt{g}\left[\mathrm{Tr}\boldsymbol{\pi}^2 - \frac{1}{2}\mathrm{Tr}^2\boldsymbol{\pi} + R\right].
\end{aligned}
\tag{5.15}
$$

5.0.7   Constraints

Recall the more strict definition of the Hamiltonian function that one writes before changing the tangent bundle velocities into the cotangent bundle momenta;

$$
\mathcal{H} = \pi^{ab}\dot{g}_{ab} - \mathcal{L}.
\tag{5.16}
$$

Then, solve equation 3.7 for $\dot{g}_{ab}$ to get

$$
\dot{g}_{ab} = 2NK_{ab} + N_{a|b} + N_{b|a},
\tag{5.17}
$$

which when the inner product is taken twice with the momentum gives, via equation 5.13;

$$
\begin{aligned}
\pi^{ab}\dot{g}_{ab} &= \sqrt{g}(K^{ab} - g^{ab}\mathrm{Tr}\boldsymbol{K})(2NK_{ab} + N_{a|b} + N_{b|a}) \\
&= 2N\sqrt{g}[\mathrm{Tr}(K^2) - (\mathrm{Tr}K)^2] + \pi^{ab}[N_{a|b} + N_{b|a}] \\
&= 2N\sqrt{g}[\mathrm{Tr}(K^2) - (\mathrm{Tr}K)^2] + 2\pi^{ab}N_{a|b}.
\end{aligned}
\tag{5.18}
$$

However, the last term may be treated by parts to be written

$$
\pi^{ab}N_{a|b} = (\pi^{ab}N_a)_{|b} - \pi^{ab}_{|b}N_b.
\tag{5.19}
$$

But the first term on the right is a divergence in the 3-surface, which via Stoke's theorem contributes a boundary term when integrated over the 3-surface. But the three surface is considered not to have a boundary, in that identifications of point subsets (would-be boundaries) have been made to give the surface a compact topological connectedness. Therefore, this term is suppressed in the action integral, but the other term is retained since it is not a divergence.

Noting that the equation 5.18 has terms in common with the Lagrangian function under the action integral sign of equation 5.3, we subtract these terms from the equation 5.16, and write the result as

$$
\mathcal{H} = N\mathcal{H}_\perp + N_i\mathcal{H}^i.
\tag{5.20}
$$

where

$$\mathcal{H}_\perp = \frac{1}{\sqrt{g}}(\mathrm{Tr}\boldsymbol{\pi}^2 - \frac{1}{2}\mathrm{Tr}^2\boldsymbol{\pi}) - \sqrt{g}R, \qquad (5.21)$$

and

$$\mathcal{H}^i = -2\pi^{ij}_{|j}, \qquad (5.22)$$

where also we have transformed the terms in $\boldsymbol{K}$ into terms in $\boldsymbol{\pi}$ by the equation 5.13, i.e.,

$$\mathrm{Tr}\boldsymbol{\pi}^2 = g(\mathrm{Tr}\boldsymbol{K}^2 + \mathrm{Tr}^2\boldsymbol{K}), \qquad (5.23)$$

and

$$\mathrm{Tr}\boldsymbol{\pi} = 2\sqrt{g}\mathrm{Tr}\boldsymbol{K}. \qquad (5.24)$$

$\mathcal{H}_\perp$ is usually referred to as the "superHamiltonian", which is a scalar function of the coordinates g_{ij} and the conjugate momenta π_{ij}. It is the generator of evolution along the normal vector $\boldsymbol{n}$. $\mathcal{H}^i = -2\pi^{ij}_{|j}$ is referred to as the "supermomentum", and is the generator of evolution within the 3-surface. Solving equation 5.16 for $\mathcal{L}$ and using equation 5.20 these equations result in an integral expression for the action;

$$I = \int d^3x \ dt [\pi^{ij}\dot{g}_{ij} - N\mathcal{H}_\perp(\pi^{ij}g_{ij}) - N_i\mathcal{H}^i(\pi^{ij}g_{ij})]. \qquad (5.25)$$

By varying the action integral with respect to the shift vector N_i, three constraint equations appear in the form

$$\mathcal{H}^i = -2\pi^{ij}_{|j} = 0. \qquad (5.26)$$

This set of constraints may be eliminated by utilizing a spatial concept called "super-space", as a configuration space. This space is a gauge-diffeomorphism equivalence class of pullbacks of the 3-surface metric g_{ij}, and therefore eliminates the gauge degrees of freedom in a manner similar to the more familiar freedom in re-gauging the Maxwell equations (see [46] page 467). To show this, we write the spatial gauge diffeomorphism invariance of momenta by demanding the equality

$$\int d^3x \ \ \pi^{ij}(\delta g_{ij} + w_{(i|j)}) = \int d^3x \ \ \pi^{ij}\delta g_{ij}, \tag{5.27}$$

where w_i is a vector field generator of diffeomorphisms on the 3-surface. The expression in parentheses is used often as a gauge expansion of the metric (see e.g. [46], page 75). But this means that

$$\int d^3x \ \ \pi^{ij}w_{i|j} \ = \ 0 \ = \ \int d^3x \left[(\pi^{ij}w_i)_{|j} - w_i\pi^{ij}_{|j} \right], \tag{5.28}$$

and after getting rid of the total divergence just shows the automatic satisfaction of the supermomenta vector constraint $\pi^{ij}_{|j} = 0$. This set of 3 constraints, the set of "vector constraints", is equivalent to the set of $0i$ index vacuum field equations as written by Einstein. That these constraints may be dealt with on a level of gauge transformations means that it is able to be dealt with as a mathematical problem. However, the other remaining constraint, the scalar constraint obtained by variation with respect to the Lagrange multiplier N in the action integral, is physically important.

Upon varying the action integral with respect to the lapse N, and setting it to zero, while throwing out boundary terms (see [46], appendix E, page 465), it is found that

$$\mathcal{H}_\perp(\pi^{ij}, g_{ij}) = 0, \qquad\qquad (5.29)$$

which is referred to as the "superHamiltonian constraint", or the "scalar constraint". This constraint happens to be identical to the R_{00} field equation as written by Einstein, and explains the '$\perp$' as labeling the constraint because of generativity normal to the 3-surface. This constraint also involves a gauge transformation with respect to time, and by looking at equation 5.21, is a quadratic expression in the momenta. When transitioning to quantum mechanics the quadratic term in the momenta becomes a second spatial derivative on the states, which is set equal to the time derivative of the state in the Schrodinger equation. If it is identically zero then there is no real Schrodinger equation. This constraint brings up all the notorious "problems of time" in classical and quantum gravity, and remains an obstacle to quantization of gravity (see e.g. Kheyfets et al [20]).

There being no time generator implies that the total energy would be zero and hence there would be no quantum mechanical propagator, because the phasing of the state vector in the Schrodinger picture or the phasing of the basis vectors in the Heisenberg picture would be absent, and therefore there would be no development of the states in time. This problem could be intimately linked with the problem of the lack of an external observer in quantum cosmology. It is possible that the total energy of the universe includes that of the possible observer, but could possibly be divided such that

neither one would be exactly zero.

Chapter 6

Hamiltonian Cosmology of the

Bianchi I Model

The quantization of a system often begins with either writing or developing a Hamiltonian function. Therefore let us find the Hamiltonian function for the Bianchi I model.

6.0.8   The Hamiltonian Function

We will begin with the expression 5.3 for the Lagrangian function and write for the Lagrangian

$$\mathcal{L} = \frac{1}{16\pi}\sqrt{g}N[R + \text{Tr}(K^2) - (\text{Tr}K)^2]. \tag{6.1}$$

Recalling the expression 4.27 for the scalar 3-curvature for Bianchi models, and consulting the table for the values of the structure constant trace vector $\boldsymbol{n}$ corresponding to the Bianchi I model, it is seen that this vector vanishes, implying that the scalar curvature

R for the Bianchi I model is zero, i.e.,

$$R = 0. \tag{6.2}$$

The rest of the Lagrangian may be calculated using the diagonal metric described above in equation 4.29, which we will write with a slight simplification of variable names;

$$ds^2 = -dt^2 + e^{-2\Omega}[e^{2\beta}(dx^1)^2 + e^{2\beta}(dx^2)^2 + e^{-4\beta}(dx^3)^2], \tag{6.3}$$

where β_0 has been changed to Ω, and where β is the abbreviate symbol for β_+ listed in the earlier equation 4.29 for the metric. Recall that the symmetrized model is locally rotationally symmetric in the 3-space, which is also referred to as the Kantowski-Sachs model. When the decay of anisotropy is complete, the model is homeomorphic and isomorphic to the Friedman isotropic model. One writes the diagonal spatial 3-metric tensor for the model with anisotropy as

$$\boldsymbol{g} = e^{-2\Omega}\mathrm{diag}(e^{2\beta}, e^{2\beta}, e^{-4\beta}), \tag{6.4}$$

or as

$$\boldsymbol{g} = \mathrm{diag}(e^{2\beta-2\Omega}, e^{2\beta-2\Omega}, e^{-4\beta-2\Omega}), \tag{6.5}$$

where β and Ω are parameterized functions of time t, but where time might be better considered as a temporal parameter of a mechanical symplectic phase space wherein β is

a "configuration space" dynamic variable.

The temporal parameter should be related somehow to a cosmological temporal parameter in the Lorentzian 4-manifold in which cosmological observations are made (our universe), but such parameters are arbitrary with respect to local Lorentz transformations and subject to curvature effects over large distances. After all, there is seemingly no absolute Lorentz frame that exists in a vacuum model universe. The visible matter in our universe all seems to be in about the same Lorentz frame on the average. However, it is unknown if the visible matter is the total mass content of the universe. It is also unknown whether the visible mass density is large enough to affect the dynamics of the universe. As for the so-called "dark energy" of the universe, this energy is dark and so its Lorentz frame cannot be determined as of yet, much less its true existence and nature, or if it affects the dynamics of the universe and matter therein.

The assumption of homogeneity (cosmological principle), and the assumption of synchronous system (4-d "gradient" of the time scalar being perpendicular to the 3-surface) can simplify the relation between theoretical and observed time considerably. The problem of time flowing at different rates throughout the 3-surface may be eliminated as well if there is homogeneity. If the normal to the 3-surface points in a different direction in spacetime than the tangent to the geodesic that is the average 4-path of visible matter, then the model is referred to as "tilted".

We as observers within the model are part of the matter distribution that we observe, and being "comoving observers" makes it appear that the matter distribution is approx-

imately static locally. And so such matter would have a 4-velocity that points all in one direction in 4-space, which is the direction we would call perpendicular to the 3-surface. However, if such matter were non-existent or unobservable, it would be difficult to decide if there is indeed a unique Lorentz time direction to hang one's hat on. The concept of a test mass could be used, but the initial conditions of the test mass would be hard to define, and would have to be defined with respect to and in concert with the slant of the 3-surface chosen with respect to the 4-manifold. Even though vacuum models are important in describing the universe as is observed, it is known that any amount of matter added to a vacuum model will significantly change its properties (see [41]).

If the observable matter distribution is considered to be test matter, a metric using lapse and shift may be written with respect to that distribution since the lapse and shift are usually written down so that the problem parameters will match some ansatz. Subsequently, geodesics may be calculated with respect to the metric, and then traced throughout the 4-manifold (see [41] chapter 5). The fate of geodesic curves is an interesting topic, in that some geodesics may be traced up to a boundary, where beyond the metric tensor is singular. Traveling backwards in time, the classical boundary of the singularity could be used to formulate initial conditions which have an invariant quality with respect to evolution of a vacuum model, and which should exhibit homogeneity because of the small volume and extreme conditions at very singular times. Currently, observations of microwave background structure and velocity is being compared to observations of conventional mass density and velocities in order to see if they both retain

a related quantum signature from early extremely close conditions before decoupling at lower temperatures and larger volumes. Even though it is generally accepted (because of observations such as helium and hydrogen content and the background radiation) that there were extremely high energy conditions in the past, it is still generally hoped that quantum mechanical models will make classical singularities irrelevant, just like the quantum mechanical model of the atom suppresses the problem of singularity of energy of an electron-nucleus system with zero angular momentum at zero radius. The idea of a singularity means that there would be no known physics that apply there, which could raise the question of whether singularities can or do in fact exist. Some very interesting models of astrophysical compact objects which avoid singularities in the equations is being worked out by Mottola et al at Los Alamos National Laboratory. [26]

The difference between homogeneity and isotropy within the 3-surface should be mentioned before continuing. Homogeneity implies that the same representation of the metric tensor may be found at all points of the 3-surface. However, isotropy implies that a spatial rotation about any point of the 3-surface would cause no change in observations, i.e., the rotation group about any point in space is an invariance symmetry. That is, everything should appear the same on the average no matter which direction in the sky that one points the telescope or other measurement-taking device. Noting the obvious asymmetry of the metric components in equation 6.4, it is seen that any mapping that employs the metric tensor along the third spatial dimension generator dx^3 can be quite different from mappings within the plane of the two first metric components x^1 and x^2

which are symmetrical. (This is true even though the metric tensor is the same every-where). Since the metrical anisotropy in the model is also a dynamic function of time, obvious observational effects such as a different observed Hubble flow with associated redshift in one unique direction in space is possible. Several analyses of Hubble flow and microwave background radiation data have been done for such effects, with inconclusive results.

After digressing, it is time now to return to the problem of making a Hamiltonian and Lagrangian for the Bianchi I system. Recalling equation 5.9;

$$K_{ij} = -\frac{\dot{g}_{ij}}{2N}, \tag{6.6}$$

along with the fact that we have shown that the scalar 3-curvature is identically zero for this model, the Lagrangian of equation 6.1 reduces to

$$\mathcal{L} = \frac{1}{16\pi}\sqrt{g}N[\text{Tr}(K^2) - (\text{Tr}K)^2]. \tag{6.7}$$

But we have for the volume determinant factor that

$$
\begin{aligned}
\sqrt{g} &= \sqrt{\det[\text{diag}(e^{2\beta-2\Omega}, e^{2\beta-2\Omega}, e^{-4\beta-2\Omega})]} \\
&= \sqrt{e^{-6\Omega}} \\
&= e^{-3\Omega}, \tag{6.8}
\end{aligned}
$$

and due to the diagonality of the metric tensor matrix and the representation of the extrinsic curvature K_{ij} in terms of $\dot{g}_{ij}$ that

$$
\begin{aligned}
\mathrm{Tr}(K^2) - (\mathrm{Tr}K)^2 &= (K_{ij}K_{kl}g^{ik}g^{jl} - (K_{ij}g^{ij})^2) \\[4pt]
&= \frac{1}{(2N)^2}\left(\begin{array}{l} \dot{g}_{11}\dot{g}_{11}g^{11}g^{11} + \dot{g}_{22}\dot{g}_{22}g^{22}g^{22} + \dot{g}_{33}\dot{g}_{33}g^{33}g^{33} \\[6pt] -(\dot{g}_{11}g^{11} + \dot{g}_{22}g^{22} + \dot{g}_{33}g^{33})^2 \end{array}\right) \\[4pt]
&= \frac{1}{(2N)^2}\left(\begin{array}{l} (\dot{g}_{11})^2(g^{11})^2 + (\dot{g}_{22})^2(g^{22})^2 + (\dot{g}_{33})^2(g^{33})^2 \\[6pt] -(\dot{g}_{11})^2(g^{11})^2 - (\dot{g}_{22})^2(g^{22})^2 - (\dot{g}_{33})^2(g^{33})^2 \\[6pt] -2\dot{g}_{11}\dot{g}_{22}g^{11}g^{22} - 2\dot{g}_{11}\dot{g}_{33}g^{11}g^{33} - 2\dot{g}_{22}\dot{g}_{33}g^{22}g^{33} \end{array}\right) \\[4pt]
&= \frac{-2}{(2N)^2}\left(\dot{g}_{11}\dot{g}_{22}g^{11}g^{22} + \dot{g}_{11}\dot{g}_{33}g^{11}g^{33} + \dot{g}_{22}\dot{g}_{33}g^{22}g^{33} \right) \\[4pt]
&= \frac{-2}{(2N)^2}\left(\begin{array}{l} e^{2\beta-2\Omega}(2\dot{\beta}-2\dot{\Omega})e^{2\beta-2\Omega}(2\dot{\beta}-2\dot{\Omega})e^{-2\beta+2\Omega}e^{-2\beta+2\Omega} \\[6pt] +e^{2\beta-2\Omega}(2\dot{\beta}-2\dot{\Omega})e^{-4\beta-2\Omega}(-4\dot{\beta}-2\dot{\Omega})e^{-2\beta+2\Omega}e^{4\beta+2\Omega} \\[6pt] +e^{2\beta-2\Omega}(2\dot{\beta}-2\dot{\Omega})e^{-4\beta-2\Omega}(-4\dot{\beta}-2\dot{\Omega})e^{-2\beta+2\Omega}e^{4\beta+2\Omega} \end{array}\right) \\[4pt]
&= \frac{-2}{(2N)^2}(2\dot{\beta}-2\dot{\Omega})\left[(2\dot{\beta}-2\dot{\Omega}) + 2(-4\dot{\beta}-2\dot{\Omega}) \right] \\[4pt]
&= \frac{-4}{(2N)^2}(\dot{\beta}-\dot{\Omega})(-6\dot{\beta}-6\dot{\Omega}) \\[4pt]
&= \frac{6}{N^2}(\dot{\beta}^2 - \dot{\Omega}^2).
\end{aligned}
$$

Therefore, we have that the Lagrangian of equation 6.7 is

$$
\mathcal{L} = \frac{3}{8\pi N}e^{-3\Omega}(\dot{\beta}^2 - \dot{\Omega}^2). \tag{6.9}
$$

A metric tensor in minisuperspace may be found by looking at the coefficients of the two terms of the Lagrangian above (see the following chapter for more details on minisuperspace and [35], p.124). The two terms are quadratic and kinetic in character. However

the term quadratic in $\dot{\Omega}$ is more abstract in that Ω is usually used for a timelike axis in the minisuperspace because of its quadratic entry with Lorentzian negative signature in the above representation.

The variables in minisuperspace are distinct from any coordinate axes labels in the real 4-dimensional spacetime continuum, and Ω has no direct connection with any observer's clock time in spacetime. Even so, the variable Ω may be used as a cosmological time reference because it is indicative of the scale factor of the universe, which is a dynamic observable most of the time. The time when Ω would not be a good indicator of cosmic time is at the time of maximal expansion, when the extrinsic curvature vanishes. (Recall that the extrinsic curvature is basically the Lie derivative of the metric tensor with respect to the timelike normal vector to the hypersurface leaf.) At the time of maximal expansion there would be no way to observe changes in the scale factor, so that Ω would be useless as a temporal indicator at this stage. But it is only because Ω is borrowed for the purpose of timekeeping that it would have any connection with the time of the spacetime continuum, or the proper time of any observer within the continuum. Some minisuperspace models include Ω as a dynamical variable with a conjugate momentum, and some do not. Later on both cases will be looked at. There are problems admitting Ω as a dynamical variable because of its entry in the Lagrangian with a negative sign and the fact that true kinetic quadratic terms should enter positively. Therefore, later on we follow Kheyfets [29] in considering the quadratic term in $\dot{\Omega}^2$ to be an effective time-dependent potential energy term of the Lagrangian and action integral. It would

be conceivable to include the $\dot{\Omega}^2$ term as a true dynamic variable if one were to make $\dot{\Omega} \to i\dot{\Omega}$, but that is beyond our scope here.

The other minisuperspace variables, which map to the anisotropy of space, could also be used as indicators of cosmological time, but there are also times when these too would fail as timekeeping observables. For instance, at later times there would necessarily have been a decay of anisotropy in order to match current observations of effectively zero anisotropy. And so, with no anisotropy to observe, it would be much more difficult to observe a change in isotropy, and especially acceleration or deceleration of isotropy. It is only in this sense that one of the minisuperspace coordinates could be borrowed as something to set clocks in spacetime by, or to keep time by.

The natural metric on minisuperspace being defined by the coefficients of the kinetic terms in the Lagrangian, one would write this minisuperspace metric as

$$ds^2 = -\left(\frac{3V}{8\pi}e^{-3\Omega}\right)d\Omega^2 + \left(\frac{3V}{8\pi}e^{-3\Omega}\right)d\beta^2. \tag{6.10}$$

The metric components are dependent on the Ω parameter, and so there would be no simplistic isometries in the minisuperspace, except at small values of Ω, where the metric would be almost a multiple of the Minkowski metric times a volume scale factor. At large Ω this minisuperspace metric would seemingly be ill-behaved insofar that an inverse metric tensor would diverge. A further treatment within minisuperspace is developed in the next chapter, where some transformations improve the character of the solution sub-manifolds within the minisuperspace manifold.

Recalling that $N = \dot{t}$, we will follow in this section what is called $N = 1$ cosmology [41], where we let the lapse function be equal to unity, implying the identity transformation of affine parameter. Furthermore, since we are utilizing the cosmological principle of a homogeneous spatial 3-surface, we may integrate the action integral over the 3-surface to get the volume of the 3-surface, V. Therefore, we may write the action integral in the form

$$I = \frac{3V}{8\pi} \int e^{-3\Omega}(\dot{\beta}^2 - \dot{\Omega}^2)dt, \qquad (6.11)$$

implying that the Lagrangian, after integration over the 3-surface, is

$$L = \frac{3V}{8\pi} e^{-3\Omega}(\dot{\beta}^2 - \dot{\Omega}^2), \qquad (6.12)$$

where a regular upper case capital is used to imply that the integration over the 3-surface has been done. However, the factor of scale is determined solely by the scale factor Ω and not the variables of spatial integration. Therefore, the volume here V is a topological volume which is a constant, and usually a multiple on the order of π^2. That is, the scalar volume is included in the Lagrangian via the factor exponential in Ω.

Note in the above equation that the term in $\dot{\beta}^2$ is reminiscent of the kinetic energy term in a free particle Lagrangian proportional to $\dot{q}^2$. In this way it is seen that the dynamics of a homogeneous vacuum cosmology can be similar in many respects to the dynamics of a particle. The term in $\dot{\Omega}^2$ can be regarded as a potential function, and will be shown later to initiate a time dependent Hamiltonian energy functional which must

be integrated over time to evolve the dynamics.

It is possible to fashion a dynamic Hamiltonian function from the Lagrangian inte-
grand of this action integral (see Kheyfets [29]). For such a Hamiltonian we will follow the
Legendre transformation which changes the Lagrangian function of the classical configu-
ration variable tangent bundle $(q, \dot{q})$ to the classical cotangent bundle (q, p) (see equation
5.16, and reference [14]. Letting the geometrodynamic configuration degree of freedom
variable be β, we write the Hamiltonian as

$$H = p_\beta \dot{\beta} - L, \tag{6.13}$$

where regular capital letters for the Hamiltonian and Langrangian functions have been
used symbolizing the fact that integration over the 3-surface has taken place. Furthermore
(and henceforth dropping the subscript on p) recall that the momentum function on the
cotangent bundle is defined and calculated from the expression

$$p = \frac{\partial L}{\partial \dot{\beta}}, \tag{6.14}$$

and upon performing the differentiation becomes

$$p = \frac{3V}{4\pi} e^{-3\Omega} \dot{\beta}. \tag{6.15}$$

In accordance with the usual methods, we replace $\dot{\beta}$ in equation 5.15 by solving the

above equation in terms of p_β to formulate a dynamic Hamiltonian that has the form $H = H(\dot{\beta}, \Omega, \dot{\Omega})$. This dynamic Hamiltonian explicitly has the form

$$
\begin{aligned}
H &= p_\beta \dot{\beta} - L, \\
&= p_\beta^2 \frac{4\pi}{3V} e^{3\Omega} - \frac{3V}{8\pi} e^{-3\Omega} (p_\beta \frac{4\pi}{3V} e^{3\Omega})^2 + \frac{3V}{8\pi} e^{-3\Omega} \dot{\Omega}^2,
\end{aligned}
$$

which finally comes to

$$
H = \frac{2\pi}{3V} e^{3\Omega} p_\beta^2 + \frac{3V}{8\pi} e^{-3\Omega} \dot{\Omega}^2. \tag{6.16}
$$

We will be using this dynamic Hamiltonian function to continue our analysis of the Bianchi I cosmology in the quantum mechanics.

6.0.9   A Canonical Representation Cyclic in Ω

Recall that the Lagrangian function for the Kantowski-Sachs diagonal Bianchi I model was derived in the previous section to be

$$
L = \frac{\mu}{2} e^{-3\Omega} (\dot{\beta}^2 - \dot{\Omega}^2). \tag{6.17}
$$

Where for notational purposes we have substituted $\frac{3V}{8\pi} = \frac{\mu}{2}$. The fact that Ω appears explicitly causes many complications when applying the equations of the canonical formalism, as will be found in the succeeding section. In this section a transformation of variables will be applied which will eliminate the explicit dependence of the Lagrangian, and thus the Hamiltonian, on the scale factor Ω. Therefore it will be seen that the

Lagrangian and the Hamiltonian functions will become cyclic in all degrees of freedom, including the scale factor Ω. It should be noted here that Ω will appear here as a dynamic variable. This is different than the Misner minisuperspace (where Ω is used to play the role of a time coordinate since the minisuperspace manifold is Lorentzian in metric structure and Ω takes the negative signature.)

As mentioned earlier, the scale factor variable Ω may in general be incorporated into the formalism as a dynamical variable, and as such will have a conjugate momentum p_Ω. For more detail, see Kheyfets, *The Issue of Time Evolution in Quantum Gravity* [28]. In the minisuperspace approach [32] there are three coordinates, two degrees of freedom representing cosmological anisotropy β_+ and β_-, and one coordinate representing the overall scale, Ω. However, there is no obvious way to parameterize these degrees of freedom by a time variable, because the Hamiltonian function (the Lie algebra generator in the time direction) is identically zero due to the scalar constraint from classical general relativity. It is usually the case that the anisotropic degrees of freedom $\beta_+ \beta_-$ are represented as being parameterized by the scale factor Ω because Ω enters minisuperspace with a negative signature in the minisuperspace metric, giving the space a Lorentz signature, similar to the time coordinate in 4-dimensional spacetime. Such usage for Ω is slightly ad hoc and artificial because Ω has nothing directly to do with time other than being an observable that one can associate with real time. In fact, the other two anisotropic coordinates could be used as well as temporal-like parameters simply because they are observables that change with time.

Presumably the state of the universe represented as a point in minisuperspace can be parameterized by an outside variable that could be related more directly to the real time that is countable by clocks, but there is no one way to do this that is proven to be consistent. It is possible to introduce time into the canonical formalism by way of proposing a dynamic Hamiltonian which is derived in accordance with the usual canonical methods, starting with a Lagrangian function like the one above.

In order to make the Lagrangian for the current model cyclic in all variables, it is instructive to start with a change in variables

$$
\begin{aligned}
u &= \beta + \Omega \\
v &= \beta - \Omega,
\end{aligned}
$$
(6.18)

which gives the reverse transformation

$$
\begin{aligned}
\beta &= \frac{u+v}{2} \\
\Omega &= \frac{u-v}{2}.
\end{aligned}
$$
(6.19)

This transformation leaves the Lagrangian function as

$$
\begin{aligned}
L &= \frac{\mu}{2} e^{-\frac{3}{2}(u-v)} \dot{u}\dot{v} \\
&= \frac{\mu}{2} e^{-\frac{3}{2}u} \dot{u} e^{\frac{3}{2}v} \dot{v}.
\end{aligned}
$$
(6.20)

This form suggests another transformation of variables

$$\eta \;\; = \;\; e^{-\frac{3}{2}u}$$

$$\xi \;\; = \;\; e^{\frac{3}{2}v}, \tag{6.21}$$

whence the Lagrangian becomes simply

$$L = \frac{4\mu}{9}\dot{\eta}\dot{\xi}, \tag{6.22}$$

which is cyclic in all the coordinates. From here it is also easy to compute the Hamiltonian. We have

$$p_\eta \;\; = \;\; \frac{\partial L}{\partial \dot{\eta}} = \;\; \frac{4\mu}{9}\dot{\xi}$$

$$p_\xi \;\; = \;\; \frac{\partial L}{\partial \dot{\xi}} = \;\; \frac{4\mu}{9}\dot{\eta}, \tag{6.23}$$

whence

$$\dot{\eta} \;\; = \;\; \frac{9}{4\mu}p_\xi$$

$$\dot{\xi} \;\; = \;\; \frac{9}{4\mu}p_\eta. \tag{6.24}$$

In accordance with the usual definition of a Hamiltonian, we write

$$H = p_\eta\dot{\eta} + p_\xi\dot{\xi} - L, \tag{6.25}$$

which gives

$$H = \frac{9}{4\mu} p_\eta p_\xi.$$

(6.26)

This Hamiltonian is very simple, and is somewhat reminiscent of the free particle Hamiltonian. However, the Hamiltonian is not separated into quadratic terms each of one dynamic variable. However, it is interesting to proceed.

In accordance with the usual Dirac method, it is possible now to replace the Hamiltonian variables with their quantum mechanical representations as operators;

$$
\begin{aligned}
H &\mapsto i\hbar \frac{\partial}{\partial t} \\
p_\eta &\rightarrow \frac{\hbar}{i} \frac{\partial}{\partial \eta} \\
p_\xi &\rightarrow \frac{\hbar}{i} \frac{\partial}{\partial \xi},
\end{aligned}
$$

(6.27)

and this give a partial differential equation for the wave function $\Psi(\eta, \xi, t)$ which is similar to the free-particle Schrodinger equation;

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{9}{4\mu} \frac{\partial^2 \Psi}{\partial \eta \partial \xi}.$$

(6.28)

In the relations here, the variable with respect to which the time derivatives are taken is the time which parameterizes the dynamical variables in the quantum mechanical phase space of the coordinate-momentum plane. The Newtonian time in this space may be identified with the proper time of an observer that is on a geodesic curve of a test

particle that is at rest on average with respect to its angular position in the topology. For example, a flat connected and compact 3-space is that of the 3-torus, similar in topology to that of the surface of a doughnut. A test particle that would retain its angular position on the torus would be a good fiduciary test mass particle for this purpose. Each hypersurface labeled by some t would be compact and have the topology of a 3-torus, yet have zero spatial curvature. A zero-curvature 3-torus is unlike a 2-torus embedded in a Euclidean 3-space, which does in fact have a curved surface.

A separation of variables may be done to find energy eigenfunctions of the wave function. Upon letting $\Psi(\eta, \xi, t) = F_\eta(\eta) F_\xi(\xi) F_t(t)$, and entering this expression for Ψ into the differential equation, there results

$$i\hbar F_\eta F_\xi \dot{F}_t = \frac{9}{4\mu} F'_\eta F'_\xi F_t, \tag{6.29}$$

where the primes represent derivatives with respect to the respective variable dependencies. Dividing the differential equation by $\Psi(\eta, \xi, t) = F_\eta(\eta) F_\xi(\xi) F_t(t)$ results in

$$i\hbar \frac{\dot{F}_t}{F_t} = \frac{9}{4\mu} \frac{F'_\eta}{F_\eta} \frac{F'_\xi}{F_\xi}. \tag{6.30}$$

The two sides of the equation are independent of one another for all values, and so each side must equal some constant eigenvalue. Therefore the time factor in the wave function

must be proportional to the eigenfunction

$$F_t \sim e^{i\omega t}. \tag{6.31}$$

The left and right hand sides of the differential equation being constants further results

in

$$\frac{F_\eta'}{F_\eta} = \text{const} \frac{F_\xi}{F_\xi'}, \tag{6.32}$$

which is also independent in variables from one side to the other, resulting in two more

relations for the energy eigenfunctions

$$F_\eta \quad \sim \quad e^{ik_\eta \eta}$$

$$F_\xi \quad \sim \quad e^{ik_\xi \xi}, \tag{6.33}$$

all which gives an expression for an energy eigenbasis for the wave function

$$\Psi(\eta, \xi, t) \sim e^{i(k_\eta \eta + k_\xi \xi - \omega t)}. \tag{6.34}$$

The wave function follows a preferred direction $\vec{k}$ in the transformed configuration space

determined by the constants of integration

$$\Psi(\eta, \xi, t) \sim e^{i(\vec{k} \cdot \vec{r} - \omega t)}. \tag{6.35}$$

The Hamiltonian has been transformed to be cyclic in Ω, but instead the Hamiltonian is not separable. The above resulting wave function may be transformed back into the regular variables. Considering the retransformation would involve another chapter, and so the problem is deferred until a later time.

Chapter 7

Quantizing the Bianchi I Cosmological Model

We will be utilizing the Hamiltonian function derived in the preceding chapter to first obtain a quantized Bianchi I system in accordance with the Schrodinger picture, and then subsequently in the Heisenberg picture.

7.0.10 Quantizing the Bianchi I system in the Schrodinger Picture

Time development for quantized gravitational systems is unique because of the difficult problems of time and the role of observers and observations. A consistent relationship between the temporal function used in the mechanical configuration space and the time recorded by an observer within the system is difficult to find. C. Misner, J. York, and

A. Kheyfets, among others have developed means for parameterizing both classical and quantized cosmological models with temporal functions [48, 20, 29]. We will employ the dynamic Hamiltonian function derived at the end of the last chapter to initiate a quantization procedure.

We may write the state vector as a linear combination of basis kets in the momentum representation. Let these momenta basis kets have eigenvalues k, and write for the state vector

$$|\psi\rangle = \int |k\rangle \, \langle k|\psi\rangle dk = \int |k\rangle \, \phi(k) dk. \tag{7.1}$$

For typical initial conditions (wave packet) at a time t_0 we may assign a Gaussian distribution spectrum of momenta for $\phi(k)$ such that

$$\phi(k) = C e^{-a(k-k_0)^2}, \tag{7.2}$$

and

$$|\psi\rangle = C \int |k\rangle \, e^{-a(k-k_0)^2} dk, \tag{7.3}$$

where k_0 is the initial expectation value for the momentum (at the center of the Gaussian distribution curve). a is a constant which determines the breadth of the wave packet distribution, also called the statistical spread, or the standard deviation. C is a normal-

ization constant allowing the normalized condition

$$\langle \psi | \psi \rangle = 1 = CC^* \int \langle k' | k \rangle e^{-a(k'-k_0)^2} e^{-a(k-k_0)^2} dk' dk = CC^* \int e^{-2a(k-k_0)^2} dk = CC^* \sqrt{\frac{\pi}{2a}},$$
(7.4)

which gives for C;

$$C = \left(\frac{2a}{\pi}\right)^{1/4}.$$
(7.5)

For time development of the initial state vector we will apply the the time development unitary operator. We have written the above state ket $|\psi\rangle$ as a spectrum (linear combination or distribution) of momentum kets, each with a phase and a direction in Hilbert space. The differential operators in quantum mechanics utilize the Lie group concept of using the identity element as the origin. An operator is pictured as a matrix which can change the orientation of a vector in Hilbert space. However, it is also pictured in the Lie group concept as being a matrix-valued vector, itself in a space of operators where the identity element is the unit matrix. To first order, an operator which will change the state vector by a small amount will be something close to the identity matrix. Such first order operators are written as (see [39] ch. 2)

$$U = I - i\Omega d\alpha,$$
(7.6)

where I is the unit matrix. Ω is a matrix operator as well and is multiplied by a differential of some dynamic variable such as momentum, translation (configuration) coordinate,

angular rotation variable, or temporal parameter. The imaginary number i will force any slight changes to be 90 degrees out of phase with the totally real identity matrix. We are interested now in a time development operator which we will write as

$$U(dt) = I - i\frac{\hat{H}dt}{\hbar},\tag{7.7}$$

where $\hat{H}$ is the Hamiltonian. The constant $\hbar$ is the fundamental unit of absolute physical phase. The differential of the time may be represented by taking a finite interval of time and dividing it by a large number of divisions N. Because of the compositional property [1] of such unitary operators, a compositional product of many of these differential operators may be written as (see [39] ch. 2)

$$U(t, t_0) = \lim_{N\to\infty} \left(I - i\frac{\hat{H}(t)(t - t_0)}{\hbar N} \right)^N.\tag{7.8}$$

The limit is known and the time development operator of a finite (non-infinitesimal) length of time is written as

$$U(t, t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t)dt}.\tag{7.9}$$

This non-infinitesimal time development operator may then be applied to the state vector representation 7.3 to develop the state in time. For abbreviated notation, we may write

---

[1] An application of two such unitary operators in immediate succession results in an equivalent one composed of the sum of the infinitesimal parameters of both, i.e., with the compositional property $U(\Delta t_1)U(\Delta t_2) = U(\Delta t_1 + \Delta t_2)$.

the dynamic Hamiltonian function found at the end of the last chapter as

$$H = \frac{2\pi}{3V}e^{3\Omega}p_\beta^2 + \frac{3V}{8\pi}e^{-3\Omega}\dot{\Omega}^2 = F(t)p_\beta^2 + G(t), \qquad (7.10)$$

where the functions F and G are merely abbreviations for the coefficients of p_β^2 and $\dot{\Omega}^2$.

It is seen that p_β commutes with the Hamiltonian function, and we may represent the

Hamiltonian in terms of the eigenvalues of the momentum basis ket vectors as $H = F(t)k^2 + G(t)$. Now we may apply this Hamiltonian function to develop the state ket $|\psi\rangle$

in time;

$$
\begin{aligned}
|\psi, t\rangle &= e^{-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t)dt}\,|\psi, t_0\rangle \\
&= e^{-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t)dt}C\int|k\rangle\,e^{-a(k-k_0)^2}dk \\
&= C\int e^{-\frac{i}{\hbar}\int_{t_0}^{t}H(k,t)dt}\,|k\rangle\,e^{-a(k-k_0)^2}dk \\
&= C\int e^{-\frac{i}{\hbar}\int_{t_0}^{t}[k^2 F(t)+G(t)]dt}\,|k\rangle\,e^{-a(k-k_0)^2}dk \\
&= Ce^{-\frac{i}{\hbar}\int_{t_0}^{t}G(t)dt}\int e^{-\frac{i}{\hbar}k^2\int_{t_0}^{t}F(t)dt}\,|k\rangle\,e^{-a(k-k_0)^2}dk
\end{aligned}
$$

$$(7.11)$$

The momentum representation coefficient function of the time dependent state vector

may be read off of the above equation by inspection. Leaving out the prefactor unitary

exponential number which is outside the integral sign because it is an overall phase factor

independent of k and thus not as important, we have

$$\phi(k,t) = \langle k|\psi,t\rangle = e^{-\frac{i}{\hbar}k^2\int_{t_0}^t F(t)dt}e^{-a(k-k_0)^2}.$$
(7.12)

One can observe that at $t = 0$ all of the coefficient function is real, but as time goes on the coefficient values $\phi(k,t)$ will phase at different rates in accordance with the unitary complex exponential's dependence on k and t. This will cause the different coefficient values to start phasing out of step, and cause interesting behaviours such as dispersions and interferences.

Let us ask for the β configuration space representation of the state vector's coefficients. A Fourier transform must be done via the complex integration factor $\langle\beta|k\rangle$ in order to transform from the k space into the configuration space of β.

$$
\begin{aligned}
\langle\beta|\psi,t\rangle &= Ce^{-\frac{i}{\hbar}\int_{t_0}^t G(t)dt}\int e^{-\frac{i}{\hbar}k^2\int_{t_0}^t F(t)dt}\langle\beta|k\rangle e^{-a(k-k_0)^2}dk \\
&= Ce^{-\frac{i}{\hbar}\int_{t_0}^t G(t)dt}\int e^{-\frac{i}{\hbar}\int_{t_0}^t k^2 F(t)dt}\left(\frac{e^{-ik\beta/\hbar}}{\sqrt{2\pi\hbar}}\right)e^{-a(k-k_0)^2}dk \\
&= \frac{C}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}\int_{t_0}^t G(t)dt}\int e^{-\frac{i}{\hbar}[k^2\int_{t_0}^t F(t)dt+k\beta]-a(k-k_0)^2}dk \\
&= \frac{C}{\sqrt{2\pi\hbar}}e^{-\frac{ig(t)}{\hbar}}\int e^{-\frac{i}{\hbar}[k^2 f(t)+k\beta]-a(k-k_0)^2}dk
\end{aligned}
$$
(7.13)

After integration is performed [29] the wave function as a function of β at a time t is

found to be

$$\Psi(\beta, t) = \frac{C}{\sqrt{2\pi\hbar}}[\alpha(t)]^{-1/4} \exp\left(\frac{-a[\beta - 2k_0 f(t)]^2}{4\hbar\alpha(t)}\right) \exp\left(\frac{i\gamma(\beta, t)}{\hbar}\right), \qquad (7.14)$$

where

$$
\begin{aligned}
f(t) &= \int_{t_0}^{t} F(t)dt = \frac{2\pi}{3V}\int_{t_0}^{t} e^{3\Omega}dt \\
g(t) &= \int_{t_0}^{t} G(t)dt = \frac{3V}{8\pi}\int_{t_0}^{t} \dot{\Omega}^2 e^{-3\Omega}dt, \\
\alpha(t) &= \left(a^2 + \frac{f^2(t)}{\hbar^2}\right), \\
\gamma(\beta, t) &= k_0[\beta - k_0 f(t)] + \frac{f(t)[\beta - 2k_0 f(t)]^2}{4\hbar^2\alpha(t)} - g(t) - \hbar\theta(t), \\
\cos[2\theta(t)] &= \frac{a}{\sqrt{\alpha(t)}}, \\
\sin[2\theta(t)] &= \frac{f(t)}{\sqrt{\alpha(t)}}. \qquad (7.15)
\end{aligned}
$$

By letting $t = t_0$ in the above expressions and equation 7.14 for the wave function, an initial value for the wave function as a function of β is found to be

$$\Psi(\beta, t_0) = \frac{C}{\sqrt{2\pi a\hbar}}\; e^{\frac{-\beta^2}{4a\hbar}}\; e^{\frac{i\beta k_0}{\hbar}}, \qquad (7.16)$$

which is seen to be a wave packet consisting of a sine/cosine wave modulated by a Gaussian envelope. This initial wave packet would disperse in time due to the full wave function's inverse dependence on $\alpha(t)$.

The momentum p_β commutes with the Hamiltonian function 7.10. Upon substituting

the momentum into the Heisenberg equation of motion for the change of momentum with

respect to time, it is seen that the momentum is a constant of the motion;

$$\dot{p}_\beta = [p_\beta, H] = 0. \tag{7.17}$$

The constant value for the momentum is found by taking the expectation value of p_β

with respect to the distribution of momenta at the initial time t_0. Recalling equation 7.2;

$$< p_\beta \to k >= \int \phi^*(k)k\phi(k)dk = C^2 \int k e^{-2a(k-k_0)^2} dk = k_0.$$

The expectation value for the anisotropy coordinate β is obtained by performing the

integral

$$
\begin{aligned}
< \beta(t) > \quad &= \quad \int \Psi^*(\beta,t)\beta\Psi(\beta,t)d\beta \\
&= \quad \frac{C^2}{2\pi\sqrt{\gamma(t)}} \int \beta \exp\left(\frac{-a[\beta - 2k_0 f(t)]^2}{2\hbar\alpha(t)}\right) d\beta \\
&= \quad 2k_0 f(t) \\
&= \quad 2 < p_\beta > f(t).
\end{aligned}
\tag{7.19}
$$

By recalling the original expression 6.4 for the metric line element,

$$ds^2 = -dt^2 + e^{-2\Omega}[e^{2\beta}(dx^1)^2 + e^{2\beta}(dx^2)^2 + e^{-4\beta}(dx^3)^2] \tag{7.20}$$

it is seen from the above relations that the spatial universe would collapse from a 3-dimensional anisotropic manifold into a 2-dimensional pancake-like isotropic one that would grow singularly and exponentially with $f(t)$. It is at singular situations such as this where topological transitions or deformations may take place and where the Lie group identifier of the manifold may signal a change into another Bianchi type [36]. Another type of such a singular situation may be where the 3-manifold would collapse to a point. However, the general trend is to try to avoid such singularities by arranging for calculations whereby the probability for such singular configurations is proportional to measure zero. The analogous situation in atomic quantum mechanics is that the probability of an electron with zero angular momentum existing at the point of the nucleus is proportional to measure zero. Before quantum mechanics, the classical result was singular in that the electron would exist with certainty at the nucleus, whereby the potential energy would diverge in accordance with the $1/r$ behaviour of the potential there.

The Heisenberg equation of motion for β is obtained from the commutator of β with respect to the Hamiltonian H,

$$
\begin{aligned}
\dot{\beta} &= -i[\beta, H] \\
&= -i[\beta, \frac{2\pi}{3V}e^{3\Omega}p_\beta^2] - i[\beta, \frac{3V}{8\pi}e^{-3\Omega}\dot{\Omega}^2] \\
&= -i\frac{2\pi}{3V}e^{3\Omega}[\beta, p_\beta^2] \\
&= -i\frac{2\pi}{3V}e^{3\Omega}2p_\beta[\beta, p_\beta] \\
&= \frac{2\pi}{3V}e^{3\Omega}2p_\beta \\
&= \frac{4\pi}{3V}e^{3\Omega}k_0 \\
&= 2\frac{df(t)}{dt}k_0,
\end{aligned}
\tag{7.21}
$$

which coincides with the expression for the expectation value of β in equation 7.19. It is seen that the velocity of the anisotropy coordinate increases exponentially with Ω. With present-day lack of observational evidence of significant anisotropy, it is possible that in the current epoch the value for Ω is negative, and the change with time of the anisotropy remains small as well. However, at some time in the future if Ω would become positive or on the order of unity, the velocity of the anisotropy would start to increase significantly, and the model would progress rapidly to a singular metric describing a flattened out 2-dimensional space, where the zz component of the metric would vanish, while the xx and yy components would increase exponentially, stretching out the 2-dimensional "pancake".

In the Schrodinger picture, the expression for any operator is given by the matrix-

valued integral

$$\hat{a} = \int |a\rangle \langle a| \, da, \tag{7.22}$$

where a is the eigenvalue from the action of $\hat{a}$ on $|a\rangle$,

$$\hat{a} \, |a\rangle = a \, |a\rangle . \tag{7.23}$$

In the Heisenberg picture, the base states are given phase velocities proportional to the energy of the Hamiltonian function,

$$|a\rangle_{(t)} = e^{i\hat{H}t/\hbar} \, |a\rangle . \tag{7.24}$$

Therefore, the representation for an operator in the Heisenberg picture becomes

$$
\begin{aligned}
\hat{a}(t) \quad &= \quad \hat{a} \int da \;\; a \, |a\rangle_{(t)} \, \langle a|_{(t)} \\
&= \quad \int da \; e^{i\hat{H}t/\hbar} a \, |a\rangle \, \langle a| \, e^{-i\hat{H}t/\hbar} \\
&= \quad e^{i\hat{H}t/\hbar} \left[\int da \; a \, |a\rangle \, \langle a| \right] e^{-i\hat{H}t/\hbar},
\end{aligned}
\tag{7.25}
$$

and so any operator in the Heisenberg picture is represented by

$$|a\rangle_{(t)} = e^{i\hat{H}t/\hbar} \, \hat{a} \, e^{-i\hat{H}t/\hbar}. \tag{7.26}$$

Note that in the Heisenberg form, it is very easy to see why it is that operators that

commute with the Hamiltonian are indeed constants of the motion, because the two

exponential operators will commute with the original operator and cancel. For example,

recalling that the Hamiltonian function is $\hat{H} = F(t)\hat{p}^2 + G(t)$, where $G(t)$ and $G(t)$ are

c numbers, it is easily seen that the momentum operator is a constant of the motion, as

was determined earlier with the commutator $\dot{p} = [p, H] = 0$.

The Heisenberg picture representation of the configuration space coordinate operator

$\hat{\beta}$ is more interesting because there will not be a commuting situation. One writes for

the position operator

$$
\begin{aligned}
\hat{\beta}(t) \quad &= \quad e^{i\hat{H}t/\hbar} \; \hat{\beta} \; e^{-i\hat{H}t/\hbar} \\[2mm]
&= \quad e^{i(F\hat{p}^2+G)t/\hbar} \; \hat{\beta} \; e^{-i(F\hat{p}^2+G)t/\hbar} \\[2mm]
&= \quad e^{iF\hat{p}^2t/\hbar} \; \hat{\beta} \; e^{-iF\hat{p}^2t/\hbar}.
\end{aligned}
$$

$$(7.27)$$

A change in notation could be written such that

$$F = \frac{3V}{2\pi}e^{-3\Omega} = \frac{1}{2\mu} = T \tag{7.28}$$

$$G = \frac{3V}{8\pi}e^{-3\Omega} = \frac{\mu}{2} = V, \tag{7.29}$$

whereby the Hamiltonian appears as

$$H = \frac{p^2}{2\mu} + V, \tag{7.30}$$

in which case the expression for the coordinate operator becomes

$$e^{i(\hat{p}^2/2\mu)t/\hbar} \; \hat{\beta} \; e^{-i(\hat{p}^2/2\mu)t/\hbar}. \qquad (7.31)$$

This relation expresses the well-known fact that the Bianchi I model corresponds to the free-particle class of mechanical problems. That is, the above operator is the free-particle time dependent representation of the coordinate β.

By differentiating the Heisenberg representation of the time dependent coordinate β, the Heisenberg equation of motion may be found for β;

$$
\begin{aligned}
\frac{\partial \hat{\beta}(t)}{\partial t} &= \frac{\partial}{\partial t} e^{i\hat{H}t/\hbar} \; \hat{\beta} \; e^{-i\hat{H}t/\hbar} \\[2mm]
&= (i\hat{H}\hbar)e^{i\hat{H}t/\hbar} \; \hat{\beta} \; e^{-i\hat{H}t/\hbar} + e^{i\hat{H}t/\hbar} \; \hat{\beta} \; e^{-i\hat{H}t/\hbar}(-i\hat{H}/\hbar) \\[2mm]
&= (i\hbar)e^{i\hat{H}t/\hbar} \; [\hat{H}, \hat{\beta}] \; e^{-i\hat{H}t/\hbar} \\[2mm]
&= \frac{1}{i\hbar}[e^{i\hat{H}t/\hbar} \; \hat{\beta} \; e^{-i\hat{H}t/\hbar}, \hat{H}] \\[2mm]
&= \frac{1}{i\hbar}[\hat{\beta}(t), \hat{H}] \\[2mm]
&= \frac{1}{i\hbar}[\hat{\beta}(t), \hat{p}^2]\frac{1}{2\mu} \\[2mm]
&= \frac{1}{i\hbar}\frac{2\hat{p}}{2\mu}[\hat{\beta}(t), \hat{p}] \\[2mm]
&= \hat{p}/\mu, \qquad\qquad\qquad\qquad\qquad\qquad (7.32)
\end{aligned}
$$

because of the canonical commutation relation $[\hat{\beta}, \hat{p}] = i\hbar$.

The need for a non-zero Hamiltonian as a generator in time may be seen by taking a

first-order expansion in t of the expression for $\beta(t)$;

$$
\begin{aligned}
e^{i\hat{H}dt/\hbar} \; \hat{\beta} \; e^{-i\hat{H}dt/\hbar} \quad &= \quad (1 + i\hat{H}dt/\hbar) \; \hat{\beta} \; (1 - i\hat{H}dt/\hbar) \\[2mm]
&= \quad (1 + i\hat{H}dt/\hbar)(\hat{\beta} - i\hat{\beta}\hat{H}dt/\hbar) \\[2mm]
&= \quad \hat{\beta} - i\hat{\beta}\hat{H}dt/\hbar + i\hat{H}dt/\hbar\hat{\beta} + O(dt^2) \\[2mm]
&= \quad \hat{\beta} + \frac{1}{i\hbar}[\hat{\beta},\hat{H}]dt \\[2mm]
&= \quad \hat{\beta} + \dot{\hat{\beta}}dt \\[2mm]
&= \quad \hat{\beta} + d\hat{\beta},
\end{aligned}
\tag{7.33}
$$

from which it is easily seen the problems with the development of dynamical variables if there is a Hamiltonian constraint of $H = 0$.

From the Hamiltonian it is possible to extract an expression for a mechanical phase-space intrinsic time t by applying Hamilton-Jacobi methods of classical mechanics. First the Hamiltonian is written as

$$
\begin{aligned}
H \quad &= \quad \frac{p^2}{2\mu} + \frac{1}{2}\mu\dot{\Omega}^2 \\[2mm]
&= \quad \frac{1}{2\mu}\left(\frac{\partial W}{\partial \beta}\right)^2 + \frac{1}{2}\mu\dot{\Omega}^2,
\end{aligned}
\tag{7.34}
$$

where W is Hamilton's characteristic function. Then the equation is solved with respect

to $\frac{\partial W}{\partial \beta}$, and then integrated with respect to β to obtain

$$W = \int d\beta \sqrt{\mu(2P_0 - \mu\dot{\Omega}^2)}, \qquad (7.35)$$

where the Hamiltonian has been renamed by $H \to P_0$. Then the intrinsic time t is defined as

$$
\begin{aligned}
t \quad &= \quad \frac{\partial W}{\partial P_0} \\
&= \quad \frac{\partial}{\partial P_0} \int d\beta \sqrt{\mu(2P_0 - \mu\dot{\Omega}^2)} \\
&= \quad \int \frac{d\beta \; 2\mu}{\sqrt{\mu(2P_0 - \mu\dot{\Omega}^2)}} \qquad (7.36)
\end{aligned}
$$

If the Hamiltonian constraint is effective then $P_0 \to 0$ and

$$
\begin{aligned}
t \quad &= \quad \int \frac{d\beta \; 2\mu}{\sqrt{\mu(2P_0 - \mu\dot{\Omega}^2)}} \Bigg\|_{P_0 = 0} \\
&= \quad \int \frac{d\beta \; 2\mu}{\sqrt{-\mu^2\dot{\Omega}^2}} \\
&= \quad -i \int \frac{d\beta}{\dot{\Omega}}, \qquad (7.37)
\end{aligned}
$$

where the time is seen to come out to be imaginary. However, since the constraint and final limit is such that $P_0 \to 0$, it is permissible to allow P_0 to be negative imaginary, in which case $\frac{\partial W}{\partial P_0}$ becomes imaginary as well, which in turn would cancel the number i in

the last equation above. That is,

$$
\begin{aligned}
t &= \frac{\partial W}{\partial(-i|P_0|)} \\
-it &= \frac{\partial W}{\partial(|P_0|)} \\
&= -i\int\frac{d\beta}{\dot\Omega}.
\end{aligned}
\tag{7.38}
$$

This is possible because $P_0 \to 0$ under the square root sign, yet the derivative $\frac{\partial}{\partial P_0}$ would

be completely imaginary if P_0 is, which would not matter if it went to zero in the final

limit anyway. That is, a factor of i introduced in the right hand side of equation 7.36

would cancel the imaginary factor in the right hand side of the final equation 7.37. It is

irrelevant if P_0 is imaginary if it is zero in the final limit. The intrinsic time would then

be a real number,

$$
t = \int\frac{d\beta}{\dot\Omega}.
\tag{7.39}
$$

If β and $\dot\Omega$ are independent, then after a constant of integration t_0 is added, a dynamic

equation for β results;

$$
\beta = \dot\Omega(t - t_0),
\tag{7.40}
$$

which is the result for the Bianchi I model obtained by Misner in the minisuperspace

model [32].

7.0.11   Minisuperspace Quantization

Recall the classical derivation of the action integral equation 5.25 which came to

$$I = \int d^3x \ dt \ [\pi^{ij} \dot{g}_{ij} - N\mathcal{H}_\perp(\pi^{ij} g_{ij}) - N_i \mathcal{H}^i(\pi^{ij} g_{ij})]. \tag{7.41}$$

Since the lapse and shift $N, \ N_i$ are Lagrange multipliers of the constraints $\mathcal{H}_\perp = 0 = \mathcal{H}^i$, the action integral may be written as

$$I = \int d^3x \ dt \ \pi^{ij} \dot{g}_{ij}. \tag{7.42}$$

Now, the scale of the universe is parameterized only by the factor $e^{2\Omega}$, and the model under consideration is homogeneous, and so the integral over the remaining spatial co-ordinates is only a topological number that could be set to unity or some multiple of π. For now, let us keep the integral

$$\int d^3x = 1, \tag{7.43}$$

so that

$$I = \int dt \ \pi^{ij} \dot{g}_{ij}. \tag{7.44}$$

It is possible to shorten this integral by writing $\dot{g}_{ij} \ dt = dg_{ij}$ to get

$$I = \int \pi^{ij} dg_{ij}. \tag{7.45}$$

The general form of the metric line element we are working with is

$$ds^2 = -dt^2 + e^{-2\Omega}[e^{2\beta^{11}}(dx^1)^2 + e^{2\beta^{22}}(dx^2)^2 + e^{2\beta^{33}}(dx^3)^2]. \qquad (7.46)$$

From the expression for the metric it is possible to write g_{ij} as

$$g_{ij} \sim e^{-2\Omega}\begin{pmatrix} e^{2\beta^{11}} & & \\ & e^{2\beta^{22}} & \\ & & e^{2\beta^{33}.} \end{pmatrix} = e^{-2\Omega}e^{2\beta} = e^{-2\Omega+2\beta} = e^{2Q}, \qquad (7.47)$$

where

$$\beta \sim \mathrm{diag}(\beta^{11}, \beta^{22}, \beta^{33}), \qquad (7.48)$$

and where

$$Q = -1\Omega + \beta. \qquad (7.49)$$

For general covariant spatial transformations and for the fact that the scale of the volume element is parameterized only by the factor $e^{-2\Omega}$, the determinant of $e^{2\beta}$ should be equal to unity, i.e.,

$$\det e^{2\beta} = \det e^{2\ TR(\beta)} = 1, \qquad (7.50)$$

which implies that there is a constraint on the three degrees of freedom $\beta^{11}, \beta^{22}, \beta^{33}$ such that

$$TR(\beta) = 0, \qquad (7.51)$$

which will decrease the number of degrees of freedom from three to two.

Since the trace of the matrix $\boldsymbol{\beta}$ is zero and is with only two absolute degrees of freedom, let us write the matrix as

$$\boldsymbol{\beta} \sim \begin{pmatrix} \beta_+ + \sqrt{3}\beta_- & & \\ & \beta_+ - \sqrt{3}\beta_- & \\ & & -2\beta_+ . \end{pmatrix} \tag{7.52}$$

Then dg_{ij} is given by

$$d\boldsymbol{g} = de^{2\boldsymbol{Q}} = 2\boldsymbol{g} \cdot d\boldsymbol{Q}, \tag{7.53}$$

and upon multiplying with the momentum gives

$$\pi^{ij} dg_{ij} = \boldsymbol{\pi} \cdot \boldsymbol{g} \cdot d\boldsymbol{Q}. \tag{7.54}$$

Let us also write the momentum tensor as a sum of a traceless part p^i_j and a part proportional to the identity matrix,

$$\pi^i_j = \frac{1}{2\pi}(p^i_j + \frac{1}{3}H\delta^i_j), \tag{7.55}$$

where the factor $1/2\pi$ is added in anticipation of a Fourier transform from momentum space to coordinate space, and where

$$p^k_k = 0 \qquad \text{and} \qquad H = 2\pi\pi^k_k. \tag{7.56}$$

It is easy to see that the second term that is proportional to the identity matrix gives zero when multiplied times $d\boldsymbol{\beta}$ because the latter is traceless.

The situation may be seen from the Lie group and Lie algebra point of view as well. The Euclidean spatial metric $\mathbf{1}$ may be thought of as at the "origin" where the coordinates β and Ω are equal to zero. Let

$$
\begin{aligned}
\boldsymbol{Q} &= \quad -\Omega \mathbf{1} \quad + \quad \beta_+ \boldsymbol{\sigma}_+ \quad + \quad \beta_- \boldsymbol{\sigma}_- \\[2mm]
&= \quad -\Omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \beta^+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \beta^- \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

$$\tag{7.57}$$

Then, when $\boldsymbol{Q} = \mathbf{0}$, it will be such that $\boldsymbol{g} = e^{\boldsymbol{Q}} = \mathbf{1}$. Furthermore, since the momentum tensor $\boldsymbol{\pi}$ may be divided into a trace and traceless part, and since $\boldsymbol{\pi}$ is proportional to $\dot{\boldsymbol{Q}}$, $\boldsymbol{\pi}$ may be written as

$$
\begin{aligned}
\boldsymbol{\pi} &= \quad \frac{1}{2\pi}\left(\frac{1}{3}H\mathbf{1} + \boldsymbol{p}\right) \\[2mm]
&= \quad \frac{1}{12\pi}\left(2H\mathbf{1} + p_+ \boldsymbol{\sigma}_+ + p_- \boldsymbol{\sigma}_-\right).
\end{aligned}
\tag{7.58}
$$

$$\tag{7.59}$$

With the above constructions in place, it is then possible to write the action integral

in strictly canonical form as

$$
\begin{aligned}
I &= \int \pi^{ij} dg_{ij} \\
&= \int \boldsymbol{\pi} \cdot \boldsymbol{g} \cdot d\boldsymbol{Q} \\
&= \int (\frac{1}{3}H\mathbf{1} + \boldsymbol{p})(\mathbf{1}d\Omega + d\boldsymbol{\beta}) \\
&= \int \frac{1}{3}Hd\Omega(\mathbf{1} \cdot \mathbf{1}) + \boldsymbol{p} \cdot d\boldsymbol{\beta} \\
&= \int -Hd\Omega + p_+ d\beta_+ + p_- d\beta_- .
\end{aligned}
\tag{7.60}
$$

As shown by Misner [32], it is very productive to transform the primary dynamical coordinates (the metric components and their conjugate momenta) of the cosmological model into a three dimensional Lorentz space referred to as *minisuperspace*. Recall that the Lie group parameterization for any Bianchi class A or B model in the simply or multiply transitive 3D spacial subgroups after diagonalization is given in general by

$$
g_{ii} \sim \begin{bmatrix} e^{2\beta^{11}} & & \\ & e^{2\beta^{22}} & \\ & & e^{2\beta^{33}} \end{bmatrix},
\tag{7.61}
$$

and the full general four-dimensional pseudo-Riemannian line element is given by

$$
\boldsymbol{d}s^2 = N^2(t)\boldsymbol{d}t^2 + g_{ii}\boldsymbol{\omega}^i\boldsymbol{\omega}^i,
\tag{7.62}
$$

where we recall that the form valued quantities are given by the diffeomorphism generator

form function $\boldsymbol{\omega}^i = \boldsymbol{d}x^i + N^i\boldsymbol{d}t$.

The development of minisuperspace by Misner is motivated by the transformation

$$\beta_j^i \sim -\Omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \beta^+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \beta^- \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(7.63)$$

where the variable Ω may be used for a time substitute, and the variables β_+, β_- are the anisotropy parameters. One may see the reducibility of this transformation in that the first term is a multiple of identity and the second two terms make up a traceless portion of the reduction. There are obviously three degrees of freedom displayed in the above representation, and solely contained intrinsically within the hypersurface leaf itself. It is known from analysis of Einstein's equations that the entire dynamical information concerning the dynamical development of the hypersurface is contained intrinsically within the hypersurface and it's degrees of freedom. The variable Ω has a negative coefficient of the identity matrix, and since it has a special coefficient and is a multiple of identity, it is usually thought of as a temporal parameterization variable. The other two degrees of freedom are a polarization which gives information concerning anisotropy of 3-space. The transformation redistributes the degrees of freedom within the metric tensor into an irreducible form with desirable trace and determinant characteristics, as can be seen by

looking at the metric tensor in terms of the transformed degrees of freedom,

$$
g_{ii} \sim e^{-2\Omega}
\begin{bmatrix}
e^{2(\beta_+ + \sqrt{3}\beta_-)} & & \\
& e^{2(\beta_+ - \sqrt{3}\beta_-)} & \\
& & e^{-4\beta_+}
\end{bmatrix},
\tag{7.64}
$$

where one of the degrees of freedom has emerged out front as an overall multiplier which parameterizes the conformal scale factor of the universe. It will be seen shortly how this degree of freedom also plays the role of an effective time coordinate in the minisuperspace because of its Lorentzian signature within the metric of the minisuperspace.

Minisuperspace contains constraint surfaces which are submanifolds that comprise solutions of Einstein's field equations. The constraint surface turns out to be the forward light cone symmetric with respect to the Ω axis. The Bianchi I model and Kasner model are on the conic surface, and are restricted as well to the $\beta^- = 0$, $\beta^+ < 0$ half-plane, but there are discussions in the literature of deformations between different Bianchi types. [36] The dynamic of the actual universe is equivalent to that of a point particle that travels within the constraint surfaces of the minisuperspace. It will be seen that the Bianchi I model has no potential field within the minisuperspace that can cause a deflection of the particle path in that space. A potential field in any of these models is proportional to the curvature of the cosmological hypersurface in the actual spacetime continuum. The non-Bianchi I type models do in fact have a curvature term in the constraint equation, creating a potential field which in turn can alter the path of the
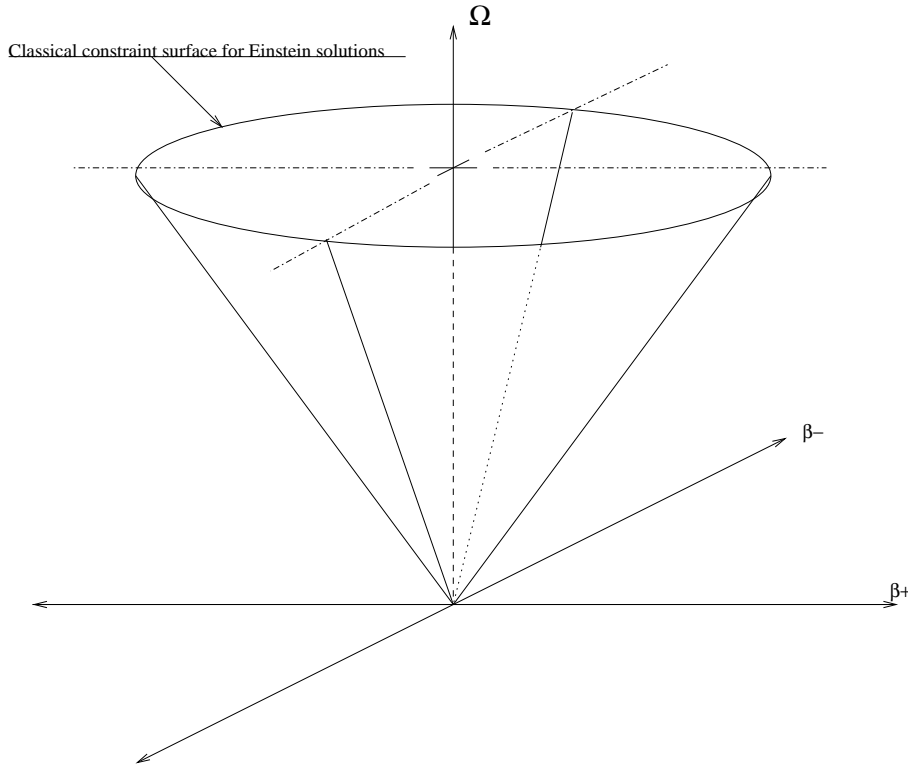
Figure 7.1: The dynamical degrees of freedom may be transformed into minisuperspace, a flat 3-D Minkowski space where the classical solutions are constrained to the forward light cone surface. The Bianchi I Kasner models are admitted over the entire light cone, whereas the Bianchi II model dynamics are further constrained to the $\beta_+ > 0$ half of the boundary cone.

dynamical point particle in the minisuperspace. The potential fields are functions of the coordinates β^+ and β^-, but the potential field scales dynamically with respect to the Ω parameter. In these models the potential "walls" move, as does the state point "particle". The good news though is that the potential field has an exponential dependence on the coordinates such that it may be thought of as a "moving wall" which will deflect the particle in a predictable and simple fashion. (See [41] chapter 11.)

A brief review of the Hamiltonian formulation of the action integral and resultant constraints from variational principles will be outlined in order to motivate the mapping

of the dynamics into the minisuperspace environment.

The Hilbert action integral for gravitation is given by

$$S(g_{\mu\nu}) = \int d^4x L, \tag{7.65}$$

where L is the local Lagrangian density. The method of producing the field equations by variation of the action integral was published Hilbert simultaneously as the field equations were published by Einstein. The action formulation is equivalent to the Einstein equations because the variationional calculus applied to the action integral results in the complete field equations of general relativity. The correct Lagrangian that gives the field equations in this manner is proportional to the full 4-dimensional curvature Ricci scalar with a local 4-volume tensor density factor as

$$L = {}^{(4)}R\sqrt{{}^{(4)}g}. \tag{7.66}$$

But we recall that the volume element density may be written in terms of the ADM $3+1$ decomposition as

$$\sqrt{{}^{(4)}g} = N\sqrt{h}, \tag{7.67}$$

where h is the determinant of the induced spatial metric formulated by means of a unit normal vector to the local hypersurface, i.e., $h_{ab} = g_{ab} + n_a n_b$, and N is the timelike lapse function. We further recall that the Ricci curvature scalar may be separated into ADM

$3 + 1$ variables as

$$^{(4)}R = [R + \text{Tr}(K^2) - (\text{Tr}K)^2],\qquad(7.68)$$

where R is the local spatial curvature of the 3-dimensional hypersurface foliation, and K is the extrinsic curvature tensor described in the previous chapters, i.e.,

$$K_{ij} = \frac{1}{2N}(\dot{h}_{ij} - N_{i|j} - N_{j|i}),\qquad(7.69)$$

where the vertical bar notation signifies covariant differentiation with respect to the covariant derivative operator that is associated with the metric $h_{\mu\nu}$, and $h_{\mu\nu}$ is adapted to the hypersurface orientation via the normal vector n^μ. The canonical momenta conjugate to the spatial metric components may be found through the ordinary classical equation for canonical momenta,

$$\pi^{ij} = \frac{\partial L}{\partial \dot{h}_{ij}},\qquad(7.70)$$

which, after substituting the definition of K_{ab} into the Lagrangian definition gives

$$\pi^{ij} = \sqrt{h}(K^{ij} - \text{Tr}K h^{ij}).\qquad(7.71)$$

Recall here that the intrinsic scalar curvature R is like a potential term in that it is not a function of the velocities $\dot{g}_{ij}$, meaning that the conjugate momenta are independent of R. Therefore the momenta are dependent only on the extrinsic curvature K.

One writes down the canonical covariant definition of the Hamiltonian function

$$H = \pi^{ij}\dot{h}_{ij} - L, \tag{7.72}$$

and subsequently replaces $\dot{h}_{ij}$ using equation 7.69, and also replaces the extrinsic curvature tensors K_{ij} with the canonical momenta π_{ij} via equation 7.71, all resulting in the expression for the Hamiltonian density,

$$H = \sqrt{h}\left\{N\left(\frac{\pi^{ij}\pi_{ij}}{h} - \frac{(\pi_i^i)^2}{2h} - R\right) - 2N_a\left(\frac{\pi^{ij}}{\sqrt{h}}\right)_{|i} + 2\left(\frac{N_j\pi^{ij}}{\sqrt{h}}\right)_{|i}\right\}. \tag{7.73}$$

One notes that the last term in the previous expression is a total divergence without any accompanying Lagrange multiplying factors, which, when integrated over the hypersurface foliation only gives a boundary term via Stoke's theorem, and so is not treated in the variation process because there is no variation at the boundaries by definition. A variation inside the boundaries *is* performed however with respect to the Lagrange multipliers lapse N and shift N_j. The two main constraint equations of general relativity result from the variation, the well known scalar Hamiltonian constraint and the diffeomorphism vector constraint;

$$
\begin{aligned}
\mathcal{H} &= 0 &=& \quad \pi_j^i\pi_i^j - \frac{1}{2}(\pi_i^i)^2 - hR \\
\mathcal{H}^i &= 0^i &=& \quad \pi_{|j}^{ij}.
\end{aligned} \tag{7.74}
$$

It turns out to be very productive to reduce the canonical momenta into a traceless matrix added with a trace term multiplier of the identity matrix, i.e.,

$$\pi^i_j = \frac{1}{2\pi}(p^i_j + \frac{1}{3}\delta^i_j H),$$

(7.75)

where

$$p^i_i = \mathrm{Tr}\,\boldsymbol{p} = 0, \qquad\qquad H = 2\pi\,\mathrm{Tr}\boldsymbol{\pi} = 2\pi\pi^i_i.$$

(7.76)

Then the Hamiltonian scalar constraint, the first of equations 7.74, reads as

$$H^2 = 6p^i_j p^j_i - 24\pi^2 hR.$$

(7.77)

If one makes a transformation of momenta proportional to the transformation of coordinates of anisotropy degree of freedom like

$$\pi^i_j \sim \frac{H}{6\pi}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{p_+}{12\pi}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \frac{p_-}{12\pi}\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(7.78)

then by making the substitutions one may verify the following identity for the canonical

action integral;

$$
\begin{aligned}
I &= \int p_+ d\beta_+ + p_- d\beta_- - H d\Omega \\
&= \pi \int \pi^{ij} dg_{ij},
\end{aligned}
\tag{7.79}
$$

where the last expression is the usual action integral but where the constraints have been suppressed by the transformation. Variation on this integral will produce the field equations [33]. The first expression here is written in a form that easily gives the Hamiltonian equations of motion. A slight manipulation and shorthand gives,

$$
\begin{aligned}
I &= \int (p_\pm \dot{\beta}_\pm - H) d\Omega \\
&= \int f(p_\pm, \beta_\pm, \dot{p}_\pm, \dot{\beta}_\pm) d\Omega,
\end{aligned}
\tag{7.80}
$$

where Ω plays the role of temporal parameter. In the Hamiltonian formulation of dynamics the coordinates and conjugate momenta have equal status, and so there are two sets of Lagrangian equations of motion that emerge from a variation of the integral,

$$
\left[\frac{\partial f}{\partial p_\pm} - \frac{d}{d\Omega} \left(\frac{\partial f}{\partial \dot{p}_\pm} \right) \right] = 0 = \left[\frac{\partial f}{\partial \beta_\pm} - \frac{d}{d\Omega} \left(\frac{\partial f}{\partial \dot{\beta}_\pm} \right) \right].
\tag{7.81}
$$

Substitution of the above integrand into these equations will produce the standard Hamil-

tonian equations of motion

$$\left[\dot{\beta}_{\pm} = \frac{\partial H}{\partial p_{\pm}} \right] , \left[\dot{p}_{\pm} = -\frac{\partial H}{\partial \beta_{\pm}} \right] , \left[\dot{H} = \frac{\partial H}{\partial \Omega} \right] . \tag{7.82}$$

This transformation having achieved such clean separation as in the form of the action

of equation 7.79 as $dI = p_{+}d\beta_{+} + p_{-}d\beta_{-} - Hd\Omega$, it is possible to write a corresponding

metric function within the minisuperspace manifold as

$$ds^2 = -d\Omega^2 + d\beta_{+}^2 + d\beta_{-}^2, \tag{7.83}$$

as for a 3-dimensional Lorentz hyperbolic signature space with a causal structure and with

a flat 3-dimensional Minkowski metric function η_{ij}. Furthermore, the Hamilton-Jacobi

theoretical structure may be seen in writing the above integrand as

$$\begin{aligned} dS &= p_{+}d\beta_{+} + p_{-}d\beta_{-} - Hd\Omega \\ &= \frac{\delta S}{\delta \beta_{+}}d\beta_{+} + \frac{\delta S}{\delta \beta_{-}}d\beta_{-} - \frac{\delta S}{\delta \Omega}d\Omega, \end{aligned} \tag{7.84}$$

where S is the dynamical phase in accordance with the classical Hamilton-Jacobi theory.

This is the symplectic structure which identifies the momentum with the dynamical

change in phase with respect to the configuration variables on the pq phase plane, and the

Hamiltonian function being identified with the change in dynamical phase with respect

to the effective time parameter, i.e.

$$p_i = \frac{\delta S}{\delta q^i}, \qquad \text{and} \qquad H = \frac{\delta S}{\delta t} \tag{7.85}$$

Now, rewriting the constraint equation 7.77 in terms of the transformed variables, one regains the constraint equation in minisuperspace

$$
\begin{aligned}
H^2 &= p_+^2 + p_-^2 - 24\pi^2 h R \\
&= p_+^2 + p_-^2 - \mathcal{R}, \tag{7.86}
\end{aligned}
$$

where the first two terms are regarded as kinetic-type terms and the last term regarded as an effective potential term. One then recalls the Hamilton-Jacobian procedure of writing differential equations for the dynamical phase produced by the constraint equations, and writes

$$\left(\frac{\delta S}{\delta \Omega}\right)^2 = \left(\frac{\delta S}{\delta \beta_+}\right)^2 + \left(\frac{\delta S}{\delta \beta_-}\right)^2 - \mathcal{R}. \tag{7.87}$$

as the Hamilton-Jacobi equation in minisuperspace. This equation motivates the transition to quantum mechanics by way of changing the above equation into a Klein-Gordon equation. This is done by way of a few replacements within the Hamilton-Jacobi equation. The replacements are done by replacing the dynamical phase by the probability distribution wave function Ψ of quantum mechanics, replacing the momenta with partial derivative operators with an added factor i to change the phase by 90 degrees, and replacing the Hamiltonian function with a similar out of phase partial derivative with

respect to the effective time variable, i.e.

$$S \to \Psi, \qquad p_\pm \to \frac{1}{i}\frac{\partial}{\partial \beta^\pm} \qquad H \to -\frac{1}{i}\frac{\partial}{\partial \Omega}, \qquad (7.88)$$

where a quantum of area on the phase plane given by Planck's constant has been written as equal to unity. When these replacements are done, a Klein-Gordon field theoretic wave-type hyperbolic differential equation results;

$$-\frac{\partial^2 \psi}{\partial \Omega^2} + \frac{\partial^2 \psi}{\partial \beta_+^2} + \frac{\partial^2 \psi}{\partial \beta_-^2} - \mathcal{R}\psi = 0. \qquad (7.89)$$

The form of this equation differs from the Hamilton-Jacobi equation a little more than appears at first glance. The terms change from first-order derivatives to the second power into second-order derivatives to the first power, and the function ψ multiplies the effective potential term. The above equation is known as the Wheeler-DeWitt equation.

We recall that in the Bianchi I model the curvature $\mathcal{R}$ is zero. Therefore, the Hamiltonian function is written as

$$H = \sqrt{p_+^2 + p_-^2}, \qquad (7.90)$$

and it is possible to find the solution submanifold as a 2-dimensional surface within the 3-dimensional minisuperspace manifold. We write the metric $ds^2 = -d\Omega^2 + d\beta_+^2 + d\beta_-^2$ as in equation 7.83, but divided by $d\Omega^2$, i.e.

$$\dot{s}^2 = -1 + \dot{\beta}_+^2 + \dot{\beta}_-^2, \qquad (7.91)$$

and then use Hamilton's equations 7.82 which were proven for this metric by a variation of the Lagrangian. We recall from those equations that

$$
\begin{aligned}
\dot{\beta}_{\pm} &= \frac{\partial H}{\partial p_{\pm}} \\
&= \frac{\partial}{\partial p_{\pm}} \sqrt{p_{+}^2 + p_{-}^2} \\
&= \frac{p_{\pm}}{\sqrt{p_{+}^2 + p_{-}^2}},
\end{aligned}
\tag{7.92}
$$

and when substituted into equation 7.91 for the metric gives the invariant line element $ds^2 = 0$, implying that the solution manifold surface is the null surface "light cone" in the minisuperspace. Such findings reinforce the idea that the dynamics of the universe may be simplified to the analogous problem of a relativistic particle constrained to or near to the light cone in Minkowski space.

The Wheeler-DeWitt equation becomes, in the Bianchi I model with zero curvature,

$$
-\frac{\partial^2 \psi}{\partial \Omega^2} + \frac{\partial^2 \psi}{\partial \beta_{+}^2} + \frac{\partial^2 \psi}{\partial \beta_{-}^2} = 0.
\tag{7.93}
$$

The Wheeler-DeWitt equation differs significantly from the Schrodinger equation because it is second order differentiated with respect to the effective time coordinate Ω, which is only chosen as an effective time coordinate because of its being multiplied by the negative component of a Minkowski metric, giving a Lorentz structure to the minisuperspace manifold. The Schrodinger equation enjoys having positive definite solutions for inner products of its solutions, whereas the Wheeler-DeWitt equation often may not.

However, a solution for the Bianchi I Wheeler-DeWitt equation may be written down in terms of an expansion of plane-type waves within the minisuperspace manifold as

$$\psi(\vec{\beta}) = \int \phi(\vec{p}) e^{i\vec{p}\cdot\vec{r}} d^2p, \tag{7.94}$$

where vector $\vec{\beta} = (\beta_+, \beta_-)$ may be admitted simply because of the isomorphism of the minisuperspace to a flat vector space [8]. When the unitary time development operator is applied to the initial wave function distribution, that distribution as a function of time is as

$$
\begin{aligned}
\psi(\vec{\beta}, \Omega) \quad &= \quad e^{-i\hat{H}\Omega} \psi(\vec{\beta}, 0) \\
&= \quad e^{-i\Omega\sqrt{(\hat{p}_+^2 + \hat{p}_-^2)}} \psi(\vec{\beta}, 0) \\
&= \quad e^{-i|\vec{p}|\Omega} \psi(\vec{\beta}, \Omega) \\
&= \quad e^{-i|\vec{p}|\Omega} \int \phi(\vec{p'}) e^{i\vec{p'}\cdot\vec{r}} d^2p', \\
&= \quad \int \phi(\vec{p}) e^{i(\vec{p'}\cdot\vec{r} - i|\vec{p'}|\Omega)} d^2p'.
\end{aligned}
\tag{7.95}
$$

Notice that this wave function will satisfy both the Schrodinger equation $H\psi = i\frac{\partial\psi}{\partial\Omega}$, as well as the Klein-Gordon Wheeler-DeWitt equation 8.15. Figure 7.2 shows the Hamiltonian function as a function of the momenta conjugate to the minisuperspace coordinates. This space is the cotangent bundle space that is canonically conjugate to the coordinate space of β_+, β_-. Gauge symmetries with respect to Ω allow one to set the phase of all basis kets $|p_+, p_-\rangle$ equal to zero at the plane $\Omega = 0$. The derivative with respect to the
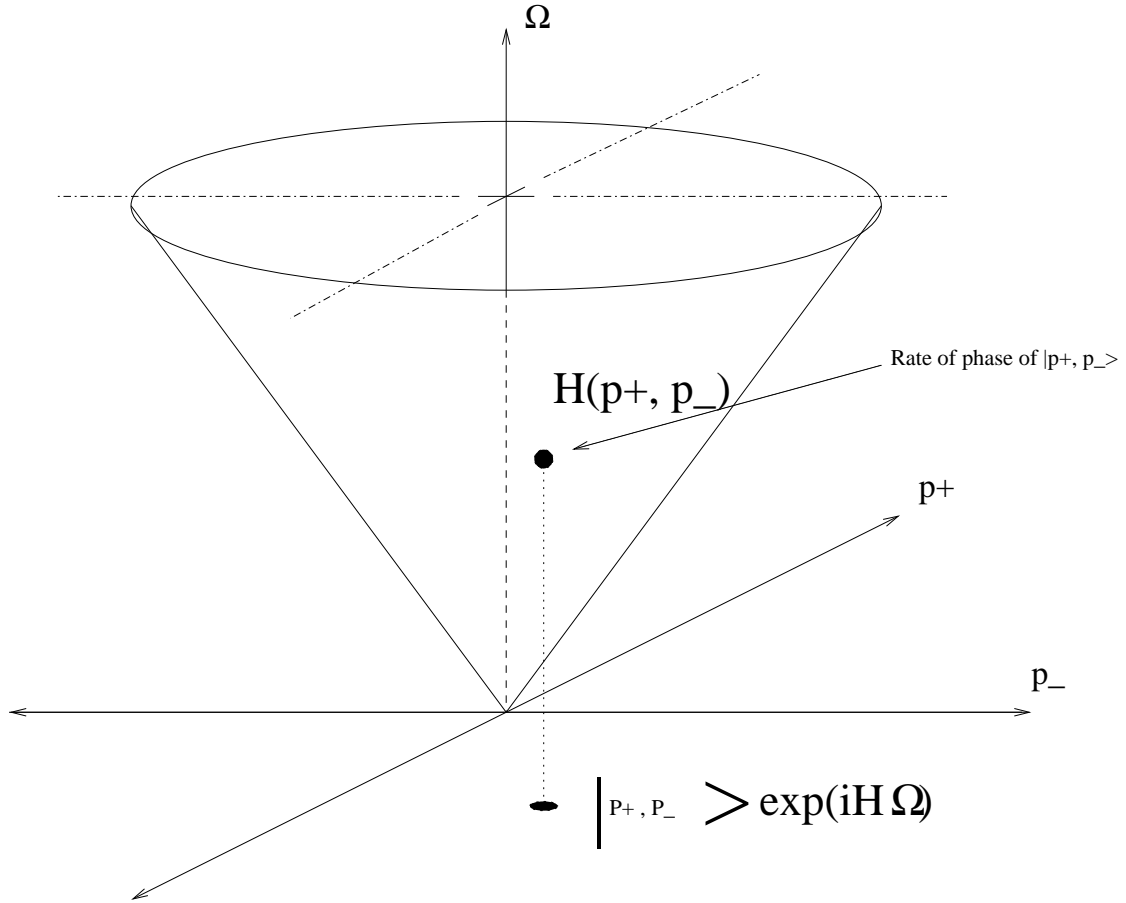
Figure 7.2: The momentum minisuperspace has an energy surface $H = \sqrt{p_+^2 + p_-^2}$ which gives the Heisenberg dynamics in the Bianchi I model. The basis vectors for the momentum eigenstates $|p_+, p_-\rangle$ phase at a rate proportional to the height of the energy surface, or equivalently, at a rate proportional to their distance from the origin.

effective minisuperspace time coordinate Ω of the time dependent basis kets is just the

minisuperspace Hamiltonian function. Since the momenta $p_\pm$ commute with the Hamil-

tonian $H = \sqrt{p_+^2 + p_-^2}$, the dynamics is trivialized if one works in the Heisenberg picture.

All of the dynamics is contained within the time dependence of the basis vectors as

$$|p_+, p_-, \Omega\rangle = e^{iH\Omega} |p_+, p_-, 0\rangle\,, \tag{7.96}$$

and the time dependence of the distribution (unit area continuously distributed coeffi-

cients of the basis vectors, or wave function) is thereby suppressed. A unitary develop-

ment operator with respect to the effective minisuperspace time coordinate Ω may be

easily fashioned from the momenta basis vectors as

$$\hat{\mathcal{U}}(\Omega) = \int dp'_+ dp'_- \, |p'_+, p'_-\rangle \, e^{i\sqrt{p_+'^2 + p_-'^2}} \, \langle p'_+, p'_-|\,. \tag{7.97}$$

Accordingly, one is then able to write the unitary development operator in the coordinate

β basis as

$$\mathcal{U}(\vec{\beta}, \vec{\beta}', \Omega) = \langle \vec{\beta}| \, \hat{\mathcal{U}}(\Omega) \, |\vec{\beta}'\rangle\,, \tag{7.98}$$

where $\vec{\beta} = (\beta_+, \beta_-)$. Or, in the Heisenberg picture, the effective time dependence of the

coordinate basis kets may be expressed as

$$|\vec{\beta}, \Omega\rangle = \hat{\mathcal{U}}(\Omega) \, |\vec{\beta}\rangle = \int dp'_+ dp'_- \, |p'_+, p'_-\rangle \, e^{i\sqrt{p_+'^2 + p_-'^2}} \, \langle p'_+, p'_-|\beta_+, \beta_-\rangle\,. \tag{7.99}$$

The second term under the square root sign in the above expression is somewhat similar to the mass term in the Hamiltonian for a relativistic particle, and therefore much of the theory of relativistic particles may be brought into play.

The probability distribution in the coordinate minisuperspace would be normalized to unit integral probability of the squared norm of the wave function, and the wave function would be initially peaked in some region of minisuperspace. Classically the points representing solutions to Einstein's field equations are constrained to the shell that is the "light cone" boundary in minisuperspace. The Lorentzian metric of minisuperspace is such that classically there is no "proper time" or "invariant interval" between solutions. However, the probability distribution given by quantum mechanics is spread out in the minisuperspace manifold, and only approximates the classical solution space in the limit of correspondence. Therefore behaviors such as tunneling are permissible, as well as off shell solutions. Tunneling could occur between different Bianchi models, signifying fundamental changes in topology without the system passing through a singularity. Furthermore, a point in minisuperspace corresponding to a singular state of the system might have measure zero, meaning that the probability for the system to be found in a singular state would be very low.

The evolution of the state of the system can be approximated by looking at the expectation value of the effective coordinates of the system, i.e. the anisotropy coordinates β_+, β_- parameterized by the conformal scale factor Ω. In the Heisenberg representation

the coordinate operator is represented as a function of effective time as

$$
\begin{aligned}
\hat{\beta}_{\pm}(\Omega) \quad &= \quad e^{iH\Omega}\hat{\beta}_{\pm}e^{-iH\Omega} \\
&= \quad e^{i\Omega\sqrt{(p_+^2+p_-^2)}}i\frac{\partial}{\partial p_{\pm}}e^{-i\Omega\sqrt{(p_+^2+p_-^2)}} \\
&= \quad \Omega\frac{p_{\pm}}{\sqrt{(p_+^2+p_-^2)}},
\end{aligned}
\qquad (7.100)
$$

which, when the expectation value with respect to the state function 7.95 gives,

$$
< \boldsymbol{\beta} >= \hat{\boldsymbol{p}}\Omega,
\qquad (7.101)
$$

where $\hat{\boldsymbol{p}}$ is a unit vector pointing in the direction of the momentum vector $\boldsymbol{p} = (p_+, p_-)$.

7.0.12   Interaction Picture Dynamics

The early stage of the universe is a very important topic in quantum cosmology because of the desire to get around the initial singularity which occurs in the classical analysis. As mentioned earlier, this would be a victory for quantum gravity as was the elimination by quantum mechanics of the infinite energy of an electron with zero angular momentum in the atom. We here undertake an analysis of the early stages of the universe by considering the geometrodynamic Hamiltonian at early conditions, an analysis which lends itself to the interaction picture of quantum mechanics. The interaction picture is the picture which is intermediate between the Schrodinger picture and the Heisenberg picture.

We motivate an analysis on the quantized Bianchi I model in an interaction picture

Table 7.1: The three main dynamical "pictures" in quantum mechanics

	Heisenberg Picture	Interaction Picture	Schrodinger Picture
State Ket	Doesn't change	Evolution determined by V_I	Evolution determined by H_{tot}
Observable	Evolution determined by H_{tot}	Evolution determined by H_0	Doesn't change

by recalling the Hamiltonian function derived in the previous chapter and given also previously as equation 7.10;

$$H = \frac{2\pi}{3V}e^{3\Omega}p_\beta^2 + \frac{3V}{8\pi}e^{-3\Omega}\dot{\Omega}^2. \tag{7.102}$$

The first term of the Hamiltonian is the kinetic term which is quadratic in the momentum, and made time dependent by the factor $e^{3\Omega}$, where the scale factor Ω is not a dynamic or canonical coordinate variable in the geometrodynamic paradigm. However, Ω is a function of (or parameterized by) the geometrodynamic phase time t.

The current cosmological observations show that the current level of anisotropy in the universe is minimal or nonexistent, which implies that a Bianchi class cosmological model with dynamic anisotropy must have undergone a decay of anisotropy during the course of its evolution. It stands to reason that with a large anisotropy factor β, the universe must have been in an early stage with high anisotropy and small scale. A large value of Ω implies that the universe was in the early stage with small scale by recalling

the form of the metric

$$ds^2 = -dt^2 + e^{-2\Omega}[e^{2\beta}(dx^1)^2 + e^{2\beta}(dx^2)^2 + e^{-4\beta}(dx^3)^2], \qquad (7.103)$$

where the spatial coordinate part of the metric is scaled down by large Ω. Along with a large value of Ω, a large negative value of $\dot{\beta} = d\beta/dt$ is implied as well in order to account for rapid decay of anisotropy. Recalling that the momentum is proportional to $\dot{\beta}$, it is seen that the momentum was large in early times, as was Ω. In view of the large value of momentum, the second term of the Hamiltonian is comparatively small, which would motivate an interaction picture analysis, as the Hamiltonian could be written down as

$$H = H_0 + V, \qquad (7.104)$$

where V is a small perturbative term which can be regarded as a potential term which interacts perturbatively with the primary kinetic momentum term H_0. Furthermore, if the second term is regarded as small, then the factor $\dot{\Omega}$ must be necessarily small in comparison. If such is the case, we may in this situation treat Ω as a constant in the first approximation, and we may write down the Hamiltonian as

$$H = \frac{p^2}{2\mu} + \frac{\mu}{2}\dot{\Omega}^2, \qquad (7.105)$$

where

$$\mu = \frac{3V}{4\pi}e^{-3\Omega} \sim \text{const.} \tag{7.106}$$

The smallness of μ at early times (large values of Ω) further motivates the interaction picture analysis by scaling down the potential term while bolstering the kinetic term. Recall that the factor V in the expression enters from an integration over the spatial coordinates. However, also recall that the overall scale is completely determined by the scale factor $e^{-\Omega}$, and the remaining coordinates which it scales are just prototype coordinates that run from 0 to something like 1 or an integer multiple of $/pi$, like the angular coordinates in an expression such as for the length of a circle which is scaled by a radius factor, i.e. $C = R \cdot 2\pi\theta$. Therefore, the value V is a type of number that depends on the topological configuration of the spatial manifold, and is usually set to a constant such as $4\pi^3$. The value might be slightly different depending on if the spatial manifold is given the topology of a 3-torus or a 3-sphere or some other topological identification. Sometimes an integration over the spatial coordinates when the scale factor is set to unity is called the coordinate volume. The coordinate volume may depend on the units being employed, the overall scale which is gauged to unity, and the choice of coordinates. There are enough choices to allow a freedom to set the coordinate volume to almost any constant.

The solution of the problem of the unperturbed Hamiltonian is now very simple

because it is that of the free particle in one dimension

$$H = H_0 = \frac{p^2}{2\mu}. \tag{7.107}$$

It is necessary to solve this equation for the eigenvalues of the unperturbed Hamiltonian. After substituting $H \to i\hbar\frac{\partial}{\partial t}$ and $p \to i\hbar\frac{\partial}{\partial \beta}$ there results the differential equation

$$i\hbar\frac{\partial \Psi}{\partial t} = -\hbar^2 \frac{\partial^2 \Psi}{\partial \beta^2}, \tag{7.108}$$

which has the well known solution of basis states

$$\Psi(\beta, t) = \langle \beta | \Psi(t) \rangle = e^{i(k\beta - Et/\hbar)}, \tag{7.109}$$

where $\hbar k$ is the eigenvalue of the momentum (k is the wave-number) and

$$E = |H_0| = \frac{\hbar^2 k^2}{2\mu} = \frac{|p^2|}{2\mu}. \tag{7.110}$$

The energy eigenvalues of the Hamiltonian comprise a continuous and dense spectrum on the energy line because the Hamiltonian corresponds to that of a free particle. And since the momentum commutes with the Hamiltonian, these eigenvalues correspond to a continuous spectrum on the momentum line as well. By linear superposition, a wave packet made up of a linear combination of such states of momenta centered around some large initial momentum $\hbar k_0$ may be constructed. However, it is desired to consider the

probability of a transition from some initial state of momentum to some final state, or more exactly, to a neighbourhood of final states that are centered around some momentum or energy.

Labeling the second perturbative term in the Hamiltonian by $V = \frac{3V}{8\pi}e^{-3\Omega}\dot{\Omega}^2 = \frac{\mu}{2}\dot{\Omega}^2$, the complete Hamiltonian may be written as

$$H = H_0 + V. \tag{7.111}$$

After replacing $H \to i\hbar\frac{\partial}{\partial t}$ and applying the operators to the wave function, the following relation results;

$$i\hbar\frac{\partial}{\partial t}\left|\Psi\right\rangle = H_0\left|\Psi\right\rangle + V\left|\Psi\right\rangle. \tag{7.112}$$

It is desirable to find the effects which are due to the second small term, which is known also as the interaction term. Therefore the above equation might be better written in a way to isolate the interaction term V on the right hand side and then collect terms under

one differentiation with respect to t on the left hand side;

$$i\hbar\frac{\partial}{\partial t}\left|\Psi\right\rangle - H_0\left|\Psi\right\rangle \quad = \quad V\left|\Psi\right\rangle$$

$$i\hbar\frac{\partial}{\partial t}\left|\Psi\right\rangle + i\hbar(iH_0/\hbar)\left|\Psi\right\rangle \quad = \quad V\left|\Psi\right\rangle$$

$$i\hbar\frac{\partial}{\partial t}\left|\Psi\right\rangle + i\hbar\left(e^{-iH_0t/\hbar}\frac{\partial}{\partial t}e^{iH_0t/\hbar}\right)\left|\Psi\right\rangle \quad = \quad V\left|\Psi\right\rangle$$

$$i\hbar e^{iH_0t/\hbar}\frac{\partial}{\partial t}\left|\Psi\right\rangle + i\hbar\left(\frac{\partial}{\partial t}e^{iH_0t/\hbar}\right)\left|\Psi\right\rangle \quad = \quad e^{iH_0t/\hbar}V\left|\Psi\right\rangle$$

$$i\hbar\frac{\partial}{\partial t}(e^{iH_0t/\hbar}\left|\Psi\right\rangle) \quad = \quad e^{iH_0t/\hbar}V\left|\Psi\right\rangle$$

$$i\hbar\frac{\partial}{\partial t}(e^{iH_0t/\hbar}\left|\Psi\right\rangle) \quad = \quad e^{iH_0t/\hbar}Ve^{-iH_0t/\hbar}e^{iH_0t/\hbar}\left|\Psi\right\rangle$$

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle \quad = \quad v\left|\psi\right\rangle \qquad (7.113)$$

where

$$\left|\psi\right\rangle \quad = \quad e^{iH_0t/\hbar}\left|\Psi\right\rangle$$

$$\text{and} \qquad v \quad = \quad e^{iH_0t/\hbar}Ve^{-iH_0t/\hbar}. \qquad (7.114)$$

In the above derivation, it is first recognized that H_0 is the constant eigenvalue of the unitarily time-transformed operator $e^{-iH_0t/\hbar}\frac{\partial}{\partial t}e^{iH_0t/\hbar}$. Next, an integration by parts is recognized and performed, and then time dependence is distributed to the other terms to impart a similar phase rate to all the terms, in which case the phasing becomes transparent if all kets and operators have such a rate depending on H_0.

In an effort to solve for the state ket that appears in the above differential equation

$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = v\left|\psi\right\rangle$, an integration with respect to the time variable may be attempted;

$$\left|\psi(t)\right\rangle = \int_{t_0}^{t} \frac{dt}{i\hbar} v\left|\psi(t)\right\rangle + \left|\psi_0\right\rangle. \tag{7.115}$$

But a problem presents itself in that the same ket to be solved for appears under the integral sign as well as explicitly on the left hand side of the equation. However, if the eigenket is not changing too rapidly, one may see that the integral term in the above equation is proportional to the time and so the initial ket $\left|\psi_0\right\rangle$ will be the significant term to a first approximation. So it is permissible to substitute the initial ket $\left|\psi_0\right\rangle$ for the time dependent one; $\left|\psi_0\right\rangle \simeq \left|\psi(t)\right\rangle$ under the integral sign for a first order approximation;

$$\left|\psi(t)\right\rangle = \int_{t_0}^{t} \frac{dt}{i\hbar} v\left|\psi_0\right\rangle dt + \left|\psi_0\right\rangle, \tag{7.116}$$

in which case now it is possible to integrate the equation in principle.

If the early universe were in a state comprised of a linear combination of states centered around a particular momentum p_0, then a wave packet with a Gaussian distribution may be written for the initial state such that

$$\left|\psi_0\right\rangle = C\int_{-\infty}^{\infty} dp\, e^{-\alpha_o(p-p_0)^2} \left|p\right\rangle, \tag{7.117}$$

where the constant C may be found by normalizing $|\psi_0\rangle$ such that

$$
\begin{aligned}
1 &= \langle\psi_0|\psi_0\rangle \\
&= |C|^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq\,dq'\, e^{-\alpha_o q^2} e^{-\alpha_o q'^2} \langle q+p_0|q+p_0\rangle \\
&= |C|^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq\,dq'\, e^{-\alpha_o q^2} e^{-\alpha_o q'^2} \delta(q+p_0-q'-p_0) \\
&= |C|^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq\,dq'\, e^{-\alpha_o q^2} e^{-\alpha_o q'^2} \delta(q-q') \\
&= |C|^2 \int_{-\infty}^{\infty} dq\,dq'\, e^{-2\alpha_o q^2} \\
&= |C|^2 \frac{1}{2}\sqrt{\frac{\pi}{2\alpha_o}} \\
\Rightarrow \quad |\psi_0\rangle &= \left(\frac{8\alpha_o}{\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} dp\; e^{-\alpha_o(p-p_0)^2} |p\rangle\,.
\end{aligned}
\tag{7.118}
$$

The next step is to put this initial wave packet normalized state into the time depen-

dent perturbation formula derived previously, equation 7.116. A natural question to ask

would be about the behaviour of the momentum distribution as a function of time. Recall

that the behaviour of the cosmological wave function in the Bianchi I model is basically

that of a free particle, with perturbations resulting from introducing a matter distribu-

tion or some scalar field. It is advantageous to study a scalar field in the Bianchi I model

that results in an inflationary universe, which agrees more favourably with observations

in today's universe of large scale isotropy and homogeneity. The observed magnitude

of the redshift implies the universe is expanding at a disproportionally rapid rate with

respect to the amount of homogeneity that is present. That is, it appears as if there has

been a degree of equilibrium reached throughout the universe, which can usually only

occur if there is sufficient time to reach such a shared condition, or if the homogeneity

is an artifact of a much smaller universe that has suddenly expanded. The inflationary

model admits large scale homogeneity because of the possibility of the universe essentially

expanding instantaneously to a scale larger by many orders of magnitude, and the instan-

taneous expansion would allow the retention of the homogeneity that was in existence

when the universe was very compact and able to share information across the very small

extent of the universe. According to some theories, the homogeneity will survive the

inflation, but with some thumbprints of quantum wave functions that were in existence

in the small realm of the universe before the inflation occurred. Inflation in a Bianchi

I model has been studied extensively by several workers, including Rothman, Madsen,

Anninos, Ryan, Matzner, Ellis, and Futamase [23, 13, 24, 38]. Figure 7.3 depicts the

behavior of the scale of the universe at early times when the universe is very compact,

up until the inflation occurs when the scale of the universe increases by many orders

of magnitude within a very short period of time. Due to the rapid inflation, the factor

$\dot{\Omega}^2$ in the potential term of the Hamiltonian of equation 7.105 can be written as being

proportional to a Dirac delta function, i.e.,

$$V \sim \dot{\Omega}^2 \sim \delta(t - t_0). \tag{7.119}$$

That is, one may write the time dependent interaction picture potential term of the
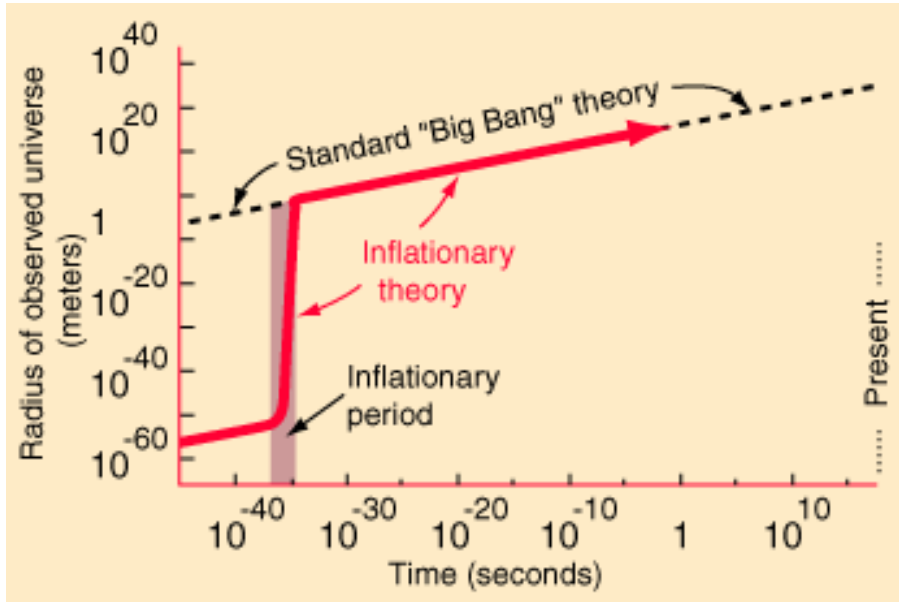
Figure 7.3: Inflation causes the scale of the universe to increase by many orders of magnitude within a very short period of time. The archtypical model of this situation is a step function with respect to time (also known as a Heaviside function), the time derivative of which is the delta function $\delta(t - t_0)$.

Hamiltonian in equation 7.114 as

$$v(t) = e^{iH_0t/\hbar}\delta(t - t_0)\hat{\eta}e^{-iH_0t/\hbar}, \tag{7.120}$$

where $\hat{\eta}$ is some operator which has units of $\hbar$. Subsequently introducing this expression

into the time dependent perturbation theory equation 7.116 will give

$$|\psi(t)\rangle = \int_{t_0}^{t} \frac{dt}{i\hbar}e^{iH_0t/\hbar}\delta(t - t_0)\hat{\eta}e^{-iH_0t/\hbar}|\psi_0\rangle\,dt + |\psi_0\rangle\,. \tag{7.121}$$

A natural question to ask would be one concerning the effect of the shock of inflation

on the distribution of momenta of the initial wave packet. Therefore, let us calculate

the distribution of momenta $\psi(p)$ after inflation, using the time dependent perturbation equation.

$$
\begin{aligned}
\psi(p) \quad &= \quad \langle p|\psi\rangle \\[2mm]
&= \quad \langle p| \left\{ \int_{t_0}^{t} \frac{dt}{i\hbar} v \, |\psi_0\rangle + |\psi_0\rangle \right\} \\[2mm]
&= \quad \langle p| \left\{ \int_{t_0}^{t} \frac{dt}{i\hbar} e^{iH_0 t/\hbar}\delta(t-t_0)\hat{\eta} e^{-iH_0 t/\hbar} \, |\psi_0\rangle + |\psi_0\rangle \right\} \\[2mm]
&= \quad \langle p| \left\{ \begin{array}{l} \displaystyle \int_{t_0}^{t} \frac{dt}{i\hbar} e^{iH_0 t/\hbar}\delta(t-t_0)\hat{\eta} e^{-iH_0 t/\hbar} C \int_{-\infty}^{\infty} dp' \; e^{-\alpha_o(p'-p_0)^2} \, |p'\rangle \\[4mm] \displaystyle \hspace{3cm} +C \int_{-\infty}^{\infty} dp' \; e^{-\alpha_o(p'-p_0)^2} \, |p'\rangle \end{array} \right\} \\[4mm]
&= \quad \left\{ \begin{array}{l} \displaystyle \frac{C}{i\hbar} \int_{-\infty}^{\infty} dp' \; e^{-\alpha_o(p'-p_0)^2} \, \langle p| e^{iH_0 t_0/\hbar}\hat{\eta} e^{-iH_0 t_0/\hbar} \, |p'\rangle \\[4mm] \displaystyle \hspace{2cm} +C \int_{-\infty}^{\infty} dp' \; e^{-\alpha_o(p'-p_0)^2} \, \langle p|p'\rangle \end{array} \right\} . \qquad (7.122)
\end{aligned}
$$

One now recalls that the unperturbed Hamiltonian operator is $H_0 = p^2/2m$, of which $|p\rangle$ is an eigenstate, and so

$$
\begin{aligned}
e^{-iH_0 t_0/\hbar} \, |p'\rangle \quad &= \quad e^{-ip'^2 t_0/2m\hbar} \, |p'\rangle \\[2mm]
\langle p| \, e^{iH_0 t_0/\hbar} \quad &= \quad \langle p| \, e^{ip^2 t_0/2m\hbar} \\[2mm]
\langle p|p'\rangle \quad &= \quad \delta(p-p'). \qquad (7.123)
\end{aligned}
$$

And so the matrix element in the perturbative term becomes

$$
\langle p| \, e^{iH_0 t_0/\hbar}\hat{\eta} e^{-iH_0 t_0/\hbar} \, |p'\rangle = \langle p| \, \hat{\eta} \, |p'\rangle \; e^{i(p^2-p'^2)t_0/2m\hbar}. \qquad (7.124)
$$

Furthermore, if $\hat{\eta}$ was proportional to the unitary operator, then the matrix element would be proportional to the Dirac delta function $\delta(p - p')$. However, one expression for the Dirac delta function is

$$\delta(p - p') = \lim_{\epsilon \to 0} \frac{e^{-(p-p')^2/4\epsilon}}{\sqrt{4\pi\epsilon}}, \tag{7.125}$$

which apparently is a Gaussian function taken to a singular limit of the Dirac delta function. Therefore, we will use the non-singular form of the above representation, in deference to the fact that the matrix element $\langle p | \hat{\eta} | p' \rangle$ would go to the limit of $\delta(p - p')$ as $\hat{\eta} \to 1$. It is also obvious that for physical states, the matrix element would fall off to zero as p and p' would be separated further apart, the dependence of which would be reflected in a Gaussian function for $\langle p | \hat{\eta} | p' \rangle$. Therefore, an expression for the matrix element may be proposed to be approximated by

$$\langle p | \hat{\eta} | p' \rangle \sim \eta_0 \frac{e^{-(p-p')^2/4\epsilon}}{\sqrt{4\pi\epsilon}}. \tag{7.126}$$

In order to match the notation for the rest of the factors in the perturbative term, let us rewrite the matrix element with a change of notation $(4\epsilon)^{-1} \to \alpha_m$. Also, since the matrix element may be complex, a unitary phase factor may be applied, and the matrix element can be approximated by

$$\langle p | \hat{\eta} | p' \rangle \simeq \eta_0 e^{i\theta} \sqrt{\alpha_m} \; e^{-\alpha_m (p-p')^2}, \tag{7.127}$$

where α_m has the same units as α_o in the original Gaussian distribution of momenta, and performs the same role of describing the spread of the bell curve, but where the subscript reminds us that it is an artifact of the matrix element. Finally, to further make the notation consistent, let's rename the imaginary multipliers in the exponent of equation 7.128 so that;

$$e^{i(p^2 - p'^2)t_0/2m\hbar} = e^{\alpha_c(p^2 - p'^2)},$$ (7.128)

with the subscript of c reminding us that $\alpha_c = it_0/2m\hbar$ is imaginary. As may be seen from figure 7.3, inflation occurs at very small values of t, and so the factor $\alpha_c = it_0/2m\hbar$ is thought of as small and imaginarily perturbative with respect to the spreading factor α_o in the original Gaussian wave packet distribution, i.e., $|\alpha_O/\alpha| < 1$.

And so picking up where we left off, we may now proceed to integrate the expression

for $\psi(p)$ after the inflationary shock;

$$
\psi(p) \quad = \quad \left\{ \begin{array}{r} \dfrac{C}{i\hbar} \displaystyle\int_{-\infty}^{\infty} dp'\ e^{-\alpha_o(p'-p_0)^2}\ \langle p|\, e^{iH_0t_0/\hbar}\hat{\eta}e^{-iH_0t_0/\hbar}\, |p'\rangle \\[4mm] +C \displaystyle\int_{-\infty}^{\infty} dp'\ e^{-\alpha_o(p'-p_0)^2}\ \langle p|p'\rangle \end{array} \right\}
$$

$$
= \quad \left\{ \begin{array}{r} \dfrac{C}{i\hbar} \displaystyle\int_{-\infty}^{\infty} dp'\ e^{-\alpha_o(p'^2-2p'p_0+p_0^2)}\eta_0 e^{i\theta}\sqrt{\alpha_m}\ e^{-\alpha_m(p^2-2pp'+p'^2)}\ e^{i(p^2-p'^2)t_0/2m\hbar} \\[4mm] +C \displaystyle\int_{-\infty}^{\infty} dp'\ e^{-\alpha_o(p'-p_0)^2}\delta(p-p') \end{array} \right\}
$$

$$
= \quad \left\{ \begin{array}{r} -iC(\eta_0/\hbar)\ e^{i\theta}\sqrt{\alpha_m}\ e^{-\alpha_o p_0^2}\ e^{-p^2(\alpha_m-\alpha_c)} \displaystyle\int_{-\infty}^{\infty} dp'\ e^{-p'^2(\alpha_o+\alpha_m+\alpha_c)}\ e^{p'(2\alpha_o p_0+2\alpha_m p)} \\[4mm] +C\ e^{-\alpha_o(p-p_0)^2} \end{array} \right\}
$$

$$
= \quad \left\{ \begin{array}{r} -iC(\eta_0/\hbar)\ e^{i\theta}\sqrt{\alpha_m}\ e^{-\alpha_o p_0^2}\ e^{-p^2(\alpha_m-\alpha_c)} \displaystyle\int_{-\infty}^{\infty} dp'\ e^{-\alpha p'^2}\ e^{-Bp'} \\[4mm] +C\ e^{-\alpha_o(p-p_0)^2}, \end{array} \right\}
$$

$$\tag{7.129}$$

where

$$
\alpha = \alpha_o + \alpha_m + \alpha_c \qquad\qquad B = 2\alpha_o p_0 + 2\alpha_m p. \tag{7.130}
$$

The remaining integral in the perturbative term is well known, and can be integrated by completing the square in the exponents. The result is

$$
\int_{-\infty}^{\infty} dp'\ e^{[-\alpha p'^2 - Bp']} = e^{B^2/4\alpha}\sqrt{\dfrac{\pi}{\alpha}}. \tag{7.131}
$$

After integrating the remaining integral, we can proceed to arrange the expression into

an expression that could be simpler to analyse. Just considering the perturbative term

in the expression for $\psi(p)$, which could be called $\Delta\psi(p)$,

$$
\begin{aligned}
\Delta\psi(p) \quad = \quad & -\frac{iC\eta_0}{\hbar} \, e^{i\theta} \sqrt{\alpha_m} \, e^{-\alpha_o p_0^2} \, e^{-p^2(\alpha_m - \alpha_c)} \int_{-\infty}^{\infty} dp' \, e^{-\alpha p'^2} \, e^{-Bp'} \\
= \quad & -\frac{iC\eta_0}{\hbar} \, e^{i\theta} \sqrt{\alpha_m} \, e^{-\alpha_o p_0^2} \, e^{-p^2(\alpha_m - \alpha_c)} \, e^{B^2/4\alpha} \sqrt{\frac{\pi}{\alpha}} \\
= \quad & -\frac{iC\eta_0}{\hbar} \sqrt{\frac{\pi\alpha_m}{\alpha}} \, e^{i\theta} \, e^{-\alpha_o p_0^2} \, e^{-p^2(\alpha_m - \alpha_c)} \, e^{\left(\frac{(2\alpha_o p_0 + 2\alpha_m p)^2}{4\alpha}\right)} \\
= \quad & -\frac{iC\eta_0}{\hbar} \sqrt{\frac{\pi\alpha_m}{\alpha}} \, e^{i\theta} \, e^{-\alpha_o p_0^2} \, e^{-p^2(\alpha_m - \alpha_c)} \, e^{\left(\frac{\alpha_o^2 p_0^2 + 2\alpha_o \alpha_m p p_0 + \alpha_m^2 p^2}{\alpha_o + \alpha_m + \alpha_c}\right)} \\
= \quad & -\frac{iC\eta_0}{\hbar} \sqrt{\frac{\pi\alpha_m}{\alpha}} \, e^{i\theta} \, e^{-\alpha_o p_0^2(1 - \alpha_o/\alpha)} \, e^{-\alpha_m p^2(1 - \alpha_m/\alpha)} \, e^{-\alpha_c p^2} \, e^{2\alpha_o \alpha_m p p_0/\alpha} \\
= \quad & -\frac{iC\eta_0}{\hbar} \sqrt{\frac{\pi\alpha_m}{\alpha}} \exp\left[i\theta - \alpha_o p_0^2 (1 - \frac{\alpha_o}{\alpha}) - \alpha_m p^2 (1 - \frac{\alpha_m}{\alpha}) + \frac{2\alpha_o \alpha_m p p_0}{\alpha} - \alpha_c p^2\right]
\end{aligned}
$$

$$(7.132)$$

Some of the various parameters may now be entered into $\Delta\psi(p)$, and some graphs

may be produced by some plotting programs. The general trend upon entering various

values of $\theta, \alpha_o, \alpha_m, \alpha_c, p_0$ results in a wave in p that peaks around p_0, but which has lower

frequency behaviour at values of p below p_0, and then the frequency increases above p_0.

A typical graph is shown in figure 7.4, where the solid curve is the real part of $\Delta\psi(p)$

and the dotted line is the imaginary part.

The relation between the real and the imaginary parts of the function $\Delta\psi(p)$ may be

expressed by the theory of Hilbert transforms. The Hilbert transform of a function $f(x)$
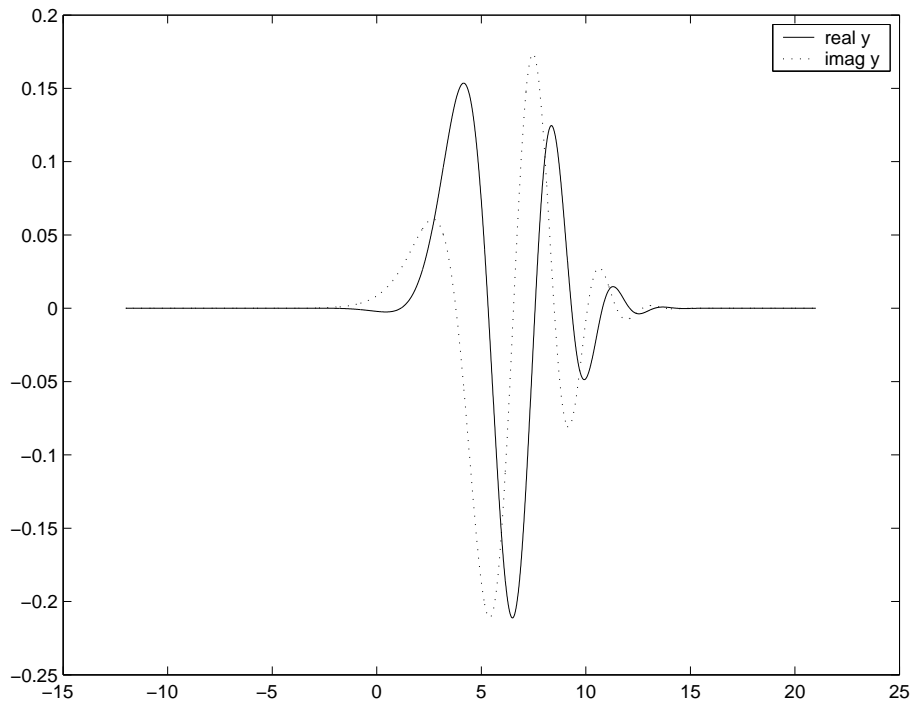
Figure 7.4: A plot is shown of the perturbative term $\Delta\psi(p)$. The wave envelope is peaked near p_0, and the envelope modulates a wave of increasing frequency as p increases from below p_0 to above p_0. The solid curve is the real part of $\Delta\psi(p)$ and the dotted line is the imaginary part.

is given by the integral expression

$$F_H(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x')\; dx'}{x' - x}, \tag{7.133}$$

which also is expressed equivalently as a convolution of the function $f(x)$ with the Hilbert kernel,

$$F_H(x) = \frac{-1}{\pi x} * f(x). \tag{7.134}$$

The singularity of the integrand is taken into account by taking the Cauchy principle value of the integral, which removes the value of the integrand at the singular point. The result is not affected because the support of the singularity is measure zero. A process consisting of two Hilbert transforms will just reverse the sign of the original function, that is $H \cdot H \cdot f(x) = -f(x)$. The total complex function $f(x) = f_R(x) + i f_I(x)$ is known as an analytic function if the imaginary and real parts are related by a Hilbert transform. The imaginary part of the function is known as the *quadrature*. The real and imaginary parts of the function f are called *harmonic conjugates*.

The function $\Delta\psi(p)$, the perturbative term from the time dependent perturbation of the initial momentum distribution $\psi(p)$ was analysed on the mathematical software MATLAB using the Hilbert transform function within MATLAB. The transform of the real part of $\Delta\psi(p)$ was found to be almost perfectly identical to the original imaginary part of $\Delta\psi(p)$. The results of the transform are given in figure 7.5.

Furthermore, the real and imaginary parts of the function being harmonic conjugates
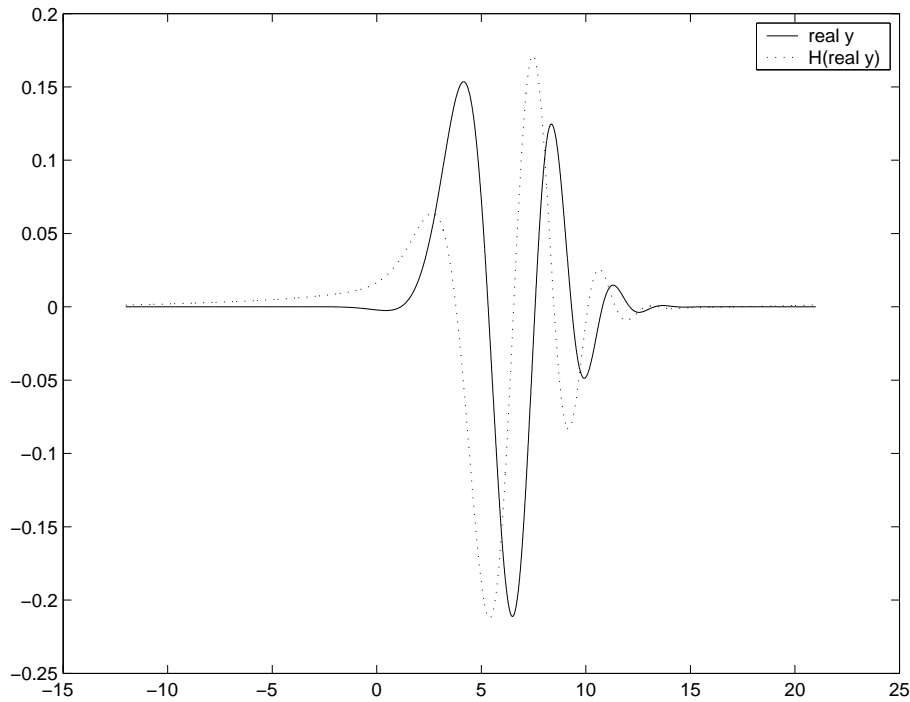
Figure 7.5: The plot shown in this figure is the real part of the perturbation function $\Delta\psi(p)$ and the MATLAB Hilbert transform of that function. The transform matches identically to that of the imaginary part of $\Delta\psi(p)$ that is pictured in figure 7.4.
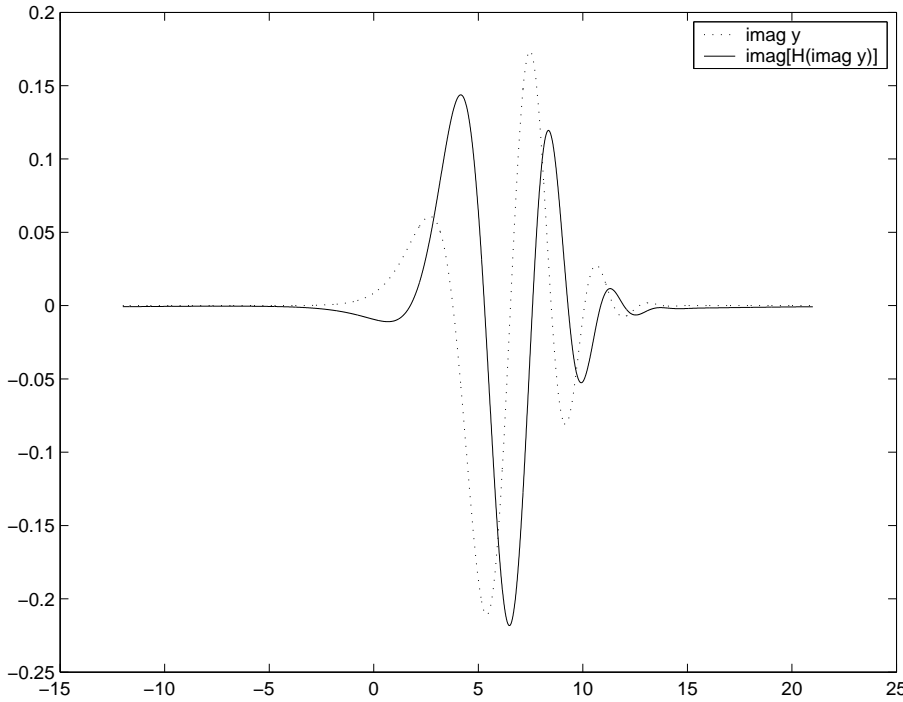
Figure 7.6: The graph here shows the original imaginary part of $\Delta\psi(p)$ along with the negative Hilbert transform of the imaginary part. This pair closely duplicates the original harmonic conjugate pair as depicted in figure 7.4.

implies that the Hilbert transform of the imaginary part should produce the real part. This is indeed the case, as may be seen in the figure 7.6.

The total momentum distribution includes the original wave function added to the perturbation, and the total distribution as a function of p can now be added together and various parameters entered into a plotting program. The plot in figure 7.7 shows the original momentum distribution density $|\psi(p)|^2$ before inflation in the dotted curve, while the solid curve shows the momentum density after inflation, $|\psi(p) + \Delta\psi(p)|^2$. The general trend in these graphs with various parameters is one of a decreasing peak in momenta, signifying a loss in momentum to inflation. However, another general trend
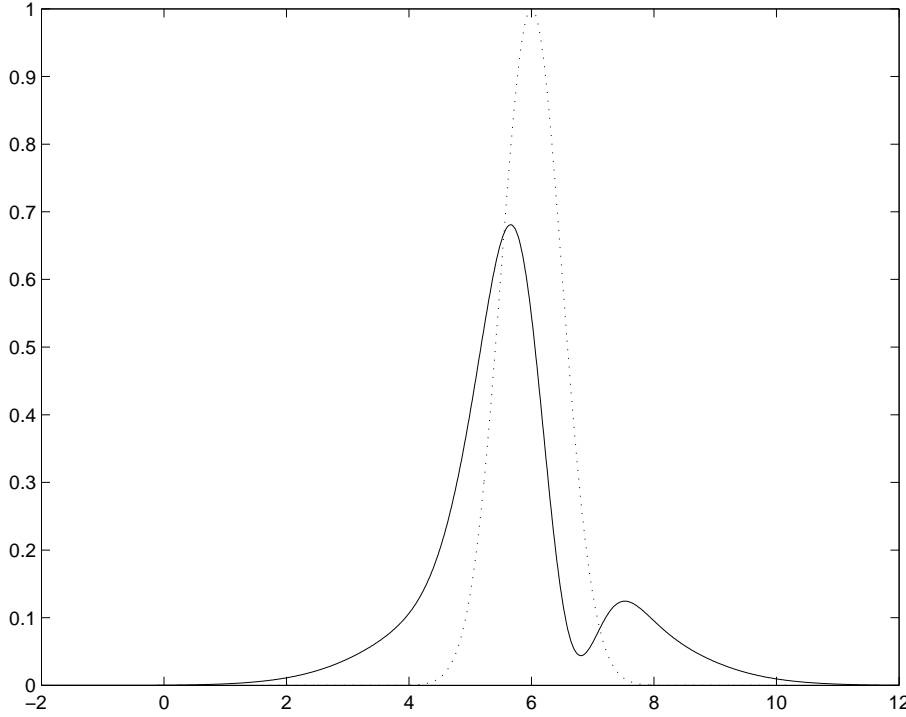
Figure 7.7: This figure shows the total momentum distribution $|\psi(p)\psi^*(p)|$. Before inflation the distribution is Gaussian, and is represented in the dotted curve. After inflation is the perturbed momenta redistribution is represented by the solid curve. The general trend is that of a peak at a lower momentum, but with an added component at a higher momentum due to the ringing effect of the inflationary shock.

is one of bumps in the density at higher momenta, signifying the introduction of some

higher momenta due to the ringing effect of the inflationary shock on the momentum

distribution.

We know that the wave function $\psi(\beta)$ in the anisotropy coordinate is related to the

distribution of momentum by way of a simple Fourier transform, simply because they

comprise a conjugate coordinate/momentum dynamical pair. Such a transform on a

Gaussian distribution of momenta $\psi_G(p)$ results in a Gaussian wave packet in coordinate

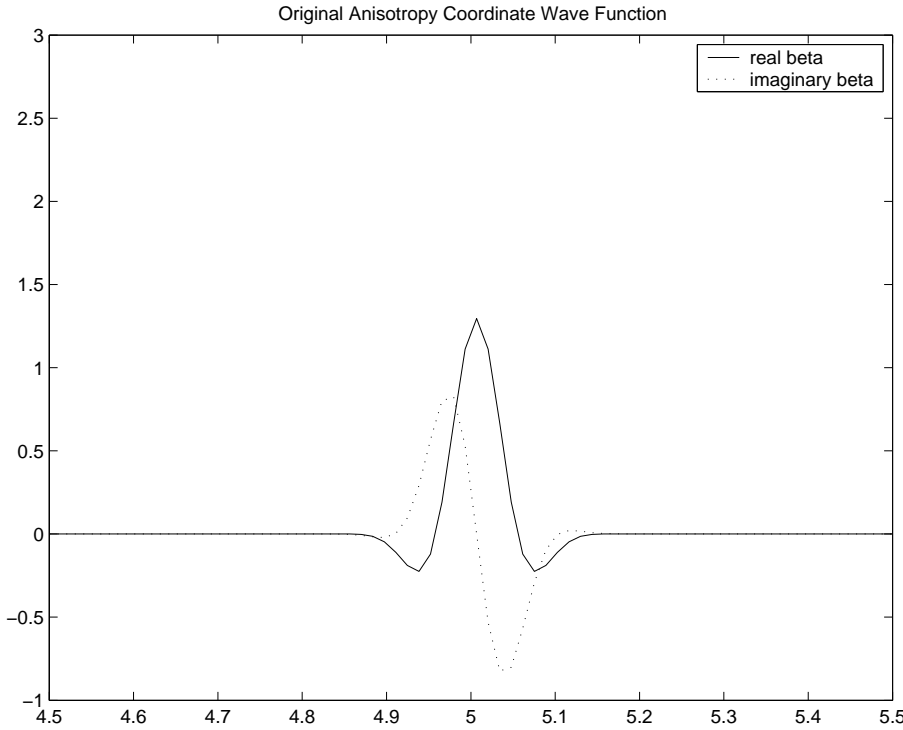space $\psi_G(\beta)$. The Fourier transform of the momenta distribution $\psi_G(p)$ is shown in

Figure 7.8: This figure depicts the shape of the Fourier transformed Gaussian distribution of momenta into that of the anisotropy coordinate wave function $\psi(\beta)$. The original distribution of momenta not being centered at $p = 0$ causes the transformed Gaussian distribution to be modulated into a real and complex waveform as shown.

figure 7.8. The modulus of the Fourier transformed distribution is Gaussian, but the Gaussian is modulated by a phase factor in β that gives it a real wave and an imaginary wave, as shown in figure 7.8. The real and imaginary waves in $\psi_G(\beta)$ are a result of the distribution of momenta not being centered at $p = 0$. The modulus of the wave function is Gaussian in nature and is shown with the real and imaginary component waves in figure 7.9.

The distribution in momenta after the inflationary stage is distorted from a Gaussian distribution, and therefore the Fourier transformed distribution will obviously be some
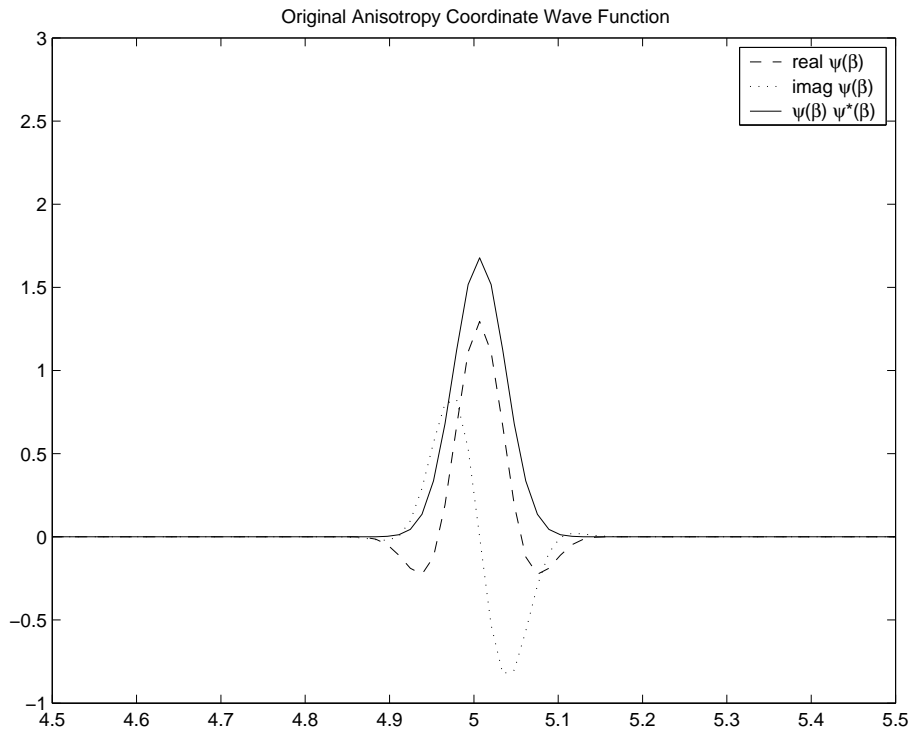
Figure 7.9: This figure shows the real and imaginary component waves of the pre-inflationary wave function $\psi_G(\beta)$. The modulus of the complex wave is represented as the Gaussian curve, which is modulated by a complex phase waveform in β.
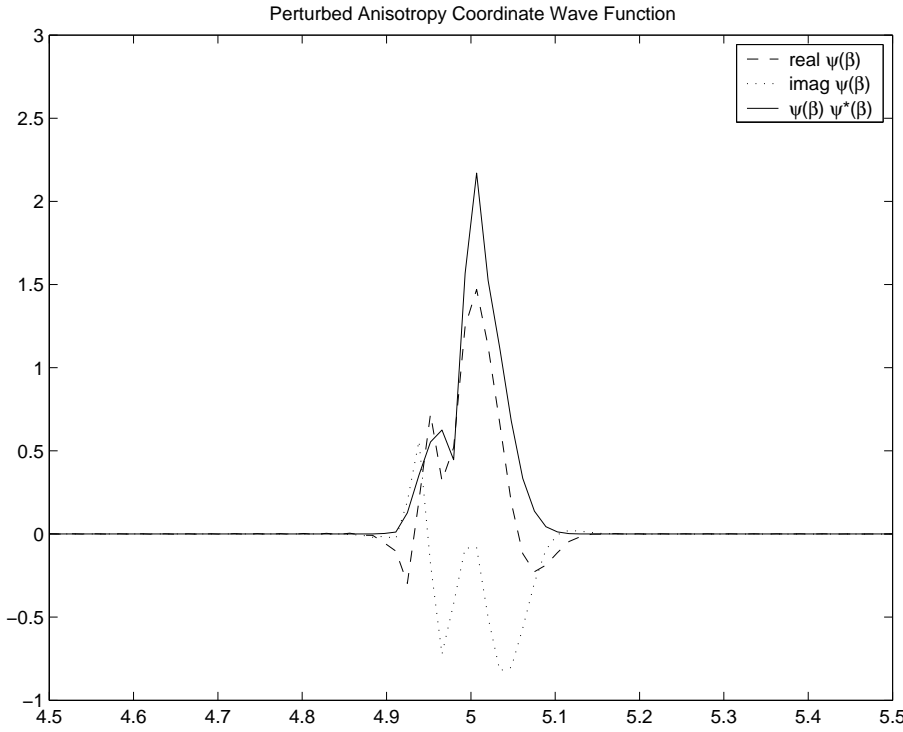
Figure 7.10: This figure shows the total post-inflationary perturbed wave function $\psi(\beta)$. This graph exhibits the dispersion of the wave function and its modulus to that of lower anisotropy.

distorted Gaussian wave function in β. The Fourier transformed post-inflationary wave function $\psi(\beta)$ is depicted in figure 7.10. The general trend for various parameters entered into equation 7.132 is that of the figure where the wave function tends to be distributed to one of lower β. The Gaussian shape is shown to be distorted with a larger antinode at lower anisotropy. These nodes tend to become more pronounced and at lower values of β upon inputting different parameters into equation 7.132, but the shape becomes more distorted from the Gaussian shape and so is less identifiable. The squared moduli $\psi(\beta)\psi^*(\beta)$ of the pre-inflationary and post-inflationary wave functions are shown in figure 7.11.
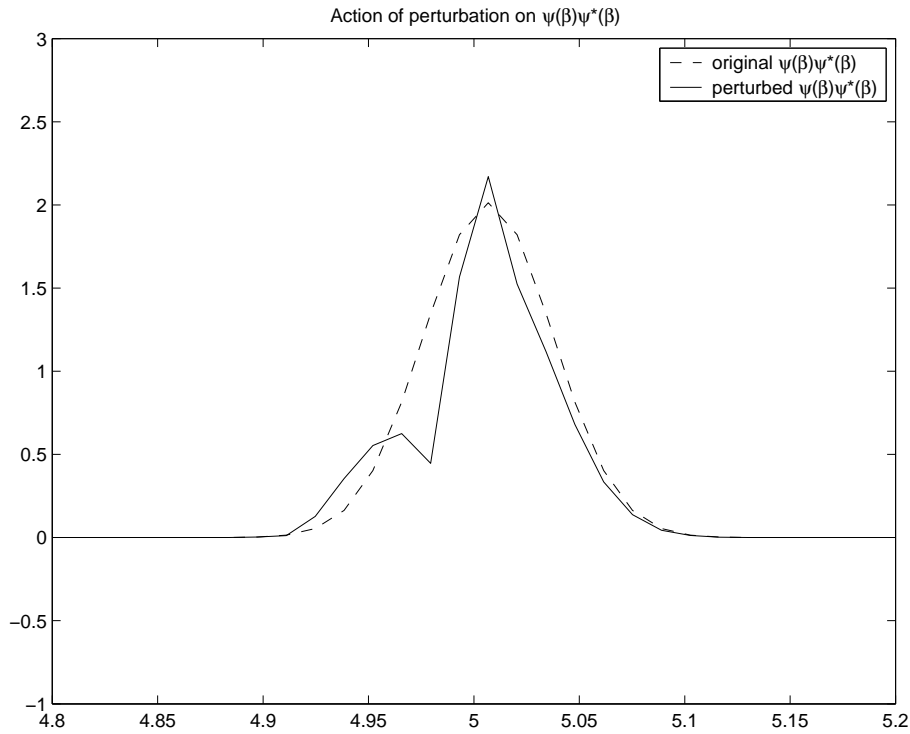
Figure 7.11: This figure shows the comparison of the behaviours of the pre-inflationary and post-inflationary squared moduli of the wave function in the anisotropy coordinate $\psi(\beta)\psi^*(\beta)$. The behaviour is that of dispersion of the wave function towards a lower anisotropy.

Chapter 8

Embedded 3-Space in a Central

Potential

We consider here a related model [19] that is a 3-surface embedded in a flat space of arbitrary dimension parameterized in polar style coordinates. That is, it is a variant upon the multidimensional Kaluza-Klein type of theoretical spaces. Followed in this section is a model of quantum gravitation analysed by by E.I. Guendelman and A. B. Kaganovich, based on work of Kheyfets et al. [27] The total dynamic spatial configuration of the model is parameterized by a cosmological time parameter which serves also as the observable cosmological time. The total spatial configuration is also a Reimannian space, in that the metric has no negative signature contributions, i.e., a non-Lorentz type. The collection of spatial dimensions having a Reimannian metric implies that the space may be referred to as a Kaluza-Klein space.

The topology of the spatial collection is assumed to be toroidally compact in that $0 \leq x_i \leq L_i$, where L_i is some finite length at which a topological identification could be made. The total number of spatial dimensions D is finite but arbitrary, and the $D + 1$ dimensional Lorentz manifold has a line element

$$ds^2 = -dt^2 + \sum_{i=1}^{D} a_i^2(t) dx_i^2. \tag{8.1}$$

The dynamic coefficients a_i are given by

$$a_i(t) = [V(t)]^{1/D} e^{\theta_i(t)}, \tag{8.2}$$

such that

$$V = \prod_{i=1}^{D} a_i. \tag{8.3}$$

The scalar curvature R for the homogeneous $D + 1$ dimensional space is given by

$$R = \frac{2}{V}\frac{d^2V}{dt^2} - \frac{D-1}{D}\left(\frac{d(\ln V)}{dt}\right)^2 + \frac{1}{D}\sum_{i<m}\left(\frac{d(\ln a_i)}{dt} - \frac{d(\ln a_m)}{dt}\right)^2. \tag{8.4}$$

The cosmological gravitational action with a cosmological constant Λ is written as a Hilbert integral as

$$S = \frac{1}{\kappa}\int \sqrt{g}(R - 2\Lambda)\ d^Dx\ dt. \tag{8.5}$$

Because the space is homogeneous the spatial variables $0 \leq x_i \leq L_i$ may be integrated

over, and the result set equal to 1 so that the action integral may be written as

$$S = - \int L \, dt, \tag{8.6}$$

where L may be read off to be

$$L = \frac{1}{V} \frac{D-1}{D} \left(\frac{dV}{dt} \right)^2 - \frac{V}{D} \sum_{i<j} \left(\frac{d(\ln a_i)}{dt} - \frac{d(\ln a_j)}{dt} \right)^2 + 2\Lambda V. \tag{8.7}$$

Performing a mapping of the variables into polar form will allow a Hamiltonian representation which has rotational symmetry with respect to a central potential. Such a reparameterization is given by

$$\begin{aligned} \rho^2 &= \frac{4(D-1)}{D}V \\ \omega^2 &= -\frac{D\Lambda}{2(D-1)} \\ z^i &= \frac{1}{1+\sqrt{D}} \sqrt{\frac{D}{2(D-1)}} \left[\sum_{n=1}^{D-1} \theta_n + (1+\sqrt{D})\theta_i \right], \end{aligned} \tag{8.8}$$

where the Lagrangian function may be written from these relations as

$$L = \left(\frac{d\rho}{dt} \right)^2 - \rho^2 \sum_{i=1}^{D-1} \left(\frac{dz^i}{dt} \right)^2 - \omega^2 \rho^2, \tag{8.9}$$

Performing a Legendre transformation on the Lagrangian function and inserting the transformed variables above,as well as adding a mass term μ representing mass density,

a Hamiltonian function is found that is written as

$$H = \left(\frac{d\rho}{dt}\right)^2 + U(\rho) - \mu, \tag{8.10}$$

where

$$U(\rho) = -\frac{\ell^2}{4\rho^2} + \omega^2 \rho^2, \tag{8.11}$$

where ℓ is an abstract angular quantity, and where

$$\ell^2 = \sum_{i=1}^{D-1} \ell_i^2 \tag{8.12}$$

and

$$\ell_i = -2\rho^2 \frac{dz^i}{dt} \tag{8.13}$$

This is a fully classical Hamiltonian, and is reminiscent of the Hamiltonian of a particle systematized in a multi-dimensional symmetric harmonic oscillator potential in a polar coordinate system.

8.1   Quantized Solutions of the Wheeler-DeWitt Equation in MiniSuperspace

As in the previous sections we consider a tangent and cotangent bundle space with a metric function, and write for the most general Lagrangian and Hamiltonian functions

in their respective spaces;

$$
\begin{aligned}
L &= f_{\alpha\beta}(q)\frac{dq^{\alpha}}{dt}\frac{dq^{\beta}}{dt} - \omega^2\rho^2 \\
H &= f_{\alpha\beta}\frac{dq^{\alpha}}{dt}\frac{dq^{\beta}}{dt} + \omega^2\rho^2 - \mu, \\
H &= f^{\alpha\beta}\pi_{\alpha}\pi_{\beta} + \omega^2\rho^2 - \mu,
\end{aligned}
$$

where

$$
f_{\alpha\beta} = diag(1, -\rho^2, ..., -\rho^2), \tag{8.14}
$$

where the metric tensor matrix may be seen to derive from equation 8.10.

The Wheeler-DeWitt equation is the quantum mechanical equivalent of the classical time-time Einstein equation, which is actually a constraint equation that says the classical Hamiltonian is equal to zero. Using the Dirac prescription of replacing classical dynamical variables with their corresponding quantum mechanical operators, the Wheeler-DeWitt equation reads,

$$
H\Psi = 0. \tag{8.15}
$$

When the above Wheeler-DeWitt equation is transformed by the replacement of dynamic variables by their respective operators (the Dirac classical-to-quantum system method), the Wheeler-DeWitt equation appears as

$$
H\Psi = 0 = \frac{1}{\rho^{D-1}}\frac{\partial}{\partial\rho}\left(\rho^{D-1}\frac{\partial\Psi}{\partial\rho}\right) - \frac{1}{\rho^2}\sum_{i=1}^{D-1}\frac{\partial^2\Psi}{\partial(z^i)^2} + 4(\mu - \omega^2\rho^2)\Psi, \tag{8.16}
$$

where a radial operator ordering in the first term has been done. For the origination of this equation, it is helpful to compare the Hamiltonians in equations 8.14 along with the Hamiltonian in equation 8.10, where q^1 is the dynamical variable ρ and the rest are the spin-like dynamical variables represented by z^i. Recall also that in the quantization from classical dynamical variables, there is the substitution

$$\pi_\alpha \to -i\hbar \frac{\partial}{\partial q^\alpha} \tag{8.17}$$

It is easy to see that the operator $\frac{\partial}{\partial z^i}$ commutes with the Hamiltonian of the above expression. Therefore, a trial wave function solution may be written as

$$\Psi(\rho, z^i) = \frac{1}{(2\pi)^{(D-1)/2}} R(\rho) e^{i \sum_{i=1}^{D-1} \ell_i z^i}, \tag{8.18}$$

where ℓ_i is the constant of the motion in the tangent bundle of $\dot{z}$ that is described in equation 8.13. When the trial function is inserted into the Wheeler-DeWitt equation, a differential equation results which may be solved in terms of a confluent hypergeometric function. The differential equation is

$$\frac{d^2 R}{d\rho^2} + \frac{D-1}{\rho}\frac{dR}{d\rho} + \frac{\ell^2}{\rho^2}R - 4\omega^2\rho^2 R = -4\mu R, \tag{8.19}$$

where the last two terms on the left hand side are the negative of the effective potential in the mechanical space. The apparent condition that ℓ^2 is a positive quantity would

imply that the effective potential would create an attractive core. The limit as ℓ^2 goes to zero would imply an harmonic system. However, it would be possible to introduce different factor orderings that could cause ℓ^2 to be less than zero, and in this case the effective potential would have a repulsive core. Observing the right hand side of the above equation reveals that the complete differential operator of the equation has 4μ as an eigenvalue.

The fact that the wavefunction must vanish in the limit as ρ becomes very large implies that the solutions R become quantized to solutions R_n according to the confluent hypergeometric function Φ, where

$$R_n(\rho) = N_n \rho^{\mu/|\omega|-2n-D/2} e^{-|\omega|\rho^2} \Phi(-n, \mu/|\omega|, -2n, 2|\omega|\rho^2), \qquad (8.20)$$

where N_n is a normalization factor. The development of this wave function is determined by the application of a unitary development operator, which would in turn be determined by the levels corresponding to the quantized levels n of the wave function. However, the interpretation of development of the wave function with respect to a time parameter would need to be analysed further, since the unitary operator would have to be derived from a Hamiltonian function in order to be a time development operator, which in turn would involve the notorious problems of time from the Hamiltonian constraint. The unitary development operator corresponding to the energy levels could be written as

$$\mathcal{U}(n) = |K_n\rangle e^{-i\kappa_n} \langle K_n|, \qquad (8.21)$$

where K_n is an eigenvalue of an observable that would be a complete set of base states countable by n. The complete set of states commuting with the Hamiltonian would imply that they could be used to form a unitary development operator with respect to time, but would be complicated by the Hamiltonian constraint. Thus the above unitary development operator would rotate the distribution of the state vector into a distribution given in terms of eigenstates corresponding to n. Therefore the problem would be to determine which observables would commute with the Hamiltonian function, which is a delicate problem when the Hamiltonian is constrained to be zero. However, we will look further into this problem in the next section where the Heisenberg equations of motion are considered, where the initial state vector is a constant of the motion. The motion however must be considered quantum mechanically because in the classical picture there is no time development due to the Hamiltonian constraint.

There is a large difference between pure vacuum solutions and solutions introducing even a small amount of matter [41]. However, once matter is introduced into this model, the behavior as $\rho \to 0$ (the singularity) can depend in this model on the amount of matter via the density μ. That is, looking at the factor ρ in equation 8.20 reveals that the wave function will vanish as $\rho \to 0$ if its exponent remains positive, i.e. if $\mu > |\omega|(2n + D/2)$. Thus, the singularity can be avoided if the mass density is large enough.

8.2 The Heisenberg Picture

The "problem of time" is discussed in depth by K. Kuchar [28]. Basically, it centers around the problem of the Hamiltonian being constrained to zero, leaving no infinitesimal generator in the direction of time (either forward or backward). This constraint appears in the time-time equation of the set of Einstein equations of general relativity. Therefore, it is somewhat difficult to produce the Heisenberg equations of motion because of the Hamiltonian being constrained zero. However, in the model being studied in this section it is possible to produce non-trivial equations of motion.

In the Schrodinger picture of quantum mechanics, the operators (such as position and momentum operators) which have observables as eigenvalues are not time dependent and do not change with time. However, in that picture the state vector evolves with time and so will produce changing eigenvalues for the static operators.

The Heisenberg equations of motion imitate the classical equations of motion (or better, vice versa), where the observables (such as position and momentum) are dynamical functions which evolve with time. Recall that for some function $A(q, p)$ in the classical phase space the differential equation for time evolution of that function is

$$\frac{dA}{dt} = [A, H]_{Poisson} = \frac{\partial A}{\partial q^i}\frac{\partial H}{\partial p^i} - \frac{\partial A}{\partial p^i}\frac{\partial H}{\partial p^i}. \tag{8.22}$$

It is generally believed that quantum mechanics is *the* basic theory, and where the Planck's constant h plays a fundamental role. But the correspondence principle is such

that as the system becomes more macroscopic, Planck's constant begins to appear small, and will look like zero in the classical limit. And so, it is better to think of the quantum mechanical commutator becoming the classical Poisson brackets, as opposed to vice versa, as in the correspondence

$$\frac{1}{i\hbar}[\hat{A}, \hat{B}] \longleftrightarrow [A, B]_{Poisson}. \tag{8.23}$$

The Hamiltonian 8.10 may be written in terms of ρ as

$$H = \frac{1}{4}p_\rho^2 - \frac{\ell^2}{4\rho^2} + \omega^2\rho^2 - \mu, \tag{8.24}$$

where $p_\rho = \frac{\partial L}{\partial \dot{q}} = 2\dot{q}$. The commutator of ρ with the Hamiltonian also gives the regular classical equation of motion for ρ,

$$\dot{\rho} = [\rho, H]_{Poisson} = \frac{1}{2}p_\rho, \tag{8.25}$$

where the mass of the effective relativistic particle is $\frac{1}{2}$ in units used here. When p_{z^i} and H are entered into the Poisson bracket equation 8.22, it is easily seen that

$$\dot{p}_{z^i} = [p_{z^i}, H]_{Poisson} = 0, \tag{8.26}$$

which was already known from symmetries, i.e. that the Hamiltonian is cyclic in z^i. With the Hamiltonian in the above form, the classical equation of motion for the dynamical

variables z^i are

$$\dot{z}^i = [z^i, H]_{Poisson} = \frac{p_z}{2\rho^2}. \tag{8.27}$$

A Schrodinger picture wave function has been obtained previously in this chapter by way of the Wheeler-Dewitt equation 8.16. But it is fruitful to work in the Heisenberg representation because the classical equations are more easily analysed quantum mechanically in that picture, and transfer from classical to quantum more naturally. The classical constraint equation forms a manifold consisting of a set of solutions to the constraint. However, in the quantum mechanical system it is possible for nonclassical effects such as quantum mechanical tunneling and so forth, which depart somewhat from the classical constraint equations. Therefore, it is necessary to consider a quantum mechanical solution space that can depart from the classical continuum manifold of solutions.

According to equation 8.23 and equation 8.27 it is observed that the commutator of the dynamical variables z^i with the Hamiltonian is nontrivial, even with the Hamiltonian constraint. Thus, the Heisenberg picture reveals that there is indeed a time dependence which is not contained as information given in the constraint equations. From the equations 8.14, and using the standard replacement in the configuration variable basis of the classical momentum with the quantum mechanical operator equivalent, along with the canonical position-momentum commutation relations

$$\pi_i \rightarrow -i\frac{\partial}{\partial z^i}$$

it is possible to retrieve the operator equations that give time dependence in this model in accordance with the equations of motion in the Heisenberg representation. That is, using equations 8.14

$$
\begin{aligned}
\dot{z}^{\gamma} &= -i[z^{\gamma}, H] \\
&= -i[z^{\gamma}, f^{\alpha\beta}\pi_{\alpha}\pi_{\beta}] \\
&= -if^{\alpha\beta}[z^{\gamma}, \pi_{\alpha}\pi_{\beta}] \\
&= -if^{\alpha\beta}\left(\pi_{\alpha}[z^{\gamma}, \pi_{\beta}] + [z^{\gamma}, \pi_{\alpha}]\pi_{\beta}\right) \\
&= -if^{\alpha\beta}\left(\pi_{\alpha}\delta^{\gamma}_{\beta} + \delta^{\gamma}_{\alpha}\pi_{\beta}\right) \\
&= -i(f^{\alpha\gamma}\pi_{\alpha} + f^{\gamma\beta}\pi_{\beta}) \\
&= -2if^{\gamma\gamma}\pi_{\gamma} \ \text{(no sum)} \\
&= -\frac{i}{2\rho^2}\pi_{\gamma} \\
&= -\frac{i}{2\rho^2}\frac{\partial}{\partial z^{\gamma}}
\end{aligned}
$$

$$(8.29)$$

One can attempt to solve for the evolution of the matrix elements of the dynamical operators in the Heisenberg representation. Naively, one could write for the change in $\langle\psi|\, z^i\, |\psi'\rangle$, keeping in mind that the state vectors are constant,

$$
\frac{\partial}{\partial t}\langle\psi|\, z^i\, |\psi'\rangle = \langle\psi|\, \frac{\partial z^i}{\partial t}\, |\psi'\rangle
$$

$$(8.30)$$

It is possible to consider performing a measurement on the system in accordance with the expectation value formalism. Some interesting questions and insights can be brought to light by looking at some simple examples in the Heisenberg picture. The cosmological example is similar in many ways to the free particle example in classical and quantum mechanics.

Consider the ordinary canonical commutator

$$[x, p] = i\hbar \tag{8.31}$$

This relation is considered as a starting point of quantum mechanical analysis [39]. Dirac has named them "fundamental quantum conditions". However, the above relation also coincides with a Heisenberg equation motion of a relativistic free particle,

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{1}{i\hbar}[x, H] \\
&= \frac{1}{i\hbar}[x, pc] \\
&= c\frac{1}{i\hbar}[x, p] \\
&= c.
\end{aligned}
\tag{8.32}
$$

Therefore a relativistic particle should be quantum mechanically fundamental as well. Erroneous results are obtained if one considers the equation of motion for a non-relativistic particle when measured with respect to normalized momentum eigenstates while assum-

ing that the Hamiltonian is Hermitian and applied in a quadratic fashion;

$$
\begin{aligned}
\langle p| \frac{dx}{dt} |p\rangle &= \langle p| \frac{1}{i\hbar}[x, \hat{H}] |p\rangle \\
&= \frac{1}{i\hbar} \langle p| [x, \frac{\hat{p}^2}{2m}] |p\rangle \\
&= \frac{1}{i\hbar 2m} \langle p| (\hat{p}^2 x - x\hat{p}^2) |p\rangle \\
&= \frac{1}{i\hbar 2m} (p^2 - p^2)\langle p|p\rangle \\
&= 0,
\end{aligned}
\tag{8.33}
$$

where the correct result should be equal to p. However, if the commutator is worked out within the brackets before applying the operator to the states, a more satisfactory result is obtained, i.e.

$$
\begin{aligned}
\langle p| \frac{dx}{dt} |p\rangle &= \langle p| \frac{1}{i\hbar}[x, H] |p\rangle \\
&= \frac{1}{i\hbar} \langle p| [x, \frac{p^2}{2m}] |p\rangle \\
&= \frac{1}{i\hbar 2m} \langle p| (p[x, p] + [x, p]p) |p\rangle \\
&= \frac{1}{i\hbar 2m} (2pi\hbar)\langle p|p\rangle \\
&= \frac{p}{m}.
\end{aligned}
\tag{8.34}
$$

Evidently, the correct result is obtained by using the basic definitions of the commutator, that appears to be associated with a relativistic particle, even though there were no relativistic assumptions about the energy of the particle. Therefore, the limiting behaviour

of mechanical processes is evidently important in the basic definitions and the application of the definitions. It is known that in order to measure the exact position of a particle, the state of a probing particle must be that of an infinite frequency or zero wavelength, and that a measurement of the momentum of a particle must be performed with a probe particle of zero frequency or zero momentum in order not to disturb the momentum state of the particle [40].

Another limiting concept is interesting with respect to gravitation. The equivalence principle appears valid only to a certain degree. It is well known that a test mass can only be so large before it affects the surrounding spacetime, and becoming a perturbation on the solution to the Einstein equations. However, it is also interesting to consider the other limit as the test particle becomes lighter and lighter. The equivalence principle states that a test mass greater than zero will follow a certain geodesic path in the spacetime manifold. The equivalence principle would seem to state that in the limit as the test mass goes to zero, all that would be remaining is the geometric path of the geodesic, which is a timelike path within the region bounded by the light cone. The equivalence principle is experimentally accurate down to very light particles. However, in the limit as the test mass goes to zero, the particle would not follow the timelike geodesic path, but would follow a null path on the light cone boundary itself. Therefore, at some point in the limit as the test mass gets smaller, the equivalence principle will fail, and the particle would evidently jump from the timelike geodesic path to a null path, two very separate and distinct curves.

In the Heisenberg picture, the expectation of a measurement of the anisotropy dynamic variables is given by the expression

$$
\begin{aligned}
\frac{d < z^i >}{dt} \quad &= \quad \left\langle \frac{dz^i}{dt} \right\rangle \\
&= \quad \int \Psi(q_\alpha) \left(\frac{-i}{2\rho^2} \frac{\partial}{\partial z^i} \right) \Psi(q_\alpha) \sqrt{|\Pi f_{\alpha\alpha}|} d^D q \\
&= \quad \left\langle \frac{-\ell_i}{2\rho^2} \right\rangle \\
&= \quad -\frac{\ell_i}{2} \left\langle \frac{1}{\rho^2} \right\rangle
\end{aligned}
\tag{8.35}
$$

in accordance with the results of the classical theory, equation 8.13.

With respect to the "almost solutions" which are not on the constraint manifold, another limiting procedure may be employed in order to obtain the correct solutions for the dynamic operator commutator equations in the Heisenberg principle. Essentially, the commutator equations may be measured with respect to states which are off of the solution manifold, and therefore have non-vanishing commutators, and then brought to the manifold by way of a limiting process, i.e.

$$
< [\Omega, H] > = \lim_{\langle \Phi| \to \langle \Psi|, |\Xi\rangle \to |\Psi\rangle} \frac{\langle \Phi| [\Omega, H] |\Xi\rangle}{\langle \Phi|\Xi\rangle}
\tag{8.36}
$$

This limiting process will obtain solutions when the limit is approached in the right fashion, without commuting with zero constraint.

Chapter 9

Conclusion

The problem of cosmological anisotropy is an intriguing one. The anisotropic model is the more general model allowed from the constraint equations. Therefore it would seem probabilistically that at least some anisotropy would exist in observations, unless there would be a physical reason for decay of anisotropy. The problem could be compared with that of the cosmological constant Λ in the Einstein equations. The constant is allowed into the equations and there are observations that suggest that Λ is indeed non-zero. However, there is a very small upper limit on Λ given by observations of motions of the planets in the solar system. Therefore, it is possible that there is some physical reason that Λ is zero or nearly zero, as opposed to any value that it could possibly take. With respect to anisotropy however, the Friedman-LeMaitre isotropic cosmological model complies with observations, yet is only a special case of Bianchi I models which admit any values of anisotropy in general.

   We have found here, that given some initial distribution of the momentum of anisotropy,

the inflationary shock will reduce the mean value of the momentum as well as reduce

the mean expectation value of the anisotropic coordinate β.   The result gives insight

into observations of minimal anisotropy in the post-inflationary universe. Previously the

problem of dissipation of anisotropy in a model with zero curvature such as Bianchi I

has been difficult due to the fact that in minisuperspace the potential well restricting the

anisotropy was formed from the scalar curvature, as we have seen.

   The effects of anisotropy may be expressed in terms of an effective energy density

quantity formed from the time derivative of the metric determinant (see [9]), where the

determinant is expressive of the scale factor.   The effective energy is expressed as

$$H^2 = (\frac{1}{3}\frac{d}{dt}\ln\sqrt{g})^2 = \frac{8\pi}{3}\rho_{an}, \tag{9.1}$$

and the equation of state corresponding to the energy density is found to be

$$\rho_{an} \sim g^{-1} = (\text{vol})^{-2}. \tag{9.2}$$

Early work on the effects of dissipation of anisotropy and associated energy was performed

by Parker and Zel'dovich (see [48]).   One is referred to this line of references to find

out more about particle creation in the early universe with respect to dissipation of

anisotropy.

   The dissipation of anisotropy we have found to be related to the dispersion relations of

wave mechanics, in that the wave function describing the probability density of anisotropy coordinate suffered dispersion, and in particular a dispersion toward lower values of anisotropy. We have found early on that dynamic anisotropy is the complement to the theory of dynamic scale transformations, and together cover a large portion of the total theory of dynamic cosmology. They are both quite different groups of motions, but they have an interplay that originates in the form of the Lagrangian function and the Hamiltonian function. The Lagrangian defined on the tangent space is useful for classical equations of motion but the transition to the Hamiltonian on the cotangent mechanical space was found to be very useful as always for transitions into quantum theory. The literature on the relations between inflation, anisotropy, and scalar motions is extensive and too large to consider completely here. However, a much more complete picture of the system of our universe has been attained in this dissertation, and routes have been opened for future work.

It has been determined that many of the important equations of the theory of quantum mechanics may be applied to the problem of quantum gravity once the Hamiltonian function has been found. In particular, the concepts of the Heisenberg picture and furthermore that of the interaction picture have proven to be exceptionally useful, especially for a Hamiltonian such as the one studied extensively here. The interaction picture has been particularly useful here due to the fact that the effective potential term in the Hamiltonian proportional to $\dot{\Omega}^2$ becomes momentarily very very large at the early time of inflation when the cosmological scale inflated by many orders of magnitude within

a very short period of time. The fact that the same term is otherwise very small in comparison has allowed us to apply the time-dependent perturbation theory of quantum mechanics, along with the useful equation of interaction of perturbation theory.

The cosmic background radiation (CMB) is thought to be an image of the very early universe when the high temperature and pressure were such that the matter was coupled to radiation. When the universe expanded and cooled just enough for a decoupling to take place, the radiation at that time was uncoupled from matter to release the background radiation. Today, after expansion and cooling, the background radiation has cooled to a temperature of about 3 degrees Kelvin, which is in the microwave frequency range of the electromagnetic spectrum. The matter which decoupled from the radiation has formed into galaxies and dark matter (the nature of which is speculative). The CMB was discovered experimentally (and serendipitously) by Penzias and Wilson by way of their famous microwave observations in 1965. The CMB was theoretically predicted originally by LeMaitre's big bang dynamic model of the universe. LeMaitre's model also predicted the Hubble redshift, which was observationally determined by Edwin Hubble. The redshift is a result of Doppler shifts of galaxies which are all receding from one another due to the overall expansion of the universe. The electromagnetic frequency signature of the universe before decoupling was that of a black body spectrum because of compact high energy conditions causing an equilibrium state of energy sharing. However, after the decoupling era and the following expansion the radiation has cooled in accordance with the $\frac{1}{R}$ law, but has retained its black body character, although shifted down in its

peak frequency.

The background radiation has a signature spectrum of black body radiation, which signifies that there was thermodynamic equilibrium at the time of decoupling. The CMB is also of extremely uniform intensity in observations over the celestial sphere, which indicates that there is a high degree of isotropy in the universe. The Bianchi models that we have studied have a theoretical anisotropy coordinate which would appear in the observational surveys of the CMB as a redshift of the CMB that would have a dependency on direction in the sky. There being a lack of such observations imply that there has been decay of anisotropy, which as we have seen can be an indicator of the epoch that the universe is in presently.

Current redshift surveys show large scale structure in the universe that theoretically evolved from primordial conditions. Figure 9.1 shows results from studies by Huchra et al which depict the departure of large scale structure from homogeneity. Large scale structures are thought to be the remnant artifacts of primeval fluctuations in the original fireball from the big bang.

Figure 9.3 represents the spatial frequency distribution of inhomogeneities in terms of spherical harmonics. The theoretical spatial frequencies of the inhomogeneities are represented by two terms in the equation for the differential temperature variations referred to as the Sachs-Wolfe and Integrated Sachs-Wolfe terms,

$$\frac{\delta T(\hat{r})}{T_0} = \frac{\Phi(\hat{r})}{3c^2} + \frac{2}{c^2} \int \frac{\partial}{\partial t}\Phi(\hat{r},t)dt, \tag{9.3}$$
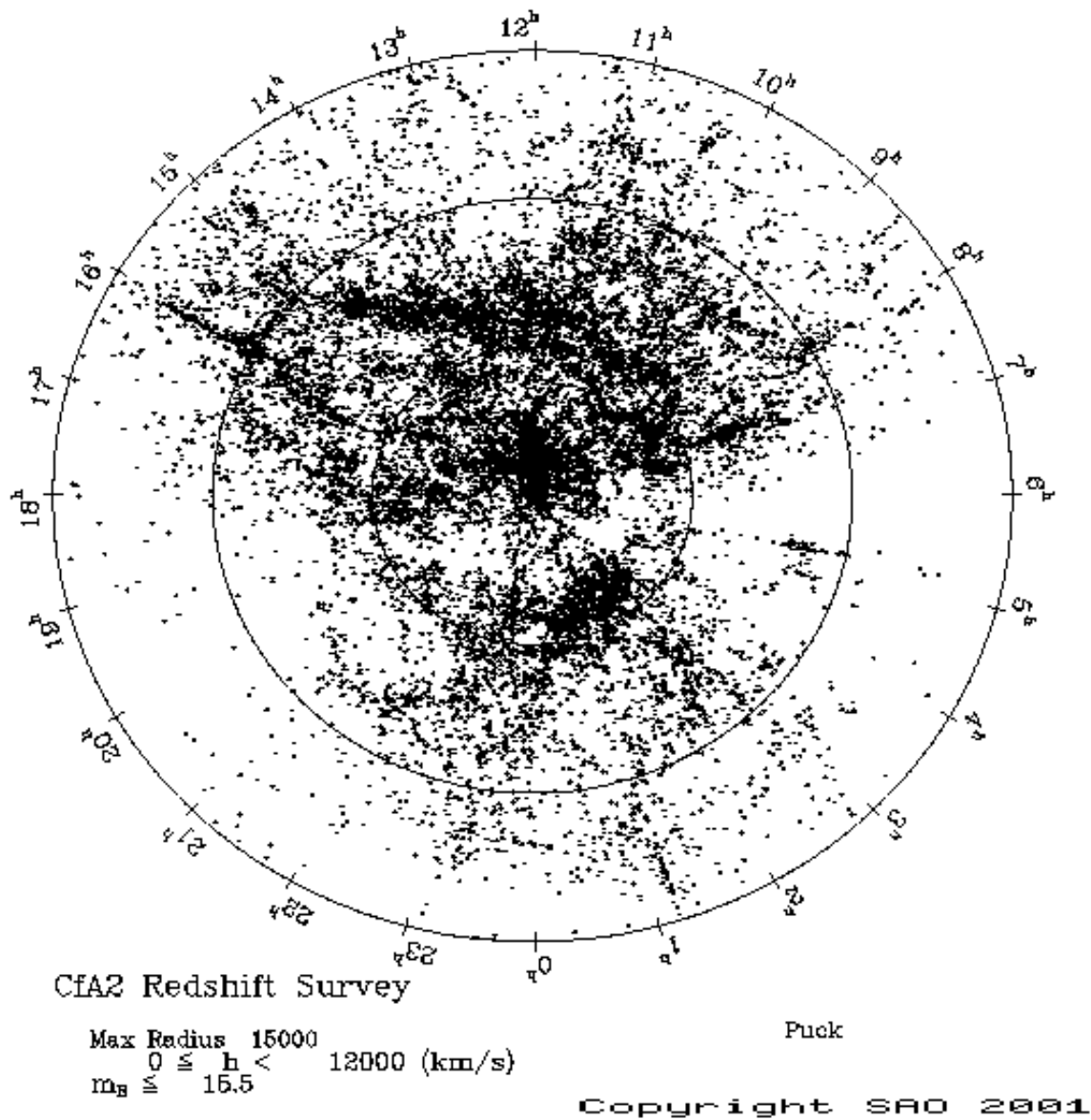
Figure 9.1: Large scale structures (inhomogeneities) are artifacts of quantum fluctuations in the compact early universe.
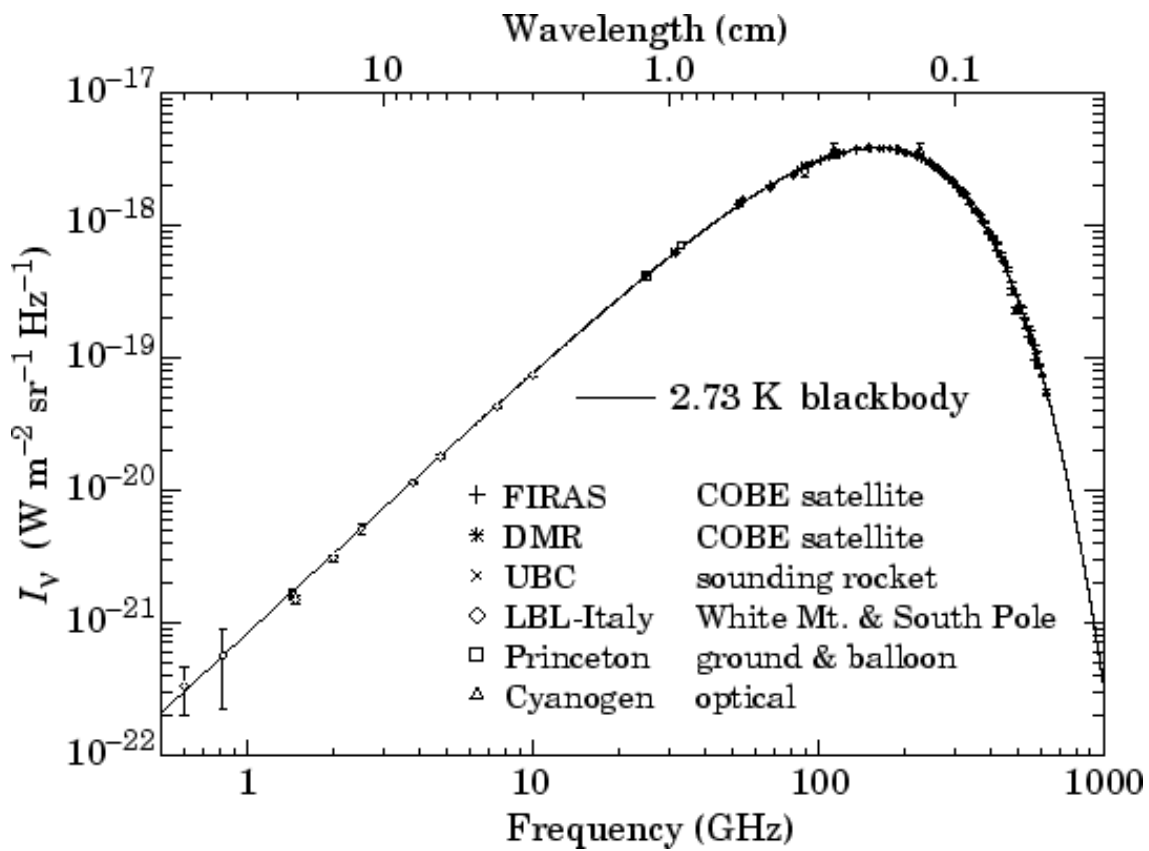
Figure 9.2: The frequency distribution of the cosmic background radiation matches that of a black body equilibrium distribution.

where $\hat{r}$ is the direction of the observation on the celestial sphere, and Φ is the potential
fluctuations that appear in the Poisson equation for the mass density $\nabla^2 \Phi \sim \rho$.

There is no greater satisfaction that can be obtained from studying physics and ge-
ometry than that from seeing the both of them as one and the same entity, referred to
by J. A. Wheeler as *geometrodynamics*. It is indeed "magic without magic", as Wheeler
put it, that a geometric 3-manifold has dynamical equations associated with it that can
determine future geometries from past ones, and vice versa.

Investigations into both classical and quantum theories of gravity and cosmology
have made considerable progress, and have proceeded too far not to have credibility as a
valid and convincing theory. There still remains challenging but at the same time exciting
problems that cosmology still faces. The theoretical work is undoubtedly leading to some
fundamental discoveries concerning the nature of time and subsequent re-definitions that
will be quite revolutionary and surprising. Cosmology has always been and will especially
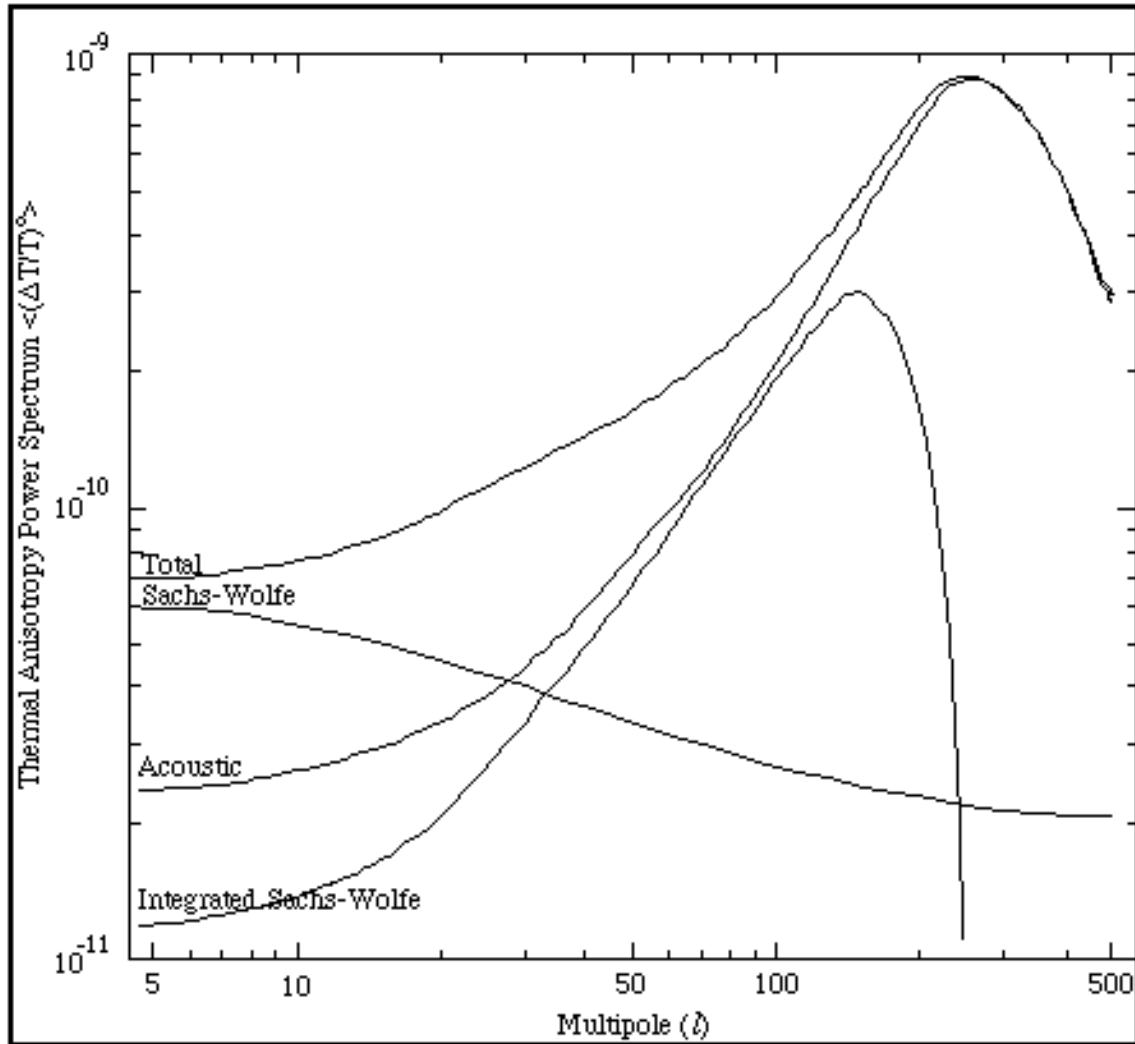be in the future a productive and fertile area for physics.

Figure 9.3: Inhomogeneities represented as spatial frequency distributions over the celestial sphere.

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