ABSTRACT

RIDER, ROBIN LYNN. The Effect of Multi-Representational Methods on Students' Knowledge of Function Concepts in Developmental College Mathematics. (Under the direction of Hollylynne Stohl and Lee V. Stiff)

The purpose of this study was to investigate the potential benefits of a multi-representational curriculum on students' understanding of and connections among graphical, tabular, and symbolic representations of algebraic concepts.

The participants of the study were 313 college students enrolled in developmental college algebra at two southern universities. This study utilized a quasi-experimental design in which instructors at one university (control) taught the course from a traditional algebraic perspective while instructors at the other university (treatment) taught the course from a functional approach simultaneously introducing multiple representations.

The effect of a multi-representational curriculum on student success and representational preference was assessed with a pretests and posttests of five problems, each with three representations; graphic, tabular, and symbolic. The problems were chosen because of their prevalence in most developmental college algebra curricula. Although both curricula were successful in increasing student achievement, students from the multi-representational curriculum scored significantly higher and were significantly more adept in using representational methods other than algebraic to solve the problems.

Qualitative interviews were also conducted with eight participants from each school to examine the connections that students were making and their ability to

move flexibly among the graphical, tabular, and algebraic representations. The interviews were analyzed using Biggs and Collis's SOLO Taxonomy. This research showed that a multi-representational curriculum could be effective in expanding students' web of connected knowledge of algebraic and functional concepts. The SOLO Taxonomy and rubric defined in this research gives teachers an effective way of measuring student learning.

THE EFFECT OF MULTI-REPRESENTATIONAL METHODS ON STUDENTS' KNOWLEDGE OF FUNCTION CONCEPTS IN DEVELOPMENTAL COLLEGE MATHEMATICS

by

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DEDICATION

This dissertation is dedicated to all my students. You have believed in me, inspired me, and touched my life forever. It was a privilege to be your teacher.

In memory of Eric Hucknell, Scott Robertson, and James Willer

BIOGRAPHY

Robin Lynn Rider was born in Millinocket, Maine on September 23, 1963, the daughter of Richard William Angotti, Sr. and Ruth Lane Jewell. After graduating from Stearns High School, Millinocket, Maine, in June of 1980, she ventured into the career world until moving to North Carolina and in 1984, entering the East Carolina University college transfer program at Carteret Technical College in Morehead City, North Carolina. She transferred to East Carolina University in Greenville, North Carolina in the fall of 1986 and received the degree of Bachelor of Science in Mathematics, Magna Cum Laude, in 1988. In the fall of that same year, she began her graduate work at East Carolina University, receiving the degree of Master of Arts in Education in August 1990.

Upon graduation, Robin taught for two years as a lecturer in the Department of Mathematics at East Carolina University. In the fall of 1992, Robin began a career as a public high school teacher in rural eastern North Carolina. She earned National Board Certification in 1998 and entered the Ph.D. program in Mathematics Education at North Carolina State University in the fall of 2001, where her research efforts have focused on the use of multi-representational methods in developmental college mathematics.

Robin Rider is currently a faculty member at East Carolina University, where she is the Assistant Director of the Center for Science, Mathematics, and Technology Education, a position she has held since March 2003. While her primary responsibilities include designing staff development activities for in-service teachers,

she has also had the opportunity to teach undergraduate and co-teach graduate level courses in Mathematics Education.

Her hobbies include spending time with her family, reading, and gardening. An avid outdoorswoman, Robin enjoys hiking in her two favorite parks, Shenandoah National Park in Luray, Virginia and Baxter State Park in Maine. A struggling golfer, she is an active member of Ayden Golf and Country Club. Robin Rider currently resides in Greenville, North Carolina with her husband, Dinh, her daughter Jessica, the family dogs, Boomer and Jackson, and two cats. Robin's son Patrick is a senior attending East Carolina University and lives nearby in Greenville.

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CHAPTER 1

Introduction

Traditionally, pre-college algebra and developmental college algebra are taught with an emphasis on algebraic manipulation. Individual concepts are taught in isolation with little connection to each other. Skemp (1987) describes this type of understanding as instrumental and characterizes it as "rules without reasons" (p. 153). Although students and teachers who understand content instrumentally can perform a procedure and produce a correct answer, they may not truly understand the how or why of what they are doing. Skemp contrasts this type of understanding with relational understanding which he characterizes as "knowing both what to do and why" (p. 153). Almost a decade after the release of national curriculum reforms at both the pre-college and college levels which advocate moving away from teaching only manipulation skills to a more robust and connected understanding of algebra (National Council of Teachers of Mathematics, NCTM, 1989; Cohen, 1995), the majority of pre-college and developmental college algebra courses still employ a traditional approach which promotes instrumental understanding.

An example of traditional teaching practices can be found when introducing functions. There, symbolic function notation is introduced and students substitute values for the input variable and produce the output variable as the answer. The teacher asks questions such as: given $f(x) = -x^2 + 4$, find $f(3)$. The teacher may then have students construct a table of values and produce a graph of that function. However, producing a graph is typically viewed as a process for obtaining a

visualization of the function, without much further exploration of that graphical representation. Students taught in this way may perceive algebra as a study of equations, made up of rules and procedures which need to be memorized (Boaler, 1998).

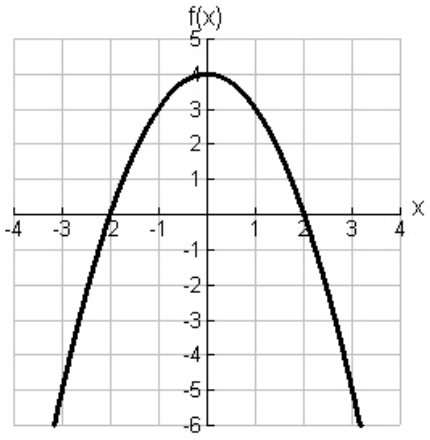
In science classrooms, by contrast, a student's experience can be quite different from a traditional mathematics classroom. Science students are often asked to interpret graphs and tables of data. Rarely does a science teacher present students with an equation. The teacher may give the students a graph of population based on time and ask the students to determine the population at three years. Essentially, the concept is the same as in the case of the traditional mathematics example: finding an output value from a given input value. The representation of the problem presented to the students is different. This difference in instructional approaches can lead students to conclude that mathematics and science are not related because they are approaching the concept of function from different representational perspectives.

One way this trend may be overcome is to teach mathematics from multiple representational perspectives. The NCTM (1989, 2000) advocates utilizing multiple representations of mathematical concepts to build connections among algebraic, tabular, and graphical representations in grades Pre K through 12. *Crossroads in Mathematics* (Cohen, 1995), a document that outlines curriculum for college mathematics below calculus, advocates the same for developmental college mathematics. When teaching functions, instead of always approaching problems from an algebraic perspective, tabular and graphical representations could also be

presented at the same time as the algebraic with connections made among the representations (Figure 1). Each representation, though mathematically equivalent, requires a different type of thought process about input and output variables (Eisenberg, 1992; Janvier, 1987). Understanding what to do and why (relational understanding) with each representation shows a higher degree of concept formation than knowing procedurally what to do (instrumental understanding) with a representation. When viewed together, the representations make input and outputs of functions more salient than any one representation alone.

1.
$f(x) = -x^2 + 4$, find $f(x)$ when $x = 3$

2. Given the graph of f(x), approximate f(x) when x = 3



3. Given the table of values for the function f(x), find f(x) when x = 3

x	**f(x)**
-5	-21
-4	-12
-3	-5
-2	0
-1	3
0	4
1	3
2	0
3	-5
4	-12
5	-21

Figure 1: An example of a question presented in 3 different forms using multiple representations of functions

Many teachers of developmental college mathematics have observed that employing a traditional teaching approach has had limited success in student achievement in both two and four-year college situations (Laughbaum, 1992).

Students in developmental mathematics courses have typically been through at least two pre-college algebra courses in secondary school and have still not mastered the material, a situation that necessitates their placement in developmental mathematics as their first college-level course. These students have shown that they do not grasp the material from a traditional perspective of manipulating algebraic representations. Although suggestions for using alternative methods abound, college departments of mathematics have been slow to embrace these methods with most instructors relying on traditional teaching methods.

One teaching approach that has been advocated is to teach developmental mathematics from a functional perspective, utilizing calculator or computer technology to rapidly produce multiple representations of functions (Conference Board of the Mathematical Sciences, CBMS, 2000). Teachers who have employed these methods believe students gain flexibility in their problem solving and make more connections among representations. These students should be more successful than their traditionally-educated counterparts, though there has been mixed empirical research to support these beliefs (Even 1998; Knuth, 2000a; Knuth, 2000b; Thompson, 1994). A major confounding variable in studies of curricula using multiple representations has been student preference for a particular representation and their unwillingness to utilize a different representation (Keller & Hirsch, 1998; Knuth, 2000a).

Empirical research needs to be conducted to examine the effects of a curricular emphasis on functions and the use of multiple representations of functions

on student achievement in developmental college mathematics. Such research should consider instructional techniques that utilize calculator or computer technology to produce representations and highlight the connections among them. The research also needs to address whether students can overcome preferences toward a specific representation and gain greater understanding of mathematical concepts by employing different representations. Such research could then inform colleges in instructional and curricular choices regarding developmental mathematics programs.

Importance of Multiple Representations

College-level developmental mathematics curricula contain concepts and skills typically taught in secondary school. Thus, curriculum recommendations for secondary school mathematics are often used to inform curriculum and instructional decisions for these lower level college courses. The NCTM (1989, 2000) has advocated a K-12 curriculum which stresses mathematical connections among representations. Even for children in grades 5-8, NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) suggest that students use graphical, numerical, and algebraic representations to investigate concepts, problems, and express results. The algebra standard for the same grade band suggests that students should be able to "represent situations and number patterns with tables, graphs, verbal rules, and equations and explore interrelationships of these representations" (p. 102). Students in grades 9-12 should be able to identify and use equivalent representations (tables, verbal rules, equations, and graphs) of the same concept to solve problems, a

process that requires students to be able to represent and analyze functions with the representations and translate among them.

In 1989, the NCTM felt that "students who are able to apply and translate among different representations of the same problem situation or of the same mathematical concept will have at once a powerful, flexible set of tools for solving problems and a deeper appreciation of the consistency and beauty of mathematics" (p. 146). The NCTM's (2000) most recent document, *Principles and Standards for School Mathematics* also reflects an increased emphasis on representations by adding a "Representation" standard and expanding on expectations for students in grades 9-12. This change reflects a belief by mathematics educators that students' understanding will be enriched from connecting multiple representations of the same mathematical concept.

Making connections among mathematical concepts and representations of those concepts is critical in developing relational understanding because "knowing how they are inter-related enables one to remember them as parts of a connected whole which is easier" (Skemp, 1987, p. 159). Not only does Skemp suggest that connections among the representations make the mathematics easier to understand but also easier to recall.

Functions

The function concept is a central theme in mathematics and serves a coalescing role in a student's understanding of mathematics (Selden & Selden, 1992). One of the predominate uses for functions is in modeling real world phenomena in

which there is a relationship or co-variation between two or more variables. The function concept is uniquely suited to this use because it can be regarded as "a set of ordered pairs, a correspondence, a graph, a dependent variable, a formula, an action, a process, or an object (entity)" (p.4). To develop a rich understanding of the function concept and its uses, one must have a rich understanding of all the ways in which it is possible to represent functions and to be able to move from one view to another.

Therefore, *"*a major goal of the secondary and collegiate curriculum should be to develop in students a *sense for function*" (Eisenberg, 1992, p. 154). A part of this sense for function is the ability to interpret different representations of functions. Eisenberg stresses that an ability to make connections between graphical and analytical representations of functions are a main component of a robust understanding of the function concept. Indeed, this idea seems to be supported by the emphasis on functions and representations of functions in the Mathematics Framework for the 2003 National Assessment of Educational Progress (NAEP) (National Center for Educational Statistics, NCES, 2003).

This idea of using and translating among representations of a concept has been defined by Lesh, Post, and Behr (1987) as a critical part of a student's understanding. A student can be thought of as having an increased understanding if:

> (1) he or she can recognize the idea embedded in a variety of qualitatively different representational systems, (2) he or she can flexibly manipulate the idea within given representational systems, and (3) he or she can accurately translate the idea from one system to another. (p. 36)

In order for students to develop an understanding that allows them to compare and translate among representational illustrations of functions, they must develop a strong process conception of functions (Dubinsky & Harel, 1992). When a student possesses a process conception of functions they view the different representations as different forms of the same entity. Students who have not yet reached the process conception view the different representations as different entities, not just different forms of the same function. With a process conception, students are able to begin with one representation and transform it to another representation and reverse the process. This process conception is complex and students may have a wide variety of levels of development of their conception of functions.

Research on students' conceptions of functions has shown significant problems in students' understanding. This lack of understanding has been linked to a curriculum that fails to make connections among representations or to teach techniques to help students deal with different representations, and strengths and weaknesses that each representation employs (Brenner et al., 1997; Knuth, 2000a; Moschkovich, Schoenfeld, & Arcavi, 1993; Piez & Voxman, 1997). Such a curriculum does not teach toward developing a process conception. Since the NCTM (1989, 2000) has promoted curricula which make connections and emphasize multiple representations of algebraic concepts, one would expect that the majority of students entering colleges would have been taught in this manner. Yet research findings indicate that students at both the secondary level and college level before

calculus are not making the connections among the representations of functions (Knuth, 2000a; Knuth, 2000b).

Reform of Developmental College Mathematics

The American Mathematical Association of Two-Year Colleges (AMATYC) produced a document similar to the NCTM (1989) *Standards* as a guide for all college courses below calculus. This document, called *Crossroads in Mathematics* (Cohen, 1995), outlined three sets of standards for developmental college mathematics courses: intellectual development, content, and pedagogy. As in the NCTM (1989, 2000) and NAEP (NCES, 2003), *Crossroads* advocated a curriculum that emphasized functions through multiple means: graphic, numeric, algebraic, and verbal. It also stressed that students be taught to translate from one representation to another. This ability to transfer and make connections among representations builds mathematical power, which was the goal of the AMATYC standards.

In 2003, over 18% of public 4-year college students and over 50% of public 2-year college students were enrolled in a developmental level mathematics course (Golberg, 2003). These students came into developmental mathematics courses having few symbolic manipulation skills and oftentimes they left the same way (Laughbaum, 2003). In many cases, there was no gain in their mathematical power, an outcome that defeated the objectives of the course. This result could be attributed in part to a curricular focus on abstract symbolic manipulation skills, especially when these skills were disconnected from other mathematical ideas.

Although many developmental mathematics courses use a traditional approach, there has been research suggesting that employing methods recommended by NCTM (1989, 2000) and AMATYC (Cohen, 1995) may improve student performance in developmental mathematics (Crowley, 2001; Hollar & Norwood, 1999; McGowan & Tall, 1999). This improved performance has been attributed to reform curriculum in developmental mathematics that integrates technology with a functional approach and de-emphasizes procedures. These studies support a contention that developmental mathematics curricula should be changed (Crowley, 2001; Laughbaum, 2003; McGowan & Tall, 1999). Research has suggested that instructional methods also significantly affect what students learn (Brophy & Good, 1986; Patterson, 2002). Therefore, in order to implement new curricular methods, instructional strategies must also change to support them.

Technology

One area of curricular change advocated by research and national standards was using technology to allow students ease of access to multiple representations of function concepts (Cohen, 1995; Crowley, 2001; McGowan & Tall, 1999; NCES, 2003; NCTM, 1989; NCTM, 2000). A graphing calculator could be regarded as a representation-producing machine. The user could transform functions expressed in symbolic form into graphical and tabular representations, quickly and easily. When used in conjunction with a curricular approach that highlights the connections among these representations, a graphing calculator could be a powerful tool for developing an understanding of functions (Doerr & Zangor, 2000).

Research has produced mixed results on the effectiveness of graphing calculators. These mixed results have been linked to the teacher and teachers' beliefs about the usefulness of technology in teaching mathematics. Research has shown that teachers' attitudes toward technology and how it is used in classrooms have a dramatic effect on how students perceive use of technology as a tool for learning (Doerr & Zangor, 2000; Patterson, 2002). In their review on the research on graphing calculators, Penglase and Arnold (1996) found that much of the research did not consider factors such as learning environments, teaching methods, or assessment measures used in classrooms incorporating graphing calculators. Consequently, they found contradictory results in the research. Research involving the use of graphing calculators in the classroom must consider these influences.

Problem Statement

After a decade of curriculum reform, students entering universities with several years of exposure to algebra in the high school curriculum still have limited knowledge of connections among different functional representations: algebraic, tabular (numeric), and graphical (Knuth, 2000a). Studies that have compared effectiveness of utilizing graphing calculators to generate representations on student success in developmental algebra have shown mixed results, but many have not taken into consideration pedagogical methods or assessment practices of teachers (Doerr & Zangor, 2000).

Research has found that techniques to help students deal with different representations, and strengths and weaknesses that each representation employs were

rarely directly taught in mathematics classrooms and it was believed that this may have contributed to misconceptions that students held about function concepts (Brenner et al., 1997; Knuth, 2000a; Moschkovich et al., 1993; Piez & Voxman, 1997). Some of the areas of misconceptions included: translating among algebraic, tabular, and graphical representations of functions, understanding the correspondence among the representations, and interpreting the graphical and tabular representations (Galbraith & Haines, 2000).

Studies suggested that students would return to an algebraic representation to solve problems even when an available alternate representation would provide an easier route to find a solution (Keller & Hirsch, 1998; Knuth, 2000b). Reasons for this choice may have included student preference for a particular representation (Knuth, 2000b; Piez & Voxman, 1997) or a curriculum that focuses on algebraic and verbal representations with limited time on tabular and graphical representations (Yerushalmy & Schwartz, 1993).

The purpose of the present study was to compare student success in a traditional developmental algebra curriculum with student success in a multi-representational curriculum that introduced the algebraic, graphical, and tabular representations simultaneously and highlighted connections among them using graphing calculators. A second part of the study examined the problem of student preference for the algebraic representation and whether that preference could be changed to employ flexibility to use the representation most efficient for solving a given task. This flexibility to choose a representation other than algebraic would

indicate a student was moving toward a process conception of function (Dubinsky & Harel, 1992) and toward relational understanding (Skemp, 1987). Since connections among representations has been identified as a major factor in students developing a robust understanding of functions, studying students' connections among representations was a critical part of the study. The specific research questions for the present study will be stated at the end of Chapter 2.

CHAPTER 2

Review of Literature

As noted in Chapter 1, the function concept has been distinguished as a primary theme in mathematics (Selden & Selden, 1992). Although students may have arrived at the college level with several years of high school mathematics, many aspects of the function concept have continued to be problematic for developmental college mathematics students to master. Current reform efforts of pre-college and college mathematics have attempted to rectify this by focusing attention on students' exposure to mathematical concepts and de-emphasizing mathematical procedures (NCTM, 2000; Cohen, 1995). This literature review examined the research related to some of the mathematical concepts that developmental college mathematics students have difficulty understanding and curriculum factors that contribute to these conceptual difficulties. This was an appropriate starting point to examine the effect of a curriculum which considers alternate methods of instruction to overcome these difficulties.

The first section of this literature review looked at research to provide a rationale for a curriculum which employed multiple representations of functions and stressed the connections among representations. Research has attributed this type of curriculum to ameliorating student difficulties in making connections among representations and affecting student preference for a particular representation. These two aspects have been major stumbling blocks in student understanding of the function concept. Thus, this literature review also looked at research in the areas of

connections and preferences. In order for any curriculum to be effective, research on teaching practices such as assessment and use of technology should be considered to determine the most effective curricular methods. This literature review also examined how teachers understanding and beliefs affected their use of reform curriculum strategies. All of these areas had an effect on the conceptual framework for the present study which follows at the end of this literature review.

Research on Multiple Representations of Functions

"The term *representation* refers both to process and to product - in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself" (NCTM, 2000, p. 67). The process referred to by NCTM was both the internal way mathematical concepts were perceived by a learner and the external way they were portrayed as a representation of the learner's conception. This process highlighted how teachers and students externalized and internalized algebraic thinking and how they made connections among different representations in algebra.

A significant body of research has been published on the use of multiple representations (graphical, tabular, and algebraic) to teach function concepts (Crowley & Tall, 1999; Hollar & Norwood, 1999; Keller & Hirsch, 1998; Knuth, 2000a; Knuth, 2000b; McGowan & Tall, 1999). In order for students to use representations effectively, further research should be done to examine which representations are more interesting and useful from the point of view of the student (Janvier, 1987; Keller & Hirsch, 1998). Since "different representations support different ways of thinking about and manipulating mathematical objects" (NCTM,

2000, p. 361) research on the effective use of multiple representations in fostering mathematical thinking has required delving into the way students' perceptions of each representation affects their ability to solve a problem.

The NCTM (2000) suggested that students should be able not only to "select, apply, and translate among mathematical representations to solve problems" (p. 66) but also to "understand how mathematical ideas interconnect and build on one another to produce a coherent whole" (p. 63). Many mathematics educators and researchers shared a common belief that a curriculum utilizing multiple representations enhanced student understanding of mathematical topics. According to Keller and Hirsch (1998), using multiple representations had the following potential benefits: "providing multiple concretizations of a concept, selectively emphasizing and de-emphasizing different aspects of complex concepts, and facilitating cognitive linking of representations thereby creating a whole that is more than the sum of its parts" (p.1).

Some mathematics educators considered the study of functions to be the most important concept in algebra (Harel & Dubinsky, 1992; Yerushalmy & Schwartz, 1993). This importance formed the basis for a belief that functions should be woven throughout algebra utilizing an array of representations (Yerushalmy & Schwartz, 1993). It was found that a key factor in student success in developmental mathematics was developing an ability to move flexibly among the multiple representations of function concepts (McGowan & Tall, 2001). Students who were able to move flexibly

among representations and employ alternate procedures were more successful than students who chose only one type of representation with which to work.

Expectations of teaching an algebra curriculum which emphasized functions with multiple representations, even conventional representations of graph, table, and algebraic, were such that students would be able to use these representations as tools. These representational tools would allow students to choose the most appropriate representation for a problem situation and to understand why the chosen representation was selected over another. “All these expectations suppose that the learner has 'grasped' the representations; that he knows the possibilities, the limits, and the effectiveness of each" (Dufour-Janvier, Bednarz, & Belanger, 1987, p. 111)**.** Dugdale (1993) suggested that students who have developed a collection of different techniques and could apply those techniques to typical problems would be more capable of dealing with new situations because they were not limited to just one approach. Yerushalmy and Schwartz (1993) argued that “allowing students to use a rich set of operations, some of which operate on functions symbolically and some of which operate on functions graphically, builds a deeper and richer understanding of the mathematics" (p. 42). These suggestions were corroborated by the findings of McGowan and Tall (2001) that students were more successful when they could utilize alternate procedures and efficiently translated among representations.

Student learning of functions was enhanced not only by utilizing multiple representations but also by examining differences among representations (Keller & Hirsch, 1998; Noble, Nemirovsky, Wright, & Tierney, 2001) and how one

representation made a concept more salient than another representation. Emphasizing differences as well as similarities allowed students to connect representations to each other, broadening their understanding of functions (Dufour-Janvier et al., 1987; Noble et al., 2001). For example, one concept that is problematic for students is the domain and range of a square root function. By considering the symbolic representations of the functions $y = \sqrt{(x^2)}$ and $y = (\sqrt{x})^2$, a teacher can emphasize verbally that the first equates to $y = |x|$ whereas the second equates to $y = x$ *iff* $x \geq 0$. The power of multiple representations allows students to view a graph of each of the functions to visualize the difference and then examine the corresponding tabular representation (Figure 2). The concepts of domain and range are more salient from a graphical and tabular perspective, which gives students a powerful referent for the absolute value when compared with square root functions of squared variables and for the restrictions on the domain of square root functions.

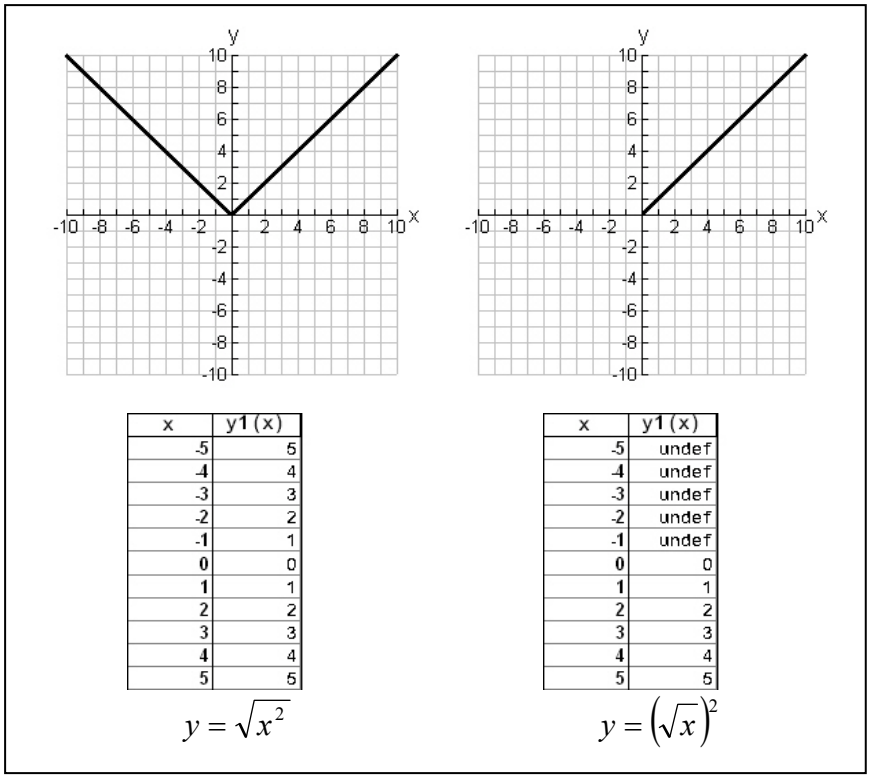


Figure 2: Multiple representations of two square root functions

Although there have been many strong arguments toward a curriculum involving multiple representations, particularly when teaching functions, there have also been some concerns among researchers in adopting such a curriculum. Thompson (1994) advocated that "we orient ourselves toward developing *conceptual* curricula - curricula that are mathematically sound, but nevertheless are constructed from the start with an eye to building students' understandings, and are constructed to assess skill as an expression of understanding" (p. 24). He noted that a curriculum that involves multiple representations but failed to make a connection among those representations becomes equivalent to a traditional curriculum in which each of the representations became just another procedure to be learned.

Connections Among Representations

Using multiple representations to make connections among graphical, tabular, and symbolic descriptions of problem situations has been regarded as an aspect critical to the development of functional understanding (NCTM, 2000). Developing connections among representations of functions was stressed by mathematics educators and researchers in addition to procedural knowledge and underlying concepts (Stein, Baxter, & Leinhardt, 1990).

Thompson (1994) suggested that functions are not represented by what was commonly thought of multiple representations but rather representations provided a method for highlighting connections. An example clarifying Thompson's point is the treatment of the Factor Theorem of polynomials. The Factor Theorem states that a polynomial $P(x)$ has a factor of '$x - c$' if and only if $P(c) = 0$. When the factor theorem is typically taught, teachers emphasize that '$x – c$' is a factor of $P(x)$ if and only if c is a zero of $P(x)$ by writing the theorem out symbolically. This has little meaning for developmental mathematics students trying to factor polynomial functions algebraically. Examining the factors of a polynomial using multiple representations highlights the zeros in the tabular and graphical representations and connections are made to the symbolic factors, making the Factor Theorem more salient than it is in symbolic form only (Figure 3).

Determine the factors of $f(x) = x^3 - 2x^2 - 5x + 6$

$f(x) = (x + 2)\cdot(x - 3)\cdot(x - 1)$

x	y1(x)
-3	-24
-2	0
-1	8
0	6
1	0
2	-4
3	0
4	18

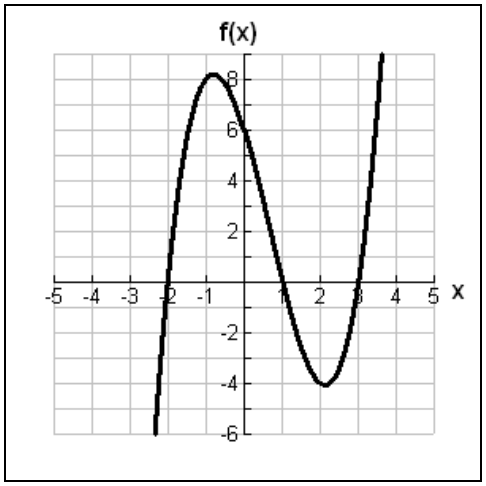


Figure 3: An example of utilizing multiple representations to highlight connections when teaching the Factor Theorem

The study of links or connections that students make among and between representations has been an area in the study of functions in which research was lacking (Knuth, 2000a). In a significant publication on the research of graphical functions, Romberg, Carpenter, and Fennema (1993) suggested that "a coherent body of knowledge about how the connections are developed among tables, graphs, and the algebraic expressions related to functions is desperately needed" (p. ix). A decade later, there still seemed a need for this knowledge contribution.

Although multiple representations have been assumed to be stressed in secondary education, many students entering college had little or no understanding of the connections among representational forms (Knuth, 2000a; Galbraith & Haines,

2000). Thus, at a collegiate level, developmental mathematics students were especially prone to having difficulty making connections among multiple representations of functions. Yerushalmy and Schwartz (1993) attributed this lack of connection to the order in which the representations are taught.

> Within the traditional process of learning algebra in the secondary schools, the learning of graphs of functions usually occurs after a long period of numerical and symbolic manipulations and is normally introduced as a final stage of the subject. We think it is quite likely that certain difficulties observed in the understanding of functions in various representations (numerical, visual, and symbolic representations) might be grounded in this form of learning (p. 43).

There has been little research on whether a curriculum where each of the three representations - graphical, tabular, and algebraic - were introduced simultaneously throughout the curriculum was any better at helping students make the connection among representations than a typical curriculum. Knuth (2000a) contended that an issue in students' ability to make connections dealt with the fact that the typical connection often consists of beginning with an algebraic representation and ending with the graphical representation. Knuth suggested that understanding both directions, algebraic-to-graph and graph-to-algebraic, was fundamental to developing the flexibility to move among the representations.

This type of flexible thinking has been described as reversibility (Krutetskii, 1969). The learner was able to not only look at the direct connection but to also reverse the process. Such insight did not come automatically. For students to develop the skills necessary to visually interpret graphs, it was suggested that instruction should be designed to facilitate that understanding (Eisenberg, 1992; Leake, 1996).

However, teachers' reluctance for using visual methods may have hindered students' willingness to use a graphical representation to solve problems. Janvier (1987) contended that different cognitive processes are involved in translating from one representation to another in a different order. As a result, the direction of the connections among representations may have made one representation easier to use than another for different individuals.

McGowan and Tall (2001) characterized these connections among representations as conceptual (the representations can be thought with and thought about) as opposed to procedural (doing something with the representation). This related to Skemp's (1987) terminology of relational (conceptual) verses instrumental (procedural) understanding. A student who displayed flexibility and could move back and forth among representations displayed conceptual understanding whereas a student who could only move one way had an instrumental understanding and knew a procedure to follow to produce an answer. McGowan and Tall (2001) also suggested that an inability to move flexibly among representational forms caused a conceptual gap that would keep a student from progressing further until that gap was bridged.

One weakness in students' understanding of functional representations was an understanding of the strengths and weaknesses that each representation afforded (Brenner et al., 1997; Knuth, 2000a; Moschkovich et al., 1993; Piez & Voxman, 1997). One possible reason suggested by Knuth (2000a) was that "the majority of students' work with functions is restricted to the domain of algebraic representations, and, as a consequence, students do not develop ability to flexibly employ, select, and

move between algebraic and graphical representations" (p. 506). Since different representations emphasized different features of the function concept, the ability to move flexibly among representations was critical in students being able to choose the representation that will facilitate their ability to most efficiently solve a problem (Dufour-Janvier et al., 1987; Dugdale, 1993; Kaput, 1987; Keller & Hirsch, 1998; Knuth, 2000a; Moschkovich et al., 1993). For example, in the Factor Theorem example previously highlighted in Figure 3, the graph displays the end behavior of a cubic polynomial as well as the shape, approximate zeros, and local extremes. The tabular representation shows the exact zeros as well as a numerical change in the independent and dependent variables. The algebraic representation shows the actual factors of the polynomial. As students gain experience with the different representations and the connections among them, their flexibility in moving among them and choosing the best representation for the problem situation will grow.

Recognizing the equality of procedures that can be used with different representations was described as the first ability needed to have a "property-oriented" view of functions (Galbraith & Haines, 2000; Slavit, 1996). "It is the *property-oriented* view that is most needed when linking algebraic and graphical representations, and that the difficulties in this area that have been reported in so many studies suggests that such a view is substantially absent from student understanding" (Galbraith & Haines, 2000, p. 654). For example, if a student understood that the x intercept of the graph of a polynomial function was equivalent to solving the polynomial when set equal to zero this would demonstrate a property-

oriented view. This would allow the student to make the connections between the two procedures and the relevant properties of each representation needed to solve the task.

Although support has been strong for students' use of multiple representations in learning mathematical and algebraic concepts (Brenner et al, 1997; Janvier, 1987; Noble et al, 2001), there have been legitimate concerns about using too many representations and causing more confusion than understanding (Dufour-Janvier et al., 1987; Noble et al, 2001). If the connections were not highlighted and students were only shown the representations as separate procedures, the number of procedures they needed to perform increased (Thompson, 1994). Thus, their knowledge of functions was not enhanced, but constrained by the representations and this may have greatly influenced students' preference for using different representations.

Student Preference for Representations

Student preference for the algebraic representation has been identified as a major influence in their unwillingness to use an alternate representation even when one was readily available and perhaps easier to use (Knuth, 2000a; Piez & Voxman, 1997). This partiality to the algebraic representation has been observed even when a curriculum emphasized multiple representations of functions (Knuth, 2000a; Thompson, 1994). Student preference for a particular representation was influenced by many factors such as experience, viability of representation to solve the problem,

and curriculum emphasis (Knuth 2000a; Keller & Hirsch, 1998; Yershalmy & Schwartz, 1993).

Yershalmy and Schwartz (1993) found that one of the most significant influences on students' preference of representation used to solve a problem was an instructional emphasis on the manipulation of the algebraic representation. Students often viewed graphs as end-products or something extra that was unconnected to the algebraic representation. Traditional curricula emphasized an equation-to-graph direction in teaching functional representations. This method may have hindered a student's ability in seeing the graph as a viable means for solving a problem (Knuth, 2000b). This was emphasized in a study by Keller and Hirsch (1998) where "the choice of representation may well have been constrained by the individual's perception as to whether the necessary procedural skills were in hand to successfully manipulate the choice of representation" (p. 3).

Experience strongly influenced what representation a student chose to use to solve a problem (Keller & Hirsch, 1998). "Because each representational format has varying limitations or strengths in different contexts, it is beneficial to have the choice of which representations to employ and the knowledge needed to make such a choice" (Lloyd & Wilson, 1998, p. 253). For students to have this choice and this knowledge, they must have had experience with each different type of representation. Keller and Hirsch (1998) pointed out that there were many factors influencing students' preferences for representations such as ability level, student confidence in

symbolic manipulation, experience, perception of the viability of the representation as an option, and complexity of the symbolic information.

Keller and Hirsch (1998), when studying student preference, used a multiple choice preference test, where the answer choices designated which representation the students would use to solve the problem. Keller and Hirsch felt that the initial choice a student had for a representation was the first step in understanding the connections that students made among the representations. Due to the nature of the testing instrument, this study did not address whether the students were able to solve the problems. Thus, a question left unanswered was whether the students were making the choice of representation based on their knowledge of actually how to solve the problem or just choosing a convenient answer choice without any contemplation of how to solve the problem.

Being able to read information from a graphical representation was one significant area in which Piez and Voxman (1997) found that students had difficulty. Because of this difficulty, students tended to avoid graphical representations. Students had difficulty understanding what the coordinates of points represented and lacked the ability to grasp a single variable solution from a Cartesian coordinate system. Through classroom interactions and informal assessment, Piez and Voxman felt that students could gain flexibility in using different representations, especially, with students with weak mathematical skills.

Research on Teaching Practices

Graphing Calculators

A large body of research has been published that suggested using graphing calculators as tools in mathematics classrooms had the potential to increase students' understanding of functions, the connections among representations of functions, and the flexibility to move among the representations (Dick, 1992; Fey, 1992; Hollar & Norwood, 1999; O'Callaghan, 1998). Graphing calculators allowed student and teacher to easily produce multiple representations of a function. This could be used to highlight features of each representation and to illuminate invariance among them, building students' flexibility to move among different representations. The teacher was instrumental in these endeavors by using the technology to produce simultaneous representations and discussing the strengths and weaknesses in each as well as the connections among them. Studies in which teachers continued to use traditional methodology when incorporating graphing calculators showed that the calculator only provides more visual examples but did little to increase student understanding (Simmt, 1997).

Many researchers felt that graphing calculators could be a catalyst for mathematical discourse especially when used in conjunction with a projection device to allow a focal point for a whole class discussion (Doerr & Zangor, 2000; Drijvers & Doorman, 1996; Forster & Taylor, 2000; Goos, Galbraith, Renshaw, & Geiger, 2003; Harvey, Waits & Demana, 1995; Heid & Baylor, 1992). This discourse between teacher and student and among students could be a powerful tool for teachers to

informally assess mathematical understanding in students (Adams, 1997). This discourse did not occur, however, without changing roles for both students and teachers. It required students to take a more active part in learning and teachers to act more as a facilitator and partner than as the authoritarian (Forster & Taylor, 2000; O'Callaghan, 1998). One of the strongest influences on the way calculators are used as tools in classrooms was the belief of the teacher that the tool can enhance mathematical understanding. This belief stemmed from teachers perspectives about mathematics and mathematics education (Doerr & Zangor, 2000; Patterson & Norwood, in press).

In a careful review of the literature, Penglase and Arnold (1996) highlighted the fact that many studies on the use of graphing calculators in the classroom failed to look at the instructional context in which the tool was used. Researchers found that most studies on the effects of incorporating graphing calculators into curriculum did not take into consideration how and why students were using the calculators (Adams, 1997; Ruthven, 1990) nor the relationship between teachers' mathematical and pedagogical beliefs and attitudes about using calculators (Drijvers & Doorman, 1996; Hollar & Norwood, 1999; Van Streun, 2000). It has been suggested that the entire context of the curriculum be addressed, including assessment practices, when researching the effectiveness of a technological curriculum versus a pencil and paper curriculum (Doerr & Zangon, 2000; Penglase & Arnold, 1996).

Van Streun (2000) suggested that "the use of the graphics calculator turned out to be profitable for the weaker students only" (p. 718). Weaker students made

greater gains on assessment instruments than did their counterparts, closing the gap between the weakest and strongest students. Van Streun attributed this success to students' being able to solve problems using a variety of methods. Regular use of graphing calculators seemed to produce a shift in the type of representation used to solve problems, but that students still used symbolic manipulations if it was more efficient for solving the problem.

Others found that students using the graphing calculator had a better understanding of functions because of the increased repertoire of representational use, their ability to tie together ideas, and by their overall success (Hollar & Norwood, 1999; O'Callaghan, 1998; Van Streun, 2000). A contributing factor to this increased understanding when using graphing calculators may have been a multi-representational curriculum that was new to both the students and teacher. "The richer conceptions … students formed about functions can be traced directly to the corresponding instruction, examples, and activities used throughout their curriculum" (O'Callaghan, 1998, p. 37). Teachers had to make daily adjustments and the emphasis in instruction shifted from teacher-directed to a learning community of which the teacher was a facilitator.

The use of graphing calculators has had an effect on classroom practices of teachers. Research showed that, at the 2-year college level, when teaching without a graphing calculator, teachers typically used direct instruction to review homework and cover new content (Adams, 1997). The questions teachers asked were mainly short answer, low-level questions addressed to the whole class with very few

questions directed at individual students. When utilizing the graphing calculator classroom discourse was changed dramatically. There were typically longer verbal exchanges between teacher and students and among students. Teachers tended to leave the chalkboard more often and walked around the room. They looked at what students were doing on the calculator and addressed individuals as well as the class as a whole. The graphing calculator was a tool for teachers to provide less step-by-step instructions and ask more questions to prod students to fill in the gaps in their reasoning. Adams (1997) concluded that the graphing calculator can act as a catalyst for the teacher to interact with students, both spatially (by moving away from the board and closer to the students) and verbally. It also allowed for teachers to observe and address individual students work and was a vehicle which spawned deeper mathematical discussions (O'Callaghan, 1998). Thus, graphing calculators played a role in changing teachers' classroom instructional and assessment practices.

Impact of Teacher Understanding and Teaching Practices on Reform Curriculum

Several studies proposed that teachers' conceptions about curriculum content could impede their ability to effectively implement mathematics reform (Lloyd & Wilson, 1998; Stein et al., 1990; Wilson, 1994). "If we are to support teachers in making long-term instructional changes, it is crucial to continue to investigate the process through which the current reform agenda is interpreted and personalized by teachers involved in the implementation of innovative curricula"(Lloyd & Wilson, 1998, p. 272). Though students entering college may have taken several years of algebra and been very successful at the secondary level, college calculus teachers still

complained about the weakness of their functional skills in preparing them for calculus (Harvey et al., 1995). The call among mathematics education professionals was for functions to be treated as relationships among quantities in all of their forms, symbolic, graphical, and tabular, instead of a set of procedures. This represented a major shift in the way mathematics was traditionally taught and was contrary to the way most mathematics teachers learned mathematics (Klein, 2001). The crux of this shift was a move from a procedural understanding of the function concept to a process conception of functions (Dubinsky & Harel, 1992).

Success in traditional algebra required symbolic manipulation skills and word-problem representational skills. Traditional algebra classrooms "may focus on symbol manipulation skills at the expense of representational skills" (Brenner et al, 1997, p. 684). However, teachers may be unprepared for making curriculum changes to place less emphasis on symbolic manipulation in traditional algebra classrooms.

Good teaching practices have been shown to lead to meaningful connections throughout mathematics. "Mathematics teachers help students learn to use representations flexibly and appropriately by encouraging them as they create and use representations to support their thinking and communication" (NCTM, 2000, p. 284). A teacher facilitated and built on the students' experiences and made sense of conceptions through his or her own conceptual images. A strong teacher's concept of co-variation and graphical representation of functions, a vital part of algebraic thinking, was instrumental when the teacher utilized multiple representations to help students understand patterns in data (Lloyd & Wilson, 1998). "A teacher's repertoire

of examples and problem situations for functions is one of the key bases from which meaningful instruction may be constructed" (p. 252). A teacher with a strong conceptual background encouraged students to examine real-world problems and made connections between them and mathematical functions. Lloyd and Wilson found that "teachers' conceptions contribute to instruction characterized by emphases on conceptual connections, powerful representations, and meaningful discussions" (p. 270). Similar studies (Stein et al., 1990; Wilson, 1994) found that a teachers' ability to lay the foundations for mathematical understanding was influenced by the richness of their own knowledge.

Teachers' with limited content knowledge over-utilized procedures and used those procedures as the centerpieces for lessons (Stein et al., 1990). "The concern is that these heavily emphasized rules may contribute to student understanding that is structurally weak because it is organized around relatively unmeaningful rules" (p. 660). Although the teacher featured in their study covered the unit on functions as was outlined in the text, Stein et al. suggested that he "missed opportunities for fostering meaningful connections between key concepts and representations" (p. 659). A relational and reform-oriented textbook mis-matched with an instrumental teacher may not be significantly different than if that teacher was teaching from an instrumental (procedural) text (Skemp, 1987). Stein et al. (1990) suggested that, in many of the teacher's lessons, "these two representations [algebraic and graphical] remained as isolated islands bridged only by ordered pairs and the practice of self checking" (p. 658).

Pre-service teachers also viewed functions and mathematics "as a collection of concrete procedures to be applied in isolated contexts to obtain correct answers to well-defined problems" (Wilson, 1994, p. 361). Even with experience in a reform curriculum, teachers did not necessarily change this fundamental view of mathematics. Skemp (1987) classified this view as instrumental and stated that the mis-match between a teacher who taught from an instrumental point of view and a child trying to learn from a relational point of view could be extremely damaging to the mathematical education of that child. Wilson (1994) suggested that it would be difficult for a teacher who believed that mathematics was about obtaining correct answers by using a standard set of procedures, which the teacher themselves did not recognize the relationship of, to "encourage students to be creative and explore mathematical relationships" (p. 367). Thus, it would be difficult for this type of teacher to embrace a reform-minded curriculum. More than just text and standards that aim at students having relational understanding were needed to change the teaching style of teachers (Skemp, 1987).

To teach algebraic thinking effectively by using multiple representations, studies found that teachers had to begin to look at algebraic concepts in new ways. In light of the research on how teachers' knowledge of subject matter affected their classroom practices (Stein et al., 1990; Wilson, 1994) research has suggested that some teachers may even need to re-learn the content of what they teach using multiple representations, the connections among different representations, and how those connections stimulate understanding of algebraic concepts.

Changing Assessment Practices

With the impact of written curriculum, technological tools, and changing classroom practices, instituting reform curriculum entailed new methods for both formal and informal assessment techniques (Adams, 1997; Archbald & Grant, 2000). "When the focus and form of assessment are different from that of instruction, assessment subverts students' learning by sending them conflicting messages about what mathematics is valued" (NCTM, 1995, p 13). Thus, any reform curriculum must have incorporated reform of assessment practices. Assessment has been seen a key to informing instruction. It shaped how teachers determine such things as: if their lessons achieved their goals, if students correctly developed their understanding of concepts, if remediation was needed, and showed students' strengths and weaknesses. Assessment also provided students with feedback on their understanding, or lack thereof, of concepts.

Studying the assessment practices of teachers, Archbald and Grant (2000) found that even incorporating teacher education, group planning, and collaboration with university partners was not enough to change all teachers' assessment practices. It was found that problems dealing with representation were incorporated, on average, in 4% of the test items that teachers educated in reform curriculum produced. The majority of all test items given by teachers who were teaching using reform curriculum were either fact/visual recognition or single-path/single-solution problems. When the data was disaggregated by teacher, it showed enormous variability in the types of problems asked by classroom teachers who were all

teaching the same curriculum. If an assumption was made that assessment was aligned with course content, a question that arose was whether course content was not reformed or whether tests did not reflect the content. It would have been difficult for teachers teaching a reform curriculum to accurately assess student learning with traditional testing instruments. Reform-oriented curriculum must be incorporated with reform-oriented test items.

Constructivist Mathematics Teaching

Understanding how students learn a new concept and applying teaching strategies to strengthen that understanding were stated as characteristics of effective teaching (Simon, 1995; Tzur, Simon, Heinz, & Kinzel, 2001). Constructivism has been defined as a theory of learning that embraces the idea that students come into the classroom, not as empty vessels, but at various states of conceptual understanding. Individual students continued to build their mathematical frameworks from the point where they started; therefore, at the end of a particular course, they were still at different states of understanding or knowing. From the constructivist perspective, mathematical learning was seen as a reorganization of ideas already held to incorporate new information, thereby adding to the framework and building conceptual knowledge (Von Glaserfeld, 1987a).

Constructivism as a method of describing how students learn did not prescribe a method of teaching (Simon, 1995). Simon outlined the characteristics of effective teaching in the Mathematical Teaching Cycle. This cycle consisted of 3 broad parts:

1. Assessing what students know

2. Identifying the learning goal (mathematical concept) and hypothesizing a path by which students will come to understand that goal (Hypothetical Learning Trajectory)
3. Planning activities that are likely to bring concept formation and implementing them (planning and teaching). At the end of this step, the cycle continues back at step one.

This Mathematical Teaching Cycle gave a practical framework for teachers to understand and evaluate student learning which informed their classroom teaching.

Because of the tremendous influence of teaching on implementation of reform curriculum (Schoen, Cubulla, Finn & Fi, 2003), teaching practices should be considered in any study that measured the success of such a curriculum. Reform curriculum teachers must strive to be effective teachers of that curriculum which means employing the Mathematical Teaching Cycle. To implement a curriculum involving multiple representations, teachers must have assessed what the students know about each representation, identified the mathematical concept they want the students to learn using the representations, and hypothesized how the students will come to know that concept. Then they must have planned the activities involving the representations that will facilitate students' understanding of a concept and assessed for student understanding using all the representations.

Simon's (1995) framework could be used to guide the practices of a community of instructors teaching developmental mathematics from a functional multi-representational perspective. This type of framework could assist teachers in

changing their teaching practices, making student-based instructional decisions, and hopefully implementing a reform-oriented curriculum from the perspective of developing relational understanding.

Conceptual Framework

Each of the pieces of research mentioned above contained factors which related to studying the effects of a curriculum incorporating multiple representations. Curriculum was viewed more than just a syllabus of topics; it was made up of mathematical content, teaching methods, and assessment practices (Howson, Keitel, & Kilpatrick, 1981). Informing the framework for the present study were: content issues such as functions and multiple representations, teaching methods such as the use of graphing calculators and simultaneous introduction of the representations, teacher attitudes about implementing a new curriculum, and assessment practices.

The conceptual framework for studying students' understanding was grounded in the theory of constructivism. The majority of researchers in the domain of representations shared this theory of how knowledge is developed (Janvier, 1987). Viewed with this lens, conceptual knowledge is constructed by the assimilation of new relationships and stored as a linked network of concepts. Procedural knowledge is gained by practice involving performing a routine in response to a certain stimulus (Galbraith & Haines, 2000). The present study specifically addressed students' conceptual understanding of functions by evaluating the strength of their connections among algebraic, graphical, and tabular representations. These connections are fundamental to functional understanding and developing function sense (Eisenburg,

1992). At the end of the study, if it was shown that the students could successfully solve problems using alternative representations and make connections among representations, it could be concluded that they have made progress toward concept formation.

The present study was based on the theories proposed by Hiebert & Carpenter (1992), Biggs & Collis (1982), and Skemp (1979, 1987). Although not all of these theorists are constructivists, their theories combined form the basis on which the present research was conducted. A key aspect to this research was determining if students have increased their conceptual knowledge of functions. This could be indicated if students could make connections among the representations, implying a move toward relational understanding (Skemp, 1987). If students could solve a problem with more than one representation but viewed those representations as separate entities, this would indicate they have expanded their instrumental understanding in that they had more than one procedure to solve a problem, but not an increased conceptual knowledge because they did not link the representations (Skemp, 1987).

Hiebert and Carpenter (1992) developed a framework for defining students' understanding of representations. They contended that mathematical ideas are communicated in the form of external representations which then must be internally processed by students. The form of external representation (e.g., graphs, tables, or algebraic representations) communicated to the student affected how the student interacted with the representation internally. How the student used that information to

generate or utilize an external representation "reveals something of how that student represented that information internally (p. 66).

Hiebert and Carpenter (1992) suggested that students connect internal representations by creating networks of connected knowledge. These networks are viewed as webs of knowledge, where links among representations of a function were like the threads of the web connecting one representational "node" to another (Figure 4). Because students were at various stages of knowledge construction of a function concept, some representational nodes were connected and some were not.

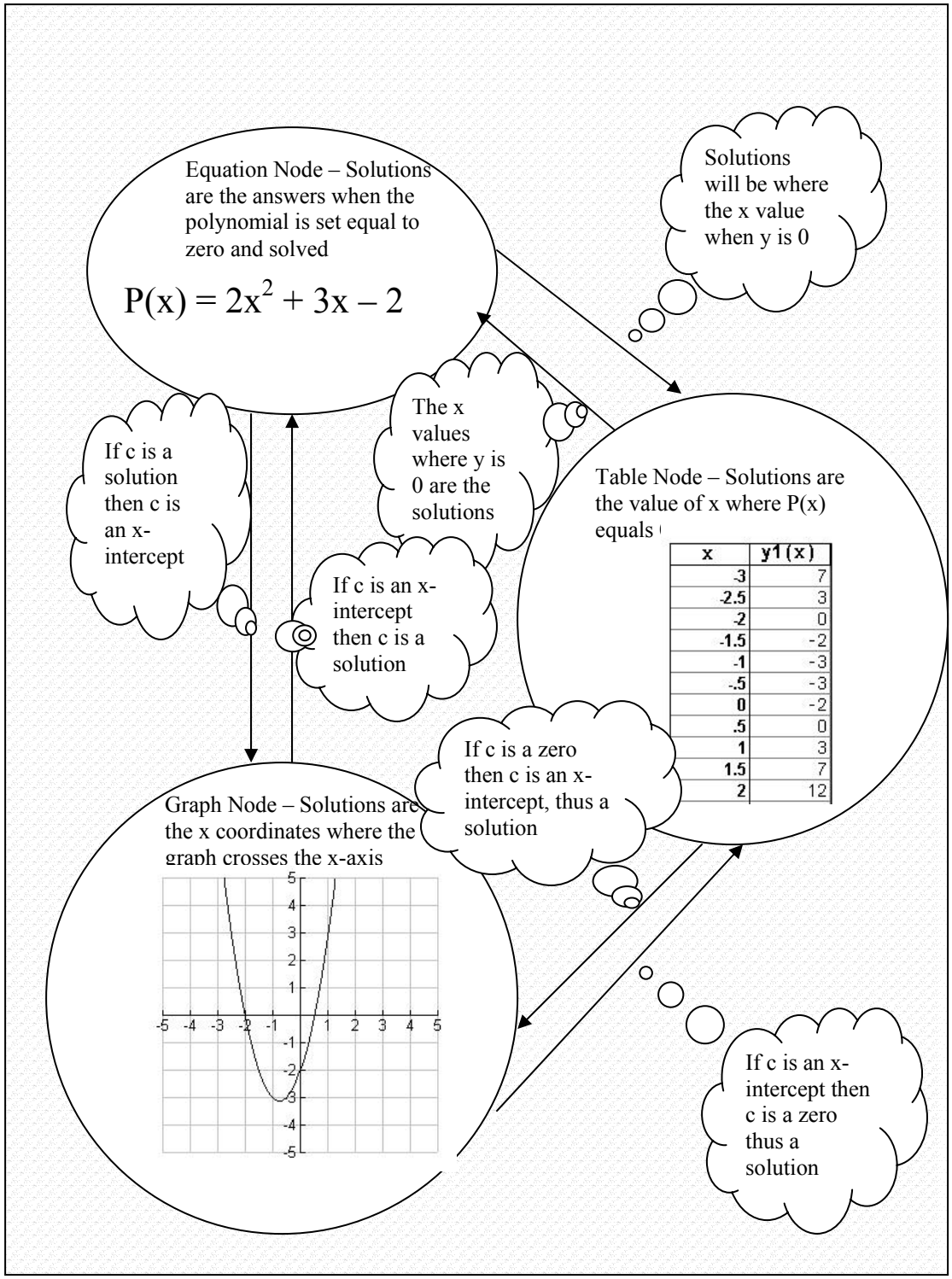


Figure 4: Webs of linked representations of a quadratic function

Skemp (1979) described "varifocal learning theory" in which magnifying each node showed in detail that within the node was a web or connected schema. Among the nodes of the graphical, tabular, and symbolic representation there were connected threads of understanding. By examining the node of symbolic representation alone, there was also a connected schema within that node. For example, when examining a polynomial function, P(x), detailed algebraically (within the algebraic node) one could solve P(x) for zero, find the coordinates of the vertex by completing the square, etc (see Figure 4). Within the graphical node, the zeros would be x-intercepts of the graphical representation; the coordinates of the vertex would be where the function reached its max or min, etc. An internal connection among the nodes enabled the learner to see, for example, the zeros of the function algebraically and the x-intercepts of the graph as a single concept.

The SOLO Taxonomy (Biggs & Collis, 1982) was used for assessing a students' web of connected knowledge. SOLO is an acronym for Structure of Observed Learning Outcomes. The SOLO Taxonomy classifies answers to written assessment questions into five qualitative categories, evaluated according to the nature and strength of connections made: *prestructural, unistructural, multistructural, relational*, and *extended abstract*. The SOLO Taxonomy has been used in college algebra to determine student's cognitive development (Crowley, 2001), in statistics education to evaluate students' understanding of chance (Watson & Moritz, 1998), in evaluating student learning outcomes in organic chemistry (Hodges & Harvey, 2003), in evaluating learning outcomes in counseling education (Burnett, 1999), and in many

other disciplines. One of the benefits of the SOLO Taxonomy has been attributed to its applicability to a wide range of content domains (Pusey, 2003).

Prestructural concept formation is manifested by the student having no knowledge of the element being assessed. A student at this level might have pieces of isolated information but which make no sense with the problem situation. At the *unistructural* level a student focuses on a single portion of the task. They may have made associations that were obvious but did not understand their significance. The *multistructural* level describes concept development in which the student focuses on several aspects of the task, but the connections among them are not apparent. At the *relational* level the student sees the parts as they related to the whole. The *extended abstract* level is reached when a student could make connections within the task and could also transfer the ideas beyond the problem situation (Figure 5).

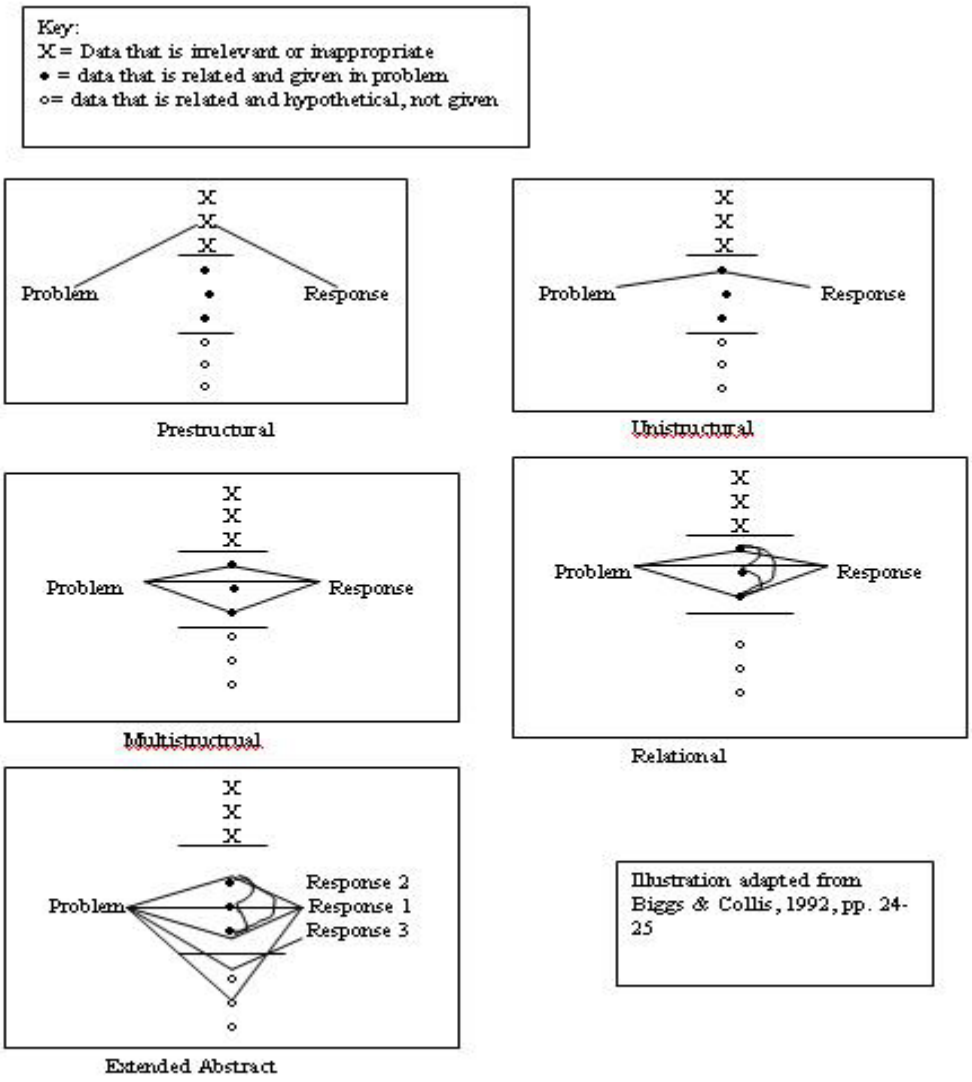


Figure 5: The levels in the SOLO Taxonomy

When applying the SOLO Taxonomy (Biggs & Collis, 1982) to the study of multiple representations of functions, students at the *prestructural, unistructual,* and *multistructural* levels would be operating within a single representation node (e.g., symbolic). This understanding could be classified as instrumental (Skemp, 1987) since the student had a procedure and may have been able to produce a correct answer. Although at the *multistructural* level they had an understanding of different

representations, there was no connection among any representational nodes. The *relational* level seemed to be where conceptual understanding begins as the individual connected separate representational nodes into a web of connected knowledge. At the *relational* level and the *extended abstract* levels the individual was connecting the different representational nodes as well as working successfully within each node. Understanding at the relational and extended abstract levels correlated to Skemp's (1987) definition of relational understanding. The student moved beyond just a procedural knowledge and actually understood why the representations were related. At the *relational* and *extended abstract* levels the connection among the nodes was very tight and compressed, allowing the individual to see the concept as a single entity, move flexibly among and between representations of the concept, and recognize the features of different representations that would facilitate effective and efficient solution strategies to a problem (Crowley, 2001).

The SOLO Taxonomy has been correlated to process-object theories by viewing the multistructural level as higher than just a procedure, but not having reached a process stage since the connections are absent (Pegg & Tall, 2002). The prestructural level correlated to a pre-procedure stage, the unistructural level correlated to procedure stage, and the relational level correlated to a process level.

Questions and Specific Hypotheses

The present study was different from previous studies in several ways. Keller and Hirsch (1998) examined student preferences for representations that have concentrated on graphical, tabular, and algebraic representations of functions but did

not measure student success in solving the problem with that preference. The present study examined student preference as well as their success in solving the problem with that preference. The present study also differed from previous studies (Knuth, 2000a; Knuth, 2000b) in that each representation appeared to be a separate problem, instead of one problem embedded with different representations that may confound student preference.

Also, much research has been done on the study of functions. Most research has been limited to specific classes of functions, linear, quadratic, etc. rather than functions that are common to a specific subject area and level of student. Other studies have examined the difference in two curricula but not in their effects on student preferences for specific representations and connections that students make among those representations (Keller & Hirsch, 1998; Kieren, 1993; Knuth, 2000a; Moschkovich et al., 1993; Noble et al., 2001; Piez & Voxman, 1997).

Curricular Impact

The present study compared student outcomes from two different college-level developmental mathematics curricula (traditional and multi-representational) to determine if a curriculum that emphasizes multiple representations of functions would have a more substantial increase in students' success rate in solving problems involving linear functions, polynomial functions, systems of equations, quadratic equations, and identifying functions. In regard to student preference, the study attempted to discern if there was a difference between students in the two curricula in choosing to solve problems with algebraic representations or if students used a

different representation. For this dimension of the study the following questions were addressed:

1. What was the impact of a multi-representational developmental college mathematics curriculum on the success of students in solving routine problems as compared to a traditional algebraic-focused curriculum?
2. How did a multi-representational curriculum affect students' preference to use a different representation and their reliance on symbolic manipulations to solve routine problems, as compared to students in a traditional curriculum?

Students' Understanding

The other focus of the study was on the connections students made among the representations, their flexibility to solve the problem using a different representation, and the extent of their development of a well-connected web of knowledge as an indicator of relational understanding. The questions addressed here were:

3. What impact did a curriculum with an instructional emphasis on different representations have on the number of connections students made among representations?
4. What influence did a curriculum emphasizing multiple representations have on the flexibility of students to solve a problem with a different representation, even if the algebraic method is the first one they choose? Did students in a curriculum involving multiple representations broaden their repertoire of representations?

5. To what extent were students' webs of knowledge of different function-related concepts well-connected? What were the differences or similarities in the level of this connectedness among students in a curriculum emphasizing multiple representations and students in a traditional curriculum?

The researcher believed that curriculum designed to stress connections among representations would enable students to see the alternative representations (other than algebraic) as viable choices for solving problems. This would give them greater flexibility in choosing alternate representations to solve problems and increase their understanding of functions and the interaction among the different representations of functions, thus strengthening their conceptual understanding by moving them toward a process conception of functions (Dubinsky & Harel, 1992). The researcher hypothesized that students who have been thus taught will be more successful in solving typical problems because of their increased ability to solve problems with an alternate representation, decreasing their reliance on algebraic representations as the only viable choice. It was hypothesized that students in the multi-representational curriculum would also develop a more well-connected web of knowledge about functions including multistructural and relational levels on the SOLO Taxonomy (Biggs & Collis, 1982).

CHAPTER 3

Methodology

The methodology for the present study was developed after consideration of past research in the field and was influenced by studies of connections (Knuth, 2000a) and preference (Keller & Hirsch, 1998). This chapter first describes the pilot study which was conducted to evaluate the effectiveness of the test instrument in supplying the information needed to answer the study questions. The pilot study then informed the current study detailed here. The current study was expanded from the pilot study to include both quantitative and qualitative measures of analysis to give a more complete picture of the effectiveness of a curriculum involving the utilization of multiple representations.

Pilot Study

In order to assess the effect of the curriculum on student preference for a particular representation and if that preference could be changed, a test instrument (Appendix A) was developed containing five problems, each with three representations; graphic, tabular, and symbolic. The five problems consisted of (a) finding the slope and y-intercept of a linear function, (b) solving a linear system of equations, (c) finding the factors of a polynomial function of degree three with integer roots, (d) determining if a relation was a function, and (e) solving a quadratic equation with rational zeros. These problems were chosen because of their prevalence in most developmental college algebra curricula. Each problem was listed, with the

three representations, on a separate page of the instrument with the representations appearing on each page in random order.

A pilot study was done with students (n=13) from one section of developmental mathematics that emphasized multiple representations. Students were given the testing instrument prior to having the course and again at the end of the course. If a student did not know how to begin solving the problem with any of the representations on a page, they were asked to circle the representation that they were most familiar with and would choose to use if they had the mathematical ability. The representation that the student chose to do or circled was coded as their choice.

Due to the unknown nature of the distribution of the data and the relatively small sample, the data was analyzed using non-parametric statistical methods, which allowed for milder assumptions regarding the population distribution of data, such as not having to assume the population was normally distributed, and provided for greater efficiency when the data were heavy tailed (Hollander & Wolfe, 1999). Thus the pretest and posttest data was paired by student and analyzed for a difference in the number of times an algebraic representation was chosen to solve a problem versus an alternative representation using Wilcoxon signed rank tests.

Analysis of the data determined the pretest showed no significant difference between students' use of algebraic and a graph or a table (testing a two-sided alternative for a difference obtained a p-value of .485). The posttest showed a significant change in method of solving the problem. The difference between the number of times a student chose to solve a problem algebraically was significantly

less than alternative methods (testing a one-sided alternative for a difference between number of times a student used algebraic methods vs. the number of times a student used an alternative method was less than 0 obtained a p-value of .002).

Although the pilot study showed a difference in the representation that a student chose to solve a problem, it did not assess the success of the curriculum in terms of student achievement, as it was not analyzed for correctness of student response. The pilot study questions were changed for the current study to incorporate whether a student had been successful in solving the problem.

A question raised by the pilot study was whether the impact of the reform curriculum was any better than a traditional curriculum on students' success in developmental college mathematics. The study outlined here attempted to build on the pilot study by examining the success of students in two curricula, the multi-representational and a traditional algebraic curriculum. Since there was an algebraic representation on each page of the testing instrument, students in both programs should have been similarly successful at correctly solving one problem on each page. If there was no difference of the new curriculum on success rate, then both curricula should produce statistical results that were not significantly different.

The pilot study also showed that the quantitative research would not be enough to determine if students were making a connection among representations. The researcher decided that qualitative interviews should be conducted to better understand the connections that students were making among the representations. The interview protocol was piloted in a study in the next semester. Interviews were

conducted with three students, and interviews were transcribed and analyzed using Dubinsky's APOS framework: action, process, object, and schema (Dubinsky & Harel, 1992). The analysis attempted to determine the level of concept formation of the interviewee. It was determined that the design of the study did not allow for the level of granularity needed to use the APOS framework, thus the researcher adopted the SOLO Taxonomy (Biggs & Collis, 1982) which was more adequate for assessing levels of understanding. The levels in the taxonomy assess for connections among concepts as part of levels of understanding.

Modifications were made in the protocol to ask students to complete as many problems as possible to determine the flexibility of the student's use of the representations. The protocol was also modified to ask questions that would enable the student to demonstrate a SOLO (Biggs & Collis, 1982) level of extended abstract by incorporating questions which went beyond the basic problems listed on the page. Traditionally in algebraic-based teaching it was assumed that students who understand and were successful in solving problems symbolically would make the connections among their solutions and the graphical and tabular representations of those solutions. If this was true then the students from both curricula should be able to move flexibly among the representations regardless of the representation they choose to initially solve the problem.

Current Study

The research was conducted with students enrolled in intermediate algebra, a developmental course bearing no credit towards a degree, at two major universities in

the eastern United States. The students at each university were enrolled in their respective developmental mathematics classes on the basis of their college entrance exam scores in mathematics or on the basis of similar placement test scores. Both universities had similar criteria for placing students in remedial mathematics, thus both groups were comparable in ability.

Instructional Support at University 1

University 1 taught intermediate algebra with a traditional approach, emphasizing algebraic concepts. The text used by the teachers at University 1 had a traditional emphasis on algebraic procedures similar to other traditional developmental algebra textbooks currently on the market (Tobey & Slater, 2001). This university did not allow students to use graphing technology, but did allow students to use scientific calculators for computation. The courses at both universities were similar in concepts taught, class size, and student populations. The universities differed in instructor qualification with University 1 classes taught by more experienced and more highly qualified teachers (all had obtained master's degrees with at least 18 graduate credit hours in mathematics). The instructors at University 2 were typically masters or doctoral-level graduate students employed as teaching assistants.

Instructors at University 1 were given the textbook and syllabus with details of specific course content (Appendix B) to be covered corresponding to each class period as well as student assignments designated for each class period. Due to constraints on office space, many of the instructors were only on campus during class

time. Therefore, these instructors had little to no interaction or collaboration with each other on teaching methods. Although they met students by appointment, they had no designated office hours.

Instructional Support at University 2

University 2 used a reform curriculum based on a functional approach using multiple representations and connections among representations. The textbook used at University 2 had emphases on graphical, tabular, and algebraic methods (Norwood, 2002). This university required students to have a graphics calculator available to them throughout the course. Because teacher influence has been found to have a major impact on instituting reform curriculum (Patterson, 2002; Schoen et al., 2003), instructors at University 2, who were utilizing the reform curriculum, attended weekly meetings led by the researcher. At these meetings, they discussed the most effective methods of teaching function concepts utilizing representations, and how to use graphing technology to support that teaching. These meetings included having the instructors think about the different phases of Simon's (1995) Mathematics Teaching Cycle, particularly in making instructional decisions based on students' current conceptions as shared by several instructors. The instructors wrote regular reflections regarding their teaching and attitudes toward the curriculum, and were observed on a regular basis by the researcher. Although assumed to be an influence in the implementation of the multi-representational curriculum, this teacher data was not part of the research analysis for this current study.

The instructors at University 2, in their weekly meetings, discussed, and agreed upon the concepts to be taught each week, loosely following a syllabus (Appendix C) constructed by the researcher. They also discussed and created activities and teaching strategies to move students toward an understanding of the concepts. For example, at the end of the chapter on linear functions, there was an application section. To help students make connections among the verbal, graphical, symbolic, and tabular representations, the instructors developed six word problems and created corresponding equations, graphs, and tables and put them on overhead transparencies. The transparencies were then cut and the representations separated. Students were divided into groups and given four non-corresponding representations (one graph, one table, one symbolic, one verbal). The groups had to generate the other three representations for each of the transparencies they were given. After finishing, one group was asked to choose one of their overhead transparency representations and put it on the overhead projector. The class was then asked to find the corresponding representations. When a group had a corresponding representation, they would come up and put it on the overhead and justify why the representations matched. Once all four corresponding representations were found and there was class agreement on the matching, another group would be asked to start by placing a representation of one of the remaining problems on the overhead, and the process would start again. This activity was intended to increase the connections among the representations by using small group and whole class discussions and having students justify their reasoning.

During a typical week the instructors taught the material, instituted the class activities, and informally assessed the level of concept formation of their students. At the next weekly meeting the instructors discussed the growth or lack of students' understanding, new activities to move them along in their construction of understanding of algebra from a function perspective, how they were assessing student understanding, and any issues regarding use of the technology. Thus the Mathematics Teaching Cycle (Simon, 1995) continued, individually for each instructor, but also for the group of instructors.

Part I: Quantitative Methods

This portion of the study attempted to focus on two aspects of student performance in solving functional problems. One aspect was the effect of both curricula on student achievement within each school and comparing the two schools. The other aspect was whether students gain flexibility in being able to solve a problem using a representation other than algebraic.

Test Instrument

The instrument used to collect data was the same as in the pilot test (Appendix A). This instrument consists of five problems each showing the three representations (graphical, tabular, and symbolic). The problems were chosen because of their prevalence in both developmental algebra curricula from both universities. The types of questions used on the instrument (linear, quadratic, and polynomial functions and relations) were chosen because they appear early in the curricula of both universities (before the posttest was given). The researcher chose the specific questions because

most students who have had high school algebra were exposed to linear equations, linear systems, factoring polynomial functions, determining functional relations, and solving quadratic equations for at least two years in both Algebra I and Algebra II, which were required for college admission. Whereas, fewer students have been exposed to solving absolute value equations for example, which was usually only taught in Algebra II, thus they have had less experience with it. The questions were chosen with the assumption that the students had prior knowledge of the concepts for at least two previous courses. The students at both universities were given pretests and posttests using the instrument and asked to solve the problems using one of the representations.

Instructors of both universities administered the instrument during the first week of the course (pretest) and again two weeks before exams, after roughly 12 weeks of instruction (posttest). If a student did not know how to solve a problem they were asked, in the directions, to choose the representation that they were most familiar with and would choose to solve if they had the mathematical ability.

Participants

Students at both universities participated in the study voluntarily. Students (identified by ID-number only) at both universities were assessed at the beginning of the semester with a departmental placement exam and with the pretest for this research (Appendix A). The pretest data was collected from 414 students at University 1 and 145 students at University 2. Due to attrition and absenteeism, posttest data was collected on 308 students from University 1 and 121 students from

University 2. Since the study compared pretest and posttest data, students who were in attendance on one collection day but not the other could not be used in the study. There was also unusable data due to incorrect ID numbers or students who wrote their name on the test instrument instead of ID numbers. Thus there were 213 usable samples from University 1 and 100 from University 2.

Data Analysis

Participants' pretest and posttest data were paired based on ID number. Answers for each question on the pretests and posttests were initially coded based on type of representation used to solve the problem (G = Graphical, T = Tabular, A= Algebraic) and correctness of the solution (Y = Yes, N = Not Correct, NA = Not answered). The data were also graded on a numerical 20-point scale, with partial credit given according to a standard rubric (Appendix D). The scores were first analyzed by comparing the difference between the pretest and posttest scores by university, showing whether each curriculum independently was successful. The researcher compared the pretest-posttest differences between universities to determine if there was a significant difference in the student success between universities.

Because of the unknown (and unlikely normal) distribution of the data on the pilot study and the relative small sample of participants, the researcher had employed nonparametric statistical methods. However, due to the large number of participants in the current study and grading pretests and posttests on a numerical scale, parametric statistical methods were employed. The present study examined the difference between pretests and posttests, paired by student, in two areas: the student

score (posttest-pretest) and the difference in the type of representation chosen (algebraic vs. tabular or graphic). This was to determine if students' success rate increased as measured in solving problems correctly and if they gained flexibility using alternate representations other than an algebraic. Differences in pretest and posttest scores were measured for each school using a one sided (greater than 0) paired t-test for a mean difference (posttest-pretest) to examine student success, thus, the effectiveness of each curriculum individually. Differences in proportions of alternate representations used from pretest to posttest were measured using McNemar dependent proportions tests (Hollander & Wolfe, 1999).

Differences in average pretest scores were measured between schools using a two-sided two-sample t-test, to test if the schools were different at the outset. The study also measured the difference in the average posttest scores to see if there was a significant difference between each curriculum at the end of the course using a one-sided (University 1 < University 2) two-sample t-test for the differences in the posttests between schools. This evaluated the effects of adopting a teaching approach emphasizing simultaneous use of multiple representations, determining if there was a significantly lower success rate with students in the traditional curriculum.

The study also measured the flexibility of students to use representations other than algebraic using McNemar's test (Hollander & Wolfe, 1999) for a difference in proportions of alternate representations on the pretest and posttests within each curriculum and a two-proportion Z-test for two independent samples between schools. McNemar's test counts the number of discordant pairs in which the students

switched the type of representation used from pretest to posttest. If there was no association between representation use and curriculum then the number of students who switched from algebraic representations on the pretest to alternative representations on the posttest should have been the same as the number of students who switched from alternative representations on the pretest to algebraic representations on the posttest. Differences were measured between schools using a two-sided two-sample Z-test for the differences in the proportion of students using alternate methods on the pretests between schools, to test if the schools were different at the outset. The study also measured the posttests proportions to see if there was a significant difference between each curriculum at the end of the course using a one sided (University 1 < University 2) two sample Z-test for the difference in the posttests proportions between schools. This evaluated the effects of adopting a teaching approach emphasizing simultaneous use of multiple representations, determining if there was a significantly lower flexibility with students in the traditional curriculum in the use of an alternate representation.

Confidence intervals and effect sizes were also reported for the present study. The 5th edition of the *Publication Manual of the American Psychological Association* (APA, 2001) suggests that confidence intervals may be the best reporting strategy for statistical data. In addition to hypothesis tests and corresponding p-values, an index of effect sizes is also recommended by the APA. In mathematics education, it has been suggested that reporting effect sizes can add additional support for the conclusions gathered from the significance testing as well as for the practical importance of the

study (Capraro, 2004). Confidence intervals and effect sizes also contribute to the field by providing a benchmark for future studies with the same parameters, allowing for results to be corroborated.

Part II: Qualitative Methods

In order to answer the research question of whether students could successfully make connections among the three representations at the end of the course and whether there was a difference in the connections made by students in the two curricula, interviews were conducted with a sample of students (n = 8) from each curriculum (total of 16 interviews). The interview tasks were the same five questions used in the quantitative methods (Appendix A).

Requests for interview applicants were made during classes at both universities. Instructors from each university were asked to select average students from those willing to participate. Once eight students from each university had been identified, the selected students signed up for a one hour block of time for their interview on the day and time that was most convenient for them. Interviews were held in a neutral location away from the students' classrooms and were videotaped for further evaluation.

Design of Protocol

In order to maintain consistency among interviewees and gain as much information as possible about the connections the students were making among the representations, a semi-structured task-based interview process was employed (Goldin, 2000). An interview protocol (Appendix E) guided the interviews, with

deviations from the protocol explored in reaction to particular participant responses. The protocol was designed to determine student understanding as they initially approached the problem, their flexibility in utilizing the other representations, their connections among the representations and their ability to go beyond the initial problem. Thus, the questions in the protocol ask for students to:

- solve a problem presented using a representation and describe their process and thinking
- solve the problem presented with the other two representations
- describe what their answers from each problem represent
- describe the relationship among the representations on the page
- answer additional questions as needed

Because of the nature of task-based interviews, if an opportunity for deeper understanding of a student's thinking occurred, the researcher deviated from the protocol.

Data Collection

Eight students were chosen from each curriculum from the pool of students who were willing to be interviewed. The participants were selected from instructor recommendations. All students participating in the interviews were characterized by their instructors as "typical" students. Interviews were conducted by the researcher approximately one week after the posttests were given in class to all students and a week before the final course exam. Interviews lasted between 20 and 50 minutes and were videotaped for later examination.

Data Analysis

The analysis was conducted in three phases. During the initial phase, the interviews were transcribed and were viewed while the researcher made annotated notes in the transcripts. In phase two, the researcher used the annotated transcripts to construct concept diagrams (Crowley, 2001) for each question a student answered. Sub-diagrams for each of the representations the student used per question were also created. Thus, each student had a minimum of five and a maximum of fifteen concept diagrams if they used all three representations for each question. These concept diagrams were employed to help determine how the student solved each of the five problems using the different representations.

For example, the concept diagram of a pilot study student as they solved problem two on the test instrument is shown (Figure 6). The student started by solving the algebraic equation, but made an arithmetic mistake in the process. The student arrived at an incorrect answer using a correct procedure. The student also solved the problem correctly with each of the other representations which afforded more direct, compressed solutions. The student was unconcerned that they had arrived at different answers and treated the problems as separate entities. The student made no connection among the representations and showed no understanding of the relationship among the three representational nodes (Figure 7). This student appeared to have a *multistructural* (Biggs & Collis, 1982) understanding of this problem.

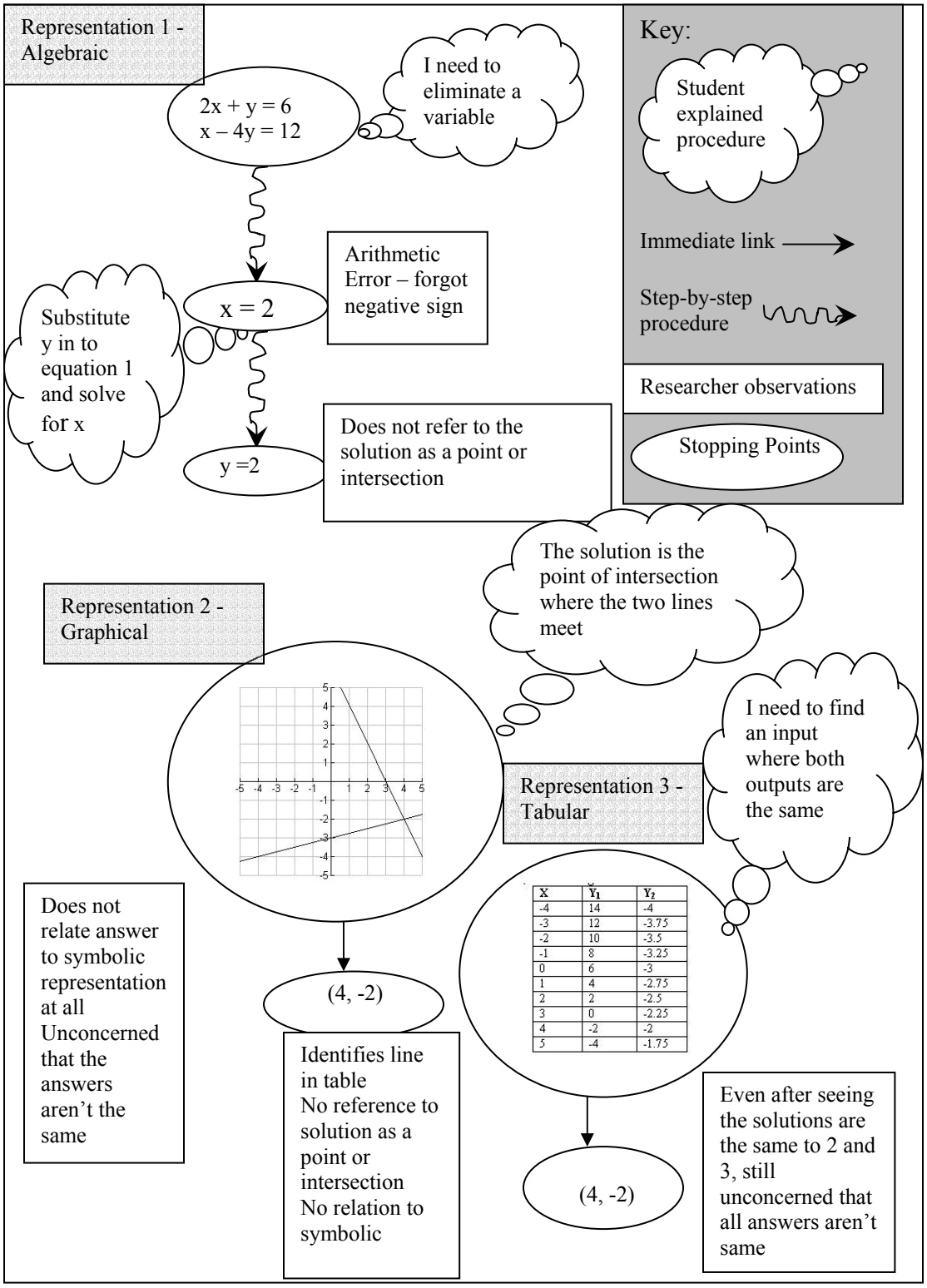


Figure 6: Concept diagram of student's response

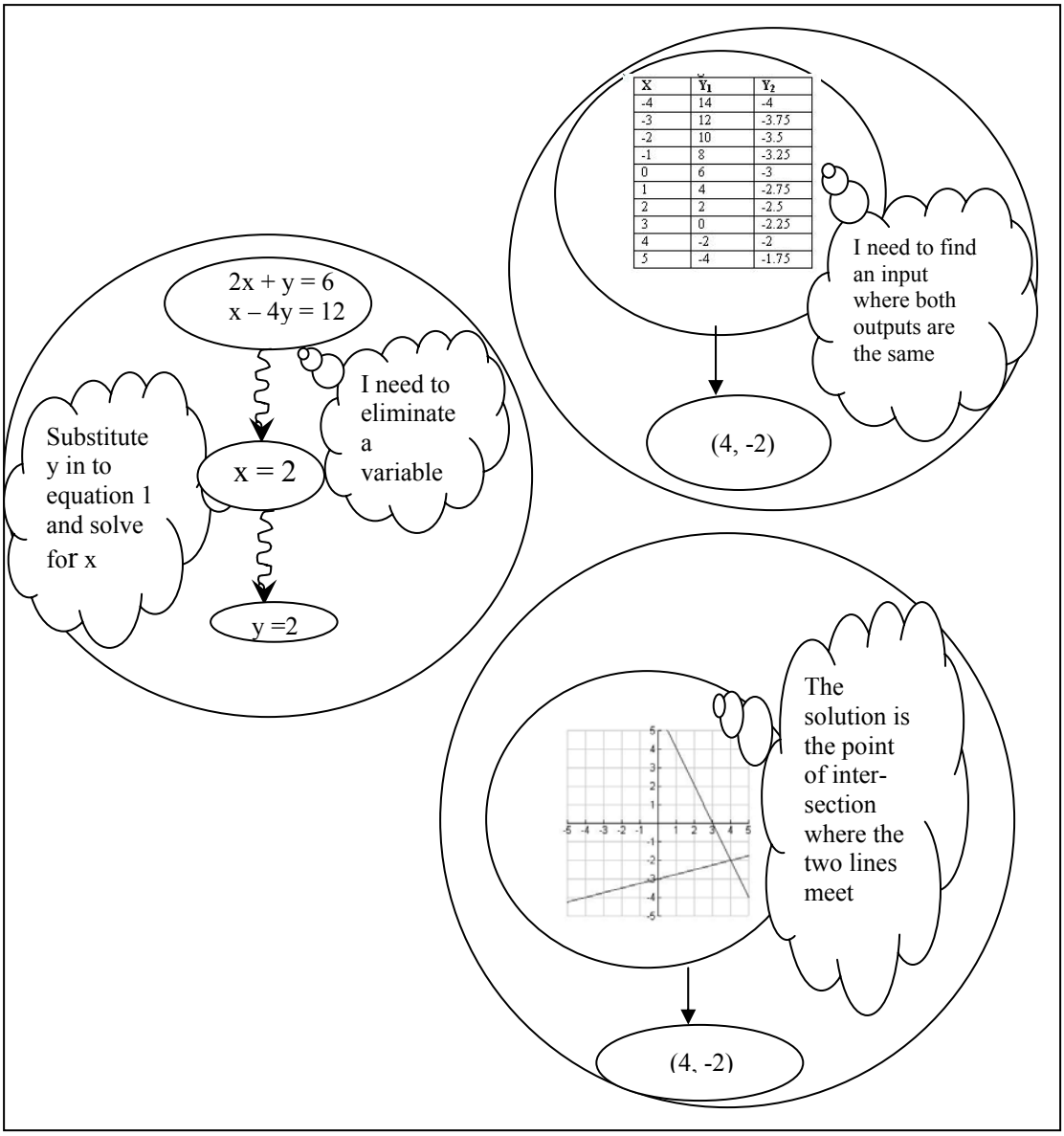


Figure 7: Three unconnected representations

Phase three of the analysis began once the diagrams in phase two were complete. The student identities were stripped and the diagrams for each student were assigned a randomly generated number (1-16). The data was then compressed further

into a summary sheet (Figure 8) for each problem (5 per student) that provided a concise way of comparing students' work.

Student Number_____

Problem Number_____

Order of Representation	Representation Used	Connected To	Answer Correct with Representation (Inaccuracies explained below)
	G	G	Y N
	T	T	Y N
	A	A	Y N

Comments:

Figure 8: Student summary sheet

By carefully analyzing the data from these problem summary sheets, the students were classified on a numerical scale (0-4) based on the levels of the SOLO taxonomy (Biggs & Collis, 1982). Considering the notion of a web of connected knowledge with three representation nodes, the general structure of the SOLO Taxonomy was applied to functional understanding through the use of three representations as follows:

0. Prestructural (lacks knowledge of the assessed component) – may have pieces of unconnected knowledge but make no sense to the student – student cannot successfully solve the problem with any of the representations
1. Unistructural (focuses on a single aspect) – has an understanding of one aspect but not of its significance to the whole or relationship to other aspects – student can solve the problem from one of the representations
2. Multistructural (focuses on several separate aspects) – has an understanding of more than one aspect but not of their significance to the whole or relationship to other aspects - student has gained flexibility in solving problems with more than one representation, but little structural knowledge of how those representations relate to each other
3. Relational (relates different aspects together) – Has an understanding of more than one aspect and of their significance to the whole and relationship to other aspects within the task – student has gained flexibility and structural understanding in solving problems with more than one representation and can see the relationship among the representations – conceptual compression begins.
4. Extended Abstract (seeing the concept from an overall viewpoint) – student makes connections within the task and can go beyond it to generalize the underlying principles

The specific rubric used for each of the five problems applied the levels of the taxonomy to each question on the instrument and gave examples of how that concept appeared in student responses to the specific task (Appendix F). The rubric was created before the analysis began.

The discrete levels in the SOLO taxonomy were often not sufficient for classifying students' learning since various students were at the low or high ends of a level. A low level response was indicated by a – next to the number for the level, a high level response was indicated by a + next to the number for the level. Numerically, the – and + corresponded to a drop or a gain of 0.25. A student at a high relational level was classified at a 3+ and received a score of 3.25, whereas a student who was a low unistructural level was classified as a 1- and given a score of 0.75. In a few cases students were classified as between two levels and rated as ±0.5. For example a student who was found to be between multistructural and relational levels was rated a 2.5. This "between level" phenomenon is consistent with the findings of others who have applied the SOLO Taxonomy in other content domains (e.g., Drier, 2000). After each student was rated on each question, the students were identified and sorted by university.

Hypothesis

It was hypothesized that students from University 1 would start with the algebraic representation because they would not recognize the graph and the table as viable tools for solving the problems, but should see the connections among the other representations listed on the page after solving the problem.

The researcher expected that the students in the multi-representational curriculum at University 2 would be at a higher stage of concept formation as outlined using the SOLO Taxonomy (Biggs & Collis, 1982) framework than the students in the traditional curriculum. This would indicate that the students had increased their structural understanding of the function concept and made connections among the different representations of functions.

Possible Constraints

One factor that may contribute to a difference between schools that could not be controlled for was the school effect. There may have been differences in the results between the two schools due to differing populations of students at each school. Since the students were placed in developmental college mathematics based on similar criteria, this effect was minimized as much as possible in the study design.

All classroom research has a constraint of teacher effect. There was no way to totally control for the difference of instructional methods and beliefs among individual teachers. The present study attempted to control for teacher effect at University 2 as much as possible by having weekly planning meetings and classroom observations by the researcher. However, as noted, this practice was not implemented at University 1. The use of graphing calculators at University 2 and not at University 1 is also an uncontrolled factor in the study design. However, students at both universities were not permitted to use a graphing calculator during the pretests or posttests or the task-based interviews.

CHAPTER 4

Results

Both quantitative and qualitative data were collected to investigate the effects of a multi-representational curriculum on student success in developmental algebra. This chapter presents the analyses of the results of both types of data collected from the pretests and posttests and interviews from students at each university as outlined in Chapter 3. The first part of this chapter details the quantitative analysis and is separated into two sections. The first section provides the overall results of the study and seeks to answer the research questions:

1. What was the impact of a multi-representational developmental college mathematics curriculum on the success of students in solving routine problems as compared to a traditional algebraic-focused curriculum?
2. How did a multi-representational curriculum affect students' preference to use a different representation and their reliance on symbolic manipulations to solve routine problems, as compared to students in a traditional curriculum?

The second section of quantitative results is a question-by-question analysis of the data which seeks to answer the research questions listed above for each specific question area.

The second part of this chapter displays the qualitative results from the sixteen interviews. This part of the analysis attempted to answer the following research questions by using the concept diagramming (Crowley & Tall, 1999), student

summary sheets, and SOLO Taxonomy (Biggs & Collis, 1982) rubric as outlined in Chapter 3:

3. What impact did a curriculum with an instructional emphasis on different representations have on the number of connections students made among representations?
4. What influence did a curriculum emphasizing multiple representations have on the flexibility of students to solve a problem with a different representation, even if the algebraic method was the first one they chose? Did students in a curriculum involving multiple representations broaden their repertoire of representations?
5. To what extent were students' webs of knowledge of different function-related concepts well-connected? What were the differences or similarities in the level of this connectedness among students in a curriculum emphasizing multiple representations and students in a traditional curriculum?

Part I: Quantitative Methods

This portion of the study focused on two aspects of student performance in solving functional problems. One aspect was the effect of each curriculum on student achievement within each school and the difference in the effect between the curricula when comparing the two schools. The other aspect was whether students gain flexibility in being able to solve a problem using a representation other than algebraic within each curriculum and whether there was a difference between the two schools.

Overall Results

Pretests and posttests were scored according to the rubric (Appendix D) and the results paired by student. Recall that the maximum score on the test was 20. The differences in posttests and pretests were found and normal probability plots of the differences for each university showed little significant deviation from normality except in the extremes or tails of the distributions (Figures 9 and 10). A Kolmogorov-Smirnov test for a null hypothesis that the differences follow a normal distribution versus the alternative that the differences were not normally distributed produced p-values greater than 0.15 for each set of differences. Since these p-values were larger than any reasonable chosen α level, the test failed to reject the null hypothesis that the data were normally distributed. Thus, normal procedures were appropriate and a paired t-test was performed on each set of differences. A 95% confidence interval and effect size was also calculated.

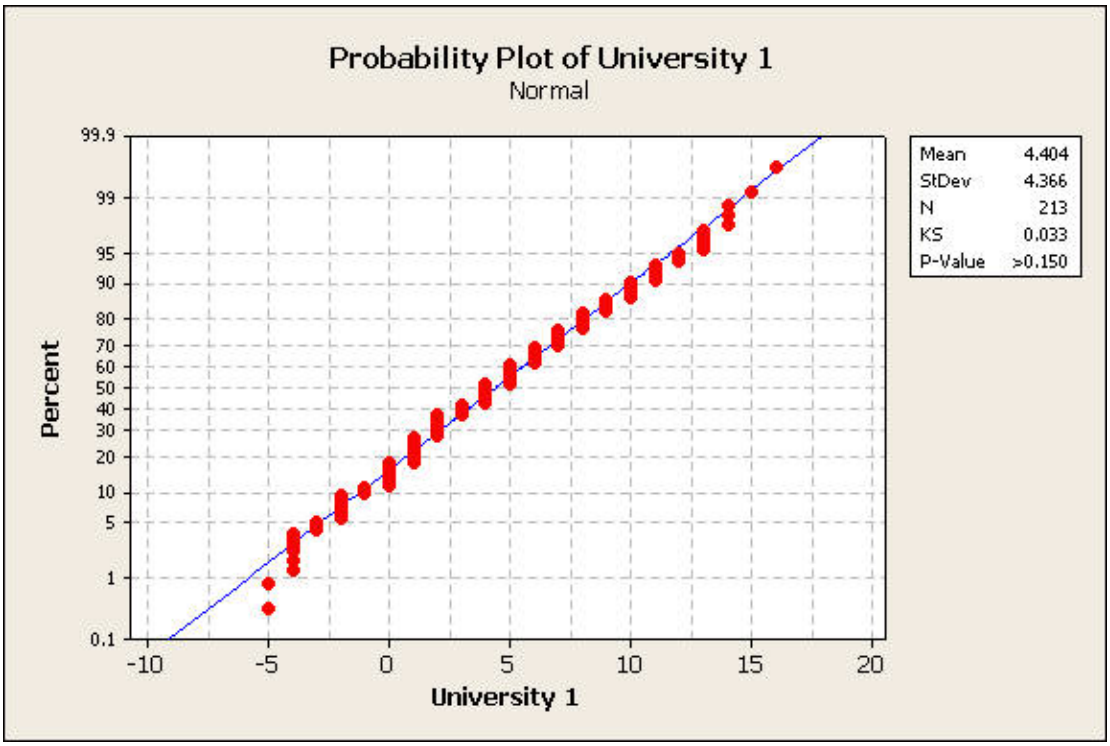


Figure 9: Normal probability plot of the differences in pretest and posttest scores for University 1

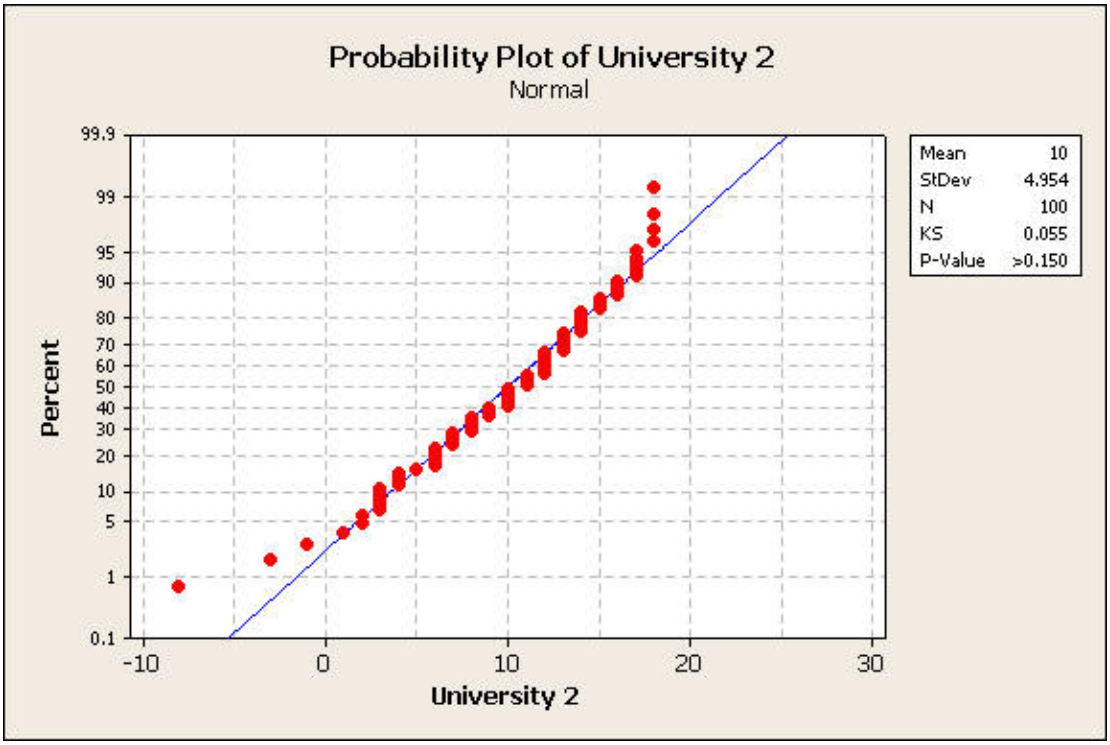


Figure 10: Normal probability plot of the differences in pretest and posttest scores for University 2

The researcher first employed a paired t-test for each university, with the pretest scores being subtracted from the posttest scores, for a null hypothesis that there was no difference in the posttest minus pretest scores against an alternative that the difference was positive. The results (Table 1) showed that each curriculum had a positive effect on the success of students to solve basic function problems with p-values less than 0.001. The mean posttest-pretest difference for University 1 was found to have been 4.404. A two-sided 95% confidence interval for the posttest-pretest difference for University 1 was (3.8141, 4.9935), implying that, without zero in this interval, there was a positive effect of the traditional curriculum of University 1 on the success of students to solve basic function problems. The standardized mean difference effect size for the success of University 1's traditional curriculum was 1.008, showing that there was large effect on student achievement at this university. The mean posttest-pretest difference for University 2 was found to have been 10.000. A two-sided 95% confidence interval for the posttest-pretest difference for University 2 was (9.017, 10.983), implying that there was a positive effect of the curriculum of University 2 utilizing a multi-representational curriculum on the success of students to solve basic function problems. The standardized mean difference effect size for the success of University 2's multi-representational curriculum was 2.018. This showed an extremely large effect on student success at University 2.

Table 1: Results of paired t-tests by university

University 1

```
Paired T for Post-Univ1 - Pre-Univ1

             N      Mean     StDev    SE Mean
Post-Univ1  213   8.71831   4.74992   0.32546
Pre-Univ1   213   4.31455   4.02464   0.27576
Difference  213   4.40376   4.36608   0.29916

T-Test of mean difference = 0 (vs > 0): T-Value = 14.72 P-Value = 0.000
95% CI for mean difference: (3.81405, 4.99346)
```

University 2

```
Paired T for Post-Univ2 - Pre-Univ2

            N      Mean     StDev    SE Mean
Post-Univ2 100   14.6500   4.4820   0.4482
Pre-Univ2  100    4.6500   4.2221   0.4222
Difference 100   10.0000   4.9543   0.4954


T-Test of mean difference = 0 (vs > 0): T-Value = 20.18 P-Value = 0.000
95% CI for mean difference: (9.0170, 10.9830)
```

Although both curricula were found to have an impact on student success, the purpose of the study was to determine the effectiveness of the multi-representational curriculum of University 2 versus the traditional curriculum of University 1. Therefore, the data were analyzed for a difference between the scores of the two groups both on pretest and posttest scores. Although the data were heavily right skewed (the majority of the data was low) on the pretests for both universities, the skewness and kurtosis values for the pretest score and posttest score for both groups were between -1 and 1, indicating no severe departures from normality. This fact combined with the size of the samples made the use of normal distribution procedures

appropriate and, thus, a two-sample t-test was employed for comparison of the pretests and of the posttests. Since two-sample t-test procedures assume equal variances, the samples were tested for equal variances using Levene's Test for homogeneity of variances, which tested a null hypothesis of equal variances against an alternative of unequal variances. Due to the skewness of the data, Levene's Test was appropriate because of its applicability to any continuous distribution. Levene's Test does not rely on the data resulting from a normal population to determine equality of variances. Both pretest and posttest scores met the assumption of homogeneity of variance evidenced by Levene's Tests ($F(1, 311) = 0.02$, $p = .877$; $F(1, 311) = 2.03$, $p = 0.156$). The tests failed to reject the null hypothesis of equal variances, thus it was reasonable to assume equal variances for two-sample t-test procedures.

The two-sample t-test procedures were conducted on the pretest scores for each university and on the posttest scores, assuming equal variances in each case (Table 2). The test on the pretest scores found that there was no significant difference between the two universities on the scores of the pretest ($t(311) = -0.68$, $p = 0.499$). The two sided 95% confidence interval for the difference between the two universities reported in Table 2 shows that 0 was contained in the interval, thus no difference between the groups could be concluded. This reinforces that there was no significant difference between the students' knowledge of these five function concepts at the beginning of the semester.

Testing the posttest scores for a null hypothesis of no difference between the two universities versus the alternative that University 1 was less than University 2 showed that the traditional curriculum of University 1 produced significantly lower student scores ($t(311) = -10.49$, $p < 0.001$). This result indicated that the students at University 2 were more successful on the posttest. The two-sided 95% confidence interval for the mean difference in the posttest scores at the universities showed that the average score for University 1 was between 4.817 and 7.043 points lower than University 2. The effect size, measured by Cohen's *d*, for this test was found to be 1.27 with a 95% confidence interval of (1.01, 1.53). The effect size of 1.27 indicated that the mean score from University 2 was approximately at the 90th percentile for scores from University 1 (Cohen, 1988) and demonstrates considerably higher academic achievement by the students in the multi-representational curriculum at University 2 as compared with the traditionally taught students at University 1.

Table 2: Minitab results of two sample t-tests comparing universities

Pretests

```
Two-sample T for Pre-Univ1 vs Pre-Univ2

           N     Mean  StDev  SE Mean
Pre-Univ1 213    4.31  4.02    0.28
Pre-Univ2 100    4.65  4.22    0.42


Difference = mu (Pre-Univ1) - mu (Pre-Univ2)
Estimate for difference: -0.335446
T-Test of difference = 0 (vs not =): T-Value = -0.68 P-Value = 0.499
DF = 311
Two-sided 95% CI for difference = (-1.314, .63429)
Both use Pooled StDev = 4.0885
```

Posttests

```
Two-sample T for Post-Univ1 vs Post-Univ2

            N   Mean  StDev  SE Mean
Post-Univ1 213  8.72  4.75   0.33
Post-Univ2 100 14.65  4.48   0.45


Difference = mu (Post-Univ1) - mu (Post-Univ2)
Estimate for difference: -5.93169
T-Test of difference = 0 (vs <): T-Value = -10.49 P-Value = 0.000 DF
= 311
Two-sided 95% CI for difference = (-7.043, -4.817)
Both use Pooled StDev = 4.6663
```

These results showed that the multi-representational curriculum had a significant impact on the success of developmental college mathematics students. Although all the students in both curricula had been taught how to do problems similar to the pretest and posttest with at least one representation (see syllabi in Appendix B and C), the students in the multi-representational curriculum were much more inclined to solve a problem correctly on the posttest. A passing grade of 60% on the test would require a student to successfully earn 12 out of 20 possible points. In

the traditional curriculum at University 1, the mean score was 8.72 or 43.6%, whereas the multi-representational curriculum students at University 2 had a mean score of 14.65 or 73.25%.

To measure the effect of both curricula on student flexibility to employ non-traditional methods as their primary means of solving a problem, the data were first paired by student to examine the effect of each curriculum separately. Since the pretest and posttest data for each school represented dependent groups, the data were measured for differences in posttest – pretest using McNemar's test (Hollander & Wolfe, 1999) to compare paired proportions. McNemar's test counts the number of discordant pairs in which the students switched the type of representation used from pretest to posttest (Tables 3 & 4). If there was no association between representation use and curriculum then the number of students who switched from algebraic representations on the pretest to alternative representations on the posttest should have been the same as the number of students who switched from alternative representations on the pretest to algebraic representations on the posttest. Although both groups were found to have a statistically significant effect of curriculum on student representation use (Univ. 1: χ^2_{1df} = 5.191, p= 0.022; Univ. 2: χ^2_{1df} =108.896, p<0.0001), students in the traditional curriculum were more inclined to change from an alternative representation to an algebraic one. The students in the multi-representational curriculum were overwhelmingly switching from algebraic to alternative representational use. This result highlighted the dominant effect of the algebraic representation in a traditionally taught curriculum.

Table 3. Concordant and discordant pairs of representations at University 1

Pretest to Posttest	Question 1	Question 2	Question 3	Question 4	Question 5	Total
Alternative to Alternative (AA)	54	6	27	108	4	199
Alternative to Algebraic (AB)	43	13	39	21	20	136
Algebraic to Alternative (BA)	31	14	15	22	18	100
Algebraic to Algebraic (BB)	85	180	132	62	171	630

Table 4. Concordant and discordant pairs of representations at University 2

Pretest to Posttest	Question 1	Question 2	Question 3	Question 4	Question 5	Total
Alternative to Alternative (AA)	30	5	24	59	12	130
Alternative to Algebraic (AB)	21	11	4	1	2	39
Algebraic to Alternative (BA)	19	42	43	39	59	202
Algebraic to Algebraic (BB)	30	42	29	1	27	129

The proportion of alternate representations used was then calculated for both groups for pretests and posttests. Using an independent, two-proportion Z-test for the difference in the proportion of students using alternate methods at each university on the pretest, the data showed that there was no significant difference between the two groups at the beginning of the curriculum, $Z = 0.926$, $p = 0.355$. The proportion of students at University 1 using alternate methods on the pretest was 0.315. The proportion of students at University 2 using alternate methods on the pretest was

0.338. A two-sided 95% confidence interval for the difference in proportions on the pretest was found to be (-0.0734, 0.027). Since this interval contains 0 it could be concluded that there was no significant difference in the groups at the beginning of the course.

Comparing the posttest results of alternate methods using a one-sided two-proportion Z-test for an alternative hypothesis that University 1 would have a smaller proportion of students using alternate representations than University 2 showed a significant difference in the two universities, $Z = -14.412$, $p < 0.0001$. The proportion of students at University 1 using alternate representations on the posttest was 0.281. The proportion of students at University 2 using alternate representations on the posttest was 0.664. A two-sided 95% confidence interval for the difference in proportions between the two groups on the posttests was (-0.433, -0.334).

The proportion of problems answered using alternative representations in the traditional curriculum at University 1 slightly decreased whereas in the multi-representational curriculum at University 2, the proportion of alternate representations used was almost double. Thus students appeared to gain ability in using alternate representations when taught using a curriculum which highlights the graphical and tabular as well as the algebraic representation.

Results by Question

In order to assess the effectiveness of each curriculum on a question by question basis, the data was examined to see if both curricula gained significantly on each concept tested from pretest to posttest. The posttest – pretest differences on all

questions exhibited non-normality because of discreteness; each question was graded out of four possible points according to the rubric in Appendix D. Because of the varying degrees of departure from normality when comparing results by question, nonparametric statistical methods were employed for this analysis. Although these tests are less powerful than t-test procedures when the data are normally distributed, they can be significantly more powerful when the data are not normally distributed as in this case (Hollander & Wolfe, 1999).

For each question three nonparametric hypothesis tests were completed. A distribution-free Wilcoxon rank test to indicate a shift in location due to the treatment for each school was performed to test the effectiveness of each curriculum separately on each concept on the test instrument. A distribution free Mann-Whitney test was performed on the pretest scores between both schools to test for differences in the schools on a particular question before the treatment. Another distribution free Mann-Whitney test was performed on the posttest scores to test for differences in the schools as a result of the treatment on each specific test item. The summary data for each of the pretests for the two schools are listed in Table 5 with the posttest summary data listed in Table 6.

Table 5. General descriptive statistics of pretest scores by question

Measure (Scale 0-4)	University 1 n = 213				University 2 n = 100			
	Mean	Median	Std. Dev	Prop. of Alt. Reps	Mean	Median	Std. Dev	Prop. of Alt. Reps.
Question 1	1.643	2	1.503	0.46	1.800	2	1.537	0.51
Question 2	1.080	0	1.404	0.09	1.080	0	1.447	0.16
Question 3	0.277	0	0.716	0.31	0.230	0	0.680	0.28
Question 4	0.915	0	1.543	0.61	0.820	0	1.507	0.60
Question 5	0.399	0	1.101	0.11	0.720	0	1.400	0.14

Table 6. General descriptive statistics of posttest scores by question

Measure (Scale 0-4)	University 1 n = 213				University 2 n = 100			
	Mean	Median	Std. Dev.	Prop. of Alt. Reps.	Mean	Median	Std. Dev.	Prop. of Alt. Reps.
Question 1	2.484	2	1.478	0.40	3.170	4	1.064	0.49
Question 2	2.155	3	1.356	0.09	2.950	3	1.218	0.47
Question 3	0.939	0	1.388	0.20	2.340	2	1.526	0.67
Question 4	1.178	0	1.678	0.61	3.010	4	1.667	0.98
Question 5	1.962	2	1.893	0.10	3.180	4	1.431	0.71

Measures of Success

Comparing the posttest and pretest data for each question using a Wilcoxon Signed Rank Test that the median difference was equal to zero versus an alternative that the median difference was greater than zero showed that each school gained significantly in posttest achievement (Tables 7 & 8). Thus each curriculum was effective in increasing student achievement in each question area. The area and school with the smallest gain was the traditional curriculum (University 1) in determining if a relation was a function (Question 4). The Wilcoxon Statistic shows the number of Walsh averages above the hypothesized median. The sample size was reduced by the number of observations that were equal to the hypothesized value.

Table 7. Results of Wilcoxon signed rank test for University 1 by question

Measure (Scale 0-4)	Estimated Difference of Medians (Posttest – Prestest)	Wilcoxon Statistic	N for Test*	P-Value
Question 1	1.0	7566.5	136	<0.001
Question 2	1.0	7332.5	124	<0.001
Question 3	0.5	2663.0	77	<0.001
Question 4	0.0	1394.0	63	0.004
Question 5	2.0	5923.0	111	<0.001

*N = 213

Table 8. Results of Wilcoxon signed rank test for University 2 by question

Measure (Scale 0-4)	Estimated Difference in Medians (Posttest-Pretest)	Wilcoxon Statistic	N for Test*	P-Value
Question 1	1.5	2789.5	77	<0.001
Question 2	2	3290.5	83	<0.001
Question 3	2	2978.5	77	<0.001
Question 4	2	2066.5	65	<0.001
Question 5	2	2925.5	77	<0.001

*N = 100

Comparison of the pretest scores of the universities for each question showed no difference between the two schools on Questions 1 through 4, testing a null hypothesis that there was no difference against an alternative that there was a difference in pretest scores (All p-values adjusted for ties. Question 1: W = 32832.5, p = 0.3858; Question 2: W = 33538.0, p = 0.884; Question 3: W = 33718.0, p = .5355; Question 4: W = 33791.5, p = .5495). Differences in pretest scores were significant on Question 5 (W = 32235.0, p = 0.0141). University 1 scored an average grade of 0.399 on this question (scale 0-4) while University 2 scored an average grade of 0.72. Both universities had a median grade of 0 on Question 5. Although the differences on this question were statistically significant, they were not practically significant in terms of concluding that students at either school were successful in answering the question. Thus it can be concluded that both schools were similar in

their understanding of the five concepts on the test instrument before the treatment began.

Comparison of the posttest scores of both universities showed that University 1 was significantly lower on each test instrument item (All p-values adjusted for ties. Question 1: $W = 30802.0$, $p = 0.001$; Question 2: $W = 29690.0$, $p < 0.0001$; Question 3: $W = 29690.0$, $p < 0.0001$; Question 4: $W = 28031.5$, $p < 0.0001$; Question 5: $W = 29871.5$, $p < 0.0001$). Thus the students in the multi-representational curriculum outperformed, as a whole, the students in the traditional curriculum on every test item.

Measures of Flexibility

Each question was then examined using three tests. McNemar's test (Hollander & Wolfe, 1999) was used to test for a difference in the proportion of students who switched from algebraic representations on the pretest to alternative representations on the posttest within each school and an independent two sample Z-test for the difference in the proportion of students using alternative representations to solve each problem on both the pretests and the posttests between schools. These tests were to assess whether the multi-representational curriculum had an effect on student reliance on the algebraic methods to solve problems. McNemar's test was appropriate for testing differences in pretests and posttests scores within each school because the results were paired by student. Independent two-sample proportion tests were appropriate for measuring between school differences due to the large samples, the

assumption that the schools were independent, and the sample proportions in each problem (np>5 and n(1-p)>5).

Question 1. It was interesting to note that the proportion of students solving problem 1 with alternate methods changed very little between the pretests and posttests at both universities. McNemar's test showed no significant difference in the proportion of students switching from working with an algebraic representation to an alternate representation and those starting with an alternate representation on the pretest and using an algebraic on the posttest (Univ. 1: χ^2_{1df} = 1.635, p = 0.201; Univ. 2: χ^2_{1df} =0.025, p=0.8744).

Question 1 was the problem on the test instrument with the highest scores on both the pretests and posttests. This may be due to the emphasis that this task (slope and y-intercept of linear functions) gets in the pre-college curriculum. A two-proportion Z-test for the difference in proportion of students using alternate representations in Question 1 found no difference between the two schools on the pretests or the posttests in the number of students using alternate representations on this question (Pretest: Z = -0.902, p = 0.367; Posttest: Z = -1.516, p = 0.129).

Question 2. Although there was a slight difference in the proportions of students using alternate representations on the pretest for question 2, a comparison of the average score which was 1.08 (scale 0-4) for both groups and the median which was 0 for both groups makes the difference between 9% from University 1 and 16% from University 2 attempting to use alternate methods of little practical import. The proportion of students using a graphical or tabular representation for solving a system

of equations increased dramatically on the posttest in the multi-representational curriculum at University 2 (Pretest: Z = -1.853, p = 0.064; Posttest: Z = -7.564, p <0.0001).

McNemar's test for changes within each university reveals that in the traditional curriculum, there was virtually no difference in the number of students switching from algebraic on the pretest to alternative on the posttest versus alternative on the pretest and algebraic on the posttest (χ^2_{1df} = 0, p = 1) . There were significant differences in the types of representations used by students in the multi-representational curriculum with the proportion of students switching from algebraic representations on the pretest to alternative representations on the posttest at 42% (Univ. 1: χ^2_{1df} = 16.981, p < 0.0001).

Solving a system of equations using a graph by finding the intersection or a table of values where an input produces the same output for both equations when those two representations were given would seem, to people who are proficient in mathematics to be the most direct way to solve the problem. It could be suggested that the students in the traditional curriculum did not understand that the solution to a system of equations was the point of intersection of two lines (graphically) or that the dependent variable must be equal for the same independent variable (tabular) and thus their understanding of systems of linear equations was to perform a procedure in solving the system algebraically.

Question 3. Comparing the proportion of students switching representational approaches within each university found that both universities showed a significant

student shift in choice of representation used to solve question 3 (Univ. 1: χ^2_{1df} = 9.796, p = 0.0015; Univ. 2: χ^2_{1df} =30.728, p < 0.0001). However, at University 1 the shift was toward moving away from attempting to solve the problem with an alternative representation toward solving the problem algebraically. The multi-representational curriculum at University 2 had a large percentage (43%) of students shifting from an algebraic representation on the pretest to an alternative representation on the posttest.

It appeared by the small proportion of students that attempted to use alternative representations from the traditional curriculum at University 1, that even though they have been taught the factor theorem, students in a traditional algebraic curriculum did not see the x-intercepts of the graph or the zeros in the tabular representation as the most direct route for solving a problem that requires factoring a polynomial. The two universities were not significantly different on the pretest, but there was a significant increase in the proportion of students in the multi-representational curriculum using alternate methods on the posttest (Pretest: Z = 0.537, p = 0.591; Posttest: Z = -8.187, p < 0.0001). This suggested that students in the multi-representational curriculum understood that the zeros or x-intercepts have an association to the factors of the polynomial. When considered in conjunction with the average and median scores of 2.34 and 2 respectively for the multi-representational curriculum as compared to 0.939 and 0 for the traditional curriculum, it would seem to indicate that the ability to make this association and to solve the problem using an

alternate approach enabled the students in the multi-representational curriculum to experience a higher degree of success on the factoring concept.

Question 4. Question 4 has been typically taught with a graphical representation using the vertical line test or by examining the table for repeated values of the independent variable. It was no surprise that a high percent of students from both curricula (61% and 60%) chose an alternate representation on the pretest and that both groups were not significantly different ($Z = 0.095$, $p = 0.924$). What was surprising was that the proportion of students choosing alternate representations basically stayed the same on the posttest for the traditional curriculum (61.0%), while almost all students in the multi-representational curriculum (98%) chose an alternative representation on the posttest. This led to a significant difference between the two groups on the posttests ($Z = -6.856$, $p < 0.0001$). One explanation of this may be the under-development of students' sense of graphical and tabular interpretation promoted by an algebraic curriculum at University 1.

McNemar's test for a difference between pretest and posttest proportions found that there was no difference in the proportions of students switching from algebraic on the pretest to an alternative on the posttest and the proportions of students switching from alternative to algebraic ($\chi^2_{1df} = 0$, $p = 1$). At University 2, the difference in proportions was highly significant ($\chi^2_{1df} = 34.225$, $p < 0.0001$).

Also surprising was the low score for University 1 on this question. The students in the traditional curriculum at University 1 scored an average of 1.178 points out of 4 on this question with a median of 0 on the posttest. The students in the

multi-representational curriculum at University 2 scored an average of 3.010 with a median of 4. This dramatic difference could possibly be attributed to the emphasis on functions at University 2.

Question 5. Examining within each curriculum found a significant increase using McNemar's test for the proportion of students using an alternative representation to solve a quadratic equation in the multi-representational curriculum (χ^2_{1df} =51.41, p < 0.0001). However, in the traditional curriculum, there was no significant difference from pretest to posttest in the proportion of students changing the type of representation they use (χ^2_{1df} =0.026, p = 0.871).

Pretest comparisons showed no difference in the proportion of alternate representations used on the pretest between the universities (Z = -0.690, p = 0.490) for Question 5. Posttest comparisons showed a dramatic difference in the number of alternate representations used in the multi-representational curriculum (Z = -10.952, p <0.0001). With 71% of students in the multi-representational curriculum using alternate representations on the posttest, it appears that these students made a connection between the zeros of a quadratic equation and the x-intercepts of a graph or the zeros in the tabular representation. Most students (90%) in the traditional curriculum attempted the problem with the algebraic representation.

Summary

These findings suggest that, at the very least, students in the multi-representational curriculum have gained in procedural knowledge and in their flexibility to use alternate representations as viable means for solving problems over

students in a traditional curriculum. Examining the relationship between increase in the proportion of alternate representations used and the increase in average score on problems that were not traditionally taught from alternate representations (Question 4 was eliminated from the data), the study found that there was no correlation in the control group between the proportion of alternate representations and the scores (Table 9). Examining the same relationship in the multi-representational curriculum found that there was a significant correlation in the treatment group between the proportion of alternate representations used and the scores (Table 10). This suggested a curriculum strategy of using a functional approach combined with multiple representations and the connections among the representations may have enhanced the success of developmental mathematics students.

Table 9. ANOVA results for regression of proportions of alternate representations used vs. average score by problem for control group (University 1)

Source	DF	SS	MS	F	P
Regression	1	0.0651	0.0651	0.08	0.781
Residual Error	6	4.6255	0.7707		
Total	7	4.6895			

Table 10. ANOVA results for regression of proportions of alternate representations used vs. average score by problem for treatment group (University 2)

Source	DF	SS	MS	F	P
Regression	1	6.2123	6.2123	11.69	0.014
Residual Error	6	3.1897	0.5316		
Total	7	9.4020			

Part II: Qualitative Methods

This portion of the study focused on the connections students made among the three representations at the end of the course and the similarities and differences in the connections made by students in each curriculum. Qualitative interviews were conducted with a sample of students from each curriculum. The qualitative interview tasks were the same five questions on the instrument used in the quantitative methods.

The interviews were transcribed and then the student responses and methods for solving each of the problems were represented diagrammatically (Crowley & Tall, 1999) as discussed in Chapter 3. Thus each student response for each question and each representation was represented with a diagram. In order to prevent researcher bias, the student diagrams which detailed responses and methods for solving problems were stripped from identification and randomly coded with a student number from 1-16. Because of the magnitude of the data, for ease of interpretation, these diagrams were then represented by a problem summary sheet as discussed in Chapter 3. This allowed the data to be coded for the order of the representations used, the type of representations used, the connections among the representations that each student made and the accuracy of the student's work. The randomly assigned student numbers (1-16) allowed the data to be rated on the SOLO taxonomy (Biggs & Collis, 1982) without regard to what school the student was from.

After careful analysis of the data from the problem summary sheets, the students were classified on a numerical scale based on the levels of the SOLO taxonomy (Biggs & Collis, 1982) according to the rubric in Appendix F. Student

ratings were determined before the identity of the university was indicated. After the coding was complete, the students were then matched to their respective universities for comparison.

Examples of Analysis

The results of the qualitative analysis revealed the differences among the levels of student learning between the two curricula. The diagrammed responses helped to zoom in on the pertinent responses the student made regarding a specific problem. Rather than providing a detailed analysis of all sixteen cases with five problems each, two examples of the analysis of two students for two different questions were shown here. Full results were summarized in tables in the question-by-question analysis later in this chapter.

Example 1: Student 6, *Problem 2*. Student 6 was taught in the traditional curriculum at University 1. The diagram of Student 6's understanding of problem two on the test instrument (see Appendix A) showed that although her text (Tobey & Slater, 2002) defined the solution to a system of equations as an ordered pair that satisfies all the equations in the system, she did not associate that definition with the intersection of two lines on the graph (Figure 11). The text and the teacher demonstrated how to solve a system of equations three ways: by graphing, substitution, and elimination. Student 6 had a strategy for solving the problem algebraically by using the elimination method but made an error when multiplying the second equation by -2 and only multiplied the x term by -2. She arrived at a solution

for y of 18/5. From there she explained how she should find x, but said she could not do the procedure because x was a fraction.

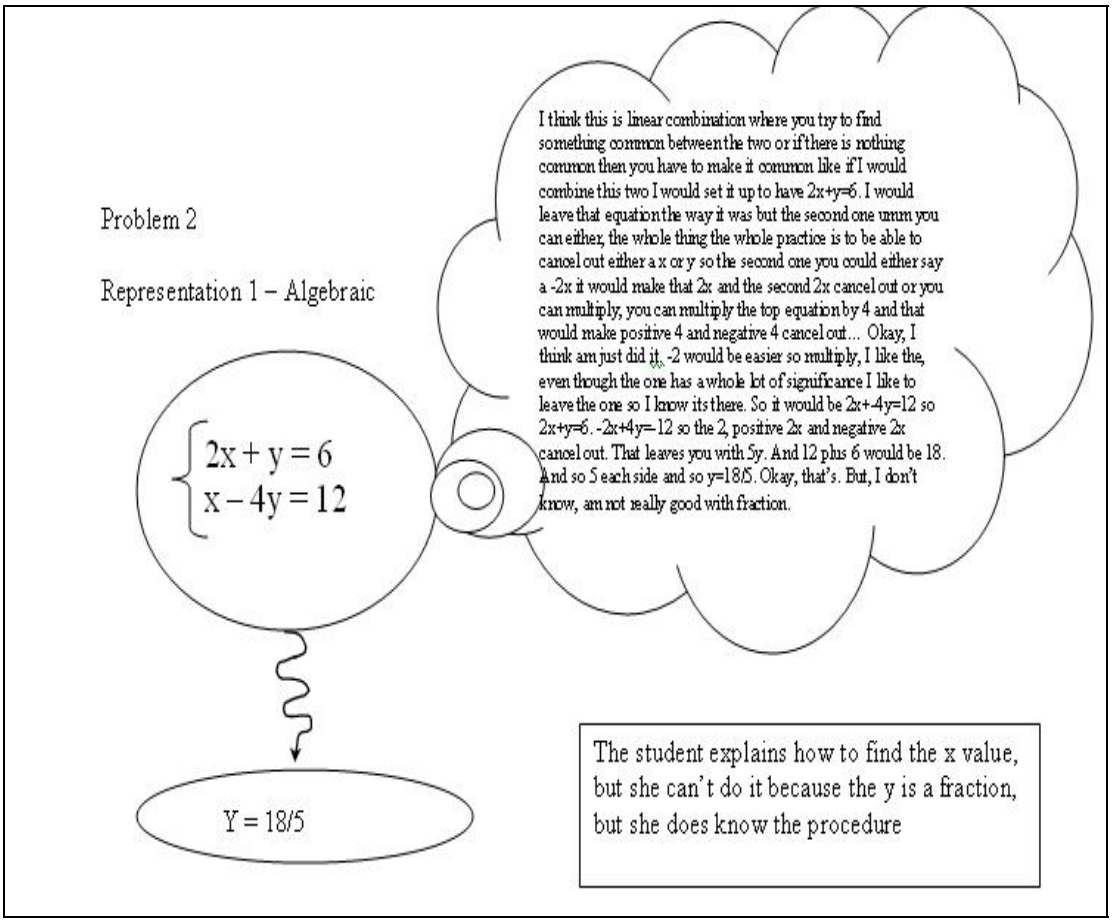


Figure 11: Concept diagram for Student 6, Problem 2, and Representation 1

Student 6's response when she was questioned about how to solve the system of equations with the graphical representation showed her lack of understanding that the solution to the system was the point of intersection of two lines:

Student 6: This one *(points to the graph)* no. I don't know what it's asking for. I know it's asking for a solution but I don't know how to go about

getting a solution. If I had to guess I'd say that you just have to pick two points.

Researcher: Pick two points? Anywhere?

S6: Yea, just anywhere that this line falls on *(points to the graph of 2x + y = 6)*. Like say we have (3,0) and (2,2).

R: Okay

S6: My guess would be there is a formula that you can use. I think that might rewrite y to the second minus y to the first over x to the second minus x to the first. That will be my best guess but really I really don't know how to go about getting it. I don't know if it's even the correct formula to use or not.

Although this student might have recognized that the two ordered pairs she named were solutions to one of the equations, she did not demonstrate an understanding that the point of intersection would satisfy both equations, and thus would be a solution to the system. It was also interesting that her statement of the formula she suggested was similar to a formula for finding slope of a line $\left(\text{e.g., } \frac{y_2 - y_1}{x_2 - x_1}\right)$ —a formula she might use to determine slope of a line when she has two points. For the problem at hand, this computation represents an unnecessary procedure.

Student 6 further shows her desire for a formula or an equation, thus a procedural approach to solving the problem, by her attempt to make sense of the table of values:

R: Okay, how about the table? Any ideas there?

S6: Find the solutions to the system of equations represented. I don't really know what this is asking for. It's just a whole mess of numbers to me.

R: Okay, there are a lot of numbers there. Do you understand what the question is asking for?

S6: [inaudible]

R: But what is confusing you is?

S6: I mean there is like no variable to solve for or anything, I mean…

R: So you need a variable to be able to solve?

E4: Yea and I won't know where to start with this.

Based on these responses the student was classified as unistructural in her learning about systems of linear functions with misconceptions about arithmetic, namely fractions. The summary of her diagrammed response is shown in Figure 12.

Student Number 6

Problem Number 2

Order of Representation	Representation Used	Connected To	Answer Correct with Representation (Inaccuracies explained below)
	G	G	Y N
	T	T	Y N
1	(A)	A	Y (N)

Comments: *Student made some procedural mistakes and got a fraction for the answer to y. At this point, she gave up and could not solve for x because she is "not really good with fractions". She does explain the procedure for what she should do.*

The student cannot do anything with either of the two representations.

Figure 12: Summary sheet for Student 6, Problem 2

Student 6 was at a unistructural level of understanding. Since she also had some procedural misunderstandings, she was classified as a low level of unistructural understanding. Although this student was taught the graphical interpretation of a system of equations, she cannot produce the answer to the system when provided with a graphical representation.

Example 2: Student 8, Problem 5. Student 8 was taught in the multi-representational curriculum at University 2. The student's procedures for solving problem five with all three representations and connections among representations became more salient for the researcher by using the concept diagram (Figures 13-15).

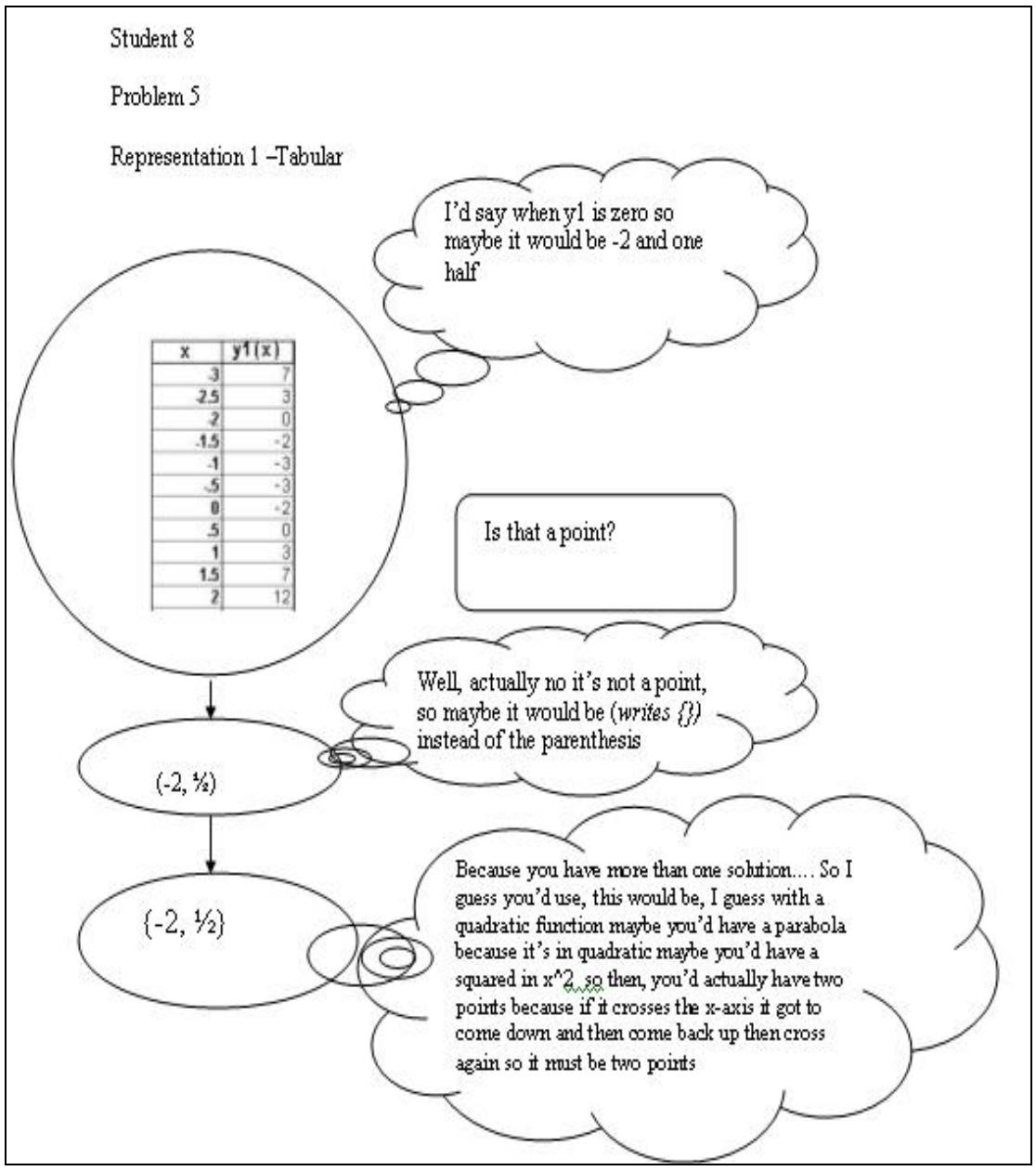


Figure 13: Concept diagram for Student 8, Problem 5, and Representation 1

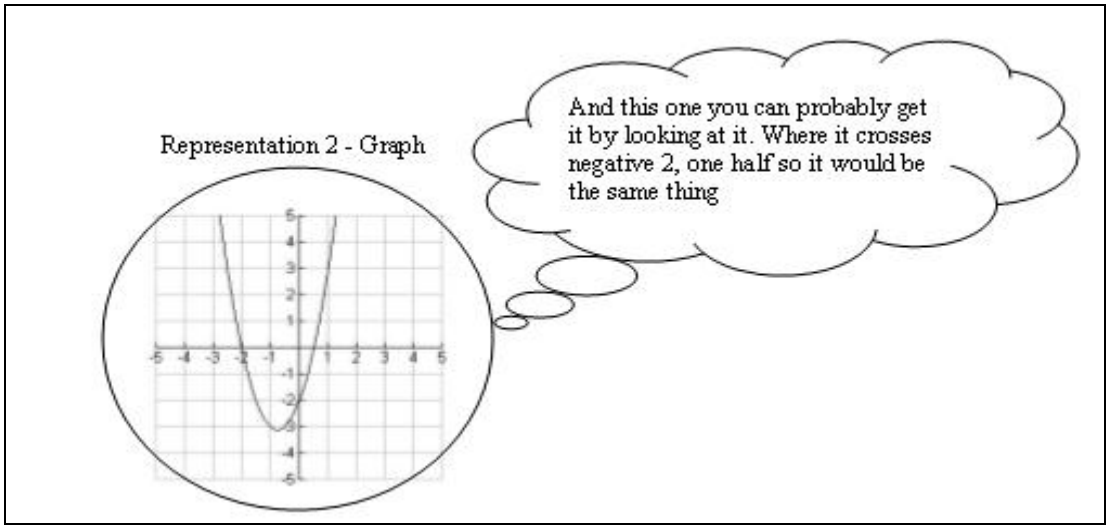


Figure 14: Concept diagram for Student 8, Problem 5, and Representation 2

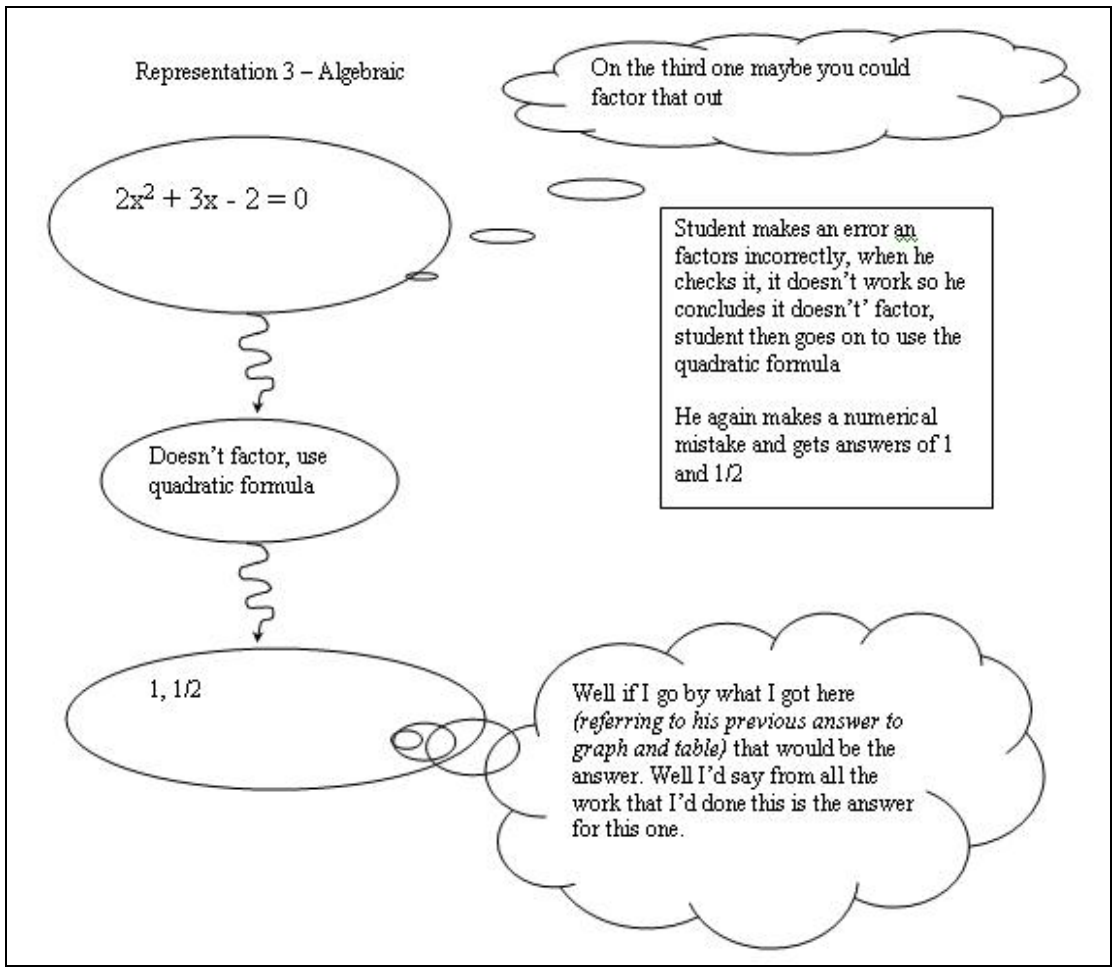


Figure 15: Concept diagram for Student 8, Problem 5, and Representation 3

Mapping the diagrammed responses to the SOLO taxonomy (Biggs & Collis, 1982) using the problem summary sheet allowed for the synthesizing of the information contained in the responses in such a way as to classify what the student knew. This permitted the researcher to understand what misconceptions the student needed to overcome to move to the next level of understanding. For example, mapping the diagrammed response of Student 8 using the problem summary sheet is shown below (Figure 16).

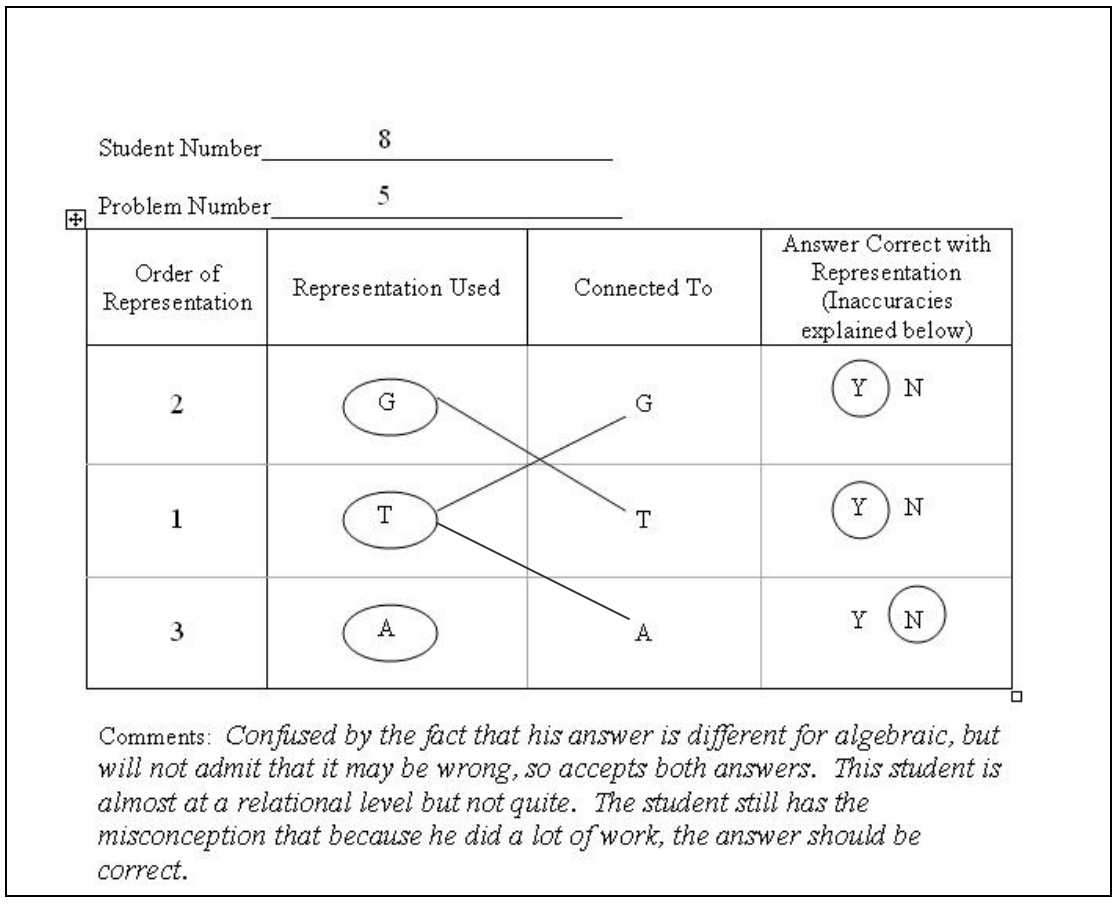


Figure 16: Summary sheet for Student 8, Problem 5

Because the student allowed two different answers, he seemed to view the algebraic problem as unrelated to the other two representations instead of different representations of the same problem. Student 8 understood the relationship between the table and the graph when he described from the table that it would be the graph of a parabola and that it would cross the x-axis in two points. When he went on to the graph he said it would be "the same thing." However, in response to the different answer he got when solving the algebraic representation; the student concluded that there were two different answers, one answer to the graph and the table, one for the algebraic.

Overall Summary of Results

Summarizing the analysis for the students for each problem in tabular form for order of representation, type of representation used, number of connections made, number of correct answers, and an explanation of the misconceptions allowed for the numerical rating of each student on the SOLO Taxonomy (Biggs & Collis, 1982) by problem. For example, student 6, problem 2, was classified as A [Algebraic] for order of representation, 1 for number of representations used, 0 for the number of connections made, and 0 for the number of correct answers. Based on these data and careful reflection on the student's responses, the student was rated a 1- (0.75) on the SOLO Taxonomy scale, a low unistructural understanding. Student 8, problem 5, was classified as TGA [(1) Tabular, (2) Graphical, and (3) Algebraic] for order of representation, 3 for number of representations used, 2 for the number of connections made, and 2 for the number of correct answers. Based on these data and careful

reflection on the student's inability to accept that one of the three answers produced was incorrect, the student was rated a 2.5 on the SOLO taxonomy scale, in between multistructural and relational.

Graphing the results from each question by university on side-by-side box-plots, a technique used by Drier (2000), allowed a visual comparison of the differences between universities (Figure 17). At the end of the courses, the eight students in the multi-representational curriculum tended to have higher SOLO (Biggs & Collis, 1982) levels that were mostly above the multistructural level, except for the wide range of SOLO levels assigned for question 4 (determining if a relation was a function). In contrast, the eight students from University 1's traditional curriculum were mainly in the prestructural through multistructural level across all questions.

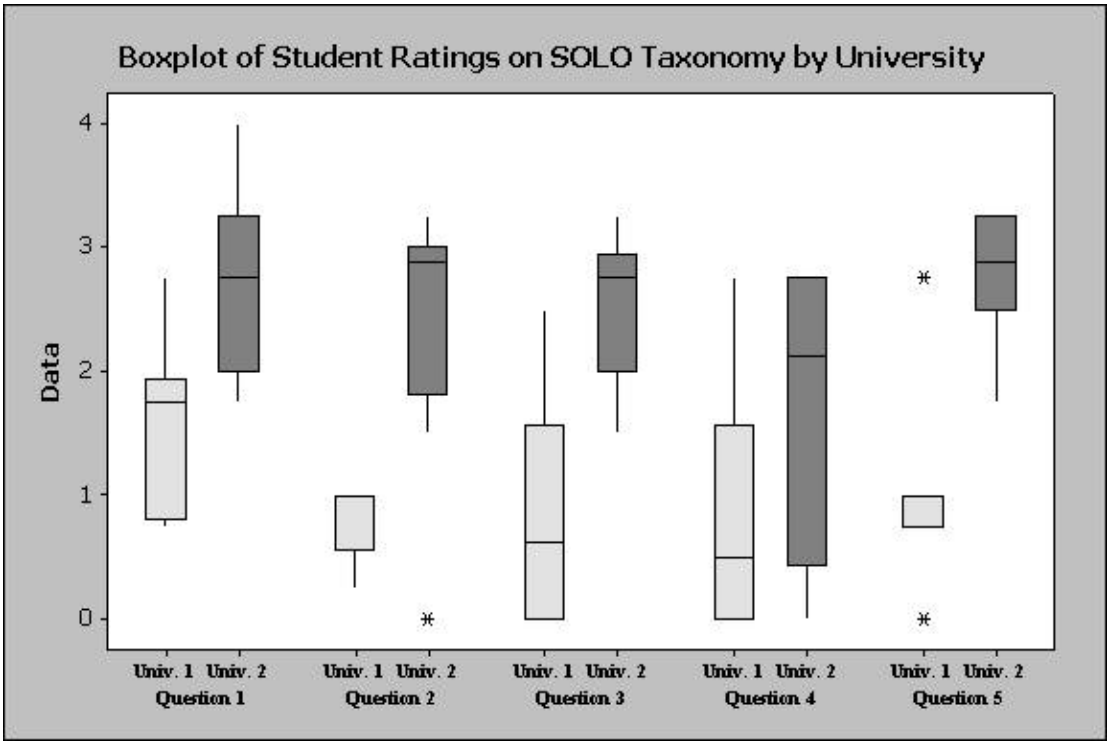


Figure 17: Boxplots of student levels on the SOLO Taxonomy by school and question

Analysis of Interview Data by Question

Summary tables were created for all sixteen participants for each question and then each table was split into two tables by school. The summary tables highlighted the impact of the differences between each curriculum on student learning.

Question 1. Students interviewed from the traditional curriculum (Table 11) tended to use two representations, but did not connect those representations. The algebraic and the graphical were the predominate methods for finding the slope and y-intercept of a linear function, with most students in both curricula starting with the algebraic. Almost all the students had some kind of misconception about the slope or the y-intercept.

Even though six out of eight students in the multi-representational curriculum started with the algebraic representation (Table 12), they tended to use all three representations and make strong connections among the representations. Four of the students had misconceptions about slope or y-intercept. When asked what would happen if the y-intercept would change, most students who interpreted the graphical representation said that the slope would also change. The students in both curricula tended to hold the x-intercept stationary and move the y-intercept. They would then conclude that the slope changed with the y-intercept. One student from University 2 not only understood how the graph and equation would change but was also successful in describing how the table would change, that each of the y-values would change by a fixed amount. That student was classified as extended abstract on this question.

The students in the traditional curriculum had a median rating on the SOLO Taxonomy (Biggs & Collis, 1982) of 1.75, thus were low multistructural. By comparison, the median level for the students in the multi-representational curriculum was 2.75, or low relational. Thus the curriculum seemed successful in helping students attain a high level of conceptual understanding about the relationship among the algebraic, tabular and graphical representations of the slope and y-intercept of a linear function.

Table 11. Summary results for University 1 - Problem 1 – Slope and y-intercept of a linear function

Student Number	Order of Representations Used	Number of Representations Used	Number of Connections Made	Number Correct	Classification and Notes	Rating
3	AG	2	1	.5	Attempts Graph but uses wrong information, only gets y-intercept correct on Algebraic **Unistructural** (Algebraic, with misconceptions)	1-
4	G	1	0	1	**Unistructural** (Graph)	1
6	AG	2	0	1	**Multistructural** (Algebraic, Graph, with misconceptions)	2-
9	AG	2	2	2	**Relational** (Algebraic & Graph)	3-
11	GT	2	0	1.5	**Multistructual** (Graph & Table, with misconceptions of y-intercept in table)	2-
12	AG	2	0	1	High **Multistructural** (Algebraic and Graph, with Misconceptions, procedural errors)	2
13	A	1	0	.5	**Unistructural** (Algebraic, with Misconceptions in slope	1-
16	TA	2	0	1	**Multistructural** (Table & Algebraic, with procedural errors)	2-

Table 12. Summary results for University 2 – Problem 1 – Slope and y-intercept of a linear function

Student Number	Order of Representations Used	Number of Representations Used	Number of Connections Made	Number Correct	Classification/Notes	Rating
1	ATG	3	6	3	Strong **Relational** (All)	3+
2	AG	2	0	2	**Multistructural** (Algebraic, Graph)	2-
5	ATG	3	6	0	**Relational** (all), has misconception throughout the representations (has x and y reversed)	3-
7	AG	2	2	1.5	**Relational** (Algebraic & Graph, with misconceptions)	3-
8	AGT	3	0	2.5	**Multistructural** (All, with some misconceptions)	2-
10	GTA	3	6	2.5	**Relational** (All, with misconceptions of intercepts)	3-
14	GAT	3	6	3	High **Relational/Extended Abstract** (All)	4
15	AGT	3	6	3	Strong **Relational** (All)	3+

Question 2. The majority of students in the traditional curriculum (Table 13) solved the system of linear equations algebraically. These students were taught how to solve a system graphically, by substitution and by elimination. Graphically, these students have been taught that the solution of a system of equations was the point of intersection of two lines. These eight students did not display this knowledge with

their inability to interpret the graphical representation of the problem. None of the students in the traditional curriculum attempted the tabular representation of this question.

The majority of students (6 of 8) in the multi-representational curriculum (Table 14) chose the algebraic representation first and used the graphical and tabular representations to reinforce their confidence in the correctness of their answers. In contrast to the traditional curriculum, the majority of these students were able to solve the problem using the tabular representation.

The students in the traditional curriculum had a median rating on the SOLO Taxonomy (Biggs & Collis, 1982) of 0.875, or low unistructural, whereas the students in the multi-representational curriculum had a median rating of 2.875 or low relational. Therefore the students in the multi-representational curriculum appeared to have a higher level of conceptual understanding of the connections among the algebraic, graphical and tabular representations of systems of linear equations.

Table 13. Summary results for University 1 – Problem 2 – Solving a system of linear equations

Student Number	Order of Representations Used	Number of Representations Used	Number of Connections Made	Number Correct	Classification and Notes	Rating
3	A	1	0	1	**Unistructural** (Algebraic)	1
4	G	1	0	1	**Unistructural** (Graph)	1
6	A	1	0	0	**Unistructural** (Algebraic, with MAJOR arithmetic mistakes)	1-
9	AG	2	0	.5	**Unistructural** (Algebraic with arithmetic mistake, Student uses wrong information from graph, uses the intercepts).	1
11	GA	2	2	0	**Prestructural** (student finds the intercepts for both algebraic and graph which reinforces the misconception that the problem is correct)	0+
12	A	1	0	1	**Unistructural** (Algebraic, student says the solution of x = 4 and y = -2 is two answers, never recognizes it as an ordered pair or a point of intersection)	1
13	A	1	0	.5	**Unistructural** (Algebraic, MAJOR misconceptions)	.5
16	A	1	0	1	**Unistructural** (Algebraic, writes the answer as a ordered pair, but when asked what that represents, it is just how the answer is written)	1

Tab*le 14.* Summary results for University 2 – Problem 2 – Solving a system of linear equations

Student Number	Order of Representations Used	Number of Representations Used	Number of Connections Made	Number Correct	Classification and Notes	Rating
1	GA	2	2	2	**Relational** (Algebraic & Graph)	3-
2	AGT	3	2	3	**Relational** (Algebraic & Graph). Treats table as separate object	3-
5	ATG	3	6	3	**Relational** (All)	3+
7	ATG	3	6	2	**Relational** (All, student is bothered when algebraic answer does not produce the same answer as in graph and table. Is able to find her mistake)	3
8	TGA	3	0	1	**Multistructural** (Table & Graph, with MAJOR misconceptions Student only identifies one coordinate with the graph and the table. With the algebraic with student uses the x intercept of one equation as the answer (prestructural)	1.5
10	ATG	3	0	0	**Prestructural** (student does random incorrect procedures with each representation)	0
14	AGT	3	6	2.5	**Relational** (All, realizes algebraic must be wrong since answers don't match)	3
15	AGT	3	6	2.5	**Relational** (All, realizes algebraic is wrong when using the graph and table	3

Question 3. Results from University 1 (Table 15) show that students in the traditional curriculum relied heavily on algebraic procedures. Most students displayed little or no understanding of the factor theorem and its connections to the graphical and tabular representations of the polynomial function. As in the previous problem, none of the students from this traditional curriculum attempted to interpret the tabular representation. Even though these students had recently done a considerable amount of algebraic factoring of which this problem was typical, the large majority were not successful in their attempt to factor the polynomial.

Interestingly, all the students in the multi-representational curriculum (Table 16) attempted the tabular representation, with most being successful at identifying the zeros of the polynomial. Many students were still confused about factors and zeros and even though they were able to identify the zeros, they did not necessarily write the factors. Their misconceptions were either that the zeros were the factors or that when writing the factors the correct format was $x + c$ instead of $x - c$. Half of the students in the multi-representational curriculum did not attempt the algebraic representation.

The students in the traditional curriculum had a median rating on the SOLO Taxonomy (Biggs & Collis, 1982) of 0.625, thus were very low unistructural. By comparison, the median level for the students in the multi-representational curriculum was 2.75, or low relational. Thus the curriculum seemed successful in moving students to a higher level of conceptual understanding about the relationship among

the algebraic, tabular, and graphical representations of the factors of a polynomial function.

Table 15. Summary results for University 1 – Problem 3 – Factoring a polynomial

Student Number	Order of Representations Used	Number of Representations Used	Number of Connections Made	Number Correct	Classification and Notes	Rating
3	A	1	0	.5	**Unistructural** (Algebraic, with procedural errors)	1-
4	NONE	0	0	0	**Prestructural**	0
6	A	1	0	0	**Pre/Unistructural** (Algebraic, student realizes that they should factor by grouping but cannot perform the procedure)	.5
9	NONE	0	0	0	**Prestructural**	0
11	AG	2	2	1	**Relational** (Algebraic and graph, with misconceptions about factors, gets solutions instead of leaving as factors)	2.5
12	AG	2	1	1	**Multistructural** (Algebraic and graph, student correctly factors algebraic but then solves for the zeros, this lets her tie it in with the x intercepts)	2-
13	NONE	0	0	0	**Prestructural**	0
16	A	1	0	1	**Unistructural** (Algebraic)	1

Table 16. Summary results for University 2 – Problem 3 – Factoring a polynomial

Student Number	Order of Representations Used	Number of Representations Used	Number of Connections Made	Number Correct	Classification and Notes	Rating
1	TG	2	2	2	**Relational** (Table and Graph)	3-
2	GT	2	6	2	**Relational** (All, the student doesn't solve the problem algebraically but explains a method for solving it by using the graph or table to find the zeros and then rewrite as factors)	3-
5	TAG	3	6	2	**Relational** (All, student realizes from the answers to the table and graph that he has an extra factor in his algebraic answer)	3
7	TG	2	0	0	**Uni/Multistructural** (Table and Graph, Student knows to look at the zeros but performs some undirected manipulations and gets the answer as the opposite of the zeros.	1.5
8	GTA	3	2	3	**Relational** (Graph & Table, Unclear whether student relates algebraic)	3-
10	TG	2	0	1	**Multistructural** (Table and graph, with misconception, writes zeros as answers instead of factors)	2-
14	GTA	3	6	1.5	**Relational** (ALL, with misconceptions about factors, "the factor is the solution")	3-
15	GTA	3	6	3	**Relational** (All)	3+

Question 4. Question 4 yielded the most unexpected results. From experience, typically the concept that most students remembered from secondary school mathematics was the vertical line test for determining if a relation was a function in graphical form. The researcher expected that most students from both curricula would answer question 4 correct using the graphical representation and the vertical line test. However, this was not the case. More students could not answer and did not even attempt to answer question 4 than any other problem. Of the students who did attempt to answer the question, the majority used either the graph and/or the tabular representation. Only two students, one from each curriculum, attempted the algebraic representation. Neither was successful in using the symbolic expression to identify if it was a function, nor did they make any connections from the algebraic to the other two representations. This shows a lack in both curricula and in the secondary school curricula of identification of functions from their algebraic representations.

The students in the traditional curriculum (Table 17) made very few connections among the representations, with most using the graphical representation to solve the problem. This was not unexpected because the identification of functions was stressed in a traditional curriculum from a graphical perspective. It was unexpected that half of the students in the traditional curriculum could not answer the question at all.

The students in the multi-representational curriculum (Table 18) used the table and graph, with neither being prevalent over the other. They could not identify any features of the algebraic representation that could indicate whether it was a function

or not. The multi-representational curriculum, though stressing the other two representations, may not have given enough attention to the algebraic when identifying functions. This may have been a result of teacher influence. The issue of how to teach the algebraic representation for identifying a function was never raised in the weekly meetings.

The students in the traditional curriculum had a median rating on the SOLO Taxonomy (Biggs & Collis, 1982) of 0.500, thus were between a prestructural and a unistructural understanding of determining if a relation was a function. By comparison, the median level for the students in the multi-representational curriculum was 2.125, or high multistructural.

Table 17. Summary results for University 1 – Problem 4 – Identifying a function

Student Number	Order of Representations Used	Number of Representations Used	Number of Connections Made	Number Correct	Classification and Notes	Rating
3	NONE	0	0	0	**Prestructural**	0
4	G	1	0	1	**Unistructural** (Graph)	1
6	NONE	0	0	0	**Prestructural**	0
9	GTA	3	4	2	**Relational** (All, student does not answer the algebraic problem but explains the strategy for solving it by graphing and then looking at the graph or the table)	3-
11	TG	2	0	2	**Multistructural** (Table and graph, does not imply any connections)	2-
12	G	1	0	1	**Unistructural** (Graph)	1
13	NONE	0	0	0	**Prestructural**	0
16	NONE	0	0	0	**Prestructural**	0

Table 18. Summary results for University 2 – Problem 4 – Identifying a function

Student Number	Order of Representations Used	Number of Representations Used	Number of Connections Made	Number Correct	Classification and Notes	Rating
1	TG	2	0	2	**Multistructural** (Table & Graph)	2-
2	TG	2	1	2	**High Multistructural/Low Relational** (Table and Graph, student does say they would graph the algebraic to look at the graph and the table, but doesn't connect the three)	2.5
5	TGA	3	6	2	**Relational** (All, with some undirected manipulation on the algebraic)	3-
7	GT	2	0	2	**Multistructural** (Graph & Table, cannot make judgement about relationship)	2-
8	NONE	0	0	0	**Prestructural** (student writes down the domain and range, but cannot determine if it is a function or not)	0
10	GT	2	0	0	**Prestructural** (student says it isn't a function, but wrongly justifies why they are saying that)	0
14	TG	2	2	2	**Relational** (Table and Graph)	3-
15	GT	2	2	2	**Relational** (Graph and Table	3-

Question 5. Students in the traditional curriculum (Table 19) relied heavily on the algebraic representation to find the solutions to a quadratic equation. Some students factored while others used the quadratic formula. The relatively high degree of success with only a few procedural errors was attributed to the order of the curriculum. The concept of quadratic equations was the most recent concept taught in this curriculum before the posttest was given of all the questions on the testing instrument. The students in the traditional curriculum made little connection with the zeros of the graph or the table representing the quadratic equation as a function of y.

Students in the multi-representational curriculum (Table 20) showed a strong understanding of the connections among the representations. All but one student used all three representations, with the majority making the connections among those representations. It was concluded that the multi-representational curriculum was effective in helping students attain a high level of concept development as measured on the SOLO Taxonomy (Biggs & Collis, 1982) scale for this problem.

The students in the traditional curriculum had a median rating on the SOLO Taxonomy (Biggs & Collis, 1982) of 1.00, thus were primarily at a unistructural level. By comparison, the median level for the students in the multi-representational curriculum was 2.875, or low relational. Thus the multi-representational curriculum seemed more successful than the traditional curriculum for helping students attain a higher level of conceptual understanding about the relationship among the algebraic, tabular and graphical representations of the solutions to a quadratic equation.

Table 19. Summary results for University 1 – Problem 5 – Solving a quadratic equation with rational zeros

Student Number	Order of Representations Used	Number of Representations Used	Number of Connections Made	Number Correct	Classification and Notes	Rating
3	A	1	0	1	**Unistructural** (Algebraic)	1
4	A	1	0	1	**Unistructural** (Algebraic)	1
6	A	1	0	1	**Unistructural** (Algebraic)	1
9	NONE	0	0	0	**Prestructural**	0
11	A	1	0	0	**Unistructual** (Algebraic, with procedural errors in factoring)	1-
12	AG	2	2	2	**Relational** (Algebraic and Graph)	3-
13	A	1	0	0	**Unistructural** (Algebraic with procedural errors)	1-
16	A	1	0	0	**Unistructural** (Algebraic)	1

Table 20. Summary results for University 2 – Problem 5 – Solving a quadratic equation with rational zeros

Student Number	Order of Representations Used	Number of Representations Used	Number of Connections Made	Number Correct	Classification and Notes	Rating
1	GTA	3	4	3	**Relational** (All)	3
2	AGT	3	6	3	Strong **Relational** (All)	3+
5	TGA	3	6	3	Strong **Relational** (All)	3+
7	GTA	3	0	0	**Multistructural** (All, procedural errors, unconcerned that there are different answers)	2-
8	TGA	3	2	2	**Multi/Relational** (All, Confused by the answer being different for the algebraic but not quite at a level where he admits his algebraic may be wrong so accepts both answers)	2.5
10	TG	2	2	2	**Relational** (Table & Graph, does say that to solve the algebraic they would graph it)	3-
14	AGT	3	2	2	**Multi/Relational** (All)	2.5
15	TGA	3	6	3	Strong **Relational** (All, checks and relates all answers together, explains how answers are found in each of the representations)	3+

Classification of Individual Students

It is important to note that the SOLO taxonomy assigns levels to students' learning outcomes for a particular question, not as a classification for the student in general (Watson, Chick, & Collis, 1988). By examining the side-by-side boxplots of each student, the level of variability in student responses can be seen (Figure 18). Because a student was classified at a relational level on one question does not mean that he or she would be at a relational level on all questions related to functions. Thus, based on the questions asked in this study, it would have been incorrect to classify a student at a particular level for their overall understanding of functions.

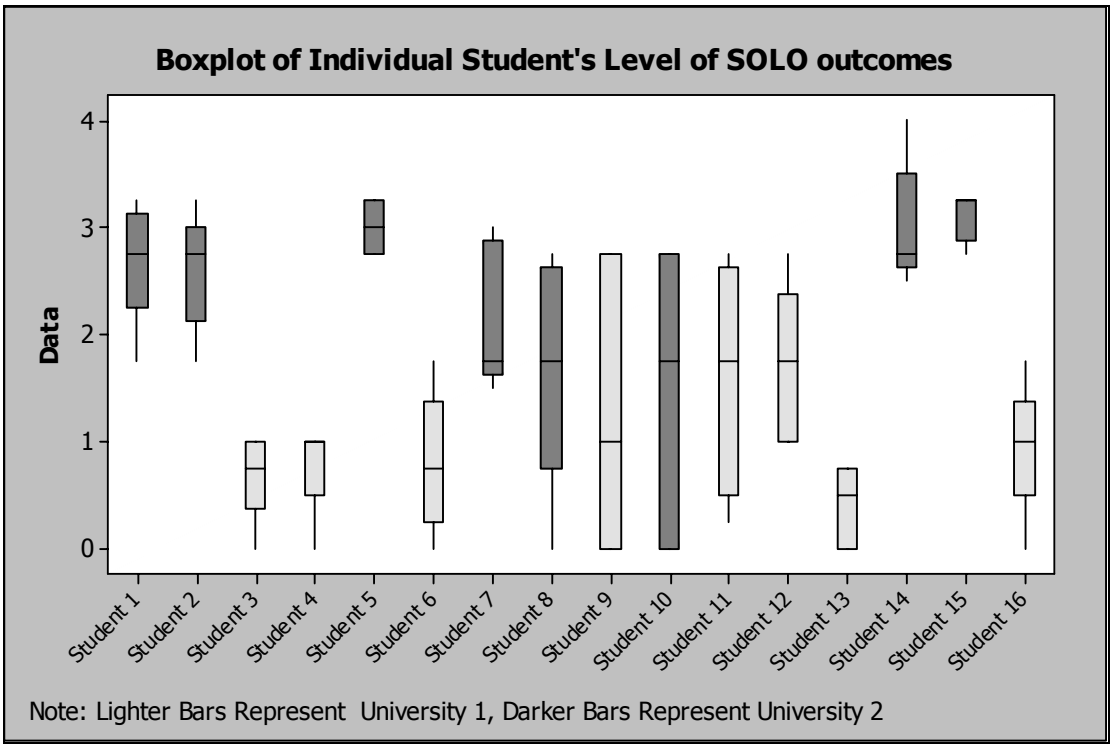


Figure 18: Boxplot of individual student's levels of SOLO outcomes

Students 8, 9, 10, and 11 (two from University 1 and two from University 2) had the most variability in their understanding of functions across the five questions.

This may indicate that they were only beginning to develop low relational understanding (level 2.75) of functions in certain contexts (e.g., linear functions, identifying a function). Five of the eight students from University 2 had a small range of variability between multistructural and relational, with one student attaining an extended abstract level (4) understanding on Question 1 concerning slope and y-intercept of a linear function. Five of the eight students from University 1 had a range of variability between prestructural, unistructural, and an occasional low multistructural level.

The SOLO Taxonomy did not distinguish whether a particular student was functioning at a relational level of understanding based on curriculum strategies or if the student would have a relational understanding without certain curriculum strategies. However, when the data for all the students in the study were examined by question, both of these issues have been addressed. By combining the students, individual differences get absorbed and the group functioning on a higher or lower level does show differences in curriculum strategies on a particular question as shown previously (Figure 17).

Results acquired from the quantitative and qualitative analyses give evidence to support the use of a functional approach, multi-representational curriculum to augment developmental mathematics students understanding of function concepts. Conclusions and recommendations based on these results are presented in Chapter 5.

CHAPTER 5

Summary, Conclusions, Discussion, and Recommendations

The purpose of this research was to investigate the potential benefits of teaching developmental college mathematics from a functional perspective utilizing and making connections among three common representations (graphical, tabular, and algebraic) of functions. There were several factors behind the rationale for this study: (1) the need for students to have relational understanding (Skemp, 1987); (2) a shift in understanding of functions from a symbolic, procedural perspective to a view of functions as a co-variation between two or more quantities is necessary if the function concept is viewed as the unifying theme in algebra (Selden & Selden, 1992); (3) this shift in understanding can be enhanced by multiple representations and the connections among the representations (Hiebert & Carpenter, 1992); and (4) current teaching methods in developmental college algebra do not typically use multiple representations or make these connections (Laughbaum, 2003).

This chapter is organized into three parts. First, a brief summary of the study is followed by conclusions, based on the quantitative and qualitative results reported in Chapter 4, which are presented in relation to the research questions and related research hypothesis. The next section of the chapter is a discussion of limitations and recommendations of the study on the use of multiple representations in teaching developmental college algebra and on the use of the SOLO Taxonomy as a diagnostic tool for student understanding. Recommendations for teaching and for future research make up the last part of this chapter.

Summary and Conclusions

The participants of the study were 313 college students enrolled in developmental college algebra at two southern universities. The present study utilized a quasi-experimental design in which instructors at one university (control) taught the course from a traditional algebraic perspective while instructors at the other university (experimental) taught the course from a functional approach simultaneously introducing multiple representations of functions.

The two groups were taught the same topics over the same time period. The traditional curriculum used at University 1 emphasized algebraic representations. Students in this curriculum (n = 213) were taught symbolic manipulation skills and assessed during the course on their knowledge using algebraic representations. Students in the reform curriculum at University 2 (n = 100) were also taught symbolic manipulation skills along with the graphical and tabular representations. Connections were highlighted among the representations and students were assessed throughout the course using a variety of representations. Students in the traditional curriculum could use scientific but not graphing calculators. Students in the reform curriculum were required to have a graphing calculator.

At the beginning of the course, both groups were given pretests to measure their success as well as the type of representation they chose to solve a problem. No significant differences were found between the two groups either on success, measured in the student score graded out of 20 points, or the proportion of alternate representations (non-algebraic) used by students. When analyzing the data question

by question, there was a significant difference in students' success on the pretest in finding the zeros of a quadratic function, with the experimental group having higher scores than the control group. Since the mean pretest score for this question for both groups was well below 20% correct, this result had little practical significance. There was also a significant difference on the pretest in the proportion of alternate representations used by students in the experimental group on the specific question of solving a system of linear functions with 9% of the control group choosing an alternative representation versus 16% of the experimental group.

At the end of the semester, a posttest was given to both groups of students and qualitative interviews were conducted with 16 participants, 8 from each school, to examine the connections that students in each curriculum were making among the representations and their ability to move flexibly from representation to representation, a sign of stronger conceptual understanding. The following section addresses the findings of the study in relation to the research questions.

Research Question 1

> What was the impact of a multi-representational developmental college mathematics curriculum on the success of students in solving routine problems as compared to a traditional algebraic-focused curriculum?

Hypothesis: A multi-representational developmental college mathematics curriculum would increase student success in solving routine problems when compared to a traditional algebraic-focused curriculum.

An analysis of pretest scores showed no initial significant difference in the students' scores between the two curricula. There were significant differences between the two groups on posttest scores. The students in the multi-representational developmental college mathematics curriculum scored significantly higher on the posttest with a mean score of 14.65 out of 20, or 73.25 %, whereas the students in the traditional curriculum scored an average of 8.72 out of 20, or 43.6%. The effect size for the difference between the two curricula was 1.27, with a 95% confidence interval of (1.01, 1.53). The effect size of 1.27 indicated that the mean score from the multi-representational curriculum was approximately at the 90th percentile for scores from the traditional curriculum (Cohen, 1988). This showed a very large effect on student success in developmental mathematics by implementing a curriculum which employs a functional approach and emphasizes multiple representations and the connections among them.

It was concluded that the multi-representational curriculum had a significant impact on the success of students in solving routine problems. Students in this curriculum had at least added to their "cognitive kit-bags" (Crowley, 2001) and procedurally understood how to solve the problems on the testing instrument. Consideration must be given to teaching methods, activities that promote classroom discourse, changing assessment practices, using technology, and making connections among the representations in order for these results to be replicated.

Research Question 2

How did a multi-representational curriculum affect students' preference to use a different representation and their reliance on symbolic manipulations to solve routine problems, as compared to students in a traditional curriculum?

Hypothesis: A multi-representational developmental college mathematics curriculum would decrease students' reliance on symbolic manipulations to solve routine problems when compared to students in a traditional curriculum.

Past studies have suggested that students will continue to rely on the algebraic representation even when another representation was available and may have been procedurally easier (Knuth, 2000b; Thompson, 1994). McNemar's test (Hollander & Wolfe, 1999) for testing the difference in proportions when the proportions are dependent, as they are for repeated measures within universities, found that both groups had a statistically significant proportion of students changing their preference of representation from pretest to posttest. However, students in the traditional curriculum of University 1 changed from using more alternative representations on the pretest, to using less on the posttest. These students relied more on algebraic representations after the course, which was not surprising since this was the representation emphasized throughout the course. Contrary to previous research, the students in the multi-representational curriculum showed significant willingness to change from an algebraic representation to a tabular or graphical representation. This corroborates the belief that a reliance on particular representations may be influenced by curricular and instructional choices (Knuth, 2000b; Thompson, 1994).

There was no difference between schools on the pretest in the proportion of alternative representations with which students used to solve problems; however, a two-sided 95% confidence interval for the difference in proportions between the two groups (multi-representational - traditional) on the posttests was (0.334, 0.433). This implies that students in the reform curriculum were significantly more likely to employ a representation other than the algebraic. A multi-representational curriculum appeared to increase the flexibility of students to use a different representation and decrease their reliance on algebraic representations. This conclusion was confirmed by the qualitative interviews. Each student interviewed who was enrolled in the multi-representational curriculum had the confidence to be able to use more than one representation on every problem except one student on Question 4 who could not answer the problem at all. This ability showed a broader repertoire of representations in their functional understanding. Again, this shows that they have at least added to their cognitive kit-bags of procedural techniques (Crowley, 2001).

Research Question 3

> What impact did a curriculum with an instructional emphasis on different representations have on the number of connections students made among representations?

Hypothesis: Students taught in with a multi-representational curriculum where the graphical, tabular, and algebraic representations are introduced simultaneously, with no instructional preference given to a particular representation will make more connections among representations than students in a traditional curriculum.

The problem summary sheet and classification of students using the SOLO Taxonomy (Biggs & Collis, 1982) showed that the sample of eight students in the multi-representational curriculum made more connections among representations than their counterparts from the traditional curriculum. This showed that these students have gone beyond just adding to their procedural understanding and have formed a tighter, more compressed (Tall & Barnard, in progress) cognitive understanding of functions.

Research Question 4

> What influence did a curriculum emphasizing multiple representations have on the flexibility of students to solve a problem with a different representation, even if the algebraic method is the first one they choose? Did students in a curriculum involving multiple representations broaden their repertoire of representations?

Hypothesis: Students taught in with a multi-representational curriculum where the graphical, tabular, and algebraic representations are introduced simultaneously, with no instructional preference given to a particular representation will not be limited to an equation-to-graph direction, but will be able to flexibly move among the representations.

Of the 16 students interviewed, students in the multi-representational curriculum scored much higher ratings on the SOLO Taxonomy (Biggs & Collis, 1982) than did students in the traditional curriculum, most in fact exhibiting a relational level of understanding across several problems. Also significant was that students from the multi-representational curriculum did not necessarily start with the

algebraic representation, but often with the tabular and graphical representations. The students in this curriculum started with a table thirteen times, the graph twelve times and the algebraic fourteen times. Thus, it does not appear that as a group they have a preference for one representation over another. This ability to use more than one representation and to make connections among the representations showed a broadened repertoire of techniques. This broader understanding of representational techniques may transfer to situations which rely on graphical or tabular interpretations.

Research Question 5

To what extent were students' webs of knowledge of different representational forms of function-related concepts well-connected? What were the differences or similarities in the level of this connectedness among students in a curriculum emphasizing multiple representations and students in a traditional curriculum?

Hypothesis: Students in a multi-representational curriculum have a more well-connected web representational of forms of function-related concepts than students in a traditional curriculum.

Traditionally taught students in the present study recognized little of function concepts in either the graphical or the tabular representations. Students in the multi-representational curriculum had a well-connected network of representational knowledge of function concepts, demonstrated by their mostly multistructural and higher SOLO levels of understanding for each of the five aspects of functions (i.e., linear functions, systems of linear equations, factors of polynomials, function

identification, and quadratic equations). One representation that both groups had difficulty using was an algebraic representation to determine if a relation was a function. Of the sixteen students interviewed only a few even attempted to manipulate that representation and none got it correct. The algebraic representation appeared unconnected for both curricula when deciding if a relation was a function.

Overall Conclusions

Trivial concepts to mathematicians are not necessarily trivial to developmental mathematics students. Most mathematicians would consider the connections among the representations on the test instrument in the present study to be trivial. However, by examining the lack of connections that students in the traditional curriculum made among representations, in fact their inability to interpret the graphical and tabular representations showed that, to these students, these connections are not only non-trivial but nonexistent. The assumption that these connections will be apparent was false for this group of individuals. Having been taught the representations without preference given to the algebraic, the students in the multi-representational curriculum were not only more successful in their representational competence, but gained significant mathematical understanding.

The results of the present study can only be replicated if consideration is given to factors that were controlled in the study. One of the most influential factors was that of the teachers. The two curricula may have been equivalent without educating teachers from University 2 on how to teach using representations effectively, introducing them at the same time without preference toward a particular

representation, and to make the connections among the representations. Other factors which must be considered include assessment practices, technology use, and activities which promote classroom discourse. If these things are considered and accounted for, then these results should be able to be replicated in most developmental algebra classrooms.

An important detail in the present study was that each representation was presented simultaneously throughout the multi-representational curriculum, without preference being given to any one representation. This allowed students the freedom to decide which representation is most effective in certain situations and with which representation they were most comfortable. If students did not know whether an assessment question would be given from a graphical, tabular, or algebraic perspective, it necessitated that they know how to operate with all the representations equally, even if they had a preference for a particular one. If students were not comfortable with a representation, then they needed to know how to translate into an alternate representation that they were comfortable with, thus making connections among the representations and showing a higher level of understanding.

By educating developmental and pre-college students in this manner, they could have more understanding of graphical and tabular representations. This could lay the foundation for their development of mathematical understanding in subsequent courses like Pre-Calculus where they study graphical transformations of trigonometric functions or in Calculus when analyzing graphs of the derivative. It could also be beneficial in other subjects such as science where problems take the

form of data analysis and graphical interpretations. It may even help their understanding of graphs and tables in the popular press, like newspapers. More research on the transfer into these areas of applications is needed.

Students taught in a multi-representational curriculum may begin to view algebra as more than just the study of equations and functions and as more than just equations into which numbers are substituted by employing methods that are not limited to memorizing rules and procedures. Most mathematics educators believe that relational understanding is a better goal to aspire to than instrumental understanding, yet still the majority of pre-college and developmental mathematics classes are taught with an approach that leads to knowing what to do, but not necessarily why.

Discussion

Discussion of the Use of Multiple Representations

Skemp (1987) metaphorically describes instrumental and relational understanding with an example of different routes or paths in an unfamiliar city. Instrumental understanding is like having one route that will get a person from home to office, whether or not it is the best route is not the issue, but that they have a strategy for how to accomplish the commute. Relational understanding is likened to exploring a town and developing an internal cognitive map, which allows a person to take numerous routes, change direction, and get from any starting point to any finishing point. Traditional approaches to algebra and functions can be likened to this same metaphor; a set of rules and procedures to be mastered to get from point A to point B. Whereas, relational understanding is developing a cognitive map, an

understanding of algebra and functions that lets a student develop numerous strategies and routes so that starting at any point, they can arrive at any other point.

The students in the present study who were enrolled in a traditional curriculum typically displayed instrumental understanding, they had one route (algebraic) to get them from problem to answer which may or may not have been successful. When faced with a different route (graphical or tabular) they were confused and/or had no mental map by which to travel the route from problem to answer using an alternate representation, that is, they were "lost". The students in the multi-representational curriculum were more successful at going from any one representation to another, traveling from one starting point to another, whether by their own choice or interviewer prompting. Even when they did not relate the representations together (i.e., multistructural level), most still had an alternate method for solving problems. There would be less chance for students with a relational or extended abstract SOLO level of understanding to "get lost" even when faced with new problems since they would have developed multiple strategies for approaching new problems.

Traditionally, algebra has been taught in an equation-to-graph direction with tabular representations being the intermediate step. The reverse path of graph-to-equation gets little focus in the traditional curriculum (Eisenberg, 1992). The path which begins with the tabular representation is even less emphasized by teachers, and therefore by students. These alternate paths have long been thought of as implicit skills that students will develop though not explicitly taught. On many standardized

testing instruments, this reversibility is what tests students' depth of understanding of algebraic concepts such as functions (e.g., NAEP, 2003).

Students in developmental mathematics have typically had at least two opportunities to implicitly understand the connections among algebraic representations in secondary school courses. The fact that they are enrolled in developmental mathematics would lead one to conclude that they did not implicitly comprehend what people who are adept in mathematics take for granted, the connections among representations. These results showed that after again learning algebra and function concepts from a traditional approach for possibly a third time, the students from University 1 were no better at developing these connections than previously. This leads to the implication that the traditional approach does not work for the majority of developmental students to increase such connected knowledge. That is not to say that it does not work for anyone, because many mathematicians and scientists have been taught in traditional ways and understand these connections.

Students in a curriculum that approached the teaching of algebra from a functional perspective, utilizing the typical representations of graph, table, and algebraic and emphasizing connections among these representations, displayed a more connected use of representations of algebraic and functional concepts. This was evident by the fact that even at the end of the course, the students in the reform curriculum remembered how to do problems that they had essentially learned at the beginning of the course, a sign of relational understanding (Skemp, 1987). Several of the traditional students interviewed remarked that they "needed to study", that they

had “forgotten this stuff”. None of the students in the multi-representational curriculum made those types of comments.

Not all students in the multi-representational curriculum were successful and the present study does not attempt to say that they were. However, concept formation does not happen in one step, it happens with many experiences and understandings that when connected, form a stronger concept image than previously attained and develops into a complete realization of the concept (Vinner & Dreyfus, 1989). Thus only teaching algebraic methods to students who have not understood algebraic methods on two previous attempts is not an effective way to increase their concept formation. Developmental mathematics students need new experiences and more interaction with different representations to form associations among algebraic and function concepts that are organized and lasting.

Before technology, producing alternate representations of function concepts was tedious and time consuming, thus mathematicians relied on algebraic methods. However, with the advent of the graphing calculator, students have access to an inexpensive and powerful representation-producing tool. Teachers can utilize this tool, combined with classroom discourse and non-traditional assessment practices, to strengthen students’ understanding of concepts by connecting the representations in meaningful ways.

This requires teachers to think about mathematics differently than they were taught. There is “great psychological difficulty for teachers of reconstructing their existing and longstanding schemas” (Skemp, 1987, p. 161). Handing a teacher a

graphing calculator and telling them to use it in their teaching does not mean that a teacher will use it in a way to encourage relational understanding (Patterson & Norwood, in press). Similarly, using a reform-oriented textbook which aims at relational understanding does not mean that a teacher is going to teach for that type of understanding (Skemp, 1987). However, it was shown in the present study that educating teachers on methods for using technology and written curriculum materials in reform minded ways might have contributed to students' success in developmental algebra. It is reasonable to assume that this would be transferable to similar courses at the secondary level as well.

Discussion of the Use of the SOLO Taxonomy

Overall, the SOLO Taxonomy (Biggs & Collis, 1982) was useful in classifying learning outcomes of students on particular problems and making comparisons between the two curricula. It was extremely beneficial in identifying levels of student performance by which remediation strategies could then be employed. For example, if a teacher realized that the majority of students were functioning on a multistructural level on tasks related to linear functions, he/she can develop strategies for moving students toward a relational level of understanding. By using the problem summary sheet, breakdown in connections among representations became visible. The summary sheet also allows for an analysis of the correspondence in the connections among representations and the success a student has with a representation. This is important because if the student made the connection among representations, but got the problem incorrect using all the representations, there was

obviously some conceptual misunderstanding woven through all of the representations. However, if the student made connections between two of the representations and got both of those correct, while incorrectly using the third representation, that student may only need help with that representational skill and conceptualization of how that representation connects with the other two.

Issues in using the SOLO Taxonomy (Biggs & Collis, 1982) arose when it was found during analysis that there are different levels within each level of the taxonomy, therefore, the levels are not discrete. This creates a need for classifying students as weak or low in a certain category, and some as strong or high in a certain category. This can be done based on the number of connections the students make among representations, the number of representations they can use, and the correctness of their procedure and answers.

Overall, for classroom teachers, the SOLO Taxonomy (Biggs & Collis, 1982) is an effective method of interpreting student responses. For the teacher, developing the classification rubric for each of the levels is extremely beneficial in that it necessitates that the teacher consider what it means for students to understand at each level and what typical responses students should give. This is important because most assessment in classrooms is informal and happens continuously throughout a class period. By developing a rubric of typical responses that classifies student learning at each level of the taxonomy, the teacher has an easy way of assessing informal classroom discourse, effectively allowing them to modify their instruction.

Recommendations

Implications for Teaching

This research shows that a multi-representational curriculum can be effective in expanding students' web of connected knowledge of algebraic and functional concepts if certain considerations are made. Consideration must be given to teacher education on how to teach effectively with multiple representations. Teachers who have never experienced a curriculum in which all the representations are introduced simultaneously and the connections highlighted would not necessarily ever think to teach in that manner. When a teacher presents a function, makes a table of values, and then graphs the function, they could argue that they are teaching with multiple representations. They need to become aware of the order in which the representations are introduced and the critical role it plays on how students view each representation as a viable means for solving problems.

Consideration also must be given to how curricula are assessed. Students will usually only learn what they are expected to learn. If all assessment is traditional in nature, from an algebraic perspective, then students will learn to value the algebraic representation and will develop a preference for that representation. For students to learn how to solve problems with the other representations, they need to be assessed using the table and the graph. Thus, tests need to contain more than just algebraic representations.

With any curriculum, attention needs to be paid to what students are actually learning and whether the goals of the curriculum have been met. The SOLO

Taxonomy (Biggs & Collis, 1982) and rubric defined in this research gives teachers an effective way of measuring student learning. By reflecting on informal assessment like classroom discussions, teachers can analyze responses and statements for levels of understanding using the taxonomy. This gives them a way of identifying remediation needs of particular students and of an entire class.

Future Research Considerations

The present study should be made longitudinal in scope to determine the effect of teaching using these methods on subsequent mathematics or even science courses. A suggestion would be to start with a secondary first-year algebra course and follow those students through their secondary mathematics courses and possibly, to their first college mathematics course. This would allow researchers to see if students would find it easier in higher level courses like pre-Calculus and Calculus where most students have difficulty because of the increase in graphical and tabular interpretation in these courses. Would their immersion with all the representations at an earlier stage allow them to make the transition to upper level mathematics with more success than traditionally educated students?

A second consideration for future research deals with the SOLO Taxonomy (Biggs & Collis, 1982). "The SOLO Taxonomy cannot do better than the potential of the original question in giving useful information on the level of learning outcome" (Watson, Chick & Collis, 1988, p. 280). Since most of the questions on the interview protocol focus on the relational level and below, the present study has few opportunities to observe students who are classified as relational but may have been

at the extended abstract level. Thus, future studies should consider more questions which demand a higher level of understanding.

Toward a Theory

Skemp (1979) described "varifocal learning theory" in which magnifying each node showed in detail that within the node was a web or connected schema. The present study looked at five different aspects of functions (i.e., linear functions, systems of linear equations, factors of polynomials, function identification, and quadratic equations) considered as five of many nodes that may be a part of students' comprehensive understanding of functions. The methods used allowed the researcher to examine students' understanding within those five nodes for the three most common representational forms; graphs, tables, and algebraic representations. The results of the analysis provided a way to categorize and characterize connections that students made among the three representational forms as an indicator of their understanding of that particular aspect of functions (e.g., linear functions). The present study did not examine the connections that students were making among the five aspects of functions represented in the study. These aspects and the connections between them represent nodes in a larger, zoomed out, picture of functional understanding. As shown in the results, a student at a relational level in terms of linear functions might not be at a relational level in the understanding of other aspects of functions (e.g., quadratic function, systems of equations). However, if one did examine the connections a student is able to make across different aspects of functions, it may be possible to determine their level of understanding of the domain

of functions as a whole. For example, a student who can operate successfully and has a relational level in several aspects of functions (e.g., functions, quadratic, square root) but does not connect salient aspects of the different concepts or use knowledge about one family of functions to inform their work in another context, may only have a multi-structural SOLO level for their understanding of functions. Zooming out even further, this student may be at a unistructural level in their understanding of algebraic concepts if they view functions as a separate concept and unconnected or un-useful for other concepts in algebra (Figure 19).

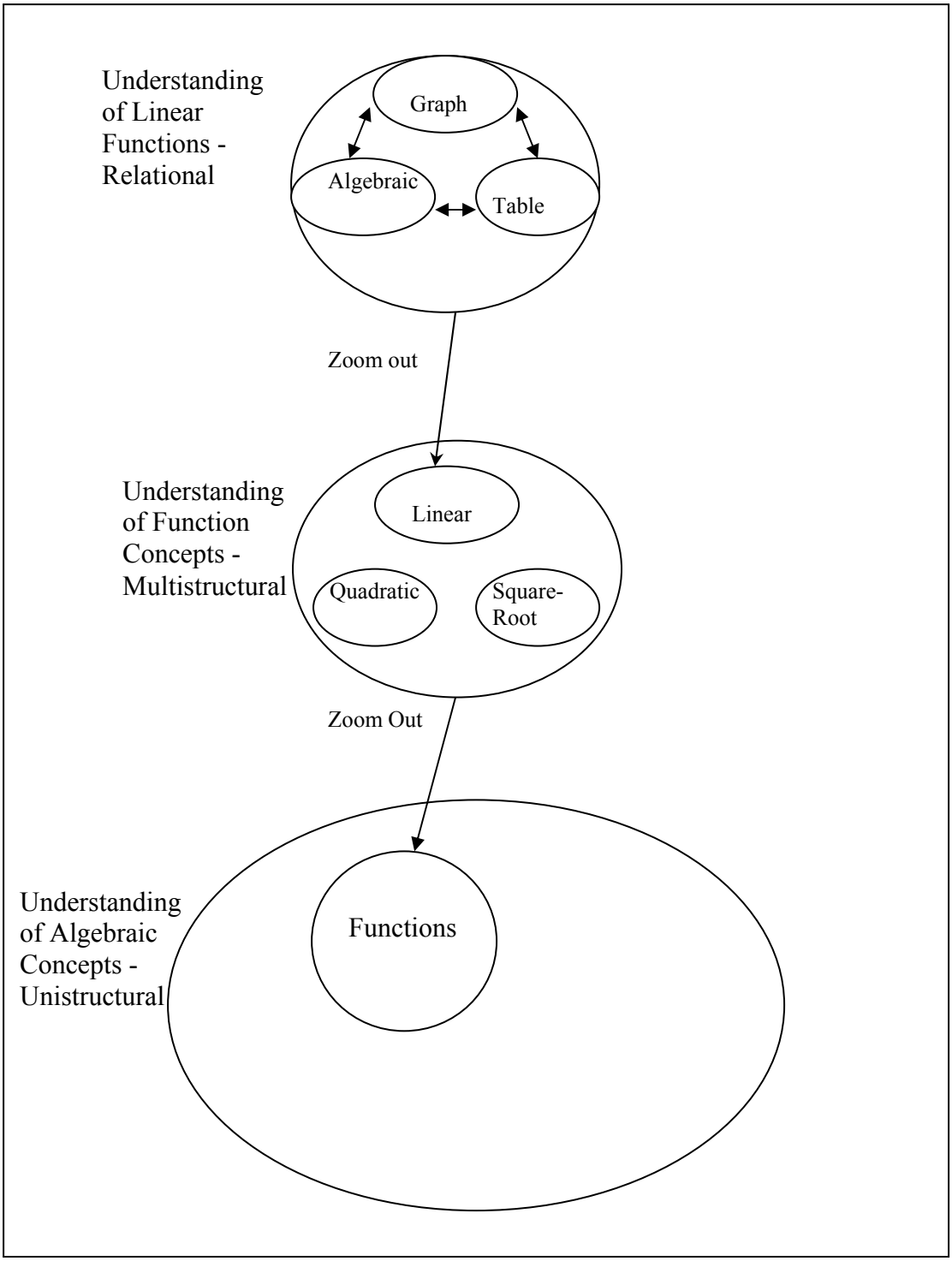


Figure 19. An example of a zoomed out lens on students understanding of aspects of algebraic concepts

The present study focused on one level of understanding, the relationship among procedures across different representational forms of five separate aspects of functions. Further research is needed to determine if teaching with multi-representational methods would build students' understanding across different families of functions, for example finding zeros of linear, quadratic, square-root, and absolute value functions. Examining the multiple representations of different functions may make the invariance of a procedure like finding zeros across functions evident (Slavit, 1997).

Pegg and Tall (2002) linked the SOLO levels to the range of growth from procedure to process common to process-object theories of understanding. By viewing a multi-procedural level as a discrete stage between procedural and process, the process-object theory level of *procedure* corresponds to unistructural on the SOLO Taxonomy, a *multi-procedure* corresponds to multistructural, and a *process* level corresponds to relational. Thus, the students in the present study who were at a relational SOLO level of understanding about a particular function concept could be described as having a process conception of that particular function concept, according to Pegg and Tall. Further research needs to determine if there is a correspondence between the level of student understanding of different representational forms on individual function concepts and overall understanding of functions. This research could lead to a theory of how representational knowledge is connected to mathematical understanding.

A shift in viewing representational forms as separate entities to different forms of the same concept has been termed "conceptual compression" (Gray & Tall, 1994). Correlated to the SOLO Taxonomy, this compression would begin to occur as students reach the relational level of understanding and become even more compressed as the student reached the extended abstract level. As a student reaches the relational level in all five aspects of functions described in this research, further research needs to examine how the conceptual compression occurring within each separate function concept is effecting students' conceptual compression and move toward relational understanding of the entire node of functional understanding. Research on how students at higher levels on the SOLO Taxonomy use relational understanding of the representational nodes of a function concept to understand a different function concept would contribute to research on student development of function sense (Eisenberg, 1992).

Final Thoughts

Early in the educational process, many students classify themselves either as a "math person" or a "non-math person." (This is not to say that this researcher believes in such classifications, but that these are common classifications used in our society.) Most developmental college mathematics students would probably classify themselves as a "non-math person." One distinction between these math and non-math groups is their confidence in their ability to interpret different representational forms of mathematical concepts. "A representation does not represent by itself – it needs interpreting and, to be interpreted, it needs an interpreter" (Von Glaserfeld,

1987b, p.216). The challenge for the mathematics education research community is to find ways for students who consider themselves a "non-math person" to correctly interpret and make connections between equivalent representational forms. This research hopes to build on the knowledge base of how students construct interpretations of representational forms of functions and suggest how educators can help students strengthen their functional understanding, especially those in developmental college mathematics courses.

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APPENDICES

Appendix A – Test Instrument

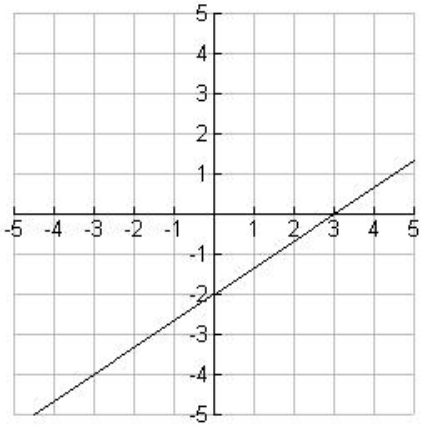
I. Choose one of the following to solve:

1. Find the slope and y-intercept of the line going through the follow

x	y1(x)
-6	-6
-3	-4
0	-2
3	0
6	2
9	4

2. Find the slope and y-intercept of the equation 2x - 3y = 6

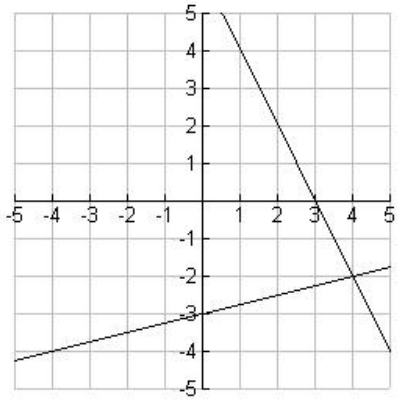
3. Find the slope and y-intercept of the line in the graph:



II. Choose one of the following to solve:

1. Find the solution to the system of equations: $\begin{cases} 2x + y = 6 \\ x - 4y = 12 \end{cases}$

2. Find the solution to the system of equations seen in the graph:

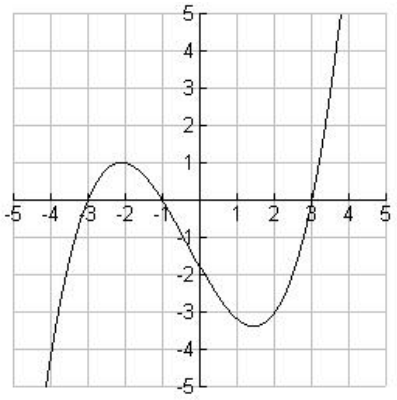


3. Find the solution to the system of equations represented by Y_1 and Y_2 in the following table:

X	**Y_1**	**Y_2**
-4	14	-4
-3	12	-3.75
-2	10	-3.5
-1	8	-3.25
0	6	-3
1	4	-2.75
2	2	-2.5
3	0	-2.25
4	-2	-2
5	-4	-1.75

III. Choose one of the following to solve:

1. Find the real variable factors of the polynomial function in the following graph:



2. Find the real variable factors of the polynomial function which goes through the points in the following table (assume all real roots are shown):

x	y1(x)
-4	-4.2
-3	0
-2	1
-1	0
0	-1.8
1	-3.2
2	-3
3	0
4	7

3. Find all real variable factors of the polynomial: $x^3 + x^2 - 9x - 9$

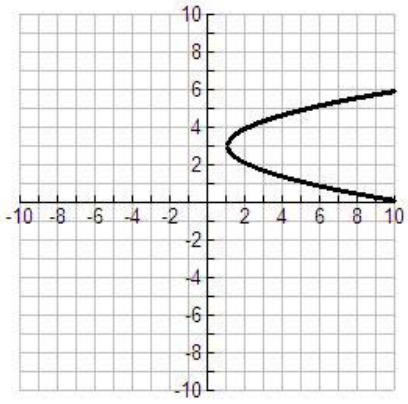
IV. Choose one of the following to solve:

1. Determine if the relation $x = (y-3)^2 + 1$ is a function. Justify your answer.

2. Determine if the table depicts a relation that is a function. Justify your answer.

X	Y1
10	0
5	1
2	2
1	3
2	4
5	5
10	6

3. Determine if the graph depicts a relation that is a function. Justify your answer.

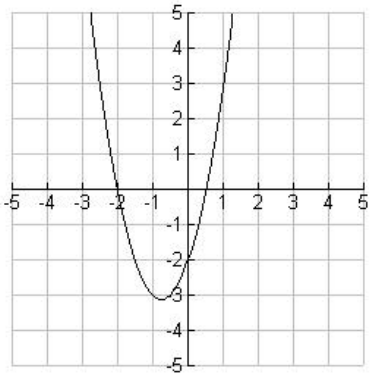


V. Choose one of the following to solve:

1. A table of values for a quadratic function is given, find the solutions when the function equals 0:

x	y1(x)
-3	7
-2.5	3
-2	0
-1.5	-2
-1	-3
-.5	-3
0	-2
.5	0
1	3
1.5	7
2	12

2. Find the solutions when the function equals 0 to the quadratic function pictured in the following graph:



3. Find the solutions to the quadratic equation: $2x^2 + 3x - 2 = 0$

Appendix B - Syllabus of Concepts Taught at University 1

Text: Tobey, J. & Slater, J. (2002). *Intermediate algebra* (4th ed). New York: Prentice Hall.

TEST #1
Material Covered:
Chapter 1, **Basic Concepts**

1. The Real Number System
2. Operations with Real Numbers
3. Powers, Square Roots, and the Order of Operations
4. Integer Exponents and Scientific Notation
5. Operations with Variables and Grouping Symbols
6. Evaluating Variable Expressions and Formula

Chapter 10, **Additional Properties of Functions**

1. Function Notation

TEST #2
Material Covered:
Chapter 2, **Linear Equations and Inequalities**

1. First-Degree Equations with One Unknown
2. Literal Equations and Formulas
3. Absolute Value Equations
4. Using Equations to Solve Word Problems
5. Solving More Involved Word Problems
6. Linear Inequalities
7. Compound Inequalities
8. Absolute Value Inequalities

TEST #3
Material Covered:
Chapter 3, **Equations and Inequalities in Two Variables and Functions**

1. Graphing Linear Equations with Two Unknowns
2. Slope of a Line
3. Graphs and the Equations of a Line

Chapter 4, **Systems of Linear Equations and Inequalities**

1. Systems of Equations in Two Variables

Chapter 5, **Polynomials**

1. Introduction to Polynomials and Polynomial Functions: Addition, Subtraction, and Multiplication
2. Division of Polynomials
4. Removing Common Factors: Factoring by Grouping
5. Factoring Trinomials
6. Special Cases of Factoring

7. Factoring a Polynomial Completely
8. Solving Equations and Exercises Using Polynomials

TEST #4
Material Covered:
Chapter 6, **Rational Expressions and Equations**

1. Rational Expressions and Functions: Simplifying, Multiplying, and Dividing
2. Addition and Subtraction of Rational Expressions
4. Rational Equations
5. Applications: Formulas and Advanced Ratio Exercises

Supplemental Worksheet

TEST #5
Material Covered:
Chapter 7, **Rational Exponents and Radicals**

1. Rational Exponents
2. Radical Expressions and Functions
3. Simplifying, Adding, and Subtracting Radicals
4. Multiplication and Division of Radicals
5. Radical Equations
6. Complex Numbers

Chapter 8, **Quadratic Equations and Inequalities**

2. The Quadratic Formula and Solutions to Quadratic Equations

Appendix C - Syllabus of Concepts Taught at University 2

Text: Norwood, K. S. (2002). *Intermediate algebra through multiple representations* (Preliminary ed.). Boston, MA: Pearson Custom Publishing.

TEST #1

Material Covered:

Chapter 1, **Review of Algebra**

1. Sets, Number Systems, and Properties of Real Number
2. Operations with Real Numbers
3. Order of Operations
4. Properties of Exponents and Scientific Notation
5. Expressions and Equations
6. An Introduction to Inequalities and Absolute Value

Chapter 2, **Functions and Graphs**

1. The Coordinate Plane
2. Graphing Linear Equations
3. Slopes and Rates
4. Relations and Functions
5. Function Notation, Evaluation, and Composition
6. Analyzing Graphs of Functions

Chapter 3, **Linear Functions and Inequalities**

1. Solving Linear Functions
2. Literal Equations and Formulas
3. Solving Linear Inequalities
4. Solving Absolute Value Equations and Inequalities
5. Mathematical Modeling and Linear Regression

TEST #2

Material Covered:

Chapter 4, **Systems of Equations and Inequalities**

1. The Graph/Table Method
2. The Substitution Method
3. The Addition/Elimination Method
4. Systems of Linear Inequalities
5. Three Equations in Three Variables
6. Matrices and Determinants
7. Systems of Linear Equations and Linear Programming

Chapter 5, **Polynomial Functions**

1. Models of Polynomial Functions
2. Adding and Subtracting Polynomial Functions
3. Multiplying Polynomial Functions and Special Products
4. Dividing Polynomial Functions and the Factor and Remainder Theorems

5. Greatest Common Factor, Factor by Grouping, and Special Factoring
6. Factoring Trinomials
7. Solving Quadratic Equations by Factoring and Graphs/Tables

Chapter 6, **Rational Expressions and Functions**

1. Rational Expressions and Simplifying Rational Expressions
2. Multiplication and Division of Rational Expressions
3. Sums and Differences of Rational Expressions
4. Complex Fractions
5. Solving Rational Equations and Models of Rational Equations
6. Solving Rational Inequalities and Models of Rational Inequalities

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TEST #3

Material Covered:

Chapter 7, **Rational Exponents, Radicals, and Complex Numbers**

1. n^{th} Roots and Rational Exponents
2. Simplifying Radical Expressions
3. Adding, and Subtracting Radical Expressions
4. Multiplying and Dividing Radicals Expressions
5. Solving Radical Equations and Functions
6. Imaginary Numbers

Chapter 8, **Quadratic Functions**

1. Solving Quadratic Equations
2. Completing the Square and the Quadratic Formula
3. Solving Equations Quadratic in Form
4. Solving Quadratic Inequalities
5. Mathematical Modeling Using Quadratic Regression

Appendix D - Rubric for Grading Pretests and Posttests

Each Question is 4 points

Question 1 - Finding the slope and y-intercept of a linear equation
Table: 1 point for correctly calculating the change in y and x or recognizing the pattern, 1 point for correct numerator and denominator, and 2 points for the correct y -intercept (0 points if they use the x-intercept).
Algebraic: 2 points for correctly solving for y (1 point for correctly subtracting 2x, one point for dividing by -3), 1 point for correctly identifying the slope, 1 point for correctly identifying the y-intercept.
Graph: 2 points each for correct slope and y-intercept (0 points if they use the x intercept or if they put run/rise)

Question 2 - Solving a system of linear equations
Algebraic: If they solve by substitution, then it is 1 point for solving for a variable and for correctly substituting it into the other equation, 1 point for correctly solving it and 1 point for correctly substituting and finding the other variable, and 1 point for writing the answer as an ordered pair. If they solve it by elimination, They get 1 points for correctly eliminating a variable, 1 point for solving for the variable, 1 point for substituting back into the equation and 1 point for writing the answer as an ordered pair.
Graph: 4 points for the point of intersection written as an ordered pair (-1 if the they don't write it as an ordered pair)
Table: 4 points for the point of intersection written as an ordered pair (-1 if the they don't write it as an ordered pair)

Question 3 - Finding real variable factors
Graph: 2 points for identifying zeros, 2 points for writing them correctly as factors (-2 if they get the signs reversed)
Table: 2 points for identifying zeros, 2 points for writing them correctly as factors (-2 if they get the signs reversed)
Algebraic: 2 points for grouping first two terms and last two terms and factoring out the common factor of each, 1 point for factoring the common factor that resulted from step one, 1 point for factoring the difference of squares (sign errors -1 at each step)

Question 4 - Identifying a function (or not)
All representations: 2 points for correct answer, 2 points for correct justification

Question 5 - Finding solutions to a quadratic equation
Table: 2 points each correct answer (-1 for each one if they right the answers as factors) (0 points if they use the y intercept)
Graph: 2 points each correct answer (-1 for each one if they right the answers as factors) (0 points if they use the y intercept)
Algebraic: If they solve by factoring, 2 points for correctly factoring and setting each factor equal to zero (they may do this in their head), 1 point for each correct answer. If they use the quadratic formula, 2 points for correctly applying the formula and simplifying, 1 point for each answer.

Appendix E – Interview Protocol

PROTOCOL

Purpose: To examine whether students are flexible in solving problems with the three representations end of the course and whether there was a difference in the flexibility to solve with a different representation made by the students in the two curriculums. To examine the connections the students make between the representations and to determine the level of concept development based on the SOLO taxonomy.

Interview Protocol: "My name is Robin Rider and I am doing a research study to examine student thinking as they work through various problems in algebra. Do you understand that this interview will be videotaped and I will make a transcript, but that your identity will be kept confidential?

You took a pretest and a posttest in your class that had various functional representations. Today what I want to do is look at that survey again and get a better idea of your thinking behind each problem. I am going to ask you again to solve one problem on each page and while you are solving it I want you to tell me what you are doing. There is no right or wrong problem to solve, I am just trying to get a better understanding of students thinking as they do a problem. I am also going to ask you questions as we go along to help clarify things that I still may not understand. Do you have any questions?"

Exercise 1-5:

Solve one of the problems on the page and tell me what you are doing as you solve it.

1. Why did you choose this particular one to solve?
2. Could you solve another one for me and tell me what you are doing as you solve it?
3. Could you solve the last one for me and tell me what you are doing as you solve it?
4. How does that relate to what you were just doing (in the other representation)?
5. What does that answer represent?

Additional questions:

Exercise 1

1. Suppose in any one of them let's take the graph for example, suppose I were to take that same line and I am going to change the y-intercept, how does that effect the graph?

2. Does changing the y-intercept have an effect on the table? If so what is it? How about the equation?

Exercise 2:

1. If I said the solution was (1,3) how would that effect the graph? The table?

Exercise 3:

1. If I said the x-intercepts were -4, -2, and 5 what effect would there be on the graph, table equation?

Exercise 5:

1. If the equation had 4 as a factor, how would that affect the solution?

Appendix F - Rubric for Analysis of Qualitative Interviews
The Solo-Taxonomy Applied to Multiple Representations of Functions

Table 21. Question 1- Finding the Slope and Y-intercept of a line

Phase of Concept Development	**Examples of how emerges when evaluating study task**
Pre-structural (lacks knowledge of the assessed component) – may have pieces of unconnected knowledge but make no sense to the student	Student may do some procedural work on the problem but ultimately has no understanding of how to solve the problem. For example • May use the equation to find intercepts but substituting zero, but has no knowledge what they represent • May use the graph to find slope (or intercept) but have no knowledge of what it represents
Uni-structural (focuses on a single aspect) – has an understanding of one aspect but not of its significance to the whole or relationship to other aspects	Student has knowledge of how to solve the problem using one representation. No connections to the other representations. For example • Can solve the problem algebraically (may make minor errors in arithmetic) but cannot solve the problem using the table or the graph. The student may have a pre-structural understanding of one or both of the other representation, i.e. can find the slope from the graph, but not the y-intercept • May use the equation to find intercepts by substituting zero, but has no knowledge of how to find slope • May use the graph to find slope (or intercept) but have no knowledge of how to find intercept (or slope) • May use table to find slope but does not or cannot recognize how to find y-intercept

Table 21 Continued

Multi-structural (focuses on several separate aspects) – has an understanding of more than one aspect but not of their significance to the whole or relationship to other aspects. Student has gained flexibility in solving problems with more than one representation, but little structural knowledge of how those representations relate to each other.	Student has knowledge of how to solve the problem using more than one representation, but makes little or no connection between the representations. For example: • Can solve the problem using the algebraic and/or graph and/or table, but cannot connect them other than to notice that they produced the same answer. • May or may not be able to solve the problem using the third representation (may have a pre-structural understanding of the third representation) • Will be unconcerned if an algebraic or arithmetic mistake in solving one representation produces a different answer than solving another representation. Since the representations are not connected, they are seen as separate problems
Relational (Relates different aspects together) – Has an understanding of more than one aspect and of their significance to the whole and relationship to other aspects within the task. Flexibility and structural understanding – Conceptual compression begins.	Student has knowledge of how to solve the problem using more than one representation and can make connections between at least two of the representations. For example: • Can solve the problem using at least two of the representations and connects the two, may use one to check the other and if they have made arithmetic or counting errors, can find them by checking. May not be able to solve the third representation or if they can solve it, cannot link it to the other two. • Views the two and/or three problems as one problem, just represented different ways
Extended Abstract (Seeing the concept from an overall viewpoint) – student makes connections within the task and can go beyond it to generalize the underlying principles	Student can take the concept beyond the task presented. For example: • When asked what happens to each representation if the y-intercept is changed, student is able to describe correctly how it effects the other representations. A students might be able to do this with only one representation (i.e. the graph and not the table) thus showing an extended abstract concept of on representation but not the other

Table 22. Question 2- Solving a system of equations

Phase of Concept Development	**Examples of how emerges when evaluating study task**
Pre-structural (lacks knowledge of the assessed component) – may have pieces of unconnected knowledge but make no sense to the student	Student may do some procedural work on the problem but ultimately has no understanding of how to solve the problem. For example • May solve one equation for a variable (substitution method) but cannot proceed further • May try to use the intercepts instead of the point of intersection as the solution • May use the zeros in the table to try to find the solution
Uni-structural (focuses on a single aspect) – has a understanding of one aspect but not of its significance to the whole or relationship to other aspects	Student has knowledge of how to solve the problem using one representation. No connections to the other representations. For example • Can solve the problem algebraically (may make minor errors in arithmetic) but cannot solve the problem using the table or the graph. The student may have a pre-structural understanding of one or both of the other representations, i.e. can recognize the intercepts from the graph, but not know that the solution is the point of intersection • May see the algebraic solutions to the symbolic manipulations as two solutions instead of a point of intersection
Multi-structural (focuses on several separate aspects) – has an understanding of more than one aspect but not of their significance to the whole or relationship to other aspects. Student has gained flexibility in solving problems with more than one representation, but little structural knowledge of how those representations relate to each other.	Student has knowledge of how to solve the problem using more than one representation, but makes little or no connection between the representations. For example: • Can solve the problem using the algebraic and/or graph and/or table, but cannot connect the two other than to notice that they produced the same answer. • May have a pre-structural understanding of one of the representations • Will be unconcerned if an algebraic or arithmetic mistake in solving one representation produces a different answer than solving another representation. Since the representations are not connected, they are seen as separate problems

Table 22 Continued

Relational (Relates different aspects together) – Has an understanding of more than one aspect and of their significance to the whole and relationship to other aspects within the task. Flexibility and structural understanding – Conceptual compression begins.	Student has knowledge of how to solve the problem using more than one representation and can make connections between at least two of the representations. For example: • Can solve the problem using at least two of the representations and connects the two, may use one to check the other and if they have made arithmetic or counting errors, can find them by checking. May not be able to solve the third representation or if they can solve it, cannot link it to the other two. • Views the two and/or three problems as one problem, just represented different ways
Extended Abstract (Seeing the concept from an overall viewpoint) – student makes connections within the task and can go beyond it to generalize the underlying principles	Student can take the concept beyond the task presented. For example: • When asked what happens to each representation if the solution were (1,3), student is able to describe correctly how it effects the other representations. A students might be able to do this with only one representation (i.e. the graph and not the table) thus showing an extended abstract concept of on representation but not the other.

Table 23. Question 3- Factoring a cubic polynomial

Phase of Concept Development	**Examples of how emerges when evaluating study task**
Pre-structural (lacks knowledge of the assessed component) – may have pieces of unconnected knowledge but make no sense to the student	Student may do some procedural work on the problem but ultimately has no understanding of how to solve the problem. For example • May identify the x-intercepts and y-intercepts from the graph or table but have no understanding of what they represent • May try to factor but not by grouping, cannot arrive at a factorization
Uni-structural (focuses on a single aspect) – has a understanding of one aspect but not of its significance to the whole or relationship to other aspects	Student has knowledge of how to solve the problem using one representation. No connections to the other representations. For example • Can factor the problem algebraically (may make minor errors in arithmetic) but cannot use the intercepts in the table or the graph to identify factors.
Multi-structural (focuses on several separate aspects) – has an understanding of more than one aspect but not of their significance to the whole or relationship to other aspects. Student has gained flexibility in solving problems with more than one representation, but little structural knowledge of how those representations relate to each other.	Student has knowledge of how to solve the problem using more than one representation, but makes little or no connection between the representations. For example: • Can solve the problem using the algebraic and/or graph and/or table, but cannot connect the two other than to notice that they produced the same answer • May have a pre-structural understanding of one of the representations • Will be unconcerned if an algebraic or arithmetic mistake in solving one representation produces a different answer than solving another representation. Since the representations are not connected, they are seen as separate problems

Table 23 Continued

Relational (Relates different aspects together) – Has an understanding of more than one aspect and of their significance to the whole and relationship to other aspects within the task. Flexibility and structural understanding – Conceptual compression begins.	Student has knowledge of how to solve the problem using more than one representation and can make connections between at least two of the representations. For example: • Can solve the problem using at least two of the representations and connects the two, may use one to check the other and if they have made arithmetic or counting errors, can find them by checking. May not be able to solve the third representation or if they can solve it, cannot link it to the other two. • Views the two and/or three problems as one problem, just represented different ways
Extended Abstract (Seeing the concept from an overall viewpoint) – student makes connections within the task and can go beyond it to generalize the underlying principles	Student can take the concept beyond the task presented. For example: • When asked what happens to each representation if the x-intercepts are changed the student is able to describe correctly how it effects the other representations. A students might be able to do this with only one representation (i.e. the graph and not the table) thus showing an extended abstract concept of on representation but not the other.

Table 24. Question 4-Identifying a function

Phase of Concept Development	**Examples of how emerges when evaluating study task**
Pre-structural (lacks knowledge of the assessed component) – may have pieces of unconnected knowledge but make no sense to the student	Student may do some procedural work on the problem but ultimately has no understanding of how to solve the problem. For example • May use a horizontal line in the graphical representation • May try to solve the algebraic equation for y, but have no knowledge of how to do that or what it means if they do it correctly • May look at the y-values in the table as opposed to the x-values
Uni-structural (focuses on a single aspect) – has a understanding of one aspect but not of its significance to the whole or relationship to other aspects	Student has knowledge of how to solve the problem using one representation. No connections to the other representations. For example • May be able to use the vertical line test to determine if it is a function but has no knowledge of how that relates to the other two representations.
Multi-structural (focuses on several separate aspects) – has an understanding of more than one aspect but not of their significance to the whole or relationship to other aspects. Student has gained flexibility in solving problems with more than one representation, but little structural knowledge of how those representations relate to each other.	Student has knowledge of how to solve the problem using more than one representation, but makes little or no connection between the representations. For example: • Can identify a function using the algebraic and/or graph and/or table, but cannot connect the two other than to notice that they produced the same answer • May have a pre-structural understanding of one of the representations • Will be unconcerned if an algebraic or arithmetic mistake in solving one representation produces a different answer than solving another representation. Since the representations are not connected, they are seen as separate problems

Table 24 Continued

Relational (Relates different aspects together) – Has an understanding of more than one aspect and of their significance to the whole and relationship to other aspects within the task. Flexibility and structural understanding – Conceptual compression begins.	Student has knowledge of how to solve the problem using more than one representation and can make connections between at least two of the representations. For example: • Can solve the problem using at least two of the representations and connects the two, may use one to check the other and if they have made arithmetic or counting errors, can find them by checking. May not be able to solve the third representation or if they can solve it, cannot link it to the other two. • Views the two and/or three problems as one problem, just represented different ways
Extended Abstract (Seeing the concept from an overall viewpoint) – student makes connections within the task and can go beyond it to generalize the underlying principles	• Student can take the concept beyond the task presented.

Table 25. Question - Solving a quadratic equation

Phase of Concept Development	**Examples of how emerges when evaluating study task**
Pre-structural (lacks knowledge of the assessed component) – may have pieces of unconnected knowledge but make no sense to the student	Student may do some procedural work on the problem but ultimately has no understanding of how to solve the problem. For example • May identify the x-intercepts from the graph or table but not be able to identify them as the solutions to the quadratic equation set equal to zero • May factor the algebraic form, but not realize that the factors need to be set equal to zero and solved
Uni-structural (focuses on a single aspect) – has a understanding of one aspect but not of its significance to the whole or relationship to other aspects	Student has knowledge of how to solve the problem using one representation. No connections to the other representations. For example • Can solve the problem algebraically by factoring or using the quadratic formula (may make minor errors in arithmetic) but cannot use the intercepts in the table or the graph to identify factors.
Multi-structural (focuses on several separate aspects) – has an understanding of more than one aspect but not of their significance to the whole or relationship to other aspects. Student has gained flexibility in solving problems with more than one representation, but little structural knowledge of how those representations relate to each other.	Student has knowledge of how to solve the problem using more than one representation, but makes little or no connection between the representations. For example: • Can solve the problem using the algebraic and/or graph and/or table, but cannot connect the two other than to notice that they produced the same answer • May have a pre-structural understanding of one of the representations • Will be unconcerned if an algebraic or arithmetic mistake in solving one representation produces a different answer than solving another representation. Since the representations are not connected, they are seen as separate problems

Table 25 Continued

Relational (Relates different aspects together) – Has an understanding of more than one aspect and of their significance to the whole and relationship to other aspects within the task. Flexibility and structural understanding – Conceptual compression begins.	Student has knowledge of how to solve the problem using more than one representation and can make connections between at least two of the representations. For example: • Can solve the problem using at least two of the representations and connects the two, may use one to check the other and if they have made arithmetic or counting errors, can find them by checking. May not be able to solve the third representation or if they can solve it, cannot link it to the other two. • Views the two and/or three problems as one problem, just represented different ways
Extended Abstract (Seeing the concept from an overall viewpoint) – student makes connections within the task and can go beyond it to generalize the underlying principles	Student can take the concept beyond the task presented. For example: • When asked what effect a factor of 4 would have on the representations, the student is able to describe correctly how it effects the other representations. A students might be able to do this with only one representation (i.e. the graph and not the table) thus showing an extended abstract concept of on representation but not the other.