Abstract

DAI, YUE. Game Theoretic Approach to Supply Chain Management. (Under the direction of Dr. Shu-Cherng Fang, Dr. Xiuli Chao, and Dr. Henry L.W. Nuttle.)

This dissertation studies the competitive behavior of firms in supply chain management and revenue management contexts. A game theoretic approach is employed. We analyze capacity allocation and pricing strategies and derive equilibrium solutions for multiple competing firms. We also study channel coordination mechanisms to bring the competing firms together for chain-wide optimality and conduct sensitivity analysis of equilibrium solutions.

First we consider a single-period distribution system with one supplier and two retailers. When a stockout occurs at one retailer the customer may go to the other retailer. The supplier may have infinite or finite capacity. In the latter case, if the total quantity ordered (claimed) by the retailers exceeds the supplier's capacity, an allocation policy is invoked to assign the capacity to the retailers. We show that a unique Nash equilibrium exists when the supplier has infinite capacity. While, when the capacity is finite, a Nash equilibrium exists only under certain conditions. For the finite capacity case, we also use the concept of Stackelberg game to develop optimal strategies for both the leader and the follower. In addition to the decentralized inventory control problem, we study the centralized inventory control problem and obtain the optimal allocation that maximizes the expected profit of the entire supply chain. We also design perfect coordination mechanisms, i.e., a decentralized cost structure resulting in a Nash equilibrium with chain-wide

profits equal to those achieved under a fully centralized system.

As an extension to the capacity allocation models above, we then consider two firms where each firm has a local store and an online store. Customers may shift among these stores upon encountering a stockout. Each firm makes the capacity allocation decision to maximize its profit. We consider two scenarios of a single-product single-period model and derive corresponding existence and stability conditions of a Nash equilibrium. We then conduct sensitivity analysis of the equilibrium solution with respect to price and cost parameters. Finally we extend the results to a multi-period model in which each firm decides its total capacity and allocates this capacity between its local and online stores. A myopic solution is derived and shown to be a Nash equilibrium solution of a corresponding sequential game.

Finally, we consider the pricing strategies of multiple firms providing same service and competing for a common pool of customers in a revenue management context. The demand at each firm depends on the selling prices charged by all firms, each of which satisfies demand up to a given capacity limit. We use game theory to analyze the systems under both deterministic and general stochastic demand. We derive the existence and uniqueness conditions for a Nash equilibrium and calculate the explicit Nash equilibrium point when the demand at each firm is a linear function of price.

GAME THEORETIC APPROACH TO SUPPLY CHAIN MANAGEMENT

by

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To my family

Biography

DAI, YUE, daughter of Dai Yanjue and Yang Xiuqin, was born on July 26, 1974 in Hubei, China. She attended the Huazhong University of Science and Technology, located in Wuhan, China from 1992 to 1999, and received her B.S. degree in Automatic Control in 1996 and M.S. degree in System Engineering in 1999, respectively. In Fall 1999, she enrolled into the Ph.D. program of Industrial Engineering at North Carolina State University, Raleigh, NC. She is married to Liao, Ming, who is graduating from Duke University.

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Chapter 1

Introduction

A supply chain consists of multiple firms (vendors, retailers, distributor, etc.) linked by a flow of materials, information, and funds. The firms in a supply chain may or may not belong to the same corporate entity, which results in decision making conflicts. For instance, retailers who sell same or substitutable products compete for the common pool of customers. This competition can be for a limited supply of product or in the setting of selling prices. The purchasing and pricing decisions of one retailer affect the profit of other competing retailers, which results in a strategic interaction among the decision making of all retailers. In this dissertation, we apply game theory to analyze the supply chain management problems when a supply chain consists of multiple agents with possibly conflicting objectives.

1.1 Capacity allocation problem

In many distribution systems, a single supplier provides products to several retailers. If orders of all retailers are uncertain and capacity is costly, the supplier may not be willing to have capacity that is high enough to cover all orders at any point in time. Therefore when the total order from retailers exceeds the supplier's capacity, the supplier must allocate his/her supply based on some rules. This kind of problem is called a capacity allocation problem. In a supply chain where several retailers sell same product, customers, whose demand cannot be satisfied by one retailer, may go to another retailer. This behavior is termed "market search" (Anupindi and Bassok (1999)), i.e., customers "search the market" before leaving the system.

In the first part of this dissertation, we analyze capacity allocation problems with market search in which retailers compete for a supplier's capacity on one hand and compete for customers on the other hand. Our objective is to provide game theoretic analysis for decentralized system when retailers order simultaneously or sequentially and to suggest a management solution to optimize the performance of the whole supply chain through the coordination of retailers. For this purpose, we analyze two models with market search. In the first model, retailers have general cost structure and make ordering decisions to maximize their own profits. The order strategy of one retailer affects the order strategies of all other retailers, which results in a strategic interaction among the decision making of all retailers, and we use game theory to analyze the optimal order strategies of retailers. We consider the case when all retailers order simultaneously and the case when one retailer has the privilege of ordering first. For both cases, we obtain the necessary and sufficient conditions of the existence of an equilibrium solution.

Generally speaking, in a distribution system, an equilibrium solution reached by all retailers may be sub-optimal in terms of the system-wide profit. Designing an easily acceptable incentive structure that can be implemented in practice is a challenging and important task. Therefore, through the analysis of the second capacity allocation model with market search, we describe how to coordinate the retailers so that the performance of the whole supply chain is optimized. First, we consider decentralized control where retailers are competitive and obtain the equilibrium solution, which is the individual optimum for each retailer. Then we consider centralized control where all retailers are cooperative and obtain the central optimum, which maximizes the profit of whole supply chain. Finally, we design a coordination mechanism to make the decentralized system solution give a chain-wide profit equal to that achieved under a centralized system.

1.2 An extension to the capacity allocation problem

There is a large body of literature on the capacity allocation problem that treats single-period models in which each retailer has a single demand class. In the second part of this dissertation, we examine several extensions of the work. We start with a single-period capacity allocation model in which each firm has multiple demand classes, specifically, each firm has a local store and an online store. Upon encountering a stockout, customers may shift from the local store to the online store of the same firm, or they may switch to the other firm. Each firm makes its capacity decision to maximize its profit. As an extension to the basic problem, we allow each firm make two capacity decisions simultaneously: its total capacity and the allocation between its local and online stores. We consider two versions of this single-period model and derive the corresponding existence and stability conditions of an equilibrium solution. We then extend our results to a multi-period model, deriving a myopic solution for the resulting sequential game. Our model formulation and analysis draw on and contribute to the literature on capacity allocation problems.

1.3 Revenue management

Revenue management is the practice of controlling the availability and/or pricing of accommodations in different booking classes with the goal of maximizing expected revenues or profits. It is widely applied in capacity-constrained service industries such as the airlines, hotels, car rentals, and cruise-liners. Generally speaking, revenue management aims at maximizing the revenue or profit of one firm (e.g., airline). However, in reality, several firms may compete for the common pool of customers. In the third part of this dissertation, we analyze the pricing strategies for multiple firms competing for customers from a common pool (e.g., airlines with identical aircraft and fares). The firms aim to maximize their own profit (subject to the capacity constraint) by setting prices to attract potential customers. Since the pricing strategy of one firm affects the demand streams of other firms, there is a strategic interaction among the firms' pricing decisions; therefore game theory is applied to analyze this problem.

1.4 Organization of the dissertation

This dissertation is organized as follows. Literature surveys on related work are given in Chapter 2. In Chapter 3, we analyze the basic capacity allocation model with market search, in which retailers have general cost structure and the supplier may have infinite or finite capacity. In Chapter 4, we study both the decentralized and centralized systems and design a channel coordination mechanism to optimize the performance of the supply chain.

Extensions to the basic capacity allocation problem are the focus of Chapter 5. We consider single and multi-period capacity allocation models in which each firm has multiple demand classes and makes simultaneous decisions on total capacity and capacity allocation.

In all the above models, the competition is for a limited supply of product. In the real world, it is reasonable to expect that firms will compete not only for inventory, but also on price. In Chapter 6, we consider the pricing strategies of multiple firms providing same service, competing for a common pool of customers in a revenue management context. We analyze both systems in which firms face either a deterministic demand function or a general stochastic demand function and derive the existence and uniqueness conditions for a Nash equilibrium.

Finally, in Chapter 7, we summarize the findings and give directions for other research in the area of game theoretic analysis of supply chain management and revenue management.

Chapter 2

Literature Review

2.1 Game theory

Game theory has been widely used in supply chain management (Mesterton (2000)). The different parties in a supply chain are called *players* in game theory. The profit function of a player is called his/her *payoff function.* A player's *best response* is his/her best strategy given the strategies of all other players. The concept of "Nash equilibrium" is used to represent a solution to a game in which all players make decisions simultaneously. A set of strategies constitutes a Nash equilibrium if each player's strategy maximizes his/her own payoff function given the strategies of other players. Chapter 9 of Heyman and Sobel (1984) talks about a *sequential game*, which is a multi-player decision process in which each player makes a sequence of decisions. Each player's decision sequence influences the evolution of the process and affects the time streams of rewards to all players. A sequential game is said to have a *myopic solution* if its data can be used easily to specify a one-period game such that *ad infinitum* repetition of a Nash equilibrium of the one-period game comprises an equilibrium for the sequential game. Different from games above in which all players make decision simultaneously, Stackelberg game is used to refer to the case when one of the players makes his/her decision before the others do. Specifically, in a Stackelberg game, one player, called the *leader*, makes a decision first and announces it, then the other players, called *followers*, make their decisions.

Parlar (1988) is perhaps the first author to treat an inventory problem using game theory. He examines an extension of the classical newsvendor problems in which two vendors sell substitutable product. In his two-player model, substitution occurs with a certain probability. Parlar proves the existence of a unique Nash equilibrium. Lippman and McCardle (1994) also study an extension of the classical newsvendor problem in which the salvage value of excess inventory and penalty for unmet demand are assumed to be zero. Under this assumption, they examine the equilibrium inventory levels and the rules to reallocate excess demand. They provide conditions under which a Nash equilibrium exists for the case with two or more newsvendors. Mahajan and Ryzin (1999) study a model with n retailers that provide substitutable goods, assuming that the demand process is a stochastic sequence of heterogeneous consumers who choose dynamically from the available goods (or choose not to purchase) based on a utility maximization criterion. Raju and Zhang (1999) analyze the Stackelberg game in which one of the retailers is dominant and capable of unilaterally setting a retail price which will be adopted by all other retailers.

2.2 Channel coordination in supply chain management

A supply chain consists of multiple players (vendors, retailers, distributor, etc.) linked by a flow of materials, information, and funds. Total expected supply chain profit will be maximized if all decisions are made by a single decision maker with access to all available information. This is referred to as the *optimal case* or *first-best case*, and is often associated with *centralized control*. However, in reality, the firms in a supply chain often do not belong to one corporate entity, which results in decision making conflicts. Typically no single decision is in a position to control the entire supply chain, and each player has his/her own incentives and state of information. We refer to this as a *decentralized control* structure. Under decentralized control, each player needs to know how to behave in order to maximize his/her own profit. Under centralized control, a system manager needs to know how to design a mechanism to optimize the performance of the whole supply chain. In order to increase the total profit of a decentralized supply chain and improve the performance of the players, one strategy is to form contracts among players by modifying their payoffs. Some contracts provide a means to bring the total profit resulting from decentralized control to the centralized optimal profit. This is referred to as *channel coordination*.

Cachon and Zipkin (1999) investigate a two-stage (supplier and retailers) serial supply chain with stationary stochastic demand and fixed transportation time over an infinite horizon. They compare the base stock policies chosen under the competitive regime to those selected so to minimize total supply chain costs. Furthermore, they use a linear contract between the supplier and the retailer to modify the payoffs of the players and make the total profit close to the global optimum. In contrast, Klastorin et al. (2002) use price discounts to influence buyers' ordering behavior and coordinate a two-echelon distribution system. The supplier offers a price discount to any retailer who places an order which coincides with the beginning of retailer's cycle. They show that this policy can lead to more efficient supply chains under certain conditions, and present a straightforward method for finding the optimal price discount in the decentralized supply chain. Reviews by Goyal and Gupta (1989) and Weng (1995) show how coordination can be achieved in integrated lot-sizing models with deterministic demand. Their work and that of others provide valuable insights into how and when price discount schemes can be used to achieve jointly optimal outcomes. Lariviere and Porteus (2001) consider a simple supply-chain contract in which a manufacturer sells to a retailer facing a newsvendor problem and the long contract parameter is the wholesale price. They show that the manufacturer's profit and sales quantity increase with market size, but the resulting wholesale price depends on how the market grows.

2.3 Capacity allocation problem with market search

In many distribution systems, a single supplier provides products to several retailers. If retailers' orders are uncertain and capacity is costly, the supplier may not be willing to have capacity that is high enough to cover all orders at any point in time. Therefore when the total order from retailers exceeds the supplier's capacity, the supplier must allocate his/her supply based on some sort of rules. This kind of problem is called the capacity allocation problem. In this dissertation we refer to the quantity of product requested by a retailer as an *order*, while the quantity of product the retailer actually gets is called an *allocation*. Three allocation rules are commonly used (see Cachon and Lariviere (1999)) to allocate the limited capacity to retailers when the total order from the retailers exceeds the supplier's capacity: proportional, linear, and uniform. Under these three allocation rules, if the total order does not exceed the capacity, each retailer receives what he/she ordered. Otherwise, the capacity is totally allocated and each retailer gets at most what he/she ordered. Cachon and Lariviere (1999) compare these three allocation schemes in the same context of one supplier, two retailers, and one period. They state that the supply chain must balance two objectives: (i) increase the supplier's profits by maximizing the supplier's capacity utilization; and (ii) increase the retailer's profits by ensuring that the allocation of supply closely matches the retailer's true needs.

The terminology "market search", to the best of our knowledge, was first introduced by Anupindi and Bassok (1999). A related concept is "substitutable product", refers to a scenario where retailers sell similar products (McGillivray and Silver (1978), Pasternack and Drezner (1991) and Drezner (1995)) and when one product is out of stock, the customers may substitute another in its place. McGillivray and Silver (1978) consider the substitutability of two products in an EOQ context. They investigate the effects of substitutability on inventory control policies and develop a heuristic approach for establishing the order-up-to levels. Netessine and Rudi (2002) analyze the centralized and decentralized inventory controls of a supply chain in which an infinite capacity supplier provides substitutable products to n retailers.

Anupindi and Bassok (1999) appears to be the only paper that considers a capacity allocation problem with market search. They compare two systems: one in which the retailers hold stocks separately and the other in which they cooperate by holding a centralized stock at a single location. They find that whether one system is better than the other depends on the probability of customers' switching to another retailer upon encountering a stockout.

2.4 Revenue management

Revenue management, also called yield management, is the practice of controlling the availability and/or pricing of accommodations in different booking classes with the goal of maximizing expected revenues or profits (Gallego and van Ryzin (1997)). It is widely applied in capacity-constrained service industries such as the airlines, hotels, car rentals, and cruise-liners. Historically, revenue management started as an operations function, focusing only on capacity allocation given exogenous demand. The problem of seat inventory control across multiple fare classes has been studied by many researchers since 1972 (see for example, Littlewood (1972), Bhatia and Parekh (1973), and Ladany and Bedi (1977)). However, more and more researchers and practitioners have come to realize that the pricing decisions cannot be separated from traditional capacity-oriented yield management decisions. Among them, Weatherford (1997) presents a formulation of the simultaneous pricing/allocation decision that assumes normally distributed demands, and models mean demand as a linear function of price. In Gallego and van Ryzin (1994, 1997), demand is formulated as a stochastic point process with an intensity that is a function of the vector of prices for the products and the time at which these prices are offered. Their basic result is that simple heuristics adapted from the solution to the problem in which demand processes are replaced by their expectations are asymptotically optimal for the stochastic control problem. Based on this, the optimal dynamic pricing that maximizes expected revenues over a finite horizon is provided using intensity control theory. Feng and Gallego (2000) address the problem of deciding the optimal timing of price changes within a given menu of allowable, possibly time dependent, price paths each of which is associated with a general Poisson process with Markovian, time dependent, predictable intensities. They develop an efficient algorithm to compute the optimal value functions and the optimal pricing policy. For a comprehensive and up-to-date overview of the revenue management we refer to McGill and van Ryzin (1999) containing a bibliography of over 190 references.

2.5 Pricing problems

There is an extensive literature in the pricing strategies among competitive firms. Bernstein et al. (1999) consider a two-echelon distribution system with price-dependent demand. The deterministic demand of each retailer is dependent on the prices charged by all firms. Alternatively, the price each retailer can charge for his/her product depends on the sales volumes targeted by all of the retailers. The authors characterize perfect coordination mechanisms for the distribution system. They also consider the Stackelberg game when the supplier acts like the leader and chooses the wholesale prices so as to maximize his/her own profits. In another paper, Bernstein and Federgruen (1999) compare the performance of a centralized system, a decentralized system and a Vendor Managed Inventories (VMI) system. The deterministic demand is assumed to be a linear function of price and an EOQ-like cost structure is applied.

In addition to deterministic demands, there are many papers analyzing competitive oligopoly models with stochastic demands. Birge et al. (1998) and van Mieghem and Dada (1999) study models with endogenously determined prices and retailer stocking levels determined in advance of demand realizations. Birge et al. characterize the equilibrium behavior in a two-retailer scenario with constant wholesale prices. They assume that the stochastic demands are uniformly distributed. Van Mieghem and Data (1999) consider a two-stage decision model where retailers make three decisions: capacity investment, production quantity, and price. They analyze and compare the impacts of postponing the capacity decision, the production decision, and the pricing decision. Bernstein and Federgruen (2002) consider a periodic review, infinite horizon model where competitive retailers play a pricing game. In every period, each retailer faces a random demand volume. Two kinds of demand structures are considered: multiplicative and non-multiplicative.

In addition to the papers mentioned above, McGuire and Staelin (1983) consider a supply chain with two identical retailers, with linear demand functions and linear procurement costs, who compete on the basis of price. They assume the two retailers are supplied by two manufacturers who may be vertically integrated with their retailer. Petruzzi and Data (1999) examine an extension of the newsvendor problem in which stocking quantity and selling price are set simultaneously. They provide a comprehensive review that synthesizes the then existing results for the single-period problem and develop a number of additional results. Sudhir (2001) analyzes the competitive pricing behavior in the U.S. auto market. He uses a random utility approach, which is dependent on prices, to estimate competitive interactions among firms in markets with many competing products.

Chapter 3

Capacity Allocation Problem with Market Search: General Cost Structure

3.1 Introduction

We consider a single-period scenario for a distribution system in which a single supplier provides one product to two retailers. When the total quantity of orders from retailers exceeds the supplier's capacity, some rules are followed to allocate the capacity to the two retailers. The quantity of product that a retailer actually receives is called an *allocation*. Note that in general a retailer's allocation is different from his/her order. The customer demand at each retailer is random, and when a demand cannot be met by one retailer because of a stockout, the customer may go to the other retailer. This phenomenon is often referred to as "market search."

Since the two retailers compete for both supply and demand, the ordering decision at one retailer affects the demand of the competing retailer, thereby creating a strategic interaction among the retailers' inventory decisions. In this paper game theory is used to study this problem. We are able to derive necessary and sufficient conditions for the existence of a Nash equilibrium. We also show that, in case the supplier's capacity is unlimited, a unique Nash equilibrium always exists. However, when the supplier's capacity is finite, the Nash equilibrium exists only under certain conditions. For situations where Nash equilibrium does not exist, we consider the problem as a Stackelberg game and find optimal strategies for both the leader and the follower.

As mentioned in Section 2.3, Anupindi and Bassok (1999) appears to be the only paper that considers a capacity allocation problem with market search. The authors compared two systems: one in which the retailers hold stocks separately and the other in which retailers cooperate to centralize stocks at a single location. Our model is different from that used in Anupindi and Bassok (1999) and other papers cited in Section 2.3. In our model, the two retailers hold stocks separately and have a general cost structure. Our goal is to analyze the ordering strategies of the two retailers when they compete with each other for both the supplier's capacity and for customers. In addition, we consider the cases when all retailers make order simultaneously and when one retailer has the privilege of ordering first.

The rest of the chapter is organized as follows: In Section 3.2 we present details of the model. Section 3.3 analyzes retailers' game behavior assuming that the supplier's capacity is infinite, while Section 3.4 considers the situation in which the supplier has finite capacity. We show that only under certain conditions does a Nash equilibrium exist. In case a Nash equilibrium does not exist, we study the retailers' strategies through Stackelberg game in Section 3.5. We conclude the chapter with a discussion in Section 3.6.

3.2 The model

We consider a single-product, single-period distribution system with one supplier and two retailers. The demand during the period at each retailer is random. Customers encountering a stockout at retailer i visit retailer j ($j \neq i$) with probability a_{ij} before leaving the system. The market search structure of this distribution system is represented by a 2×2 matrix A with $a_{11} = a_{22} = 0$ and $0 \leq a_{ij} \leq 1$ for $i \neq j$, referred to as the *market search matrix*. Thus, for retailer i, the total demand consists of customers who visit retailer i first, and customers who switch from retailer j due to a stockout. The former is called *local demand*, the latter *distant demand*, and the sum of these two, i.e., the total demand facing retailer i, *effective demand* at retailer i.

At the beginning of the season, the retailers place orders, and the supplier allocates his/her product to each retailer. At the end of the season, the holding cost or stockout penalty is incurred depending on whether there is unsold stock or a stockout. Therefore for each retailer, the decision problem is a newsvendor-like problem. Since the decision of one retailer affects the total demand at the other retailer, a game arises as the two retailers make their ordering decisions. We assume that each retailer has knowledge of the distributions of the local demands, the market search matrix, and the capacity. Furthermore, we assume that each retailer is a rational player who chooses an order quantity to maximize his/her expected payoff.

In this chapter, we assume that the allocation rules of the supplier are exogenous and known to each retailer. Thus, retailers apply the allocation rules to make their ordering decisions, and they aim on making their allocations most beneficial for them. Therefore, we focus on the analysis of allocation rather than that of order.

We use the following notation throughout the chapter:

For the supplier:

K: the capacity;

c: the unit production cost;

w_i: the unit wholesale price to retailer i, $i = 1, 2$.

For retailer i, $i = 1, 2$:

s_i: the unit selling price of the product to customers;

D_i: a continuous random variable for the stochastic local demand;

$f_i(d_i)$: the probability density function of D_i;

$F_i(d_i)$: the cumulative distribution function of D_i;

a_{ij}: an element of the market search matrix A;

y_i: the allocation, i.e., the inventory of retailer i at the beginning of the season;

R_i: the effective demand at retailer i, i.e.,

$$\begin{aligned} R_i &= D_i + \sum_{j=1}^{2} a_{ji}(D_j - y_j)^+ \\ &= D_i + \sum_{j=1}^{2} a_{ji}(y_j - D_j)^- \end{aligned}$$

where $(x)^+ = \max\{x, 0\}$ and $(x)^- = \max\{-x, 0\}$;

$c_i(\cdot)$: the cost function including holding and stockout penalty;

$\pi_i(y_1, y_2)$: the expected payoff given the allocations to retailers are y_1 and y_2, respectively.

In our analysis, we further assume that $s_i > w_i$, $i = 1, 2$, for obvious reasons, and $c_i(\cdot)$ is a convex function satisfying

$$c_i'(0) \leq 0 \text{ and } c_i'(+\infty) > 0. \tag{3.1}$$

The first inequality implies that the minimizer of $c_i(\cdot)$ is not negative, and the second inequality implies that the cost function $c_i(\cdot)$ is not decreasing on $(-\infty, \infty)$.

3.3 Analysis of the infinite capacity problem

In this section, we study the ordering (claiming) strategy of each retailer assuming the supplier has infinite capacity. Note that in this case each retailer's allocation is equal to his/her order. Given retailer 1's order is y_1, the payoff of retailer 1 includes the cost of purchasing $w_1 y_1$, the cost of holding and stockout penalty $c_1(y_1 - R_1) = c_1(y_1 - (D_1 + a_{21}(D_2 - y_2)^+)$, and the revenue of selling $s_1 \min\{y_1, R_1\} = s_1(y_1 - (y_1 - R_1)^+)$. Thus retailer 1's expected payoff function is

$$\begin{aligned}
\pi_1(y_1, y_2) &= E[s_1 \min\{y_1, R_1\} - c_1(y_1 - R_1) - w_1 y_1] \\
&= E[(s_1 - w_1)y_1 - s_1(y_1 - R_1)^+ - c_1(y_1 - R_1)] \\
&= (s_1 - w_1)y_1 + E[g_1(y_1 - R_1)],
\end{aligned}$$

where

$$g_1(y_1 - R_1) \triangleq -s_1(y_1 - R_1)^+ - c_1(y_1 - R_1).$$

Since the function $(x)^+$ is convex in x and $c_1(\cdot)$ is also convex, clearly g_1 is a concave function and, consequently, $\pi_1(y_1, y_2)$ is concave in y_1 for any given y_2. Since (3.1) holds for the cost function, for any given y_2, the optimal allocation for retailer 1 can be obtained by setting the derivative (with respect to y_1) of $\pi_1(y_1, y_2)$ to zero, i.e.,

$$s_1 - w_1 + E[g_1'(y_1 - D_1 - a_{21}(D_2 - y_2)^+)] = 0. \tag{3.2}$$

Let $y_1 = r_1(y_2)$ be the solution of (3.2), i.e., the optimal allocation for retailer 1 given retailer 2's allocation is y_2. Also let $y_2 = v_1(y_1)$ be the inverse function of $y_1 = r_1(y_2)$. By taking the derivative with respect to y_1 on both sides of (3.2), we obtain

$$E[g_1''(y_1 - D_1 - a_{21}(D_2 - y_2)^+)(1 + a_{21} I(y_2 \le D_2) v_1'(y_1))] = 0,$$

where

$$I(y_2 \le D_2) = \begin{cases} 1, & if\ y_2 \le D_2, \\ 0, & otherwise. \end{cases}$$

Thus

$$v_1'(y_1) = \frac{-1}{a_{21}} \frac{E[g_1''(y_1 - D_1 - a_{21}(D_2 - y_2)^+)]}{E[g_1''(y_1 - D_1 - a_{21}(D_2 - y_2)^+)I(y_2 \le D_2)]}.$$

It follows that

$$v_1'(y_1) < \frac{-1}{a_{21}} \le -1. \tag{3.3}$$

Similarly, for retailer 2, given y_1 is known, we have

$$\begin{aligned} \pi_2(y_1, y_2) &= E[s_2 \min\{y_2, R_2\} - c_2(y_2 - R_2) - w_2 y_2] \\ &= (s_2 - w_2)y_2 + E[g_2(y_2 - R_2)], \end{aligned}$$

where g_2 is defined as

$$g_2(y_2 - R_2) \triangleq -s_2(y_2 - R_2)^+ - c_2(y_2 - R_2).$$

For any given allocation y_1 of retailer 1, the optimal allocation for retailer 2 satisfies

$$s_2 - w_2 + E[g_2'(y_2 - D_2 - a_{12}(D_1 - y_1)^+)] = 0. \tag{3.4}$$

Let $r_2(y_1)$ be the optimal allocation of retailer 2, given retailer 1's allocation is y_1, as defined by (3.4). Taking derivative with respect to y_2 on both sides of (3.4) yields

$$E[g_2''(y_2 - D_2 - a_{12}(D_1 - y_1)^+)(r_2'(y_1) + a_{12}I(y_1 \le D_1))] = 0,$$

where

$$I(y_1 \le D_1) = \begin{cases} 1, & if\ y_1 \le D_1, \\ 0, & otherwise. \end{cases}$$

Therefore, we have

$$r_2'(y_1) = -\frac{a_{12}E[g_2''(y_2 - D_2 - a_{12}(D_1 - y_1)^+)I(y_1 \le D_1)]}{E[g_2''(y_2 - D_2 - a_{12}(D_1 - y_1)^+)]}.$$

It follows that

$$r_2'(y_1) > -a_{12} \ge -1. \tag{3.5}$$

Figure 3.1 shows the optimal allocation functions of $v_1(y_1)$ and $r_2(y_1)$. When $y_2 = 0$, retailer 1 faces a newsvendor problem with stochastic demand $D_1 + a_{21}D_2$. We denote retailer 1's optimal allocation in this case by $\overline{y}_1$. When $y_2 = \infty$, retailer 1 faces a newsvendor problem with stochastic demand D_1. We denote retailer 1's optimal allocation in this case by $\underline{y}_1$. Clearly, retailer 1's optimal allocation y_1 takes $\underline{y}_1$ as a lower bound and $\overline{y}_1$ an upper bound. The curve $v_1(y_1)$ of the optimal allocation function for retailer 1 starts at the point $(\underline{y}_1, \infty)$ and ends at the point $(\overline{y}_1, 0)$. In Figure 3.1, if retailer 2's allocation is y_2', then the corresponding y_1' is the optimal allocation for retailer 1. Similarly, retailer 2's optimal allocation y_2 has an upper bound $\overline{y}_2$ and a lower bound $\underline{y}_2$. The curve $r_2(y_1)$ represents the optimal allocation function for retailer 2. If retailer 1's allocation is y_1'', then the corresponding y_2'' is the optimal allocation for retailer 2.

Theorem 3.1 *When the supplier has infinite capacity, there exists a unique Nash equilibrium (y_1^{Nash}, y_2^{Nash}), which can be obtained by solving the following system of equations*

$$\begin{cases} s_1 - w_1 + Eg_1'(y_1 - D_1 - a_{21}(D_2 - y_2)^+) = 0, \\ s_2 - w_2 + Eg_2'(y_2 - D_2 - a_{12}(D_1 - y_1)^+) = 0. \end{cases} \tag{3.6}$$

Proof Retailer 1's optimal allocation function, labeled as $v_1(y_1)$ in Figure 3.1, is a strictly decreasing curve starting at $(\underline{y}_1, \infty)$ and ending at $(\overline{y}_1, 0)$. Retailer 2's optimal allocation

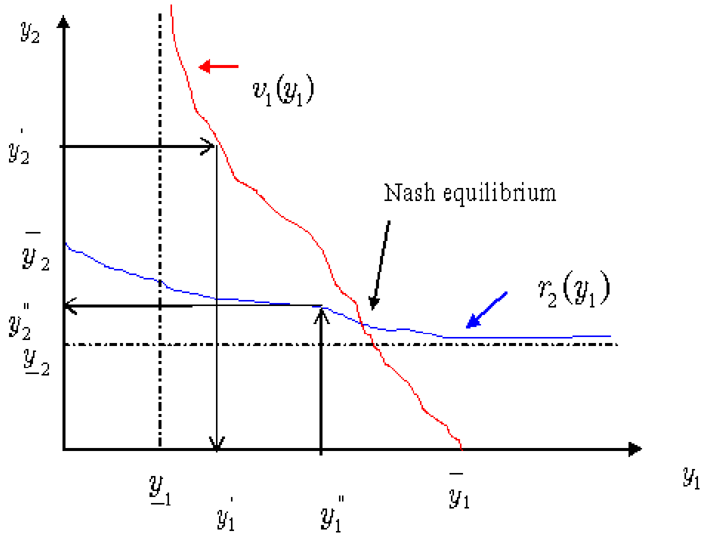

Figure 3.1: Nash equilibrium when the supplier provides infinite capacity.

function, labeled as $r_2(y_1)$ in Figure 3.1, is also a strictly decreasing curve starting at $(0, \overline{y}_2)$ and ending at $(\infty, \underline{y}_2)$. These two optimal allocation functions must intersect, and hence a Nash equilibrium exists. To prove uniqueness, we show that $v_1(y_1)$ and $r_2(y_1)$ have at most one intersection. By (3.3) and (3.5), we have $v_1'(y_1) < -1 < r_2'(y_1) < 0$. Therefore, $r_2(y_1) - v_1(y_1)$ is strictly increasing and $r_2(y_1) - v_1(y_1) = 0$ can have at most one solution. Consequently, these two curves must have a unique intersection that is the Nash equilibrium. ∎

Example 3.2 *Suppose that for each retailer the cost function consists of the holding cost and stockout penalty in the following form:*

$$c_i(y_i - R_i) = h_i(y_i - R_i)^+ + p_i(y_i - R_i)^-$$

where h_i is the unit holding cost of the product and p_i the unit stockout penalty cost. Clearly, $c_i(y_i - R_i)$ is a convex function of $(y_i - R_i)$ satisfying (3.1). In this case, following

(3.6), the Nash equilibrium (y_1^{Nash}, y_2^{Nash}) *can be obtained by solving*

$$\begin{cases} s_1 - w_1 + p_1 - (s_1 + h_1 + p_1)(F_2(y_2)F_1(y_1) + \int_0^{y_1} \int_{y_2}^{\frac{y_1 + a_{21} y_2 - d_1}{a_{21}}} dF_2(d_2) dF_1(d_1)) = 0, \\ s_2 - w_2 + p_2 - (s_2 + h_2 + p_2)(F_2(y_2)F_1(y_1) + \int_0^{y_2} \int_{y_1}^{\frac{y_2 + a_{12} y_1 - d_2}{a_{12}}} dF_1(d_1) dF_2(d_2)) = 0. \end{cases}$$

3.4 Analysis of the capacitated problem

In this section, we study the ordering strategies of retailers assuming the supplier has a finite capacity K. Recall that in this case, based on the three commonly used allocation rules, the allocation of each retailer is less than or equal to his/her order, and we focus on the analysis of allocation. For convenience, in the (y_1, y_2) plane, we call the line formed by $y_1 + y_2 = K$ the *supply line*, and denote the intersection point of $v_1(y_1)$ and $r_2(y_1)$ by (y_1^{Nash}, y_2^{Nash}). Moreover, like in Figure 3.2, if the curve $v_1(y_1)$ intersects the supply line, we denote the intersection point by A; if curve $r_2(y_1)$ intersects the supply line, we denote the intersection point as B. If the curves do not intersect the supply line, as in Figure 3.3, we denote the boundary point $(K, 0)$ by A, and $(0, K)$ by B. A simple result is given below.

Theorem 3.3 *If* $y_1^{Nash} + y_2^{Nash} \leq K$*, then there exists a unique Nash equilibrium allocation.*

Proof At any point on the supply line, at least one retailer wants to apply his/her optimal allocation function and deviate from that point. Thus, there is no Nash equilibrium on the supply line. Below the supply line, as shown in Figure 3.2, the point (y_1^{Nash}, y_2^{Nash})

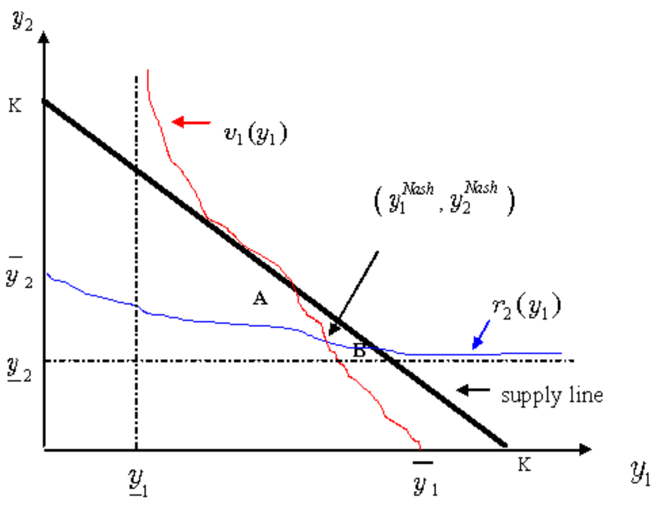

Figure 3.2: One Nash equilibrium.

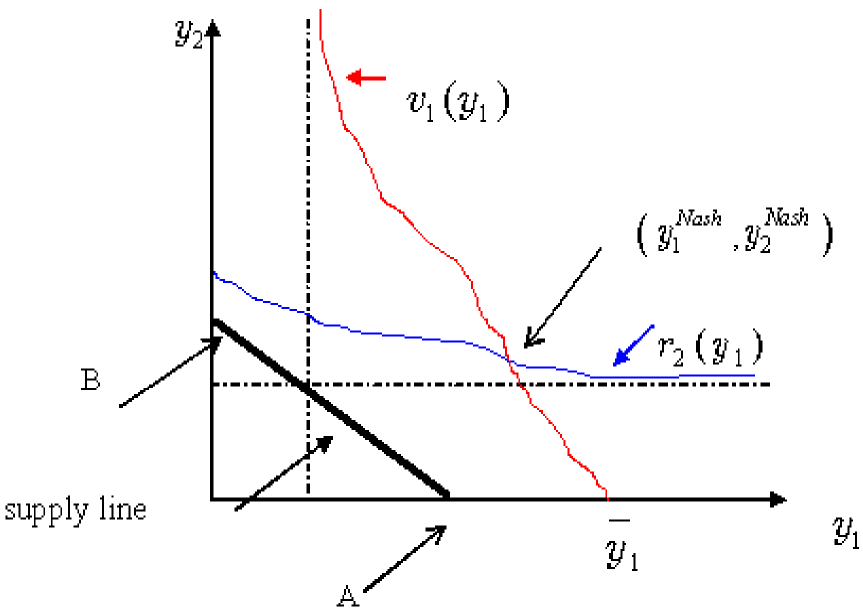

Figure 3.3: The capacity is too limited.

is a feasible allocation, and no player is willing to deviate from this point unilaterally. Therefore, the point (y_1^{Nash}, y_2^{Nash}) is the unique Nash equilibrium allocation. ■

A more interesting case is

$$y_1^{Nash} + y_2^{Nash} > K.$$

In this case, any allocation rule discussed in Section 2.3 can be applied. Two observations are made here:

(i) If retailer 2 makes an order (claim) of $y_2^c < K$, then retailer 1 can either make an order (claim) of $y_1^c \leq K - y_2^c$ or $y_1^c > K - y_2^c$. In the former case, since the total order does not exceed K, each retailer gets what he/she orders. Therefore, the allocation (y_1, y_2) falls on the line segment between the points $(0, y_2^c)$ and $(K - y_2^c, y_2^c)$, as shown in Figure 3.4. In the latter, since the total order quantity exceeds K, the supplier's capacity is depleted with the allocation (y_1, y_2) falling on the supply line. When any of the three commonly used allocation rules applies (see Section 2.3), retailer 2's allocation y_2 cannot be more than y_2^c. Hence retailer 1 can go with the allocation rule to make the allocation fall on the line segment between the points $(K - y_2^c, y_2^c)$ and $(K, 0)$, as shown in Figure 3.4. In other words, for any given y_2^c, retailer 1 can only push the allocation downward on the supply line. Similarly, for any given y_1^c, retailer 2 can only push the allocation upward on the supply line.

(ii) If retailer 2 makes an order (claim) of $y_2^c > K$, then retailer 1 can make the allocation (y_1, y_2) fall at any place on the supply line.

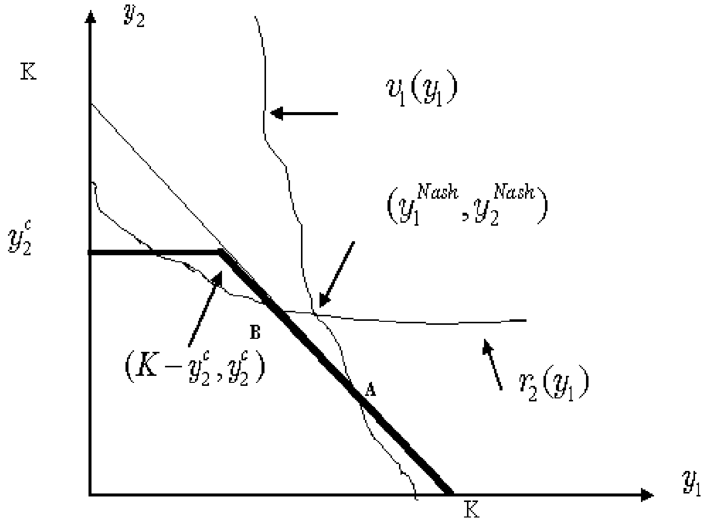

Figure 3.4: Two observations.

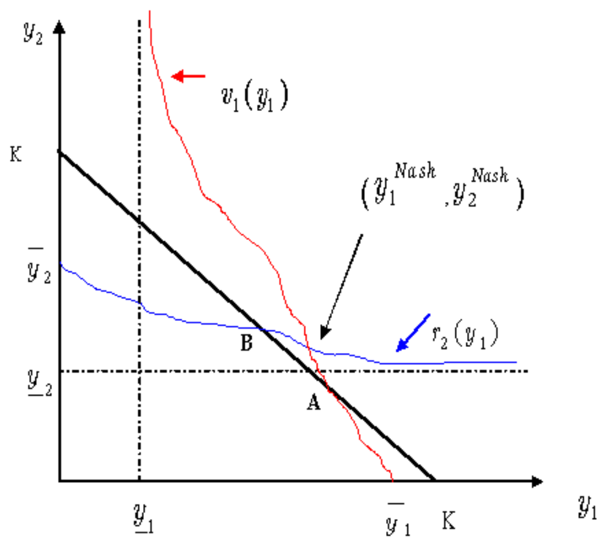

Figure 3.5: $y_1^{Nash} + y_2^{Nash} > K$

Theorem 3.4 *Assume that $y_1^{Nash} + y_2^{Nash} > K$. There exists a Nash equilibrium allocation if and only if there exists $(y_1,\ y_2)$ on the line segment AB such that y_1 is the optimal solution of*

$$\begin{aligned} \max \qquad & \pi_1(y, K-y) \\ s.t. \quad & \max\{K - y_2, 0\} \le y \le K \end{aligned} \tag{3.7}$$

where $\pi_1(y, K-y) = (s_1 - w_1)y + E[g_1(y - D_1 - a_{21}(D_2 - K + y)^+)]$, and y_2 is the optimal solution of

$$\begin{aligned} \max \qquad & \pi_2(K-y, y) \\ s.t. \quad & \max\{K - y_1, 0\} \le y \le K \end{aligned} \tag{3.8}$$

where $\pi_2(K-y, y) = (s_2 - w_2)y + E[g_2(y - D_2 - a_{12}(D_1 - K + y)^+)]$.

Proof We first prove sufficiency. Consider a point $(y_1,\ y_2)$ lying on the line segment AB (see Figure 3.5). If y_1 is an optimal solution of (3.7), then retailer 1 cannot increase his/her payoff by pushing the allocation downward on the supply line. Similarly, if y_2 is an optimal solution of (3.8), then retailer 2 cannot increase his/her payoff by pushing the allocation upward on the supply line. Since each retailer makes the decision that is optimal given the behavior of the other retailer, the point $(y_1,\ y_2)$ must be a Nash equilibrium.

We then prove necessity. Note that the optimal allocation functions $v_1(y_1)$ and $r_2(y_1)$ have no intersection below the supply line. Hence it is impossible to have a Nash equilibrium below the supply line. Consequently, if a Nash equilibrium exists, it must fall on the supply line. Now, if the Nash equilibrium does not fall on the line segment AB, one retailer can use the corresponding optimal allocation function to increase his/her payoff and deviate from the equilibrium. Therefore, if a Nash equilibrium exists, it must fall on the line segment AB. Moreover, for retailer 1 not having any incentive to deviate from the equilibrium, y_1 must be an optimal solution of (3.7). Similarly, for retailer 2 not having any incentive to deviate from the equilibrium, y_2 must be an optimal solution of (3.8). This proves the theorem. ■

Remark 3.5 *If $K = \infty$, then Theorem 3.4 reduces to Theorem 3.1, with (y_1, y_2) being the solution of (3.6).*

3.5 Stackelberg game

As we see from the last section, when the supplier has finite capacity, the Nash equilibrium may not exist. In this situation, we use the framework of Stackelberg game to study the claiming (ordering) strategies of the retailers. In a Stackelberg game, one retailer, called *leader*, makes a claim first, then the other retailer, called *follower*, makes his/her claim. Based on both claims, the supplier allocates the finite capacity according to a given allocation rule to the retailers, and the game is over.

Remember we use y_i^c and y_i ($i = 1, 2$) for retailer i's claim (order) and allocation, respectively. In the following analysis, we first study the follower's claiming (ordering) strategy assuming the leader's strategy is known, and then we study the leader's claiming strategy based on the follower's potential strategy. Recall that retailers apply the allocation rules to make their ordering decisions, and they aim on making their allocations most beneficial for them. We focus on the analysis of optimal allocation.

3.5.1 The follower's strategy

Without loss of generality, we assume that retailer 2 is the leader and retailer 1 follows. Suppose the leader's claim (order) y_2^c is known, the follower makes a claim y_1^c with an attempt to maximize his/her payoff. Recall that point A is the intersection of the curve $v_1(y_1)$ and the supply line. Denote its coordinate by (y_1^A, y_2^A) in the (y_1, y_2) plane. We face two possible cases: (i) $y_2^c \geq y_2^A$ and (ii) $y_2^c < y_2^A$.

Case (i): As shown in Figure 3.6, if $y_2^c \geq y_2^A$, the the follower (retailer 1) has two choices. The first choice is to make a claim such that the allocation falls below the supply line. The second choice is to make a claim such that the allocation falls on the supply line. For the first choice, due to the concavity of the profit function, for any $y_1 \leq K - y_2^c$, we have $\pi_1(K - y_2^c, y_2^c) > \pi_1(y_1, y_2^c)$. In other words, for the follower (retailer 1), no point below the supply line is better than the point $(K - y_2^c, y_2^c)$ that sits on the supply line. (As an example, in Figure 3.6, point D is not so good as point C for retailer 1.) Therefore, we need only to consider the second choice through which the allocation falls on the supply line. Consequently, the follower's decision is actually given by (3.7), i.e.,

$$\begin{aligned} \max \quad & \pi_1(y, K - y) \\ s.t. \quad & \max\{K - y_2^c, 0\} \leq y \leq K. \end{aligned}$$

Denote an optimal solution of the above problem by $y_1^{line\_opt}$. The point *line_opt* with coordinates $(y_1^{line\_opt}, K - y_1^{line\_opt})$ is the follower's optimal allocation downward on the supply line given the leader's claim is known. Thus, once the leader's claim y_2^c is given,

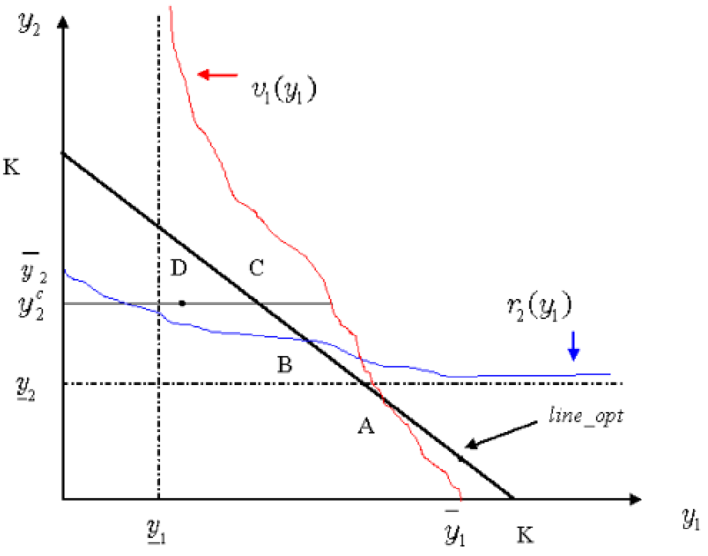

Figure 3.6: $y_2^c \geq y_2^A$

the follower makes a claim $y_1^c > K - y_2^c$ such that the underlying allocation rule results in an allocation of $(y_1^{line\_opt}, K - y_1^{line\_opt})$. By doing so, the follower achieves an optimal payoff $\pi_1(y_1^{line\_opt}, K - y_1^{line\_opt})$.

Case(ii): As shown in Figure 3.7, if $y_2^c < y_2^A$, then the follower (retailer 1) again has the two choices: one resulting in an allocation falling below the supply line while the other resulting in an allocation falling on the line. Recall that function $r_1(y_2)$ is the inverse function of $v_1(y_1)$, i.e., given y_2, $r_1(y_2)$ is the optimal allocation for retailer 1 when the supplier has infinity capacity. Thus, given the leader's claim y_2^c is known, if it goes with the first choice, the follower (retailer 1) may make a claim of $y_1^c = r_1(y_2^c)$ and end up with a payoff of $\pi_1(r_1(y_2^c), y_2^c)$, which is the follower's best payoff when the supplier's capacity is not fully utilized. If it goes with the second choice, as discussed before, retailer 1 will end up with a payoff of $\pi_1(y_1^{line\_opt}, K - y_1^{line\_opt})$. Consider both of these two choices,

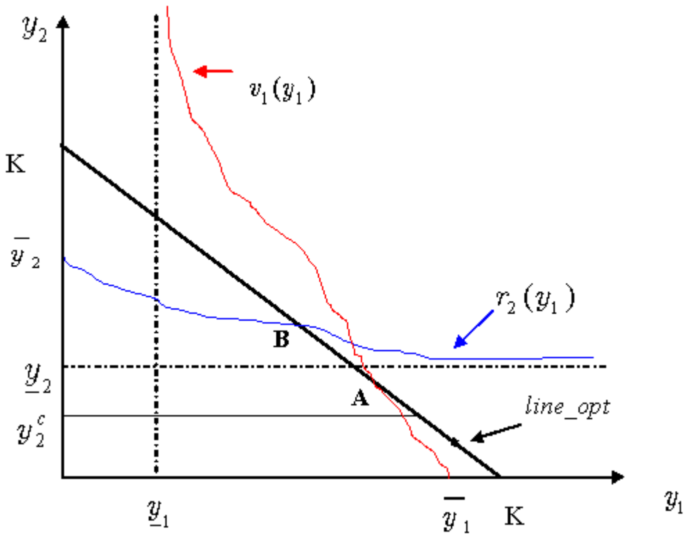

Figure 3.7: $y_2^c < y_2^A$

the follower's best strategy is to compare $\pi_1(r_1(y_2^c), y_2^c)$ with $\pi_1(y_1^{line\_opt}, K - y_1^{line\_opt})$ and choose the one with a higher payoff.

3.5.2 The leader's strategy

Once the follower's claiming strategy is figured out, we can study the leader's strategy in the Stackelberg game. Remember that the follower's (retailer 1's) payoff on the supply line plays an important role. In particular the follower's payoff function becomes $\pi_1(y_1, K - y_1)$ on this line. Denote the point (on the line) that maximizes $\pi_1(y_1, K - y_1)$ by $opt1$ with coordinates $(y_1^{opt1}, K - y_1^{opt1})$, as shown in Figure 3.8. We know from previous analysis that, for any $y_2^c > K - y_1^{opt1}$, the follower can make a claim such that the allocation falls at the point $opt1$.

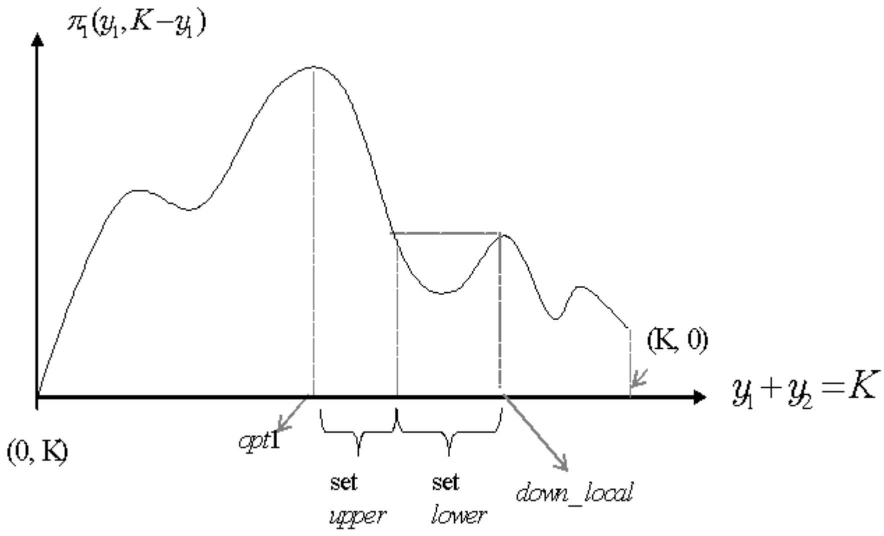

Figure 3.8: Retailer 1's payoff on the supply line.

Focus on the line segment between the points *opt*1 and $(K, 0)$. As shown in Figure 3.8, the follower gets the largest payoff π_1 on the line segment at the point *opt*1.

(i) If there exists a local maximum whose payoff is the second highest on the line segment. We denote this point by *down_local* with coordinates $(y_1^{down\_local}, y_2^{down\_local})$. Since the function $\pi_1(y_1, K-y_1)$ is continuous, some points near the point *opt*1 have higher payoffs than the point *down_local* does. Therefore, we can partition the segment between the points *opt*1 and *down_local* into two sets: set *upper* and set *lower*. Set *upper* consists of all points whose π_1 values are greater than or equal to $\pi_1(y_1^{down\_local}, K - y_1^{down\_local})$, and set *lower* is formed by the rest.

To analyze the leader's strategy, we use Figure 3.9 and Table 1 to highlight retailer 2's payoff when his/her claim is above point *down_local*, i.e., $y_2^c \geq K - y_1^{down\_local}$. As shown in the figure, we partition the line segments between the points $(0, K)$ and *down_local*

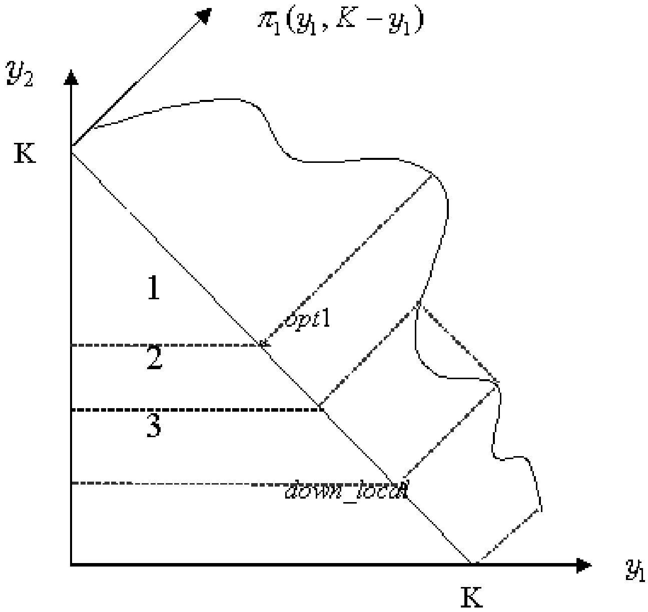

Figure 3.9: Region 1, 2 and 3 when retailer 2's claim is above point *down_local*.

Region	$y_2^c \geq y_2^A$	$y_2^c < y_2^A$
1	$\pi_2(y_1^{opt1}, K - y_1^{opt1})$	$\begin{cases} \pi_2(y_1^{opt1}, K - y_1^{opt1}), \\ \quad if\ \pi_1(y_1^{opt1}, K - y_1^{opt1}) > \pi_1(r_1(y_2^c), y_2^c) \\ \pi_2(r_1(y_2^c), y_2^c), \qquad otherwise \end{cases}$
2	$\pi_2(K - y_2^c, y_2^c)$	$\pi_2(r_1(y_2^c), y_2^c)$
3	$\pi_2(y_1^{down\_local}, K - y_1^{down\_local})$	$\begin{cases} \pi_2(y_1^{down\_local}, K - y_1^{down\_local}), \\ \quad if\ \pi_1(y_1^{down\_local}, K - y_1^{down\_local}) > \pi_1(r_1(y_2^c), y_2^c) \\ \pi_2(r_1(y_2^c), y_2^c), \qquad otherwise \end{cases}$

Table 3.1: Retailer2's payoff when his/her claim is in region 1, 2, and 3

into three regions: all points above $opt1$ are in region 1, all points between $opt1$ and the other end-point of set *upper* are in region 2, and the rest points are in region 3. Also remember that the curve $v_1(y_1)$ intersects the supply line at the point A.

If the leader's (retailer 2's) claim falls in region 1, the follower can increase his/her payoff by pushing the allocation downward on the supply line. From our previous analysis of the follower's strategy, we know that, if $y_2^c \geq y_2^A$, retailer 1 will push the allocation downward on the supply line. Since the point $opt1$ is the best point for retailer 1 in this direction, retailer 1 will push the allocation to this point. In this case, the leader will achieve $\pi_2(y_1^{opt1}, K - y_1^{opt1})$, as shown in the second row of the table. On the other hand, if $y_2^c < y_2^A$, retailer 1 will compare $\pi_1(y_1^{opt1}, K - y_1^{opt1})$ with $\pi_1(r_1(y_2^c), y_2^c)$. Therefore, if $\pi_1(y_1^{opt1}, K - y_1^{opt1})$ is bigger, the leader will get $\pi_2(y_1^{opt1}, K - y_1^{opt1})$; otherwise, $\pi_2(r_1(y_2^c), y_2^c)$. This completes the second row of the table. Similar reasoning process leads to the leader's optimal strategy and his/her associated payoff as shown on the third and fourth rows of the table for regions 2 and 3, respectively.

Note that the above process may be repeated to cover the region between *down_local* and $(K, 0)$ by taking the point *down_local* as a new $opt1$ and finding a local maximum with the second highest π_1, a new *down_local*, on the remaining region.

(ii) If there is no local maximum with the second highest π_1, i.e., the point *down_local* does not exist, then π_1 must be nonincreasing on the line segment between the points $opt1$ and $(K, 0)$. In this case, retailer 1 cannot increase his/her payoff by pushing the allocation downward on the supply line. The analysis is the same as that of region 2 in

Figure 3.9.

3.6 Concluding remarks

In this chapter, we apply game theory to study the optimal ordering strategies of two retailers in a distribution system who compete for both supply capacity and for customers. When stockout occurs at one retailer, a portion of customers will switch to the other retailer. We focus on a single-period inventory model and derive necessary and sufficient conditions for the existence of a unique Nash equilibrium solution. In case the Nash equilibrium does not exist, we use the framework of Stackelberg game to analyze optimal strategies for both the leader and the follower.

We conjecture that in the Stackelberg game with one of those commonly used allocation rules, it is more beneficial to be the follower. Being the follower, we can actually show that retailer 1's payoff is guaranteed to be no less than the payoffs on the curve segment between points A and $(\overline{y}_1, 0)$ in Figure 3.6 and Figure 3.7. However, there is no such guarantee for the leader whose ordering strategy heavily depends on the follower's payoffs on the supply line. In addition, the follower can always make an extreme claim to get all capacity K by knowing y_2^c and the commonly used allocation rules. Since the follower has more information than the leader, he/she can take advantage of this situation. Then, what is the incentive to be the leader in this scenario?

The "one-supplier two-retailer" model studied in this chapter can be extended to a "one-supplier n-retailer" model. The following lemma given by Nikaido and Isora (1955) may be used to find existence conditions for a Nash equilibrium.

Lemma 3.6 *If each player's payoff function is continuous in all variables and concave in his/her own decision variable, the game has at least one Nash equilibrium which is determined by setting the first partial derivative of each player's payoff function with respect to his/her own decision variable to be zero.*

As to uniqueness conditions for the Nash equilibrium, we can apply Theorem 3 in Chapter 6 of Moulin's book (Moulin (1986)).

Chapter 4

Capacity Allocation Problem with Market Search: Channel Coordination

4.1 Introduction

In this chapter, we consider another capacity allocation model. As in the previous chapter, customers do market search. A major difference lies in the cost and revenue structures. In the previous chapter, the holding and stockout penalty costs of each retailer are represented by a convex function, while in this chapter more complicated holding and stockout penalty structures are used. As for the revenue structure, in the previous chapter, a retailer sells the product at the same price to all customers no matter where they appear first. In this chapter, market search occurs with a different revenue level with the product being sold at different prices for customers taking a retailer as the first choice and customers shifting from the other retailer.

In addition, compared to the papers cited in Section 2.3, our model has more general revenue structure, and our goal is to improve the performance of the whole supply chain including the supplier and the retailers. In this chapter, we consider centralized control as well as decentralized control. By taking the whole supply chain as a centralized system, we are able to find the optimal allocations that maximize the total profit of whole supply chain. Based on this analysis, we apply the concept of channel coordination to create independent decisions for each retailer that collectively optimize the performance of the supply chain.

The rest of the chapter is organized as follows: In Section 4.2 we present a detailed description of the model. Section 4.3 analyzes the game behavior of the retailers in the decentralized control, while Section 4.4 considers a centralized model. In Section 4.5 we apply channel coordination to optimize the performance of the whole supply chain. Conclusions and future research directions are given in Section 4.6.

4.2 The model

We use the following notation throughout the chapter:

For the supplier:

K: the capacity of the supplier;

c: the production cost of the product;

w_i: the wholesale price to retailer i, $i = 1, 2$;

For retailer i, $i = 1, 2$:

s_i: the selling price for local demand;

t_i: the selling price for distant demand;

h_i: the holding cost of the product left at the end of the season;

p_i: the stockout penalty cost;

D_i: a continuous random variable, denoting the stochastic local demand;

$f_i(d_i)$: the probability density function of D_i;

$F_i(d_i)$: the cumulative distribution function of D_i, denote 1-$F_i(d_i)$ as $\overline{F}_i(d_i)$;

a_{ij}: an element of market search matrix, $0 \leq a_{ij} \leq 1$, when $j \neq i$; $a_{ii} = 0$.

y_i: the allocation, i.e., the inventory retailer i at the beginning of the season.

R_i: the effective demand which is defined as:

$$\begin{aligned} R_i &= D_i + \sum_{j=1}^{2} a_{ji}(D_j - y_j)^+ \\ &= D_i + \sum_{j=1}^{2} a_{ji}(y_j - D_j)^-. \end{aligned}$$

$\pi_i(y_1, y_2)$: the expected payoff given retailers' allocations y_1 and y_2.

Following Pasternack and Drezner (1991), we make the assumptions as follows:

(A4.1) $s_i > w_i, i = 1, 2;$

(A4.2) $t_i > w_i, i = 1, 2;$

(A4.3) $s_i - p_j \geq t_i - p_i, i, j = 1, 2, i \neq j;$

(A4.4) $s_i - h_j \geq t_j - h_i, i, j = 1, 2, i \neq j.$

Assumptions (A4.1) and (A4.2) are obvious. Assumptions (A4.3) and (A4.4), as observed in Pasternack and Drezner (1991), are based on the phenomenon that the retailers first satisfy their local demands, then use remaining inventory, if any, for distant demand. The rationale behind assumption (A4.3) is as follows: Taking the whole supply chain as a centralized system, suppose that retailer i has only one unit left and faces a local demand and a distant demand at the same time. In the interest of maximizing the payoff of whole supply chain, retailer i should satisfy his/her local demand first. The rationale behind assumption (A4.4) is also based on the consideration of whole supply chain: Suppose both retailers have one unit left, and a local demand for retailer i appears. This demand should be satisfied by retailer i as a local demand, instead of by retailer j ($j \neq i$) as a distant demand. Assumptions (A4.3) and (A4.4) make sure that in centralized control (see Section 4.4), to maximize the total profit of the whole supply chain, the retailers also satisfy their local demands first, like what they do in the decentralized control. When we cannot distinguish the local versus distant demands, i.e., $s_i = t_i$, under assumptions (A4.3) and (A4.4), it becomes a system with multiple identical retailers, which is a special case of our model.

4.3 Decentralized control

In this section, we study the ordering strategies of both retailers when they act to maximize their own profits. First we assume the supplier has infinite capacity. Note that in this case each retailer's allocation is equal to his/her order. Given retailer 1's order is y_1, the payoff of retailer 1 includes:

(i) Purchase cost: $w_1 y_1$;

(ii) Holding and penalty cost: $[h_1(y_1 - R_1)^+ + p_1(y_1 - D_1)^-]$, where $R_1 = D_1 + a_{21}(D_2 - y_2)^+$;

(iii) Selling revenue: $s_1 \min\{y_1, D_1\} + t_1 \min\{(y_1 - D_1)^+, a_{21}(y_2 - D_2)^-\}$, where $(x)^- = \max\{-x, 0\}$.

Thus retailer 1's expected payoff function is:

$$\begin{aligned}
\pi_1(y_1, y_2) &= E[s_1 \min\{y_1, D_1\} - h_1(y_1 - R_1)^+ - p_1(y_1 - D_1)^- - w_1 y_1 \\
&\quad + t_1 \min\{(y_1 - D_1)^+, a_{21}(y_2 - D_2)^-\}] \\
&= E[s_1 y_1 - w_1 y_1 + p_1(y_1 - D_1) - h_1(y_1 - D_1 - a_{21}(y_2 - D_2)^-)^+ \\
&\quad - (s_1 + p_1 - t_1)(y_1 - D_1)^+ - t_1((y_1 - D_1)^+ - a_{21}(y_2 - D_2)^-)^+].
\end{aligned}$$

Lemma 4.1 *For given y_2, $\pi_1(y_1, y_2)$ is concave in y_1.*

Proof If function $g(x)$ is convex in x, and function $f(\cdot)$ is convex and nondecreasing, then $f(g(x))$ is convex in x (Zipkin (2000)). We know that $(\cdot)^+$ is a convex and nondecreasing function and $(y_1-D_1)^+-a_{21}(y_2-D_2)^-$ is convex in y_1. Therefore $t_1((y_1-D_1)^+-a_{21}(y_2-D_2)^-)^+$ is convex in y_1, implying that $-t_1((y_1-D_1)^+-a_{21}(y_2-D_2)^-)^+$ is concave in y_1. From assumption (A4.3), we know $s_1+p_1>t_1$, so $-(s_1+p_1-t_1)(y_1-D_1)^+$ is concave in y_1. The sum of several concave functions is still concave, thus $\pi_1(y_1, y_2)$ is concave in y_1. ■

The partial derivative of $\pi_1(y_1, y_2)$ with respect to y_1 is

$$\begin{aligned}
&\frac{\partial \pi_1(y_1, y_2)}{\partial y_1} \\
&= -h_1 \int_0^{y_1} \int_{y_2}^{\frac{y_1+a_{21}y_2-d_1}{a_{21}}} dF_2(d_2) dF_1(d_1) \\
&- h_1 \int_0^{y_1} \int_0^{y_2} dF_2(d_2) dF_1(d_1) + (s_1+p_1)\overline{F}_1(y_1) \\
&+ t_1 \int_0^{y_1} \int_{\frac{y_1+a_{21}y_2-d_1}{a_{21}}}^{\frac{y_1+a_{21}y_2}{a_{21}}} dF_2(d_2) dF_1(d_1) \\
&+ t_1 \int_0^{y_1} \int_{\frac{y_1+a_{21}y_2}{a_{21}}}^{\infty} dF_2(d_2) dF_1(d_1) - w_1.
\end{aligned} \tag{4.1}$$

It is not difficult to prove that $\frac{\partial \pi_1(y_1,y_2)}{\partial y_1}|_{y_1=0} > 0$ and $\frac{\partial \pi_1(y_1,y_2)}{\partial y_1}|_{y_1=+\infty} < 0$. By Lemma 4.1, for any given allocation to retailer 2, y_2, the optimal allocation for retailer 1 can be obtained by solving

$$\frac{\partial \pi_1(y_1, y_2)}{\partial y_1} = 0. \tag{4.2}$$

Let $y_1 = r_1(y_2)$ be the solution of (4.2), i.e., $r_1(y_2)$ denotes the optimal allocation of retailer 1 given y_2. Let $y_2 = v_1(y_1)$ be the inverse function of $y_1 = r_1(y_2)$. Notice that

$y_2 = v_1(y_1)$ and $y_1 = r_1(y_2)$ represent the same curve in the (y_1, y_2) plane. We refer to this curve as $v_1(y_1)$.

Lemma 4.2 *Curve $v_1(y_1)$ is strictly decreasing in the (y_1, y_2) plane.*

Proof Let $v_1'(y_1)$ denote the slope of the curve $v_1(y_1)$. By implicit differentiation, we have

$$v_1'(y_1) = -\frac{\partial^2 \pi_1(y_1, y_2)}{\partial y_1^2} \Big/ \frac{\partial^2 \pi_1(y_1, y_2)}{\partial y_1 \partial y_2}.$$

After some calculations, we obtain

$$\begin{aligned}\frac{\partial^2 \pi_1(y_1, y_2)}{\partial y_1^2} &= -(s_1 + p_1 - t_1) f_1(y_1) - (h_1 + t_1) f_1(y_1) F_2(y_2) \\ &\quad - \frac{h_1 + t_1}{a_{21}} \int_0^{y_1} f_2(\frac{y_1 - d_1}{a_{21}} + y_2) dF_1(d_1),\end{aligned} \tag{4.3}$$

$$\frac{\partial^2 \pi_1(y_1, y_2)}{\partial y_1 \partial y_2} = -(h_1 + t_1) \int_0^{y_1} f_2(\frac{y_1 - d_1}{a_{21}} + y_2) dF_1(d_1). \tag{4.4}$$

From (4.3) and (4.4), we obtain

$$v_1'(y_1) = -\frac{1}{a_{21}} - \frac{(s_1 + p_1 - t_1) f_1(y_1) + (h_1 + t_1) f_1(y_1) F_2(y_2)}{(h_1 + t_1) \int_0^{y_1} f_2(\frac{y_1 - d_1}{a_{21}} + y_2) dF_1(d_1)} < 0,$$

which proves Lemma 4.2. Notice that $v_1'(y_1) < -1$. ■

For retailer 2, we have similar result: The partial derivative of $\pi_2(y_1, y_2)$ with respect to y_2 is

$$\frac{\partial \pi_2(y_1,y_2)}{\partial y_2} = -h_2 \int_{y_1}^{\frac{y_1 a_{12}+y_2}{a_{12}}} \int_0^{a_{12}y_1+y_2-a_{12}d_1} dF_2(d_2)dF_1(d_1) \tag{4.5}$$
$$- h_2 \int_0^{y_1}\int_0^{y_2} dF_2(d_2)dF_1(d_1) + (s_2+p_2)\overline{F}_2(y_2)$$
$$+ t_2 \int_{y_1}^{\frac{y_1 a_{12}+y_2}{a_{12}}} \int_{a_{12}y_1+y_2-a_{12}d_1}^{y_2} dF_2(d_2)dF_1(d_1)$$
$$+ t_2 \int_{\frac{y_1 a_{12}+y_2}{a_{12}}}^{\infty} \int_0^{y_2} dF_2(d_2)dF_1(d_1) - w_2.$$

Thus for any given allocation to retailer 1, y_1, the optimal allocation for retailer 2 can be obtained by solving

$$\frac{\partial \pi_2(y_1,y_2)}{\partial y_2} = 0. \tag{4.6}$$

Let $r_2(y_1)$ denote the optimal allocation of retailer 2 for any given y_1, as defined by (4.6).

Lemma 4.3 *Curve $r_2(y_1)$ is strictly decreasing in the (y_1, y_2) plane.*

Proof Denoting the slope of curve $r_2(y_1)$ as $r_2'(y_1)$, we have

$$r_2'(y_1) = -\frac{\partial^2 \pi_2(y_1,y_2)}{\partial y_1 \partial y_2} / \frac{\partial^2 \pi_2(y_1,y_2)}{\partial y_2 \partial y_2},$$

$$\frac{\partial^2 \pi_2(y_1,y_2)}{\partial y_1 \partial y_2} = -a_{12}(h_2+t_2) \int_{y_1}^{\frac{y_1 a_{12}+y_2}{a_{12}}} f_2(y_1 a_{12} + y_2 - a_{12}d_1)dF_1(d_1), \tag{4.7}$$

$$\frac{\partial^2 \pi_2(y_1,y_2)}{\partial y_2 \partial y_2} = -(h_2+t_2) \int_{y_1}^{\frac{y_1 a_{12}+y_2}{a_{12}}} f_2(y_1 a_{12} + y_2 - a_{12}d_1)dF_1(d_1) \tag{4.8}$$
$$- (s_2+p_2-t_2)f_2(y_2) - (h_2+t_2)f_2(y_2)F_1(y_1).$$

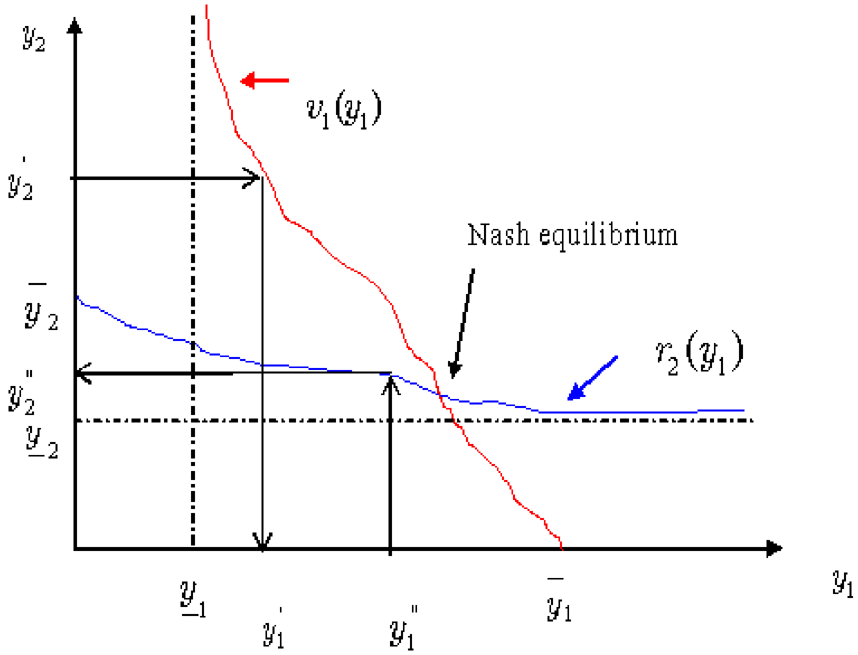

Figure 4.1: Nash equilibrium when the supplier has infinite capacity.

From (4.7) and (4.8), we obtain

$$
\begin{aligned}
& r_2'(y_1) \\
&= -\frac{a_{12}(h_2+t_2)\int_{y_1}^{\frac{y_1 a_{12}+y_2}{a_{12}}} f_2(y_1 a_{12}+y_2-a_{12}d_1)dF_1(d_1)}{(s_2+p_2-t_2)f_2(y_2)+(h_2+t_2)f_2(y_2)F_1(y_1)+(h_2+t_2)\int_{y_1}^{\frac{y_1 a_{12}+y_2}{a_{12}}} f_2(y_1 a_{12}+y_2-a_{12}d_1)} \\
&< 0,
\end{aligned}
$$

which proves Lemma 4.3. Notice that $-1 < r_2'(y_1) < 0$. ∎

Figure 4.1 shows the optimal allocation functions, $v_1(y_1)$ and $r_2(y_1)$. Comparing Figure 3.1 in Chapter 3 and Figure 4.1 in this chapter, we can see that they look similar, except that functions $v_1(y_1)$ and $r_2(y_1)$ have different representations.

Theorem 4.4 *When each retailer gets exactly what he/she orders, there exists a unique*

Nash equilibrium (y_1^{Nash}, y_2^{Nash}), *which can be obtained by solving*

$$\begin{cases} \frac{\partial \pi_1(y_1,y_2)}{\partial y_1} = 0, \\ \frac{\partial \pi_2(y_1,y_2)}{\partial y_2} = 0. \end{cases} \tag{4.9}$$

Proof Retailer 1's optimal allocation function, labelled as $v_1(y_1)$ in Figure 4.1, is a strictly decreasing curve starting at $(\underline{y}_1, \infty)$ and ending at $(\overline{y}_1, 0)$. Retailer 2's optimal allocation function, labelled as $r_2(y_1)$ in Figure 4.1, is also a strictly decreasing curve starting at $(0, \overline{y}_2)$ and ending at $(\infty, \underline{y}_2)$. Therefore, these two optimal allocation functions must intersect, which proves the existence of a Nash equilibrium. Since $v_1'(y_1) < r_2'(y_1)$, $v_1(y_1)$ and $r_2(y_1)$ have a unique intersection, which is the unique Nash equilibrium. ■

Following the structure of the previous chapter, we could analyze the cases when the supplier has limited supply and when the Stackelberg game is applied. In fact, all results are the same, so we do not repeat them here.

4.4 Centralized control

In centralized control, the total supply chain, including the supplier and two retailers, is owned by one company, and all decisions are made on behalf of the company by a single decision maker to maximize the expected profit of the supply chain. In centralized control, there is no stockout penalty for demand satisfied within the supply chain. Only when customers leave the system unsatisfied is the stockout penalty incurred. This includes customers who visit only one retailer and leave unsatisfied and customers who visit both retailers and leave unsatisfied. In the latter case, the amount of the penalty incurred in the supply chain is assumed to be the stockout penalty cost of the retailer visited first by the customer. Now recall the following two assumptions of Section 4.2:

(A4.3) $s_i - p_j \geq t_i - p_i, i, j = 1, 2, i \neq j$;

(A4.4) $s_i - h_j \geq t_j - h_i, i, j = 1, 2, i \neq j$.

With these assumptions, we try to answer the following question: Assuming the whole supply chain is in centralized control, in order to maximize the total profit, what are optimal allocations for both retailers?

As before, let y_i denote the allocation for retailer i. In centralized control, to maximize the total profit, the supplier should only provide what is needed by the two retailers. Namely K should equal $y_1 + y_2$. The cost and revenue includes:

(i) Total production cost: $cK = c(y_1 + y_2)$;

(ii) Total holding cost:

$$h_1(y_1 - D_1 - a_{21}(y_2 - D_2)^-)^+ + h_2(y_2 - D_2 - a_{12}(y_1 - D_1)^-)^+;$$

(iii) Total penalty cost:

$$\begin{aligned}
&p_1(y_1 - D_1)^-(1 - a_{12}) + p_1(a_{12}(y_1 - D_1)^- - (y_2 - D_2)^+)^+ \\
&+ p_2(y_2 - D_2)^-(1 - a_{21}) + p_2(a_{21}(y_2 - D_2)^- - (y_1 - D_1)^+)^+;
\end{aligned}$$

(iv) Total selling revenue:

$$s_1 \min\{y_1, D_1\} + t_1 \min\{(y_1 - D_1)^+, a_{21}(y_2 - D_2)^-\}$$
$$+ s_2 \min\{y_2, D_2\} + t_2 \min\{(y_2 - D_2)^+, a_{12}(y_1 - D_1)^-\}.$$

With some algebraic manipulation, the expected payoff function of the system, denoted $\pi(y_1, y_2)$, is:

$$\begin{aligned}
\pi(y_1, y_2) = E[&s_1 \min\{y_1, D_1\} + t_1 \min\{(y_1 - D_1)^+, a_{21}(y_2 - D_2)^-\} \\
&+ s_2 \min\{y_2, D_2\} + t_2 \min\{(y_2 - D_2)^+, a_{12}(y_1 - D_1)^-\} \\
&- cy_1 - h_1(y_1 - D_1 - a_{21}(y_2 - D_2)^-)^+ \\
&- cy_2 - h_2(y_2 - D_2 - a_{12}(y_1 - D_1)^-)^+ \\
&- p_1(y_1 - D_1)^-(1 - a_{12}) - p_1(a_{12}(y_1 - D_1)^- - (y_2 - D_2)^+)^+ \\
&- p_2(y_2 - D_2)^-(1 - a_{21}) - p_2(a_{21}(y_2 - D_2)^- - (y_1 - D_1)^+)^+].
\end{aligned}$$

Theorem 4.5 $\pi(y_1, y_2)$ *is a concave function.*

Proof After taking some calculations, we have

$$
\begin{aligned}
\frac{\partial \pi(y_1,y_2)}{\partial y_1} &= -c + (s_1+p_1)\overline{F}_1(y_1) - h_1 F_1(y_1)F_2(y_2) \\
&- h_1 \int_0^{y_1} \int_{y_2}^{\frac{y_1+a_{21}y_2-d_1}{a_{21}}} dF_2(d_2)dF_1(d_1) \\
&- a_{12}(h_2+p_1+t_2) \int_{y_1}^{\frac{y_1a_{12}+y_2}{a_{12}}} \int_0^{y_1a_{12}+y_2-a_{12}d_1} dF_2(d_2)dF_1(d_1) \\
&+ (p_2+t_1) \int_0^{y_1} \int_{\frac{y_1+a_{21}y_2-d_1}{a_{21}}}^{\frac{y_1+a_{21}y_2}{a_{21}}} dF_2(d_2)dF_1(d_1) \\
&+ (p_2+t_1) \int_0^{y_1} \int_{\frac{y_1+a_{21}y_2}{a_{21}}}^{\infty} dF_2(d_2)dF_1(d_1)),
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
\frac{\partial \pi(y_1,y_2)}{\partial y_2} &= -c + (s_2+p_2)\overline{F}_2(y_2) - h_2 F_1(y_1)F_2(y_2) \\
&- h_2 \int_{y_1}^{\frac{y_1a_{12}+y_2}{a_{12}}} \int_0^{y_1a_{12}+y_2-a_{12}d_1} dF_2(d_2)dF_1(d_1) \\
&- a_{21}(h_1+p_2+t_1) \int_0^{y_1} \int_{y_2}^{\frac{y_1+a_{21}y_2-d_1}{a_{21}}} dF_2(d_2)dF_1(d_1) \\
&+ (t_2+p_1) \int_{y_1}^{\frac{y_1a_{12}+y_2}{a_{12}}} \int_{y_1a_{12}+y_2-a_{12}d_1}^{y_2} dF_2(d_2)dF_1(d_1) \\
&+ (t_2+p_1) \int_{\frac{y_1a_{12}+y_2}{a_{12}}}^{\infty} \int_0^{y_2} dF_2(d_2)dF_1(d_1).
\end{aligned} \tag{4.11}
$$

Denoting $(h_1+t_1+p_2)\int_0^{y_1} f_2(\frac{y_1+a_{21}y_2-d_1}{a_{21}})dF_1(d_1)$ and $(h_2+t_2+p_1)\int_{y_1}^{\frac{y_1a_{12}+y_2}{a_{12}}} f_2(y_1a_{12}+y_2-a_{12}d_1)dF_1(d_1)$ as nonnegative values $V1$ and $V2$ respectively, we have

$$\begin{aligned}
\frac{\partial^2\pi(y_1,y_2)}{\partial y_1\partial y_2} &= -V1-a_{12}V2,\\
\frac{\partial^2\pi(y_1,y_2)}{\partial y_1\partial y_1} &= -\frac{1}{a_{21}}V1-(a_{12})^2V2\\
&\quad -f_1(y_1)(s_1+p_1-p_2-t_1)\overline{F}_2(y_2)\\
&\quad -f_1(y_1)(h_1+s_1+p_1-a_{12}(h_2+p_1+t_2))F_2(y_2),\\
\frac{\partial^2\pi(y_1,y_2)}{\partial y_2\partial y_2} &= -a_{21}V1-V2\\
&\quad -f_2(y_2)(s_2+p_2-p_1-t_2)\overline{F}_1(y_1)\\
&\quad -f_2(y_2)(h_2+s_2+p_2-a_{21}(h_1+p_2+t_1))F_1(y_1).
\end{aligned}$$

From assumptions (A4.3) and (A4.4), we know that the third and fourth terms of $\frac{\partial^2\pi(y_1,y_2)}{\partial y_1\partial y_1}$ and $\frac{\partial^2\pi(y_1,y_2)}{\partial y_2\partial y_2}$ are both non-positive. In other words, they can be denoted as "$-V3$" and "$-V4$" respectively, where $V3$ and $V4$ are nonnegative values. Therefore, we obtain

$$\begin{aligned}
&\frac{\partial^2\pi(y_1,y_2)}{\partial y_1\partial y_1}\frac{\partial^2\pi(y_1,y_2)}{\partial y_2\partial y_2}\\
&= (-\frac{1}{a_{21}}V1-(a_{12})^2V2-V3)(-a_{21}V1-V2-V4)\\
&\geq (-\frac{1}{a_{21}}V1-(a_{12})^2V2)(-a_{21}V1-V2)\\
&\geq (-V1-a_{12}V2)^2\\
&= (\frac{\partial^2\pi(y_1,y_2)}{\partial y_1\partial y_2})^2.
\end{aligned}$$

Since $\frac{\partial^2\pi(y_1,y_2)}{\partial y_1\partial y_1}<0$ and $\frac{\partial^2\pi(y_1,y_2)}{\partial y_2\partial y_2}<0$, the Hessian matrix is negative semidefinite. Thus $\pi(y_1,y_2)$ is a concave function. ■

Based on Theorem 4.5, we know that the maximizer of $\pi(y_1, y_2)$, (y_1^*, y_2^*), can be obtained by solving

$$\begin{cases} \frac{\partial \pi(y_1,y_2)}{\partial y_1} = 0, \\ \frac{\partial \pi(y_1,y_2)}{\partial y_2} = 0. \end{cases}$$

Correspondingly, in the optimal case, the capacity of the supplier, K, equals $y_1^* + y_2^*$.

4.5 Channel coordination

In this section, we apply the concept of channel coordination to optimize the performance of the whole supply chain. Generally speaking, channel coordination may be achieved by three steps: First, under decentralized control, apply game theory to determine how the players will behave when they each seek to maximize their own profits, and whether a Nash equilibrium exists. Next, determine the optimal solution under centralized control. Third, if the decentralized and centralized solutions differ, investigate how to modify the players' payoffs so that the new decentralized solution matches the centralized solution. These three steps are commonplace in supply chain inventory management research. We will also follow these steps to coordinate our one-supplier, two-retailer supply chain. The first two steps have been discussed in the previous sections.

Under centralized control in Section 4.4, we consider the whole supply chain as an entity, and the money flow within the system is not involved. Therefore, the optimal solution (y_1^*, y_2^*) is independent with the wholesale prices, w_1 and w_2. Based on the analysis of decentralized control in Section 4.3, we know that the Nash equilibrium does depend on the wholesale prices. Channel coordination can be obtained by determining the wholesale prices, w_1 and w_2, so as to make the optimal solution (y_1^*, y_2^*) a Nash equilibrium. Recall that in centralized system, $K = y_1^* + y_2^*$. In this case, to be a Nash equilibrium in decentralized system, (y_1^*, y_2^*) must satisfy (4.9), i.e.,

$$\begin{cases} \frac{\partial \pi_1(y_1,y_2)}{\partial y_1}|_{y_1=y_1^*,y_2=y_2^*} = 0, \\ \frac{\partial \pi_2(y_1,y_2)}{\partial y_2}|_{y_1=y_1^*,y_2=y_2^*} = 0. \end{cases}$$

Combining (4.1), (4.5), (4.10) and (4.11), we find that the wholesale prices that achieve the channel coordination objective are:

$$\begin{aligned} w_1 &= c + a_{12}(h_2 + p_1 + t_2)\int_{y_1^*}^{\frac{y_1^* a_{12}+y_2^*}{a_{12}}} \int_0^{y_1^* a_{12}+y_2^*-a_{12}d_1} dF_2(d_2)dF_1(d_1) \qquad (4.12) \\ &- p_2\Big(\int_0^{y_1^*} \int_{\frac{y_1^*+a_{21}y_2^*-d_1}{a_{21}}}^{\frac{y_1^*+a_{21}y_2^*}{a_{21}}} dF_2(d_2)dF_1(d_1) + \int_0^{y_1^*} \int_{\frac{y_1^*+a_{21}y_2^*}{a_{21}}}^{\infty} dF_2(d_2)dF_1(d_1)\Big), \\ w_2 &= c + a_{21}(h_1 + p_2 + t_1)\int_0^{y_1^*} \int_{y_2^*}^{\frac{y_1^*+a_{21}y_2^*-d_1}{a_{21}}} dF_2(d_2)dF_1(d_1) \\ &- p_1\Big(\int_{y_1^*}^{\frac{y_1^* a_{12}+y_2^*}{a_{12}}} \int_{y_1^* a_{12}+y_2^*-a_{12}d_1}^{y_2^*} dF_2(d_2)dF_1(d_1) + \int_{\frac{y_1^* a_{12}+y_2^*}{a_{12}}}^{\infty} \int_0^{y_2} dF_2(d_2)dF_1(d_1)\Big). \end{aligned}$$

We must also consider the satisfaction of the assumptions (A4.1) $s_i > w_i$ and (A4.2) $t_i > w_i, i = 1, 2$. As shown in Figure 4.1, based on the results of the newsvendor model, we know that given w_1, y_1 has a lower bound $\underline{y}_1$ which solves

$$F_1(y_1) = \frac{s_1 - w_1 + p_1}{s_1 + h_1 + p_1},$$

and an upper bound $\overline{y}_1$ which solves

$$(s_1 + p_1 - t_1)F_1(y_1) + (t_1 + h_1)F_1(y_1)F_2(\frac{y_1 - d_1}{a_{21}}) = s_1 - w_1 + p_1.$$

Similarly, y_2 has a lower bound $\underline{y}_2$ and an upper bound $\overline{y}_2$. In Figure 4.2, the feasible region for a Nash equilibrium (y_1^{Nash}, y_2^{Nash}) is the rectangle formed by the points

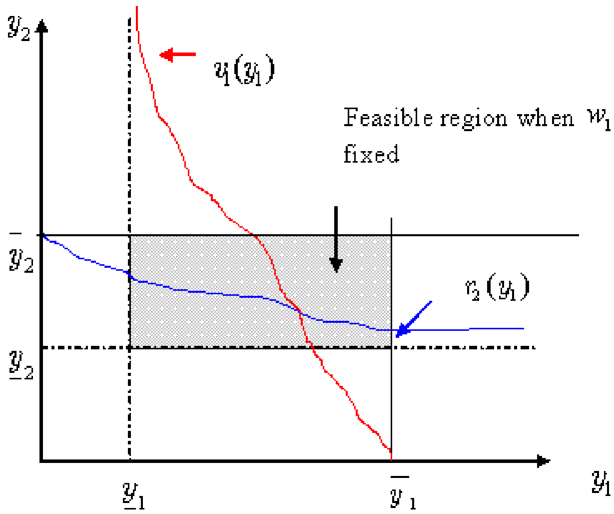

Figure 4.2: Feasible region of the Nash equilibrium when w_1 and w_2 are fixed.

$(\underline{y}_1, \underline{y}_2)$, $(\underline{y}_1, \overline{y}_2)$, $(\overline{y}_1, \overline{y}_2)$, and $(\overline{y}_1, \underline{y}_2)$. We can see that the lower bound $\underline{y}_1$ decreases if w_1 increases and the upper bound $\overline{y}_1$ increases if w_1 decreases. w_1 is required to be less than $min\{s_1, t_1\}$ and greater than 0, so when w_1 is a variable, the minimal lower bound of y_1, denoted $\underline{y}_1^{\min}$, solves

$$F_1(y_1) = \frac{s_1 - min\{s_1, t_1\} + p_1}{s_1 + h_1 + p_1},$$

and the maximal upper bound of y_1, denoted $\overline{y}_1^{\max}$, solves

$$(s_1 + p_1 - t_1)F_1(y_1) + (t_1 + h_1)F_1(y_1)F_2(\frac{y_1 - d_1}{a_{21}}) = s_1 + p_1.$$

Similarly, we can obtain the minimal lower bound of y_2, denoted $\underline{y}_2^{\min}$, and the maximal upper bound of y_2, denoted $\overline{y}_2^{\max}$. Thus, based on the assumptions $s_i > w_i > 0$ and $t_i > w_i > 0, i = 1, 2$, the feasible region of Nash equilibrium (y_1^{Nash}, y_2^{Nash}) is the rectangle

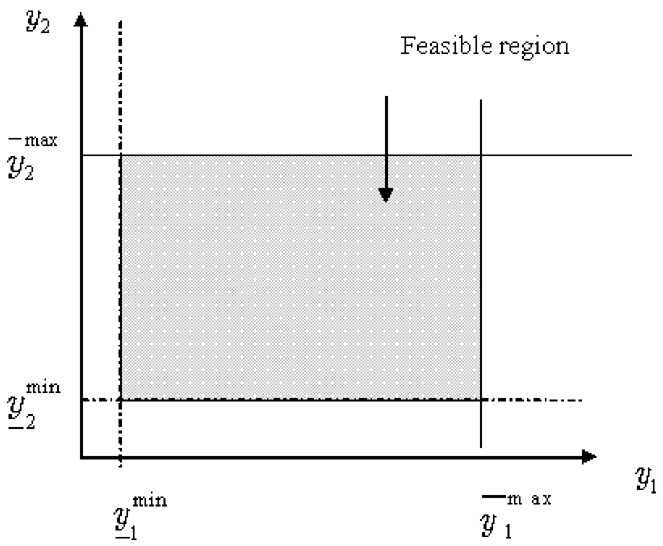

Figure 4.3: Feasible region of a Nash equilibrium when $min\{s_1, t_1\} > w_i > 0, i = 1, 2$.

formed by the points $(\underline{y}_1^{\min}, \underline{y}_2^{\min})$, $(\underline{y}_1^{\min}, \overline{y}_1^{\max})$, $(\overline{y}_1^{\max}, \overline{y}_1^{\max})$, and $(\overline{y}_1^{\max}, \underline{y}_2^{\min})$ (see Figure 4.3). Therefore, if (y_1^*, y_2^*) falls in this region, it is a feasible Nash equilibrium, i.e., the wholesale prices from (4.12) satisfies that $s_i > w_i > 0$ and $t_i > w_i > 0$, $i = 1, 2$; otherwise it is not.

4.6 Concluding remarks

In this chapter, we consider a distribution system with one supplier and two retailers. If the total order quantity from retailers exceeds the capacity of the supplier, an allocation policy has to be used to allocate the limited capacity to each retailer. When stockout occurs at one retailer the unsatisfied demand may switch to the other retailer. In this way, the inventory level at one retailer affects the demand of the competing retailer. A game theoretic approach is used to analyze this problem.

We consider a single period inventory model and study both the decentralized and centralized controls. In the former case, we obtain necessary and sufficient conditions for the existence of a Nash equilibrium; while for the latter case, we derived an optimal allocation that maximizes the expected total profit. By applying the concept of channel coordination, we design perfect coordination mechanisms, i.e., a decentralized cost structure resulting in a Nash equilibrium with chain-wide profits equal to those achieved under a fully centralized system.

There are some potentially interesting extensions. While this chapter focuses on a single-period (newsvendor-like) inventory model, further research on other inventory models, such as the EOQ model of Drezner et al. (1995) and Chand et al. (1994) and the base-stock model of van Ryzin and Mahajan (1998), may lead to new findings. In a multi-period model where backorder is allowed, each retailer makes a sequence of decisions, and the decision of one period affects the decision of all following periods. Sequential game in Heyman and Sobel (1984) may be used to analyze this multi-person decision process.

Chapter 5

Capacity Allocation with Traditional and Internet Channels

5.1 Introduction

Internet, as a relatively inexpensive electronic medium, dramatically reduces the transaction costs and increases information availability. By utilizing the Internet, in-stock items can be made available to more customers, and orders can be placed in real time. Internet-based electronic marketplace has become an integral part of the modern economy. In this study, we consider two competitive firms, each of which has a local store and an online store, namely, its web page. Customers can either visit the local store or order from the web page. We assume that the local store and the online store hold separate stocks. When a stockout occurs at a local store customers may go to the online store belonging to the same firm, or visit the local store of the other firm. However, as commonly observed, when a stockout occurs at an online store, customers usually will not visit the local store belonging to the same firm, but instead they may go to the online store of the other firm. One question facing each firm is how to allocate its finite capacity between its local and online stores to maximize its profit as a whole.

Because customers may shift from one firm to another when stockout occurs, the capacity allocation of one firm affects the decision of its rival, thereby creating a strategic interaction. In this chapter game theory is used for analysis. We first consider a single-product single-period model and assume that the capacity of each firm is given and known. We study two scenarios of this capacity allocation game. Remember that when a stockout occurs at a local store customers may go to the rival's local store or go to the online store belonging to the same firm. In Scenario 1, it is assumed that those customers who have visited both local and online stores of the same firm will leave the system. In Scenario 2, we assume that these customers with unmet demand in one firm may go to the rival's online store before leaving the system. For both scenarios, we present some existence and stability conditions of a Nash equilibrium and conduct sensitivity analysis of the equilibrium solution with respect to price and cost parameters. Based on the analysis, we consider a more general case that each firm has to decide its total capacity and allocate between its local and online stores. We extend the results to a multi-period model and show that a myopic solution is a Nash equilibrium solution for a corresponding multi-period game.

Our study is related to revenue management in the sense that each firm has a finite capacity and decides its allocation to maximize the profit. In contrast to the papers cited in Section 2.4, instead of focusing on maximizing the revenue or profit of a single firm, this chapter applies game theory to analyze the capacity strategies for multiple competitive firms. We propose a single-period model and a multi-period model and derive Nash equilibrium solutions to corresponding games.

This chapter is also related to Netessine and Shumsky (2001), which examines the seat inventory control problem with two fare classes for two competing airlines. Each airline chooses an optimal booking limit for the lower-fare class while taking into account any overflow of passengers from its rival. They show that under certain conditions this "revenue management game" has a Nash equilibrium, and in some special cases, the Nash equilibrium is unique. They also compare the total number of seats allocated to each fare class with, and without, competition. Our model is different from that of Netessine and Shumsky on the following aspects: (i) We consider the possible overflow from one demand class to another within a firm, while Netessine and Shumsky's model does not allow. (ii) In Netessine and Shumsky, all unsatisfied demand at one airline overflow to the rival. In our model, this overflow occurs in probability. (iii) In addition to a single-period model in which each firm allocates its capacity, we extend our results to a multi-period model in which each firm decides its total capacity and allocates this capacity simultaneously.

The rest of this chapter is organized as follows. In Section 5.2 we consider a single-product single-period model and assume that the total capacity level of each firm is given and known. We study two scenarios of this capacity allocation game and derive corresponding existence and stability conditions of a Nash equilibrium. In Section 5.3

we conduct the sensitivity analysis of an equilibrium solution with respect to the price and cost parameters. In Section 5.4, we extend our results to a multi-period model in which each firm makes simultaneous decisions on its total capacity and the allocation of this capacity. A myopic solution is derived for the corresponding sequential game. We conclude this chapter in Section 5.5.

5.2 Single-period model

We consider two firms selling the same product. Each firm has a local store and an online store. To simplify the exposition, we start with a single-period model and assume the total capacity of each firm is given and known. Customers may switch among the stores upon encountering a stockout as Figure 5.1 shows. For each firm, a challenging problem is to allocate its finite capacity between its local and online stores in order to maximize its total profit.

The initial demand at the local store of firm i is referred to as firm i's *local demand*, while the demand at the online store as firm i's *online demand*. Figure 5.1 shows the customer shifting behavior. When a stockout occurs at a local store, customers may go to the online store belonging to the same firm, or visit the local store of its rival firm. However, when a stockout occurs at an online store, customers may only go to the rival's online store. Therefore all stores are faced with a random initial demand as well as demand shifting from other stores. The total demand at a store is referred to as its *effective demand*.

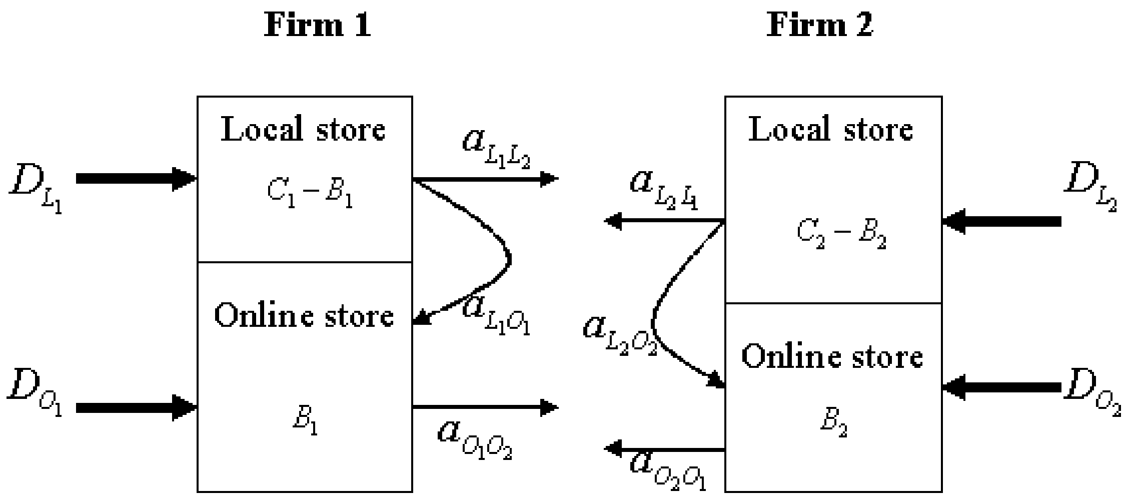

Figure 5.1: Customer shifting

In the sequel, we use the following notation:

s_{L_i}: the selling price at the local store of firm $i = 1, 2$;

s_{O_i}: the selling price at the online store of firm $i = 1, 2$;

h_{L_i}: the holding cost at the local store of firm $i = 1, 2$;

h_{O_i}: the holding cost at the online store of firm $i = 1, 2$;

p_{L_i}: the stockout penalty cost at the local store of firm $i = 1, 2$;

p_{O_i}: the stockout penalty cost at the online store of firm $i = 1, 2$;

D_{L_i}: a continuous random variable, denoting firm i's stochastic local demand, $i = 1, 2$;

D_{O_i}: a continuous random variable, denoting firm i's stochastic online demand, $i = 1, 2$;

$a_{L_iL_j}$: the probability of an unsatisfied local customer at firm i visiting the local store of (the other) firm j, $i, j = 1, 2, j \neq i$;

$a_{L_iO_i}$: the probability of an unsatisfied local customer at firm i visiting its online store, $i = 1, 2$;

$a_{O_iO_j}$: the probability of an unsatisfied online customer at firm i visiting the online store of (the other) firm j, $i, j = 1, 2, j \neq i$;

C_i: the total capacity of firm $i = 1, 2$;

B_i: the capacity of firm i allocated to its online store, $i = 1, 2$;

R_{L_i}: the effective demand at the local store of firm $i = 1, 2$;

R_{O_i}: the effective demand at the online store of firm $i = 1, 2$;

$\pi_i(B_i, B_j)$: the expected payoff function of firm i, $i, j = 1, 2, j \neq i$.

As most, if not all, papers in traditional revenue management, we assume that the local and online demands are exogenous and independent. The decision variable for firm i is the capacity allocated to its online store, i.e., B_i. Meanwhile, we know that R_{L_i}, R_{O_i} and $\pi_i(B_i, B_j)$ depend on $B_i, i, j = 1, 2, j \neq i$. We further assume that $a_{L_iL_j} + a_{L_iO_i} \leq 1, i, j = 1, 2, j \neq i$ for obvious reasons.

To study the interactive allocation strategies of the two firms and show the existence of a Nash equilibrium, we shall make use of the following definition of submodularity and related results introduced by Topkis (1978) and (1979).

Definition 5.1 *A function $f(x_1, x_2)$ is submodular in (x_1, x_2) if $f(x_1^{small}, x_2) - f(x_1^{large}, x_2)$ is nondecreasing in x_2 for all $x_1^{small} \leq x_1^{large}$. Function $f(x_1, x_2)$ is supermodular if $-f(x_1, x_2)$ is submodular. If a function is both supermodular and submodular, it is a valuation.*

Lemma 5.2 *Let function $f(x_1, x_2)$ be twice differentiable. Then $f(x_1, x_2)$ is submodular in (x_1, x_2) if and only if $\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \leq 0$.*

Lemma 5.3 *Function $f(x_1, x_2)$ is submodular in (x_1, x_2) if and only if it is supermodular in $(x_1, -x_2)$.*

Lemma 5.4 *Let $g(B_i)$ be a nondecreasing function in B_i and $m(B_j)$ be a nonincreasing function in B_j. Then the function $min\{g(B_i),\ m(B_j)\}$ is submodular in (B_i, B_j).*

Lemma 5.5 *Let function $g(B_i, B_j)$ be monotone in both B_i and B_j and submodular in (B_i, B_j). Also let $m(\cdot)$ be a nondecreasing concave function. Then the composition function $m(g(B_i, B_j))$ is submodular in (B_i, B_j).*

We consider two scenarios of the single-period model. Remember that when a stockout occurs at a local store customers may go to the rival's local store or go to the online store

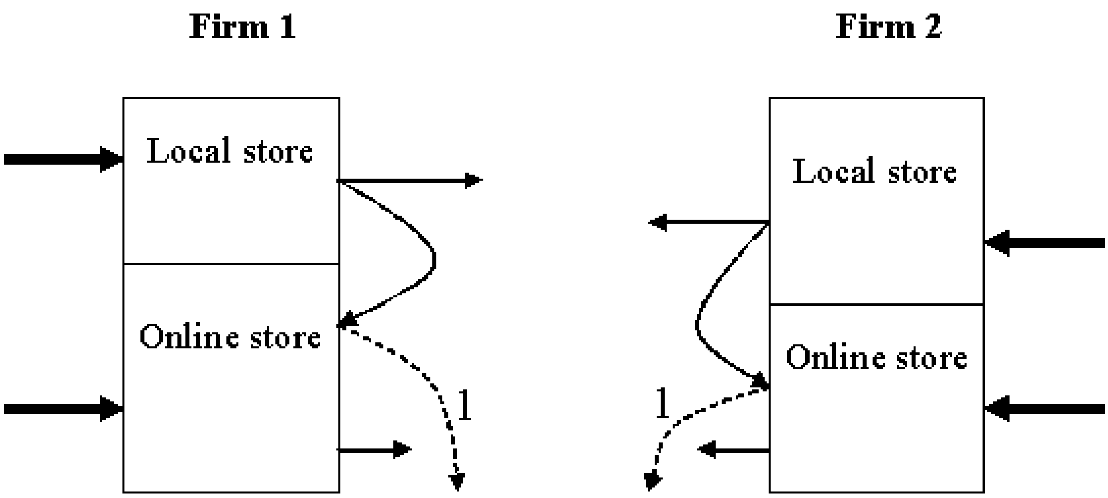

Figure 5.2: Scenario 1

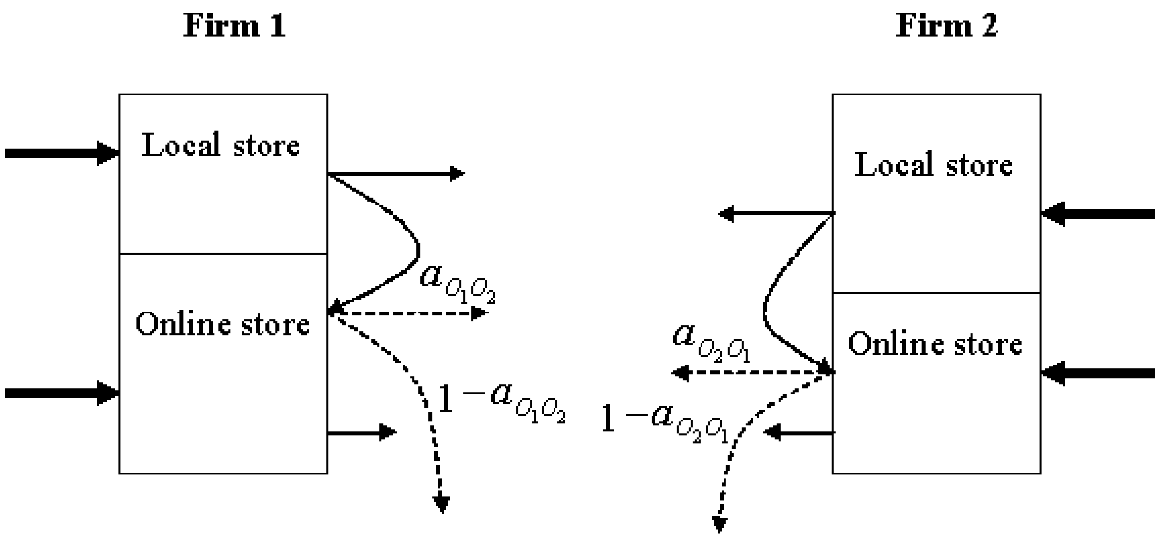

Figure 5.3: Scenario 2

belonging to the same firm. In Scenario 1, it is assumed that those customers who have visited both local and online stores of the same firm will leave the system as Figure 5.2 shows. In Scenario 2, we assume that these customers with unmet demand in one firm will follow the same shifting probabilities as the online demand of this firm. Namely, they may go to the rival's online store before leaving the system as Figure 5.3 shows.

5.2.1 Scenario 1

In Scenario 1, based on the customer shifting structure in Figures 5.1 and 5.2, we can write down the effective demands at the local and online stores of firm i:

$$R_{L_i} = D_{L_i} + a_{L_jL_i}(D_{L_j} - (C_j - B_j))^+,$$

$$R_{O_i} = D_{O_i} + a_{O_jO_i}(D_{O_j} - B_j)^+ + a_{L_iO_i}(D_{L_i} - (C_i - B_i))^+,$$

where $(x)^+ = max\{x, 0\}$.

For firm 1, the expected payoff function is:

$$\pi_1(B_1, B_2) = E[s_{L_1} \min\{R_{L_1}, C_1 - B_1\} - p_{L_1}(R_{L_1} - (C_1 - B_1))^+ - h_{L_1}(R_{L_1} - (C_1 - B_1))^- \tag{5.1}$$

$$+ s_{O_1} \min\{R_{O_1}, B_1\} - p_{O_1}(R_{O_1} - B_1)^+ - h_{O_1}(R_{O_1} - B_1)^-]$$

$$= E[s_{L_1} \min\{R_{L_1}, C_1 - B_1\} - p_{L_1}(R_{L_1} - (C_1 - B_1))$$

$$- (p_{L_1} + h_{L_1})(R_{L_1} - (C_1 - B_1))^-$$

$$+ s_{O_1} \min\{R_{O_1}, B_1\} - (p_{O_1} + h_{O_1})(R_{O_1} - B_1)^+ + h_{O_1}(R_{O_1} - B_1)],$$

where $(x)^- = max\{-x, 0\}$.

Theorem 5.6 *In our two-firm model, each payoff function $\pi_i(B_1, B_2), i = 1, 2,$ is submodular in (B_1, B_2).*

Proof First we prove $\pi_1(B_1, B_2)$ is submodular in (B_1, B_2) by showing that each term in the second equation of (5.1) is submodular in (B_1, B_2). By Lemma 5.4, we know that $\min\{R_{L_1}, C_1 - B_1\}$ is submodular in (B_1, B_2). From Definition 5.1, we know that $(R_{L_1} - (C_1 - B_1))$ is a valuation. By Lemma 5.5, we know $-(R_{L_1} - (C_1 - B_1))^-$ is submodular. To see the submodularity of $\min\{R_{O_1}, B_1\}$, notice that

$$\begin{aligned}\min\{R_{O_1}, B_1\} = &\min\{D_{O_1} + a_{O_2O_1}(D_{O_2} - B_2)^+, B_1 - a_{L_1O_1}(D_{L_1} - (C_1 - B_1))^+\} \\ &+ a_{L_1O_1}(D_{L_1} - (C_1 - B_1))^+,\end{aligned}$$

where $\min\{D_{O_1} + a_{O_2O_1}(D_{O_2} - B_2)^+, B_1 - a_{L_1O_1}(D_{L_1} - (C_1 - B_1))^+\}$ is submodular by Lemma 5.4 and $a_{L_1O_1}(D_{L_1} - (C_1 - B_1))^+$ is a valuation. By Lemma 5.5 we can prove that $-(p_{O_1} + h_{O_1})(R_{O_1} - B_1)^+$ is submodular. The last term in (5.1), $h_{O_1}(R_{O_1} - B_1)$, is a valuation. Therefore, as the sum of several submodular functions, $\pi_1(B_1, B_2)$ is submodular in (B_1, B_2). Similarly, we can prove that $\pi_2(B_1, B_2)$ is submodular in (B_1, B_2). ■

Theorem 3.1 of Topkis (1979) asserts that if the strategy space is a complete lattice, the joint payoff function is upper-semicontinuous, and each player's payoff function is supermodular (submodular), then there exists a pure strategy Nash equilibrium. For example, in a two-person game, if both players' payoffs are supermodular, then each player's best response is increasing in his/her rival's strategy. When the best responses exhibit this monotonicity property, the players' strategies are said to be strategic complements, and the existence of a Nash equilibrium is easy to establish (see Lippman (1994)).

Theorem 5.7 *For Scenario 1 of our two-firm model, there exists a Nash equilibrium* (B_1^{Nash}, B_2^{Nash})*, which can be obtained by solving the following system of equations:*

$$\begin{cases} \frac{\partial \pi_1(B_1,B_2)}{\partial B_1} = 0, \\ \frac{\partial \pi_2(B_1,B_2)}{\partial B_2} = 0. \end{cases}$$

Proof From Theorem 3.1 of Topkis (1979) and the submodularity of payoff functions, we can prove the existence of a Nash equilibrium. ■

Now, we the stability of a Nash equilibrium solution. The concept of stability of a Nash equilibrium was introduced by Moulin (1986) using the concept of Cournot tatonnement. A Cournot tatonnement is a sequence formed by the best responses of all players. For example, in our two-firm game, let $r_i(B_j)$ be the best response function of firm i given the strategy of firm j, B_j, then the Cournot tatonnement is the following sequence

$$(B_1, B_2) \rightarrow (r_1(B_2), r_2(B_1)) \rightarrow (r_1(r_2(B_1)), r_2(r_1(B_2))) \rightarrow \dots$$

A Nash equilibrium is locally stable if the Cournot tatonnement starting within a local area of this Nash equilibrium converges to it.

Lemma 5.8 *(Moulin (1986)) If* $|\frac{\partial^2 \pi_i(B_i,B_j)}{\partial^2 B_i}|_{B_i=B_i^{Nash},B_j=B_j^{Nash}} > |\frac{\partial^2 \pi_i(B_i,B_j)}{\partial B_i \partial B_j}|_{B_i=B_i^{Nash},B_j=B_j^{Nash}}$, $i,j = 1,2, j \neq i$, *then the Nash equilibrium is locally stable.*

Taking derivatives using the Leibnitz formula is complicated. We follow the approach of Rudi (2001) (see also Netessine (2001), Netessine and Rudi (2000), and Netessine and

Shumsky (2001)) to check Moulin's condition. With some simple calculations, we have

$$\begin{aligned}
\frac{\partial \pi_i(B_i,B_j)}{\partial B_i} = \; & h_{L_i} - h_{O_i} - (s_{L_i} + p_{L_i} + h_{L_i}) \Pr(R_{L_i} > C_i - B_i) \\
& + (s_{O_i} + h_{O_i}) a_{L_iO_i} \Pr(D_{L_i} > C_i - B_i) \\
& + (s_{O_i} + p_{O_i} + h_{O_i})(1 - a_{L_iO_i}) \Pr(R_{O_i} > B_i, D_{L_i} > C_i - B_i) \\
& + (s_{O_i} + p_{O_i} + h_{O_i}) \Pr(R_{O_i} > B_i, D_{L_i} \leq C_i - B_i),
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
& \frac{\partial^2 \pi_i(B_i,B_j)}{\partial B_i \partial B_j} \\
& = -(s_{L_i} + p_{L_i} + h_{L_i}) a_{L_jL_i} f_{R_{L_i}|D_{L_j} > C_j - B_j}(C_i - B_i) \Pr(D_{L_j} > C_j - B_j) \\
& \quad - (s_{O_i} + p_{O_i} + h_{O_i})(1 - a_{L_iO_i}) a_{O_jO_i} f_{R_{O_i}|D_{O_j} > B_j, D_{L_i} > C_i - B_i}(B_i) \Pr(D_{O_j} > B_j, D_{L_i} > C_i - B_i) \\
& \quad - (s_{O_i} + p_{O_i} + h_{O_i}) a_{O_jO_i} f_{R_{O_i}|D_{O_j} > B_j, D_{L_i} \leq C_i - B_i}(B_i) \Pr(D_{O_j} > B_j, D_{L_i} \leq C_i - B_i) \\
& \leq 0
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
\frac{\partial^2 \pi_i(B_i,B_j)}{\partial^2 B_i} = \; & -(s_{L_i} + p_{L_i} + h_{L_i}) f_{R_{L_i}}(C_i - B_i) \\
& -(s_{O_i} + p_{O_i} + h_{O_i})(1 - a_{L_iO_i})^2 f_{R_{O_i}|D_{L_i} > C_i - B_i}(B_i) \Pr(D_{L_i} > C_i - B_i) \\
& -(s_{O_i} + p_{O_i} + h_{O_i}) f_{R_{O_i}|D_{L_i} \leq C_i - B_i}(B_i) \Pr(D_{L_i} \leq C_i - B_i) \\
& +(s_{O_i} + h_{O_i}) a_{L_iO_i} f_{D_{L_i}}(C_i - B_i).
\end{aligned} \tag{5.4}$$

Theorem 5.9 *If $a_{L_iO_i} + a_{O_jO_i} \leq 1$ in our two-firm model, a sufficient condition for the Nash equilibrium, (B_1^{Nash}, B_2^{Nash}), to be locally stable is*

$$\Pr(D_{L_j} \leq C_j - B_j^{Nash}) > \frac{(s_{O_i} + h_{O_i}) a_{L_iO_i}}{s_{L_i} + p_{L_i} + h_{L_i}}, i, j = 1, 2, j \neq i. \tag{5.5}$$

Proof It is easy to see that

$$\begin{aligned}
& f_{R_{L_i}}(C_i - B_i) \\
&= f_{R_{L_i}|D_{L_j}>C_j-B_j}(C_i - B_i)\Pr(D_{L_j} > C_j - B_j) + f_{R_{L_i}|D_{L_j}\le C_j-B_j}(C_i - B_i)\Pr(D_{L_j} \le C_j - B_j) \\
&= f_{R_{L_i}|D_{L_j}>C_j-B_j}(C_i - B_i)\Pr(D_{L_j} > C_j - B_j) + f_{D_{L_i}}(C_i - B_i)\Pr(D_{L_j} \le C_j - B_j).
\end{aligned}$$

Therefore, from (5.4), if

$$\Pr(D_{L_j} \le C_j - B_j) > \frac{(s_{O_i} + h_{O_i})a_{L_iO_i}}{s_{L_i} + p_{L_i} + h_{L_i}},$$

then

$$\frac{\partial^2 \pi_i(B_i, B_j)}{\partial^2 B_i} \le 0$$

It follows that

$$\begin{aligned}
& |\tfrac{\partial^2 \pi_i(B_i,B_j)}{\partial^2 B_i}| - |\tfrac{\partial^2 \pi_i(B_i,B_j)}{\partial B_i \partial B_j}| \\
&> (s_{O_i} + p_{O_i} + h_{O_i})(1 - a_{L_iO_i})(1 - a_{L_iO_i} - a_{O_jO_i}) f_{R_{O_i}|D_{L_i}>C_i-B_i}(B_i)\Pr(D_{L_i} > C_i - B_i) \\
&\quad + (s_{L_i} + p_{L_i} + h_{L_i}) a_{L_jL_i} f_{R_{L_i}|D_{L_j}>C_j-B_j}(C_i - B_i)\Pr(D_{L_j} > C_j - B_j) \\
&\quad + (s_{L_i} + p_{L_i} + h_{L_i}) f_{R_{L_i}}(C_i - B_i) - (s_{O_i} + h_{O_i}) a_{L_iO_i} f_{R_{L_i}|D_{L_j}\le C_j-B_j}(C_i - B_i) \\
&> (s_{O_i} + p_{O_i} + h_{O_i})(1 - a_{L_iO_i})(1 - a_{L_iO_i} - a_{O_jO_i}) f_{R_{O_i}|D_{L_i}>C_i-B_i}(B_i)\Pr(D_{L_i} > C_i - B_i) \\
&\quad + f_{R_{L_i}|D_{L_j}\le C_j-B_j}(C_i - B_i)((s_{O_i} + h_{O_i}) a_{L_iO_i} - (s_{L_i} + p_{L_i} + h_{L_i})\Pr(D_{L_j} \le C_j - B_j)).
\end{aligned}$$

It is easy to see that if $a_{L_iO_i} + a_{O_jO_i} \le 1$ and $\Pr(D_{L_j} \le C_j - B_j^{Nash}) > \frac{(s_{O_i}+h_{O_i})a_{L_iO_i}}{s_{L_i}+p_{L_i}+h_{L_i}}$, then $|\frac{\partial^2 \pi_i(B_i,B_j)}{\partial^2 B_i}|_{B_i=B_i^{Nash},B_j=B_j^{Nash}} > |\frac{\partial^2 \pi_i(B_i,B_j)}{\partial B_i \partial B_j}|_{B_i=B_i^{Nash},B_j=B_j^{Nash}}$. By using Lemma 5.8, we can complete the proof. ■

Remark 5.10 *If $a_{L_iO_i} = 0$ for all i, then*

$$|\frac{\partial^2 \pi_i(B_i, B_j)}{\partial^2 B_i}| > |\frac{\partial^2 \pi_i(B_i, B_j)}{\partial B_i \partial B_j}|$$

always holds. Following Theorem 3 in Chapter 5 of Moulin (1986), we know that a unique and globally stable Nash equilibrium exists. This special case has been studied by Netessine and Shumsky (2001).

Remark 5.11 *It is easy to see that the two-firm model which has high s_{L_i}, p_{L_i} and h_{L_i} and low s_{O_i}, h_{O_i} and $a_{L_iO_i}$ is more likely to have a stable Nash equilibrium solution.*

5.2.2 Scenario 2

For this scenario, we can write the effective demands at the local and online stores of firm i as:

$$R_{L_i} = D_{L_i} + a_{L_jL_i}(D_{L_j} - (C_j - B_j))^+,$$
$$R_{O_i} = D_{O_i} + a_{O_jO_i}(D_{O_j} + a_{L_jO_j}(D_{L_j} - (C_j - B_j))^+ - B_j)^+ + a_{L_iO_i}(D_{L_i} - (C_i - B_i))^+$$

Analogous to Scenario 1, the expected payoff function of firm 1 is:

$$\begin{aligned}\pi_1(B_1, B_2) = E[&s_{L_1} \min\{R_{L_1}, C_1 - B_1\} - p_{L_1}(R_{L_1} - (C_1 - B_1)) \\ &- (p_{L_1} + h_{L_1})(R_{L_1} - (C_1 - B_1))^- \\ &+ s_{O_1} \min\{R_{O_1}, B_1\} - (p_{O_1} + h_{O_1})(R_{O_1} - B_1)^+ + h_{O_1}(R_{O_1} - B_1)],\end{aligned}$$

which has the same representation as in Scenario 1, except that the form of R_{O_1} is different.

Theorem 5.12 *For Scenario 2 of our two-firm model, there exists a Nash equilibrium*

(B_1^{Nash}, B_2^{Nash})*, which can be obtained by solving the following system of equations:*

$$\begin{cases} \frac{\partial \pi_1(B_1,B_2)}{\partial B_1} = 0, \\ \frac{\partial \pi_2(B_1,B_2)}{\partial B_2} = 0. \end{cases}$$

Proof To prove $\pi_1(B_1, B_2)$ is submodular in (B_1, B_2), we need only to prove that the terms with R_{O_1} are still submodular in (B_1, B_2). Notice that

$$\begin{aligned} \min\{R_{O_1}, B_1\} = \min\{& D_{O_1} + a_{O_2O_1}(D_{O_2} + a_{L_2O_2}(D_{L_2} - (C_2 - B_2))^+ - B_2)^+, \\ & B_1 - a_{L_1O_1}(D_{L_1} - (C_1 - B_1))^+\} + a_{L_1O_1}(D_{L_1} - (C_1 - B_1))^+. \end{aligned}$$

By Lemma 5.4, we know $\min\{R_{O_1}, B_1\}$ is submodular. Meanwhile, since $(R_{O_1} - B_1)$ is a submodular function and decreasing in both B_1 and B_2, by Lemma 5.5, we know $-(p_{O_1}+h_{O_1})(R_{O_1}-B_1)^+$ is submodular. Therefore, $\pi_1(B_1, B_2)$ is submodular in (B_1, B_2). Similarly, we can prove that $\pi_2(B_1, B_2)$ is submodular in (B_1, B_2). From Theorem 3.1 of Topkis (1979) and the submodularity of payoff functions, we can prove the existence of a Nash equilibrium. ■

Theorem 5.13 *In Scenario 2, if* $a_{L_iO_i} + a_{O_jO_i} \leq 1$*, a sufficient condition for the Nash equilibrium,* (B_1^{Nash}, B_2^{Nash})*, to be locally stable is*

$$\Pr(D_{L_j} \leq C_j - B_j^{Nash}) > \frac{(s_{O_i} + h_{O_i})a_{L_iO_i}}{s_{L_i} + p_{L_i} + h_{L_i}}, i, j = 1, 2, j \neq i.$$

Proof Since R_{O_i} is not related to B_i, we know that $\frac{\partial \pi_i(B_i,B_j)}{\partial B_i}$ and $\frac{\partial^2 \pi_i(B_i,B_j)}{\partial^2 B_i}$ are the same as (5.2) and (5.4) respectively. Analogous to Scenario 1, denoting $a_{O_jO_i}(D_{O_j} + a_{L_jO_j}(D_{L_j} - (C_j - B_j))^+$ as D'_{O_j}, we obtain:

$$\begin{aligned} \frac{\partial^2 \pi_i(B_8,B_j)}{\partial B_i \partial B_j} = & -(s_{L_i} + p_{L_i} + h_{L_i})a_{L_jL_i} f_{R_{L_i}|D_{L_j} > C_j - B_j}(C_i - B_i)\Pr(D_{L_j} > C_j - B_j) \\ & -(s_{O_i} + p_{O_i} + h_{O_i})(Y + Z) \end{aligned}$$

where

$$Y = (1-a_{L_iO_i})a_{O_jO_i}f_{R_{O_i}|D_{L_i}>C_i-B_i,D'_{O_j}>B_j,D_{L_j}\le C_j-B_j}(B_i)$$
$$\Pr(D_{L_i}>C_i-B_i,D'_{O_j}>B_j,D_{L_j}\le C_j-B_j)$$
$$+a_{O_jO_i}(1-a_{L_iO_i})^2 f_{R_{O_i}|D_{L_i}>C_i-B_i,D'_{O_j}>B_j,D_{L_j}>C_j-B_j}(B_i)$$
$$\Pr(D_{L_i}>C_i-B_i,D'_{O_j}>B_j,D_{L_j}>C_j-B_j),$$

$$Z = a_{O_jO_i}f_{R_{O_i}|D_{L_i}\le C_i-B_i,D'_{O_j}>B_j,D_{L_j}\le C_j-B_j}(B_i)$$
$$\Pr(D_{L_i}\le C_i-B_i,D'_{O_j}>B_j,D_{L_j}\le C_j-B_j)$$
$$+(1-a_{L_iO_i})a_{O_jO_i}f_{R_{O_i}|D_{L_i}>C_i-B_i,D'_{O_j}>B_j,D_{L_j}>C_j-B_j}(B_i)$$
$$\Pr(D_{L_i}>C_i-B_i,D'_{O_j}>B_j,D_{L_j}>C_j-B_j)$$

We know that if $\Pr(D_{L_j}\le C_j-B_j) > \frac{(s_{O_i}+h_{O_i})a_{L_iO_i}}{s_{L_i}+p_{L_i}+h_{L_i}}$, then $\frac{\partial^2\pi_i(B_i,B_j)}{\partial^2 B_i}\le 0$. After some calculations, analogous to Scenario 1, we have

$$|\tfrac{\partial^2\pi_i(B_i,B_j)}{\partial^2 B_i}| - |\tfrac{\partial^2\pi_i(B_i,B_j)}{\partial B_i\partial B_j}|$$
$$> (s_{O_i}+p_{O_i}+h_{O_i})(1-a_{L_iO_i})(1-a_{L_iO_i}-a_{O_jO_i})f_{R_{O_i}|D_{L_i}>C_i-B_i}(B_i)\Pr(D_{L_i}>C_i-B_i)$$
$$+(s_{L_i}+p_{L_i}+h_{L_i})a_{L_jL_i}f_{R_{L_i}|D_{L_j}>C_j-B_j}(C_i-B_i)\Pr(D_{L_j}>C_j-B_j)$$
$$+(s_{L_i}+p_{L_i}+h_{L_i})f_{R_{L_i}}(C_i-B_i)-(s_{O_i}+h_{O_i})a_{L_iO_i}f_{R_{L_i}|D_{L_j}\le C_j-B_j}(C_i-B_i)$$
$$> (s_{O_i}+p_{O_i}+h_{O_i})(1-a_{L_iO_i})(1-a_{L_iO_i}-a_{O_jO_i})f_{R_{O_i}|D_{L_i}>C_i-B_i}(B_i)\Pr(D_{L_i}>C_i-B_i)$$
$$+f_{R_{L_i}|D_{L_j}\le C_j-B_j}(C_i-B_i)((s_{O_i}+h_{O_i})a_{L_iO_i}-(s_{L_i}+p_{L_i}+h_{L_i})\Pr(D_{L_j}\le C_j-B_j))$$
$$> 0$$

This completes the proof. ■

5.3 Sensitivity analysis

In this section, we analyze the sensitivity of an equilibrium solution with respect to system parameters, such as selling price, holding cost and stockout penalty cost. Based on the above results, we know that in both scenarios the Nash equilibrium is characterized by the following optimality conditions:

$$\begin{cases} G_1(B_1,B_2) \triangleq \frac{\partial \pi_1(B_1,B_2)}{\partial B_1} = 0, \\ G_2(B_1,B_2) \triangleq \frac{\partial \pi_2(B_1,B_2)}{\partial B_2} = 0, \end{cases}$$

where

$$\begin{aligned} \tfrac{\partial \pi_i(B_i,B_j)}{\partial B_i} = \quad & h_{L_i} - h_{O_i} - (s_{L_i} + p_{L_i} + h_{L_i}) \Pr(R_{L_i} > C_i - B_i) \\ & +(s_{O_i} + h_{O_i}) a_{L_i O_i} \Pr(D_{L_i} > C_i - B_i) \\ & +(s_{O_i} + p_{O_i} + h_{O_i})(1 - a_{L_i O_i}) \Pr(R_{O_i} > B_i, D_{L_i} > C_i - B_i) \\ & +(s_{O_i} + p_{O_i} + h_{O_i}) \Pr(R_{O_i} > B_i, D_{L_i} \le C_i - B_i), i,j = 1,2, j \neq i. \end{aligned}$$

Recall that the two scenarios differ in the form of R_{O_i}. In this section, we discuss some properties that hold for both scenarios.

Theorem 5.14 *Assume that the stability condition (5.5) holds at an equilibrium solution. The following results hold for both Scenarios 1 and 2:*

(i) Relative to selling price, B_i^{Nash} is decreasing in s_{L_i} and s_{O_j}, while it is increasing in s_{L_j} and s_{O_i}.

(ii) Relative to holding cost, B_i^{Nash} is increasing in h_{L_i} and h_{O_j}, while it is decreasing in h_{L_j} and h_{O_i}.

(iii) Relative to stockout penalty cost, B_i^{Nash} is decreasing in p_{L_i} and p_{O_j}, while it is increasing in p_{L_j} and p_{O_i}.

Proof We only show the relationship between B_i^{Nash} and selling prices s_{L_i} and s_{L_j}. Other cases follow the same logic. By the implicit function theorem applied at (B_1^{Nash}, B_2^{Nash}), we have

$$\begin{bmatrix} \frac{\partial B_1}{\partial s_{L_1}} & \frac{\partial B_1}{\partial s_{L_2}} \\ \frac{\partial B_2}{\partial s_{L_1}} & \frac{\partial B_2}{\partial s_{L_2}} \end{bmatrix}$$
$$= - \begin{bmatrix} \frac{\partial G_1}{\partial B_1} & \frac{\partial G_1}{\partial B_2} \\ \frac{\partial G_2}{\partial B_1} & \frac{\partial G_2}{\partial B_2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial B_1}{\partial s_{L_1}} & \frac{\partial B_1}{\partial s_{L_2}} \\ \frac{\partial B_2}{\partial s_{L_1}} & \frac{\partial B_2}{\partial s_{L_2}} \end{bmatrix}$$
$$= \frac{1}{\left(\frac{\partial G_1}{\partial B_1}\frac{\partial G_2}{\partial B_2} - \frac{\partial G_1}{\partial B_2}\frac{\partial G_2}{\partial B_1}\right)} \begin{bmatrix} -\frac{\partial G_2}{\partial B_2}\frac{\partial G_1}{\partial s_{L_1}} & \frac{\partial G_1}{\partial B_2}\frac{\partial G_2}{\partial s_{L_2}} \\ \frac{\partial G_2}{\partial B_1}\frac{\partial G_1}{\partial s_{L_1}} & -\frac{\partial G_1}{\partial B_1}\frac{\partial G_2}{\partial s_{L_2}} \end{bmatrix}$$

Recall from (5.3), $\frac{\partial G_i}{\partial B_j} = \frac{\partial^2 \pi_i(B_i,B_j)}{\partial B_i \partial B_j} < 0$. Under the stability condition (5.5),

$$\frac{\partial G_i}{\partial B_i} = \frac{\partial^2 \pi_i(B_i, B_j)}{\partial^2 B_i} \leq 0,$$

and

$$\frac{\partial G_1}{\partial B_1}\frac{\partial G_2}{\partial B_2} - \frac{\partial G_1}{\partial B_2}\frac{\partial G_2}{\partial B_1} = |\frac{\partial^2 \pi_1(B_1, B_2)}{\partial^2 B_1}| - |\frac{\partial^2 \pi_1(B_1, B_2)}{\partial B_1 \partial B_2}| > 0.$$

Further, we have

$$\frac{\partial G_i}{\partial s_{L_i}} = -\Pr(R_{L_i} > C_i - B_i) \leq 0, i = 1, 2.$$

Hence, we know $\frac{\partial B_i}{\partial s_{L_i}}$ is nonnegative and $\frac{\partial B_i}{\partial s_{L_j}}$ is non-positive at the Nash equilibrium (B_1^{Nash}, B_2^{Nash}). In other words, B_i^{Nash} is decreasing in s_{L_i} and increasing in s_{L_j}. ■

The results of Theorem 5.14 go with our intuition. For example, the rationale behind (i) is as follows: If the selling price of the local store at firm i, s_{L_i}, increases, then at equilibrium more capacity should be allocated to that store, and thus less capacity is allocated to the online store of firm i. Consequently, more online demand is unsatisfied at firm i and shift to the online store of firm j. Therefore, at equilibrium, firm j allocates more capacity to its online store. Intuitively, a small change in s_{O_i} causes an opposite effect on the equilibrium solution.

5.4 Extensions of the basic model

We next extend the basic model. First, we consider the case that each firm has to decide its total capacity and allocates this capacity between its local and online stores. In other words, we have two decision variables to consider. Second, we extend the single-period model to a multi-period model, in which each firm makes a sequence of decisions. We would like to derive a myopic solution of a corresponding sequential game.

5.4.1 Two decision variables

Assume that each firm decides its total capacity and allocates this capacity between its local and online stores. For the sake of brevity, we only consider Scenario 1. Recall that C_i is the total capacity of firm i, B_i is the capacity of firm i allocated to its online store, and (thus) $C_i - B_i$ is the capacity of firm i allocated to its local store. Denoting the expected payoff function of firm i as $\pi_i(B_i, C_i, B_j, C_j)$, for firm 1, we have

$$\begin{aligned}\pi_1(B_1, C_1, B_2, C_2) &= E[s_{L_1}R_{L_1} + h_{L_1}(R_{L_1} - (C_1 - B_1)) - (s_{L_1} + p_{L_1} + h_{L_1})(R_{L_1} - (C_1 - B_1))^+ \\ &\quad + s_{O_1}R_{O_1} + h_{O_1}(R_{O_1} - B_1) - (s_{O_1} + p_{O_1} + h_{O_1})(R_{O_1} - B_1)^+],\end{aligned} \tag{5.6}$$

which is the same as (5.1). The first derivative of $\pi_1(B_1, C_1, B_2, C_2)$ with respect to B_1 is the same as (5.2), and the first derivative with respect to C_1 is

$$\begin{aligned}\frac{\partial \pi_1}{\partial C_1} = & -h_{L_1} + (s_{L_1} + p_{L_1} + h_{L_1}) \Pr(R_{L_1} > C_1 - B_1) \\ & -(s_{O_1} + h_{O_1})a_{L_1O_1} \Pr(D_{L_1} > C_1 - B_1) \\ & +(s_{O_1} + p_{O_1} + h_{O_1})a_{L_1O_1} \Pr(R_{O_1} > B_1, D_{L_1} > C_1 - B_1)\end{aligned}$$

With some simple calculations, we obtain the Hessian matrix,

$$\begin{pmatrix} \frac{\partial^2 \pi_1}{\partial^2 B_1} & \frac{\partial^2 \pi_1}{\partial B_1 C_1} \\ \frac{\partial^2 \pi_1}{\partial C_1 B_1} & \frac{\partial^2 \pi_1}{\partial^2 C_1} \end{pmatrix}$$

where

$$\begin{aligned}\frac{\partial^2 \pi_1}{\partial B_1 C_1} = & \ (s_{L_1} + p_{L_1} + h_{L_1}) f_{R_{L_1}}(C_1 - B_1) - (s_{O_1} + h_{O_1})a_{L_1O_1} f_{D_{L_1}}(C_1 - B_1) \\ & - (s_{O_1} + p_{O_1} + h_{O_1})a_{L_1O_1}(1 - a_{L_1O_1}) f_{R_{O_1}|D_{L_1} > C_1 - B_1}(B_1) \Pr(D_{L_1} > C_1 - B_1), \\ \frac{\partial^2 \pi_1}{\partial^2 C_1} = & -(s_{L_1} + p_{L_1} + h_{L_1}) f_{R_{L_1}}(C_1 - B_1) + (s_{O_1} + h_{O_1})a_{L_1O_1} f_{D_{L_1}}(C_1 - B_1) \\ & - (s_{O_1} + p_{O_1} + h_{O_1})a^2_{L_1O_1} f_{R_{O_1}|D_{L_1} > C_1 - B_1}(B_1) \Pr(D_{L_1} > C_1 - B_1),\end{aligned}$$

and $\frac{\partial^2 \pi_1}{\partial^2 B_1}$ is the same as (5.4).

A sufficient condition for $\pi_1(B_1, C_1, B_2, C_2)$ to be jointly concave in (B_1, C_1) is that the Hessian matrix is negative semidefinite, i.e.,

$$\frac{\partial^2 \pi_1}{\partial^2 B_1}\frac{\partial^2 \pi_1}{\partial^2 C_1} - (\frac{\partial^2 \pi_1}{\partial B_1 C_1})^2 > 0$$

Denoting

$$f_{R_{O_1}|D_{L_1}>C_1-B_1}(B_1)\Pr(D_{L_1} > C_1 - B_1),$$

which is positive, as M and

$$f_{R_{O_1}|D_{L_1}\le C_1-B_1}(B_1)\Pr(D_{L_1} \le C_1 - B_1),$$

which is also positive, as N, we have

$$\begin{aligned}
&\frac{\partial^2\pi_1}{\partial^2 B_1}\frac{\partial^2\pi_1}{\partial^2 C_1} - (\frac{\partial^2\pi_1}{\partial B_1 C_1})^2 \\
&= N((s_{L_1} + p_{L_1} + h_{L_1})f_{R_{L_1}}(C_1 - B_1) - (s_{O_1} + h_{O_1})a_{L_1O_1}f_{D_{L_1}}(C_1 - B_1)) \\
&\quad + M((s_{L_1} + p_{L_1} + h_{L_1})f_{R_{L_1}}(C_1 - B_1) - (s_{O_1} + h_{O_1})a_{L_1O_1}f_{D_{L_1}}(C_1 - B_1) \\
&\quad - 2a^2_{L_1O_1}(s_{L_1} + p_{L_1} + h_{L_1})f_{R_{L_1}}(C_1 - B_1)) + (s_{O_1} + p_{O_1} + h_{O_1})a^2_{L_1O_1}MN
\end{aligned}$$

Therefore, if

$$(2a^2_{L_1O_1} - 1)(s_{L_1} + p_{L_1} + h_{L_1})f_{R_{L_1}}(C_1 - B_1) + (s_{O_1} + h_{O_1})a_{L_1O_1}f_{D_{L_1}}(C_1 - B_1) \le 0, \quad (5.7)$$

then $\pi_1(B_1, C_1, B_2, C_2)$ is jointly concave in (B_1, C_1). Symmetrically, we have the following sufficient condition for firm 2 to have a concave payoff function:

$$(2a^2_{L_2O_2} - 1)(s_{L_2} + p_{L_2} + h_{L_2})f_{R_{L_2}}(C_2 - B_2) + (s_{O_2} + h_{O_2})a_{L_2O_2}f_{D_{L_2}}(C_2 - B_2) \le 0, \quad (5.8)$$

By using Tsay and Agrawal (2000), we have the following theorem.

Theorem 5.15 *For both Scenarios 1 and 2 of our two-firm model with two decision variables, if (5.7) and (5.8) are satisfied, then there exists a Nash equilibrium, which can* *be obtained by solving the following system of equations:*

$$\begin{cases} \frac{\partial \pi_1(\cdot)}{\partial B_1} = 0, \\ \frac{\partial \pi_1(\cdot)}{\partial C_1} = 0, \\ \frac{\partial \pi_2(\cdot)}{\partial B_2} = 0, \\ \frac{\partial \pi_2(\cdot)}{\partial C_2} = 0. \end{cases}$$

Proof If (5.7) and (5.8) are satisfied, then $\pi_i(B_i, C_i, B_j, C_j)$ is jointly concave in $(B_i, C_i), i, j = 1, 2, j \neq i$. As stated in Tsay and Agrawal (2000), the equilibrium solution of our two-firm model with two decision variables can be obtained by solving the system of four equations consisting of the two first order conditions for each of the two firms. ■

Now we show that conditions (5.7) and (5.8) are valid, namely they can be satisfied under certain conditions. For example, for (5.7), note that it is equivalent to

$$a_{L_1O_1} \leq -V + \sqrt{V^2 + \frac{1}{2}},$$

where $V = \frac{(s_{O_1}+h_{O_1})f_{D_{L_1}}(C_1-B_1)}{4(s_{L_1}+p_{L_1}+h_{L_1})f_{R_{L_1}}(C_1-B_1)} \geq 0$.

Since $-V + \sqrt{V^2 + \frac{1}{2}}$ is decreasing in V, we denote $\frac{(s_{O_1}+h_{O_1})}{4(s_{L_1}+p_{L_1}+h_{L_1})} \max\limits_{0 \leq y \leq C_1} \frac{f_{D_{L_1}}(y)}{f_{R_{L_1}}(y)}$ as $\overline{V}$. Thus, if

$$a_{L_1O_1} \leq -\overline{V} + \sqrt{\overline{V}^2 + \frac{1}{2}}, \tag{5.9}$$

then $\pi_1(B_1, C_1, B_2, C_2)$ is jointly concave in (B_1, C_1). Note that the right hand side of (5.9) is decreasing in the selling price at the online store of firm 1, s_{O_1}, and increasing in the selling price at the local store of firm 1, s_{L_1}. This goes with our intuition, since $a_{L_1O_1}$ is the probability of an unsatisfied local customer at firm 1 visiting its online store.

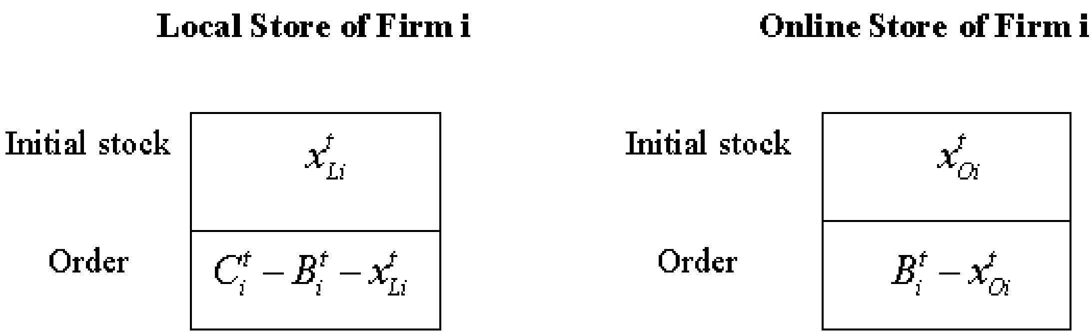

Figure 5.4: The initial stock and order of firm i at the beginning of period t.

5.4.2 Multi-period model

Now we consider a multi-period model in which each player makes a sequence of decisions. At the beginning of each period t ($t = 1, 2, \ldots$), firm i has initial on-hand inventories $x^t_{L_i}$ and $x^t_{O_i}$ at its local and online stores, respectively. It makes decisions C^t_i and B^t_i, where C^t_i is the total stock of firm i, B^t_i is the allocation to its online store and (thus) $C^t_i - B^t_i$ is the allocation to its local store, as shown in Figure 5.4. We assume that the inventory replenishment is instantaneous, so B^t_i and $C^t_i - B^t_i$ are the actual inventory available at the online and local stores of firm i, respectively. The local demand, online demand, and effective demands are also period-related, and we denote them by $D^t_{L_i}, D^t_{O_i}, R^t_{L_i}$ and $R^t_{O_i}$, respectively. We assume that $D^t_{L_i}$ and $D^t_{O_i}$ are random variables with independent identical distribution. The resulting inventory levels of firm

i at the beginning of period $t+1$ become:

$$x_{L_i}^{t+1} = (C_i^t - B_i^t - R_{L_i}^t)^+,$$

$$x_{O_i}^{t+1} = (B_i^t - R_{O_i}^t)^+.$$

Let $\beta_i < 1$ be the discount factor per period for firm i, and u_i be the unit cost that firm i pays for the product. Therefore, firm i's expected total profit is:

$$\begin{aligned}
\Pi_i = E[&\sum_{t=1}^{\infty}(\beta_i)^{t-1}[s_{L_i}\min\{R_{L_i}^t, C_i^t - B_i^t\} - p_{L_i}(R_{L_i}^t - (C_i^t - B_i^t))^+ - h_{L_i}(R_{L_i}^t - (C_i^t - B_i^t))^- \\
&+ s_{O_i}\min\{R_{O_i}^t, B_i^t\} - p_{O_i}(R_{O_i}^t - B_i^t)^+ - h_{O_i}(R_{O_i}^t - B_i^t)^- - u_i(C_i^t - B_i^t - x_{L_i}^t) - u_i(B_i^t - x_{O_i}^t)]] \\
= E[&\sum_{t=2}^{\infty}(\beta_i)^{t-1}[s_{L_i}\min\{R_{L_i}^t, C_i^t - B_i^t\} - p_{L_i}(R_{L_i}^t - (C_i^t - B_i^t))^+ - h_{L_i}(R_{L_i}^t - (C_i^t - B_i^t))^- \\
&+ s_{O_i}\min\{R_{O_i}^t, B_i^t\} - p_{O_i}(R_{O_i}^t - B_i^t)^+ - h_{O_i}(R_{O_i}^t - B_i^t)^- \\
&- u_i(C_i^t - B_i^t - (C_i^{t-1} - B_i^{t-1} - R_{L_i}^{t-1})^+) - u_i(B_i^t - (B_i^{t-1} - R_{O_i}^{t-1})^+)] \\
&+ s_{L_i}\min\{R_{L_i}^1, C_i^1 - B_i^1\} - p_{L_i}(R_{L_i}^1 - (C_i^1 - B_i^1))^+ - h_{L_i}(R_{L_i}^1 - (C_i^1 - B_i^1))^- \\
&+ s_{O_i}\min\{R_{O_i}^1, B_i^1\} - p_{O_i}(R_{O_i}^1 - B_i^1)^+ - h_{O_i}(R_{O_i}^1 - B_i^1)^- - u_i(C_i^1 - x_{L_i}^1 - x_{O_i}^1)] \\
= u_i(x_{L_i}^1 + x_{O_i}^1) + E[&\sum_{t=1}^{\infty}(\beta_i)^{t-1}[s_{L_i}\min\{R_{L_i}^t, C_i^t - B_i^t\} - p_{L_i}(R_{L_i}^t - (C_i^t - B_i^t))^+ \\
&- h_{L_i}(R_{L_i}^t - (C_i^t - B_i^t))^- + s_{O_i}\min\{R_{O_i}^t, B_i^t\} - p_{O_i}(R_{O_i}^t - B_i^t)^+ - h_{O_i}(R_{O_i}^t - B_i^t)^- \\
&- u_i C_i^t + \beta_i u_i((C_i^t - B_i^t - R_{L_i}^t)^+ + (B_i^t - R_{O_i}^t)^+)]] \\
= u_i(x_{L_i}^1 + x_{O_i}^1) + E[&\sum_{t=1}^{\infty}(\beta_i)^{t-1}[(s_{L_i} - \beta_i u_i)R_{L_i}^t + h_{L_i}(R_{L_i}^t - (C_i^t - B_i^t)) \\
&- (s_{L_i} - \beta_i u_i + p_{L_i} + h_{L_i})(R_{L_i}^t - (C_i^t - B_i^t))^+ + (s_{O_i} - \beta_i u_i)R_{O_i}^t + h_{O_i}(R_{O_i}^t - B_i^t) \\
&- (s_{O_i} - \beta_i u_i + p_{O_i} + h_{O_i})(R_{O_i}^t - B_i^t)^+ - (1 - \beta_i)u_i C_i^t]]
\end{aligned}$$

Denoting $s_{L_i} - \beta_i u_i$ by s'_{L_i} and $s_{O_i} - \beta_i u_i$ by s'_{O_i}, we have

$$\Pi_i = u_i(x^i_{L_i} + x^i_{O_i}) + \sum_{t=1}^{\infty} (\beta_i)^{t-1} G^t_i(B^t_i, C^t_i, B^t_j, C^t_j)$$

where

$$G^t_i(B^t_i, C^t_i, B^t_j, C^t_j) = E[s'_{L_i} R^t_{L_i} + h_{L_i}(R^t_{L_i} - (C^t_i - B^t_i)) - (s'_{L_i} + p_{L_i} + h_{L_i})(R^t_{L_i} - (C^t_i - B^t_i))^+ \tag{5.10}$$

$$+ s'_{O_i} R^t_{O_i} + h_{O_i}(R^t_{O_i} - B^t_i) - (s'_{O_i} + p_{O_i} + h_{O_i})(R^t_{O_i} - B^t_i)^+ - (1 - \beta_i) u_i C^t_i],$$

which is very close to (5.6).

We now apply the theory of sequential games developed in Heyman and Sobel (1984) to analyze this multi-period model. In general, a multi-period game is difficult to solve. Hence we apply the concept "myopic solution" to simplify it. A multi-period game is said to have a myopic solution if its data can be used easily to specify a single-period game such that *ad infinitum* repetition of a Nash equilibrium of the single-period game comprises an equilibrium for the multi-period game. We would like to derive a myopic solution to our multi-period game.

We refer to the single-period game, in which each firm has payoff function (5.10) without superscript t, as game Φ. For our sequential game characterized by (5.10), from Section 9-4 of Heyman and Sobel, we know that if the following three conditions are satisfied, then a myopic solution exists and this sequential game can be simplified into single-period game Φ. Those sufficient conditions are:

(A5.1) The demands in all periods are random variables with independent identical distribution;

(A5.2) The discount factor per period for firm i is less than one, i.e., $\beta_i < 1$;

(A5.3) The equilibrium solution of the game Φ is a feasible solution to the multi-period game in each period t.

For our model, conditions (A5.1) and (A5.2) are satisfied automatically. Condition (A5.3) is equivalent to:

$$x_{L_i}^{t+1} \le (C_i^t - B_i^t) = (\overline{C}_i - \overline{B}_i),$$

$$x_{O_i}^{t+1} \le B_i^t = \overline{B}_i, i = 1, 2,$$

where $(\overline{B}_i, \overline{C}_i, \overline{B}_j, \overline{C}_j)$ is an equilibrium solution of the game Φ. Since $(\overline{B}_i, \overline{C}_i, \overline{B}_j, \overline{C}_j)$ is an equilibrium solution of the game Φ, we know that $(B_i^1, C_i^1, B_j^1, C_j^1) \le (\overline{B}_i, \overline{C}_i, \overline{B}_j, \overline{C}_j)$ is feasible. Further, we have

$$x_{L_i}^{t+1} = (C_i^t - B_i^t - R_{L_i}^t)^+ = (\overline{C}_i - \overline{B}_i - R_{L_i}^t)^+ \le \overline{C}_i - \overline{B}_i,$$

$$x_{O_i}^{t+1} = (B_i^t - R_{O_i}^t)^+ = (\overline{B}_i - R_{O_i}^t)^+ \le \overline{B}_i, i = 1, 2.$$

Hence, condition (A5.3) is also satisfied.

Therefore, we have the following theorem.

Theorem 5.16 *For our multi-period game characterized by (5.10), a myopic solution exists which simplifies this multi-period game into the single-period game Φ, in which firm i makes decisions on B_i and C_i and has payoff function (5.10) without superscript t, i.e.,*

$$G_i(B_i, C_i, B_j, C_j) = E[s'_{L_i} R_{L_i} + h_{L_i}(R_{L_i} - (C_i - B_i)) - (s'_{L_i} + p_{L_i} + h_{L_i})(R_{L_i} - (C_i - B_i))^+$$
$$+ s'_{O_i} R_{O_i} + h_{O_i}(R_{O_i} - B_i) - (s'_{O_i} + p_{O_i} + h_{O_i})(R_{O_i} - B_i)^+ - (1 - \beta_i) u_i C_i]$$

Finding a Nash equilibrium solution for game Φ is almost the same problem as the one discussed in Subsection 5.4.1 so that the myopic equilibrium is characterized as:

$$\begin{cases} \frac{\partial G_1(\cdot)}{\partial B_1} = 0, \\ \frac{\partial G_1(\cdot)}{\partial C_1} = 0, \\ \frac{\partial G_2(\cdot)}{\partial B_2} = 0, \\ \frac{\partial G_2(\cdot)}{\partial C_2} = 0. \end{cases}$$

5.5 Concluding remarks

In this study, we consider the capacity allocation problem for two firms selling the same product. Each firm has a local store and an online store. When a stockout occurs at a local store customers may go to the online store belonging to the same firm, or visit the rival's local store. However, when a stockout occurs at an online store, customers usually will not visit the local store belonging to the same firm, but instead they may go to the rival's online store. One question facing each firm is how to allocate its finite capacity between the local and online stores to maximize its profit as a whole.

Because customers may shift from one firm to the other when stockout occurs, one firm's allocation affects the decision of the rival, thereby creating a strategic interaction. In this chapter game theory is used for analysis. We first consider a single-product single-period model and assume the total capacity of each firm is given and known. We study two scenarios of this model and derive corresponding existence and stability conditions of a Nash equilibrium. We also conduct sensitivity analysis of the equilibrium solution with respect to price and cost parameters. We then study the case in which each firm decides its total capacity and allocates this capacity between its local and online stores simultaneously and derive the existence condition of a Nash equilibrium. Finally we extend the results to a multi-period model, and show that a myopic solution is a Nash equilibrium for the corresponding sequential game.

In our model, we have assumed that the demands at local and online stores are exogenous. A potential avenue of research is to study the case in which the demands are affected by the quality of service, such as the percentage of demand satisfied, or by the selling prices. It is also possible to analyze the case when the probabilities of customer shifting, i.e, parameters like $a_{L_iL_j}$, are related to these factors.

Another extension would be to analyze collaboration between these two firms, such as the side payment rules to make both of them better off by using a single web page. We can also study the behavior of a monopolist who owns both firms, namely a centralized model, as in Netessine and Shumsky (2001) and compare the results with the behavior of two firms in competition, namely a decentralized model.

Chapter 6

Pricing Game in Revenue Management with Multiple Firms

6.1 Introduction

In this chapter, we study the pricing strategies of multiple firms in a revenue management context. We consider a scenario in which there are multiple firms selling same product or providing same service. Each firm has a given capacity and competes for customers from a common pool. The firms aim to maximize their profit subject to their capacity constraint by setting prices to attract potential customers. Since the pricing strategy of one firm affects the demand streams of other firms, there exists a strategic interaction among the firms' pricing decisions; therefore game theory is applied to analyze this problem. We present the existence and uniqueness conditions of a Nash equilibrium when firms face either a deterministic demand function or a general stochastic demand function. In particular, we calculate the explicit Nash equilibrium point when the demand at each firm is a known linear function of price. We also perform sensitivity analysis of the equilibrium prices with respect to cost and capacity parameters.

Our study is related to revenue management in that each firm in our model has finite capacity and makes a pricing decision to maximize its profit. In contrast to most revenue management models, which aim to maximize the revenue or profit of one firm, this chapter analyzes the pricing strategies for multiple firms competing for customers from a common pool. Potential applications for our research may be found in the airline and hotel industries. For example, consider several airlines offering direct flights between the same origin and destination, with departures and arrivals at similar times. Since the number of seats on a flight is fixed, each airline makes pricing decisions to maximize its profit or revenue under a capacity constraint. Similarly, consider several hotels of the same quality located in the same city. Each of them tries to attract customers from a same pool by making pricing decisions for a fixed number of rooms. In this chapter we focus on the class of non-dynamic pricing strategies. We remark that it was shown in Gallego and van Ryzin (1994) that the one-price strategy is near the optimal dynamic pricing strategy. Using a game theoretic formulation we demonstrate the existence of equilibrium prices when the demands are deterministic and also when demands are random following a general stochastic demand function.

As mentioned in Section 2.5, there are other papers which treat pricing strategies among competitive firms, such as Bernstein et al. (1999) and Bernstein and Federgruen

(1999). Our model is different from those used in Bernstein et al. and Bernstein and Federgruen. In our model, each firm has limited capacity and the demands are assumed to have general properties which should be satisfied by most real substitutable products. Furthermore, in addition to deterministic demand, we also consider the case when the price-dependent demands are random. A stochastic ordering relationship is used to express the dependency between random demand and pricing.

Our model is also related to papers analyzing competitive oligopoly models with stochastic demands, such as Birge et al. (1998), van Mieghem and Dada (1999), and Bernstein and Federgruen (2002). As was previously mentioned in Section 2.5, these authors consider a certain kinds of demand structures, for example, multiplicative and non-multiplicative. In contrast, we model the random demands with general stochastic functions. Based on common properties of substitutable products, Topkis (1979) states three properties of the deterministic demand functions which are consistent with common observation. These properties are assumed in many subsequent papers. In this chapter, we generalize these assumptions to a stochastic scenario. Compared with the demand structures mentioned above, our demand structure is more general.

The rest of this chapter is organized as follows. In Section 6.2 we analyze the pricing game with deterministic demands. Then the case with linear demands is studied, and the unique Nash equilibrium is calculated. In Section 6.3, we study the existence and uniqueness of a Nash equilibrium when the demands are stochastic. We analyze the sensitivity of equilibrium prices with respect to the cost and capacity parameters in Section 6.4 and conclude the chapter with a discussion in Section 6.5.

6.2 Deterministic demand

In this section, we consider the case in which the demand of each firm is given in a deterministic form. For the sake of brevity and to facilitate some of the derivations, we confine ourselves to a two-firm model. All results apply for a model with n firms.

We use the following notation:

c_i: the capacity of firm i;

w_i: the unit cost of the product/service at firm i;

p_i: the selling price of the product/service at firm i;

$\pi_i(p_1, p_2)$: the payoff function of firm i.

$d_i(p_1, p_2)$: the deterministic demand of firm i.

In this section we assume that $d_i(p_1, p_2)$ is a continuous and twice differentiable function of p_1 and p_2. The demand actually satisfied at firm i depends on its demand $d_i(p_1, p_2)$ and its capacity c_i. The payoff function of firm i is

$$\pi_i(p_1, p_2) = (p_i - w_i) \min\{c_i, d_i(p_1, p_2)\}.$$

To study the interactive pricing strategies of the two firms and analyze the existence of a Nash equilibrium, we make use of the definition of submodularity and related results introduced by Topkis (1979) (see Chapter 5).

In our model firms offer substitutable services and compete for customers. We assume three properties of the demand functions:

(A6.1) $\frac{\partial d_i(p_1,p_2)}{\partial p_i} < 0$;

(A6.2) $\frac{\partial d_i(p_1,p_2)}{\partial p_j} > 0, j \neq i$;

(A6.3) $-d_i(p_1, p_2)$ is submodular in (p_1, p_2).

Assumptions (A6.1) and (A6.2) are obvious. Assumption (A6.3) is equivalent to: For $p_1^{small} \leq p_1^{large}$, $d_1(p_1^{small}, p_2) - d_1(p_1^{large}, p_2)$ becomes larger as p_2 becomes smaller. For example, an increase in the price of beef will cause a greater reduction in the demand for beef when the price of chicken is lower. These assumptions, introduced by Topkis (1979), are commonly observable with substitutable services and have been assumed by many papers (Bernstein et al. (1999), Bernstein and Federgruen (1999), and Bernstein and Federgruen (2002)) .

Similarly, as in Lippman and McCardle (1997), for mathematical convenience we let $z_2 = -p_2$ be the decision variable of firm 2. z_2 is the negative of the price charged by firm 2. Then using Lemma 5.3, the three assumptions above can be rewritten as:

(A6.1$'$) $\frac{\partial d_1(p_1,z_2)}{\partial p_1} < 0, \frac{\partial d_1(p_1,z_2)}{\partial z_2} < 0$;

(A6.2$'$) $\frac{\partial d_2(p_1,z_2)}{\partial p_1} > 0, \frac{\partial d_2(p_1,z_2)}{\partial z_2} > 0$;

(A6.3$'$) $-d_i(p_1, z_2)$ is supermodular in (p_1, z_2); in other words, $d_i(p_1, z_2)$ is submodular in (p_1, z_2).

Lemma 6.1 *(Topkis (1978)) Suppose $g(x_1, x_2)$ is monotone in both x_1 and x_2 and is a* *supermodular function in (x_1, x_2). Also suppose that $f(\cdot)$ is an increasing convex function. Then $f(g(x_1, x_2))$ is a supermodular function in (x_1, x_2).*

Theorem 6.2 *The payoff function of each firm is supermodular in (p_1, p_2).*

Proof For firm 1 we have

$$\pi_1(p_1, z_2) = (p_1 - w_1) \min\{c_1, d_1(p_1, z_2)\}.$$
$$= (p_1 - w_1)c_1 - (p_1 - w_1)(c_1 - d_1(p_1, z_2))^+$$

It follows that

$$\begin{aligned} \frac{\partial \pi_1(p_1,z_2)}{\partial p_1} &= c_1 - (c_1 - d_1(p_1, z_2))^+ - (p_1 - w_1)\frac{\partial (c_1 - d_1(p_1,z_2))^+}{\partial p_1}, \\ \frac{\partial^2 \pi_1(p_1,z_2)}{\partial p_1 \partial z_2} &= -\frac{\partial (c_1 - d_1(p_1,z_2))^+}{\partial z_2} - (p_1 - w_1)\frac{\partial^2 (c_1 - d_1(p_1,z_2))^+}{\partial p_1 \partial z_2}. \end{aligned}$$

From the assumption (A6.1$'$) we know that $\frac{\partial d_1(p_1,z_2)}{\partial z_2} < 0$, so $\frac{\partial (c_1 - d_1(p_1,z_2))^+}{\partial z_2} \geq 0$. Meanwhile, from assumptions (A6.1$'$) and (A6.3$'$) we know that $c_1 - d_1(p_1, z_2)$ is increasing in both p_1 and z_2 and supermodular in (p_1, z_2). Further, since $(\cdot)^+$ is a convex increasing function, by Lemma 6.1 we know that $(c_1 - d_1(p_1, z_2))^+$ is supermodular in (p_1, z_2), which is equivalent to $\frac{\partial^2 (c_1 - d_1(p_1,z_2))^+}{\partial p_1 \partial z_2} \geq 0$. Therefore, we obtain $\frac{\partial^2 \pi_1(p_1,z_2)}{\partial p_1 \partial z_2} \leq 0$, which proves that $\pi_1(p_1, z_2)$ is submodular in (p_1, z_2) and consequently supermodular in (p_1, p_2). Similarly, we can prove that $\pi_2(p_1, p_2)$ is supermodular in (p_1, p_2). ∎

Theorem 6.3 *In the two-firm pricing game with deterministic demands, a Nash equilibrium exists.*

Proof The result follows from the supermodularity of the payoff functions and Theorem 3.1 of Topkis (1979). ∎

Now we consider the uniqueness of the Nash equilibrium. Moulin (1986) states that uniqueness is stronger than local stability in that if a Nash equilibrium is globally stable, then it is unique. A sufficient condition for the uniqueness of a Nash equilibrium is that the slopes of the players' best responses never exceed 1 in absolute value. In a two-player game, this is equivalent to the following lemma:

Lemma 6.4 *If a Nash equilibrium exists and* $|\frac{\partial^2 \pi_i(x_i,x_j)}{\partial x_i \partial x_j}| < |\frac{\partial^2 \pi_i(x_i,x_j)}{\partial x_i^2}|$*, for all* $(x_i, x_j), i, j = 1, 2, i \neq j$*, then the Nash equilibrium is unique.*

In this lemma, the left (right) side of the inequality can be interpreted as the effect of the firm j's (firm i's) decision on the best response of firm i. If firm i itself can act to counter firm j's effect on its best response, the equilibrium is stable. After some calculations, we obtain

$$
\begin{aligned}
|\frac{\partial^2 \pi_1(p_1, z_2)}{\partial p_1^2}| &= |2\frac{\partial (c_1 - d_1(p_1, z_2))^+}{\partial p_1} + (p_1 - w_1)\frac{\partial^2 (c_1 - d_1(p_1, z_2))^+}{\partial p_1^2}|, \\
|\frac{\partial^2 \pi_1(p_1, z_2)}{\partial p_1 \partial z_2}| &= \frac{\partial (c_1 - d_1(p_1, z_2))^+}{\partial z_2} + (p_1 - w_1)\frac{\partial^2 (c_1 - d_1(p_1, z_2))^+}{\partial p_1 \partial z_2}.
\end{aligned}
\tag{6.1}
$$

Combining (6.1) and Lemma 6.4, we obtain a sufficient condition for the uniqueness of the Nash equilibrium.

So far, we have obtained existence and uniqueness conditions for a Nash equilibrium. However, how to calculate it in closed form is another issue. In what follows we show how to calculate the Nash equilibrium when the demand at each firm is a linear function of price; i.e.,

$$d_i(p_i, p_j) = a_i - b_i p_i + \beta_{ij} p_j \text{ with } a_i, b_i, \beta_{ij} > 0 \text{ for } i, j = 1, 2 \text{ and } i \neq j. \tag{6.2}$$

We assume that $b_i > \beta_{ij}$; otherwise both firms can increase their demands by simultaneously raising their prices. Note that the assumptions (A6.1), (A6.2), and (A6.3) mentioned above are satisfied.

To calculate the Nash equilibrium we first consider firm 1's best response given the price of firm 2, p_2. We know that firm 1's payoff function is

$$\pi_1(p_1, p_2) = \begin{cases} (p_1 - w_1)d_1(p_1, p_2) & \text{when } d_1(p_1, p_2) < c_1, \\ (p_1 - w_1)c_1 & \text{otherwise.} \end{cases}$$

Let $p_1^{'}$ solve $d_1(p_1, p_2) = c_1$, i.e.,

$$p_1^{'} = \frac{a_1 - c_1 + \beta_{12}p_2}{b_1}.$$

We now consider the two cases $p_1 > p_1^{'}$ and $p_1 \leq p_1^{'}$ separately. If $p_1 > p_1^{'}$, i.e., $d_1(p_1, p_2) < c_1$, the payoff function is

$$\pi_1(p_1, p_2) = (p_1 - w_1)(a_1 - b_1p_1 + \beta_{12}p_2), \tag{6.3}$$

which is concave in p_1 given p_2. Therefore the best response is obtained by making the first derivative of (6.3) equal 0, i.e.,

$$a_1 - 2b_1p_1 + \beta_{12}p_2 + b_1w_1 = 0. \tag{6.4}$$

In the (p_1, p_2) plane, (6.4) is a line with slope $\frac{2b_1}{\beta_{12}}$ and it passes through the point $(\frac{a_1+b_1w_1}{2b_1}, 0)$. This line is the best response of firm 1 when its capacity is not constraining. We denote the line as NoCapaBR1. If $p_1 \leq p_1^{'}$, i.e., $d_1(p_1, p_2) \geq c_1$, the payoff function is

$$\pi_1(p_1, p_2) = (p_1 - w_1)c_1,$$

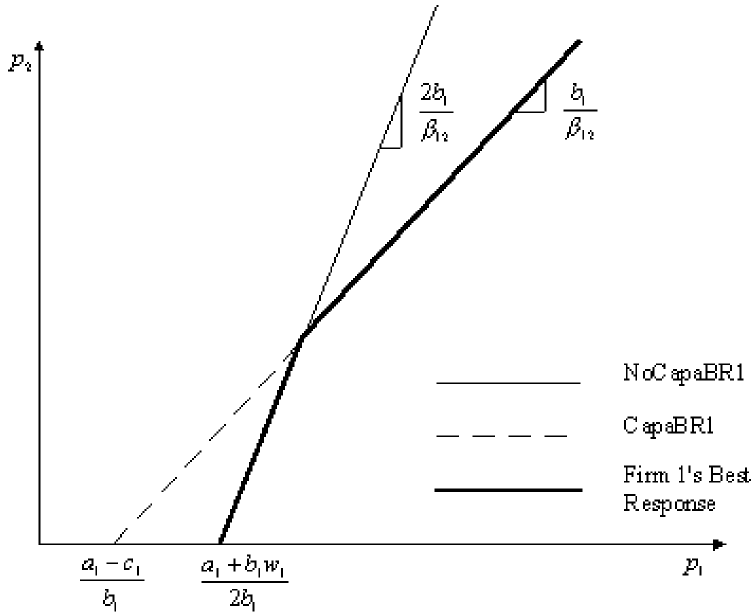

Figure 6.1: Firm 1's best response when $c_1 \geq \frac{a_1 - b_1 w_1}{2}$.

which is increasing in p_1. Therefore the best response for firm 1 is

$$p_1 = p_1^{'} = \frac{a_1 - c_1 + \beta_{12} p_2}{b_1}. \tag{6.5}$$

which is a line with slope $\frac{b_1}{\beta_{12}}$ which passes through the point $(\frac{a_1 - c_1}{b_1}, 0)$. We denote this line as CapaBR1, which is the best response of firm 1 when its capacity constraint is active.

Based on the analysis above, we can derive the best response of firm 1. There are two cases. If $\frac{a_1 + b_1 w_1}{2b_1} \geq \frac{a_1 - c_1}{b_1}$, i.e, $c_1 \geq \frac{a_1 - b_1 w_1}{2}$, firm 1's best response includes a segment from NoCapaBR1 and a segment from CapaBR1 (see Figure 6.1); otherwise, firm 1's best response is line CapaBR1 (see Figure 6.2).

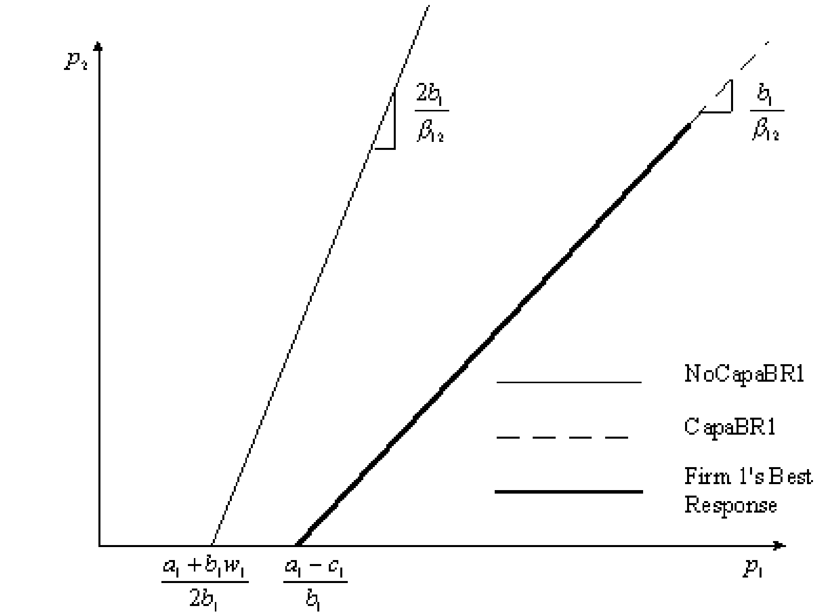

Figure 6.2: Firm 1's best response when $c_1 < \frac{a_1 - b_1 w_1}{2}$.

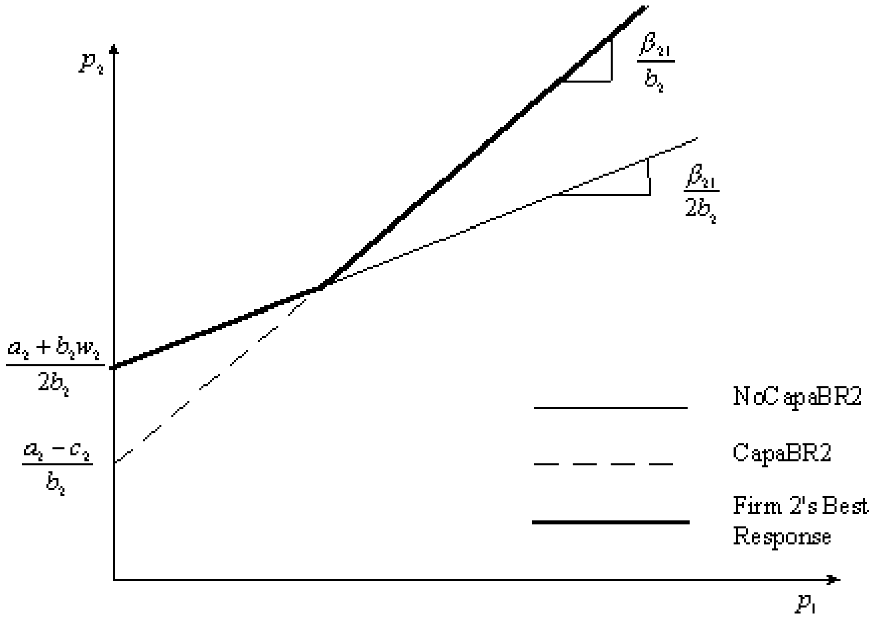

Figure 6.3: Firm 2's best response when $c_2 \geq \frac{a_2 - b_2 w_2}{2}$.

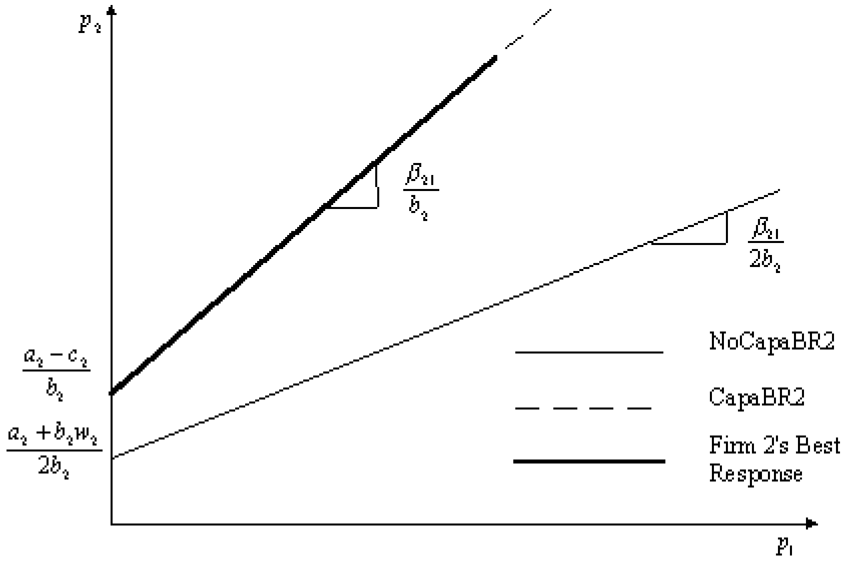

Figure 6.4: Firm 2's best response when $c_2 < \frac{a_2-b_2w_2}{2}$.

Similarly for firm 2, denoting the line

$$a_2 - 2b_2p_2 + \beta_{21}p_1 + b_2w_2 = 0$$

as NoCapaBR2 and the line

$$p_2 = \frac{a_2 - c_2 + \beta_{21}p_1}{b_2}$$

as CapaBR2, we can derive firm 2's best response. If $\frac{a_2+b_2w_2}{2b_2} \geq \frac{a_2-c_2}{b_2}$, i.e, $c_2 \geq \frac{a_2-b_2w_2}{2}$, firm 2's best response includes segments from NoCapaBR2 and CapaBR2 (see Figure 6.3); otherwise, firm 2's best response is line CapaBR2 (see Figure 6.4).

The intersection of two best response curves is the Nash equilibrium. The values of capacities c_1 and c_2 affect where the two best responses intersect. There are four cases as follows.

- Case 1: $c_1 < \frac{a_1-b_1w_1}{2}$, $c_2 < \frac{a_2-b_2w_2}{2}$

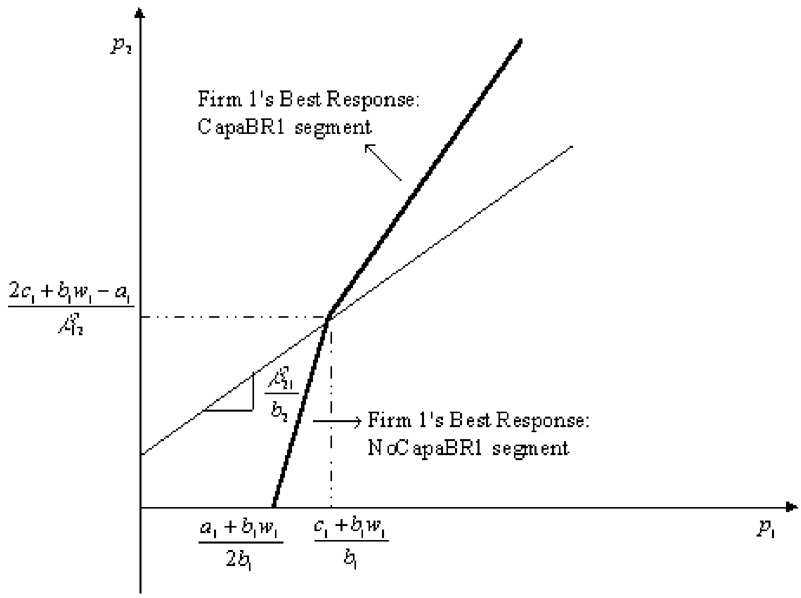

Figure 6.5: If $(2b_1b_2 - \beta_{12}\beta_{21})c_1 + \beta_{12}b_1c_2 = m_1$, CapaBR2 passes the intersection of line NoCapaBR1 and line CapaBR1.

In this case the best responses of the two firms are, respectively, line CapaBR1 and line CapaBR2. It follows that the Nash equilibrium is

$$\begin{cases} p_1 = \frac{a_1b_2+\beta_{12}a_2-\beta_{12}c_2-b_2c_1}{b_1b_2-\beta_{12}\beta_{21}}, \\ p_2 = \frac{a_2b_1+\beta_{21}a_1-\beta_{21}c_1-b_1c_2}{b_1b_2-\beta_{12}\beta_{21}}. \end{cases} \tag{6.6}$$

- Case 2: $c_1 \geq \frac{a_1-b_1w_1}{2}$ and $c_2 < \frac{a_2-b_2w_2}{2}$

In this case the best response of firm 1 is shown in Figure 6.1, while that of firm 2 is line CapaBR2 shown in Figure 6.4. After some calculations, we obtain that if

$$(2b_1b_2 - \beta_{12}\beta_{21})c_1 + \beta_{12}b_1c_2 = m_1,$$

where $m_1 = a_1b_1b_2 + \beta_{12}\beta_{21}b_1w_1 + \beta_{12}a_2b_1 - b_1^2b_2w_1$, then the firm 2's best response, line CapaBR2, passes through the intersection of line NoCapaBR1 and line

CapaBR1, shown in Figure 6.5. It follows that if

$$(2b_1b_2 - \beta_{12}\beta_{21})c_1 + \beta_{12}b_1c_2 \geq \ m_1, \tag{6.7}$$

the intersection of line NoCapaBR1 and line CapaBR2 and the Nash equilibrium is

$$\begin{cases} p_1 = \frac{a_1b_2+b_1b_2w_1+\beta_{12}a_2-\beta_{12}c_2}{2b_1b_2-\beta_{12}\beta_{21}}, \\ p_2 = \frac{2a_2b_1+\beta_{21}a_1+\beta_{21}w_1b_1-2b_1c_2}{2b_1b_2-\beta_{12}\beta_{21}}; \end{cases} \tag{6.8}$$

otherwise, the intersection of line CapaBR1 and line CapaBR2, given by (6.6), is the Nash equilibrium (same as in Case 1).

- Case 3: $c_1 < \frac{a_1-b_1w_1}{2}$ and $c_2 \geq \frac{a_2-b_2w_2}{2}$

In this case the best response of firm 1 is line CapaBR1 shown in Figure 6.2, and that of firm 2 is shown in Figure 6.3. Similar to Case 2, we obtain that if

$$(2b_1b_2 - \beta_{12}\beta_{21})c_2 + \beta_{21}b_2c_1 = m_2$$

where $m_2 = a_2b_1b_2 + \beta_{12}\beta_{21}b_2w_2 + \beta_{12}a_1b_2 - b_2^2b_1w_2$, then the firm 2's best response, line CapaBR2, passes through the intersection of line NoCapaBR1 and line CapaBR1. It follows that if

$$(2b_1b_2 - \beta_{12}\beta_{21})c_2 + \beta_{21}b_2c_1 \geq \ m_2, \tag{6.9}$$

the intersection of line CapaBR1 and line NoCapaBR2 and the Nash equilibrium is

$$\begin{cases} p_1 = \frac{2a_1b_2+\beta_{12}a_2+\beta_{12}w_2b_2-2b_2c_1}{2b_1b_2-\beta_{12}\beta_{21}}, \\ p_2 = \frac{a_2b_1+b_1b_2w_2+\beta_{21}a_1-\beta_{21}c_1}{2b_1b_2-\beta_{12}\beta_{21}}; \end{cases} \tag{6.10}$$

otherwise, the intersection of lines CapaBR1 and CapaBR2, given by (6.6), is the Nash equilibrium (again same as in Case 1).

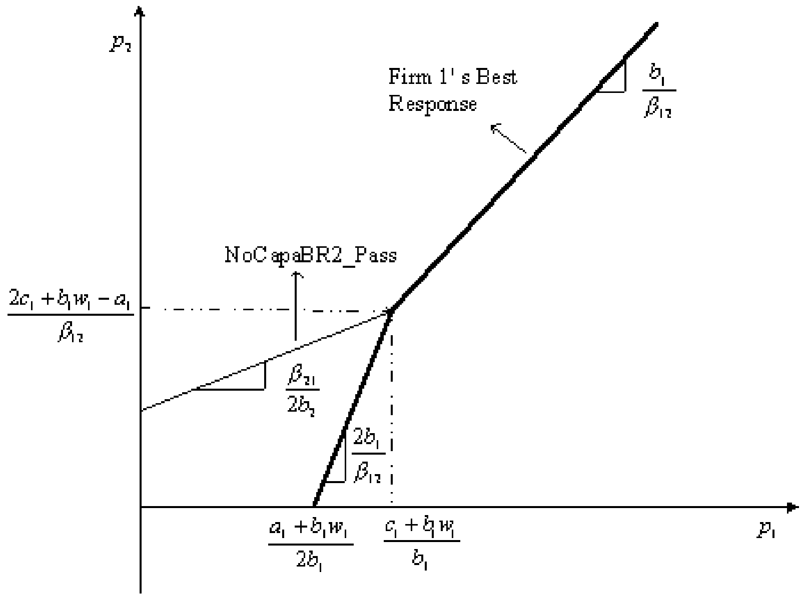

Figure 6.6: Line NoCapaBR2_Pass

- Case 4: $c_1 \geq \frac{a_1-b_1w_1}{2}$ and $c_2 \geq \frac{a_2-b_2w_2}{2}$.

  In this case the best response of firm 1 includes the segments from NoCapaBR1 and CapaBR1 shown in Figure 6.1, while that of firm 2 includes the segments from NoCapaBR2 and CapaBR2 shown in Figure 6.3.

  We modify the value of c_1 to make line NoCapaBR2 pass through the point $(\frac{c_1+b_1w_1}{b_1}, \frac{2c_1+b_1w_1-a_1}{\beta_{12}})$, i.e., the intersection of line NoCapaBR1 and line CapaBR1 (see Figure 6.6). In this case, we denote the value of c_1 as $c_1^{'}$ and refer to the line NoCapaBR2 as NoCapaBR2_Pass. After some calculations, we obtain

$$c_1^{'} = \frac{b_1(2a_1b_2 + \beta_{12}b_2w_2 + \beta_{12}a_2 + \beta_{12}\beta_{21}w_1 - 2b_1b_2w_1)}{4b_1b_2 - \beta_{12}\beta_{21}}.$$

  Notice that the best response of firm 2 has segments from NoCapaBR2 and CapaBR2. If $c_1 < c_1^{'}$, the NoCapaBR2 segment falls above NoCapaBR2_Pass, as in

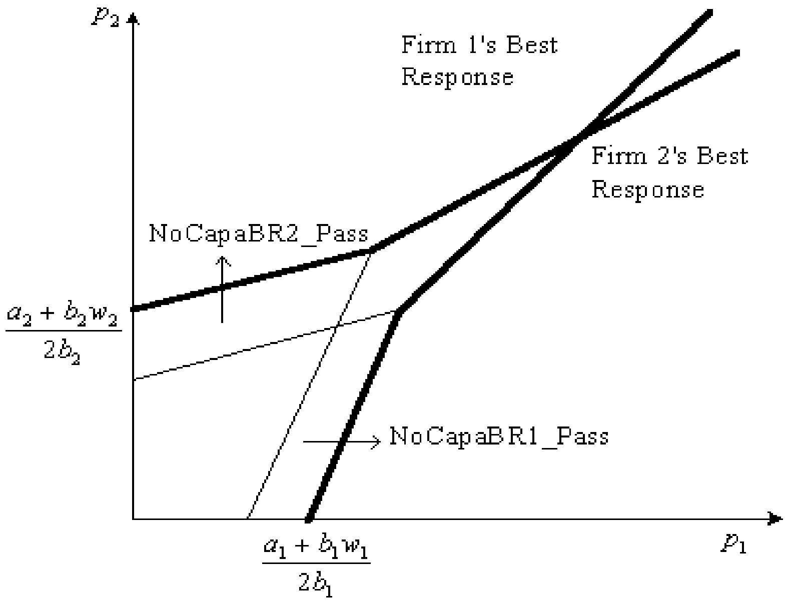

Figure 6.7: In Case 4-1, the Nash equilibrium is the intersection of line CapaBR1 and line CapaBR2.

Figure 6.7; otherwise, the NoCapaBR2 segment falls below NoCapaBR2_Pass, as in Figure 6.8 and Figure 6.9.

Similarly, if line NoCapaBR1 passes through the intersection of lines NoCapaBR2 and CapaBR2, we call it NoCapaBR1_Pass and denote the value of c_2 in this case as $c_2^{'}$. We obtain

$$c_2^{'} = \frac{b_2(2a_2b_1 + \beta_{21}b_1w_1 + \beta_{21}a_1 + \beta_{12}\beta_{21}w_1 - 2b_1b_2w_1)}{4b_1b_2 - \beta_{12}\beta_{21}}.$$

If $c_2 < c_2^{'}$, the NoCapaBR1 segment of firm 1's best response falls on the right side of NoCapaBR1_Pass, as in Figure 6.7 and Figure 6.8; otherwise, the NoCapaBR1 segment falls on the right side of NoCapaBR1_Pass, as in Figure 6.9.

It is easy to verify that $c_1^{'} > \frac{a_1 - b_1w_1}{2}$ and $c_2^{'} > \frac{a_2 - b_2w_2}{2}$. Therefore, based on the values of c_1 and c_2, there are four cases.

- Case 4-1: $\frac{a_1-b_1w_1}{2} \leq c_1 < c_1^{'}$ and $\frac{a_2-b_2w_2}{2} \leq c_2 < c_2^{'}$

  Based on the analysis above, the best responses of firm 1 and firm 2 are shown in Figure 6.7. It follows that the Nash equilibrium is the intersection of line CapaBR1 and line CapaBR2, given in (6.6).

- Case 4-2: $c_1 \geq c_1^{'}$ and $\frac{a_2-b_2w_2}{2} \leq c_2 < c_2^{'}$

  There are two cases (see Figure 6.8). Similar to Case 2, we know that if

$$(2b_1b_2 - \beta_{12}\beta_{21})c_1 + \beta_{12}b_1c_2 \geq \ m_1,$$

  the Nash equilibrium is the intersection of line NoCapaBR1 and line CapaBR2 (see the below graph in Figure 6.8), which is (6.8); otherwise, the Nash equilibrium is the intersection of line CapaBR1 and line CapaBR2 (see the upper graph in Figure 6.8), given in (6.6).

- Case 4-3: $\frac{a_1-b_1w_1}{2} \leq c_1 < c_1^{'}$ and $c_2 \geq c_2^{'}$

  Similar to Case 3, we know that if

$$(2b_1b_2 - \beta_{12}\beta_{21})c_2 + \beta_{21}b_2c_1 \geq \ m_2,$$

  the Nash equilibrium is the intersection of line CapaBR1 and line NoCapaBR2, given in (6.10); otherwise the Nash equilibrium is the intersection of line CapaBR1 and line CapaBR2, given in (6.6).

- Case 4-4: $c_1 \geq c_1^{'}$ and $c_2 \geq c_2^{'}$

  The best responses of retailer 1 and retailer 2 are shown in Figure 6.9. It follows that the Nash equilibrium is the intersection of lines NoCapaBR1 and

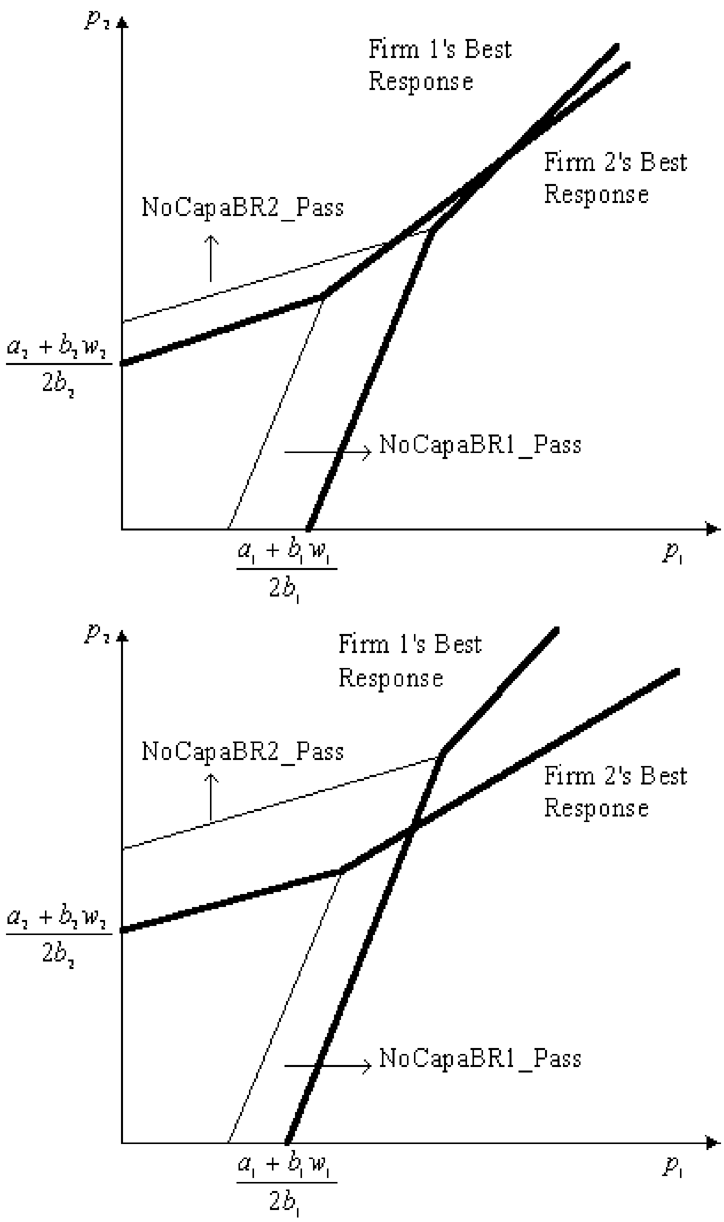

Figure 6.8: In Case 4-2, CapaBR2 may intersect with CapaBR1 or NocapaBR1.

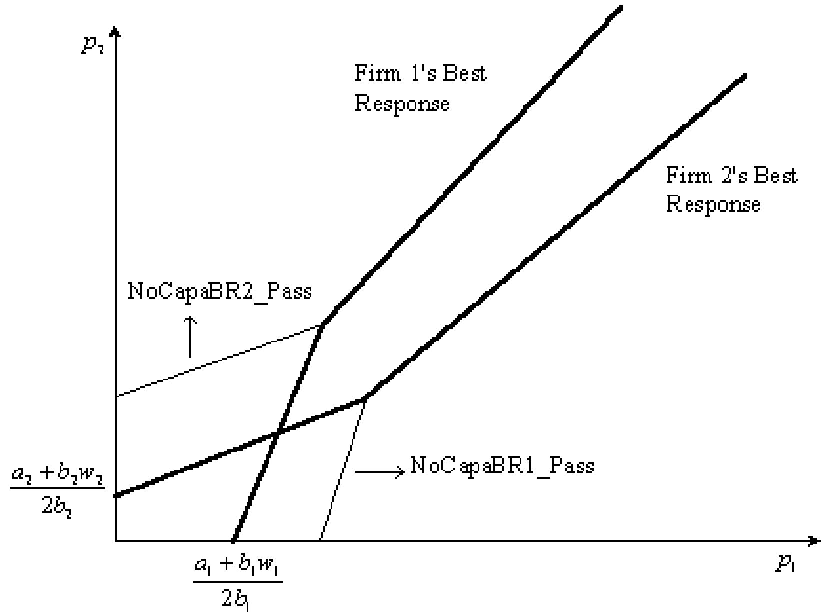

Figure 6.9: In Case 4-4, the Nash equilibrium is the intersection of line NoCapaBR1 and line NoCapaBR2

NoCapaBR2, which is

$$\begin{cases} p_1 = \frac{2a_1b_2+\beta_{12}a_2+\beta_{12}b_2w_2+2b_1b_2w_1}{4b_1b_2-\beta_{12}\beta_{21}}, \\ p_2 = \frac{2a_2b_1+\beta_{21}a_1+\beta_{21}w_1b_1+2b_1b_2w_2}{4b_1b_2-\beta_{12}\beta_{21}}. \end{cases} \tag{6.11}$$

From the above analysis, we see that the assumption $b_i > \beta_{ij}$ guarantees that the best responses of firm 1 and firm 2 intersect only once, which results in a unique Nash equilibrium. Furthermore, the unique Nash equilibrium has four possible values: (6.6), (6.8), (6.10) and (6.11). The capacities of the two firms play a major role in determining the value of the Nash equilibrium in this pricing game. When the capacities of both firms are small (see Case 1 and Case 4-1), they are constraining, and therefore the Nash equilibrium is dependent on both capacities (see (6.6)). When one of the firms has a comparatively large capacity (see Case 2, Case 3, Case 4-2 and Case 4-3), if its capacity is big enough to compensate for the capacity limitation of the other firm (conditions

(6.7) and (6.9)), then the firm with the large capacity is unconstrained, and therefore the Nash equilibrium is independent of the comparatively large capacity ((6.8) and (6.10)); if the large capacity is not large enough to compensate for the other, both firms are constrained, which results in a Nash equilibrium dependent on both capacities. Only when both firms have comparatively large capacities are both firms unconstrained (see Case 4-4). In this case the Nash equilibrium is independent of the capacities (see (6.11)).

In order for the equilibrium prices to be feasible, the price of each firm must be greater than its unit cost, i.e.,

$$\begin{cases} p_1 > w_1, \\ p_2 > w_2. \end{cases}$$

In the appendix, we prove that the Nash equilibria of the above cases are feasible under the following condition

$$\begin{cases} a_1 - b_1 w_1 + \beta_{12} w_2 > 0, \\ a_2 - b_2 w_2 + \beta_{21} w_1 > 0. \end{cases} \tag{6.12}$$

From (6.2), we know that when both firms charge their respective unit cost, they both have positive demand. This goes with our intuition.

6.3 Stochastic demand

In this section, we study the case where the demands of both firms are stochastic. The notation is the same as in the last section except that for the demand. We represent the stochastic demand of firm i with a continuous random variable $D_i(p_1, p_2)$. To model the dependence of $D_i(p_1, p_2)$ on deterministic decision variables p_1 and p_2, we assume that the c.d.f. of $D_i(p_1, p_2)$ is a function of p_1 and p_2 and denote it as $F_i^{(p_1,p_2)}(x)$. We use

$\pi_i(p_1, p_2)$ to represent the *expected* payoff function of firm i.

Recall from Section 6.2 that when both demands are deterministic, the demand of one firm increases (decreases) when the price of the other firm increases (decreases). In this section, the demands are random. The demand of one firm changes from one random variable to another when the price the other firm changes. To compare these two random variables, we apply the concept of stochastic ordering (see Ross (1983) or Shaked and Shanthikumar (1994)). Let U and V be two random variables. If

$$\Pr\{U > x\} \leq \Pr\{V > x\} \text{ for all } x,$$

then U is said to be smaller than V in stochastic ordering (denoted by $U \leq_{st} V$).

Assumptions in Section 6.2 express common properties of substitutable services when the demands are deterministic. We now modify those assumptions for the stochastic scenario.

Analogous to assumption (A6.1) in Section 6.2, we assume:

$$\begin{aligned} D_1(p_1^{large}, p_2) &\leq_{st} D_1(p_1^{small}, p_2), \qquad p_1^{large} \geq p_1^{small}, \\ D_2(p_1, p_2^{large}) &\leq_{st} D_2(p_1, p_2^{small}), \qquad p_2^{large} \geq p_2^{small}. \end{aligned}$$

Denoting $1 - F_i^{(p_1,p_2)}(x)$ as $\overline{F}_i^{(p_1,p_2)}$, this assumption is equivalent to:

$$\begin{aligned} \text{(A6.1}''\text{)} \qquad & \overline{F}_1^{(p_1^{large},p_2)}(x) \leq \overline{F}_1^{(p_1^{small},p_2)}(x), \qquad p_1^{large} \geq p_1^{small}, \\ & \overline{F}_2^{(p_1,p_2^{large})}(x) \leq \overline{F}_2^{(p_1,p_2^{small})}(x), \qquad p_2^{large} \geq p_2^{small}. \end{aligned}$$

Similarly, analogous to assumption (A6.2) in Section 6.2, we make the following assumption (A6.2″):

$$\text{(A6.2}''\text{)} \qquad \overline{F}_1^{(p_1,p_2^{small})}(x) \leq \overline{F}_1^{(p_1,p_2^{large})}(x), \qquad p_2^{large} \geq p_2^{small},$$
$$\overline{F}_2^{(p_1^{small},p_2)}(x) \leq \overline{F}_2^{(p_1^{large},p_2)}(x), \qquad p_1^{large} \geq p_1^{small}.$$

Assumption (A6.3) in Section 6.2 states that $d_1(p_1^{small}, p_2) - d_1(p_1^{large}, p_2)$ is larger as p_2 becomes smaller for $p_1^{small} \leq p_1^{large}$. When demands are stochastic, we assume correspondingly that $\overline{F}_1^{(p_1^{small},p_2)}(x) - \overline{F}_1^{(p_1^{large},p_2)}(x)$ is larger as p_2 becomes smaller for $p_1^{small} \leq p_1^{large}$; in other words, we make the following assumption:

(A6.3″) $\overline{F}_i^{(p_1,p_2)}(x)$ is supermodular in (p_1, p_2).

The expected payoff function of firm 1 is

$$\begin{aligned}
\pi_1(p_1,p_2) &= (p_1 - w_1)E[\min\{c_1, D_1(p_1,p_2)\}] \\
&= (p_1 - w_1)c_1 + (p_1 - w_1)\int_0^{c_1}(c_1 - x)d\overline{F}_1^{(p_1,p_2)}(x) \\
&= (p_1 - w_1)\int_0^{c_1}\overline{F}_1^{(p_1,p_2)}(x)dx
\end{aligned}$$

After some calculations, we obtain

$$\begin{aligned}
\frac{\partial \pi_1(p_1,p_2)}{\partial p_1} &= \int_0^{c_1}\overline{F}_1^{(p_1,p_2)}(x)dx + (p_1 - w_1)\int_0^{c_1}\frac{\partial \overline{F}_1^{(p_1,p_2)}}{\partial p_1}(x)dx, \\
\frac{\partial^2 \pi_1(p_1,p_2)}{\partial p_1 \partial p_2} &= \int_0^{c_1}\frac{\partial \overline{F}_1^{(p_1,p_2)}(x)}{\partial p_2}dx + (p_1 - w_1)\int_0^{c_1}\frac{\partial^2 \overline{F}_1^{(p_1,p_2)}(x)}{\partial p_1 \partial p_2}dx.
\end{aligned}$$

Theorem 6.5 *In our two-firm pricing game with stochastic demands, a Nash equilibrium exists.*

Proof From assumption (A6.1″), we know that $\frac{\partial \overline{F}_1^{(p_1,p_2)}(x)}{\partial p_2} \geq 0$. Assumption (A6.3″) is equivalent to $\frac{\partial^2 \overline{F}_1^{(p_1,p_2)}(x)}{\partial p_1 \partial p_2} \geq 0$. Therefore, we obtain that $\frac{\partial^2 \pi_1(p_1,p_2)}{\partial p_1 \partial p_2} \geq 0$, which proves the supermodularity of $\pi_1(p_1, p_2)$. Similarly, we can prove that $\pi_2(p_1, p_2)$ is supermodular in (p_1, p_2). Then by Theorem 3.1 of Topkis (1979), we conclude the existence of a Nash equilibrium. ■

Similar to the deterministic case in Section 6.2, the uniqueness condition of the Nash equilibrium can be obtained by applying Lemma 6.4.

6.4 Sensitivity analysis

In this section, we analyze the sensitivity of equilibrium prices with respect to capacity c_i and unit cost w_i. For the deterministic case with linear demand function, this analysis is straightforward, because the linear relationship prevails. Here, we focus on the scenario with stochastic demand.

First we study the effects of the capacities, c_1 and c_2, on the Nash equilibrium. Based on the results above, we know the Nash equilibrium is characterized by the following optimality conditions:

$$\begin{cases} \frac{\partial \pi_1(p_1,p_2)}{\partial p_1} = 0, \\ \frac{\partial \pi_2(p_1,p_2)}{\partial p_2} = 0, \end{cases}$$

where

$$\frac{\partial \pi_1(p_1,p_2)}{\partial p_1} = \int_0^{c_1} \overline{F}_1^{(p_1,p_2)}(x)dx + (p_1 - w_1)\int_0^{c_1} \frac{\partial \overline{F}_1^{(p_1,p_2)}}{\partial p_1}(x)dx,$$

$$\frac{\partial \pi_2(p_1,p_2)}{\partial p_2} = \int_0^{c_2} \overline{F}_2^{(p_1,p_2)}(x)dx + (p_2 - w_2)\int_0^{c_2} \frac{\partial \overline{F}_2^{(p_1,p_2)}}{\partial p_2}(x)dx.$$

We denote the above formula as

$$\begin{cases} G_1(\cdot) = 0, \\ G_2(\cdot) = 0. \end{cases}$$

By implicit function theory, we have

$$\begin{bmatrix} \frac{\partial p_1}{\partial c_1} & \frac{\partial p_1}{\partial c_2} \\ \\ \frac{\partial p_2}{\partial c_1} & \frac{\partial p_2}{\partial c_2} \end{bmatrix} = - \begin{bmatrix} \frac{\partial G_1}{\partial p_1} & \frac{\partial G_1}{\partial p_2} \\ \\ \frac{\partial G_2}{\partial p_1} & \frac{\partial G_2}{\partial p_2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial G_1}{\partial c_1} & \frac{\partial G_1}{\partial c_2} \\ \\ \frac{\partial G_2}{\partial c_1} & \frac{\partial G_2}{\partial c_2} \end{bmatrix} \tag{6.13}$$

$$= \frac{1}{\left(\frac{\partial G_1}{\partial p_1}\frac{\partial G_2}{\partial p_2} - \frac{\partial G_1}{\partial p_2}\frac{\partial G_2}{\partial p_1}\right)} \begin{bmatrix} -\frac{\partial G_2}{\partial p_2} & \frac{\partial G_1}{\partial p_2} \\ \\ \frac{\partial G_2}{\partial p_1} & -\frac{\partial G_1}{\partial p_1} \end{bmatrix} \begin{bmatrix} \frac{\partial G_1}{\partial c_1} & 0 \\ \\ 0 & \frac{\partial G_2}{\partial c_2} \end{bmatrix}$$

$$= \frac{1}{\left(\frac{\partial G_1}{\partial p_1}\frac{\partial G_2}{\partial p_2} - \frac{\partial G_1}{\partial p_2}\frac{\partial G_2}{\partial p_1}\right)} \begin{bmatrix} -\frac{\partial G_2}{\partial p_2}\frac{\partial G_1}{\partial c_1} & \frac{\partial G_1}{\partial p_2}\frac{\partial G_2}{\partial c_2} \\ \\ \frac{\partial G_2}{\partial p_1}\frac{\partial G_1}{\partial c_1} & -\frac{\partial G_1}{\partial p_1}\frac{\partial G_2}{\partial c_2} \end{bmatrix}$$

where

$$\frac{\partial G_1}{\partial p_1} = 2\int_0^{c_1} \frac{\partial \overline{F}_1^{(p_1,p_2)}}{\partial p_1}(x)dx + (p_1 - w_1)\int_0^{c_1} \frac{\partial^2 \overline{F}_1^{(p_1,p_2)}}{\partial^2 p_1}(x)dx,$$

$$\frac{\partial G_1}{\partial p_2} = \int_0^{c_1} \frac{\partial \overline{F}_1^{(p_1,p_2)}}{\partial p_2}(x)dx + (p_1 - w_1)\int_0^{c_1} \frac{\partial^2 \overline{F}_1^{(p_1,p_2)}}{\partial p_1 \partial p_2}(x)dx,$$

$$\frac{\partial G_2}{\partial p_1} = \int_0^{c_2} \frac{\partial \overline{F}_2^{(p_1,p_2)}}{\partial p_1}(x)dx + (p_2 - w_2)\int_0^{c_2} \frac{\partial^2 \overline{F}_2^{(p_1,p_2)}}{\partial p_1 \partial p_2}(x)dx,$$

$$\frac{\partial G_2}{\partial p_2} = 2\int_0^{c_2} \frac{\partial \overline{F}_2^{(p_1,p_2)}}{\partial p_2}(x)dx + (p_2 - w_2)\int_0^{c_2} \frac{\partial^2 \overline{F}_2^{(p_1,p_2)}}{\partial^2 p_2}(x)dx,$$

$$\frac{\partial G_1}{\partial c_1} = \overline{F}_1^{(p_1,p_2)}(c_1) + (p_1 - w_1)\frac{\partial \overline{F}_1^{(p_1,p_2)}}{\partial p_1}(c_1),$$

$$\frac{\partial G_2}{\partial c_2} = \overline{F}_2^{(p_1,p_2)}(c_2) + (p_2 - w_2)\frac{\partial \overline{F}_2^{(p_1,p_2)}}{\partial p_2}(c_2).$$

Theorem 6.6 *At equilibrium a small change in c_i leads to the changes in p_1 and p_2 such that*

$$\frac{\partial p_i}{\partial c_i} / \frac{\partial p_j}{\partial c_i} = \frac{-2\int_0^{c_j} \frac{\partial \overline{F}_j^{(p_1,p_2)}}{\partial p_j}(x)dx - (p_j - w_j)\int_0^{c_j} \frac{\partial^2 \overline{F}_j^{(p_1,p_2)}}{\partial^2 p_j}(x)dx}{\int_0^{cj} \frac{\partial \overline{F}_j^{(p_1,p_2)}}{\partial p_i}(x)dx + (p_j - w_j)\int_0^{c_j} \frac{\partial^2 \overline{F}_j^{(p_1,p_2)}}{\partial p_i \partial p_j}(x)dx}, \quad j \neq i.$$

Proof From (6.13) we know that $\frac{\partial p_i}{\partial c_i} / \frac{\partial p_j}{\partial c_i} = -\frac{\partial G_j}{\partial p_j} / \frac{\partial G_j}{\partial p_i}$, so the quantity relation is easy to establish. ■

Notice that supermodularity is a weaker condition than concavity. We now replace assumption (A6.3$''$) with:

(A6.4) $\overline{F}_i^{(p_1,p_2)}(x)$ is concave in (p_1, p_2).

Assumption (A6.4) implies that for any given p_j, $\overline{F}_i^{(p_1,p_2)}(x)$ is concave in p_i, $i, j = 1, 2, i \neq j,$. Based on assumption (A6.4), we obtain the following result.

Theorem 6.7 *Under the uniqueness condition of Lemma 6.4 and assumption (A6.4), at equilibrium an increase in c_i leads to decreases in both p_1 and p_2 if and only if $\overline{F}_i^{(p_1,p_2)}(c_i) + (p_i - w_i)\frac{\partial \overline{F}_i^{(p_1,p_2)}}{\partial p_i}(c_i) \leq 0$.*

Proof By assumption (A6.4), we know that $\frac{\partial G_i}{\partial p_i} \leq 0$, which is equivalent to $\frac{\partial^2 \pi_i(x_1,x_2)}{\partial x_i^2} \leq 0$. In this case, the uniqueness condition leads to $(\frac{\partial G_1}{\partial p_1}\frac{\partial G_2}{\partial p_2} - \frac{\partial G_1}{\partial p_2}\frac{\partial G_2}{\partial p_1}) > 0$. Combining this with the fact that $\frac{\partial G_i}{\partial p_j} \geq 0$, we have that $\frac{\partial p_i}{\partial c_i}$ and $\frac{\partial p_j}{\partial c_i}$ are non-positive if and only if $\frac{\partial G_i}{\partial c_i} \leq 0$. ■

Next we analyze the changes in equilibrium prices with respect to changes in the unit cost, w_i. By implicit function theory, we obtain

$$\begin{bmatrix} \frac{\partial p_1}{\partial w_1} & \frac{\partial p_1}{\partial w_2} \\ \frac{\partial p_2}{\partial w_1} & \frac{\partial p_2}{\partial w_2} \end{bmatrix} = \frac{1}{(\frac{\partial G_1}{\partial p_1}\frac{\partial G_2}{\partial p_2} - \frac{\partial G_1}{\partial p_2}\frac{\partial G_2}{\partial p_1})} \begin{bmatrix} -\frac{\partial G_2}{\partial p_2}\frac{\partial G_1}{\partial w_1} & \frac{\partial G_1}{\partial p_2}\frac{\partial G_2}{\partial w_2} \\ \frac{\partial G_2}{\partial p_1}\frac{\partial G_1}{\partial w_1} & -\frac{\partial G_1}{\partial p_1}\frac{\partial G_2}{\partial w_2} \end{bmatrix}$$

where

$$\frac{\partial G_1}{\partial w_1} = -\int_0^{c_1} \frac{\partial \overline{F}_1^{(p_1,p_2)}}{\partial p_1}(x)dx \geq 0,$$

$$\frac{\partial G_2}{\partial w_2} = -\int_0^{c_2} \frac{\partial \overline{F}_2^{(p_1,p_2)}}{\partial p_2}(x)dx \geq 0.$$

Theorem 6.8 *At equilibrium a small change in w_i leads to the changes in p_1 and p_2 such that*

$$\frac{\partial p_i}{\partial w_i}/\frac{\partial p_j}{\partial w_i} = \frac{-2\int_0^{c_j}\frac{\partial \overline{F}_j^{(p_1,p_2)}}{\partial p_j}(x)dx - (p_j - w_j)\int_0^{c_j}\frac{\partial^2 \overline{F}_j^{(p_1,p_2)}}{\partial^2 p_j}(x)dx}{\int_0^{c_j}\frac{\partial \overline{F}_j^{(p_1,p_2)}}{\partial p_i}(x)dx + (p_j - w_j)\int_0^{c_j}\frac{\partial^2 \overline{F}_j^{(p_1,p_2)}}{\partial p_i \partial p_j}(x)dx}, j \neq i.$$

Proof This result is easy to establish from the fact that $\frac{\partial p_i}{\partial w_i}/\frac{\partial p_j}{\partial w_i} = -\frac{\partial G_j}{\partial p_j}/\frac{\partial G_j}{\partial p_i}$. ■

Theorem 6.9 *Under uniqueness condition and assumption (A6.4), at equilibrium an increase in w_i leads to increases in both p_1 and p_2.*

Proof Under the uniqueness condition of Lemma 6.4 and assumption (A6.4), $\frac{\partial G_i}{\partial p_i} \leq 0$ and $(\frac{\partial G_1}{\partial p_1}\frac{\partial G_2}{\partial p_2} - \frac{\partial G_1}{\partial p_2}\frac{\partial G_2}{\partial p_1}) \geq 0$. Combining this with the facts that $\frac{\partial G_i}{\partial p_j} \geq 0$ and $\frac{\partial G_i}{\partial w_i} \geq 0$, we see that $\frac{\partial p_i}{\partial w_i}$ and $\frac{\partial p_j}{\partial w_i}$ are both nonnegative. ■

Remark 6.10 This result is closely related to those mentioned in Katz (1989) and Bernstein and Federgruen (2002). The fact that equilibrium retail prices increase with the charged (set of) wholesale price(s) is assumed by Katz (p. 678-679). Similar results are derived by Bernstein and Federgruen (p. 18-19) under multiplicative and non-multiplicative demand structures.

6.5 Concluding remarks

In this chapter, we apply game theory to study the pricing strategies of multiple competing firms in a revenue management context. The demand at each firm depends on the selling prices charged by all firms, each of which can satisfy demand up to a given capacity limit. We derive the existence and uniqueness conditions of a Nash equilibrium when the demands are deterministic and random according to general stochastic functions. In particular, we study the case when the deterministic demand is a linear function of price. The unique Nash equilibrium is derived and its feasibility is verified. Stochastic ordering is used to model the dependence between deterministic prices and random demands. We also analyze the changes in equilibrium prices as a result of changes in cost and capacity parameters.

In the one-period model studied in this chapter, a lost sales occurs when the demand exceeds the retailer's capacity. A potential topic for future research would be to analyze a similar multi-period model with backorders. In this case, each retailer makes a sequence of decisions, and the decision of one period affects those of all following periods. Sequential game (Chapter 9 of Heyman and Sobel (1984)) may be used to analyze this multi-period decision process.

Our model can be extended to a one-supplier, n-retailer distribution system, where each retailer faces demand as defined in this chapter and makes decisions about his/her stocking quantity and selling price simultaneously. A potential avenue of research is to study decentralized control, centralized control and coordination mechanisms for this distribution system.

Chapter 7

Summary and Future Research

7.1 Summary

In this dissertation, we have applied game theory to analyze the supply chain management problems. An understanding of the ways in which conflicting objectives affect optimal inventory/capacity decisions can greatly improve the performance of a supply chain. We extend the exiting literature by considering multiple competitive firms, each of which makes multiple decisions either simultaneously or sequentially.

In Chapter 3, we began by analyzing a one-supplier, two-retailer distribution system with general cost structure. When a stockout occurs at one retailer customers may go to the other retailer. We studied a single-period model in which the supplier may have infinite or finite capacity. In the latter case, if the total quantity ordered (claimed) by the retailers exceeds the supplier's capacity, an allocation policy is used to assign the limited capacity to the retailers. We used game theory to analyze the inventory control decisions for the retailers. We showed that a unique Nash equilibrium exists when the capacity at the supplier is infinite. However, when the capacity is finite, only under certain conditions does the Nash equilibrium exist. We also used the concept of Stackelberg game to develop optimal strategies for both the leader and the follower.

In a distribution system, an equilibrium solution reached by all retailers may be suboptimal in terms of the system-wide profit. Designing an easily acceptable incentive structure that can be implemented in practice is a challenging and important issue. For this purpose, in Chapter 4, we analyzed a capacity allocation model with cost and revenue structures that differ from those of the model in Chapter 3. We studied both the decentralized and centralized inventory control problems. For the decentralized problem we derived necessary and sufficient conditions for the existence of a unique Nash equilibrium. For centralized inventory control we obtained an optimal allocation that maximizes the expected profit of the entire supply chain. We also designed a perfect coordination mechanism, i.e., a decentralized cost structure resulting in a Nash equilibrium with chain-wide profits equal to those achieved under a fully centralized system.

In Chapter 5, we examined several extensions of the basic capacity allocation problem that involves a single-period model, in which each retailer has one demand class. We started with a single-period capacity allocation model in which each firm has multiple demand classes. We considered two scenarios of this model and derived corresponding existence and stability conditions of a Nash equilibrium. We also conducted sensitivity analysis of the equilibrium solution with respect to price and cost parameters. From the well-studied single-period problem, we moved ahead to modelling and analyzing a multi-period problem in which each firm decides its total capacity and allocates this capacity between its local and online stores. We analyzed the resulting sequential game, derived a myopic solution, and showed that it is a Nash equilibrium.

In Chapter 3, Chapter 4, and Chapter 5, the competition between firms is for a limited supply of product. It is reasonable to expect that the firms compete not only on the basis of inventory, but also on the basis of price. In Chapter 6, we analyzed the case when multiple firms set selling price to attract potential customers. We considered the pricing strategies of multiple firms providing same service and competing for a common pool of customers in a revenue management context. We analyzed systems in which firms face either a deterministic demand function or a stochastic demand function. For both cases, we derived the existence and uniqueness conditions of a Nash equilibrium solution. In particular, we calculated the explicit Nash equilibrium solution when the demand at each firm is a linear function of price. In addition, we performed sensitivity analysis of the equilibrium prices with respect to cost and capacity parameters.

7.2 Directions for further work

A potential topic for future research is to extend the two-retailer models studied in Chapter 3 and Chapter 4 of this dissertation to n-retailer models. The lemma given by Nikaido and Isora (1955) can be used to find the conditions for the existence of a Nash equilibrium. To address uniqueness conditions for the Nash equilibrium, we can apply

Theorem 3 in Chapter 6 of Moulin (1986).

For the capacity allocation problems analyzed in this dissertation, the demand at each firm is exogenous. A potential avenue of research is to study the case in which the demands are affected by the quality of service, such as the percentage of demand satisfied, or by selling prices. It would also be interesting to analyze the case when the probabilities of customer shifting, i.e, market search matrix elements, are related to these factors.

In Chapter 4, we designed a channel coordination mechanism through wholesale prices to optimize the performance of the supply chain. Beyond the price-only contracts used, a potential topic is to explore other contracts that are designed to improve the performance of a supply chain. Such contracts might include the reallocation of decision rights (VMI, RMI), rules for sharing the costs of inventory and backorders (buyback, quantity flexibility), and policies governing pricing to the end-customer or between supply chain partners (pricing).

Another possible extension would be to incorporate overbooking into our pricing models in Chapter 6. Overbooking (e.g., ticketing seats beyond the capacity of an aircraft to allow for the probability of no-shows) has been extensively studied in revenue management research. In most published scenarios pricing is used as the mechanism to alter customer demand, but it has to be done jointly with the other managerial decisions, such as inventory decision and booking limit decision.

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Appendix

Appendix

In this appendix, we prove that under the condition (6.12) in Section 6.2, i.e.,

$$\begin{cases} a_1 - b_1 w_1 + \beta_{12} w_2 > 0, \\ a_2 - b_2 w_2 + \beta_{21} w_1 > 0, \end{cases}$$

the Nash equilibria in all cases in Section 6.2 are feasible, i.e.,

$$\begin{cases} p_1 > w_1, \\ p_2 > w_2. \end{cases}$$

Proof Case 1: Substituting c_1 by $\frac{a_1 - b_1 w_1}{2}$ (its upper bound in this case), and c_2 by $\frac{a_2 - b_2 w_2}{2}$ (its upper bound in this case) in (6.6), we obtain a lower bound on the Nash equilibrium $(\underline{p_1}, \underline{p_2})$ and

$$\begin{cases} \underline{p_1} - w_1 = \frac{a_1 b_2 + \beta_{12} a_2 + \beta_{12} b_2 w_2 - b_1 b_2 w_1 + 2\beta_{12}\beta_{21} w_1}{2(b_1 b_2 - \beta_{12}\beta_{21})}, \\ \underline{p_2} - w_2 = \frac{a_2 b_1 + \beta_{21} a_1 + \beta_{21} b_1 w_1 - b_1 b_2 w_2 + 2\beta_{12}\beta_{21} w_2}{2(b_1 b_2 - \beta_{12}\beta_{21})}. \end{cases}$$

Clearly, under the condition (6.12), it is a feasible Nash equilibrium.

Case 2: When $(2b_1 b_2 - \beta_{12}\beta_{21})c_1 + \beta_{12} b_1 c_2 \geq m_1$, substituting c_2 by $\frac{a_2 - b_2 w_2}{2}$ in (6.8), we obtain a lower bound on the Nash equilibrium $(\underline{p_1}, \underline{p_2})$ and

$$\begin{cases} \underline{p_1} - w_1 = \frac{2a_1 b_2 + \beta_{12} a_2 + \beta_{12} b_2 w_2 - 2b_1 b_2 w_1 + 2\beta_{12}\beta_{21} w_1}{2(2b_1 b_2 - \beta_{12}\beta_{21})}, \\ \underline{p_2} - w_2 = \frac{2a_2 b_1 + \beta_{21} a_1 + \beta_{21} b_1 w_1 - 2b_1 b_2 w_2 + 2\beta_{12}\beta_{21} w_2}{2(2b_1 b_2 - \beta_{12}\beta_{21})}. \end{cases}$$

We know that

$$\begin{aligned} &2a_1 b_2 + \beta_{12} a_2 + \beta_{12} b_2 w_2 - 2b_1 b_2 w_1 + \beta_{12}\beta_{21} w_1 \\ &= 2b_2(a_1 - b_1 w_1 + \beta_{12} w_2) + \beta_{12}(a_2 - b_2 w_2 + \beta_{21} w_1) \\ &> 0, \end{aligned} \tag{7.1}$$

which proves the feasibility of the Nash equilibrium.

When $(2b_1b_2 - \beta_{12}\beta_{21})c_1 + \beta_{12}b_1c_2 < m_1$, substituting c_2 by $\frac{m_1-(2b_1b_2-\beta_{12}\beta_{21})c_1}{\beta_{12}b_1}$ in the formula for p_1 in (6.6), we obtain a lower bound on firm 1's equilibrium price, which is

$$\underline{p_1} = \frac{c_1}{b_1} + w_1 > w_1.$$

Substituting c_1 by $\frac{m_1-\beta_{12}b_1c_2}{(2b_1b_2-\beta_{12}\beta_{21})}$ in the formula for p_2 in (6.6), we obtain

$$p_2 = \frac{2a_2b_1 + \beta_{21}a_1 + \beta_{21}b_1w_1 - 2b_1c_2}{2b_1b_2 - \beta_{12}\beta_{21}}.$$

Substituting c_2 by $\frac{a_2-b_2w_2}{2}$, we obtain a lower bound on firm 2's equilibrium price $\underline{p_2}$ and

$$\underline{p_2} - w_2 = \frac{a_2b_1 + \beta_{21}a_1 + \beta_{21}b_1w_1 - b_1b_2w_2 + \beta_{12}\beta_{21}w_2}{2b_1b_2 - \beta_{12}\beta_{21}} > 0.$$

The feasibility of the Nash equilibrium in Case 3 can be proved in a similar manner.

Case 4-1: Substituting c_1 by $c_1^{'}$ and c_2 by value $c_2^{'}$ in (6.6), we obtain a lower bound on the Nash equilibrium $(\underline{p_1}, \underline{p_2})$ and

$$\begin{cases} \underline{p_1} - w_1 = \frac{2a_1b_2+\beta_{12}a_2+\beta_{12}b_2w_2-2b_1b_2w_1+\beta_{12}\beta_{21}w_1}{4b_1b_2-\beta_{12}\beta_{21}}, \\ \underline{p_2} - w_2 = \frac{2a_2b_1+\beta_{21}a_1+\beta_{21}b_1w_1-2b_1b_2w_2+\beta_{12}\beta_{21}w_2}{4b_1b_2-\beta_{12}\beta_{21}}. \end{cases} \tag{7.2}$$

We can prove its feasibility following the arguments for Case 2.

Case 4-2: When $(2b_1b_2 - \beta_{12}\beta_{21})c_1 + \beta_{12}b_1c_2 \geq m_1$, substituting c_2 by value $c_2^{'}$ in (6.8), we obtain a lower bound on the Nash equilibrium $(\underline{p_1}, \underline{p_2})$ which is the same as (7.2).

When $(2b_1b_2 - \beta_{12}\beta_{21})c_1 + \beta_{12}b_1c_2 < m_1$, substituting c_2 by $\frac{m_1-(2b_1b_2-\beta_{12}\beta_{21})c_1}{\beta_{12}b_1}$ in the formula for p_1 in (6.6), we obtain a lower bound on firm 1's equilibrium price, which is

$$\underline{p_1} = \frac{c_1}{b_1} + w_1 > w_1.$$

Substituting c_1 by $\frac{m_1-\beta_{12}b_1c_2}{(2b_1b_2-\beta_{12}\beta_{21})}$ in the formula for p_2 in (6.6), we obtain

$$p_2 = \frac{2a_2b_1 + \beta_{21}a_1 + \beta_{21}b_1w_1 - 2b_1c_2}{2b_1b_2 - \beta_{12}\beta_{21}}.$$

Substituting c_2 by $c_2^{'}$, we obtain a lower bound on firm 2's equilibrium price and

$$\underline{p_2} - w_2 = \frac{2a_2b_1 + \beta_{21}a_1 + \beta_{21}b_1w_1 - 2b_1b_2w_2 + \beta_{12}\beta_{21}w_2}{2b_1b_2 - \beta_{12}\beta_{21}} > 0.$$

The feasibility of the Nash equilibrium in Case 4-3 can be proved in a similar manner.

Case 4-4: From (6.11), we obtain

$$\begin{cases} p_1 - w_1 = \frac{2a_1b_2+\beta_{12}a_2+\beta_{12}b_2w_2-2b_1b_2w_1+\beta_{12}\beta_{21}w_1}{4b_1b_2-\beta_{12}\beta_{21}} > 0, \\ p_2 - w_2 = \frac{2a_2b_1+\beta_{21}a_1+\beta_{21}b_1w_1-2b_1b_2w_2+\beta_{12}\beta_{21}w_2}{4b_1b_2-\beta_{12}\beta_{21}} > 0. \end{cases}$$