Abstract

TULLIE, TRACEY ANDREW. Variance Reduction for Monte Carlo Simulation of European, American or Barrier Options in a Stochastic Volatility Environment. (Under the direction of Jean-Pierre Fouque.)

In this work we develop a methodology to reduce the variance when applying Monte Carlo simulation to the pricing of a European, American or Barrier option in a stochastic volatility environment. We begin by presenting some applicable concepts in the theory of stochastic differential equations. Secondly, we develop the model for the evolution of an asset price under constant volatility. We next present the replicating portfolio and equivalent martingale measure approaches to the pricing of a European style option. Modeling an asset price utilizing constant volatility has been shown to be an inefficient model[8, 16]. One way to compensate for this inefficiency is the use of stochastic volatility models, which involves modeling the volatility as a function of a stochastic process[26]. A class of these models is presented and a discussion is given on how to price European options in this framework.

After developing the methods of how to price, we begin our discussion on Monte Carlo simulation of European options in a stochastic volatility environment. We start by describing how to simulate Monte Carlo for a diffusion process modeled as a stochastic differential equation. The essential element to our variance reduction technique, which is known as importance sampling, is hereafter presented. Importance sampling requires a preliminary approximation to the expectation of interest, which we obtain by a fast mean-reversion expansion of the pricing partial differential equation[22, 6].

A detailed discussion is given on this fast mean-reversion expansion technique, which was first presented in [10]. We shall compare utilizing this method of expansion with that developed in [11], which is known as small noise expansion, and demonstrate numerically the efficiency of the fast mean-reversion expansion, in particular in the presence of a skew. We next wish to apply our variance reduction technique to the pricing of an American and barrier option. A discussion is given on how to price these options under constant volatility and in the presence of stochastic volatililty. Applying the importance sampling variance reduction method to a barrier option is similar to that of a European option since there exists a closed form solution to the price of this option in the context of constant volatility[4, 15]. However, in the case of an American option Monte Carlo simulation and applying importance sampling are more complex. We present an algorithm to compute an American option price via Monte Carlo and describe an approximation technique to obtain a preliminary estimate to the pricing function under constant volatility. Hence, we are able to apply our variance reduction methodology to pricing of an American option. We subsequently present numerical results for both of these options.

Variance Reduction for Monte Carlo Simulation of European, American or Barrier Options in a Stochastic Volatility Environment

by

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Biography

Tracey Andrew Tullie was born in Landsthul, Germany to Samuel C. and Ruby M. Tullie July 18, 1973. Born the son of a father who was in the Air Force, Tracey spent most of his life in Montgomery, Alabama. He attended high school at Jefferson Davis High School and went on to college at Alabama State University. Tracey received his bachelor of science in mathematics with honors in May of 1995. In August of 1995 he went on to pursue his Master of Science in mathematics at North Carolina State University, which was completed in December of 1997. In August of 2002 he received his Doctor of Philosophy in applied mathematics at North Carolina State University and began a position at the level of assistant professor at North Carolina Agricultural and Technical State University.

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Chapter 1

Introduction

Asset prices, such as a stock for a company, are often modeled using stochastic differential equations. The simplest of these models involves modeling the return on the stock as the mean growth rate plus a random term. The random term consists of a parameter, called the volatility, and an increment of Brownian motion. This particular model is often referred to as geometric Brownian motion or the lognormal model of an asset.

Although the lognormal model serves as a good basis for modeling an asset, there has been a wide-range of research done to improve upon this model. Empirical evidence suggests that the volatility may be modeled as a function of a stochastic process[14]. This class of models is known as stochastic volatility models. In particular, a mean-reverting stochastic process is a suitable model for volatility. Mean-reversion refers to a linear pull back term in the mean growth rate of the volatility process.

Contracts that are based on an underlying asset are called derivatives. We are primarily interested in contracts known as options. Three types of options will be discussed in this work. A European option gives its holder the right to buy or sell stock for a predetermined price at a specified maturity date. An American option gives its holder the right to buy or sell stock for predetermined price at any time before or on a specified maturity date. Barrier options are options where the right to exercise is forfeited if the underlying asset crosses a certain value or the option only comes into existence only if the asset crosses a certain level. Computation of the premium of such contracts is the focal point of this work.

Fisher Black and Myron Scholes formulated a method, based on no-arbitrage, which gives a closed-form solution for the price of a European style option when the underlying asset is modeled as a geometric Brownian motion and no dividends are paid on the stock[3, 19]. A closed-form solution for the price of a Barrier option under the lognormal model has also been developed. However, there is very rarely a closed-form solution for the price of any type of option when a stochastic volatility model is used to describe the evolution of an asset. Solving partial differential equations with two space dimensions or Monte Carlo simulation are two methods used to compute the premium for these types of contracts[5, 9, 13, 1].

Computing option premiums utilizing Monte Carlo simulation has become popular among many financial institutions[27, 21]. Therefore, numerical techniques that provide variance reduction for Monte Carlo methods are in demand. We introduce a variance reduction methodology for Monte Carlo simulation of a European, American or Barrier option premium when the underlying asset has volatility modeled as a mean-reverting process. The variance reduction technique consists of two major components: Importance sampling and asymptotic analysis.

A preliminary approximation for the expectation of interest is the main feature of the importance sampling technique. Two methods of expansion will be used in order to obtain an apriori estimate of the expectation of interest. The first, introduced [11], corresponds to a regular perturbation of the pricing B-S partial differential equation. The second, the focal point of this work and introduced in [10], is based on fast mean-reverting stochastic volatility asymptotics. It corresponds to a singular perturbation of the pricing partial differential equation.

The dissertation is organized as follows. In Chapter 2 we provide background on stochastic differential equations. The multi-dimensional model for a diffusion process is established as well as the existence and uniqueness of a solution. Also, we discuss several results of diffusion theory that include Ito's lemma, the martingale representation theorem, infinitesimal generators, the Feyman-Kac formula and Girsanov's theorem. These are essential tools needed when modeling with stochastic differential equations in finance. In addition, we develop the Euler scheme for simulation of stochastic differential equations. In chapter 3 we formulate mathematically how an asset is modeled. We also establish how the price of a European style option is obtained utilizing replicating portfolios or an equivalent martingale measure. In chapter 4, a discussion is given on why volatility should be modeled as a function of a stochastic process. In addition, pricing a European option when an asset has volatility modeled as a function of a stochastic process is also addressed. In chapter 5, we describe how Monte Carlo simulation is utilized for pricing a European option. Also, we define the Importance Sampling procedure that is used for variance reduction of the Monte Carlo method. In chapter 6, we develop the asymptotic analysis for the price of an European option in a stochastic volatility environment. This involves two methods of expansion of the pricing partial differential equation. In chapter 7, we implement the importance sampling variance reduction technique to price a European option when the underlying has volatility modeled as a mean-reverting process. In chapter 8, we formulate mathematically an American option and describe how to compute the premium of this type of contract under constant volatility. This computation involves evaluating an expectation or formulating a linear complementarity problem. Also, we present the asymptotic results when the underlying asset has volatility modeled as a mean-reverting process. Next, we establish how the price of an American option may be computed using Monte Carlo. Variance reduction for Monte Carlo simulation of an American option utilizing Importance sampling and asymptotic expansion is subsequently considered. In chapter 9, we define the Barrier option mathematically and describe how to price an option this type when volatility is taken to be constant in the underlying asset. Next, we provide the asymptotic results when the underlying asset has volatility modeled as a function of a stochastic process. Lastly, we perform variance reduction for Monte Carlo simulation of a barrier option when the underlying has volatility modeled as a mean-reverting process.

Chapter 2

Stochastic Differential Equations

Stochastic differential equations are used to model processes that have continuous time paths and have a source of randomness contained within them. This source of randomness is commonly referred to as *white noise* and is modeled as increments of Brownian motion. The term Brownian refers to the Scottish botanist Robert Brown who, in 1828, observed an irregular motion when pollen grains were suspended in liquid. We begin our discussion on stochastic differential equations by defining mathematically Brownian motion. Secondly, we construct a multi-dimensional model and establish an existence and uniqueness result. Subsequently, we present some theoretical results on diffusion processes that appear frequently in financial modeling. Lastly, we introduce the Euler scheme for simulating stochastic differential equations and develop a convergence criterion for this method.

2.1 Brownian Motion

Brownian motion is a real-valued stochastic Gausssian process with continuous trajectories that have independent and stationary increments. We shall denote the trajectories of the Brownian motion by $t \to W_t$. In particular, the one-dimensional standard Brownian motion has the following properties:

- for any $0 < t_1 < \ldots < t_k$, the random variables $(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_k} - W_{t_{k-1}})$ are independent,
- $\mathbb{E}(W_t|\mathcal{F}_0) = 0$,
- for any $s \leq t$, $\mathbb{E}(W_t - W_s|\mathcal{F}_s) = 0$ and $\mathbb{E}((W_t - W_s)^2|\mathcal{F}_s) = t - s$,

where we denote the probability space that our Brownian motion is defined and expectation $\mathbb{E}\{\cdot\}$ is computed by $(\Omega, \mathcal{F}, \mathbb{P})$. Ω may be taken as the space of all continuous trajectories ω such that $W_t(\omega) = \omega(t)$. $\mathcal{F}$ is a σ-algebra which contains sets of the form $\{\omega \in \Omega : |\omega(s)| < M, s \leq t\}$ and $\{\mathcal{F}_t, t \geq 0\}$ is defined as an increasing family of sub-σ algebras of $\mathcal{F}$ such that W_t is $\mathcal{F}_t$ measurable and that also contains sets of probability 0 in $\mathcal{F}$. This completion is important because if two random variables are equal almost surely then if one of the random variables is $\mathcal{F}_t$ measurable, then so is the other. $\mathcal{F}_t$ may be thought of as the history of W_s, $s \leq t$, up to time t[24]. Lastly, $\mathbb{P}$ is the Wiener measure, which is the probability distribution of the standard Brownian motion.

Similarly, let $W_t = (W_t^1, W_t^2, \cdots, W_t^n)$ represent an n-dimensional Brownian motion with independent components associated with an increasing family of sub-σ algebras $\{\mathcal{F}_t\}$. Each W_t^i for $i = 1, 2, \cdots, n$, is a scalar Brownian motion with respect to $\{\mathcal{F}_t, t \geq 0\}$ with the usual properties. In addition, we include the property $\mathbb{E}[(W_t^i - W_s^i)(W_t^j - W_s^j)|\mathcal{F}_s] = (t - s)\delta_{ij}$, for $s \leq t$ and $i, j = 1, 2, \cdots, n$, where δ_{ij} is the Kronecker delta function.

2.2 Multi-dimensional Model

An n-dimensional stochastic differential equation may be formulated as follows

$$\begin{aligned} dX_t &= a(t, X_t)dt + b(t, X_t)dW_t, \qquad t \in [0, T] \\ X_0 &= \hat{X} \end{aligned} \tag{2.1}$$

where $X_t \in \mathbb{R}^n$ represents the state of the process being modeled at any time t and $dW_t \in \mathbb{R}^n$ is an increment of Brownian motion. In addition, $a : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is called the drift vector of the process and $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{m \times n}$ is called the diffusion matrix.

Equation (2.1) may equivalently be written as an integral equation of the following form

$$X_t = X_0 + \int_0^t a(\tau, X_\tau)d\tau + \int_0^t b(\tau, X_\tau)dW_\tau, \tag{2.2}$$

for any $0 \le t \le T$, which may be interpreted component-wise as

$$X_t^i = X_0^i + \int_0^t a^i(\tau, X_\tau^i)d\tau + \sum_{j=1}^n \int_0^t b^{ij}(\tau, X_\tau^i)dW_\tau^j, \tag{2.3}$$

for $i = 1, 2, \cdots, n$. The first term is a standard Lebesgue or Riemann integral for each sample path ω while the second integral represents a stochastic Ito integral. An essential element to the proof of the existence and uniqueness of a solution to (2.1) is the Ito integral, which we define rigorously.

We wish to provide a concise definition for an integral of the form

$$\int_0^t q(s, \omega)dW_s \qquad t \in [0, T] \tag{2.4}$$

We start by establishing a class for which (2.4) is defined and then define the Ito integral for a step function in this class.

Definition 2.2.1 *Let $\Gamma = \Gamma(0,T)$ be the class of functions*

$$f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$$

such that

- $f(t,\omega)$ *is $\mathcal{B} \times \mathcal{F}$- measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$.*
- $f(t,\omega)$ *is $\mathcal{F}_t$-adapted.*
- $\mathbb{E}\left[\int_0^T f(t,\omega)^2 dt\right] < \infty.$

Let $\phi(t,\omega) \in \Gamma$ be a step function corresponding to a partition $0 = t_0 < t_2 < \cdots < t_n = T$ and random functions $\phi_1,\ \phi_2, \cdots, \phi_n$, then the stochastic integral of ϕ is defined as

$$\int_0^T \phi(s,\omega) dW_s = \sum_{j=1}^{n} \phi_n(\omega)\{W_{t_{j+1}}(\omega) - W_{t_j}(\omega)\}$$

Definition 2.2.2 (Ito Integral) *Let $q \in \Gamma$. Then the Ito integral of q is defined by*

$$\int_0^t q(\tau,\omega) dW_\tau(\omega) = \lim_{m\to\infty} \int_0^t q_m(\tau,\omega) dW_\tau(\omega),$$

where $\{q_m\}$ is a sequence of step functions and such that

$$\mathbb{E}\left[\int_0^t (q(\tau,\omega) - q_m(\tau,\omega))^2 d\tau\right] \to 0 \quad \text{as } m \to \infty$$

Having defined formally the Ito integral, we next state without proof the following existence and uniqueness theorem for the stochastic differential equation defined in (2.1).

Theorem 2.2.1 (Existence and uniqueness theorem) *[23] Let a and b be Borel measurable functions satisfying*

$$|a(t,u)| + |b(t,u)| \le M(1 + |u|); \quad u \in \mathbb{R}^n,\ t \in [0,T]$$

for some constant M and such that

$$|a(t,u)-a(t,v)|+|b(t,u)-b(t,v)|\le D|u-v|;\quad u,v\in\mathbb{R}^n,\ t\in[0,T],$$

for some constant D. Let $\hat{X}$ be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty$ generated by W_t and such that

$$\mathbb{E}\left[|\hat{X}|^2\right]<\infty,$$

then (2.1) has a unique t-continuous solution $X_t(\omega)$ with the property that X_t is $\mathcal{F}_t$-measurable and

$$\sup_{0\le t\le T}\mathbb{E}\left[|X_t|^2\right]<\infty.$$

2.3 Diffusion Theory

We next present some theoretical results that are prevalent in modeling diffusion processes in finance. We begin by stating the well-known Ito formula.

2.3.1 Ito's Formula

Ito's formula gives an explicit formula for computing the differential for a function of a stochastic diffusion process. This is analogous to the chain rule in elementary differential calculus.

Theorem 2.3.1 (Ito's formula) *[23] Let X_t be an n-dimensional stochastic process which evolves as (2.1) and let $g:[0,\infty]\times\mathbb{R}^n\to\mathbb{R}^p$ be $\mathcal{C}^2(\mathbb{R}^n)$. Then the process $Y_t(\omega)=g(t,X(t))$ has differential defined component-wise as*

$$dY_t^k=\frac{\partial g^k}{\partial t}(t,X_t)dt+\sum_i\frac{\partial g^k}{\partial x_i}(t,X_t)dX_t^i+\frac{1}{2}\sum_{i,j}\frac{\partial^2 g^k}{\partial x_i\partial x_j}(t,X_t)dX^idX^j,\qquad(2.5)$$

where $dW_t^idW_t^j=\delta_{ij}dt,\ dW_t^idt=dtdW_t^i=0.$

2.3.2 Martingale Representation Theorem

Before presenting the martingale representation theorem we define a martingale with respect to an increasing family of σ-subalgebras $\{\mathcal{R}_t, t \geq 0\}$.

Definition 2.3.1 (Martingle) *An n-dimensional stochastic process M_t on a probability space $(\Omega, \mathcal{R}, \mathbb{P})$ is called a martingale with respect to $\{\mathcal{R}_t, t \geq 0\}$ if the following hold*

- *M_t is $\mathcal{R}_t$-measurable for all t.*
- *$\mathbb{E}[|M_t|] < \infty$ for all t.*
- *$\mathbb{E}[M_s|\mathcal{R}_t] = M_t$ for all $s \geq t$.*

The martingale representation theorem is utilized to show that any $\mathcal{F}_t$-martingale can be represented as an Ito integral and is presented as follows

Theorem 2.3.2 *[Martingale representation theorem][23] Let W_t be an n-dimensional Brownian motion. Suppose M_t is an $\mathcal{F}_t$-martingale, where $\mathcal{F}_t$ is the natural filtration of W_t and that $M_t \in L^2(\mathbb{P})$ for all $t \geq 0$. Then there exists a unique stochastic process $q(t, \omega)$ such that for all $t \geq 0$*

$$M_t(\omega) = \mathbb{E}[M_0] + \int_0^t q(s, \omega) dW_s \quad a.s., t \geq 0$$

2.3.3 Generator of an Ito Diffusion

Given an n-dimensional Ito diffusion of the form

$$\begin{aligned} dX_t &= a(t, X_t)dt + b(t, X_t)dW_t, \quad t \in [0, T] \qquad (2.6) \\ X_0 &= \hat{X}, \end{aligned}$$

we wish to characterize this process by a second order partial differential equation operator $\mathcal{L}$. We call $\mathcal{L}$ the infinitesimal generator of the process X_t and it is defined as follows

Definition 2.3.2 *[Infinitesimal generator of an Ito process][23] Let $\{X_t\}$ be an Ito diffusion in $\mathbb{R}^n$. The infinitesimal generator $\mathcal{L}$ of X_t is defined by*

$$\mathcal{L}h(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}[h(X_t)] - h(x)}{t},$$

where the set of functions for which the limit exist for all $x \in \mathbb{R}^n$ is denoted by $\mathcal{D}_{\mathcal{L}}$.

The infinitesimal generator of the Ito process given by (2.6) is given by

$$\mathcal{L}h(x) = \sum_i a^i(t,x) \frac{\partial h}{\partial x_i} + \frac{1}{2} \sum_{i,j} (bb')^{ij}(t,x) \frac{\partial^2 h}{\partial x_i \partial x_j}, \tag{2.7}$$

where $h \in \mathcal{D}_{\mathcal{L}} = \mathcal{C}^2(\mathbb{R}^n)$ and has compact support.

2.3.4 Kolmogorov's backward equation and the Feyman-Kac formula

We now present two results, which are applications of the infinitesimal generator defined in the previous section, that describe the evolution of the conditional expectation of a function of an Ito process.

Theorem 2.3.3 *[Kolgomorov's backward equation][23] Let X_t be an Ito diffusion of the form (2.6) with infinitesimal generator $\mathcal{L}$ given by (2.7).*

1. *Define*

$$u(t,x) = \mathbb{E}[f(X_T)|X_t = x]. \tag{2.8}$$

Then $u(t,x)$ satisfies the Kolmogorov equation given by

$$\frac{\partial u}{\partial t} + \mathcal{L}u = 0; \qquad 0 \leq t \leq T \tag{2.9}$$

$$u(T,x) = f(x) \tag{2.10}$$

where we assume $f \in \mathcal{C}^2(\mathbb{R}^n)$.

2. *Moreover, if $w(t,x) \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^n)$ that satisfies (2.9) and (2.10), then $w(t,x) = u(t,x)$, given by (2.8).*

The following theorem is a important extension of the Kolmogorov backward equation.

Theorem 2.3.4 *[Feyman-Kac formula][23] Let X_t be an Ito diffusion of the form (2.6) with infinitesimal generator $\mathcal{L}$ given by (2.7).*

1. *Define*

$$u(t,x) = \mathbb{E}\left[\exp\left(\int_t^T r(X_s)ds\right) f(X_T)|X_t = x\right]. \tag{2.11}$$

Then

$$\frac{\partial u}{\partial t} + \mathcal{L}u + ru = 0; \qquad 0 \leq t \leq T \tag{2.12}$$

$$u(T,x) = f(x); \tag{2.13}$$

where we assume $f \in \mathcal{C}^2(\mathbb{R}^n)$ and $r \in \mathcal{C}(\mathbb{R}^n)$.

2. *Moreover, if $w(t,x) \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^n)$ that satisfies (2.12) and (3.37), then $w(t,x) = u(t,x)$, given by (2.11).*

2.3.5 Girsanov's Transformation

The following result states that altering the drift vector of an Ito diffusion does not dramatically change the probability law of the process.

Theorem 2.3.5 *[Girsanov's theorem][23] Let Z_t be an Ito process of the form*

$$dZ_t = \beta(t,\omega)dt + \theta(t,\omega)dW_t; \qquad t \in [0,T]. \tag{2.14}$$

Suppose there exist processes $\gamma(t,\omega) \in \Gamma(0,T)$ and $\alpha(t,\omega) \in \Gamma(0,T)$ such that

$$\theta(t,\omega)\gamma(t,\omega) = \beta(t,\omega) - \alpha(t,\omega) \tag{2.15}$$

and assume $\gamma(t,\omega)$ satisfies the Novikov condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \gamma(s,\omega)\cdot\gamma(s,\omega)ds\right)\right] < \infty. \tag{2.16}$$

Define

$$M_t = \exp\left(-\int_0^t \gamma(t,\omega)\cdot dW_s - \frac{1}{2}\int_0^t \gamma(t,\omega)\cdot\gamma(t,\omega)ds\right); \qquad t \leq T. \tag{2.17}$$

Since $\mathbb{E}[M_T] = 1$, we may define a new probability measure Q as follows

$$d\mathbb{Q}(\omega) = M_T(\omega)d\mathbb{P}(\omega) \quad on \quad \mathcal{F}_T. \tag{2.18}$$

The process

$$\hat{W}_t = \int_0^t \gamma(s,\omega)ds + W_t; \qquad t \leq T \tag{2.19}$$

is a Brownian motion with respect to $\mathbb{Q}$ and in terms of $\hat{W}_t$ the process Z_t has the representation

$$dZ_t = \alpha(t,\omega)dt + \theta(t,\omega)d\hat{W}_t. \tag{2.20}$$

2.4 Euler's Method

Many stochastic differential equations in practical applications do not have an explicit solution. Therefore, there has been a great deal of research done on numerical simulation of these types of equations. In this work, we shall utilize the Euler scheme since we are implementing a variance reduction methodology for Monte Carlo simulation and this scheme is sufficient to test for variance reduction. Efficient implementation of the variance reduction methodology using higher order schemes is a subject of future work. We begin by describing the convergence criterion for an approximation to an Ito process. Secondly, we describe the stochastic Taylor expansion technique. Subsequently, we provide the time discrete approximations for the Euler numerical method and discuss its convergence properties. Detailed descriptions of various other numerical methods are provided in [18].

2.4.1 Convergence criterion

Strong Convergence

In many applications, such as filtering, it is important that the sample paths be close to the true Ito process. This indicates that there should be a strong form of convergence. We say that the approximation $\tilde{X}_T^{\delta}$ converges in the strong sense to the true process, X_T, with order γ if there exists a constant L such that

$$\mathbb{E}[|X_T - \tilde{X_T}^{\delta}|] \leq L\delta^{\gamma}$$

for any time discretization with a bounded step size δ.

Weak Convergence

In many practical applications, such as Monte Carlo simulation, it is important that the sample paths be close in distribution to the true Ito process. This indicates that there should be a weak form of convergence. We say that the approximation $\tilde{X}_T^\delta$ converges in the weak sense to the true process, X_T, with order β if for any $f \in \mathcal{C}^2$ there exists a constant L such that

$$|\mathbb{E}[f(X_T)] - \mathbb{E}[f(\tilde{X}_T)^\delta]| \leq L\delta^\beta$$

for any time discretization with a bounded step size δ.

2.4.2 Stochastic Taylor expansion

We next present stochastic Taylor expansion, which will lead to the formulation of the Euler numerical scheme. We shall derive the one-dimensional stochastic Taylor expansion for an Ito diffusion in integral form, which is given in full detail [18]. The multi-dimensional case follows similarly.

$$X_t = X_0 + \int_0^t a(\tau, X_\tau) d\tau + \int_0^t b(\tau, X_\tau) dW_\tau \qquad t \in [0, T], \tag{2.21}$$

$$X_0 = \hat{X} \tag{2.22}$$

For any $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g \in \mathcal{C}^{1\times 2}(\mathbb{R} \times \mathbb{R})$ the Ito formula gives

$$\begin{aligned} g(t, X_t) &= g(0, X_0) + \int_0^t \left(\frac{\partial g(\tau, X_\tau)}{\partial t} + (a(\tau, X_\tau) \frac{\partial}{\partial x} g(\tau, X_\tau) + \frac{1}{2} b^2(\tau, X_\tau) \frac{\partial^2}{\partial x^2} g(\tau, X_\tau) \right) d\tau \\ &\quad + \int_0^t b(\tau, X_\tau) \frac{\partial}{\partial x} g(\tau, X_\tau) dW_\tau \\ &= g(0, X_0) + \int_0^t \mathcal{K}^0 g(\tau, X_\tau) d\tau + \int_0^t \mathcal{K}^1 g(\tau, X_\tau) dW_\tau, \end{aligned} \tag{2.23}$$

where we have introduced the following operators

$$\mathcal{K}^0 = \frac{\partial}{\partial t} + a(s, X_s) \frac{\partial}{\partial x} + \frac{1}{2} b^2(s, X_s) \frac{\partial^2}{\partial x^2}$$

and

$$\mathcal{K}^1 = b(s, X_s)\frac{\partial}{\partial x}.$$

We perform an expansion of the functions $a(t, X_t)$ and $b(t, X_t)$ in (2.21) about $t = 0$ by using (2.23) as follows

$$\begin{aligned} X_t &= X_0 + \int_0^t \left(a(0, X_0) + \int_0^\tau \mathcal{K}^0 a(z, X_z)dz + \int_0^\tau \mathcal{K}^1 a(z, X_z)dW_z \right) d\tau \\ &\quad + \int_0^t \left(b(0, X_0) + \int_0^\tau \mathcal{K}^0 b(z, X_z)dz + \int_0^\tau \mathcal{K}^1 b(z, X_z)dW_z \right) dW_\tau \\ &= X_0 + a(0, X_0)\int_0^t d\tau + b(0, X_0)\int_0^t dW\tau + R, \end{aligned} \tag{2.24}$$

where R is the remainder defined by

$$\begin{aligned} R &= \int_0^t \int_0^\tau \mathcal{K}^0 a(z, X_z)dzd\tau + \int_0^t \int_0^\tau \mathcal{K}^1 a(z, X_z)dW_z d\tau \\ &\quad \int_0^t \int_0^\tau \mathcal{K}^0 b(z, X_z)dzdW_\tau + \int_0^t \int_0^\tau \mathcal{K}^1 b(z, X_z)dW_z dW_\tau \end{aligned} \tag{2.25}$$

Equations (2.24) and (2.25) together represent the simplest form of stochastic Ito-Taylor expansion. We may continue the expansion by applying Ito's formula to the function $\mathcal{K}^1 b(z, X_z)$ in (2.25) as follows

$$\begin{aligned} X_t &= X_0 + a(0, X_0)\int_0^t d\tau + b(0, X_0)\int_0^t dW_\tau \\ &\quad + \mathcal{K}^1 b(0, X_0)\int_0^t \int_0^\tau dW_z dW_\tau + \bar{R}, \end{aligned} \tag{2.26}$$

where $\bar{R}$ is given by

$$\begin{aligned} \bar{R} &= \int_0^t \int_0^\tau \mathcal{K}^0 a(z, X_z)dzd\tau + \int_0^t \int_0^\tau \mathcal{K}^1 a(z, W_z)dW_z d\tau \\ &\quad + \int_0^t \int_0^\tau \mathcal{K}^0 b(z, X_z)dzdW_\tau + \int_0^t \int_0^\tau \int_0^z \mathcal{K}^0\mathcal{K}^1 b(u, X_u)dudW_z dW_\tau \\ &\quad + \int_0^t \int_0^\tau \int_0^z \mathcal{K}^1\mathcal{K}^1 b(u, X_u)dW_u dW_z dW_\tau \end{aligned}$$

Equation (2.26) leads to the Milstein approximation, where we refer to [18] for more details, while (2.24) leads the Euler approximation, which we describe here.

2.4.3 Euler Numerical Scheme

Euler Scheme

The Euler numerical method consists of truncating the remainder term of the Ito-Taylor expansion given by (2.24) keeping only the time and Ito integrals that appear first in (2.24).

Suppose we have a discretization of the time interval $[0, T]$ of the form

$$0 = t_1 < t_2 < t_3 < \cdots < t_S = T$$

The one-dimensional Euler method is defined as follows

$$X_{s+1} = X_s + a(s, X_s)\Delta_s + b(s, X_s)\Delta W_s \tag{2.27}$$

for $s = 0, 1, 2, \cdots, S-1$ with initial value given by $X_0 = \hat{X}$,
where $\Delta_s = t_{s+1} - t_s$ and $\Delta W_s = W_{t_{s+1}} - W_{t_s}$. The random variables ΔW_s are independent and $\mathcal{N}(0, \Delta_s)$, which may be simulated using a pseudo-random number generator on any computer.

The multi-dimensional scheme may similarly be defined component-wise as follows

$$X^i_{s+1} = X^i_s + a^i(s, X_s)\Delta_s + \sum_{j=1}^{m} b^{ij}(s, X_s)\Delta W^j_n \quad , \tag{2.28}$$

for $i = 1 \cdots n$ and $s = 0, 1, 2, \cdots, S-1$.

In addition

$$\Delta W^j = W^j_{t_{n+1}} - W^j_{t_n} \tag{2.29}$$

represents the jth component of the n-dimensional increment of Brownian motion on $[t_n, t_{n+1}]$. Also, W^l an W^m are independent for $l \neq m$.

We shall now state, without proof, sufficient conditions for the Euler scheme to have strong order of convergence $\frac{1}{2}$.

Theorem 2.4.1 (Strong convergence of Euler's method) *[18] Suppose that X_t represents the true Ito process while $\tilde{X}_t^\delta$ represents the Euler approximation of the process with time step δ. Then if*

$$\mathbb{E}([|X_0|^2]) < \infty$$

$$\mathbb{E}(|X_0 - \tilde{X_0}^\delta|)^{\frac{1}{2}} \leq L_1\delta^{\frac{1}{2}},$$

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq L_2|x-y|,$$

$$|a(t,x)| + |b(t,x)| \leq L_3(1+|x|)$$

and

$$|a(s,x) - a(t,x)| + |b(s,x) - b(t,x)| \leq L_4(1+|x|)|s-t|^{\frac{1}{2}}$$

for all $s,\ t \in [0,T]$ and $x,\ y \in \mathbb{R}^n$, where the constants $L_1,\ L_2,\ L_3,\ L_4$ do not depend on δ. Then for $\tilde{X}^\delta$ the estimate

$$\mathbb{E}(|X_T - \tilde{X}_T^\delta|) \leq L_5\delta^{\frac{1}{2}}$$

holds. Hence the Euler approximation has strong order convergence $\frac{1}{2}$.

Lastly, we present sufficient conditions for the Euler approximation $\tilde{X}_T^\delta$ to be a weak approximation of order 1 to the true Ito process X_T.

Theorem 2.4.2 (Weak convergence of Euler's method) *[18] Given X_t and $\tilde{X}_t^\delta$. Then if a and b are Lipschitz continuous with components $a^k,\ b^{ij} \in \mathcal{C}^{4\times 1}$ for all*

$i = 1, \cdots, n$ and $j = 1, \cdots, m$ and the following linear growth bound is satisfied

$$|a(t,x) + b(t,x)| < L(1 + |x|)$$

for all $x \in \mathbb{R}^n$ and $t \in [0,T]$. Then for all $f : \mathbb{R}^n \to \mathbb{R}$ with $f \in \mathcal{C}^2(\mathbb{R}^n)$, there exist a constant K, which does not depend on δ, such that

$$|\mathbb{E}[f(X_T)] - \mathbb{E}[f(\tilde{X}_T^\delta)]| \leq K\delta^1.$$

Hence the Euler approximation, with the above assumptions, has weak convergence of order one.

Chapter 3

Pricing European Options

We begin our discussion of the Black and Scholes analysis for pricing European options by defining the models of two assets involved in a market. For simplicity, we first consider the one-dimensional case. Upon development of a market model, we formally define mathematically a European option. We shall consider two methodologies for the computation of the premium of a European option. The first method is based on replicating self-financing portfolios and a no-arbitrage argument. The second consists of finding an equivalent martingale measure under which the discounted risky asset price is a martingale. This is also known as risk-neutral valuation. We shall show that utilizing classical stochastic differential equation theory that the second method is just a probabilistic interpretation of the first.

3.1 Market Model

The Black and Scholes analysis considers a riskless asset such as a bond and a risky asset, which we assume is a stock index. The two assets are modeled as follows

3.1.1 Riskless Asset

A bond price, B_t, is modeled with the following ordinary differential equation

$$\frac{dB_t}{B_t} = rdt \qquad t \in [0, T]. \tag{3.1}$$

This may be interpreted as the infinitesimal return on the bond is given by rdt. The parameter r represents the instantaneous rate of return, which is commonly called the instantaneous interest rate. Using the separation of variables technique for ordinary differential equations, the solution to (3.1) is given by

$$B_t = B_0 e^{rt}, \tag{3.2}$$

where B_0 represents the initial investment in the bond.

3.1.2 Risky Asset

The corresponding Black and Scholes model for the stock price, S_t, is a stochastic differential equation as follows

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \qquad t \in [0, T]. \tag{3.3}$$

Equation (3.3) has the interpretation that the infinitesimal return on the stock has mean μdt and centered random fluctuations independent of the past up to time t. The parameter μ is the constant mean return rate and the random fluctuations are modeled by σdW_t, where dW_t is an increment of Brownian motion. The parameter σ is a positive constant, which we call the volatility of the stock. For simplicity we do not model the dividends paid in the time interval we are considering.

Equation (3.3) may be expressed in differential form as

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{3.4}$$

while in integral form we have

$$S_t = S_0 + \int_0^t \mu S_\tau d\tau + \int_0^t \sigma S_\tau dW_\tau, \tag{3.5}$$

where the last integral is the stochastic Ito integral described in section 2.2.2. Since $\mu(t,s) = \mu s$ and $\sigma(t,s) = \sigma s$ are linearly growing at infinity and S_0 is the initial condition that is square integrable and independent of the Brownian motion, then by theorem 2.2.1 this is enough to guarantee existence and uniqueness of an adapted square integrable solution S_t.

Utilizing Ito's formula we may obtain an explicit solution to (3.3). We may suspect that the solution would involve $\log S_t$ since from elementary calculus $\int \frac{ds}{s} = \log s$. We compute the differential of $\log S_t$ utilizing the one-dimensional form of Ito's formula given by (2.5) with $g(t,s) = \log s$, $\mu(t,s) = \mu s$ and $\sigma(t,s) = \sigma s$

$$d \log S_t = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t.$$

Therefore, the logarithm of the stock price is given by

$$\log S_t = \log S_0 + (\mu - \frac{\sigma^2}{2})t + \sigma W_t.$$

Hence, we have the following solution for the stock price

$$S_t = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right).$$

Since $\frac{S_t}{S_0}$ is the exponential of a process that is normally distributed with mean $(\mu - \frac{1}{2})t$ and variance $\sigma^2 t$ at time t, then we say $\frac{S_t}{S_0}$ has a lognormal distribution or S_t is a geometric Brownian motion.

Notice in this model that if the stock price ever becomes 0, then it remains there until the terminal time. Hence, bankruptcy is a permanent state in this model.

However, if S_0 is nonzero, then does not go to zero in finite time with probability one because $\frac{1}{t}W_t$ tends to zero as t tends to infinity, with probability one [10].

In figure 3.1 we show a sample path or trajectory of a stock price modeled by a geometric Brownian motion. Notice how the stock price has a mean growth rate plus "noise" attributed to the random fluctuations.

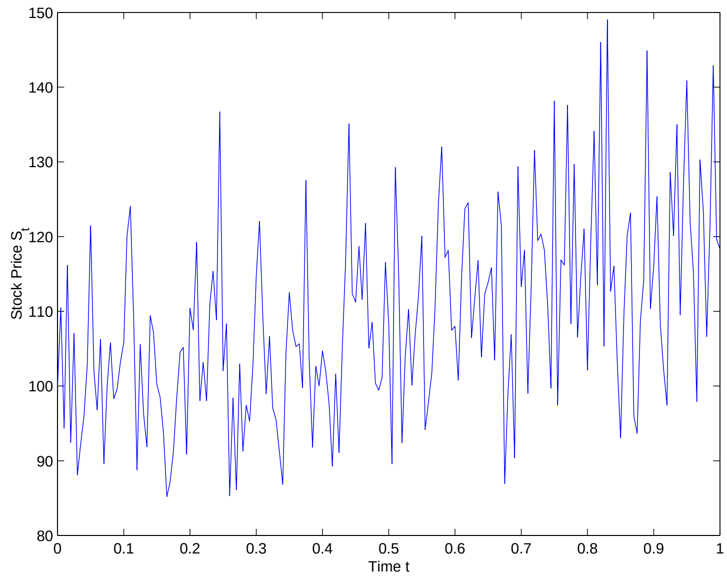

Figure 3.1: *A sample path or trajectory of a stock price modeled by a geometric Brownian given by (3.3), with* $S_0 = 100$ *and* $\mu = .15$*,* $\sigma = .10$ *and* $T = 1$*.*

3.2 European Style Option

We now formally define a European call or put option and formulate a functional form for the payoff of each of these contracts.

3.2.1 Call Option

A European call option is a contract that gives its owner the right, but not the obligation, to purchase one unit of an underlying asset for a predetermined strike price K at a specified maturity T. If S_T represents the asset price at time T, then the value of the contract at time T, its payoff, is given by

$$\phi(S_T) = (S_T - K)^+ = \begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{if } S_T < K \end{cases}$$

3.2.2 Put Option

Analogously, a European put option is a contract that gives its owner the right, but not the obligation, to sell one unit of an underlying asset for a predetermined strike price K at a specified maturity T. If S_T represents the asset price at time T, then the value of the contract at time T, its payoff, is given by

$$\phi(S_T) = (K - S_T)^+ = \begin{cases} K - S_T & \text{if } S_T < K \\ 0 & \text{if } S_T > K \end{cases}$$

The question that remains is "What premium should be paid in advance in order to enter into one of these contracts that pays $\phi(S_T)$ at the terminal time?" We present two methods by which this premium may be computed. We start with the replicating portfolio strategy.

3.3 Replicating Portfolio Pricing Strategy

The Black and Scholes analysis for pricing a European option leads to a precise trading strategy in the underlying stock and bond whereby the terminal price of the portfolio is equivalent to the price of the option at the terminal time, regardless of the path of the stock[7]. Thus, implementing this trading strategy when an investor is short an option protects him against all risk of eventual loss because a loss at the final time in one part of the portfolio will be balanced by a gain in the other part. This is called a replicating strategy and it provides an insurance policy for an investor who is short an option. Since the trading strategy will require continuous adjustments in each part of the portfolio, we call it a dynamic hedging strategy, where hedge means to eliminate risk. The primary step in the Black and Scholes methodology is the constructing of this hedging strategy and arguing based on no-arbitrage that the replicated portfolio is the fair price of the option at any given time. We begin by defining replicating self-financing portfolio and then derive the Black-Scholes pricing partial differential equation.

3.3.1 Replicating Self-Financing Portfolios

We assume that S_t represents the price process that is modeled as a geometric Browninan motion and $\phi(S_T)$ the payoff of the stock at terminal time T. A trading strategy is a pair of adapted processes, (α_t, β_t), which represent the number of stock and bond units held at time t. We shall assume that α_t is square integrable and β_t is integrable so that the stochastic integral involving α_t and the usual integral involving β_t are well defined.

Given that the price of a bond at time t is $B_t = B_0 e^{rt}$, the value of a portfolio with α_t units of stock and β_t units of bond is given by $\alpha_t S_t + \beta_t e^{rt}$. For simplicity, we have assumed that $B_0 = 1$. We say that the portfolio will replicate an option at time T if its value is almost surely equal to the payoff of the option

$$\alpha_T S_T + \beta_T e^{rT} = \phi(S_T). \tag{3.6}$$

Also, we say that the portfolio is self-financing if the only changes in the portfolio over time are due to the variations in the stock and the bond. For example, if more of the stock is bought, then it would be paid for by selling bonds.[10] Using the integration by parts formula to compute $d(\alpha_t S_t)$, we may express the self-financing principle mathematically as follows

$$d(\alpha_t S_t + \beta_t e^{rt}) = \alpha_t dS_t + r\beta_t e^{rt} dt, \tag{3.7}$$

where we have assumed the following because the change in the portfolio is due only to the movement of the stock or bond

$$S_t d\alpha_t + e^{rt} d\beta_t + d < \alpha, S >_t = 0. \tag{3.8}$$

The self-financing property given by (3.7) may be written in integral form as

$$\alpha_t S_t + \beta_t e^{rt} = \alpha_0 + \beta_0 + \int_0^t a_\tau S_\tau + \int_0^t r\beta_\tau e^{r\tau} d\tau \tag{3.9}$$

3.3.2 Black-Scholes Pricing Partial Differential Equation

We denote $P(t, s)$ as the price of a European option at any time t and stock price s with payoff given by $\phi(S_T)$. We make the assumption that the pricing function $P(t, s)$ is regular enough to apply Ito's formula. Our goal is to construct a self-financing portfolio (α_t, β_t) that will replicate the option at maturity time T.

Arbitrage refers to there being an opportunity to make an instantaneous risk-free profit. Therefore, to ensure there are no arbitrage opportunities we require the value of the portfolio at any time t to equal the value of the option as follows:

$$\alpha_t S_t + \beta_t e^{rt} = P(t, S_t), \qquad t \in [0, T]. \tag{3.10}$$

An arbitrage opportunity would exist, for example, if the left side of (3.10) were less than right side, by an investor selling the over-priced option immediately and investing in the under-priced stock-bond strategy. This yields an instant profit and there is no risk to future loss because the portfolio replicates the option at the terminal time.[10]

We begin the derivation of the Black-Scholes pricing partial differential equation by using Ito's formula to differentiate (3.10). Applying Ito's formula and the self-financing property yields the following equation

$$(\alpha_t \mu S_t + \beta_t r e^{rt})dt + \alpha_t \sigma S_t dW_t = \left(\frac{\partial P}{\partial t} + \mu S_t \frac{\partial P}{\partial s} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 P}{\partial s^2}\right) dt + \sigma S_t \frac{\partial P}{\partial s} dW_t, \tag{3.11}$$

where the partial derivatives of P are evaluated at (t, S_t). By equating the coefficients of the dW_t terms of (3.11) we obtain

$$\alpha_t = \frac{\partial P}{\partial s}(t, S_t) \tag{3.12}$$

and from (3.10) we have

$$\beta_t = (P(t, S_t) - \alpha_t S_t)e^{-rt}. \tag{3.13}$$

Equating dW_t terms in (3.10) gives

$$r\left(P - S_t \frac{\partial P}{\partial s}\right) = \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 P}{\partial s^2}. \tag{3.14}$$

Equation (3.14) describes the evolution of the price of an option and is called the Black-Scholes pricing partial differential equation. In operator notation we may express (3.14) as

$$\mathcal{L}_{BS}(\sigma)P = 0, \tag{3.15}$$

where the operator $\mathcal{L}_{BS}(\sigma)$ is defined as

$$\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + r\left(s\frac{\partial}{\partial s} - \cdot\right). \tag{3.16}$$

This equation holds for $t \leq T$ and $s > 0$, since in our model the stock price remains positive. Since we know the payoff of the option, then (3.15) may be solved backward in time with the final condition $P(T, s) = \phi(s)$. This yields a unique solution, which is the value of the self-financing replicating portfolio at any time t. Once the price of the option is obtained, we may compute the trading strategy (α_t, β_t) by using equations (3.12) and (3.13).

A European call option may be priced by solving equation (3.15) with the terminal condition being $P(T, s) = (s - K)^+$. We shall denote the price of the call option by $C_{BS}(t, s)$. An explicit solution for C_{BS} is given as

$$C_{BS}(t, s) = sN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{3.17}$$

where

$$d_1 = \frac{log(s/K) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \tag{3.18}$$

$$d_2 = d_1 - \sigma\sqrt{T - t} \tag{3.19}$$

and

$$N(z) = \frac{1}{2\pi}\int_{-\infty}^{z} e^{-y^2/2} dy \tag{3.20}$$

The popularity of this formula in the finance industry since the mid-1970's is attributed to the fact that the price may be explicitly computed once the volatility parameter σ has been estimated from data. We will also denote $C_{BS}(t, s)$ by $C_{BS}(t, s, T, K, \sigma)$ to explicitly show the additional dependency of the function on terminal time T, strike price K and and volatility σ.

Figure 3.2 gives a plot of the pricing function $C_{BS}(0, s, 100, 1, .15)$ versus the present stock value s.

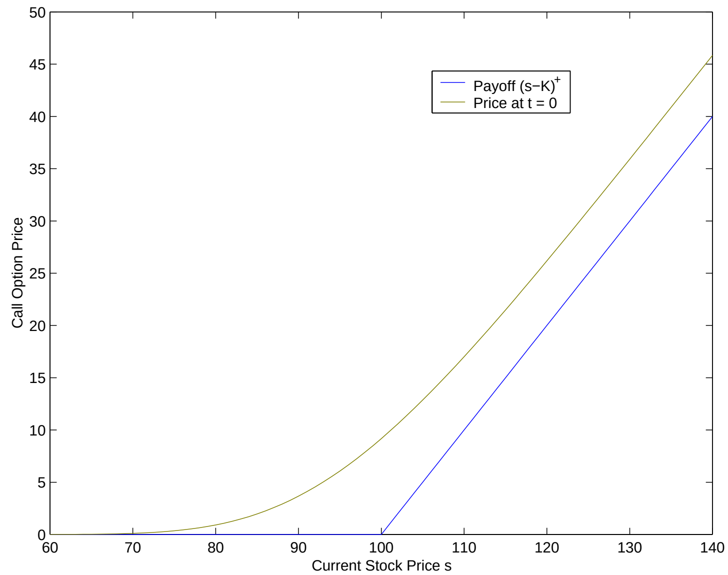

Figure 3.2: *Black-Scholes call option pricing function at time* $t = 0$*, with* $\sigma = .15$*,* $K = 100$*,* $T = 1$*,* $\sigma = .15$ *and* $r = .06$

We begin our discussion on the price of a put option, which we denote by $P_{BS}(t, s)$, by introducing the relationship between a call and put option, which is called put-call parity

$$C_{BS}(t, S_t) - P_{BS}(t, S_t) = S_t - Ke^{-r(T-t)}. \tag{3.21}$$

Equation (3.21) follows from no-arbitrage arguments. For example, buying a call, selling a put and one unit of stock and investing the difference in a bond, creates a profit at time T regardless of the path of the stock.[10]

Combining equations (3.21) and (3.17) give the price of the put option as follows

$$P_{BS}(t, s) = Ke^{-r(T-t)}N(-d_2) - sN(-d_1), \tag{3.22}$$

where d_1, d_2 and $N(z)$ are given by (3.18), (3.19) and (3.20) respectively.

Figure 3.3 gives a plot of the pricing function $P_{BS}(0, s, 100, 1, .15)$ versus the present stock value s.

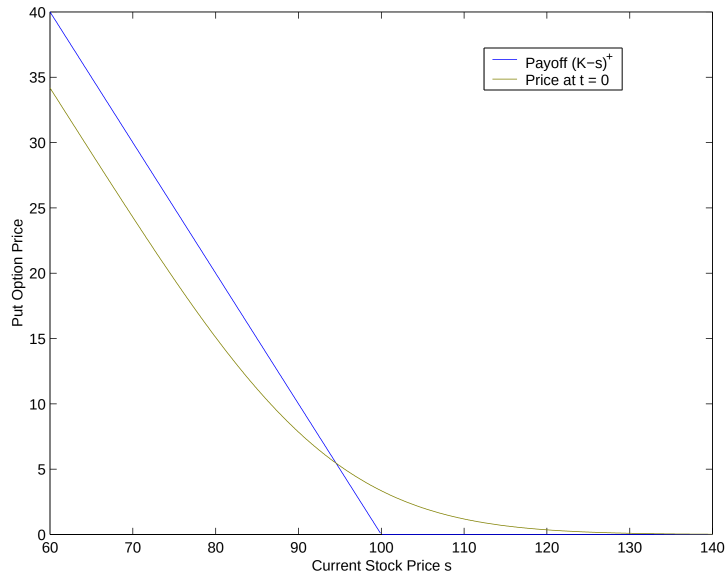

Figure 3.3: *Black-Scholes put option pricing function at time* $t = 0$*, with* $\sigma = .15$*,* $K = 100$*,* $T = 1$*,* $\sigma = .15$ *and* $r = .06$

3.4 Pricing under an Equivalent Martingale Measure

In this section we wish to compute the price of a European option from a probabilistic point of view. It would seem reasonable that the price or premium of a European option would be the expected payoff of the stock discounted back to the present time. More precisely, suppose the stock process evolves as a geometric Brownian motion and let $S_t = s$, then the price at time $t = 0$ of the option is

$$P(0, s) = \mathbb{E}[e^{-rT}\phi(S_T)|S_t = s]. \tag{3.23}$$

In equation (3.23), the expectation is computed with respect to the probability measure associated with the stock. This is also known as the physical measure, which we denote by $\mathbb{P}$. However, pricing an option in this manner leads to an arbitrage opportunity because the discounted stock price process is not a martingale under this measure. Therefore, we find an equivalent measure under which the discounted stock price process is a martingale. We call this new measure, which we denote by $\mathbb{P}^*$, an equivalent martingale measure. We first show that the discounted stock price process is a martingale under this new measure. This leads to the discounted self-financing portfolio process being a martingale under this measure and hence we may obtain the price of the option at any time t.

3.4.1 Equivalent Martingale Measure

Let $\hat{S}_t = e^{-rt} S_t$ denote the discounted price process. Using Ito's formula, we see that the evolution of the discounted stock price process is as follows

$$d\hat{S}_t = (\mu - r)\hat{S}_t dt + \sigma \hat{S}_t dW_t. \tag{3.24}$$

We wish to apply Girsanov's theorem to the above equation in order to obtain a measure under which $\hat{S}_t$ is a martingale. We proceed as follows.

Let

$$\gamma\sigma = \mu - r$$

and define

$$M_t = exp\left(-\int_0^t \gamma dW_s - \frac{1}{2}\int_0^t \gamma^2 ds\right) = exp\left(-\gamma W_t - \frac{1}{2}\gamma^2 t\right); \quad t \leq T.$$

Since $\mathbb{E}[M_T] = 1$, we may a define new probability measure as follows

$$d\mathbb{P}^*|_{\mathcal{F}_t} = M_t d\mathbb{P}|_{\mathcal{F}_t}.$$

By Girsanov's theorem(2.3.5) the process

$$W_t^* = \int_0^t \gamma ds + dW_t = \gamma t + W_t$$

is a Brownian motion with respect to $\mathbb{P}^*$ and in terms of W_t^* the process $\hat{S}_t$ has the representation

$$d\hat{S}_t = \sigma \hat{S}_t dW_t^*,$$

which shows that $\hat{S}_t$ is a martingale under the measure $\mathbb{P}^*$.

3.4.2 Self-Financing Portfolios

We shall show that the value of a self-financing portfolio under the measure $\mathbb{P}^*$, which we term the risk-neutral measure, is a martingale. In addition, we establish a relationship between martingales and no-arbitrage.

We denote a portfolio that consists of α_t units of stock and β_t units of bonds as V_t:

$$V_t = \alpha_t S_t + \beta_t B_t. \tag{3.25}$$

Utilizing the self-financing property we now show that the discounted portfolio price, which we denote as $\hat{V}_t = e^{-rt} V_t$, is a martingale under $\mathbb{P}^*$ as follows:

$$\begin{aligned} d\hat{V}_t &= -re^{-rt} V_t dt + e^{-rt} dV_t \\ &= -re^{-rt}(\alpha_t S_t + \beta_t e^{rt})dt + e^{-rt}(\alpha_t dS_t + r\beta_t e^{rt} dt) \\ &= -re^{rt}\alpha_t S_t dt + e^{-rt}\alpha_t dS_t \\ &= \alpha_t d(e^{-rt} S_t) \\ &= \alpha_t d\hat{S}_t \\ &= \sigma \alpha_t \hat{S}_t dW_t^*. \end{aligned} \tag{3.26}$$

In addition, a similar computation shows that if a discounted portfolio is a martingale, then it is self-financing.

We now demonstrate the relationship between martingales and no-arbitrage. Suppose that (α_t, β_t) is a self-financing arbitrage strategy. In other words

$$V_T \geq e^{rT} V_0 \quad \mathbb{P} - a.s. \tag{3.27}$$

with

$$\mathbb{P}[V_T > e^{rT} V_0] > 0, \tag{3.28}$$

so that this trading strategy always makes at least money-in-the-bank. However,

$$\mathbb{E}^*[V_T] = e^{rT} V_0$$

because $V_T = e^{rT} \hat{V}_T$ and $\hat{V}_T$ is a martingale. Therefore, equations (3.27) and (3.28) can not hold because they should hold for $\mathbb{P}$ and $\mathbb{P}^*$ since $\mathbb{P}$ and $\mathbb{P}^*$ are equivalent.[10]

3.4.3 Risk-Neutral Valuation

We make the same assumption as in (3.6) that (α_t, β_t) is a self-financing portfolio that replicates an option at terminal time T as follows:

$$\alpha_T S_T + \beta_T e^{rT} = \phi(S_T) \tag{3.29}$$

In section 3.3.2 a no-arbitrage argument shows that the price of a European option at time t is the value of this portfolio. The previous section shows that the discounted portfolio process is a martingale under the risk-neutral measure $\mathbb{P}^*$ that states

$$\hat{V}_t = \mathbb{E}^*[\hat{V}_T | \mathcal{F}_t],$$

and hence

$$V_t = \mathbb{E}^*[e^{-r(T-t)} \phi(S_T) | \mathcal{F}_t]. \tag{3.30}$$

Utilizing the markov property of S_t, which states that conditioning with respect to the filtration $\mathcal{F}_t$ is the same as conditioning with respect S_t, (3.30) may be rewritten as:

$$V_t = \mathbb{E}^*[e^{-r(T-t)}\phi(S_T)|S_t]. \tag{3.31}$$

Denoting by $P(t,s)$, the price of a European option at time t for an observed stock price $S_t = s$ is given as

$$P(t,s) = \mathbb{E}^*[e^{-r(T-t)}\phi(S_T)|S_t = s]. \tag{3.32}$$

W_t^* is a standard Brownian motion under the risk-neutral measure P^*, the increment $W_T^* - W_t^*$ is distributed as $\mathcal{N}(0, T-t)$, hence (3.34) may be written as the Gaussian integral

$$P(t,s) = \frac{1}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{\infty} e^{-r(T-t)}\phi(se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma z})e^{-\frac{z^2}{2(T-t)}}dz. \tag{3.33}$$

3.4.4 Risk-Neutral Expectations and Partial Differential Equations

In this section we wish to establish the connection between risk-neutral expectation and partial differential equations.

From the previous section, we know that the price of a European option at any time t is given by

$$P(t,s) = \mathbb{E}^*[e^{-r(T-t)}\phi(S_T)|S_t = s]. \tag{3.34}$$

Using the definition of infinitesimal generator given by definition 2.3.2 the infinitesimal generator of S_t under the risk neutral measure $\mathbb{P}^*$ is given by

$$\mathcal{L} = rs\frac{\partial}{\partial s} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2}{\partial s^2} \tag{3.35}$$

Utilizing the Feyman-Kac formula, which is given by theorem 2.11, we may formulate the following partial differential equation

$$\frac{\partial P}{\partial t} + \mathcal{L}P - rP = 0, \qquad t \in [0, T] \tag{3.36}$$

with

$$P(T, s) = \phi(s). \tag{3.37}$$

Equations (3.36) and (3.37) are exactly the Black and Scholes pricing partial differential equation given by (3.15).

Chapter 4

Pricing European Options under Stochastic Volatility

Thus far we have assumed that the stock price process being modeled is a geometric Brownian motion as follows

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{4.1}$$

where we have assumed a constant mean rate of return μ and constant volatility σ. We shall relax the assumption that volatility is constant and allow it to be randomly varying for the following reason: a well-known discrepancy between Black-Scholes predicted European option prices and market traded option prices, which is known as the smile curve, is resolved by stochastic volatility models[10]. We begin this chapter by defining implied volatility, the smile curve and explain why stochastic volatility models are a viable option. Secondly, we formulate mathematically stochastic volatility models and provide some of the more common types; in particular a mean-reverting stochastic volatility model. Lastly, we address how to price a European option in a stochastic volatility environment.

4.1 Implied Volatility and the Smile Curve

Given an observed European call option price C_{OBS} for a contract, the implied volatility, which we denote by I, is the value of the parameter σ that must be input in the Black-Scholes pricing formula to match this price

$$C_{BS}(t, s, K, T, I) = C_{OBS} \tag{4.2}$$

Implied volatility is a great way to compare model predicted option prices with observed prices. It is also the unit by which traders quote option prices with the conversion to price computed using the Black-Scholes formula.

In general, $I = I(t, s; K, T)$, but if observed option prices equaled the Black-Scholes prices, then the implied volatility would be constant across all option contracts and hence modeling the volatility with a constant parameter would be an accurate model; however this is not the case, which signifies the limitation of modeling with constant volatility.

The smile effect is a widely known phenomenon testifying to the inaccuracy of the Black and Scholes model[12]. This means that implied volatilities of observed prices are not constant across the range of options, but vary with respect to their strike price and time-to-maturity of the option. The graph of $I(K)$ tends to be downward sloping for at- and around-the-money call option prices ($95\% \leq K/s \leq 105\%$) and then curves upwards for far out-of-the-money ($K >> s$) call option prices [10]. This is known as a downward sloping skew. Therefore, using implied volatility from an at-the-money call option will result in a premium charged for in-the-money call options and out-of-the-money put options. This tells us the market prices as though the geometric Brownian motion model fails to capture probabilities of large downward stock price movements and adds a premium to Black-Scholes prices to account for this.

We have shown that the market prices options contrary to the Black-Scholes pricing formula and we next present stochastic volatility models which capture the downward sloping skew property of the implied volatility.

4.2 Stochastic Volatility Models

A stochastic volatility model is a model in which the evolution of the stock price is described in the following manner

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_t, \tag{4.3}$$

where μ represents the mean return rate and $\sigma(\cdot)$ represents the volatility, which is driven by another stochastic process Y_t. We shall assume that $\sigma(\cdot)$ is positive, bounded and bounded away from zero: $0 \leq \sigma_1 \leq \sigma(\cdot) \leq \sigma_2$ for two constants σ_1 and σ_2. The volatility is driven by an Ito process, Y_t, satisfying another stochastic differential equation driven by a second Brownian motion. In order to account for the downward sloping skew we allow these Brownian motions to be dependent.

The process, Y_t, which drives the volatility is commonly modeled as a mean-reverting process. The term mean-reverting refers to the fact that the process returns to the average value of its invariant distribution-the long run distribution of the process. In terms of financial modeling, mean-reverting often refers to a linear pull-back term in the drift of the volatility process. Usually, Y_t takes the following form:

$$dY_t = \alpha(m - Y_t)dt + \cdots d\hat{Z}_t\,, \tag{4.4}$$

where $\hat{Z}_t$ is a Brownian motion correlated with W_t. The rate of mean reversion is represented by the parameter α and the mean level of the invariant distribution of Y_t is given by m. We consider here the simplest model, which is known as the Ornstein-Uhlenbeck process:

$$dY_t = \alpha(m - Y_t)dt + \beta d\hat{Z}_t \, , \tag{4.5}$$

where $\beta > 0$ is a constant and $\hat{Z}_t$ is a Brownian motion expressed as

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2}\, Z_t \, ,$$

where Z_t is a standard Brownian motion independent of W_t. The parameter $\rho \in (-1, 1)$ is the constant instantaneous correlation coefficient between $\hat{Z}_t$ and W_t defined by

$$d < W, \hat{Z} >_t = \rho dt. \tag{4.6}$$

The process Y_t may be explicitly written in terms of its initial value, which we denote by y, as

$$Y_t = m + (y - m)e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)} d\hat{Z}_s. \tag{4.7}$$

We see by equation (4.7) that Y_t is distributed as

$$Y_t \sim \mathcal{N}(m + (y - m)e^{-\alpha t}, \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})). \tag{4.8}$$

Hence, the distribution of Y_t as $t \to \infty$, which is the invariant distribution, is given by $\mathcal{N}(m, \frac{\beta^2}{2\alpha})$. Denoting the variance of the invariant distribution by $\nu^2 = \frac{\beta^2}{2\alpha}$, we may rewrite (4.5) as

$$dY_t = \alpha(m - Y_t)dt + \nu\sqrt{2\alpha}\, d\hat{Z}_t \, . \tag{4.9}$$

Rewriting equation (4.5) in the above form is effective for when we perform asymptotic analysis.

A couple of other common driving processes for Y_t are the following

Lognormal:

$$dY_t = c_1 Y_t dt + c_2 Y_t d\hat{Z}_t,$$

which is not mean-reverting and

Cox-Ingersall-Ross:

$$dY_t = \kappa(\hat{m} - Y_t)dt + \upsilon\sqrt{Y}_t d\hat{Z}_t.$$

4.3 Pricing in a Stochastic Volatility Environment

We now focus on pricing European options when the volatility is modeled as a function of a stochastic process. One approach, which is detailed in [10], is to construct a hedged portfolio of assets which can be priced by the no-arbitrage principle. With this approach we hedge with the underlying stock and another option because volatility is not a tradeable asset. The second approach, which we describe here, is the construction of an equivalent martingale measure so that the discounted stock price process is a martingale. In addition, we also present the partial differential equation representation of the option price that is obtained utilizing the Feyman-Kac formula, which is the same partial differential equation formulated through the no-arbitrage principle.

4.3.1 Equivalent Martingale Measure Approach

We denote $P(t, s, y)$ to be the price of a European option at time t, stock price s and volatility level $\sigma(y)$ that has payoff $\phi(S_T)$. Although volatility is not directly observable we shall assume for this work that the level at time t has been estimated.

We wish to obtain a pricing formula when the volatility is a function of a mean-reverting OU process. We formulate the two-dimensional model as

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma(Y_t) S_t dW_t \\ dY_t &= \alpha(m - Y_t)dt + \nu\sqrt{2\alpha}(\rho dW_t + \sqrt{1-\rho^2}dZ_t), \end{aligned} \tag{4.10}$$

where W_t and Z_t are independent Brownian motions.

We may obtain, as in section 3.4.1, an equivalent martingale measure utilizing Girsanov's theorem; however, there will be a family of equivalent martingale measures since any shift in the Brownian motion Z_t will still result in the discounted price process being a martingale.

We apply Girsanov's theorem by defining the following:

$$\psi_t = \begin{pmatrix} \frac{\mu - r}{\sigma(Y_t)} \\ \gamma_t \end{pmatrix}, \quad N_t = \begin{pmatrix} W_t \\ Z_t \end{pmatrix},$$

where γ_t is any adapted square integrable process. We assume that ψ_t satisfies the Novikov condition given in theorem 2.3.5 and define

$$M_t = exp\left(-\int_0^t \psi_s \cdot dN_s - \frac{1}{2}\int_0^t \psi_s \cdot \psi_s ds\right).$$

Since $\mathbb{E}[M_T] = 1$, we may define a new probability measure $\mathbb{P}^{*(\gamma)}$ as follows

$$d\mathbb{P}^{*(\gamma)}|_{\mathcal{F}_t} = M_t d\mathbb{P}|_{\mathcal{F}_t}.$$

By Girsanov's theorem we conclude the process

$$N_t^* = \int_0^t \psi_s ds + N_t$$

is a Brownian motion with respect to $\mathbb{P}^{*(\gamma)}$, where N_t^* has the representation

$$N_t^* = \begin{pmatrix} W_t^* \\ Z_t^* \end{pmatrix}.$$

Under the measure $\mathbb{P}^{*(\gamma)}$, equation (4.10) may be rewritten as

$$\begin{aligned} dS_t &= rS_t dt + \sigma(Y_t) S_t dW_t^* \\ dY_t &= [\alpha(m - Y_t) - \nu\sqrt{2\alpha}\,\Lambda(y)]dt + \nu\sqrt{2\alpha}\,(\rho dW_t^* + \sqrt{1-\rho^2}\,dZ_t^*) \end{aligned} \tag{4.11}$$

where

$$\Lambda(y) = \frac{\rho(\mu - r)}{\sigma(y)} + \gamma(y)\sqrt{1-\rho^2}. \tag{4.12}$$

Any admissible choice of γ_t leads to a risk-neutral measure $\mathbb{P}^{*(\gamma)}$ and the no-arbitrage price computed as

$$P(t, x, y) = \mathbb{E}^{*(\gamma)}[e^{-r(T-t)}\phi(S_T)|S_t = s, Y_t = y]. \tag{4.13}$$

The process (γ_t) is called the market price of volatility risk from the second source of randomness Z_t that drives the volatility. We refer to [10] for a full detailed explanation of the market price of volatility risk. Notice that it parameterizes the space of risk-neutral measures $\{\mathbb{P}^{*(\gamma)}\}$. We note the market chooses the equivalent martingale measure $\{\mathbb{P}^{*(\gamma)}\}$.

4.3.2 Partial differential equation approach

By the Feynman-Kac formula, the pricing function given by equation (8.27) satisfies the following partial differential equation with two space dimensions:

$$\begin{aligned} \frac{\partial P}{\partial t} &+ \frac{1}{2}\sigma^2(y)s^2\frac{\partial^2 P}{\partial s^2} + \rho\nu\sqrt{2\alpha}\,s\sigma(y)\frac{\partial^2 P}{\partial s \partial y} + \nu^2\alpha\,\frac{\partial^2 P}{\partial y^2} \\ &+ r(s\frac{\partial P}{\partial s} - P) + \Big[(\alpha(m-y)) - \nu\sqrt{2\alpha}\,\Lambda(y)\Big]\,\frac{\partial P}{\partial y} = 0\,, \end{aligned} \tag{4.14}$$

where $\Lambda(y)$ is given by (9.5). This is same partial differential equation that is obtained through hedging and using a no-arbitrage argument as shown in [10]. In order to find $P(t,s,y)$, this PDE is solved backward in time with the terminal condition $P(T,s,y) = \phi(s)$ which is $(s-K)^+$ in the case of a call option. We introduce the following convenient operator notation:

$$\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} \tag{4.15}$$

$$\mathcal{L}_1 = \rho\nu\sqrt{2}\, s\sigma(y)\frac{\partial^2}{\partial s \partial y} - \nu\sqrt{2}\,\Lambda(y)\frac{\partial}{\partial y} \tag{4.16}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2(y)s^2\frac{\partial^2}{\partial s^2} + r\left(s\frac{\partial}{\partial s} - \cdot\right), \tag{4.17}$$

where

- $\alpha\mathcal{L}_0$ is the infinitesimal generator of the OU process Y_t.
- $\mathcal{L}_1$ contains the mixed partial derivative due to the correlation ρ between the W^* and Z^*. It also contains the first order derivative with respect to y due to the market prices of risk.
- $\mathcal{L}_2$ is the Black-Scholes operator with volatility $\sigma(y)$, also denoted by $\mathcal{L}_{BS(\sigma(y))}$.

Equation (9.11) may be written in the compact form:

$$\left(\alpha\mathcal{L}_0 + \sqrt{\alpha}\mathcal{L}_1 + \mathcal{L}_2\right) P = 0\,, \tag{4.18}$$

to be solved with the payoff terminal condition at maturity T.

Chapter 5

Monte Carlo and Importance Sampling

In this chapter we present the importance sampling variance reduction technique for Monte Carlo simulation when modeling with stochastic differential equations. The reader is referred to [22] and [11] for more details.

Let $(V_t)_{0\leq t\leq T}$ be an n-dimensional stochastic process which evolves as follows

$$dV_t \quad = \quad b(t,V_t)dt + a(t,V_t)d\eta_t \,, \tag{5.1}$$

where η_t is a standard n-dimensional $\mathbb{P}$-Brownian motion and $b(\cdot,\cdot) \in \mathbb{R}^n$, $a(\cdot,\cdot) \in \mathbb{R}^{n\times n}$ which satisfy the usual regularity and boundedness assumptions to ensure existence and uniqueness of the solution. Given a real function $\phi(v)$, which is bounded for instance, we define the following function $u(t,v)$

$$u(t,v) = \mathbb{E}\{\phi(V_T)|V_t = v\} \,.$$

A Monte Carlo simulation consists of approximating $u(t,v)$ in the following manner

$$u(t,v) \approx \frac{1}{N}\sum_{k=1}^{N} \phi(V_T^{(k)}) \,, \tag{5.2}$$

where $(V_s^{(k)}, k = 1, \cdots, N)$ are independent realizations of the process V_s for $t \le s \le T$ and $V_t^{(k)} = v$.

There is an alternative way to construct a Monte Carlo approximation of $u(t, v)$. Given a square integrable $\mathbb{R}^n$-valued, η_t-adapted process of the form $h(t, V_t)$, we consider the following process Q_t

$$Q_t = \exp\left\{ \int_0^t h(s, V_s) \cdot d\eta_s + \frac{1}{2} \int_0^t \|h(s, V_s)\|^2 ds \right\} .$$

If $\mathbb{E}(Q_T^{-1}) = 1$, then $(Q_t)_{0 \le t \le T}$ is a positive martingale and a new probability measure, $\tilde{\mathbb{P}}$, may be defined by the density

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = (Q_T)^{-1} .$$

With respect to this new measure, $u(t, v)$ may be written as

$$u(t, v) = \tilde{\mathbb{E}}\{\phi(V_T) Q_T | V_t = v\} . \tag{5.3}$$

By Girsanov's theorem, the process$(\tilde{\eta}_t)_{0 \le t \le T}$ defined by $\tilde{\eta}_t = \eta_t + \int_0^t h(s, V_s) ds$ is a standard Brownian motion under the new measure $\tilde{\mathbb{P}}$. In terms of the Brownian motion $\tilde{\eta}_t$, the processes V_t and Q_t may be rewritten as

$$dV_t = (b(t, V_t) - a(t, V_t) h(t, V_t)) dt + a(t, V_t) d\tilde{\eta}_t \tag{5.4}$$
$$Q_t = \exp\left\{ \int_0^t h(s, V_s) \cdot d\tilde{\eta}_s - \frac{1}{2} \int_0^t \|h(s, V_s)\|^2 ds \right\} \tag{5.5}$$

which will be used in the simulations for the approximation of (5.3) by

$$u(t, v) \approx \frac{1}{N} \sum_{k=1}^{N} \phi(V_T^{(k)}) Q_T^{(k)} . \tag{5.6}$$

The **importance sampling variance reduction method** consists of determining a function $h(t, v)$ that leads to a smaller variance for the Monte Carlo approximation given in (9.27) than the variance for (5.2).

Applying Ito's formula to $u(t,V_t)Q_t$ and using the Kolmogrov's backward equation for $u(t,v)$ one gets

$$\begin{aligned} d(u(t,V_t)Q_t) &= u(t,V_t)Q_t h(t,V_t)\cdot d\tilde{\eta}_t + Q_t a^T(t,V_t)\nabla u(t,V_t)\cdot d\tilde{\eta}_t \\ &= Q_t(a^T\nabla u + uh)(t,V_t)\cdot d\tilde{\eta}_t\,. \end{aligned}$$

where a^T denotes the transpose of a, and ∇u the gradient of u with respect to the space variable v.

In order to obtain $u(0,v)$, for instance, one can integrate between 0 and T and deduce

$$u(T,V_T)Q_T = u(0,V_0)Q_0 + \int_0^T Q_t(a^T\nabla u + uh)(t,V_t)\cdot d\tilde{\eta}_t\,,$$

which reduces to

$$\phi(V_T)Q_T = u(0,v) + \int_0^T Q_t(a^T\nabla u + uh)(t,V_t)\cdot d\tilde{\eta}_t\,.$$

Therefore, the variances for the two Monte Carlo simulations (5.2) and (9.27) are given by

$$\begin{aligned} \mathrm{Var}_{\tilde{\mathbb{P}}}\left(\phi(V_T)Q_T\right) &= \tilde{\mathbb{E}}\left\{\int_0^T Q_t^2\|a^T\nabla u + uh\|^2 dt\right\} \\ \mathrm{Var}_{\mathbb{P}}\left(\phi(V_T)\right) &= \mathbb{E}\left\{\int_0^T \|a^T\nabla u\|^2 dt\right\}\,. \end{aligned}$$

If $u(t,v)$ were known, then the problem would be solved and the optimal choice for h, which gives a zero variance, would be

$$h = -\frac{1}{u}a^T\nabla u\,. \tag{5.7}$$

The main idea of the variance reduction methodology is find an appropriate approximation apriori for the unknown u in the previous formula so that the following holds

$$\mathrm{Var}_{\tilde{\mathbb{P}}}\left(\phi(V_T)Q_T\right) < \mathrm{Var}_{\mathbb{P}}\left(\phi(V_T)\right)$$

In the following chapter we present two ways by which we may obtain an approximation apriori to the expectation of interest when we analyze stochastic volatility models. Upon determining a procedure by which to find an approximation to the expectation of interest, we shall apply the importance sampling technique to the pricing of a European option when the stock is modeled with stochastic volatility.

Chapter 6

Asymptotic Analysis

In this chapter we provide two approaches to the asymptotic expansion of the stochastic volatility model used to compute the price of a European option. We utilize these two approaches to find an initial approximation for the price of a European option. The first approach, introduced in [11], involves a regular perturbation of the pricing partial differential equation. The second approach, introduced by [10],and detailed here, corresponds to a singular perturbation of the pricing partial differential equation.

6.1 Small Noise Expansion

We consider the following stochastic volatility model under the the risk-neutral measure $\mathbb{P}^{*(\gamma)}$

$$\begin{aligned} dS_t &= rS_t dt + \sigma(Y_t) S_t dW_t^* \\ dY_t &= [\alpha(m - Y_t) - \nu\sqrt{2\alpha}\,\Lambda(y)]dt + \nu\sqrt{2\alpha}\,(\rho dW_t^* + \sqrt{1-\rho^2}\,dZ_t^*) \end{aligned} \tag{6.1}$$

where

$$\Lambda(y) = \frac{\rho(\mu - r)}{\sigma(y)} + \gamma(y)\sqrt{1-\rho^2}. \tag{6.2}$$

Any admissible choice of $\gamma(y)$ leads to a risk-neutral measure $\mathbb{P}^{*(\gamma)}$ and the no-arbitrage European price computed as

$$P(t,x,y) = \mathbb{E}^{*(\gamma)}[e^{-r(T-t)}\phi(S_T)|S_t = s, Y_t = y]. \tag{6.3}$$

By the Feyman-Kac formula we know that (8.27) solves the following partial differential equation in compact form

$$\left(\alpha\mathcal{L}_0 + \sqrt{\alpha}\mathcal{L}_1 + \mathcal{L}_2\right) P = 0\,, \tag{6.4}$$

where the operators $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by (4.15). Equation (6.4) is to be solved with the payoff terminal condition as $P(T,s,y) = \phi(s-K)^+$.

If $\alpha = 0$, then (6.4) becomes

$$\mathcal{L}_2 P = 0\,.$$

Since $\mathcal{L}_2$ is simply the Black-Scholes operator with constant volatility $\sigma(y)$, then an approximation $P_{BS(\sigma(y))}$ of P is given by the Black-Scholes formula

$$P_{BS(\sigma(y))}(t,s) = sN(d_1) - Ke^{rT}N(d_2)\,, \tag{6.5}$$

where $N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-z^2/2}dz$, $d_1 = \frac{\ln(s/K)+(r+\sigma^2(y)/2)(T-t)}{\sigma(y)\sqrt{T-t}}$, $d_2 = d_1 - \sigma(y)\sqrt{T-t}$.

The function $P_{BS(\sigma(y))}(t,s)$ is the first term in the small noise expansion of $P(t,s,y)$ around $\alpha = 0$ or, in other words, when volatility is slowly varying and, in the limit, Y_t being "frozen" at its initial point y. A complete proof of the expansion result with higher order terms is given in [11] as well as numerical results showing that the important gain in variance reduction when utilizing this expansion as an approximation to the pricing function in importance sampling is obtained by using the leading order term $P_{BS(\sigma(y))}$ alone as an approximation of P.

6.2 Fast Mean-Reverting Expansion

In this section we present fast mean-reverting asymptotics, which corresponds to a singular perturbation of the pricing partial differential equation. However, we first present the concept of effective volatility, which will be utilized in the development of the expansion.

6.2.1 Effective Volatility

The process Y_t has an invariant distribution which admits the density $\Phi(y)$ obtained by solving the adjoint equation

$$\mathcal{L}_0^* \Phi = 0 \,,$$

where $\mathcal{L}_0^*$ denotes the adjoint of the infinitesimal generator $\mathcal{L}_0$ given by (4.15). In the case of the Ornstein-Uhlenbeck process, which we consider in this work, the invariant distribution is $\mathcal{N}(m, \nu^2)$ and the density is explicitly given by

$$\Phi(y) = \frac{1}{\sqrt{2\pi\nu^2}} \exp\left(-\frac{(y-m)^2}{2\nu^2} \right) .$$

Let $\langle \cdot \rangle$ denote the average with respect to this invariant distribution

$$\langle g \rangle = \int_{-\infty}^{\infty} g(y) \Phi(y) dy \,.$$

Given a bounded function g, by the ergodic theorem, the long-time average of $g(Y_t)$ is close to the average with respect to the invariant distribution

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t g(Y_s)ds =< g > \ .$$

In our case the "real time" for the process Y_t is the product αt and the long time behavior is the same in distribution as a large rate of mean-reversion, and therefore

$$\frac{1}{t}\int_0^t g(Y_s)ds \approx< g >,$$

for α large and any fixed $t > 0$. In particular, in the context of stochastic volatility models, we consider the **mean-square-time-averaged volatility** $\overline{\sigma^2}$ defined by

$$\overline{\sigma^2} = \frac{1}{T-t}\int_t^T \sigma^2(Y_s)ds\,.$$

The result above shows that, for α large

$$\overline{\sigma^2} \approx \langle \sigma^2 \rangle \equiv \bar{\sigma}^2\,, \tag{6.6}$$

which defines the constant **effective volatility** $\bar{\sigma}$. This quantity is easily estimated from the observed fluctuations in returns. We refer to [10] for more details.

6.2.2 Fast Mean-Reverting Asymptotics

We begin our discussion on fast mean-reverting asymptotics by letting $\alpha = \frac{1}{\epsilon}$ throughout our model. The Feyman-Kac representation of the European option pricing function may be expressed in terms of ϵ in operator form as

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P = 0\,, \tag{6.7}$$

with the payoff terminal condition as $P(T,s,y) = \phi(s-K)^+$.

Fast mean-reversion corresponds to epsilon becoming small. This is the same as the rate of mean-reversion, α, becoming large, which may be interpreted as the intrisic decorrelation in volatility being small. Equation (6.7) is called a singular perturbation because of the diverging terms when $\epsilon \to 0$, keeping the time derivative in $\mathcal{L}_2$ of order one.

The main idea of the asymptotic analysis is to expand the pricing function in powers of $\sqrt{\epsilon}$ as follows

$$P(t,s,y) = P_0(t,s,y) + \sqrt{\epsilon}P_1(t,s,y) + \epsilon P_2(t,s,y) + \epsilon\sqrt{\epsilon}P_3(t,s,y) + \cdots, \tag{6.8}$$

where P_0, P_1 and P_2 are functions to determined such that $P_0(T,s,y) = \phi(s)$. We will only find representations for the first two terms $P_0 + \sqrt{\epsilon}P_1$ and we shall impose the terminal condition $P_1(T,s,y) = 0$. Substituting (6.8) into (6.7) we obtain

$$\begin{aligned} \frac{1}{\epsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) + (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) & \\ +\sqrt{\epsilon}(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) + \cdots &= 0 \end{aligned} \tag{6.9}$$

We find a representation for P_0 and $\sqrt{\epsilon}P_1$ by analyzing the equations obtained when equating terms of the same order. We begin with the terms of order $\frac{1}{\epsilon}$ in (6.9). Equating terms of order $\frac{1}{\epsilon}$ we have

$$\mathcal{L}_0 P_0 = 0. \tag{6.10}$$

Since the operator $\mathcal{L}_0$ includes only derivatives in the space variable y, then we choose P_0 to be constant with respect to that variable and hence P_0 must be a function of only t and s

$$P_0(t,s,y) = P_0(t,s). \tag{6.11}$$

Similarly, we equate terms of order $\frac{1}{\sqrt{\epsilon}}$ in (6.9) to obtain

$$\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0. \tag{6.12}$$

From (6.11) we know that $\mathcal{L}_1 P_0 = 0$ because $\mathcal{L}_1$ consists of derivatives with respect to y. Therefore, we are left with $\mathcal{L}_0 P_1 = 0$. Using the same argument as in (6.10) we have P_1 as a function of only t and s

$$P_1(t,s,y) = P_1(t,s). \tag{6.13}$$

Note that the first two terms of the expansion will not depend upon the present volatility level.

We continue the expansion by equating terms of order one of (6.9). The order one terms give

$$\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0. \tag{6.14}$$

We know that $\mathcal{L}_1 P_1 = 0$ because P_1 does not depend on y. Hence (6.14) reduces to

$$\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0. \tag{6.15}$$

The s variable being fixed, $\mathcal{L}_2 P_0$ is a function of y since $\mathcal{L}_2$ involves $\sigma(y)$. Considering only the dependency of y, equation (6.15) is of the form

$$\mathcal{L}_0 \psi(y) + g(y) = 0, \tag{6.16}$$

which is known as a Poisson equation for $\psi(y)$ with respect to the operator $\mathcal{L}_0$ in the y variable. This equation does not admit a solution on a suitable space unless $g(y)$ is centered with respect to the invariant distribution of the Markov process Y_t with infinitesimal generator given by $\mathcal{L}_0$. We denote the density of the invariant distribution of Y_t by $\Phi(y)$, which is given by

$$\Phi(y) = \frac{1}{\sqrt{2\pi}\nu} e^{-\frac{(y-m)^2}{2\nu^2}}. \tag{6.17}$$

The centering condition necessary so that there exist a solution to (6.16) is

$$< g >= \int g(y)\Phi(y)dy = 0. \tag{6.18}$$

The Zero-Order Term

We now present how the zero-order term of the expansion given by (6.8) may be obtained. Since (6.15) is a Poisson equation, then it only admits a solution if $\mathcal{L}_2 P_0$ satisfies the centering condition

$$< \mathcal{L}_2 P_0 >= 0. \tag{6.19}$$

P_0 is a function of only s and t and since we average with respect to the y variable (6.19) may be reduced to

$$< \mathcal{L}_2 > P_0 = 0. \tag{6.20}$$

From the definition of $\mathcal{L}_2$ we may deduce that $< \mathcal{L}_2 >= \mathcal{L}_{BS}(\bar{\sigma})$, where $\bar{\sigma}$ corresponds to the effective volatility defined by (6.6). Therefore, the zero-order term $P_0(t,s)$ is the solution of the the Black-Scholes partial differential equation

$$\mathcal{L}_{BS}(\bar{\sigma}) P_0 = 0, \tag{6.21}$$

with terminal condition given by $P_0(T,s) = \phi(s)$.

Knowing the centering condition is satisfied we may express $\mathcal{L}_2 P_0$ as

$$\mathcal{L}_2 P_0 = \mathcal{L}_2 P_0 - < \mathcal{L}_2 P_0 >= \frac{1}{2}(\sigma(y)^2 - \bar{\sigma}^2) s^2 \frac{\partial^2 P_0}{\partial s^2}. \tag{6.22}$$

The solution of the Poisson equation (6.15), P_2, may now be expressed as

$$\begin{aligned} P_2(t,s,y) &= -\frac{1}{2}\mathcal{L}_0^{-1}(\sigma(y)^2 - \bar{\sigma}^2) s^2 \frac{\partial^2 P_0}{\partial s^2} \\ &= -\frac{1}{2}(\psi(y) + c(t,s)) s^2 \frac{\partial^2 P_0}{\partial s^2}, \end{aligned} \tag{6.23}$$

where $\psi(y)$ is a solution of the Poisson equation

$$\mathcal{L}_0 \psi = \sigma^2 - < \sigma^2 >, \tag{6.24}$$

and $c(t,s)$ is a constant in y.

When we compute $P_1(t,s)$ we shall utilize ψ', which is derived in [10] and given as

$$\psi'(y) = \frac{1}{\nu^2\Phi(y)}\int_{-\infty}^{y}(\sigma^2(s) - <\sigma^2>)\Phi(s)ds \tag{6.25}$$

The First Correction

We continue the process of equating terms in (6.8) by setting the terms of order $\sqrt{\epsilon}$ equal to zero, which gives

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0. \tag{6.26}$$

The above equation is also a Poisson equation that must have the centering condition

$$<\mathcal{L}_1 P_2 + \mathcal{L}_2 P_1>= 0 \tag{6.27}$$

in order to admit the solution P_3 on a suitable space.

Using equation (6.23) for P_2, the fact that P_1 and c do not depend on y and $<\mathcal{L}_2>= \mathcal{L}_{BS}(\bar{\sigma})$ we conclude P_1 solves

$$\mathcal{L}_{BS}(\bar{\sigma})P_1 = \frac{1}{2} <\mathcal{L}_1\psi(y)> s^2\frac{\partial^2 P_0}{\partial s^2}. \tag{6.28}$$

The operator $<\mathcal{L}_1\psi(y)>$ may be computed as

$$<\mathcal{L}_1\psi(y)\bullet>= \sqrt{2}\rho\nu <\sigma(y)\psi'(y)> s\frac{\partial}{\partial s} - \sqrt{2\nu} <\Lambda(y)\psi'(y)>\bullet, \tag{6.29}$$

which leads to the following partial differential equation for $P_1(t,s)$

$$\mathcal{L}_{BS}(\bar{\sigma})P_1 = \frac{\sqrt{2}}{2}\rho\nu <\sigma\psi'> s^3\frac{\partial^3 P_0}{\partial s^3} + \left(\sqrt{2}\rho\nu <\sigma\psi'> - \frac{\sqrt{2}}{2}\nu <\Lambda\psi'>\right)s^2\frac{\partial^2 P_0}{\partial s^2}, \tag{6.30}$$

with the terminal condition $P_1(T,s) = 0$.

We now introduce the first correction, which is defined as follows

$$\tilde{P}_1(t,s) = \sqrt{\epsilon} P_1(t,s). \tag{6.31}$$

The first correction satisfies the following partial differential equation

$$\mathcal{L}_{BS}(\sigma)\tilde{P}_1 = V_2 s^2 \frac{\partial^2 P_0}{\partial s^2} + V_3 s^3 \frac{\partial^3 P_0}{\partial s^3}, \tag{6.32}$$

where V_2 and V_3 are small coefficients. In terms of α they may be expressed as

$$V_2 = \frac{\nu}{\sqrt{2\alpha}}(2\rho < \sigma\psi' > - < \Lambda\psi' >), \tag{6.33}$$

$$V_3 = \frac{\rho\nu}{\sqrt{2\alpha}} < \sigma\psi' > . \tag{6.34}$$

We may express V_2 and V_3 in terms of the original model parameters by computing $< \sigma\psi' >$ and $< \Lambda\psi' >$ utilizing integration by parts

$$\begin{aligned} < \sigma\psi' > &= < \frac{\sigma}{\nu^2 \Phi} \int_{-\infty}^{y} (\sigma^2(s) - < \sigma^2 >)\Phi(s) ds > \qquad (6.35) \\ &= \frac{1}{\nu^2} \int_{-\infty}^{\infty} \sigma(\tau) \left(\int_{-\infty}^{\tau} (\sigma^2(s) - < \sigma^2 >)\Phi(s) ds \right) d\tau \\ &= -\frac{1}{\nu^2} < \Sigma(\sigma^2 - < \sigma^2 >) >, \end{aligned}$$

where Σ denotes the antiderivative of σ. Similarly, we obtain $< \Lambda\psi' >$ as

$$\begin{aligned} < \Lambda\psi' > &= \rho(\mu - r) < \frac{\psi'}{\sigma} > + \sqrt{1-\rho^2} < \gamma\psi' > \qquad (6.36) \\ &= -\frac{\rho(\mu - r)}{\nu^2} < \tilde{\Sigma}(\sigma^2 - < \sigma^2 >) > - \frac{\sqrt{1-\rho^2}}{\nu^2} < \Gamma(\sigma^2 - < \sigma^2 >) >, \end{aligned}$$

where $\tilde{\Sigma}$ and Γ represent the antiderivatives of $\frac{1}{\sigma}$ and γ respectively. We may relate V_2 and V_3 to the original model parameters as follows

$$V_2 = \frac{1}{\nu\sqrt{2\alpha}} < [-2\rho\Sigma + \rho(\mu - r)\tilde{\Sigma} + \sqrt{1-\rho^2}\Gamma](\sigma^2 - < \sigma^2 >) >, \tag{6.37}$$

$$V_3 = \frac{-\rho}{\nu\sqrt{2\alpha}} < \Sigma(\sigma^2 - < \sigma^2 >) > . \tag{6.38}$$

The first correction satisfies the Black-Scholes partial differential equation with a zero terminal condition and a small source term computed from the leading order term $P_0(t,s)$. It has a closed form solution that is given by

$$\tilde{P}_1(t,s) = -(T-t)\left(V_2 s^2 \frac{\partial^2 P_0}{\partial s^2} + V_3 s^3 \frac{\partial P_0}{\partial s^3}\right). \tag{6.39}$$

By combining P_0 and $\tilde{P}_1$ we obtain a $O(\sqrt{\epsilon})$ approximation of the pricing function for a European option, which is given explicitly by

$$P(t,s,y) = P_0(t,s) - (T-t)\left(V_2 s^2 \frac{\partial^2 P_0}{\partial s^2} + V_3 s^3 \frac{\partial P_0}{\partial s^3}\right). \tag{6.40}$$

Chapter 7

Variance Reduction for European Option

In this chapter we apply the Importance Sampling variance reduction technique to computing the price of a European option. We shall use the two methods of expansion described in the previous chapter to obtain a preliminary estimate of the expectation utilized to compute the premium. Secondly, we present some numerical results obtained from implementing the methodology in Matlab.

7.1 Application of Importance Sampling to Pricing Model

We apply the importance sampling variance reduction technique to the stochastic volatility model (9.4) used for computing European call options. In matrix form the evolution under the risk neutral measure $\mathbb{P}^*$ is given by

$$dV_t = b(V_t)dt + a(V_t)d\eta_t \,, \tag{7.1}$$

where we have set

$$\eta_t = \begin{pmatrix} W_t^* \\ \\ Z_t^* \end{pmatrix}, V_t = \begin{pmatrix} S_t \\ \\ Y_t \end{pmatrix},$$

and

$$a(v) = \begin{pmatrix} s\sigma(y) & 0 \\ \\ \nu\rho\sqrt{2\alpha} & \nu\sqrt{2\alpha(1-\rho^2)} \end{pmatrix}, b(v) = \begin{pmatrix} rs \\ \\ \alpha(m-y) - \nu\sqrt{2\alpha}\,\Lambda(y) \end{pmatrix}.$$

The price of a call option at time 0 is computed by

$$P(0,v) = \mathbb{E}^*\{e^{-rT}\phi(V_T)|V_0 = v\}, \tag{7.2}$$

where $v = (s,y)$ and $\phi(v) = (s-K)^+$.

We now apply the importance sampling technique described in chapter 5.

Define $\tilde{\eta}_t = \eta_t + \int_0^t h(s,V_s)ds$, which is a Brownian motion under the probability $\tilde{\mathbb{P}}^*$ which admits the density Q_T^{-1} as described in chapter 5

$$Q_T^{-1} = \exp\left\{-\int_0^t h(s,V_s)\cdot d\tilde{\eta}_s + \frac{1}{2}\int_0^t \|h(s,V_s)\|^2 ds\right\}.$$

Under the new measure, the price of the call option at time 0 is then computed by

$$P(0,v) = \tilde{\mathbb{E}}^*\{e^{-rT}\phi(V_T)Q_T|V_0 = v\}, \tag{7.3}$$

where the expectation is taken with respect to the measure $\tilde{\mathbb{P}}^*$.

By (8.50), if $P(0,v)$ were known, the optimal choice for h that gives the minimal variance is

$$h = -\frac{1}{P}\begin{pmatrix} s\sigma(y) & \nu\rho\sqrt{2\alpha} \\ \\ 0 & \nu\sqrt{1-\rho^2}\sqrt{2\alpha} \end{pmatrix}\begin{pmatrix} \frac{\partial P}{\partial s} \\ \\ \frac{\partial P}{\partial y} \end{pmatrix}. \tag{7.4}$$

Once we have found an approximation of P by using small noise expansion or fast mean-reversion expansion, then we may determine h in order to approximate (9.25) via Monte Carlo simulation

$$P(0,v) \approx \frac{1}{N}\sum_{k=1}^{N} \phi(V_T^{(k)})Q_T^{(k)} \tag{7.5}$$

under the evolution

$$\begin{aligned} dV_t &= (b(V_t) - a(V_t)h(t,V_t))dt + a(V_t)d\tilde{\eta}_t && (7.6)\\ Q_t &= \exp\left\{\int_0^t h(s,V_s)\cdot d\tilde{\eta}_s - \frac{1}{2}\int_0^t \|h(s,V_s)\|^2 ds\right\}. && (7.7) \end{aligned}$$

When we utilize small noise expansion as an apriori estimate for the price of a call option, the function $h(t,v)$ given by (9.26) will have the following form

$$h(t,v) = -\frac{1}{P_{BS(\sigma(y))}}\begin{pmatrix} s\sigma(y)\frac{\partial P_{BS(\sigma(y))}}{\partial s} \\ 0 \end{pmatrix}, \tag{7.8}$$

where $P_{BS(\sigma(y))}(t,s)$ and $\frac{\partial P_{BS(\sigma(y))}}{\partial s}(t,s) = N(d_1)$ are given by (6.5).

Similarly, using the zero order term of the fast mean-reversion expansion leads to the function $h(t,v)$ of the following form

$$h(t,v) = -\frac{1}{P_{BS(\bar{\sigma})}}\begin{pmatrix} s\sigma(y)\frac{\partial P_{BS(\bar{\sigma})}}{\partial s} \\ 0 \end{pmatrix}, \tag{7.9}$$

where the effective volatility, $\bar{\sigma}$, is given by (6.6).

The order $O(\sqrt{\epsilon})$ term of the expansion of the price $P(t,s,y)$ is given by

$$\tilde{P}_1(t,s) = -(T-t)\left(V_2 s^2 \frac{\partial^2 P_{BS(\bar{\sigma})}}{\partial s^2} + V_3 s^3 \frac{\partial^3 P_{BS(\bar{\sigma})}}{\partial s^3}\right). \tag{7.10}$$

We denote the order $O(\sqrt{\epsilon})$ approximation to the pricing function by P_{FMR} and it is given by

$$P_{FMR} = P_{BS(\bar{\sigma})} - (T-t)\left(V_2 s^2 \frac{\partial^2 P_{BS(\bar{\sigma})}}{\partial s^2} + V_3 s^3 \frac{\partial^3 P_{BS(\bar{\sigma})}}{\partial s^3}\right), \tag{7.11}$$

where V_2 and V_3 are given by (6.37) and (6.38). Therefore, h takes the following form

$$h(t,v) = -\frac{1}{P_{FMR}} \begin{pmatrix} s\sigma(y)\frac{\partial P_{FMR}}{\partial s} \\ 0 \end{pmatrix}. \tag{7.12}$$

Since we have control over how to choose this function h, we choose it in the following manner so that the Novikov condition will hold

$$h^{(M)} = \min(\max(-M, h), M),$$

where M is large. Therefore, Girsanov's theorem may be applied and Q_t has finite variance.

7.2 Numerical Results

In this section we present some numerical results from applying the importance sampling variance reduction methodology to the pricing of a European call option. We shall test the variance methodology developed by implementing the technique on the three types of call options: in-the-money ($S_0 > K$), at-the-money($S_0 = K$) and out-of-the-money ($S_0 < K$), where S_0 represents the stock price at time 0 and K represents the strike price. Since each method of expansion is characterized by the rate of mean-reversion, we present results for various values of α ranging from slow mean reversion $\alpha = .5$ to fast mean-reversion $\alpha = 100$.

In table 7.1 we present the model parameters utilized when performing the simulations. We note in our choice of parameters that $\sigma(y)$ is bounded and that the effective volatility, $\bar{\sigma}$, is computed using

$$\bar{\sigma}^2 =< (\max\{.0001, \min\{\exp(y), 5\}\})^2 >= \int_{-\infty}^{\infty} (\max\{.0001, \min\{\exp(y), 5\}\})^2 \, \Phi(y) dy,$$

Table 7.1: Model Parameters for the European Pricing Problem

Parameter	Value
m	-2.6
r	$.1$
ν	1
ρ	$-.3$
$\sigma(y)$	$\max\{.0001, \min\{\exp(y), 5\}\}$
μ	$.1$
$\gamma(y)$	0
$\Lambda(y)$	0
$\bar{\sigma}$	$.1971$
V_2	$\frac{.0090}{\sqrt{\alpha}}$
V_3	$\frac{.0045}{\sqrt{\alpha}}$

where

$$\Phi(y) = \frac{1}{2\pi} \exp\left(-\frac{(y+2.6)^2}{2} \right).$$

For simplicity we choose $\mu = r$ and $\gamma = 0$. The parameters V_2 and V_3 are computed from (6.37) and (6.38) as follows

$$\begin{aligned} V_2 &= \frac{-2\rho}{\nu\sqrt{2\alpha}} < \Sigma(\sigma^2 - < \sigma^2 >) >, \\ V_3 &= \frac{1}{2} V_2, \end{aligned}$$

where $\Sigma(y) = \max\{.0001, \min\{\exp(y), 5y\}\}$.

We use the Euler scheme, which is detailed in chapter 2, to approximate the

Table 7.2: Empirical variance for an in-the-money European call option.

S_0	Y_0	K	T	α	BMC	$P_{BS(\sigma(y))}$	$P_{BS(\bar{\sigma})}$	P_{FMR}
110	-2.32	100	1	.5	.0164	.0026	.0028	.0021
110	-2.32	100	1	1	.0205	.0046	.0044	.0013
110	-2.32	100	1	5	.0232	.0081	.0036	.0012
110	-2.32	100	1	10	.0237	.0083	.0028	.0008
110	-2.32	100	1	25	.0257	.0115	.0010	.0007
110	-2.32	100	1	50	.0288	.0150	.0007	.0006
110	-2.32	100	1	100	.0319	.0184	.0004	.0003

diffusion process V_t given by (9.28). Suppose the time interval $[0, T]$ is discretized as

$$0 = t_1 < t_2 < t_3 < \cdots < t_N = T,$$

then the Euler scheme may be explicitly written as

$$\begin{aligned} S_{n+1} &= (rS_n - \sigma(Y_n)S_n h_1(n, S_n, Y_n))\Delta_s + \sigma(Y_n)S_n\Delta W_n^1 \qquad (7.13) \\ Y_{n+1} &= (\alpha(m - Y_n) - \rho\nu\sqrt{2\alpha}h_1(n, S_n, Y_n))\Delta_s + \nu\sqrt{2\alpha}(\rho\Delta W_n^1 + \sqrt{1-\rho^2}\Delta W_n^2), \end{aligned}$$

where $n = 0, 1, 2, \cdots, N-1$ with initial values $S_0 = s$ and $Y_0 = y$. In addition, $\Delta_s = t_{s+1} - t_s$ and $\Delta W_s = W_{t_{s+1}} - W_{t_s}$. In our simulation we shall take the time step to be a fixed increment: $\Delta_s = 10^{-3}$. The function h_1 represents the first component of the h vector given in the three methods of expansion. In the case of basic Monte Carlo, where we do not apply importance sampling, $h_1 = 0$. Lastly, the number of sample paths computed in our simulations is 10000.

In table 7.2 we present the empirical variance of each Monte Carlo simulation for an in-the-money call option when using basic Monte Carlo or using importance sampling with the two forms of expansion. BMC refers to computing the price under the measure $\mathbb{P}^*$, while $P_{BS(\sigma(y))}$ refers to using importance sampling with small noise expansion. The quantities $P_{BS}(\bar{\sigma})$ and P_{FMR} refer to using fast mean-reversion expansion as an initial estimate of the expectation in importance sampling.

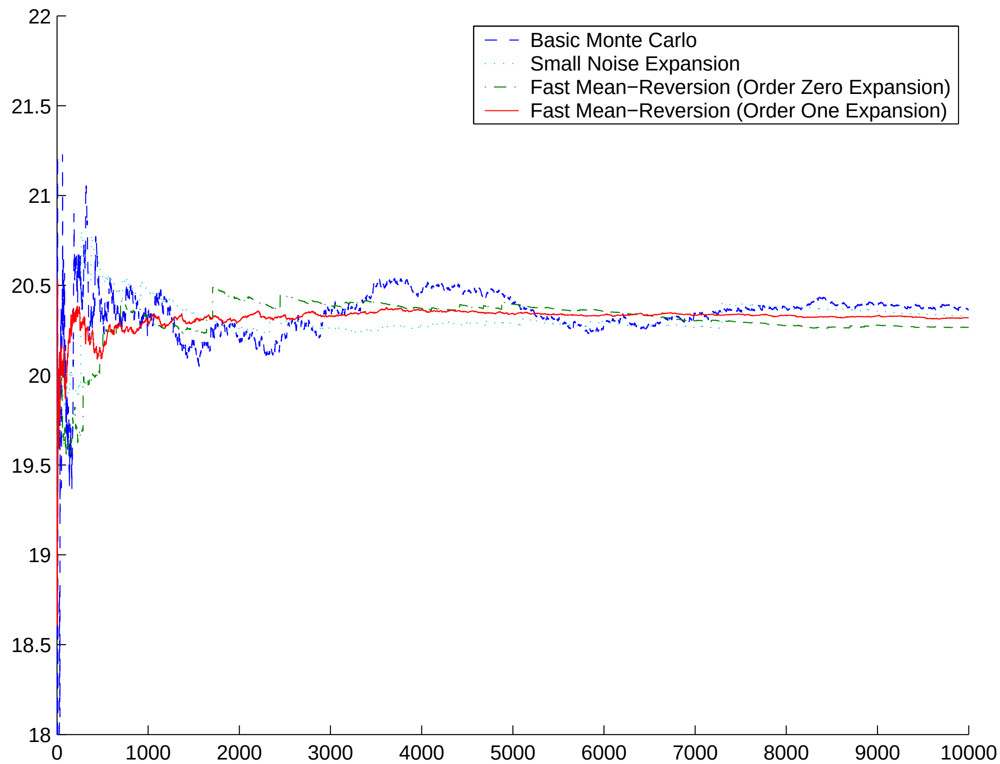

Figure 7.1: *Monte Carlo simulations of an in-the-money European call option with a rate of mean-reversion* $\alpha = 1$.

Figures 9.1 and 9.2 present the results of our Monte Carlo simulations as a function of the number of realizations. The simulations are for an in-the-money call option. The two illustrations given are for $\alpha = 1$ and $\alpha = 10$. It is clear from table 7.2 and the figures that the basic Monte Carlo estimator performs extremely poorly when compared to the other three estimators. Also, notice that when $\alpha = 1$ the variance for small noise expansion and the order zero fast mean-reversion expansion are

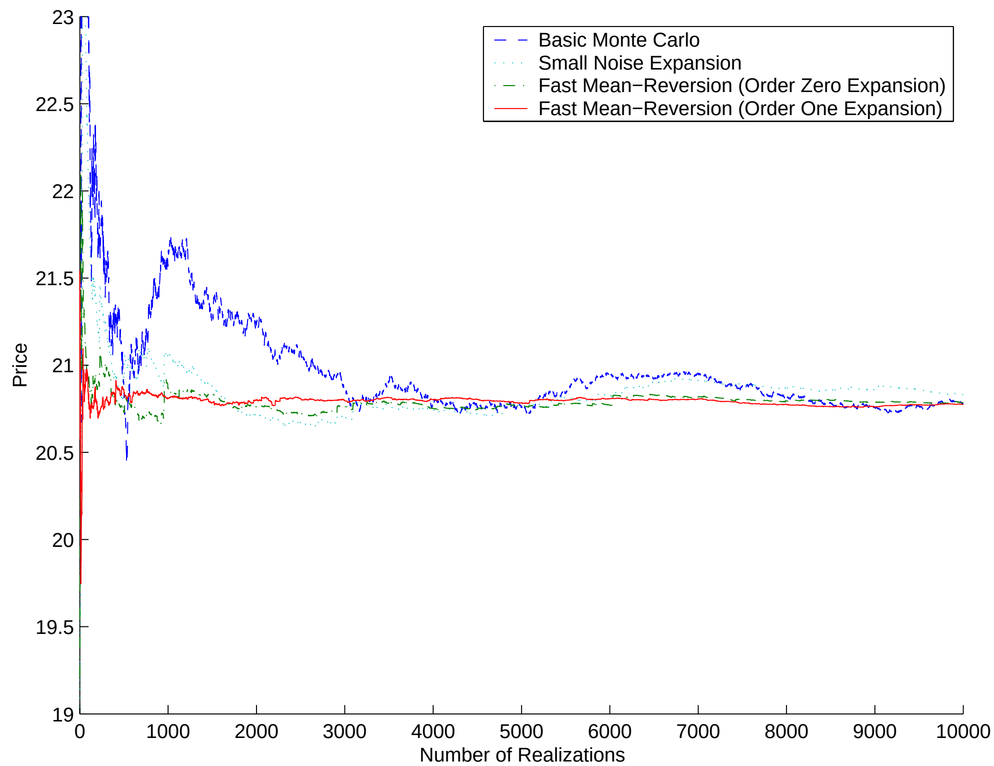

Figure 7.2: *Monte Carlo simulations of an in-the-money European call option with a rate of mean-reversion* $\alpha = 10$.

approximately the same; however, when the first correction is added to the approximation we obtain a greater reduction in the variance. Additionally, when the rate of mean-reversion is extremely large, the order zero and order one approximations are about the same.

We also wish to present results for an at-the-money call option and an out-of-the call option so that we have tested our methodology on the three types of call options.

In tables 7.3 and 7.4 we present the empirical variance for each method of expansion and basic Monte Carlo for an at-the-money and out-of-the-money European call option. As in the case of an in-the-money European call option basic Monte Carlo estimator performs extremely poorly when compared to the other three estimators.

Table 7.3: Empirical variance for an at-the-money European call option.

S_0	Y_0	K	T	α	BMC	$P_{BS(\sigma(y))}$	$P_{BS(\bar{\sigma})}$	P_{FMR}
100	-2.32	100	1	.5	.0264	.0019	.0025	.0022
100	-2.32	100	1	1	.0279	.0034	.0038	.0017
100	-2.32	100	1	5	.0288	.0068	.0028	.0015
100	-2.32	100	1	10	.0240	.0093	.0015	.0012
100	-2.32	100	1	25	.0347	.0092	.0011	.0009
100	-2.32	100	1	50	.0388	.0170	.0010	.0008
100	-2.32	100	1	100	.0319	.0184	.0007	.0005

Also, we once again notice that when the first correction is added to the expansion we obtain greater reduction of the variance when $\alpha = 1$. Additionally, when the rate of mean-reversion is extremely large, the order zero and order one approximations are about the same, which occurs in the case of an in-the-money European call option.

Table 7.4: Empirical variance for an out-the-money European call option.

S_0	Y_0	K	T	α	BMC	$P_{BS(\sigma(y))}$	$P_{BS(\bar{\sigma})}$	P_{FMR}
90	-2.32	100	1	.5	.0110	.0017	.0023	.0018
90	-2.32	100	1	1	.0205	.0038	.0036	.0015
90	-2.32	100	1	5	.0235	.0055	.0028	.0011
90	-2.32	100	1	10	.0337	.0093	.0021	.0009
90	-2.32	100	1	25	.0369	.0112	.0012	.0008
90	-2.32	100	1	50	.0355	.0144	.0009	.0007
90	-2.32	100	1	100	.0388	.0174	.0004	.0002

Chapter 8

American Option

In this chapter we wish to apply the importance sampling variance reduction technique developed in chapters 5 and 6 to the pricing of an American option. We begin by formally defining an American option and how to compute its price. Secondly, we present a computational inexpensive procedure, developed by Barone-Adesi and Whaley, which computes the price of an American option under constant volatility. We utilize this method to compute a preliminary estimate of the price of an American option when applying importance sampling. Next, we consider the pricing of an American option when the underlying asset is modeled with stochastic volatility and develop the asymptotic analysis in this framework. A numerical technique utilized to compute an American option price via Monte Carlo simulation, which was developed by Longstaff and Schwartz, is subsequently presented. Finally, we apply the importance sampling variance reduction technique when pricing an American option in a stochastic volatility environment and present some numerical results.

8.1 Pricing an American option

An American style option is a contract that gives its owner the right, but not the obligation to exercise the option at any time before the terminal time of the contract. The two types of American options we consider are the call, where to exercise means to buy one unit of the underlying asset for the strike price and the put, where the owner has the right to sell one unit of stock at the predetermined strike price. We shall denote the time the contract is exercised in either case as τ. At time t, the holder must decide whether to exercise the option given information in the σ-algebra $\mathcal{F}_t$. In other words, the owner observes the stock up to the present time and then decides whether or not to exercise the option. The exercise time τ is called a random time that is $\mathcal{F}_t$ measureable i.e. the event $\{\tau \leq t\}$ belongs to $\mathcal{F}_t$ for any $t \leq T$, where T represents the terminal time of the contract. We regard τ as a stopping time with respect to the filtration $\mathcal{F}_t$.

We again denote the payoff function as ϕ, which is given as

$$\phi(s) = (s-K)^+ = \begin{cases} s-K & \text{if } s > K \\ 0 & \text{if } s < K \end{cases}$$

in the case of the call and

$$\phi(s) = (K-s)^+ = \begin{cases} K-s & \text{if } K > s \\ 0 & \text{if } K < s \end{cases}$$

in the case of the put and where K represents the strike price of the contract. Hence, the value of the payoff function at the stopping time τ is given by $\phi(S_\tau)$.

Due to the early exercise feature of the American option, pricing is more complicated than pricing a European option. The price of an American option gives its holder more rights than that of a European option and therefore its price is greater than or equal to that of a European option that has the same payoff function and terminal date. Using the theory of optimal stopping it can be shown that the price of an American option with payoff function ϕ is obtained by maximizing the expected discounted payoff over all possible stopping times[3]. The expected value is computed under the risk-neutral probability P^* as we should expect. The price of an American option may be computed using

$$P(t,s) \;\; = \;\; \max_{t \leq \tau \leq T} \mathbb{E}^* \left[e^{-r(\tau - t)} \phi(S_\tau) | S_t = s \right]. \tag{8.1}$$

It can be shown through no-arbitrage arguments that for nonnegative interest rates and no dividends paid that the price of an American call option is equivalent to the price of a European call option[3]. Therefore, we shall perform all of our analysis on the American put option. The price of an American put option is computed as

$$P^a(t,s) \;\; = \;\; \max_{t \leq \tau \leq T} \mathbb{E}^* \left[e^{-r(\tau - t)} (K - S_\tau)^+ | S_t = s \right]. \tag{8.2}$$

By taking $\tau = t$ we deduce that $P^a(t,s) \geq (K-s)^+$ which we should suspect since if $P^a(t,s) < (K-s)^+$ then there is an arbitrage opportunity. This is illustrated by the following argument.

Figure 8.1 gives an example of a European put pricing function. Suppose that $P(t,s) < \phi(s)$ and consider the consequence of exercising the option. An arbitrage opportunity may created as follows: we buy the underlying asset in the market for s and purchase an option for P; if we immediately exercise the option by the selling the asset for K, a profit of $K - (P+s)$ is obtained.[5]

An alternate method for computing the premium for an American option is the formulation of a free boundary problem. The phrase **free boundary** refers to the fact that we must determine for a fixed time which values of s should we exercise the option. These values of s or boundary at any time are not known apriori. Therefore,

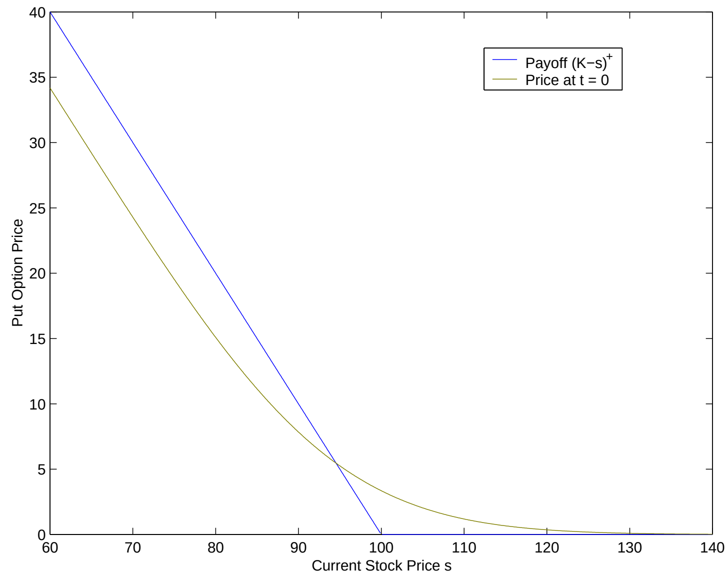

Figure 8.1: *Black-Scholes put option pricing function at time* $t = 0$*, with* $\sigma = .15$*,* $K = 100$*,* $T = 1$*,* $\sigma = .15$ *and* $r = .06$

we do not know where in advance to apply the boundary conditions. Hence the name **free boundary**.

No-arbitrage arguments state the price of an American put option must be greater than equal to its payoff and the infinitesimal return from an option must be less than or equal to the return from a bank deposit. Also, at any time the price must equal the payoff or the Black-Scholes equation must be satisfied. From these assumptions a linear complementarity problem may be constructed as follows

$$P^a \geq \phi, \tag{8.3}$$

$$\frac{\partial P^a}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 P^a}{\partial s^2} + rs\frac{\partial P^a}{\partial s} \leq rP^a, \tag{8.4}$$

$$(P^a - \phi)\left(\frac{\partial P^a}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 P^a}{\partial s^2} + rs\frac{\partial P^a}{\partial s} - rP^a\right) = 0. \tag{8.5}$$

Inequality (8.5) would be strict equality in the case of a European option. In the case of the American option it is optimal to hold the option when the Black and

Scholes equation is satisfied. Otherwise, when we have strict inequality in equation (8.5) it is optimal to exercise the option because the infinitesimal return on the option is less than the return from a bank deposit.

We may now formulate the free boundary problem as follows. We begin by dividing the S axis into two distinct regions for each time t. We shall denote the boundary between the two regions as s^*. At any fixed time, the first region where $0 \leq s < s^*$ and early exercise is optimal, the following hold

$$P^a = K - s, \qquad \frac{\partial P^a}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P^a}{\partial s^2} + rs\frac{\partial P^a}{\partial s} < rP^a.$$

In the other region where $s^* < s < \infty$ and early exercise is not optimal, the following hold

$$P^a > K - s, \qquad \frac{\partial P^a}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P^a}{\partial s^2} + rs\frac{\partial P^a}{\partial s} = rP^a.$$

Therefore, we have the following partial differential equation with a free boundary

$$\begin{aligned} P^a(t,s) &= K - s, && s < s^*(t) \\ \frac{\partial P^a}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 P^a}{\partial s^2} + rs\frac{\partial P^a}{\partial s} - rP^a &= 0, && s > s^*(t), \end{aligned} \tag{8.6}$$

with

$$\begin{aligned} P^a(T,s) &= (K-s)^+ \\ s^*(T) &= K. \end{aligned} \tag{8.7}$$

Also, we shall impose that P^a and $\frac{\partial P^a}{\partial s}$ are continuous across the boundary $s^*(t)$, so that

$$P^a(t, s^*(t)) = K - s^*(t), \tag{8.8}$$

$$\frac{\partial P^a}{\partial s}(t, s^*(t)) = -1. \tag{8.9}$$

We may think of the conditions given on the free boundary by (8.8) and (8.9) as a condition that determines the price of the option on the boundary and as a condition that determines the location of the free boundary.

8.2 Analytic Quadratic Approximation of an American Put Premium

In this section we present an efficient analytic approximation to the price of an American put option under constant volaility, which was first introduced by Giovanni Barone-Adesi and Robert Whaley. One method that has often been utilized to compute the price of an American put option is finite difference; however finite difference methods are often computationally expensive. To ensure a high degree of accuracy, it is necessary to partition the asset price and time dimensions into a fine grid and enumerate every possible path, which is very cumbersome and can only be accomplished on a high speed computer[2]. We shall present a brief review of how the quadratic approximation is obtained and we refer to [2] for full details.

An American option may be thought of as a European option with an early exercise feature added to it. Hence, the premium of an American option may be computed as the price of a European option plus an early exercise premium.

We define the early exercise premium for a put option as

$$e^p(t,s) \quad = \quad P^a(t,s) - P_{BS}(t,s), \tag{8.10}$$

where P^a and P_{BS} represent the price of an American and European put option respectively. The essential element behind the quadratic approximation method is that since the European and the American option price movements in the region where the option is not exercised are both modeled by

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 P}{\partial s^2} + rs\frac{\partial P}{\partial s} - rP \quad = \quad 0, \tag{8.11}$$

then the early exercise premium, by linearity, must also satisfy (8.11). The partial differential equation for the early exercise premium is therefore

$$\frac{\partial e^p}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 e^p}{\partial s^2} + rs\frac{\partial e^p}{\partial s} - re^p = 0. \tag{8.12}$$

We consider the decomposition of the early exercise premium as

$$e^p(t,s) = G(t)f(s, G(t)), \tag{8.13}$$

where G is an invertible function of t. Equation (8.12) may be written in terms of the early exercise premium as

$$\frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial s^2} + rs\frac{\partial f}{\partial s} - rf\left[1 - \left(\frac{G'}{rG}\right)\left[1 - \frac{G\frac{\partial f}{\partial G}}{f}\right]\right] \quad = \quad 0. \tag{8.14}$$

We shall choose G and using an approximation along with some assumptions we will be able to determine f. Hence, we will have determined the early exercise premium.

We choose $G(t) = 1 - e^{-r(T-t)}$. Substituting G into (8.14) and simplifying gives

$$\frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial s^2} + rs\frac{\partial f}{\partial s} - \frac{rf}{G} - (1-G)r\frac{\partial f}{\partial G} \quad = \quad 0. \tag{8.15}$$

We begin our approximation procedure by assuming that $(1-G)r\frac{\partial f}{\partial G} = 0$. This is justified by the following reason. For commodity options with very short (long) times to expiration, this assumption is reasonable since, as $T-t$ approaches 0 (∞), $\frac{\partial f}{\partial G}$ approaches 0 (G approaches 1), and the last term on the left side of (8.15) disappears[2]. The approximation of the early exercise premium partial differential equation becomes

$$\frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial s^2} + rs\frac{\partial f}{\partial s} - \frac{rf}{G} \quad = \quad 0. \tag{8.16}$$

Equation (8.16) is a second order differential equation with two linearly independent solution of the form λs^{κ}. The general solution of (8.16) is given by

$$f(s) \quad = \quad \lambda_1 s^{\kappa_1} + \lambda_2 s^{\kappa_2}, \tag{8.17}$$

where κ_1 and κ_2 are given by

$$\kappa_1 \quad = \quad \frac{-(M-1) - \sqrt{(M-1)^2 + 4M/G}}{2} \tag{8.18}$$

$$\kappa_2 \quad = \quad \frac{-(M-1) + \sqrt{(M-1)^2 + 4M/G}}{2}, \tag{8.19}$$

where $M = \frac{2r}{\sigma^2}$. Note that since $\frac{M}{G} > 0$ then, $\kappa_1 < 0$ and $\kappa_2 > 0$.

With κ_1 and κ_2 known, we are left to determine λ_1 and λ_2. The early exercise premium of the American put must approach 0 as s approaches positive infinity. Since $\kappa_2 > 0$, then the term $\lambda_2 s^{\kappa_2}$ will violate this boundary condition and thus we set $\lambda_2 = 0$. Therefore, the approximate value of an American put option is given by

$$P^a(t,s) = P_{BS}(t,s) + G\lambda_1 s^{\kappa_1}. \tag{8.20}$$

To find λ_1, we shall impose the following two conditions

- The exercise value of the option on the free boundary s^* is set equal to the value of the option on s^*.
- The slope of the exercise value, -1, is set equal to the slope of the option at s^*.

The above conditions may be expressed mathematically as a system of nonlinear equations as follows

$$K - s^* \quad = \quad P_{BS}(t,s^*) + G\lambda_1 s^{*\ \kappa_1} \tag{8.21}$$

$$-1 \quad = \quad -N(-d_1(s^*)) + G\lambda_1\kappa_1 s^{*\ \kappa_1 - 1}, \tag{8.22}$$

where $N(\cdot)$ is given by (3.20) and $d_1(s^*)$ is given by evaluating (3.18) at $s = s^*$.

Upon obtaining s^* and λ_1, the price of an American put may be computed using

$$\begin{aligned} P^a(t,s) &= P_{BS}(t,s) + A_1(s/s^*)^{\kappa_1} \quad s > s^* & (8.23)\\ P^a(t,s) &= K - s \qquad\qquad\qquad\qquad s \leq s^*, & (8.24)\end{aligned}$$

where $A_1 = -(s^*/\kappa_1)[1 - N(-d_1(s^*))]$.

This quadratic approximation is an attractive method because the price may be easily obtained once we have computed s^* using a nonlinear solver such as Newton's method. The numerical results given in [2] show that a solution of s^* may be obtained in three iterations of Newton's method. We shall utilize this quadratic approximation to compute a preliminary estimate of the American put option when we apply importance sampling to pricing an American option in the presence of stochastic volatility.

8.3 American Option under Stochastic Volatilty

In this section we present the problem of pricing an American option when the stock price is modeled with stochastic volatility. We also present the small noise and fast mean-reverting asymptotics when applied to the American problem.

8.3.1 Pricing under Stochastic Volatility

We again consider the following stochastic volatility model under the the risk-neutral measure $\mathbb{P}^{*(\gamma)}$

$$\begin{aligned} dS_t &= rS_t dt + \sigma(Y_t) S_t dW_t^* \\ dY_t &= [\alpha(m - Y_t) - \nu\sqrt{2\alpha}\,\Lambda(y)]dt + \nu\sqrt{2\alpha}\,(\rho dW_t^* + \sqrt{1-\rho^2}\,dZ_t^*) \quad (8.25)\end{aligned}$$

where

$$\Lambda(y) = \frac{\rho(\mu - r)}{\sigma(y)} + \gamma(y)\sqrt{1-\rho^2}. \tag{8.26}$$

The market selects a unique risk-neutral measure $\mathbb{P}^{*(\gamma)}$ and the no-arbitrage American put price may be computed as

$$P^a(t,x,y) = \max_{t\leq\tau\leq T} \mathbb{E}^{*(\gamma)}[e^{-r(\tau-t)}(K-S_\tau)^+|S_t = s, Y_t = y], \tag{8.27}$$

where the maximum is taken over all possible stopping time $\tau \in [0,T]$.

The pricing function $P^a(t,s,y)$ once again satisfies a free boundary problem that is similar to (8.6) with the additional space variable y. The free boundary s^* is now a function of time t and y, which must be computed as part of the problem. The free boundary problem is formulated as

$$\begin{aligned}
P^a(t,s,y) &= K - s, && s < s^*(t,y) \\
\frac{\partial P^a}{\partial t} &+ \frac{1}{2}\sigma^2(y)s^2\frac{\partial^2 P^a}{\partial s^2} + \rho\nu\sqrt{2\alpha}\, s\sigma(y)\frac{\partial^2 P^a}{\partial s \partial y} + \nu^2\alpha\,\frac{\partial^2 P^a}{\partial y^2} && \\
&+ r(s\frac{\partial P^a}{\partial s} - P^a) + \Big[(\alpha(m-y)) - \nu\sqrt{2\alpha}\,\Lambda(y)\Big]\frac{\partial P^a}{\partial y} = 0\,, && s > s^*(t,y)
\end{aligned} \tag{8.28}$$

with

$$\begin{aligned}
P^a(T,s,y) &= (K-s)^+ \\
s^*(T,y) &= K.
\end{aligned} \tag{8.29}$$

Additionally, P^a, $\frac{\partial P^a}{\partial s}$ and $\frac{\partial P^a}{\partial y}$ are continuous across the boundary

$$\begin{aligned}
P^a(t,s^*(t,y),y) &= (K - s^*(t,y))^+ \\
\frac{\partial P^a}{\partial s}(t,s^*(t,y),y) &= -1 \\
\frac{\partial P^a}{\partial y}(t,s^*(t,y),y) &= 0
\end{aligned} \tag{8.30}$$

8.3.2 Small Noise Expansion

Utilizing the convenient compact operator notation given by (4.15), the evolution of the price in the continuation region is given by

$$\left(\alpha\mathcal{L}_0 + \sqrt{\alpha}\mathcal{L}_1 + \mathcal{L}_2\right) P^a(t,s,y) = 0. \tag{8.31}$$

Expanding the above partial differential equation about $\alpha = 0$ once again yields

$$\mathcal{L}_2 P^a(t,s,y) = 0.$$

Recall that $\mathcal{L}_2$ is simply the Black-Scholes operator with constant volatility $\sigma(y)$. Since the process Y_t is degenerate when $\alpha = 0$ then neither the pricing function nor the free boundary will not depend on y: $P^a(t,s,y) = P^a(t,s)$, $s^*(t,y) = s^*(t)$. Hence, the free boundary problem becomes

$$\begin{aligned} P^a(t,s) &= K - s, & s < s^*(t) \\ \frac{\partial P^a}{\partial t} &+ \frac{1}{2}\sigma^2(y)s^2\frac{\partial^2 P^a}{\partial s^2} + +r\left(s\frac{\partial P^a}{\partial s} - P^a\right) = 0\,, & s > s^*(t) \end{aligned}$$

with

$$\begin{aligned} P^a(T,s) &= (K-s)^+ \\ s^*(T) &= K. \end{aligned}$$

Additionally, P^a, $\frac{\partial P^a}{\partial s}$ and $\frac{\partial P^a}{\partial y}$ are continuous across the boundary

$$\begin{aligned} P^a(t,s^*(t)) &= (K - s^*(t))^+ \\ \frac{\partial P^a}{\partial s}(t,s^*(t)) &= -1. \end{aligned}$$

This is exactly the American put pricing problem with constant volatility level $\sigma(y)$. There is no explicit solution for $P_0^a(t,s)$ and $s_0^*(t)$, but utilizing the quadratic approximation method developed by Barone-Adesi and Whaley, we can compute an efficient solution that is computationally inexpensive.

8.3.3 Fast Mean-Reversion Expansion

As with the European option we shall make the substitution $\alpha = \frac{1}{\epsilon}$ into equation (9.10), which yields

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P^a(t,s,y) = 0\,. \tag{8.32}$$

We wish to construct an asymptotic expansion of the pricing function and the free boundary as follows

$$P^a(t,s,y) = P_0^a(t,s,y) + \sqrt{\epsilon}P_1^a(t,s,y) + \epsilon P_2^a(t,s,y) + \cdots\,, \tag{8.33}$$

$$s^*(t,y) = s_0^*(t,y) + \sqrt{\epsilon}s_1^*(t,y) + \epsilon s_2^*(t,y) + \cdots \tag{8.34}$$

We shall construct an order $O(\sqrt{\epsilon})$ approximation similar to that of the European pricing function. We begin by substituting (8.33) and (8.34) into (8.32). We next look at the resulting equations of each order of $\sqrt{\epsilon}$ in both the hold and exercise region. The term $s_0^*(t,y)$ is used as the boundary for each problem and thus the extension or truncation of the hold region to the s_0^* boundary is assumed to introduce only an $O(\sqrt{\epsilon})$ error into each term $P_i^a(t,s,y)$ of the expansion of the price[3].

Since the partial differential equation in the hold region is the same as with the European option, then substituting (8.33) and (8.34) into (8.32) again yields

$$\begin{aligned}\frac{1}{\epsilon}\mathcal{L}_0P_0^a + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0P_1^a + \mathcal{L}_1P_0^a) + (\mathcal{L}_0P_2^a + \mathcal{L}_1P_1^a + \mathcal{L}_2P_0^a)&\\ +\sqrt{\epsilon}(\mathcal{L}_0P_3^a + \mathcal{L}_1P_2^a + \mathcal{L}_2P_1^a) + \cdots &= 0.\end{aligned} \tag{8.35}$$

Substitution in the exercise region yields

$$P_0^a + \sqrt{\epsilon}P_1^a + \epsilon P_2^a + \cdots = K - s. \tag{8.36}$$

Lastly, we substitute the approximation to the price up to the $\sqrt{\epsilon}$ term of the expansion into the boundary conditions as follows

$$\begin{aligned}
P_0^a(t,s^*(t,y),y)+\sqrt{\epsilon}(s_1^*(t,y)\frac{\partial P_0^a}{\partial s}(t,s^*(t,y),y) \\
+P_1(t,s^*(t,y),y)) &= K-s^*(t,y)-\sqrt{\epsilon}s_1^*(t,y) && (8.37)\\
\frac{\partial P_0^a}{\partial s}(t,s_0^*(t,y),y)+\sqrt{\epsilon}(s_1^*(t,y)\frac{\partial^2 P_0^a}{\partial s^2}(t,s^*(t,y),y) \\
+\frac{\partial P_1}{\partial s}(t,s^*(t,y),y)) &= -1 && (8.38)\\
\frac{\partial P_0^a}{\partial y}(t,s_0^*(t,y),y)+\sqrt{\epsilon}(s_1^*(t,y)\frac{\partial^2 P_0^a}{\partial s\partial y}(t,s^*(t,y),y) \\
+\frac{\partial P_1^a}{\partial y}(t,s^*(t,y),y)) &= -1 && (8.39)
\end{aligned}$$

The terminal condition gives $P_0^a(T,s,y)=(K-s)^+$ and $P_1^a(T,s,y)=0$, and the condition $P^a=(K-s)^+$ in the exercise region gives that $P_0^a(t,s,y)=(K-s)^+$ and $P_1^a(t,s,y)=0$ in that region.

Zero Order Term

Looking at terms of $O(\frac{1}{\epsilon})$ in the hold region and terms of $O(1)$ in the exercise region and the boundary conditions gives the following problem

$$\begin{aligned}
\mathcal{L}_0 P_0^a(t,s,y) &= 0 && s>s_0^*(t,y)\\
P_0^a(t,s,y) &= (K-s)^+ && s<s_0^*(t,y)\\
P_0^a(t,s_0^*(t,y),y) &= (K-s_0^*(t,y))^+ \\
\frac{\partial P_0^a}{\partial s}(t,s_0^*(t,y),y) &= -1.
\end{aligned}$$

Since the generator $\mathcal{L}_0$ does not depend on y the same argument as in section 6.2 holds and hence P_0^a does not depend on y on both sides of the boundary. Further, it also can not depend on y on the surface s_0^* and hence the free boundary is a function of s alone: $s_0^*(t,y)=s_0^*(t)$.

We next analyze terms of $O(\frac{1}{\sqrt{\epsilon}})$ in the hold region and terms of $O(\sqrt{\epsilon})$ in the exercise region and the boundary conditions. The operator $\mathcal{L}_1$ contains y derivatives in each term and hence $\mathcal{L}_1 P_0 = 0$. Therefore, we obtain the problem

$$
\begin{aligned}
\mathcal{L}_0 P_1(t,s,y) &= 0 \qquad\qquad s > s_0^*(t)\\
P_1^a(t,s,y) &= 0 \qquad\qquad s < s_0^*(t)\\
P_1^a(t,s_0^*(t),y) &= 0\\
s_1^*(t,y)\frac{\partial^2 P_0^a}{\partial s^2}(t,s_0^*(t)) + \frac{\partial P_1}{\partial s}(t,s_0^*(t),y) &= 0.
\end{aligned}
$$

By the same argument utilized for P_0^a we see that $P_1^a(t,s,y)$ is only a function of t and s: $P_1^a(t,s,y) = P_1^a(t,s)$.

From the $O(1)$ terms in (8.35) and the $O(\epsilon)$ term in (8.36) we obtain

$$\mathcal{L}_0 P_2^a(t,s,y) + \mathcal{L}_2 P_0^a(t,s) = 0 \qquad s > s_0^*(t) \tag{8.40}$$

$$P_2^a(t,s,y) = 0 \qquad s < s_0^*(t), \tag{8.41}$$

since $\mathcal{L}_2 P_1^a = 0$. The equation obtained in the hold region is a poisson equation. A solution on a suitable space to this equation exists only if $\mathcal{L}_2 P_0^a$ is centered with respect to the invariant distribution of the OU process Y_t as follows

$$< \mathcal{L}_2 P_0^a > = 0. \tag{8.42}$$

Since P_0^a does not depend on y then $< \mathcal{L}_2 P_0^a > = < \mathcal{L}_2 > P_0^a$. From section 6.2 we know that $< \mathcal{L}_2 >$ is computed as

$$< \mathcal{L}_2 > = \mathcal{L}_{BS}(\bar{\sigma}) = \frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}^2 s^2 \frac{\partial^2}{\partial s^2} + r\left(s\frac{\partial}{\partial s} - \cdot\right), \tag{8.43}$$

where $\bar{\sigma}^2 = < \sigma^2 >$.

Therefore, $P_0^a(t,s)$ and $s_0^*(t)$ satisfy the following problem

$$\begin{aligned} P_0^a(t,s) &= K - s, \qquad s < s_0^*(t) \\ \frac{\partial P_0^a}{\partial t} + \frac{1}{2}\bar{\sigma}^2 s^2 \frac{\partial^2 P_0^a}{\partial s^2} + rs\frac{\partial P_0^a}{\partial s} - rP_0^a &= 0, \qquad\quad s > s_0^*(t), \end{aligned}$$

with

$$\begin{aligned} P_0^a(T,s) &= (K-s)^+, \\ s_0^*(T) &= K, \\ P_0^a(t, s_0^*(t)) &= K - s_0^*(t), \\ \frac{\partial P_0^a}{\partial s}(t, s_0^*(t)) &= -1. \end{aligned}$$

This is exactly the problem of pricing an American put option with constant volatility $\bar{\sigma}$. There is again no explicit solution for $P_0^a(t,s)$ and $s_0^*(t)$, but utilizing the quadratic approximation method we may compute an efficient solution.

Order $\sqrt{\epsilon}$ Term

We next wish to compute the function $\sqrt{\epsilon}P_1^a(t,s,y)$, which is the second term in the $O(\sqrt{\epsilon})$ approximation of the pricing function $P^a(t,s,y)$.

We analyze the $O(\sqrt{\epsilon})$ terms in the hold region and the $O(\epsilon)$ terms in the exercise region, which gives

$$\begin{aligned} \mathcal{L}_0 P_3^a(t,s,y) + \mathcal{L}_1 P_2^a(t,s,y) + \mathcal{L}_2 P_1^a(t,s) &= 0 \qquad s > s_0^*(t), \\ P_3^a(t,s,y) &= 0 \qquad s < s_0^*(t). \end{aligned}$$

The equation in the hold region is a Poisson equation for P_3^a, which has a solution on a suitable space only if $\mathcal{L}_1 P_2^a(t,s,y) + \mathcal{L}_2 P_1(t,s)$ is centered with respect to the invariant distribution of the OU process. This is given by

$$< \mathcal{L}_1 P_2^a + \mathcal{L}_2 P_1 >= 0.$$

Utilizing the same techniques as in section 6.2 the second term of the expansion, which is defined to be $\tilde{P}_1^a = \sqrt{\epsilon} P_1^a$, evolves in the hold region according to

$$\mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_1^a = V_3 s^3 \frac{\partial^3 P_0^a}{\partial s^2} + V_2 s^2 \frac{\partial^2 P_0^a}{\partial s^2}, \tag{8.44}$$

where $P_0^a(t,s)$ is the Black-Scholes American put price with constant volatility and V_2 and V_3 are given by (6.37) and (6.38) respectively.

Therefore, we may formulate the complete problem of $\tilde{P}_1^a$ as follows

$$\begin{aligned} \tilde{P}_1^a(t,s) &= 0 && s < s_0^*(t), \\ \mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_1^a(t,s) &= V_3 s^3 \frac{\partial^3 P_0^a}{\partial s^2} + V_2 s^2 \frac{\partial^2 P_0^a}{\partial s^2} && s > s_0^*(t), \end{aligned}$$

with the boundary condition given by

$$\tilde{P}_1^a(t, s_0^*(t)) = 0.$$

Since the boundary $s_0^*(t)$ is computed when we compute P_0^a, the problem formulated above is therefore a fixed boundary problem. However, this boundary is determined up to an error of $\sqrt{\epsilon}$, so there is an $O(\sqrt{\epsilon})$ error in $\tilde{P}_1^a$ within an $O(\sqrt{\epsilon})$ neighborhood of s_0^a.[3] We note the solution of the partial differential equation given by (8.44) is

$$-(T-t)\left(V_3 s^3 \frac{\partial^3 P_0^a}{\partial s^2} + V_2 s^2 \frac{\partial^2 P_0^a}{\partial s^2}\right).$$

However, this solution does not satisfy the zero boundary condition given by $\tilde{P}_1^a$. Hence, we compute the solution numerically after obtaining P_0^a. When implementing the importance sampling variance reduction technique on an American put we will not consider using $\tilde{P}_1^a$ since this may be computationally expensive in Monte Carlo if for instance we use finite difference approximation method. However, finding an analytic approximation to $\tilde{P}_1^a$ is a topic on future research.

8.4 Monte Carlo Simulation of an American Option

In this section we present the Least Squares Monte Carlo technique, which was developed by Francis Longstaff and Eduardo Schwartz, that estimates the following American put pricing function via Monte Carlo simulation

$$P^a(t,s) = \max_{t\leq\tau\leq T} \mathbb{E}^*[e^{-(\tau-t)}\phi(S_\tau)|S_t = s]. \tag{8.45}$$

We begin by discussing the theoretical formulation of the method and then discuss the implementation.

8.4.1 Theoretical Valuation

The objective of the LSM method is to provide a pathwise approximation to the optimal stopping rule that maximizes the value of the American option.[20]

We begin by denoting the payoff of the option at time z on sample path ω conditioned on not being exercised before or on time v by

$$J(\omega, z; v, T),$$

where $t \leq v < z \leq T$. We note in this function that if the option is exercised at time $z = z_1$ on any path ω, then there is no payoff for any $z > z_1$ since the option is only exercised once during its life. In addition, we shall partition the possible exercise times as follows

$$t = t_0 < t_1 < t_2 < t_3 < \cdots < t_{N-2} < t_{N-1} < t_N = T.$$

The holder of the option at time T exercises the option if it is in the money or the option expires worthless. At any time t_n before T, the holder must decide whether to exercise the option immediately or continue to hold the option and subsequently revisit this decision at the next possible exercise time. The payoff upon exercising the option at the time t_n is known to its owner. The payoff from continuing to hold the option at t_n is not known; however no-arbitrage theory suggests that the value of continuation is given by taking the expectation with respect to the risk-neutral measure P^* of the remaining payoffs $J(\omega, z; t_n, T)$ discounted back to t_n[25]. We may represent the value of continuation as follows

$$F(\omega; t_n) = \mathbb{E}^* \left[\sum_{j=n+1}^{N} e^{-r(t_j - t_n)} J(\omega, t_j; t_n, T) | S_{t_n} \right]. \tag{8.46}$$

With this representation, the problem of optimal exercise reduces to comparing the immediate exercise value with this conditional expectation, and then exercise as soon as the immediate exercise value is greater than or equal to the conditional expectation.[5] Hence, we able to compute the optimal exercise time along each path. Upon finding the optimal time along each path we discount the payoff back to time t and then average over all sample paths. Thus the value the American option is computed.

8.4.2 Implementation

The initial step of the implementation of the LSM technique is to simulate M trajectories of the asset process from the initial time t to its terminal date T as if we were computing a Monte Carlo approximation for a European option. We store these simulated paths of the process because they will be used to find an approximation for the conditional expectation $F(\omega; t_n)$.

We assume that the conditional expectation $F(\omega; t_n)$ can be represented as a linear combination of countable set of $\mathcal{F}_{t_n}$- measurable basis functions. There are numerous examples of which basis to choose. Schwarz and Longstaff show the polynomial basis is an accurate choice of the basis functions

$$F(\omega; t_n) = \sum_{j=0}^{\infty} a_j s^j,$$

where s represents the value of the underlying asset.

To find the optimal stopping rule along each simulated path we work backwards from time $t_N = T$. Therefore, we assume initially that the optimal stopping time along each path is the terminal time. Beginning at time t_{N-1} we approximate the conditional expected value of continuation $F(\omega; t_{N-1})$, denoted by $\hat{F}(\omega; t_{N-1})$, by regressing the discounted values of $J(\omega, z; t_{N-1}, T)$ onto the basis functions for the paths where the option is in the money at time t_{N-1}. Using only in-the-money paths allows us to use fewer basis functions when computing the least squares regression. Once $\hat{F}(\omega; t_{N-1})$ has been computed, we determine whether early exercise is optimal by comparing the payoff from immediate exercise with $\hat{F}(\omega; t_{N-1})$ evaluated at $S_{t_{N-1}}(\omega)$. We then update our optimal stopping rule along each path by letting the new stopping time be t_{N-1} if the value of immediate exercise is greater than $\hat{F}(\omega; t_{N-1})$ evaluated at $S_{t_{N-1}}(\omega)$. We continue this process by rolling back to time t_{N-2}. At t_{N-2} we compute $\hat{F}(\omega; t_{N-2})$ by regressing the discounted values of $J(\omega, z; t_{N-2}, T)$ onto the basis functions for the paths where the option is in the money at time t_{N-2}. We note here that values of $J(\omega, z; t_{N-2}, T)$ are discounted back one period if the optimal stopping time along a given path is t_{N-1} and discounted 2 periods if the optimal stopping time is t_N. Once $\hat{F}(\omega; t_{N-2})$ has been computed, we determine whether early exercise is optimal by comparing the payoff from immediate exercise with $\hat{F}(\omega; t_{N-2})$ evaluated at $S_{t_{N-2}}(\omega)$. We then update our optimal stopping rule along each path by letting the new stopping time be t_{N-2} if the value of immediate exercise is greater than $\hat{F}(\omega; t_{N-2})$ evaluated at $S_{t_{N-2}}(\omega)$. The recursion proceeds by rolling back to time t and hence we will have determined the optimal exercise time along each path. Upon finding the optimal time along each path we discount the payoff back to time t and then average over all sample paths. Hence, we have computed the value of an American option.

8.5 Variance Reduction for an American Put Option

In this section we wish to apply the importance sampling variance reduction technique to the pricing of an American put option. As in the case with the European option, we utilize small noise and fast mean-reversion expansion to obtain a preliminary estimate to the expectation of interest. However, in this case we compute the preliminary estimate using the analytic quadratic approximation technique developed in section 8.2. Secondly, we present some numerical results obtained from implementing the method in Matlab.

8.5.1 Importance Sampling Applied to American Pricing Model

We apply the importance sampling variance reduction technique to the stochastic volatility model (9.4) used for computing American put options. In matrix form the evolution under the risk neutral measure $\mathbb{P}^*$ is given by

$$dV_t = b(V_t)dt + a(V_t)d\eta_t\,, \tag{8.47}$$

where we have set

$$\eta_t = \begin{pmatrix} W_t^* \\ Z_t^* \end{pmatrix}\,,\ V_t = \begin{pmatrix} S_t \\ Y_t \end{pmatrix}\,,$$

and

$$a(v) = \begin{pmatrix} s\sigma(y) & 0 \\ \nu\rho\sqrt{2\alpha} & \nu\sqrt{2\alpha(1-\rho^2)} \end{pmatrix}, \; b(v) = \begin{pmatrix} rs \\ \alpha(m-y) - \nu\sqrt{2\alpha}\,\Lambda(y) \end{pmatrix}.$$

The price of a put option at time 0 is computed by

$$P^a(0,v) = \max_{0\le\tau\le T} \mathbb{E}^*\{e^{-r\tau}\phi(V_\tau)|V_0 = v\}, \tag{8.48}$$

where $v = (s,y)$ and $\phi(v) = (K - S_\tau)^+$.

We now apply the importance sampling technique described in chapter 5.

Define $\tilde{\eta}_t = \eta_t + \int_0^t h(s,V_s)ds$, which is a Brownian motion under the probability $\tilde{\mathbb{P}}^*$. Since τ is a random terminal time, we know $\tilde{\mathbb{P}}^*$ still admits the density Q_τ^{-1} as described in chapter 5

$$Q_\tau^{-1} = \exp\left\{-\int_0^\tau h(s,V_s)\cdot d\tilde{\eta}_s + \frac{1}{2}\int_0^\tau \|h(s,V_s)\|^2 ds\right\}.$$

Under the new measure, the price of the put option at time 0 is then computed by

$$P^a(0,v) = \max_{0\le\tau\le T} \tilde{\mathbb{E}}^*\{e^{-r\tau}\phi(V_\tau)Q_\tau|V_0 = v\}, \tag{8.49}$$

where the expectation is taken with respect to the measure $\tilde{\mathbb{P}}^*$.

Recall that the importance sampling variance reduction method consists of determining a function $h(t,v)$ that leads to a smaller variance for the Least Squares Monte Carlo approximation computed using (8.48) than the variance for (8.47).

Applying Ito's formula to $P^a(t,V_t)Q_t$ and using the Kolmogrov's backward equation for $P^a(t,v)$ one gets

$$\begin{aligned} d(P^a(t,V_t)Q_t) &= P^a(t,V_t)Q_t h(t,V_t)\cdot d\tilde{\eta}_t + Q_t a^T(t,V_t)\nabla P^a(t,V_t)\cdot d\tilde{\eta}_t \\ &= Q_t(a^T\nabla P^a + P^a h)(t,V_t)\cdot d\tilde{\eta}_t\,. \end{aligned}$$

where a^T denotes the transpose of a, and ∇P^a the gradient of P^a with respect to the space variable v.

In order to obtain $P^a(0,v)$, for instance, one can integrate between 0 and τ and deduce

$$P^a(\tau, V_\tau)Q_\tau = P^a(0, V_0)Q_0 + \int_0^\tau Q_t(a^T\nabla P^a + P^a h)(t, V_t)\cdot d\tilde{\eta}_t\,,$$

which reduces to

$$\phi(V_\tau)Q_\tau = P^a(0, v) + \int_0^\tau Q_t(a^T\nabla P^a + P^a h)(t, V_t)\cdot d\tilde{\eta}_t\,.$$

Therefore, the variances for the two Monte Carlo simulations (5.2) and (9.27) are given by

$$\begin{aligned}\mathrm{Var}_{\tilde{\mathbb{P}}}\left(\phi(V_\tau)Q_\tau\right) &= \tilde{\mathbb{E}}\left\{\int_0^\tau Q_t^2\|a^T\nabla P^a + P^a h\|^2 dt\right\}\\ \mathrm{Var}_{\mathbb{P}}\left(\phi(V_\tau)\right) &= \mathbb{E}\left\{\int_0^\tau \|a^T\nabla u\|^2 dt\right\}\,.\end{aligned}$$

If $P^a(t,v)$ were known, then the problem would be solved and the optimal choice for h, which gives a zero variance, would be

$$h = -\frac{1}{P^a}a^T\nabla P^a\,, \tag{8.50}$$

which may be explicitly written as

$$h = -\frac{1}{P^a}\begin{pmatrix} s\sigma(y) & \nu\rho\sqrt{2\alpha} \\ 0 & \nu\sqrt{1-\rho^2}\sqrt{2\alpha}\end{pmatrix}\begin{pmatrix}\frac{\partial P^a}{\partial s} \\ \frac{\partial P^a}{\partial y}\end{pmatrix}\,. \tag{8.51}$$

Once we have found an approximation of P^a by using small noise expansion or fast mean-reversion expansion, then we may determine h in order to approximate (8.49) via Least Squares Monte Carlo simulation under the evolution

$$dV_t = (b(V_t) - a(V_t)h(t, V_t))dt + a(V_t)d\tilde{\eta}_t \tag{8.52}$$

$$Q_t = \exp\left\{\int_0^t h(s, V_s)\cdot d\tilde{\eta}_s - \frac{1}{2}\int_0^t \|h(s, V_s)\|^2 ds\right\}\,. \tag{8.53}$$

Since there is no closed form solution for a preliminary estimate of P^a in either case of expansion, we utilize the quadratic approximation method developed in section 8.2 to compute an efficient estimate to the price.

In either case we still define the function $h^{(M)}$ as follows

$$h^{(M)} = \min(\max(-M, h), M),$$

where M is large. Hence, the Novikov condition will hold and thus Girsanov' theorem may be applied and Q_t has finite variance.

8.6 Numerical Results

In this section we present some numerical results from implementing the importance sampling variance reduction methodology to the pricing of a American put option. We shall test the variance methodology developed by implementing the technique on the three types of put options: in-the-money ($S_0 < K$), at-the-money($S_0 = K$) and out-of-the-money ($S_0 > K$), where S_0 represents the stock price at time 0 and K represents the strike price. Since each method of expansion is characterized by the rate of mean-reversion, we present results for various values of α ranging from slow mean reversion $\alpha = .5$ to fast mean-reversion $\alpha = 100$.

In table 8.1 we present the model parameters utilized when performing the simulations. We note in our choice of parameters that $\sigma(y)$ is bounded and bounded away from 0 and that the effective volatility, $\bar{\sigma}$, is computed using

$$\bar{\sigma}^2 =< (\max(\min(\exp(y), 5), .0001))^2 >= \int_{-\infty}^{\infty} (\max(\min(\exp(y), 5), .0001))^2 \, \Phi(y) dy,$$

where

$$\Phi(y) = \frac{1}{2\pi} \exp\left(-\frac{(y + 2.6)^2}{2} \right).$$

Table 8.1: Model Parameters for the American Put Pricing Problem

Parameter	Value
m	-2.6
r	$.1$
ν	1
ρ	$-.3$
$\sigma(y)$	$\max(.0001, \min(\exp(y), 5))$
μ	$.1$
$\gamma(y)$	0
$\Lambda(y)$	0
$\bar{\sigma}$	$.1971$

In tables 8.2, 8.3 and 8.4 we present the empirical variance for each method of expansion and basic Monte Carlo for an in-the-money, at-the-money and out-of-the-money American put option. We see as in the case of the European call option basic Monte Carlo estimator performs extremely poorly when compared to the other two estimators. Additionally, when the rate of mean-reversion is large, we obtain a greater reduction of the variance when utilizing the fast mean-reversion expansion as opposed to the small noise expansion. Lastly, figures 8.1 and 8.2 show sample runs of the simulation.

Table 8.2: Empirical variance for an at-the-money American put option.

S_0	Y_0	K	T	α	BMC	$P^a_{\sigma(y)}$	$P^a_{\bar{\sigma}}$
100	-1.62	100	1	.5	.0010	.00029	.00067
100	-1.62	100	1	1	.0015	.00058	.00056
100	-1.62	100	1	5	.0025	.0007	.00045
100	-1.62	100	1	10	.0040	.0010	.00022
100	-1.62	100	1	25	.0155	.0085	.00021
100	-1.62	100	1	50	.0175	.0010	.00019
100	-1.62	100	1	100	.0219	.0011	.00017

Table 8.3: Empirical variance for an in-the-money American put option.

S_0	Y_0	K	T	α	BMC	$P^a_{\sigma(y)}$	$P^a_{\bar{\sigma}}$
90	-1.62	100	1	.5	.0014	.00039	.00052
90	-1.62	100	1	1	.0019	.00036	.00038
90	-1.62	100	1	5	.0029	.00042	.00028
90	-1.62	100	1	10	.0044	.00063	.00019
90	-1.62	100	1	25	.0128	.00093	.00015
90	-1.62	100	1	50	.0185	.0011	.00014
90	-1.62	100	1	100	.0220	.0016	.00014

Table 8.4: Empirical variance for an out-of-the-money American put option.

S_0	Y_0	K	T	α	BMC	$P^a_{\sigma(y)}$	$P^a_{\bar{\sigma}}$
110	-1.62	100	1	.5	.0018	.00031	.00057
110	-1.62	100	1	1	.0015	.00038	.00043
110	-1.62	100	1	5	.0025	.00040	.00035
110	-1.62	100	1	10	.0044	.00053	.00019
110	-1.62	100	1	25	.0125	.00096	.00016
110	-1.62	100	1	50	.0175	.0011	.00012
110	-1.62	100	1	100	.0219	.0014	.00010

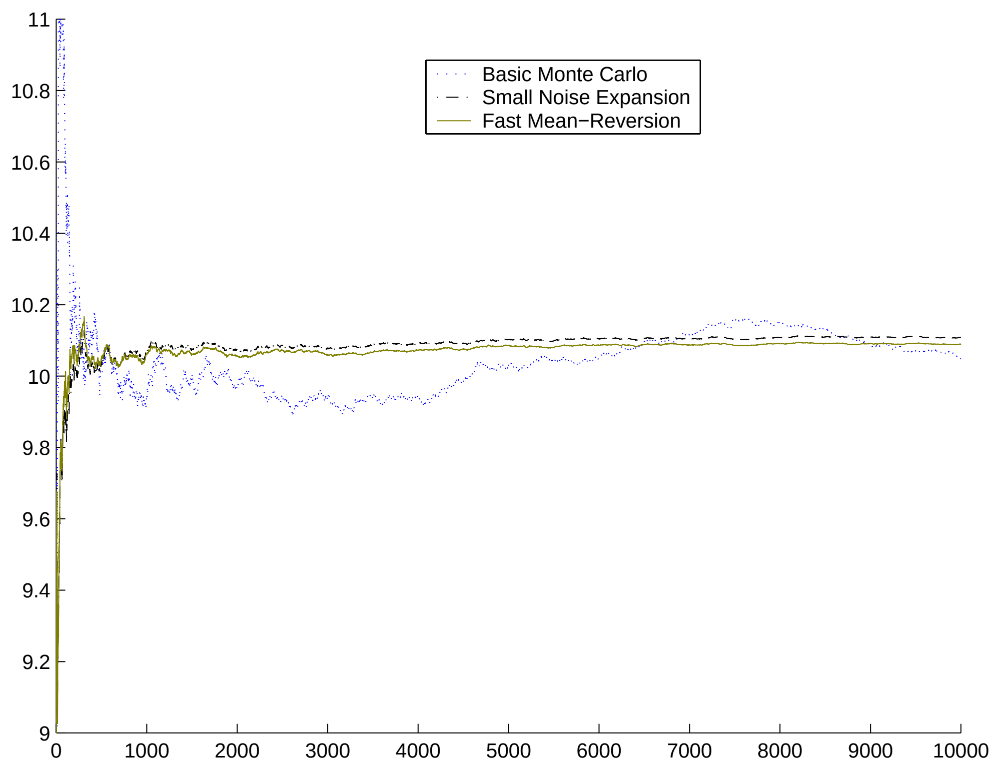

Figure 8.2: *Monte Carlo simulations of an in-the-money American put option with a rate of mean-reversion* $\alpha = 1$.

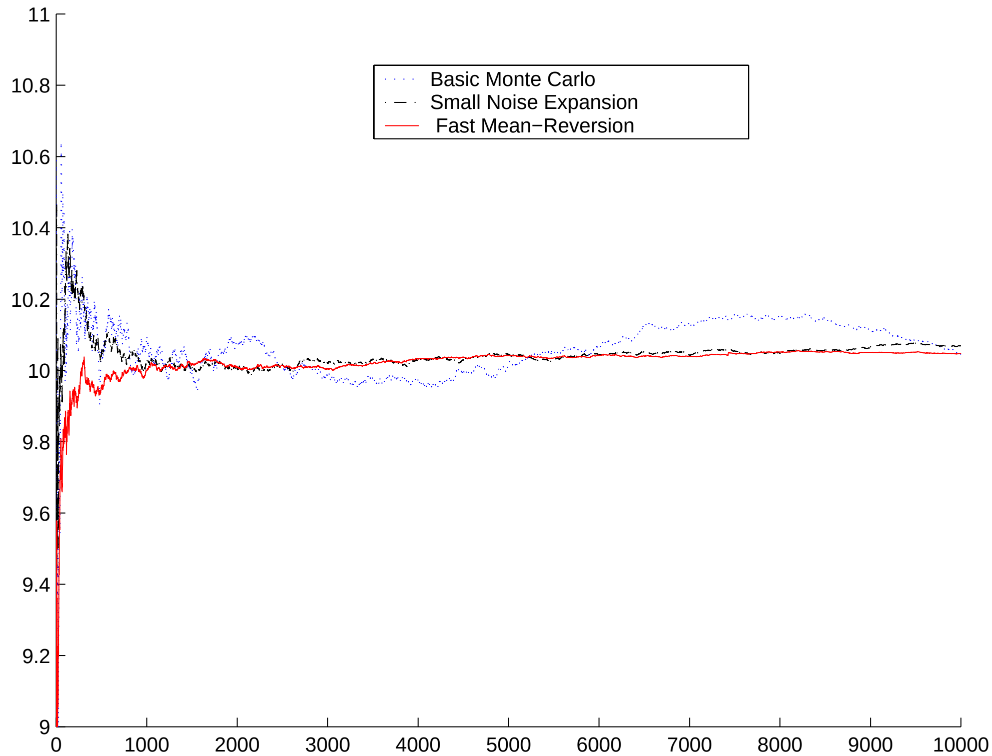

Figure 8.3: *Monte Carlo simulations of an in-the-money American put option with a rate of mean-reversion* $\alpha = 10$.

Chapter 9

Barrier Option

In this final chapter we shall apply our variance reduction methodology to the pricing of a European style barrier option. We begin by formally defining a barrier option and how it is priced when the underlying asset is modeled with constant volatility. Next, we consider the pricing of this option under a stochastic volatility model and develop the asymptotic analysis in this framework. Finally, we apply the importance sampling variance reduction technique when pricing a barrier option under stochastic volatility and present some numerical results.

9.1 Pricing a Barrier option

The four basic forms of these options include 'down-and-out', 'down-and-in', 'up-and-out' and 'up-and-in'. We shall concentrate on the down-and-out type in this work. A 'down-and-out' barrier option of European style gives its owner the right, but not the obligation to buy(Call) or sell(Put) one unit of an underlying asset for a predetermined price K as long as the asset remains above a predetermined price or barrier B during the period of the option. If the price of the asset falls below B, then the option expires worthless. We again denote the payoff function as ϕ, which is given as

$$\phi(s) = (s-K)^+ = \begin{cases} s-K & \text{if } s > K \\ 0 & \text{if } s < K \end{cases}$$

in the case of the call and

$$\phi(s) = (K-s)^+ = \begin{cases} K-s & \text{if } K > s \\ 0 & \text{if } K < s \end{cases}$$

in the case of the put and where K represents the strike price of the contract. Hence, the value of the payoff function at the terminal time T is given by $\phi(S_T)$. For simplicity, we shall consider the down-and-out call option and assume $B < K$.

Given that the underlying asset has constant volatility, the no-arbitrage price of a down-and-out call option with payoff given above is computed as

$$P(t,s) = \mathbb{E}^*[e^{r(T-t)}(S_T - K)^+ \mathbf{1}_{(\min_{t\leq T} S_t > B)} | S_t = s]. \tag{9.1}$$

We may again utilize the Feyman-Kac formula to represent (9.1) as a partial differential equation. The representation is same as a vanilla European call option with the additional boundary condition that the price of the option along the barrier B is zero. Hence we may formulate the evolution of the pricing function as follows

$$\begin{aligned} \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 P}{\partial s^2} + r\left(s\frac{\partial P}{\partial s} - P\right) &= 0, \quad (9.2) \\ P(T,s) &= \phi(s), \\ P(t,B) &= 0. \end{aligned}$$

Using the method of images, which is described in [27], a closed form solution for (9.2) is given by

$$P(t,s) = C_{BS}(t,s) - \left(\frac{s}{B}\right)^{1-\frac{2r}{\sigma^2}} C_{BS}\left(t, \frac{B^2}{s}\right), \tag{9.3}$$

where C_{BS} represents the price for a European call option.

9.2 Barrier Option under Stochastic Volatility

In this section we present the problem of pricing a down-and-out barrier call option when the stock price is modeled with stochastic volatility. We also present the small noise and fast mean-reverting asymptotics when applied to the barrier problem.

9.2.1 Pricing under Stochastic Volatility

We again consider the following stochastic volatility model under the the risk-neutral measure $\mathbb{P}^{*(\gamma)}$

$$\begin{aligned} dS_t &= rS_t dt + \sigma(Y_t) S_t dW_t^* \\ dY_t &= [\alpha(m - Y_t) - \nu\sqrt{2\alpha}\,\Lambda(y)]dt + \nu\sqrt{2\alpha}\,(\rho dW_t^* + \sqrt{1-\rho^2}\,dZ_t^*) \end{aligned} \tag{9.4}$$

where

$$\Lambda(y) = \frac{\rho(\mu - r)}{\sigma(y)} + \gamma(y)\sqrt{1-\rho^2}. \tag{9.5}$$

The market selects a unique risk-neutral measure $\mathbb{P}^{*(\gamma)}$ and the no-arbitrage down-and-out call price may be computed as

$$P(t,s,y) = \mathbb{E}^*[e^{r(T-t)}(S_T - K)^+ \mathbf{1}_{(\min_{t \le T} S_t > B)} | S_t = s, Y_t = y]. \tag{9.6}$$

Once again utilizing the Feynman-Kac formula, the pricing function given by equation (9.6) satisfies the following partial differential equation with two space dimensions:

$$\begin{aligned}\frac{\partial P}{\partial t} &+ \frac{1}{2}\sigma^2(y)s^2\frac{\partial^2 P}{\partial s^2} + \rho\nu\sqrt{2\alpha}\,s\sigma(y)\frac{\partial^2 P}{\partial s\partial y} + \nu^2\alpha\,\frac{\partial^2 P}{\partial y^2} \\ &+ r\left(s\frac{\partial P}{\partial s} - P\right) + \left[(\alpha(m-y)) - \nu\sqrt{2\alpha}\,\Lambda(y)\right]\frac{\partial P}{\partial y} = 0\,, \end{aligned} \tag{9.7}$$

with boundary and final conditions given by

$$P(T,s,y) = \phi(s), \tag{9.8}$$

$$P(t,B,y) = 0. \tag{9.9}$$

9.2.2 Small Noise Expansion

Utilizing the convenient compact operator notation given by (4.15), the evolution of the price is given by

$$\left(\alpha\mathcal{L}_0 + \sqrt{\alpha}\mathcal{L}_1 + \mathcal{L}_2\right)P(t,s,y) = 0. \tag{9.10}$$

Expanding the above partial differential equation about $\alpha = 0$ once again yields

$$\mathcal{L}_2 P(t,s,y) = 0.$$

Recall that $\mathcal{L}_2$ is simply the Black-Scholes operator with constant volatility $\sigma(y)$. Since the process Y_t is degenerate when $\alpha = 0$ then the pricing function will not depend on y: $P(t,s,y) = P(t,s)$. Hence, the evolution of the pricing function is as follows

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2(y)s^2\frac{\partial^2 P}{\partial s^2} + r\left(s\frac{\partial P}{\partial s} - P\right) + \left[(\alpha(m-y)) - \nu\sqrt{2\alpha}\,\Lambda(y)\right]\frac{\partial P}{\partial y} = 0, \tag{9.11}$$

with boundary and final conditions given by

$$P(T,s) = \phi(s), \tag{9.12}$$

$$P(t,B) = 0. \tag{9.13}$$

This is exactly the down-and-out barrier pricing problem with constant volatility level $\sigma(y)$. The closed form solution is given by

$$P(t,s) = C_{BS}(t,s) - \left(\frac{s}{B}\right)^{1-\frac{2r}{\sigma(y)^2}} C_{BS}\left(t, \frac{B^2}{s}\right), \tag{9.14}$$

where C_{BS} represents the price of a European call option. We shall utilize (9.14) as an initial approximation to the pricing function when implementing importance sampling.

9.2.3 Fast Mean-Reversion Expansion

The fast mean-reverting asymptotics of the barrier option are derived in the same fashion as in the European case except we must keep track of the boundary condition $P(t,B) = 0$. Our fast mean-reverting approximation is once again given by

$$P(t,s) = P_0(t,s) + \sqrt{\epsilon} P_1(t,s). \tag{9.15}$$

The zero-order term, P_0, is given by the Black-Scholes price with volatility level $\bar{\sigma}$

$$P(t,s) = C_{BS}(t,s) - \left(\frac{s}{B}\right)^{1-\frac{2r}{\bar{\sigma}^2}} C_{BS}\left(t, \frac{B^2}{s}\right). \tag{9.16}$$

The evolution of the correction $\tilde{P}_1 = \sqrt{\epsilon} P_1$ may described by the following partial differential equation

$$\mathcal{L}_{BS}(\sigma)\tilde{P}_1 = V_3 s^3 \frac{\partial^3 P_0}{\partial s^3} + V_2 s^2 \frac{\partial^2 P_0}{\partial s^2}, \tag{9.17}$$

which is the same as the European case; however, we have the additional boundary condition $\tilde{P}_1(t,B) = 0$. The solution to this equation is not simply given by

$$-(T-t)\left[V_3 s^3 \frac{\partial^3 P_0}{\partial s^3} + V_2 s^2 \frac{\partial^2 P_0}{\partial s^2}\right],$$

as in the case of the European option because it does not satisfy the boundary condition $\tilde{P}_1(t,B) = 0$. There is still a closed form solution to this partial differential equation and is presented hereafter. We transform the equation into a backward heat equation and then utilizing the probabilistic interpretation of the transformed equation a solution to (9.17) is obtained.

Let

$$w(t,s) = \tilde{P}_1 + (T-t)\left[V_3 s^3 \frac{\partial^3 P_0}{\partial s^3} + V_2 s^2 \frac{\partial^2 P_0}{\partial s^2}\right],$$

for $s \geq B$. Then $w(t,s)$ solves the simpler problem

$$\begin{aligned} \mathcal{L}_{BS}(\bar{\sigma})w &= 0 \qquad s > B,\ t < T \\ w(T,s) &= 0 \\ w(t,B) &= f(t), \end{aligned} \tag{9.18}$$

where

$$f(t) = (T-t)\lim_{s\to B^+}\left(V_3 s^3 \frac{\partial^3 P_0}{\partial s^3} + V_2 s^2 \frac{\partial^2 P_0}{\partial s^2}\right)(t,s).$$

An explicit formulation of f is given by

$$f(t) = (T-t)F(t,B),$$

where

$$\begin{aligned} F(t,s) &= V_3 s^3 \frac{\partial^3 C_{BS}}{\partial s^3} + V_2 s^2 \frac{\partial^2 C_{BS}}{\partial s^2} - \left(\frac{s}{B}\right)^{1-k} \\ &\cdot \left[V_2 \frac{B^4}{s^2}\frac{\partial^2 C_{BS}}{\partial s^2}\left(t, \frac{B^2}{s}\right) V_3 \frac{B^6}{s^3}\frac{\partial^3 C_{BS}}{\partial s^3}\left(t,\frac{B^2}{s}\right) + q\left(t,\frac{B^2}{s}\right)\right], \end{aligned} \tag{9.19}$$

and

$$
\begin{aligned}
q(t,s) &= \kappa_0 C_{BS}(t,s) + \kappa_1 s \frac{\partial C_{BS}(t,s)}{\partial s} + \kappa_2 s^2 \frac{\partial^2 C_{BS}}{\partial s^2}(t,s) \\
\kappa_0 &= k(k-1)(V_2 - V_3(k+1)) \\
\kappa_1 &= 2kV_2 - 3k\left(\frac{2r}{\bar{\sigma}^2} + 1\right) V_3 \\
\kappa_2 &= -3(k+1)V_3 \\
k &= \frac{2r}{\bar{\sigma}^2}
\end{aligned}
$$

We use the following assignments to transform (9.18) into a backward heat equation

$$
\begin{aligned}
z &= log(s) \\
w(t,s) &= exp\left[\left(-\frac{1}{2\bar{\sigma}^2}(r - \frac{1}{2}\bar{\sigma}^2)^2 - r\right)(T-t) - \frac{1}{\bar{\sigma}^2}(r - \frac{1}{2}\bar{\sigma}^2)z\right] v(t,z)
\end{aligned}
$$

Defining $C = \log B$, then $v(t,z)$ solves

$$
\begin{aligned}
\frac{\partial v}{\partial t} + \frac{1}{2}\bar{\sigma}^2 \frac{\partial^2 v}{\partial z^2} &= 0, \quad z > l, t < T & (9.20) \\
v(t,L) &= \tilde{f}(t), & (9.21)
\end{aligned}
$$

where

$$
\tilde{f}(t) = exp\left[\left(\frac{1}{2\bar{\sigma}^2}(r - \frac{1}{2}\bar{\sigma}^2)^2 + r\right)(T-t)\right] B^{r/\bar{\sigma}^2 - \frac{1}{2}} g(t)
$$

The solution to (9.20) has a probabilistic interpretation that is given by

$$
v(t,z) = \mathbb{E}[\tilde{g}(t)\mathbf{1}_{\tau \le T} | W_t = z > L], \tag{9.22}
$$

where W_t is a Brownian motion with $< B >_t = \bar{\sigma}^2 t$ and τ is the first time after t that it hits L[17]. Utilizing the distribution of the hitting time τ, v has closed solution given by the integral form

$$
v(t,z) = \frac{1}{\bar{\sigma}\sqrt{2\pi}} \int_t^T \frac{(z-L)}{(x-t)^{3/2}} e^{-(z-L)^2/2\bar{\sigma}^2(x-t)} \tilde{g}(x) dx.
$$

From this we may deduce the following solution for $\tilde{P}_1$

$$
\begin{aligned}
\tilde{P}_1(t,s) &= -(T-t)\left[V_3 s^3 \frac{\partial^3 P_0}{\partial s^3} + V_2 s^2 \frac{\partial^2 P_0}{\partial s^2}\right] \\
&\quad + s^{-\frac{1}{\bar{\sigma}^2}(r-\frac{1}{2}\bar{\sigma}^2)} \exp\left[\left(-\frac{1}{2\bar{\sigma}^2}(r-\frac{1}{2}\bar{\sigma}^2)^2 - r\right)(T-t)\right] v(t, log(s))
\end{aligned}
$$

We have developed the $O(\sqrt{\epsilon})$ approximation for a down-and out barrier option that could be used along with importance sampling for variance reduction. However, we were unsuccessful in our attempt to implement the $O(\sqrt{\epsilon})$ term along with the the zero order term. Therefore, we shall only consider the implementation of the zero order term of the fast mean-reverting approximation to the pricing function when utilizing importance sampling.

9.3 Variance Reduction for Barrier Option

In this chapter we apply the Importance Sampling variance reduction technique to computing the price of a down-and-out call option. We shall use the two methods of expansion described in the previous chapter to obtain a preliminary estimate of the expectation utilized to compute the premium. Secondly, we present some numerical results obtained from implementing the methodology in Matlab.

9.4 Application of Importance Sampling to Pricing Model

We apply the importance sampling variance reduction technique to the stochastic volatility model (9.4) used for computing European call options. In matrix form the evolution under the risk neutral measure $\mathbb{P}^*$ is given by

$$dV_t = b(V_t)dt + a(V_t)d\eta_t \,, \tag{9.23}$$

where we have set

$$\eta_t = \begin{pmatrix} W_t^* \\ Z_t^* \end{pmatrix} , \; V_t = \begin{pmatrix} S_t \\ Y_t \end{pmatrix} ,$$

and

$$a(v) = \begin{pmatrix} s\sigma(y) & 0 \\ \nu\rho\sqrt{2\alpha} & \nu\sqrt{2\alpha(1-\rho^2)} \end{pmatrix} , \; b(v) = \begin{pmatrix} rs \\ \alpha(m-y) - \nu\sqrt{2\alpha}\,\Lambda(y) \end{pmatrix} .$$

The price of a down-and-out call option at time 0 is computed by

$$P(0,v) = \mathbb{E}^*\{e^{-rT}\phi(V_T)\mathbf{1}_{\min_{t\leq T} S_t > B}|V_0 = v\} \,, \tag{9.24}$$

where $v = (s,y)$ and $\phi(v) = (s-K)^+$.

We now apply the importance sampling technique described in chapter 5.

Define $\tilde{\eta}_t = \eta_t + \int_0^t h(s,V_s)ds$, which is a Brownian motion under the probability $\tilde{\mathbb{P}}^*$ which admits the density Q_T^{-1} as described in chapter 5

$$Q_T^{-1} = \exp\left\{-\int_0^t h(s,V_s)\cdot d\tilde{\eta}_s + \frac{1}{2}\int_0^t \|h(s,V_s)\|^2 ds\right\} .$$

Under the new measure, the price of a down-and-out call option at time 0 is then computed by

$$P(0,v) = \tilde{\mathbb{E}}^*\{e^{-rT}\phi(V_T)\mathbf{1}_{\min_{t\leq T} S_t > B} Q_T|V_0 = v\} \,, \tag{9.25}$$

where the expectation is taken with respect to the measure $\tilde{\mathbb{P}}^*$.

Once again, if $P(0,v)$ were known, the optimal choice for h that gives the minimal variance is

$$h = -\frac{1}{P}\begin{pmatrix} s\sigma(y) & \nu\rho\sqrt{2\alpha} \\ 0 & \nu\sqrt{1-\rho^2}\sqrt{2\alpha} \end{pmatrix}\begin{pmatrix} \frac{\partial P}{\partial s} \\ \frac{\partial P}{\partial y} \end{pmatrix} . \tag{9.26}$$

Once we have found an approximation of P by using small noise expansion or fast mean-reversion expansion, then we may determine h in order to approximate (9.25) via Monte Carlo simulation

$$P(0,v) \approx \frac{1}{N}\sum_{k=1}^{N} \phi(V_T^{(k)}) Q_T^{(k)} \mathbf{1}_{\min_{t\le T} S_t^{(k)} > B} \tag{9.27}$$

under the evolution

$$\begin{aligned} dV_t &= (b(V_t) - a(V_t)h(t,V_t))dt + a(V_t)d\tilde{\eta}_t & (9.28)\\ Q_t &= \exp\left\{\int_0^t h(s,V_s)\cdot d\tilde{\eta}_s - \frac{1}{2}\int_0^t \|h(s,V_s)\|^2 ds\right\}. & (9.29)\end{aligned}$$

When we utilize small noise expansion as an apriori estimate for the price of a call option, the function $h(t,v)$ given by (9.26) will have the following form

$$h(t,v) = -\frac{1}{P^b_{BS(\sigma(y))}} \begin{pmatrix} s\sigma(y)\frac{\partial P^b_{BS(\sigma(y))}}{\partial s} \\ 0 \end{pmatrix}, \tag{9.30}$$

where $P^b_{BS(\sigma(y))}(t,s)$ is the price of a down-and-out call option with volatility level given by $\sigma(y)$.

Similarly, using the zero order term of the fast mean-reversion expansion leads to the function $h(t,v)$ of the following form

$$h(t,v) = -\frac{1}{P^b_{BS(\bar{\sigma})}} \begin{pmatrix} s\sigma(y)\frac{\partial P^b_{BS(\bar{\sigma})}}{\partial s} \\ 0 \end{pmatrix}, \tag{9.31}$$

where $\bar{\sigma}$ again represents the effective volatility.

Since we have control over how to choose this function h, we choose it in the following manner so that the Novikov condition will hold

$$h^{(M)} = \min(\max(-M,h),M),$$

where M is large. Therefore, Girsanov's theorem may be applied and Q_t has finite variance.

9.5 Numerical Results

In this section we present some numerical results from applying the importance sampling variance reduction methodology to the pricing of a down-and out call option. We shall test the variance methodology developed by implementing the technique on the three types of down-and-out European call options: in-the-money ($S_0 > K$), at-the-money($S_0 = K$) and out-of-the-money ($S_0 < K$), where S_0 represents the stock price at time 0 and K represents the strike price. Since each method of expansion is characterized by the rate of mean-reversion, we present results for various values of α ranging from slow mean reversion $\alpha = .5$ to fast mean-reversion $\alpha = 100$.

In table 7.1 we present the model parameters utilized when performing the simulations. We note in our choice of parameters that $\sigma(y)$ is bounded and that the effective volatility, $\bar{\sigma}$, is computed using

$$\bar{\sigma}^2 =< (\max(\min(\exp(y), 5), .0001))^2 >= \int_{-\infty}^{\infty} (\max(\min(\exp(y), 5), .0001))^2 \, \Phi(y)dy,$$

where

$$\Phi(y) = \frac{1}{2\pi} \exp\left(-\frac{(y+2.6)^2}{2}\right).$$

In tables 9.2, 9.3 and 9.4 we present the empirical variance for each method of expansion and basic Monte Carlo for an in-the-money, at-the-money and out-of-the-money down-and-out call option. We see as in the case of the European call option and American put option that basic Monte Carlo estimator performs extremely poorly when compared to the other two estimators. Additionally, when the rate of mean-reversion is large, we obtain a greater reduction of the variance when utilizing the fast mean-reversion expansion as opposed to the small noise expansion. We note in

Table 9.1: Model Parameters for the Barrier Pricing Problem

Parameter	Value
m	-2.6
r	$.1$
ν	1
ρ	$-.3$
$\sigma(y)$	$\max(.0001, \min(\exp(y), 5))$
μ	$.1$
$\gamma(y)$	0
$\Lambda(y)$	0
$\bar{\sigma}$	$.1971$

the table that B represents the barrier. Lastly, we also provide sample simulations in figures 9.1 and 9.2.

Table 9.2: Empirical variance for an in-the-money down-and-out European call option.

S_0	Y_0	K	B	T	α	BMC	$P_{BS(\sigma(y))}$	$P_{BS(\bar{\sigma})}$
110	-1.82	100	80	1	.5	.0164	.0026	.0038
110	-1.82	100	80	1	1	.0205	.0046	.0044
110	-1.82	100	80	1	5	.0232	.0081	.0036
110	-1.82	100	80	1	10	.0237	.0083	.0028
110	-1.82	100	80	1	25	.0257	.0115	.0010
110	-1.82	100	80	1	50	.0288	.0150	.0007
110	-1.82	100	80	1	100	.0319	.0184	.0004

Table 9.3: Empirical variance for an at-the-money down-and-out European call option.

S_0	Y_0	K	B	T	α	BMC	$P_{BS(\sigma(y))}$	$P_{BS(\bar{\sigma})}$
100	-1.82	100	80	1	.5	.0178	.0032	.0045
100	-1.82	100	80	1	1	.0235	.0058	.0059
100	-1.82	100	80	1	5	.0232	.0093	.0044
100	-1.82	100	80	1	10	.0227	.0097	.0038
100	-1.82	100	80	1	25	.0237	.0117	.0027
100	-1.82	100	80	1	50	.0298	.0120	.0014
100	-1.82	100	80	1	100	.0280	.0140	.0010

Table 9.4: Empirical variance for an out-of-the-money down-and-out European call option.

S_0	Y_0	K	B	T	α	BMC	$P_{BS(\sigma(y))}$	$P_{BS(\bar{\sigma})}$
90	-1.82	100	80	1	.5	.0152	.0034	.0037
90	-1.82	100	80	1	1	.0205	.0045	.0041
90	-1.82	100	80	1	5	.0288	.0070	.0032
90	-1.82	100	80	1	10	.0238	.0077	.0022
90	-1.82	100	80	1	25	.0257	.0095	.0013
90	-1.82	100	80	1	50	.0248	.0150	.0010
90	-1.82	100	80	1	100	.0250	.0184	.0007

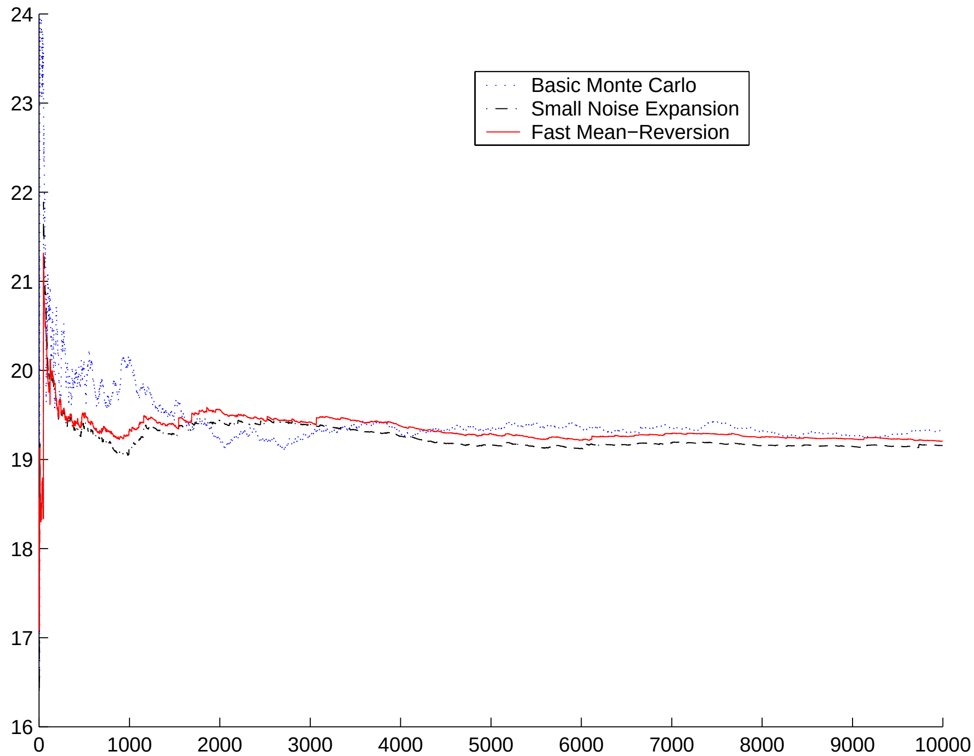

Figure 9.1: *Monte Carlo simulations of an in-the-money down-and-out call option with a rate of mean-reversion* $\alpha = 1$.

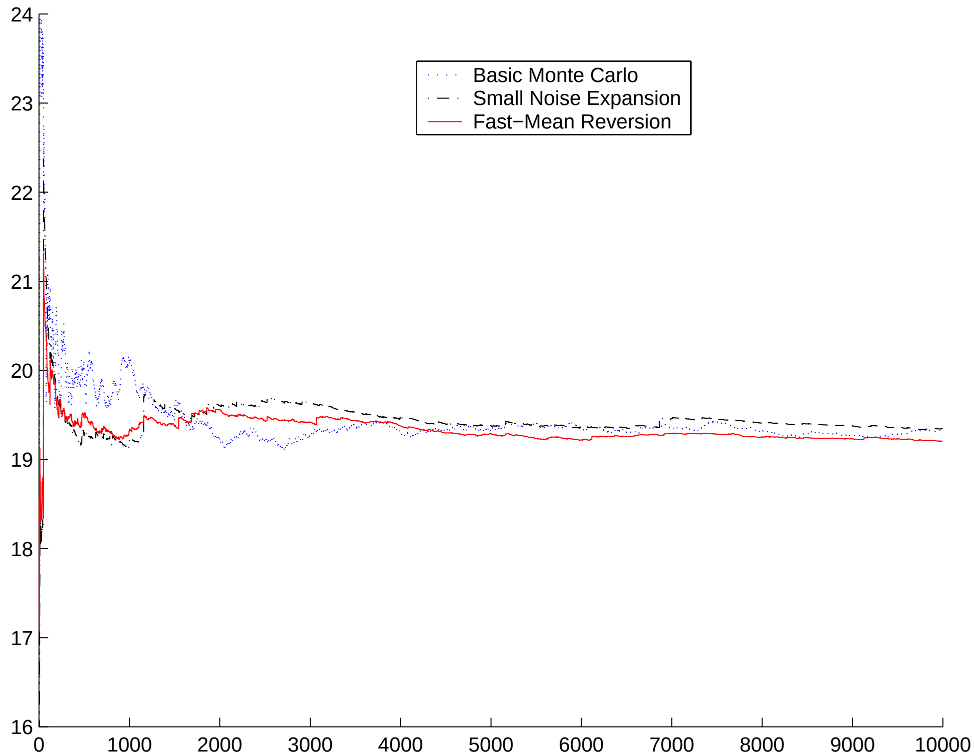

Figure 9.2: *Monte Carlo simulations of an in-the-money down-and-out call option with a rate of mean-reversion* $\alpha = 10$.

Chapter 10

Conclusions/Future Work

In this work we consider the pricing of European, American and barrier options in a stochastic volatility environment. We have presented a technique, which is based on applying fast mean-reverting asymptotics to importance sampling to reduce the variance of Monte Carlo simulation in this framework. We began by applying the method to a standard European style option. Numerical results show that applying this technique results in substantial reduction in the variance as shown by tables 7.2, 7.3 and 7.4, even if the rate of mean-reversion is slow. In particular, in the presence of a skew, utilizing the first correction in the preliminary estimate of the expectation results in an even greater reduction of variance when the rate of mean-reversion is slow. This contrasts with the method of expansion proposed in [11], which states that adding the next term of the small noise expansion does not improve the variance reduction significantly.

Although Monte Carlo simulation of an American option is more complex than that of a European, we have shown that importance sampling applied with small noise expansion or fast mean-reverting expansion may be implemented to again obtain a substantial reduction in the variance as illustrated by tables 8.2, 8.3 and 8.4. Lastly, we implemented this methodology on a barrier option and the results presented in tables 9.2, 9.3 and 9.4 once again again show the efficiency of the method. However, in both the American and barrier case the first correction has not been implemented. These are subjects of future research.

Another area of future research is the introduction of jumps in the model. Jumps may be introduced in the model in different ways. For instance, one may consider jumps in volatility. In that case the fast mean-reversion may be performed as shown in [10]. This again leads an effective volatility used to compute to approximate price. Another way to introduce jumps is to consider possible jumps in the underlying asset itself, combined with stochastic volatility. Fast mean-reverting asymptotics may be performed, leading to a model with jumps and constant effective volatility. If prices can be computed effeciently with this simplified model then our variance reduction technique may be applied. Lastly, another area of future research is the application of this variance reduction technique to fixed income markets. In particular, we will consider the computation of bond option prices.

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