Abstract

AXELLE CLAUDE PERSON. Solving homogeneous linear differential equations of order 4 in terms of equations of smaller order. (Under the direction of Michael F. Singer.)

In this thesis we consider the problem of deciding if a fourth order linear differential equation can be solved in terms of solutions of lower order equations. There is a group theoretic criteria which can be turned into a decision procedure for solving this problem. Once the decision has been made that a certain type of equation can be solved in terms of lower order equations we also give methods for producing the lower order equations used for solving it.

SOLVING HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF ORDER 4 IN TERMS OF EQUATIONS OF SMALLER ORDER

BY

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Biography

Axelle Claude Person was born in Rennes, France on December 14th, 1974. She graduated from the lycée Emile Zola in Rennes in June of 1992 and received the degree of Bachelor's of Science in Mathematics from Université de Rennes 1 in June of 1996. She then prepared for the Agrégation that she passed in July of 1997, after which she attended graduate school at Université de Rennes 1 working on her DEA in Algebra and Geometry that she received in the Summer of 1998. Since August 1998 she has been working on her co-advisory Ph'D in Mathematics at North Carolina State University and Université de Rennes 1. She also got her teaching certification in June of 2001 after a year of teaching 10th grade at Lycée Jacques Cartier in St Malo.

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Chapter 1

Résumé

Ce travail porte sur l'étude des équations différentielles linéaires homogènes à coefficients dans un corps différentiel k de corps des constantes algébriquement clos. En particulier nous nous intéressons aux équations $L(y) = a_4y^{(4)} + a_3y^{(3)} + a_2y^{(2)} + a_1y' + a_0y = 0$ d'ordre 4 et cherchons à les résoudre en exprimant leurs solutions à l'aide de solutions d'équations d'ordre inférieur à 4.

Le problème de résoudre des équations différentielles en terme d'ordre moindre remonte à Fano qui s'intéressa en 1900 aux relations entre les solutions d'une équation dans (12). M.F.Singer donna une preuve de ces résultats qui utilise la théorie des groupes dans (32), et en déduit un critère portant sur la nature du groupe de Galois $\mathcal{G}(L)$ associé à une équation, qui permet de décider si celle-ci se résout en terme d'équations d'ordre moindre.

Une manière de résoudre en terme d'ordre moindre est de factoriser. On sait factoriser

des équations différentielles linéaires sur le corps des coefficients (voir (5) et (19)) si bien que nous considérons ici des équations irréductibles. Ulmer et Singer ont donné des algorithmes efficaces pour trouver des facteurs d'ordre 1 sur une extension algébrique du corps des coefficients ((37) et (40)). Récemment Compoint et Weil se sont également intéressés à trouver des factorisations dans la clôture algébrique du corps des coefficients ((9)). D'autres cas de résolutions se présentent également; par exemple les solutions d'une équation peuvent s'écrire comme le produit de solutions de deux équations d'ordre 2.

L'idée est d'utiliser les travaux de Hessinger qui s'inspirent d'un théorème de Chevalley permettant de calculer la composante connexe de $\mathcal{G}(L)$ en factorisant un certain nombre d'équations associées à des constructions obtenues à partir de L. Une fois la nature du groupe determinée il est alors possible de développer des algorithmes calculant les solutions d'une équation qui se résout en terme d'ordre moindre.

Dans cette thèse on décrit toutes les situations possibles de résolution d'une équation d'ordre 4 en terme d'ordre moindre. Pour chaque cas nous expliquons pourquoi la structure du groupe de Galois nous permet d'être sûr que l'on pourra résoudre en terme d'ordre moindre, puis nous donnons une méthode pour produire les équations d'ordre inférieur en question. Nous suivons donc le plan suivant:

- Le chapitre 3 est une introduction succincte à la théorie de Galois différentielle. Dans un premier temps nous y introduisons les notions de corps différentiel, groupe de Galois différentiel et extensions de Picard-Vessiot. Puis nous rappelons le lien entre D-modules et équations ou systèmes différentiels ainsi que l'effet d'un changement de base sur une équation. Ensuite nous rappelons quelques propriétés de factorisation d'opérateurs différentiels, ainsi que le critère de M.F. Singer pour la résolution en terme d'ordre moindre, rappelons l'importance des contructions et finalement donnons quelques résultats de résolution dans le cas d'équations d'ordre 2 et 3.

- Dans le chapitre 4 nous rappelons des notions de base sur les algèbres de Lie puis donnons la liste des sous-algèbres irréductibles de $\mathbf{sl}_4(\mathcal{C})$ et leurs représentations. Nous y donnons la classification des sous-groupes de $Sl_4(\mathcal{C})$ par Hessinger puis traduisons cette classification en terme d'équation différentielle dans le théorème 9. Nous y rappelons également la méthode utilisée par Hessinger pour distinguer les sous-groupes primitifs infinis de $Sl_4(\mathcal{C})$ puis consacrons chaque section suivante à l'étude des cas nous intéressant. Dans chaque cas nous expliquons pourquoi l'on peut résoudre en terme d'ordre moindre et donnons une procédure pour calculer les équations d'ordre 2 utilisées pour la résolution.

  - Le premier cas est celui où l'algèbre de Lie est réductible. Dans ce cas l'équation factorise sur une extension algébrique du corps des coefficients. Ce cas contient les équations réductibles ainsi que celles qui possèdent des solutions liouvilliennes (i.e. peuvent se résoudre en terme d'équations d'ordre 1). Ces cas ont déjà fait l'objet d'études approfondies (voir (39) et (40)), et nous nous contentons de donner un exemple de méthode de résolution.

  - Le deuxième cas est celui où l'algèbre est $\mathbf{sl}_2(\mathcal{C})$ avec une représentation d'ordre 4 irréductible. Dans ce cas l'équation est équivalente à la troisième puissance symétrique d'une équation d'ordre deux.
  - Dans le troisième cas l'algèbre de Lie est $\mathbf{so}_4(\mathcal{C})$ et l'équation est équivalente au produit symétrique de deux équations d'ordre 2.
  - Le dernier cas traite la situation où l'algèbre est $\mathbf{sl}_2(\mathcal{C})$ mais avec une représentation réductible cette fois, cas dans lequel nous écrivons l'équation comme un plus petit multiple commun de deux équations d'ordre 2 sur une extension algébrique de degré $4, 6$ ou 12 du corps des coefficients.

- Le chapitre 5 est consacré à l'application des méthodes développées dans le chapitre 4 sur des exemples. La structure de ce chapitre respecte l'ordre d'étude du chapitre précédent et pour chaque exemple nous expliquons la technique de construction, comment calculer le groupe de Galois et les problèmes rencontrés à cette étape, puis produisons les équations d'ordre deux nécessaires à la résolution.

- Finalement dans le chapitre 6 nous résumons ce qui précède en soulignant les principales difficultés rencontrées pour la résolution de ce problème.

Chapter 2

Introduction

The problem of solving a differential equation in terms of equations of lower order goes back to Fano ((12)) whose results were reproven by M.F. Singer ((32)) from a group theoretical perspective. Thanks to the work of S.A. Hessinger ((17)) one can decide if a linear differential equation of order four is solvable in terms of lower order equations using group theoretical considerations. Her results can be used for developing algorithms to calculate the solutions of a fourth order equation that is solvable in terms of second order equations. The most popular problem so far has been to study the equations whose solutions can be written as the products of solutions of two second order linear differential equations (see (4), (41)), but other cases need to be considered. For example the equation $L_4(y) = y^{(4)} - 10xy^{(2)} - 10y' + 9x^2y$ is the third symmetric power of the Airy equation $L(y) = y'' - xy$, hence its solutions are $\{{y_1}^3, {y_1}^2y_2, y_1{y_2}^2, {y_2}^3\}$ where $\{y_1, y_2\}$ is a basis of solutions of the Airy equation.

In this paper we give all the situations one can encounter while solving a fourth order linear differential equation in terms of lower order. For each situation we explain the criteria that allows us to decide of the solvability, and then we show that one can produce the corresponding lower order equations.

The first part is an introduction to Differential Galois Theory and other algebraic results that will be needed for solving our problem, as well as known results for the case of equations of order less than four. The second part recalls how the study of Lie algebras gives a decision criteria for solvability and enumerates a list of cases one can encounter. Then each following part is devoted to one particular case for which we explain the reasons for solvability and give a procedure to compute the equations of order 2.

The first case we consider is the one where the Lie Algebra is reducible, for which we give an example where our equation factors over an quadratic extension. The second case is when the Lie algebra is $\mathbf{sl}_2(\mathcal{C})$ with irreducible representation, which is when we write the fourth order equation equivalent to the third symmetric power of a second order equation. The third case is when the Lie Algebra is $\mathbf{so}_4(\mathcal{C})$ and the equation is equivalent to the symmetric product of 2 second order equations, and the last case is when the Lie Algebra is $\mathbf{sl}_2(\mathcal{C})$ with reducible representation from which the equation can be written as the least common left multiple of 2 equations of order 2 over an algebraic extension of degree 4, 6 or 12 of the field of coefficients.

Chapter 3

Differential Galois theory and known results

In this section we will present the basic notions of differential Galois theory needed for solving our problem, as well as some results about the representation theory of Lie Algebras. Then we give a brief summary of how to solve equations of second and third order using lower order equations.

3.1 Differential Galois theory

Definition 1 *Let k be a field.*

A **derivation** *on k is an operation $\delta : k \mapsto k$ such that $\forall a, b \in k$, $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$.*

A **differential field** (k, δ) *is a field* k *together with a derivation* δ *on* k. *Given a differential field* (k, δ), *the set* $\{c \in k | \delta(c) = 0\}$ *is a subfield of* k *called the* **field of constants** *of* k, *denoted* $Const(k)$, *or more simply* $\mathcal{C}$.

A **differential field extension** *of* (k, δ) *is a differential field* (K, Δ) *such that* K *is a field extension of* k *and* Δ *is an extension of the derivation* δ *to a derivation on* K.

In this thesis we will always assume that k is of characteristic 0 and that $\mathcal{C}$ is algebraically closed (e.g. $(\bar{\mathcal{Q}}(x), \frac{d}{dx})$).

We will write $y^{(n)}$ instead of $\delta^n(y)$ and $y', y'', \cdots$ for $\delta(y)$, $\delta^2(y)$, $\cdots$.

If $F \subset E$ are differential fields and S is a subset of E, we denote by $F < S >$ the smallest subfield of E containing F and S. It is the field generated over F by the elements of S and their derivatives.

Whenever we refer to a differential equation $L(y) = 0$ we will mean an ordinary homogeneous linear differential equation $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ with coefficients in a differential field k. To this equation one associates the differential operator $L = \delta^{(n)} + a_{n-1}\delta^{(n-1)} + \cdots + a_0$.

Definition 2 *Let* K_1 *and* K_2 *be two differential extensions of* k. *A* **differential k-isomorphism** *between* K_1 *and* K_2 *is a field isomorphism that leaves* k *fixed and commutes with* δ.

The **Differential Galois group** $\mathcal{G}(K/k)$ *of a differential field extension* K *of* k *is the* *set of all differential k-automorphisms of K.*

A **fundamental system of solutions** of $L(y) = 0$ is a set $\{y_1, \cdots, y_n\}$ of solutions of $L(y) = 0$ that are linearly independent over $\mathcal{C}$, which is the case if and only if the **wronskian** $Wr(y_1, \cdots, y_n) = det(W)$ is not equal to zero, where $W = ({y_i}^{(j)})_{1 \leq i \leq n, 0 \leq j \leq n-1}$. The set of solutions of $L(y) = 0$ is a vector space over $\mathcal{C}$, of dimension at most n ((24) p.21).

Definition 3 *Let $L(y) = 0$ be a homogeneous linear differential equation of order n with coefficients in a differential field k. A differential field extension K of k is called a* **Picard-Vessiot extension** *(PVE) of k for $L(y) = 0$ if the following hold:*

- *$K = k < y_1, \cdots, y_n >$, where $\{y_1, \cdots, y_n\}$ is a fundamental system of solutions of $L(y) = 0$.*
- *$Const(K) = Const(k)$.*

If $\mathcal{C}$ is algebraically closed, given any such equation $L(y) = 0$, there exists a PVE K for $L(y) = 0$ which is unique up to differential k-isomorphisms and may be viewed as the splitting field for the equation $L(y) = 0$. We denote by $\mathcal{G}(L)$ the differential Galois group $\mathcal{G}(K/k)$.

Let $V = \{y \in K | L(y) = 0\}$ be the n-dimensional $\mathcal{C}$-vector space of solutions of $L(y) = 0$. If $\sigma \in \mathcal{G}(L)$ and $y_1, \cdots, y_n$ is a basis for V, then $\sigma(y_i) = \sum_{j=1}^{n} c_{ij} y_j$ where $c_{ij} \in \mathcal{C}$. This gives a faithful representation of $\mathcal{G}(L)$ as a subgroup of $Gl_n(\mathcal{C})$ by identifying σ with (c_{ij}).

We will assume that the reader is familiar with the notions of linear algebraic groups and Zariski topology on a variety. If needed see (22).

Theorem 1 *((24)* **Galois correspondence***) Let $k \subset K$ be differential fields with K a PVE of k and $\mathcal{C}$ algebraically closed. Let $G = \mathcal{G}(K/k) \subset Gl_n(\mathcal{C})$. Then G is a linear algebraic group and there is a Galois correspondence between Zariski closed subgroups of G and differential subfields E of K with $k \subset E \subset K$.*

As a result many properties of the equation $L(y) = 0$ and of its solutions can be found in the structure of the linear algebraic group $\mathcal{G}(L)$, in particular the two following results will be useful:

Definition 4 *Let k be a differential field and $L(y) = 0$ a linear differential equation with coefficients in k. A solution y of $L(y) = 0$ is said to be*

- **rational** *if $y \in k$.*
- **exponential** *if $y'/y \in k$.*

Algorithms for finding rational and exponential solutions can be found in (6). But already by looking at the Galois group one can decide whether we only have rational solutions:

Lemma 1 *A linear differential equation $L(y) = 0$ has all its solutions rational if and only if its Galois group is trivial.*

Definition 5 *A differential field extension (K, Δ) of (k, δ) is a* **liouvillian extension** *if there is a tower of fields*

$$k = K_0 \subset K_1 \subset \cdots \subset K_m = K$$

where K_{i+1} is a simple field extension $K_i(\nu_i)$ of K_i such that one of the following holds:

- *ν_i is algebraic over K_i, or*
- *$\delta(\nu_i) \in K_i$ (extension by an integral), or*
- *$\delta(\nu_i)/\nu_i \in K_i$ (extension by the exponential of an integral).*

A function contained in a liouvillian extension of k is called a **liouvillian function** *over k.*

Theorem 2 *((25)) A differential equation $L(y) = 0$ with coefficients in k has*

- *only solutions which are algebraic over k if and only if $\mathcal{G}(L)$ is a finite group,*
- *only liouvillian solutions over k if and only if the component of the identity $\mathcal{G}(L)^0$ of $\mathcal{G}(L)$ in the Zariski topology is solvable.*

$$\begin{array}{ccl} K & \longleftrightarrow & \{id\} \\ \cup & & \ \cap \\ k_0 & \longleftrightarrow & Gal(K/k_0) = G^0 solvable \\ \cup & & \ \cap \\ k & \longleftrightarrow & Gal(K/k) \end{array}$$

3.2 D-modules, equivalence and cyclic vectors

To a linear differential equation $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y$ one associates the operator $L = \delta^n + a_{n-1}\delta^{n-1} + \cdots + a_0$ and talks about the differential Galois group and the

Picard-Vessiot extension associated with the operator L as well. Given a differential field k we will work in **the ring of linear differential operators with coefficients in** k, $k[\delta]$, which is the ring of noncommutative polynomials in the variable δ with coefficients in k where δ satisfies $\delta a = a\delta + a'$ for all $a \in k$. The integer n is called the order of L, denoted $ord(L)$.

Definition 6 *((29) p.38) A differential module or $\mathcal{D}$-module $\mathcal{M}$ over k is a finite dimensional k-vector space which is also a left module for the ring $k[\delta]$.*

To an operator $L = \delta^n + a_{n-1}\delta^{n-1} + \cdots + a_0$ one can associate a first order system $Y' = AY$ where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & \cdots & -a_{n-1} \end{bmatrix}$$

The matrix A is called the **companion matrix** of L.

Definition 7 *((29) p.39) Given $A \in Hom(k^n, k^n)$ we define the differential module $\mathcal{M}_A$ associated with $Y' = AY$ via the formula*

$$\delta e_i = -\sum_j a_{j,i} e_j$$

where $\mathcal{B} = \{e_1, \cdots, e_n\}$ *is the standard basis of* k^n *and* $A = (a_{i,j})$.

If $\mathcal{B}' = \{e_1{}', \cdots, e_n{}'\}$ is another basis of k^n and $\mathcal{B}' = P\mathcal{B}$ then one verifies that $A_{\mathcal{B}'} = P^{-1}AP + P^{-1}P'$.

To a differential equation L one can always associate a differential module $\mathcal{M}_L$ given by the companion matrix. Conversely given a $\mathcal{D}$-module $\mathcal{M}$ and a k-basis $e_1, \cdots, e_n$ of $\mathcal{M}$ one has

$$\delta e_i = -\textstyle\sum_j a_{j,i} e_j$$

and $A = (a_{i,j})$ defines a differential system associated with $\mathcal{M}$. Using a cyclic vector ((29) p.44) one can then associate to $\mathcal{M}$ a matrix equivalent to A that is in companion form, hence a differential equation to $\mathcal{M}$.

Example 1 *Let*

$$A = \begin{bmatrix} 0 & \frac{-(-1-x+x^2)}{-1+x} & 1 & 0 \\ -1 & \frac{-1}{-1+x} & 0 & 1 \\ x & 0 & 0 & \frac{-(-1-x+x^2)}{-1+x} \\ 0 & x & -1 & \frac{-1}{-1+x} \end{bmatrix}$$

and consider the system $Y' = AY$. *The vector* $v = [1, 0, 0, 0]$ *is a cyclic vector for* A *since*

v, v', v'' are linearly independent, and the corresponding companion matrix for A is

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2(4x^3-4x^2+6x-63)}{((-1+x)(4x^3-4x^2-4x-15))} & \frac{2(20x^4-24x^3-14x^2-165x+69)}{((-1+x)(4x^3-4x^2-4x-15))} & \frac{2(8x^4-8x^3-12x^2-30x+9)}{(4x^3-4x^2-4x-15)} & \frac{-4(x^3+x^2-5x-16)}{((-1+x)(4x^3-4x^2-4x-15))} \end{bmatrix}$$

Definition-Proposition 1 *Let $Y' = AY$ and $V' = BV$ be 2 matrix differential equations of the same order such that there exists an invertible matrix Z with coefficients in k and $Y = ZV$. Then Z verifies $Z' = AZ - ZB$.*

Furthermore Z can be found by solving the linear system:

$$W' = (A \otimes I - I \otimes B^T)W$$

where W is the vector formed by the rows of the matrix Z.

Two operators/systems satisfying the above conditions are said to be **equivalent**. If we look at the companion matrices A_1 and A_2 for L_1 and L_2 respectively with L_1 and L_2 having the same order n, we have that $L_1(y) = 0$ and $L_2(v) = 0$ are equivalent if and only if there exists a $B \in GL_n(k)$ such that the substitution $Y = BV$, which leads to $V' = (B^{-1}A_1B - B^{-1}B')V$, has the property that $A_2 = B^{-1}A_1B - B^{-1}B'$.

In solving linear differential equations in terms of equations of lower order, the notion of equivalence for linear differential equations will be needed. Indeed one may not be able to express the solutions of our given equation in a more simple form but one may find an equation equivalent to it for which this is possible, in which case one will apply a **gauge transformation** to our equation corresponding to the matrix Z in the previous definition:

Definition 8 *Let L_1 and L_2 be 2 homogeneous linear differential operators with coefficients in a differential field k and solutions $\{u_1, \cdots, u_n\}$, $\{v_1, \cdots, v_n\}$ respectively such that L_1 is equivalent to L_2. A gauge transformation from the solution space of $L_1(y) = 0$ to the solution space of $L_2(y) = 0$ is defined by $a_0, \cdots, a_{n-1}$ in k such that:*

$$v_1 = a_0u_1 + a_1{u_1}' + \cdots + a_{n-1}{u_1}^{(n-1)}$$
$$\vdots$$
$$v_n = a_0u_n + a_1{u_n}' + \cdots + a_{n-1}{u_n}^{(n-1)}$$

Remark 1 1. *Notice that the Galois group G of L_1 and L_2 in the previous definition will be the same since the linear combination does not change the group action on the solutions. In particular a gauge transformation will not affect the form of the solutions of our equation, for instance if L_1 has Liouvillian solutions, so does L_2.*

2. *In particular L_1 is equivalent to L_2 if and only if $\mathcal{M}_1$ is isomorphic to $\mathcal{M}_2$, which*

is the case if and only if the solutions spaces of $L_1(y) = 0$ and $L_2(y) = 0$ are isomorphic as G-modules (see lemma 2.41 of (29) p.56).

To explicitely find a gauge transform we will work with systems:

Example 2 *The equation $L_2(y) = y'' - xy = 0$*

is equivalent to $\tilde{L}_2(v) = v'' - \frac{1}{x-1}v' - \frac{x^2-x-1}{x-1}v = 0$ by the gauge transform $v = y + y'$.

Indeed if

$$A = \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix}$$

is the companion matrix for L_2 and

$$B = \begin{bmatrix} 0 & 1 \\ -\frac{x^2-x-1}{x-1} & -\frac{1}{x-1} \end{bmatrix}$$

the companion matrix for $\tilde{L}_2$, then using the Maple package ISOLDE ((1)) one finds a rational solution for $W' = (A \otimes I - I \otimes B^T)W$ to be $W = [-\frac{1}{x-1}, \frac{1}{x-1}, \frac{x}{x-1}, \frac{-1}{x-1}]$. The matrix

$$Z = \begin{bmatrix} -\frac{1}{x-1} & \frac{1}{x-1} \\ \frac{x}{x-1} & \frac{-1}{x-1} \end{bmatrix}$$

is invertible with inverse

$$Z^{-1} = \begin{bmatrix} 1 & 1 \\ x & 1 \end{bmatrix}$$

and from $V = Z^{-1}Y$ comes $v = y + y'$.

3.3 Factorization

One of the main tools we will use is factorization in $k[\delta]$:

The following is well introduced in the second chapter of (29). It is known ((29) p.38) that for any left ideal $I \subset k[\delta]$ there exists an $L_1 \in k[\delta]$ such that $I = k[\delta]L_1$. Similarly for any right ideal $J \subset k[\delta]$ there exists an $L_2 \in k[\delta]$ such that $I = L_2 k[\delta]$.

Consequently one can define the **Least Common Left Multiple**, $LCLM(L_1, L_2)$ of $L_1, L_2 \in k[\delta]$ as the unique monic generator of $k[\delta]L_1 \cap k[\delta]L_2$ and the **Greatest Common Left Divisor**, $GCLD(L_1, L_2)$ of $L_1, L_2 \in k[\delta]$ as the monic generator of $k[\delta]L_1 + k[\delta]L_2$. In the following we will denote $LCLM(L_1, L_2)$ by $[L_1, L_2]_l$. Similar definitions exist for the **Greatest Common Right Divisor** and the **Least Common Right Multiple**.

Lemma 2 *((29) p.57) Let $L \in k[\delta]$. Let K be the Picard-Vessiot extension of k associated with $L(y) = 0$ and G be its corresponding Galois group. There is a bijective correspondance between monic right factors of L in $k[\delta]$ and G-invariant subspaces of V, the solution space of $L(y) = 0$ in K.*

Since the liouvillian solutions of a linear differential equation form a G-invariant space we have:

Lemma 3 *((29) p.35) Let k be a differential field.*

Let $L(y) = 0$ be a scalar differential equation with coefficients in k. If $L(y) = 0$ has a *nonzero solution liouvillian over k, then the operator L has a right factor L_1 of order at least one with coefficients in k such that all solutions of $L_1(y) = 0$ are liouvillian over k.*

Definition 9 *An operator $L \in k[\delta]$ is said to be* **reducible over** *k if it can be written as $L = L_1L_2$ where $ord(L_1), ord(L_2) < ord(L)$. The associated equation is then called reducible as well, and if this is not the case, then both are* **irreducible in** *k.*

Proposition 1 *((29) p.57) Let $L(y) = 0$ be a linear differential equation with coefficients in k. Let K be the corresponding Picard-Vessiot extension and let G be its Galois group. The following are equivalent:*

1. *The differential module $\mathcal{M}_L \simeq k[\delta]/k[\delta]L$ contains a proper, non-zero submodule.*
2. *The operator L is reducible over k.*
3. *The solution space V of $L(y) = 0$ in K^n is a reducible G-module.*

One can decompose an operator $L \in k[\delta]$ as the product of irreducible operators but this decomposition need not be unique. For example, if $k = \mathcal{C}(x)$, $x' = 1$, $\delta^2 = \delta\delta = (\delta + \frac{1}{x+a})(\delta - \frac{1}{x+a})$ for any $a \in \mathcal{C}$. Thanks to the Jordan-Hölder Theorem we do have a weaker form of uniqueness though. We say that a tower of differential modules $\{0\} = \mathcal{M}_r \subset \cdots \subset \mathcal{M}_1$ is a composition series if successive quotients $\mathcal{M}_i/\mathcal{M}_{i+1}$ are simple, that is, have no proper nonzero submodules. Two composition series $\{0\} = \mathcal{M}_r \subset \cdots \subset \mathcal{M}_1$ and $\{0\} = \mathcal{N}_s \subset \cdots \subset \mathcal{N}_1$ are said to be equivalent if $r = s$ and, after a possible permutation of indices $i \mapsto i'$ we have that $\mathcal{M}_i/\mathcal{M}_{i+1} \simeq \mathcal{N}_{i'}/\mathcal{N}_{i'+1}$.

Proposition 2 *((29) p.58)*

1. *For any differential module all composition series are equivalent.*

2. *For any $L \in k[\delta]$ of positive order, we may write $L = L_1 \cdots L_r$ where the L_i are irreducible and of positive order. If $L = \tilde{L}_1 \cdots \tilde{L}_s$ is another factorisation then $r = s$ and after a permutation of indices $i \mapsto i'$ we have that L_i and $\tilde{L}_{i'}$ are equivalent.*

Definition 10 *Let k be a differential field and $L \in k[\delta]$. We say that L is* **completely reducible** *if L is a nonzero k-multiple of the least common left multiple of a set of irreducible operators.*

Let us now introduce some group theoretic definitions that allow to partially classify linear algebraic groups. Notice the correspondence between the above definitions for the differential operators and the definitions below concerning subgroups of $Gl_n(\mathcal{C})$.

Definition 11 *A subgroup G of $Gl(V)$ is said to act* **irreducibly** *if and only if the only G-invariant subspaces of V are $\{0\}$ and V; we usually refer to such a group as irreducible. Notice in particular that a linear differential equation will be irreducible (i.e. will not have factors) if and only if its associated Galois group is.*
The G-module V is **completely reducible** *if there are minimal G-invariant subspaces $V_1, \cdots, V_k$ such that $V = V_1 \oplus \cdots \oplus V_k$.*

Note that an irreducible representation is completely reducible. Those definitions of complete reducibility are linked in the following lemma:

Lemma 4 *((29) p.59) Let k be a differential field with constant field $\mathcal{C}$ and $L, L_1, \cdots, L_m \in k[\delta]$. Let K be a Picard-Vessiot extension of k containing a full set of solutions of each $L(y) = 0$, $L_1(y) = 0$, $\cdots$, $L_m(y) = 0$. The operator L is a k-multiple of the least common left multiple of $L_1, \cdots, L_m$ if and only if the solutions spaces V_i of $L_i(y) = 0$ in K span the solution space of $L(y) = 0$.*

In particular when an operator L is completely reducible there exists a minimal representation of L as a least common left multiple $L = [L_1, \cdots, L_r]_l$.
Also if a group G has an irreducible representation, then any representation of G is completely reducible. As a result we will always be able to decompose constructions from an irreducible operator as the least common left multiple of some irreducible operators.

3.4 Solving in terms of lower order equations

Let us finally define precisely what we understand by "solving a differential equation in terms of lower order equations":

Definition 12 *The differential equation $L(y) = 0$ defined over k is said to be* **solvable in terms of linear differential equations of lower order** *if the associated Picard-Vessiot extension K of k lies in a tower of fields $k = K_0 \subset K_1 \subset \cdots \subset K_n$ such that for each $i = 1, \cdots, r$ either:*

- *K_i is a finite algebraic extension of K_{i-1}, or*

- *K_i is generated over K_{i-1} by a solution of a (not necessarily homogeneous) linear differential equation of order less than n with coefficients in K_{i-1}.*

Example 3 *Notice that finding the liouvillian solutions for a differential equation implies solving it in terms of equations of order one since a liouvillian extension is one contained in a tower of field that are generated by solutions of equations of order one.*

A general result to solve this problem is given in (33):

Theorem 3 *Let k be a differential field with algebraically closed field of constants $\mathcal{C}$ and let $L(y) = y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1y'+a_0y = 0$ with $a_i \in k$. The equation $L(y) = 0$ is* **not** *solvable in terms of equations of lower order if and only if the associated Picard-Vessiot extension has Galois group G whose connected component G^0 of the identity is:*

1. *simple, and*
2. *there does not exist a linear algebraic group H with finite-to-one homomorphism $\alpha : H \to G^0$ such that H has a nontrivial representation of dimension less than n.*

In fact we will use the Lie algebra version of this theorem, but for this purpose we need some notions on representation theory of Lie algebras and algebraic groups that will be introduced in the next section.

Let $L(y) = y^{(4)} + a_3y^{(3)} + \cdots + a_0y = 0$ be a homogeneous linear differential equation of order 4 with coefficients in a differential field k of characteristic zero and algebraically closed constant field $\mathcal{C}$. Let M be the Picard-Vessiot extension of k associated with L and suppose the Galois Group of L over k is contained in $Sl_4(\mathcal{C})$. If this is not the case one can always use the substitution $z = y \exp\left(-1/4 \int a_3\right)$ to form an equation whose coefficient of $z^{(3)}$ is zero, hence whose Galois group is a subgroup of $Sl_4(\mathcal{C})$ ((24) p.41). Since this change of variables corresponds to multiplying all the solutions of $L(y) = 0$ by $\exp\left(-1/4 \int a_3\right)$, one understands easily that the new equation is solvable in terms of lower order if and only if the original one is.

Definition 13 *A linear differential equation of order n with its $n-1$ coefficient equal to zero is said to be in* **normal form** *or* **normalized***.*

3.5 Constructions

1. **Symmetric powers**

   For the following one needs to recall a few facts about symmetric products of linear differential operators:

   Let $L_1(y)$ and $L_2(y)$ be two homogeneous linear differential equations with coefficients in k and let M_1 and M_2 be the Picard-Vessiot extensions associated with $L_1(y)$ and $L_2(y)$.

   Lemma 5 *((34) p.671-673)*

(a) *There exists a Picard-Vessiot extension M_3 of k containing copies of M_1 and M_2.*

(b) *Let M be a Picard-Vessiot extension of k containing copies of M_1 and M_2 and let V be the $\mathcal{C}$-vector space spanned by $\{\mu_1\mu_2 | L_1(\mu_1) = 0$ and $L_2(\mu_2) = 0\}$. Then V is the solution space of a monic homogeneous linear differential equation $L_3(y) = 0$, with coefficients in k. Furthermore, $L_3(y)$ does not depend on M.*

The operator L_3 is called the **symmetric product** *of L_1 and L_2, denoted $L_1 ⓢ L_2$. Since the operator ⓢ is associative we can define $L^{ⓢn}$ for $n \geq 1$ by $L^{ⓢ1} = L$ and $L^{ⓢn} = L^{ⓢn-1} ⓢ L$.*

Then $L^{ⓢn}$ is called the nth symmetric power of L.

(c) *Let $L(y) = 0$ be a homogeneous linear differential equation with coefficients in a differential field k, with algebraically closed field of constants $\mathcal{C}$.*

- *If $L(y)$ has order n then $L^{ⓢm}$ has order at most $\binom{m+n-1}{n-1}$.*
- *If $L(y)$ has order 2 and $\{y_1, y_2\}$ is a basis for the solution space of $L(y) = 0$, then $\{{y_1}^m, {y_1}^{m-1}y_2, \cdots, {y_2}^m\}$ is a basis for the solution space of $L^{ⓢm}$. In particular $L^{ⓢm}$ will have order precisely $m+1$.*

2. **Exterior powers**

Another type of construction that we will need is the exterior powers.

The mth exterior power of a differential equation L with solutions $\{y_1, \cdots, y_n\}$ is the equation having for its space of solutions the vector space generated by the $m \times m$ wronskians of solutions of L. For a precise definition of the exterior power of an operator and how to construct them we refer to (29) section 2.4.

If V is a G-module for some group G, then G acts on $\bigwedge^m V$ as well and this action is given by $\sigma(y_{i_1} \wedge \cdots \wedge y_{i_m}) = \sigma(y_{i_1}) \wedge \cdots \wedge \sigma(y_{i_m})$.

Proposition 3 *((29) p.112) Let L be a linear differential operator with coefficients in a differential field k and let K be the associated Picard-Vessiot extension with Galois goup G. Let V be the solution space of $L(y) = 0$ in K and let $\bigwedge^m V$ be the m^{th} alternating power of V.*

(a) If $\{y_1, \cdots, y_n\}$ is a basis of V, then the map defined by

$y_{i_1} \wedge \cdots \wedge y_{i_m} \mapsto wr(y_{i_1}, \cdots, y_{i_m})$, *for all* $1 \leq i_1 < \cdots < i_m \leq n$

defines a G-morphism from $\bigwedge^m V$ onto the solution space of $\bigwedge^m(L)y = 0$. Therefore, if the order of $\bigwedge^m(L)$ is $\binom{n}{m}$, $\bigwedge^m V$ is isomorphic to the solution space of $\bigwedge^m(L)y = 0$.

(b) V contains a G-invariant subspace of dimension m if and only if there exist linearly independent elements $v_1, \cdots, v_m \in V$ such that $v_1 \wedge \cdots \wedge v_m$ spans a G-invariant line in $\bigwedge^m V$.

(c) If $\bigwedge^m(L)$ has order $\binom{n}{m}$, then L has a right factor of order m if and only if $\bigwedge^m(L)$ has an exponential solution w so that $w = Wr(z_1, \cdots, z_m)$ for linearly

independent solutions $z_1, \cdots, z_m$ of $L(y) = 0$. If this is the case, then L has a right factor of the form $\tilde{L} = \delta^m + (w'/w)\delta^{m-1} + \cdots$.

Remark 2 *Maple V contains commands in the DEtools package to calculate exterior powers of operators.*

In particular if $L = \delta^4$ then $\bigwedge^2(\delta^4) = \delta^5$ therefore the d^{th} exterior power of an operator of order n can have order less than $\binom{n}{d}$

3.6 Results for second and third order equations

As we pointed out earlier, the key idea in differential Galois theory is to study properties of the Galois group to find results about the corresponding equation. In the following we give examples of how this is done for equations of order 2 and 3.

1. **Second order equations**

   In 1978 Kovacic produced an algorithm for finding Liouvillian solutions of equations of order 2 (cf (27)). Subsequently, in 1979 M.F.Singer gave an algorithm to find Liouvillian solutions of equations of order $n \geq 2$. The following theorems take care of the case of equations of order 2.

Definition 14 *Let G be a subgroup of $Gl_n(\mathcal{C})$ acting irreducibly on the vector space V of dimension n over $\mathcal{C}$. Then G is called* **imprimitive** *if, for $k > 1$, there exist* *subspaces* $V_1, \cdots, V_k$ *such that* $V = V_1 \oplus \cdots \oplus V_k$ *and, for each* $g \in G$*, the mapping* $V_i \to g(V_i)$ *is a permutation of the set* $S = \{V_1, \cdots, V_k\}$*.*

The set S *is called a* **system of imprimitivity of** G*.*

If all the subspaces V_i *are one-dimensional, then* G *is said to be* **monomial***.*

An irreducible group $G \subset Gl_n(\mathcal{C})$ *which is not imprimitive is called* **primitive***.*

Here is a classification of algebraic subgroups of $Sl_2(\mathcal{C})$:

Theorem 4 *((29) p.135) Let* G *be an algebraic subgroup of* $Sl_2(\mathcal{C})$*. Then one of the following four cases can occur:*

(a) G *is reducible*

(b) G *is imprimitive and conjugate to a subgroup of*

$$\mathcal{D} = \{ \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} | c \in \mathcal{C}, c \neq 0\} \cup \{ \begin{bmatrix} 0 & c \\ -c^{-1} & 0 \end{bmatrix} | c \in \mathcal{C}, c \neq 0\}$$

(c) G *is primitive and* $G/\mathcal{Z}(G)$ *is isomorphic to* A_4*,* S_4 *or* A_5 *where* $\mathcal{Z}(G)$ *is the center of* G*.*

(d) $G = Sl_2(\mathcal{C})$*.*

Using this results one can show:

Theorem 5 *((27) p.136) Let k be a differential field of constants K and let $L(y) = y'' + ry$ with $r \in k$. Precisely 4 cases can occur.*

(a) $L(y) = 0$ has a solution y such that $y'/y \in k$.

(b) $L(y) = 0$ has a solution y such that y'/y is algebraic of degree 2 over k and (a) does not hold.

(c) $L(y) = 0$ has a solution y such that y'/y is algebraic of degree 4,6 or 12 over k and (a) and (b) do not hold.

(d) $L(y) = 0$ has no Liouvillian solutions.

2. **Third order equations**

The following proposition shows how to recognize the Galois group of a third order linear differential equation and decide whether it has a Liouvillian solution:

Proposition 4 *((27) p.140) Let $L(y) = 0$ be an irreducible third order linear differential equation with coefficients in a differential field k with algebraically closed field of constants whose differential Galois group $\mathcal{G}(L)$ is unimodular. $L(y) = 0$ has a Liouvillian solution if and only if*

(a) $L^{Ⓢ4}$ has order less that 15 or factors, and

(b) one of the following holds:

i. $L^{Ⓢ2}$ has order 6 and is irreducible, or

ii. $L^{Ⓢ3}$ has a factor of order 4.

Also in (34) M.F.Singer shows how to solve a third order homogeneous linear differential equation using second order linear differential equations. After giving a list of the Galois groups for which this happens he proves the following result:

Theorem 6 *((34) p.680) Let k be a differential field with algebraically closed field of constants. Let $L(y) = y''' - py' - qy$ with $p, q \in k$ and let M be the associated Picard-Vessiot extension of k. Then $L(y) = 0$ can be solved in terms of second order linear differential equations if and only if one of the following holds:*

(a) All solutions of $L(y) = 0$ are algebraic over k.

(b) There exist an extension K of k of degree $1, 2, 3$ or 6 and L_1 and L_2 of order 1 and 2 respectively with coefficients in K such that

$$L(y) = L_2(L_1(y)) \text{ or}$$

$$L(y) = L_1(L_2(y))$$

(c) There exist a subfield K of M with $[K : k] = 1$ or 3, an algebraic extension N of K with $[N : K] = 1$ or 2 and elements a_0, a_1, a_2, b, c in N such that if $\{u, v\}$ is a basis for the solution space of $y'' + by' + cy = 0$, then

$$y_1 = a_0u^2 + a_1(u^2)' + a_2(u^2)''$$

$$y_2 = a_0uv + a_1(uv)' + a_2(uv)''$$

$$y_3 = a_0v^2 + a_1(v^2)' + a_2(v^2)''$$

is a basis for the solution space of $L(y) = 0$.

Chapter 4

Study of the Lie sub-algebras of $\mathbf{sl}_4(\mathcal{C})$

Solving a linear differential equation in terms of equations of order 1 is the same problem as finding its liouvillian solutions. For this one already knows algorithms explained in (38), (37) and (39). As a result we will focus on the solvability in terms of second order equations. So our goal is to decide when the solutions of a 4th order homogeneous linear differential equation can be written using solutions of equations of order 2 or 3, i.e. when a linear differential equation of order 4 can be solved in terms of second or third order linear differential equations.

The fields we work with will always have characteristic zero.

We will assume that the reader is familiar with the definitions of Lie algebras and their representations (see (22) for a good introduction to these notions).

Given a linear algebraic group G over a field k, denote $A = k[G]$ its coordinate field. The action of G on A is given by: $(\lambda_x f)(y) = f(x^{-1}y)$. The vector space $\mathcal{L}(G) = \{\delta \in DerA | \delta\lambda_x = \lambda_x\delta \ \forall x \in G\}$ is called the Lie algebra associated with G, denoted $\mathbf{g}$.

One can see the Lie algebra $\mathbf{g} = \mathcal{L}(G)$ associated with G as its tangent space at identity $\mathcal{T}_e(G) = \mathcal{T}_e(G^0)$.

The exponential map $exp : \mathbf{g} \to G$ is useful for studying the relationship between G and $\mathbf{g}$ ((13) p.114-119).

In particular:

- If $\phi : G \to G'$ is a morphism of algebraic groups, then $d\phi_e : \mathbf{g} \to \mathbf{g}'$ is a linear application between 2 vector spaces.
- A linear algebraic group and its Lie algebra have the same dimension.
- If $G \to Gl(V)$ is a group representation of G, then we have a corresponding representation $\mathcal{L}(G) : \mathbf{g} \to \mathbf{gl}(V)$ of same dimension.

Definition 15 *A Lie algebra* $\mathbf{g}$ *is* **simple** *if it is not abelian and it does not have a non trivial ideal besides* $\mathbf{g}$ *itself.*

A Lie algebra $\mathbf{g}$ *is* **semi-simple** *if its radical (i.e. its maximal solvable ideal)* $Rad(\mathbf{g})$ *is trivial.*

In particular for any Lie algebra $\mathbf{g}$, $\mathbf{g}/Rad(\mathbf{g})$ is semi-simple, and a semi-simple Lie algebra can always be written as the direct sum of simple ones.

One can give a list of the simple Lie algebras up to isomorphisms (see (13) p.131):

- $\mathbf{sl}_n(\mathcal{C}) = \{A \in \mathbf{gl}_n(\mathcal{C}) | trA = 0\}$ with $n \geq 2$, of dimension $n^2 - 1$.

- $\mathbf{sp}_{2n} = \{A \in \mathbf{gl}_{2n}(\mathcal{C}) | A^t J + JA = 0\}$ with $n \geq 2$, where

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

  of dimension $2n^2 + n$.

- $\mathbf{o}_n = \{A \in \mathbf{gl}_n(\mathcal{C}) | A^t + A = 0\}$ with $n \geq 7$.

To which one needs to add exceptional Lie algebras of greater dimension.

The link between the representations of a linear algebraic group and those of its Lie algebra is not necessarily obvious. In the case where G is connected, any quotient of G by a finite group will have $\mathbf{g}$ for its Lie algebra. Also:

Theorem 7 *((23) p.89) Let G be a connected linear algebraic group.*

The group G is semi-simple if and only if its Lie algebra $\mathbf{g}$ is semi-simple.

If G is simple, then $\mathbf{g}$ is simple.

And if G is connected and simply connected there is a one-to-one correspondence between the irreducible representations of G and those of its Lie algebra (cf (18)).

Here is now the Lie algebra version of theorem 3:

Theorem 8 *((32) p.118)*

Let k be a differential field of characteristic zero with algebraically closed field of constants $\mathcal{C}$. Let $L(y) = 0$ be a homogeneous linear differential equation of order n, $n \geq 3$, with coefficients in k and let K be the associated Picard-Vessiot extension. Assume that $G(K/k) \subset Sl_n(\mathcal{C})$. Then $L(y) = 0$ can be solved in terms of equations of lower order iff one of the following holds:

1. $\mathbf{g}(\mathrm{K/k}) \subset \mathbf{sl}_{\mathrm{n}}(\mathcal{C})$ *leaves a nontrivial subspace of $\mathcal{C}^n$ invariant.*

2. $\mathbf{g}(\mathrm{K/k})$ *is semisimple but not simple.*

3. $\mathbf{g}(\mathrm{K/k})$ *is simple and there exists a nonzero Lie algebra homomorphism* $\rho\colon \mathbf{g}(\mathrm{K/k}) \to \mathbf{gl}_{\mathrm{m}}(\mathcal{C})$ *for some $m < n$ (i.e. we can find a non trivial representation of $\mathbf{g}$ of degree strictly less than n).*

Furthermore if 1. holds then there exist homogeneous linear differential equations L_{n-i} and L_i of order $n-i$ and i $(i > 0)$ with coefficients in the algebraic closure k^* of k in K such that $L(y) = L_{n-i}(L_i(y))$.
If 2. or 3. hold, then there exist linear homogeneous differential equations $L_i(y)$, $1 \leq i \leq m$ (with $m = 1$ if 3. holds), each having order less than n and coefficients in an algebraic extension k_0 of k such that K lies in an algebraic extension of $M_1 \cdots M_m$, where M_i is the Picard-Vessiot extension of k_0 associated with $L_i(y) = 0$.

This result is a very powerful tool in deciding theoretically the solvability of differential equations in terms of lower order equations, which we will do now in the case of an equation of order 4, but we also want to give methods to actually solve these equations.

The Lie algebra corresponding to $Sl_4(\mathcal{C})$ is $\mathbf{sl}_4(\mathcal{C})$. According to theorem 8, to solve our problem, we need a list of the irreducible Lie subalgebras of $\mathbf{sl}_4(\mathcal{C})$ (see (23) section 1 and 19). There are 4 simple Lie algebras:

$$\mathbf{sl}_4(\mathcal{C})$$

$$\mathbf{sp}_4(\mathcal{C})$$

$$\mathbf{sl}_3(\mathcal{C})$$

$$\mathbf{sl}_2(\mathcal{C})$$

and one semi-simple Lie algebra:

$$\mathbf{o}_4(\mathcal{C}) = \mathbf{sl}_2(\mathcal{C}) \oplus \mathbf{sl}_2(\mathcal{C})$$

From this list and from (17) p.496 we find 2 irreducible cases that are of interest for our problem : when the Lie algebra is $\mathbf{sl}_2(\mathcal{C})$, corresponding to case 3. of theorem 8, and when it is $\mathbf{sl}_2(\mathcal{C}) \oplus \mathbf{sl}_2(\mathcal{C})$, corresponding to case 2.

Indeed if V denotes the standard representation space for $\mathbf{sl}_2(\mathcal{C})$, then the only four dimensional space on which $\mathbf{sl}_2(\mathcal{C})$ acts irreducibly is Sym^3V. The Lie Algebra $\mathbf{sl}_3(\mathcal{C})$ does not have a representation of order four while the only irreducible representations of $\mathbf{sl}_4(\mathcal{C})$ of dimension four are the standard and the dual one. The only four-dimensional vector space on which the symplectic Lie algebra $\mathbf{sp}_4(\mathcal{C})$ acts irreducibly is also the standard representation space. And finally the only irreducible representation of $\mathbf{o}_4(\mathcal{C})$ is $V \otimes V$.

We will also be interested in the case where the Lie algebra is $\mathbf{sl}_2(\mathcal{C})$ with a reducible representation in $\mathbf{sl}_4(\mathcal{C})$, but irreducible Galois group. The reducible equations and those having liouvillian solutions (see (36)) are contained in case 1.

On the group level one can classify the corresponding subgroups of $Sl_4(\mathcal{C})$ as follows:

Proposition 5 *((17)) Let G be an algebraic subgroup of $Sl_4(\mathcal{C})$, $\mathcal{C}$ an algebraically closed field of characteristic zero, and let G^0 be its component of the identity. One of the following holds:*

1. *G is reducible.*

2. *G is finite.*

3. *G is irreducible but G^0 is reducible, in which case either*

   (a) *G is imprimitive, monomial and has a subgroup $H \subset diag(Sl_4(\mathcal{C}))$ such that G/H is isomorphic to a transitive subgroup of S_4, or*

   (b) *G is imprimitive, non-monomial and has a normal subgroup of index 2, or*

   (c) *G is primitive and G^0 is conjugate to*

$$\{\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in Sl_2(\mathcal{C})\}$$

in which case $G = HG^0$ *with*

$$H = \{ \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \tilde{H}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \}$$

where $\tilde{H}$ *is a finite primitive subgroup of* $G_2 = \{A \in Gl_2(\mathcal{C}) : det(A^2) = 1\}$.

4. G^0 *is irreducible.*

Proof.

1. The first 2 cases that can obviously hold.

2. For the last 2 cases we refer to the classification of the irreducible subgroups of $Sl_4(\mathcal{C})$ given in (17):

   Case 3.(b) is Theorem 2.2.2 of (17) p.505.

   For case 3.(c) we refer to Theorem 2.1.1 p.494 of (17) and the case where G^0 is irreducible is Theorem 2.1.2 p.495 of (17) for which one of the following representation of G holds:

   (a) $G \subset \cup_{\omega^4=1}\omega G^0$, where G^0 is one of

   - $Sl_2(\mathcal{C})$ acting on $Sym^3(\mathcal{C}^2)$
   - $Sp_4(\mathcal{C})$

(b) $G \subset (\cup_{\omega^4=1}\omega G^0) \cup J(\cup_{\omega^4=1}\omega G^0)$, where

$$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and G^0 is the usual representation of $SO_4(\mathcal{C})$.

(c) $G = Sl_4(\mathcal{C})$ acting on W, where W is one of

- the usual representation space, $V \simeq \mathcal{C}^4$
- the dual space, V^*, of V.

□

As theorem 8 suggests, we want to translate these results in terms of fourth order equations using the properties of the groups. Let us give the general situations we can encounter while dealing with the problem of solving a fourth order equation with second order equations.

When the Galois group of an equation is reducible, one knows algorithms to factor it that are implemented in Maple (use DFactor (19)). So the equations we consider from now on will be assumed irreducible over their field of coefficients, i.e. will have an irreducible Galois group.

Theorem 9 *Let k be a differential field with algebraically closed field of constants $\mathcal{C}$. Let $L_4(y) = 0$ be a fourth order differential equation, irreducible over k, unimodular, and let K be the associated Picard-Vessiot extension of k. Let G be the Galois group of L_4 over k and G^0 be its component of the identity. If $L_4(y) = 0$ is solvable in terms of second order linear differential equations then one of the following occurs:*

1. *When G is finite, all the solutions of $L_4(y) = 0$ in K are algebraic over k.*

2. *When G is imprimitive monomial and G^0 is irreducible, there exists a subfield M of K with $[M : k] = 1, 2, 3, 4, 6, 8, 12$ or 24 such that*

$$L_4(y) = L_3(L_1(y))$$

   where L_1 and L_3 are linear differential operators of order one and three respectively with coefficients in M. In this case the equation has a basis of liouvillian solutions.

3. *When G is imprimitive, nonmonomial and G^0 is reducible, $L_4 = [L_2, \tilde{L_2}]_l$ where L_2 and $\tilde{L_2}$ are second order linear differential operators with coefficients in a field M quadratic over k.*

4. *When G is primitive and G^0 reducible, there exist an extension M of k and linear differential operators L_1 and L_2 with coefficients in M such that $L_4 = [L_1, L_2]_l$ and $[M : k] = 1, 2, 3, 4, 6$ or 12.*

5. *When G^0 is irreducible and conjugate to $\rho(Sl_2(\mathcal{C}))$ where ρ is the third symmetric product representation of $Sl_2(\mathcal{C})$. In this case there exist a subfield N of K with*

$[N : k] \leq 2$ and elements a_0, a_1, a_2, a_3 of degree at most 2 over N and b, c in N such that if $\{u, v\}$ is a basis for the solution space of $y'' + by' + cy = 0$, then

$$y_1 = a_0 u^3 + a_1(u^3)' + a_2(u^3)'' + a_3(u^3)'''$$

$$y_2 = a_0 u^2 v + a_1(u^2 v)' + a_2(u^2 v)'' + a_3(u^2 v)'''$$

$$y_3 = a_0 u v^2 + a_1(u v^2)' + a_2(u v^2)'' + a_3(u v^2)'''$$

$$y_4 = a_0 v^3 + a_1(v^3)' + a_2(v^3)'' + a_3(v^3)'''$$

is a basis for the solution space of $L_4(y) = 0$.

6. *When G^0 is irreducible and isomorphic to $Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})/\{\pm(I, I)\}$ in which case there exist a subfield k_0 of K with $[k_0 : k] = 1, 2$, a quadratic extension N of k_0, and elements $a_0, a_1, a_2, a_3, r_1, r_2, s_1, s_2 \in N$ such that if $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are bases for the solution spaces of $y'' + r_1 y' + s_1 y = 0$ and $y'' + r_2 y' + s_2 y = 0$ respectively, then*

$$y_1 = a_0 u_1 v_1 + a_1(u_1 v_1)' + a_2(u_1 v_1)'' + a_3(u_1 v_1)'''$$

$$y_2 = a_0 u_2 v_2 + a_1(u_2 v_2)' + a_2(u_2 v_2)'' + a_3(u_2 v_2)'''$$

$$y_3 = a_0 u_1 v_2 + a_1(u_1 v_2)' + a_2(u_1 v_2)'' + a_3(u_1 v_2)'''$$

$$y_4 = a_0 u_2 v_1 + a_1(u_2 v_1)' + a_2(u_2 v_1)'' + a_3(u_2 v_1)'''$$

is a basis of solutions for $L_4(y) = 0$.

In the following sections we will prove this theorem and show how to perform the necessary computations. Then in the next chapter we give examples of how to use those methods.

4.1 A theorem of Chevalley

In (17) the author gives a way of computing the Galois group of a linear differential equation of order four by looking at its action on modules of the form $\bigwedge^2 V$ and $Sym^m V$ where $V \simeq (\mathcal{C}^*)^4$. Thanks to her work one can decide the nature of the Galois group of a given fourth order linear differential equation and once this is done our work is to solve it in terms of lower order equations when this is possible.

Now let us explain the decision procedure to determine the Galois group using the table below; it is explained p.523-527 of (17) and relies on the following result by Chevalley:

Theorem 10 *((23) p.80) Let G be an algebraic group, H a closed subgroup. Then there is a rational representation $\phi : G \mapsto Gl(V)$ and a one dimensional subspace L of V such that*

$$H = \{x \in G | \phi(x)L = L\}$$

$$\mathbf{h} = \{x \in \mathbf{g} | d\phi(x)L \subset L\}$$

In other words, given any subgroup H of the Galois group, one can find a construction in which H is the stabilizer of a line. As a result studying the action of the subgroups of $Sl_4(\mathcal{C})$ on constructions should enable us differentiate between those subgroups.

In the following examples we will use the following table (table 4 p.532 of (17)) to identify the Galois groups of the equations we construct, i.e. we assume we already know that we have an infinite primitive group G, and all we need to do is figure out the nature of the connected component of the identity G^0.

Table 4.1: Decompositions for the Infinite Primitive Groups

Group	$\bigwedge^2 V$	$Sym^2 V$	$Sym^3 V$	$Sym^4 V$	$Sym^6 V$
$G^0 = Sl_2(\mathcal{C})$	1,5	3,7	4,6,10	1,5,7,9,13	$3,7^2$,9,11, 13,15,19
$G^0 = Sl_4(\mathcal{C})$	6	10	20	35	84
$G^0 = Sp_4(\mathcal{C})$	1,5	10	20	35	84
$G^0 = So_4(\mathcal{C})$	3^2	1,9	4,16	1,9,25	1,9,25,49
$G = {A_4}^{Sl_2} \otimes Sl_2(\mathcal{C})$	3^2	1,9	$4,8^2$	$1,5^2$,9,15	$1,5^2$,7,9, $15,21^2$
$G = {S_4}^{Sl_2} \otimes Sl_2(\mathcal{C})$	3^2	1,9	4,16	1,9,10,15	1,7,9,10, $15,21^2$
$G = {A_5}^{Sl_2} \otimes Sl_2(\mathcal{C})$	3^2	1,9	4,16	1,9,25	1,9,21,25,28

For example, given a linear differential equation of order 4, $L(y) = 0$, with solution space W, we will compute its second exterior power. If the latter has order 6 then its solution space is isomorphic to $\bigwedge^2 W$ and by looking for a rational solution one can decide whether the associated equation has a factor of order one, i.e. whether it has Galois group $Sl_2(\mathcal{C})$ or $Sp_4(\mathcal{C})$. Then to distinguish between the two one needs to compute the second symmetric power of L, find its degree and try to factor it in case the degree is 10 (if it turns out to be 3 or 7 then we have $Sl_2(\mathcal{C})$ for Galois group); if it factors then the Galois group is $Sl_2(\mathcal{C})$, if not it is $Sp_4(\mathcal{C})$.

However most of the time we will be working in an algebraic extension of k and the Galois group will be bigger than its connected component, as a result Hessinger's method to determine the Galois group will force us to work in an algebraic extension which is not always possible in practice, as we notice in the applications with example 5.2.2.

For each coming section we will discuss the nature of the Galois group and say how to find the lower order equations.

4.2 When the Lie Algebra is reducible

Since we assumed that $L_4(y)$ is normalized, the Galois group of K over k is a subgroup of $Sl_4(\mathcal{C})$ and apply proposition 5.

This situation corresponds to case 1. of theorem 8 and cases 2. and 3. of theorem 9. They are direct consequences of the properties of factorization of an equation over an algebraic extension. Concerning case 2. one needs to calculate the liouvillian solutions of the equation. It can be decided thanks to an algorithm given in (21), whether a differential equation has liouvillian solutions. In particular we know we have a liouvillian solution when the Galois group is imprimitive, monomial or when the group is imprimitive, nonmonomial and $G^0 = \{id\}$ (i.e. the blocks from proposition 5 3. (b) correspond to central extensions of finite groups).

Suppose we have shown that $L_4(y) = 0$ has no liouvillian solution.

Then G is imprimitive, nonmonomial, has a normal subgroup H of index two and its fixed field $\check{H}$ is a quadratic extension of k. The group G is described in (17) p.506 and over the quadratic extension $\check{H}$ of k we can always find a decomposition of the solution space V of $L_4(y) = 0$ into the direct sum of two twodimensional vector spaces V_2 and $\tilde{V}_2$. This tells us from proposition 4 that the equation L_4 can reduce over $\check{H}$ as $L_4 = [L_2, \tilde{L}_2]_l$ where L_2 and $\tilde{L}_2$ are differential operators of order two with coefficients in $\check{H}$ and solution spaces V_2 and $\tilde{V}_2$ respectively.

Example 4 *Consider the equation*

$$L(y) = y^{(4)} - \frac{-4x^2+3+8x^3}{x(-4x^2-4x-3+4x^3)}y^{(3)} + \frac{1}{4}\left(\frac{3+44x^3+12x+16x^4-12x^2}{x^2(4x^3-4x^2-4x-3)}\right)y''$$
$$-\frac{4x^2+9+8x}{4x^3-4x^2-4x-3}y' - \frac{4x^4-4x^3-8x^2-11x-9}{4x^3-4x^2-4x-3}y$$

We want to factor L *over a quadratic extension* E *of* $\bar{\mathcal{Q}}(x)$ *and write it as the least common left multiple of 2 equations of order 2 with coefficients in* E*:* $L = [L_2, \tilde{L_2}]_l$*. In order to find* E *one can calculate the second exterior power* $\bigwedge^2 L$ *of* L*, the solutions of which can be seen as the wronskians of solutions of* L*, hence will be in the field of coefficients of* L_2 *and* $\tilde{L}_2$*. Then by factoring* $\bigwedge^2 L$ *one finds a factor of order 4 and two factors of order 1, the solutions of which are 1 and* $\sqrt{x}$*, hence* E *being a quadratic extension of* k *is necessarily* $\mathcal{Q}(x,\sqrt{x})$*. Once we know* E*, replacing* x *by* t^2 *in* L *and using the command DFactorLCLM in Maple tells us that* L *is the least common left multiple of*

$$L_2(y) = 4xy'' - \frac{2(x-\sqrt{x})}{x-1}y' + \frac{2(x^{5/2}-2x^{3/2}+x-\sqrt{x})}{x-1}y$$

and

$$\tilde{L_2}(y) = 4xy'' - \frac{2(x+\sqrt{x})}{x-1}y' - \frac{2(x^{5/2}-2x^{3/2}-x-\sqrt{x})}{x-1}y$$

Recall that a finite Galois group corresponds to algebraic solutions for the equation, which takes care of case 1. in theorem 9.

If $\mathbf{g}$ is irreducible, we may not be able to factor L_4 but we will see other ways to express the solutions of our equation using equations of order two. This is what we will focus on now.

4.3 When G^0 is $Sl_2(\mathcal{C})$

The case we are dealing with here is case 5. in theorem 9 and case 3. in theorem 8, i.e. the case where the Lie algebra is $\mathbf{sl}_2(\mathcal{C})$ with G^0 irreducible.

In the following we will denote by $\mathcal{V}_m$ the space $Sym^m(\mathcal{C}^2)$, i.e. the irreducible $Sl_2(\mathcal{C})$-module of dimension m.

The only simple irreducible subalgebra of $\mathbf{sl}_4(\mathcal{C})$ that has a representation of order less than 4 is $\mathbf{sl}_2(\mathcal{C})$ acting on $\mathcal{V}_3$. Furthermore this symmetric power is, up to isomorphism, the only irreducible representation of degree 4 of $\mathbf{sl}_2(\mathcal{C})$ ((13) p.50).

The group $Sl_2(\mathcal{C})$ acts as a group of linear substitutions on the polynomial ring $\mathcal{C}[X, Y]$ as follows:

Given

$$\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sl_2(\mathcal{C})$$

apply the linear substitution $X \to aX + bY$, $Y \to cX + dY$.

If R_3 is the space of homogeneous polynomials of degree 3 in 2 variables, then R_3 is an irreducible $Sl_2(\mathcal{C})$-module hence isomorphic to $\mathcal{V}_3$.

Consider the map $i_3\colon P^1 \to P^3$ such that $i_3([X, Y]) = [X^3, X^2Y, XY^2, Y^3]$.

Then for $\sigma \in Sl_2(\mathcal{C})$, $i_3(\sigma[X, Y]) = \sigma.(i_3[X, Y])$.

It maps P^1 onto the twisted cubic C_3 defined by the 3 quadrics :

$$\begin{aligned} &Z_0Z_2 - {Z_1}^2 \\ &Z_0Z_3 - Z_1Z_2 \\ &Z_1Z_3 - {Z_2}^2 \end{aligned} \tag{4.1}$$

In this case G is primitive and G^0 is irreducible.

Lemma 6 *((32) p.129, (13) p.154)*

Let $G \subset Sl_4(\mathcal{C})$ be the group of automorphisms that preserve the variety V defined by equations (4.1). Then $G \subset \rho(Sl_2(\mathcal{C})) \cdot H$ where H is the center of $Sl_4(\mathcal{C})$, i.e. a four elements subgroup, and ρ is the map:

$$\rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & a^2d + 2abc & b^2c + 2abd & b^2d \\ ac^2 & bc^2 + 2adc & ad^2 + 2bdc & bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3 \end{bmatrix}$$

i.e. the representation of $Sl_2(\mathcal{C})$ in $Aut(R_3)$.

Proof. Let $\pi \colon \mathcal{C}^4 \to P^3$ be the map identifying lines in $\mathcal{C}^4$ with points in projective 3-space P^3. The variety $\pi(V)$ is the twisted cubic C_3, hence a non-singular conic isomorphic to P^1.

Let H be the group of constant matrices in G. We can consider G/H as acting on $\mathcal{C}_3$ and since $PSl_2(\mathcal{C})$ is the group of automorphisms of P^1 it can also be seen as the group of projective motions of C_3 hence there exists a map ϕ from G into $PSl_2(\mathcal{C})$. Also notice that $Sl_2(\mathcal{C})$ is isomorphic to $\rho(Sl_2(\mathcal{C})) \subset G$ and that $\rho(\pm I) = \pm I \in H$, hence the map ϕ is injective on $\rho(Sl_2(\mathcal{C}))$. Now if $g \in G$ is in the kernel of ϕ, then g must act on V by scalar multiplication. Since V contains a basis of $\mathcal{C}^4$, g must be a scalar matrix. Therefore $G \subset \rho(Sl_2(\mathcal{C})) \cdot H$

□

We will show that an equation having Galois group $G \subset \rho(Sl_2(\mathcal{C})) \cdot H$ where H is the center of $Sl_4(\mathcal{C})$ is equivalent to the third symmetric power of a second order equation by using the fact that its solutions can be sent on the twisted cubic.

First let us consider the special case of an equation that is equal to the symmetric power of a second order linear differential equation:

Proposition 6 *Let $L_4(y) = y^{(4)} + p_3 y^{(3)} + p_2 y'' + p_1 y' + p_0$ be a homogeneous linear differential equation of order 4 with $p_0, p_1, p_2, p_3 \in k$, a differential field with algebraically closed field of constants $\mathcal{C}$.*

1. (a) *There exists a homogeneous linear differential equation $L(y) = 0$ of order 2 with coefficients in some differential extension of k such that $L_4 = L^{\circledS 3}$ iff*

   - $p_1 = \dfrac{{p_3}^3}{36} + \dfrac{7 p_3 {p_3}'}{36} + p_3 \dfrac{p_2 - \frac{11}{36}{p_3}^2 - \frac{2}{3}{p_3}'}{2}$

$$+{p_2}' - \frac{11p_3{p_3}'}{18} - \frac{2}{3}{p_3}'' + \frac{{p_3}''}{6}$$

- $p_0 = 18r^2s + 6r's + 9s^2 + 15rs' + 3s''$

  where $r = \dfrac{p_3}{6}$ *and* $s = \dfrac{p_2 - \frac{11{p_3}^2}{36}\frac{2}{3}{p_3}'}{10}$

in which case $L(y) = y'' + ry' + sy$*, which has coefficients in* k*.*

(b) *If* $p_3 = 0$ *then this holds iff* $p_1 = {p_2}'$ *and* $p_0 = \frac{9}{100}{p_2}^2$*; in particular* $r = 0$

2. *Let* K *be the Picard-Vessiot extension of* k *associated with* $L_4(y) = 0$ *and assume the Galois group of* K *over* k *is a subgroup of* $Sl_4(\mathcal{C})$*. There exists a homogeneous linear differential equation* $L(y) = 0$ *of order 2 with coefficients in* k *such that* $L_4 = L^{Ⓢ3}$ *iff there exists a fundamental system of solutions of* L_4 *lying on the twisted cubic* C_3*.*

Proof. ((32) p.129-130) The first 2 results are due to computations, so let us focus on the third one.

Suppose $L_4 = L^{Ⓢ3}$ and $\{u, v\}$ is a fundamental set of solutions of $L(y) = 0$. Then $\{u^3, u^2v, uv^2, v^3\}$ is a basis for the solution space of $L_4(y) = 0$ lying on C_3.

Conversely suppose a fundamental set of solutions $\{y_1, \cdots, y_4\}$ of $L_4(y) = 0$ lies on C_3 and let M be the Picard-Vessiot extension of k associated to L_4. Let G be the Galois group of M over k that can be considered as a subgroup of $Sl_4(\mathcal{C})$ with respect to this basis. By lemma 6 $G \subset \rho(Sl_2(\mathcal{C})) \cdot H$.

Let $u = \sqrt[3]{y_1}$, $v = \sqrt[3]{y_4}$, and select these cube roots such that $y_3 = uv^2$ and $y_2 = u^2v$ (note that this determines u and v up to a multiplication by the same cube root of unity since the solutions lie on $\mathcal{C}_3$, hence satisfy equations (4.1)). Let $\tilde{M}$ be a universal differential field extension of M, and let $\tilde{\mathcal{C}}$ be its field of constants. Let σ be a k_0-isomorphism from $M < u, v >$ into $\tilde{M}$, where k_0 is the fixed field of G^0. σ will restrict to a k_0-isomorphism of M into $\tilde{M}$ so there exist $a, b, c, d \in \tilde{\mathcal{C}}$ with $ad - bc = 1$ such that $\sigma(u^3) = (au + bv)^3$ and $\sigma(v^3) = (cu + dv)^3$, and so $\sigma(u) = \mu(au + bv)$ and $\sigma(v) = \xi(cu + dv)$ where $\xi^3 = \mu^3 = 1$. Therefore the coefficients of

$$L(y) = \tfrac{Wr(y,u,v)}{Wr(u,v)}$$

are left fixed by any k_0-isomorphism of $M < u, v >$ into $\tilde{M}$ hence $L_4(y)$ is the third symmetric power of a homogeneous second order linear differential equation with coefficients in k_0. Then by case 1. we know that those coefficients are in k. $\square$

Example 5 *Let* $L_4(y) = y^{(4)} - 10xy^{(2)} - 10y' + 9x^2y$. *Since* $p_3 = 0$ *we can check that* $p_1 = {p_2}'$ *and* $p_0 = \frac{9}{100}{p_2}^2$ *to find that* L_4 *is the third symmetric power of the Airy equation* $L(y) = y'' - xy$ *as stated in the introduction.*

But it can happen that the solutions of the equation L_4 do not lie on $\mathcal{C}_3$, in which case one may be able to transform it in order to write the transform as the second symmetric power of an equation of order 2, i.e. have the solutions of the transform lie on $\mathcal{C}_3$.

For our purpose we adapted the result by Michael Singer on third order equations to the case of an equation not necessarily normalized:

Lemma 7 *((34) p.678 proposition 4.2) Let F be a differential field with algebraically closed field of constants $\mathcal{C}$. Let K be the Picard-Vessiot extension of F associated with $L_3(y) = y''' - my'' - py' - qy = 0$ with $m, p, q \in F, q \neq 0$. Assume that there exists a basis y_1, y_2, y_3 of the solutions space of $L_3(y) = 0$ such that with respect to this basis the Galois group G of K over F is the irreducible representation of $SL_2(\mathcal{C})$ of degree 3. Then*

1. *the elements $I_0 = {y_2}^2 - y_3y_1$, $I_1 = (y_2')^2 - y_3'y_1$ and $I_2 = (y_2'')^2 - y_3''y_1''$ lie in F.*

2. *If b_0, b_1, b_2 are any non-zero solutions, algebraic over F of*

$$I_0{b_0}^2 + I_1{b_1}^2 + I_2{b_2}^2 + b_0b_1I_0' + b_1b_2I_1' + \tfrac{1}{q}(I_2' - 2mI_2 - pI_1') = 0$$

then

$$z_1 = b_0y_1 + b_1y_1' + b_2y_1''$$

$$z_2 = b_0y_2 + b_1y_2' + b_2y_2''$$

$$z_3 = b_0y_3 + b_1y_3' + b_2y_3''$$

will be a basis for the solution space of ${L_2}^{ⓢ2}(y) = 0$, where $L_2(y) = 0$ is a homogeneous second order linear differential equation with coefficients in $F(b_0, b_1, b_2)$.

Proposition 7 *Let F be a differential field with an algebraically closed field of constants $\mathcal{C}$ and $\overline{F}$ the algebraic closure of F. Let K be a Picard-Vessiot extension of F associated with a fourth order operator L_4 with coefficients in F and $G = \mathrm{SL}_2(\mathcal{C})$ its Galois group. Assume that the solution space of L_4 is isomorphic to $\mathcal{V}_4$ as a G-module. Then*

1. $\mathrm{Sym}^2(\mathcal{V}_4) = \mathcal{V}_7 \oplus \mathcal{V}_3$. *Therefore, ${L_4}^{ⓢ2}(y)$ has at most one monic factor of order* 3.

2. *If* ${L_4}^{ⓢ2}(y)$ *has order* 7 *then* $L_4(y)$ *is the third symmetric power of a second order differential operator defined over* F*.*

3. *If* ${L_4}^{ⓢ2}(y)$ *does not have order* 7*, then it has a unique monic right factor* L_3 *of order* 3 *which is equivalent over* $\overline{F}$ *to the second symmetric power of a second order operator* L_2 *having Galois group* $\mathrm{SL}_2(\mathcal{C})$ *and defined over* $\overline{F}$*. Furthermore, if* F *is an algebraic extension of* $\mathcal{C}(x)$*, then* L_2 *may be chosen to have coefficents in* F *and the equivalence will be defined over* F *as well. Let* $\overline{L}_2$ *be any second order operator over* E*, an algebraic extension of* F*, with Galois group* $\mathrm{SL}_2(\mathcal{C})$ *whose second symmetric power is equivalent over* E *to* L_3*, then the third symmetric power of* $\overline{L}_2$ *is equivalent to* L_4 *over a field* E_0 *with* $[E_0 : E] \leq 2$*.*

The proof of this result was found by Michael Singer and goes as follows: *Proof.* Conclusion 1. follows from the representation theory of SL_2 . Indeed looking at the character of $\mathrm{Sym}^2(\mathcal{V}_4)$ at an element of $SL_2(\mathcal{C})$ gives us the decomposition $\mathrm{Sym}^2(\mathcal{V}_4) = \mathcal{V}_7 \oplus \mathcal{V}_3$ (cf (17) p.510). Hence the symmetric power of the solution space V of L_4 has unique invariant subspaces of dimension 3 and 7. The invariant subspace of dimension 3 is spanned by the elements $z_0z_2 - z_1^2, z_0z_3 - z_1z_2, z_1z_3 - z_2^2$ for suitable basis elements z_0, z_1, z_2, z_3 in V. Since these vanish, we can apply Proposition 6 and achieve conclusion 2.

If ${L_4}^{ⓢ2}(y)$ has order 10, then representation theory implies that there will be a unique monic factor L_3 of order 3 whose solution space is isomorphic to $\mathcal{V}_3$. The techniques of Proposition 4.2 of (34) apply (see Lemma 7).

For an appropriate basis $\{y_1, y_2, y_3\}$ of the solution space of L_3 we have that the elements $\mathcal{I}_i = (y_2^{(i)})^2 - y_3^{(i)} y_1^{(i)},\ i = 0, 1, 2$ all lie in F. Therefore any b_i as described in Lemma 7. define an equivalence of L_3 and a second symmetric power of a second order operator L_2. Note that if F is an algebraic extension of $\mathcal{C}(x)$ then it is a C_1 field, i.e. any form f with coefficients in F of degree d in n variables, with $n > d$, has a non-trivial zero in F ((14) p.3-4), and so any homogeneous quadratic equation in 3 variables will have a nonzero solution in F. Therefore in this case, the b_i can be chosen to lie in F.

Let H be the Galois group of L_2 and let $\rho : H \to \mathrm{GL}_3(\mathcal{C})$ be the representation of H on the second symmetric power. We know that $\rho(H)$ is the three dimensional irreducible representation of $\mathrm{SL}_2(\mathcal{C})$ so $\mathrm{SL}_2 \subset H \subset \mathrm{GL}_2$. Furthermore the kernel of ρ is $\{\pm I\}$. Therefore $H = \mathrm{SL}_2$.

Now, let $\overline{L}_2$ be an operator as in 3. with coefficients in E, an algebraic extension of F. We shall abuse notation and now let K denote the Picard-Vessiot extension of E associated with L_4. Since the Galois group of L_4 over F is connected it remains the same over E. Furthermore, K will contain the Picard-Vessiot extension of the second symmetric power of $\overline{L}_2$. Let $K_1 \supset E$ be a Picard-Vessiot extension of E associated with $\overline{L}_2$ and $K_1 \cdot K$ the compositum of K_1 and K. We have the following diagram:

$$
\begin{tikzcd}
 & K_1 \cdot K & \\
K \arrow[ur] & & K_1 \arrow[ul] \\
 & K_1 \cap K \arrow[ul] \arrow[ur] & \\
 & E \arrow[u] &
\end{tikzcd}
$$

Since $K_1 \cap K$ contains the Picard-Vessiot extension of the second symmetric power of $\overline{L}_2$, its Galois group contains $\mathrm{PSL}_2(\mathcal{C})$. Therefore $[K_1 : K_1 \cap K] \leq 2$ and since Galois theory tells us that $G(K_1 \cdot K/K_1) \simeq G(K/K \cap K_1)$, we have $[K_1 \cdot K : K] \leq 2$. This implies that the index of $\mathrm{SL}_2(\mathcal{C})$ in the Galois group of $K_1 \cdot K$ is at most 2 and therefore that the algebraic closure E_0 of E in $K_1 \cap K$ is of degree at most 2 over E. The Galois group of $K_1 \cdot K$ over E_0 is $\mathrm{SL}_2(\mathcal{C})$ and the solution space of the third symmetric power of $\overline{L}_2$ and of L_4 are isomorphic as $\mathrm{SL}_2(\mathcal{C})$-modules. Therefore, these operators are equivalent over E_0 by remark 2. $\square$

Corollary 1 *Let k be an algebraic extension of $\mathcal{C}(x)$ and L_4 a fourth order operator with coefficients in k. Assume that the differential Galois group G of L_4 has connected component $G^0 = \mathrm{SL}_2(\mathcal{C})$ and that the solution space of L_4 is isomorphic to $\mathcal{V}_4$ as a G^0-module. Then either*

1. *The second symmetric power of L_4 has order 7 in which case L_4 is a third symmetric*

power of a second order operator over k, or

2. *The second symmetric power of L_4 has a unique monic third order factor over k. This factor is equivalent to the second symmetric power of a second order operator L_2 all defined over E with $[E : k] \leq 2$ and L_4 is equivalent to the third symmetric power of L_2 over an algebraic extension of E of degree at most 2.*

Furthermore, if $G = G^0$, then in conclusion 2. we have $E = k$.

Proof. If G is as above, then we must have that $G = G^0 \cdot H$ where H is the center of $\mathrm{SL}_4(\mathcal{C})$. Let F be the fixed field of G^0, then $[F : k] = |G : G^0| = 2$. We now apply the previous result. $\square$

One can make the above result effective as follows:

- Form the second symmetric power of L_4. If it has order 7, use Proposition 6 to write L_4 as a third symmetric power. If not, factor $L_4^{\circledS 2}$ over k.

- Let L_3 be the third order factor. Use the algorithm of (32) to find an operator L_2 whose second symmetric power is equivalent to L_3. This may be over a field E of degree 2 over k.

- Form $L_2^{\circledS 3}$ and find an equivalence of this operator to L_4. This latter task can be performed the way described in Proposition 1. If we let A be the companion matrix for $L_4(y) = 0$, B be the companion matrix for the third symmetric power of $L_2(v) = 0$, and Z be the change of basis matrix such that $Y = ZV$ then we must find a solution to the differential system $Z' = AZ - ZB$ in some quadratic extension E of k and this latter task is known to be algorithmically solvable.

4.4 When G^0 is $SO_4(\mathcal{C})$

In this section we prove case 6. in theorem 9. We consider an equation L with semi-simple Lie algebra $\mathbf{sl}_2(\mathcal{C}) \oplus \mathbf{sl}_2(\mathcal{C})$ having a representation (the tensor product representation) in $\mathbf{sl}_4(\mathcal{C})$ induced by the following representation of $Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})$ into $Sl_4(\mathcal{C})$:

$$\rho\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = \begin{bmatrix} a_1a_2 & a_1b_2 & b_1a_2 & b_1b_2 \\ a_1c_2 & a_1d_2 & b_1c_2 & b_1d_2 \\ c_1a_2 & c_1b_2 & d_1a_2 & d_1b_2 \\ c_1c_2 & c_1d_2 & d_1c_2 & d_1d_2 \end{bmatrix}$$

with kernel $\{\pm(I,I)\}$. Hence we have an isomorphism of $Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})/\{\pm(I,I)\}$ into $Sl_4(\mathcal{C})$.

This is case 2. of theorem 8, hence we know that an equation associated to such a Lie algebra will be solvable in terms of lower order equations.

We say that a subgroup G of $Sl_4(\mathcal{C})$ induces automorphisms on a projective variety when its quotient by the center of $Sl_4(\mathcal{C})$ is a group of projective motions of that variety. Again let us characterize our group by its action on a quadric:

Lemma 8 *The group G of elements in $PSl_4(\mathcal{C})$ that induce automorphisms of the projective variety $\mathcal{Q} = \{(v_1, v_2, v_3, v_4) | v_1v_2 - v_3v_4 = 0\}$ is isomorphic to*

$Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})/\pm(I,I) \cdot Z_2$ where Z_2 is a cyclic group of order 2.

Proof. Consider the Segre embedding $s : P^1 \times P^1 \longrightarrow P^3$ defined by $s([\mu_1,\mu_2],[\xi_1,\xi_2]) = [\mu_1\xi_1,\mu_2\xi_2,\mu_1\xi_2,\mu_2\xi_1]$. Also let π_1 and π_2 be the first and second projection of $P^1 \times P^1$. Then the quadric $y_1y_2 - y_3y_4 = 0$ in P^3, called the Segre variety $\mathcal{Q}$, can be parametrized in the following way:

$$y_1 = \mu_1\xi_1$$

$$y_2 = \mu_2\xi_2$$

$$y_3 = \mu_1\xi_2$$

$$y_4 = \mu_2\xi_1$$

The quadric $\mathcal{Q}$ is a determinantal variety $\{Z : det \begin{bmatrix} Z_0 & Z_1 \\ Z_2 & Z_3 \end{bmatrix} = 0\}$ of matrices of rank one. By setting both μ_i =constant or both ξ_i =constant we can distinguish 2 families of lines on $\mathcal{Q}$, namely $s({\pi_1}^{-1}(P^1))$ and $s({\pi_2}^{-1}(P^1))$ called the rulings of $\mathcal{Q}$. In other words we can see $[\mu_1,\mu_2]$ as a parameter; when it is fixed $s({\pi_1}^{-1}([\mu_1,\mu_2])) = s([\mu_1,\mu_2],P^1)$ is a linear subspace of $\mathcal{Q}$ and our first ruling is the set of all such linear subspaces.

In fact any line on the quadric is contained in one of those 2 families ((15) pp.113). Indeed we claim that any vector space $W \subset Hom(\mathcal{C}^2,\mathcal{C}^2)$ of matrices of rank 1 (which corresponds to a linear subspace in P^3) has all its elements having either the same image or the same kernel. Suppose for instance the latter holds. This tells us that all matrices in W have the same row space spanned by some (ξ_1,ξ_2). If we see such a matrix as

$$M = \begin{bmatrix} \mu_1\xi_1 & \mu_1\xi_2 \\ \mu_2\xi_1 & \mu_2\xi_2 \end{bmatrix}$$

then we clearly have that $M \in s({\pi_2}^{-1}([\xi_1, \xi_2])$. The same argument can be applied to matrices with same image (i.e., same column space) that will belong to the other ruling.

Let us now prove the claim. Suppose that A and B in W have different kernel and different image. Then there exist $(v, w) \in Ker(A) \times Ker(B)$ such that $v \notin Ker(B)$ and $w \notin Ker(A)$. This implies that $B(v)$ and $A(w)$ are independent since otherwise $B(v) \in Im(A) \cap Im(B)$ and the rank being 1 gives a contradiction (distinct images). But then any linear combination $L_{\alpha\beta} = \alpha A + \beta B$ with $\alpha\beta \neq 0$ is of rank 2, a contradiction with the fact that W is a vector space. Hence either A and B have same kernel and any line $L_{\alpha,\beta} = \alpha A + \beta B$ has the same kernel, or they have same image and so does $L_{\alpha,\beta}$. This implies that either all elements of W have same kernel, or they all have same image ; indeed given any three elements A, B, C in W, two pairs will have the same kernel or two pairs will have the same image, hence this holds for all three elements.

As a result any line on $\mathcal{Q}$ belongs to either one of our 2 families.

We now consider a projective transformation of the quadric $\mathcal{Q}$ and look how it operates on a line.

The image of a family of lines will again be a family of lines, hence a transformation of the quadric either exchanges the two families of lines, or leaves them fixed.

Suppose we are in the latter case. Then we can see such a transformation as acting on the copies (μ_1, μ_2) and (ξ_1, ξ_2) separately in the following way:

$$\tilde{\mu_i} = \sum a_{ij}\mu_j$$

$$\tilde{\xi_k} = \sum b_{kl}\xi_l$$

where $A = (a_{ij})$ and $B = (b_{kl})$ are elements of $Sl_2(\mathcal{C})$.

Conversely given 2 matrices of $Sl_2(\mathcal{C})$, they act on (μ_1, μ_2) and (ξ_1, ξ_2) to give us a linear transformation of the points of the quadric:

$$\tilde{\mu_i}\tilde{\xi_k} = \sum a_{ij} b_{kl} \mu_j \xi_l$$

that we can extend to a projective transformation by setting

$$y_1 = z_{11} \; y_2 = z_{22}$$

$$y_3 = z_{12} \; y_4 = z_{21}$$

and considering $\tilde{z_{ik}} = \sum a_{ij} b_{kl} z_{jl}$.

Hence we have shown that the group of projective transformations of P^3 that leave $\mathcal{Q}$ fixed but do not exchange the 2 families of lines on the quadric $\mathcal{Q}$ is the image of the direct product $Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C}) \subset Sl_4(\mathcal{C})$, hence isomorphic to $Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})/\{\pm(I, I)\}$. Now the action of exchanging the two families of lines is just a transposition, hence generates a cyclic group of order two, which gives us the conclusion of the lemma. □

Corollary 2 *The group G of elements in $Sl_4(\mathcal{C})$ that induce automorphisms of the variety $\tilde{\mathcal{Q}} = \{(v_1, v_2, v_3, v_4) | v_1 v_2 - v_3 v_4 = 0\}$ is isomorphic to $Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})/\{\pm(I, I)\} \cdot Z_2 \cdot H$ where Z_2 is a cyclic group of order 2 and H is the center of $Sl_4(\mathcal{C})$.*

For further references on Segre embeddings and determinantal varieties, we refer to (15) p.25-27 and p.98-113.

Remark 3 *In the above proof note that the action of* $Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})$ *on the variety defined by* $y_1y_2 - y_3y_4 = 0$ *induces an action on the corresponding projective variety in the following way: The first copy of* $Sl_2(\mathcal{C})$ *will act on the pairs* $[\frac{y_1}{y_4}, 1] = [\frac{\mu_1}{\mu_2}, 1]$ *while the second copy acts on* $[1, \frac{y_1}{y_3}] = [1, \frac{\xi_1}{\xi_2}]$.

We will use lemma 8 to locate the solutions of our equation. The following propositions deal with the special case where L is equal to the symmetric product of two equations of order two, which can happen when the solutions lie on the quadric $\mathcal{Q}$. First let us explain what happens when the two equations are normalized:

Proposition 8 *Let* k *be a differential field with algebraically closed field of constants* $\mathcal{C}$ *and let* $L(y) = y^{(4)} + p_3y^{(3)} + p_2y^{(2)} + p_1y' + p_0$ *with* p_0, p_1, p_2, p_3 *in* k *be unimodular. If* $L = L_1 ⓢ L_2$ *with* $L_1(y) = y'' + ry$ *and* $L_2(y) = y'' + sy$ *then*

$$\frac{3}{2}{p_2}' + p_2p_3 = p_1 \tag{4.2}$$

$$l\delta\left(\frac{-1}{2}{p_2}'' + p_0 - p_3\frac{{p_2}'}{2}\right) = -2p_3 \tag{4.3}$$

where $l\delta$ *stands for the logarithmic derivative and*

$$\begin{aligned} r(x) &= \tfrac{p_2}{4} + \tfrac{1}{2}(\tfrac{-{p_2}'}{2} + p_0 - p_3\tfrac{{p_2}'}{2})^{1/2} \\ s(x) &= \tfrac{p_2}{4} - \tfrac{1}{2}(\tfrac{-{p_2}'}{2} + p_0 - p_3\tfrac{{p_2}'}{2})^{1/2} \end{aligned}$$

(note that L_1 and L_2 have a symmetric role) .

If $p_3 = 0$, the above formulas become $\frac{3}{2}{p_2}' = p_1$ *and* $l\delta\left(\frac{-1}{2}{p_2}'' + p_0\right) = 0$.

Proof. Suppose $L_1(y) = y'' + ry$ and $L_2(y) = y'' + sy$. Then $L_1 \circledS L_2 = y^{(4)} - \frac{r'-s'}{r-s}y^{(3)} + 2(r+s)y^{(2)} + \{3(r'+s') - \frac{2(r+s)(r'-s')}{r-s}\}y' - \{\frac{r'^2-s'^2}{r-s} - (r-s)^2 - (r''+s'')\}y = 0$ and by equating the coefficients with those of $L(y)$ we find the given relations (corresponding to first check and second check in (20)). □

If one can write $L = L_1 \circledS L_2$ with L_1 or L_2 not normalized we use:

Lemma 9 *Let k be a differential field with algebraically closed field of constants $\mathcal{C}$, and let $K = k(Sl_2(\mathcal{C}))$ be the function field of $Sl_2(\mathcal{C})$. Let $f \in K$ such that $\forall \sigma \in Sl_2(\mathcal{C})$, $\sigma(f) = \frac{af+b}{cf+d}$, with $a, b, c, d \in k$. Then there exist $u, v \in K$ such that $f = \frac{u}{v}$ with $\sigma(u) = au + bv$ and $\sigma(v) = cu + dv$.*

Proof. Suppose first that k is algebraically closed.

It follows from Corollary 4.5 of (11) that if G is a semi-simple, simply connected group, then its coordinate ring $k[G]$, where k is algebraically closed, is a unique factorisation domain.

We can apply this to $Sl_2(\mathcal{C})$ and write $f = \frac{u}{v}$ with u and v relatively prime in $k[Sl_2(\mathcal{C})]$.

Then we have for any $\sigma \in Sl_2(\mathcal{C})$:

$$\frac{\sigma(u)}{\sigma(v)} = \sigma(f) = \frac{a(\frac{u}{v}) + b}{c(\frac{u}{v}) + d} = \frac{au + bv}{cu + dv}$$

Note that since $gcd(u, v) = 1$ we also have $\forall \sigma \in Sl_2(\mathcal{C})$ as above, $gcd(\sigma(u), \sigma(v)) = 1$, otherwise $\sigma(u) = \lambda f$ and $\sigma(v) = \lambda g$ implies $u = (\sigma)^{(-1)}(\lambda)(\sigma)^{(-1)}(f)$ and $v = (\sigma)^{(-1)}(\lambda)(\sigma)^{(-1)}(g)$ hence, $k[Sl_2(\mathcal{C})]$ is a unique factorisation domain, $(\sigma)^{(-1)}(\lambda)$ is invertible and so is λ. Also $gcd(au + bv, cu + dv) = 1$. Indeed if $\lambda | au + bv$ and $\lambda | cu + dv$, since σ is invertible we have $u = A(au + bv) + B(cu + dv)$, $v = C(au + bv) + D(cu + dv)$ and so λ divides v and u.

Hence since

$$\sigma(u)(cu + dv) = \sigma(v)(au + bv)$$

we necessarily have

$$\sigma(u) = \gamma_\sigma(au + bv)$$

$$\sigma(v) = \gamma_\sigma(cu + dv)$$

for some $\gamma_\sigma \in k[Sl_2(\mathcal{C})]$. Since $\sigma(u)$ and $\sigma(v)$ are relatively prime, any common divisor is a unit in $k[Sl_2(\mathcal{C})]$. In other words γ_σ is invertible in $k[Sl_2(\mathcal{C})]$, and using a result found in (31) it must be of the form $\gamma_\sigma = c.\chi$ with $c \in k$ and χ a character on $Sl_2(\mathcal{C})$, hence $\chi = 1$. We can conclude that $\gamma_\sigma \in k^*$. Therefore γ_σ is independent of σ. Since $\gamma_{Id} = 1$, we have γ_σ is the element 1.

Now if k is not algebraically closed Magid pointed to us that one can still show that $k[Sl_2(\mathcal{C})]$ is factorial using Picard group consideration. Indeed factorial is equivalent to the triviality of the Picard group in this case and if E is the algebraic closure of k, then $E[Sl_2(\mathcal{C})] = E \otimes_k k[Sl_2(\mathcal{C})]$ is a Galois commutative ring extension of $k[Sl_2(\mathcal{C})]$ with Galois group $\Gamma = Gal(E/k)$. Since E is algebraically closed, $Pic(E[Sl_2(\mathcal{C})]) = 0$ and there is an isomorphism $H^1(\Gamma, Units(E[Sl_2(\mathcal{C})])) \to Pic(k[Sl_2(\mathcal{C})])$ ((8), Cor 5.5). Following (31) we know $Units(E[G])$ are constant multiples of characters, hence constants, and so $H^1(\Gamma, Units(E[Sl_2(\mathcal{C})])) = H^1(\Gamma, Units(E))$ and the latter is trivial by Hilbert's theorem 90. So $Pic(k[Sl_2(\mathcal{C})]) = 0$ and so the result still holds.

□

This result unables us to prove:

Proposition 9 *Let k be a differential field with algebraically closed field of constants $\mathcal{C}$ and let $L(y) = y^{(4)} + p_3y^{(3)} + p_2y^{(2)} + p_1y' + p_0y$ with p_0, p_1, p_2, p_3 in k be unimodular.*

1. *If L is equal to $L_1 \circledS L_2$ for some second order homogeneous linear differential equations L_1 and L_2 with coefficients in k with L_1 and L_2 not equivalent to each other over any quadratic extension of k and each having Galois group $Sl_2(\mathcal{C})$, then the Galois group of L is $Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})/\pm(I, I)$.*

2. *If k is an algebraic extension of $\mathcal{C}(x)$ and there exists a basis $y_1, \cdots, y_4$ of the solution space of L such that $y_1y_2 - y_3y_4 = 0$ and such that the corresponding representation of the Galois group is exactly $Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})/\pm(I, I) \cdot Z_2$ where Z_2 is a cyclic group of order 2, then $L = L_1 \circledS L_2$ where L_1 and L_2 have coefficients in an extension of degree at most 4 over k and both have $Sl_2(\mathcal{C})$ as their Galois group.*

Proof.

1. Suppose $L = L_1 \circledS L_2$ with L_1, L_2 not equivalent and each having Galois group $Sl_2(\mathcal{C})$.

   The solution space of the equation $[L_1, L_2]$ is of the form $V_1 \oplus V_2$ by lemma 4, hence the Galois group of the Picard-Vessiot extension K for the equation $[L_1, L_2]_l$ is then a subgroup of $G = Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})$. We claim that this Galois group is not a proper subgroup. If it were, then one could apply Kolchin's theorem (see (26) p.1152 and especially the case we are dealing with on p.1155) and conclude that L_1 would be equivalent to L_2.

   Let $F \subset K$ be the Picard-Vessiot extension for the symmetric product $L_1 \circledS L_2$ and let $y_1 z_1$, $y_1 z_2$, $y_2 z_1$, $y_2 z_2$ be the basis for the solution space. Note that $y_1 z_1 / y_1 z_2 = z_1/z_2 \in F$ and $Wr(z_1, z_2)/(z_2^2) = (z_1/z_2)' \in F$. Since $Wr(z_1, z_2) \in k$ this implies that $z_2^2 \in F$ and the degree of K over F is 2. Since $\pm(I, I)$ leave F fixed, we have that the Galois group of F over k is as claimed.

2. Let k_0 be the fixed field of Z_2.

   Then $\mathcal{G}(M/k_0)$ is connected and equal to $Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})/\{\pm(I, I)\}$. Let $G = Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})$. The field $k_0(G)$ is a quadratic extension of $k_0(\mathcal{G}(M/k_0))$, hence the derivation on $k_0(\mathcal{G}(M/k_0))$ extends uniquely to a derivation on $k_0(G)$.

   Let us show now that the group action commutes with this derivation, i.e. for every $g \in G$ and every $z \in k_0(G)$, $(g \cdot z)' = g \cdot (z')$. To see this first note that $k_0(G) = k_0(\mathcal{G}(M/k_0))(\sqrt{f})$ for some $f \in k_0(\mathcal{G}(M/k_0))$. We already know that $(g \cdot z)' = g \cdot (z')$ for all $g \in G$ and all $z \in k_0(\mathcal{G}(M/k_0))$, and since the group action is distributive over the multiplication (it is an homomorphism) we have:

$$(g \cdot f)' = (g \cdot \sqrt{f}^2)' = [(g \cdot \sqrt{f})^2]' = 2(g \cdot \sqrt{f})(g \cdot \sqrt{f})'$$
$$(g \cdot f)' = g \cdot f' = g \cdot [(\sqrt{f})^2]' = g \cdot [2\sqrt{f}(\sqrt{f})'] = 2(g \cdot \sqrt{f})(g \cdot (\sqrt{f})')$$

therefore $(g \cdot \sqrt{f})' = g \cdot (\sqrt{f})'$ and $k_0(G)$ is a Picard-Vessiot extension of k_0 since every $g \in G$ acts as a differential automorphism of $k_0(G)$.

As underlined in remark 3, the group G acts on $[\frac{y_1}{y_4}, 1]$ as $Sl_2(\mathcal{C})$. Similarly the second copy of $Sl_2(\mathcal{C})$ in G will act on $[1, \frac{y_1}{y_3}]$. Hence we can see $\frac{y_1}{y_4} = f_1$ and $\frac{y_1}{y_3} = f_2$ as elements of $k_0(Sl_2(\mathcal{C}) \times \{1\}) \simeq k_0(Sl_2(\mathcal{C}))$ and $k_0(\{1\} \times Sl_2(\mathcal{C})) \simeq k(Sl_2(\mathcal{C}))$ to apply lemma 9 and find $u_1, u_2 \in k_0[Sl_2(\mathcal{C}) \times \{1\}]$ and $v_1, v_2 \in k_0[\{1\} \times Sl_2(\mathcal{C})]$ such that $f_1 = \frac{u_1}{u_2}$ and $f_2 = \frac{v_1}{v_2}$.

The functions u_1, u_2, v_1, v_2 live in $k_0(G)$, quadratic extension of $k_0(G/H)$ and let λ be such that:

$$\frac{y_1}{u_1 v_1} = \frac{y_2}{u_2 v_2} = \frac{y_3}{u_1 v_2} = \frac{y_4}{u_2 v_1} = \frac{1}{\lambda}.$$

Now let

$$\begin{aligned} \tilde{L_1}(u) &= \frac{Wr(u, u_1, u_2)}{Wr(u_1, u_2)} = u'' + a_1(x)u' + b_1(x)u \\ \tilde{L_2}(v) &= \frac{Wr(v, v_1, v_2)}{Wr(v_1, v_2)} = v'' + a_2(x)v' + b_2(x)v \end{aligned} \tag{4.4}$$

Notice that the Galois group of $\tilde{L_1}$ and $\tilde{L_2}$ is $Sl_2(\mathcal{C})$, meaning those two equations are unimodular.

Let also $\tilde{L} = \tilde{L_1} ⓢ \tilde{L_2} = Dx^4 + \tilde{p_3} Dx^3 + \cdots$, with coefficients in some extension F of degree at most 2 over k_0 (the Galois group acts on $(< u_1, u_2 >, < v_1, v_2 >)$ the way it acts on $([\frac{y_1}{y_4}, 1], [1, \frac{y_1}{y_3}])$, either leaving the pairs fixed or exchanging them). Then the solutions of $L(y) = 0$ multiplied by λ are equal to the solutions of $\tilde{L}(y) = 0$, hence the change of variable $y \to \lambda y$ in L should give us $\tilde{L}$. Conversely the change of variables $y \to \frac{y}{\lambda}$ in $\tilde{L}$ gives us L; in particular if we equate the coefficients we find that the logarithmic derivative of λ is in F $(l\delta(\lambda) = \frac{1}{4}(p_3 - \tilde{p_3}))$.

So suppose $\lambda(x) = \exp \int \alpha(x)dx$ with $\alpha(x)$ in F.

Then we get the following system of solutions for L

$$u_1 e^{-\int \frac{\alpha(x)}{2} dx} v_1 e^{-\int \frac{\alpha(x)}{2} dx}$$

$$u_1 e^{-\int \frac{\alpha(x)}{2} dx} v_2 e^{-\int \frac{\alpha(x)}{2} dx}$$

$$u_2 e^{-\int \frac{\alpha(x)}{2} dx} v_1 e^{-\int \frac{\alpha(x)}{2} dx}$$

$$u_2 e^{-\int \frac{\alpha(x)}{2} dx} v_2 e^{-\int \frac{\alpha(x)}{2} dx}$$

and if we make the change of variables

$$u \to e^{\int -\frac{\alpha(x)}{2} dx} u \text{ and } v \to e^{\int -\frac{\alpha(x)}{2} dx} v$$

in equations (4.4) (calling them L_1 and L_2 after this operation), then $L = L_1 ⓢ L_2$ is a symmetric product with equations L_1 and L_2 having coefficients in F with $[F : k] \leq 4$.

$$\begin{array}{ccc}
k_0(G) & & \\
\cup & & \\
k_0(G/H) = K & \longleftrightarrow & \{id\} \\
\cup & & \cap \\
k_0 & \longleftrightarrow & Gal(K/k_0) \\
\cup & & \cap \\
k & \longleftrightarrow & Gal(K/k) \simeq (Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})/\pm(I,I)) \times Z_2
\end{array}$$

□

Lemma 10 *((20) p.2) Let L be a fourth order operator with coefficients in $\mathcal{C}(x)$ such that $L = L_1 Ⓢ L_2$ where L_1 and L_2 have coefficients in $\mathcal{C}(x)$. Then one can assume that $L_1 = \delta^2 + a\delta + b + \sqrt{c}$ and $L_2 = \delta^2 + a\delta + b - \sqrt{c}$ for some $a, b, c \in \mathcal{C}(x)$.*

Proof. Let $L_1 = (\delta^2 + a_1\delta + b_1)$ and $L_2 = (\delta^2 + a_2\delta + b_2)$.

Recall that taking $(\delta - d)Ⓢ(\delta^4 + p_3\delta^3 + \cdots) = \delta^4 + (p_3 - 4d)\delta^3 + \cdots$ is equivalent to multiplying the solutions of $L(y) = 0$ by $e^{\int d}$ and enables us to eliminate the term in δ^3 in L without changing the solvability in terms of lower order equations (take $d = \frac{p_3}{4}$). Now suppose $L = L_1 Ⓢ L_2$ and let us multiply the solutions of $L_1(y) = 0$ by $e^{\int d}$ while multiplying those of $L_2(y) = 0$ by $e^{-\int d}$; again we have $L = (L_1 Ⓢ (\delta - d)) Ⓢ (L_2 Ⓢ (\delta + d))$. Then for $d = \frac{a_2 - a_1}{4}$ we get $L = (\delta^2 + a\delta + B_1) Ⓢ (\delta^2 + a\delta + B_2)$ where $a = \frac{a_1 + a_2}{2}$ and by taking $b = \frac{B_1 + B_2}{2}$ and $c = (\frac{B_1 - B_2}{2})^2$ we get $L = (\delta^2 + a\delta + b + \sqrt{c}) Ⓢ (\delta^2 + a\delta + b - \sqrt{c}) = L_1 Ⓢ L_2$. □

Here is how one can make this effective:

Suppose we are given an equation $L(y) = y^{(4)} + p_3 y^{(3)} + p_2 y'' + p_1 y' + p_0 y = 0$ with Galois group $Sl_2(\mathcal{C})\times Sl_2(\mathcal{C})/\pm(I,I)\cdot Z_2$ such as in proposition 9. We know that L can be written as the symmetric product of two equations of order 2 : $L=(\delta^2+a\delta+B_1)ⓢ(\delta^2+a\delta+B_2)$, and we want to find those using proposition 8.

Notice that $(\delta-a)ⓢ(L_1ⓢL_2)=((\delta-a/2)ⓢL_1)ⓢ((\delta-a/2)ⓢL_2)$ and given L, let $\tilde{L}=Lⓢ(\delta-a)$. If L can be written as the symmetric product $L=(\delta^2+a\delta+B_1)ⓢ(\delta^2+a\delta+B_2)$, then this new equation $\tilde{L}$ can be written as the symmetric product of 2 equations without second term, hence verifies proposition 8. So all we need to do is find a.

Let $a=\frac{1}{4}p_3-\frac{i'(x)}{2i(x)}$ with i undetermined. Mark Van Hoeij shows in (20) that using equation (4.3) of proposition 9 applied to $\tilde{L}$ (i.e. the coefficients of $\tilde{L}$ obtained from L using the above a should verify equation (4)), one finds the following third order linear differential equation L_3 of which i should be a solution with $i^4\in\mathcal{C}(x)$:

$$L_3=20\delta^3+(8p_2-12p_3{}'-3p_3^2)\delta+12p_2{}'-8p_1+4p_2p_3-10p_3{}''-p_3^3-9p_3p_3{}'$$

Then for each possible i one can check whether equation (4.2) of proposition 9 holds and use the corresponding formulas to find $r(x)$ and $s(x)$, hence $\tilde{L_1}$ and $\tilde{L_2}$. The last step consists of replacing $\tilde{L_1}$ and $\tilde{L_2}$ by $L_1=\tilde{L_1}ⓢ(\delta+\frac{a}{2})$ and $L_2=\tilde{L_2}ⓢ(\delta+\frac{a}{2})$.

Notice that $\tilde{L_1}$ and $\tilde{L_2}$ both have no term in y', hence their Galois group is $Sl_2(\mathcal{C})$. Therefore it will also be the case for L_1 and L_2 since $a\in k$.

It can happen that an equation is only equivalent to a symmetric product, but not equal to one, in which case one needs to transform it before applying proposition 9, i.e. find a gauge transformation that sends the solutions on $\mathcal{Q}$:

Proposition 10 *Let k be a differential extension of $\mathcal{C}(x)$ and $L_4(y) = y^{(4)} - p_3y^{(3)} - p_2y'' - p_1y' - p_0y = 0$ with $p_0, p_1, p_2, p_3 \in k$ be unimodular with Picard-Vessiot extension K. Assume there exists a basis y_1, y_2, y_3, y_4 of the solution space of $L_4(y) = 0$ such that with respect to this basis the Galois group G of K over k is $Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})/\pm(I, I) \cdot Z_2$ acting irreducibly. Then*

1. *The elements*

$$I_i = {y_1}^{(i)}{y_2}^{(i)} - {y_3}^{(i)}{y_4}^{(i)}, \text{ where } i = 0, \cdots, 3$$

   lie in k.

2. *If b_0, b_1, b_2, b_3 are nonzero solutions of a certain quadratic polynomial with coefficients in k, then*

$$z_1 = b_0y_1 + b_1{y_1}' + b_2{y_1}'' + b_3{y_1}^{(3)}$$
$$z_2 = b_0y_2 + b_1{y_2}' + b_2{y_2}'' + b_3{y_2}^{(3)}$$
$$z_3 = b_0y_3 + b_1{y_3}' + b_2{y_3}'' + b_3{y_3}^{(3)}$$
$$z_4 = b_0y_4 + b_1{y_4}' + b_2{y_4}'' + b_3{y_4}^{(3)}$$

   will be a basis for the solution space of $L_2 \circledS \tilde{L_2}(y) = 0$ where L_2 and $\tilde{L_2}$ are homogeneous second order linear differential operators with coefficients in $k(b_0, b_1, b_2, b_3)$.

Proof.

1. Let $\sigma \in G$ and consider its action on the solutions of $L_4(y) = 0$: $\sigma(y_i) = \sum_{j=1}^{4} a_{ij}y_j$. Clearly the group G acts on the derivatives of the y_i's the same way and since it leaves I_0 fixed it leaves the I_i's fixed for $i = 0, \cdots, 3$, hence those lie in k.

2. Let

$$z_i = b_0 y_i + b_1 y_i{}' + b_2 y_i{}'' + b_3 y_i{}^{(3)}$$

and expand the expression $z_1 z_2 - z_3 z_4$. We get the quadratic polynomial

$$b_0{}^2 I_0 + b_1{}^2 I_1 + b_2{}^2 I_2 + b_3{}^2 I_3 + b_0 b_1 I_0{}' + b_1 b_2 I_1{}' + b_2 b_3 I_2{}' + b_0 b_2 A + b_0 b_3 B + b_1 b_3 C$$

where

$$A = \tfrac{1}{p_0}(I_2{}'' - 2I_3 - p_3 I_2{}' - 2p_2 I_2 - p_1 I_1{}'),\ B = A' - I_1{}' \text{ and}$$
$$C = \tfrac{1}{p_1}(I_3{}' - 2p_3 I_3 - p_2 I_2{}' - p_0 B)$$

with coefficients involving the I_i's and the coefficients of L_4, all lying in k.

Since k is a C_1 field there exists (b_0, b_1, b_2, b_3) a nonzero solution of this polynomial, hence the z_i's lie on the quadric defined by the equation $z_1 z_2 - z_3 z_4 = 0$. We need to show that these are linearly independent.

Suppose there exist $(c_1, c_2, c_3, c_4) \in \mathcal{C}^4 \backslash \{(0,0,0,0)\}$ such that

$$0 = c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_4.$$

Then $c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$ is a nonzero common solution for $L_4(y) = 0$ and $\tilde{L_4}(y) = b_3 y^{(3)} + b_2 y'' + b_1 y' + b_0 y = 0$, a contradiction since L_4 is irreducible.

Now consider the equation

$$L(y) = \frac{Wr(y, z_1, z_2, z_3, z_4)}{Wr(z_1, z_2, z_3, z_4)}$$

It has coefficients in $k(b_0, b_1, b_2, b_3)$ since G acts linearly on the span of $\{z_1, z_2, z_3, z_4\}$ and we can apply proposition 9 to conclude that L is the symmetric product of two second order equations.

□

So in case $G = Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})/\pm(I, I) \cdot Z_2$, we will apply proposition 10 to find a solution b_0, b_1, b_2, b_3 in a quadratic extension N of k_0 where k_0 is the fixed field of G^0, this time at most quadratic over k. Using Cramer's rule we can find $a_0, a_1, a_2, a_3 \in N$ such that $y_i = a_0 z_i + a_1 z_i{}' + a_2 z_i{}'' + a_3 z_i{}^{(3)}$ for $i = 1, \cdots 4$.

Notice that if k is not an algebraic extension of $\mathcal{C}(x)$ then there is an algebraic extension F of k for which the above result holds but we do not know how to bound the degree of F over k.

Mark Van Hoeij's procedure:

Finally let us recall how Mark Van Hoeij decides whether an equation is the symmetric product of two equations of order 2.

The procedure is the following: ((20))

Input:$L(y) = y^{(4)} + A_3 y^{(3)} + A_2 y^{(2)} + A_1 y' + A_0 y$ with $A_0, A_1, A_2, A_3 \in \mathbb{C}(x)$. Normalize L and write $L(y) = y^{(4)} + p_2 y^{(2)} + p_1 y' + p_0 y$.

Output:$a, b, c \in \mathbb{C}(x)$ such that $L = (\delta^2 + a\delta + b + \sqrt{c}) \circledS (\delta^2 + a\delta + b - \sqrt{c})$.

Step 1: Compute $L_3 = \frac{5}{2}\delta^3 + p_2\delta + \frac{3}{2}p_2{}' - p_1$.

Step 2: Compute all solutions i of L_3 whose fourth power is in $\mathbb{C}(x)$.

Step 3: For each non zero i compute $\frac{i'}{i}$ and test if $(\delta + \frac{1}{2}\frac{i'}{i}) \circledS L$ satisfies equation 4.2 of proposition 9.

Step 5: If so find r and s as in proposition 9 using the coefficients of $(\delta + \frac{1}{2}\frac{i'}{i}) \circledS L$ and replace them by $(\delta - \frac{1}{4}\frac{i'}{i}) \circledS L_1$ and $(\delta - \frac{1}{4}\frac{i'}{i}) \circledS L_2$ which is the output.

Remark 4 *Mark Van Hoeij's procedure decides if $L = L_1 \circledS L_2$ for $L, L_1, L_2 \in \mathbb{C}(x)[\delta]$ without knowing the nature of the Galois group. In our method we consider the case where we know the Galois group is $G = Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})/(\pm(I, I)) \cdot Z_2 \cdot H$, in which L can be written as $L = L_1 \circledS L_2$ with $L_1, L_2 \in k[\delta]$ and $[k : \mathcal{C}(x)] \leq 8$. Also Mark Van Hoeij's procedure has to be adapted to $\mathcal{C}\bar{(}x)$.*

4.5 When G is of the form $H \otimes Sl_2(\mathcal{C})$ with H a finite primitive subgroup of $Sl_2(\mathcal{C})$

We consider the case of a fourth order equation $L_4(y) = y^{(4)} + p_3 y^{(3)} + p_2 y'' + p_1 y' + p_0 y$ with primitive Galois group $G \subset Sl_4(\mathcal{C})$ and reducible component of the identity G^0, which corresponds to case 4. in theorem 9. The corresponding Lie algebra is still $\mathbf{sl}_2(\mathcal{C})$ but this time we are looking at a reducible representation, precisely the direct sum of two two-dimensional spaces $\mathcal{C}^2 \otimes \mathcal{C}^2 = V_1 \oplus V_2$. Therefore we are dealing with case 1. of theorem 8 and L will factor over the algebraic closure of k as the following theorem states:

Theorem 11 *Let k be a differential field and let L be a linear differential operator with coefficients in k such that the Galois group G of L over k has the following representation:*

G^0 is conjugate to

$$\left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in Sl_2(\mathcal{C}) \right\}$$

and $G = H \cdot G^0$ where H is the finite group

$$H = \left\{ \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \tilde{H}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

with $\tilde{H}$ *a finite primitive subgroup of* $G_2 = \{A \in Gl_2(\mathcal{C}) : detA^2 = 1\}$.

Then there exist L_1 *and* L_2 *operators of order 2 with coefficients in an extension of* k *of degree at most 4, 6 or 12 such that* $L = [L_1, L_2]_l$.

In (9) Compoint and Weil give algorithms to characterize and compute such a factorization of an operator over an algebraic closure of the field of coefficients. Here we use a different technique than they do, following an algorithm due to Beke (see (2)).

The finite primitive subgroups of G_2 are ${A_4}^{Sl_2}, {S_4}^{Sl_2}, {A_5}^{Sl_2}$ and degree two extensions of these groups by $\sqrt{-1}I$.

Let K be the Picard-Vessiot extension of k associated with $L_4(y) = 0$ and let k_0 be the fixed field of G^0, of degree $|H|$ over k. Then the form of G^0 shows that the solution space V of $L_4(y) = 0$ over k_0 can be decomposed into the direct sum of two vector spaces V_1 and V_2. Hence we can use lemma 4 to say that $L_4 = a[L_1, L_2]_l$ where $a \in k_0$ and $[L_1, L_2]_l$ is the least common left multiple of two operators L_1 and L_2 of order 2 with coefficients in k_0 and solution space V_1 and V_2 respectively.

Note that since the central elements are diagonal, the orders of the groups A_4, S_4, A_5 being respectively 12, 24 and 60 determine a bound on the coefficients of L_1 and L_2.

We want to sharpen this result and find a smaller field extension E between k and k_0 such that L_4 is the least common left multiple of 2 operators with coefficients in E, which will give us smaller bounds on the coefficients of L_1 and L_2. Then we also want to find these 2 equations.

For this we need to find a subgroup N of H such that the action of the group $N \cdot G^0$ preserves our solution space decomposition into $V_1 \bigoplus V_2$. The situation is the following:

$$\begin{array}{lcll} K & \longleftrightarrow & \{id\} & \\ \cup & & \cap & \\ k_0 & \longleftrightarrow & G^0 & \\ \cup & & \cap & \\ E & \longleftrightarrow & N \cdot G^0 & \\ \cup & & \cap & \text{minimize} \\ k & \longleftrightarrow & G = H \cdot G^0 & \end{array}$$

We show below that it will be the case for any abelian subgroup of H since those are diagonal (true for any finite, abelian subgroup).

Indeed let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathcal{C}^4$, in which

$$G^0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in Sl_2(\mathcal{C}) \right\}$$

We will need the following lemma:

Lemma 11 *((17) p.493) Let*

$$G = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in Sl_2(\mathcal{C}) \right\}$$

The centralizer of G in $Gl_4(\mathcal{C})$ is

$$\left\{ \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} : (ad - bc) \neq 0 \right\}$$

This result enables us to prove:

Lemma 12 *Let L_4 be a linear differential equation with coefficients in a differential field k such that its Galois group is of the form $H \cdot G^0$. Then $L_4 = [L_1, L_2]_l$ and a bound on the degree of the coefficients of L_1 and L_2 is found by looking at maximal diagonal subgroups of H.*

Proof. Let N be an abelian subgroup of H. In other words let $\tilde{N}$ be an abelian subgroup of $\tilde{H}$ (see previous theorem). The group N is diagonalizable in a basis $\{f_1, f_2, f_3, f_4\}$. Hence there exists a matrix $M \in Sl_2(\mathcal{C})$ such that $M\tilde{n}M^{-1}$ is diagonal for every $\tilde{n} \in \tilde{N}$. Suppose

$$M = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$$

Let

$$h = \begin{bmatrix} u_1 I & u_2 I \\ u_3 I & u_4 I \end{bmatrix}$$

be the change of basis matrix in $Sl_4(\mathcal{C})$. By lemma 11, for every $ng \in N \cdot G^0$, $h \cdot ng \cdot h^{-1} = h \cdot n \cdot h^{-1} h \cdot g \cdot h^{-1}$ has again the block diagonal form given by the decomposition $V = V_1 \oplus V_2$.

So let us consider a maximal diagonal subgroup H^d of H. The fixed field E of

$$H^d \cdot G^0 = \left\{ \begin{bmatrix} aA & 0 \\ 0 & a^{-1}A \end{bmatrix} : A \in Sl_2(\mathcal{C}) \right\}$$

has degree $[H : H^d] = [\tilde{H} : \tilde{H}^d]$ over k and again the solution space V can be written $V = V_1 \bigoplus V_2$ over E which tells us that we can find 2 differential operators L_1 and L_2 of order 2 with coefficients in E of which L_4 is the least common left multiple. $\square$

If $H \simeq A_4$,then the maximal diagonal subgroups are of index 4, hence to factor L_4 one follows the procedure explained in (29) p.102-105 knowing that the second order factors we are looking for have their coefficients in an extension of k of degree 4.

In the case where $\tilde{H} \simeq {S_4}^{Sl_2}$ then the maximal diagonal subgroups of S_4 are of index 6, hence there is an extension of degree at most 6 over which $L_4 = [L_1, L_2]$.

And when $\tilde{H} \simeq {A_5}^{Sl2}$, then the maximal diagonal subgroups of A_5 being of index 12, we can find the coefficients of L_1 and L_2 in an extension of k of degree at most 12.

Which shows that theorem 11 holds.

Now the goal is to get information about how to find these 2 operators. For this we follow a procedure given in (29) chapter 4.

We want group theory to tell us more about the coefficients of L_1 and L_2. First note that in the case where $\tilde{H}$ is a subgroup of $Sl_2(\mathcal{C})$, G can be seen as a subgroup of $SO_4(\mathcal{C}) \simeq Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})$ acting on $\mathcal{C}^2 \otimes \mathcal{C}^2$ via:

$$\begin{array}{ccc} \{1\} \times Sl_2(\mathcal{C}) & \longrightarrow & SO_4(\mathcal{C}) \\ A & \longmapsto & \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \end{array}$$

giving the action of G^0, and

$$\begin{array}{ccc} Sl_2(\mathcal{C}) \times \{1\} & \longrightarrow & SO_4(\mathcal{C}) \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \longmapsto & \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} \end{array}$$

for the action of H.

Algorithms for factoring linear differential operators over the field of coefficients are known but the difficulty here will be to factor over an algebraic extension, while implementation exists only over $\mathbb{C}(x)$. For this we will use Beke's algorithm (see (2)) or rather its improved version by Tsarev ((42)).

Following (29) p.111-116, in order to factor L_4 we look at its second exterior power and in particular the space $\bigwedge^2 V$ where V is the solution space of $L_4(y) = 0$. Indeed one wants to locate the wronskians of solutions of L_1 and L_2 so that we know where to look for their coefficients. According to Beke's algorithm, when the second exterior power is of maximal order, it is enough to find the degree of the first coefficient of a factor over k, i.e. the degree of the logarithmic derivative of its wronskian.

Lemma 13 *Let $L_4(y) = 0$ be a linear differential equation qith coefficient in a differential k such that its Galois group is of the form $H \cdot G^0$. Assume that its second exterior power is of order 6. Then $\bigwedge^2(L_4)$ is the least common left multiple of 2 operators of order 3, $\bar{L_H}$ and $\bar{L_{SL}}$ that are equivalent to the second symmetric powers of 2 operators of order 2 having Galois group H and $Sl_2(\mathcal{C})$ respectively. Furthermore the wronskians of solutions of a second order factor for L_4 are solutions of L_H.*

Proof. This result is a consequence of the representations given in Table 4.1 for the groups $H \cdot G^0 \subset SO_4(\mathcal{C})$.

To find those S.Hessinger explains she looked at the action of the whole group $Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})$, then drew conclusions on the subgroups. In (13) p.274 we find the decomposition of the representation of the Lie algebra $\mathbf{so}_4(\mathcal{C})$ on $\bigwedge^2 V$, where $V = \mathcal{C}^4$, to be the direct sum of 2 irreducible three-dimensional representations that are the second symmetric powers (also equal to the adjoint representation) of two two-dimensional representations of $Sl_2(\mathcal{C})$, which corresponds to the decomposition of $\mathbf{so}_4(\mathcal{C})$ as $\mathbf{sl}_2(\mathcal{C}) \oplus \mathbf{sl}_2(\mathcal{C})$. On the group level this corresponds to the decomposition of $SO_4(\mathcal{C})$ as the image of $Sl_2(\mathcal{C}) \times Sl_2(\mathcal{C})$ by the above maps.

Hence in each case $\bigwedge^2 V$ decomposes over k as the direct sum of two irreducible three-dimensional representations $\bar{W}_1$ and $\bar{W}_2$ such that $Sl_2(\mathcal{C})$ acts on $\bar{W}_1$ trivially while having $\bar{W}_2$ as a second symmetric power representation, and H acts on $\bar{W}_2$ trivially while having $\bar{W}_1$ as a second symmetric power representation.

So let us consider the second exterior power of L_4 and assume we know that it is of order 6, i.e. its solutions space is isomorphic to $\bigwedge^2 V$. We are then able to write $\bigwedge^2(L_4)$ as the least common left multiple of 2 operators $\bar{L_1}$ and $\bar{L_2}$ with coefficients in k and solutions spaces $\bar{W}_1$ and $\bar{W}_2$. Then one has that $\bar{L_1}$ and $\bar{L_2}$ are equivalent to the second symmetric powers of 2 operators of order 2.

Furthermore, since $V = V_1 \bigoplus V_2$ over E, we have $\bigwedge^2 V = \bigwedge^2 V_1 \bigoplus \bigwedge^2 V_2 \bigoplus V_1 \otimes V_2$, which can be further decomposed into the direct sum of three one-dimensional wedge spaces and one three-dimensional symmetric space $W_1 \bigoplus W_2$ where:

$$W_1 =< e_1 = v_1 \wedge v_2 > \bigoplus < e_3 = u_1 \wedge u_2 > \bigoplus < e_2 = v_1 \wedge u_2 - v_2 \wedge u_1 >$$

$$\text{and } W_2 =< e_4 = v_1 \otimes u_1, e_6 = v_2 \otimes u_2, e_5 = v_1 \wedge u_2 + v_2 \wedge u_1 >$$

for $V_1 =< v_1, v_2 >$ and $V_2 =< u_1, u_2 >$.

This decomposition tells us that the operator $\bigwedge^2(L_4)$ can also be written as the least common left multiple of 2 equations of order 3: $\bigwedge^2(L_4) = [\tilde{L}_1, \tilde{L}_2]_l$. With $\tilde{L}_1$ having solution space W_1 and $\tilde{L}_2$ having solution space W_2, and both equations have coefficients in E.

One can check that the component $G^0 \simeq Sl_2(\mathcal{C})$ will act on W_1 trivially, while having W_2 as a second symmetric power representation, and the group H has a central action on W_2 while having W_1 as a second symmetric power representation.

Indeed let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sl_2(\mathcal{C})$$

such that

$$\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \in G^0$$

is an element of G^0 with respect to the basis $\{u_1, u_2, v_1, v_2\}$. We want to consider its action on $\{u_1 \wedge u_2, v_1 \wedge v_2, v_1 \wedge u_2 - v_2 \wedge u_1, v_1 \otimes u_1, v_2 \otimes u_2, v_1 \wedge u_2 + v_2 \wedge u_1\}$. We find the matrix

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^2 & b^2 & 2ab \\ 0 & 0 & 0 & c^2 & d^2 & 2cd \\ 0 & 0 & 0 & ac & bd & (ad+bc) \end{bmatrix}$$

giving the action of G^0 on the exterior power of V.

Similarly if

$$\begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} \in H$$

represents a matrix in H with respect to the basis $\{u_1, u_2, v_1, v_2\}$, i.e. $(ad - bc)^2 = 1$, then its action on $\bigwedge^2 V$ is given by the matrix:

$$\begin{bmatrix} \\ a^2 & b^2 & 2ab & 0 & 0 & 0 \\ c^2 & d^2 & 2cd & 0 & 0 & 0 \\ ac & bd & (ad+bc) & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \pm 1 \end{bmatrix}$$

Note that if $H = {A_4}^{Sl_2}$, ${S_4}^{Sl_2}$, or ${A_5}^{Sl_2}$ those lie in $Sl_2(\mathcal{C})$ hence their action on W_2 is trivial.

Clearly the wronskians of elements of V_i, $i = 1, 2$ lie in W_1. So we want to find $\tilde{L}_1$, or at least its solutions to get the wronskians of solutions of L_1 and L_2 and eventually find those 2 equations. The Galois group associated with $\tilde{L}_1$ is isomorphic to $\tilde{H}/\pm I$ since the representation we are looking at is a symmetric power (the kernel is killed).

□

We can show that the characters associated with each of the three one-dimensional spaces in W_1 are distinct from each other. Indeed recall that an element $y \in K\ \{0\}$

is an exponential over k if and only if there is a character $\xi : G \to \mathcal{C}$ of G such that $\sigma(y) = \xi(\sigma)y$ for all $\sigma \in G$ (cf lemma 4.8 in (29)). So let

$$\begin{bmatrix} aI & 0 \\ 0 & \pm a^{-1}I \end{bmatrix} \in H^d$$

and

$$\begin{bmatrix} b & 0 & 0 & 0 \\ 0 & b^{-1} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} \in G^0$$

i.e.

$$\begin{bmatrix} ab & 0 & 0 & 0 \\ 0 & ab^{-1} & 0 & 0 \\ 0 & 0 & \pm a^{-1}b & 0 \\ 0 & 0 & 0 & \pm a^{-1}b^{-1} \end{bmatrix} \in H^d \cdot G^0$$

.

Then we can find the characters of the representation $\bigwedge^2 V$:

$$\begin{bmatrix} a^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & b^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & b^{-2} \end{bmatrix}$$

and clearly the characters of the alternate spaces are distinct. Hence the equation $\tilde{L}_1(y) = 0$ has three distinct exponential solutions (i.e. three distinct factors of order one), two of which can be expressed as wronskians of solutions of order two differential equations.

4.6 Solving forms in many variables

In the examples that we discuss in the next chapter we will need to solve homogeneous quadratic polynomials in three or four variables to find a gauge transformation as stated in Proposition 7 and Proposition 10. Since the field of coefficients we work with is $\mathcal{C}(x)$ the idea is to use Theorem 3.6 of (14) together with its proof p.22 that gives us a method for solving forms with coefficients in a C_1 field.

The field $\mathcal{C}$ being algebraically closed, it is C_0. Let $P(b_1, \cdots, b_n)$ be a quadratic polynomial in n variables with coefficients in $\mathcal{C}(x)$. We will consider the case where $n > 2$, for which the conditions of Theorem 3.4 of (14) p.18 due to Lang and Nagata are satisfied. Let r be the highest degree of the coefficients of P. Then for $s > \frac{(r+1)-n}{n-2}$ there must exist a non-trivial common zero of the form

$$b_\mu = \bar{b}_{\mu 0} + \bar{b}_{\mu 1} x + \bar{b}_{\mu 2} x^2 + \cdots + \bar{b}_{\mu s} x^s$$

$\mu = 1 \cdots n$ to the $2s + r$ forms in $n(s+1)$ variables P_ν where:

$$P(b) = P_0(\bar{b}) + P_1(\bar{b})x + \cdots + P_{2s+r}(\bar{b})x^{2s+r}$$

for $b = (b_0, \cdots b_n)$ and $\bar{b} = (\bar{b}_{00}, \bar{b}_{10}, \cdots, \bar{b}_{ns})$. This gives a non-trivial solution of P in $\mathcal{C}[x]$.

Example 6 *Consider*

$$
\begin{aligned}
p_0 =& \\
&+\frac{4(1024x^{20}-5120x^{17}-320x^{16}-640x^{15}+14720x^{14}+1088x^{13}+208x^{12}-22208x^{11})}{(7+40x^7+7x-83x^4-12x^3+28x^5-20x^8-8x^2-192x^9+64x^{12}+232x^6)}\\
&+\frac{4(-3272x^{10}+2288x^9+10832x^8+2768x^7+1450x^6-2708x^5-1100x^4+200x^3-320x^2-28x-23)}{(7+40x^7+7x-83x^4-12x^3+28x^5-20x^8-8x^2-192x^9+64x^{12}+232x^6)}\\
p_1 =& \\
&+\frac{4(46-5x-308x^4-636x^3+2758x^7-854x^6-2144x^9-256x^{12})}{(7+40x^7+7x-83x^4-12x^3+28x^5-20x^8-8x^2-192x^9+64x^{12}+232x^6)}\\
p_2 =& \\
&+\frac{4(1200x^5-1920x^8+840x^{11}+24x^2-1176x^{10}+640x^{15})}{(7+40x^7+7x-83x^4-12x^3+28x^5-20x^8-8x^2-192x^9+64x^{12}+232x^6)}\\
&-\frac{4(320x^16-1088x^{13}-36x^{12}+72x^{11}+968x^{10}-12x^9-227x^8)}{(7+40x^7+7x-83x^4-12x^3+28x^5-20x^8-8x^2-192x^9+64x^{12}+232x^6)}\\
&-\frac{4(304x^7+206x^6-143x^5-406x^4+38x^3+46x^2-17x+11)}{(7+40x^7+7x-83x^4-12x^3+28x^5-20x^8-8x^2-192x^9+64x^{12}+232x^6)}\\
p_3 =& \\
&-\frac{(7-16x+140x^4-332x^3-160x^7+280x^6+1392x^5-1728x^8+768x^{11}-36x^2)}{(7+40x^7+7x-83x^4-12x^3+28x^5-20x^8-8x^2-192x^9+64x^{12}+232x^6)}
\end{aligned}
$$

and

- $I_0 = 1$.
- $I_1 = x^4 - \frac{2}{5}x + \frac{3}{20}$.
- $I_2 = x^8 - \frac{4}{7}x^5 + \frac{5}{12}x^4 - \frac{5}{21}x^3 - \frac{1}{6}x + \frac{5}{84}$.
- $I_3 = x^{12} - \frac{54}{85}x^9 + \frac{247}{340}x^8 - \frac{42}{85}x^7 - \frac{146}{85}x^6 - \frac{41}{85}x^5 + \frac{11}{34}x^4 + \frac{1}{17}x^3 + \frac{19}{85}x^2 - \frac{1}{17}x + \frac{1}{170}$.

We want a solution for the quadratic polynomial in the b_i*'s:*

$$P = {b_0}^2 I_0 + {b_1}^2 I_1 + {b_2}^2 I_2 + {b_3}^2 I_3 + b_0 b_1 {I_0}' + b_1 b_2 {I_1}' + b_2 b_3 {I_2}' + b_0 b_2 A + b_0 b_3 B + b_1 b_3 C$$

where

$$A = \frac{1}{p_0}({I_2}'' - 2I_3 - p_3 {I_2}' - 2p_2 I_2 - p_1 {I_1}'),\ B = A' - {I_1}' \text{ and}$$

$$C = \frac{1}{p_1}({I_3}' - 2p_3 I_3 - p_2 {I_2}' - p_0 B)$$

The highest degree for the coefficients of P *is 67(after taking the numerator). The general method gives* $s > 32$*, which requires us to consider a system of* 131 *forms in* 132 *variables. An alternative to this method is to set* $b_3 = 0$ *and* $b_2 = 1$ *in* P *and solve the resulting polynomial* $\bar{P}$ *in* b_0 *and* b_1*. We then look for* b_0 *such that the discriminant of* $\bar{P}$ *in* b_1 *is a square. We get that* b_0 *must be a root of an irreducible polynomial of degree 110. This gives us a solution for* P *in* $\mathcal{C}(x)$*.*

Chapter 5

Applications

In the following we apply the previous methods on examples illustrating each case of study. We explain how each example was constructed, then how we determine the nature of the Galois group, and finally we try to find the lower order equations used to solve it. Most of the following calculations were done using the machines of the French UMS Medicis.

5.1 Examples for $Sl_2(\mathcal{C})$ with irreducible representation

1. • **Constructing the example**

   Here we give an equation having Galois group $Sl_2(\mathcal{C})$ that we found by using the method described in (28) to solve the inverse problem.

Let

$$A_0 = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

and

$$A_1 = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

From theorem 3.6 of (28) together with its proof we know that the differential system $Y' = (A_0 + x^2 A_1)Y$ has Galois group $Sl_2(\mathcal{C})$. Using the cyclic vector $[1, 0, 0, 0]$ one gets the following corresponding equation:

$$L_4(y) =$$

$$y^{(4)} + (-20x - 10 - 10x^4)y^{(2)} + (-20 - 40x^3)y' + (36x^5 + 36x + 9 + 18x^4 + 9x^8)y$$

- **Calculating the Galois group with Table 4.1**

  The operator L_4 is irreducible and unimodular.

  Using Table 4.1 we show its Galois group is $Sl_2(\mathcal{C})$ since its second exterior power is of order 5 and its second symmetric power of order 7.

- **Solving in terms of lower order equations**

  This example is a third symmetric power of an equation of order 2 and by

Proposition 6 we show it is the third symmetric power of $L_2(y) = y^{(2)} + (-2x - 1 - x^4)y$.

2. • **Constructing the example**

Let us apply the gauge transformation $z = y + y'$ to the above equation $L_4(y)$. We get:

$$\begin{aligned}
L_4(y) = y^{(4)} &- \frac{4(30x^2 + 45x^4 + 8x^3 + 4 + 18x^7)}{16x + 8x^4 + 20 + 9x^8 + 40x^3 + 36x^5} y^{(3)} \\
&- \frac{2(100x^2 + 340x^5 + 280x + 270x^9 + 450x^4 + 360x^6 + 85x^8 + 45x^{12} + 164x^7 + 92 + 184x^3)}{16x + 8x^4 + 20 + 9x^8 + 40x^3 + 36x^5} y'' \\
&+ \frac{4(270x^2 + 205x^4 - 128x^3 - 164 + 82x^7 + 540x^5 - 80x - 500x^6)}{16x + 8x^4 + 20 + 9x^8 + 40x^3 + 36x^5} y' \\
&+ \frac{9(84 + 96x + 9x^{16} + 72x^{13} + 428x^4 - 140x^5 - 152x^3 + 25x^8)}{16x + 8x^4 + 20 + 9x^8 + 40x^3 + 36x^5} y \\
&+ \frac{9(26x^{12} + 156x^9 + 288x^6 + 344x^{10} + 488x^7 - 56x^2)}{16x + 8x^4 + 20 + 9x^8 + 40x^3 + 36x^5} y
\end{aligned}$$

• **Calculating the Galois group with Table 4.1**

We checked that the operator L_4 is irreducible and unimodular.

Its second exterior power has maximal order 6 with one rational solution. Its second symmetric power has maximal order 10 and factors into 2 equations of order 3 and 7 respectively. Again according to Table 4.1 the Galois group of L_4 has for its connected component $Sl_2(\mathcal{C})$.

• **Solving in terms of lower order equations**

This time the first formula of Proposition 6 isn't true, so we need to transform L_4 with a gauge transformation to find an equation equivalent to L_4 that passes the tests of Proposition 6.

According to Proposition 7 this gauge transformation exists. This corresponds to finding a gauge transformation such that the second symmetric power of our new equation is of order 7 exactly. The idea is to consider the factor of order 3 of the second symmetric power of L_4 and show it is equivalent to the second symmetric power of a second order equation using Singer's work on third order equations (see (34)).

The factor of order three that we get after factoring the second symmetric power of L_4 is

$$L_3(y) = y^{(3)} + (-4x^4 - 8x - 4)y' - (8x^3 + 4)y$$

and it turns out to be precisely the second symmetric power of

$$L_2(y) = y^{(2)} + (-2x - 1 - x^4)y$$

So let

$$\tilde{L}_4(v) = v^{(4)} + (-20x - 10 - 10x^4)v^{(2)} + (-20 - 40x^3)v' + (36x^5 + 36x + 9 + 18x^4 + 9x^8)v = {L_2}^{ⓢ3}(v)$$

The equations L_4 and $\tilde{L}_4$ are equivalent by the gauge transform $z = y + y'$. Indeed as in Lemma 1 let us consider the companion matrices for L_4 and $\tilde{L}_4$, respectively A and B. In this case finding a cyclic vector is expensive since one needs to inverse a 16×16 matrix, so we used the Maple package ISOLDE ((1)), and in particular the online interactive demonstration, to find the rational solutions of $W' = (A \otimes I - I \otimes B^T)W$ and the corresponding matrix Z such that $Z' = AZ - ZB$ where $Y = ZV$. Finally we get $Z^{-1} =$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -9x^8 - 36x^5 - 18x^4 - 36x - 9 & 40x^3 + 20 & 10x^4 + 20x + 10 & 1 \end{bmatrix}$$

and the gauge transform as announced.

5.2 Examples for $SO_4(\mathcal{C})$

The following two examples were constructed by P.Gaillard.

5.2.1 The Mitschi-Singer method

1. • **Constructing the example**

   Also M.F. Singer and C. Mitschi give a procedure in (28) in order to solve the inverse problem for any connected semisimple group which P. Gaillard was able to apply to $SO_4(\mathcal{C})$.

   Let

$$A_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$A_1 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

From Theorem 3.6 of (28) together with its proof we know that the differential system $Y' = (A_0 + x^2 A_1)Y$ has Galois group $Sl_2(\mathcal{C})$. Using the cyclic vector $[0, 0, 1, 0, 0, 1, 1, 0]$ one gets the following corresponding equation:

$$L_4(y) = y^{(4)} - \frac{1}{x}\left(\frac{-1+4x^3}{-1+x^3}\right)y^{(3)} + (-20x^4+8x-4)y^{(2)} - \frac{4}{x}\left(\frac{x-20x^4-4x^3+1+10x^7}{-1+x^3}\right)y' + 4\left(\frac{16x^{12}+1+58x^6-48x^9}{x(-1+x^3)}\right)y$$

- **Calculating the Galois group using Table 4.1**

  We checked that L_4 is irreducible and unimodular.

  This equation was obtained by factoring an equation of order 8 the Galois group of which we know is $SO_4(\mathcal{C})$. Hence the Galois group of L can be either $SO_4(\mathcal{C})$ or a subgroup of $SO_4(\mathcal{C})$, i.e. $PGl_2(\mathcal{C})$. We now test L_4 using Table 4.1.

Its second symmetric power is of order 9. Its second exterior power is of order 6 and has two factors of order 3, which wouldn't be the case if the Galois group was $PSl_2(\mathcal{C})$. To determine whether the group is $SO_4(\mathcal{C})$ using Hessinger's method one would need to construct, then factor the sixth symmetric power of L_4 which can be an operator of order 84, hence this task is too expensive.

- **Solving in terms of lower order equations**

  Let us use Proposition 9 to write L_4 as the symmetric product of 2 equations of order 2.

  Its coefficients satisfy (4.3) and (4.2), hence it is the symmetric product of equations without second term. One finds L_4 to be the symmetric product of the equations:

$$L_1(y) = y'' + (-9x^4 + 6x - 1)y$$

$$L_2(y) = y'' + (-x^4 - 2x - 1)y$$

2. - **Constructing the example**

  It is interesting to notice in the previous example that the second exterior power of L_4 is the least common left multiple of ${L_1}^{ⓢ2}$ and ${L_2}^{ⓢ2}$. Now one considers an equation that is not the symmetric product of 2 equations of order 2, but is equivalent to it, which will not be as easy to solve. We took the previous equation and applied a gauge transform on it to construct this one:

$$\begin{aligned}
L_4(y) = {} & y^{(4)} \\
& + \frac{4(46 - 5x - 308x^4 - 636x^3 + 2758x^7 - 854x^6 - 2144x^9 - 256x^{12})}{(7 + 40x^7 + 7x - 83x^4 - 12x^3 + 28x^5 - 20x^8 - 8x^2 - 192x^9 + 64x^{12} + 232x^6)} y' \\
& + \frac{4(1200x^5 - 1920x^8 + 840x^{11} + 24x^2 - 1176x^{10} + 640x^{15})}{(7 + 40x^7 + 7x - 83x^4 - 12x^3 + 28x^5 - 20x^8 - 8x^2 - 192x^9 + 64x^{12} + 232x^6)} y' \\
& - \frac{4(320x^{16} - 1088x^{13} - 36x^{12} + 72x^{11} + 968x^{10} - 12x^9 - 227x^8)}{(7 + 40x^7 + 7x - 83x^4 - 12x^3 + 28x^5 - 20x^8 - 8x^2 - 192x^9 + 64x^{12} + 232x^6)} y'' \\
& - \frac{4(304x^7 + 206x^6 - 143x^5 - 406x^4 + 38x^3 + 46x^2 - 17x + 11)}{(7 + 40x^7 + 7x - 83x^4 - 12x^3 + 28x^5 - 20x^8 - 8x^2 - 192x^9 + 64x^{12} + 232x^6)} y'' \\
& - \frac{(7 - 16x + 140x^4 - 332x^3 - 160x^7 + 280x^6 + 1392x^5 - 1728x^8 + 768x^{11} - 36x^2)}{(7 + 40x^7 + 7x - 83x^4 - 12x^3 + 28x^5 - 20x^8 - 8x^2 - 192x^9 + 64x^{12} + 232x^6)} y^{(3)} \\
& + \frac{4(1024x^{20} - 5120x^{17} - 320x^{16} - 640x^{15} + 14720x^{14} + 1088x^{13} + 208x^{12} - 22208x^{11})}{(7 + 40x^7 + 7x - 83x^4 - 12x^3 + 28x^5 - 20x^8 - 8x^2 - 192x^9 + 64x^{12} + 232x^6)} y \\
& + \frac{4(-3272x^{10} + 2288x^9 + 10832x^8 + 2768x^7 + 1450x^6 - 2708x^5 - 1100x^4 + 200x^3 - 320x^2 - 28x - 23)}{(7 + 40x^7 + 7x - 83x^4 - 12x^3 + 28x^5 - 20x^8 - 8x^2 - 192x^9 + 64x^{12} + 232x^6)} y
\end{aligned}$$

- **Calculating the Galois group using Table 4.1**

  We checked that L_4 is unimodular and irreducible.

  The second exterior power of L_4 factors as the least common left multiple of two equations of order 3 and its second symmetric power has a rational solution. Again it would be too expensive to use Hessinger's method to determine the Galois group in this case.

- **Solving in terms of lower order equations**
  The second exterior power of L_4 is the least common left multiple of:

$$\begin{aligned}
L_1 = {} & \delta^3 - \frac{8(2688x^{11} - 5184x^8 - 368x^7 + 1680x^6 + 1920x^5 + 100x^4 - 792x^3 - 18x^2 + 16x + 21)}{(1792x^{12} - 4608x^9 - 368x^8 + 1920x^7 + 2560x^6 + 160x^5 - 1584x^4 - 48x^3 + 64x^2 + 168x + 7)} \delta^2 \\
& - \frac{4(16128x^{16} - 52224x^{13} - 1520x^{12} + 9216x^{11} + 48000x^{10})}{(1792x^{12} - 4608x^9 - 368x^8 + 1920x^7 + 2560x^6 + 160x^5 - 1584x^4 - 48x^3 + 64x^2 + 168x + 7)} \delta \\
& - \frac{4(-960x^9 - 16160x^8 - 5152x^7 + 2856x^6 + 520x^5 - 2425x^4 + 956x^3 - 38x^2 + 146x + 45)}{(1792x^{12} - 4608x^9 - 368x^8 + 1920x^7 + 2560x^6 + 160x^5 - 1584x^4 - 48x^3 + 64x^2 + 168x + 7)} \delta \\
& + \frac{4(32256x^{15} - 92928x^{12} - 6368x^{11} + 12672x^{10} + 84480x^9 + 5040x^8)}{(1792x^12 - 4608x^9 - 368x^8 + 1920x^7 + 2560x^6 + 160x^5 - 1584x^4 - 48x^3 + 64x^2 + 168x + 7)} \\
& + \frac{4(-16304x^7 - 19000x^6 - 8064x^5 + 6256x^4 - 94x^3 + 806x^2 + 220x + 101)}{(1792x^{12} - 4608x^9 - 368x^8 + 1920x^7 + 2560x^6 + 160x^5 - 1584x^4 - 48x^3 + 64x^2 + 168x + 7)}
\end{aligned}$$

and

$$\begin{aligned}L_2 =&\delta^3 - \frac{8(1152x^{11} - 1728x^8 - 48x^7 + 1120x^6 - 864x^5 - 180x^4 - 128x^3 + 54x^2 + 48x + 7)}{(768x^{12} - 1536x^9 - 48x^8 + 1280x^7 - 1152x^6 - 288x^5 - 256x^4 + 144x^3 + 192x^2 + 56x - 105)}\delta^2 \\ &- \frac{4(768x^{16} + 720x^{12} + 1664x^{11} - 16896x^{10} - 1920x^9 + 3792x^8)}{(768x^{12} - 1536x^9 - 48x^8 + 1280x^7 - 1152x^6 - 288x^5 - 256x^4 + 144x^3 + 192x^2 + 56x - 105)}\delta \\ &- \frac{4(5792x^7 - 2936x^6 - 4296x^5 + 4175x^4 + 124x^3 - 134x^2 + 698x - 299)}{(768x^{12} - 1536x^9 - 48x^8 + 1280x^7 - 1152x^6 - 288x^5 - 256x^4 + 144x^3 + 192x^2 + 56x - 105)}\delta \\ &+ \frac{12(512x^{15} + 768x^{12} + 736x^{11} + 896x^{10} - 12032x^9 - 2352x^8)}{(768x^{12} - 1536x^9 - 48x^8 + 1280x^7 - 1152x^6 - 288x^5 - 256x^4 + 144x^3 + 192x^2 + 56x - 105)} \\ &+ \frac{12(4496x^7 - 1448x^6 - 832x^5 + 1488x^4 - 1082x^3 - 174x^2 + 260x + 115)}{(768x^{12} - 1536x^9 - 48x^8 + 1280x^7 - 1152x^6 - 288x^5 - 256x^4 + 144x^3 + 192x^2 + 56x - 105)}\end{aligned}$$

To solve $L_4(y) = 0$ one needs to find a degree two invariant that will be found by computing the rational solutions of the second symmetric power of L_4, L_4', L_4'' and ${L_4}^{(3)}$ where L_4', L_4'' and ${L_4}^{(3)}$ are operators such that if $L_4(y) = 0$, then $L_4'(y') = 0$, $L_4''(y'') = 0$ and ${L_4}^{(3)}(y^{(3)}) = 0$.

We found those rational solutions using the Maple package Bernina ((7)):

- $I_0 = 1$ for $L_4(y) = 0$.
- $I_1 = x^4 - \frac{2}{5}x + \frac{3}{20}$ for ${L_4}'(y) = 0$.
- $I_2 = x^8 - \frac{4}{7}x^5 + \frac{5}{12}x^4 - \frac{5}{21}x^3 - \frac{1}{6}x + \frac{5}{84}$ for ${L_4}''(y) = 0$.
- $I_3 = x^{12} - \frac{54}{85}x^9 + \frac{247}{340}x^8 - \frac{42}{85}x^7 - \frac{146}{85}x^6 - \frac{41}{85}x^5 + \frac{11}{34}x^4 + \frac{1}{17}x^3 + \frac{19}{85}x^2 - \frac{1}{17}x + \frac{1}{170}$ for ${L_4}^{(3)}(y) = 0$.

We want a solution for the quadratic polynomial in the b_i's:

$${b_0}^2 I_0 + {b_1}^2 I_1 + {b_2}^2 I_2 + {b_3}^2 I_3 + b_0 b_1 {I_0}' + b_1 b_2 {I_1}' + b_2 b_3 {I_2}' + b_0 b_2 A + b_0 b_3 B + b_1 b_3 C$$

where

$$A = \frac{1}{p_0}({I_2}'' - 2I_3 - p_3 {I_2}' - 2p_2 I_2 - p_1 {I_1}'),\ B = A' - {I_1}' \text{ and}$$

$$C = \frac{1}{p_1}({I_3}' - 2p_3 I_3 - p_2 {I_2}' - p_0 B)$$

Following the method described in section 4.6 we find the b_is in $\mathcal{C}(x)$, but the result is too large to be included.

5.2.2 The Beukers-Heckman method

- **Constructing the example**

We wanted to build the example of a homogeneous linear differential equation of order 4 with Galois group $SO_4(\mathcal{C})$, ie solve the inverse problem in this case. F.Beukers and G.Heckman in (3) describe the Galois groups of the hypergeometric equation, and P.Gaillard followed their procedure to build the following equation with Galois group $O_4(\mathcal{C})$. He checked that the hypergeometric group with parameters $1, -1, i, -i, e^{2i\pi/17}, e^{-2i\pi/17}, e^{2i\pi/3}, e^{-2i\pi/3}$ was primitive and not a scalar shift of a finite group to apply Theorem 6.5 of (3) and conclude it was indeed $O_4(\mathcal{C})$. Then he constructed the corresponding hypergeometric equation:

$$L(y) = y^{(4)} + \frac{\frac{1}{2}(8+x)}{x(x-1)} y^{(3)} - \frac{1}{41616} \frac{(244828+2601x)}{x^2(x-1)} y^{(2)} - \frac{1}{83232} \frac{-313856+2601x}{x^3(x-1)} y' - \frac{256}{289} \frac{1}{x^4(x-1)} y$$

So that the Galois group of L is a subgroup of $Sl_4(\mathcal{C})$ we want L to be unimodular, i.e. that the second coefficient of L is a rational function. Since it is not the case here we use a change of variables to make the equation unimodular.

The new unimodular equation we get is:

$$
\begin{aligned}
L_4(y) = & y^{(4)} + \frac{\frac{1}{83232}(49419x^2 + 389282x - 508928)}{x^2(x-1)^2} y^{(2)} \\
& - \frac{1}{18496} \frac{18496x^3 + 336829x^2 - 547510x + 147968}{x^3(x-1)^3} y' \\
& + \frac{1}{10653696} \frac{8304993x^4 + 242993284x^3 - 438833712x^2 + 188425344x - 30736384}{x^4(x-1)^4} y
\end{aligned}
$$

- **Calculating the Galois group using Table 4.1**

  We checked that L_4 is an irreducible homogeneous hypergeometric unimodular equation. By construction its Lie algebra is $\mathbf{so}_4(\mathcal{C})$.

  The second symmetric power of L_4 is of order 10 but does not have a rational solution, hence according to Table 4.1 the Galois group of L_4 is not $SO_4(\mathcal{C})$. However the connected component of the Galois group of L_4 is $SO_4(\mathcal{C})$, so we should be able to write L_4 as the symmetric product of 2 equations of order 2 over an algebraic extension of k.

- **Solving in terms of lower order equations**

  In this case we will have problems factoring as we need to work in an extension.

5.3 Examples for $Sl_2(\mathcal{C})$ with reducible representation

1. • **Constructing the example**

   In (30) the authors give an equation for the finite group ${A_4}^{Sl_2}$ and take its symmetric product with an equation having $Sl_2(\mathcal{C})$ for its Galois group.

   Let us use the symmetric product of the Airy equation $L_1(y) = y'' - xy$, that has Galois group $Sl_2(\mathcal{C})$ with an equation that has Galois group ${A_4}^{Sl_2}$: $L_2(y) = y'' - (\frac{-3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)})y$.

   We get:

$$
\begin{aligned}
L_4(y) =& y^{(4)} - \frac{144x^6 - 432x^5 + 432x^4 - 208x^3 + 81x^2 - 135x + 54}{(144x^5 + 32x^2 - 27x + 144x^3 - 288x^4 + 27)(x-1)x} y^{(3)} \\
& - \frac{1}{72} \frac{144x^5 - 288x^4 + 144x^3 - 32x^2 + 27x - 27}{(x-1)^2x^2} y^{(2)} \\
& - \frac{1}{144} \Big(\frac{20736x^{11} - 103680x^{10} + 207360x^9 - 138240x^8}{(144x^5 + 32x^2 - 27x + 144x^3 - 288x^4 + 27)(x-1)^3x^3} y' \\
& + \frac{135360x^7 + 386064x^6 - 429232x^5 + 248400x^4}{(144x^5 + 32x^2 - 27x + 144x^3 - 288x^4 + 27)(x-1)^3x^3} y' \\
& + \frac{-50085x^3 - 7560x^2 + 5103x - 1458}{144x^5 + 32x^2 - 27x + 144x^3 - 288x^4 + 27)(x-1)^3x^3} y' \Big) \\
& + \frac{1}{20736} \Big(\frac{2985984x^{15} - 17915904x^{14} + 44789760x^{13} - 54743040x^{12}}{(144x^5 + 32x^2 - 27x + 144x^3 - 288x^4 + 27)(x-1)^4x^4} y \\
& + \frac{17231616x^{11} + 47215872x^{10} - 77068800x^9 + 4726456x^8}{(144x^5 + 32x^2 - 27x + 144x^3 - 288x^4 + 27)(x-1)^4x^4} y \\
& + \frac{9777456x^7 - 48894400x^6 + 45476640x^5 - 18085680x^4}{(144x^5 + 32x^2 - 27x + 144x^3 - 288x^4 + 27)(x-1)^4x^4} y \\
& + \frac{722925x^3 + 2240217x^2 - 898857x + 229635}{(144x^5 + 32x^2 - 27x + 144x^3 - 288x^4 + 27)(x-1)^4x^4} y \Big)
\end{aligned}
$$

   • **Calculating the Galois group using Table 4.1**

   We checked that L_4 is irreducible and unimodular.

   Its second exterior power has degree 6 and factors into $[\bar{L_H}, \bar{L_{SL}}]$, where $\bar{L_{SL}}$ is the second symmetric power of the Airy equation and $\bar{L_H}$ is the second symmetric power of L_2. Its second symmetric power has degree 9. To use Table 4.1 and decide that the Galois group is ${A_4}^{Sl_2(\mathcal{C})} \otimes Sl_2(\mathcal{C})$ one needs to construct then factor the fourth symmetric power of L_4 which can be an operator of order 45, hence this method will again be too expensive.

- **Solving in terms of lower order equations**

  To find the second order equation of which L_4 is the least common left multiple one needs to work in an extension of degree 4.

2. - **Constructing the example**
  Now consider the following equation that we constructed by applying a gauge transform to L_4:

$$
\begin{aligned}
L(y) &= y^{(4)} \\
&- \frac{1}{144}\,\frac{(429981696x^{21} - 3869835264x^{20} + 12469469184x^{19} - 14571601920x^{18} - 6264594432x^{17} + 3723522048x^{16})}{A1}\,y' \\
&- \frac{1}{144}\,\frac{(196856967168x^{15} - 640821694464x^{14} + 1110703684608x^{13} - 1206718748928x^{12} + 758753678592x^{11})}{A1}\,y' \\
&- \frac{1}{144}\,\frac{(91138362624x^{10} - 281042039008x^{9} + 232783630704x^{8} - 49832845488x^{7} - 38364692688x^{6} + 37097558307x^{5})}{A1}\,y' \\
&- \frac{1}{144}\,\frac{(17592716430x^{4} + 6392579130x^{3} - 1780130520x^{2} + 266606235x - 12400290)}{A1}\,y' \\
&- \frac{1}{72}\,\frac{(429981696x^{20} - 4299816960x^{19} + 19349176320x^{18} - 50761728000x^{17} + 84879581184x^{16})}{A2}\,y'' \\
&- \frac{1}{72}\,\frac{(93150756864x^{15} + 67746004992x^{14} - 34115696640x^{13} + 16328521728x^{12})}{A2}\,y'' \\
&- \frac{1}{72}\,\frac{(16796431872x^{11} + 32433387264x^{10} - 57319661952x^{9} + 65900416352x^{8} - 39481962624x^{7})}{A2}\,y'' \\
&+ \frac{1}{72}\,\frac{(2212100064x^{6} + 13500094392x^{5} - 10123466265x^{4} + 4472371260x^{3} - 1967971950x^{2} + 719216820x - 105402465)}{A2}\,y'' \\
&- \frac{3(459270 - 2066715x + 4533165x^{2} + 37506672x^{6} - 4084695x^{4} - 4431105x^{3} - 276293376x^{11} + 202155264x^{10} - 147690240x^{9})}{A3}\,y^{(3)} \\
&- \frac{3(135533232x^{8} - 103723984x^{7} - 24883200x^{15} + 92565504x^{14} - 202051584x^{13} + 284891904x^{12} + 3938544x^{5} + 2985984x^{16})}{A3}\,y^{(3)} \\
&+ \frac{1}{20736}\,\frac{(-19380087226368x^{19} + 86638015217664x^{18} - 837019575x + 50183973630x^{2} - 12500427866112x^{22})}{A4}\,y \\
&+ \frac{1}{20736}\,\frac{(22927054012416x^{21} - 18948863361024x^{20} + 61917364224x^{25} - 743008370688x^{24} + 4024628674560x^{23}}{A4}\,y \\
&+ \frac{1}{20736}\,\frac{(27195298169040x^{6} + 2955137746455x^{4} - 540342636750x^{3} + 176449800796416x^{11} - 33269203481344x^{10})}{A4}\,y \\
&\frac{1}{20736}\,\frac{(-59645104839216x^{9} + 78283810352448x^{8} - 54860276808000x^{7} + 292838436864x^{15} - 161235787161600x^{14})}{A4}\,y \\
&+ \frac{1}{20736}\,\frac{(279913147163136x^{13} - 284525448620544x^{12} - 10349321577147x^{5})}{A4}\,y \\
&+ \frac{1}{20736}\,\frac{(114629356486656x^{16} - 137422311284736x^{17} + 167403915)}{A4}\,y
\end{aligned}
$$

with

$A1 = x^3(x-1)^2(-314731008x^{11}+385502976x^{12}+2985984x^{16}-263761920x^{13}-26873856x^{15}+110481408x^{14}+46103040x^{10}-299702736x^8+918540x^3-2194290x^2+12588048x^5+4073085x^4+219529520x^7+216281088x^9+918540x-229635-91888784x^6)$

$A2 = x^2(-314731008x^{11}+385502976x^{12}+2985984x^{16}-263761920x^{13}-26873856x^{15}+110481408x^{14}+46103040x^{10}-299702736x^8+918540x^3-2194290x^2+12588048x^5+4073085x^4+219529520x^7+216281088x^9+918540x-229635-91888784x^6)(x-1)$

$A3 = x(-314731008x^{11}+385502976x^{12}+2985984x^{16}-263761920x^{13}-26873856x^{15}+110481408x^{14}+46103040x^{10}-299702736x^8+918540x^3-2194290x^2+12588048x^5+4073085x^4+219529520x^7+216281088x^9+918540x-229635-91888784x^6)$

$A4 = x^4(-314731008x^{11}+385502976x^{12}+2985984x^{16}-263761920x^{13}-26873856x^{15}+110481408x^{14}+46103040x^{10}-299702736x^8+918540x^3-2194290x^2+12588048x^5+4073085x^4+219529520x^7+216281088x^9+918540x-229635-91888784x^6)(x^3-3x^2+3x-1)$

- **Calculating the Galois group using Table 4.1**
  We checked that L_4 is irreducible and unimodular.
  For the same reasons as above Hessinger's method will be too expensive.
  The second exterior power of L is of order 6 and can be written as the Least

Common Left Multiple of

$$
\begin{aligned}
L_1 &= \delta^3 \\
&- 3\,\frac{(995328x^{16} - 8957952x^{15} + 36827136x^{14} - 90574848x^{13} + 145956096x^{12} - 156093696x^{11})}{B1}\,\delta^2 \\
&- 3\,\frac{(104448768x^{10} - 29619456x^{9} - 11247088x^{8} - 1177360x^{7} + 29595888x^{6} - 30828816x^{5})}{B1}\,\delta^2 \\
&- 3\,\frac{(12656439x^{4} + 657153x^{3} - 3589677x^{2} + 2066715x - 459270)}{B1}\,\delta^2 \\
&+ \frac{1}{36}\,\frac{(246841344x^{17} - 2248445952x^{16} + 9377980416x^{15} - 23790440448x^{14} + 41056395264x^{13} - 51286493952x^{12})}{B2}\,\delta \\
&+ \frac{1}{36}\,\frac{(48520485888x^{11} - 36678357504x^{10} + 26663021184x^{9} - 26490082864x^{8} + 30057900768x^{7} - 25798214448x^{6})}{B2}\,\delta \\
&+ \frac{1}{36}\,\frac{(14366156112x^{5} - 5370901425x^{4} + 1997457084x^{3} - 1020773502x^{2} + 372008700x - 55801305)}{B2}\,\delta \\
&- \frac{1}{72}\,\frac{(557383680x^{18} - 5150822400x^{17} + 22482468864x^{16} - 60784128000x^{15} + 113666899968x^{14})}{B3} \\
&- \frac{1}{72}\,\frac{(-154163955456x^{13} + 147989865984x^{12} - 77291278848x^{11} - 35689321728x^{10})}{B3} \\
&- \frac{1}{72}\,\frac{(126995560976x^{9} - 142439373504x^{8} + 94068221616x^{7} - 38322916128x^{6})}{B3} \\
&- \frac{1}{72}\,\frac{(10021106403x^{5} - 3289063806x^{4} + 2083524282x^{3} - 868204008x^{2} + 167403915x - 12400290)}{B3}
\end{aligned}
$$

where

$B1 = (995328x^{15} - 8957952x^{14} + 38817792x^{13} - 106721280x^{12} + 202459392x^{11} - 272007936x^{10} + 258228480x^{9} - 171456768x^{8} + 75330960x^{7} - 11966480x^{6} - 12683952x^{5} + 9975312x^{4} - 2002077x^{3} - 797769x^{2} + 688905x - 229635)x(x-1)$

$B2 = (995328x^{15} - 8957952x^{14} + 38817792x^{13} - 106721280x^{12} + 202459392x^{11} - 272007936x^{10} + 258228480x^{9} - 171456768x^{8} + 75330960x^{7} - 11966480x^{6} - 12683952x^{5} + 9975312x^{4} - 2002077x^{3} - 797769x^{2} + 688905x - 229635)x^2(x-1)^2$

$B3 = 995328x^{15} - 8957952x^{14} + 38817792x^{13} - 106721280x^{12} + 202459392x^{11} - 272007936x^{10} + 258228480x^{9} - 17145676x^{8} + 75330960x^{7} - 11966480x^{6} - 12683952x^{5} + 9975312x^{4} - 2002077x^{3} - 797769x^{2} + 688905x - 229635)x^3(x-1)^3$

and

$$
\begin{aligned}
L_2 &= \delta^3 \\
&- 3\,\frac{(8957952x^{16} - 76640256x^{15} + 295612416x^{14} - 675827712x^{13} + 1007652096x^{12} - 1020867840x^{11})}{C1}\,\delta^2 \\
&- 3\,\frac{(+717656832x^{10} - 384238848x^{9} + 237325200x^{8} - 210980048x^{7} + 163801008x^{6} - 84609360x^{5})}{C1}\,\delta^2 \\
&- 3\,\frac{(+29799927x^{4} - 6890751x^{3} - 1702701x^{2} + 2066715x - 459270)}{C1}\,\delta^2 \\
&- 2\,\frac{(17915904x^{20} - 185131008x^{19} + 877879296x^{18} - 2489647104x^{17} + 4641587712x^{16} - 5881102848x^{15})}{C2}\,\delta \\
&- 2\,\frac{(5016333312x^{14} - 2608475904x^{13} + 343132704x^{12} + 671471840x^{11} - 409487584x^{10} - 576364192x^{9})}{C2}\,\delta \\
&- 2\,\frac{(1483855782x^{8} - 1617425982x^{7} + 1033689276x^{6} - 383492151x^{5} + 74082195x^{4} - 16594956x^{3} + 12890178x^{2} - 4822335x + 688905)}{C2}\,\delta \\
&+ 2\,\frac{(8957952x^{19} - 86593536x^{18} + 373248000x^{17} - 909398016x^{16} + 1282003200x^{15} - 782224128x^{14})}{C3} \\
&+ 2\,\frac{(-651723264x^{13} + 2063902464x^{12} - 2604269808x^{11} + 2507732176x^{10} - 2440692624x^{9})}{C3} \\
&+ 2\,\frac{(2415691664x^{8} - 1953186885x^{7} + 1079339283x^{6} - 352917162x^{5} + 52269543x^{4} - 6511509x^{3} + 8220933x^{2} - 3735396x + 688905)}{C3}
\end{aligned}
$$

where

$C1 = 8957952x^{15} - 74649600x^{14} + 289640448x^{13} - 674500608x^{12} + 1024047360x^{11} - 1028816640x^{10} + 654098688x^9 - 202394880x^8 - 61437168x^7 + 115729552x^6 - 70200432x^5 + 22953456x^4 - 4832541x^3 + 617463x^2 + 688905x - 229635)/x/(x-1)$

$C2 = 2 + 1024047360x^{11} - 1028816640x^{10} + 654098688x^9 - 202394880x^8 - 61437168x^7 + 115729552x^6 - 70200432x^5 + 22953456x^4 - 4832541x^3 + 617463x^2 + 688905x - 229635)/x^2/(x-1)^2$

$C3 = 8957952x^15 - 74649600x^14 + 289640448x^13 - 674500608x^12 + 1024047360x^11 - 1028816640x^10 + 654098688x^9 - 202394880x^8 - 61437168x^7 + 115729552x^6 - 70200432x^5 + 22953456x^4 - 4832541x^3 + 617463x^2 + 688905x - 229635)/x^2/(x-1)^2$

- **Solving in terms of lower order equations**

  We want to find two order 2 equations L_{12} and L_{22} the second symmetric power of which are equivalent to L_1 and L_2. For this one needs to compute the rational solutions of $L_1^{\circledS 2}$, $L_1'^{\circledS 2}$ and $L_1''^{\circledS 2}$ one the one hand, and those of $L_2^{\circledS 2}$, $L_2'^{\circledS 2}$ and $L_2''^{\circledS 2}$ one the other hand.

We find them using Bernina. For L_1 we get:

- $I_0 = \frac{(x^5-\frac{5}{2}x^4+2x^3-\frac{13}{18}x^2+\frac{3}{16}x-\frac{3}{16})}{(x^4-2x^3+x^2)}$
- $I_1 = \frac{(x^{10}-6x^9+17x^8-\frac{248}{9}x^7+\frac{1885}{72}x^6-\frac{461}{36}x^5+\frac{959}{648}x^4+\frac{13}{6}x^3-\frac{1637}{768}x^2+\frac{87}{128}x+\frac{9}{256})}{(x^8-4x^7+6x^6-4x^5+x^4)}$
-

$$I_2 = \frac{(x^{12}-\frac{1083}{176}x^{11}+\frac{3021}{176}x^{10}-\frac{11101}{396}x^9+\frac{23615}{792}x^8-\frac{29751}{1408}x^7}{(x^{12}-6x^{11}+15x^{10}-20x^9+15x^8-6x^7+x^6)}$$
$$\frac{(\frac{851821}{114048}x^6+\frac{9527}{2816}x^5-\frac{15227}{2816}x^4+\frac{94365}{45056}x^3+\frac{8793}{45056}x^2-\frac{16281}{45056}x+\frac{4131}{45056})}{(x^{12}-6x^{11}+15x^{10}-20x^9+15x^8-6x^7+x^6)}$$

and for L_2 we get:

- $I_0 = \frac{(x^5-\frac{5}{2}x^4+2x^3-\frac{13}{18}x^2+\frac{3}{16}x-\frac{3}{16})}{(x^4-2x^3+x^2)}$
- $I_1 = \frac{(x^{10}-6x^9+17x^8-\frac{248}{9}x^7+\frac{1885}{72}x^6-\frac{461}{36}x^5+\frac{959}{648}x^4+\frac{13}{6}x^3-\frac{1637}{768}x^2+\frac{87}{128}x+\frac{9}{256})}{(x^8-4x^7+6x^6-4x^5+x^4)}$
-

$$I_2 = \frac{(x^{15}-8x^{14}+29x^{13}-\frac{545}{9}x^{12}+\frac{1855}{24}x^{11}-\frac{1019}{18}x^{10}+\frac{18235}{1296}x^9+\frac{20473}{1296}x^8}{(x^{12}-6x^{11}+15x^{10}-20x^9+15x^8-6x^7+x^6)}$$
$$+\frac{-\frac{462503}{20736}x^7+\frac{84299}{5184}x^6-\frac{3655}{384}x^5+\frac{1393}{256}x^4-\frac{605}{256}x^3+\frac{17}{64}x^2+\frac{27}{128}x-\frac{9}{256})}{(x^{12}-6x^{11}+15x^{10}-20x^9+15x^8-6x^7+x^6)}$$

Then one needs to solve the quadratic polynomial in (b_0, b_1, b_2):

$$P = I_0{b_0}^2 + I_1{b_1}^2 + I_2{b_2}^2 + b_0b_1I_0' + b_1b_2I_1' + \frac{1}{-p_0}(I_2' + 2p_2I_2 + p_1I_1')$$

in each case. We know there is a non-trivial solution as stated in section 4.6, but again using that method will not be efficient since the degrees of the coefficients of P are sparse around 30 and we will end up having to eliminate the variables using resultants and discriminants as in the Mitschi-Singer example for $SO_4(\mathcal{C})$, then solving a polynomial in b_0 of degree 143.

Chapter 6

Summary

The work done in the previous chapters shows that one can decide whether a given linear differential equation $L_4(y) = 0$ of order 4 can be solved in terms of equations of lower order, and if so produce those lower order equations that one can solve to find the solutions of $L_4(y) = 0$. Let us give a summary of the method one should follow. First of all let us recall that one can always assume that $L_4(y) = 0$ is of the form $y^{(4)} - p_2 y^{(2)} - p_1 y' - p_0 y = 0$.

1. Test the reducibility of L_4 using the results on factorisation of differential operators (see (5), (6) and (19)).

   If L_4 is reducible over the coefficient field, use Maple to find the factors of lower order. If not go to the next step.

2. Check whether $L_4(y) = 0$ has liouvillian solutions using the work of M.F.Singer and F.Ulmer in (37). If so find them using (21). If not check for a factorization of

L_4 over a quadratic extension of the field of coefficients as in Example 2.

3. If all the above failed use Table 4.1 in the following order:

   Using Maple compute and factor the second exterior power and the second symmetric power of L_4. For this finding the eigenring and then calculating the factors might be faster.

   (a) If $\bigwedge^2 L_4$ is of order 5 or has a rational solution and if ${L_4}^{ⓢ2}$ is of order 7 or has a factor of order 3, then L_4 is equivalent to the third symmetric power of an operator of order 2, say L_2. Use the equations in Proposition 6 to check whether $L_4 = {L_2}^{ⓢ3}$ and if so Proposition 6 gives L_2 as well. In case the equations in Proposition 6 do not hold find an equation of order 2 the second symmetric power of which is equivalent to the factor of order 3 L_3 of ${L_4}^{ⓢ2}$ using Lemma 7 and take its third symmetric power. To do this one can use the Maple package Bernina ((7)) to find the rational solutions of the second symmetric product of L_3, L_3' and L_3''. This third symmetric power is equivalent to L_4 by a gauge transform that one can calculate using Lemma 1 and the Maple package ISOLDE ((1)).

   (b) If $\bigwedge^2 L_4$ has 2 factors of order 3 or $L^{ⓢ2}$ has a rational solution, factor $L^{ⓢ3}$.

      - If $L^{ⓢ3}$ has 2 factors of order 8 L_4 is the least common left multiple of 2 equations of order 2 with coefficients in a degree 4 extension of the field of coefficients. To find those use Beke's algorithm ((2)). Else factor $L^{ⓢ4}$. This task can be expensive since it might already have order 35.

- If $L^{\circledS 4}$ has one irreducible factor of order 10 L_4 is the least common left multiple of 2 equations of order 2 with coefficients in a degree 6 extension of the field of coefficients. Again use Beke's algorithm to factor. Else factor $L^{\circledS 6}$, which can also be too expensive as $L^{\circledS 6}$ can be of order 84.

- If $L^{\circledS 6}$ has one irreducible factor of order 21 L_4 is the least common left multiple of 2 equations of order 2 with coefficients in a degree 12 extension of the field of coefficients. For this use Beke's algorithm.

(c) If none of the above hold use Proposition 10 to write L_4 equivalent to the least common left multiple of 2 operators of order 2 with coefficients in an extension of degree 4, by first finding the rational solutions of the second symmetric power of L_4, L_4', L_4'' and ${L_4}^{(3)}$ using Bernina, then solving the corresponding quadratic polynomial that is over determined, which gives us the gauge transform for the equivalence.

4. If none of the above hold, then L_4 is not solvable in terms of lower order equations.

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