

ABSTRACT

MANSI, KATE ELIZABETH. Reasoning and Geometric Proof in Mathematics Education: A Review of the Literature. (Under the direction of Dr. Hollylynne Stohl.)

The purpose of this literature review is to examine the role that reasoning and geometric proof play in the teaching and learning of mathematics. Specifically, I explore four questions: 1) What reasoning capabilities do students need to be ready for proof? 2) What evidence is there to show that high school students are not successful with proof and hold misconceptions about the nature of proof? 3) How can teachers' beliefs and understandings contribute to students' proof abilities? 4) What can be done to promote mathematical reasoning and improve students' proof writing skills?

Through a comparison of the theories of Piaget and van Hiele, I discuss how students acquire mathematical and geometric reasoning skills and how this relates to their readiness to produce formal proofs. I then discuss research findings, which indicate that students are not typically at a high enough van Hiele level to be successful with proof by the time they get to high school. Further research is presented which examines common geometric and proof misconceptions among students, and how this relates to proof achievement. Teacher proof-conceptions and achievement are also discussed, citing studies with elementary, middle, and high school preservice and inservice teachers, and how this may affect students' proof performance. Finally, I discuss ways in which preservice and inservice teachers can help their students improve their mathematical and geometric reasoning skills, thus furthering their proof comprehension and achievement.

REASONING AND GEOMETRIC PROOF IN
MATHEMATICS EDUCATION: A REVIEW OF THE LITERATURE

by

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BIOGRAPHY

My name is Kate Elizabeth Mansi. I was born in Bridgeport, Connecticut on April 7, 1978. I joined an older sister, Jill. Our brother, Tyler, who was adopted from Korea in January 1985, later joined us. My father was an officer in the Coast Guard, so we lived in several places as I was growing up. We spent the most time on Governor's Island off the tip of lower Manhattan in New York City. We moved to Elizabeth City, North Carolina in August 1989. My parents, brother, and sister, along with her family, still live there.

After graduating from Northeastern High School in June 1996, I attended Elon College as a North Carolina Teaching Fellow. During my sophomore year, I spent my spring semester in London and was able to backpack through Europe during my stay. I regard the entire experience as the best four months of my life. I came home with a new sense of independence and self-confidence, which has benefited me tremendously in my profession, studies, and personal life.

I graduated from Elon in May 2000 and began my teaching career at Southern Alamance High School in Alamance County, NC. After my first year, I decided to move to Raleigh. I was hired to teach Geometry and Introductory Math at Sanderson High School, and have been there for the past two years. My experience at Sanderson has been rewarding and challenging. I am very grateful to have ended up at such a wonderful school with a tremendous faculty and administration.

At the advice of my former advisor at Elon, I began graduate school in August 2001. I was determined to finish in two years, and fortunately have been successful with that goal. I am getting married on July 19, 2003 to an officer in the United States Air Force. My

fiance, Jonathan, is a physicist at Kirtland Air Force Base in Albuquerque, New Mexico. I will be following the Mansi family tradition of a military lifestyle. With my love of travel and passion for new and exciting things, I am sure that this lifestyle is ideal for me. In the fall, I will begin a new career, hopefully in the field of education.

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CHAPTER 1

INTRODUCTION

Proof is a topic that is often met with resistance by even the most advanced high school student in mathematics. Most students do not really comprehend why they must learn how to write a proof, and many have a preconceived notion of proof as some “necessary evil” that they must conquer in high school Geometry. The concept of proof in mathematics is often first introduced in high school Geometry and not seen in a broader view as the more formal aspect to reasoning and justification. When students are asked to comment on proof, some common responses are “I hated proofs,” “Why did I need to prove something that seemed so obvious?” and “I would rather trust the brilliant mathematician who came up with the theorem” (Sowder & Harel, 1998, p. 670). These words could probably be heard in any typical Geometry classroom. Many teachers, myself included, do not always understand why students should learn to write proofs, why they struggle with proof writing, or how to best help them achieve proof-writing skills.

As a classroom teacher responsible for preparing many Geometry students to succeed in writing proofs, I was compelled to investigate the educational issues and complexities involved with learning to write proofs. Several critical questions framed my curiosity for this literature review:

- Why do students have negative feelings about proof?
- Are these feelings common among students of Geometry?

- Is the concept of proof so far out of reach for Geometry students (typically age 14-16) that we should postpone teaching it until later, or even leave it out of the curriculum altogether?
- Should proof only be emphasized in Geometry classes?
- Most importantly, if I am expected to teach proof in my Geometry classes, what can I do to help students better understand it?

The research reviewed has helped me understand that it is not necessarily the writing of two-column proofs that is important in helping students learn and comprehend mathematics. Rather, it is the reasoning and understanding required in proof writing that is important to student comprehension and doing of mathematics. A complete understanding of proof is not possible without sufficient mathematical reasoning skills.

A Broad Perspective on Critical Issues in Teaching Reasoning and Proof

Proof is viewed by mathematicians as central to the discipline and practice of mathematics (Knuth, 2002a). However, its place in mathematics education is the source of constant debate. Some argue that proof no longer has a place in the mathematics curriculum (Hanna, 2002a). Others argue that to take proof out of mathematics would be to strip the discipline of its most fundamental and essential practice (Bruckheimer & Arcavi, 2001; Hanna, 2000a; Knuth, 2002a; Steen, 1999; Wu, 1996). This ongoing debate stems from the fact that proof is rarely explored in present-day mathematics education outside of a high school Geometry course. In the North Carolina Standard Course of Study (1998), the first mention of proof is in the high school Geometry curriculum. However, according to the

National Council of Teachers of Mathematics (NCTM, 2000), reasoning and proof are topics that should be explored as early as in elementary school. This does not imply that five-year-olds should learn to write formal proofs. The NCTM simply recommends that students be required to formulate conjectures, communicate, explain, and justify their reasoning and to develop these thinking skills throughout their mathematical experiences in K-12 (2000). If this recommendation is followed, students should be better prepared to write formal proofs by the time they get to high school. However, this paper will present evidence that most students are not prepared to write proofs at that time. It appears that students may not be getting the necessary mathematical reasoning skills before they get to high school.

Local and National Curriculum Guidelines

According to the North Carolina Department of Public Instruction (NCDPI), a goal of mathematics education is to develop “strong mathematical problem solving and reasoning abilities” in students across all levels. Proof is first mentioned in the North Carolina Standard Course of Study in the Geometry curriculum (NCDPI, 1998). Competency Goal 2 states that students should be able to use properties of geometry to solve problems and write proofs. This is further broken down by Competency Goal 2.07, which states that students should be able to “write direct (two-column, paragraph, or flow) and indirect proofs” (p. 53). Proof is mentioned again in Competency Goals 2.08, 2.12, 2.13, 2.14, and 2.16, which state that students should be able to write proofs related to angle and segment relationships, properties of quadrilaterals, triangles, circles, and congruence.

Though proof is emphasized consistently throughout the NC Standard Course of Study for Geometry, it does not appear to be emphasized elsewhere in the mathematics

curriculum (NCDPI, 1998). The NCDPI makes the importance of proof in Geometry clear. However, although proof has a definite place in the Geometry curriculum, not all students are necessarily mastering proof writing in their geometry courses.

In the *Principles and Standards of School Mathematics (PSSM)* the NCTM (2000) makes recommendations for how to promote reasoning and proof in the elementary, middle, and high school levels. The Reasoning and Proof Standard of the *PSSM* states that, in grades K-12, students should be able to:

- Recognize reasoning and proof as fundamental aspects of mathematics
- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proof
- Select and use various types of reasoning and methods of proof (p. 56)

At the elementary level, students should be placed in situations in which they are able to make, refine, and test their own conjectures. Students should be able to use concrete materials to test their conjectures. This should continue into high school, at which point students need to learn how to express their ideas in terms of mathematical language and symbols. Students can further learn about reasoning by discussing conjectures formed by their peers. They should learn to develop examples and counterexamples, and be able to articulate their reasoning by presenting to groups. In high school, students should be able to put their arguments into clear, written form. Both teachers and students should be in the habit of asking “why?” This critical question is essential for students to develop mathematical reasoning skills.

The NCTM (2000) does not imply that elementary students should be able to write formal proofs. Proof may not be something that can be mastered in one course in high school

Geometry. Therefore, a framework for proof that includes extensive instruction in mathematical reasoning should be developed from the very beginning of mathematics education. If students have solid mathematical reasoning skills, then they should be better prepared for proof when we teach it in high school Geometry.

The Possible Decline of Proof In Mathematics Education

Even though states like NC extensively emphasize proof in high school Geometry, some believe that proof is not as prevalent in today's high school curriculum as it was 20-30 years ago. Hanna (2000a) cites three factors that he believes have contributed to the decline of proof in the secondary mathematics curriculum. One of these factors is in the recommendations made by the NCTM's *Curriculum and Evaluation Standards* (1989). Hanna claims that in 1989, the NCTM implied that the only students who need to be taught proof are those that intend to study mathematics in college. The second factor that Hanna claims led to the decline of proof is that many educators see proof as unnecessary, and that heuristic techniques are seen as more useful in developing reasoning and justification skills. This is a view popularized not only by the NCTM Standards in 1989, but also by the British National Curriculum (Noss, 1994). Finally, Hanna (2000a) makes the claim that dynamic geometry software has replaced the need for teaching formal proof. In other words, with an increase of technological tools in instruction, he feels that deductive proof has been abandoned "in favor of a dynamic visual approach to mathematical justification" (p. 23).

Hanna is not alone in his belief that there has been a decline in the teaching of proof in the secondary curriculum. Hadas, Hershkowitz, and Schwartz (2000) also make a case for the decline of proof as a result of dynamic geometry software. Is there evidence that shows

secondary students are not ready for writing and understanding formal proofs?

Purpose of this Literature Review

As a high school geometry teacher, I have witnessed first hand that the alarming majority of my students lack the skills necessary for success in proof writing. My frustration in teaching proof to students who struggle with mathematical reasoning has been the motivating source behind this literature review. Students in NC are given End of Course exams covering the entire Geometry curriculum, which puts a great deal of pressure on teachers and students to accomplish every goal on the curriculum. Proof is emphasized in the NC Geometry curriculum to a great extent. However, based on research and confirmed by my own teaching experience, students appear to not be ready for proof when we teach it in high school Geometry. This literature review seeks to answer four questions.

- What reasoning capabilities do students need to be ready for proof?
- What evidence is there to show that high school students are not successful with proof and hold misconceptions about the nature of proof?
- How can teachers' beliefs and understandings contribute to students' proof abilities?
- What can be done to promote mathematical reasoning and improve students' proof writing skills?

Each question will be addressed separately in the following four chapters. Relevant research findings will be reported and summarized within each question. Of course, the results discussed in prior chapters will be built upon and referred to in later chapters to weave the important findings together into an integrated discussion. The final chapter (Chapter 6) will

include general conclusions and implications.

CHAPTER 2

WHAT REASONING CAPABILITIES DO STUDENTS NEED TO BE READY FOR PROOF?

The NCTM (2000) makes its view of the importance of mathematical reasoning clear, emphasizing that being able to reason is essential to understanding mathematics. In fact, by the end of high school, students should be able to use mathematical reasoning to “produce mathematical proofs, and should appreciate the value of such arguments” (p. 56). Mathematical reasoning and proof should not be confined only to a Geometry course, but should be emphasized throughout school mathematics. However, we are likely to notice a student’s reasoning deficiencies the most in Geometry, as it is traditionally the first subject in which students are asked explicitly to “prove” a mathematical statement (Mason & Moore, 1997).

Research shows that mathematical reasoning has a cognitive structure in terms of how students acquire the ability to reason. It may be that we are expecting students to be able to mathematically reason before they are ready for it. The focus of this chapter is on how students acquire mathematical reasoning skills and what it implies for readiness to construct geometric proofs. I will first discuss issues of mathematical reasoning in general, then geometric reasoning, in particular. The final aspect of this chapter includes results from several studies that relate levels of geometric reasoning to proof.

Development of Mathematical Reasoning

Webster (1982) defines reason as “the ability to think coherently and logically and draw inferences or conclusions from facts known or assumed” (p. 1183). Applying this definition to research on mathematics education, I define mathematical reasoning as the ability to think coherently and logically and draw inferences or conclusions from mathematical facts known or assumed. Mathematical reasoning should be developed long before we ask students to write proofs (Battista & Clements, 1995; Edwards, 1997; Hanna, 2000b; Knuth, 2002a; Mistretta, 2000; Perham & Perham, 1997). Students’ overall ability to reason about mathematical ideas and make justifications for why a mathematical concept makes sense or why a procedure should be used is a powerful and necessary part of learning mathematics. Students who are not forming these reasoning and justification abilities throughout their learning of mathematics will most likely struggle with the notion of proof. This section discusses the types of reasoning students should be engaged in before they are at a level to begin writing proofs.

Edwards (Edwards, 1997) uses a metaphor to explain “the territory before proof” in students. The “territory before proof” involves “ways of thinking, talking, and acting that support the goal of seeking and establishing mathematical certainty” (p. 189). In other words, the territory before proof is the mathematical reasoning students engage in that leads to the development of formal proof-writing skills. Students’ justifications and reasoning skills are linked to their everyday mathematical activities. Within these everyday mathematical activities, Edwards has proposed five types of reasoning activities that are commonly noticed before the territory of proof. These reasoning activities are based hierarchically and include:

- Noticing and constructing patterns

- Describing the pattern
- Conjecturing
- Inductive Reasoning
- Deductive reasoning

The earliest noticeable reasoning skill in children is that of noticing and constructing patterns or rules to explore certain problems. After children have mastered this skill, they begin describing patterns by putting the rule into words, informally or formally. An informal description is a verbal or pictorial description, while a formal description uses mathematical notation and symbols. Whether the description is formal or informal, students may represent their rule by demonstrating specific examples or by presenting a generalization. After students are able to describe the pattern or rule, they are able to make conjectures. This means that they are able to say that the pattern or rule is true for a general case, though they do not yet see the necessity of “proving” it. They may try to work out several examples to verify that their rule applies in several cases, and then conjecture that it will always be true. Students may attempt a justification using reasoning skills. This leads to the fourth reasoning activity noticed before proof.

Students begin to inductively reason when they test specific cases to see if their rule or pattern still holds. They base their justification of the truth of their conjecture on the use of empirical examples. For example, when proving that adding two even numbers always results in an even number, a student may use as his justification “I tried it, and it worked”. Often times, inductive reasoning leads to a sense of certainty that a conjecture is valid. However, students must make the transition to deductive reasoning in order to prove, or validate, that their conjecture is always true. This is the final reasoning activity seen before

proof. When students are invoking deductive reasoning, they find a way to show that their conjecture is true for a general case, based on mathematical axioms and theorems. Once students are able to reason deductively, they are ready for proof (Edwards, 1997).

Educators tend to think of critical thinking and problem solving as synonymous (Steen, 1999), with both processes involving mathematical reasoning. Many people can still perform mathematics skills without mathematical reasoning skills, but research shows that these people have only a very superficial knowledge of the subject. In order to have a clear picture of the capabilities students need to be able to construct proofs, we must closely examine how students mathematically reason, as well as how they geometrically reason. Both types of reasoning seem to go hand in hand in that as one develops, so should the other. The next section further describes the development of geometric reasoning in students.

Development of Geometric Reasoning

Many researchers argue that mathematical reasoning is best gauged by a student's performance with geometric tasks (Mason & Moore, 1997; Wu, 1996). Mathematical reasoning is necessary to geometric reasoning. That is, as mathematical reasoning is successfully developed, geometric reasoning follows (Battista & Clements, 1992). There are two predominate theories about the development of geometric reasoning in students, that of Piaget and van Hiele.

The Research of Piaget and Inhelder

Piaget structured his theories about development in several domains around four

stages of cognitive development. These stages are sensorimotor (infancy), preoperational (early childhood through preschool), concrete operational (childhood through adolescence), and formal operational (early adulthood) (as cited in Pusey, 2003). Piaget claimed that these stages were physiological in nature, in that a child progresses through each stage at certain points of their biological development. One major focus of Piaget's work examined how children organize and construct ideas about geometry, as well as how they form a representation of space.

The research of Piaget and Inhelder (1967) focused on a child's conception and representations of space. Their theory was comprised of two parts, the first being how a child constructs their own representation of space. Second, Piaget and Inhelder claim that a child's organization of geometric ideas follows a definite, logical order. They found that preschool children could discriminate objects based on topological features, but could not discriminate between curvilinear and rectilinear objects until later in their development. Preschool age children also had difficulty drawing copies of geometric shapes, lending to the assumption that hand-eye coordination also impacts a child's conception of space. According to Piaget and Inhelder, children can only truly begin to discriminate among Euclidean shapes around the age of 4.

Piaget and Inhelder's work suggest that geometric ideas develop over time, becoming more synthesized and integrated throughout the child's development (Battista & Clements, 1992). Children progress through levels of cognitive development as part of a natural order. Their current level of development is dictated by their age, not by instructional techniques or activities (Pusey, 2003). According to Piaget and Inhelder's research, preschool children appear to have some competency in establishing a valid representation of space that could be

built upon in the classroom. A child's conception of space is not necessarily formed based on pure mathematical logic. On the contrary, it is more likely formed through experience and through the child's perception of "reality."

The Research of Van Hiele

Contrary to the findings of Piaget and Inhelder, Pierre van Hiele's theory suggests that students progress through levels of geometric thought that are not based on cognitive development. Van Hiele claims that there are six levels of geometric thought through which students progress in learning geometry. For students to function in any given level, they must have mastered the reasoning needed at the preceding level. These levels are hierarchically based. Progression from one level to another is based more on instruction and experiences than on age or physical development. High school students, and even adults can be reasoning at a very low van Hiele level (Jones & Swafford, 1997). Each van Hiele level (VHL) has unique characteristics, and what one conceptualizes at one level is not necessarily a robust enough conceptualization to operate at a higher level. For example, what a student considers to be only a rectangle in one level may be reconsidered as a parallelogram in general at a higher level. The following is a brief description of the van Hiele levels of geometric reasoning (1986).

Level 0: Pre-recognition. Children at VHL 0 are able to perceive geometric shapes, but recognize only a few of the shape's visual characteristics. Although they may be able to distinguish between curvilinear and rectilinear shapes, they cannot necessarily separate shapes by a subset of characteristics in each group. For example, they will not be able to distinguish between a square and a triangle, or a circle and an oval. They are not yet able to

form visual images and therefore cannot identify certain common shapes. Students are able to reason about objects that are “the same shape” by only focusing on certain visual or tactile properties.

Level 1: Visual. At VHL 1, students are able to identify shapes by physical attributes. For example, a student may be able to tell you that something is a rectangle without being able to tell you any of a rectangle’s properties. It is a rectangle because it looks like a rectangle. Therefore, students may also classify a parallelogram as a rectangle. They are able to form a visual, mental representation of geometric figures, yet they have not yet mastered class inclusion. Students may be able to distinguish one figure from another simply based on appearance, but not on specific properties. By calling a figure a “rhombus,” a student is saying that, “This figure has the shape I have learned to call ‘rhombus’” (van Hiele, 1986, p. 109).

Level 2: Descriptive/Analytic. Students who have reached VHL 2 are able to recognize shapes by specific properties. Now, along with the mental, visual representation acquired in VHL 1, students also have a mental representation based on properties. For example, the rhombus that students at VHL 1 classified based only on a visual representation is now classified by some property of a rhombus, such as having four congruent sides. Students begin to recognize that certain properties hold only for certain figures. However, students still don’t have class inclusion. They may say that a square is not a rhombus because it is a square. In essence, they are not able to conceptualize the relationship of properties between figures, and can only reason about properties within a figure.

Level 3: Abstract/Relational. Class inclusion is developed at VHL 3. A square may be considered a rhombus with additional properties. Students begin to see how some

properties are inter-related and can make informal deductions about classes of figures. They begin to provide logical arguments in support of conjectures, and organize properties hierarchically. Students may see how some properties imply others, leading to logically sound definitions and organization.

Level 4: Formal Deduction. At VHL 4, students can now develop original proofs. They are able to see how theorems in an axiomatic system are related and can produce a logically sound argument and conclusion. Students reason “formally, by logically interpreting geometric statements such as axioms, definitions, and theorems” (Battista & Clements, 1992, p.427).

Level 5: Rigor/Mathematical. Students at VHL 5 are able to reason outside of Euclidean geometry and explore other axiomatic systems, such as hyperbolic and elliptical geometry. They are able to make connections and see relationships between different axiomatic systems.

Van Hiele’s research is widely used as an indicator of students’ geometry readiness. Research shows that the van Hiele levels have been successful in classifying students in terms of their conceptual understanding of geometric concepts (Battista & Clements, 1992). Though the theory states that the levels are discrete, more recent research suggest that students can reason at more than one level, given certain situations (Pusey, 2003). Furthermore, it is possible for students to be in-between levels, making it difficult to classify them according to VHL. Although students may be reasoning at more than one level, typically the acquisition of the lower level is more complete than the acquisition of the higher level. Several studies have been conducted to determine an appropriate method of classification for students who are transitioning between levels (Pusey, 2003). Despite these

findings, the van Hiele theory remains a fairly accurate indicator of students' geometric thinking abilities. It also holds many implications for proof readiness among students.

Geometric Reasoning and Proof Readiness

The research of Piaget and van Hiele hold several implications for how students learn proof. Mathematicians use proof to “establish truth” based on logical and deductive reasoning (Battista & Clements, 1992). Proof for a mathematician is used to establish conceptual understanding, and not thought of as simply a series of technical steps (Hanna, 2000b). Several studies have considered the link between proof abilities and levels of geometric reasoning.

Piaget (1987) claims that students progress through three levels (PL) in the development of their justification and proof skills. Unlike van Hiele's levels of geometric understanding, Piaget's levels for proof and justification coincide with the biological development of the student. Students up to 7 or 8 years are at PL 1. At this level, students treat interrelated events separately. They do not see any necessity in making their point clear to others, nor do they see any sense in seeing another's point of view. Near the end of this level, thought becomes more directed as children begin to integrate thoughts and ideas. Children at this level are capable of elementary deduction. Students progress through PL 2 between the ages of 7 or 8 through 11 or 12. At this level, children begin to make predictions and justify their reasoning, however their predictions can still be incorrect, as they are based on empirical results. They are able to connect subsequent events based on what happened in the previous event. Implications and conjectures are only based on observation, not

assumption. PL 3 occurs at age 11 or 12 and beyond. Students at this level are capable of formal deductive reasoning. Students see the necessity of logical and deductive reasoning and understand the importance of justifying their conjectures (Battista & Clements, 1992).

We can now look at the van Hiele levels (VHL) from another perspective, that of learning proof. According to van Hiele, children reason differently at different levels. Progression through these levels is dependent on the ability to consistently reason successfully in preceding levels, rather than on biological development, as implied by Piaget's theory. At VHL 1 (visual), students' reasoning capabilities are limited to observation. They see a picture and notice its properties, but fail to make connections between those properties, or see how they can make conjectures based on those properties. In VHL 2 (descriptive/analytic) students see that a shape has specific properties. They are able to classify shapes based on these properties. However, they are not yet able to see how these properties imply others. Students are only beginning to see relationships between classes of figures. It is in VHL 3 that students are able to see how certain properties are interrelated. They are able to link pieces of information. As students become increasingly able to pose and defend their conjectures in a coherent fashion, they begin to demonstrate a readiness for proof. Their logical and deductive skills are developed at this level. Successful reasoning at VHL 3 can prepare students for the formal deductive reasoning needed in VHL 4 to build solid mathematical arguments and write proofs.

It is important to note that not all students are at VHL 3 at the same time. A few may reach VHL 3 by high school, but most will not. Mason and Moore (1997) used van Hiele's theory to assess geometry readiness among mathematically talented middle school students. Their research had established that VHL 1 thinking was prevalent in grades K-8, and that

geometric understanding depends on a student's van Hiele level, as well as the student's level of logical reasoning ability and the quality of their basic geometry content knowledge. In most schools, the only requirement to be placed in a Geometry course is the successful completion of Algebra 1. Therefore, many mathematically talented students go straight from 8th grade Algebra to 9th grade Geometry. In examining the performance of previously identified, academically talented students in geometry, Mason and Moore (1997) found that students who were not yet at VHL 2 were not sufficiently prepared to be successful in Geometry. Academically talented students did not necessarily perform at the same van Hiele level across the board. Therefore, Mason and Moore suggest that students who are still at VHL 1 after completing Algebra 1 should progress into Algebra 2 rather than Geometry.

Mason and Moore's findings are consistent with Senk's (1989) research, which analyzed proof-readiness in high school students. Students in a high school geometry course were tested in the fall for their van Hiele level and basic geometry content knowledge. At the end of the school year, students were tested again for their van Hiele level, knowledge of geometry, and proof-writing ability. Senk found that students who start Geometry at VHL 0 have very little chance of learning to write proofs. Students at VHL1 have about a 33% chance of learning to write proofs, while students at VHL 2 have a 50% chance of learning to write proofs. The students who have the most potential in learning to write proofs start high school Geometry reasoning at VHL 3 or above. Furthermore, of the students who did learn to write proof, those who began the course at VHL 3 significantly outperformed those who had started at VHL 2. It was determined that students at VHL 3 had a 57% mastery of proof by the end of the course. Though there were not many students reasoning at VHL 4, those who began the year at this level had an 85% mastery of proof. This provides support to van

Hiele's theory that students have a mastery of proof at VHL 4, while VHL 3 is where students begin to learn to prove formally and informally. Senk recommends that students who enroll in a high school Geometry course should be at VHL 2 or above. Those students at VHL 2, with proper instruction, may still reach VHL 3 by the end of the course, thus giving them a good chance of success in proof writing. It is important that students attain the necessary level of geometric thought before they enroll in a proof-intensive Geometry course (Mason & Moore, 1997).

Without the necessary reasoning skills, students will not be successful in a proof-intensive geometry course. Through an examination of research by Piaget and van Hiele, students can be classified according to stages of development in mathematical and geometric reasoning. This classification holds many implications for students' proof-writing abilities. It is important to understand, however, that applying the research findings of Piaget and van Hiele does not necessarily guarantee that students will arrive at their first Geometry course at the stage necessary for success in both geometry and proof writing. The focus of the next chapter is on evidence that most students have not mastered the skills necessary for success in these areas.

CHAPTER 3

WHAT EVIDENCE IS THERE TO SHOW THAT HIGH SCHOOL STUDENTS ARE NOT SUCCESSFUL WITH PROOF AND HOLD MISCONCEPTIONS ABOUT THE NATURE OF PROOF?

This chapter will focus on research-based evidence that students do not have the necessary reasoning capabilities to be successful in pre-college courses that emphasize formal and deductive proof. First, I will examine proof achievement among high school students. This involves a discussion of geometric misunderstandings and the van Hiele levels of students enrolled in a Geometry course. Both of these factors contribute to student achievement in both geometry and proof. Then I will discuss proof misconceptions among students enrolled in a Geometry course as well as misconceptions of students who have already completed a high school Geometry course. Understanding these misconceptions will help us better understand why students are having difficulty in proof writing.

Geometry and Proof Achievement of High School Students

International comparison studies have shown that American students do poorly in geometry in comparison to their international counterparts. For example, American elementary and middle grades students are outperformed in Geometry achievement tests by students in other nations (Cai & Hwang, 2002). In a study by Stigler, Lee, and Stevenson (1990), fifth grade students from Japan and Taiwan performed twice as high as American fifth-grade students in geometric tasks. These tasks included tests of visualization, such as

paper folding, and achievement tests of basic geometry content knowledge. As high school Geometry tends to be where proof is first introduced to students, it is important to understand student achievement in geometry in order to understand achievement in proof. First, I will examine some geometric misunderstandings that high school students have. Next, evidence will be presented that shows students do not have the necessary capabilities to write proofs. Finally, proof achievement will be examined in terms of van Hiele levels of geometric reasoning.

Geometric Misunderstandings Among Students

Several studies within the United States have shown that many students do not even understand basic geometric ideas. Usiskin (1987) found that only half of all American high school students enroll in a geometry course. Of those students enrolled in Geometry, only 63% could correctly pick out triangles when shown a group of figures. Furthermore, according to the National Assessment of Educational Progress in 1982, only 64% of 17-year-old high school students who have taken Geometry knew that a rectangle was a parallelogram, and only 16% could find the area of a figure made up of two rectangles (Battista & Clements, 1992).

Based on prior research, Battista and Clements (1992) noted common misconceptions among students enrolled in high school Geometry. Students often believe that:

- An angle must have one horizontal ray.
- To be a side of a figure, a segment must be vertical.
- A segment is not a diagonal if it is vertical or horizontal.

- A square is not a square if its base is not horizontal.
- The only way a figure can be a triangle is if it is equilateral.
- The height of a triangle or parallelogram is a side adjacent to the base.
- If a shape has four sides, then it is a square.

If students hold these geometric misconceptions, it seems their abilities and readiness to construct proofs, which involve these, and other, geometric properties would be greatly affected. Furthermore, a student who holds these misconceptions is likely to be classified as reasoning at VHL 1. As discussed in Chapter 2, in order to be successful in proof writing students should start high school Geometry at VHL 2 or above (Mason & Moore, 1997; Senk, 1989).

Students' Current Level of Proof Performance

Senk (1985) conducted a study in which proof-writing skills were tested among 1,520 students enrolled in Geometry, one month before the end of the year. Only 3% of the students received perfect scores on the test. Consider the following results.

- 70% of the students were able to complete a six-step proof if all they had to do was supply either the statement or the reason of each step
- 51% were successful when there was one line in which both the statement and the reason had to be supplied
- 32% could supply a complete proof of the diagonals in a rectangle being congruent
- only 6% could supply a complete proof of a theorem that did not follow from triangle-congruence postulates of theorems.

These results led Senk to the conclusion that only about 30% of high schools students, after one full year in a Geometry course, could reach a 75% mastery level in proof writing.

Providing support to Senk's (1985) research is a study by Brumfield (Brumfield, 1973) in which high school students who were about to take an advanced calculus course were tested on their knowledge of postulates, theorems, axioms, and definitions in geometry. When asked to write down as many postulates, theorems, or axioms that they could recall from geometry, 50% listed nothing, while 31% listed statements that were not even postulates. Many students mixed theorem with axioms, definitions, and false statements. When asked to pick any theorem and write a proof for it, 81% of the students did not even attempt a proof. Brumfield concluded that students, even mathematically advanced students, get little meaning out of a proof-intensive Geometry course.

Proof Achievement and van Hiele Level

Recall Senk's (1985) research on van Hiele levels (VHL) in regard to students' proof performance. It was found that students who start Geometry at VHL 0 have very little chance of learning to write proofs. Students at VHL 1 have about a 33% chance of learning to write proofs, while students at VHL 2 have a 50% chance of learning to write proofs. The students who have the most potential in learning to write proofs start Geometry at VHL 3. In a separate study, Senk (1989) evaluated the VHL of 241 beginning Geometry students. She found that 27% of the students had not yet reached VHL 1, 51% had mastered VHL 1, 15% had mastered VHL 2, 7% had mastered VHL 3, and only 1% had mastered VHL 4. Therefore, we might reasonably conclude that about 78% of beginning geometry students have a 33% or less chance of learning to write proofs. Also recall that Senk (1985) found

that 30% of students could reach a 75% mastery level of learning to write proofs. Now that we take students' VHL into consideration, we can understand where this result comes from. Many students are not at a high enough VHL to be successful in proof writing. Similarly, Usiskin (1982) found that of 2,699 students enrolled in Geometry courses, almost 40% of the students were below VHL 3 at the conclusion of the course.

Mistretta (2000) conducted a field trial of a geometry unit intended to raise van Hiele levels in eighth grade students. First, she assessed each student's van Hiele level by giving a pretest consisting of level 0, 1, and 2 questions in multiple choice and short answer form. Answers to questions were marked correct based on the van Hiele level of reasoning addressed in the particular question. Twenty-two percent of the students were classified at level 0 and 35% were "nonclassifiable." Forty-three percent were classified at VHL 1. None of the students were classified at VHL 2 or above. Mistretta also administered an opinion survey to evaluate students' attitudes toward geometry. Sixty-one percent of the students said that geometry was difficult, complicated, or confusing. When asked about past geometry experiences, students expressed boredom, claiming that the study of geometry consisted mainly of memorizing formulas and theorems without really understanding them. In individual interviews, students showed weakness in applying reasoning typical of VHL 1 and 2, not having clear understandings of area and perimeter, especially in regards to some irregular shape. Students were not aware of relationships between properties of triangles, quadrilaterals etc. Mistretta's research was conducted with students who are typically close to enrolling in a Geometry course. If we apply Senk's (1989) theory to this data, then these students would have only a 33% chance or less of learning to write proofs. Yet, these are the students that we would push into a proof-intensive Geometry course in high school.

Although mathematically talented students may start high school Geometry at a higher VHL, their performance with geometric concepts and proofs can generally be predicted by a student's VHL. Recall Mason and Moore's (1997) research of the VHL of 64 students identified as "mathematically talented." They found that, although the mathematically talented students tended to be at a higher VHL than the students in the Senk and Usiskin studies, the probabilities of whether or not they could learn to write proofs remained the same. Of the mathematically talented students, only 5% had not mastered VHL 1, therefore having less than a 33% chance of learning to write proofs. Twenty-five percent of the students had mastered VHL 1, therefore having a 33% chance of learning to write proofs. Seventy percent of the mathematically talented students were at VHL 2 or above, giving them a 50% or above chance of learning to write proofs. In the mathematically talented students, Mason and Moore concluded that given proper instruction in mathematical reasoning, students at VHL 2 or above should be capable of learning to write proofs. However, it must be noted that 30% of the mathematically talented students in the study were below VHL 2, thus having less than a 33% chance of learning to write proofs. It seems that even among mathematically talented students, VHL is still a predictor of proof and geometry achievement and that these students face the same problems with proof as do students not identified as mathematically talented.

Most students are obviously not getting the necessary skills before entering into a high school Geometry course. Many high school students have geometric misunderstandings that are characteristic of VHL 1 thinking. Based on research findings, students at this level do not have the necessary skills to be successful in a proof-intensive Geometry course. Research indicates that a majority of students are entering high school Geometry courses at

level 1 or below, setting the stage for a lack of success in proof writing. Even after a course in geometry, students do not have a solid understanding of what constitutes a valid proof. The next section provides information regarding students' conceptions of proof and their general beliefs about the nature of proof.

Proof Conceptions

It is becoming increasingly apparent that there is a lack of student achievement in proof. Evidence presented in earlier sections demonstrates that students' mathematical and geometric reasoning seems to be contributing to their lack of achievement in proof. We can gain further insight into this problem if we look at students' conceptions of proof. One way to analyze whether or not students are ready for proof when we teach it is to examine students' proof conceptions before and after they have taken a proof-intensive geometry course. After taking a proof-intensive course, have students really learned how to create a formal, deductive proof? Examining students' beliefs about what constitutes a proof can help us to understand why students do not perform well on proof tasks. In this section, I will first examine common proof misconceptions among students. Then evidence will be provided that indicates students' proof performance is affected by these misconceptions.

Common Proof Misconceptions Among Students

In addition to having geometric misunderstandings, students also have proof misconceptions that can provide insight into student achievement in proof. Students tend to accept the visual appearance of particular drawings as proof (Schoenfeld, 1986). In other

words, if a picture looks like a rectangle, then students think it is a rectangle without having to prove it. Furthermore, students have a hard time accepting a general proof as complete without testing empirical examples (Fischbein & Kedem, 1982). Finally, students often misunderstand the use of counterexamples.

Schoenfeld (1986) studied college students' development of conjectures based on compass and straightedge constructions. He found that students often make conjectures, and then feel that their conjecture is validated because their construction "looks" accurate. They are relying on visual appearance and accepting this as proof. Students will tend to accept invalid conjectures as true, and often to accept valid conjectures as false. In this study, Schoenfeld first asked students to solve a construction problem after having constructed a proof that offered the answer to the problem. Nearly one-third of the students provided a solution to the construction that completely violated the findings of the proof they had just constructed. It seems that these students are relying too heavily on pictorial representations as a validation of their conjectures. If the picture does not "look" right, then they say their conjecture is false. Students who say, "This is a right angle because the picture looks that way" are relying on a similar misunderstanding.

Fischbein and Kedem (1982) studied high school students' perceptions of proofs as general arguments of the truth of a statement. It was found that students, even after having constructed a valid proof, insisted that surprises were still possible, and continued to empirically test conjectures. They were not satisfied with the general argument provided by the proof. It's as if they needed to see the proof "work" in a numerical or geometric setting in order to believe the truth of their conjecture. It seems that students are not even sure of the necessity or meaning of proof. Students are missing the point of proof being a general

argument to show that an arbitrary case is true.

Galbraith (1981) found that one-third of high school students do not understand the concept of counterexamples. They do not realize that the counterexample must satisfy the hypothesis and violate the conclusion of a conjecture. Students feel that one counterexample is not enough to disprove a statement, and will continue to try to find additional counterexamples. Again, this shows that students do not understand the purpose of proof, or rather the arbitrariness of proof. If they knew that a proof showed that an arbitrary case was true, then they should realize that it takes only one counterexample to disprove a statement.

Studies Examining Proof Misconceptions

Healy and Hoyles (2000) studied proof conceptions of British students in Algebra. In the United Kingdom, the mathematics curriculum is divided into five “targets,” Number and Algebra being among them. In this curriculum, reasoning and proof are processes emphasized from the beginning, similar to the revised recommendations from the NCTM (2000). This has been the source of much debate between educators and mathematicians, the latter claiming that rote memorization and rigor are necessary for the acquisition of basic skills (Healy & Hoyles, 2000). Educators, on the other hand, emphasize “understanding” and heuristics as opposed to the acquisition of proof writing skills, which are often taught as a rote process.

Healy and Hoyles (2000) conducted an analysis of conceptions of proof held by students in the UK curriculum with reference to factors of environment, teacher content knowledge, and school. They investigated characteristics of arguments recognized as proofs by high-achieving students in arithmetic/algebra and geometry, as well as the reasoning

behind the students' judgments. Students were given a questionnaire designed to provide an overview of their views of proof, its role, and its generality, as well as an indication of students' competence in proof writing. The questionnaire sought to examine what methods students would use to construct certain proofs and to what method they thought their teacher would give the highest marks. Students were also asked questions regarding the degree to which each statement in a proof convinced them of the truth. Teacher questionnaires were also given, focusing on which methods they thought their students would use in a proof as opposed to which method they would give the highest marks.

Results indicate that students are better at choosing correct mathematical proofs than they are at creating them, and students constructed better arguments for familiar conjectures. Producing empirical examples was the most popular form of argument used by students, though they didn't think this type of argument would earn the highest marks from their teachers. This indicates that students knew more was expected of them, but didn't know how to achieve those expectations. Students were more likely to use empirical examples if they were already convinced of the truth of the statement. This is aligned with Fischbein and Kedem's (1982) study, which found that students were not satisfied with a proof's truth until empirical examples were given. In other words, if students felt that a statement or conjecture was false, they were reluctant to give an empirical justification. Arguments that used algebra were another popular form of justification, however students admitted that it would be hard to explain using algebraic arguments to someone who didn't understand algebra. When using algebraic arguments, students weren't always able to convince themselves of the validity of their justification. This was also true with narrative arguments. When students gave narrative arguments, they felt the need to provide an empirical example as well, as if the

narrative argument alone was not sufficient (Healy & Hoyles, 2000).

Although most of the students in this study were unable to construct proofs, they still valued explanatory arguments. Students recognized that empirical arguments, though easier to construct, do not prove. They were also aware that a valid proof must be general, however they still consistently gave empirical examples as justifications. While the students tended to understand narrative proofs the best, they held the belief that the more complicated-looking the proof was, the higher marks they would receive from their teacher (Healy & Hoyles, 2000).

It seems that students, while aware of a “proof technique,” are still unable to construct a proof of their own. If they are able to construct some sort of proof, whether it’s narrative or algebraic, they are still not convinced of the proof’s accuracy. Although students know that a proof must be general, they still lean toward empirical examples to convince themselves of the truth of a conjecture. The fact that students in the Healy and Hoyles (2000) study did not think that a narrative proof was sufficient on its own and that more complicated-looking proofs are the most widely accepted, indicate that students do not know exactly what constitutes a mathematical proof.

Edwards (1999) conducted a study of 10 first year high school students in which the task was to decide whether a statement pertaining to odd and even numbers was true or false, giving a reason for their conclusion. All students were successful in determining whether statements were true or false, and were also able to provide examples in support of their decisions. Students could provide counterexamples to false statements. However, when asked why a statement was true, the typical response was “I tried it, and it worked” (p. 494). None of the students offered an algebraic proof of any kind. Three of the students attempted

all or part of an informal proof based on the structure of odd and even numbers.

Edwards' (1999) results indicate that students don't see any need to justify a statement until they are convinced that the statement is true. However, they may think that providing empirical examples is a sufficient justification. Some students go beyond empirical examples and try to create a justification, which may be mathematically sound. These findings are similar to those found in a separate study by Edwards (1997). In the 1997 study, two groups of students were examined, neither of which had any experience with transformational geometry. Their task involved combining transformations. Students were given a combination of transformations, and then asked if they could get the same effect with only one transformation, or "rule". Edwards found that students tend to over-generalize. For example, many students thought that a double reflection would still result in a reflection, because a double translation still resulted in a translation (this was also the case for a double rotation). When students were asked to explain why certain conjectures were true, they were able to provide "formulas" based on their activities with reflections, translations, and rotations. However, students could still not provide any type of formal proof on their own. When the investigator offered one, the students were able to accept and understand this explanation. Still, students were satisfied with inductive reasoning. They tested their conjectures on one or two cases and did not feel that any general proof was necessary. Again, this mirrors the results of Fischbein and Kedem's (1982) study. A very common problem among high school students seems to be that students are not convinced by a proof. To them, it is too general. They believe its truth only after empirical examples are given. Students do not understand what constitutes a proof, and seem more satisfied giving empirical examples.

According to Weber (2001), even students at the college level think that the only valid proof follows some traditional form, such as two-column proofs. In addition to this misconception, students often misapply theorems in proofs. Even if they do not misapply theorems, students often come to a point in a proof where they “simply do not know what to do” (p. 102). Weber’s research focuses on why this happens. His study focuses on undergraduate mathematics majors’ construction of proofs versus doctoral mathematics students’ construction of proofs in a given topic in abstract algebra.

The undergraduates had a higher failure rate in proof attempts than the doctoral students. In 57% of the undergrads’ failed proof attempts, they failed to apply syntactic knowledge and used nearly four times as many irrelevant inferences than doctoral students. Syntactic knowledge refers to the facts needed regarding the mathematical situation in order to complete the proof. In this case, the undergraduates indicated that they had the proper syntactic knowledge, based on a true-false exam given earlier in the study. However, they did not know that this was the knowledge needed to complete the proof. This data indicates that an understanding of mathematical proof and syntactic knowledge is not enough to be a “competent prover” (p. 107). Undergraduates also failed to apply factual knowledge, perhaps because the number of possible paths to take with one proof overwhelmed them. Weber’s (2001) findings indicate that the primary cause of the undergraduates’ difficulties with proofs was that they lacked “strategic knowledge.” They would often write down as many rules they could in hopes that one would lead them in the right direction in the proof.

Students’ proof performance is affected by the proof misconceptions discussed above. Even those students who appear to be at a high level of geometric reasoning (college students, for example) have a hard time understanding what constitutes a proof. A recurring

theme in this research seems to be students' reliance on empirical examples as a means of convincing themselves of the truth of a conjecture. Oddly, they pick out the most "complicated looking" proof as the one that is likely to be correct, but then admit that they could not produce it themselves (Healy & Hoyles, 2000). Yet, if proof is "what convinces me," then to these students, proof is empirical examples. In a later chapter, I will discuss how we can capitalize on the use empirical examples to demonstrate to students the need for proof.

The evidence regarding students' lack of ability in formal proof writing is overwhelming. You can walk into the majority of high school Geometry classes and witness many students struggling to construct accurate proofs. Many teachers, including myself, tend to attribute students' low performance in proof as a consequence of poor study habits. However, as discussed in this chapter and the previous one, the ability to construct mathematical proofs is affected by several factors. Rather than attributing poor proving skills to poor study habits, a closer examination of students' reasoning capabilities may be more appropriate. Many students reach high school with weak mathematical reasoning skills and low van Hiele levels. As evidenced in research reported in Chapter 2, to be successful in writing proofs, a student should be reasoning at least at VHL 2 or above when they enter high school Geometry. Students who are not at this level demonstrate weak reasoning capabilities (Senk, 1989). The majority of students are entering high school Geometry below VHL 2, and are thus unsuccessful in writing proofs. The focus of the next chapter will be to understand other factors that contribute to students' struggle with proof writing.

CHAPTER 4

HOW CAN TEACHERS' BELIEFS AND UNDERSTANDING CONTRIBUTE TO STUDENTS' PROOF ABILITIES?

Students' low level reasoning abilities are not the only factors in their unsuccessful attempts at writing proofs. If a surprising number of students are entering Geometry without the necessary reasoning capabilities, then it is necessary to consider the material that is being taught prior to Geometry, and how it is being taught. Reasoning skills can be acquired in the elementary grades. In fact, the NCTM (2000) emphasizes that geometry should not be taught as a separate discipline only in high school, but should be laced into the mathematics curriculum in all grades from kindergarten through high school. The Reasoning and Proof Standard emphasizes the importance of student-developed conjectures and justifications. Yet if students are still making it to high school without the necessary skills to be successful in Geometry, then the instruction and experiences in elementary and middle school is probably lacking. This could be because of a weak curriculum, or could be due to improper implementation of the curriculum on behalf of the teacher. In the 1990 and 1996 National Assessment of Educational Progress, 34% and 29% of fourth grade teachers, respectively, reported they would spend "a little or none" instructional time devoted to geometry. In eighth grade, 22% of the teachers reported devoting "a little or none" time to geometry in both 1990 and 1996 (data reported from the National Center for Educational Statistics at <http://nces.ed.gov/nationsreportcard/naepdata>).

Even if students get adequate geometry instruction in the lower grades, they may still experience problems in high school. Proof is traditionally taught very formally, as a series of

statements and reasons based on rote memorization and a very specific “process.” It is hard for students to develop an understanding of the necessity and value of proof if their skills are based completely on rote memorization. Furthermore, many high school teachers, though they teach proof, are unclear as to the generality and importance of proof themselves. (Knuth, 2002b). Few would argue the important role a teacher has in the development of students’ mathematical reasoning, geometry reasoning, and proof conceptions and construction capabilities. Thus, in considering why students may be unprepared for proof and lack appropriate reasoning skills, it is important to review literature related to preservice and inservice teachers’ mathematical preparation and understandings, and their understandings and beliefs about proofs and geometry.

Understanding and Beliefs About Proof Among Preservice Teachers

There are several factors that contribute to a preservice mathematics teacher’s perceptions and beliefs about mathematics education, and in particular, proof. Prior experience in mathematics classes can affect a teacher’s pedagogical approaches to teaching mathematics, as can their mathematical ability. Though many times these factors can enhance mathematics instruction, we must examine the instances in which these factors hinder mathematics instruction. In order to gain a thorough understanding of teacher’s beliefs about proof, we must first examine both preservice elementary and high school teachers’ mathematical backgrounds and experiences.

Preservice Teachers’ Mathematical Background

As presented in earlier chapters, mathematical reasoning skills are paramount to proof writing skills. According to the NCTM (2000), a wealth of mathematical reasoning capabilities can be gained at the elementary level. However, most elementary school teachers do not necessarily have an advanced mathematics degree. On the contrary, most elementary school teachers have degrees that most closely resemble a Liberal Arts degree, in which a wide range of disciplines are only superficially examined (Gellert, 2000). Therefore, many elementary school teachers, who are expected to teach mathematics, do not have a strong mathematics background. This can hinder their confidence, knowledge, and instruction, thus preventing students from gaining important skills that will be necessary in higher levels of mathematics. Aside from not having a strong mathematical background, elementary school teachers also tend to have a history of struggling with mathematical content (Gellert, 2000). In addition, a large number of elementary school teachers have shown negative attitudes toward mathematics and mathematics education (Carroll, 1995). These attitudes seem to be the result of poor experiences in mathematics classes and weak mathematical self-concepts (Sloan, Daane, & Giesen, 2002).

Gellert (2000) conducted an in-depth analysis of 42 preservice elementary school teachers' attitudes and beliefs toward mathematics. He focused on the following questions:

- What materials for mathematics instruction do prospective elementary teachers want to use?
- What kind of mathematics do they want to teach?
- What are possible implications of these emerging conceptions?

These 42 student teachers were asked to keep journals. Their journal entries were in response to researcher-proposed prompts as well as personal reflections.

A thought that was echoed by nearly all of the preservice teachers was that the criterion for good mathematics teaching was how much fun the subject could be made for the students. One teacher wrote:

From personal experience, I already know how I do not want to teach mathematics. What I am learning is only the idea of how to teach mathematics to students in elementary schools in a nice and amusing way (p. 258).

Games were suggested to “hide” mathematical content. Quizzes were often structured to spark competition between students as a motivator to learn basic facts. Most alarming in these findings is that these teachers, more than anything else, felt that mathematics was something so unpleasant that it needed to be disguised. For example, one teacher suggested that students be offered a reward for completing more mundane tasks. “On a worksheet, there could be problems for addition and subtraction grouped in blocks. When students are finished with one block, they can color the figures” (p. 260). Notice that coloring is meant to bare no reference to mathematical content, it is simply a reward for finishing a task.

Preservice teachers also had some trouble distinguishing between “important and unimportant” information. Arithmetic was seen as important. Abstract information (in their opinion), such as probability, was seen as unimportant. All seemed to agree that students needed mathematics to survive in society. However, these preservice teachers believed the only understanding that students need to have is arithmetic in order to “make sense of reports” that they may see on the news or in history classes, or just for use in “shopping situations” (p. 261).

No one would argue that the use of games as a tool for teaching mathematics is not worthwhile. However, these preservice teachers did not know how to tie these “fun little

games” into the mathematical content. Instead, they were intended to make a subject that was seen as “unappealing” less of a necessary evil. By doing this, teachers are indirectly conveying their negative feelings toward mathematics to their students. These preservice teachers demonstrated a need to disguise mathematics to the point that “students won’t even recognize that they are doing math” (p. 259). If students don’t recognize that they are doing math, how will they make the connection that they are actually learning something? If this is the mentality of some elementary school teachers, then we can begin to see why students may not be gaining the necessary reasoning capabilities at the elementary level. Many preservice elementary teachers seem to underestimate the importance of mathematical content (Gellert, 2000).

The fact that preservice elementary teachers are having trouble determining the importance of mathematical content holds some implications for teacher education. Studies have shown that there is a discrepancy between what is learned in a mathematics-methods courses and actual teaching practices. Raymond (1997) studied relationships between beginning elementary school teachers’ beliefs and practices. Participants were six first year teachers from the same teacher education program. The first stage of the study involved interviews to assess participants’ beliefs about mathematics and mathematics pedagogy. Through classroom observations, Raymond examined how participants taught mathematics. Finally, the consistency between the participants’ beliefs and actual teaching practices were analyzed. Results indicate that participants have a fairly traditional view of mathematics being fact and procedure-driven, with memorization an important aspect of learning. However, their beliefs about learning mathematics tended to be more non-traditional. All advocated the use of hands-on activities, group work, problem solving, and manipulatives.

Yet, the classroom observations show that these teachers weren't using any of these non-traditional methods in their instruction. The teachers, when asked about this discrepancy, seemed oblivious to it. One teacher in particular felt that her use of a "problem of the day" covered the problem solving aspect of mathematics instruction. Classroom management and control seemed to be the teachers top concern in formatting their instruction. None of the participants felt that their teacher education program had much of an influence on their current classroom practices (Raymond, 1997). This implies that mathematics education courses need to provide preservice teachers with a solid framework for how to implement their "non-traditional" beliefs in their classroom. For the time being, beginning teachers are more focused on management.

Of course, we cannot only focus on elementary school teachers and elementary teacher preparation. Even though high school teachers, for the most part, are required to have an advanced degree in mathematics, they may still have some misconceptions as to how to teach mathematics. Ensor (2001) observed a year-long secondary mathematics methods course at a university in South Africa. She analyzed student work and interviewed the instructors as well as the students. The following year, she observed seven of these students who were now in their first year of teaching. She found that there was a discrepancy between what the students learned in their methods course and what they taught in their classrooms. The practices they learned in their methods course were intended to raise students' level of conceptual understanding by engaging them in hands-on and critical-thinking activities. However, when these teachers began their first year of teaching, they tended to prefer a teacher-centered approach that was more "traditional" in nature. Emphasis was placed on rules, algorithms, and memorization, rather than on conceptualization and mathematical

reasoning on the part of the student (Ensor, 2001).

These research findings on teachers' preparation and approaches to teaching mathematics hold many implications for the teaching and learning of proof. Research indicates that good mathematical reasoning skills are imperative to proof-writing performance (Battista & Clements, , 1995; Edwards, 1997, 1999; Fischbein & Kedem, 1982; Izen, 1998; Jones & Swafford, 1997; Mistretta, 2000). If teachers are not teaching these skills, students are less likely to be successful with proof. In addition, pre-service teachers' perceptions of mathematical proof can affect the way they teach proof.

Preservice Teachers' Preparedness to Teach Proof

It is not only recently that students have struggled with proof. Many preservice teachers are not sure what constitutes a proof, and even underestimate the value of teaching their future students how to construct a proof (Mingus & Grassl, 1999). Furthermore, in a preservice teacher's undergraduate coursework, there is not often a class devoted to the teaching of proof, let alone the teaching of geometry. The only proof experience most preservice teachers have in college is in writing proofs in advanced mathematics courses.

Mingus and Grassl (1999) explored issues pertaining to preservice teachers' beliefs about proofs. Their work focused on the teachers' experience and exposure to proof, their beliefs as to what constitutes a proof, the role of proof in mathematics, and their belief as to when proof should be introduced in grades K-12. A survey was given to 51 preservice teachers, including both elementary and secondary mathematics education majors. They found that most undergraduates, even those that were majoring in mathematics, feared proofs. Of those surveyed, 69% had only one experience with mathematical proof, and that

was in their high school Geometry class. Surprisingly, half of these respondents were secondary mathematics majors. Though their exposure to proof was very limited, these respondents did admit that their comfort level with proof may have increased had they used it more, or seen some value in it. Many of the respondents (33%) suggested that proof ideas should be introduced as early as possible, becoming more formal in high school. Of these respondents, 69% wanted proof to be introduced prior to tenth grade. Perhaps not surprisingly, most of these respondents were secondary preservice teachers.

As to what constitutes a proof, most of the elementary majors' responses indicated that proof was something used only to describe geometric relationships. Secondary majors felt that proof was more explanatory in nature, used as logical and convincing arguments to demonstrate the validity of a conjecture. This interesting dichotomy reflects most elementary school teachers' limited experiences with proof. They have only been exposed to proofs once in their mathematical experience – usually a high school geometry class. It's no wonder that they don't see proof as necessary in any other course but Geometry. However, even secondary majors have usually only been exposed to proof once prior to college. Though they may now understand the value of proof in their more advanced courses, they may still not see the value of proof in grades K-12 (Mingus & Grassl, 1999). They also are likely to still hold some of the same proof misconceptions as their students, as discussed in Chapter 3.

College-level Preparation for Teachers

If Geometry is currently the course in which proof is most emphasized, then we should hope that Geometry teachers are well prepared to teach students appropriate proof skills. However, most secondary mathematics education majors are only required to take one

college-level course in Geometry (Grover & Connor, 2000). Therefore, they have in total, two geometry courses from which to gain knowledge – one in high school, and one in college. Is this enough preparation to teach Geometry, in general, and proof, in particular?

In the United States, teachers do not typically receive any sort of formal mentoring after they are hired as full-time teachers. Therefore, the course they take in geometry as an undergraduate plays a very important role in how they will teach geometry to their students. However, a member of the university's mathematics department, rather than a member of the mathematics education department, typically teaches these college-level Geometry courses. Although teachers may learn more about advanced geometry, they may not be gaining an understanding of how to best teach geometry and proof to their students.

Grover and Connor (2000) sought to examine syllabi and questionnaires from college-level Geometry courses to determine the overall content and content preparing to teach high school Geometry. Syllabi were examined on the basis of mathematical development, overlap with the NCTM's *PSSM* (2000), and characteristics of the learning environment. Only about one-third of these courses overlapped with the NCTM *PSSM*. Of these, only about 35% had significant overlap with the NCTM *PSSM*. In fact, 27% of the instructors were unfamiliar with the NCTM *PSSM* recommendations. Over 40% were lecture-intensive courses, and only 22% used a small amount of group work as an instructional tool. Exams and homework dominated the assessment tools used.

It does not seem that these courses are being structured to help preservice teachers develop quality teaching methods. The content of the course is strictly geometry, not how to teach it. If “increased knowledge of both geometry and students’ cognition influences what geometry teachers teach, how they teach it, and certain professional characteristics they

exhibit when teaching geometry” (Jones & Swafford, 1997, p. 470), then these undergraduate courses may be necessary, but are not sufficient for preparing teachers to teach a proof-intensive high school Geometry course.

Just as students’ inability to construct proofs can partially be attributed to teachers’ inability to provide appropriate instruction, the shortcomings in teachers’ preparedness can certainly be affected by their experience in teacher education programs. These experiences, and others, contribute to the many beliefs and perceptions that teachers hold concerning reasoning and proof.

Inservice Teachers’ Proof Understanding, Beliefs, and Practices

Current mathematics teachers were once preservice teachers. If their education bears some resemblance to what was described above, then it is likely that they are not currently teaching geometry and proof in a very effective manner. Several studies have specifically addressed inservice teachers’ understanding and beliefs about proof, including their readiness to do complete proofs as indicated by the van Hiele level of geometric reasoning. In addition, the NCTM (1989, 2000) has made specific recommendations about the importance of reasoning and proof that has most likely influenced practice in the last 15 years.

In the Chapters 2 and 3, it was emphasized that students, in order to be successful with proof, must enter high school Geometry at VHL 2 or above. Several researchers have also assessed teachers’ VHL. Mayberry (1983) analyzed 19 pre-service teachers and found that only about 50% were at VHL 2 or above. Jones and Swafford (1997) examined middle grades teachers before an intensive four-week program consisting of a course in geometry

and research on the van Hiele theory. The vast majority (79%) of the teachers were below VHL 3. Recall from Chapter 3 that students below VHL 3 only have a 50% chance of learning to write proofs in a Geometry course. This implies that there are probably many teachers who have not yet mastered proof writing. Therefore, it should come as no surprise that students are not making it to VHL 3 by the time they reach high school.

Knuth (2002b) focused his research with teachers that are already at VHL 3 and beyond. He interviewed 16 inservice mathematics teachers about their conceptions of proof. He found that, although teachers may recognize the varying roles that proof plays in mathematics, they might not necessarily view proof as a necessary tool for learning mathematics. Knuth examined two aspects of teachers' conceptions: 1) their conceptions of proof and 2) their conceptions of teaching proof in secondary mathematics.

All of the teachers viewed proof as a means of verification, or as a way to establish the truth of a statement. Some teachers said that truth is established by a logical or deductive argument, others said that truth is established by means of a convincing argument. Many teachers had a hard time generalizing. They recognized a valid proof, but were still not sure that it "always" worked, even though the proof was shown for a general case. This indicates that teachers do not always have a complete understanding of the generality of proof. Like their students, they are not convinced of a statements' truth just by seeing a proof. They want to test the conjecture with empirical examples (Knuth, 2002b). This is similar to results with students of geometry (Healy & Hoyles, 2000) as presented in earlier chapters.

Most teachers in Knuth's (2002b) study did not see proof as a means of explanation. They thought it was important to understand how the logical steps lead to a proof, but did not necessarily see proofs as "explaining." This is a critical component to proof. Usually, one

constructs a proof only after they are convinced that a statement is already true. The proof explains why a statement is valid. Much like their students, teachers may not view proof as “what convinces me” (Battista & Clements, 1995). Teachers did, however, view proof as a way of communicating mathematics and as a way of “systematizing” or showing how “math is a building block” and “everything is based upon what was proven before” (Knuth, 2002b, p. 390).

Knuth then provided the 16 teachers with examples of proofs and non-proofs. Overall, they were successful in distinguishing between proofs and non-proofs, though they rated many empirically based arguments as proofs. Thirteen of the teachers based their determination of an argument’s validity on whether the argument was mathematically sound, as opposed to a particular method used. Twelve of the teachers required that an argument deemed to be a proof must have sufficient detail in order to be awarded the highest rating. Overall, results indicate that what teachers find convincing is more based on form than on substance, and that the generality of proof is doubted without empirical examples (Knuth, 2002b).

Misconceptions as to what constitutes a proof may be widespread among high school mathematics teachers. Teachers do not necessarily have accurate conceptions of proof and, thus are not prepared to teach proof. Yet, is proof even being taught?

Recall that Hanna (2000a) claims that proof is not being emphasized enough in mathematics classrooms. We have already seen that it is not likely that it is being taught in the elementary schools, or that mathematical reasoning is even emphasized. However, at the high school level, even among teachers who have an understanding of proof and what constitutes one, proof is not being taught to the capacity that it should be (Hanna, 2000b).

There is a debate among mathematicians and educators. Mathematicians claim that proof, in and of itself, is an important skill to be learned. This is often misconstrued as a view of proof as a process to be memorized and not understood. Traditionally, proof is taught in a two-column, statement-and-reason form. Students are to memorize definitions, postulates, axioms, and theorems and use them to create a formal proof of a given conjecture (Wu, 1996). When you present proof in this light, it does appear to be based almost solely on rote memorization and tedious work. It is understandable that educators have tried to modify this instructional practice. However, as Hanna (2000a) notes, it is important that we not leave proof out of the curriculum all together.

If we follow the advice of Hanna and the recommendations of the NCTM *PSSM* (2000) to teach appropriate proof skills in high school, then we must remember that evidence shows that students are not ready for formal proof in high school Geometry. Research presented in this chapter shows that many teachers are not prepared to teach proof. This phenomenon may help explain why students are not ready for proof, since their teachers may not be properly designing instruction for effective and appropriate learning about proof. When teachers are not properly prepared to teach mathematics as early as the elementary grades, a trickle-down effect occurs in which students are the receivers of second-rate mathematics instruction. What should teachers do for the students that don't have sufficient reasoning skills? The next chapter describes methods that can be used to teach mathematical reasoning to promote learning and understanding of mathematics and proof.

CHAPTER 5

WHAT CAN BE DONE TO PROMOTE MATHEMATICAL REASONING AND IMPROVE STUDENTS' PROOF WRITING SKILLS?

As discussed in prior chapters, students do not seem to have the necessary skills to be successful in proof writing. In many cases, students do not even have the necessary skills to be successful in geometry. A variety of factors contribute to this phenomenon. Many students enrolled in a high school Geometry course are at too low of a van Hiele level (i.e., below VHL 3) to be able to handle the demands of a proof-intensive geometry course. In the previous chapter, evidence was presented that indicates students may not be prepared for proof because their teachers likely are not fully prepared to teach them reasoning and proof skills in K-12. In many cases, teachers suffer the same misconceptions about geometry and proof as their students, beginning with elementary teachers. Teacher education may need to be changed in order to prepare teachers at all levels, and particularly at the elementary level, to provide instruction which promotes mathematical reasoning. Further research should explore options for preservice mathematics teachers that will enable them to understand students' needs in terms of mathematical and geometric reasoning.

In this chapter, I will explore ways in which current teachers can alter their instruction to increase their students' mathematical reasoning skills, thereby promoting a better understanding of geometry and proof. I will first discuss research-based suggestions designed to help teachers increase their understanding of mathematical reasoning and proof. Turning the focus more toward high school, I will then discuss ways in which teachers can present information to increase mathematical reasoning skills and set the stage for proof.

Finally, I will look at how technology can aid in teaching reasoning and proof-writing skills. Dynamic geometry software is increasingly being used with promising results to promote geometric understanding.

Preparing Teachers to Teach Geometry and Proof

In the previous chapter, evidence was presented which indicated that inservice and preservice teachers, from the elementary to the secondary level, might not be sufficiently prepared to teach mathematical reasoning skills. This section will explore ways in which preservice and inservice teachers can be better prepared to promote mathematical reasoning skills in their classrooms.

Important Elements in Preparing Preservice Teachers

Implications of the previous chapter indicate that teacher education may need to be restructured in order to better prepare preservice teachers for teaching in general, and teaching mathematical reasoning skills, in particular. This is particularly important for elementary teacher education, as research has indicated that many preservice elementary teachers have a history of negative attitudes toward mathematics and high levels of mathematics anxiety. However, there is also evidence that even middle grades and secondary preservice teachers do not have a complete understanding of how to promote mathematical reasoning and proof skills among their students. Research in this area describes ways in which teacher education courses can change to help prepare teachers to teach mathematical reasoning skills.

In a study done by Bischoff, Hatch, and Watford (1999), the “state of readiness” for teaching of 10 middle grades mathematics preservice teachers was analyzed through interviews and observations. “State of readiness” is defined as the teacher’s “ability to plan, implement, and reflect on an integrated mathematics lesson” (p. 394). These preservice teachers were enrolled in a mathematics methods course at the time of the study. They were given a specific objective and asked to create a lesson based on the objective and present this to a sixth-grade class.

Although each preservice teacher showed some potential to eventually become an effective teacher, only one demonstrated a high state of readiness. Most of the teachers were concerned with lesson formatting, rather than student understanding. Only one teacher tried a hands-on activity, while the others just focused on the algorithms and drill-type exercises. In personal reflections, the preservice teachers did not see this as a problem. They measured their success by how much control they had over the students. Furthermore, the preservice teachers admitted during interviews that they did not feel comfortable with the material they were expected to teach. They were worried that a student would ask a question that they could not answer, indicating that they did not have a very high level of confidence in teaching the mathematics in their lesson (Bischoff, Hatch, & Watford, 1999).

This study indicates that preservice teachers need to feel confident in their knowledge of the material in order to successfully teach it and promote mathematical reasoning skills. Mathematics methods courses, as well as content-specific courses, need to help preservice teachers develop this confidence. Bischoff, Hatch, and Watford (1999) claim that teacher education programs must include instruction in self-evaluation and reflection. This is evidenced by the preservice teachers who felt they were successful simply because they were

able to stick to their plan, not because students learned any mathematics. Previous research also indicated that there is a strong relationship between mathematics beliefs and instructional practices among elementary, middle, and secondary preservice teachers (Raymond, 1997). Teacher education programs should help preservice teachers to understand their own beliefs as they relate to instructional practice.

According to Ensor (2001), preservice teachers' lesson plans should be reviewed by a mathematics education specialist so that feedback and suggestions can be given before the lesson is implemented. Furthermore, the mathematics faculty needs to work more closely with the education faculty to ensure that preservice teachers understand both the topic they are teaching and how to successfully teach the topic. Typically, content-specific courses and education courses are not jointly taught (Grover & Connor, 2000). Preservice teachers may be required to take a mathematics course, such as Geometry, that is taught by a member of the mathematics faculty and not by a member of the education faculty. Thus, preservice teachers may fail to see the connection between the material and how to teach it. This is also true with the use of technology in instruction. Most teacher-education programs require a course in technology as an instructional tool, but it is not always taught by a member of the education department (Niess, 2001). Thus, preservice teachers may also be having trouble implementing technology into their instruction. In a later section, I will discuss the importance of technology in developing mathematical reasoning.

In order for effective instruction in mathematical reasoning to occur, teachers must first be aware of their own beliefs and how they affect instructional practice. Currently, teachers are entering the profession with either negative beliefs about mathematics or a shallow understanding of how to effectively teach outside of the traditional algorithm-based

instruction. Teacher education programs must enable preservice teachers to be aware of their beliefs, to reflect on their practice, and to be knowledgeable about the content they are teaching. Further research needs to be done to explore specific ways in which teacher education programs can accomplish this task. To gain a deeper understanding of how to prepare teachers to teach geometry and proof, it is also necessary to discuss ways in which inservice teachers can increase their geometric understanding.

A Program Designed to Raise Geometric Understanding Among Inservice Teachers

It is believed that the more a teacher knows about a subject, the better she will teach it. In order to enhance geometric achievement among students, you first must improve teachers' content knowledge and their instructional practice (Jones & Swafford, 1997). Forty-nine fourth through eighth grade inservice teachers enrolled in a four-week summer program consisting of a content course in geometry and a research seminar on the van Hiele theory. The geometry course consisted of an exploration of two- and three-dimensional shapes with emphasis on analysis and informal deduction. The instructional approach taken was mainly a problem-solving approach. Each session began with the presentation of one problem, which was then analyzed and discussed in groups. A follow-up, whole-class discussion then followed, in which solutions were presented, connections identified, and van Hiele levels of thought were discussed. Teachers then created an instructional unit and assessment plan for their particular grade level. The research seminar presented the van Hiele theory and research findings pertaining to students. Teachers had the option of interviewing a student and assessing his or her VHL, or of analyzing their textbooks by separating activities by VHL (Jones & Swafford, 1997).

At the beginning of the program, geometry content knowledge and VHL were tested among the teachers. The content knowledge test consisted of problems about points, lines, planes, angles, properties of polygons, congruence and similarity, and perimeter, area, and volume. The van Hiele assessment used was based on Usiskin's (1982) test used in many other studies presented in prior chapters. At the end of the four-week course, teachers were tested in these two areas again to see if there had been an improvement as a result of the course. Teachers were also asked to provide lesson plans pertaining to certain geometry topics both before and after the intervention. Eight teachers were then chosen for follow up interviews and observations.

Pretest and posttest scores indicate a significant improvement in geometry content knowledge, particularly among the elementary teachers. Seventy-two percent of the teachers increased their van Hiele levels by at least one level, and 50% increased by two levels. No teacher decreased in level. There was also an improvement in lesson planning. Before the intervention, teachers' goals seemed split between promoting VHL1 and VHL 2 understanding. After the intervention, lesson plans provided goals that promoted more VHL 2 and 3 reasoning. Three percent of the lesson plans promoted reasoning beyond VHL3. Furthermore, in the second lesson plan, most teachers suggested some type of pre-assessment in order to understand what the students already knew (Jones & Swafford, 1997).

As a result of this intervention program, teachers are likely to spend more time on geometry instruction. From the lesson plans, it is apparent that they became more likely to try new ideas and more confident in their geometric-thinking ability. Most importantly, they increased their content-knowledge of geometry as well as their own van Hiele levels. The experience provided by this program will allow these teachers to provide more effective

instruction to their students. By attempting to improve instruction at the elementary and middle grade levels, students will be better prepared for a course in Geometry when they enter high school. However, not all elementary and middle grades teachers are taking advantage of programs like this. Therefore, the focus should now shift to how high school teachers should structure their courses to enable students to be successful in Geometry and proof writing.

Promoting Students' Communication and Justification Skills

In the previous chapter, evidence that preservice teachers were not sufficiently prepared to teach mathematical reasoning, particularly at the elementary level, was presented. At the inservice level, teachers can strive to be better instructors of mathematics and help students who struggle with mathematical reasoning, and in turn, with geometry and proof. Research in this area describes ways in which high school Geometry teachers can help make proof meaningful to their students. I have cited two major aspects involved in promoting proof skills among students: 1) the promotion of communication skills and 2) the promotion of justification skills.

Promoting Communication Among High School Students

Evidence has been presented in previous chapters that good proving skills result from good reasoning skills. Brendefur and Frykholm (Brendefur & Frykholm, 2000) further suggest that good reasoning and proof skills result from good communications skills. If students are able to effectively communicate, then proof writing will come more naturally,

especially if we approach proof as a means of explaining why something is true. In teaching proof, we should first elicit students' own explanation, in their own words, and then lead them to mathematical language, or in other words, formal proof (Edwards, 1999). Teachers can use questioning techniques to promote good communications skills.

Brendefur and Frykholm (2000) studied two pre-service teachers during their student teaching experiences to analyze how effective questioning can help to promote mathematical reasoning and understanding. In most classroom settings, teachers dominate discussions through lecture. This does not allow students a chance to discuss and explore their own ideas and strategies. Two types of classroom communication are discussed: contributive and reflective. Contributive communication is defined as “the interactions among students and between teacher and students in which the conversation is limited to assistance or sharing, often with little or no deep thought” (p. 127). On the contrary, reflective communication is defined as “teacher and students use mathematical conversations with each other as springboards for deeper investigations, such that what the student and teacher do in action subsequently becomes an explicit object of discussion” (p. 127). Students further reflect on their activities and ideas as further discussion ensues. Reflective communication is what teachers of geometry should strive towards. This type of communication enables students to communicate their ideas freely. As students communicate more, they become more aware of how to effectively convey their thoughts and ideas, eventually leading into the development of a formal proof.

Promoting this type of communication is not an easy task. One of the student teachers in the study had success only after months of frustration. In trying to promote reflective communication, she often felt as though it was hard to maintain good classroom

management. Though she felt that students could learn a lot from each other through openly communicating in the classroom, she struggled with wanting to control the direction of their communication. The students also didn't appear to be ready for this new type of communication. They were used to the more lecture-intensive style of their original teacher. However, as the student teacher continued to struggle toward reflective communication, she finally reached a balance. In her classroom, students were prepared to openly articulate their ideas. While discussing solutions, new ideas were likely to arise, further advancing the discussion to other topics. When asked on a test to explain a solution, students were able to give detailed responses (Brendefur & Frykholm, 2000).

Shifting instruction away from contributive communication is an idea that can be met with resistance by experienced teachers who are used to presenting information in this "traditional" manner. However, in order to prepare students for proof, it is important that reflective communication is emphasized, as it promotes communication skills among students. If students are better able to communicate their ideas, then they will be more successful with proof.

Communication skills in the classroom can also be obtained through the use of open problems. Furinghetti, Olivero, and Paola (2001) claim that the transition to proof is often abrupt, causing students who had previously performed at a superior level in mathematics to struggle with proof writing. The problems given to students, and the way in which students are required to participate in the problem-solving process, affect their transition to proof. Rather than say "prove that....", teachers should involve students in tasks that foster explorations of a topic and axioms pertaining to it that will help develop mathematical learning skills. Furinghetti, Olivero, and Paola define open and closed problems as such:

In a closed problem, both the starting and the goal situation are closed, i.e., they are exactly explained in the task. If the starting situation and/or the goal situation are both opened (i.e., not closed), we have an open problem (p. 320).

In solving an open problem, students are required to create and validate their conjectures. Students begin by first “discovering” a result and convincing themselves perhaps through a series of empirical examples. Then, they use axioms to explain why the result happened, thereby opening the door to proof. When students prove something they discover on their own, it’s more meaningful than proving something they are given. Attention needs to be focused on how to produce a result, rather than just proving the correct result.

Proof has been defined as a way of explaining why something is true (Hanna, 2000b). In order for students to be able to effectively explain why a conjecture is true, they must have good communication skills. Questioning techniques and the use of open problems can help to promote reflective communication among students. Once students are able to effectively communicate, they are a step closer to being competent in proof writing. In order for students to be fully capable of writing proofs, they must also have strong justification skills. Therefore, it is necessary to study how teachers can promote justification skills in their students.

Promoting Justification Skills In High School Students

“Proving, or justifying a result involves ascertaining – that is convincing oneself – and persuading – that is convincing others” (Sowder & Harel, 1998). In line with Hanna’s (2000b) view that proof, for high school students, should be explanatory in nature, Sowder and Harel claim that student-generated conjectures lead naturally to questions of justification to set the stage for mathematical reasoning. As students create and communicate their

conjectures, they will naturally see the need to justify their conjectures. Classroom activities should set the stage for this communication. In terms of justification, Sowder and Harel suggest proof schemes (justifications in general) which students use in justifying conjectures. There are three categories of proof schemes that Sowder and Harel suggest: externally based, empirical, and analytic.

In an externally based proof scheme, what convinces the student, and what the student uses to convince others, comes from some outside source. Some students may base their proof schemes on an authoritarian perspective by relying too much on teachers, textbooks, or peers to justify “correct” results. Students also tend to rely on ritualistic aspects about the way a proof should “look” (e.g., two columns) or merely follow a set of rules to complete a proof in a ritualistic fashion. Students also use symbolic proof schemes when they treat symbols independent of the relationship to the quantities from which they arose (e.g. $2^5 + 2^3 = 4^{15}$).

In using empirical proof schemes, students make justifications based on given examples of another proof scheme. Students may use a perceptual proof scheme if they arrive at conclusions based on one, or several drawings and convince others using drawings. In using this type of scheme, students may forget to account for the arbitrary case. In an examples-based proof scheme, students follow patterns or examples to form a conjecture, but may still not be able to reason through an arbitrary case.

Students use analytic proof schemes once they are competent in proof writing. These are more advanced levels of justifications. In using a transformational proof scheme, justifications are based on general aspects of the situation and involve reasoning toward settling the conjecture. Students use an axiomatic proof scheme when they realize that

results are logical consequences of preceding ones. Undefined terms are assumed in developing conjectures.

As presented in Chapter 4, students often hold proof misconceptions. Students tend to feel that a pictorial representation is a sufficient proof when it is not, or that empirical examples constitute a valid proof (Schoenfeld, 1986; Fischbein & Kedem, 1982). These misconceptions are characteristic of empirical proof schemes. When students begin their experience with proof in high school geometry, they may be operating at any one or more of these proof schemes. While traditional instruction is usually based at an axiomatic proof scheme, it is unrealistic to think that students are initially at this level. It is important that teachers assess which level their students are currently operating at and modify their instruction to accommodate for this. The ultimate goal is to get students to use an axiomatic proof scheme. Sowder and Harel (1998) offer some suggestions as to how teachers can help students progress from their current proof scheme to an axiomatic proof scheme. Rather than jumping immediately into proof, students should gradually be introduced to it through mathematical reasoning activities.

When students are operating with an authoritarian proof scheme, they tend to focus more on results than on reasoning. Teachers can accommodate for this by emphasizing an environment in which reasoning is as important as results. In other words, rote memorization should be eliminated and the focus should be more on reasoning. Teachers can guide students away from authoritarian and ritual proof schemes by decreasing the emphasis on two-column proofs. Paragraph and flow proofs should also be used. Otherwise, students may think that the only correct proof is in two-column form. In addition to using these different types of proof, teachers can also use algebraic proofs as an introduction to

geometric proof. This will help students to develop good symbolic reasoning, thus eliminating the problems with symbolic proof schemes as previously discussed.

When students are using an empirical, perceptual proof scheme, they may rely on just one type of drawing that demonstrates the validity of a conjecture only for a very specific case. Students may not be able to account for the arbitrary case. Teachers can use dynamic geometry software, such as The Geometer's Sketchpad (GSP, Jackiw, 1995), to help students better analyze the arbitrary case. The value of the "drag" effect in examining empirical examples will be discussed in the following section. To help students progress from an examples-based proof scheme, teachers can present students with problems that initially invite this approach, but result in the pattern "breaking down" in order to show students the dangers of this approach (Sowder & Harel, 1998).

An analysis of Sowder and Harel's (1998) proof schemes can help teachers to effectively analyze their student's proof-writing capabilities. Mills (2002) studied the proof schemes of students in her high school Geometry class and analyzed in which context they tended to utilize these schemes. Through student interviews and journal responses, as well as a videotaped analysis of particular lessons, Mills asserted that students could make the transition from an external, symbolic proof scheme to an analytical, axiomatic proof scheme after they learned to plan their approach toward completing a proof. Initially, most of the 27 students involved in the study began writing a proof as though "they were attempting to solve a single variable equation, where one does not have to think two or three steps ahead in order to complete the problem" (p. 62). Mills' emphasis on "formulating a plan" caused students to stop and consider the information presented in the problem before they wrote a proof. As they became used to this strategy, they were able to move toward an axiomatic proof scheme.

Justification sets the stage for proof. When students are able to effectively justify their conjectures, they are ready to begin the process of proof writing (Hanna, 2000b; Sowder & Harel, 1998). Geometry instruction has typically begun assuming that students were operating under an axiomatic proof scheme and reasoning at VHL 4. Previously reported research demonstrates that students are rarely at this level of reasoning. Thus, it is important that teachers modify instruction to accommodate for students operating under a variety of different proof schemes. With proper instruction, students can eventually be led to the world of axiomatic proof schemes.

Once students are effectively able to communicate their ideas and justify their conjectures, they will be ready to begin writing proofs. However, we still cannot expect students to be able to write a flawless proof without adequate guidance from the teacher. Though they may have good communication and justification skills, it is still important that we emphasize the use of proof as a means of explanation. Furthermore, we have to be able to guide students toward seeing the need for proof. A dynamic geometry environment can be a valuable tool for accomplishing this task.

Using Dynamic Geometry to Promote Proof Writing

Dynamic geometry software (DGS) has the potential to open many doors for geometry and proof explorations in the classroom. DGS refers to interactive software in which students essentially create compass and straightedge constructions, which can then be “dragged,” altering the size of the construction, but not affecting the axioms or theorems used in the construction. With an increase in the availability of technology, DGS environments

are becoming more prevalent in the classroom. Using DGS has many advantages for the teaching and learning of geometry and proof in high school. However, it is important that lessons using DGS are properly implemented. Students must be able to understand the connection between the DGS environment and the axioms and theorems that make up Euclidean geometry. Critics claim that the increased use of DGS has not only led to a decline in the teaching and learning of proof, but also to a decline in mathematical reasoning (Bruckheimer & Arcavi, 2001; Hanna, 2000a, 2000b; Wu, 1996). This may be due to students not seeing the connection between what they are doing on the computer to what they are learning from their teachers.

In a case study of conceptions of proof using DGS, preservice teachers were asked to solve two geometric problems with the use of a DGS and attempt to create formal proofs (Pandiscio, 2002). All of the preservice teachers argued that with the use of the DGS, proof became unnecessary. One preservice teacher asked, “Why would my students want to bother proving this theorem when they can see that it is true right in front of them?” (p. 218). The “drag” effect of many software packages, including GSP, (Jackiw, 1995) allows students to see many empirical examples instantaneously. By dragging, students can alter the size and orientation of constructions and notice that a certain theorem always seems to be true. While this is a valuable means of convincing a student that a conjecture is true, it does not prove that the conjecture is true. Students operating under an empirical, perceptual proof scheme (Sowder & Harel, 1998) may mistakenly take this drag effect as a means of proving. “It always works, so I’ve proven it” is a common misconception among students using DGS (Izen, 1998, 719).

The preservice teachers in Pandiscio’s (2002) study struggled with this as well. They

felt that their students, having seen that the theorem worked for these empirical examples, would not see any use in proving. If not approached carefully, the drag effect can not only keep students from operating in more advanced proof schemes, but can also convince students that there is no need for proof. However, it is also possible to engage students in explorations, which, despite the drag effect in DGS environments, foster a need for proof. If the technology is used as a tool for convincing, and classroom or group discussion ensues to “explain,” then proof can be a result of explorations in DGS environments.

The function of proof has been the source of debate for some time. De Villiers (1990) claims that there are six main functions of proof: verification/conviction, explanation, discovery, systematization, communication, and intellectual challenge. Students can use DGS to verify a conjecture. Therefore, when asked to write a proof for the function of verification, students will see no need for it. However, when challenged to explain why their conjecture appears to work, students quickly see that a deductive argument is necessary. The empirical examples they’ve witnessed in a DGS serve only to confirm, and students recognize this when they use the function of proof as explanation (De Villiers, 1999).

DGS environments can help students to write proofs if teachers help students to make the proper connections. The drag effect of software such as GSP can both help and hinder students proving capabilities. It is helpful to students to be able to quickly convince themselves that a conjecture is true, but they must also understand the importance of the explanation. The following section presents two activities that help students to make these connections.

Using DGS to Promote Proof Writing Skills and to Prompt a Need for Proof

Prior to the availability of DGS, students had to create constructions with paper and pencil, leaving room for errors. Furthermore, students were often confused when they saw that the construction they made was completely different from the construction of one of their peers (Perham & Perham, 1997). When students use DGS, they can see many different constructions at once, so they tend to see the computer-generated picture as more arbitrary and representing a general case. This perception of arbitrariness can help students consider generalizations for a class of figures, rather than for a specific-sized construction.

Perham and Perham (1997) conducted a study with tenth-grade geometry students to promote geometric reasoning and proof writing. They used manipulatives, DGS, and graphing calculators to develop and test conjectures about the centroid of a triangle. In the first activity, students constructed the medians to each of the sides of a paper triangle. They were able to balance the triangle at the centroid (intersection of the medians) on the tip of a pencil and cut the triangle into the three smaller triangles. Upon observing that each triangle had equal weights, they conjectured that the three triangles had equal areas. In the second activity, students constructed a triangle and its centroid using a DGS. By setting the program to calculate the area of each of the three triangles, they saw that their areas remained equal to each other no matter how they dragged the construction. Several students admitted that they trusted this dynamic construction in more than they did their manipulative activity with the paper triangles. They were now convinced that their conjecture of equal areas was true. The third activity that students engaged in was using a TI-83 calculator to construct a triangle and its centroid on the coordinate plane (this could also be done in a DGS). Students examined the equations of the three medians and solved a linear system to arrive at the coordinates of the centroid. They created an algorithm to support their initial conjecture that the areas of the

three triangles were the same.

After these experimental activities, students were to create a proof of their conjecture. When the teacher asked them to consider the construction of the centroid and the algorithm they created with the TI-83, students were able to develop a formal proof of their conjecture. (Perham & Perham, 1997). Though students initially formulated a conjecture without the use of technology, they were more convinced of their conjecture after using the DGS. Rather than being handed a theorem and told to, “prove it,” students were given time to form a conjecture on their own and convince themselves of its truth before attempting to prove it. In the process of convincing themselves of the conjecture’s truth, they essentially discovered several of the steps necessary to form the proof of the conjecture. Though this activity was valuable in helping the students write the proof, it may not have necessarily created a student-driven need for proof. In other words, students may still have not seen the necessity of writing the proof once they were convinced of their conjecture.

Hadas, Hershkowitz, and Schwarz (2000) claim that if students are not writing proofs, it is probably not because they don’t know how, but because they don’t see the need. This is evidenced by the large number of students who, even after they have proven something, still feel that they need to show empirical examples to give further verification (Fischbein & Kedem, 1982). In using DGS, conviction can come quickly, which has led to the question of whether or not proof should be taught in the first place. The DGS itself may not help students to understand the need for proof, but can lead to generalizations and discoveries. It is up to the teachers to prompt the need for “why?”

Hadas, Hershkowitz, and Schwartz (2000) conducted a study in which 50 Geometry students were given two activities, which required creating constructions on GSP. The first

activity was meant to create a “surprise by contradiction” between the students’ hypotheses and their findings, which would lead into the need for an explanation. The second activity also created contradictions, but led to uncertainty and surprise, as students were not sure whether the given situation could geometrically exist. In the first activity, students made a conjecture about the interior angle sum of a convex polygon as the number of sides increases. Only 9 students gave complete deductive explanations. The second activity was to make a conjecture about the measure of the exterior angle sum of a convex polygon as the number of sides increases. The contradiction occurred in that all of the students conjectured that the sum would increase as the number of sides did, just as it had for the interior angle sum. Students were surprised to find that the exterior angle sum was 360 degrees for every example they tried on GSP. This sparked an interest in the students, causing them to want to understand “why?” In the second activity, 23% of the students gave no explanation, 4% gave an inductive explanation, 11% gave a partial deductive explanation, and 56% gave a complete deductive explanation.

These two activities show how a DGS can allow opportunities for students to realize the necessity of proof. These activities were based on surprise, thus capturing the interest of the students. When things did not work out the way they expected it too, students saw a need to understand why. The design of this activity brought proofs into “the realm of student activity and argument; that is, they engaged naturally in true mathematical activity” (Hadas et al., 2000, p. 149).

Dynamic geometry software, though criticized as bringing about the recent decline in the teaching of proof in high school geometry (Hanna, 2000a), can enhance reasoning and proof skills (De Villiers, 1999). DGS allow students to explore empirical examples as a

means of convincing themselves that a theorem or conjecture is true. It is important, however, that teachers have clear proof objectives. When misused, dynamic geometry can merely serve the purpose of convincing, and not of explaining. A teacher needs to structure activities with DGS in such a way as to prompt a need for proof. As seen in some of the activities discussed earlier, the construction process in a DGS can also come into play as students begin writing proofs. By paying attention to how something was constructed, students were able to quickly see what steps were needed in their proof. Thus, DGS, while prompting a need for proof, can also aid in proof writing.

Teacher preparation is paramount to improving mathematical and geometric reasoning skills, as well as proof-writing skills. The NCTM (2000) emphasizes the need to promote mathematical reasoning from the beginning of a students' educational experience. The NCTM also recommends the use of DGS to promote reasoning skills and geometric understanding. Teachers must be able to reflect on their instructional practices in order to verify that their students are learning the necessary skills. Emphasis should be placed on understanding and conceptualizing, rather than on algorithms and rote-memorization. Communication and justification skills need to be fostered across all levels of mathematics, but especially as students begin to explore geometry and proof. Once students are at a level in which proof writing can begin, dynamic geometry can be used to prompt the need for proof, and to help students formulate a proof based on empirical examples.

CHAPTER 6

CONCLUSION AND IMPLICATIONS

The NCTM (2000) is clear in its expectation that mathematical reasoning be taught in K-12. The Reasoning and Proof Standard in the *PSSM* (2000) states that all students should be exposed to the development of conjectures and should be expected to communicate their thoughts and justify their reasoning. Despite the NCTM *PSSM* recommendations, research shows that many students are not mastering these concepts at any level of mathematics education.

In order for students to be ready to write geometric proofs, there are certain skills that need to be mastered. The van Hiele theory claims that students progress through six levels of geometric thought. These levels are hierarchically based, and students typically do not progress to the next level without being able to reason successfully at lower levels, though some research indicates that students may reason at two different levels when they are transitioning between levels (Pusey, 2003). Research shows that in order to be successful with geometry and proof, students should be entering geometry at VHL 2 or above (Senk, 1989). However, the majority of students are entering Geometry below VHL 2 (Usiskin, 1982). Thus, it is not that these students just “don’t get proof.” Rather, they are just not ready for proof.

Mason and Moore (1997) suggest that students who are not yet at VHL 2 be placed in Algebra 2 rather than Geometry. This suggestion deserves serious consideration. If students’ level of geometric reasoning is as low as VHL 2, they may experience more success in an algebra-based course than in Geometry. Students would then take Geometry once they

had increased their VHL. A drawback of this, however, is the fact that students may still not be getting appropriate instruction in mathematical and geometric reasoning in an Algebra 2 course. For this reason, secondary mathematics teachers should strive to integrate the Algebra and Geometry curriculums. According to Sowder and Harel (1998), algebraic proofs can serve as an introduction to geometric proofs, thus helping students develop good symbolic reasoning which in turns guides them toward an axiomatic proof scheme. If algebraic proofs were used more in Algebra 1 and 2, students would be more prepared for proof in Geometry. Furthermore, students would begin to see that proof is important to all mathematical disciplines, not just geometry.

This literature review reported research that indicated that many teachers are not prepared to teach mathematical reasoning and proof. This could be due to many different factors. In some cases, teachers simply hold some misconceptions about mathematical reasoning, while in others, the teachers don't even have a conceptual understanding of many of the topics they are required to teach (Bischoff et al., 1999). Furthermore, many new teachers tend to focus mainly on classroom order and management more than they do on student understanding and conceptualization (Ensor, 2001).

Teacher education programs need to address this problem. Most elementary school teachers only take one mathematical methods course in their training. This course is taught by a member of the education department with very little interaction with the mathematics department. If teachers are having content-specific problems, then these courses should provide the preservice teachers the opportunity to better understand the mathematics content, as well as how to teach it. Even at the secondary level, preservice teachers are not provided with a course specifically in teaching high school Geometry. Usually, secondary

mathematics education majors are required to take a Geometry course, but a member of the mathematics department most often teaches it. Furthermore, this course focuses on the material and not on how to teach it. This type of course, though valuable, does not completely help prepare preservice teachers to teach geometry. There should be a closer interaction between the mathematics and mathematics education departments in teaching courses like this. Teacher education courses should also require preservice teachers to continually reflect on their instructional practice and student understanding. If preservice teachers are more aware of their students' understanding, they will be better able to create a lesson based on the level of understanding of their students.

Once students reach geometry, there are many ways in which teachers can make the transition to proof smooth. If students are able to effectively communicate, then proof writing will come more naturally (Brendefur & Frykholm, 2000). Thus, rather than jumping right into a two-column proof, teachers should take some time to work on reflective communication. This can be fostered through appropriate questioning techniques and the use of open problems. In solving open problems, students are required to create and validate their conjectures. They should be encouraged to work together to “discover” a result, convince themselves of its truth, and explain why it happened. These types of problems should be routine in the Geometry classroom.

Sowder and Harel (1998) offer some suggestions on how to help students develop justification skills. First, teachers must be able to assess what proof scheme their students are operating under (authoritarian, ritual, symbolic, perceptual, examples-based, transformational, or axiomatic). When students are operating with an authoritarian proof scheme, they tend to focus more on results than on reasoning. Teachers can accommodate

for this by emphasizing an environment in which reasoning is as important as results. In other words, rote memorization should be eliminated and the focus should be more on reasoning. Teachers can guide students away from authoritarian and ritual proof schemes by decreasing the emphasis on two-column proofs. Paragraph and flow proofs should also be used. Otherwise, students may think that the only correct proof is in two-column form. In addition to using these different types of proof, teachers can also use algebraic proofs as an introduction to geometric proof. This will help students to develop good symbolic reasoning, thus eliminating the problems with symbolic proof schemes as previously discussed.

When students are using an empirical, perceptual proof scheme, they may rely on just one type of drawing that demonstrates the validity of a conjecture only for a very specific case. Students may not be able to account for the arbitrary case. Teachers can use dynamic geometry software, such as GSP (Jackiw, 1995), to help students to better analyze the arbitrary case. To help students progress from an examples-based proof scheme, teachers can present students with problems that initially invite this approach, but result in the pattern “breaking down” in order to show students the dangers of this approach.

Dynamic geometry environments should be used in the classroom to help students develop their justification skills. Many times, constructions made in a DGS provide a “skeleton” of a proof. For example, by constructing a perpendicular bisector in GSP, students can actually see the steps to a proof of the Perpendicular Bisector Theorem. However, students will not see this connection without the appropriate guidance from the teacher. Consider the following example in which students are asked questions about the construction of a perpendicular bisector in GSP:

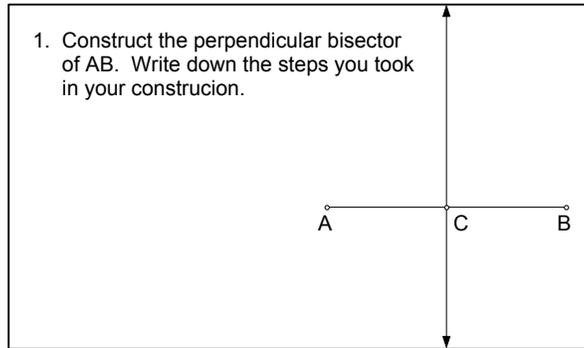


Figure 6.1. *GSP Activity, Question 1.*

Students are first asked to construct a perpendicular bisector of segment AB. A class discussion should ensue as students describe methods they used in their construction. Students are likely to have constructed a midpoint, followed by a perpendicular line to AB through the midpoint. The teacher should ask the students how the definition of the perpendicular bisector helped them to determine how to construct it.

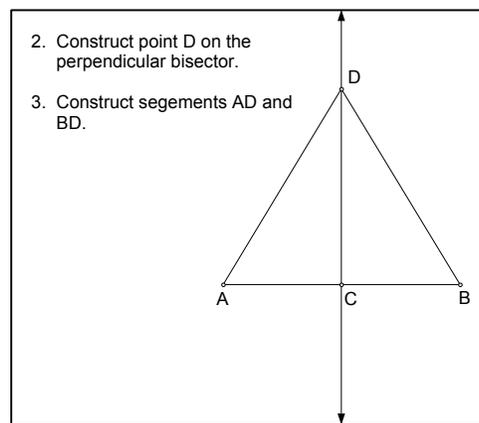


Figure 6.2. *GSP Activity, Questions 2 and 3.*

After students have constructed a perpendicular bisector, they are asked to construct a point on the perpendicular bisector. They then connect that point to the two endpoints of the segment. This step will eventually lead to the “discovery” of the perpendicular bisector

theorem.

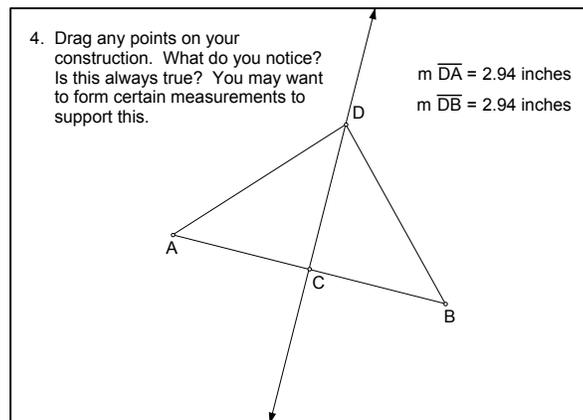


Figure 6.3. GSP Activity, Question 4.

As students move different points on their construction, they will notice that the length of AD is always equal to the length of BD. They may decide to take some measurements to convince themselves. At this point, rather than allowing students to assume that this is always true because they have seen several empirical examples, the teacher should ask students to explain *why* this phenomenon is happening. Guiding questions such as “What type of triangle is ADB?” and “What appears to be true about triangles ADC and BDC? How do we know this is true? How did the perpendicular bisector make this happen?” will help students to see that the two congruent triangles create congruent segments AD and BD. After this discussion, students will have already produced most of the steps necessary for a proof of the perpendicular bisector theorem, as seen below:

Statements	Reasons
1. DC is the perpendicular bisector of AB	1. Given
2. DC is perpendicular to AB C is midpoint of AB	2. Definition of perpendicular bisector
3. AC is congruent to BC	3. Definition of midpoint
4. $\angle DCA$ and $\angle DCB$ are right angles	4. Definition of perpendicular lines
5. $\angle DCA$ is congruent to $\angle DCB$	5. Definition of right angles
6. DC is congruent to DC	6. Reflexive Property
7. Triangles ADC and BDC are congruent	7. SAS Congruence
8. DA is congruent to DB	8. CPCTC

Figure 6.4. *Proof of the Perpendicular Bisector Theorem*

It is important that teachers help their students see the connections between what they are doing on the computer and what they are learning in Geometry. Teachers have to be careful to use DGS in such a way as to create a need for proof. DGS make it possible to explore many empirical examples at once, leading some students to believe that what they see on screen is actually a proof. This can be done simply by asking students to “explain” why their conjecture is true, rather than asking them to prove it.

Students should not be expected to immediately be at an axiomatic proof scheme. However, this is where Geometry instruction usually starts. Perhaps proof instruction should be delayed until the second half of the school year. If students are entering high school Geometry below VHL 2, then they will not be at an axiomatic proof scheme at the beginning of the year. If proof instruction were delayed, students would have more of an opportunity to develop their conjecturing and justification skills before attempting to write formal proofs. During the first half of the year, instruction could focus on reasoning and informal proofs, allowing students to develop their skills in other proof schemes. By understanding students’ justification skills, teachers will be able to alter instruction to accommodate for whichever

proof scheme they are currently operating under.

In order to be ready for proof in high school Geometry, students need to have sufficient mathematical and geometric reasoning skills. Researchers have seen low proof performance among the majority of students in high school Geometry, probably because students are not entering Geometry with the necessary skills and reasoning capabilities. Evidence shows that this could be due to a lack of teacher preparation to teach mathematical reasoning and the minimal attention to mathematical reasoning and justification in earlier grades. Further research should examine what teacher education programs can do to better prepare teachers of all grade levels to foster mathematical reasoning and justification habits of minds. Once students are in Geometry, teachers need to promote an environment that fosters communication and explanation, providing a framework for the learning of proof.

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