Abstract

PETERSEN, RICHARD FRANCIS. Transformation Semigroups Over Groups. (Under the direction of Mohan Putcha.)

The semigroup analogue of the symmetric group, S_n, is the full transformation semigroup, T_n. T_n is the set of all mappings from the set $\{1, 2, ..., n\}$ to itself. This semigroup has been studied in great detail, especially in connection with automata theory.

The wreath product of a group G by S_n has been studied for almost one hundred years. In this thesis, we study the wreath product of a group G by T_n. These wreath products are expressed as $GwrS_n$ and $GwrT_n$, respectively. Many interesting theorems and properties for wreath products will be discussed. For example, the result of John Howie that every element in $T_n - S_n$ can be expressed as a product of idempotents, is generalized to show that any element of $GwrT_n - GwrS_n$ can be expressed as a product of idempotents. It will also be shown that $GwrT_n$ is unit regular.

Chapter five begins with a review of Green's relations for a moniod, M. Green's relations for T_n are also reviewed and $\mathcal{R}$ and $\mathcal{L}$-classes for the wreath product $GwrT_n$ are determined. Finally, in the last two chapters, the conjugacy class structures of $GwrT_n$ are determined. Just as the conjugacy classes of $GwrS_n$ are indexed by colored partitions, we show that the conjugacy classes of $GwrT_n$ are indexed by certain colored directed graphs.

Transformation Semigroups Over Groups

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Dedication

This paper is dedicated to Danielle Nicole Haynes.

Biography

Richard F. Petersen was born March 6, 1979 in Lowell, MA. He attended elementary schools in California, New York, and Massachusetts. Afterwards, he attended junior high and high schools in Michigan and New Jersey. In 1997, he graduated with honors from Pemberton Township High School in Pemberton,NJ.

In 1999, Richard obtained an Associate of Science in Liberal Arts and Sciences from Burlington County College in Pemberton,NJ. He went on to attend Elizabeth City State University in Elizabeth City, NC where he earned a Bachelor of Science in Mathematics in 2002. While at ECSU, he worked on a research project on chaos in communication systems with Dr. Dipendra Sengupta. During the summer of 2002, the late Dr. Georgia Lawrence hired Richard to teach his first mathematics course, at ECSU.

After ECSU, Richard began graduate school in Applied Mathematics at N.C. State. He was a member of the teaching assistant program and taught numerous calculus and pre-calculus courses. In 2004, Richard completed his master's project on gamma functions, under the direction of Dr. L.O. Chung. He then decided to stay at N.C. State and pursue a Ph.D in Mathematics. After studying for a year, he passed the qualifying exams in August of 2005. In 2006, he began working on the research for his thesis, with Dr. Mohan Putcha.

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Table Of Contents

List of Tables ix

1 - Groups 1

1.1 - Basic Definitions 1

1.2 - Group Examples 2

1.3 - Focusing On S_n and $\widetilde{S_n}$ 3

1.4 - Notation 3

1.4.1 - Two Line Notation 3

1.4.2 - One Line Notation 4

1.4.3 - Cycle Notation 4

1.4.4 - Matrix Notation 5

1.4.5 - Directed Graph Notation 6

1.5 - Conjugacy Classes In S_n 7

1.6 - Looking Ahead To Chapter 2 8

2 - Semigroups 9

2.1 - Basic Definitions 9

2.2 - Semigroup Examples 10

2.3 - Notation 11

2.3.1 - Two Line Notation 11

2.3.2 - One Line Notation 11
2.3.3 - Matrix Notation 11
2.3.4 - Directed Graph Notation 12
2.4 - Orders of Full Transformation Semigroups 12
2.5 - Regular Semigroups 13
2.6 - Rank and Range of Elements in T_n 14
2.7 - Idempotents in T_n and $\widetilde{T_n}$ 14
2.8 - Looking Ahead To Chapters 3 and 4 17
3 - The Wreath Product $GwrS_n$ 18
3.1 - Defining $GwrS_n$ 18
3.2 - $GwrS_n$ Is A Group 18
3.3 - Notation and Examples 19
3.4 - Multiplication of Elements in $GwrS_n$ 21
4 - The Wreath Product $GwrT_n$ 23
4.1 - Defining $GwrT_n$ 23
4.2 - $GwrT_n$ Is A Monoid 23
4.3 - Notation and Examples 24
4.4 - Multiplication Of Elements In $GwrT_n$ 26
4.5 - $GwrT_n$ Is Unit Regular 27
4.6 - Idempotents In $GwrT_n$ 28
5 - Green's Relations 33

5.1 - Green's Relations On A Monoid M 33

5.2 - Green's Relations For Idempotents 34

5.3 - Green's Relations For T_n 34

5.4 - Green's Relations For Idempotents In $GwrT_n$ 39

5.5 - Looking Ahead To Chapter 6 44

6 - Conjugacy Classes in S_n and T_n 45

6.1 - Conjugacy Classes Of S_n Revisited 45

6.2 - Conjugacy Classes Of T_n 46

6.2.1 - The Conjugacy Classes Of T_1 and T_2 49

6.2.2 - The Conjugacy Classes Of T_3 51

6.2.3 - The Conjugacy Classes Of T_4 53

6.2.4 - More About Generalized The Cycle Structure 61

7 - Conjugacy Classes in $GwrS_n$ 63

7.1 - N-Cycles In $GwrS_n$ 63

7.2 - Example In $GwrS_{12}$ 66

7.3 - The Conjugacy Class Formula For $GwrS_n$ 67

8 - Conjugacy Classes in $GwrT_n$ 72

8.1 - Conjugation Examples 72

8.2 - The Two Lemmas 76

8.3 - The Conjugacy Class Formula For $GwrT_n$ 79

8.4 - Colored Directed Graphs 84

9 - Future Endeavors 87

References 88

List of Tables

Table 5.3.1 - $\mathcal{D}_1$-class of T_3 35

Table 5.3.2 - $\mathcal{D}_2$-class of T_3 35

Table 5.3.3 - $\mathcal{D}_3$-class of T_3 35

Table 5.3.4 - Eggbox Diagram for T_3 36

Table 5.3.5 - $\mathcal{D}_1$-class of $\widetilde{T_3}$ 37

Table 5.3.6 - $\mathcal{D}_2$-class of $\widetilde{T_3}$ 38

Table 5.3.7 - Eggbox Diagram for $\widetilde{T_3}$ 39

1 Groups

It is necessary to review a bit of group theory before proceeding on to the new material in this thesis. The following definitions will be useful to keep in mind.

1.1 Basic Definitions

Definition 1.1.1 *A* **group** *is a set G together with a law of composition, $*$, which has the following properties: For $a, b, c \in G$,*

(1) $a * (b * c) = (a * b) * c$ (associativity)

(2) $1 \in G$ (identity)

(3) If $a \in G$, then $a^{-1} \in G$ (inverses)

If the law of composition, $*$, is commutative (i.e., $a * b = b * a$, for $a, b \in G$), then G is said to be an Abelian group. We may also wish to consider special kinds of subsets of G called subgroups.

Definition 1.1.2 *A* **subgroup** *is a subset H of a group G which has the following properties, for $a, b \in H$,*

(1) If $a \in H$ and $b \in H$, then $a * b \in H$ (closure under the operation $*$)

(2) $1 \in H$ (identity element)

(3) If $a \in H$, then $a^{-1} \in H$ (inverses)

1.2 Group Examples

The following are classic examples of groups:

Example 1.2.1 *General Linear Group*

$GL_n = \{$ $n \times n$ *matrices* A *with det* $A \neq 0\}$

Example 1.2.2 *Special Linear Group*

$SL_n = \{$ $n \times n$ *matrices* B *with det* $B = 1\}$

SL_n *is a subgroup of* GL_n.

Example 1.2.3 *Dihedral Group*

D_n = *group generated by two elements,* x *and* y*, such that the relations* $x^n = 1$, $y^2 = 1$, *and* $yx = x^{-1}y$ *hold.*

Example 1.2.4 *Symmetric Group*

S_n = *the set of all bijections from* $\{1, 2, ..., n\} \to \{1, 2, ..., n\}$.

More generally, S_X *is the group of all permutations of any set* X.

Example 1.2.5 *Signed Permutation Group*

$\widetilde{S_n}$ = *the set of all bijections from* $\{1, 2, ..., n\} \to \{\pm 1, \pm 2, ..., \pm n\}$.

Example 1.2.6 *Weyl Groups*

The group of type A_{n-1} *corresponds to the symmetric group,* S_n*, and the group of type* B_n *corresponds to the signed permutation group,* $\widetilde{S_n}$.

1.3 Focusing on S_n and $\widetilde{S_n}$

The symmetric group, S_n, and the signed permutation group, $\widetilde{S_n}$, are of the most importance for future chapters, so we shall focus on them in greater detail for the rest of this chapter. These results are well known and have appeared in numerous papers and books.

Definition 1.3.1 *The* **order** *of a group G is the number of elements in G. We denote the order of G by the symbol, $|G|$.*

Theorem 1.3.2 $|S_n| = n!$

Example 1.3.3 $|S_6| = 6! = 720$

Corollary 1.3.4 $|\widetilde{S_n}| = 2^n \cdot n!$

Example 1.3.5 $|\widetilde{S_6}| = 2^6(6!) = (64)(720) = 46,080$

1.4 Notation

We have a few useful ways to represent elements in S_n. The following notations will be used where appropriate.

1.4.1 Two Line Notation

Let $\pi \in S_n$; we can write $\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}$.

Example 1.4.1 *If $\pi \in S_4$ and $\pi(1) = 4$, $\pi(2) = 2$, $\pi(3) = 1$, $\pi(4) = 3$, then* $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$.

1.4.2 One Line Notation

In two line notation, the top line is always the same, so we may omit it and just write the bottom line. Let $\pi \in S_n$; we can write
$\pi = \begin{pmatrix} \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}$.

Example 1.4.2 *If $\pi \in S_4$ and $\pi(1) = 4$, $\pi(2) = 2$, $\pi(3) = 1$, $\pi(4) = 3$, then* $\pi = \begin{pmatrix} 4 & 2 & 1 & 3 \end{pmatrix}$.

1.4.3 Cycle Notation

Sagan gives a nice description of this cycle notation in his book on the symmetric group [10]. Given $i \in \{1, 2, ..., n\}$, the elements of the sequence $i, \pi(i), \pi^2(i), \pi^3(i), ...$ cannot all be distinct. Taking the first power p, such that $\pi^p(i) = i$, we have the cycle $\begin{pmatrix} i & \pi(i) & \pi^2(i) & \cdots & \pi^{p-1}(i) \end{pmatrix}$. Equivalently, the cycle $\begin{pmatrix} i & j & k & \cdots & l \end{pmatrix}$ means that π sends i to j, j to k, ..., and l back to i.

Example 1.4.3 *If $\pi \in S_4$ and $\pi(1) = 4$, $\pi(2) = 2$, $\pi(3) = 1$, $\pi(4) = 3$, then* $\pi = (143)(2)$ *in cycle notation.*

Definition 1.4.4 *A k-**cycle**, or **cycle of length** k, is a cycle containing k elements.*

Example 1.4.5 $\pi = (143)(2)$ *consists of a cycle of length 3 and a cycle of length 1.*

Example 1.4.6 $\sigma = (12)(34)(5678)$ *consists of two cycles of length 2 and a cycle of length 4.*

Definition 1.4.7 *The* **cycle type**, *or simply the* **type**, *of* π *is an expression of the form* $\begin{pmatrix} 1^{m_1} & 2^{m_2} & 3^{m_3} & \cdots & n^{m_n} \end{pmatrix}$, *where* m_k *is the number of cycles of length* k *in* π.

Example 1.4.8 $\pi = (143)(2)$ *has cycle type* $\begin{pmatrix} 1^1 & 2^0 & 3^1 & 4^0 \end{pmatrix}$.

Example 1.4.9 $\sigma = (12)(34)(5678)$ *has cycle type* $\begin{pmatrix} 1^0 & 2^2 & 3^0 & 4^1 & 5^0 & 6^0 & 7^0 & 8^0 \end{pmatrix}$.

Another way to give the cycle type is as a partition:

Definition 1.4.10 *A* **partition** *of* n *is a sequence* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$, *where* λ_i *are weakly decreasing and* $\sum_{i=1}^{l} \lambda_i = n$. *Thus,* k *is repeated* m_k *times in the partition version of the cycle type of* π.

Example 1.4.11 *For* $\pi = (143)(2)$, $\lambda = (3, 1)$.

Example 1.4.12 *For* $\sigma = (12)(34)(5678)$, $\lambda = (4, 2, 2)$.

1.4.4 Matrix Notation

Let $\pi \in S_n$; we can indicate $\pi(j) = i$ by placing a 1 in the (i, j)-entry of an n×n matrix.

Example 1.4.13 $\pi = (143)(2)$, *which means* $\pi(1) = 4$, $\pi(2) = 2$, $\pi(3) = 1$, $\pi(4) = 3$, *can be written in matrix notation as* $\pi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

1.4.5 Directed Graph Notation

Consider an element $\pi \in S_n$, where $\pi = \begin{pmatrix} i & j & k & \cdots & l \end{pmatrix}$ in cycle notation. Draw n vertices and label them i, j, k, ..., l. Indicate $\pi(i) = j$ by drawing a directed line segment from i to j.

Example 1.4.14 *For $\pi = (143)(2)$ the directed graph is,*

$$\begin{tikzcd} 1 \arrow[dr] & 2 \arrow[loop right] \\ 3 \arrow[u] & 4 \arrow[l] \end{tikzcd}$$

Each of these four notations will be used throughout this thesis. Later on, some variations of these notations will be used.

Example 1.4.15 *Similar notation works for elements in $\widetilde{S_n}$.*
If $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & 4 & -1 & 3 \end{pmatrix} \in \widetilde{S_4}$, we can represent this element in matrix form by placing ± 1 in the (i, j)-entry to indicate $j \to \pm i$.

So, $\pi = \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.
We can also make a slight modification on our directed graph notation to represent π as

$$\begin{tikzcd} 1 \arrow[r, "-1"] & 2 \arrow[d, "1"] \\ 3 \arrow[u, "-1"] & 4 \arrow[l, "1"] \end{tikzcd}$$

The cycle notation we used for S_n is not very useful for elements in $\widetilde{S_n}$.

1.5 Conjugacy Classes In S_n

Definition 1.5.1 *In any group G, elements g and h are* **conjugates** *if $g = khk^{-1}$, for some $k \in G$.*

Definition 1.5.2 *The set of all elements conjugate to a given $g \in G$ is called the* **conjugacy class of g**.

In S_n, if $\pi = \begin{pmatrix} i_1 & i_2 & i_3 & \cdots & i_l \end{pmatrix}\begin{pmatrix} i_m & i_{m+1} & i_{m+2} & \cdots & i_n \end{pmatrix}$ in cycle notation, then for any $\sigma \in S_n$,

$$\sigma\pi\sigma^{-1} = \begin{pmatrix} \sigma(i_1) & \sigma(i_2) & \sigma(i_3) & \cdots & \sigma(i_l) \end{pmatrix}\begin{pmatrix} \sigma(i_m) & \sigma(i_{m+1}) & \sigma(i_{m+2}) & \cdots & \sigma(i_n) \end{pmatrix}.$$

Conjugacy is an equivalence relation, so the distinct conjugacy classes partition G. This means that if G has t conjugacy classes, $C^1, C^2, ..., C^t$, then $C^i \cap C^j = \emptyset$, for $i \neq j$, and $\bigcup_i C^i = G$.

In S_n, two permutations are in the same conjugacy class if and only if they have the same cycle type. There is a natural correspondence between partitions of n and the conjugacy classes of S_n.

Example 1.5.3 *We can write the number* 1 *only as* 1*, so* S_1 *has only* 1 *conjugacy class.*

We can write the number 2 *as* $2+0$ *and* $1+1$*, so* S_2 *has* 2 *conjugacy classes.*

We can write the number 3 *as* $3+0$*,* $2+1$ *and* $1+1+1$*, so* S_3 *has* 3 *conjugacy classes.*

We can write the number 4 *as* $4+0$*,* $3+1$*,* $2+2$*,* $1+1+2$ *and* $1+1+1+1$*, so* S_4 *has* 5 *conjugacy classes.*

We can write the number 5 *as* $5+0$*,* $1+4$*,* $2+3$*,* $1+1+3$*,* $1+2+2$*,* $1+1+1+2$*, and* $1+1+1+1+1$*, so* S_5 *has* 7 *conjugacy classes.*

We will consider the conjugacy classes of $\widetilde{S_n}$ in a later chapter. See the chapters on conjugacy classes in wreath products for those results.

1.6 Looking Ahead To Chapter 2

This chapter was written so that the reader may get a feel for the notation and properties in later chapters. In the next chapter, we shall consider semigroups. It is of interest to note the differences between groups and semigroups and develop numerous properties of semigroups.

2 Semigroups

In this chapter we will examine numerous properties and examples of semigroups. We begin with the following well known definitions.

2.1 Basic Definitions

Definition 2.1.1 *A* **monoid** *is a set M together with a law of composition, *, which has the following properties: For $a, b, c \in M$,*

(1) $a * (b * c) = (a * b) * c$ *(associativity)*

(2) $1 \in M$ *(identity)*

Definition 2.1.2 *A* **semigroup** *is a set S together with a law of composition, *, which is associative.*

Definition 2.1.3 *Let S be a semigroup and $T \subseteq S$. T is a* **subsemigroup***, if T is closed under the semigroup operation, *.*

Definition 2.1.4 *Let S be a semigroup. We say an element, $\sigma \in S$, is* **idempotent** *if and only if $\sigma^2 = \sigma$. We denote the set of all idempotents in S by $E(S)$.*

Definition 2.1.5 *Let M be a monoid. Let G be its unit group. If $a, b \in M$, then a and b are* **conjugate***, denoted $a \sim b$, if and only if $xax^{-1} = b$, for some $x \in G$.*

2.2 Semigroup Examples

Example 2.2.1 *The set of all integers, $\mathbb{Z}$, under multiplication, is a monoid.*

Example 2.2.2

The set of all positive integers, under addition, is a semigroup.

Example 2.2.3

The set of all nonnegative matrices, under matrix multiplication, is a semigroup.

Example 2.2.4 *Full Transformation Semigroup*

T_n *= the set of all mappings from* $\{1, 2, ..., n\} \to \{1, 2, ..., n\}$.

T_n *is the semigroup analogue of* S_n*. We notice that* S_n *is the unit group of* T_n.

Example 2.2.5 *Signed Full Transformation Semigroup*

$\widetilde{T_n}$ *= the set of all mappings from* $\{1, 2, ..., n\} \to \{\pm 1, \pm 2, ..., \pm n\}$.

$\widetilde{T_n}$ *is the semigroup analogue of* $\widetilde{S_n}$*. We see that* $\widetilde{S_n}$ *is the unit group of* $\widetilde{T_n}$.

The primary focus of this chapter will be the full transformation semigroups, T_n and $\widetilde{T_n}$.

2.3 Notation

The following notations are of the most use for representing elements in full transformation semigroups. These notations should look similar to those used in the last chapter.

2.3.1 Two Line Notation

Let $\pi \in T_n$. We can write $\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}$.

Example 2.3.1 *If* $\pi \in T_5$ *and* $\pi(1) = 1$, $\pi(2) = 1$, $\pi(3) = 4$, $\pi(4) = 4$, $\pi(5) = 5$, *then* $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 4 & 5 \end{pmatrix}$.

2.3.2 One Line Notation

Once again, we may remove the top line in the two line notation to get $\pi = \begin{pmatrix} \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}$.

Example 2.3.2 *If* $\pi \in T_5$ *and* $\pi(1) = 1$, $\pi(2) = 1$, $\pi(3) = 4$, $\pi(4) = 4$, $\pi(5) = 5$, *then* $\pi = \begin{pmatrix} 1 & 1 & 4 & 4 & 5 \end{pmatrix}$.

2.3.3 Matrix Notation

Let $\pi \in T_n$; we can indicate $\pi(j) = i$ by placing a 1 in the (i, j)-entry of an n×n matrix.

Example 2.3.3 *$\pi = (11445) \in T_5$, which means $\pi(1) = 1$, $\pi(2) = 1$, $\pi(3) = 4$, $\pi(4) = 4$, $\pi(5) = 5$, can be written in matrix notation as*

$$\pi = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.3.4 Directed Graph Notation

Consider an element $\pi \in T_n$, where $\pi = \begin{pmatrix} i & j & k & \cdots & l \end{pmatrix}$ in one-line notation. Draw n vertices and label them i, j, k, ..., l. Indicate $\pi(i) = j$ by drawing a directed line segment from i to j.

Example 2.3.4 *For $\pi = (1111) \in T_4$, the directed graph is,*

$$\begin{tikzcd} 1 \arrow[loop left] & 2 \arrow[l] \\ 3 \arrow[u] & 4 \arrow[ul] \end{tikzcd}$$

2.4 Orders of Full Transformation Semigroups

Definition 2.4.1 *The* **order** *of a semigroup S is the number of elements in S. We denote the order of S by the symbol, $|S|$.*

Theorem 2.4.2 $|T_n| = n^n$

Example 2.4.3 $|T_6| = 6^6 = 46,656$

Corollary 2.4.4 $|\widetilde{T_n}| = 2^n \cdot n^n = (2n)^n$

Example 2.4.5 $|\widetilde{T_6}| = (2 \cdot 6)^6 = 12^6 = 2,985,984$

2.5 Regular Semigroups

Definition 2.5.1 *An element y of a semigroup S is called* **regular** *if there exists $x \in S$, such that $yxy = y$.*

Definition 2.5.2 *A semigroup S is called a* **regular semigroup** *if all of its elements are regular.*

Definition 2.5.3 *A semigroup is said to be* **unit regular** *if for each $y \in S$ there is a unit u such that $yuy = y$.*

Theorem 2.5.4 *T_n is a regular semigroup.*

Proof:

(This proof comes from [9].) For every $\alpha \in T_n$, it is easily seen that the relationship $yxy = y$ holds for every $x \in T_n$, such that $x(\alpha)$ $(\alpha \in \{1, 2, ..., n\})$ is equal to some one of the elements β, for which $y(\beta) = \alpha$ and arbitrarily, for $\alpha \notin y(\{1, 2, ..., n\})$. $\square$

Theorem 2.5.5 *$\widetilde{T_n}$ is a regular semigroup.*

The proof of this result will be given in Chapter 4. It is a special case of a theorem for wreath products.

2.6 Rank and Range of Elements in T_n

Definition 2.6.1 *Let* $\hat{n} = \{1, 2, ..., n\}$ *and* $\sigma \in T_n$*. The* **range** *of* σ*, denoted* $Rng(\sigma)$*, is* $\sigma(\hat{n})$*.*

Definition 2.6.2 *Let* $\hat{n} = \{1, 2, ..., n\}$ *and* $\sigma \in T_n$*. The* **rank** *of* σ*, denoted* $Rnk(\sigma)$*, is* $|\sigma(\hat{n})|$*.*

Example 2.6.3 *Consider the elements* (111)*,* (122)*, and* (123) *from* T_3*.* $Rng((111)) = \{1\}$*,* $Rng((122)) = \{1, 2\}$*, and* $Rng((123)) = \{1, 2, 3\}$*.*

Example 2.6.4 *Once again, consider the elements* (111)*,* (122)*, and* (123) *from* T_3*.* $Rnk((111)) = 1$*,* $Rnk((122)) = 2$*, and* $Rnk((123)) = 3$*. We see that the ranks of the elements are just the orders of the sets from the above example. Another way to find the rank is by writing the elements in matrix form and using techniques from linear algebra.*

2.7 Idempotents in T_n and $\widetilde{T_n}$

Definition 2.7.1 *We say an element* $\sigma \in T_n$ *is* **idempotent** *if and only if* $\sigma^2 = \sigma$*. We denote the set of all idempotents in* T_n *by* $E(T_n)$*.*

We will usually denote idempotent elements by the letter, e. Another way to say that $e \in E(T_n)$ is $e(i) = i$, for $i \in Rng(e)$.

Example 2.7.2 *Consider the element* $(223) \in T_3$. *We can write this element in matrix form as,* $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

We can see this element is idempotent via matrix multiplication,

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We could also see that $Rng((223)) = \{2, 3\}$, *and we do in fact have* $e(2) = 2$ *and* $e(3) = 3$. *So, our alternate definition for idempotent elements is also satisfied in this case.*

Example 2.7.3 *Some other idempotents in* T_3 *are:*

(1) The element, (111), *which we can write in matrix form as,* $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(2) The element, (122), *which we can write in matrix form as,* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

(3) The element, (121), *which we can write in matrix form as,* $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Example 2.7.4 *For* T_2, *the set of all idempotent elements is* $E(T_2) = \{(11), (22), (12)\}$, *where elements are written in one line notation.*

Example 2.7.5 *For $\widetilde{T_2}$, the set of all idempotent elements is $E(\widetilde{T_2}) = \{(11), (22), (12), (1-1), (-22)\}$, where elements are written in one line notation.*

Example 2.7.6 *For T_3, the set of all idempotent elements is $E(T_3) = \{(111), (222), (333), (123), (113), (121), (122), (133), (223), (323)\}$, where elements are written in one line notation.*

In 1966, John M. Howie proved that the subsemigroup $T_n - S_n$ is generated by the idempotents of T_n. The following is a rewording of the theorem in [6].

Theorem 2.7.7 *Every element of $T_n - S_n$ can be expressed as a product of idempotent elements.*

The proof of this theorem may be found in [6] on pages 708-709. He proves the theorem using notation that is not found in this thesis. Therefore, it is left to the interested reader to sort through Howie's proof.

Example 2.7.8 *In T_3, we can write the non-idempotent, $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, as a product of idempotents in the following way:*

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

It is also possible to express every element in $\widetilde{T_n} - \widetilde{S_n}$ as a product of idempotents. This result will be proved in Chapter 4. It is a special case of a theorem about the idempotents of wreath products.

Example 2.7.9 *In $\widetilde{T_3}$, we can write the non-idempotent,* $\begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$*, as a product of idempotents in the following way:*

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2.8 Looking Ahead To Chapters 3 and 4

In the next two chapters, we will work with the wreath products of a group G by the symmetric group S_n and a group G by the full transformation semigroup T_n. We will see how to construct $\widetilde{S_n}$ and $\widetilde{T_n}$ via wreath products. Many of the same theorems and properties from the first two chapters will carry over to the wreath products.

3 The Wreath Product $GwrS_n$

We will show many interesting properties of $GwrS_n$ in this chapter. First, we will construct $GwrS_n$ and show that it is a group. We follow the exposition in [8].

3.1 Defining $GwrS_n$

Consider a group G and let $\hat{n} = \{1, 2, ..., n\}$. The product of G with itself n times, $G \times G \times \cdots \times G$, will be denoted $G^{\hat{n}}$. $G^{\hat{n}}$ is the set of all mappings from $\hat{n}$ into G. In other words, $G^{\hat{n}} = \{f | f : \hat{n} \to G\}$.

We put $GwrS_n = G^{\hat{n}} \times S_n = \{(f, \pi) | f : \hat{n} \to G, \pi \in S_n\}$ and, for $f \in G^{\hat{n}}$ and $\pi \in S_n$, we define $f_\pi = f \circ \pi$. We define a multiplication on $G^{\hat{n}}$ as follows: For $f, f' \in G^{\hat{n}}$, $(ff')(i) = f(i)f'(i)$, where $i \in \hat{n}$. Using this, we define a law of composition on $GwrS_n$,

$$(f, \pi)(f', \pi') = (f_{\pi'} f', \pi\pi') = ((f \circ \pi')f', \pi\pi')$$

3.2 $GwrS_n$ Is A Group

If we define $e \in G^{\hat{n}}$ by $e(i) = 1_G$, where $i \in \hat{n}$, then the identity element of $GwrS_n$ will be $1_{GwrS_n} = (e, 1_{S_n})$. For $f \in G^{\hat{n}}$, the mapping $f^{-1} \in G^{\hat{n}}$ is defined by $f^{-1}(i) = (f(i))^{-1}$, for $i \in \hat{n}$. Using this, we define the inverse of $(f, \pi) \in GwrS_n$ to be $(f, \pi)^{-1} = (f_\pi{}^{-1}, \pi^{-1}) = (f^{-1} \circ \pi^{-1}, \pi^{-1})$.

So far, we have shown that $GwrS_n$ has an identity element and that every element, $(f,\pi) \in GwrS_n$, has an inverse element, $(f,\pi)^{-1} \in GwrS_n$.

For elements $(f,\pi),(g,\sigma),(h,\gamma) \in GwrS_n$, it can be shown, with a little work, that $(f,\pi)[(g,\sigma)(h,\gamma)] = [(f,\pi)(g,\sigma)](h,\gamma)$. Therefore, the law of composition for $GwrS_n$ is associative. This shows us that $GwrS_n$ forms a group, called the **wreath product of G by S_n**, using the given law of composition.

Since $GwrS_n$ is a group, it can also be shown that $[(f,\pi)(g,\sigma)]^{-1} = (g,\sigma)^{-1}(f,\pi)^{-1}$, where $(f,\pi),(g,\sigma) \in GwrS_n$. The order of $GwrS_n$ is $|GwrS_n| = |G|^n \cdot |S_n| = |G|^n \cdot n!$, if G is finite.

3.3 Notation and Examples

Example 3.3.1 *$\mathbb{Z}_2wrS_n$ is the wreath product of the group of sign changes by S_n. This is just the set of all bijections from $\{1,2,...,n\} \to \{\pm1,\pm2,...,\pm n\}$. So, $\mathbb{Z}_2wrS_n = \widetilde{S_n}$. In Chapter 1, we claimed that $|\widetilde{S_n}| = 2^n \cdot n!$. We see this is the case in another way, $|\mathbb{Z}_2wrS_n| = |\mathbb{Z}_2|^n \cdot |S_n| = 2^n \cdot n!$.*

Example 3.3.2 *We see that* $\mathbb{Z}_2wrS_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \right.$
$\left. \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}$

So, $|\mathbb{Z}_2wrS_2| = 2^2 \cdot 2! = 8$.

We can represent the elements in $GwrS_n$ in a few different ways. Variations on the previous graph notation and matrix notation will be of the greatest use in this chapter.

Example 3.3.3 *Consider $GwrS_3$ and let $g_1, g_2, g_3 \in G$. We can represent the mapping,* $1 \xrightarrow{g_1} 3$, $2 \xrightarrow{g_3} 1$, $3 \xrightarrow{g_2} 2$, *by a graph,*

$$\begin{tikzcd} 1 \arrow[d, "g_1"'] & 2 \arrow[l, "g_3"'] \\ 3 \arrow[ur, "g_2"'] & \end{tikzcd}$$

or as a matrix, $\begin{pmatrix} 0 & g_3 & 0 \\ 0 & 0 & g_2 \\ g_1 & 0 & 0 \end{pmatrix}$,
where $j \xrightarrow{g} i$ *is represented by placing $g \in G$ in the (i, j)-entry of the $n \times n$ matrix.*

Example 3.3.4 *Consider $GwrS_5$ and let $h \in G$. We can represent the mapping,* $1 \xrightarrow{h} 2$, $2 \xrightarrow{h} 1$, $3 \xrightarrow{h} 3$, $4 \xrightarrow{h} 5$, $5 \xrightarrow{h} 4$, *by a graph,*

$$\begin{tikzcd} 1 \arrow[r, bend left, "h"] & 2 \arrow[l, bend left, "h"] \end{tikzcd}$$

$$\begin{tikzcd} 3 \arrow[loop left, "h"] \end{tikzcd}$$

$$\begin{tikzcd} 4 \arrow[r, bend left, "h"] & 5 \arrow[l, bend left, "h"] \end{tikzcd}$$

or as a matrix, $\begin{pmatrix} 0 & h & 0 & 0 & 0 \\ h & 0 & 0 & 0 & 0 \\ 0 & 0 & h & 0 & 0 \\ 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & h & 0 \end{pmatrix}$.

Example 3.3.5 *Consider $KwrS_4$ and let $k_1, k_3, k_4, k_5 \in K$. We can represent the mapping,* $1 \xrightarrow{k_1} 1,\ 2 \xrightarrow{k_3} 3,\ 3 \xrightarrow{k_4} 4,\ 4 \xrightarrow{k_5} 2$, *by a graph,*

$$\begin{tikzcd} 1 \arrow[loop left, "k_1"] & 2 \arrow[dl, "k_3"'] \\ 3 \arrow[r, "k_4"'] & 4 \arrow[u, "k_5"'] \end{tikzcd}$$

or as a matrix, $\begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_5 \\ 0 & k_3 & 0 & 0 \\ 0 & 0 & k_4 & 0 \end{pmatrix}.$

We may also use wreath product notation to define the following:

$\hat{S} = \{(1, \pi) | \pi \in S_n\} \cong S_n$ and

$\hat{G} = \{(f, 1) | f \in G^{\hat{n}}\} \cong G^{\hat{n}} \cong G \times G \times G \times \cdots \times G.$

3.4 Multiplication of Elements in $GwrS_n$

We can think of multiplication of elements in $GwrS_n$ in a couple of different ways:

(1) $i \xleftarrow{g} j \xleftarrow{g'} k$, where $g, g' \in G$.

(2) We can take the product of a matrix with an element $g \in G$ in the (i, j)-entry and a matrix with an element $g' \in G$ in the (j, k)-entry. This produces a matrix with an element $gg' \in G$ in the (i, k)-entry.

Example 3.4.1 *In $GwrS_3$, let $g_1, g_2, g_3 \in G$ and consider two elements, $z, z' \in GwrS_3$, where*

$z : 1 \xrightarrow{g_1} 2,\ 2 \xrightarrow{g_2} 3,\ 3 \xrightarrow{g_3} 1,$ *is represented by the matrix,* $\begin{pmatrix} 0 & 0 & g_3 \\ g_1 & 0 & 0 \\ 0 & g_2 & 0 \end{pmatrix}$,

and

$z' : 1 \xrightarrow{g_3} 1,\ 2 \xrightarrow{g_2} 3,\ 3 \xrightarrow{g_1} 2,$ *is represented by the matrix,* $\begin{pmatrix} g_3 & 0 & 0 \\ 0 & 0 & g_1 \\ 0 & g_2 & 0 \end{pmatrix}$.

Then, $zz' = \begin{pmatrix} 0 & 0 & g_3 \\ g_1 & 0 & 0 \\ 0 & g_2 & 0 \end{pmatrix} \begin{pmatrix} g_3 & 0 & 0 \\ 0 & 0 & g_1 \\ 0 & g_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & g_3 g_2 & 0 \\ g_1 g_3 & 0 & 0 \\ 0 & 0 & g_2 g_1 \end{pmatrix}$,

or $zz' : 1 \xrightarrow{g_1 g_3} 2,\ 2 \xrightarrow{g_3 g_2} 1,\ 3 \xrightarrow{g_2 g_1} 3.$

and $z'z = \begin{pmatrix} g_3 & 0 & 0 \\ 0 & 0 & g_1 \\ 0 & g_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & g_3 \\ g_1 & 0 & 0 \\ 0 & g_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & g_3^2 \\ 0 & g_1 g_2 & 0 \\ g_2 g_1 & 0 & 0 \end{pmatrix}$,

or $z'z : 1 \xrightarrow{g_2 g_1} 3,\ 2 \xrightarrow{g_1 g_2} 2,\ 3 \xrightarrow{g_3^2} 1.$

4 The Wreath Product $GwrT_n$

Now, we will consider the wreath product, $GwrT_n$. First, we will construct $GwrT_n$. Then, we will show how $GwrT_n$ differs from $GwrS_n$.

4.1 Defining $GwrT_n$

Consider a group G and let $\hat{n} = \{1, 2, ..., n\}$. The product of G with itself n times, $G \times G \times \cdots \times G$, will be denoted $G^{\hat{n}}$. $G^{\hat{n}}$ is the set of all mappings from $\hat{n}$ into G. In other words, $G^{\hat{n}} = \{f | f : \hat{n} \to G\}$.

We put $GwrT_n = G^{\hat{n}} \times T_n = \{(f, \pi) | f : \hat{n} \to G, \pi \in T_n\}$. We define a multiplication on $G^{\hat{n}}$ as follows: For $f, f' \in G^{\hat{n}}$, $(ff')(i) = f(i)f'(i)$, where $i \in \hat{n}$. Using this, we define a law of composition on $GwrT_n$,

$$(f, \pi)(f', \pi') = ((f \circ \pi')f', \pi\pi')$$

4.2 $GwrT_n$ Is A Monoid

If we define $e \in G^{\hat{n}}$ by $e(i) = 1_G$, where $i \in \hat{n}$, then the identity element of $GwrT_n$ will be $1_{GwrT_n} = (e, 1_{T_n})$. For elements $(f, \pi), (g, \sigma), (h, \gamma) \in GwrT_n$, we see that $(f, \pi)[(g, \sigma)(h, \gamma)] = [(f, \pi)(g, \sigma)](h, \gamma)$. Therefore, $GwrT_n$ has an identity element and the law of composition is associative.

In $GwrT_n$, we do not have inverses, $(f, \pi)^{-1}$, defined for every element $(f, \pi) \in GwrT_n$. Thus, $GwrT_n$ is not a group. We can only say that $GwrT_n$ is a monoid. We still call $GwrT_n$ the **wreath product of G by T_n**.

4.3 Notation and Examples

The order of $GwrT_n$ is $|GwrT_n| = |G|^n \cdot |T_n| = |G|^n \cdot n^n = (n \cdot |G|)^n$, if G is finite. We should also note that $GwrS_n$ is the unit group of $GwrT_n$.

Example 4.3.1 *If $G = \mathbb{Z}_3$, then $|GwrT_4| = (4 \cdot 3)^4 = 12^4 = 20,786$.*
Notice how this compares to $|GwrS_4| = 3^4 \cdot 4! = 1,944$. $GwrT_4$ has many more elements than $GwrS_4$.

We will represent the elements of $GwrT_n$ in the same manner as the elements in $GwrS_n$. The reader should refer back to Chapter 3 for a discussion on such notation.

Example 4.3.2 *$\mathbb{Z}_2wrT_n$ is the wreath product of the group of sign changes by T_n. This is just the set of all mappings from $\{1, 2, ..., n\} \rightarrow \{\pm 1, \pm 2, ..., \pm n\}$. So, $\mathbb{Z}_2wrT_n = \widetilde{T_n}$.*
In Chapter 2, we claimed that $|\widetilde{T_n}| = (2n)^n$. We see this is the case in another way, $|\mathbb{Z}_2wrT_n| = |\mathbb{Z}_2|^n \cdot |T_n| = 2^n \cdot n^n = (2n)^n$.

Example 4.3.3 *We see that* $\mathbb{Z}_2 wr T_2 = \Big\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$
$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$
$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix},$
$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \Big\}$

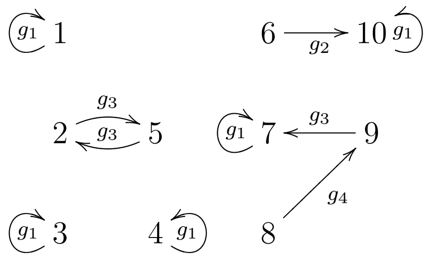
So, $|\mathbb{Z}_2 wr T_2| = 2^2 \cdot 2^2 = 16.$

Example 4.3.4 *Consider* $GwrT_3$ *and let* $g_1, g_2, g_3 \in G$. *We can represent the mapping,* $1 \xrightarrow{g_1} 1$, $2 \xrightarrow{g_2} 1$, $3 \xrightarrow{g_3} 1$, *by a graph,*

$$\begin{tikzcd} 1 \arrow[loop left, "g_1"] & 2 \arrow[l, "g_2"'] \\ 3 \arrow[u, "g_3"'] & \end{tikzcd}$$

or as a matrix, $\begin{pmatrix} g_1 & g_2 & g_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,
where $j \xrightarrow{g} i$ *is represented by placing* $g \in G$ *in the* (i, j)*-entry of the* $n \times n$ *matrix.*

Example 4.3.5 *Let $g_1, g_2, g_3, g_4 \in G$, and consider the following element in $GwrT_{10}$ where,*



We can represent this element in matrix form as,

$$\begin{pmatrix}
g_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & g_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & g_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g_1 & 0 & g_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & g_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & g_2 & 0 & 0 & 0 & g_1
\end{pmatrix}.$$

4.4 Multiplication Of Elements In $GwrT_n$

We can show multiplication of two elements in $GwrT_n$ using matrix multiplication, as we did in Chapter 3 for $GwrS_n$.

Example 4.4.1 *In $GwrT_3$, where $g_1, g_2, g_3 \in G$, consider two elements, $z, z' \in GwrT_3$, where*

$z : 1 \xrightarrow{g_1} 1,\ 2 \xrightarrow{g_2} 3,\ 3 \xrightarrow{g_3} 3,$ *is represented by the matrix,* $\begin{pmatrix} g_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & g_2 & g_3 \end{pmatrix}$,

and

$z' : 1 \xrightarrow{g_3} 3,\ 2 \xrightarrow{g_1} 2,\ 3 \xrightarrow{g_2} 2,$ *is represented by the matrix,* $\begin{pmatrix} 0 & 0 & 0 \\ 0 & g_1 & g_2 \\ g_3 & 0 & 0 \end{pmatrix}$.

Then, $zz' = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & g_2 & g_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_1 & g_2 \\ g_3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g_3^2 & g_2g_1 & g_2^2 \end{pmatrix}$,

or $zz' : 1 \xrightarrow{g_3^2} 3,\ 2 \xrightarrow{g_2g_1} 3,\ 3 \xrightarrow{g_2^2} 3.$

and $z'z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_1 & g_2 \\ g_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & g_2 & g_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_2^2 & g_2g_3 \\ g_3g_1 & 0 & 0 \end{pmatrix}$,

or $z'z : 1 \xrightarrow{g_3g_1} 3,\ 2 \xrightarrow{g_2^2} 2,\ 3 \xrightarrow{g_2g_3} 2.$

4.5 $GwrT_n$ Is Unit Regular

Using wreath product notation, we can define the following:

$\hat{T} = \{(1, \pi) | \pi \in T_n\} \cong T_n$ and

$\hat{G} = \{(f, 1) | f \in G^{\hat{n}}\} \cong G^{\hat{n}} \cong G \times G \times G \times \cdots \times G.$

Theorem 4.5.1 *$GwrT_n$ is unit regular. (We can write $GwrT_n = \hat{T}\hat{G}$)*

Proof:

We must prove that $(f, \pi) = (1, \pi)(f, 1)$.

Using the definition of the composition of two elements we get,

$(1, \pi)(f, 1) = ((1_G \circ 1_{T_n})f, \pi 1_{T_n}) = ((1_G \circ 1_{T_n})f, \pi) = (f, \pi)$,

since $((1_G \circ 1_{T_n})f)(k) = (1_G \circ 1_{T_n})(k)f(k) = 1 \cdot f(k) = f(k)$, for all $k \in \hat{n}$. $\square$

Corollary 4.5.2 *$GwrT_n$ is regular.*

Example 4.5.3 *Consider* $\begin{pmatrix} g_1 & g_2 \\ 0 & 0 \end{pmatrix}$*, which is an element of $GwrT_2$, where $g_1, g_2 \in G$. We can write,* $\begin{pmatrix} g_1 & g_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$.

4.6 Idempotents In $GwrT_n$

Idempotents in $GwrT_n$ are of the form, $\hat{e} = (f, e)$, where $\hat{e} \in E(GwrT_n)$ if and only if $e^2 = e$ and $f(i) = 1$, for all $i \in Rng(e)$.

Example 4.6.1 *Some idempotents in $GwrT_3$ are:* $\begin{pmatrix} 1 & g & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & 1 & 1 \end{pmatrix}$, *and* $\begin{pmatrix} 1 & 0 & g \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$*. You should notice that these look strikingly similar to the idempotents:* $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, *and* $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *in T_3.*

Now, we can show that Howie's Theorem from [6] holds for $GwrT_n$ as well. We see that the units of $GwrT_n$ make up $GwrS_n$.

Theorem 4.6.2 *Every non-unit of $GwrT_n$ can be written as a product of idempotents.*

Proof:

From the previous theorem, we know that $\hat{\sigma} = (f, \sigma) = (1, \sigma)(f, 1)$, where $\sigma \notin \ S_n$. Due to [6], we know that $(1, \sigma)$ can be represented as a product of idempotents. So, it suffices to prove that if $\hat{e} = \hat{e}^2 \neq \hat{1}$, $g = (f, 1) \in \hat{G}$, then $\hat{e}g$ is a product of idempotents. Starting with $\hat{e}g$, we can conjugate by g to get $g(\hat{e}g)g^{-1} = g\hat{e}$. This means that it is sufficient to prove that $g\hat{e}$ is a product of idempotents.

Remember, $\hat{G} = \{(f, 1) | f \in G^{\hat{n}}\} \cong G^{\hat{n}} \cong G \times G \times \cdots \times G$ and

$\hat{G}_i = \{(f, 1) | f \in G_i\} \cong G$, so $\hat{G} \cong \hat{G}_1 \times \hat{G}_2 \times \cdots \times \hat{G}_n$. This means that, for $g \in \hat{G}$, it suffices to prove that $g\hat{e}$ is a product of idempotents, for $g \in \hat{G}_i$.

Let $i \notin Rng(e)$ $(e(i) \neq i$, so $f(i) \neq 1)$. Then, for $\hat{e} = (f, e)$, $g = (h, 1) \in \hat{G}_i$, where $h \in G_i$ and $h(s) = 1$, when $s \neq i$. This means that $g\hat{e} = (h, 1)(f, e) = ((h \circ e)f, e) = (f, e) = \hat{e}$, since $(h \circ e)f(k) = h(e(k))f(k) = f(k)$ and $e(k) \neq i$ implies $h(e(k)) = 1$. So, $g\hat{e} = \hat{e}$.

So, assume that $i \in Rng(e)$. This means that $e(i) = i$, so $f(i) = 1$.

Define $\hat{e_0} = (1, e_0)$, where e_0 is defined as follows:

$$e_0(k) = \begin{cases} k & \text{if } k \in Rng(e) \\ i & \text{if } k \notin Rng(e) \end{cases}$$

So, $\hat{e} = \hat{e_0}\hat{e}$, which implies that $g\hat{e} = g\hat{e_0}\hat{e}$. This means that it suffices to show that $g\hat{e_0}$ is a product of idempotents. Once again, let $g = (f, 1) \in \hat{G}_j$, for $j \neq i$. We must prove that $g\hat{e_0} = (f, e_0)\hat{e_0}$.

Now, $g\hat{e_0} = (f, 1)\hat{e_0} = (f, 1)(1, e_0) = ((f \circ e_0)1_G, e_0) = (f \circ e_0, e_0)$ and $(f, e_0)(1, e_0) = ((f \circ e_0)1_G, e_0e_0) = (f \circ e_0, e_0^2) = (f \circ e_0, e_0)$, which tells us $g\hat{e_0} = (f, e_0)\hat{e_0}$.

Now to complete the proof of the theorem, let $k \notin Rng(e)$. Let (ik) be the permutation which switches the i-th and k-th rows. It must be shown that, $(f, e_0) = (1, e_0)[(ik)(f, e_0)] = (1, e_0)(f, (ik)e_0)$ and then it must be shown that, $(f, (ik)e_0)$ is an idempotent. First, we will compute $(1, e_0)(f, (ik)e_0)$.

So, $(1, e_0)(f, (ik)e_0) = (1, e_0)(f, (ik)e_0) = ((1_G \circ (ik)e_0)f, e_0(ik)e_0)$

$= (1_Gf, e_0(ik)e_0) = (f, e_0)$

Now, we can use this fact to show that $(f, (ik)e_0)$ is an idempotent. We must prove that,

$(f, (ik)e_0)(f, (ik)e_0) = (ik)(f, e_0)(ik)(f, e_0) = (ik)(f, e_0)$, which just amounts to showing that $(f, e_0)(ik)(f, e_0) = (f, e_0)$.

So, $(f, e_0)(1, (ik))(f, e_0) = (f, e_0)(1, (ik))(1, e_0)(1, (ik))(f, e_0)$

$= ((f \circ (ik))1_G, e_0(ik))(1, e_0)(1, (ik))(f, e_0)$

$= (1, e_0(ik))(1, e_0)(1, (ik))(f, e_0) = ((1_G \circ e_0)1_G, e_0(ik)e_0)(1, (ik))(f, e_0) =$

$((1, e_0)(1, (ik))(f, e_0)$

$= ((1_G \circ (ik))1_G, e_0(ik))(f, e_0) = ((1, e_0(ik))(f, e_0) = ((1_G \circ e_0)f, e_0(ik)e_0) =$

$(1_G f, e_0) = (f, e_0)$

This means that, $(f, e_0)(ik)(f, e_0) = (f, e_0)$ and thus,

$(ik)(f, e_0)(ik)(f, e_0) = (ik)(f, e_0)$, which is the final thing we needed to show. This completes the proof of the theorem. $\square$

Example 4.6.3 *Consider $GwrT_3$ and let $g_1, g_2, g_3 \in G$, we can write the non-idempotent,* $\begin{pmatrix} g_1 & g_2 & g_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$*, as a product of idempotents in the following way:*

$$\begin{pmatrix} 1 & 0 & g_2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g_2^{-1}g_1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & g_1^{-1}g_3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example 4.6.4 *Once again, consider $GwrT_3$ and let $g \in G$, we can write the non-idempotent,* $\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$*, as a product of idempotents in the following way:*

$$\begin{pmatrix} 1 & 0 & g \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

5 Green's Relations

We will start by defining Green's relations on a monoid, M, as in [4], [7]. Then, we will move on to determine Green's relations on $GwrT_n$.

5.1 Green's Relations On A Monoid M

Definition 5.1.1 *Two elements in M are* $\mathcal{R}$**-related***, denoted $a\mathcal{R}b$, if $aM = bM$.*

Definition 5.1.2 *Two elements in M are* $\mathcal{L}$**-related***, denoted $a\mathcal{L}b$, if $Ma = Mb$.*

Definition 5.1.3 *Two elements in M are* $\mathcal{J}$**-related***, denoted $a\mathcal{J}b$, if $MaM = MbM$.*

$\mathcal{J}$ is a two sided analogue of $\mathcal{R}$ and $\mathcal{L}$. These relations allow us to write M in terms of $\mathcal{R}$-classes, $\mathcal{L}$-classes, and $\mathcal{J}$-classes.

Definition 5.1.4 *The intersection of an $\mathcal{R}$-class and an $\mathcal{L}$-class is called an $\mathcal{H}$-class (i.e., $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$). Two elements in M are* $\mathcal{H}$**-related***, denoted $a\mathcal{H}b$, if and only if $a\mathcal{R}b$ and $a\mathcal{L}b$.*

Definition 5.1.5 *The join of an $\mathcal{R}$-class and an $\mathcal{L}$-class is called a $\mathcal{D}$-class (i.e., $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} = \mathcal{L} \vee \mathcal{R}$). Two elements in M are* $\mathcal{D}$**-related***, denoted $a\mathcal{D}b$, if and only if there exists $z \in M$ such that $a\mathcal{L}z$ and $z\mathcal{R}b$.*

For finite monoids, M, $\mathcal{J} = \mathcal{D}$.

5.2 Green's Relations for Idempotent Elements

Let $e, f \in M$ be idempotents. Two idempotents are $\mathcal{R}$-related, denoted $e\mathcal{R}f$, if $ef = f$ and $fe = e$. They are $\mathcal{L}$-related, denoted $e\mathcal{L}f$, if $fe = f$ and $ef = e$. They are $\mathcal{H}$-related, denoted $e\mathcal{H}f$, if $e = f$. If $e^2 = e$, then the $\mathcal{H}$-class of e is the unit group of eMe.

5.3 Green's Relations For T_n

Let $\sigma, \theta \in T_n$; $\sigma\mathcal{R}\theta$ if and only if $Rng(\sigma) = Rng(\theta)$, and $\sigma\mathcal{L}\theta$ if and only if σ and θ have the same fibres.

Definition 5.3.1 *Recall that a* **fibre** *of a map $f : X \to Y$ is*
$f^{-1}(y) = \{x \in X | f(x) = y\}$.

$\mathcal{R}$-classes are in one-to-one correspondence with subsets of $\hat{n}$ and $\mathcal{L}$-classes are in one-to-one correspondence with partitions of $\hat{n}$.

Example 5.3.2 *We can examine the Green's relations on T_3 by forming its $\mathcal{D}$-picture, as in [4], where:*
*(1) The headings for the rows are the subsets of $\{1, 2, 3\}$. The headings for the columns are the partitions of $\{1, 2, 3\}$. Each element with a, *, is an idempotent element.*
(2) (abc) means $1 \to a$, $2 \to b$, $3 \to c$. (i.e., This is the one-line notation discussed in Chapter 2 for elements of T_n.)

(3) The $\mathcal{D}_r$-class contains the rank r elements of T_3.

The $\mathcal{D}_1$-class contains the rank 1 elements.

Table 5.3.1 - $\mathcal{D}_1$-class of T_3

$\mathcal{D}_1$	$\{1,2,3\}$
$\{1\}$	$(111)^*$
$\{2\}$	$(222)^*$
$\{3\}$	$(333)^*$

The $\mathcal{D}_2$-class contains the rank 2 elements.

Table 5.3.2 - $\mathcal{D}_2$-class of T_3

$\mathcal{D}_2$	$\{1\}\{2,3\}$	$\{2\}\{1,3\}$	$\{3\}\{1,2\}$
$\{1,2\}$	$(122)^*$ (211)	$(121)^*$ (212)	(112) (221)
$\{1,3\}$	$(133)^*$ (311)	(131) (313)	$(113)^*$ (331)
$\{2,3\}$	(233) (322)	(232) $(323)^*$	(332) $(223)^*$

The $\mathcal{D}_3$-class contains the rank 3 elements.

Table 5.3.3 - $\mathcal{D}_3$-class of T_3

$\mathcal{D}_3$	$\{1\}\{2\}\{3\}$
$\{1,2,3\}$	$(123)^*$ (132) (213) (231) (312) (321)

The six elements which make up the $\mathcal{D}_3$-class are the elements of S_3.

Thus, T_3 has the following eggbox structure:

Table 5.3.4 - Eggbox Diagram for T_3

1				
1				
1				
	2	*2*	*2*	
	2	*2*	*2*	
	2	*2*	*2*	
				6

We see that we get, $|T_3| = 6 + 9(2) + 3(1) = 27 = 3^3$, *like we should.*

Each box in the $\mathcal{D}$-class diagram is an $\mathcal{H}$-class of T_3. Each row is an $\mathcal{R}$-class of T_3 and each column is an $\mathcal{L}$-class of T_3. By a theorem in [4], T_3 is a regular semigroup since each $\mathcal{R}$-class and each $\mathcal{L}$-class contain at least one idempotent. The $\mathcal{H}$-classes of T_3 which contain an idempotent are subgroups of T_3, by a result in [4].

Using $\mathcal{D}$-class diagrams, we see that we could write out the elements of T_3 in terms of $\mathcal{R}$-classes, $\mathcal{L}$-classes, and $\mathcal{J}$-classes. Since T_3 is finite, the $\mathcal{J}$-classes are simply the $\mathcal{D}$-classes described in the diagrams above.

In the next example, we will look at the $\mathcal{D}$-class diagrams for $\widetilde{T_3}$. We will see that the eggbox structure of $\widetilde{T_3}$ is quite different from the eggbox structure of T_3.

Example 5.3.3 *The $\mathcal{D}$-classes of $\widetilde{T_3}$ will be described below.*

The $\mathcal{D}_1$-class contains the rank 1 elements.

Table 5.3.5 - $\mathcal{D}_1$-class of $\widetilde{T_3}$

$\mathcal{D}_1$	$\{1,2,3\}$			
$\{1\}$	*(1 1 1)** *(-1 -1 -1)*	*(1 -1 1)** *(-1 1 -1)*	*(1 1 -1)** *(-1 -1 1)*	*(1 -1 -1)** *(-1 1 1)*
$\{2\}$	*(2 2 2)** *(-2 -2 -2)*	*(2 -2 2)* *(-2 2 -2)**	*(2 2 -2)** *(-2 -2 2)*	*(2 -2 -2)* *(-2 2 2)**
$\{3\}$	*(3 3 3)** *(-3 -3 -3)*	*(3 -3 3)** *(-3 3 -3)*	*(3 3 -3)* *(-3 -3 3)**	*(3 -3 -3)* *(-3 3 3)**

The $\mathcal{D}_2$-class contains the rank 2 elements.

Table 5.3.6 - $\mathcal{D}_2$-class of $\widetilde{T_3}$

$\mathcal{D}_2$	$\{1\}\{2,3\}$	$\{1\}\widetilde{\{2,3\}}$	$\{2\}\{1,3\}$	$\{2\}\widetilde{\{1,3\}}$	$\{3\}\{1,2\}$	$\{3\}\widetilde{\{1,2\}}$
$\{1,2\}$	(1 2 2)*	(1 2 -2)*	(1 2 1)*	(1 2 -1)*	(1 1 2)	(1 -1 2)
	(-1 2 2)	(-1 2 -2)	(-1 2 -1)	(-1 -2 1)	(-1 -1 2)	(-1 1 2)
	(1 -2 -2)	(1 -2 2)	(1 -2 1)	(1 -2 -1)	(1 1 -2)	(1 -1 -2)
	(-1 -2 -2)	(-1 -2 2)	(-1 -2 -1)	(-1 2 1)	(-1 -1 -2)	(-1 1 -2)
	(2 1 1)	(2 -1 1)	(2 1 2)	(2 1 -2)	(2 2 1)	(-2 2 1)
	(-2 1 1)	(2 1 -1)	(-2 1 -2)	(-2 1 2)	(-2 -2 1)	(2 -2 1)
	(2 -1 -1)	(-2 -1 1)	(2 -1 2)	(-2 -1 2)	(2 2 -1)	(2 -2 -1)
	(-2 -1 -1)	(-2 1 -1)	(-2 -1 -2)	(2 -1 -2)	(-2 -2 -1)	(-2 2 -1)
$\{1,3\}$	(1 3 3)*	(1 -3 3)*	(1 3 1)	(1 3 -1)	(1 1 3)*	(1 -1 3)*
	(-1 3 3)	(-1 3 -3)	(-1 3 -1)	(-1 -3 1)	(-1 -1 3)	(-1 1 -3)
	(1 -3 -3)	(1 3 -3)	(1 -3 1)	(1 -3 -1)	(1 1 -3)	(-1 1 3)
	(-1 -3 -3)	(-1 -3 3)	(-1 -3 -1)	(-1 3 1)	(-1 -1 -3)	(1 -1 -3)
	(3 1 1)	(3 1 -1)	(3 1 3)	(-3 1 3)	(3 3 1)	(-3 3 1)
	(-3 1 1)	(3 -1 1)	(-3 1 -3)	(3 1 -3)	(-3 -3 1)	(3 -3 1)
	(3 -1 -1)	(-3 -1 1)	(3 -1 3)	(-3 -1 3)	(3 3 -1)	(3 -3 -1)
	(-3 -1 -1)	(-3 1 -1)	(-3 -1 -3)	(3 -1 -3)	(-3 -3 -1)	(-3 3 -1)
$\{2,3\}$	(2 3 3)	(2 3 -3)	(2 3 2)	(2 3 -2)	(3 3 2)	(3 -3 2)
	(-2 3 3)	(-2 3 -3)	(-2 3 -2)	(-2 -3 2)	(-3 -3 2)	(-3 3 -2)
	(2 -3 -3)	(-2 -3 3)	(2 -3 2)	(2 -3 -2)	(3 3 -2)	(-3 3 2)
	(-2 -3 -3)	(2 -3 3)	(-2 -3 -2)	(-2 3 2)	(-3 -3 -2)	(3 -3 -2)
	(3 2 2)	(3 2 -2)	(3 2 3)*	(-3 2 3)*	(2 2 3)*	(-2 2 3)*
	(-3 2 2)	(-3 2 -2)	(-3 2 -3)	(3 2 -3)	(-2 -2 3)	(2 -2 3)
	(3 -2 -2)	(-3 -2 2)	(3 -2 3)	(-3 -2 3)	(2 2 -3)	(2 -2 -3)
	(-3 -2 -2)	(3 -2 2)	(-3 -2 -3)	(3 -2 -3)	(-2 -2 -3)	(-2 2 -3)

In the above table, $\{1\}\widetilde{\{2,3\}}$ *is the dual of* $\{1\}\{2,3\}$.

The $\mathcal{D}_3$-class contains the elements of $\widetilde{S_3}$, which are all of rank 3.

We see that $|\widetilde{T_3}| = 24 + 144 + 48 = 216 = 6^3$

Thus, $\widetilde{T_3}$ has the following eggbox structure:

Table 5.3.7 - Eggbox Diagram for $\widetilde{T_3}$

2	2	2	2							
2	2	2	2							
2	2	2	2							
				8	8	8	8	8	8	
				8	8	8	8	8	8	
				8	8	8	8	8	8	
										48

Each box in the $\mathcal{D}$-class diagram is an $\mathcal{H}$-class of $\widetilde{T_3}$. Each row is an $\mathcal{R}$-class of $\widetilde{T_3}$ and each column is an $\mathcal{L}$-class of $\widetilde{T_3}$. By a theorem in [4], $\widetilde{T_3}$ is a regular semigroup since each $\mathcal{R}$-class and each $\mathcal{L}$-class contain at least one idempotent.

5.4 Green's Relations For Idempotents In $GwrT_n$

Let $\hat{e} = (f, e)$ and $\hat{e}' = (f', e')$ be idempotents in $GwrT_n$. Remember, for idempotents $(f, e) \in GwrT_n$, $e^2 = e \in T_n$ and $f(i) = 1$, for all $i \in Rng(e)$.

We determine the $\mathcal{R}$-relatedness of elements in the same manner as in T_n. Recall, that $\hat{e}$ is $\mathcal{R}$-related to $\hat{e}'$, denoted $\hat{e}\mathcal{R}\hat{e}'$ if and only if $Rng(\hat{e}) = Rng(\hat{e}')$. The number of $\mathcal{R}$-classes of rank k in $GwrT_n$ is the same as the number of $\mathcal{R}$-classes of rank k in T_n. So, the number of $\mathcal{R}$-classes of rank k in $GwrT_n$ is $\begin{pmatrix} n \\ k \end{pmatrix}$.

This counting argument is proven in [3], on page 61. He uses the notation of, [4], which is the reverse of the notation found in this paper. This difference in notation is due to the fact that they write maps on the right and we write maps on the left.

Looking back to our $\mathcal{D}$-class diagram for T_3, we see that there are $\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3$ $\mathcal{R}$-classes of rank 1. These are the rows of the $\mathcal{D}_1$-class. There are $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3$ $\mathcal{R}$-classes of rank 2. These are the rows of the $\mathcal{D}_2$-class. There is $\begin{pmatrix} 3 \\ 3 \end{pmatrix} = 1$ $\mathcal{R}$-class of rank 3. This is the row of the $\mathcal{D}_3$-class. The same results will hold for $GwrT_3$. This fact will allow us to write $GwrT_n$ in terms of $\mathcal{R}$-classes.

When describing the $\mathcal{L}$-classes, the following theorem will be quite useful. The theorem works because $GwrT_n$ is unit regular, so every element is $\mathcal{L}$-related to an idempotent.

Theorem 5.4.1 *(Rank One Case) If $\hat{e} = (f, e)$ is an idempotent and $e' \in T_n$ is an idempotent such that $e\mathcal{L}e'$, then $\hat{e}\mathcal{L}\hat{e}' = (f', e')$, for some f'.*

Proof:

Let $e, e \in T_n$, where $e\mathcal{L}e'$. We partition $\hat{n}$ as $\hat{n} = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_t$. Then, for $\alpha_i, \alpha_i' \in A_i$, we have $e : A_i \to \alpha_i$ and $e' : A_i \to \alpha_i'$. Now, for $\hat{e} = (f, e)$, $f(\alpha_i) = \alpha_i$. For $\alpha \in \hat{n}$, if $\alpha \in A_i$, we define $f'(\alpha) = f(\alpha_i')^{-1} f(\alpha)$. We must show that $\hat{e}' = (f', e')$ is an idempotent.

We already know that $(e')^2 = e'$, so all we need to show is $f'(\alpha) = 1$, for $\alpha \in Rng(e')$. We know that $Rng(e') = \alpha_i'$, so $f'(\alpha) = f(\alpha_i')^{-1} f(\alpha_i') = 1$, for $i = 1, 2, ..., t$. Therefore, $\hat{e}' = (f', e')$ is an idempotent. Now, we must show that $\hat{e}\mathcal{L}\hat{e}'$.

So, $\hat{e}\hat{e}' = (f, e)(f', e') = ((f \circ e')f', ee')$. Since $e\mathcal{L}e'$ in T_n, $ee' = e$ and $((f \circ e')f')(\alpha) = (f \circ e')(\alpha)f'(\alpha) = (f \circ e')(\alpha)(1) = f(e'(\alpha)) = f(\alpha)$, for $\alpha \in Rng(e')$. So, $((f \circ e')f', ee') = (f, e)$, which implies $\hat{e}\hat{e}' = \hat{e}$.

Also, $\hat{e}'\hat{e} = (f', e')(f, e) = ((f' \circ e)f, e'e)$. Since $e\mathcal{L}e'$ in T_n, $e'e = e'$ and $((f' \circ e)f)(\beta) = (f' \circ e)(\beta)f(\beta) = (f' \circ e)(\beta)(1) = f'(e(\beta)) = f'(\beta)$, for $\beta \in Rng(e)$. So, $((f' \circ e)f, e'e) = (f', e')$, which implies $\hat{e}'\hat{e} = \hat{e}'$. So, $\hat{e}\mathcal{L}\hat{e}'$. Therefore, our choice of f' was correct. $\square$

The above proof only covers the case of rank 1 matrices. The following example is one for which the theorem works.

Example 5.4.2 *In $GwrT_4$:*

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathcal{L} \begin{pmatrix} 0 & 0 & 0 & 0 \\ y_1 & 1 & y_3 & y_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\textit{means that} \left\{ \begin{array}{c} x_2 y_1 = 1 \Rightarrow y_1 = x_2^{-1} \\ x_2 y_3 = x_3 \Rightarrow y_3 = x_2^{-1} x_3 \\ x_2 y_4 = x_4 \Rightarrow y_4 = x_2^{-1} x_4 \\ y_2 = 1 \end{array} \right\}$$

Both matrices used to represent the elements,

$$\begin{tikzcd} 1 \arrow[loop left, "1"] & 2 \arrow[l, "x_2"'] \\ 3 \arrow[u, "x_3"'] & 4 \arrow[ul, "x_4"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 1 \arrow[r, "y_1"'] & 2 \arrow[loop right, "1"] \\ 3 \arrow[ur, "y_3"'] & 4 \arrow[u, "y_4"'] \end{tikzcd}$$

were rank one matrices. The previous theorem is also true for matrices of rank greater than one.

Theorem 5.4.3 *(General Case) If $\hat{e} = (f, e)$ is an idempotent and $e' \in T_n$ is an idempotent such that $e\mathcal{L}e'$, then $\hat{e}\mathcal{L}\hat{e}' = (f', e')$, for some f'.*

Proof:

Once again, let $e, e \in T_n$, where $e\mathcal{L}e'$. We partition $\hat{n}$ as $\hat{n} = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_t$. Then, for $\alpha_i, \alpha_i' \in A_i$, we have $e : A_i \to \alpha_i$ and $e' : A_i \to \alpha_i'$.

Now, for $\hat{e} = (f, e)$, we can write $\hat{e} = \hat{e}_1 \oplus \hat{e}_2 \oplus \cdots \oplus \hat{e}_t$. This just says that the matrix which represents $\hat{e}$ has a block decomposition. Similarly, we may write, $\hat{e}' = \hat{e}_1' \oplus \hat{e}_2' \oplus \cdots \oplus \hat{e}_t'$.

Each block matrix in the block decomposition of $\hat{e}$ and $\hat{e}'$ is of rank one. So, we are now able to relate each block of $\hat{e}$ to the corresponding blocks of $\hat{e}'$. So, for $\hat{e}_i = (f_i, e_i)$, which is an idempotent with $e_i\mathcal{L}e_i'$ in T_n, the rank one case of the theorem tells us that $\hat{e}_i\mathcal{L}\hat{e}_i' = (f_i', e_i')$, for some f_i'.

This gives us, $\hat{e}_1\mathcal{L}\hat{e}_1'$, $\hat{e}_2\mathcal{L}\hat{e}_2'$, ... ,$\hat{e}_t\mathcal{L}\hat{e}_t'$. Since $\hat{e} = \hat{e}_1 \oplus \hat{e}_2 \oplus \cdots \oplus \hat{e}_t$ and $\hat{e}' = \hat{e}_1' \oplus \hat{e}_2' \oplus \cdots \oplus \hat{e}_t'$, we have what we needed to prove, in the general case. $\square$

Theorem 5.4.4 *Two idempotents, $\hat{e} = (f, e)$ and $\hat{e}_1 = (f_1, e)$, are $\mathcal{L}$-related if and only if $f = f_1$.*

Proof:

Let $\hat{e} = (f, e)$ and $\hat{e}_1 = (f_1, e)$ be two idempotents which are $\mathcal{L}$-related. Then, $\hat{e}\hat{e}_1 = \hat{e}$ and $\hat{e}_1\hat{e} = \hat{e}_1$.

This means that, $(f, e)(f_1, e) = ((f \circ e)f_1, ee) = ((f \circ e)f_1, e)$ and $(f_1, e)(f, e) = ((f_1 \circ e)f, ee) = ((f_1 \circ e)f, e)$.

So, $((f \circ e)f_1, e) = (f, e)$ and $((f_1 \circ e)f, e) = (f_1, e)$. This tells us that, $(f \circ e)f_1 = f$ and $(f_1 \circ e)f = f_1$, which implies $f = f_1$. $\square$

Given e of rank k, the number of possibilities for f is $|G|^{n-k}$. This is because $f(i) = 1$, for $i \in Rng(e)$ and $|Rng(e)| = k$ and f is arbitrary on $\hat{n} - Rng(e)$. Hence we have the following theorem, which gives the number of $\mathcal{L}$-classes.

Theorem 5.4.5 *The number of $\mathcal{L}$-classes of $GwrT_n$ of rank k is equal to $|G|^{n-k}$ times the number of partitions of $\hat{n}$ with k parts.*

5.5 Looking Ahead To Chapter 6

One could continue on and describe the $\mathcal{R}$-classes and $\mathcal{L}$-classes for $GwrT_n$ in greater detail, as well as, the notions of $\mathcal{H}$-relatedness and $\mathcal{J}$-relatedness in $GwrT_n$. At some point in the future, I may do this, but in the next chapters we shall focus on the conjugacy classes in the wreath products instead.

6 Conjugacy Classes In S_n and T_n

In this chapter, we will examine the conjugacy classes of S_n and T_n and see many examples.

6.1 Conjugacy Classes Of S_n Revisited

Once again, let $\hat{n} = \{1, 2, ..., n\}$. Consider the symmetric group, S_n, which consists of the permutations of $\hat{n}$. Let $\alpha, \beta \in \hat{n}$ and $\sigma \in S_n$. We say $\alpha \sim \beta$ if $\sigma^i(\alpha) = \beta$, for some i. This relation allows us to decompose into cycles.

Example 6.1.1 *In S_7, consider* $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & 3 & 5 & 6 & 1 & 2 \end{pmatrix}$.

In cycle notation this element can be represented as, $(1456)(27)(3)$.

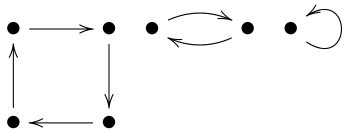
Notice that: $1 \sim 4$ *since* $\sigma(1) = 4$, $1 \sim 5$ *since* $\sigma^2(1) = 5$,

$1 \sim 6$ *since* $\sigma^3(1) = 6$, $2 \sim 7$ *since* $\sigma(2) = 7$, *and* $3 \sim 3$ *since* $\sigma(3) = 3$.

So, we have the decomposition of $\hat{n} = \{1, 2, 3, 4, 5, 6, 7\}$ *into* $\{1, 4, 5, 6\}$, $\{2, 7\}$, *and* $\{3\}$. *This can be shown in graph form as follows:*

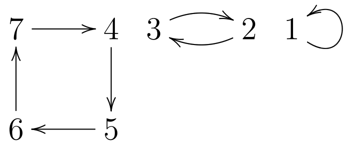
$$\begin{tikzcd} 1 \arrow[r] & 4 \arrow[d] & 2 \arrow[r, bend left] & 7 \arrow[l, bend left] & 3 \arrow[loop right] \\ 6 \arrow[u] & 5 \arrow[l] & & & \end{tikzcd}$$

We see that we have a decomposition into cycles. We may move the labels around to produce other elements with the same cycle structure. So, we could produce other elements of the form,



All of these elements would have the same cycle type and would be contained in the same conjugacy class of S_7*.*

For example,



has the same cycle structure as our original example. Thus, $(4567)(32)(1)$ *would be in the same conjugacy class as* $(1456)(27)(3)$*, in* S_7*.*

6.2 Conjugacy Classes Of T_n

Next, we will consider what happens in the full transformation semigroup, T_n. We will be able to produce a similar type of cycle decomposition. Consider the full transformation semigroup, T_n, which consists of the mappings from $\hat{n} \to \hat{n}$. Let $\alpha, \beta \in \hat{n}$ and $\sigma \in T_n$. We say $\alpha \sim \beta$ if $\sigma^i(\alpha) = \sigma^j(\beta)$, for some i, j.

Theorem 6.2.1 *The relation,* $\alpha \sim \beta$ *if* $\sigma^i(\alpha) = \sigma^j(\beta)$*, for some* i, j*, is an equivalence relation.*

Proof:

To show the relation is an equivalence relation, we must show it is reflexive, symmetric, and transitive.

Reflexive: Let $\alpha \in \hat{n}$. Then, $\alpha \sim \alpha$ since $\sigma^i(\alpha) = \sigma^j(\alpha)$, for i, j. This occurs when $i = j$. Thus, the relation is reflexive.

Symmetric: Let $\alpha, \beta \in \hat{n}$ and let $\alpha \sim \beta$. Then, $\sigma^i(\alpha) = \sigma^j(\beta)$, for some i, j. Then $\sigma^j(\beta) = \sigma^i(\alpha)$, for some i, j, so $\beta \sim \alpha$. Thus, the relation is symmetric.

Transitive: Let $\alpha, \beta, \gamma \in \hat{n}$. Let $\alpha \sim \beta$ and $\beta \sim \gamma$. Then, $\sigma^i(\alpha) = \sigma^j(\beta)$, for some i, j and $\sigma^k(\beta) = \sigma^l(\gamma)$, for some k, l.

Then, $\sigma^{i+k}(\alpha) = \sigma^{j+k}(\beta) = \sigma^j(\sigma^k(\beta)) = \sigma^j(\sigma^l(\gamma)) = \sigma^{j+k}(\gamma)$, for some i, j, k, l. Thus, $\alpha \sim \gamma$ and the relation is transitive.

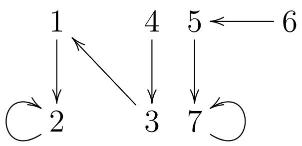
So the relation is reflexive, symmetric, and transitive, which proves that it is an equivalence relation. $\square$

This is the analogue of the cycle decomposition in S_n. We call this the generalized cycle decomposition. In other words, the equivalence relation yields a decomposition into connected pieces.

Example 6.2.2 *In* T_7*, consider* $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 2 & 1 & 3 & 7 & 5 & 7 \end{pmatrix}$.

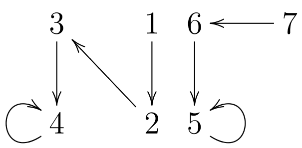
Notice that: $1 \sim 2$ *since* $\sigma(1) = 2 = \sigma(2)$*,* $1 \sim 3$ *since* $\sigma(1) = 2 = \sigma^2(3)$*,* $1 \sim 4$ *since* $\sigma(1) = 2 = \sigma^3(4)$*,* $5 \sim 6$ *since* $\sigma(5) = 7 = \sigma^2(6)$*,* *and* $5 \sim 7$ *since* $\sigma(5) = 7 = \sigma(7)$*. So, we have the decomposition of* $\hat{n} = \{1, 2, 3, 4, 5, 6, 7\}$ *into* $\{1, 2, 3, 4\}$ *and* $\{5, 6, 7\}$.

This can be shown in graph form as follows:



Notice that we have a decomposition into generalized cycles. Relabeling while keeping the same generalized cycle structure (or "type") will give us the other elements in the same conjugacy class as our example.

So, another element in the same conjugacy class of T_7 *would be:*



which we could write as $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 4 & 5 & 5 & 6 \end{pmatrix}$.

Example 6.2.3 *The following are examples of generalized cycle decompositions in T_{10}:*

(1) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 5 & 6 & 1 & 1 & 7 & 9 & 6 & 8 & 8 \end{pmatrix}$,

which we represent in graph form as,

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] & 3 \arrow[r] & 6 \arrow[r] & 7 \arrow[d] \\ 4 \arrow[u] & 5 \arrow[ul] & 10 \arrow[r] & 8 \arrow[u] & 9 \arrow[l] \end{tikzcd}$$

(2) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 1 & 1 & 6 & 5 & 7 & 10 & 8 & 8 \end{pmatrix}$,

which we represent in graph form as,

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] & 5 \arrow[r, bend left] & 6 \arrow[l, bend left] & 7 \arrow[loop left] & 8 \arrow[d, bend left] & 9 \arrow[l] \\ 4 \arrow[u] & 3 \arrow[ul] & & & & 10 \arrow[u, bend left] & \end{tikzcd}$$

6.2.1 The Conjugacy Classes Of T_1 and T_2

The elements in each conjugacy class will be represented using one-line notation. An unlabeled graph will also be shown to describe the generalized cycle type of each element in the conjugacy class.

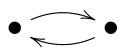
The only conjugacy class of T_1 is $C^1 = \{(1)\}$, which consists of elements of the form,

$$\begin{tikzcd} \bullet \arrow[loop left] \end{tikzcd}$$

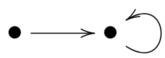
In T_2,

The conjugacy class, $C^1 = \{(12)\}$, consists of elements of the form,

The conjugacy class, $C^2 = \{(21)\}$, consists of elements of the form,



The conjugacy class, $C^3 = \{(11), (22)\}$, consists of elements of the form,

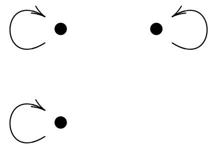


So, T_2 has three conjugacy classes.

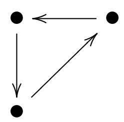
6.2.2 The Conjugacy Classes Of T_3

Again, the elements in each conjugacy class will be represented using one-line notation. An unlabeled graph will also be shown to describe the generalized cycle type of each element in the conjugacy class.

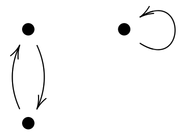
The conjugacy class, $C^1 = \{(123)\}$, consists of elements of the form,



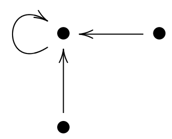
The conjugacy class, $C^2 = \{(231), (312)\}$, consists of elements of the form,



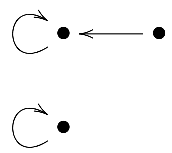
The conjugacy class, $C^3 = \{(132), (213), (321)\}$, consists of elements of the form,



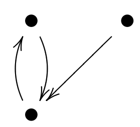
The conjugacy class, $C^4 = \{(111), (222), (333)\}$, consists of elements of the form,



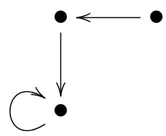
The conjugacy class, $C^5 = \{(223), (323), (121), (122), (133), (113)\}$, consists of elements of the form,



The conjugacy class, $C^6 = \{(332), (331), (311), (232), (212), (211)\}$, consists of elements of the form,



The conjugacy class, $C^7 = \{(112), (131), (221), (233), (313), (322)\}$, consists of elements of the form,

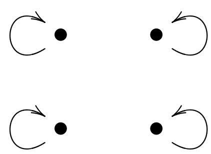


We see that T_3 has seven conjugacy classes.

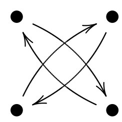
6.2.3 The Conjugacy Classes Of T_4

Once again, the elements in each conjugacy class will be represented using one-line notation. An unlabeled graph will also be shown to describe the generalized cycle type of each element in the conjugacy class.

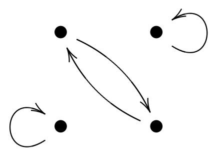
The conjugacy class, $C^1 = \{(1234)\}$, consists of elements of the form,



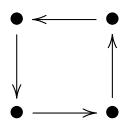
The conjugacy class, $C^2 = \{(4321), (2143), (3412)\}$, consists of elements of the form,



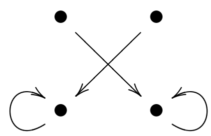
The conjugacy class, $C^3 = \{(1324), (4231), (1243), (1432), (2134), (3214)\}$, consists of elements of the form,



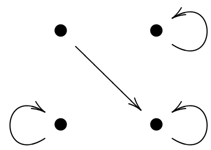
The conjugacy class, $C^4 = \{(2413), (3142), (2341), (3421), (4123), (4312)\}$, consists of elements of the form,



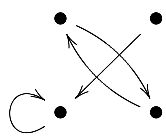
The conjugacy class, $C^5 = \{(1221), (1331), (4224), (4334), (1133), (1212), (2244), (3434), (1144), (1414), (2233), (3232)\}$, consists of elements of the form,



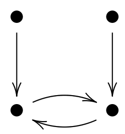
The conjugacy class, $C^6 = \{(1224), (1231), (1324), (1334), (1134), (1214), (1232), (1233), (1244), (2234), (3234), (1434)\}$, consists of elements of the form,



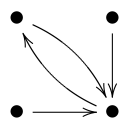
The conjugacy class, $C^7 = \{(1321), (4221), (4331), (4324), (1143), (1412), (2133), (2144), (2243), (3212), (3414), (3432)\}$, consists of elements of the form,



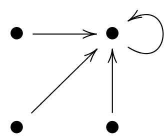
The conjugacy class, $C^8 = \{(2112), (2442), (3113), (3443), (2121), (3311), (4343), (4422), (2323), (4141), (3322), (4411)\}$, consists of elements of the form,



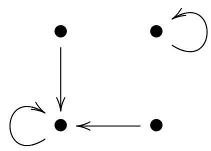
The conjugacy class, $C^9 = \{(2322), (4111), (4441), (3323), (2111), (2122), (2422), (3111), (3313), (3343), (4442), (4443)\}$, consists of elements of the form,



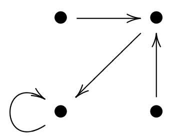
The conjugacy class, $C^{10} = \{(1111), (2222), (3333), (4444)\}$, consists of elements of the form,



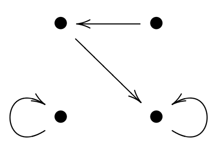
The conjugacy class, $C^{11} = \{(1114), (1444), (2232), (3233), (1131), (1211),$ $(1222), (1333), (2224), (3334), (4244), (4434)\}$, consists of elements of the form,



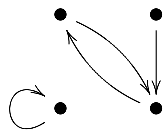
The conjugacy class, $C^{12} = \{(1441), (2332), (3223), (4114), (1122), (1313),$ $(2211), (2424), (3344), (4242), (4433), (3131)\}$, consists of elements of the form,



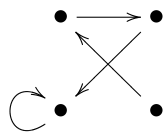
The conjugacy class, $C^{13} = \{(1223), (1241), (1332), (1431), (2334), (4134),$ $(4214), (3224), (1132), (1213), (1242), (1433), (2214), (2434), (3134), (3244),$ $(1124), (1314), (1344), (1424), (2231), (3231), (4232), (4233)\}$, consists of elements of the form,



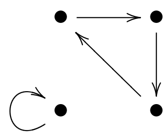
The conjugacy class, $C^{14} = \{(1322), (1323), (2324), (3324), (4241), (4211), (4431), (4131), (1442), (1443), (2132), (3114), (2114), (3243), (2432), (3213), (1422), (2131), (4243), (3314), (2124), (4432), (3211), (4243)\}$, consists of elements of the form,



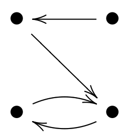
The conjugacy class, $C^{15} = \{(1341), (1421), (2331), (3221), (4223), (4332), (4314), (4124), (1123), (1312), (2241), (2344), (3431), (3424), (4133), (4212), (1142), (1413), (2213), (2414), (2433), (3132), (3144), (3242)\}$, consists of elements of the form,



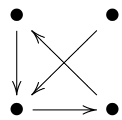
The conjugacy class, $C^{16} = \{(1342), (3124), (3241), (4132), (1423), (2314), (2431), (4213)\}$, consists of elements of the form,



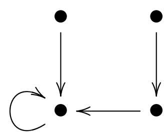
The conjugacy class, $C^{17} = \{(2123), (2141), (2343), (3312), (3411), (4143), (3422), (4412), (2113), (2142), (2412), (2443), (3112), (3143), (3413), (3442), (2321), (3321), (4121), (4341), (4323), (4322), (4421), (4311)\}$, consists of elements of the form,



The conjugacy class, $C^{18} = \{(2311), (2421), (3121), (3341), (4122), (4313), (4342), (4423), (2312), (2441), (3123), (3423), (3441), (4112), (4113), (2342), (2313), (2423), (2411), (3122), (3141), (3342), (4142), (4413)\}$, consists of elements of the form,



The conjugacy class, $C^{19} = \{(1112), (1113), (2242), (2444), (3133), (3433), (3444), (2212), (1121), (1311), (2221), (4222), (4333), (4344), (4424), (3331), (1141), (1411), (2223), (2333), (3222), (3332), (4144), (4414)\}$, consists of elements of the form,



We see that T_4 has nineteen conjugacy classes.

6.2.4 More About The Generalized Cycle Structure

Now, we will examine the generalized cycle structure in greater detail. Let $Y_0 = \hat{n}$ and $Y_i = \sigma^i(Y_0)$, where $\sigma \in T_n$. We have $Y_0 \supset Y_1 \supset \cdots \supset Y_k = Y_{k+1}$. We define, $Y = Y_k$, to be the core of σ. We should note that $\sigma|_Y \in S_Y$.

As we have seen in the above examples, σ produces a generalized cycle decomposition, $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_r$, into r connected components. $\sigma_i = \sigma|_{X_i}$ is connected with core $Z_i = X_i \cap Y$. Moreover, $\sigma|_{Z_i}$ is a cycle. This means that r is the number of cycles of $\sigma|_Y$.

We could represent the previous paragraphs of information pictorially. If we make a "bullseye" diagram for the Y_i's which shows, $Y_0 \supset Y_1 \supset \cdots \supset Y_k = Y_{k+1}$, then, the center of the "bullseye" diagram would be $Y = Y_k$, the core. We would use σ to divide up into connected pieces, X_1,X_2,X_3,...,X_r, where $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_r$. This would illustrate the fact that $\sigma_i = \sigma|_{X_i}$ is connected with core, $Z_i = X_i \cap Y$.

Example 6.2.4 *Consider the following element, $\sigma \in T_9$,*

$$\begin{tikzcd}
1 \arrow[d, bend left] & 3 \arrow[l] & & 8 \arrow[d] & & \\
2 \arrow[u, bend left] & 4 \arrow[l] & 5 \arrow[l] & 6 \arrow[r] & 7 \arrow[ul] & 9 \arrow[l]
\end{tikzcd}$$

Using the notation described on the previous page, we see that,

$Y_0 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $Y_1 = \sigma(Y_0) = \{1, 2, 4, 6, 7, 8\}$, *and the core of* σ *is* $Y = Y_2 = \sigma^2(Y_0) = \{1, 2, 6, 7, 8\}$. *Notice that we have,* $Y_0 \supset Y_1 \supset Y_2 = Y$.

σ *produces a generalized cycle decomposition,* $X = X_1 \sqcup X_2$, *where* $X_1 = \{1, 2, 3, 4, 5\}$ *and* $X_2 = \{6, 7, 8, 9\}$.

σ_1 *is connected with core* $Z_1 = X_1 \cap Y = \{1, 2\}$ *and* σ_2 *is connected with core* $Z_2 = X_2 \cap Y = \{6, 7, 8\}$.

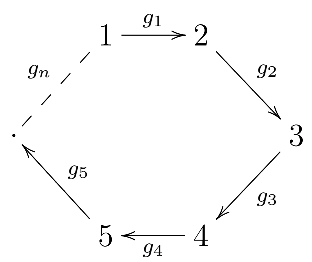
Now that we understand the cycle decompositions in S_n and T_n, we may examine what happens in $GwrS_n$ and $GwrT_n$.

7 Conjugacy Classes in $GwrS_n$

Consider the wreath product of any group G with S_n, denoted $GwrS_n$. Let G have t conjugacy classes, $[g_1], [g_2], ..., [g_t]$. It is a well known result that we can associate a color with each cycle.

7.1 N-Cycles in $GwrS_n$

We can reduce any n-cycle in $GwrS_n$ to a cycle with only one label. So, for an n-cycle,



which is represented by the matrix, $\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & g_n \\ g_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & g_2 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & g_{n-1} & 0 \end{pmatrix}$,

we can reduce to the standard (or canonical) form,

$$
\begin{tikzcd}
& 1 \arrow[r, "1"] & 2 \arrow[dr, "g_n g_{n-1} g_{n-2}\cdots g_2 g_1"] & \\
\cdot \arrow[ur, dashed, no head, "1"] & & & 3 \arrow[dl, "1"] \\
& 5 \arrow[ul, "1"] & 4 \arrow[l, "1"] &
\end{tikzcd}
$$

via multiple conjugations by different matrices.

Also, by conjugation, we are able to put the $g_n g_{n-1} g_{n-2} \cdots g_2 g_1$ anywhere we wish along the n-cycle, and have all 1's elsewhere. The conjugacy class of $g_n g_{n-1} g_{n-2} \cdots g_2 g_1$ is referred to as the type of the cycle, or the color.

Example 7.1.1 *In $GwrS_4$, consider the element,*

$$
\begin{tikzcd}
1 \arrow[r, "g_1"] & 2 \arrow[d, "g_2"] \\
4 \arrow[u, "g_4"] & 3 \arrow[l, "g_3"]
\end{tikzcd}
$$

which is represented by the matrix, $\begin{pmatrix} 0 & 0 & 0 & g_4 \\ g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_3 & 0 \end{pmatrix}$

(1) Conjugate by, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_4 \end{pmatrix}$*:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & g_4 \\ g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_4^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & g_4 \\ g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_4g_3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_4^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_4g_3 & 0 \end{pmatrix}$$

(2) Conjugate by, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & g_4g_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$*:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & g_4g_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_4g_3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (g_4g_3)^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & g_4 \\ g_1 & 0 & 0 & 0 \\ 0 & g_4g_3g_2 & 0 & 0 \\ 0 & 0 & g_4g_3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (g_4g_3)^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ g_1 & 0 & 0 & 0 \\ 0 & g_4g_3g_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(3) Conjugate by, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_4g_3g_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_4g_3g_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ g_1 & 0 & 0 & 0 \\ 0 & g_4g_3g_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (g_4g_3g_2)^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & g_4 \\ g_4g_3g_2g_1 & 0 & 0 & 0 \\ 0 & g_4g_3g_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (g_4g_3g_2)^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ g_4g_3g_2g_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which gives us the element in canonical form,

$$\begin{tikzcd} 4 \arrow[r, "1"] & 1 \arrow[d, "g_4g_3g_2g_1"] \\ 3 \arrow[u, "1"] & 2 \arrow[l, "1"'] \end{tikzcd}$$

We can move the $g_4g_3g_2g_1$ around by conjugation.

7.2 Example in $GwrS_{12}$

Every element in $GwrS_n$ is conjugate to an element in standard form, a permutation with each cycle having a color.

Example 7.2.1 *In $GwrS_{12}$, the element,*

$$\begin{tikzcd}
1 \arrow[dr, "g_1"] & & 4 \arrow[dr, "g_4"] & & 7 \arrow[r, bend left, "g_7"] & 8 \arrow[l, bend left, "g_8"] \\
3 \arrow[u, "g_3"] & 2 \arrow[l, "g_2"'] & 6 \arrow[u, "g_6"] & 5 \arrow[l, "g_5"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
9 \arrow[loop left, "g_9"] & 10 \arrow[r, bend left, "g_{10}"] & 11 \arrow[l, bend left, "g_{11}"] & 12 \arrow[loop right, "g_{12}"]
\end{tikzcd}$$

is conjugate to the following element in standard form,

$$\begin{tikzcd}
1 \arrow[dr, "g_3 g_2 g_1"] & & 4 \arrow[dr, "g_6 g_5 g_4"] & & 7 \arrow[r, bend left, "g_8 g_7"] & 8 \arrow[l, bend left, "1"] \\
3 \arrow[u, "1"] & 2 \arrow[l, "1"'] & 6 \arrow[u, "1"] & 5 \arrow[l, "1"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
9 \arrow[loop left, "g_9"] & 10 \arrow[r, bend left, "g_{11} g_{10}"] & 11 \arrow[l, bend left, "1"] & 12 \arrow[loop right, "g_{12}"]
\end{tikzcd}$$

7.3 The Conjugacy Class Formula For $GwrS_n$

The fact that we can make any element in $GwrS_n$ conjugate to an element in standard form, characterizes the conjugacy classes of $GwrS_n$ by their cycle structures (or "types"). There are as many conjugacy classes as there are "types". The following theorem from [8], gives a formula for determining the number of conjugacy classes in $GwrS_n$.

Theorem 7.3.1 *If $t \in \mathbb{N}$ is equal to the number of conjugacy classes of G and if $p(m)$ denotes the number of partitions of m, then the number of conjugacy classes of $GwrS_n$ is $\sum\limits_{(n)} p(n_1)p(n_2)\cdots p(n_t)$, where the sum is taken over all t-tuples $(n_1, n_2, ..., n_t)$ over $\mathbb{N}_0$, such that $\sum n_i = n$.*

Example 7.3.2 *Consider $\mathbb{Z}_2 wr S_2$. The conjugacy classes are*

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

So, we have five conjugacy classes in $\mathbb{Z}_2 wr S_2$.

We will arrive at the same conclusion using the previous theorem. Notice that we can partition $n = 2$ in three ways, as $0 + 2$, $1 + 1$, and $2 + 0$.

For $2 = 2 + 0$, $p(2)p(0) = 2$.

For $2 = 1 + 1$, $p(1)p(1) = 1$.

For $2 = 0 + 2$, $p(0)p(2) = 2$.

This is because S_2 has two conjugacy classes and S_1 has one conjugacy class.

So, $\sum p(n_1)\cdots p(n_t) = p(2)p(0) + p(1)p(1) + p(0)p(2) = 2 + 1 + 2 = 5$.

Again, we see that the number conjugacy classes in $\mathbb{Z}_2 wr S_2$ is five.

Example 7.3.3 *If G has 3 conjugacy classes, then $GwrS_2$ has 9 conjugacy classes since,*

$2 = 0 + 0 + 2$, *so* $p(0)p(0)p(2) = 1 \cdot 1 \cdot 2 = 2$

$2 = 0 + 2 + 0$, *so* $p(0)p(2)p(0) = 1 \cdot 2 \cdot 1 = 2$

$2 = 2 + 0 + 0$, *so* $p(2)p(0)p(0) = 2 \cdot 1 \cdot 1 = 2$

$2 = 1 + 1 + 0$, *so* $p(1)p(1)p(0) = 1 \cdot 1 \cdot 1 = 1$

$2 = 1 + 0 + 1$, *so* $p(1)p(0)p(1) = 1 \cdot 1 \cdot 1 = 1$

$2 = 0 + 1 + 1$, *so* $p(0)p(1)p(1) = 1 \cdot 1 \cdot 1 = 1$

and 2+2+2+1+1+1=9.

Example 7.3.4 *If G has 5 conjugacy classes, then $GwrS_3$ has 65 conjugacy classes since,*

$3 = 0 + 0 + 0 + 0 + 3$, *so* $p(0)p(0)p(0)p(0)p(3) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 3 = 3$

$3 = 0 + 0 + 0 + 3 + 0$, *so* $p(0)p(0)p(0)p(3)p(0) = 1 \cdot 1 \cdot 1 \cdot 3 \cdot 1 = 3$

$3 = 0 + 0 + 3 + 0 + 0$, *so* $p(0)p(0)p(3)p(0)p(0) = 1 \cdot 1 \cdot 3 \cdot 1 \cdot 1 = 3$

$3 = 0 + 3 + 0 + 0 + 0$, *so* $p(0)p(3)p(0)p(0)p(0) = 1 \cdot 3 \cdot 1 \cdot 1 \cdot 1 = 3$

$3 = 3 + 0 + 0 + 0 + 0$, *so* $p(3)p(0)p(0)p(0)p(0) = 3 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 3$

$3 = 2 + 1 + 0 + 0 + 0$, *so* $p(2)p(1)p(0)p(0)p(0) = 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 2$

$3 = 2 + 0 + 1 + 0 + 0$, *so* $p(2)p(0)p(1)p(0)p(0) = 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 2$

$3 = 2 + 0 + 0 + 1 + 0$, *so* $p(2)p(0)p(0)p(1)p(0) = 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 2$

$$3 = 2+0+0+0+1, \ so \ p(2)p(0)p(0)p(0)p(1) = 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 2$$

$$3 = 1+2+0+0+0, \ so \ p(1)p(2)p(0)p(0)p(0) = 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 = 2$$

$$3 = 0+2+1+0+0, \ so \ p(0)p(2)p(1)p(0)p(0) = 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 = 2$$

$$3 = 0+2+0+1+0, \ so \ p(0)p(2)p(0)p(1)p(0) = 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 = 2$$

$$3 = 0+2+0+0+1, \ so \ p(0)p(2)p(0)p(0)p(1) = 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 = 2$$

$$3 = 1+0+2+0+0, \ so \ p(1)p(0)p(2)p(0)p(0) = 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 = 2$$

$$3 = 0+1+2+0+0, \ so \ p(0)p(1)p(2)p(0)p(0) = 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 = 2$$

$$3 = 0+0+2+1+0, \ so \ p(0)p(0)p(2)p(1)p(0) = 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 = 2$$

$$3 = 0+0+2+0+1, \ so \ p(0)p(0)p(2)p(0)p(1) = 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 = 2$$

$$3 = 1+0+0+2+0, \ so \ p(1)p(0)p(0)p(2)p(0) = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 1 = 2$$

$$3 = 0+1+0+2+0, \ so \ p(0)p(1)p(0)p(2)p(0) = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 1 = 2$$

$$3 = 0+0+1+2+0, \ so \ p(0)p(0)p(1)p(2)p(0) = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 1 = 2$$

$$3 = 0+0+0+2+1, \ so \ p(0)p(0)p(0)p(2)p(1) = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 1 = 2$$

$$3 = 1+0+0+0+2, \ so \ p(1)p(0)p(0)p(0)p(2) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 = 2$$

$$3 = 0+1+0+0+2, \ so \ p(0)p(1)p(0)p(0)p(2) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 = 2$$

$$3 = 0+0+1+0+2, \ so \ p(0)p(0)p(1)p(0)p(2) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 = 2$$

$$3 = 0+0+0+1+2, \ so \ p(0)p(0)p(0)p(1)p(2) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 = 2$$

$$3 = 1+1+1+0+0, \ so \ p(1)p(1)p(1)p(0)p(0) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

$$3 = 1+1+0+1+0, \ so \ p(1)p(1)p(0)p(1)p(0) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

$$3 = 1+1+0+0+1, \ so \ p(1)p(1)p(0)p(0)p(1) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

$$3 = 1+0+1+1+0, \ so \ p(1)p(0)p(1)p(1)p(0) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

$3 = 1 + 0 + 1 + 0 + 1,$ *so* $p(1)p(0)p(1)p(0)p(1) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$

$3 = 1 + 0 + 0 + 1 + 1,$ *so* $p(1)p(0)p(0)p(1)p(1) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$

$3 = 0 + 1 + 1 + 1 + 0,$ *so* $p(0)p(1)p(1)p(1)p(0) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$

$3 = 0 + 0 + 1 + 1 + 1,$ *so* $p(0)p(0)p(1)p(1)p(1) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$

$3 = 0 + 1 + 0 + 1 + 1,$ *so* $p(0)p(1)p(0)p(1)p(1) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$

$3 = 0 + 1 + 1 + 0 + 1,$ *so* $p(0)p(1)p(1)p(0)p(1) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$

and $(5 \cdot 3) + (20 \cdot 2) + (10 \cdot 1) = 65.$

Next, we will consider $GwrT_n$ and see that an analogous theorem will hold for determining the number of conjugacy classes.

8 Conjugacy Classes In $GwrT_n$

8.1 Conjugation Examples

Example 8.1.1 *Consider the following element in $GwrT_4$,*

$$\begin{tikzcd}
1 \arrow[dr, bend left, "g_4"] & 2 \arrow[l, "g_1"'] \\
3 \arrow[u, "g_2"] & 4 \arrow[ul, bend left, "g_3"]
\end{tikzcd}$$

which is represented by the matrix, $\begin{pmatrix} 0 & g_1 & g_2 & g_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_4 & 0 & 0 & 0 \end{pmatrix}$.

The core of the element is $\{1, 4\}$. We will reduce the labels from the core outward via conjugation.

(1) Conjugate by, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_3 \end{pmatrix}$*:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_3 \end{pmatrix} \begin{pmatrix} 0 & g_1 & g_2 & g_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_3^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & g_1 & g_2 & g_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_3g_4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_3^{-1} \end{pmatrix} = \begin{pmatrix} 0 & g_1 & g_2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_3g_4 & 0 & 0 & 0 \end{pmatrix}$$

(2) Conjugate by, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & g_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$*:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & g_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & g_1 & g_2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_3g_4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & g_2^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & g_1 & g_2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_3g_4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & g_2^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & g_1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_3g_4 & 0 & 0 & 0 \end{pmatrix}$$

(3) Conjugate by, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$*:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & g_1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_3g_4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_1^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & g_1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_3g_4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_1^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_3g_4 & 0 & 0 & 0 \end{pmatrix}$$

which gives us the element in canonical form,

$$\begin{tikzcd} 1 \arrow[dr, bend left, "1"] & 2 \arrow[l, "1"'] \\ 3 \arrow[u, "1"] & 4 \arrow[ul, bend left, "h"] \end{tikzcd}$$

where $h = g_3g_4$*.*

Notice that all labels have been reduced to 1's, except for one label in the core.

Example 8.1.2 *In $GwrT_4$, consider the element,*

$$\begin{tikzcd}
1 \arrow[dr, "g_3"'] & 2 \arrow[l, "g_1"'] \\
3 \arrow[ur, "g_4"] & 4 \arrow[u, "g_2"']
\end{tikzcd}$$

which is represented by the matrix, $\begin{pmatrix} 0 & g_1 & 0 & 0 \\ 0 & 0 & g_4 & g_2 \\ 0 & 0 & 0 & 0 \\ g_3 & 0 & 0 & 0 \end{pmatrix}$.

The core of the element is $\{1, 2, 4\}$. We will reduce the labels from the core outward via conjugation.

(1) Conjugate by, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$*:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & g_1 & 0 & 0 \\ 0 & 0 & g_4 & g_2 \\ 0 & 0 & 0 & 0 \\ g_3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_1^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & g_1 & 0 & 0 \\ 0 & 0 & g_1g_4 & g_1g_2 \\ 0 & 0 & 0 & 0 \\ g_3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_1^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g_1g_4 & g_1g_2 \\ 0 & 0 & 0 & 0 \\ g_3 & 0 & 0 & 0 \end{pmatrix}$$

(2) Conjugate by, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_1g_2 \end{pmatrix}$*:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_1g_2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g_1g_4 & g_1g_2 \\ 0 & 0 & 0 & 0 \\ g_3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (g_1g_2)^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g_1g_4 & g_1g_2 \\ 0 & 0 & 0 & 0 \\ g_1g_2g_3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (g_1g_2)^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g_1g_4 & 1 \\ 0 & 0 & 0 & 0 \\ g_3 & 0 & 0 & 0 \end{pmatrix}$$

(3) Conjugate by, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & g_1g_4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$*:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & g_1g_4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g_1g_4 & 1 \\ 0 & 0 & 0 & 0 \\ g_1g_2g_3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (g_1g_4)^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g_1g_4 & 1 \\ 0 & 0 & 0 & 0 \\ g_1g_2g_3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (g_1g_4)^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ g_1g_2g_3 & 0 & 0 & 0 \end{pmatrix}$$

which gives us the element in canonical form,

$$\begin{tikzcd} 1 \arrow[dr, "h"] & 2 \arrow[l, "1"'] \\ 3 \arrow[ur, "1"] & 4 \arrow[u, "1"'] \end{tikzcd}$$

where $h = g_1g_2g_3$.

Notice that all labels have been reduced to 1's, except for one label in the core.

Example 8.1.3 *If we work from the core outward, we can reduce the element (in $GwrT_4$),*

$$\begin{tikzcd} 1 \arrow[r, "g_1"] & 2 \arrow[r, "g_2"] & 3 \arrow[r, bend left, "g_4"] & 4 \arrow[l, bend left, "g_3"] \end{tikzcd}$$

to its canonical form,

$$\begin{tikzcd} 1 \arrow[r, "1"] & 2 \arrow[r, "1"] & 3 \arrow[r, bend left, "1"] & 4 \arrow[l, bend left, "g_4 g_3"] \end{tikzcd}$$

Example 8.1.4 *If we work from the core outward, we can reduce the element (in $GwrT_4$),*

$$\begin{tikzcd} 1 \arrow[d, "1"] & 2 \arrow[d, "1"] \\ 3 \arrow[r, bend left, "g_1"] & 4 \arrow[l, bend left, "g_2"] \end{tikzcd}$$

to its canonical form,

$$\begin{tikzcd} 1 \arrow[d, "1"] & 2 \arrow[d, "1"] \\ 3 \arrow[r, bend left, "1"] & 4 \arrow[l, bend left, "g_2 g_1"] \end{tikzcd}$$

8.2 The Two Lemmas

Let $(f, \sigma) \in GwrT_n$, where $\sigma \in T_n$. Let $X_0 = \hat{n}$ and $X_i = \sigma^i(X_0)$. Then $X_0 \supset X_1 \supset \cdots \supset X_k = X_{k+1}$. We define, $X = X_k$, to be the core of σ. Let $\widetilde{\sigma} = \sigma|_X \in S_X$ and $\widetilde{f} = f|_X$. Then $(\widetilde{f}, \widetilde{\sigma}) \in GwrS_X$.

We will prove that each element of $GwrT_n$ is conjugate to an essentially unique standard element. This is an element with all labels being ones, except for at most one label per cycle in the core. This is suggested by the examples in the previous section. We begin with the following lemmas:

Lemma 8.2.1 *Let* $(f, \sigma) \in GwrT_n$*. Then,* $(f, \sigma) \sim (g, \sigma)$*, for some* g*, with* $f = g$ *on* X_{i+1} *and* $g = 1$ *on* $X_i - X_{i+1}$*.*

Proof:

Let

$$h(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha \in X_i - X_{i+1} \\ 1 & \text{if } \alpha \notin X_i - X_{i+1} \end{cases}$$

Let $(h, 1)(f, \sigma)(h, 1)^{-1} = (g, \sigma)$. Then, $((h \circ \sigma)f, \sigma)(h^{-1} \circ 1, 1) = (g, \sigma)$. Then, $([((h \circ \sigma)f) \circ 1](h^{-1} \circ 1), \sigma) = (g, \sigma)$.

Now, $g(i) = [((h \circ \sigma)f) \circ 1](h^{-1} \circ 1)(i) = [((h \circ \sigma)f) \circ 1](i)(h^{-1} \circ 1)(i)$ $= [(h \circ \sigma)f](i)h^{-1}(i) = (h \circ \sigma)(i)f(i)h^{-1}(i) = h(j)f(i)h^{-1}(i)$, because $1_{T_n}(i) = i$ and $\sigma(i) = j$.

For $i \notin X_i - X_{i+1}$, we have $h(i) = 1$, so $g(i) = 1(j)f(i)1(i) = f(i)$. Therefore, $f(i) = g(i)$ for $i \notin X_i - X_{i+1}$, or $f|_{X_{i+1}} = g|_{X_{i+1}}$.

For $i \in X_i - X_{i+1}$, we have $h(i) = f(i)$, so $g(i) = f(j)f(i)f^{-1}(i) = f(j)$, where $i \neq j$.

Recall, that $GwrT_n = G^{\hat{n}} \times T_n$, where $G^{\hat{n}} \cong G_1 \times G_2 \times \cdots \times G_n$ and $G_i = \{f \in G^{\hat{n}} | f(j) = 1, i \neq j\}$. Since $f \in G^{\hat{n}}$ and $i \neq j$, we must have $f(j) = 1$ and thus, $g(i) = 1$. So, $g = 1$, for $i \in X_i - X_{i+1}$. □

Lemma 8.2.2 *Let $(f, \sigma) \in GwrT_n$. Then, $(f, \sigma) \sim (g, \sigma)$, for some g with $g = f$ on X, $g = 1$ on $\hat{n} - X$.*

Proof:

Suppose that $(f, \sigma) \sim (g, \sigma)$, with $g = f$ on X and $g = 1$ on $X_j - X$, for some j. This is trivially true for $j = k$, with $g = f$. By the previous lemma, there exists h with $g = h$ on X_j and $h = 1$ on $X_{j-1} - X_j$.

So, we have $(f, \sigma) \sim (g, \sigma) \sim (h, \sigma)$, where $h = 1$ on $X_{j-1} - X$ and $h = f$ on X. By reverse induction, the above property holds for all values $j \leq k$. Thus, $(f, \sigma) \sim (g, \sigma)$, for some g with $g = f$ on X and $g = 1$ on $X_0 - X = \hat{n} - X$. $\square$

By using the two lemmas and section 7.1, we can give a generalized cycle decomposition for elements in $GwrT_n$. For each piece, we are able to reduce to at most one label in the core. We can have a color assigned to each such generalized cycle. This is what is referred to as the standard (or canonical) form of an element in $GwrT_n$. Every element in $GwrT_n$ is conjugate to an element in standard form.

Example 8.2.3 *In $GwrT_{10}$, the element,*

$$\begin{tikzcd}
1 \arrow[r, "g_1"] & 2 \arrow[d, "g_2"] & 5 \arrow[r, bend left, "g_5"] & 6 \arrow[l, bend left, "g_6"] & 7 \arrow[loop left, "g_7"] & 8 \arrow[d, bend left, "g_8"] & 9 \arrow[l, "g_9"] \\
4 \arrow[u, "g_4"] & 3 \arrow[ul, "g_3"] & & & & 10 \arrow[u, bend left, "g_{10}"] &
\end{tikzcd}$$

is conjugate to the following element in standard form,

$$\begin{tikzcd}
1 \arrow[r, "1"] & 2 \arrow[d, "g_3g_2g_1"] & 5 \arrow[r, bend left, "g_6g_5"] & 6 \arrow[l, bend left, "1"] & 7 \arrow[loop left, "g_7"] & 8 \arrow[d, bend left, "g_{10}g_8"] & 9 \arrow[l, "1"] \\
4 \arrow[u, "1"] & 3 \arrow[ul, "1"] & & & & 10 \arrow[u, bend left, "1"] &
\end{tikzcd}$$

Example 8.2.4 *In $GwrT_n$, the element,*

$$\begin{tikzcd}
1 \arrow[r, "g_1"] & 2 \arrow[r, "g_2"] & \; - - - - - - - \arrow[r] & n-1 \arrow[r, bend left, "g_{n-1}"] & n \arrow[l, bend left, "g_n"]
\end{tikzcd}$$

is conjugate to the following element in standard form,

$$\begin{tikzcd}
1 \arrow[r, "1"] & 2 \arrow[r, "1"] & \; - - - - - - - \arrow[r] & n-1 \arrow[r, bend left, "1"] & n \arrow[l, bend left, "g_ng_{n-1}"]
\end{tikzcd}$$

8.3 The Conjugacy Class Formula For $GwrT_n$

The fact that we can make any element in $GwrT_n$ conjugate to an element in standard form, characterizes the conjugacy classes of $GwrT_n$ by their generalized cycle structures (or "types"). There are as many conjugacy classes as there are "types". The following theorem, which gives a formula for determining the number of conjugacy classes in $GwrT_n$, is analogous to the formula for the number of conjugacy classes in $GwrS_n$. (Theorem 7.3.1)

Theorem 8.3.1 *If $t \in \mathbb{N}$ is equal to the number of conjugacy classes of G and if $q(m)$ denotes the number of conjugacy classes of T_m, then the number of conjugacy classes of $GwrT_n$ is $\sum_{(n)} q(n_1)q(n_2)\cdots q(n_t)$, where the sum is taken over all t-tuples $(n_1, n_2, ..., n_t)$ over $\mathbb{N}_0$, such that $\sum n_i = n$.*

Example 8.3.2 *Consider $\mathbb{Z}_2wrT_2$. The conjugacy classes are*

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}\right\}$$

So, we have seven conjugacy classes in $\mathbb{Z}_2wrT_2$.

We will arrive at the same conclusion using the previous theorem. Notice that we can partition $n = 2$ in three ways, as $0 + 2$, $1 + 1$, and $2 + 0$.

For $2 = 2 + 0$, $q(2)q(0) = 3$.

For $2 = 1 + 1$, $q(1)q(1) = 1$.

For $2 = 0 + 2$, $q(0)q(2) = 3$.

This is because T_2 has three conjugacy classes and T_1 has one conjugacy class.

So, $\sum q(n_1) \cdots q(n_t) = 3 + 1 + 3 = 7$. *Again, we see that the number conjugacy classes in* $\mathbb{Z}_2 wr T_2$ *is seven.*

Example 8.3.3 *Consider* $\mathbb{Z}_2 wr T_3$. *Notice that we can partition* $n = 3$ *in four ways, as* $0+3$, $2+1$, $1+2$ *and* $3+0$.

For $3 = 3+0$, $q(3)q(0) = 7$.

For $3 = 2+1$, $q(2)q(1) = 3$.

For $3 = 1+2$, $q(1)q(2) = 3$.

For $3 = 0+3$, $q(0)q(3) = 7$.

This is because T_3 *has seven conjugacy classes,* T_2 *has three conjugacy classes, and* T_1 *has one conjugacy class.*

So, $\sum q(n_1) \cdots q(n_t) = 7 + 3 + 3 + 7 = 20$. *So, we see that the number conjugacy classes in* $\mathbb{Z}_2 wr T_3$ *is twenty.*

Example 8.3.4 *Consider $\mathbb{Z}_2 wr T_4$. Notice that we can partition $n = 4$ in five ways, as $0+4$, $3+1$, $1+3$, $2+2$ and $4+0$.*

For $4 = 4+0$, $q(4)q(0) = 19$.

For $4 = 3+1$, $q(3)q(1) = 7$.

For $4 = 1+3$, $q(1)q(3) = 7$.

For $4 = 2+2$, $q(2)q(2) = 9$.

For $4 = 0+4$, $q(0)q(4) = 19$.

This is because T_4 has nineteen conjugacy classes, T_3 has seven conjugacy classes, T_2 has three conjugacy classes, and T_1 has one conjugacy class.

So, $\sum q(n_1)\cdots q(n_t) = 19+7+7+9+19 = 61$. So, we see that the number conjugacy classes in $\mathbb{Z}_2 wr T_4$ is sixty one.

Example 8.3.5 *If G has 3 conjugacy classes, then $GwrT_2$ has 12 conjugacy classes since,*

$2 = 0 + 0 + 2$, *so* $q(0)q(0)q(2) = 1 \cdot 1 \cdot 3 = 3$

$2 = 0 + 2 + 0$, *so* $q(0)q(2)q(0) = 1 \cdot 3 \cdot 1 = 3$

$2 = 2 + 0 + 0$, *so* $q(2)q(0)q(0) = 3 \cdot 1 \cdot 1 = 3$

$2 = 1 + 1 + 0$, *so* $q(1)q(1)q(0) = 1 \cdot 1 \cdot 1 = 1$

$2 = 1 + 0 + 1$, *so* $q(1)q(0)q(1) = 1 \cdot 1 \cdot 1 = 1$

$2 = 0 + 1 + 1$, *so* $q(0)q(1)q(1) = 1 \cdot 1 \cdot 1 = 1$

and 3+3+3+1+1+1=12.

Example 8.3.6 *In general,*

If G has t conjugacy classes, where $t \geq 2$, then $GwrT_2$ has
$3t + \begin{pmatrix} t \\ 2 \end{pmatrix}$ *conjugacy classes.*

If G has t conjugacy classes, where $t \geq 3$, then $GwrT_3$ has
$7t + 3t(t-1) + \begin{pmatrix} t \\ 3 \end{pmatrix}$ *conjugacy classes.*

If G has t conjugacy classes, where $t \geq 4$, then $GwrT_4$ has
$19\begin{pmatrix} t \\ 1 \end{pmatrix} + 7t(t-1) + 7\begin{pmatrix} t \\ 2 \end{pmatrix} + 9\begin{pmatrix} t \\ 3 \end{pmatrix} + \begin{pmatrix} t \\ 4 \end{pmatrix}$ *conjugacy classes.*

It becomes difficult to compute the number of conjugacy classes for $GwrT_n$, when $n > 4$. This is mainly because we would need to know the number of conjugacy classes for T_n, with $n > 4$, in order to perform such computations. At the present time, I am not aware of a nice way to find the number of conjugacy classes for T_5, let alone for T_{100}.

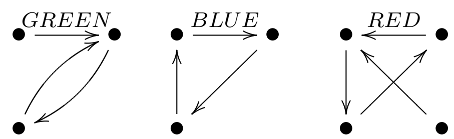
We could draw the graphs for all 3125 elements in T_5 and put all of the graphs that "look the same" together. This would give us the number of conjugacy classes in T_5. Unfortunately, it would be quite demanding to do so and would help to deplete our forests of even more trees. Hopefully, I can find a formula which will determine the number of conjugacy classes in T_n for all n, at some point in the future.

8.4 Colored Directed Graphs

In general, G has t conjugacy classes which we can represent as, $[g_1], [g_2], ..., [g_t]$. We can associate a unique color with each conjugacy class, say $C_1, C_2, ..., C_t$. We can partition n as $n = n_1 + n_2 + \cdots + n_t$ and associate the color C_i with a graph from T_{n_i}. This will produce a standard element in $GwrT_n$ represented by a collection of colored directed graphs.

So, the conjugacy classes of G give the coloring of the directed graphs and the conjugacy classes of T_n are used to determine the number of conjugacy classes in $GwrT_n$.

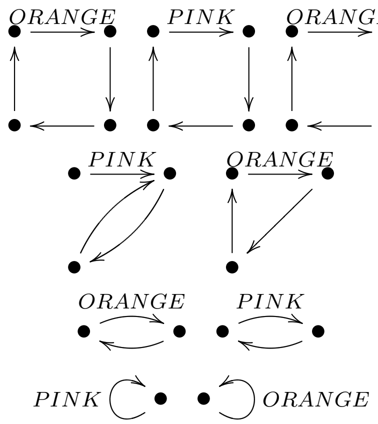
Example 8.4.1 *Consider S_3wrT_{10}. In S_3, the conjugacy class representatives for the three conjugacy classes can be chosen as, (1), (12) and (123). We can associate a color with each conjugacy class in S_3, say green, blue and red.*
So, $[(1)]$ -green, $[(12)]$ -blue, and $[(123)]$ - red
If we partition $10 = 3+3+4$, then a typical standard element would look like,



We get colored directed graphs.

Example 8.4.2 *Consider $\mathbb{Z}_2wrT_{24}$. In $\mathbb{Z}_2$, the conjugacy class representatives for the two conjugacy classes can are, $\{1\}$ and $\{-1\}$. We can associate a color with each conjugacy class in $\mathbb{Z}_2$, say pink and orange.*
So, $[\{1\}]$ -pink, $[\{-1\}]$ -orange

If we partition $24 = 4+4+4+3+3+2+2+1+1$, *then a typical standard element would look like,*



Finally, we note that any idempotent in $GwrT_n$ is conjugate to an element with all labels being ones. Since Carscadden [3] has shown that the number of conjugacy classes of idempotents in T_n is equal to the number of partitions of n, we have the following corollary:

Corollary 8.4.3 *The number of conjugacy classes of idempotents in* $GwrT_n$ *is equal to the number of partitions of* n.

9 Future Endeavors

In the future, I would like to examine the wreath product, $SwrT_n$, where S is a semigroup. It would be very interesting to study the properties of this wreath product. I would like to see what similarities and differences exist between $GwrT_n$ and $SwrT_n$.

It may be possible to place conditions on S in order to make $SwrT_n$ (unit) regular. I also wonder whether my theorem about writing non-idempotents as a product of idempotents will hold for $SwrT_n$. It may only work for special cases. Then, I would like to study conjugacy classes in $SwrT_n$.

There are quite a few things left to research about $GwrT_n$ as well. I would like to study the representations and write out the character tables for $GwrT_n$. I look forward to making many more discoveries in mathematics. This thesis is just the beginning of my study of wreath products.

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