Abstract

Soto, Marco Antonio. Actuator Saturation Control of LPV Systems and Systems with Rate and Amplitude Saturation. (Under the direction of Dr. Fen Wu)

In this thesis, we consider the design of anti-windup compensators for exponentially unstable systems with actuator saturation of amplitude and rate, as well as linear parameter varying systems. A set of synthesis conditions for anti-windup compensators are developed for each of the system types, in which the effects of actuator saturation are modelled as sector-bounded nonlinearities, using traditional linear fractional transformations. The performance criteria are the minimization of the induced $\mathcal{L}_2$ norm from disturbance input to error output, as well as the minimization of controller windup due to actuator saturation. Explicit construction formulae are provided for the direct construction of these anti-windup compensators. An exponentially unstable linearized model of an F8 aircraft is used to validate the results of the control analysis. We present the main advantage of the two-step anti-windup controller design procedure; the ease of implementation and the maintenance of high performance criteria in design.

ACTUATOR SATURATION CONTROL OF LPV SYSTEMS AND SYSTEMS WITH RATE AND AMPLITUDE SATURATION

by

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To my parents, *Carlos* and *Teresa*, and my love *Heather*.

Biography

Marco Antonio Soto was born May 11 1977, in New York City. He moved to Cary, NC where he attended Apex High School. After some time in Wilmington, NC, he moved back to Raleigh to attend North Carolina State University. In 2000, he graduated with a BS in Mechanical Engineering. After moving to Coleman Falls, VA to complete his Co-op, he returned to Raleigh, NC and received a BS in Pure Mathematics in 2001. He enrolled in graduate school at NCSU in the spring of 2001.

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Chapter 1

Introduction

An often neglected component in the design of control systems (is) the specifics of the sensor and actuator dynamics. In particular, the saturation limits of the actuators in the system are usually ignored when a controller is being synthesized. However, actuator saturation is an element in all real physical systems, as real actuators cannot supply an infinitely large amount of output. The result of this saturation is a difference between the control input demanded by the controller, and the realized output of the actuator. Obviously, this can lead to degradation in the performance of the control system, and in some cases, instability of the control system. Recent attempts to correct this situation have opened up a new focus of research in control systems. This area of interest is referred to as anti-windup control or anti-windup compensation, due to the fact that the saturation phenomenon causes integral terms in the control equations to increase rapidly, or, "wind-up" [2].

Some critical applications where this type of constraint is pertinent include the maximum flow capacity in a fuel valve, the peak saturation voltage in a operational amplifier, and the deflection angle of a control surface in an aircraft, among others. In the particular example of flight control, the presence of actuator saturation constraints limits the approaches to control. Aircraft control surfaces have limitations both in their maximum deflections and in their rate of variation due to geometric and aerodynamic constraints. Actuator limitations have been attributed to the crash of several aircraft. These limitations have also been identified as a major contribution to catastrophic pilot induced oscillations [23]. Also, the failure of several aircraft have been attributed to the saturation of actuators. In the case of the F-22 crash in April 1992 [7], the control surface rate was limited. A similar occurrence caused the Gripen crash in August of 1992 [30].

The goal, then, of any anti-windup control scheme, would be to stabilize the system in the presence of the saturation, and to approach as closely as possible the performance of the system in the absence of such saturation. Typically, this saturation is modelled as a nonlinearity which creates difficulties in the synthesis of a controller to handle such discrepancy. A common method of analyzing these problems involves modelling the plant as a linear time-invariant (LTI) plant connect to a sector-bounded nonlinearity via an upper linear fractional transformation (LFT). With this complete, a small-gain theorem (or its variant) is applied to the system for the stability and performance analysis. This often amounts to an overly conservative estimation of the conditions. This is a single-step procedure, in which the goals of stability, performance, and saturation control are united into a single control synthesis problem, which would then be solved by traditional means. The alternative approach is the two-step procedure, in which the saturation control has been divided into nominal control and anti-windup compensation designs. In the following subsection, we will discuss examples of both types of design.

The single-step procedure can, at times, become a more complicated approach to anti-windup controller synthesis. This is due to the fact that the combination of various criteria leads to a more conservative procedure for design. That is, if the stability and performance criteria of the original controller design are augmented by the addition of saturation control goals, the approach taken to synthesize the controller must be amended. This usually involves the enlargement of already spacious LMIs to include the new goals. Furthermore, the new conditions sometimes suffer from a loss of convexity. So while the single-step procedure involves fewer steps, the computations can become more complicated than effective.

The two-step design procedure has the handicap of requiring more controller synthesis LMIs and more calculations in general. However, this procedure has several distinct advantages over the single-step procedure. The main advantage is that it can be used as an augmentation of a nominal control scheme which suffers from actuator saturation. If it is unclear wether or not a control system will suffer a loss of performance or stability due to saturation, a nominal controller can be designed without regarding anti-windup problems. If this controller is insufficient, then the proposed synthesis approach can be taken. This implementation usually only requires a software or computer change. Thus, it is usually the case that no additional hardware will be necessary in order to implement the anti-windup compensator. Simply, a new addition to the control law will be added, only requiring that the effects of saturation can be measured by existing sensors. That is, that the realized control output can be measured by the existing sensor, in order to be compared with the nominal output.

The other major advantage of the two-step design procedure lies in the construction of the original controller. When designing the nominal controller without regard to saturation, the designer is free to choose as strict a set of performance criteria as desired. Since the concerns of actuator saturation are not present, a well-performing optimal controller can be sought and implemented. If this proves to be insufficient in the face of actuator saturation, then a compensator can be designed. Therefore, this approach does not suffer from the inherent conservatism involved in the design of a single-step anti-windup compensator.

1.1 Background

As stated earlier, recent years have seen a variety of control techniques for saturated LTI systems. In [22], a generalized framework for many anti-windup control schemes was proposed. Previous to that work, this area of research suffered from the lack of unity among control and performance objectives. The work of Teel and Kapoor in [33] brought a definition of anti-windup compensation in terms of $\mathcal{L}_2$ stability and performance. Another key idea is the formulation of the synthesis of anti-windup controllers in terms of LMI problems. This was undertaken in [24, 31, 20]. The main advantage of this approach is the adaptation of the well known Circle stability criteria to the anti-windup compensation problem. In [15], the concept of a null-controllable region is used in conjunction with one-step methods as an approach to anti-windup compensator design for LTI systems. Also, several approaches to systems with rate and magnitude saturation have been undertaken for LTI systems. These also involve the study of the null-controllable region in designing a single step anti-windup compensator as in [17].

Often, it is more appropriate to model a physical system as a Linear Time-Varying (LTV) system. This is the case when the parameters that make up the state-space model of the system are functions of time. A specific class of these systems are the Linear Parameter-Varying (LPV) systems. These are systems whose state-space data are functions of some parameter which is varying with time. (The parameter is assumed varying in a bounded set and its current vaule is measurable). In [1, 35, 26] LMI methods for the $\mathcal{H}_\infty$ control of these LPV systems are established. These gain-scheduling methods are used typically to design controllers for these systems.

In [38], a one-step approach is used in conjunction with classical LPV methods. The key idea in this work is that the saturation effects can be modelled as an LPV block attached to the system and not a nonlinearity block. While the system is still considered nonlinear, the modelling of the actuator saturation is now a gain-scheduling block. This allows for the controller to be designed in a one step process, and also allows for the use of classical LPV methods such as the ones in [1] in solving the resulting controller synthesis problems.

Another one-step approach is taken in [16]. Here, the actuators are subject to amplitude and rate saturation, and a discrete control scheme was developed using state feedback and dynamic output feedback. A linear quadratic regulator design was modified to compensate the saturated system. The saturation was modelled as a sector-bounded nonlinearity, as in this work. Also the model for rate saturation of the actuators was developed, which will be used in the simulations of the F8 aircraft rate saturation.

A two-step approach involving the modelling of the saturation as a sector-bounded nonlinearity was taken in [11]. In this paper, the design of an anti-windup *compensator* was undertaken. Instead of designing a controller which would single-handedly stabilize the system in the face of actuator saturation, the author sought to design a compensator for a nominal controller which stabilized the system and had a relatively good measure of performance. The synthesis was based on the linear matrix inequality (LMI) forms of the small gain theorem and bounded real lemma. The anti-windup compensator was designed in such a way as to act as a gain upon the difference between the saturated and unsaturated control input. This gain was then used as an output which acted to update the controller states and the controller output. The approach was to recast the design of anti-windup compensator as a classical $\mathcal{H}_\infty$ problem. In this case, the analysis and synthesis conditions would be convex and solvable LMIs. The strongest assumption made in this paper was that the nominal plant must be stable.

The work of Wu and Lu [39] builds upon this concept by removing the stability requirement. The work of Teel, [32] also includes the conditions for synthesizing anti-windup compensators for exponentially unstable systems. Thus, an anti-windup compensator design scheme for stable or unstable systems was created. This approach also involved the use of convex and solvable LMIs in order to synthesize and analyze the stability and performance of the closed loop system.

There are two natural extensions to the previous work. This work involves dismantling the problem of the maximum saturation magnitude of an actuator. Another type of saturation which occurs during actuation is that of the rate of response of the actuator. This is also limited by the physical constraints on the system. The main difference in these types of saturation is that the rate saturation is more difficult to measure. However, with a clever design of the control system architecture, this problem becomes more simple. This rate and magnitude saturation problem can then be addressed in a similar manner to the magnitude saturation problem, by employing a two-step design procedure using convex and solvable LMIs.

The second conclusion involves linear time variant (LTV) problems. More specifically, the class of LTV plants which can be described by a parameter variation. This class includes the class of linear parameter varying (LPV) plants. Can a similar design procedure be used to address LPV plants undergoing actuator saturation? The typical response to designing a controller for an LPV plant is the gain-scheduling method used in Packard [26] and Apkarian [1] among other papers. We shall seek to design a gain-scheduled anti-windup which depends on the same parameters as the gain-scheduled controller and LPV plant. This way, a system can be stabilized against a sector-bounded nonlinearity (saturation) and parameter variation.

When using LMI methods to synthesize controller solutions to $\mathcal{H}_\infty$ problems, the typical procedure is to first solve some LMI which corresponds to the existence of a stabilizing controller using a procedure to eliminate the controller variables from the inequality. With this accomplished, the scaling matrix solutions are back-substituted into the original LMI in order to solve for the controller gains. In [13], an explicit construction scheme was developed which is equivalent to the above procedure. This scheme allows us to explicitly solve for the controller gains after evaluating the synthesis LMI. This approach was also used in [39] in order to construct an explicit anti-windup compensator.

We will apply our proposed rate and magnitude anti-windup compensator to the control of a linearized F-8 aircraft model. This model will contain an unstable pole which was added in to demonstrate the capability of the proposed design to compensate for windup in the presence of instability.

1.2 Thesis Objective

The objectives of this thesis are all related to the general problem of anti-windup compensation for actuator saturation problems with the specific example of an exponentially unstable model of an F8 aircraft.

First, we hope to provide modifications to well-known stability and performance theorems in order to make them compatible with the goals of anti-windup compensation. That is to say, we would like to cast the problem of saturation control in a way that is consistent with familiar LMI methods.

The second goal is to provide synthesis and analysis conditions for systems suffering actuator saturation in two general areas. The first is the general class of systems which are constrained by both rate and magnitude saturations. The second such case is input saturated systems that are LPV gain-scheduled systems being controlled by a nominal gain-scheduled controllers.

With synthesis and analysis conditions obtained, we would like to provide explicit formulae for the construction of such controllers, in order to avoid the overuse of complicated LMIs.

Finally, an example of the usage of a rate and magnitude controller will be provided. We hope to control an F8 aircraft model which suffers from rate and magnitude saturation with the proposed anti-windup controller design.

1.3 Thesis Outline

Chapter 1 has a literature review of the previous work on anti-windup compensation and actuator saturation. This chapter also outlines the objectives and content of this thesis.

Also, the motivation for the necessity of this type of anti-windup compensator design is included.

Chapter 2 contains some information on the mathematical tools used to throughout this work. Specifically, the robust control framework, including LFT's, LMI's and matrix definitions, is established as a foundation for this work. Afterwards, the theorems and lemmas used to establish the stability and performance criteria, as well as the synthesis and analysis conditions, are given without proof.

Chapter 3 is a robust analysis of sector-bounded nonlinearities for both cases studied in this thesis. In particular, the problem of stabilizing a system in the presence of a sector-bounded nonlinearity is given the treatment of a standard $\mathcal{H}_\infty$ problem using an LMI framework.

Chapter 4 provides a thorough derivation of the synthesis conditions for the rate and magnitude anti-windup compensator, as well as the gain-scheduled anti-windup compensator. This is done through the use of the ideas contained in the previous chapters. With synthesis conditions established, and proven to be feasible and convex, the construction procedure for the rate and magnitude anti-windup compensator will be given. Also, the construction procedure for the gain-scheduled compensator will be derived in a similar fashion. These procedures are explicit construction schemes as in [13], as opposed to the feasibility approach used in [11].

Chapter 5 contains the numerical results of applying the proposed anti-windup compensator design to the F8 model. First, the effects of the magnitude saturation is shown, as well as the effects of the rate saturation. These two nonlinearities will be applied to the F8 model with a nominal $\mathcal{H}_\infty$ controller. The performance of the system will then be compared to the performance of the system with the proposed anti-windup compensator design added in.

Finally, the Chapter 6 will contain a summary of the main results, as well as provide a commentary on the future work in this area.

Chapter 2

Mathematical Preliminary

We desire to construct the problem of anti-windup compensator simulation using traditional $hinf$ control procedures. In order to do this, we must introduce several mathematical concepts which are key to this type of analysis. This chapter provides several of those results without proof. We shall present the $\mathcal{L}_2$ norm used as a performance measure here, as well as the Scaled Bounded Real Lemma and an overview of the previous results on LPV systems and gain-scheduled $\mathcal{H}_\infty$ control theory.

2.1 Signals, Norms, Operators, and Matrix Definitions

The main objective in any control system design is usually twofold. The first is to achieve stability, and the second is to achieve some performance criterion. In physical systems, these typically involve reducing some tracking error, or minimizing the settling time or peak overshoot of a system. By examining these control objectives in the light of a complex function space, we can eliminate talk of parameters with different units and different contexts and unite the goals under the lens of a single type of function. By exploring the norms of these signals, we can recast our performance objectives so that they all meet a single type of criterion.

A *Hilbert space* is a complete inner product space with its norm induced by its inner product [40]. A useful infinite dimensional Hilbert space is $\mathcal{L}_2(j\mathbf{R})$, which consists of all square integrable and Lebesgue measurable functions F defined on the interval $[a, b]$ with its inner product and induced norm defined as,

$$\langle F, G\rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} trace\left[F^*(j\omega)G(j\omega)\right]dw, \quad and \quad \|F\|_2 := \sqrt{\langle F, F\rangle}$$

$\mathcal{H}_\infty$ is a closed subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right-half plane. The $\mathcal{H}_\infty$ norm is defined as,

$$\|F\|_\infty := \sup_w \bar{\sigma}\left(F(j\omega)\right)$$

Here, the $\bar{\sigma}$ represents the maximum singular value. So in other words, the $\mathcal{H}_\infty$ norm of a system matrix is the largest of the maximum singular values over all frequency. In essence, it is the largest amplification from input to output that the system will experience. This is the norm we shall use in order to specify the performance criteria. Generally, we shall seek to minimize the $\mathcal{H}_\infty$ norm of certain signals (i.e. the tracking error, or the norm from disturbance to plant output).

A Matrix A is said to be *positive definite* (denoted as $A > 0$) if $x^*Ax > 0$ for all $x \neq 0$. Similarly, a Matrix A is said to be *positive semidefinite* if $x^*Ax \geq 0$ for all $x \neq 0$. This will be a condition in many of the Linear Matrix Inequalities that we wish to solve. A negative definite (semidefinite) matrix is defined in a similar way.

A pair of matrices (C, A) is said to be *detectable* if $A + LC$ is stable for some L.

Similarly, consider a matrix pair (A, B). Then the following theorem holds.

Theorem 1 *The following statements are equivalent.*

(i) (A, B) *is stabilizable*

(ii) The matrix $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ *has full row rank for all* λ *with nonzero real component.*

(iii) For all λ *and* x *such that* $x^*A = x^*\lambda$ *and* $Re\lambda \geq 0, x^*B \neq 0$.

(iv) There exists a matrix F *such that* $A + BF$ *is Hurwitz stable.*

The last result will prove useful in the study of the Lyapunov equation.

2.2 Linear Fractional Transformations

A Linear Fractional Transformation (LFT) is a useful tool in the control analysis and control synthesis. It is a way that we can rewrite the interconnection of multiple matrices in a block diagram into a more compact package. The definition of lower and upper LFT's follow.

If M is a complex matrix partitioned in the following form,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

then, the linear fractional transformations are defined as,

$$\mathcal{F}_l(M, \Delta_l) := M_{11} + M_{12}\Delta_l(I - M_{22}\Delta_l)^{-1}M_{21} \tag{2.1}$$

$$\mathcal{F}_u(M, \Delta_u) := M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12} \tag{2.2}$$

An LFT, $\mathcal{F}_l(M, \Delta)$, is said to be *well-defined* (*well-posed*) if $I - M_{22}\Delta$ is invertible. Also, an LFT, $\mathcal{F}_u(M, \Delta)$, is *well-defined* if $(I - M_{11}\Delta)^{-1}$ exists.

In order to see the physical meaning of LFTs in control science, we take M to be a proper control transfer matrix. With this interpretation the LFT simply represents a closed loop transfer matrix. In this case, the LFT can be considered to close the loop with the transfer matrix and controller (usually lower LFTs) or to close the loop with a plant and its associated uncertainty (usually upper LFTs). More detailed information on LFT may be found in [40].

In the anti-windup synthesis, we will use an LFT to represent the connection between the plant (with controller) and the anti-windup compensator. We will also define the relation between the associated nonlinearity and the plant as an upper LFT. This way, we can examine the stability of the plant against the nonlinearity.

2.3 Linear Matrix Inequalities

Linear Matrix Inequalities (LMIs) are useful tools in control analysis and control synthesis. Indeed, our control synthesis and control analysis requirements can be (and will be) written as LMIs. Most notably we have the so-called Algebraic Riccati Equation (ARE), the Lyapunov equation, and the LMI form of the bounded real lemma. In this section, we will examine the the first two equations, and the bounded real lemma will be inspected.

In the process of control analysis, the Lyapunov equation is useful for examining the stability properties of a system matrix. Consider given real matrices A and H in the Lyapunov equation:

$$A^*Q + QA + H = 0 \tag{2.3}$$

The relationship between the solution of this equation, Q and the stability of the matrix A can be summarized in the following two theorems.

Theorem 2 *Assuming that A is stable in the Lyapunov equation, the following statments are then true:*

(i) $Q = \int_0^\infty e^{A^*t} H e^{A^t} dt$

(ii) $Q > 0$ *if* $H > 0$ *and* $Q \geq 0$ *if* $H \geq 0$.

(iii) if $H \geq 0$ *then (H,A) is observable if and only if* $Q > 0$

A natural corollary of this theorem is that this equation can be arranged in order to learn about the controllability and observability of the (A, B, C) triple.

In many cases, we are given the solution to the Lyapunov equation and desire to conclude the stability of matrix A. In that case, the following theorem proves useful.

Theorem 3 *Given solution Q to the Lypaunov equation, then the following results hold*

(i) The real part of $\lambda(A) \leq 0$ *if* $Q > 0$ *and* $H \geq 0$

(ii) A is stable if $Q > 0$ *and* $H > 0$

(iii) A is stable if $Q \geq 0$, $H \geq 0$ *and (H,A) is detectable.*

The usefulness of the last consequence of this theorem will be apparent in the LMI form of the bounded real lemma, where we will use it to prove some useful results.

2.3.1 Scaled Bounded Real Lemma

The *bounded real lemma*, proposed by Khargonekar and Zhou [21], is used in converting the $\mathcal{H}_\infty$ norm constraint of an LTI system into an equivalent linear matrix inequality condition. This effectively converts the complicated problem of calculating a maximum over all frequencies into a single system of inequalities. Its usefulness in control theory is to examine performance in a control analysis problem, or to examine feasibility in a control synthesis problem. The scaled bounded real lemma, presented below, is useful for converting the $\mathcal{H}_\infty$ norm constrain of LPV systems, and ultimately providing analysis and synthesis conditions for gain-scheduled control schemes of parameter-varying systems.

Lemma 2.3.1 *Consider an uncertain parameter structure Δ, the associated set of positive definite similarity scalings defined by $L_\Delta = \{L > 0 : L\Delta = \Delta L\} \subset \mathbf{R}^{r\times r}$, and a square continuous-time transfer function $T(s) = D + C(sI - A)^{-1}B$. Then the following statements are equivalent,*

1. *A is stable and there exist $L \in L_\Delta$ such that*

$$\|L^{1/2}(D + C(sI - A)^{-1}B)L^{-1/2}\|_\infty < \gamma$$

2. *There exist positive definite solutions X and $L \in L_\Delta$ to the matrix inequality,*

$$\begin{pmatrix} A^TX + XA & XB & C^T \\ B^TX & -\gamma L & D^T \\ C & D & -\gamma L^{-1} \end{pmatrix} < 0.$$

This form of the bounded real lemma is presented in [1] and [26].

2.4 Linear Parameter-Varying Systems

Linear parameter-varying (LPV) systems are a special class of linear time-variant systems (LTV). In an LTV plant, the state-space matrices are functions of time, as in the plant below

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

The implementation of the solution to the $\mathcal{H}_\infty$ control problem which corresponds to this type of plant is often impractical as it involves integration of Riccati differential equations in real time [1]. However, small gain LTI techniques can be applied to plants whose time dependance has the form

$$\dot{x}(t) = A(\Theta(t))x(t) + B(\Theta(t))u(t) \tag{2.4}$$
$$y(t) = C(\Theta(t))x(t) + D(\Theta(t))u(t) \tag{2.5}$$

where $\Theta(t)$ is a vector of time-varying plant parameters. The state-space matrices then become functions of Θ. This is the class of LPV systems. Furthermore, we would like to restrict our attention to the class of LPV systems where the state-space matrices are linear fractional functions of the gain-scheduling parameter Θ.

The typical approach for these LPV plants involves the small gain theorem by treating the parameter variations as an uncertainty block, and by then designing a single robust controller for the resulting family of systems as in [26]. However, this approach is usually overly conservative [1].

The approach used in [1] involves designing robust controllers around each operating point and to then switch between controller according to some gain-scheduling policy [26]. In this work, we will use a gain-scheduling policy to switch between anti-windup compensators for the LPV system.

The form of the controller which will be corrected by the gain-scheduled anti-windup compensator is given below:

$$\begin{bmatrix} \dot{x}_k \\ u \\ z_k \end{bmatrix} = \begin{bmatrix} A_k & B_{k1} & B_{k\theta} \\ C_{k1} & D_{k11} & D_{k1\theta} \\ C_{k\theta} & D_{k\theta 1} & D_{k\theta\theta} \end{bmatrix} \begin{bmatrix} x_k \\ y \\ w_k \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$
$$w_k = \Theta z_k$$

Notice the input v. This is the correcting information from the anti-windup compensator.

2.4.1 Gain-Scheduled $\mathcal{H}_\infty$ Control Theory

In this subsection, we shall describe the nature of the gain-scheduled $\mathcal{H}_\infty$ control problem. The key notion for this type of problem is that of converting the parameter-dependant structure of the plant and controller into that of the classical uncertainty structure of a standard $\mathcal{H}_\infty$ problem.

The linear fractional dependence on Θ in an LPV plant is essentially represented by an upper LFT connection as follows [1]

$$\begin{bmatrix} z \\ y \end{bmatrix} = F_u(P, \Theta) \begin{bmatrix} w \\ u \end{bmatrix} \tag{2.6}$$

Where P is the LTI plant and Θ is the block diagonal time-variant operator which specifies the relation of θ to the plant dynamics. In actuality,

$$\Theta = \text{diag}(\theta_1 I_{r1}, \theta_2 I_{r2}, ..., \theta_K I_{rK}) \tag{2.7}$$

In order to maintain a square format for this matrix, $r_i > 1$ whenever the parameter θ_i is repeated [8].

The plant with this parameter dependance can be written in LFT form as

$$\begin{bmatrix} z \\ e \\ y \end{bmatrix} = \begin{bmatrix} P_{\theta\theta} & P_{\theta 1} & P_{\theta 2} \\ P_{1\theta} & P_{11} & P_{12} \\ P_{2\theta} & P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ d \\ u \end{bmatrix}$$
$$w = \Theta z$$

The common interpretation of this set of equations is that z and w are pseudo inputs and outputs, respectively. Therefore, the controller design problem becomes that of finding a controller of the form

$$u = F_l(K, \Theta)y \tag{2.8}$$

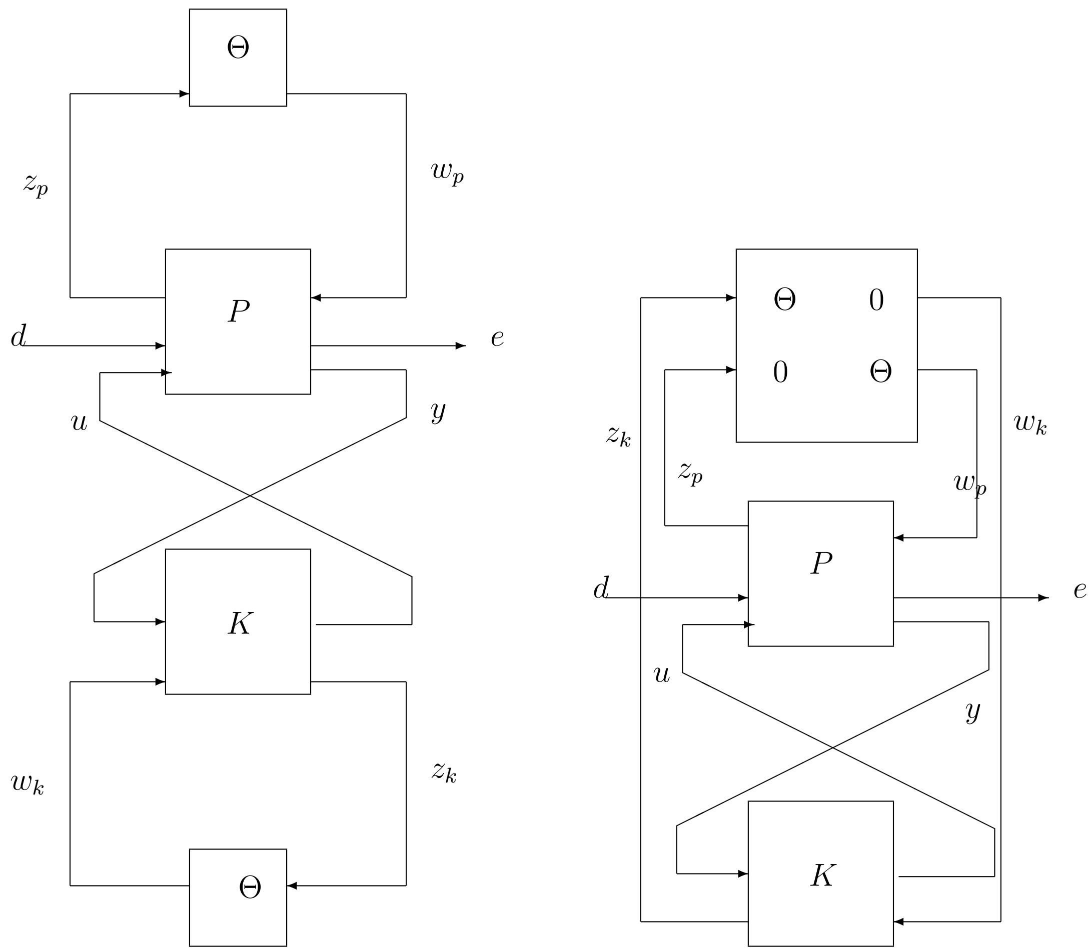
The controller will also have an LFT dependance on the gain-scheduling parameter θ. In this case, the parameter is the scheduling variable which gives the rule of updating the controller's information based on the measurements of Θ [1].

The LFT interconnection of plant and controller is shown in figure 2.1.

The closed-loop transfer function from disturbance d to output e is given by

$$T(P, K, \Theta) = F_l(F_u(P, \Theta), F_l(K, \Theta)) \tag{2.9}$$

In order to better analyze the systems with traditional control techniques, we shall rearrange the system matrices in order to match the upper LFT structure used in classical $\mathcal{H}_\infty$ theory. The new system, G will be the interconnection of plant and controller excluding the parameter dependance. This new system G will then be connected via upper LFT to the augmented block repeated uncertainty structure $\begin{bmatrix} \Theta & 0 \\ 0 & \Theta \end{bmatrix}$. This structure will hereafter be denoted as $\Theta \oplus \Theta$.



(a) LPV Synthesis Framework (b) Equivalent transformation

Figure 2.1: LPV Control Systems Diagrams

In light of this rearrangement, the LPV problem can now be interpreted as a robust performance problem for the new system, with a norm-bounded uncertainty $\Theta \oplus \Theta$. By using small gain theory [40], we can define a solution. First, it is necessary to obtain a set of positive definite similarity scalings associated with the block uncertainty structure. As in [1], we shall use the set

$$L_{\Delta} = \{L > 0 : L\Theta = \Theta L, \forall \Theta \in \Delta\} \subset \mathbf{R}^{r \times r}$$

where

$$r = \sum_{i=1}^{K} r_i \tag{2.10}$$

This set has the following important properties [26]

(i) $I_r \in L_\Delta$.

(ii) $L \in L_\Delta \Rightarrow L^T \in L_\Delta$.

(iii) $L \in L_\Delta \Rightarrow L^{-1} \in L_\Delta$.

(iv) $L_1 \in L_\Delta, L_2 \in L_\Delta \Rightarrow L_1 L_2 \Theta = \Theta L_1 L_2, \forall \Theta \in \Delta$.

(v) L_Δ is a convex subset of $\mathbf{R}^{r \times r}$.

Given this set, it is easy to show that the set of scalings which commute with the repeated structure, $\Theta \oplus \Theta$ is

$$\begin{aligned} L_{\Delta \oplus \Delta} = \Bigg\{ \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix} > 0 : L_1, L_3 \in L_\Delta \quad \text{and} \\ L_2 \Theta = \Theta L_2, \quad \forall \Theta \in \Delta \} \end{aligned}$$

We shall use this result in forming our synthesis condition, as our closed loop plant will have a block repeated uncertainty structure as above. This result, in conjunction with the so-called Elimination Lemma [5] will allow us to eliminate the uncertainty structure of the anti-windup compensator in the closed loop plant. Note that the key difference in this set is that the matrix L_2 is commutable with gain-scheduling parameter Θ, but is not a member of the set L_Δ. Therefore, it is not necessary for the matrix L_2 to be positive definite.

Chapter 3

Robust Analysis of Sector-Bounded Nonlinearities

In this chapter, we shall analyze the saturation nonlinearities in both the LTI and LPV systems. We shall extend the well-known Circle criterion results to apply to the systems examined in this work. The LTI extension will be similar to the one found in Wu [39], with the addition of the rate saturation block. The LPV extension will be another extension of [39] but combined with similar results involving scaling matrices as in Apkarian [1].

3.1 LTI System with Sector-Bounded Nonlinearity

In this section, we shall provide the framework for the LTI system with sector-bounded nonlinearity that we shall use throughout this thesis to represent the system with actuator saturation.

Consider a LTI system interconnected with an input sector-bounded nonlinearity.

$$\begin{bmatrix} \dot{x} \\ u \\ e \end{bmatrix} = \begin{bmatrix} A & B_0 & B_1 \\ C_0 & D_{00} & D_{01} \\ C_1 & D_{10} & D_{11} \end{bmatrix} \begin{bmatrix} x \\ q \\ d \end{bmatrix} \tag{3.1}$$

$$q = \psi(u) \tag{3.2}$$

Note the addition of the nonlinearity output q which has the same dimension as input u. The nonlinearity ψ defines a conic sector $[0, k]$ which is a form of the constraint on the input u.

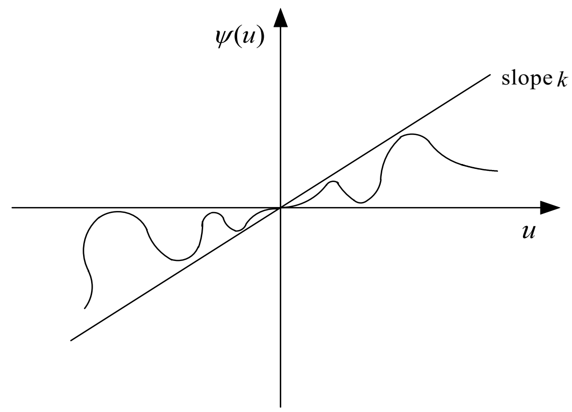


Figure 3.1: Sector-Bounded Nonlinearity

Another way of stating the constraint on nonlinearity ψ is to examine the input/output relationship that nonlinearity ψ imposes on the u, q pair. That is to say

$$q^T W(ku - q) \geq 0$$

for any diagonal matrix $W = \text{diag}\{w_1, w_2, ..., w_{n_u}\} > 0$ [39]. This matrix must be *positive definite*. This will be used to describe the input saturation for the system. Section 3.2 will provide a modified Circle criterion for deriving the synthesis and analysis conditions for the stability of this plant.

For the case where there are multiple types of saturation nonlinearities (such as when both rate and magnitude saturation are present in the system), the criterion must be modified slightly. Now instead of a constant k relating the input to output, a matrix $\bar{k}$ must be used. The format of this matrix is

$$\bar{k} = \begin{bmatrix} k_m I_{n_u} & 0 \\ 0 & k_r I_{n_u} \end{bmatrix}$$

.

Here, k_m describes the constraint on the magnitude of actuation, and k_r describes the constraint on the rate of actuation. The input/output relationship is now described by the following matrix inequality:

$$\tilde{q}^T W(\bar{k}\tilde{u} - \tilde{q}) \geq 0$$

for any diagonal matrix $W = \text{diag}\{w_1, w_2, ..., w_{2n_u}\} > 0$. Here, $\tilde{q} \in \mathbf{R}^{2n_u}$ is the vector containing feedback information about the two types of actuator saturation, that is, the difference between the unsaturated and saturated values of control input u and rate of control input $\dot{u}_{ss}$. The vector $\tilde{u} \in \mathbf{R}^{2n_u}$ is the vector of inputs to the two saturation blocks.

3.2 Modified Circle Criterion for Nonlinear Stability of LTI Plants

In this thesis, we will model the effects of actuator saturation as a nonlinearity. We would like to be able to guarantee that our closed-loop system is stable against the effects of some nonlinearity. This way, we can have a true synthesis condition. The following result was proven by Wu and Lu [39], and will be used to state our stability goals in terms of another LMI.

Theorem 4 *Given $\gamma > 0$ and the nonlinear system (3.1) - (3.2) if there exist a positive definite matrix $P \in \mathbf{S}_{+}^{n\times n}$ and a diagonal matrix $W > 0$, such that*

$$\begin{bmatrix} A^T P + PA & PB_0 + C_0^T \bar{k} W & PB_1 & C_1^T \\ B_0^T P + W\bar{k}C_0 & W\bar{k}D_{00} + D_{00}^T \bar{k} W - 2W & W\bar{k}D_{01} & D_{10}^T \\ B_1^T P & D_{01}^T \bar{k} W & -\gamma I_{n_d} & D_{11}^T \\ C_1 & D_{10} & D_{11} & -\gamma I_{n_e} \end{bmatrix} < 0 \tag{3.3}$$

Then the nonlinear system is quadratically stable against nonlinearity $\psi \in$sect$[0, k]$ and $\|e\|_2 < \gamma \|d\|_2$.

We shall prove this result using S-Theory [5].

Proof: Consider a Lyapunov function of the form $V(x) = x^T P x$ for the nonlinear system, then a sufficient condition for the performance and stability properties of the nonlinear LPV system can be established via S-Procedure [5] from the following inequality

$$\dot{V} + \frac{1}{\gamma} e^T e - \gamma d^T d + 2\tilde{q}^T W(\bar{k}\tilde{u} - \tilde{q}) < 0$$

$$\begin{aligned}
&\dot{V} + \frac{1}{\gamma} e^T e - \gamma d^T d + 2\tilde{q}^T W(\bar{k}\tilde{u} - \tilde{q}) \\
&= \dot{x}^T P x + x^T P \dot{x} + \frac{1}{\gamma} e^T e - \gamma d^T d + 2\tilde{q}^T W(\bar{k}\tilde{u} - \tilde{q}) \\
&= \begin{bmatrix} x^T & \tilde{q}^T & d^T \end{bmatrix} \times \\
&\left\{ \begin{bmatrix} A^T P + PA & PB_0 + C_0^T \bar{k} W & PB_1 \\ B_0^T P + W\bar{k} C_0 & W\bar{k} D_{00} + D_{00}^T \bar{k} W - 2W & W\bar{k} D_{01} \\ B_1^T P & D_{01}^T \bar{k} W & -\gamma I_{n_d} \end{bmatrix} \right. \\
&\left. + \frac{1}{\gamma} \begin{bmatrix} C_1^T \\ D_{10}^T \\ D_{11}^T \end{bmatrix} \begin{bmatrix} C_1 & D_{10} & D_{11} \end{bmatrix} \right\} \begin{bmatrix} x \\ \tilde{q} \\ d \end{bmatrix} < 0
\end{aligned}$$

The last inequality is equivalent to the LMI condition 3.3 through Schur complement. *Q.E.D.*

In a later chapter, we shall recast our goals so that they match this LMI. That is, application of this theorem to both the nonlinear system for magnitude and rate saturation, as well as the LPV system used for gain-scheduling, will provide a synthesis and analysis conditions for the anti-windup compensators. Afterwards, we shall seek to find a solution which guarantees the system stability and performance margin.

3.3 Modified Circle Criterion for Nonlinear Stability of LPV Plants

In order to derive synthesis and analysis conditions for a LPV plant under the effects of actuator saturation, another modified version of the Circle criterion must be derived. This version takes into account the parameter dependance of the nominal plant, nominal controller, and anti-windup compensator.

Clearly, this theorem applies to a general LPV plant which is also constrained by a saturation nonlinearity which is described as a sector bounded nonlinearity. The equations for this plant are as follows.

$$\begin{bmatrix} \dot{x} \\ u \\ z \\ e \end{bmatrix} = \begin{bmatrix} A & B_0 & B_\theta & B_1 \\ C_0 & D_{00} & D_{0\theta} & D_{01} \\ C_\theta & D_{\theta 0} & D_{\theta\theta} & D_{\theta 1} \\ C_1 & D_{10} & D_{1\theta} & D_{11} \end{bmatrix} \begin{bmatrix} x \\ q \\ w \\ d \end{bmatrix} \tag{3.4}$$

$$q = \psi(u) \tag{3.5}$$

$$w = \Theta z \quad \|\Theta\| < 1 \tag{3.6}$$

$$\Theta = \mathrm{diag}(\theta_i I_i, ..., \theta_r I_r) \tag{3.7}$$

Theorem 5 *Given $\gamma > 0$, and the nonlinear system(3.4) - (3.7), if there exists $P \in \mathbf{S}_+^{n\times n}$, diagonal matrix $W > 0$, and a matrix $T \in L_\Delta$ such that*

$$\begin{bmatrix} A^TP + PA & PB_0 + kC_0W & PB_\theta & PB_1 & C_\theta^T & C_1^T \\ B_0^TP + kWC_0 & k(WD_{00} + D_{00}^TW) - 2W & kWD_{0\theta} & kWD_{01} & D_{\theta 0}^T & D_{10}^T \\ B_\theta^TP & kD_{0\theta}^TW & -T & 0 & D_{\theta\theta}^T & D_{1\theta}^T \\ B_1^TP & kD_{01}^TW & 0 & -\gamma I & D_{\theta 1}^T & D_{11}^T \\ C_\theta & D_{\theta 0} & D_{\theta\theta} & D_{\theta 1} & -T^{-1} & 0 \\ C_1 & D_{10} & D_{1\theta} & D_{11} & 0 & -\gamma I \end{bmatrix} < 0 \tag{3.8}$$

Then the nonlinear system is quadratically stable against $\psi \in \mathrm{sect}[0, k]$ and $\|\theta\| < 1$ and $\|e\|_2 < \gamma\|d\|_2$.

Proof: The proof here is similar to the one for Theorem 4. Consider a Lyapunov function of the form $V(x) = x^TPx$ for the nonlinear system, then a sufficient condition for the performance and stability properties come from the following inequality

$$\dot{V} + \frac{1}{\gamma}e^Te - \gamma d^Td + 2q^TW(ku - q) + z^TTz - w^TTw < 0$$

Expanding the terms in this inequality yields the complete inequality as follows:

$$\begin{aligned}
&\dot{V} + \frac{1}{\gamma}e^T e - \gamma d^T d + 2q^T W(ku - q) + z^T T z - w^T T w \\
&\quad = \dot{x}^T P x + x^T P \dot{x} + \frac{1}{\gamma}e^T e - \gamma d^T d + 2q^T W(ku - q) + z^T T z - w^T T w \\
&\quad = \begin{bmatrix} x^T & q^T & w^T & d^T \end{bmatrix} \times
\end{aligned}$$

$$\left\{ \begin{bmatrix} A^T P + PA & PB_0 + kC_0 W & PB_\theta & PB_1 \\ B_0^T P + kWC_0 & k(WD_{00} + D_{00}^T W) - 2W & kWD_{0\theta} & kWD_{01} \\ B_\theta^T P & kD_{0\theta}^T W & -T & 0 \\ B_1^T P & kD_{01}^T W & 0 & -\gamma I \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} C_\theta^T & C_1^T \\ D_{\theta 0}^T & D_{10}^T \\ D_{\theta\theta}^T & D_{1\theta}^T \\ D_{\theta 1}^T & D_{11}^T \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & \frac{1}{\gamma} I \end{bmatrix} \begin{bmatrix} C_\theta & D_{\theta 0} & D_{\theta\theta} & D_{\theta 1} \\ C_1 & D_{10} & D_{1\theta} & D_{11} \end{bmatrix} \right\} \times \begin{bmatrix} x \\ q \\ w \\ d \end{bmatrix} < 0$$

Which is equivalent to the LMI (3.8) via Schur complement. *Q.E.D.*

Chapter 4

Anti Windup Synthesis Conditions

In this chapter, we shall apply our extended Circle criterion results from chapter 3 to our anti-windup controller problems. We will begin by constructing the closed-loop system for the two cases. With that accomplished, we shall apply the theorems in order to derive synthesis conditions. Finally, we shall provide explicit controller formulas for the two types of system.

4.1 Rate and Magnitude Saturation Problem

The first problem we shall examine will be the problem containing actuators which have both their maximum amplitude and maximum rate of response constrained. We shall start by examining the system equations for such a system. Then we shall derive the augmented system for synthesis. Our stability theorem will then be applied to the augmented closed-loop system to provide synthesis conditions for the new system. Finally, we shall provide an explicit construction scheme for the anti-windup compensator for this case.

The LTI framework to be used throughout this work is the nominal plant, P described by

$$\begin{bmatrix} \dot{x}_p \\ e \\ y \end{bmatrix} = \begin{bmatrix} A_p & B_{p1} & B_{p2} \\ C_{p1} & D_{p11} & D_{p12} \\ C_{p2} & D_{p21} & D_{p22} \end{bmatrix} \begin{bmatrix} x_p \\ d \\ \sigma(u) \end{bmatrix} \tag{4.1}$$

Here, the plant state vector $x_p \in \mathbf{R}^{n_p}, y \in \mathbf{R}^{n_y}$ is the control measurement, and σ(u) is the saturated control input. The disturbance input $d \in \mathbf{R}^{n_d}$ and the controlled error output is $e \in \mathbf{R}^{n_e}$. Furthermore, it is assumed that (A_p, B_{p2}, C_{p2}) triple is stabilizable and detectable, and that the matrices $\begin{bmatrix} B_{p2}^T & D_{p12}^T \end{bmatrix}$ and $\begin{bmatrix} C_{p2} & D_{p21} \end{bmatrix}$ have full row rank.

The first assumption provides a guarantee that the nominal controller K is capable of stabilizing the open-loop plant sans any saturation to the inputs. The controller, K, is described by the following dynamic equations:

$$\begin{bmatrix} \dot{x}_k \\ u \end{bmatrix} = \begin{bmatrix} A_k & B_{k1} & B_{k2} \\ C_k & D_{k1} & D_{k2} \end{bmatrix} \begin{bmatrix} x_k \\ d \\ y \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{4.2}$$

Here, the vector $\begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ is the correcting information provided to the controller by the anti-windup compensator. Also, the vector $x_k \in \mathbf{R}^{n_k}$ is the vector of controller states.

One additional assumption is to restrict the matrix $D_{p22} = 0$. This assumption is not necessary, and only serves to simplify some of the construction calculations used later in this work. In many physical systems, it is the case that $D_{p22} = 0$, but if this is not the case, an auxiliary input can be made so that the this assumption holds true.

4.1.1 Rate and Magnitude Saturated Closed-Loop System

Figure 4.1 shows the makeup of the rate and magnitude saturated system. This is the closed-loop version of the system, including the anti-windup compensator gain. The first saturation nonlinearity block is the magnitude saturation block. This will limit the maximum output of the actuator. The rate saturation block is somewhat more complicated than the magnitude block. The design of this block matches the one used in [16].

In this configuration, the term k_b is a constant which specifies the bandwidth of the actuator. The subtraction term at the beginning of the block is multiplied by this bandwidth constant to produce a pseudo-derivative. This pseudo-derivative is then saturated by a rate saturation block which is set to saturate whenever the magnitude of this derivative term exceeds the saturation limit (set by the designer). After saturation, the term $\dot{u}_{ss}$ is the derivative of the actuator response which has been subject to both rate and magnitude saturations. A final pass through an integration block returns the twice saturate output u_{ss}, which is what will actually be experienced by the controller.

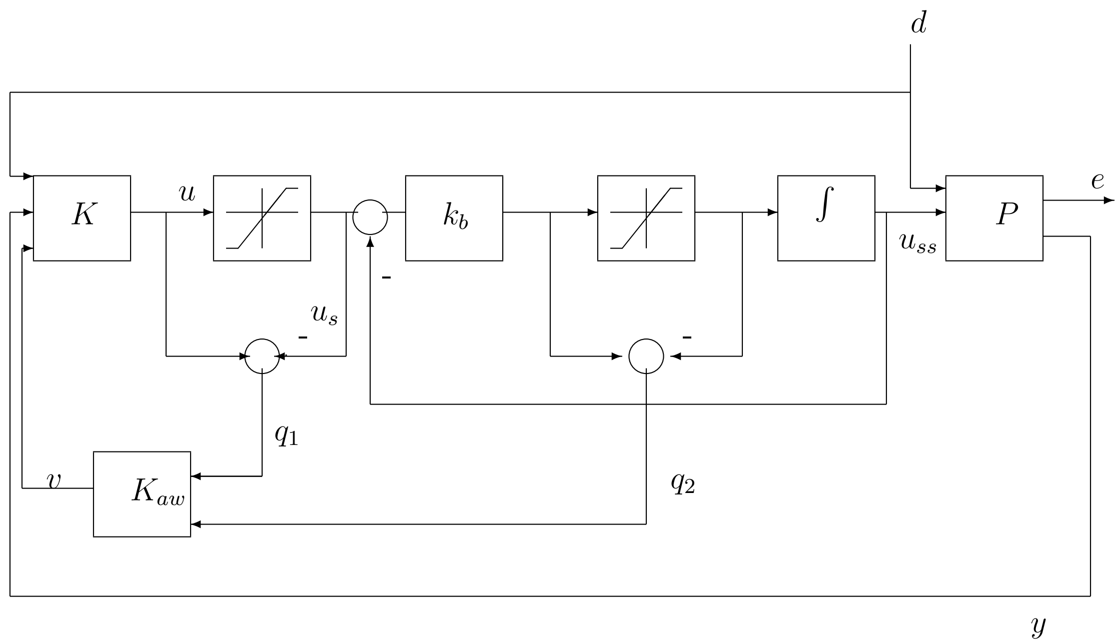


Figure 4.1: Anti-windup Controller Structure

The saturation blocks are represented by the sector-bounded nonlinearity as explained in subsection 3.1. In order to transform the system in figure 4.1 into the standard LFT form as seen in figure 4.3, we must find equations for the total closed-loop system.

The uncertainty block Δ seen in figure 4.3 is the representation of the sector-bounded nonlinearity. In accordance with standard robust control theory procedures, the nonlinearities have been lumped into an uncertainty block Δ. Later, we will construct a robust controller which will robustly stable against the nonlinearities contained in the uncertainty block using our stability (and performance analysis) theorems. At this point, it suffices to note that the uncertainty block will be substituted with a deadband nonlinearity $\Delta = I - \frac{\sigma(u)}{u}$. This will allow us to meet the goals of stating the problem in the classic robust control framework.

The uncertainty Δ resides in the conic-sector $[0, 1]$. We shall reduce this sector to sect$[0, k]$ with $0 < k < 1$. This will restrict the magnitude of any control input signal u_i to be less than $(\frac{1}{1-k})u_i^{max}$. This will then be a regional stability problem. However, it is this restriction that will extend the control scheme to be applicable to open-loop unstable

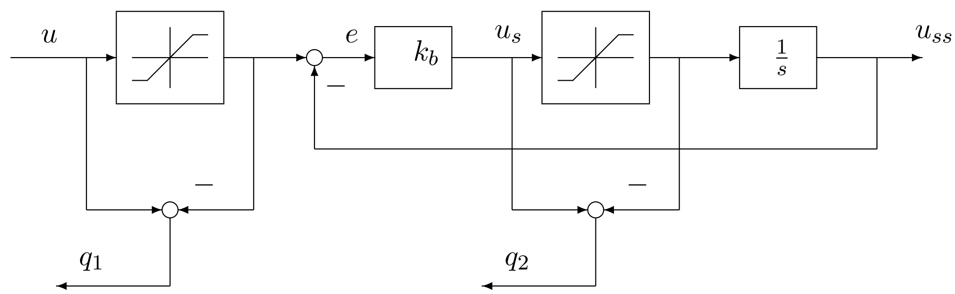


Figure 4.2: Rate Saturation Block

systems.

4.1.2 Augmenting the Nominal Plant

Recall the form of the LTI plant with sector-bounded nonlinearity given in equations (3.1) - (3.2). We would like to augment this in order to include the effects of both rate and magnitude saturation.

Wu and Lu have described the synthesis conditions for the magnitude saturation plant [39]. We wish to find the equivalent synthesis conditions for the rate and magnitude saturation problem. Finally, the closed-loop system will be in LFT form as given in figure 4.3.

Just as in [39] and [11], we must form the system matrices for the augmented plant by breaking the loop in the diagram at q_1 and q_2. Then, we must define the augmented state vector. This vector will include he states of the original system plant, but will also contain the state u_{ss}, which is the saturated input to the plant.

Define the state vector for the augmented plant, x_{pa} as:

$$x_{pa} = \begin{bmatrix} x_p \\ u_{ss} \end{bmatrix} \tag{4.3}$$

Where $u_{ss} \in \mathbf{R}^{n_u}$. Using this new augmented plant vector, and the original plant and controller state-space data, we shall rewrite the state equations for the augmented plant.

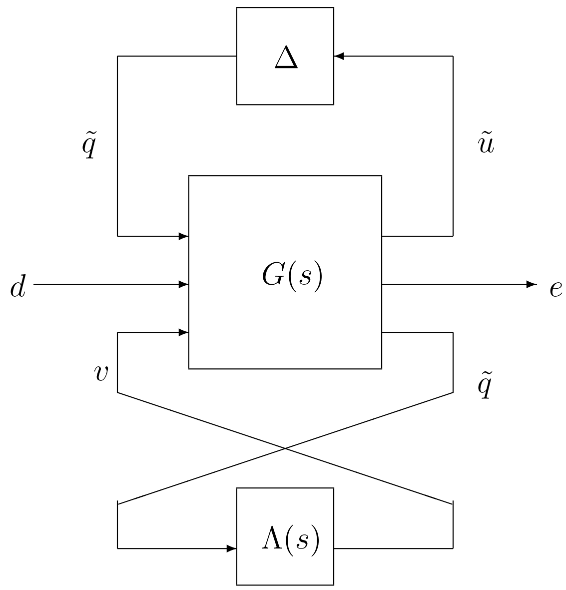


Figure 4.3: Anti-Windup System Equivalent Transformation

From (3.1) and (4.3) we have:

$$\dot{x}_p = \begin{bmatrix} A_p & B_{p2} \end{bmatrix} \begin{bmatrix} x_p \\ u_{ss} \end{bmatrix} + B_{p1} d \tag{4.4}$$

From figure 4.2, we see that $u - \text{sat}(u) = q_1$. Also note that $q_2 = k_b e - \dot{u}_{ss}$ and $e = \text{sat}(u) - u_{ss}$. Substitution of these equations gives us $q_2 = k_b u - k_b q_1 - k_b u_{ss} - u_s$. Finally, rearranging these terms gives us an equation for $\dot{u}_{ss}$:

$$\dot{u}_{ss} = -k_b u_{ss} - k_b q_1 - q_2 + k_b u \tag{4.5}$$

Or, in the matrix form for the state-space model:

$$\dot{u}_{ss} = \begin{bmatrix} 0 & -k_b I \end{bmatrix} \begin{bmatrix} x_p \\ u_{ss} \end{bmatrix} + \begin{bmatrix} -k_b I & -I \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + k_b u \tag{4.6}$$

It is important to note that in these equations, the constant term k_b is the bandwidth constant. This constant will be the factor which limits the bandwidth of the rate saturated actuator. That is to say that this constant will determine the maximum rate at which the actuator can respond without being saturated. This constant is distinct from k_m, and does *not* enter into the control synthesis conditions as part of the diagonal matrix $\bar{k}$.

Finally, the augmented plant output vector can be defined. In matrix form, it can be written as:

$$\begin{bmatrix} u \\ u_s \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -k_b I \end{bmatrix} \begin{bmatrix} x_p \\ u_{ss} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -k_b I & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} d + \begin{bmatrix} I \\ k_b I \end{bmatrix} u \tag{4.7}$$

This equation represents the last step in augmenting the plant for the new open-loop system. We have defined all of the new state-space matrices, and will present them below:

$$\tilde{A}_p = \begin{bmatrix} A_p & B_{p2} \\ 0 & -k_b I \end{bmatrix}, \quad \tilde{B}_{p0} = \begin{bmatrix} 0 & 0 \\ -k_b I & -I \end{bmatrix}, \quad \tilde{B}_{p1} = \begin{bmatrix} B_{p1} \\ 0 \end{bmatrix}, \quad \tilde{B}_{p2} = \begin{bmatrix} 0 \\ k_b I \end{bmatrix}$$

$$\tilde{C}_{p0} = \begin{bmatrix} 0 & 0 \\ 0 & -k_b I \end{bmatrix}, \quad \tilde{C}_{p1} = \begin{bmatrix} C_{p1} & D_{p12} \end{bmatrix}, \quad \tilde{C}_{p2} = \begin{bmatrix} C_{p2} & D_{p22} \end{bmatrix}$$

$$\tilde{D}_{p00} = \begin{bmatrix} 0 & 0 \\ -k_b I & 0 \end{bmatrix}, \quad \tilde{D}_{p01} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \tilde{D}_{p02} = \begin{bmatrix} I \\ k_b I \end{bmatrix}$$

$$\tilde{D}_{p10} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \tilde{D}_{p11} = D_{p11}, \quad \tilde{D}_{p12} = 0$$

$$\tilde{D}_{p20} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \tilde{D}_{p21} = D_{p21}, \quad \tilde{D}_{p22} = 0$$

These matrices correspond to the new nominal open-loop system given below

$$\begin{bmatrix} \dot{x}_{pa} \\ \tilde{u} \\ e \\ y \end{bmatrix} = \begin{bmatrix} \tilde{A}_p & \tilde{B}_{p0} & \tilde{B}_{p1} & \tilde{B}_{p2} \\ \tilde{C}_{p0} & \tilde{D}_{p00} & \tilde{D}_{p01} & \tilde{D}_{p02} \\ \tilde{C}_{p1} & \tilde{D}_{p10} & \tilde{D}_{p11} & \tilde{D}_{p12} \\ \tilde{C}_{p2} & \tilde{D}_{p20} & \tilde{D}_{p21} & 0 \end{bmatrix} \begin{bmatrix} x_{pa} \\ \tilde{q} \\ d \\ u \end{bmatrix}$$

$$\tilde{q} = \Delta \tilde{u}$$

4.2 Rate and Magnitude Synthesis Condition

Figure 4.2 shows the closed-loop system of the rate and magnitude saturations, including anti-windup compensator. The anti-windup control diagram can be transformed to its equivalent form by substituting the actuator magnitude and rate saturations with deadband nonlinearities, as shown in Fig. 4.3. For this purpose, let us define

$$\Delta_m = 1 - \frac{\mathrm{sat}(u)}{u}, \quad \Delta_r = 1 - \frac{\mathrm{sat}(u_s)}{u_s}$$

Then Δ_m, Δ_r will be deadband nonlinearity associated with magnitude and rate saturation, respectively. Both nonlinearities reside in the conic sector $[0, 1]$. In order to extend the anti-windup control scheme to open-loop unstable systems, we will constrain the nonlinearity Δ_m for each input channel to $\mathrm{sect}[0, k_m]$ with $0 < k_m < 1$. This essentially requires the magnitude of each control input signal u_i to be less than $\left(\frac{1}{1-k_m}\right) u_i^{max}$, and leads to regional stability problem. Then the input/output constraint for the uncertainty $\Delta := \mathrm{diag}\{\Delta_m, \Delta_r\}$ will be

$$\tilde{q}^T W(\bar{k}\tilde{u} - \tilde{q}) \geq 0$$

with $\bar{k} = \mathrm{diag}\{k_m I_{n_u}, I_{n_u}\}$ and W is a diagonal matrix, as described in chapter 3. Later on, we will cast the anti-windup control design as a robust control problem against the deadband nonlinearities.

Our objective is to design an anti-windup compensator Λ such that the adversary effect of input magnitude and rate saturations will be minimized in terms of $\mathcal{H}_\infty$ norm. The anti-windup compensator is in the form of

$$\begin{bmatrix} \dot{x}_{aw} \\ \hline v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} \begin{bmatrix} x_{aw} \\ \tilde{q} \end{bmatrix} \tag{4.8}$$

with the state $x_{aw} \in \mathbf{R}^{n_{aw}}$, which will be determined later on. Note the the input signal for anti-windup compensator $\tilde{q}$ contains the information of saturation degree for both magnitude and rate saturations.

First, we shall evaluate the new closed-loop plant matrices. These are the matrices that result from breaking the loop at q_1 and q_2. It is these matrices which will also be included in the new set of synthesis instructions.

$$\tilde{A} = \begin{bmatrix} \tilde{A}_p + \tilde{B}_{p2} D_{k2} \tilde{C}_{p2} & \tilde{B}_{p2} C_k \\ B_{k2} \tilde{C}_{p2} & A_k \end{bmatrix}$$

$$\tilde{B}_0 = \begin{bmatrix} \tilde{B}_{p0} \\ 0 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} \tilde{B}_{p1} + \tilde{B}_{p2}(D_{k1} + D_{k2}\tilde{D}_{p21}) \\ B_{k1} + B_{k2}\tilde{D}_{p21} \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 & \tilde{B}_{p2} \\ I & 0 \end{bmatrix}$$

$$\tilde{C}_0 = \begin{bmatrix} \tilde{C}_{p0} + \tilde{D}_{p02} D_{k2} \tilde{C}_{p2} & \tilde{D}_{p02} C_k \end{bmatrix}, \quad \tilde{C}_1 = \begin{bmatrix} \tilde{C}_{p1} & 0 \end{bmatrix}$$

$$\tilde{D}_{00} = \tilde{D}_{p00}, \quad \tilde{D}_{01} = \tilde{D}_{p01} + \tilde{D}_{p02}(D_{k1} + D_{k2}\tilde{D}_{p21}), \quad \tilde{D}_{02} = \begin{bmatrix} 0 & \tilde{D}_{p02} \end{bmatrix}$$

$$\tilde{D}_{10} = \tilde{D}_{p10}, \quad \tilde{D}_{11} = \tilde{D}_{p11}, \quad \tilde{D}_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

These matrices constitute the state-space make-up of the nominal closed-loop system G. The input/output form for this closed loop system is as follows.

$$\begin{bmatrix} \dot{x} \\ \tilde{u} \\ e \\ \tilde{q} \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}_0 & \tilde{B}_1 & \tilde{B}_2 \\ \tilde{C}_0 & \tilde{D}_{00} & \tilde{D}_{01} & \tilde{D}_{02} \\ \tilde{C}_1 & \tilde{D}_{10} & \tilde{D}_{11} & \tilde{D}_{12} \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \tilde{q} \\ d \\ \hline v_1 \\ v_2 \end{bmatrix} \tag{4.9}$$

$$\tilde{q} = \Delta \tilde{u} \tag{4.10}$$

This is the state-space data for the nominal closed-loop system G. If we denote $x_{cl}^T = \begin{bmatrix} x^T & x_{aw}^T \end{bmatrix}$, then the final closed-loop system $T = \mathcal{F}_l(G, \Lambda)$ can be described by the following input/output relationship.

$$\begin{bmatrix} \dot{x}_{cl} \\ \tilde{u} \\ e \end{bmatrix} = \begin{bmatrix} \tilde{A}_{cl} & \tilde{B}_{0,cl} & \tilde{B}_{1,cl} \\ \tilde{C}_{0,cl} & \tilde{D}_{00,cl} & \tilde{D}_{01,cl} \\ \tilde{C}_{1,cl} & \tilde{D}_{10,cl} & \tilde{D}_{11,cl} \end{bmatrix} \begin{bmatrix} x_{cl} \\ \tilde{q} \\ d \end{bmatrix} \tag{4.11}$$

$$\tilde{q} = \Delta \tilde{u} \tag{4.12}$$

and has its state-space date related to the interconnected system G and the anti-windup compensator as follows

$$\begin{bmatrix} \tilde{A}_{cl} & \tilde{B}_{0,cl} & \tilde{B}_{1,cl} \\ \tilde{C}_{0,cl} & \tilde{D}_{00,cl} & \tilde{D}_{01,cl} \\ \tilde{C}_{1,cl} & \tilde{D}_{10,cl} & \tilde{D}_{11,cl} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_0 & \mathcal{B}_1 \\ \mathcal{C}_0 & D_{00} & D_{01} \\ \mathcal{C}_1 & D_{10} & D_{11} \end{bmatrix} + \begin{bmatrix} \mathcal{P}_1^T \\ \mathcal{P}_2^T \\ \mathcal{P}_3^T \end{bmatrix} \begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} \begin{bmatrix} \mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{Q}_3 \end{bmatrix}$$

$$= \left[\begin{array}{cc|c|c} \tilde{A} & 0 & \tilde{B}_0 & \tilde{B}_1 \\ 0 & 0 & 0 & 0 \\ \hline \tilde{C}_0 & 0 & \tilde{D}_{00} & \tilde{D}_{01} \\ \hline \tilde{C}_1 & 0 & \tilde{D}_{10} & \tilde{D}_{11} \end{array}\right] + \left[\begin{array}{cc} 0 & \tilde{B}_2 \\ I & 0 \\ \hline 0 & \tilde{D}_{02} \\ \hline 0 & \tilde{D}_{12} \end{array}\right] \begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} \left[\begin{array}{cc|c|c} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{array}\right] \tag{4.13}$$

As is necessary, the anti-windup compensator and nominal closed-loop system have an affine relationship. This allows us to find a true synthesis condition, as we can apply our stability theorem to the system both with and without the controller. If we obtain a feasible solution to the synthesis LMIs, we can then apply the theorem to the closed-loop system *with* the controller, and solve the resulting LMIs to find the controller gains. The following subsection will provide these synthesis conditions for the closed-loop system.

4.2.1 Synthesis Theorem

Recall from Theorem 5 the synthesis conditions for the magnitude anti-windup compensator. Using the augmented plant matrices, we will derive a new synthesis condition for the augmented anti-windup problem.

Theorem 6 *Given a scalar* $0 < k_m < 1$*, the augmented open-loop system with a stabilizing nominal controller* K_{nom}*. If there exist a pair of positive-definite matrices* $R_{11} \in \mathbf{S}_+^{\tilde{n}_p \times \tilde{n}_p}$,

$S \in \mathbf{S}_{+}^{n\times n}$ and a diagonal matrix $V = \operatorname{diag}\{V_m, V_r\} > 0$ satisfying

$$\begin{bmatrix} R_{11}\tilde{A}_p^T + \tilde{A}_p R_{11} - \frac{2(1-k_m)}{k_m^2}\tilde{B}_{p2}V_m\tilde{B}_{p2}^T & \star & \star & \star \\ -\begin{bmatrix}0 & I_{n_u}\end{bmatrix}R_{11} - \frac{2(1-k_m)}{k_m^2}V_m\tilde{B}_{p2}^T - \frac{1}{k_b^2}V_r\tilde{B}_{p2}^T & -\frac{2(1-k_m)}{k_m^2}V_m - \frac{1}{k_b^2}V_r & \star & \star \\ \tilde{C}_{p1}R_{11} & 0 & -\gamma I_{n_e} & \star \\ \tilde{B}_{p1}^T & 0 & \tilde{D}_{p11}^T & -\gamma I_{n_d} \end{bmatrix} < 0 \tag{4.14}$$

$$\begin{bmatrix} S\tilde{A} + \tilde{A}^T S & S\tilde{B}_1 & \tilde{C}_1^T \\ \tilde{B}_1^T S & -\gamma I_{n_d} & \tilde{D}_{11}^T \\ \tilde{C}_1 & \tilde{D}_{11} & -\gamma I_{n_e} \end{bmatrix} < 0 \tag{4.15}$$

$$\begin{bmatrix} R_{11} & \begin{bmatrix} I_{\tilde{n}_p} & 0 \end{bmatrix} \\ \begin{bmatrix} I_{\tilde{n}_p} \\ 0 \end{bmatrix} & S \end{bmatrix} \geq 0 \tag{4.16}$$

then there exists an $\tilde{n}_p$th-order anti-windup compensator Λ to stabilize the closed-loop system quadratically and have the performance $\|e\|_2 < \gamma\|d\|_2$ when the condition $|u_i| \leq \left(\frac{1}{1-k_m}\right)u_i^{max}, \ i = 1, 2, \cdots, n_u$.

Now we must prove that the solution of the LMI in Theorem 6 will always exist. The goal is to use the result of Theorem 4 to show that the anti-windup compensator will stabilize the system in the face of the nonlinearity produced by saturation effects.

Proof: Denote $\Lambda = \begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix}$ and $V = W^{-1}$. We apply Theorem 4 to the closed-loop system T, and have the following inequality

$$\Psi + \mathcal{P}^T\Lambda\mathcal{Q} + \mathcal{Q}^T\Lambda^T\mathcal{P} < 0 \tag{4.17}$$

with

$$\Psi = \begin{bmatrix} \mathcal{A}^T X_{cl} + X_{cl}\mathcal{A} & X_{cl}\mathcal{B}_0 + \mathcal{C}_0^T \bar{k} W & X_{cl}\mathcal{B}_1 & \mathcal{C}_1^T \\ \mathcal{B}_0^T X_{cl} + W\bar{k}\mathcal{C}_0 & W\bar{k}\tilde{D}_{00} + \tilde{D}_{00}^T \bar{k} W - 2W & W\bar{k}\tilde{D}_{01} & \tilde{D}_{10}^T \\ \mathcal{B}_1^T X_{cl} & \tilde{D}_{01}^T \bar{k} W & -\gamma I & \tilde{D}_{11}^T \\ \mathcal{C}_1 & \tilde{D}_{10} & \tilde{D}_{11} & -\gamma I \end{bmatrix},$$

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_1 X_{cl} & \mathcal{P}_2 \bar{k} W & 0 & \mathcal{P}_3 \end{bmatrix},$$

$$\mathcal{Q} = \begin{bmatrix} \mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{Q}_3 & 0 \end{bmatrix}$$

Partition the matrix X_{cl} compatibly to the states of interconnected system G and anti-windup compensator K_{aw} as $n = \tilde{n}_p + n_k$ and n_{aw}, and let

$$X_{cl} = \begin{bmatrix} S & N \\ N^T & ? \end{bmatrix}$$

$$X_{cl}^{-1} = \begin{bmatrix} R & M \\ M^T & ? \end{bmatrix} = \left[\begin{array}{cc|c} R_{11} & R_{12} & \multirow{2}{*}{M} \\ R_{12}^T & R_{22} & \\ \hline \multicolumn{2}{c|}{M^T} & ? \end{array}\right]$$

where $MN^T = I - RS$. According to the Elimination Lemma [5, 26], the inequality (4.17) is equivalent to

$$\mathcal{N}_{\mathcal{P}}^T \Phi \mathcal{N}_{\mathcal{P}} < 0 \qquad \text{and} \qquad \mathcal{N}_{\mathcal{Q}}^T \Phi \mathcal{N}_{\mathcal{Q}} < 0 \tag{4.18}$$

where $\mathcal{N}_{\mathcal{P}}$ and $\mathcal{N}_{\mathcal{Q}}$ are the null spaces of matrices $\mathcal{P}$ and $\mathcal{Q}$, which are

$$\mathcal{N}_{\mathcal{P}} = \text{diag}\left\{X_{cl}^{-1}, W^{-1}, I, I\right\} \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -\frac{1}{k_m}\tilde{B}_{p2}^T & -\frac{1}{k_m} I & 0 & 0 \\ 0 & \frac{1}{k_b} I & 0 & 0 \\ \hline 0 & 0 & 0 & I \\ \hline 0 & 0 & I & 0 \end{array}\right], \qquad \mathcal{N}_{\mathcal{Q}} = \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right]$$

Through lengthy algebraic manipulations, it can be shown that

$$\mathcal{N}_{\mathcal{P}}^T \Phi \mathcal{N}_{\mathcal{P}} =$$

$$\left[\begin{array}{cc|cc|c|c} I & 0 & -\frac{1}{k_m}\tilde{B}_{p2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{k_m}I & \frac{1}{k_b}I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & I & 0 \end{array}\right] \begin{bmatrix} R\tilde{A}^T + \tilde{A}R & (V\tilde{B}_0^T + \bar{k}\tilde{C}_0 R)^T & \tilde{B}_1 & R\tilde{C}_1^T \\ V\tilde{B}_0^T + \bar{k}\tilde{C}_0 R & \bar{k}\tilde{D}_{00}V + V\tilde{D}_{00}^T\bar{k} - 2V & \bar{k}\tilde{D}_{01} & V\tilde{D}_{10}^T \\ \tilde{B}_1^T & \tilde{D}_{01}^T\bar{k} & -\gamma I & \tilde{D}_{11}^T \\ \tilde{C}_1 R & \tilde{D}_{10}V & \tilde{D}_{11} & -\gamma I \end{bmatrix}$$

$$\times \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -\frac{1}{k_m}\tilde{B}_{p2}^T & -\frac{1}{k_m}I & 0 & 0 \\ 0 & \frac{1}{k_b}I & 0 & 0 \\ \hline 0 & 0 & 0 & I \\ \hline 0 & 0 & I & 0 \end{array}\right]$$

$$= \begin{bmatrix} R_{11}\tilde{A}_p^T + \tilde{A}_p R_{11} - \frac{2(1-k_m)}{k_m^2}\tilde{B}_{p2}V_m\tilde{B}_{p2}^T & \star & \star & \star \\ -\begin{bmatrix} 0 & I \end{bmatrix} R_{11} - \frac{2(1-k_m)}{k_m^2}V_m\tilde{B}_{p2}^T - \frac{1}{k_b^2}V_r\tilde{B}_{p2}^T & -\frac{2(1-k_m)}{k_m^2}V_m - \frac{1}{k_b^2}V_r & \star & \star \\ \tilde{C}_{p1}R_{11} & 0 & -\gamma I & \star \\ \tilde{B}_{p1}^T & 0 & \tilde{D}_{p11}^T & -\gamma I \end{bmatrix} < 0$$

$$\mathcal{N}_{\mathcal{Q}}^T \Phi \mathcal{N}_{\mathcal{Q}} =$$

$$\left[\begin{array}{cc|c|c|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right] \begin{bmatrix} S\tilde{A} + \tilde{A}^T S & S\tilde{B}_0 + \tilde{C}_0^T\bar{k}W & S\tilde{B}_1 & \tilde{C}_1^T \\ \tilde{B}_0^T S + W\bar{k}\tilde{C}_0 & W\bar{k}\tilde{D}_{00} + \tilde{D}_{00}^T\bar{k}W - 2W & W\bar{k}\tilde{D}_{01} & \tilde{D}_{10}^T \\ \tilde{B}_1^T S & \tilde{D}_{01}^T\bar{k}W & -\gamma I & \tilde{D}_{11}^T \\ \tilde{C}_1 & \tilde{D}_{10} & \tilde{D}_{11} & -\gamma I \end{bmatrix}$$

$$\times \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right]$$

$$= \begin{bmatrix} S\tilde{A} + \tilde{A}^T S & S\tilde{B}_1 & \tilde{C}_1^T \\ \tilde{B}_1^T S & -\gamma I & \tilde{D}_{11}^T \\ \tilde{C}_1 & \tilde{D}_{11} & -\gamma I \end{bmatrix} < 0$$

which are the same as the conditions (4.14) and (4.15), respectively.

Given the definition for matrices X_{cl} and X_{cl}^{-1}, the coupling condition between R and S would be

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \qquad \text{and} \qquad \operatorname{rank}(R - S^{-1}) \leq n_{aw}$$

Since only the $(1,1)$ element of R matrix is constrained in the LMIs (4.14)-(4.16), it is always possible to augment matrix R_{11} to R in satisfying the above coupling condition. For example, one may choose

$$R = \begin{bmatrix} R_{11} & \begin{bmatrix} I & 0 \end{bmatrix} S^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \begin{bmatrix} 0 & I \end{bmatrix} S^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & I \end{bmatrix} S^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \end{bmatrix}$$

The resulting R matrix is positive-definite because of the condition (4.16). Also $R - S^{-1} \geq 0$ is satisfied for selected R matrix. The rank condition is trivially satisfied if one chooses $n_{aw} = \tilde{n}_p$. So we obtain the desired synthesis condition for the anti-windup compensator. *Q.E.D.*

4.2.2 Rate and Magnitude Compensator Construction

With the solutions R_{11} and S obtained from the compensator synthesis equations, the anti-windup compensator state-space matrices can be constructed by determining the closed-loop solution to the original LMI which resulted from the stability theorem.

$$\Psi + \mathcal{P}^T \Lambda \mathcal{Q} + \mathcal{Q}^T \Lambda^T \mathcal{P} < 0$$

This approach is taken in [11] and is a standard procedure for $\mathcal{H}_\infty$ synthesis via LMIs. However, an explicit approach to anti-windup compensator construction was taken in [39]. This approach is derived from the general explicit controller construction used by Gahinet in [13]. The advantage of using this explicit approach is the avoidance of possible numerical ill-conditioning when solving the feasibility LMI. Furthermore, the anti-windup compensator is connected directly to the plant and nominal controller gains [39].

Theorem 7 *(Rate and Magnitude Compensator Construction)*
Given the solutions R_{11}, S, γ and $V = W^{-1}$ of the LMIs (4.14)-(4.15). Let $MN^T = I_n - RS$ with $M, N \in \mathbf{S}^{n\times(n_p+n_u)}$ and $E^T = \begin{bmatrix} I_{(n_p+n_u)} & 0 \end{bmatrix}$, then an (n_p+n_u)th-order anti-windup compensator can be constructed through the following scheme:

1. *Compute a feasible $\hat{D}_{aw} \in \mathbf{R}^{n_u\times 2n_u}$ such that*

$$\Pi = \begin{bmatrix} -W\bar{k}(\tilde{D}_{00} + \tilde{D}_{p02}\hat{D}_{aw}) - (\tilde{D}_{00} + \tilde{D}_{p02}\hat{D}_{aw})^T\bar{k}W + 2W & -W\bar{k}\tilde{D}_{01} & -\tilde{D}_{10}^T \\ -\tilde{D}_{01}^T\bar{k}W & \gamma I_{n_d} & -\tilde{D}_{11}^T \\ -\tilde{D}_{10} & -\tilde{D}_{11} & \gamma I_{n_e} \end{bmatrix} > 0,$$

2. *Compute the least-square solutions of the following linear equations for $\hat{B}_{aw} \in \mathbf{R}^{n\times 2n_u}$, $\hat{C}_{aw} \in \mathbf{R}^{n_u\times(n_p+n_u)}$*

$$\left[\begin{array}{c|ccc} 0 & I_{2n_u} & 0 & 0 \\ \hline I_{2n_u} & & & \\ 0 & & -\Pi & \\ 0 & & & \end{array}\right] \left[\begin{array}{c} \hat{B}_{aw}^T \\ \hline ? \end{array}\right] = -\left[\begin{array}{c} 0_{2n_u\times n} \\ \hline \tilde{B}_0^T S + W\bar{k}\tilde{C}_0 \\ \tilde{B}_1^T S \\ \tilde{C}_1 \end{array}\right]$$

$$\left[\begin{array}{c|ccc} 0 & \tilde{D}_{p02}^T\bar{k}W & 0 & 0 \\ \hline W\bar{k}\tilde{D}_{p02} & & & \\ 0 & & -\Pi & \\ 0 & & & \end{array}\right] \left[\begin{array}{c} \hat{C}_{aw} \\ \hline ? \end{array}\right] = -\left[\begin{array}{c} \tilde{B}_{p2}^T \\ \hline (\tilde{B}_0^T + W\bar{k}\tilde{C}_0R)E + \hat{D}_{aw}^T\tilde{B}_{p2}^T \\ \tilde{B}_1^T E \\ \tilde{C}_1 RE \end{array}\right],$$

and the matrix $\hat{A}_{aw} \in \mathbf{R}^{n\times(n_p+n_u)}$ as

$$\hat{A}_{aw} = -\tilde{A}^T E - \begin{bmatrix} S\tilde{B}_0 + \hat{B}_{aw} + \tilde{C}_0^T\bar{k}W & S\tilde{B}_1 & \tilde{C}_1^T \end{bmatrix} \Pi^{-1} \times \begin{bmatrix} (\tilde{B}_0^T + W\bar{k}C_0R)E + \hat{D}_{aw}^T\tilde{B}_{p2}^T + W\bar{k}\tilde{D}_{p02}\hat{C}_{aw} \\ \tilde{B}_1^T E \\ \tilde{C}_1 RE \end{bmatrix}$$

3. *Convert the transformed anti-windup compensator gain to its original state-space data*

by

$$\begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} = \begin{bmatrix} N & S\tilde{B}_2 \\ 0 & \begin{bmatrix} 0 & I_{n_u} \end{bmatrix} \end{bmatrix}^{-1} \left(\begin{bmatrix} \hat{A}_{aw} & \hat{B}_{aw} \\ \hat{C}_{aw} & \hat{D}_{aw} \end{bmatrix} - \begin{bmatrix} S\tilde{A}RE & 0 \\ 0 & 0_{n_u \times 2n_u} \end{bmatrix} \right) \begin{bmatrix} M^T E & 0 \\ 0 & I_{2n_u} \end{bmatrix}^{-1}$$

Proof: Define

$$Z_1 = \begin{bmatrix} I_n & RE \\ 0 & M^T E \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} S & E \\ N^T & 0 \end{bmatrix}$$

Then it can be shown that $X_{cl} Z_1 = Z_2$. Also we have the following congruent transformation

$$Z_1^T X_{cl} Z_1 = \begin{bmatrix} S & E \\ E^T & E^T RE \end{bmatrix}$$

$$\begin{bmatrix} Z_1^T X_{cl} A_{cl} Z_1 & Z_1^T X_{cl} B_{0,cl} & Z_1^T X_{cl} B_{1,cl} \\ C_{0,cl} Z_1 & D_{00,cl} & D_{01,cl} \\ C_{1,cl} Z_1 & D_{10,cl} & D_{11,cl} \end{bmatrix} = \left[\begin{array}{cc|c|c} S\tilde{A} & 0 & S\tilde{B}_0 & S\tilde{B}_1 \\ E^T \tilde{A} & E^T \tilde{A} RE & E^T \tilde{B}_0 & E^T \tilde{B}_1 \\ \hline \tilde{C}_0 & \tilde{C}_0 RE & \tilde{D}_{00} & \tilde{D}_{01} \\ \hline \tilde{C}_1 & \tilde{C}_1 RE & \tilde{D}_{10} & \tilde{D}_{11} \end{array} \right]$$
$$+ \left[\begin{array}{cc} I_n & 0 \\ 0 & \tilde{B}_{p2} \\ \hline 0 & \tilde{D}_{p02} \\ \hline 0 & 0 \end{array} \right] \begin{bmatrix} \hat{A}_{aw} & \hat{B}_{aw} \\ \hat{C}_{aw} & \hat{D}_{aw} \end{bmatrix} \left[\begin{array}{cc|c|c} 0 & I_{\tilde{n}_p} & 0 & 0 \\ 0 & 0 & I_{2n_u} & 0 \end{array} \right]$$

where

$$\begin{bmatrix} \hat{A}_{aw} & \hat{B}_{aw} \\ \hat{C}_{aw} & \hat{D}_{aw} \end{bmatrix} = \begin{bmatrix} S\tilde{A}RE & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} N & S\tilde{B}_2 \\ 0 & \begin{bmatrix} 0 & I \end{bmatrix} \end{bmatrix} \begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} \begin{bmatrix} M^T E & 0 \\ 0 & I \end{bmatrix}$$

Multiply diag $\left\{Z_1^T, I, I, I\right\}$ from left side, and its conjugate transpose from right side of eq. (4.17), we get

$$
\left[\begin{array}{cc}
S\tilde{A}+\tilde{A}^T S & * \\
E^T\tilde{A}+\hat{A}_{aw}^T & E^T(\tilde{A}R+R\tilde{A}^T)E+\tilde{B}_{p2}\hat{C}_{aw}+\hat{C}_{aw}^T\tilde{B}_{p2}^T \\
\tilde{B}_0^T S+\hat{B}_{aw}^T+W\bar{k}\tilde{C}_0 & (\tilde{B}_0^T+W\bar{k}\tilde{C}_0 R)E+\hat{D}_{aw}^T\tilde{B}_{p2}^T+W\bar{k}\tilde{D}_{p02}\hat{C}_{aw} \\
\tilde{B}_1^T S & \tilde{B}_1^T E \\
\tilde{C}_1 & \tilde{C}_1 RE
\end{array}\right.
$$

$$
\left.\begin{array}{ccc}
* & * & * \\
* & * & * \\
\left\{\begin{array}{c} W\bar{k}(\tilde{D}_{00}+\tilde{D}_{p02}\hat{D}_{aw}) \\ (\tilde{D}_{00}+\tilde{D}_{p02}\hat{D}_{aw})^T\bar{k}W-2W \end{array}\right\} & * & * \\
\tilde{D}_{01}^T\bar{k}W & -\gamma I & * \\
\tilde{D}_{10} & \tilde{D}_{11} & -\gamma I
\end{array}\right] < 0 \tag{4.19}
$$

By Schur complement, it is equivalent to

$$
\begin{aligned}
&\begin{bmatrix} S\tilde{A}+\tilde{A}^T S & * \\ E^T\tilde{A}+\hat{A}_{aw}^T & E^T(\tilde{A}R+R\tilde{A}^T)E+\tilde{B}_{p2}\hat{C}_{aw}+\hat{C}_{aw}^T\tilde{B}_{p2}^T \end{bmatrix} \\
&\quad + \begin{bmatrix} S\tilde{B}_0+\hat{B}_{aw}+\tilde{C}_0^T\bar{k}W & S\tilde{B}_1 & \tilde{C}_1^T \\ E^T(\tilde{B}_0^T+W\bar{k}\tilde{C}_0 R)^T+\tilde{B}_{p2}\hat{D}_{aw}+\hat{C}_{aw}^T\tilde{D}_{p02}^T\bar{k}W & E^T\tilde{B}_1 & (\tilde{C}_1 RE)^T \end{bmatrix}\Pi^{-1} \\
&\quad \times \begin{bmatrix} \tilde{B}_0^T S+\hat{B}_{aw}^T+W\bar{k}C_0 & (\tilde{B}_0^T+W\bar{k}\tilde{C}_0 R)E+\hat{D}_{aw}^T\tilde{B}_{p2}^T+W\bar{k}\tilde{D}_{p02}\hat{C}_{aw} \\ \tilde{B}_1^T S & \tilde{B}_1^T E \\ \tilde{C}_1 & \tilde{C}_1 RE \end{bmatrix} < 0
\end{aligned} \tag{4.20}
$$

Q.E.D.

The derivation of the anti-windup controller formula basically follows the procedures outlined in [13] and [39]. It is easy to show that the lower (3×3) matrix of the inequality (4.19) is negative definite, this will determine a feasible $\hat{D}_{aw}$. Let the $(2,1)$ element equal to zero, and we can solve for $\hat{A}_{aw}$. This also leads to decoupled LMIs from the inequality (4.20). Then $\hat{B}_{aw}, \hat{C}_{aw}$ terms can be solved from the $(1,1)$ and $(2,2)$ elements of the decoupled inequality (4.20). Note that both inequalities have regular solutions [13].

The $(1, 1)$ element of the above matrix inequality corresponds to LMI (4.15) after elimination of the variables $\hat{B}_{aw}$ and $\hat{D}_{aw}$. It can also be shown that the $(2, 2)$ element is equivalent to LMI (4.14) by eliminating $\hat{C}_{aw}, \hat{D}_{aw}$.

This explicit form of the construction can also be applied to stable systems. The main difference is that the open loop case would not involve matrices W and $\bar{k}$ [39]. This introduces the need to find a feasible $\hat{D}_{aw}$ and W matrices simultaneously during the first step. The following steps will be the same by setting $\bar{k} = I$.

4.3 LPV Anti-Windup Synthesis condition

In section 4.2, we have shown that an anti-windup compensator exists to compensate rate and magnitude saturation for exponentially unstable systems. We have also provided an explicit construction scheme for such a compensator. In this section, a similar synthesis condition will be provided for the existence of a gain-scheduled anti-windup compensator. This will be followed by the construction procedure for such a controller.

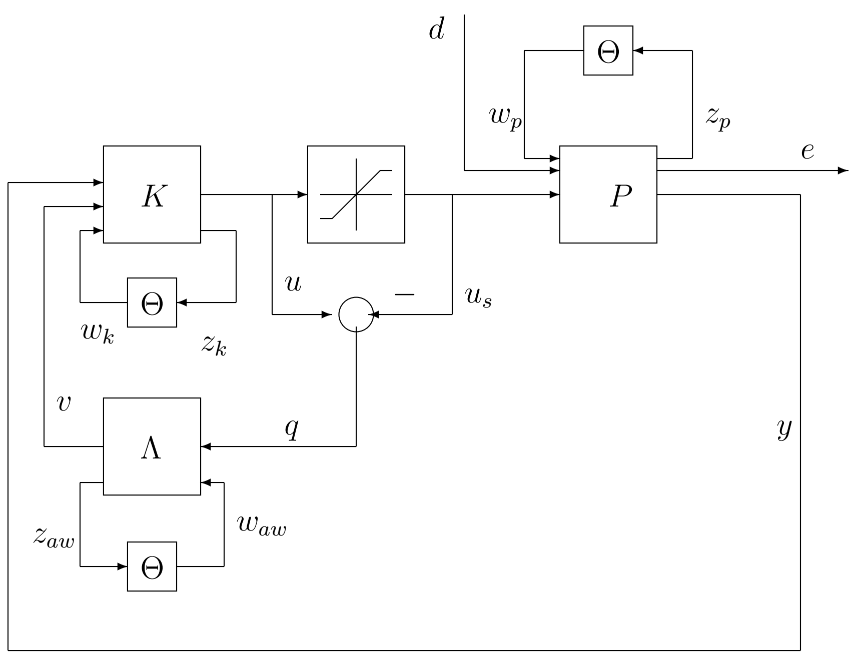


Figure 4.4: LPV Gain-Scheduled Saturated System

4.3.1 LFT Framework

Recall the form of the standard LPV system, in equation 2.5. We shall rewrite our plant, P nominal controller, K and anti-windup compensator, Λ to fit this form. That is, we shall introduce a parameter, Θ which will be the gain-scheduling parameter, which will be related to the three systems by LFT form.

First, the nominal plant shall be rewritten as

$$\begin{bmatrix} \dot{x}_p \\ z_p \\ e \\ y \end{bmatrix} = \begin{bmatrix} A_p & B_{p\theta} & B_{p1} & B_{p2} \\ C_{p\theta} & D_{p\theta\theta} & D_{p\theta 1} & D_{p\theta 2} \\ C_{p1} & D_{p1\theta} & D_{p11} & D_{p12} \\ C_{p2} & D_{p2\theta} & D_{p21} & D_{p22} \end{bmatrix} \begin{bmatrix} x_p \\ w_p \\ d \\ \sigma(u) \end{bmatrix} \tag{4.21}$$

$$w_p = \Theta z_p \tag{4.22}$$

Here, the usual dimensions are used, with one new addition, the vectors $w_p, z_p \in \mathbf{R}^{n_w}$. These vectors are pseudo inputs and outputs of the plant, that is, they are the vectors describing the LFT dependency of the plant on gain-scheduling parameter, Θ.

The matrix Θ is a block diagonal time-varying operator which specifies how the time varying parameter θ effects the dynamics of the plant [1]. In particular

$$\Theta = \text{blockdiag}(\theta_1 I_{m_1}, ..., \theta_r I_{m_r}) \tag{4.23}$$

where $r_i > 1$ when the parameter θ_i is repeated. This allows us to maintain a block diagonal structure for the matrix Θ.

In a similar fashion, the nominal controller must also be augmented to exemplify the dependence on the gain-scheduling parameter. Therefore, the new controller will have dynamic equation as follows.

$$\begin{bmatrix} \dot{x}_k \\ u \\ z_k \end{bmatrix} = \begin{bmatrix} A_k & B_{k1} & B_{k\theta} \\ C_{k1} & D_{k11} & D_{k1\theta} \\ C_{k\theta} & D_{k\theta 1} & D_{k\theta\theta} \end{bmatrix} \begin{bmatrix} x_k \\ y \\ w_k \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$w_k = \Theta z_k$$

We have a plant with saturation input described by the nonlinear function ψ and scheduled by the gain-scheduling parameter Θ. We also have a controller with gain-scheduling parameter, Θ and being corrected by anti-windup compensator input v. The next step is to combine these two systems into a nominal closed-loop system, G which will then be corrected by the anti-windup compensator. The form of the anti-windup compensator, Λ, is given below.

$$\begin{bmatrix} \dot{x}_{aw} \\ \hline v \\ z_{aw} \end{bmatrix} = \begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} \begin{bmatrix} x_{aw} \\ \hline q \\ w_{aw} \end{bmatrix}$$

$$w_{aw} = \Theta z_{aw}$$

4.3.2 Closed-Loop LPV System Construction

Our objective is to design an anti-windup compensator, Λ such that the effect of input saturation will be minimized in terms of the $\mathcal{H}_\infty$ norm of the closed-loop system. The compensator given above has state $x_{aw} \in \mathbf{R}^{n_{aw}}$.

Let the system G be the interconnection of the open-loop plant P and the nominal controller K, but excluding the anti-windup compensator. Then its dynamic equation will be

$$\begin{bmatrix} \dot{x} \\ u \\ z \\ e \\ q \end{bmatrix} = \begin{bmatrix} A & B_0 & B_\theta & B_1 & B_2 \\ C_0 & D_{00} & D_{0\theta} & D_{01} & D_{02} \\ C_\theta & D_{\theta 0} & D_{\theta\theta} & D_{\theta 1} & D_{\theta 2} \\ C_1 & D_{10} & D_{1\theta} & D_{11} & D_{12} \\ 0 & I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ q \\ w \\ d \\ \hline v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$q = \Delta u$$

$$w = (\Theta \oplus \Theta) z$$

where $x \in \mathbf{R}^n$ with $n = n_p + n_k$ and $\Delta \in$ sect[0,k].

We will also use a scaling matrix in the synthesis and analysis conditions for the gain-scheduled anti-windup compensator. This scaling matrix T must commute with the uncertainty structure $\Theta \oplus \Theta$. In other words, this must be a member of the previously defined set L_Δ. The special structure of this scaling matrix in order to enforce this condition is given below.

$$T = \left[\begin{array}{cccc|cccc} T_{11}^1 & 0 & 0 & 0 & T_{12}^1 & 0 & 0 & 0 \\ 0 & T_{11}^2 & 0 & 0 & 0 & T_{12}^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & T_{11}^r & 0 & 0 & 0 & T_{12}^r \\ \hline T_{12}^{1T} & 0 & 0 & 0 & T_{22}^1 & 0 & 0 & 0 \\ 0 & T_{12}^{2T} & 0 & 0 & 0 & T_{22}^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & T_{12}^{rT} & 0 & 0 & 0 & T_{22}^r \end{array}\right]$$

Notice that the structure of T is no longer block-diagonal. This is due to the augmentation of the second uncertainty set, the one which corresponds to the that of the nominal controller. This matrix is still a member of the set L_Δ. It is this special structure which will commute with the uncertainty set.

The equations for the closed-loop state space matrices are as follows:

$$A = \begin{bmatrix} A_p + B_{p2}D_{k11}C_{p2} & B_{p2}C_{k1} \\ B_{k1}C_{p2} & A_k \end{bmatrix}$$
$$B_0 = \begin{bmatrix} -B_{p2} \\ 0 \end{bmatrix}, \quad B_\theta = \begin{bmatrix} B_{p\theta} + B_{p2}D_{k11}D_{p2\theta} & B_{p2}D_{k1\theta} \\ B_{k1}D_{p2\theta} & B_{k\theta} \end{bmatrix}$$
$$B_1 = \begin{bmatrix} B_{p1} + B_{p2}D_{k11}D_{p21} \\ B_{k1}D_{p21} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & B_{p2} & 0 \\ I & 0 & 0 \end{bmatrix}$$
$$C_0 = \begin{bmatrix} D_{k11}C_{p2} & C_{k1} \end{bmatrix}, \quad C_\theta = \begin{bmatrix} C_{p\theta} + D_{p\theta 2}D_{k11}C_{p2} & D_{p\theta 2}C_{k1} \\ D_{k\theta 1}C_{p2} & C_{k\theta} \end{bmatrix}$$
$$C_1 = \begin{bmatrix} C_{p1} + D_{p12}D_{k11}C_{p2} & D_{p12}C_{k1} \end{bmatrix}$$
$$D_{00} = 0, \quad D_{0\theta} = \begin{bmatrix} D_{k11}D_{p2\theta} & D_{k1\theta} \end{bmatrix},$$
$$D_{01} = D_{k11}D_{p21}, \quad D_{02} = \begin{bmatrix} 0 & I & 0 \end{bmatrix}$$
$$D_{\theta 0} = \begin{bmatrix} -D_{p\theta 2} \\ 0 \end{bmatrix}, \quad D_{\theta\theta} = \begin{bmatrix} D_{p\theta\theta} + D_{p\theta 2}D_{k11}D_{p2\theta} & D_{p\theta 2}D_{k1\theta} \\ D_{k\theta 1}D_{p2\theta} & D_{k\theta\theta} \end{bmatrix}$$
$$D_{\theta 1} = \begin{bmatrix} D_{p\theta 1} + D_{p\theta 2}D_{k11}D_{p21} \\ D_{k\theta 1}D_{p21} \end{bmatrix}, \quad D_{\theta 2} = \begin{bmatrix} 0 & D_{p\theta 2} & 0 \\ 0 & 0 & I \end{bmatrix}$$
$$D_{10} = -D_{p12}, \quad D_{11} = D_{p11} + D_{p12}D_{k11}D_{p21})$$
$$D_{12} = \begin{bmatrix} 0 & D_{p12} & 0 \end{bmatrix} \quad D_{1\theta} = \begin{bmatrix} D_{p1\theta} + D_{p12}D_{k11}D_{p2\theta} & D_{p12}D_{k1\theta} \end{bmatrix}$$

We shall then assign $x_{cl}^T = \begin{bmatrix} x^T & x_{aw}^T \end{bmatrix}$ as the matrix of states for the closed-loop system. We shall denote the closed loop system as $T_{cl} = \mathcal{F}_u(\mathcal{F}_l(G, K_{aw}), \Theta \oplus \Theta \oplus \Theta)$, which is described by the following dynamic equations

$$\begin{bmatrix} \dot{x}_{cl} \\ u \\ z_{cl} \\ e \end{bmatrix} = \begin{bmatrix} A_{cl} & B_{0,cl} & B_{\theta,cl} & B_{1,cl} \\ C_{0,cl} & D_{00,cl} & D_{0\theta,cl} & D_{01,cl} \\ C_{\theta,cl} & D_{\theta 0,cl} & D_{\theta\theta,cl} & D_{\theta 1,cl} \\ C_{1,cl} & D_{10,cl} & D_{1\theta,cl} & D_{11,cl} \end{bmatrix} \begin{bmatrix} x_{cl} \\ q \\ w_{cl} \\ d \end{bmatrix} \tag{4.24}$$
$$q = \Delta u \tag{4.25}$$
$$w_{cl} = (\Theta \oplus \Theta \oplus \Theta) z_{cl} \tag{4.26}$$

The state-space data of this system is related to the interconnected nominal system G and the anti-windup compensator Λ in the following form.

$$\begin{bmatrix} A_{cl} & B_{0,cl} & B_{\theta,cl} & B_{1,cl} \\ C_{0,cl} & D_{00,cl} & D_{0\theta,cl} & D_{01,cl} \\ C_{\theta,cl} & D_{\theta 0,cl} & D_{\theta\theta,cl} & D_{\theta 1,cl} \\ C_{1,cl} & D_{10,cl} & D_{1\theta,cl} & D_{11,cl} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_0 & \mathcal{B}_\theta & \mathcal{B}_1 \\ \mathcal{C}_0 & D_{00} & \mathcal{D}_{0\theta} & D_{01} \\ \mathcal{C}_\theta & \mathcal{D}_{\theta 0} & \mathcal{D}_{\theta\theta} & \mathcal{D}_{\theta 1} \\ \mathcal{C}_1 & D_{10} & \mathcal{D}_{1\theta} & D_{11} \end{bmatrix} + \begin{bmatrix} \mathcal{P}_1^T \\ \mathcal{P}_2^T \\ \mathcal{P}_3^T \\ \mathcal{P}_4^T \end{bmatrix} \begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix}$$
$$\times \begin{bmatrix} \mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{Q}_3 & \mathcal{Q}_4 \end{bmatrix}$$
$$= \left[\begin{array}{cc|c|cc|c} A & 0 & B_0 & B_\theta & 0 & B_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline C_0 & 0 & D_{00} & D_{0\theta} & 0 & D_{01} \\ \hline C_\theta & 0 & D_{\theta 0} & D_{\theta\theta} & 0 & D_{\theta 1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline C_1 & 0 & D_{10} & D_{1\theta} & 0 & D_{11} \end{array}\right] + \left[\begin{array}{ccc} 0 & B_2 & 0 \\ I & 0 & 0 \\ \hline 0 & D_{02} & 0 \\ \hline 0 & D_{\theta 2} & 0 \\ 0 & 0 & I \\ \hline 0 & D_{12} & 0 \end{array}\right] \begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix}$$
$$\times \left[\begin{array}{cc|c|cc|c} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{array}\right]$$

Here, we have shown that the closed-loop state-space data has an affine dependance on the anti-windup compensator gain. This allows us to use the result of theorem 5 to derive analysis and synthesis conditions for our closed loop LPV plant.

4.3.3 Anti-Windup Compensator Synthesis Condition

The following theorem provides a synthesis condition for the anti-windup compensator.

Theorem 8 *Given a scalar $0 < k < 1$, the LPV open-loop system P with a stabilizing gain-scheduling nominal LPV controller K. If there exist a pair of positive-definite matrices $R_{11} \in \mathbf{S}_+^{n_p \times n_p}$, $S \in \mathbf{S}_+^{n \times n}$, a diagonal matrix $V = diag\{v_1, v_2, ..., v_{n_u}\} > 0$, and invertible* *scaling matrices $J_{11}, L \in L_\Delta$ satisfying*

$$\begin{bmatrix} R_{11}A_p^T + A_pR_{11} - \frac{2(1-k)}{k^2}B_{p2}VB_{p2}^T & \star & \star & \star & \star \\ C_{p\theta}R_{11} - \frac{2(1-k)}{k^2}D_{p\theta 2}VB_{p2}^T & -J_{11} - \frac{2(1-k)}{k^2}D_{p\theta 2}VD_{p\theta 2}^T & \star & \star & \star \\ C_{p1}R_{11} - \frac{2(1-k)}{k^2}D_{p12}VB_{p2}^T & -\frac{2(1-k)}{k^2}D_{p12}VD_{p\theta 2}^T & -\gamma I - \frac{2(1-k)}{k^2}D_{p12}VD_{p12}^T & \star & \star \\ J_{11}B_{p\theta}^T & J_{11}D_{p\theta\theta}^T & J_{11}D_{p1\theta}^T & -J_{11} & \star \\ B_{p1}^T & D_{p\theta 1}^T & D_{p11}^T & 0 & -\gamma I \end{bmatrix} < 0 \tag{4.27}$$

$$\begin{bmatrix} SA + A^TS & SB_\theta & SB_1 & C_\theta^TL & C_1^T \\ B_\theta^TS & -L & 0 & D_{\theta\theta}^TL & D_{1\theta}^T \\ B_1^TS & 0 & -\gamma I & D_{\theta 1}^TL & D_{11}^T \\ LC_\theta & LD_{\theta\theta} & LD_{\theta 1} & -L & 0 \\ C_1 & D_{1\theta} & D_{11} & 0 & -\gamma I \end{bmatrix} < 0 \tag{4.28}$$

$$\begin{bmatrix} R_{11} & \begin{bmatrix} I & 0 \end{bmatrix} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & S \end{bmatrix} \geq 0 \tag{4.29}$$

$$\begin{bmatrix} J_{11} & \begin{bmatrix} I & 0 \end{bmatrix} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & L \end{bmatrix} \geq 0 \tag{4.30}$$

then there exists an n_p-order gain-scheduled anti-windup compensator to stabilize the closed-loop system quadratically and have the $\mathcal{L}_2$ performance level $\|e\|_2 < \gamma\|d\|_2$ when the conditions $|u_i| \leq \frac{1}{1-k}u_i^{max}$, i = 1,2,...,n_u holds.

Remark 4.3.1 *This result can be considered a simultaneous generalization of the results in [11], [39], and [1]. The form of the LMI is similar to the ones in [11] and [39] with the addition of two rows and two columns and an extra coupling condition. These new terms reflect the dependance of the system on the gain-scheduling parameter θ. Removal of the* *new terms will cause result in set of conditions identically matching those of [39], where there is no parameter dependence. Similarly, removal of the terms which do not correspond to parameter dependance will realize conditions similar to the familiar conditions of LPV synthesis as in [1].*

Proof: Denote $\Lambda = \begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix}$, $V = W^{-1}$, $T = \begin{bmatrix} L & ? \\ ? & ? \end{bmatrix}$ and $T^{-1} = \begin{bmatrix} J & ? \\ ? & ? \end{bmatrix} = \left[\begin{array}{cc|c} J_{11} & J_{12} & ? \\ J_{12}^T & J_{22} & \\ \multicolumn{2}{c|}{?} & ? \end{array}\right]$. Then apply Theorem 5 to the closed-loop system T_{cl}, and the following inequality results

$$\Psi + \mathcal{P}^T \Lambda \mathcal{Q} + \mathcal{Q}^T \Lambda^T \mathcal{P} < 0 \tag{4.31}$$

where

$$\Psi = \begin{bmatrix} \mathcal{A}^T X_{cl} + X_{cl}\mathcal{A} & X_{cl}\mathcal{B}_0 + k\mathcal{C}_0^T W & X_{cl}B_\theta & X_{cl}B_1 & \mathcal{C}_\theta^T & \mathcal{C}_1^T \\ \mathcal{B}_0^T X_{cl} + kW\mathcal{C}_0 & k(WD_{00} + D_{00}^T W) - 2W & kW\mathcal{D}_{0\theta} & kWD_{01} & D_{00}^T & D_{10}^T \\ \mathcal{B}_\theta^T X_{cl} & k\mathcal{D}_{0\theta}^T W & -T & 0 & \mathcal{D}_{\theta\theta}^T & \mathcal{D}_{1\theta}^T \\ \mathcal{B}_1^T X_{cl} & kD_{01}^T W & 0 & -\gamma I_{n_d} & \mathcal{D}_{\theta 1}^T & D_{11}^T \\ \mathcal{C}_\theta & D_{00} & \mathcal{D}_{\theta\theta} & \mathcal{D}_{\theta 1} & -T^{-1} & 0 \\ \mathcal{C}_1 & D_{10} & \mathcal{D}_{1\theta} & D_{11} & 0 & -\gamma I_{n_e} \end{bmatrix}$$

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_1 X_{cl} & k\mathcal{P}_2 W & 0 & 0 & \mathcal{P}_3 & \mathcal{P}_4 \end{bmatrix}$$

$$\mathcal{Q} = \begin{bmatrix} \mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{Q}_3 & \mathcal{Q}_4 & 0 & 0 \end{bmatrix}$$

We will use the given partitions for T and T^{-1}. For the matrix X_{cl}, we must partition according to the states of the interconnected system G and the anti-windup compensator K_{aw} as $n = n_p + n_k$ and n_{aw}, letting

$$X_{cl} = \begin{bmatrix} S & N \\ N^T & ? \end{bmatrix}$$

$$X_{cl}^{-1} = \begin{bmatrix} R & M \\ M^T & ? \end{bmatrix} = \left[\begin{array}{cc|c} R_{11} & R_{12} & M \\ R_{12}^T & R_{22} & \\ \hline \multicolumn{2}{c|}{M^T} & ? \end{array}\right]$$

Where $MN^T = I - RS$. The (2,2) entry of both of these matrices is left as an unknown, as it will not be factored into any of the equations due to the structure of the LMIs.

The Elimination Lemma [26] states that we can eliminate the terms corresponding to the closed loop system including the anti-windup compensator. In order to do this, the LMI must be transformed. This lemma allows us to create a pair LMIs which are equivalent to LMI 4.31. These two LMIs would then be solved simultaneously. According to the lemma, the equivalent pair of LMIs are:

$$\mathcal{N}_{\mathcal{P}}^T \Psi \mathcal{N}_{\mathcal{P}} < 0 \qquad \text{and} \qquad \mathcal{N}_{\mathcal{Q}}^T \Psi \mathcal{N}_{\mathcal{Q}} < 0 \tag{4.32}$$

Here, $\mathcal{N}_{\mathcal{P}}$ and $\mathcal{N}_{\mathcal{Q}}$ are the respective null spaces of the matrices $\mathcal{P}$ and $\mathcal{Q}$, which have been calculated as

$$\mathcal{N}_{\mathcal{P}} = \text{diag}\left\{X_{cl}^{-1}, W^{-1}, I, I, I, I\right\} \left[\begin{array}{ccccc} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline -\frac{1}{k}B_{p2}^T & -\frac{1}{k}D_{p\theta 2}^T & -\frac{1}{k}D_{p12}^T & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & I \\ \hline 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 & 0 \end{array}\right]$$

$$\mathcal{N}_{\mathcal{Q}} = \left[\begin{array}{ccccc} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & I \end{array}\right]$$

After carefully carrying out the multiplications in (4.32), it can be shown that

$$\mathcal{N}_{\mathcal{P}}^T \Psi \mathcal{N}_{\mathcal{P}} =$$

$$\begin{bmatrix} R_{11}A_p^T + A_pR_{11} - \frac{2(1-k)}{k^2}B_{p2}VB_{p2}^T & \star & \star & \star & \star \\ C_{p\theta}R_{11} - \frac{2(1-k)}{k^2}D_{p\theta 2}VB_{p2}^T & -J_{11} - \frac{2(1-k)}{k^2}D_{p\theta 2}VD_{p\theta 2}^T & \star & \star & \star \\ C_{p1}R_{11} - \frac{2(1-k)}{k^2}D_{p12}VB_{p2}^T & -\frac{2(1-k)}{k^2}D_{p12}VD_{p\theta 2}^T & -\gamma I - \frac{2(1-k)}{k^2}D_{p12}VD_{p12}^T & \star & \star \\ J_{11}B_{p\theta}^T & J_{11}D_{p\theta\theta}^T & J_{11}D_{p1\theta}^T & -J_{11} & \star \\ B_{p1}^T & D_{p\theta 1}^T & D_{p11}^T & 0 & -\gamma I \end{bmatrix}$$

and that

$$\mathcal{N}_{\mathcal{Q}}^T \Psi \mathcal{N}_{\mathcal{Q}} = \begin{bmatrix} SA + A^TS & SB_\theta & SB_1 & C_\theta^T L & C_1^T \\ B_\theta^T S & -L & 0 & D_{\theta\theta}^T L & D_{1\theta}^T \\ B_1^T S & 0 & -\gamma I & D_{\theta 1}^T L & D_{11}^T \\ LC_\theta & LD_{\theta\theta} & LD_{\theta 1} & -L & 0 \\ C_1 & D_{1\theta} & D_{11} & 0 & -\gamma I \end{bmatrix}$$

which are identically equal to the conditions outlined earlier.

Given the stated definitions for the closed-loop scaling matrices X_{cl} and X_{cl}^{-1}, the coupling condition between R and S should be

$$\begin{bmatrix} R_{11} & \begin{bmatrix} I & 0 \end{bmatrix} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & S \end{bmatrix} \geq 0 \qquad \text{and} \qquad \text{rank}(R - S^{-1}) \leq n_{aw}$$

Similarly, the coupling condition between J and L should be

$$\begin{bmatrix} J_{11} & \begin{bmatrix} I & 0 \end{bmatrix} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & L \end{bmatrix} \geq 0$$

Since only R_{11} is constrained in the synthesis LMIs, it is always possible to augment this block into the full R matrix satisfying the above coupling conditions. Similarly, the coupling condition for J_{11} and L can always be met due to the fact that only the first block, J_{11} is constrained. As shown in [1], the off-diagonal terms of the scaling matrix must only meet a lesser constraint in order to satisfy the coupling condition. *Q.E.D.*

This theorem and subsequent proof provides us with a way to solve the synthesis condition for the anti-windup compensator. With the solutions for the various scaling matrices obtained, the anti-windup compensator can be constructed through the feasibility process of substituting the obtained scaling matrices back into the LMI from equation (4.31). This will be the final step in solving for the anti-windup compensator gains.

4.3.4 LPV Anti-Windup Compensator Construction

As shown in subsection 4.2.2, the anti-windup compensator can be obtained by substituting the solutions R_{11} and S back into the LMI

$$\Psi + \mathcal{P}^T \Theta \mathcal{Q} + \mathcal{Q}^T \Theta^T \mathcal{P} < 0$$

and the solving the subsequent LMI for the anti-windup compensator gains. However, similar to the previous problem, we can explicitly construct the anti-windup compensator gains by taking an approach similar to the ones used in [13] and [39], as well as the rate and magnitude LTI anti-windup compensator. The process is outlined in the following theorem.

Theorem 9 *(Gain-Scheduled Compensator Construction)*
Given the solutions $R_{11}, S, J_{11}, L, \gamma$ and $V = W^{-1}$ of the LMIs (4.27)-(4.28). Let $MN^T = I_n - RS$ with $M, N \in \mathbf{S}^{n \times n_p}$ and $H^T = \begin{bmatrix} I_{n_p} & 0 \end{bmatrix}$, then an n_p-order anti-windup compensator can be constructed through the following scheme:

1. *Compute a feasible* $\hat{D}_{aw} = \begin{bmatrix} \hat{D}_{aw11} & \hat{D}_{aw1\theta} \\ \hat{D}_{aw\theta 1} & \hat{D}_{aw\theta\theta} \end{bmatrix} \in \mathbf{R}^{(n_u+n_w)\times(n_u+n_w)}$ *such that*

$$\Pi =$$

$$\begin{bmatrix} \left\{ \begin{matrix} -Wk(D_{00}+\hat{D}_{aw11}) \\ -(D_{00}+\hat{D}_{aw11})^T kW + 2W \end{matrix} \right\} & \star & \star & \star & \star \\ -\begin{bmatrix} D_{0\theta}^T \\ \hat{D}_{aw1\theta}^T \end{bmatrix} kW & T & \star & \star & \star \\ -D_{01}^T kW & 0 & \gamma I_{n_d} & \star & \star \\ -\begin{bmatrix} D_{\theta 0} + D_{p\theta 2}\hat{D}_{aw11} \\ \hat{D}_{aw\theta 1} \end{bmatrix} & -\begin{bmatrix} D_{\theta\theta} & D_{p\theta 2}\hat{D}_{aw1\theta} \\ 0 & \hat{D}_{aw\theta\theta} \end{bmatrix} & -\begin{bmatrix} D_{\theta 1} \\ 0 \end{bmatrix} & T^{-1} & \star \\ -(D_{10}+D_{p12}\hat{D}_{aw11}) & -\begin{bmatrix} D_{1\theta} & D_{p12}\hat{D}_{aw1\theta} \end{bmatrix} & -D_{11} & 0 & \gamma I_{n_e} \end{bmatrix}$$

$$> 0$$

2. *Compute the least-square solutions of the following linear equations for* $\hat{B}_{aw} = \begin{bmatrix} \hat{B}_{aw1} & \hat{B}_{aw\theta} \end{bmatrix} \in \mathbf{R}^{n\times(n_u+n_w)}, \hat{C}_{aw} = \begin{bmatrix} \hat{C}_{aw1} \\ \hat{C}_{aw\theta} \end{bmatrix} \in \mathbf{R}^{(n_u+n_w)\times n_p}$

$$\left[\begin{array}{c|c} 0 & \begin{matrix} \begin{bmatrix} I \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ I \end{bmatrix} & 0 & 0 & 0 & 0 \end{matrix} \\ \hline \begin{matrix} \begin{bmatrix} I & 0 \end{bmatrix} \\ 0 \\ \begin{bmatrix} 0 & I \end{bmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & -\Pi \end{array}\right] \left[\begin{array}{c} \hat{B}_{aw}^T \\ \hline ? \end{array}\right] = -\left[\begin{array}{c} 0_{(n_u+n_w)\times n} \\ \hline B_0^T S + WkC_0 \\ B_\theta^T S \\ 0 \\ B_1^T S \\ C_\theta \\ 0 \\ C_1 \end{array}\right]$$

$$\left[\begin{array}{c|cccccccc} 0 & \begin{bmatrix} kW \\ 0 \end{bmatrix} & 0 & 0 & 0 & \begin{bmatrix} D_{p\theta 2}^T \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} D_{p12}^T \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} Wk & 0 \end{bmatrix} & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & -\Pi & & \\ \begin{bmatrix} D_{p\theta 2} & 0 \end{bmatrix} & & & & & & \\ \begin{bmatrix} 0 & I \end{bmatrix} & & & & & & \\ \begin{bmatrix} D_{p12} & 0 \end{bmatrix} & & & & & & \end{array}\right] \left[\begin{array}{c} \hat{C}_{aw} \\ \hline ? \end{array}\right] =$$

$$-\left[\begin{array}{c} \begin{bmatrix} B_{p2}^T \\ 0 \end{bmatrix} \\ \hline (B_0^T + WkC_0R)H + \hat{D}_{aw11}^T B_{p2}^T \\ B_\theta^T H \\ \hat{D}_{aw1\theta}^T B_{p2}^T \\ B_1^T H \\ C_\theta RH \\ 0 \\ C_1RH \end{array}\right],$$

and the matrix $\hat{A}_{aw} \in \mathbf{R}^{n \times n_p}$ *as*

$$\begin{aligned} \hat{A}_{aw} &= -A^T H - \\ &\begin{bmatrix} SB_0 + C_0^T kW + \hat{B}_{aw1} & SB_\theta & \hat{B}_{aw\theta} & SB_1 & C_\theta^T & 0 & C_1^T \end{bmatrix} \Pi^{-1} \\ &\times \begin{bmatrix} (B_0^T + WkC_0R)H + \hat{D}_{aw11}^T B_{p2}^T + Wk\hat{C}_{aw1} \\ B_\theta^T H \\ \hat{D}_{aw1\theta}^T B_{p2}^T \\ B_1^T H \\ C_\theta RH + D_{p\theta 2} \begin{bmatrix} I & 0 \end{bmatrix} \hat{C}_{aw} \\ \hat{C}_{aw\theta} \\ C_1RH + D_{p12}\hat{C}_{aw1} \end{bmatrix} \end{aligned}$$

3. *Convert the transformed anti-windup compensator gain to its original state-space data*

by

$$\begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} = \left[\begin{array}{c|cc} N & SB_2 & 0 \\ \hline 0 & \begin{bmatrix} 0 & I_{n_u} & 0 \end{bmatrix} & 0 \\ 0 & 0 & I_{n_w} \end{array}\right]^{-1} \left(\begin{bmatrix} \hat{A}_{aw} & \hat{B}_{aw} \\ \hat{C}_{aw} & \hat{D}_{aw} \end{bmatrix} - \begin{bmatrix} SARH & 0 \\ 0 & 0_{(n_u+n_w)\times(n_u+n_w)} \end{bmatrix} \right) \begin{bmatrix} M^T H & 0 \\ 0 & I_{n_u+n_w} \end{bmatrix}^{-1}$$

Proof: Define $H^T = \begin{bmatrix} I_n & 0 \end{bmatrix}$, $n = n_p + n_k$ and $n_w = n_{pw} + n_{kw}$, then

$$Z_1 = \begin{bmatrix} I_n & RH \\ 0 & M^T H \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} S & H \\ N^T & 0 \end{bmatrix}$$

It can be shown that $X_{cl} Z_1 = Z_2$. We also have the following congruent transformation

$$Z_1^T X_{cl} Z_1 = \begin{bmatrix} S & H \\ H^T & H^T RH \end{bmatrix}$$

$$\begin{bmatrix} Z_1^T X_{cl} A_{cl} Z_1 & Z_1^T X_{cl} B_{0,cl} & Z_1^T X_{cl} B_{\theta,cl} & Z_1^T X_{cl} B_{1,cl} \\ C_{0,cl} Z_1 & D_{00,cl} & D_{0\theta,cl} & D_{01,cl} \\ C_{\theta,cl} Z_1 & D_{\theta 0,cl} & D_{\theta\theta,cl} & D_{\theta 1,cl} \\ C_{1,cl} Z_1 & D_{10,cl} & D_{1\theta,cl} & D_{11,cl} \end{bmatrix}$$

$$= \left[\begin{array}{cc|c|cc|c} SA & 0 & SB_0 & SB_\theta & 0 & SB_1 \\ H^T A & H^T ARH & H^T B_0 & H^T B_\theta & 0 & H^T B_1 \\ \hline C_0 & C_0 RH & D_{00} & D_{0\theta} & 0 & D_{01} \\ \hline C_\theta & C_\theta RH & D_{\theta 0} & D_{\theta\theta} & 0 & D_{\theta 1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline C_1 & C_1 RH & D_{10} & D_{1\theta} & 0 & D_{11} \end{array}\right]$$

$$+ \left[\begin{array}{c|cc} I_n & 0 & 0 \\ 0 & B_{p2} & 0 \\ \hline 0 & I_{n_u} & 0 \\ \hline 0 & D_{p\theta 2} & 0 \\ 0 & 0 & I_{n_w} \\ \hline 0 & D_{p12} & 0 \end{array}\right] \begin{bmatrix} \hat{A}_{aw} & \hat{B}_{aw} \\ \hat{C}_{aw} & \hat{D}_{aw} \end{bmatrix} \left[\begin{array}{cc|c|cc|c} 0 & I_{n_p} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{n_u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{array}\right]$$

where

$$
\begin{bmatrix} \hat{A}_{aw} & \hat{B}_{aw} \\ \hat{C}_{aw} & \hat{D}_{aw} \end{bmatrix} = \begin{bmatrix} SARH & 0 \\ 0 & 0_{(n_u+n_w)\times(n_u+n_w)} \end{bmatrix}
$$
$$
+ \left[\begin{array}{c|cc} N & SB_2 & 0 \\ \hline 0 & \begin{bmatrix} 0 & I_{n_u} & 0 \end{bmatrix} & 0 \\ 0 & 0 & I_{n_w} \end{array}\right] \begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} \begin{bmatrix} M^T H & 0 \\ 0 & I_{n_u+n_w} \end{bmatrix}
$$

Multiplication by diag $\left\{Z_1^T, I, I, I, I, I\right\}$ from the left, and its conjugate transpose from right side of equation (4.31), yields

$$
\left[\begin{array}{cc}
A^T S + SA & \star \\
\hat{A}_{aw}^T + H^T A & B_{p2}\hat{C}_{aw1} + \hat{C}_{aw1}^T B_{p2}^T + H^T(RA^T + AR)H \\
\hat{B}_{aw1}^T + B_0^T S + kWC_0 & (B_0^T + kWC_0)H + \hat{D}_{aw11}^T B_{p2}^T + kW\hat{C}_{aw1} \\
B_\theta^T S & B_\theta^T H \\
\hat{B}_{aw\theta}^T & \hat{D}_{aw1\theta}^T B_{p2}^T H \\
B_1^T S & B_1^T H \\
C_\theta & C_\theta RH + D_{p\theta 2}\hat{C}_{aw1} \\
0 & \hat{C}_{aw\theta} \\
C_1 & C_1 RH + D_{p12}\hat{C}_{aw1}
\end{array}\right.
$$
$$
\left.\begin{array}{ccccc}
\star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star \\
\left\{\begin{array}{c} -kW(D_{00} + D_{aw11} \\ -k(D_{00} + \hat{D}_{aw11})^T W + 2W \end{array}\right\} & \star & \star & \star & \star \\
\begin{bmatrix} kD_{0\theta}^T \\ \hat{D}_{aw1\theta}^T \end{bmatrix} & -T & \star & \star & \star \\
kD_{01}^T W & 0 & -\gamma I_{n_d} & \star & \star \\
\begin{bmatrix} D_{\theta 0} + D_{p\theta 2}\hat{D}_{aw11} \\ \hat{D}_{aw\theta 1} \end{bmatrix} & \begin{bmatrix} D_{\theta\theta} & D_{p\theta 2}\hat{D}_{aw1\theta} \\ 0 & \hat{D}_{aw\theta\theta} \end{bmatrix} & \begin{bmatrix} \hat{D}_{\theta 1} \\ 0 \end{bmatrix} & -T^{-1} & \star \\
D_{10} + D_{p12}\hat{D}_{aw11} & \begin{bmatrix} D_{1\theta} & D_{p12}\hat{D}_{aw1\theta} \end{bmatrix} & D_{11} & 0 & -\gamma I_{n_e}
\end{array}\right] < 0 \tag{4.33}
$$

Which is equivalent via Schur complement to

$$\begin{bmatrix} A^TS+SA & \star \\ \hat{A}_{aw}^T+H^TA & B_{p2}\hat{C}_{aw1}+\hat{C}_{aw1}^TB_{p2}^T+H^T(RA^T+AR)H \end{bmatrix}$$

$$+\left[\begin{matrix} SB_0+kC_0^TW+\hat{B}_{aw1} & SB_\theta & \hat{B}_{aw\theta} \\ H^T(B_0+kC_0W)+B_{p2}\hat{D}_{aw11}+\hat{C}_{aw1}^TkW & H^TB_\theta & H^TB_{p2}\hat{D}_{aw1\theta} \end{matrix}\right.$$

$$\left.\begin{matrix} SB_1 & C_\theta^T & 0 & C_1^T \\ H^TB_1 & H^TRC_\theta^TD_{p\theta 2} & \hat{C}_{aw\theta}^T & \hat{C}_{aw1}^TD_{p12}^T \end{matrix}\right]\Pi^{-1}\times\begin{bmatrix} \star \\ \hline \star \end{bmatrix}<0 \quad (4.34)$$

As in the rate and magnitude case, the lower (5,5) element of the inequality (4.33) is inherently negative definite. This is used to determine the feasible $\hat{D}_{aw}$. After taking the Schur complement with respect to that (5,5) element, the new (2,1) element is set to be equal to zero. This determines the feasible $\hat{A}_{aw}$. This also leads to a set of decoupled LMI's for the remaining $\hat{B}_{aw}$ and $\hat{C}_{aw}$ matrices from the LMI (4.34). *Q.E.D.*

As with the previous case, this construction scheme and derivation can be applied to systems which are open-loop stable. The only difference is the lack of scaling matrix W. Feasible $\hat{D}_{aw}$ and W matrices will be solved together in the first step. The rest of the procedure remains unchanged by letting $k=1$.

Chapter 5

Anti-Windup Compensator Design and Simulation Results

With the synthesis conditions derived, we can now apply our proposed anti-windup compensator synthesis to the regulation problem of the F8 aircraft. First, we shall state the model for the open loop F8 aircraft. This model is identical to the one used in [38]. We shall then show an weighted open-loop interconnection model for the aircraft, which will be used in controller and anti-windup compensator synthesis. Following this will be the nominal robust controller which stabilizes the system, but whose output exceeds that of the saturation limit. Finally, we shall show the results achieved by the anti-windup compensator for a variety of nonlinear saturation conditions.

5.1 F8 Aircraft Model

The F8 aircraft model is a 4-state model with two inputs and two outputs. The two inputs are the elevator angle $\delta_e(t)$, and the flaperon angle $\delta_f(t)$ which are both measured in degrees. These two input will be restricted by the nonlinear saturation constraint to have magnitude no larger than $15\,^\circ$. The two outputs of the system are the pitch angle θ(t) and the flight path angle γ(t), which are both measured in radians. Finally, the four states of the system are the pitch rate $q(t)$ (rad/sec), the forward velocity $v(t)$ (ft/sec), the angle of attack α(t) (radians), and the pitch angle θ(t). An unstable pole has been added to the system in order to demonstrate the capability of the anti-windup compensator to stabilize an unstable system.

The state space dynamics of the system are given in the following equations.

$$\dot{x}(t) = \begin{bmatrix} -0.8 & -0.006 & -12 & 0 \\ 0 & -0.014 & -16.64 & -32.2 \\ 1 & -0.0001 & -1.5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -19 & -3 \\ -0.66 & -0.5 \\ -0.16 & -0.5 \\ 0 & 0 \end{bmatrix} u_s(t) \tag{5.1}$$

$$y(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} x(t) \tag{5.2}$$

$$u_s(t) = \sigma(u(t)) \tag{5.3}$$

These are the open-loop dynamics of the unstable F8 aircraft plant. For controller (and anti-windup compensator) synthesis, we would like to add weighting functions to this plant in order to better define our performance criteria. To this end, we shall penalize the control effort, the reference input, and the error measurement with the following weighting functions.

$$\begin{aligned} W_e(s) &= \frac{0.5s + 25}{s + 0.05} I_2 \\ W_u(s) &= \operatorname{diag}\left\{ \frac{0.525s + 2.4585}{s + 100}, \frac{1.05s + 5}{s + 100} \right\} \\ W_r(s) &= \frac{8}{s + 8} I_2 \end{aligned}$$

When the plant is augmented with the three weighting functions, we will have the weighted interconnection used for synthesis given in figure 5.1.

5.2 Nominal Controller

The MATLAB $\mathcal{H}_\infty$-synthesis command was used to synthesize the nominal controller for this system. This controller had 10 states and achieved a γ value of 0.647.

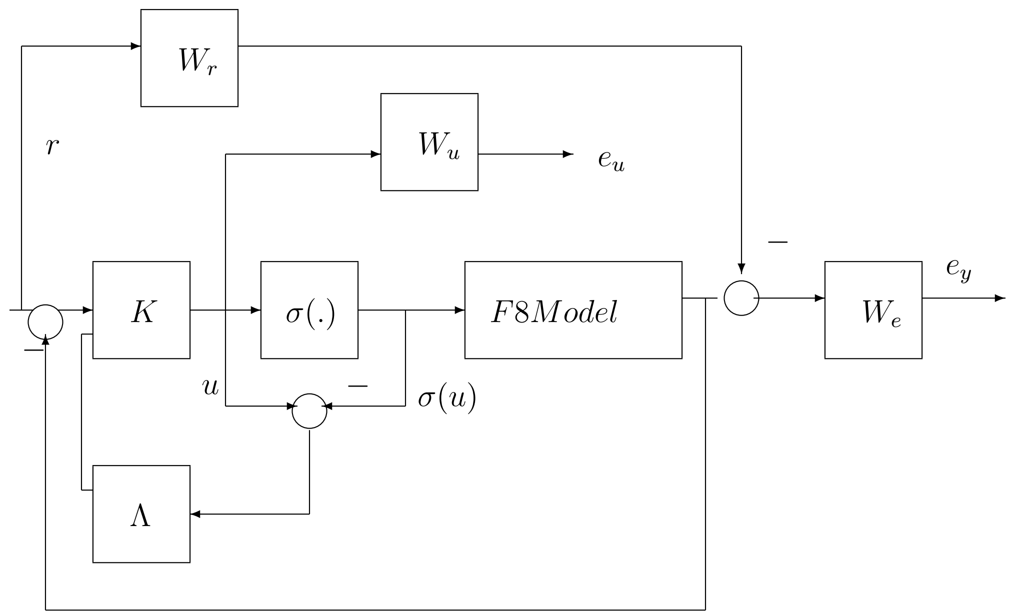


Figure 5.1: Weighted Closed-Loop Interconnection

$A_k =$

$$\begin{bmatrix}
-47.3 & -.016 & 2388 & -2831 & -188.4 & 2829 & -295.5 & 4488 & 7.77 & -1230 \\
.575 & -.014 & -.85 & 6.9 & 3.8 & -2.2 & 4.5 & -4.8 & -1.02 & 1.7 \\
3.2 & 0 & -132 & 151 & 14.2 & -154 & 21.9 & -245 & -12.5 & 61.4 \\
1 & 0 & .153 & -.243 & -.141 & -.127 & -.157 & -.142 & 0 & 0 \\
0 & 0 & .004 & 2.7 & -8.05 & -.044 & -.055 & -.049 & 0 & 0 \\
0 & 0 & -2.79 & 2.75 & -.044 & -8.04 & -.05 & -.05 & 0 & 0 \\
0 & 0 & -.016 & -1.54 & 4.47 & .018 & -.028 & -.02 & 0 & 0 \\
0 & 0 & 1.56 & -1.54 & .018 & 4.5 & .02 & -.032 & 0 & 0 \\
2.35 & .0053 & -.001 & 1466 & 107.2 & -1471 & 157.3 & 2335 & -132.4 & 626.6 \\
-55.3 & -.015 & 3181 & -3694 & -332.2 & 3750 & -513.8 & 5951 & 264.4 & -1611
\end{bmatrix}$$

$$B_k = \begin{bmatrix} -.0126 & -.0113 \\ 19.77 & 17.82 \\ -.0089 & -.008 \\ -.09 & -.081 \\ 1.4 & -.028 \\ -.029 & 1.4 \\ .0128 & .0116 \\ .0116 & .0104 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_k =$$

$$\begin{bmatrix} 6.56 & .001 & -349 & 409.7 & 29.96 & -411 & 46.8 & -652 & -9.06 & 175 \\ -10.9 & -.003 & 628.7 & -730 & -65.7 & 741 & -101.5 & 1176 & 52.24 & -298.7 \end{bmatrix}$$

$$D_k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5.3 Compensator Synthesis

Table 5.1 shows that the $\mathcal{H}_\infty$ performance for different conic sectors is always worse than the nominal performance. These results show the strong adverse effect of the saturation nonlinearity on the system. Furthermore, it can be shown that the performance of the final closed-loop sstem can be significantly improved by reducing the k_m value from 1. These values for the calculated performance level are achievable as long as the assumption that the output from the nominal controller is less than $(\frac{1}{1-k_m})u_{max}$.

5.4 Simulations

Here the numerical results of the construction were simulated under several different conditions. Subsection 5.4.1 demonstrates the actuator saturation of the aircraft under the effects of the sector-bounded nonlinearity using the nominal $\mathcal{H}_\infty$ controller.

Sector range $[0, k_m]$	$\mathcal{H}_\infty$ performance γ
0.99999	27.5253
0.9999	4.9229
0.999	1.3639
0.994	1.0453
0.99	1.0452

Table 5.1: H_∞ performance level vs. sector range $[0, k_m]$

5.4.1 Actuator Saturation Simulations

SIMULINK was used in order to simulate the closed-loop response of the system to a reference tracking input. The open-loop response of the system was (as expected) exponentially unstable. This is due to the added pole at 0.14. The response to a reference tracking signal of 10 degrees is given in the figure 5.2. The first response is that of an unsaturated input. In other words, this is the response of the system to the nominal controller in the complete absence of any type of actuator saturation.

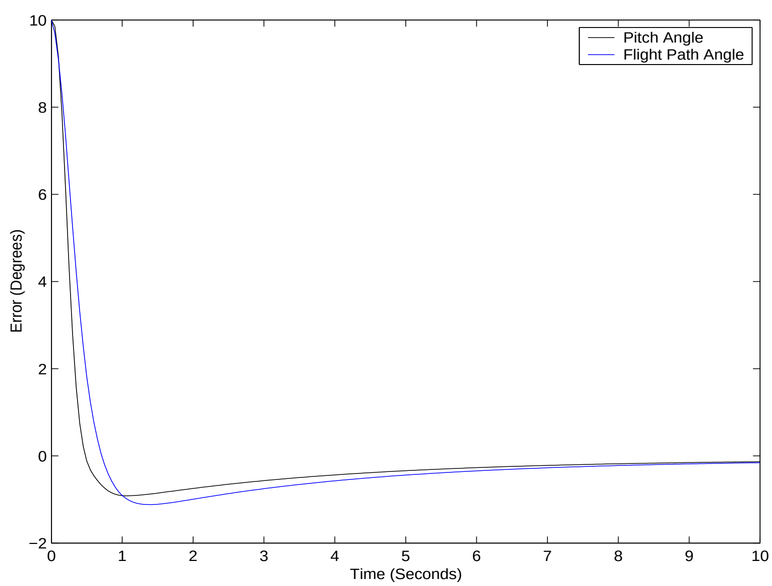


Figure 5.2: F8 Response to 10 Degree Tracking Input under Nominal Control without Saturation

As can be seen, the nominal controller stabilizes the system and reduces the error slowly, but thoroughly.

Figure 5.3 shows that the nominal control output easily exceeds the saturation limit of 15° we hope to impose upon the system. In the face of saturation, the control output has a nonlinear profile, and the performance degrades as expected. The proposed anti-windup compensator design will act as a gain on the difference between the saturated and unsaturated inputs. That is to say that the effect of the anti-windup compensator, Λ will only be present when the system inputs exceed the saturation limit. This is true for both the rate and magnitude conditions. Therefore, the saturation of actuators cannot be prevented, but the effect of this saturation will be minimized as the compensator gain will take effect immediately upon saturation.

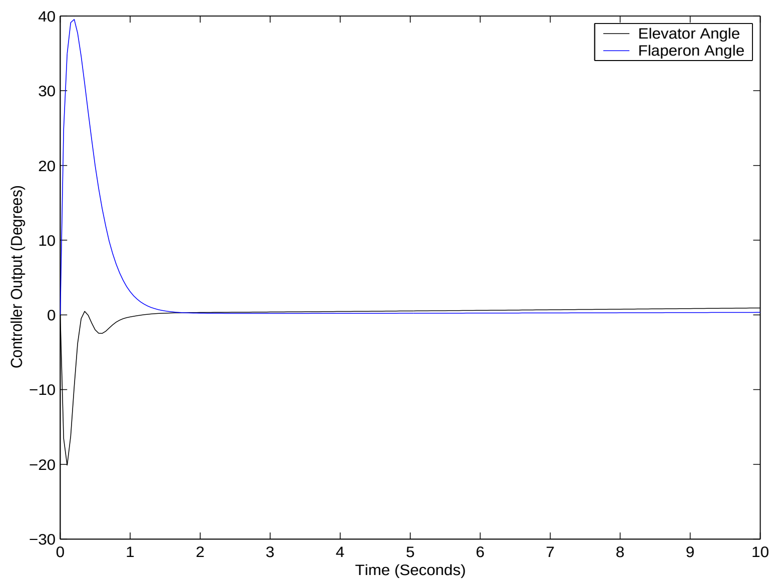


Figure 5.3: Unsaturated Input from Nominal Controller to F8 Aircraft System

We can see from figure 5.5 that the performance of the system under saturation is considerably worse, with more oscillations, and a slower settling time. This is only in the face of magnitude saturation. In order to consider the effects of rate saturation, a second SIMULINK model was constructed which penalizes the derivative of the control input with a similar saturation block limited to $60\frac{\circ}{sec}$, with the magnitude saturation remaining at 15 degrees.

It can be seen in figure 5.6, that the addition of rate saturation worsens the performance

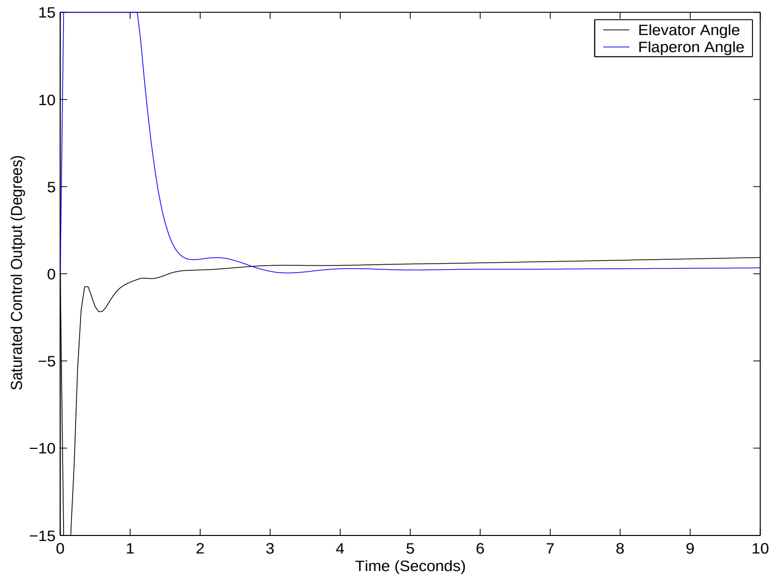


Figure 5.4: Saturated Input from Nominal Controller to F8 Aircraft System

considerably.

The settling time of the error is comparable to the settling time with only magnitude saturation, however the size of the oscillations has almost quadrupled. As shown in [23], and verified by this example, this type of saturation can be a critical contributing factor in the existence of large Pilot Induced Oscillations (PIOs).

This amplification in the size of the error oscillations is made more clear when the rate saturation of the actuators is examined.

These figures demonstrate the need for anti-windup compensation. A clear loss of performance has been noted, and this trend could lead to a loss of stability. The following section displays the effects of the anti-windup compensator on the saturated nominal system.

5.4.2 Anti-Windup Compensator Simulations

This first case of anti-windup controlled simulations is for an actuator saturation which is $60\frac{\circ}{sec}$ for rate saturation and 15° for magnitude saturation. The second case will be for $30\frac{\circ}{sec}$ rate saturation and the 15° magnitude saturation. The same compensator will be used for both cases, in order to demonstrate the robustness of the compensator design.

In designing the compensator, the saturation nonlinearity parameters were set to $k_m =$

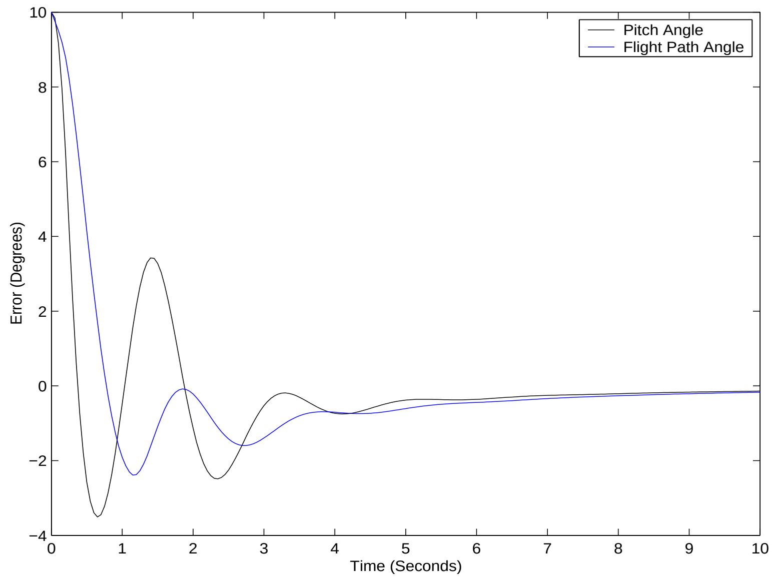


Figure 5.5: Error Measurement for F8 with Nominal Controller and Magnitude Saturation

0.9999 and $k_r = 1$. The bandwidth constant was set to $k_b = 8.727$.

Finally, the anti-windup compensator was added to the system to close the loop between saturation output $\tilde{q}$ and anti-windup compensator input v. The result was improved performance in the sense of lowering the size of the oscillations while maintaining the settling time of the error. The performance value γ of the new closed loop system was 4.9229. As stated by table 5.1, this performance value can be improved upon by reducing the restriction on k_m. While this is clearly worse than that of the nominal system with no saturation, the simulations showed that it was an improvement to the system with the nominal controller and saturation.

We can clearly see from the following figures that the compensator maintains the stability of the system, as well as improves the performance in terms of the settling time and peak overshoots, as well as number of oscillations. The effect of the anti-windup compensator seems to diminish for less strict rate saturation criterion. This is due to the fact that the compensator will actuate less when the amount of saturation decreases.

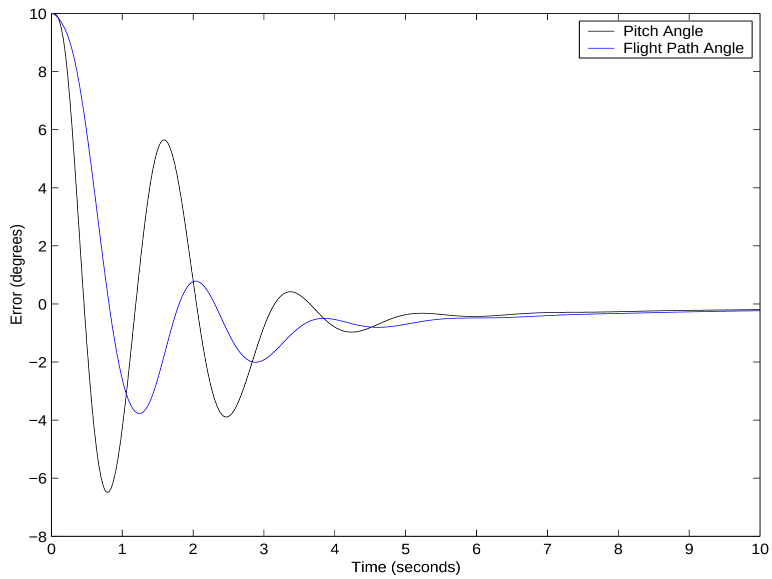


Figure 5.6: Error Measurement for F8 Aircraft with Nominal Controller and Rate and Magnitude Saturation

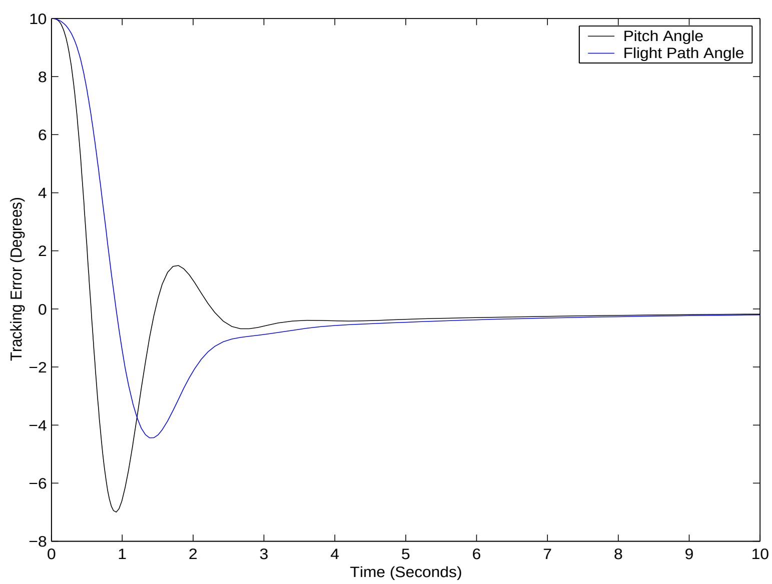


Figure 5.7: Output Error of F8 Aircraft with Anti-Windup Correction for Actuator Saturation: Case 30

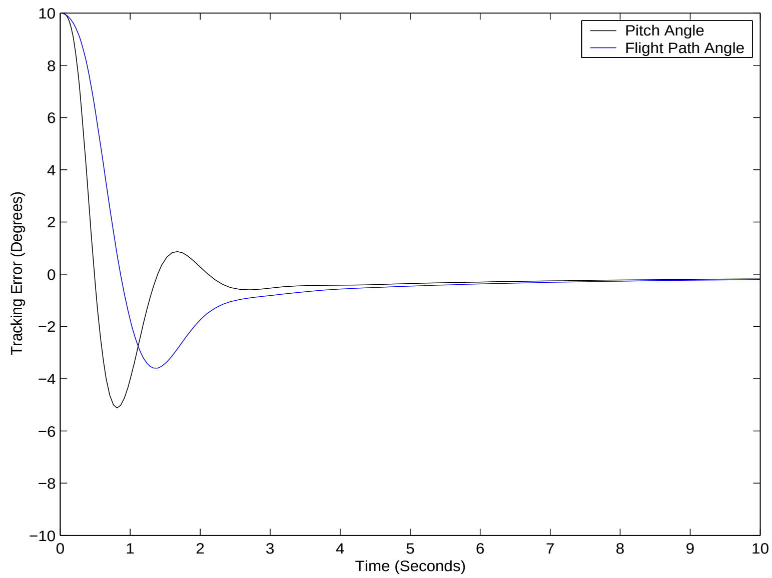


Figure 5.8: Output Error of F8 Aircraft with Anti-Windup Correction for Actuator Saturation: Case 60

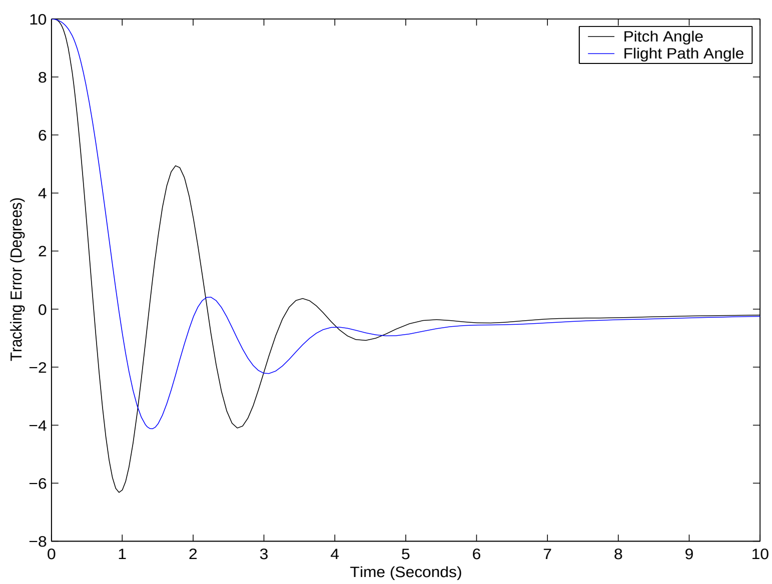


Figure 5.9: Output Error of Nominal F8 Aircraft System: Case 30

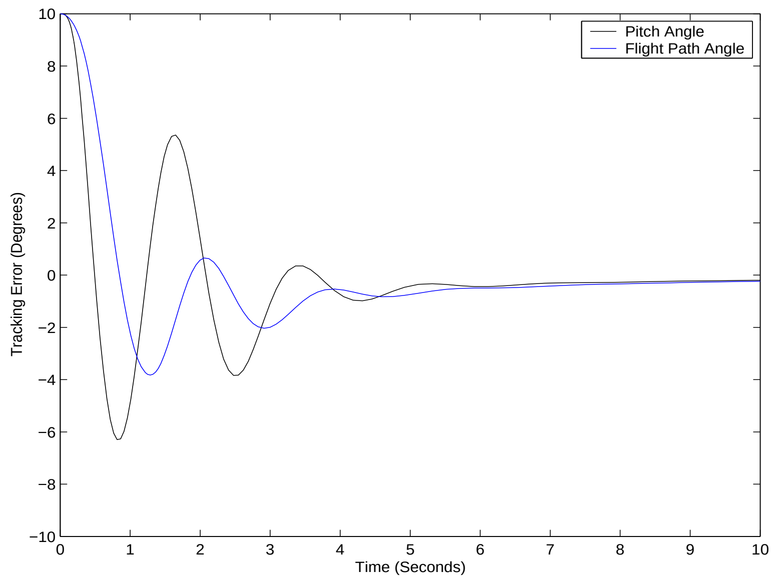


Figure 5.10: Output Error of Nominal F8 Aircraft System: Case 60

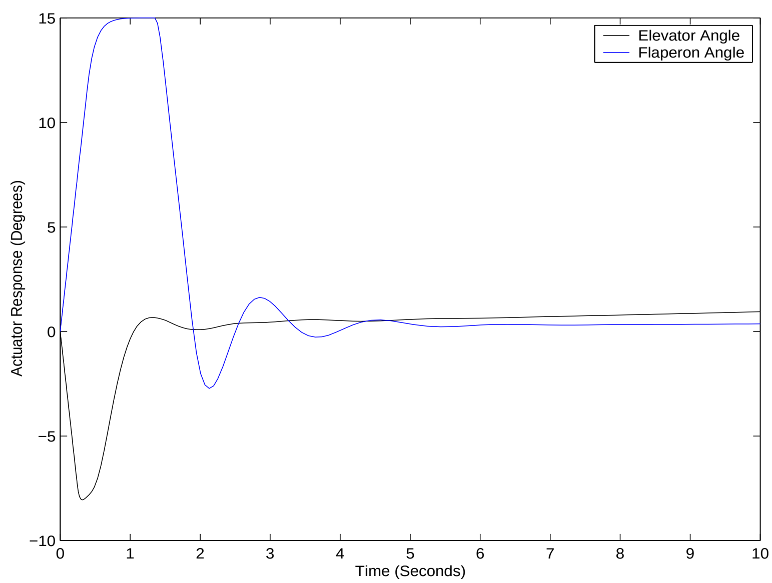


Figure 5.11: Saturated Control Input to Nominal System: Case 30

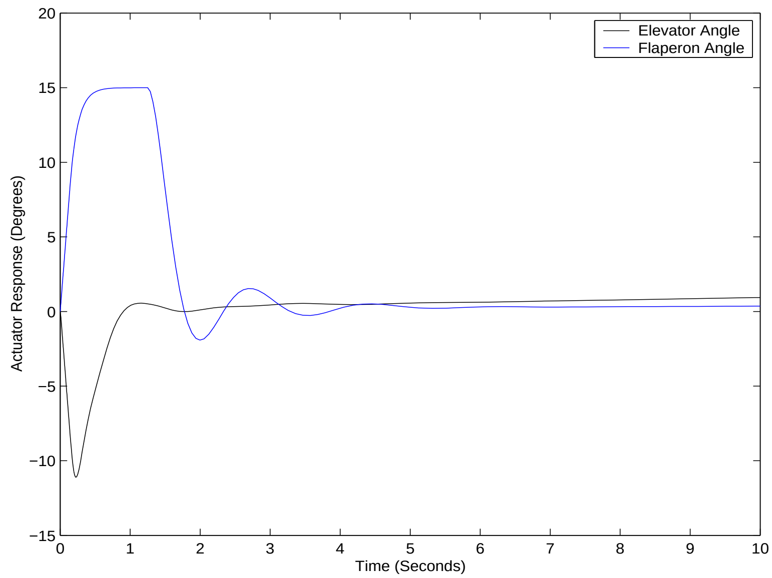


Figure 5.12: Saturated Control Input to Nominal System: Case 60

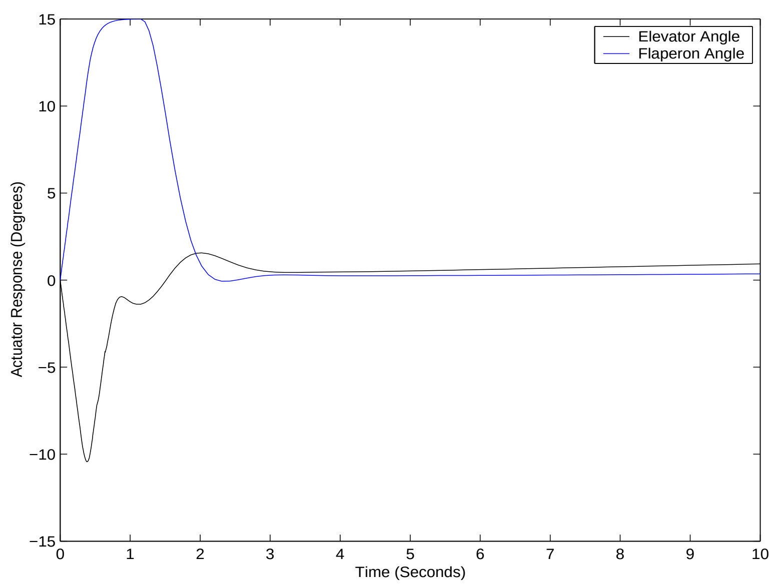


Figure 5.13: Saturated Control Input to System with Anti-Windup Compensator: Case 30

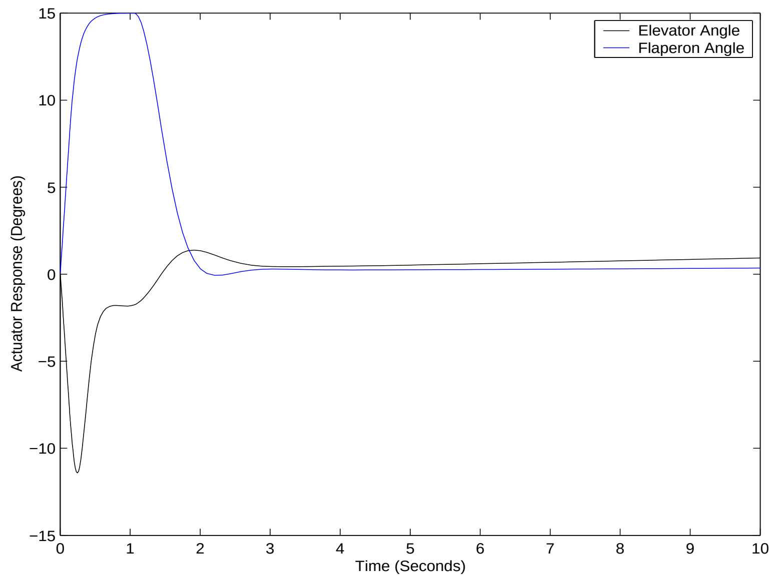


Figure 5.14: Saturated Control Input to System with Anti-Windup Compensator: Case 60

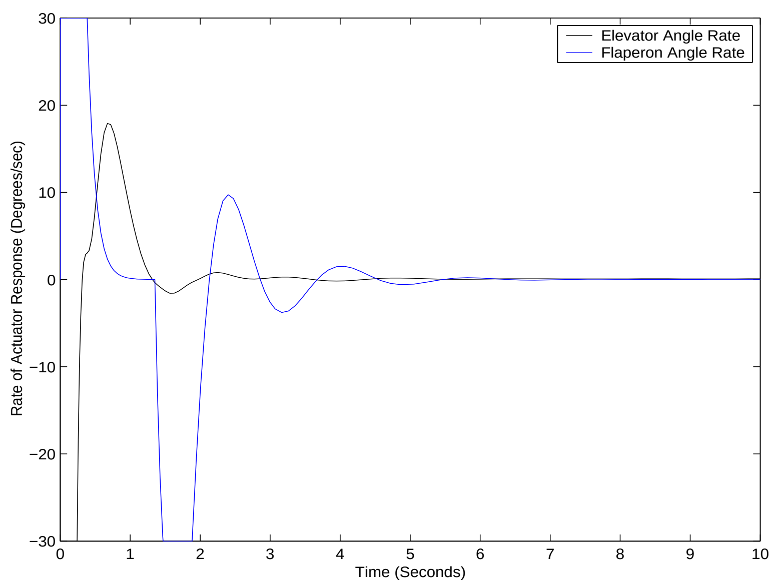


Figure 5.15: Rate of Saturated Control Input to Nominal System: Case 30

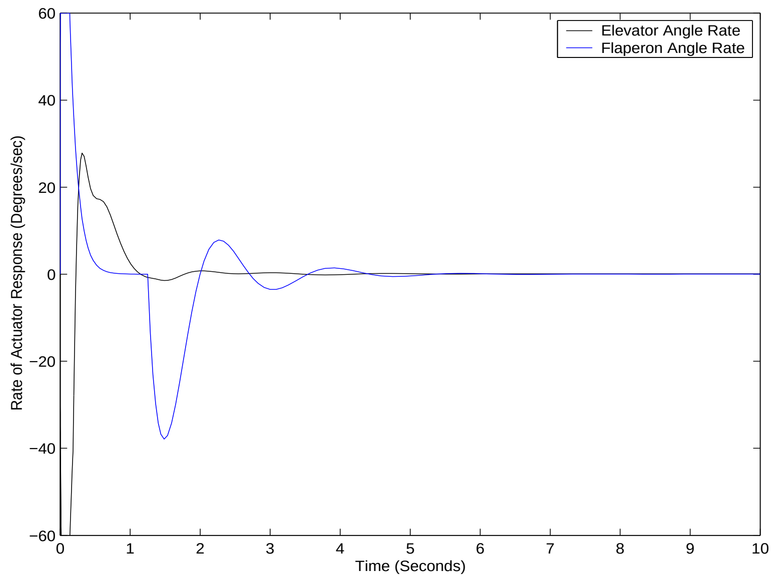


Figure 5.16: Rate of Saturated Control Input to Nominal System: Case 60

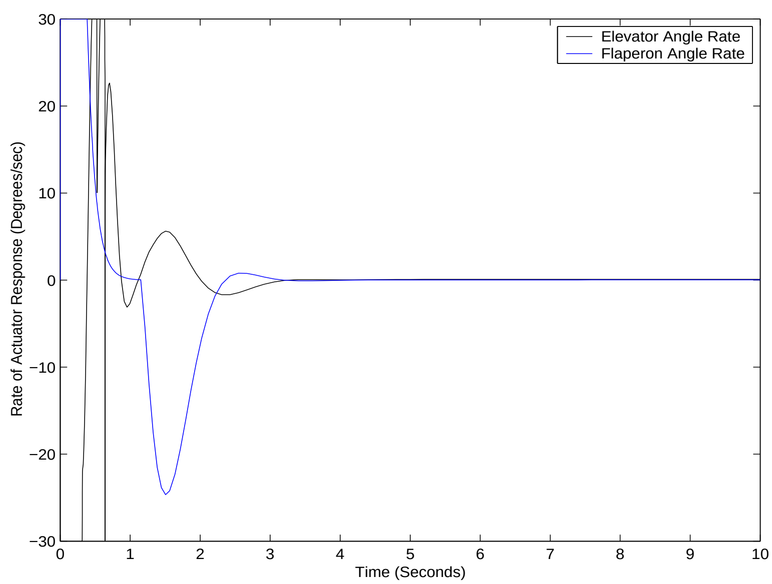


Figure 5.17: Rate of Saturated Control Input to Anti-Windup Controlled System: Case 30

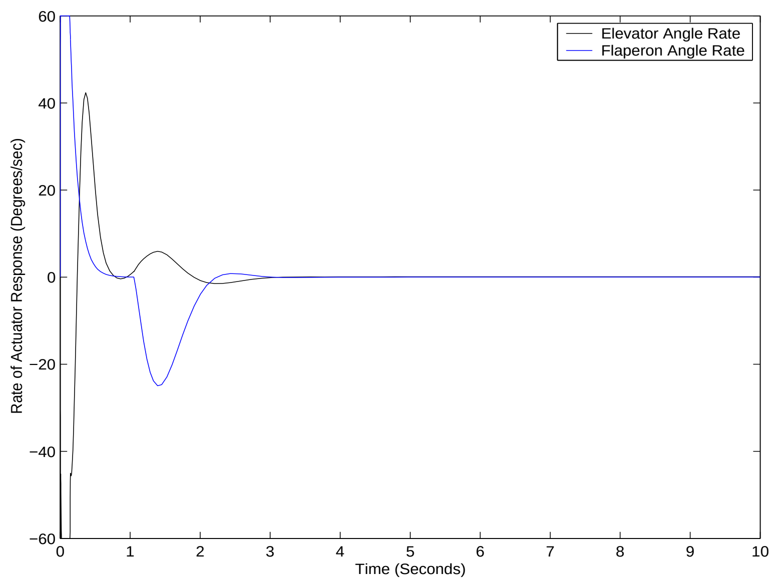


Figure 5.18: Rate of Saturated Control Input to Anti-Windup Controlled System: Case 60

Chapter 6

Conclusions

In this thesis, two-step anti-windup control procedures were studied with the particular applications of compensation for rate and amplitude saturation, and compensation for gain-scheduled linear parameter varying systems. The main goal of this research was to develop an anti-windup compensator to control exponentially unstable systems that are either LPV, or LTI with two kinds of actuator saturation. This compensator has been derived using traditional LMI methods familiar in robust control theory. The proposed anti-windup compensator design has also been used in an example which shows the capability of robustness in the control design.

Modifications were made to the well-known results of Circle criteria. The first such modification was to expand the usage of this method to systems which have multiple sector-bounded nonlinearities constraining the inputs to a certain saturation range. The second modification to the Circle criterion was an expansion to include both sector-bounded nonlinearities and norm-bounded uncertainties. This extension modified the criterion in order to ensure that an anti-windup compensator which was synthesized using this criterion would also be effectively implemented into a gain-scheduled control scheme for an LPV system.

A set of feasibility LMIs were derived for the rate and magnitude saturated system and the gain-scheduled LPV system. These LMIs were convex optimization problems. The solutions to these LMIs were shown to be obtainable for systems which met the basic well-posed constraints for robust control systems. These results were proven using S-Theory and familiar LMI variable elimination solution methods. Synthesis conditions for the two cases were also provided, based on the feasibility conditions outlined previously. These synthesis conditions were implicitly based on the solution of similar LMIs to the feasibility conditions.

A set of explicit controller construction formulae were given. In the presence of LMI controller feasibility equations, it was advantageous to provide an alternate means of constructing the anti-windup compensators. This was shown to be the case due to the fact that there is the possibility of numerical ill-conditioning when reevaluating the original feasibility LMIs to find the anti-windup compensator gains.

A model for an F8 aircraft was chosen to test the proposed design. This model had an additional unstable pole added to it in order to demonstrate the capability of the proposed anti-windup compensator design scheme. Saturation limits were set for the amplitude of the actuator response, as well as the rate of the actuator response. A nominal robust $\mathcal{H}_\infty$ controller was designed to stabilize the system and provide nominal control. The proposed anti-windup compensator design for rate and magnitude saturated systems was then implemented for this system. Numerical simulations were carried out for the augmented system with a magnitude saturation of $15°$ and a rate saturation limit of $60\frac{\circ}{sec}$ and $30\frac{\circ}{sec}$. The effects of the actuator saturation were shown by comparing the system response in both saturated and unsaturated states. Next, the anti-windup compensator was included in the simulations. The results showed that the anti-windup compensator improved the performance of the saturated system. Particularly in the area of actuator saturation. While the proposed design cannot prevent the occurrence of actuator saturation, the effects of this nonlinearity can be reduced significantly.

The results of this research may have been improved if a numerical example of the LPV synthesis could be obtained. This would prove the effectiveness of the proposed LPV gain-scheduled anti-windup compensator. Another future direction of research would be to examine how the LPV anti-windup compensator performed in the event of a destabilizing rate saturation. While the proposed design is sound in the face of exponentially unstable systems, the effects of a destabilizing rate saturation have not been examined. Another interesting possibility is that of comparing the result obtained in this work with various single step anti-windup compensator construction schemes to see how the plant was affected.

List of References

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Appendix A

MATLAB code

A.1 Rate and Magnitude Synthesis Code

```
Clear;
epsilon=.0000001;
k_r=1;
load goodk;
load wdata olicr;
bk=[zeros(10,2) bk];
dk=[zeros(2,2) dk];
K_nom=pck(ak,bk,ck,dk);
sysp=olicr;
sysk=K_nom;
nmeas=2;
nctrl=2;
k_m=0.9999;
opt='e';
band=8.737;

scalem = 2*(k_m-1)/k_m^2;
scaler = 2*(k_r-1)/k_r^2;

[dum,nop,nip,nxp] = minfo(sysp);
[dum,nok,nik,nxk] = minfo(sysk);
nep = nop-nmeas;
ndp = nip-nctrl;
nx = nxp+nctrl;
[ap,bp,cp,dp] = unpck(sysp);
bp1 = bp(:,1:ndp);
bp2 = bp(:,ndp+1:nip);
cp1 = cp(1:nep,:);
cp2 = cp(nep+1:nop,:);
dp11 = dp(1:nep,1:ndp);
dp12 = dp(1:nep,ndp+1:nip);
dp21 = dp(nep+1:nop,1:ndp);
dp22 = dp(nep+1:nop,ndp+1:nip);
[ak,bk,ck,dk] = unpck(sysk);
bk1 = bk(:,1:ndp);
bk2 = bk(:,ndp+1:nik);
dk1 = dk(:,1:ndp);
dk2 = dk(:,ndp+1:nik);

ap=[ap bp2;zeros(nctrl,nxp) -band*eye(nctrl)];
```

```
bp0=[zeros(nxp,nctrl) zeros(nxp,nctrl);-
band*eye(nctrl,nctrl) -eye(nctrl,nctrl)];
bp1=[bp1;zeros(nctrl,nctrl)];
bp2=[zeros(nxp,nctrl);band*eye(nctrl)];
cp0=[zeros(nctrl,nxp) zeros(nctrl,nctrl);zeros(nctrl,nxp) -
band*eye(nctrl)];
cp1=[cp1 dp12];
cp2=[cp2 dp22];
dp00=[zeros(nctrl,nctrl) zeros(nctrl,nctrl);-
band*eye(nctrl) zeros(nctrl,nctrl)];
dp01=zeros(2*nctrl,ndp);
dp02=[eye(nctrl);band*eye(nctrl)];
dp10=[zeros(nctrl,nctrl) zeros(nctrl,nctrl)];
dp11=dp11;
dp12=zeros(nctrl,nctrl);
dp20=[zeros(nmeas,nctrl) zeros(nmeas,nctrl)];
dp22=zeros(nmeas,nctrl);

a=[ap+bp2*dk2*cp2 bp2*ck; bk2*cp2 ak];
b0=[bp0+bp2*dk2*dp20; bk2*dp20];
b1=[bp1+bp2*(dk1+dk2*dp21);bk1+bk2*dp21];
b2=[zeros(nxp+nctrl,nxk) bp2;eye(nxk) zeros(nxk,nctrl)];
c0=[cp0+dp02*dk2*cp2 dp02*ck];
d00=[dp00+dp02*dk2*dp20];
d01=[dp01+dp02*(dk1+dk2*dp21)];
d02=[zeros(2*nctrl,nxk) dp02];
c1=[cp1+dp12*dk2*cp2 dp12*ck];
d10=dp10+dp12*dk2*dp20;
d11=dp11+dp12*(dk1+dk2*dp21);
d12=[zeros(nep,nxk) dp12];
kbar=[k_m*eye(nctrl) zeros(nctrl,nctrl);zeros(nctrl,nctrl)
k_r*eye(nctrl)];

deltak=eye(2,2);
deltap=eye(2,2);

nxp=nxp+nctrl;
nx=nxp+nxk;
k=kbar;
setlmis([]);
%
% Set optimization variables
%
idR11 = lmivar(1,[nxp 1]);    % R11
idS = lmivar(1,[nx 1]);       % S
idGAM = lmivar(1,[1 0]); % GAM
if (k_m ~= 1)
```

```
    idUm = lmivar(1,[2 0]);   % U before was (1,1), but 2
inputs....
    idUr = lmivar(1,[2 0]);
end
%
% LMI of R
%
lmiterm([1 1 1 idR11],ap,1,'s');
lmiterm([1 2 1 idR11],[zeros(nctrl,nxp-2) -eye(nctrl)],1);
lmiterm([1 2 2 idUm],scalem,1);
lmiterm([1 2 2 idUr],-2*(1/(band^2*k_r^2)),1)
lmiterm([1 3 1 0],bp1');
lmiterm([1 3 2 0],0);
lmiterm([1 3 3 idGAM],-eye(nep),1);
lmiterm([1 4 1 idR11],cp1,1);
lmiterm([1 4 2 0],zeros(nep,nxp));
lmiterm([1 4 3 0],dp11');
lmiterm([1 4 4 idGAM],-eye(nep),1);
if (k_m ~= 1)
    lmiterm([1 1 1 idUm],scalem*bp2,bp2',1);
    lmiterm([1 2 1 idUm],scalem,bp2');
    lmiterm([1 2 1 idUr],-1/(band^2*k_r),bp2');
%    lmiterm([1 3 1 idU],scale*dp12,bp2');
%    lmiterm([1 3 3 idU],scale/2*dp12,dp12','s');
end
%
% LMI of S
%
lmiterm([2 1 1 idS],1,a,'s');
lmiterm([2 2 1 idS],b1',1);
lmiterm([2 2 2 idGAM],-eye(2),1);
lmiterm([2 3 1 0],c1);
lmiterm([2 3 2 0],d11);
lmiterm([2 3 3 idGAM],-eye(2),1);
%
% Coupling condition
%
temp = [eye(nxp) zeros(nxp,nxk)];

lmiterm([-3 1 1 idR11],1,1);
lmiterm([-3 2 1 0],temp');
lmiterm([-3 2 2 idS],1,1);
%
% Additional constraint
%
if (k_m ~= 1)
    lmiterm([4 1 1 idUm],-1,1);
```

```
    lmiterm([4 2 2 idUr],-1,1);
    lmiterm([4 3 3 idUm],1,1);
    lmiterm([4 4 4 idUr],1,1);
    lmiterm([4 3 3 0],-1e4);
    lmiterm([4 4 4 0],-1e4);
end

lmis = getlmis;

nvar = lminbr(lmis);
disp(['   Total variable numbers:  ',num2str(nvar)])
nlmi = decnbr(lmis);
disp(['   Total LMI numbers:  ',num2str(nlmi)])
%
% Construct CVEC
%
nvar = decnbr(lmis);
cvec = zeros(nvar,1);
for i = 1:nvar
    [vRi,vSi,vGAMi] = defcx(lmis,i,idR11,idS,idGAM);
    cvec(i,1) = vGAMi+epsilon*(trace(vRi)+trace(vSi));
end
%
% Call LMI optimization subroutine
%
[copt,xopt] = mincx(lmis,cvec,[1e-3 300 -1 0 0]);
%
% Convert the optimization variables to matrix form
%
r11 = dec2mat(lmis,xopt,idR11);
s = dec2mat(lmis,xopt,idS);
gamma = dec2mat(lmis,xopt,idGAM);
if (k_m ~= 1)
    Um = dec2mat(lmis,xopt,idUm);
    Ur = dec2mat(lmis,xopt,idUr);
    Wm = inv(Um);
    Wr = inv(Ur);
    W=[Wm zeros(2,2);zeros(2,2) Wr];
end
minfo(W)
sinv = inv(s);
r = [r11 sinv(1:nxp,nxp+1:nx);
    sinv(nxp+1:nx,:)];

if (opt == 'e')
    temp = eye(nx)-r*s;
    [u,d,v] = svd(temp);
```

```
    minfo(u)
    tempd = sqrtm(d);
    M = u*tempd(:,1:nxp);
    N = v*tempd(:,1:nxp);
else
     temp = r*s*r-r;
     temp = (temp+temp')/2;
     [u,d,v] = svd(temp);
     N = u(:,1:nxp)*sqrtm(d(1:nxp,1:nxp));
     M = eye(nxp)+N'/r*N;
     Q = [r N;N' M];
end


if (opt == 'e')
    %
    % Explicit construction. Pick a feasible DAW
    %

    setlmis([]);

    idDAW = lmivar(2,[nctrl 2*nctrl]);% DAW
     idTAU = lmivar(1,[1 0]);           % TAU

    lmiterm([1 1 1 idDAW],W*k*dp02,1,'s');
     lmiterm([1 1 1 0],W*k*d00+d00'*k*W);
     lmiterm([1 1 1 0],-2*W);
     lmiterm([1 2 1 0],d01'*k*W);
     lmiterm([1 2 2 0],-gamma*eye(nctrl));
     lmiterm([1 3 1 idDAW],dp12,1);
     lmiterm([1 3 1 0],d10);
     lmiterm([1 3 2 0],d11);
     lmiterm([1 3 3 0],-gamma*eye(nctrl));

     lmiterm([2 1 1 idTAU],-1,1);
     lmiterm([2 2 1 idDAW],-1,1);
     lmiterm([2 2 2 0],-1);

     dmatrix = getlmis;

    nvar = decnbr(dmatrix);
     cvec = zeros(nvar,1);
     for i = 1:nvar
        [vDAWi,vTAUi] = defcx(dmatrix,i,idDAW,idTAU);
        cvec(i,1) = vTAUi;
     end
    [copt,xopt] = mincx(dmatrix,cvec,[1e-3 300 -1 0 0]);
```

```
     dawk = dec2mat(dmatrix,xopt,idDAW);
    tau = dec2mat(dmatrix,xopt,idTAU);
     %
     % Compute BAW and CAW
     %
    E = [eye(nxp);zeros(nxk,nxp)];

    pi11 = W*k*d00+d00*k*W+W*k*dp02*dawk+dawk'*dp02'*k*W-
2*W;
    pi21 = d01'*k*W;
     pi22 = -gamma*eye(ndp);
    pi31=d10;+dp12*dawk;
     pi32 = d11;
     pi33 = -gamma*eye(nep);
     Pi = [pi11 pi21' pi31';
     pi21 pi22 pi32';
     pi31 pi32 pi33];

%     b0=sel(b0,1:24,1:2);

     L1 = [(b0'+W*k*c0*r)*E+dawk'*bp2';b1'*E;c1*r*E];
     L2 = [b0'*s+W*k*c0;b1'*s;c1];
nctrl=2*nctrl;
     tempb = [eye(nctrl) zeros(nctrl,ndp)
zeros(nctrl,nep)];
     leftb = [zeros(nctrl) tempb;
          tempb' Pi];
     rightb = -[zeros(nctrl,nx); L2];
     thetab = leftb\rightb;
     bawk = (thetab(1:nctrl,:))';

     tempc = [dp02'*k*W zeros(nctrl/2,ndp) [dp12]'];
     leftc = [zeros(nctrl/2) tempc;
          tempc' Pi];
     rightc = -[bp2'; L1];
     thetac = leftc\rightc;
     cawk = thetac(1:nctrl/2,:);
     %
     % Calculate aaw
    %
     temp1 = [s*b0+bawk+c0'*k*W s*b1 c1'];
    temp2 = [(b0'+W*k*c0*r)*E+dawk'*[bp2]'+W*k*dp02*cawk;
              b1'*E;
              c1*r*E+[dp12]*cawk];
     aawk = -a'*E + temp1/Pi*temp2;
```

```
    temp1 = [aawk bawk;cawk dawk]-[s*a*r*E
zeros(nx,nctrl);zeros(nctrl/2,nxp+nctrl)];
    temp2 = [N s*b2;zeros(nctrl-2,nx) eye(nctrl-2)]\temp1;
    temp3 = temp2/[M'*E zeros(nxp,nctrl);zeros(nctrl,nxp)
eye(nctrl)];
    aaw = temp3(1:nxp,1:nxp);
    baw = temp3(1:nxp,nxp+1:nxp+nctrl);
    caw = temp3(nxp+1:nx+nctrl-2,1:nxp);
    daw = temp3(nxp+1:nx+nctrl-2,nxp+1:nxp+nctrl);

else

%     Construct controller gain through feasibility solver.

     nctrl=2*nctrl;
     A0 = [a zeros(nxp+nxk,nxp);
          zeros(nxp,nxp+nxk) zeros(nxp,nxp)];
     B0 = [b0;zeros(nxp,nctrl)];
     B1 = [b1;zeros(nxp,ndp)];
     C0 = [c0 zeros(nctrl,nxp)];
     C1 = [c1 zeros(nep,nxp)];
     H1 = [zeros(nxp+nxk,nxp) b2;
          eye(nxp) zeros(nxp,nxp)]';
     H2 = [zeros(nctrl,nxp) d02]';
     H3 = [zeros(nep,nxp) d12]';
     G1 = [zeros(nxp,nx) eye(nxp);zeros(nctrl,nx+nxp)];
     G2 = [zeros(nxp,nctrl);eye(nctrl)];
    U=[Um zeros(2,2);zeros(2,2) Ur];

     setlmis([]);

    idLAM = lmivar(2,[nx+nctrl/2,nxp+nctrl]);
     idTAU = lmivar(1,[1,1]);
    if (k_m == 1)
        idU = lmivar(1,[1 1]);
    end

    lmiterm([1 1 1 0],Q*A0'+A0*Q);
     lmiterm([1 1 1 idLAM],H1',G1*Q,'s');
     lmiterm([1 2 1 0],k*C0*Q);
     lmiterm([1 2 1 idLAM],k*H2',G1*Q);
     lmiterm([1 3 1 0],B1');
     lmiterm([1 3 2 0],d01'*k);
     lmiterm([1 3 3 0],-gamma);
     lmiterm([1 4 1 0],C1*Q);
     lmiterm([1 4 1 idLAM],H3',G1*Q);
     lmiterm([1 4 3 0],d11);
```

```
 lmiterm([1 4 4 0],-gamma);
if (k_m ~= 1)
      lmiterm([1 2 1 0],U*B0');
    lmiterm([1 2 1 -idLAM],U*G2',H1);
    lmiterm([1 2 2 0],k*d00*U+k*U*d00');
      lmiterm([1 2 2 0],-2*U);
      lmiterm([1 2 2 idLAM],k*H2',G2*U,'s');
      lmiterm([1 4 2 0],d10*U);
      lmiterm([1 4 2 idLAM],H3',G2*U);
 else
    lmiterm([1 2 1 idU],1,B0');
    lmiterm([1 2 1 -idLAM],G2',H1);
      lmiterm([1 2 2 idU],d00,1,'s');
      lmiterm([1 2 2 idU],-1,1,'s');
      lmiterm([1 2 2 idLAM],H2',G2,'s');
      lmiterm([1 4 2 idU],d10,1);
      lmiterm([1 4 2 idLAM],H3',G2);
end

temp1 = [zeros(nctrl,nxk) eye(nctrl)];
 temp2 = [zeros(nxk+nctrl,nxp-2) eye(nxk+nctrl)];
 temp3 = [zeros(nctrl,nxp) eye(nctrl)];
 lmiterm([2 1 1 idTAU],-1,1);
 lmiterm([2 2 1 -idLAM],-temp3,temp2'*temp1');
 lmiterm([2 2 2 0],-1);

construct = getlmis;

nvar = decnbr(construct);
 cvec = zeros(nvar,1);
for i = 1:nvar
    [vTAUi] = defcx(construct,i,idTAU);
    cvec(i,1) = vTAUi;
end
 [gopt,xopt] = mincx(construct,cvec,[1e-3 300 -1 0 0]);
 lambda = dec2mat(construct,xopt,idLAM);

aaw = lambda(1:nxp,1:nxp);
caw = lambda(nxp+1:nx+nctrl/2,1:nxp);
if (k_m ~= 1)
    baw = lambda(1:nxp,nxp+1:nxp+nctrl);
    daw = lambda(nxp+1:nx+nctrl/2,nxp+1:nxp+nctrl);
else
    U = dec2mat(construct,xopt,idU);
    baw = lambda(1:nxp,nxp+1:nxp+nctrl)/U;
    daw = lambda(nxp+1:nx+nctrl,nxp+1:nxp+nctrl)/U;
end
```

```
end

kaw = pck(aaw,baw,caw,daw);

load dataw a_p b_p c_p d_p
bk=sel(bk,1:10,3:4);
dk=sel(dk,1:2,1:2);
bk = [bk eye(10,10) zeros(10,2)];
dk = [dk zeros(2,10) eye(2,2)];

b = [b0 b1 b2];
c = [c0;c1;zeros(nctrl,nxp+nxk)];
d = [d00 d01 d02;
    d10 d11 d12;
    eye(nctrl) zeros(nctrl,ndp) zeros(nctrl,nxk+nctrl/2)];
G = pck(a,b,c,d);
T = starp(G,kaw);
maxre=max(real(spoles(T)));
disp(['Optimal Gamma Value: ',num2str(gamma)])
disp(['Maximum Eigenvalue of Closed Loop System:
',num2str(maxre)])

     A0 = [a zeros(nxp+nxk,nxp);
          zeros(nxp,nxp+nxk) zeros(nxp,nxp)];
     B0 = [b0;zeros(nxp,nctrl)];
     B1 = [b1;zeros(nxp,ndp)];
     C0 = [c0 zeros(nctrl,nxp)];
     C1 = [c1 zeros(nep,nxp)];
     H1 = [zeros(nxp+nxk,nxp) b2;
          eye(nxp) zeros(nxp,nxp)]';
     H2 = [zeros(nctrl,nxp) d02]';
     H3 = [zeros(nep,nxp) d12]';
     G1 = [zeros(nxp,nx) eye(nxp);zeros(nctrl,nx+nxp)];
     G2 = [zeros(nxp,nctrl);eye(nctrl)];
    U=[Um zeros(2,2);zeros(2,2) Ur];

Q=[s N;N' -N'*r*pinv(M')];

Psi = [A0'*Q+Q*A0 Q*B0+C0'*k*W Q*B1 C1';
      B0'*Q+W*k*C0 W*k*d00+d00'*k*W-2*W W*k*d01 d10';
      B1'*Q d01'*k*W -gamma*eye(ndp) d11';
      C1 d10 d11 -gamma*eye(nep)];

     H1 = [zeros(nxp+nxk,nxp) b2;
          eye(nxp) zeros(nxp,nxp)]';
     H2 = [zeros(nctrl,nxp) d02]';
```

```
     H3 = [zeros(nep,nxp) d12]';
     G1 = [zeros(nxp,nx) eye(nxp);zeros(nctrl,nx+nxp)];
     G2 = [zeros(nxp,nctrl);eye(nctrl)];
    U=[Um zeros(2,2);zeros(2,2) Ur];

    lambda=[aaw baw;caw daw];
P=[H1*Q H2*k*W zeros(nxp+nxk+2,nep) H3];
term=[G1 G2 zeros(nxp+nctrl,2) zeros(nxp+nctrl,2)];

flmi = Psi + P'*lambda*term + term'*lambda'*P;
maxref=max(real(eig(flmi)));
```