Abstract

CORNE, MATTHEW ALLAN. Instabilities of Geometrodynamic Evolution in the Hamiltonian Formulation of General Relativity. (Under the direction of A. Kheyfets.)

Difficulties with instability of numerical geometrodynamic evolution associate with the modeling of gravity waves generated by colliding black holes. This modeling is an integral part of the gravity wave detection effort (ground-based detectors LIGO and VIRGO and future space-based detector LISA). However, all algorithms have proven largely unsuccessful as they amplify (exponentially) nonphysical constraint-violating modes such that the numerical codes crash almost immediately after the modes appear. Equivalently, these models violate initial constraints (essentially enforcing conservation of energy and momentum) because of numerical errors and, subsequently, either imperfect numerical techniques or inadequate formulation of Hamilton's evolution equations cause rapid drift of the solutions off the constraint shell.

Another possibility seriously considered in these models concerns the inadequacy of Hamilton's equations off shell. This has resulted in hyperbolic reformulations of Einstein's equations that exclude acausal modes of the solutions (modes propagating between the normal and the null of a nullcone or out of the null cone), leading to more stable solutions. The development has resulted in marginally better algorithms. However, difficulties remain (this does not yield a sufficiently stable code). Improved understanding of the nature of instabilities is needed.

This research investigates instabilities caused by violation of the initial-value constraints. It considers the observation that the drift off shell of a source-free gravitational field equates with a transition from a source-free field to a field with a source. The simplest model permits, as a source, a scalar field that is not necessarily subject to standard energy conditions. Such a field resembles the one that is used by cosmologists for the description of inflation and acceleration phenomena or, historically

preceding it, the C-field introduced originally by Hoyle and Narlikar. Unlike the approach taken in cosmology, this work considers associated exponential instability in the Hamiltonian formulation (ADM and its hyperbolic modifications). Particularly, this work looks at a field theory with a Lagrangian for a pre-existing source (or sourceless, though this does not change the formulation) and considers how introducing the C-field modifies the evolution of the system.

Study of the mechanism of instability might lead to better numerical evolution schemes, but this development is beyond the scope of the suggested work.

INSTABILITIES OF GEOMETRODYNAMIC EVOLUTION IN THE HAMILTONIAN FORMULATION OF GENERAL RELATIVITY

by

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Biography

Matthew Corne was born on June 3, 1980 in Asheville, North Carolina and spent his formative years in Swannanoa, North Carolina. He received a B.S. in Physics and Minor in Mathematics, Cum Laude, from North Carolina State University in Raleigh in 2002. Currently, he lives in Raleigh.

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Contents

List of Figures

Chapter 1

Introduction

General relativity is a powerful theory. Calculations using the theory yield accurate results, although few experiments allow for study or verification of one of its predictions: gravitational waves. Development of the Earth-based detectors, LIGO and VIRGO, and the proposed space-based LISA, provide means to detect these waves. Meanwhile, development of accurate models is crucial for studying the gravitational behavior of sources such as colliding black holes and neutron stars. Further, these models yield information to compare with observations at the detectors. However, current modeling schemes yield physically unrealizeable results, given the conditions under which they are studied. Particularly, the models violate initial-value energy and momentum constraints. The wave solutions go off of the constraint shell, yielding wild solutions that blow-up (grow rapidly).

Specifically, constraint-violating modes build up; wave solutions go off of the constraint shell (set of physically admissible solutions). As time progresses, constraint-violating modes overtake constraint-obeying modes, leading to rapid growth of solutions. To keep solutions on-shell, methods consider possibilities such as adjusting the constraints at each time step, introducing algebraic and differential conditions for constraints, and even rewriting Einstein's equations. This work posits a different argument, suggesting that the growth encountered after violations dominate might be an artifact of leaving out physical information, particularly in the source term.

As a possible remedy, this work introduces a minimally-coupled scalar field in the source term. The scalar field produces particles in strongly curved spacetime. Sources of gravitational waves (colliding black holes, etc.) strongly curve space. Concerning constraint violations, a scalar field of negative energy density activates in such situations, and so the constraints cannot be violated as such. Instead, violations represent the contribution of the scalar field term. Effectively, the problem of stability turns out to be an artificial difficulty; the set of admissible solutions is increased (i.e., the constraint shell is smeared out and is less rigid).

Chapter 2 begins by tracing the development of the theory through the Lagrangian formulation, an action principle first appears which gives Einstein's equations. These equations give all of the information about the gravitational fields, more accurately described as curvature, and the geometry, and they connect this information with a source. The equations require the conservation of energy-momentum, with the terms associated with geometry and curvature vanishing under divergence. Gravitational waves represent the solutions to these equations.

Chapter 3 poses these equations more simply via the $3 + 1$ decomposition of spacetime. This makes a Cauchy problem (equivalence of classical gravitational fields to the time history of spacelike hypersurfaces' geometries), and the entire procedure of splitting the curvature into that intrinsic to the hypersurface and that extrinsic for the hypersurface embedded in 4-dimensional spacetime allows for a redressing of the problem of second-order differential equations, via variational principles, into dynamic and constraint equations. Then, the possibility of formulating an initial-value problem occurs, and it handles via the move from a canonical picture to one expressed in terms of curvature.

Chapter 4 discusses modeling of gravitational waves. These waves comprise of superpositions of plane waves. Models of gravitational waves consider both the weak-field limit and the strong-field limit. The weak-field limit considers that gravitational waves occur as perturbations of Minkowski spacetime. The strong-field limit considers curved spacetime.

Chapter 5 considers the problem of stability and presents a possible solution. The stumbling block of gravitational wave models occurs in their long-term behavior: constraint-violating modes dominate the equations, and the solutions blow-up exponentially. Avoiding wild growth requires satisfying the constraint equations. Introduction of the C-field, a particle-generating scalar field, allows for treatment of unstable solutions associated with vacuum sources. This investigation considers equivalence of unstable solutions in sourceless models with constraint-satisfying solutions in the presence of this scalar field.

Chapter 2

Einstein's Equations: Lagrangian Formulation

2.1 Geometry and the Metric

General relativity changes radically the Newtonian view of gravity as a collection of field lines in an absolute, fixed background space. A composite spacetime replaces the previous notion of absolute 3-space and absolute time, and its distortion manifests the field lines. More rigorously, a smooth 4-manifold defines classical relativistic spacetime. The definitions of a 4-manifold and a metric require introduction of several concepts (see [1], [2], [3], [4], and [5] for more details).

2.1.1 Topological Spaces

A topology on a set X is a subset $T \subset \mathcal{P}X$ (where $\mathcal{P}X$, the power set of X, is the collection of all the subsets of X) such that the following properties hold:

$$\begin{aligned} i. A_1, A_2 \in T \quad &\Rightarrow \quad A_1 \cap A_2 \in T; \\ ii. \{A_\mu | \mu \in I\} \subset T \quad &\Rightarrow \quad \bigcup_{\mu \in I} A_\mu \in T; \end{aligned}$$

$$iii.\emptyset \in T \quad \text{and} \quad X \in T. \tag{2.1}$$

A topological space (many different kinds exist) is written as (X, T). Open sets of this space correspond to the elements/subsets of T.

Considering the Euclidean space $\mathbb{R}^n$, a subset $S \subset \mathbb{R}^n$ is open when an open ball contained in S is centered around each of its elements $s \in S$. [6] An open (ϵ) ball with radius ϵ centered around a point $x' = (x'_1, ..., x'_n)$ consists of points x such that $|x - x'| < \epsilon$. Then the union of such ϵ balls gives an open set. The distance between points is given by the norm,

$$|x - x'| = [\sum_{i=1}^{n}(x_i - x'_i)^2)]^{\frac{1}{2}}. \tag{2.2}$$

A metric is a distance function satisfying several properties [5]:

$$\begin{aligned}
&\text{i.} \quad && \mathit{Positivity}: \ \forall x, x' \in X, d(x, x') \geq 0; \\
&\text{ii.} \quad && \mathit{Nondegeneracy}: \ d(x, x') = 0 \Rightarrow x = x'; \\
&\text{iii.} \quad && \mathit{Symmetry}: \ \forall x, x' \in X, d(x, x') = d(x', x); \\
&\text{iv.} \quad && \mathit{Triangle\ Inequality}: \ \forall x, x', x'' \in X, d(x, x'') \leq d(x, x') + d(x', x''). \quad (2.3)
\end{aligned}$$

This distance function maps the cartesian product of a set with itself into $\mathbb{R}$, thus defining a metric space. So, (1.2) corresponds to the Euclidean metric on $\mathbb{R}^n$. An alternative description considers differential distances. In the case of differentials, the metric is represented by

$$ds^2 = dx_1{}^2 + ... + dx_n{}^2. \tag{2.4}$$

2.1.2 Manifolds

Define $M := (X, T)$ as a topological space. Then, X is Hausdorff if for every $x, x' \in X$, $x \neq x'$, there exist neighborhoods of the points in X such that their intersection is the empty set. Let there be a set of real functions $\{g, f_1, ..., f_n\}$ on M. [7] If $\exists$ a real function $u(t_1, ..., t_n)$ defined on $\mathbb{R}^n$ such that $g = u(f_1, ..., f_n)$ on M, then g depends smoothly on the functions $f_1, ..., f_n$. For a point $m \in M$, it is equivalent that $g(p) = u(f_1(p), ..., f_n(p))$. If the set of these functions contains a nonempty subset $G := G(M)$ such that a function depending smoothly on functions in G belongs to G and such that if the function assumes an equal value to a function in G that it belongs to g, then M is a smooth premanifold.

$\mathbb{R}^n$ is a smooth premanifold. The points are n-tuples of real numbers, and smooth functions on $\mathbb{R}^n$ are infinitely differentiable functions of the points.

To further define M, introduce dimension. M is an $n - \textit{dimensional}$ smooth manifold if every m has a neighborhood diffeomorphic to some open subset of $\mathbb{R}^n$. What does it mean to be diffeomorphic?

Build up to the idea of diffeomorphism by introducing the idea of homeomorphism. Let X and Y be Hausdorff spaces. A homeomorphism is a function $f : X \rightarrow Y$ one-to-one, onto, and such that it and its inverse $f^{-1} : Y \rightarrow X$ are continuous. Going back to the topological space (X, T), a chart at $m \in X$ is a function $\alpha : A \rightarrow \mathbb{R}^d$, where the open set A contains m and α is a homeomorphism onto an open subset of $\mathbb{R}^d$. This requires the dimension of α to equal d. The chart's coordinate functions are the real-valued functions on A given by entries α's values. More specifically, they are functions (compositions of the charts with standard coordinates on $\mathbb{R}^d$) $x^i = a^i \circ \alpha : A \rightarrow \mathbb{R}$, where $a^i : \mathbb{R}^d \rightarrow \mathbb{R}$represent the standard coordinates on $\mathbb{R}^d$. So for each $p \in A$, $\alpha p = (a^1 p, ..., a^d p)$. In this way, the chart is written in terms of the coordinate functions, $\alpha = (a^1, ..., a^d)$. These terms yield coordinate definitions, with α a coordinate map, A a coordinate neighborhood, and $(x^1, ..., x^d)$ coordinates or a coordinate system at m.

Next introduce the concept of C^∞ functions. Call a real-valued function $f : B \to \mathbb{R}$ C^∞ (continuous to order ∞) if B is an open set in $\mathbb{R}^d$ and f has continuous partial derivatives of all orders and types. Then, the function $\beta : B \to \mathbb{R}^e$ is a C^∞ map if the components $a^i \circ \beta : B \to \mathbb{R}$, $i = 1, ..., e$, are C^∞.

Two $C^\infty - related$ charts $\alpha : A \to \mathbb{R}^d$ and $\beta : B \to \mathbb{R}^e$ on a topological space possess the following properties: $d = e$ and either $A \cap B = \emptyset$ or $\alpha \circ \beta^{-1}$ and $\beta \circ \alpha^{-1}$ are C^∞ maps. The domain of $\alpha \circ \beta^{-1}$ is an open set in $\mathbb{R}^d$, $\beta(A \cap B)$. A separable Hausdorff space with a $d - dimensional$ chart at every point defines a topological manifold. A C^∞ atlas requires every pair of charts to be $C^\infty - related$. An admissible chart to a C^∞ atlas is $C^\infty - related$ to every chart in the atlas. A topological manifold with all admissible charts of some C^∞ atlas comprises a C^∞ manifold.

Let an atlas exist as the single chart, the identity map $I : \mathbb{R}^d \to \mathbb{R}^d$. The coordinate functions then yield the cartesian coordinates a^i. Then, a C^∞ admissible coordinate map on $\mathbb{R}^d$ is a one-to-one C^∞ map $\alpha : A \to \mathbb{R}^d$ with A an open set and $|\frac{\partial x^i}{\partial a^i}| \neq 0$ ($x^i = a^i \circ \alpha$ are the coordinate functions). From the inverse function theorem, if f^i ($i = 1, ..., d$) are real-valued C^∞ functions on some open set of $\mathbb{R}^d$ and at some $r \in \mathbb{R}^d$ $|\frac{\partial x^i}{\partial a^i}| \neq 0$, then $\exists$ a neighborhood U of r and a neighborhood V of $(f^1 r, ..., f^d r)$ such that the map $\rho = (f^1, ..., f^d)$ takes U onto V, is one-to-one, and has a C^∞ inverse. This obtains admissible coordinates.

Let $\alpha_1 : U \to \mathbb{R}^d$ and $\alpha_2 : V \to \mathbb{R}^e$ be respectively C^∞ charts on C^∞ manifolds M and N. Assume $F : M \to N$ a continuous map. Then $\Lambda = F^1 V$ is an open subset of M. Let $\Lambda_1 = \alpha_1 \Lambda$; then Λ_1 is an open set in $\mathbb{R}^d$. The $\alpha_1 - \alpha_2$ coordinate expression for F is the map $\alpha_2 \circ F \circ {\alpha_1}^{-1} : \Lambda_1 \to \mathbb{R}^e$. If the coordinate expressions $\forall$ admissible charts are C^∞ maps on cartesian spaces, then F is a C^∞ map.

Thus, a diffeomorphism from M onto N is a one-to-one, onto, C^∞ map $F : M \to N$ such that the inverse map $F^{-1} : N \to M$ is also C^∞. And it is clear now that every m has a neighborhood diffeomorphic to an open subset in $\mathbb{R}^n$.

2.1.3 Bases: Vectors and 1-Forms

Consider vector objects residing in a vector space U. Define them as $\mathbf{u} := \frac{d\mathcal{P}}{d\lambda}$, or the derivative of a point along a curve, parametrized by λ and taken at the beginning/tail of the curve ($\lambda = 0$). Introduce basis vectors $\mathbf{e}_\mu$ and their duals ω^μ [5]. For nontrivial scalars (not all equal to zero) c_μ, a linearly dependent finite set of vectors is such that a linear combination $\sum_{\mu=1}^{i} c_\mu u_\mu = 0$. Linear independence is the opposite case; $\sum_{\mu=1}^{i} c_\mu u_\mu \neq 0$.

Define a basis as a linearly independent spanning set in this vector space. Considering a basis $\{\mathbf{e}_\mu\}$ and its dual $\{\omega^\mu\}$, then $< \omega^\nu, \mathbf{e}_\mu >= \delta^\nu{}_\mu$ gives the functional representation. The basis vectors of different indices lie parallel to surfaces defined by the duals; the basis vector of the same index as the dual pierces exactly one surface of it. This holds for any basis (coordinate and noncoordinate). For $\{x^\mu\}$ a set of coordinates of the vector space, define a coordinate (holonomic) basis $\mathbf{e}_\mu := \frac{\partial}{\partial x^\mu}$ with dual $\omega^\mu := \mathbf{d}x^\mu$. Then, at some initial point, $< \mathbf{d}x^\mu, \frac{\partial \mathcal{P}}{\partial x^\nu} >= \frac{\partial}{\partial x^\nu} x^\mu = \delta^\mu{}_\nu$ [1]. Different charts yield different coordinate bases [2].

Vectors and 1-forms expand according to bases and duals. In this way, $\mathbf{u} = \mathbf{e}_\mu u^\mu$ and $\sigma = \sigma_\mu \omega^\mu$; to obtain the components, $u^\mu =< \omega^\mu, \mathbf{u} >$, $\sigma_\mu =< \sigma, \mathbf{e}_\mu >$. These imply that $< \sigma, \mathbf{u} >= \sigma_\mu u^\mu$. For simplicity of calculation, introduce the Einstein summation convention for these components:

$$\sigma_\mu u^\mu = \sum_{\mu=0}^{n} \sigma_\mu u^\mu. \tag{2.5}$$

Consider curved spacetime (nonorthonormal basis vectors in non-Lorentz frames). Moving from a Lorentz transformation (as found in Minkowski spacetime) to a general change of basis requires the use of an arbitrary nonsingular matrix, $||L^\mu{}_\nu|| = ||L^\nu{}_\mu||^{-1}$. So, $\mathbf{e}_\nu = \mathbf{e}_\mu L^\mu{}_\nu$, and $\omega^\nu = L^\nu{}_\mu \omega^\mu$. Components of vectors and 1-forms transform also; $v^\mu = L^\mu{}_\nu v^\nu$ for vectors, and $\sigma_\nu = \sigma_\mu L^\mu{}_\nu$. For coordinate bases, $L^\mu{}_\nu = \frac{\partial x^\mu}{\partial x^\nu}$ [3], [1]. If the determinant (called the Jacobian) of this transformation matrix equals zero at a point, then the transformation is singular at the point. For noncoordinate (anholonomic) bases, however, partial derivatives do not define its basis vectors. That is, a transformation does not exist from a coordinate basis to such a basis.

Now, let $G^1(M)$ be a vector field, $m \in M$ an arbitrary point in the manifold, and $\mathbf{u}, \mathbf{v} \in G^1(M)$. If a rank-2 tensor field $\mathbf{g}(\mathbf{u}, \mathbf{v})$ is defined on M with the properties that $\mathbf{g}(\mathbf{u}, \mathbf{v}) = \mathbf{g}(\mathbf{v}, \mathbf{u})$ (nondegenerate) and the bilinear form $\mathbf{g}_p$ on the tangent bundle T_pM (the union of all the tangent spaces to M) is positive definite, then M is a Riemannian space [5]. Then, the tensor field $\mathbf{g}$ corresponds to the metric tensor field, and the bilinear form defines a scalar product in the tangent space, transforming the tangent space into a Euclidean space.

For spacetime, we want to work from a manifold M of dimension 4. However, we are not dealing with Euclidean tangent spaces anymore because of the index of the metric tensor field. The index is the number of diagonal components of a bilinear form which are equal to -1. So, metrics in relativity are known as Lorentz metrics. In the language of general relativity,

$$\begin{aligned} g &:= g_{\mu\nu}\omega^{\mu} \otimes \omega^{\nu} \text{ (general basis)} \\ &= g_{\mu\nu}\mathbf{d}x^{\mu} \otimes \mathbf{d}x^{\nu} \text{ (coordinate basis)}, \end{aligned} \tag{2.6}$$

where ω^{μ} is the dual basis of the vector field V (whose basis is $\mathbf{e}_{\mu}$), and $g_{\mu\nu}$ is the metric tensor. $\otimes$ is the tensor product; it produces new tensorial quantities of rank equal to the sum of the ranks of the inputs.

Operating with the metric tensor involves insertion of the appropriate arguments. For a tensor with lowered indices (covariant tensor), $g_{\mu\nu}$, this implies insertion of two vector arguments. The tensor will operate on them as to produce a scalar (inner product). With raised indices (contravariant tensor), $g^{\mu\nu}$, this implies insertion of two one-form arguments. And the mixed tensor, $g^{\mu}{}_{\nu}$, allows for arguments of both types. Essentially, as was stated above, a metric tensor operates to produce a scalar quantity for two arguments, and this means lengths and angles for tensors of rank 1.

Explicitly for the case of basis vectors,

$$g_{\mu\nu} \equiv \mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu) \equiv \mathbf{e}_\mu \cdot \mathbf{e}_\nu. \tag{2.7}$$

Also, the metric tensor is its own inverse, so that $||g^{\mu\nu}|| \equiv ||g_{\mu\nu}||^{-1}$. Also, redefine the Kronecker delta as $g_{\mu\nu}g^{\nu\kappa} = \delta_\mu{}^\kappa$.

This leads to an interesting aspect of general relativity known as Lorentz or general covariance. At infinitesimal scales about some point, the spacetime can be taken to be flat, as in the Minkowski case. This is an expectation from the fact that the smaller the scale, the closer the space should match the tangent space. And so this yields a Lorentz frame. Also, general covariance eliminates a preferred frame. In this respect, it develops complications in reference to Mach's Principle (see later in the paper).

2.2 Curvature and the 4-Manifold

What is curvature? Curvature in general relativity manifests as the distortion of spacetime. But how is curvature described mathematically?

A good place to start is with the concept of free-fall geodesics [1]. Before geodesics, consider free-fall trajectories. By the strong equivalence principle, in all local Lorentz frames in the universe, all nongravitational interactions must behave special relativistically. Less rigorously, this means the worldtube of a freely falling massive body (considering only that it interacts with gravity) is independent of the body's composition and structure (weak equivalence principle). Forming a spacetime with a congruence of these trajectories and using the property of locally Lorentz frames, establish an affine parameter λ as the time for events on the trajectories. This parametrization is unique only to first order since two quantities remain arbitrary: the choice of origin in time, and the units of parametrization. These trajectories are called geodesics since the trajectories from one event to another follow the straightest possible path along a tangent.

The generic definition for a covariant derivative of a tensor field at a point $\mathcal{P}(0)$ along a parametrized curve $\mathcal{P}(\lambda)$:

$$(\nabla_{\mathbf{u}}\mathbf{T})_{\text{at}\mathcal{P}(0)} = \lim_{\epsilon\to 0}\{\frac{\mathbf{T}[\mathcal{P}(\epsilon)]_{\text{parallel-transported to}\mathcal{P}(0)} - \mathbf{T}[\mathcal{P}(0)]}{\epsilon}\}. \tag{2.8}$$

The covariant derivative is a linear operator (see the properties below). Given a general basis, it follows obviously that the covariant derivative generally does not equal the partial derivative. Define the connection coefficients:

$$\Gamma^{\alpha}{}_{\mu\nu} \equiv < \omega^{\alpha}, \nabla_{\mathbf{e}_\nu}\mathbf{e}_\mu > . \tag{2.9}$$

This gives the α component of the change in the basis $\mathbf{e}_\mu$ relative to parallel transport along $\mathbf{e}_\nu$. In terms of their components,

$$\Gamma_{\alpha\beta\mu} = \frac{1}{2}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} + c_{\alpha\beta\mu} + c_{\alpha\mu\beta} - c_{\beta\mu\alpha});$$
$$\Gamma^{\alpha}{}_{\mu\nu} = g^{\alpha\kappa}\Gamma_{\kappa\mu\nu}. \tag{2.10}$$

The c's are the commutation coefficients of a basis. For tangent vector fields $\mathbf{u}$ and $\mathbf{v}$, with the picture of each as directional derivatives $\partial_{\mathbf{u}}$ and $\partial_{\mathbf{v}}$, define the commutator:

$$[\mathbf{u}, \mathbf{v}] \equiv [\partial_{\mathbf{u}}, \partial_{\mathbf{v}}] \equiv \partial_{\mathbf{u}}\partial\mathbf{v} - \partial_{\mathbf{v}}\partial_{\mathbf{u}}. \tag{2.11}$$

For two basis vectors,

$$[\mathbf{e}_\mu, \mathbf{e}_\nu] \equiv c_{\mu\nu}{}^{\kappa}\mathbf{e}_\kappa; c_{\mu\nu\kappa} \equiv c_{\mu\nu}{}^{\lambda}g_{\lambda\kappa}. \tag{2.12}$$

For a coordinate basis, $[\mathbf{u}, \mathbf{v}] = (u^\nu v^\mu{}_{,\nu} - v^\nu u^\mu{}_{,\nu})\mathbf{e}_\mu = (u^\nu v^\mu{}_{,\nu} - v^\nu u^\mu{}_{,\nu})(\frac{\partial}{\partial x^\mu})$.

Consider the case where the tensor field is a vector field. Covariant derivatives describe how quickly a vector field changes along a curve. Specifically, let $\mathbf{v}$ be a vector field, and let the curve have a tangent vector $\mathbf{u} = \frac{d}{d\lambda}$, where λ is the affine parameter along a free-fall geodesic. Then the covariant derivative of $\mathbf{v}$ along $\mathbf{u}$ is

$$\nabla_{\mathbf{u}}\mathbf{v} \equiv \frac{d\mathbf{v}}{d\lambda}. \tag{2.13}$$

This also interprets as the rate of change of $\mathbf{v}$ with respect to the affine parameter.

Covariant derivatives have several properties:

i. *Symmetry* : $\nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} = [\mathbf{u}, \mathbf{v}]$ for any vector fields $\mathbf{u}$ and $\mathbf{v}$;

ii. *Chain Rule* : $\nabla_{\mathbf{u}}(f\mathbf{v}) = f\nabla_{\mathbf{u}}\mathbf{v} + \mathbf{v}\partial_{\mathbf{u}}f$ for any function f, vector field $\mathbf{v}$, and vector $\mathbf{u}$;

iii. *Additivity* : $\nabla_{\mathbf{u}}(\mathbf{v} + \mathbf{w}) = \nabla_{\mathbf{u}}\mathbf{v} + \nabla_{\mathbf{u}}\mathbf{w}$ for any vector fields $\mathbf{v}$ and $\mathbf{w}$, and vector $\mathbf{u}$;

iv. $\nabla_{a\mathbf{u}+b\mathbf{n}}\mathbf{v} = a\nabla_{\mathbf{u}}\mathbf{v} + b\nabla_{\mathbf{n}}\mathbf{v}$ for any vector field $\mathbf{v}$, vectors or vector fields $\mathbf{u}$ and $\mathbf{n}$, and numbers or functions a and b. (2.14)

It is appropriate here to introduce the concept of parallel transport. Define parallelism to be the condition that for two vectors on two tangent spaces, they will be identical. To say that a vector field is parallel transported along a tangent vector, the following equation must hold:

$$\frac{d\mathbf{v}}{d\lambda} \equiv \nabla_{\mathbf{u}}\mathbf{v} = 0. \tag{2.15}$$

The parallel transport of a tangent vector along the curve with it as a tangent vector is equivalent to the property of a curve being a geodesic:

$$\frac{d\mathbf{u}}{d\lambda} = \nabla_{\mathbf{u}}\mathbf{u} = 0. \tag{2.16}$$

This is a second-order equation, since the tangent vector $\mathbf{u} = \frac{\partial}{\partial\lambda}$ is a derivative also.

Consider geodesic deviation. Geodesic deviation occurs when, in a congruence of geodesics, the geodesics separate in some way at common points of the affine parameter λ. The displacement $\mathbf{n} = \frac{\partial}{\partial n}$ from a point on one of these curves to the other one defines geodesic separation. $\mathbf{n}$ determines the geodesic (relatively speaking). Then, obtain the relative acceleration vector:

$$\nabla_{\mathbf{u}} \nabla_{\mathbf{u}} \mathbf{n}. \tag{2.17}$$

This describes the relative acceleration of the separation vector $\mathbf{n}$ along the tangents to the geodesics.

A definition for curvature is imminent now. Begin with a congruence of geodesics, and consider the geodesic equation, $\nabla_{\mathbf{u}} \mathbf{u} = 0$, for each. Taking the limit going to zero of the difference between geodesic equations on neighboring curves, obtain the covariant derivative along the separation vector $\mathbf{n}$:

$$\begin{aligned} \nabla_{\mathbf{n}}[\nabla_{\mathbf{u}} \mathbf{u}] &= (\nabla_{\mathbf{n}} \nabla_{\mathbf{u}} + \nabla_{\mathbf{u}} \nabla_{\mathbf{n}} - \nabla_{\mathbf{u}} \nabla_{\mathbf{n}}) \mathbf{u} \\ &= (\nabla_{\mathbf{n}} \nabla_{\mathbf{u}} + [\nabla_{\mathbf{n}}, \nabla_{\mathbf{u}}]) \mathbf{u} \\ &= 0. \end{aligned} \tag{2.18}$$

In Cartan's notation, $\mathbf{u} = \frac{d\mathcal{P}}{d\lambda}$ with $\mathcal{P} := \mathcal{P}(n, \lambda)$. Using the symmetry of the covariant derivative, write

$$\begin{aligned} \nabla_{\mathbf{n}} \mathbf{u} - \nabla_{\mathbf{u}} \mathbf{n} &= [\mathbf{n}, \mathbf{u}] \\ &= [\frac{\partial}{\partial n}, \frac{\partial}{\partial \lambda}] = \frac{\partial^2}{\partial n \partial \lambda} - \frac{\partial^2}{\partial \lambda \partial n} \\ &= 0 \\ &\Rightarrow \nabla_{\mathbf{n}} \mathbf{u} = \nabla_{\mathbf{u}} \mathbf{n} \\ &\Rightarrow (\nabla_{\mathbf{n}} \nabla_{\mathbf{u}} + [\nabla_{\mathbf{n}}, \nabla_{\mathbf{u}}]) \mathbf{u} \end{aligned}$$

$$= \nabla_{\mathbf{u}}\nabla_{\mathbf{u}}\mathbf{n} + [\nabla_{\mathbf{n}}, \nabla_{\mathbf{u}}]\mathbf{u} = 0. \qquad (2.19)$$

This is the geodesic deviation equation. The first term represents acceleration of a body relative to a fiducial observer on a geodesic. The commutator, $[\nabla_{\mathbf{n}}, \nabla_{\mathbf{u}}]$, gives the tidal gravitational forces - the spacetime curvature. Consider this commutator acting on a tensor field, $\mathbf{Z}$, with the covariant derivatives being along tensor fields $\mathbf{X}$ and $\mathbf{Y}$. Allow for $\mathbf{Z}$ to remain the same at some initial point $\mathcal{P}_0$, but allow for variations of $\mathbf{Z}$ to change at different points via a function (arbitrary except for the condition $f(\mathcal{P}_0) = 1$):

$$\mathbf{Z}_{New}(\mathcal{P}) = f(\mathcal{P})\mathbf{Z}_{Old}(\mathcal{P}). \qquad (2.20)$$

This changes the action of the commutator on $\mathbf{Z}$ since

$$\begin{aligned} [\nabla_{\mathbf{X}}, \nabla_{\mathbf{Y}}]\mathbf{Z}_{New\,\mathrm{at}\mathcal{P}_0} &- [\nabla_{\mathbf{X}}, \nabla_{\mathbf{Y}}]\mathbf{Z}_{Old\mathcal{P}_0} \\ &= [\nabla_{\mathbf{X}}, \nabla_{\mathbf{Y}}]\mathbf{Z}_{Old\mathcal{P}_0} + \mathbf{Z}_{Old}\nabla_{[\mathbf{X},\mathbf{Y}]}f - [\nabla_{\mathbf{X}}, \nabla_{\mathbf{Y}}]\mathbf{Z}_{Old\mathcal{P}_0} \\ &= \mathbf{Z}_{Old}\nabla_{[\mathbf{X},\mathbf{Y}]}f. \qquad (2.21) \end{aligned}$$

The commutator should equate to a linear operator/tensor in equation (2.14). It does not fulfill this requirement, however, since it depends on the variations of the vector fields at the evaluation point. A modification to eliminate this dependence produces a tensor, the curvature operator $\mathcal{R}(\mathbf{X}, \mathbf{Y})$:

$$\begin{aligned} \mathcal{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} &\equiv [\nabla_{\mathbf{X}}, \nabla_{\mathbf{Y}}]\mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} \\ \Rightarrow \mathcal{R}(\mathbf{X}, \mathbf{Y}) &\equiv [\nabla_{\mathbf{X}}, \nabla_{\mathbf{Y}}] - \nabla_{[\mathbf{X},\mathbf{Y}]}. \qquad (2.22) \end{aligned}$$

The object $\mathbf{R}(\sigma, \mathbf{Z}, \mathbf{X}, \mathbf{Y})$ is the Riemann curvature tensor. Written differently in terms of its arguments,

$$\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} \equiv \mathbf{R}(\sigma,\mathbf{Z},\mathbf{X},\mathbf{Y}) = < \sigma,\mathbf{Z},\mathbf{X},\mathbf{Y} > = R^{\alpha}{}_{\kappa\lambda\mu}, \tag{2.23}$$

where σ is a one-form. The last term represents the components of the Riemann tensor. Explicitly, the components take the form

$$\begin{aligned} R^{\alpha}{}_{\kappa\lambda\mu} &= \Gamma^{\alpha}{}_{\kappa\mu,\lambda} - \Gamma^{\alpha}{}_{\kappa\lambda,\mu} + \Gamma^{\alpha}{}_{\tau\lambda}\Gamma^{\tau}{}_{\kappa\mu} - \Gamma^{\alpha}{}_{\tau\mu}\Gamma^{\tau}{}_{\kappa\lambda} - \Gamma^{\alpha}{}_{\kappa\tau}c_{\lambda\mu}{}^{\tau} \\ &= g^{\alpha\sigma}(\Gamma_{\sigma\kappa\mu,\lambda} - \Gamma_{\sigma\kappa\lambda,\mu} + \Gamma_{\sigma\tau\lambda}\Gamma^{\tau}{}_{\kappa\mu} - \Gamma_{\sigma\tau\mu}\Gamma^{\tau}{}_{\kappa\lambda} - \Gamma_{\sigma\kappa\tau}c_{\lambda\mu}{}^{\tau}). \end{aligned} \tag{2.24}$$

Inserting tangent vector fields, the modification to the curvature tensor in Eq. (2.22) equals zero under commutation, so the Riemann tensor retains the form of the original geodesic deviation equation. In terms of components, the geodesic deviation equation appears as follows:

$$\frac{D^2 n^{\alpha}}{d\lambda^2} + R^{\alpha}{}_{\beta\gamma\delta}u^{\beta}n^{\gamma}u^{\delta} = 0. \tag{2.25}$$

The Riemann tensor possesses several different symmetries, both algebraic and differential. First is antisymmetry on its first two and last two indices:

$$R_{\mu\nu\tau\rho} = R_{[\mu\nu][\tau\rho]}. \tag{2.26}$$

Next is symmetry under exchange of the first and last pairs:

$$R_{[\mu\nu][\tau\rho]} = R_{[\tau\rho][\mu\nu]}. \tag{2.27}$$

Parts of the tensor that are totally antisymmetric vanish, so that

$$R_{[\mu\nu\tau\rho]} = 0 \text{ and } R_{\mu[\nu\tau\rho]} = 0. \tag{2.28}$$

In the spacetime of this universe, the Riemann tensor has only 20 independent components out of an initial 256. This aspect simplifies working with the tensor. For its differential symmetries, look at the Bianchi identities:

$$R^{\alpha}{}_{\mu[\gamma\nu;\tau]} = 0. \tag{2.29}$$

These identities, particularly, require the existence of the law of conservation of energy-momentum and its automatic fulfillment (i.e., a field with degrees of freedom to reduce the arbitrary degrees of freedom of a source).

The Ricci tensor occurs by contracting the first and third slots of Riemann. Consider a basis and its dual; the contraction will look like

$$\mathbf{R}(\omega^{\alpha}, \mathbf{a}, \mathbf{e}_{\alpha}, \mathbf{b}) = R(\mathbf{a}, \mathbf{b}). \tag{2.30}$$

In component notation,

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}. \tag{2.31}$$

The Ricci or curvature scalar is the contraction of the indices of a Ricci tensor:

$$R^{\mu}{}_{\mu} = R. \tag{2.32}$$

This scalar curvature term is the cornerstone of the Hilbert action principle, which gives the simplest geometric action in general relativity.

2.3 Calculus of Variations and the Least Action Principle

The development of field theory requires an understanding of variational calculus. Constructing a model involves starting with an action principle, varying the action, and studying variations with respect to different quantities in the action. This method yields multiple equations (equations of motion, equations of constraint, wave equations, etc.) which allow for the determination of the particular system.

More abstractly, an action is a form of functional. A functional is a quantity which is like a function of a function. More accurately, a functional J is such a quantity that it depends on some kind of function (and possibly different derivatives), and both this function and the functional itself depend on the choice of coordinates [8]. Functionals are correspondences assigning definite real quantities to functions/curves belonging to a class.

Many situations in physics require analysis of the simple functional,

$$J[f(x)] = \int_{a_1}^{a_2} F(x, f(x), f'(x), ..., f^{(n)}(x)). \tag{2.33}$$

This representation includes higher-order derivatives. In many cases (such as the Euler-Lagrange equations), only first derivatives occur. Nonetheless, a general approach proves necessary for this work. Consider the space $\mathcal{D}_n(a_1, a_2)$ consisting of all functions $f(x)$ defined on the interval $[a_1, a_2]$ and which are continuous and have continuous derivatives through order n. Let these functions satisfy the following conditions:

$$\begin{aligned} &f(a_1) = A_0, f'(a_1) = A_1, ..., f^{(n-1)}(a_1) = A_{n-1}, \\ &f(a_2) = B_0, f'(a_2) = B_1, ..., f^{(n-1)}(a_2) = B_{n-1}. \end{aligned} \tag{2.34}$$

To determine an extremum for this function, consider the variation of the functional. A necessary condition for the functional to have an extremum at a point is for its variation to vanish for all points and admissible increments $g(x)$ of the independent variable $f(x)$, or

$$\delta J[g(x)] = 0. \tag{2.35}$$

Substitute for $f(x)$ the term $f(x) + g(x)$ which also belongs to the space $\mathcal{D}_n(a_1, a_2)$. Then the variation δJ of the functional is an expression linear in the increment $g(x)$ and all of its derivatives and which differs from the increment $\triangle J = J[f(x)+g(x)] - J[f(x)]$ by an amount of order greater than 1 relative to $g(x)$ and its derivatives. For $f(x)$ and $f(x) + g(x)$ to satisfy the boundary conditions (Eq. (2.29)),

$$\begin{aligned} g(a_1) = g'(a_1) = ... = g^{(n-1)}(a_1) = 0, \\ g(a_2) = g'(a_2) = ... = g^{(n-1)}(a_2) = 0. \end{aligned} \tag{2.36}$$

Apply Taylor's theorem to obtain the increment $\triangle J$:

$$\begin{aligned} \triangle J &= \int_a^b [F(x, f+g, f'+g', ..., f^{(n)} + g^{(n)}) - F(x, f, f', ..., f^{(n)})]dx && (2.37) \\ &= \int_a^b (F_f g + F_{f'} g' + ... + F_{f^{(n)}} g^{(n)})dx + ..., && (2.38) \end{aligned}$$

where $F_{f^{(n)}} = \frac{\partial F}{\partial f^{(n)}}$, $f^{(n)} = \frac{d^{(n)} f}{dx^{(n)}}$, and $g^{(n)} = \frac{d^{(n)} g}{dx^{(n)}}$. These functionals are linear; the variation of the functional equates only with the principal linear part of the increment (as specified above), so

$$\delta J = \int_a^b (F_f g + F_{f'} g' + ... + F_{f^{(n)}} g^{(n)})dx, \tag{2.39}$$

and for the extremum $\delta J = 0$,

$$\int_a^b (F_f g + F_{f'} g' + ... + F_{f^{(n)}} g^{(n)})dx = 0. \tag{2.40}$$

To obtain a representation with no derivatives of $g(x)$ present, use integration by parts repeatedly and eliminate those derivatives by the boundary conditions given in Eq. (2.31):

$$\int_a^b \left(\frac{\partial F}{\partial f(x)} - \frac{d}{dx}\frac{\partial F}{\partial f'(x)} + ... + (-1)^n \frac{d^n}{dx^n}\frac{\partial F}{\partial f^{(n)}}\right) g(x)dx \tag{2.41}$$

$$\Rightarrow \frac{\partial F}{\partial f(x)} - \frac{d}{dx}\frac{\partial F}{\partial f'(x)} + ... + (-1)^n \frac{d^n}{dx^n}\frac{\partial F}{\partial f^{(n)}} = 0. \tag{2.42}$$

This is the more generic form of Euler's equations. In many problems, only derivatives of the functionals with respect to the first-derivatives appear; these are the Euler-Lagrange equations.

Another important topic is Lagrange multipliers [8]. Lagrange multipliers prove useful in dealing with side conditions or constraints. Constraints come in two packages: those equal to the number (dimension) of functions, and those less than the number of functions. Restated, the first condition (known as an isoperimetric problem) says that given a functional, boundary conditions for the function, and the condition that the original functional has an extremum for that function, another such functional exists that takes a fixed value where the function is not an extremum. Mathematically,

$$J[f(x)] = \int_{a_1}^{a_2} F(x, f(x), f'(x))dx; \tag{2.43}$$

$$K[f(x)] = \int_{a_1}^{a_2} G(x, f(x), f'(x))dx = s; \tag{2.44}$$

$$f(a_1) = A, \qquad f(a_2) = B, \tag{2.45}$$

where J and K are functionals and Eq. (2.43) constitutes the boundary conditions. $f(x)$ is not an extremum for $K[f(x)]$; however, a constant λ exists such that $f(x)$ is an extremum for the functional

$$\begin{aligned} &\int_{a_1}^{a_2}(F+\lambda G)dx \\ \Rightarrow \quad &\frac{\partial F}{\partial f(x)}-\frac{d}{dx}\frac{\partial F}{\partial f'(x)}+\lambda(\frac{\partial G}{\partial f(x)}-\frac{d}{dx}\frac{\partial G}{\partial f'(x)})=0. \end{aligned} \tag{2.46}$$

This generalizes to the situation with n functions:

$$J[f_1(x),...,f_n(x)]=\int_{a_1}^{a_2}F(x,f_1(x),...,f_n(x);f_1'(x),...,f_n'(x))dx;$$

$$\begin{aligned} f_i(a_1)=A_i, f_i(a_2)=B_i, \qquad &(i=1,...,n); \\ \int_{a_1}^{a_2}G_j(x,f_1(x),...f_n(x);f_1'(x),...,f_n'(x))dx \quad &= \quad s_j \; (j=1,...,k); \\ \Rightarrow \frac{\partial}{\partial f_i(x)}(F+\sum_{j=1}^{k}\lambda_j G_j)-\frac{d}{dx}\{\frac{\partial}{\partial f_i'(x)}(F+\sum_{j=1}^{k}\lambda_j G_j)\} \quad &= \quad 0. \end{aligned} \tag{2.47}$$

For the case of n derivatives in the functionals:

$$\frac{\partial}{\partial f_i(x)}(F+\sum_{j=1}^{k}\lambda_j G_j)+...+(-1)^n\frac{d^n}{d^n x}\{\frac{\partial}{\partial f_i^n(x)}(F+\sum_{j=1}^{k}\lambda_j G_j)\}=0. \tag{2.48}$$

The case of finite subsidiary conditions concerns the situation with fewer constraint equations than degrees of freedom/number of functions. Recognize the same kind of boundary conditions ($f_i(a_1)=A_i$, $f_i(a_2)=B_i$; $i=1,...,n$) for the functions contained in $J[f(x)]$. Then, the subsidiary conditions satisfy the relation [9]

$$G_j(x,f_1(x),...,f_n(x))=0,(j=1,...,k). \tag{2.49}$$

To consider the case for a general number of functions, establish boundary conditions for the functions. Consider a functional only for the class of curves existing in the $(n-k)$-dimensional manifold satisfying the boundary conditions. Let the functional have an extremum for the curve formed from these functions. Carry through k partial derivatives of the quantity $H := (F + \lambda G)$ to obtain a system of differential equations. Solve this system to obtain $\lambda(x)$:

$$\frac{\partial}{\partial f_i(x)}(F + \sum_{j=1}^{k} \lambda_j G_j) + ... + (-1)^n \frac{d^n}{d^n x}\{\frac{\partial}{\partial f_i^n(x)}(F + \sum_{j=1}^{k} \lambda_j G_j)\} = 0,$$
$$(i = 1, ..., n), (j = 1, ..., k), k < n. \qquad (2.50)$$

In principle, this formalism extends to the situation of dependence on multiple variables in $\mathbb{R}^d$, but most situations considered here (such as the thermodynamic quantities in Appendix C) do not require this development.

An important example of a functional, the action, occurs in field theory. The action describes the energetic properties of a system; it contains both the dynamics and the constraints. The most general form of the action takes the form

$$S = \int d^n x \, \mathcal{L}, \qquad (2.51)$$

where $\mathcal{L}$ is the Lagrangian density. The Lagrangian describes the energy in the system and is written in terms of the kinetic and potential terms. Consider $L := L(\phi_m, \nabla\phi_m)$, where $\phi_m := \phi_m(t, x_1, ..., x_n)$ is a set of functions. Denote the ϕ_m's as field functions. Commonly in physics, $t \equiv x_0$. Many Lagrangians depend only on such functions and their first derivatives, so the action takes the form

$$\begin{aligned} S[\phi, \nabla\phi] &= \int_{t_1}^{t_2} dx_0 \int ... \int_{(n-1)-Volume} \mathcal{L}(\phi, \nabla\phi) dx_1 ... dx_n \\ &= \int_{n-Volume} \mathcal{L}(\phi, \nabla\phi) d^n x. \end{aligned} \qquad (2.52)$$

This reduces to the most general action in 4 dimensions,

$$S[\phi, \nabla\phi] = \int \mathcal{L}(\phi, \nabla\phi) d^4 x = \int L(\phi, \nabla\phi) \, dt. \qquad (2.53)$$

The least-action principle ($\delta S = 0$) yields Euler-Lagrange equations. Note that some actions require higher-order derivatives; this follows from the above discussion. However, application of Taylor's theorem and repeated integration by parts handle the situation.

2.4 Hilbert Action Principle and Einstein's Equations

The Hilbert action principle yields appropriate initial-value data [1] for solving Einstein's equations. Written most generically,

$$S = \int d^4x[L\sqrt{-g}]. \tag{2.54}$$

$\sqrt{-g}d^4x$ together is the proper 4-volume. L is the Lagrangian. d^4x is the coordinate volume, and so $\sqrt{-g}$ densitizes the Lagrangian. When sources exist (as opposed to the existence of only vacuum), the Lagrangian splits into a geometric term and a source term. Hilbert wrote the first term, L_{Geom}, in terms of the scalar 4-curvature R:

$$L = L_{Geom} + L_{Source} = \frac{1}{16\pi}R + L_{Source}. \tag{2.55}$$

L_{Geom} depends only on the metric tensor and its derivatives. L_{Source} depends on the metric tensor as well as different source terms (commonly scalar fields but also vector and tensor fields in some field theories). R is the scalar curvature introduced before. Rewrite it as

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{2.56}$$

Vary the action; this will give $\delta S = 0$ by the principle of least action. Take this variation with respect to the contravariant metric tensor; after some computations and manipulations (see Appendix A), obtain the following:

$$\begin{aligned} \delta S &= \frac{1}{16\pi}\int d^4x[(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \\ &+ 16\pi(\frac{\delta L_{Source}}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}L_{Source}))\delta g^{\mu\nu}]\sqrt{-g} = 0. \end{aligned} \tag{2.57}$$

Setting the coefficient of the variation of the metric tensor to zero, obtain Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 16\pi(\frac{\delta L_{Source}}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}L_{Source}). \tag{2.58}$$

Introduce the definition of the energy-momentum tensor, $T_{\mu\nu}$:

$$T_{\mu\nu} = -2(\frac{\delta L_{Source}}{\delta g^{\mu\nu}} + g_{\mu\nu}L_{Source}). \tag{2.59}$$

Letting the left-hand side equate with $G_{\mu\nu}$, the Einstein tensor, obtain

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{2.60}$$

Note that this form leaves out what is known as Einstein's greatest blunder, $\Lambda g_{\mu\nu}$, where Λ is the cosmological constant. However, upon inspection,

$$\begin{aligned} G_{\mu\nu} + \Lambda g_{\mu\nu} &= 8\pi T_{\mu\nu} \\ \Rightarrow G_{\mu\nu} &= 8\pi T_{\mu\nu} - \Lambda g_{\mu\nu} \\ \Rightarrow G_{\mu\nu} &= 8\pi T'_{\mu\nu}. \end{aligned} \tag{2.61}$$

In principle (with similar consequences to the scalar field employed later in this work), this cosmological constant term can be absorbed into an energy-momentum tensor as a source. Historically, the Einstein tensor was proposed because it is divergenceless - a necessary requirement for conservation of energy-momentum since that tensor is on the other side. The covariant derivative of the metric tensor equals zero (Appendix A), so the scalar cosmological constant term Λ as written in the Einstein tensor only affects the divergences if it possesses dependences that cause its gradient to be nonzero. For classical sources, this term usually vanishes, though for anything else (e.g., minimally coupled scalar fields) it need not disappear.

2.4.1 Conditions for the Einstein Tensor

The Einstein tensor is required to be divergenceless. This follows from computations concerning the Riemann tensor, and it is of critical importance for the conservation of energy-momentum. Recall the Bianchi identities for the Riemann tensor [3]:

$$R_{\alpha\mu[\gamma\nu;\tau]} = R_{\alpha\mu\gamma\nu;\tau} + R_{\alpha\mu\tau\gamma;\nu} + R_{\alpha\mu\nu\tau;\gamma} = 0. \tag{2.62}$$

Contract these similarly to obtaining the Ricci tensor:

$$g^{\gamma\alpha}\{R_{\alpha\mu\gamma\nu;\tau} + R_{\alpha\mu\tau\gamma;\nu} + R_{\alpha\mu\nu\tau;\gamma}\} = R_{\mu\nu;\tau} - R_{\mu\tau;\nu} + R^{\gamma}{}_{\mu\nu\tau;\gamma}. \tag{2.63}$$

Contract again:

$$g^{\mu\nu}\{R_{\mu\nu;\tau} - R_{\mu\tau;\nu} + R^{\gamma}{}_{\mu\nu\tau;\gamma}\} = R_{;\tau} - R^{\nu}{}_{\tau;\nu} - R^{\gamma}{}_{\tau;\gamma} = 0. \tag{2.64}$$

By the rules of Einstein summation, $\gamma \to \nu$ without any difficulties, so that

$$R_{;\tau} - R^{\nu}{}_{\tau;\nu} - R^{\nu}{}_{\tau;\nu} = R_{;\tau} - 2R^{\nu}{}_{\tau;\nu} = 0. \tag{2.65}$$

Rewrite this equation so that the covariant derivative is over ν:

$$R_{;\tau} - 2R^{\nu}{}_{\tau;\nu} \quad = \quad g^{\nu}{}_{\tau}R_{;\nu} - 2R^{\nu}{}_{\tau;\nu} = \{g^{\nu}{}_{\tau}R - 2R^{\nu}{}_{\tau}\}_{;\nu} = 0$$

$$
\begin{aligned}
\Rightarrow \qquad & \{2R^{\nu}{}_{\tau} - g^{\nu}{}_{\tau}R\}_{;\nu} = 0 \\
\Rightarrow \qquad & 2\{R^{\nu}{}_{\tau} - \frac{1}{2}g^{\nu}{}_{\tau}R\}_{;\nu} = 0 \\
\Rightarrow \qquad & \{R^{\nu}{}_{\tau} - \frac{1}{2}g^{\nu}{}_{\tau}R\}_{;\nu} = 0.
\end{aligned} \tag{2.66}
$$

The divergence of the terms inside equals zero. Rewrite these terms using their symmetry properties:

$$
\begin{aligned}
& R^{\nu\tau} - \frac{1}{2}g^{\nu\tau}R = G^{\nu\tau} \\
\Rightarrow \quad & \{R^{\nu\tau} - \frac{1}{2}g^{\nu\tau}R\}_{;\nu} = G^{\nu\tau}{}_{;\nu} = 0.
\end{aligned} \tag{2.67}
$$

Since the Ricci tensor and the metric tensor are symmetric, the term under the divergence is symmetric. This term is, as seen above, the Einstein tensor. Then,

$$
G^{\nu\tau}{}_{;\nu} = G^{\tau\nu}{}_{;\nu} = 0. \tag{2.68}
$$

So, the Einstein tensor must be divergenceless.

Consider the Riemann tensor. The Einstein tensor can be related to the double-dual of the Riemann tensor (this is notation for relating antisymmetric tensors of different ranks). The double-dual of the Riemann tensor has components [1]

$$
G^{\mu\nu}{}_{\kappa\lambda} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}R_{\rho\sigma}{}^{\theta\tau}\frac{1}{2}\epsilon_{\theta\tau\kappa\lambda} = -\frac{1}{4}\delta_{\theta\tau\kappa\lambda}{}^{\mu\nu\rho\sigma}R_{\rho\sigma}{}^{\theta\tau}. \tag{2.69}
$$

Contract over the first and third indices of this tensor to get the Einstein tensor:

$$
G^{\phi\alpha}{}_{\phi\gamma} \equiv G^{\alpha}{}_{\gamma}. \tag{2.70}
$$

For this work, some of the components are particularly useful [1]:

$$
\begin{aligned}
G^0{}_0 &= -(R^{12}{}_{12} + R^{23}{}_{23} + R^{31}{}_{31}); \\
G^0{}_1 &= R^{02}{}_{12} + R^{03}{}_{13}.
\end{aligned} \tag{2.71}
$$

2.4.2 Conditions for the Source: Classical and Non-Classical

A complete treatment of Einstein's equations necessitates discussion of sources for the energy-momentum tensor [10]. A classical source is, roughly, a pre-existing, constant source of matter, energy, or both. Perfect fluids, dust, and vacuum satisfy this. Several pointwise energy conditions exist with varying degrees of rigidity: the strong energy condition, the weak energy condition, the dominant energy condition, the null energy condition, and the trace energy condition. Particularly, these come from the positivity property of a term in Raychauduri's equation [2]. Finally, the conservation property must hold.

Start with the trace of the Einstein tensor [2], [11]:

$$
\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \\
\Rightarrow g^{\mu\nu}G_{\mu\nu} &= g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = 8\pi g^{\mu\nu}T_{\mu\nu} \\
&= R - 4(\frac{1}{2}R) = -R = 8\pi g^{\mu\nu}T_{\mu\nu} \\
\Rightarrow R &= -8\pi T.
\end{aligned} \tag{2.72}
$$

Inserting this into Einstein's equations,

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= R_{\mu\nu} + \frac{1}{2}(8\pi T)g_{\mu\nu} = 8\pi T_{\mu\nu} \\
\Rightarrow R_{\mu\nu} &= 8\pi(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}).
\end{aligned} \tag{2.73}
$$

Consider a unit timelike vector u^μ. Then, $\rho \equiv T_{\mu\nu}u^\mu u^\nu$ is the energy density of the matter distribution measured by an observer in a comoving reference frame. For classical matter, the weak energy condition requires that the local energy density to an observer on a worldline with a tangent vector equal to the unit normal at a point on the worldline to be greater than or equal to zero:

$$\begin{aligned} T_{\mu\nu}u^\mu u^\nu &\geq 0; \\ \rho \geq 0;\ \rho + p_i &\geq 0. \end{aligned} \tag{2.74}$$

The components of a physically realizeable classical energy-momentum tensor diagonalize when expressed with respect to an orthonormal basis [2], [10]. Then, treatment of this problem yields four types of energy-momentum tensors. One is the standard for a perfect fluid. Pressures in the spacelike directions and energy density at a point are measured. Another case is for radiation traveling in one direction in space and in time. Two cases are not observed classically: the case of negative pressures being individually of greater magnitude than the energy density, and the case of vacuum generating pressure. The values ρ and p_i are the energy density and the principal pressures of the matter.

The strong energy condition places restrictions on the value of quantities in the energy-momentum tensor, putting a lower limit on the values of matter stresses [2], [12] :

$$\begin{aligned} T_{\mu\nu}u^\mu u^\nu \geq -\frac{1}{2}T; \\ \rho \geq 0; \\ \rho + \sum_{i=1}^{3} p_i \geq 0; \\ \rho + p_i \geq 0. \end{aligned} \tag{2.75}$$

This requires that negative principal pressures, or tensions, be lesser in magnitude than the energy density.

Consider the dominant energy condition. $-T^{\mu}{}_{\nu}u^{\nu}$ represents the energy-momentum 4-current density; this quantity should always be a future directed timelike or null vector, which is equivalent to saying that the speed of energy flux of matter $< c$. As previously, $T^{\mu\nu}u^{\mu}u^{\nu} \geq 0$. This means that, locally, the energy density is non-negative and that the local energy flux vector is not in a spacelike direction. Expanding the condition, it is equivalent that $\rho \geq |p_i|$.

The other energy conditions follow essentially from these three. The null energy condition already occurs in the others:

$$\rho + p_i \geq 0. \tag{2.76}$$

Finally is the trace energy condition [12]:

$$\begin{aligned} T^{\mu}{}_{\mu} &\leq 0; \\ \rho - \sum_{i=1}^{3} p_i &\geq 0. \end{aligned} \tag{2.77}$$

The regime for the weak, strong, null, and dominant energy conditions includes only purely classical sources of matter, including classical scalar field such as in Brans-Dicke theory [10], [12] . For the trace energy condition, the equations of state cannot be too rigid. Since this is not agreeable to neutron stars, however, this energy condition is not used.

Non-classical sources do not satisfy these pointwise energy conditions. Scalar fields of non-zero rest mass and quantum effects especially defy these conditions. In particular, the C-field of this work possesses a negative energy density when generating particles and expanding space.

The energy-momentum tensor must satisfy the requirement of global conservation.

That is,

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{2.78}$$

This follows simply from the Bianchi identities:

$$\begin{aligned} G^{\mu\nu} &= 8\pi T^{\mu\nu}; \\ G^{\mu\nu}{}_{;\nu} = 0 \Rightarrow G^{\mu\nu}{}_{;\nu} &= 8\pi T^{\mu\nu}{}_{;\nu} = 0 \\ \Rightarrow T^{\mu\nu}{}_{;\nu} &= 0. \end{aligned} \tag{2.79}$$

This work addresses the situation where this is violated, and a prescription is developed to compensate for such a violation.

Chapter 3

Hamiltonian Formulation: The 3 + 1 Split

3.1 Arriving at the Split

GR possesses a unique difficulty that, at any time, the metric modifies via a coordinate transformation [13]. Determining the time evolution of this field as a dynamical quantity proves challenging. Coordinate invariance leaves the physics alone, so an approach separating the true dynamical quantities (the number of independent Cauchy data) from the quantities concerning the coordinate system resolves this difficulty. Determination of the independent dynamical modes of the gravitational field arrives via the canonical form, which involves the minimal quantity of variables specifying the system's state. The canonical form requires that field equations be of first order in the time derivatives and that time is separated from the spatial quantities. In GR, linear time derivatives comprise the Palatini Lagrangian, making this form advantageous. General covariance leads to analogy with Hamiltonian mechanics, which parametrizes in a way where the conjugate pair, the Hamiltonian and the time, allows for another degree of freedom. GR's invariance under arbitrary coordinate transformations makes it already parametrized.

Recall the action principle for a system of a finite number of degrees of freedom, and introduce parametrization:

$$S = \int_{t_1}^{t_2} dt L = \int_{t_1}^{t_2} dt (\sum_{i=1}^{N} p_i \dot{q_i} - H(p,q)). \tag{3.1}$$

Define the generating function as

$$G(t) = \sum_i p_i \delta q_i - H \delta t, \tag{3.2}$$

where $\delta q_i = \delta_0 q_i + \dot{q_i} \delta t$. The first term, $\delta_0 q_i$, represents the independent variation of q_i. This generating function separates translations in spacetime to translations in space and translations in time. This follows from the hypothesis that the total variation of the action depends functionally only on the endpoints; the hypothesis also yields the standard Hamilton's equations for the p's and q's and the conservation of energy.

Consider an arbitrary parametrization, λ, so that the action takes the form

$$S = \int_{\lambda_1}^{\lambda_2} d\lambda L_\lambda \equiv \int_{\lambda_1}^{\lambda_2} d\lambda [\sum_{i=1}^{N+1} p_i q_i'], \tag{3.3}$$

where $q' \equiv \frac{dq}{d\lambda}$. This representation yields a constraint equation,

$$p_{N+1} + H(p,q) = 0. \tag{3.4}$$

Using the method of constraint multipliers, introduce another term in the action such that

$$S = \int_{\lambda_1}^{\lambda_2} d\lambda [\sum_{i=1}^{N+1} p_i q_i' - \alpha(\lambda) R]. \tag{3.5}$$

Varying the action yields the constraint equation $R(p_{N+1}, p, q) = 0$, which implies a solution that $p_{N+1} = -H$. Note that canonical form collapses because of the constraint multiplier α in the Hamiltonian; general covariance strikes the Hamiltonian.

Returning to canonical form requires manipulation via implicit differentiation and variational calculus. Insert the solution of the constraint equation:

$$
\begin{aligned}
S &= \int d\lambda[\sum_{i=1}^{N} p_i \frac{dq_i}{d\lambda} - H(p,q)\frac{dq_{N+1}}{d\lambda}] \\
&= \int dq_{N+1}\frac{\partial\lambda}{\partial q_{N+1}}[\sum_{i=1}^{N} p_i \frac{dq_i}{dq_{N+1}} - H\frac{dq_{N+1}}{d\lambda}] \\
&= \int dq_{N+1}[\sum_{i=1}^{N} p_i - H]. \qquad (3.6)
\end{aligned}
$$

This formulation allows for q_{N+1} to be an independent coordinate such that no dependence exists on an arbitrary parametrization external to the system. Practically, imposing a coordinate condition allows for elimination of parametrization. Writing the generator with this logic, obtain

$$
G = \sum_{i=1}^{N} p_i \delta q_i - H\delta q_{N+1}. \qquad (3.7)
$$

Consider parametrized field theories, where the generalized coordinates appear as four new field variables $q^{N+\kappa} = x^{\kappa}(\lambda^{\mu})$ (which in turn have four conjugate momenta $p_{N+\kappa}(\lambda^{\mu})$). Four constraint equations relate the conjugate momenta to the Hamiltonian density and the momentum density. This implies four Lagrange multipliers $\alpha_{\kappa}(\lambda^{\mu})$ for the field. The Hamiltonian is the generator of time-translation. It is invariant with respect to the lapse (true time) and the shift; this makes it invariant with respect to time-translation. This formalism extends to the metric field.

3.2 Splitting Geometry and the Metric

The formalism in Chapter 2 describes the 4-manifold as an object in which other objects of lesser or equal dimension exist. With the proper definition of curvature and introduction of the action principle and variational calculus, this machinery leads to Einstein's equations. This set of second-order equations, however, proves challenging to solve. The motivation for splitting spacetime into a foliation of spacelike hypersurfaces Σ parametrized by t, a global time function, is to yield two sets of first-order equations along with two sets of constraints. First-order equations are manifestly easier to deal with, and the constraint equations allow for specifying initial conditions, which leads to the boundary value problem.

3-geometry is fixed on two faces of a sandwich, which is a representation of two adjacent hypersurfaces. To construct a sandwich, the metrics of the 3-geometries of the lower and upper hypersurfaces are needed. Starting from the 4-geometry, the metric must be split. The formula for the proper length of a line connecting a point on the lower hypersurface to a point on the upper is required. Finally, the formula describing this point on the upper hypersurface is needed. In the language of Chapter 2, define the spacetime as $(M, \mathbf{g})$ where M is the 4-manifold and $\mathbf{g}$ is the metric. Let $\mu : M \rightarrow \Sigma$ and $t : M \rightarrow \mathbb{R}$. The mapping for time is classical; hence why it maps to the real numbers. Topologically, the 4-manifold may be written as the Cartesian product of the family of hypersurfaces and the real line, $\Sigma \times \mathbb{R}$.

Let $\mathbf{n}$ be the normal to the hypersurfaces. Introduce the spatial metric induced on Σ by $\mathbf{g}$, $\gamma_{\mu\nu}$:

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{3.8}$$

Also, let N be the lapse function and N^i be the shift vector. Define a timelike vector field $\mathbf{t}$ on M, and write its components:

$$t^\mu = N n^\mu + N^\mu, \tag{3.9}$$

where N is the lapse and N^μ is the shift. Consider normal and tangential components of $\mathbf{t}$; the timelike vector field satisfies the following relation:

$$t^\mu t_{;\mu} = 1. \tag{3.10}$$

In terms of this vector field, formalize the lapse and the shift:

$$
\begin{aligned}
N &= -t^{\mu}n_{\mu} = -t^{\mu}g_{\mu\nu}n^{\nu} \\
&= [n^{\mu}t_{;\mu}]^{-1}; && (3.11) \\
N_{\mu} &= \gamma_{\mu\nu}t^{\nu}. && (3.12)
\end{aligned}
$$

These relations follow since **n** is normal to the shift,

Consider a coordinate system $x^{\mu} = (t,\ x^{k})$ for Σ. On the adjacent hypersurface, where the parameter changes by dt, the coordinates take the form $x^{\mu} + dx^{\mu} = (t + dt, x^{k} + dx^{k})$. Finding the proper interval between the adjacent hypersurfaces requires finding the proper distance in the lower hypersurface and the proper time between the lower and upper hypersurfaces. Shift gives the proper distance, and lapse gives the proper time. To describe a point in the upper hypersurface in terms of the lower one, write

$$
x^{k}{}_{Upper}(x^{s}) = x^{k} - N^{k}(t, x^{k})dt, \tag{3.13}
$$

and to describe the proper time between them, write

$$
d\tau = N(t, x^{k})dt. \tag{3.14}
$$

(Note that $d\tau^{2} \equiv ds^{2}$.) Write the line element:

$$
\begin{aligned}
ds^{2} &= g_{\mu\nu}dx^{\mu}dx^{\nu} \\
&= g_{00}dx^{0}dx^{0} + g_{i0}dx^{i}dx^{0} + g_{0j}dx^{0}dx^{j} + g_{ij}dx^{i}dx^{j} \\
&= (N_{k}N^{k} - N^{2})dt^{2} + N_{i}dx^{i}dt + N_{j}dtdx^{j} + g_{ij}dx^{i}dx^{j} \\
&= (N_{k}N^{k} - N^{2})dt^{2} + g_{ij}(N^{j}dt + dx^{j})dx^{i} + g_{ij}N^{i}dtdx^{j} \\
&= -N^{2}dt^{2} + g_{ij}(N^{i}N^{j}dt^{2} + N^{j}dtdx^{i} + N^{i}dtdx^{j} + dx^{i}dx^{j})
\end{aligned}
$$

$$= \quad -(N\,dt)^2 + g_{ij}(dx^i + N^i\,dt)(dx^j + N^j\,dt), \tag{3.15}$$

where ${}^{(4)}g_{00} = (N_k N^k - N^2)$, ${}^{(4)}g_{i0} = N_i$, ${}^{(4)}g_{0j} = N_j$, and ${}^{(4)}g_{ij} =^{(3)} g_{ij}$, with all quantities on the lefthand side corresponding to the 4-metric and quantities on the right corresponding to the 3-metric. The critical difference between the spatial metric and the 3-metric is that the spatial metric is the metric of the hypersurface expressed in the spacetime coordinate basis, and the 3-metric is the metric of the 3-geometry (it is the term g_{ij}).

The contravariant components of the 4-geometry in this new representation follow in the standard way. This is especially important for the shift, since the object started as a vector. Using an inverse 3-metric, obtain

$$\begin{aligned}
{}^{(4)}g^{00} &= -\frac{1}{N^2};\\
{}^{(4)}g^{i0} &= \frac{N^i}{N^2};\\
{}^{(4)}g^{0j} &= \frac{N^j}{N^2};\\
{}^{(4)}g^{ij} &= \frac{g^{ij} - N^i N^j}{N^2}.
\end{aligned} \tag{3.16}$$

Finally, consider an important property of the normal to the hypersurface. Contracted with its dual, $<\mathbf{n},\mathbf{n}>= -1$, or

$$\begin{aligned}
<\mathbf{n},\mathbf{n}> &= n^\alpha \mathbf{e}_\alpha n_\alpha \mathbf{d}x^\alpha\\
&= (\frac{1}{N}\mathbf{e}_0)(-N\mathbf{d}t) + (-\frac{N^k}{N}\mathbf{e}_k)(0\mathbf{d}x^k)\\
&= -1.
\end{aligned} \tag{3.17}$$

In component language, $n_\alpha = (-N, 0, 0, 0)$, and $n^\alpha = (\frac{1}{N}, -\frac{N^k}{N})$. With this information, build the split curvature formalism.

3.3 Curvature Revisited

Curvature takes a new form in 3 + 1 decomposition. Use of coordinate bases eases calculations; the computations presented in this work take full advantage.

In this formalism, the spatial scalar curvature invariant ${}^{(3)}R$ serves as the best measure of intrinsic curvature. Considering a basis, take an infinitesimal displacement in the hypersurface,

$$\mathbf{d}\mathcal{P} = \mathbf{e}_i \mathbf{d}x^i. \tag{3.18}$$

$\mathbf{d}x^i$ is the dual basis of $\mathbf{e}_i$, and $\mathcal{P}$ is the initial reference point on the hypersurface. Let a field of tangent vectors lie in the hypersurface; they will have the same basis. Then,

$$\mathbf{V} = \mathbf{e}_i V^i, \tag{3.19}$$

and the scalar product with the base vector $\mathbf{e}_j$ is

$$(\mathbf{V} \cdot \mathbf{e}_j) = V^i(\mathbf{e}_i \cdot \mathbf{e}_j) = V^i g_{ij} = V_j, \tag{3.20}$$

where these are the covariant components of the tangent vector field.

From previously established machinery, consider parallel transport of one of these vectors. The motivation is to obtain this parallel transport within the hypersurface; from the machinery, the covariant derivative must also reside intrinsically in the 3-geometry. So,

$$\nabla_{\mathbf{e_i}} \mathbf{V} = \nabla_{\mathbf{e_i}}(\mathbf{e}_j V^j) = \mathbf{e}_j V^j{}_{,i} + (\Gamma^{\kappa}{}_{ji}\mathbf{e}_{\kappa})V^j. \tag{3.21}$$

The covariant derivative must project onto the hypersurface; this requires refinement of the previous expression to eliminate those components of connection with time. Taking the unit timelike normal $\mathbf{n}$, these components take the form

$$(V^j \Gamma^0{}_{ji})(\mathbf{e}_0 \cdot \mathbf{n}). \tag{3.22}$$

To represent the components of the covariant derivative taken with respect to the 3-geometry, write

$$V_{m|i} = \mathbf{e}_m \cdot^{(3)} \nabla_{\mathbf{e_i}} \mathbf{V} = V_{m,i} - V^k \Gamma_{kmi}, \tag{3.23}$$

where ${}^{(3)}\nabla$ here is the covariant derivative with respect to the 3-geometry (not the 4-geometry as previously). These are the covariant derivative components in Σ. In this context, then the Riemann curvature tensor may be written down for the 3-geometry and in turn the scalar curvature.

Extrinsic curvature proves an interesting and fundamentally important concept for 3 + 1 decomposition. It requires that the hypersurfaces in question be imbedded in an enveloping 4-geometry (in general, a higher dimensional space). A representation of the extrinsic curvature is that of a linear operator; for an infinitesimal displacement on the hypersurface, the infinitesimal change in the timelike normal vector represents, approximately, as

$$\mathbf{dn} = -\mathbf{K}(\mathbf{d}\mathcal{P}), \tag{3.24}$$

for some reference point on the hypersurface. The sign of the extrinsic curvature operator is positive for the case where the curvature of the hypersurface is convex and negative for a concave hypersurface.

In a coordinate representation, let $\mathbf{K}$ act on a tangent vector equivalent to the basis vector $\mathbf{e}_i$. This gives local displacement in the ith coordinate direction:

$$\begin{aligned} \nabla_i \mathbf{n} = -\mathbf{K}(\mathbf{e}_i) &= -K_i{}^j \mathbf{e}_j \\ \Rightarrow -K_i{}^j \mathbf{e}_j \cdot \mathbf{e}_k &= -K_i{}^j g_{jk} = -K_{ik}; \end{aligned}$$

$$
\begin{aligned}
\mathbf{e}_k \cdot \nabla_i \mathbf{n} = \nabla_i(\mathbf{n}\cdot\mathbf{e}_k) \quad &- \quad \mathbf{n}\cdot\nabla_i \mathbf{e}_k \\
= -\mathbf{n}\cdot\nabla_i\mathbf{e}_k = -(\mathbf{n}\cdot\mathbf{e}_\mu){\Gamma^\mu}_{ki} \quad &= \quad -(\mathbf{n}\cdot\mathbf{e}_0){\Gamma^0}_{ki} \text{ (by orthogonality)} \\
= -\mathbf{n}\cdot\nabla_k\mathbf{e}_i \quad &= \quad -K_{ki} \\
\Rightarrow K_{ik} \quad &= \quad K_{ki} \text{ (Symmetry)}
\end{aligned} \tag{3.25}
$$

Parallel transport extends in definition to transport parallel with respect to the enveloping spacetime geometry.

For this work, the representation in ADM notation is noteworthy. From the previous section, use the covariant form of the lapse and shift; the normal components look like

$$
(n_0, n_1, n_2, n_3) = (-N, 0, 0, 0). \tag{3.26}
$$

Obtaining the components of the change in the normal with regard to the parallel transport of the normal to itself,

$$
\begin{aligned}
(\mathbf{dn})_i \quad &= \quad n_{i;m}\mathbf{d}x^m \\
&= \quad [n_{i,m} - {\Gamma^\kappa}_{im} n_\kappa]\mathbf{d}x^m \\
&= \quad N{\Gamma^0}_{im}\mathbf{d}x^m \\
&\Leftrightarrow \quad -K_{im}\mathbf{d}x^m \\
&\Rightarrow \quad K_{im} = -n_{i;m} \\
= -N{\Gamma^0}_{im} \quad &= \quad -N[g^{0\sigma}\Gamma_{\sigma im}] \\
&= \quad -N[g^{00}\Gamma_{0im} + g^{0s}\Gamma_{sim}] \\
&= \quad \frac{1}{N}[\Gamma_{0im} - N^s\Gamma_{sim}] \\
&= \quad \frac{1}{2N}[N_{i,m} + N_{m,i} - g_{im,0} - 2\Gamma_{sim}N^s] \\
&= \quad \frac{1}{2N}[N_{i|m} + N_{m|i} - g_{im,0}]
\end{aligned}
$$

$$= \frac{1}{2N}[g_{i0|m} + g_{0m|i} - g_{im,0}]. \tag{3.27}$$

where the metric tensor terms in front of the connection coefficients are for the 4-geometry (i.e., they can be rewritten in terms of lapse and shift). Since these are contravariant metric terms, they take this particular form. Partial derivatives are with respect to components; the shifts change with respect to the basis in the hypersurface, and the 3-geometry changes in coordinate time. This is the extrinsic curvature as used in $3+1$ decomposition, specifically the components as projected onto the hypersurface (the spatial part).

An alternative approach to extrinsic curvature involves the Lie derivative. The Lie derivative represents another way to study curvature via parallel transport. The Lie derivative $\mathcal{L}_{\mathbf{u}}\mathbf{v}$ is defined via a commutator (generically for a noncoordinate basis):

$$\begin{aligned}
\mathcal{L}_{\mathbf{u}}\mathbf{v} &= [\mathbf{u}, \mathbf{v}] \\
&= (\mathbf{u}[v^\alpha] - \mathbf{v}[u^\alpha] + u^\mu v^\nu c_{\mu\nu}{}^\alpha)\frac{\partial}{\partial x^\alpha} \\
&= (u^\kappa v^\alpha_{,\kappa} - v^\kappa u^\alpha_{,\kappa} + u^\mu v^\nu c_{\mu\nu}{}^\alpha)\frac{\partial}{\partial x^\alpha},
\end{aligned} \tag{3.28}$$

where the partial derivative at the end corresponds to a basis vector. In a coordinate basis, the c's equal zero.

This formalism extends to tensors, also. For practical purposes (see computations from Appendix B), this work describes only up to rank 2.

Define the spatial metric: this is the spacelike hypersurface's metric as expressed in the spacetime coordinate basis. Let n^μ be the tangent field to the timelike normal curves. Then,

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{3.29}$$

Letting the curvature vector/4-acceleration of the timelike normal curves be defined by $a^\mu \equiv n^\kappa \nabla_\kappa n^\mu$ and defining the orthogonality relation $n_\mu a^\mu = 0$, state from the computation in the Appendix that

$$\begin{aligned}\pounds_{\mathbf{n}}(\gamma_{\mu\nu}) &= n_{\mu;\nu} + n_{\nu;\mu} + n_\mu a_\nu + a_\mu n_\nu \\ &= n_{\mu;\nu} + n_{\nu;\mu} + n_\mu n^\kappa n_{\nu;\kappa} + n^\kappa n_{\mu;\kappa} n_\nu. \end{aligned} \tag{3.30}$$

From the previous definition for extrinsic curvature ($K_{\mu\nu} = -n_{\mu;\nu}$) and after some computation, arrive at the statement that

$$K_{\mu\nu} = -\frac{1}{2}\pounds_{\mathbf{u}}(\gamma_{\mu\nu}). \tag{3.31}$$

Throwing out the time part of this, this yields the standard relation as shown above.

This serves more as an example to conclude the calculations for rewriting the action. The Lie derivative is a commutator; considering that curvature is the change in a vector transported around a closed route, involve basis vectors. Then,

$$\mathcal{R}(\mathbf{e}_j, \mathbf{e}_k)\mathbf{e}_i = \nabla_{\mathbf{e_j}}\nabla_{\mathbf{e_k}}\mathbf{e_i} - \nabla_{\mathbf{e_k}}\nabla_{\mathbf{e_j}}\mathbf{e_i} - \nabla_{[\mathbf{e_j},\mathbf{e_k}]}\mathbf{e_i}. \tag{3.32}$$

Since the commutator of two basis vectors is zero, the route closes. Introduce the equations of Gauss and Weingarten,

$$\nabla_{\mathbf{e_k}}\mathbf{e_i} = K_{ki}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} +^{(3)}\Gamma^m{}_{ik}\mathbf{e_m}, \tag{3.33}$$

and calculate the curvature:

$$\begin{aligned}\nabla_{\mathbf{e_j}}\nabla_{\mathbf{e_k}}\mathbf{e_i} &= \nabla_{\mathbf{e_j}}[K_{ik}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} + \Gamma^{(3)m}{}_{ik}\mathbf{e_m}] \\ &= K_{ik,j}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} - K_{ik}K_j{}^m\mathbf{e_m}\frac{1}{\mathbf{n}\cdot\mathbf{n}} + \Gamma^{(3)m}{}_{ik,j}\mathbf{e_m} \\ &+ \Gamma^{(3)m}{}_{ik}[K_{mj}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} + \Gamma^{(3)s}{}_{mj}\mathbf{e_s}];\end{aligned}$$

$$
\begin{aligned}
\nabla_{\mathbf{e_k}}\nabla_{\mathbf{e_j}}\mathbf{e_i} &= \nabla_{\mathbf{e_k}}[K_{ij}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} + \Gamma^{(3)m}{}_{ij}\mathbf{e_m}] \\
&= K_{ij,k}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} - K_{ij}K_k{}^m\mathbf{e_m}\frac{1}{\mathbf{n}\cdot\mathbf{n}} + \Gamma^{(3)m}{}_{ij,k}\mathbf{e_m} \\
&+ \Gamma^{(3)m}{}_{ij}[K_{mk}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} + \Gamma^{(3)s}{}_{mk}\mathbf{e_s}]; \\
\Rightarrow \mathcal{R}(\mathbf{e_j},\mathbf{e_k})\mathbf{e_i} &= \nabla_{\mathbf{e_j}}\nabla_{\mathbf{e_k}}\mathbf{e_i} - \nabla_{\mathbf{e_k}}\nabla_{\mathbf{e_j}}\mathbf{e_i} \\
&= (K_{ik|j} - K_{ij|k})\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} \\
&+ [(\mathbf{n}\cdot\mathbf{n})^{-1}(K_{ij}K_k{}^m - K_{ik}K_j{}^m) +^{(3)} R^m{}_{ijk}]\mathbf{e_m}. \qquad (3.34)
\end{aligned}
$$

The coefficient of the basis vector $\mathbf{e}_m$ gives the components of the curvature tensor (those components of spatial indices). Explicitly,

$$
R^m{}_{ijk} =^{(3)} R^m{}_{ijk} + (\mathbf{n}\cdot\mathbf{n})^{-1}(K_{ij}K_k{}^m - K_{ik}K_j{}^m), \qquad (3.35)
$$

and for the case of lowering the curvature tensor indices via input arguments of $\mathbf{n}$,

$$
R^n{}_{ijk} = \mathbf{g}(\mathbf{n},\mathbf{n})R_{nijk} = g^{nn}R_{nijk} = -(\mathbf{n}\cdot\mathbf{n})^{-1}(K_{ij|k} - K_{ik|j}). \qquad (3.36)
$$

These are the Gauss-Codazzi equations which yield the 4-curvature in terms of the intrinsic 3-geometry and the extrinsic curvature.

At this point, reexamine the Einstein tensor. From Chapter 2, consider the time-components of the Einstein tensor found from the double-dual of the Riemann tensor, and exploit Eq. (3.35) [1]:

$$
\begin{aligned}
G^0{}_0 &= G^n{}_n = -(R^{12}{}_{12} + R^{23}{}_{23} + R^{31}{}_{31}) \\
&= -(^{(3)}R^{12}{}_{12} +^{(3)} R^{23}{}_{23} +^{(3)} R^{31}{}_{31} + (\mathbf{n}\cdot\mathbf{n})^{-1}[(K_1{}^2K_2{}^1 - K_2{}^2K_1{}^1) \\
&+ (K_2{}^3K_3{}^2 - K_3{}^3K_2{}^2) + (K_3{}^1K_1{}^3 - K_1{}^1K_3{}^3)] \\
&= -(\frac{1}{2}{}^{(3)}R + \frac{1}{2}(\mathbf{n}\cdot\mathbf{n})^{-1}[Tr(K^2) - (TrK)^2]) \\
&= -(\frac{1}{2}[^{(3)}R - (\mathbf{n}\cdot\mathbf{n})^{-1}[(TrK)^2 - Tr(K^2)]])
\end{aligned}
$$

$$= \quad -8\pi T^0{}_0 = -8\pi\rho. \tag{3.37}$$

Similarly,

$$\begin{aligned} G^0{}_1 &= G^n{}_1 = R^{n2}{}_{12} + R^{n3}{}_{13} \\ &= -(\mathbf{n}\cdot\mathbf{n})^{-1}(K^2{}_{1|2} - K^2{}_{2|1} + K^3{}_{1|3} - K^3{}_{3|1}) \\ \Rightarrow G^n{}_i &= -(\mathbf{n}\cdot\mathbf{n})^{-1}[K^m{}_{i|m} - (TrK)_{|i}] = 8\pi J_i. \end{aligned} \tag{3.38}$$

Although these equations were determined using an orthonormal tetrad, they are equivalent if used with any frame in the hypersurface. J_i corresponds to the i-th covariant component of the momentum density associated with other sources than gravity [1], [14]. Eqs. (3.37) and (3.38) yield the Hamiltonian and momentum constraints, respectively, and these are the natural forms of the intial-value constraints.

Finally, an important quantity, along with Eq. (3.35), proves necessary for rewriting the geometric action (see Appendix B for the full computation):

$$R^i{}_{nin} = (Tr\mathbf{K})^2 - Tr\mathbf{K}^2 + \text{Covariant divergence.} \tag{3.39}$$

These quantities produce a natural splitting of spacetime into slices and their parametrization.

3.4 The Action Rewritten

With extrinsic and intrinsic curvature elucidated and with the metric split into spatial and temporal components, the action takes on a new form. Recall the Lagrangian from the Hilbert action principle, and consider the Lagrangian density:

$$\mathcal{L}_{Geom} \quad = \quad \frac{1}{16\pi}\sqrt{-g}R$$

$$
\begin{aligned}
&= \frac{1}{16\pi}\sqrt{-g}[R^{\alpha}{}_{\alpha}] = \frac{1}{16\pi}\sqrt{-g}[g^{\alpha\beta}R_{\alpha\beta}] \\
&= \frac{1}{16\pi}\sqrt{-g}[g^{\alpha\beta}R^{\mu}{}_{\alpha\mu\beta}] = \frac{1}{16\pi}\sqrt{-g}[R^{\mu\beta}{}_{\mu\beta}] \\
&= \frac{1}{16\pi}\sqrt{-g}[R^{00}{}_{00} + R^{0j}{}_{0j} + R^{i0}{}_{i0} + R^{ij}{}_{ij}]. \qquad (3.40)
\end{aligned}
$$

The first term, $R^{00}{}_{00}$, equals zero by definition. Also, under pair exchange (switching indices in both pairs), the Riemann tensor is symmetric, so

$$
R^{0j}{}_{0j} + R^{i0}{}_{i0} = 2R^{i0}{}_{i0} = 2R^{in}{}_{in}. \qquad (3.41)
$$

Using this information,

$$
\begin{aligned}
&= \frac{1}{16\pi}\sqrt{-g}[R^{ij}{}_{ij} + 2R^{in}{}_{in}] \\
&= \frac{1}{16\pi}\sqrt{-g}[g^{jj}R^{i}{}_{jij} + 2g^{nn}R^{i}{}_{nin}] = \frac{1}{16\pi}\sqrt{-g}[g^{jj}R^{i}{}_{jij} + 2(\mathbf{n}\cdot\mathbf{n})R^{i}{}_{nin}] \\
&= \frac{1}{16\pi}\sqrt{-g}[{}^{(3)}R + (\mathbf{n}\cdot\mathbf{n})^{-1}(g^{jj}K_{ji}K_{j}{}^{i} - g^{jj}K_{jj}K_{i}{}^{i}) + 2(\mathbf{n}\cdot\mathbf{n})((Tr\mathbf{K})^2 - Tr\mathbf{K}^2)] \\
&= \frac{1}{16\pi}\sqrt{-g}[{}^{(3)}R + (\mathbf{n}\cdot\mathbf{n})(Tr\mathbf{K}^2 - (Tr\mathbf{K})^2) + 2(\mathbf{n}\cdot\mathbf{n})((Tr\mathbf{K})^2 - Tr\mathbf{K}^2)] \\
&= \frac{1}{16\pi}\sqrt{-g}[{}^{(3)}R + (\mathbf{n}\cdot\mathbf{n})((Tr\mathbf{K})^2 - Tr\mathbf{K}^2)], \qquad (3.42)
\end{aligned}
$$

where ${}^{(3)}R$ is the spatial part of the Ricci scalar, and $\mathbf{n}$ is the unit timelike normal to the hypersurfaces. This calculation simplifies when a divergence integrates out to a surface term. Variations of the geometry (metric tensor) interior to this surface do not affect this surface term, and so it contributes nothing to the equations of motion - throw it out. (See Appendix B for details.) The action then becomes

$$
S = \frac{1}{16\pi}\int[[{}^{(3)}R + (\mathbf{n}\cdot\mathbf{n})((Tr\mathbf{K})^2 - Tr\mathbf{K}^2)]N\sqrt{g} + 16\pi\mathcal{L}_{Source}]d^4x, \qquad (3.43)
$$

where $\sqrt{-g} = N\sqrt{g}$ as before, and $\mathcal{L}_{Source}$ does not densitize explicitly here using $\sqrt{-g}$ since, even though it should divide by a 3-form to be a vector-valued density, other methods of densitization such as delta functions prove more useful in situations, particularly those concerning the last part of this paper when considering scalar fields acting at points/events.

Moving to the canonical (ADM) picture, introduce several quantities now:

$$\begin{aligned}\frac{1}{16\pi}\pi^{ij} &= \frac{\delta S}{\delta g_{ij}} = \frac{\delta \mathcal{L}_{Geom}}{\delta g_{ij,0}} \\ &= \frac{\delta}{\delta g_{ij,0}}[\frac{1}{16\pi}N\sqrt{g}[^{(3)}R + (\mathbf{n}\cdot\mathbf{n})((Tr\mathbf{K})^2 - Tr\mathbf{K}^2)]]. \qquad (3.44)\end{aligned}$$

Calling on Eq. (3.27), obtain

$$\begin{aligned}\frac{1}{16\pi}\pi^{ij} &= \frac{1}{16\pi}N\sqrt{g}[(\mathbf{n}\cdot\mathbf{n})(\frac{\delta}{\delta g_{ij,0}}((g^{ij}K_{ij})(g^{ij}K_{ij}) - g^{js}K_{sm}g^{mi}K_{ij}) \\ &= \frac{1}{16\pi}N\sqrt{g}[-(\frac{\delta(Tr\mathbf{K})^2}{\delta g_{ij,0}} - \frac{\delta}{\delta g_{ij,0}}g^{js}K_{sm}g^{mi}K_{ij})] \\ &= \frac{1}{16\pi}N\sqrt{g}[-(-\frac{2}{2N}Tr\mathbf{K}g^{ij} + \frac{2}{2N}g^{js}K_{sm}g^{mi})] \\ &= -\frac{1}{16\pi}\sqrt{g}[Tr\mathbf{K}g^{ij} - K^{ji}] \\ &= \frac{\sqrt{g}}{16\pi}(g^{ij}Tr\mathbf{K} - K^{ij}). \qquad (3.45)\end{aligned}$$

This quantity is the geometrodynamic field momentum, the term dynamically conjugate to the geometrodynamic field coordinate g_{ij}.

$$\frac{\mathcal{H}}{16\pi} = \frac{1}{16\pi}(\frac{1}{\sqrt{g}}(Tr\pi^2 - \frac{1}{2}(Tr\pi)^2) - \sqrt{g}^{(3)}R) \qquad \text{(superhamiltonian);}$$

$$\frac{\mathcal{H}^i}{16\pi} = \frac{-2}{16\pi}\pi^{ik}{}_{|k} \qquad \text{(supermomentum).} \qquad (3.46)$$

Rewrite the action in terms of these quantities:

$$S = \frac{1}{16\pi}\int \quad [-g_{ij}\frac{\partial \pi^{ij}}{\partial t} - N\mathcal{H}(\pi^{ij}, g_{ij}) - N_i\mathcal{H}^i(\pi^{ij}, g_{ij})$$
$$-2[\pi^{ij}N_j - \frac{1}{2}N^i Tr\pi + N^{|i}\sqrt{g}]_{,i} + 16\pi\mathcal{L}_{Source}]d^4x. \qquad (3.47)$$

The first term involving the partial derivatives of the conjugate momenta with respect to coordinate time can be written via the product rule as

$$-g_{ij}\frac{\partial \pi^{ij}}{\partial t} = -\frac{\partial}{\partial t}(g_{ij}\pi^{ij}) + \pi^{ij}\frac{\partial g_{ij}}{\partial t}, \qquad (3.48)$$

with the full time derivative falling out of the variational principle since variations in the interior geometry do not affect terms at the surface/boundary (such variations go to zero). Also, the divergence in this action, $[...]_{,i}$, disappears as it is a surface term. This gives an action principle

$$S = \frac{1}{16\pi}\int [\pi^{ij}\frac{\partial g_{ij}}{\partial t} - N\mathcal{H}(\pi^{ij}, g_{ij}) - N_i\mathcal{H}^i(\pi^{ij}, g_{ij}) + 16\pi\mathcal{L}_{Source}]d^4x. \qquad (3.49)$$

From this action follow the equations of evolution.

3.5 New Variations: Hamilton's Equations and the Constraints

Consider variations of equation (3.47) above. To get Hamilton's equations of motion, vary with respect to the field coordinate and its conjugate. To get the constraint equations (those which govern conservation of energy and momentum), vary with respect to the lapse and the shift. For the superhamiltonian and the supermomentum, the choice is clear - they both inherently depend on the field coordinates and the conjugate momenta. Lapse and shift are geometric quantities - inherent in themselves.

For the source Lagrangian, it depends on the conditions, but it will most likely have a dependence on the 4-metric and so will have a dependence on the 3-metric, the lapse, and the shift. Also, it will have dependences on its fields - it is important to analyze these equations as well, both in the context of the Lagrangian formulation of GR and also in this Hamiltonian formulation. Typically, because of the source's dependence on quantities present in the ADM decomposition, these extra terms modify the equations put forth for the superhamiltonian, the supermomentum, and one of Hamilton's equations.

Writing down these variations, first with respect to g_{ij}, then to π^{ij}, N, and N_i, obtain the desired equations:

$$
\begin{aligned}
\frac{\partial \pi^{ij}}{\partial t} &= -N\sqrt{g}(R^{ij} - \frac{1}{2}g^{ij}R) + \frac{N}{2\sqrt{g}}g^{ij}(Tr\pi^2 - \frac{1}{2}(Tr\pi)^2) \\
&\quad -\frac{2N}{\sqrt{g}}(\pi^{im}\pi_m{}^j - \frac{1}{2}\pi^{ij}N^m)_{|m} + \sqrt{g}(N^{|ij} - g^{ij}N^{|m}{}_{|m}) \\
&\quad +(\pi^{ij}N^m)_{|m} - N^i{}_{|m}\pi^{mj} - N^j{}_{|m}\pi^{mi} + \frac{\partial \mathcal{L}_{Source}}{\partial g_{ij}}; \\
\frac{\partial g_{ij}}{\partial t} &= \frac{2N}{\sqrt{g}}(\pi_{ij} - \frac{1}{2}g_{ij}Tr\pi) + N_{i|j} + N_{j|i}; \\
\frac{\mathcal{H}}{16\pi} &= \frac{\partial \mathcal{L}_{Source}}{\partial N}; \\
\frac{\mathcal{H}^i}{16\pi} &= \frac{\partial \mathcal{L}_{Source}}{\partial N_i}.
\end{aligned}
\tag{3.50}
$$

This is the canonical form. In the case of no source, then the constraint equations will equal zero. Even in the case of a source, they can be rewritten to equal zero. For the purpose of this work, however, we will consider the form where a nonzero term appears on the righthand side. The importance is in the consideration of stability, as violations of the constraints (the superhamiltonian specifically) appear as nonzero terms on the righthand side.

3.6 The Initial-Value Problem

The ADM formulation allows freedom in how the hypersurfaces push forward with respect to the time parameter. "Many-fingered time" lets different parts of the hypersurfaces move forward differently, so long as they remain spacelike. The lapse function $N(t, x^k)$, gives this freedom in the integration for each change in t. For the dynamic equations, the lapse is a prescribed quantity - nature does not determine it. The shift follows similar logic. Essentially, choice of lapse and shift yields the choice of coordinates of spacetime, leading to the appearance of the 3-metric and the extrinsic curvature of successive hypersurfaces. The 4-geometry remains unchanged, however - this says that many representations exist for the same quantities [1].

Choosing $N = 1$ and $N^i = 0$ (synchronous reference system) simplifies calculations [15]. However, doing so leads to coordinate singularities due to the focusing of normal geodesics and generally, it complicates the interpretation. The evolution of the 3-geometry and the extrinsic curvature depends on the choice of slicing, the choice of the initial spatial basis, and the way the basis changes from slice to slice (i.e., what occurs with lapse and shift). So, the best approaches take the most generic structure and avoid overspecification, initially, of the quantities.

To solve the initial-value problem, specify appropriate initial-value data: the six functions ${g_{ij}}^{(3)}(x^i)$, the six functions $\pi^{ij}(x^i)$ or $K^{ij}(x^i)$, and the source qualify. These functions satisfy the constraint equations which give the number and the form of degrees of freedom. Critical for this problem is the Cauchy formulation. Knowing the existence of the topological space $(M, \mathbf{g})$, assume the 4-metric possesses global hyperbolicity. Then, no closed or nearly closed causal paths exist; M possesses a Cauchy surface (where every inextendible causal timelike or null curve without end-points intersects a hypersurface only once); a universal time function describes this surface; the topology of M is $\Sigma \times \mathrm{R}$. Using this information, obtaining a maximal development of the data becomes plausible.

Starting with the covariant form of Einstein's equations [2],

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \tag{3.51}$$

consider the left hand side. Particularly, for unit normal vectors to the surface, using Einstein summation,

$$\begin{aligned} G_{\mu\nu}n^{\nu} &= 0 \\ \Rightarrow R_{\mu\nu}n^{\nu} - \frac{1}{2}g_{\mu\nu}Rn^{\nu} &= 0 \\ \Rightarrow R_{\mu\nu}n^{\nu} &= 0. \end{aligned} \tag{3.52}$$

From the definition of the Riemann tensor and from Eq. (3.36), this yields the momentum constraint equation in vacuum [2]:

$$\begin{aligned} \perp^{\mu}{}_{\nu}G_{\mu\kappa}n^{\kappa} &= 0 \\ \Rightarrow \perp^{\mu}{}_{\nu}R_{\mu\kappa}n^{\kappa} &= K^{k}{}_{j|k} - K^{k}{}_{k|j} = 0. \end{aligned} \tag{3.53}$$

$\perp$ is the projection operation of the spatial metric tensor on another tensor (Appendix B).

For the Hamiltonian constraint equation in vacuum, consider Eq. (3.35):

$$\begin{aligned} \gamma^{\alpha\gamma}\gamma^{\beta\delta}R_{\alpha\beta\gamma\delta} &= (g^{\alpha\gamma} + n^{\alpha}n^{\gamma})(g^{\beta\delta} + n^{\beta}n^{\delta})R_{\alpha\beta\gamma\delta} \\ = R + R_{\beta\delta}n^{\beta}n^{\delta} + R_{\alpha\gamma}n^{\alpha}n^{\gamma} &+ R_{\alpha\beta\gamma\delta}n^{\alpha}n^{\gamma}n^{\beta}n^{\delta} \\ &= R + 2R_{\kappa\lambda}n^{\kappa}n^{\lambda} \\ &= 2G_{\kappa\lambda}n^{\kappa}n^{\lambda} = 0 \\ \Rightarrow G_{\kappa\lambda}n^{\kappa}n^{\lambda} &= \frac{1}{2}R_{\alpha\beta\gamma\delta}\gamma^{\alpha\gamma}\gamma^{\beta\delta} \\ \Rightarrow \frac{1}{2}\gamma^{jm}\gamma^{ik}R_{mijk} &= \frac{1}{2}\gamma^{jm}\gamma^{ik}({}^{(3)}R_{mijk} \end{aligned}$$

$$
\begin{aligned}
&+ \quad K_{ij}K_{km} - K_{ik}K_{jm}) \\
&= \quad \frac{1}{2}({}^{(3)}R - K^{km}K_{km} + K^k{}_k K^m{}_m) \\
&= \quad \frac{1}{2}({}^{(3)}R - Tr(\mathbf{K}^2) + (Tr\mathbf{K})^2) = 0. \qquad (3.54)
\end{aligned}
$$

For the case of a matter source, recall Eqs. (3.37) and (3.38):

$$
\begin{aligned}
\mathcal{H} &= \frac{\sqrt{g}}{2}({}^{(3)}R + (Tr\mathbf{K})^2 - Tr(\mathbf{K}^2)) = 8\pi n^\mu n^\nu T_{\mu\nu}; \\
\mathcal{H}_i &= K_i{}^a{}_{|a} - (Tr\mathbf{K})_{|i} = -8\pi n^\mu \gamma_i{}^\nu T_{\mu\nu}. \qquad (3.55)
\end{aligned}
$$

Note that $\gamma^\mu{}_\nu \equiv \perp^\mu{}_\nu$. The evolution equations take the form

$$
\begin{aligned}
(\partial_t - \pounds_{N_i})K_{ij} &= -N_{|ij} + NR_{ij} - 2NK_{ia}K_j{}^a + N(Tr\mathbf{K})K_{ij} \\
&\quad -8\pi N\gamma_i{}^\nu\gamma_j{}^\mu T_{\mu\nu} - 4\pi N\gamma_{ij}(n^\mu n^\nu T_{\mu\nu} \\
&\quad -\gamma^{ab}\gamma_a{}^\nu\gamma_b{}^\mu T_{\mu\nu}). \qquad (3.56)
\end{aligned}
$$

The derivative operator in front of the extrinsic curvature is the time derivative operator normal to the spatial foliation consisting out of the partial derivative with respect to time and the Lie derivative with respect to the shift. The evolution equation for the spatial metric is

$$
(\partial_t - \pounds_{N_i})\gamma_{ij} = -2NK_{ij}. \qquad (3.57)
$$

Constructing foliations requires appropriate treatment of lapse. Maximal slicings ($Tr\mathbf{K} = 0$) in asymptotically flat spacetimes or slicings of $Tr\mathbf{K} = Const.$ in closed universes gives reasonably the maximal development. The importance of this trace is that its derivative with respect to the parametrization yields linear elliptic equations of the same form on each slice that control the lapse.

Determining the lapse follows well as a minimization problem. Consider that $\partial_\tau \gamma$ and $\partial_\tau \mathbf{K}$ define small deformations of a hypersurface (parametrized by τ) imbedded in spacetime. The lapse N defines the normal/orthogonal deflection of the hypersurface. Then, $\partial_\tau \gamma$ is the stretching-strain tensor. The components associate with the lapse; for initial nonzero $\mathbf{K}$, the lapse produces stretching, as follows from the term $-2N\mathbf{K}$. $\partial_\tau \mathbf{K}$, the bending strain tensor, yields the tangential stretching: $N_{(j|i)}$. To choose lapse, minimize the free energy of the volume bending strain, $\partial_\tau TrK$:

$$\mathcal{H}[N] = \frac{1}{2}\int_\Sigma (\partial_\tau Tr\mathbf{K})^2 d^3x. \tag{3.58}$$

Varying this with respect to the lapse causes the volume bending strain to equal zero.

After establishing the foliation, select an appropriate shift. Several reasonable approaches exist. Consider correlating the four-velocity of the source with the timelike vector field t^μ. This allows for the fixing of the spatial coordinates on each differential 3-volume element. The comoving method proves limited in vacuum and when rotation occurs.

Another method [15] uses the shift to maintain symmetry of the induced spatial metric. Again, use maximal slicing to attain the lapse. Then, select the shift such that the spatial part of the metric possesses desired properties (e.g., isotropy).

A final method uses the shift to keep differential gauge conditions on the induced spatial metric's transport along $\mathbf{t}$. Define the conformal metric:

$$\bar{\gamma}_{ij} \equiv |\gamma_{ij}|^{-\frac{1}{3}}\gamma_{ij}. \tag{3.59}$$

Then, the Lie derivative $\pounds_\mathbf{t}\bar{\gamma}_{ij}$ gives means to measure the distortion occurring for an object traveling from one slice to the next slice. Before continuing, note the Lie derivatives along $\mathbf{t}$ and N^i for the spatial metric:

$$\pounds_\mathbf{t}\gamma_{\mu\nu} = -2NK_{\mu\nu} + \pounds_{N^\mu}\gamma_{\mu\nu},$$

$$\pounds_{N^i}\gamma_{ij} = N_{j|i} + N_{i|j},$$
$$\Rightarrow \Sigma_{ij} = -2N(K_{ij} - \frac{1}{3}\gamma_{ij}TrK) + \sigma^j N_i, \tag{3.60}$$

where the tensor Σ_{ij} corresponds to the distortion. The first term represents the shear of the unit normal field, **n**, and the second term is the shear stretching in the tangential direction. From this, the goal is to minimize the shear stretching energy,

$$S[N^i] = \frac{1}{2}\int_\Sigma \Sigma_{ij}\Sigma^{ij} d^3x. \tag{3.61}$$

Choosing the shift then becomes an equilibrium problem: globally minimize the distortion.

More generally, the minimal distortion shift proves useful. Minimize the total stretching energy,

$$E = \int_\Sigma \gamma^{ij}\gamma^{kl}(\pounds_{\mathbf{t}}\gamma_{ik})(\pounds_{\mathbf{t}}\gamma_{jl})d^3x; \tag{3.62}$$

this preserves the asymptotic gauge conditions.

Chapter 4

Gravitational Waves

4.1 Definition

What are gravitational waves? One way of defining gravitational waves considers that Einstein's equations are wave equations. Vary the action with respect to the metric, get curvature on the left and source on the right, and consider what happens to the geometry. Mathematically, the solutions satisfy a wave equation.

But what does this wave look like? A physical description is necessary: the wave is a ripple of geometry propagating outward from a source. However, this picture becomes complicated when GR is considered to the fullest extent. Nonlinear effects complicate the situation: radiation damping, refraction of the waves, gravitational redshift, backscattering, and self-attraction present themselves.

Radiation damping occurs when the energy of the source decreases as waves are radiated. This is conservation of energy at work, so waves will not be strictly periodic; they diminish with time.

The energy of gravity waves can curve spacetime, also, but the strongest curvature is due to massive objects. The waves do not retain the same shape; the wavefronts refract as the waves travel through curved regions. Also, the wavelength is gravitationally redshifted.

Backscattering off of the curvature occurs. For a wave pulse, the shape and polarization will change and produce portions of the wave that will spread out behind the pulse. These portions travel slower than c. This effect also pertains to the curvature the waves produce - their self-interaction.

No way around these problems exists for the situation of strongly curved spacetime. However, spacetime far enough away from a source may be taken to be flat, so long as other sources are minimal or nonexistent, depending on the complexity of the model. This leads to a useful approximation that can be made to closely resemble the full theory in the appropriate regime.

The linearized theory of gravity gives a weak-field approximation to GR. Gravitational waves are considered in situations far enough away from the source such that spacetime is not strongly curved. Also, the waves are taken to be weaker, with the wavelength λ much less than the background radius of curvature R. This means that spacetime is taken to be Minkowski, and that Lorentz covariance holds. Instead of the traditional metric to describe gravity, a different symmetric, rank-2 tensor does the job, $\bar{h}_{\mu\nu}$. Using the standard Lorentz/Hilbert gauge,

$$\bar{h}^{\mu\kappa}_{,\kappa} = 0. \tag{4.1}$$

Use the Minkowski prescription to manipulate indices:

$$\bar{h}^{\mu\nu} = \eta^{\mu\alpha}\bar{h}_{\alpha\beta}\eta^{\beta\nu}, \tag{4.2}$$

where $\eta_{\mu\nu}$ is the flat metric.

Writing a standard wave equation for propagation,

$$\triangle\bar{h}_{\mu\nu} \equiv \eta^{\alpha\beta}\bar{h}_{\mu\nu,\alpha\beta} = \bar{h}^{;\alpha}_{\mu\nu,\alpha} = 0, \tag{4.3}$$

with $\triangle$ the d'Alembertian. From here, the full metric for spacetime can be written, noting that the global inertial frames of the previous equations are not totally inertial:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O([h_{\mu\nu}]^2), \tag{4.4}$$

where the metric perturbation $h_{\mu\nu}$ is related to $\bar{h}_{\mu\nu}$ (the gravitational field) by

$$\begin{aligned} h_{\mu\nu} &= \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}; \\ h \equiv {h_\kappa}^\kappa &= -\bar{h} = -\bar{h}^\kappa_\kappa. \end{aligned} \tag{4.5}$$

The gauge can modify coordinates, which in turn allows for correction to the metric:

$$\begin{aligned} {x^\alpha}_{\text{New}} &= {x^\alpha}_{\text{Old}} + \xi^\alpha; \\ \bar{h}_{\alpha\beta\text{New}} &= \bar{h}_{\alpha\beta\text{Old}} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}{\xi^\kappa}_{,\kappa}. \end{aligned} \tag{4.6}$$

The infinitesimal coordinate transformations, ξ_μ, change the gauge, and they obey a relation

$$\xi_{\kappa,\lambda}{}^{\lambda} = 0. \tag{4.7}$$

This gauge freedom leads into discussion of monochromatic plane waves, the simplest solutions of the linearized equations. Such waves take the form

$$\bar{h}_{\mu\nu} = \text{Re}[A_{\mu\nu}e^{ik_\alpha x^\alpha}], \tag{4.8}$$

where the constant $A_{\mu\nu}$ is the amplitude, and the constant k_α is a wave vector. These satisfy two conditions: $\mathbf{k}$ is a null vector, which is a consequence of the wave equation, and A is orthogonal to k, which is due to the subsidiary conditions. Mathematically,

$$\begin{aligned} k_\mu k^\mu &= 0\,(\text{Null vector }\mathbf{k}); \\ A_{\mu\nu}k^\nu &= 0\,(\text{Orthogonality of }\mathbf{A}\text{ and }\mathbf{k}). \end{aligned} \tag{4.9}$$

[3] These relations follow as such. Consider Einstein's equations in vacuum (Eq. 2.3). For the plane wave solution,

$$\bar{h}^{\mu\nu}_{,\kappa} = ik_\kappa \bar{h}^{\mu\nu}. \tag{4.10}$$

For Einstein's equations, this yields

$$\begin{aligned} \eta^{\kappa\lambda}\bar{h}^{\mu\nu}_{,\kappa\lambda} &= -\eta^{\kappa\lambda}k_\lambda k_\kappa \bar{h}^{\mu\nu} = 0 \\ &\Rightarrow \eta^{\kappa\lambda}k_\lambda k_\kappa \bar{h}^{\mu\nu} = 0. \end{aligned} \tag{4.11}$$

Because of the gravitational plane wave solution, this expression only vanishes when

$$\eta^{\kappa\lambda}k_\lambda k_\kappa = k^\kappa k_\kappa = 0. \tag{4.12}$$

This requires nullity of the vector and one-form quantity $\mathbf{k}$. For a hypersurface where $k_\kappa x^\kappa$ takes a constant value, the metric $\bar{\mathbf{h}}$ takes a constant value also.

Consider a photon traveling in the direction of the null 3-vector $\{k^i\}$ along a parametrized curve, $x^\kappa(\tau) = k^\kappa\tau + l^\kappa$, with the last term being a constant vector. Then,

$$k_\kappa x^\kappa(\tau) = k_\kappa k^\kappa \tau + k_\kappa l^\kappa = k_\kappa l^\kappa. \tag{4.13}$$

Since the quantity on the left is a constant, then $k_\kappa l^\kappa$ is also constant. This implies that the photon travels with the gravitational wave and since the speed of propagation of the photon is known, this yields that the gravitational wave travels with the same speed, c. Because $\mathbf{k}$ is null, then

$$k_\kappa k^\kappa = (k_0 k^0 + k_1 k^1 + k_2 k^2 + k_3 k^3) = 0;$$
$$k_0 k^0 = -\omega^2; k_1 k^1 + k_2 k^2 + k_3 k^3 = |\bar{k}|^2;$$
$$\Rightarrow \omega^2 = |\bar{k}|^2. \tag{4.14}$$

The last equation is the dispersion equation for the gravitational wave.

This wave equation holds only when imposing the gauge condition $\bar{h}^{\mu\nu}_{,\nu} = 0$. Then, for some amplitude $A^{\mu\nu}$,

$$\begin{aligned} \bar{h}^{\mu\nu}_{,\nu} &= (A^{\mu\nu} e^{ik_\mu x^\mu})_{,\nu} \\ &= A^{\mu\nu}{}_{,\nu} e^{ik_\mu x^\mu} + ik_\nu A^{\mu\nu} e^{ik_\nu x^\nu} \\ &= ik_\nu A^{\mu\nu} e^{ik_\nu x^\nu} = 0 \end{aligned}$$
$$\Rightarrow k_\nu A^{\mu\nu} = 0, \tag{4.15}$$

which gives orthogonality.

To obtain two dynamic degrees of freedom, recall the gauge. It introduces a plane wave vector,

$$\xi^\kappa \equiv -iC^\kappa e^{ik_\mu x^\mu}; \tag{4.16}$$

the four arbitrary constants representing the amplitude of this vector alter arbitrarily four of the components of A. Avoid this alteration by selecting a specific gauge.

Consider the transverse-traceless (TT) gauge. Also consider a 4-velocity, $\mathbf{u}$, to exist throughout all of the spacetime as given by $\bar{h}_{\mu\nu}$. Impose the following conditions:

$$A_{\kappa\lambda}u^{\lambda} = 0; \tag{4.17}$$

$$A^{\kappa}{}_{\kappa} = 0. \tag{4.18}$$

Choosing a comoving frame, rewrite all of the conditions without reference to wave number:

$$\begin{aligned}
h_{\kappa 0} &= 0 \text{ (nonzero spatial components only);} \\
h_{lm,m} &= 0 \text{ (divergence-free spatial components);} \\
h^{l}{}_{l} &= 0 \text{ (trace-free spatial components).}
\end{aligned} \tag{4.19}$$

This eliminates the distinction between $\mathbf{h}$ and $\bar{\mathbf{h}}$. These gauge conditions are linear in $h_{\mu\nu}$.

Consider general gravitational waves in the linearized theory. As with electromagnetic waves, gravitational waves resolve into a superposition of plane waves. Taking a specified 4-velocity as above, a gauge always exists where $h_{\mu\nu}$ satisfies the constraints above. Because of the nonzero spatial components, consider only six wave equations:

$$\triangle h_{lm} = h_{lm,\kappa}{}^{,\kappa} = 0. \tag{4.20}$$

This all leads to the transverse-traceless (TT) tensor, a symmetric tensor satisfying the previous constraints (though not necessarily the wave equations). Possessing only spatial nonzero components and being divergenceless (propagating orthogonally to its orientation) yield the transverse part. The trace-free spatial components give the traceless part.

Consider two test particles. The gravitational wave's curvature tensor oscillates. Attach a coordinate system to the worldtube of one particle. Then, as a wave passes by, the separation vector n wiggles between the two particles; the distance of one particle as measured by the other fluctuates. This information comes from the action of the curvature tensor on the separation vector. In the transverse-traceless gauge,

$$R_{l0m0} = R_{0l0m} = -R_{l00m} = -R_{0lm0} = -\frac{1}{2}h^{TT}{}_{lm,00}. \tag{4.21}$$

Fundamentally, gravitational waves induce geodesic deviation between the test particles. Directed orthogonal to the separation, deviation occurs. Plane waves traveling along the direction of separation cause no deviation.

Information about geodesic deviation for test particles allows for study of polarization of plane waves. While important in its own right, polarization yields the energy-momentum for waves. The difference between EM waves and gravity waves is in the polarization: EM waves polarize vectorially; gravity waves polarize tensorially. With gravity waves, once the polarizations are known, the waves can be written down. The amplitudes then go into computation of the energy-momentum, contributing to the large-scale background curvature.

4.2 Modeling Waves

Modeling of gravitational waves critically motivates finding solutions of the initial-value problem. The ingredients for a model include a field theory/action principle, a generic or specific metric/spacetime, and a means to compute solutions (some kind of algorithm, but this goes beyond the scope of this work). Typically, the action principle is the most simple, that given by Hilbert. Two regimes are considered: the weak-field limit and the strong-field limit. The weak-field limit considers spacetime that resembles Minkowski, and the strong-field limit considers massive objects with strong gravitational fields.

Modeling the waves involves using the ADM decomposition and its variants. Developing an appropriate foliation, specifying lapse and shift, and selecting the correct boundary conditions (depending on the problem) leads to a means of modeling. Consider the parametrized hyperbolic system of equations. It follows from the presentation in Chapter 3: write the spatial 3-metric in terms of the normal to the hypersurfaces which are foliated by the function for coordinate time [14]. With definitions for lapse and shift, adopt a coordinate system for the hypersurfaces and write the 3 + 1 line element for the metric (equation from Ch. 2).

The difficulties in modeling waves occur because the constraint-violating solutions (called unphysical in some works) of the evolution equations blow up. This manifests as a sudden and rapid increase in the amplitude of the wave. Both constraint-obeying and constraint-violating modes occur in these equations, and the gross violation of constraints eventually causes the latter solutions to overwhelm the former. This is known as the problem of stability: as the constraint-violating modes take over, the exponential growth of the solutions appears as instability.

Different approaches to this problem include rewriting Einstein's equations to accomodate extra constraints to elminate these modes and imposing constraints at every time step. The problems with stability remain, however, and the critical difficulty is that, if these waves can exist in nature and if this instability is real, why do observations indicate no rapidly growing gravitational waves? Such exponentially growing waves require a continuous feed of energy, similar to perpetual motion. But this is not present either, and it cannot be or else thermodynamics is wrong.

Chapter 5

A Possible Prescription For Stability

5.1 Motivation

Instability in numerical modeling of the dynamics of geometry arises due to modes that violate the Hamiltonian and momentum constraints. These violations drive the rapid drift of the solutions off the constraint shell. Consider a source-free gravitational field; its Hamiltonian constraint takes the following form:

$$\mathcal{H} = \frac{\sqrt{g}}{2}({}^{(3)}R + (Tr\mathbf{K})^2 - Tr(\mathbf{K}^2)) = 0. \tag{5.1}$$

For the sake of formalism, call the energy-momentum tensor in vacuum $T_{\mu\nu_{Vacuum}}$. In the Lagrangian formulation, this is equivalent to energy conservation:

$$T^{00}{}_{Vacuum;0} = 0. \tag{5.2}$$

Such measures as hyperbolic reformulation of Einstein's equations that exclude acausal modes of the solutions lead to improved stability. However, the solutions inevitably violate the constraints (particularly the Hamiltonian constraint). So, Eq. (5.1) does not equal zero and so appears to have a source present. This does not follow logically from starting in vacuum. Consider energy and momentum constraint violations in the Lagrangian formulation:

$$T^{\mu\nu}{}_{Vacuum;\nu} \neq 0. \tag{5.3}$$

A possibility for correcting this situation is to consider introduction of another term in the action that violates conservation of energy-momentum in an opposite way, so that

$$(T^{\mu\nu}{}_{Vacuum} + T^{\mu\nu}{}_{Other})_{;\nu} = 0. \tag{5.4}$$

This work considers that off-shell drift of solutions for a source-free gravitational field equates with solutions for a field with a source. A scalar field not necessarily subject to standard energy conditions suffices for the simplest possible source. Particularly, this work looks at a field theory with a Lagrangian for a pre-existing source (or sourceless, though this does not change the formulation) and considers how introducing Hoyle's C-field modifies the evolution of the system [16]. The ability of the C-field to produce particles makes it advantageous, as this property allows for a change in the Hamiltonian constraint (see sections 5.2 and 5.3). This change adjusts according to a coupling constant f, and the increasing violation of the constraint by vacuum solutions over time resembles the change in this constraint due to an increasing C-field in time.

5.1.1 Action with a Matter-Producing Scalar Field

Consider an action with a classical scalar C-field inspired by Hoyle and Narlikar [20]:

$$\begin{aligned} S[g_{\mu\nu}, C^{(a)}(x^\mu), X^\mu] &= \int [(\frac{1}{16\pi}R + L_{Matter} + L_{C-Field})\sqrt{-g}]d^4x \\ &= \int [(\frac{1}{16\pi}R + L_{Matter})\sqrt{-g}]d^4x - \sum_a m_a \int da \end{aligned}$$

$$+ \quad \frac{1}{f}\sum_a\sum_b\int\int G_{;\mu\nu}dX^{\mu}(a)dX^{\nu}(b). \qquad (5.5)$$

This is a generic action for a direct particle field with a preexisting matter distribution, and it is a modification of the Fokker action principle [20]. L_{Matter} is a source Lagrangian, and $L_{C-Field}$ is associated with the scalar field. a and b are parameters which label worldlines for different particles and possess the condition $a < b$. These parameters represent the proper time. $X^{\mu}(a)$ and $X^{\nu}(b)$ are the coordinates of points on the worldlines, parametrized by a and b. m_a is the proper mass of created particles along the worldlines a. The reason for maintaining parametrization with respect to proper time involves consideration of the proper mass; other parametrizations will change the mass. As demonstrated in this work, a condition for particle production concerns this proper mass, and convenience dictates its use. The C-field is contained within the Green's functions, and so here it is not taken as a fundamental field.

This work investigates the smooth-fluid approximation. To obtain this from the above action, see computations in Appendix C. Then,

$$\begin{aligned}
S[g_{\mu\nu}, C(x^{\mu}), X^{\mu}(a)] &= \int([\frac{1}{16\pi}R + L_{Matter} + \frac{1}{2}f\cdot(C_{,\mu}C^{,\mu})]\sqrt{-g})d^4x - \sum_a m_a\int da \\
&\quad -\sum_a m_a\int C_{,\mu}\frac{dX^{\mu}(a)}{da}da \\
&= \int([\frac{1}{16\pi}R + L_{Matter} + \frac{1}{2}f\cdot(C_{,\mu}C^{,\mu}) \\
&\quad -\sum_a m_a\int\frac{\delta^{(4)}(x^{\mu}-X^{\mu}(a))}{\sqrt{-g}}(1+C_{,\mu}\frac{dX^{\mu}(a)}{da})da]\sqrt{-g})d^4x \\
&= \int([\frac{1}{16\pi}R + L_{Matter} + \frac{1}{2}f\cdot(C_{,\mu}C^{,\mu}) \\
&\quad -\sum_a m_a\frac{\delta^{(3)}(x^{\mu}-X^{\mu}(a))}{\sqrt{-g}}(1+C_{,\mu}\frac{dX^{\mu}(a)}{da})]\sqrt{-g})d^4x. \qquad (5.6)
\end{aligned}$$

This is a theory with minimal coupling (that is, the least required coupling is that with the geometry). The scalar field manifests in this action as gradients; only derivatives of the field couple to anything. f is a coupling (time, in this case) constant for this field, and its value determines how strongly the C-field affects Einstein's equations and adjusts the conservation of energy-momentum (sections 5.2 and 5.3). The Lagrangian for the C-field will depend on the metric, lapse, and shift (ultimately, all of the stuff in the geometry); this serves as a good generic formulation. L_{Matter} is the matter Lagrangian, and it contains all sources (or vacuum) of gravitation not associated with the scalar C-field. It remains generic and depends only on the geometry; this permits flexibility in the choice of matter and matter distribution. In the last term, summation occurs over the worldlines/number of particles. x^μ represents the coordinates of a point in the spacetime not on the worldline.

5.1.2 Properties of the C-Field

C represents the scalar field; given a point with coordinates x^μ, the field $C^{(a)}(x^\mu)$ (associated with a worldline a) at that point is defined as [21]

$$C^{(a)}(x^\mu) = \frac{1}{f} \int G(x^\mu, X^\mu(a))_{;\mu} dX^\mu(a). \tag{5.7}$$

The same parametrization from above regarding the worldline holds [20]. The function $G(x^\mu, X^\mu(a))$ is a scalar Green's function; it satisfies the relationship

$$[\sqrt{-g} g^{\mu\nu} G(x^\mu, X^\mu(a))_{,\nu}]_{,\mu} = -\delta^{(4)}(x^\mu - X^\mu(a)). \tag{5.8}$$

Since the partial derivatives of the metric tensor and $\sqrt{-g}$ vanish, this equation recasts as

$$g^{\mu\nu} G(x^\mu, X^\mu(a))_{;\mu\nu} = -\frac{\delta^{(4)}(x^\mu - X^\mu(a))}{\sqrt{-g}}. \tag{5.9}$$

At a point A on the worldline, the Green's function above describing the interaction between A and another point B is

$$g^{\mu\nu} G(A, B)_{;\mu\nu} = -\frac{\delta^{(4)}(A(a) - B(b))}{\sqrt{-g}}. \tag{5.10}$$

Depending on whether a particle is created or destroyed at a point on the worldline, the field takes the value

$$C^{(a)}(x^\mu) = \mp\frac{1}{f}G(x^\mu, X^\mu(a)) \tag{5.11}$$

for creation/destruction. The parametrization of points on the worldline is taken as implicit. See Figures 5.1 and 5.2 for graphical displays of this process.

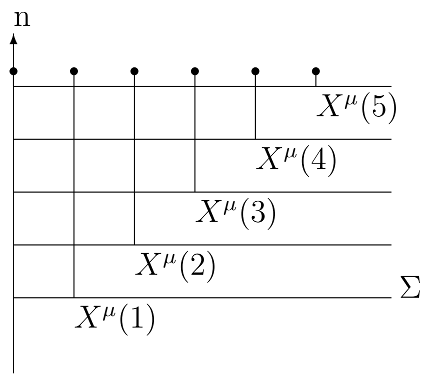

Figure 5.1: Creation of particles by C-field. **n** represents the normal to the hypersurface, and Σ represents the hypersurfaces. Particles come into existence at the points $X^\mu(a)$ on the hypersurfaces.

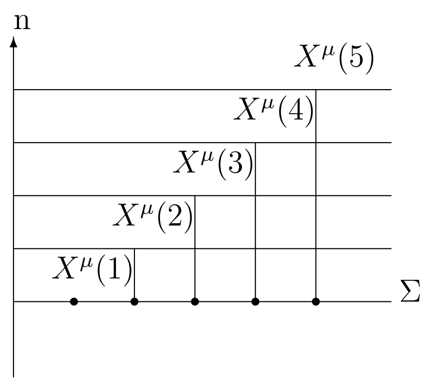

Figure 5.2: Destruction of particles by C-field. Their termination occurs at the points $X^\mu(a)$ on the hypersurfaces.

For the smooth-fluid (perfect fluids and dusts) approximation,

$$C(x^\mu) = \sum_a m_{(a)}C^{(a)}(x^\mu). \tag{5.12}$$

Consider the action given by Eq. (5.3). Write Einstein's equations (computations in Appendix C):

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi[T^{\mu\nu} - f(C^{,\nu}C^{,\mu} - \frac{1}{2}g^{\mu\nu}C_{,\kappa}C^{,\kappa})]. \quad (5.13)$$

This equation separates the energy-momentum tensor for the original matter distribution from the energy-momentum tensor for the C-field terms. Representing the conservation of energy-momentum in the Lagrangian picture involves the divergence of the right-hand side of these equations. For a standard matter distribution (such as a perfect fluid), the divergence of the energy-momentum tensor equals zero. For the matter distribution considered here, the extra terms of the C-field add in. So, where the divergence of the standard-matter energy-momentum tensor is non-zero, the divergence of the C-field terms is nonzero also, and these terms cancel each other out to preserve the zero-divergence of the righthand side. Inclusion of the C-field terms retains conservation of energy-momentum of the entire system.

Hoyle and Narlikar ([16] and [17]) considered two postulates important for the C-field. First is Weyl's postulate: worldlines of created matter form a geodesic congruence (family of curves) normal to a spacelike hypersurface. This retains only normal terms in the divergence of the energy-momentum tensor:

$$T^{\mu\nu}{}_{;\nu} = fC^{,\mu}C^{,\nu}{}_{;\nu}. \quad (5.14)$$

The second postulate says that hypersurfaces are homogeneous and isotropic. If C increases/decreases as a function of time, then either the particles possess variable proper mass, or the number of such particles will be variable throughout time. This implies for monotonically increasing C, the number of particles created at events will increase. Hoyle considered that the C-field increases with time ($C = t$) [16], [17].

Another condition relates the gradient of the C-field to the available energy. Note that the energy density can be positive or negative, since the C-field satisfies the relationship [22]

$$C_{,\mu}C^{,\mu} = p_\mu p^\mu = E^2 = {m_a}^2 \text{ (for particles of proper mass } m_a\text{)}, \qquad (5.15)$$

the same way as the Dirac Equation admits positive and negative energy solutions [24]. It pops on and off, attaining this critical energy threshold (where $E^2 = {m_a}^2$) at some point in an instant of time. For most problems (see [21] and [23]), the energy density of the C-field is negative. With a negative energy density, the C-field generates negative extrinsic curvature, which in GR associates with expansion/inflation. The space expands while the particles are being generated. It produces particles in the vicinity of strong gravitational fields when the amount of squared-energy equals the squared-proper mass.

The choice of the C-field relates to its ability to compensate for violations of energy-momentum. Other sources, such as the EM field tensor, will be unable to because they conserve energy and momentum [1], [2].

5.2 How the Scalar Field Affects the Equations of Evolution and the Constraints

Using the ADM formulation as outlined in Chapter 3, finding Hamilton's equations and the constraint equations determines some properties of this scalar field and gives information about what happens with these waves. the properties of this field are important because of their implications to the physics and, specifically, to these models. (See Appendix C for all computations.)

Consider the equations of evolution:

$$\frac{\partial g_{ij}}{\partial t} = \frac{2N}{\sqrt{g}}(\pi_{ij} - \frac{1}{2}g_{ij}Tr\Pi) + N_{i|j} + N_{j|i}; \qquad (5.16)$$

$$
\begin{aligned}
\frac{\partial \pi^{ij}}{\partial t} &= -N\sqrt{g}(R^{ij} - \frac{1}{2}g^{ij}R) + \frac{N}{2\sqrt{g}}g^{ij}(Tr\Pi^2 - \frac{1}{2}(Tr\Pi)^2) \\
&-\frac{2N}{\sqrt{g}}(\pi^{im}\pi_m{}^j - \frac{1}{2}\pi^{ij}Tr\Pi) + \sqrt{g}(N^{|ij} - g^{ij}N^{|m}{}_{|m}) \\
&+(\pi^{ij}N^m)_{|m} - N^i{}_{|m}\pi^{mj} - N^j{}_{|m}\pi^{mi} + 8\pi N\sqrt{g}g^{ij}(L_{Matter} \\
&+\frac{1}{2}f\cdot((N_kN^k - N^2)C^{,0}C^{,0} + N_iC^{,0}C^{,i} + N_jC^{,j}C^{,0} + g_{ij}C^{,j}C^{,i}) \\
&+fC^{,j}C^{,i}) + 16\pi N\sqrt{g}\frac{\delta L_{Matter}}{\delta g_{ij}} \\
&-8\pi N\sqrt{g}g^{ij}\rho_{Proper}[1 + (N_kN^k - N^2)C^{,0}\frac{dX^0(a)}{da} + N_iC^0\frac{dX^i(a)}{da} \\
&+N_jC^{,j}\frac{dX^0(a)}{da} + g_{ij}C^{,j}\frac{dX^i(a)}{da}] - 16\pi N\sqrt{g}\rho_{Proper}C^{,j}\frac{dX^i(a)}{da}. \qquad (5.17)
\end{aligned}
$$

Because none of the matter or C-field terms in the Lagrangian depend on conjugate momentum, their effects on Eq. (5.14) appear only via (5.15). For this equation, the C-field contributes strongly to the dynamics, causing this equation of motion to have a nonlinear dependence on lapse and shift (specifically, it has a cubic dependence with regard to the scalar field terms). $\rho_{Proper} \equiv \sum_a m_a \frac{\delta^{(3)}(x^\mu - X^\mu(a))}{\sqrt{-g}}$ yields the proper density of the created particles. Because the C-field creates particles at rest, the 3-momenta equal zero, so

$$
\begin{aligned}
\frac{\partial \pi^{ij}}{\partial t} &= -N\sqrt{g}(R^{ij} - \frac{1}{2}g^{ij}R) + \frac{N}{2\sqrt{g}}g^{ij}(Tr\Pi^2 - \frac{1}{2}(Tr\Pi)^2) \\
&-\frac{2N}{\sqrt{g}}(\pi^{im}\pi_m{}^j - \frac{1}{2}\pi^{ij}Tr\Pi) + \sqrt{g}(N^{|ij} - g^{ij}N^{|m}{}_{|m}) \\
&+(\pi^{ij}N^m)_{|m} - N^i{}_{|m}\pi^{mj} - N^j{}_{|m}\pi^{mi} \\
&+8\pi N\sqrt{g}g^{ij}(L_{Matter} + \frac{1}{2}f((N_kN^k - N^2)C^{,0}C^{,0}) + 16\pi N\sqrt{g}\frac{\delta L_{Matter}}{\delta g_{ij}} \\
&-8\pi N\sqrt{g}g^{ij}\rho_{Proper}[1 + (N_kN^k - N^2)C^{,0}\frac{dX^0(a)}{da}]. \qquad (5.18)
\end{aligned}
$$

Critical for the problem of stability is the Hamiltonian constraint:

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{\sqrt{g}}(Tr\Pi^2 - \frac{1}{2}(Tr\Pi)^2) - \sqrt{g}R \\
&= 16\pi\sqrt{g}(L_{Matter} + \frac{1}{2}f \cdot ((N_k N^k - N^2)C^{,0}C^{,0} + N_i C^{,0}C^{,i} \\
&\quad + N_j C^{,j}C^{,0} + g_{ij}C^{,j}C^{,i}) + \frac{N}{2}f \cdot (-2NC^{,0}C^{,0}) \\
&\quad - \rho_{Proper}[1 + (N_k N^k - N^2)C^{,0}\frac{dX^0(a)}{da} + N_i C^0 \frac{dX^i(a)}{da} + N_j C^{,j}\frac{dX^0(a)}{da}] \\
&\quad - N\rho_{Proper}[-2NC^{,0}\frac{dX^0(a)}{da}]) \\
&= 16\pi\sqrt{g}(L_{Matter} + \frac{1}{2}f((N_k N^k - N^2)C^{,0}C^{,0}) \\
&\quad + \frac{N}{2}f(-2NC^{,0}C^{,0}) - \rho_{Proper}[1 + (N_k N^k - N^2)C^{,0}\frac{dX^0(a)}{da}] \\
&\quad - N\rho_{Proper}[-2NC^{,0}\frac{dX^0(a)}{da}]). \qquad (5.19)
\end{aligned}
$$

The momentum constraint (supermomentum) is considered also:

$$
\mathcal{H}^i = -2\pi^{ij}{}_{|j} = N\sqrt{g}[(\frac{1}{2}fC^{,0}C^{,i} - \rho_{Proper}C^0\frac{dX^i(a)}{da})] = 0. \qquad (5.20)
$$

Only derivatives of the C-field appear in the constraints and the evolution equations, and the constraints possess nonlinear dependences on lapse and shift.

5.3 Thermodynamic Properties of this System

Consider an initial matter configuration of a perfect fluid (a simple gas of material particles with the same proper mass), with the production of a dust (similar to a perfect fluid except exerts no pressure on boundaries or other elements in a mixture) by the C-field. This ensures minimal interaction between anything, and the dust will not contribute to the pressure of the system.

Examine the case of variable particle number. Via Lorentz covariance on infinitesimal tangents to the hypersurface, thermodynamics as done in Minkowski spacetime extends to curved spacetime. Effectively, a single lightcone cannot describe the entire hypersurface but, on these tangents to the hypersurface, lightcones can be drawn. This does not adequately or even accurately describe the behavior of the geometry in GR, but it is necessary for extending the flat spacetime formulation to curved spacetime.

Any discussion of thermodynamics requires the equations of state. A statistical description of the system necessitates appropriate terminology, as found in Synge [25].

Working from Maxwell's thermodynamic relations, start with an entropy integral:

$$\mathcal{S}_{Modified} = -dV \int [\mathcal{N} \log \mathcal{N} + \mathcal{N}_{C-Field} \log \mathcal{N}_{C-Field}] d\Omega. \tag{5.21}$$

$\mathcal{N}$ and $\mathcal{N}_{C-Field}$ are the distribution functions for the perfect fluid and the dust, respectively. The distribution functions depend on position and momentum; this representation yields exponential relationships. Consider a nullcone originating from a point on the hypersurface. $d\Omega$ is the 3-volume formed by the projection of a part of the nullcone between the point on the hypersurface and the target region.

Varying the entropy integral with respect to the distribution functions, obtain

$$\begin{aligned} \log \mathcal{N} + 1 &= \alpha + \xi_\mu p^\mu \text{ and} \\ \log \mathcal{N}_{C-Field} + 1 &= \alpha_{C-Field} + \xi_\mu p^\mu, \end{aligned} \tag{5.22}$$

where the α's and the ξ_μ's are Lagrange multipliers.

Particles do not interact in this mixture, thus validating the separate distribution functions. A critical and necessary simplifying assumption concerns the mean 4-velocity $\bar{u}^\mu$ of this system. Define the mean 4-velocity with the Lagrange multiplier ξ_μ and the reciprocal temperature ξ:

$$\bar{u}^\mu = -\frac{\partial \xi}{\partial \xi^\mu} = \frac{\xi^\mu}{\xi}. \tag{5.23}$$

The perfect fluid and the dust share the same mean 4-velocity.

Write the distribution functions:

$$
\begin{aligned}
\mathcal{N}(x^\mu, p^\mu) &= Ae^{\xi_\mu p^\mu}; \\
\mathcal{N}_{C-Field}(x^\mu, p^\mu) &= A_{C-Field}e^{\xi_\mu p^\mu}.
\end{aligned} \tag{5.24}
$$

After some computations and manipulations, the energy-momentum tensor takes the form

$$
T_{\mu\nu} = (m\mathcal{N}_0 G(m\xi) + m_a \mathcal{N}_{0C-Field} G(m_a\xi))\bar{u}_\mu \bar{u}_\nu + g_{\mu\nu}\frac{\mathcal{N}_0}{\xi}. \tag{5.25}
$$

$\mathcal{N}_0$ and $\mathcal{N}_{0C-Field}$ are the number densities for the mixture. The action eliminates constancy for the number of created particles.

Taking the divergence of the energy-momentum tensor and using the fact that particle number for the dust is not conserved, arrive at the relation

$$
\begin{aligned}
T^{\mu\nu}{}_{;\nu} &= (m_a \mathcal{N}_{0C-Field} G(m_a\xi)\bar{u}^\mu \bar{u}^\nu)_{;\nu} \\
&= \frac{m_a \xi^\mu \xi^\nu}{\xi^2}[\mathcal{N}_{0C-Field} G(m_a\xi)]_{;\nu} = fC^{,\mu}C^{,\nu}{}_{;\nu}.
\end{aligned} \tag{5.26}
$$

This expression concerns the original energy-momentum tensor. In effect, these C-particles are built into it and, without detailing the time-dependence, shows that the energy-momentum tensor anticipates these particles. So, this makes the terms of the C-field corrections to what Einstein's equations yield with a standard classical source.

Writing the conservation equations with this new information:

$$
\begin{aligned}
fC^{,\mu}C^{,\nu}{}_{;\nu} &= f \cdot ((N_k N^k - N^2)C^{,0}{}_{;0}C^{,0} \\
&+ N_i C^{,0}{}_{;0}C^{,i} + N_j C^{,j}{}_{;j}C^{,0} + g_{ij}C^{,j}{}_{;j}C^{,i}))
\end{aligned}
$$

$$\begin{aligned}
\Rightarrow f\cdot(N_kN^k - N^2)C^{,0}{}_{;0}C^{,0} &= \frac{m_a\xi^0\xi^0}{\xi^2}[\mathcal{N}_{0C-Field}G(m_a\xi)]_{;0};\\
fN_iC^{,0}{}_{;0}C^{,i} &= \frac{m_a\xi^i\xi^0}{\xi^2}[\mathcal{N}_{0C-Field}G(m_a\xi)]_{;0};\\
fN_jC^{,j}{}_{;j}C^{,0} &= \frac{m_a\xi^0\xi^j}{\xi^2}[\mathcal{N}_{0C-Field}G(m_a\xi)]_{;j};\\
fg_{ij}C^{,j}{}_{;j}C^{,i} &= \frac{m_a\xi^i\xi^j}{\xi^2}[\mathcal{N}_{0C-Field}G(m_a\xi)]_{;j}.
\end{aligned}\tag{5.27}$$

Simplify these for particles at rest:

$$\begin{aligned}
f\cdot(N_kN^k - N^2)C^{,0}{}_{;0}C^{,0} &= \frac{m_a\xi^0\xi^0}{\xi^2}[\mathcal{N}_{0C-Field}G(m_a\xi)]_{;0};\\
fN_iC^{,0}{}_{;0}C^{,i} &= fN_jC^{,j}{}_{;j}C^{,0} = fg_{ij}C^{,j}{}_{;j}C^{,i} = 0.
\end{aligned}\tag{5.28}$$

Using this information, restate the constraints and the dynamical equations in terms of intrinsic and extrinsic curvature:

$$\begin{aligned}
\mathcal{H} &= \frac{\sqrt{g}}{2}({}^{(3)}R - Tr(K^2) + (TrK)^2)\\
&= 8\pi[(m\mathcal{N}_0G(m\xi) + m_a\mathcal{N}_{0C-Field}G(m_a\xi)) - \frac{\mathcal{N}_0}{\xi}\\
&+ \frac{f}{2}C^{,0}C_{,0}];
\end{aligned}\tag{5.29}$$

$$\mathcal{H}_i = K_i{}^a{}_{|a} - (TrK)_{|i} = 0;\tag{5.30}$$

$$\begin{aligned}
(\partial_t - \pounds_{N_i})K_{ij} &= -N_{|ij} + NR_{ij} - 2NK_{ia}K_j{}^a + N(Tr\mathbf{K})K_{ij}\\
&\quad +8\pi Ng_{ij}[\frac{1}{2}(m\mathcal{N}_0G(m\xi) + m_a\mathcal{N}_{0C-Field}G(m_a\xi))\\
&\quad -f\cdot(C^{,0}C_{,0})];
\end{aligned}\tag{5.31}$$

$$(\partial_t - \pounds_{N_i})g_{ij} \quad = \quad -2NK_{ij}. \tag{5.32}$$

These represent the constraints and evolution equations given in the standard form.

5.4 Physical Interpretation of this System

Consider evolution in the case of no source (vacuum). Over long times, the solutions blow up, violating the Hamiltonian constraint particularly. As a result, solutions go off-shell. This causes the Hamiltonian constraint, despite the initial vacuum configuration, to become nonzero. Violation of the constraints equates with deviation of solutions inside and outside of the nullcone at a point on a hypersurface. The C-field compensates for this: its placement into the source smears the constraint shell, allowing for these solutions to remain on shell. The negative energy density permitted by the C-field presents no difficulties with relaxation of classical energy conditions. Because the C-field can account for violations both within and outside the nullcone, it makes for an ideal source. With suitable adjustment of the coupling constant f, solutions can be constrained not to behave too wildly. Currently, exponential growth (the blowing up of solutions) plagues models. An advantage of this approach involves the coupling constant appearing as a time constant within an exponential function [23]. With a reasonable selection, this time constant prevents rapid explosion of solutions.

The C-field acts only when the curvature is great enough to have the necessary energy equivalent to the rest mass of a particle [22]. It switches on and off in such a situation. Specifically, it does not affect a particle's worldline; it interacts with the particle only at the ends (this follows from variation of the particle trajectories).

The critical difference between this and current gravity wave models is the inclusion of production of massive particles. Particle production does not usually associate with classical fields. To choose lapse and shift in the standard way (lapse being 1 and shift being 0) retains the C-field's influence. Certain quantities must be chosen (such as lapse and shift), while others require evaluation. With this information, the mean 4-velocity also contributes to the choice of slicing. Because the C-field creates particles at rest, frames where the spatial components of the mean 4-velocity equal zero allow for simplification of the equations.

Thermodynamics introduces number density and temperature. Consider these quantities with the C-field on the hypersurface. If the number density increases, then for constant volume the number of particles increases. This leads to the temperature: when more particles enter the system, the temperature decreases. The definition of the mean 4-velocity (the change in reciprocal temperature ξ with respect to the timelike vector ξ^μ) allows for a corresponding change in the timelike vector for any change in temperature. This simplifies the equations. Because of the many-fingered-time nature of the hypersurfaces, the C-field varies over the entire hypersurface. These equations can be used to evaluate this, although obtaining answers appears nontrivial. In this work, the created particles have uniform, non-variable proper mass along their worldlines. They are discretized (which is consistent with a dust), so the particle number varies.

More complicated sources and relaxation of conditions yield other difficulties. If the C-field produces anything but a dust, the pressure becomes variable. In all cases of matter creation, the temperature changes as the introduction of particles lowers the temperature. Also, mass might be variable along the worldlines. In this work, however, the C-field acts only at the endpoints of worldlines, so this difficulty does not occur. Relaxation of isotropy yields a nonzero momentum constraint, leading to coupling with the Hamiltonian constraint.

The negative energy density of the C-field requires relaxation of energy conditions since classical considerations involve matter distributions of positive energy densities only. Introducing a matter source - particularly one that creates new matter - requires a scalar field with behavior similar to quantum fields. Particle creation allows for violation of the classical energy conditions, although these conditions impose only after assumptions placed on the matter distribution [12].

The difficulty with current approaches to stabilization appears because of overdetermination of the equations. Removing the constraint violation from one set of terms and introducing a new set which leads to the same troubles provides only an aesthetic repair. While a great revelation of GR is the equivalence of matter and geometry, it is this equivalence which should allow the consideration of different kinds of behavior concerning the matter. The action considered here is not a radical departure from GR, in that it retains the physics given by Einstein's equations. Using a scalar field only in the production of matter and only on the righthand side conservatively adjusts the equations without requiring any modifications to the curvature. The scalar field imposes no artificial construction onto the equations; this action principle gives an ordinary energy-momentum tensor on one side of the equation with the Einstein tensor on the other side. This ordinary energy-momentum tensor anticipates the production of particles via the C-field. The implication of this requires that the sourceless case is artificial; the vacuum as conceived is not truly empty but instead possesses a minimal source (such as the C-field, perhaps).

Chapter 6

Conclusions

The choice of Hoyle's C-field as a source offers a possible resolution to the problem of stability. With suitable properties, this scalar field compensates for the constraint violations occurring in gravitational wave models. A full study of the scalar field's properties - how it affects the equations of motion, the constraints (conservation of energy-momentum), the thermodynamics of the system - yields information that can influence construction of models.

Consideration that terms produced by a scalar field source act as corrections to sourceless fields imply incomplete understanding of gravitational processes. The possibility that the C-field is a minimum required source (rather than vacuum) leads to unknown physics. Its unique properties - particularly its negative energy density that violates classical energy conditions - make it non-standard for classical theories. This means that semiclassical physics or quantum theory might enter into general relativity. The interesting thing about this scalar field is that it has not been quantized. Its behavior under quantization should be investigated, particularly if it is the classical limit of a quantum field, as it will further tie gravity with other field theories.

Another difficulty involves the form of Einstein's equations. Hyperbolic partial differential equations prove difficult to stabilize. An investigation of their stability conditions might allow for a better understanding of why these solutions blow up.

Beyond the scope of this work is the connection to a quantum theory of gravity.

Quantization is the transformation of classical quantities into quantum operators. Classical scalar fields quantize into fields consisting of operators. The quantum mechanical vacuum has a zero-point energy $\frac{1}{2}m\omega^2$; if the C-field is or relates to a quantum scalar field, then the analogy between this work and quantum considerations can be justified. The classical vacuum may turn out to be an idealization, where the C-field might act as a sort of minimal source with zero as a possibility but not rigidly fixed. A larger implication is that the presence of a quantum scalar field in the source indicates conditions that must apply to the geometry and the curvature on the quantum level. Since a quantum scalar field is written in terms of ladder operators, this representation must also influence the form of the Einstein tensor. Considerations of the lapse and shift also become important in studying quantum effects; Planck time and Planck length become standards of measurement. Also, information about the mass of a particle coming from a scalar field depends critically on the time of information transfer. A particle only knows of its mass after it interacts with such a field and if it exists within the uncertainty time, it might not know its mass, which means that it doesn't generate curvature. This might offer insight into the behavior of vacuum fluctuations and possibly that particles must interact with a field (such as Higgs) to generate curvature. A quantum theory of gravity, requirements in nature for the generation of gravitational waves, and the mechanism by which particles "know" their mass become important for this.

Numerical solutions of Einstein equations with a vacuum source seem to demonstrate a generic instability and might have a physical origin when the constraints are violated (manifested as numerical errors). The changes needed to resolve this problem are not clear presently. This work has presented the possibility that allowing the solutions to deviate from vacuum may resolve the difficulty. Admissibility of the C-field is necessary to account for the possibility that the "matter" generated by numerical errors does not necessary satisfy the energy conditions. If, in the future, this idea proves insufficient for handling the errors, further investigation on the quantum level might be necessary. In particular, interaction between quantized matter and the gravitational field might need to be reconsidered.

Appendix A

Derivation of the Lagrangian Formulation Of Einstein's Equations

This appendix gives the nuts and bolts of how to obtain Einstein's equations from an action principle. The most generic action principle to be considered in general relativity has the following form:

$$S = \int d^4x[L_{Geom}\sqrt{-g} + L_{Fields}\sqrt{-g}]. \tag{A.1}$$

L_{Geom} is the geometric Lagrangian (contained in the Hilbert action); this term gives the action its curvature. L_{Fields} contributes all other information concerning matter, energy, and sources generically speaking.

The principle of least action requires extremization of this action. So, variation of this action must equal zero:

$$\delta S = 0. \tag{A.2}$$

Applying the variation, obtain the following:

$$
\begin{aligned}
\delta S &= \int d^4x[\delta(L_{Geom}\sqrt{-g}) + \delta(L_{Fields}\sqrt{-g})] \\
&= \frac{1}{16\pi}\int d^4x[\delta(\sqrt{-g}g^{\mu\nu}R_{\mu\nu}) + 16\pi\delta(L_{Fields}\sqrt{-g})] \\
&= \frac{1}{16\pi}\int d^4x[g^{\mu\nu}R_{\mu\nu}\delta\sqrt{-g} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \\
&\quad +16\pi\sqrt{-g}\delta L_{Fields} + 16\pi L_{Fields}\delta\sqrt{-g}] \\
&= \frac{1}{16\pi}\int d^4x[R\delta\sqrt{-g} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \\
&\quad +16\pi\sqrt{-g}\delta L_{Fields} + 16\pi L_{Fields}\delta\sqrt{-g}]. \qquad \text{(A.3)}
\end{aligned}
$$

Einstein's equations follow from varying the action with respect to the metric tensor, $g^{\mu\nu}$. To achieve this, all variations must occur with respect to this metric, but the current form of this variation includes variations of $\sqrt{-g}$ and $R_{\mu\nu}$. These variations transform, via the rules of calculus, and become with respect to the metric.

Consider the first term, $\delta\sqrt{-g}$. Recall that the determinant of a matrix may be written as a product of a matrix with its cofactor matrix, in this case $A^{\mu\nu}$:

$$g = det\|g_{\mu\nu}\| = g_{\mu\nu}A^{\mu\nu}$$

$$\Rightarrow A^{\mu\nu} = g^{\mu\nu}g. \qquad \text{(A.4)}$$

Then:

$$\delta\sqrt{-g} = \frac{\partial\sqrt{-g}}{\partial g}\frac{dg}{dg_{\mu\nu}}\delta g_{\mu\nu}$$

$$= -\frac{1}{2\sqrt{-g}}\frac{\partial g_{\mu\nu}A^{\mu\nu}}{\partial g_{\mu\nu}}\delta g_{\mu\nu}$$

$$= -\frac{1}{2\sqrt{-g}} A^{\mu\nu} = -\frac{1}{2\sqrt{-g}} g^{\mu\nu} g \delta g_{\mu\nu}$$

$$= \frac{1}{2}\sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}. \tag{A.5}$$

To get the variation in the form of $\delta g^{\mu\nu}$, consider the definition of the Kronecker delta:

$$\begin{aligned} \delta^{\mu}{}_{\nu} &= g^{\mu\alpha} g_{\alpha\nu} \\ &= \begin{cases} 1 & \text{when } \mu = \nu, \\ 0 & \text{when } \mu \neq \nu. \end{cases} \end{aligned} \tag{A.6}$$

Since the Kronecker delta takes on constant values by definition, any variation of it will equal zero:

$$\delta \delta^{\mu}_{\nu} = 0$$

$$= \delta g^{\mu\alpha} g_{\alpha\nu} = \delta g^{\mu\alpha} g_{\alpha\nu} + g^{\mu\alpha} \delta g_{\alpha\nu} = 0$$

$$\Rightarrow g^{\nu\kappa} [\delta g^{\mu\alpha} g_{\alpha\nu} + g^{\mu\alpha} \delta g_{\alpha\nu}] = g^{\nu\kappa} \delta g^{\mu}_{\nu} + g^{\nu\kappa} g^{\mu\alpha} \delta g_{\alpha\nu} = 0$$

$$\Rightarrow \delta g^{\mu\kappa} + g^{\nu\kappa} g^{\mu\alpha} \delta g_{\alpha\nu} = 0$$

$$\Rightarrow \delta g^{\mu\kappa} = -g^{\nu\kappa} g^{\mu\alpha} \delta g_{\alpha\nu}$$

$$\Rightarrow \delta g_{\alpha\nu} = -g_{\mu\alpha} g_{\nu\kappa} \delta g^{\mu\kappa}$$

$$\Rightarrow \frac{1}{2}\sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} = \frac{1}{2}\sqrt{-g} g^{\mu\nu} [-g_{\kappa\mu} g_{\nu\alpha} \delta g^{\kappa\alpha}]$$

$$= -\frac{1}{2}\sqrt{-g} \delta^{\nu}_{\kappa} g_{\nu\alpha} \delta g^{\kappa\alpha}$$

$$= -\frac{1}{2}\sqrt{-g} g_{\kappa\alpha} \delta g^{\kappa\alpha}. \tag{A.7}$$

The contractions over κ and α are over dummy indices; rename them as μ and ν, respectively, so that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \tag{A.8}$$

Thus, the variation of $\sqrt{-g}$ finds representation as a variation with respect to the metric tensor.

Next, consider the variation of the Ricci tensor $\delta R_{\mu\nu}$. Because the Ricci tensor depends only on the second order derivatives of the metric tensor, changes in the metric tensor do not appear. However, the connection coefficients play a role in its variation. This can be seen by rewriting the Ricci tensor as a contracted Riemann tensor:

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}. \tag{A.9}$$

Writing the Riemann tensor in terms of the connection:

$$R^{\alpha}{}_{\mu\beta\nu} = \Gamma^{\alpha}{}_{\mu\nu;\beta} - \Gamma^{\alpha}{}_{\mu\beta;\nu} + \Gamma^{\alpha}{}_{\kappa\beta}\Gamma^{\kappa}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\kappa\nu}\Gamma^{\kappa}{}_{\mu\beta} - \Gamma^{\alpha}{}_{\mu\kappa} c_{\beta\nu}{}^{\kappa}. \tag{A.10}$$

The connections are symmetric in their two lower indices (as given in the chosen coordinate frame), as can be seen by definition. Permutation of the lower indices retrieves two of the partial derivatives of the metric, and symmetry of the metric tensor retrieves the third permuted term. In this frame, the connection coefficients take the following form (where the c's disappear in a coordinate basis):

$$\Gamma^{\kappa}{}_{\mu\nu} = g^{\kappa\lambda}\Gamma_{\lambda\mu\nu} = \frac{1}{2}g^{\kappa\lambda}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda} + c_{\lambda\mu\nu} + c_{\lambda\nu\mu} - c_{\mu\nu\lambda}). \tag{A.11}$$

Since the only concern is with changes in the connection coefficients, focus attention on the tensor, $\delta\Gamma^{\alpha}_{\mu\nu}$. For the variation of the Riemann tensor, only the partial derivatives of the connection contribute due to the transformation formula for connections:

$$\Gamma^{\kappa}{}_{\mu\nu} = \{\Gamma^{\alpha}{}_{\chi\epsilon}\frac{\partial x^{\chi}}{\partial x^{\mu}}\frac{\partial x^{\epsilon}}{\partial x^{\nu}} + \frac{\partial^2 x^{\alpha}}{\partial x^{\mu} x^{\nu}}\}\frac{\partial x^{\kappa}}{\partial x^{\alpha}}. \tag{A.12}$$

For any set of connection coefficients $\Gamma^{\alpha}{}_{\chi\epsilon}$, the tensor characteristics are eliminated by the last term. The variation of this connection, $\delta\Gamma^{\alpha}{}_{\chi\epsilon}$, subtracts this last term out between two choices of connection.

Writing out the variation of the Riemann tensor (with a motivation for varying Ricci):

$$\delta R^{\alpha}{}_{\mu\beta\nu} = \delta\Gamma^{\alpha}{}_{\mu\nu;\beta} - \delta\Gamma^{\alpha}{}_{\mu\beta;\nu} + \delta(\Gamma^{\alpha}{}_{\kappa\beta}\Gamma^{\kappa}{}_{\mu\nu}) - \delta(\Gamma^{\alpha}{}_{\kappa\nu}\Gamma^{\kappa}{}_{\mu\beta})$$

$$= \delta\Gamma^{\alpha}{}_{\mu\nu;\beta} - \delta\Gamma^{\alpha}{}_{\mu\beta;\nu} + \delta(\Gamma^{\alpha}{}_{\kappa\beta})\Gamma^{\kappa}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\kappa\beta}\delta(\Gamma^{\kappa}{}_{\mu\nu}) - \delta(\Gamma^{\alpha}{}_{\kappa\nu})\Gamma^{\kappa}{}_{\mu\beta} - \Gamma^{\alpha}{}_{\kappa\nu}\delta(\Gamma^{\kappa}{}_{\mu\beta}). \tag{A.13}$$

At the point of interest, pick a coordinate system where the connection coefficients disappear. Then, only first derivatives of the varied connections give the curvature. Thus,

$$\delta R^{\alpha}{}_{\mu\beta\nu} = \delta\Gamma^{\alpha}{}_{\mu\nu;\beta} - \delta\Gamma^{\alpha}{}_{\mu\beta;\nu}. \tag{A.14}$$

For the variation of Ricci,

$$\delta R_{\mu\nu} = \delta R^{\alpha}{}_{\mu\alpha\nu} = \delta\Gamma^{\alpha}{}_{\mu\nu;\alpha} - \delta\Gamma^{\alpha}{}_{\mu\alpha;\nu}. \tag{A.15}$$

Now, the variation of the action may be written as:

$$\begin{aligned} \delta S &= \frac{1}{16\pi}\int d^4x[(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + 16\pi(\frac{\delta L_{Fields}}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}L_{Fields}))\delta g^{\mu\nu} \\ &+g^{\mu\nu}(\delta\Gamma^{\kappa}{}_{\mu\nu;\kappa} - \delta\Gamma^{\kappa}{}_{\mu\kappa;\nu})]\sqrt{-g} = 0. \end{aligned} \tag{A.16}$$

The variation over the first derivatives of the coefficients remains. Apply the method of integration by parts:

$$\begin{aligned} &\frac{1}{16\pi}\int d^4x[g^{\mu\nu}(\delta\Gamma^{\kappa}{}_{\mu\nu;\kappa} - \delta\Gamma^{\kappa}{}_{\mu\kappa;\nu})]\sqrt{-g} \\ =\; &\frac{1}{16\pi}\int d^4x[g^{\mu\nu}\delta\Gamma^{\kappa}{}_{\mu\nu;\kappa}\sqrt{-g}] - \frac{1}{16\pi}\int d^4x[g^{\mu\nu}\delta\Gamma^{\kappa}{}_{\mu\kappa;\nu}\sqrt{-g}] \\ =\; &\frac{1}{16\pi}\{g^{\mu\nu}\delta\Gamma^{\kappa}{}_{\mu\nu}\sqrt{-g} - \int d^4x[\delta\Gamma^{\kappa}{}_{\mu\nu}(g^{\mu\nu}\sqrt{-g})_{;\kappa}] \\ &-g^{\mu\nu}\delta\Gamma^{\kappa}{}_{\mu\kappa}\sqrt{-g} + \int d^4x[\delta\Gamma^{\kappa}{}_{\mu\kappa}(g^{\mu\nu}\sqrt{-g})_{;\nu}]\} \\ =\; &\frac{1}{16\pi}\{g^{\mu\nu}\sqrt{-g}(\delta\Gamma^{\kappa}{}_{\mu\nu} - \delta\Gamma^{\kappa}{}_{\mu\kappa}) - \int d^4x[\delta\Gamma^{\kappa}{}_{\mu\nu}(g^{\mu\nu}\sqrt{-g})_{;\kappa} \\ &-\delta\Gamma^{\kappa}{}_{\mu\kappa}(g^{\mu\nu}\sqrt{-g})_{;\nu}]\} \\ =\; &\frac{1}{16\pi}\{g^{\mu\nu}\sqrt{-g}(\delta\Gamma^{\kappa}{}_{\mu\nu} - \delta\Gamma^{\kappa}{}_{\mu\kappa}) - \int d^4x[((g^{\mu\nu}\sqrt{-g})_{;\kappa} \\ &-\delta^{\nu}_{\kappa}(g^{\mu\lambda}\sqrt{-g})_{;\lambda})\delta\Gamma^{\kappa}{}_{\mu\nu}]\}. \end{aligned} \tag{A.17}$$

The first term is a term at the limits, so this variation will vanish (variation at the limits is zero). Apply symmetrization to the second term, and extremize the integral:

$$\frac{1}{16\pi}\int d^4x[((g^{\mu\nu}\sqrt{-g})_{;\kappa}-\frac{1}{2}\delta^{\mu}_{\kappa}(g^{\nu\lambda}\sqrt{-g})_{;\lambda}-\frac{1}{2}\delta^{\nu}_{\kappa}(g^{\mu\lambda}\sqrt{-g})_{;\lambda})\delta\Gamma^{\kappa}{}_{\mu\nu}]=0$$

$$\Rightarrow (g^{\mu\nu}\sqrt{-g})_{;\kappa}-\frac{1}{2}\delta^{\mu}_{\kappa}(g^{\nu\lambda}\sqrt{-g})_{;\lambda}-\frac{1}{2}\delta^{\nu}_{\kappa}(g^{\mu\lambda}\sqrt{-g})_{;\lambda}=0$$

$$\Rightarrow (g^{\mu\nu}\sqrt{-g})_{;\kappa}=\frac{1}{2}[\delta^{\mu}_{\kappa}(g^{\nu\lambda}\sqrt{-g})_{;\lambda}+\delta^{\nu}_{\kappa}(g^{\mu\lambda}\sqrt{-g})_{;\lambda}]$$

$$=\frac{1}{16\pi}\int d^4x[((g^{\mu\nu}\sqrt{-g})_{;\kappa}-\delta^{\nu}_{\kappa}(g^{\mu\lambda}\sqrt{-g})_{;\lambda})\delta\Gamma^{\kappa}{}_{\mu\nu}]. \quad \text{(A.18)}$$

Investigate the covariant derivatives of $(g^{\mu\nu}\sqrt{-g})$. Only the zero solution satisfies the forty covariant derivatives $(g^{\mu\nu}\sqrt{-g})_{;\kappa}$ for the set of forty equations $(g^{\mu\nu}\sqrt{-g})_{;\kappa}-\frac{1}{2}\delta^{\mu}_{\kappa}(g^{\nu\lambda}\sqrt{-g})_{;\lambda}-\frac{1}{2}\delta^{\nu}_{\kappa}(g^{\mu\lambda}\sqrt{-g})_{;\lambda}=0$, which implies covariant constancy. This is due to the fact that a locally flat arbitrary Lorentz frame exists. So $g^{\mu\nu}=\eta^{\mu\nu}$ in a small enough neighborhood around a point. Since the terms of $\eta^{\mu\nu}$ are constant-valued (Minkowski metric), first derivatives will be zero. From this, $\sqrt{-g}$, $g^{\mu\nu}$, $g_{\mu\nu}$, and $(g_{\mu\nu}\sqrt{-g})$ are covariantly constant also. So, the variation over the derivatives of the connection will vanish.

$$\sqrt{-g}_{;\tau}=\sqrt{-g}_{,\tau}-\Gamma^{\sigma}{}_{\sigma\tau}\sqrt{-g}$$

$$=\frac{\partial\sqrt{-g}}{\partial g}\frac{\partial g}{\partial g_{\mu\nu}}g_{\mu\nu,\tau}-\Gamma^{\sigma}{}_{\sigma\tau}\sqrt{-g}$$

$$=\frac{1}{2}\sqrt{-g}g^{\mu\nu}g_{\mu\nu,\tau}-[\frac{1}{2}g^{\alpha\sigma}(g_{\alpha\sigma,\tau}+g_{\alpha\tau,\sigma}-g_{\sigma\tau,\alpha})\sqrt{-g}]. \quad \text{(A.19)}$$

Rewrite the $\sqrt{-g}g^{\mu\nu}g_{\mu\nu,\tau}$ term via the product rule, and apply throughout the entire equation:

$$\sqrt{-g}g^{\mu\nu}g_{\mu\nu,\tau} = (\sqrt{-g}g^{\mu\nu}g_{\mu\nu})_{,\tau} - (\sqrt{-g}g^{\mu\nu})_{,\tau}g_{\mu\nu}. \tag{A.20}$$

Since the derivative of the contravariant metric density vanishes:

$$\Rightarrow \frac{1}{2}\{(\sqrt{-g}g^{\mu\nu}g_{\mu\nu})_{,\tau} - [(g^{\alpha\sigma}\sqrt{-g})(g_{\alpha\sigma,\tau} + g_{\alpha\tau,\sigma} - g_{\sigma\tau,\alpha})]\}$$

$$= \frac{1}{2}\{(\sqrt{-g}g^{\mu\nu}g_{\mu\nu})_{,\tau} - [(\sqrt{-g}g^{\alpha\sigma}g_{\alpha\sigma})_{,\tau} + (\sqrt{-g}g^{\alpha\sigma}g_{\alpha\tau})_{,\sigma} - (\sqrt{-g}g^{\alpha\sigma}g_{\sigma\tau})_{,\alpha}]\}$$

$$= \frac{1}{2}[-(\sqrt{-g}g^{\alpha\sigma}g_{\alpha\tau})_{,\sigma} + (\sqrt{g}g^{\alpha\sigma}g_{\sigma\tau})_{,\alpha}] = \frac{1}{2}[-(\sqrt{-g}\delta^{\sigma}_{\tau})_{,\sigma} + (\sqrt{-g}\delta^{\alpha}_{\tau})_{\alpha}]$$

$$= \frac{1}{2}[-\sqrt{-g}_{,\tau} + \sqrt{-g}_{,\tau}] = 0. \tag{A.21}$$

This is a consequence of the determinant's metric dependence. Covariant derivatives of the metric vanish, and so the covariant derivative of the $\sqrt{-g}$ vanishes as well.

Since the covariant derivative of the tensor density is zero from before, and the expression equals zero, then

$$g_{\nu\lambda;\kappa} = 0$$

$$= g_{\nu\lambda,\kappa} - g_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\kappa} - g_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\kappa}$$

$$\Rightarrow g_{\nu\lambda,\kappa} = g_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\kappa} + g_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\kappa}. \tag{A.22}$$

Now, consider permutations of the indices, first with λ and κ, then with ν, λ, and κ moved one slot to the left each:

$$g_{\nu\kappa,\lambda} = g_{\kappa\sigma}\Gamma^{\sigma}{}_{\nu\lambda} + g_{\nu\sigma}\Gamma^{\sigma}{}_{\kappa\lambda};$$

$$g_{\lambda\kappa,\nu} = g_{\kappa\sigma}\Gamma^{\sigma}{}_{\lambda\nu} + g_{\lambda\sigma}\Gamma^{\sigma}{}_{\kappa\nu}. \tag{A.23}$$

Using algebraic manipulation, sum the terms accordingly and group by metric tensor:

$$\begin{aligned}
& g_{\nu\lambda,\kappa} + g_{\nu\kappa,\lambda} - g_{\lambda\kappa,\nu} \\
= \; & g_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\kappa} + g_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\kappa} + g_{\kappa\sigma}\Gamma^{\sigma}{}_{\nu\lambda} + g_{\nu\sigma}\Gamma^{\sigma}{}_{\kappa\lambda} - g_{\kappa\sigma}\Gamma^{\sigma}{}_{\lambda\nu} - g_{\lambda\sigma}\Gamma^{\sigma}{}_{\kappa\nu} \\
= \; & g_{\lambda\sigma}\{\Gamma^{\sigma}{}_{\nu\kappa} - \Gamma^{\sigma}{}_{\kappa\nu}\} + g_{\nu\sigma}\{\Gamma^{\sigma}{}_{\lambda\kappa} + \Gamma^{\sigma}{}_{\kappa\lambda}\} + g_{\kappa\sigma}\{\Gamma^{\sigma}{}_{\nu\lambda} - \Gamma^{\sigma}{}_{\lambda\nu}\}.
\end{aligned} \tag{A.24}$$

Take advantage of the symmetry of the connection coefficients in their lower two indices ($\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta}$), and cancel terms as such:

$$\begin{aligned}
& g_{\nu\lambda,\kappa} + g_{\nu\kappa,\lambda} - g_{\lambda\kappa,\nu} \\
= \; & g_{\lambda\sigma}\{\Gamma^{\sigma}{}_{\nu\kappa} - \Gamma^{\sigma}{}_{\nu\kappa}\} + g_{\nu\sigma}\{\Gamma^{\sigma}{}_{\lambda\kappa} + \Gamma^{\sigma}{}_{\lambda\kappa}\} + g_{\kappa\sigma}\{\Gamma^{\sigma}{}_{\nu\lambda} - \Gamma^{\sigma}{}_{\nu\lambda}\} \\
= \; & g_{\nu\sigma}\{\Gamma^{\sigma}{}_{\lambda\kappa} + \Gamma^{\sigma}{}_{\lambda\kappa}\} = 2g_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\kappa}
\end{aligned} \tag{A.25}$$

$$\Rightarrow \Gamma^{\sigma}{}_{\lambda\kappa} = \frac{1}{2}g^{\nu\sigma}(g_{\nu\lambda,\kappa} + g_{\nu\kappa,\lambda} - g_{\lambda\kappa,\nu}). \tag{A.26}$$

This calculation recovers the equation for the connection coefficients in a coordinate basis, as required by Riemannian geometry.

Now, equating the coefficient of $\delta g^{\mu\nu}$ with zero, Einstein's equations can be recovered:

$$\Rightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + 16\pi\frac{\delta L_{Fields}}{\delta g^{\mu\nu}} - 16\pi\frac{1}{2}g_{\mu\nu}L_{Fields} = 0$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -16\pi\frac{\delta L_{Fields}}{\delta g^{\mu\nu}} + 8\pi g_{\mu\nu}L_{Fields}. \tag{A.27}$$

Setting the left hand side equal to $G_{\mu\nu}$ and the right hand side proportional to the stress-energy tensor, obtain the desired equation form:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{A.28}$$

$G_{\mu\nu}$ is the Einstein tensor, and $T_{\mu\nu}$ is the energy-momentum tensor that contains the gravitational source information.

Appendix B

Derivation of and Mathematical Tools for the Split Spacetime Formulation

This appendix bridges the gap between the action principle of the Lagrangian formulation to the splitting of space and time in the Hamiltonian formulation. The progression follows from Misner, Thorne, and Wheeler [1], Chapter 21, problems 8 - 10 and from Wald [2], Appendix E and Chapter 10.

B.1 Lie Derivatives

The Lie derivative is an important quantity for studying transport of tensorial objects along the direction of other objects. An advantage to this approach is the independence from an affine parameter of the derivative. Lie derivatives operate by converting tensor fields of type $\left\{ {r \atop s} \right\}$ into tensor fields of the same type. The following computations concern the Lie derivative of scalars, vectors, one-forms, and rank-2 covariant tensors carried out in a coordinate basis.

The Lie derivative of a scalar quantity is straightforward:

$$\pounds_{\mathbf{u}} f \equiv \mathbf{u}[f] = u^\alpha \mathbf{e}_\alpha f = f_\alpha u^\alpha. \tag{B.1}$$

For the derivative of a vector field $\mathbf{v}$ along a vector field $\mathbf{u}$:

$$\begin{aligned}
[u, v](f) = u[v(f)] - v[u(f)] = u^l e_l(v^m e_m f) - v^l e_l(u^m e_m f) \\
= u^l v^m{}_{,l} e_m f - v^l u^m{}_{,l} e_m f = u^l v^m{}_{,l} f_{,m} - v^l u^m{}_{,l} f_{,m} \\
\Rightarrow [u, v] = [u^l v^m{}_{,l} - v^l u^m{}_{,l}] e_m \\
\Rightarrow [n, v] = [n^\alpha v^\beta{}_{,\alpha} - v^\alpha u^\beta{}_{,\alpha}] e_\beta,
\end{aligned} \tag{B.2}$$

where $\mathbf{n}$ is another vector field and is only a notational change.

Consider the Lie derivative of a one-form:

$$\begin{aligned}
\pounds_n < \sigma, v > &= < \pounds_n \sigma, v > + < \sigma, \pounds_n v > \\
\Rightarrow < \pounds_n \sigma, v > &= n[< \sigma, v >] - < \sigma, [n, v] > \\
&= < \sigma, v >_{,\beta} n^\beta - \sigma_\alpha (n^\alpha v^\beta{}_{,\alpha} - v^\alpha n^\beta{}_{,\alpha}) \frac{\partial}{\partial x^\beta} dx^\alpha \\
&= \sigma_\alpha v^\alpha{}_{,\beta} n^\beta + \sigma_{\alpha,\beta} v^\alpha n^\beta - \sigma_\alpha n^\alpha v^\beta{}_{,\alpha} \delta_\beta{}^\alpha + \sigma_\alpha v^\alpha n^\beta{}_{,\alpha} \delta_\beta{}^\alpha \\
&= \sigma_\alpha (v^\alpha{}_{,\beta} n^\beta - n^\alpha v^\beta{}_{,\alpha} \delta_\beta{}^\alpha) + (\sigma_{\alpha,\beta} n^\beta + \sigma_\alpha n^\beta{}_{,\alpha} \delta_\beta{}^\alpha) v^\alpha \\
&= (\sigma_{\alpha,\beta} n^\beta + \sigma_\alpha n^\beta{}_{,\alpha} \delta_\beta{}^\alpha) v^\alpha = (\sigma_{\alpha,\beta} n^\beta + \sigma_\beta n^\beta{}_{,\alpha}) v^\alpha \\
&= (\sigma_{\alpha,\beta} n^\beta + \sigma_\beta n^\beta{}_{,\alpha}) v^\beta \delta_\alpha{}^\beta \\
&= (\sigma_{\alpha,\beta} n^\beta + \sigma_\beta n^\beta{}_{,\alpha}) v^\beta dx^\alpha \frac{\partial}{\partial x^\beta} \\
\Rightarrow \pounds_n \sigma &= (\sigma_{\alpha,\beta} n^\beta + \sigma_\beta n^\beta{}_{,\alpha}) dx^\alpha
\end{aligned} \tag{B.3}$$

Since the Lie derivative satisfies a product rule, it is a derivation (linear operator and product rule), so tensors of general types compose of tensor products of vectors and 1-forms:

$$
\begin{aligned}
\pounds_n T &= \pounds_n(\sigma \otimes \lambda) \\
&= \pounds_n \sigma \otimes \lambda + \sigma \otimes \pounds_n \lambda \\
&= [(\sigma_{\alpha,\beta} n^\beta + \sigma_\beta n^\beta{}_{,\alpha}) dx^\alpha] \otimes \lambda_\gamma \omega^\gamma + \sigma_\alpha \omega^\alpha \otimes [(\lambda_{\gamma,\beta} n^\beta + \lambda_\beta n^\beta{}_{,\gamma})] dx^\gamma \\
&= [(\sigma_{\alpha,\beta} \lambda_\gamma + \sigma_\alpha \lambda_{\gamma,\beta}) n^\beta + \sigma_\beta \lambda_\gamma n^\beta{}_{,\alpha} + \sigma_\alpha \lambda_\beta n^\beta{}_{,\gamma}] dx^\alpha \otimes dx^\gamma \\
&= [(\sigma_\alpha \lambda_\gamma)_{\,\beta} n^\beta + \sigma_\beta \lambda_\gamma n^\beta{}_{,\alpha} + \sigma_\alpha \lambda_\beta n^\beta{}_{,\gamma}] dx^\alpha \otimes dx^\gamma \\
&= [T_{\alpha\gamma,\beta} n^\beta + T_{\beta\gamma} n^\beta{}_{,\alpha} + T_{\alpha\beta} n^\beta{}_{,\gamma}] dx^\alpha \otimes dx^\gamma. \qquad \text{(B.4)}
\end{aligned}
$$

The Lie derivative possesses no affine dependence (i.e., the connections equal zero), so the partial derivatives become covariant:

$$
\pounds_n T = [T_{\alpha\gamma;\beta} n^\beta + T_{\beta\gamma} n^\beta{}_{;\alpha} + T_{\alpha\beta} n^\beta{}_{;\gamma}] dx^\alpha \otimes dx^\gamma. \qquad \text{(B.5)}
$$

B.2 Intrinsic and Extrinsic Curvature

Using this information, the six algebraically independent components of the Riemann tensor follow. Define a timelike unit normal field $\mathbf{u}$ of a spacelike hypersurface, and let Lie derivation in its direction yield a general time differentiation. Let $\gamma_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu$ be the spatial metric induced on the hypersurface by the 4-metric. The 4-acceleration of the timelike normal curves with tangent vector field u^μ is a curvature vector, $a^\mu \equiv u^\kappa \nabla_\kappa u^\mu$. Two important algebraic properties allow for simplication of many terms: $u_\mu u^\mu = -1$, and $u_\mu a^\mu = 0$.

Begin with the Lie derivative along this tangent field of the metric tensor:

$$
\pounds_u g_{\mu\nu} = [g_{\mu\nu;\kappa} u^\kappa + g_{\kappa\nu} u^\kappa{}_{;\mu} + g_{\mu\kappa} u^\kappa{}_{;\nu}]
$$

$$
\begin{aligned}
&= [g_{\kappa\nu}u^{\kappa}{}_{;\mu} + g_{\mu\kappa}u^{\kappa}{}_{;\nu}] \\
&= [u_{\nu;\mu} + u_{\mu;\nu}]. \qquad \text{(B.6)}
\end{aligned}
$$

Consider now the Lie derivative of the spatial metric; since the Lie derivative obeys additivity:

$$
\begin{aligned}
\pounds_u(g_{\mu\nu} + u_\mu u_\nu) &= \pounds_u g_{\mu\nu} + \pounds_u(u_\mu u_\nu) \\
&= \pounds_u g_{\mu\nu} + \pounds_u(u_\mu)u_\nu + u_\mu \pounds_u(u_\nu) \equiv \pounds_u \gamma_{\mu\nu} \\
&= [\gamma_{\mu\nu;\kappa}u^{\kappa} + \gamma_{\kappa\nu}u^{\kappa}{}_{;\mu} + \gamma_{\mu\kappa}u^{\kappa}{}_{;\nu}] \\
&= [g_{\mu\nu;\kappa}u^{\kappa} + u_{\mu;\kappa}u_\nu u^{\kappa} + u_\mu u_{\nu;\kappa}u^{\kappa} + g_{\kappa\nu}u^{\kappa}{}_{;\mu} \\
&\quad + u_\nu u_\kappa u^{\kappa}{}_{;\mu} + g_{\mu\kappa}u^{\kappa}{}_{;\nu} + u_\mu u_\kappa u^{\kappa}{}_{;\nu}] \\
&= [u_{\nu;\mu} + u_{\mu;\nu} + u_{\mu;\kappa}u_\nu u^{\kappa} + u_\mu u_{\nu;\kappa}u^{\kappa} \\
&\quad + u_\nu u_\kappa u^{\kappa}{}_{;\mu} + u_\mu u_\kappa u^{\kappa}{}_{;\nu}]. \qquad \text{(B.7)}
\end{aligned}
$$

Using the definition of the 4-acceleration:

$$
\begin{aligned}
&[u_{\nu;\mu} + u_{\mu;\nu} + u_\nu u^{\kappa}u_{\mu;\kappa} + u_\mu u^{\kappa}u_{\nu;\kappa} + u_\nu u_\kappa u^{\kappa}{}_{;\mu} + u_\mu u_\kappa u^{\kappa}{}_{;\nu}] \\
&\quad = [u_{\nu;\mu} + u_{\mu;\nu} + u_\nu a_\mu + u_\mu a_\nu + u_\nu u_\kappa u^{\kappa}{}_{;\mu} + u_\mu u_\kappa u^{\kappa}{}_{;\nu}]. \qquad \text{(B.8)}
\end{aligned}
$$

Rewriting the last terms in the expression as $u_\alpha u_\rho u^{\rho}{}_{;\beta} = u_\alpha u_\rho \nabla_\beta u^\rho = u_\rho u^\beta g_{\beta\alpha}\nabla_\beta u^\rho = u_\rho a^\rho g_{\alpha\beta}$, recover each term's equality with 0 from the identity $u_\mu a^\mu = 0$ so that

$$
\begin{aligned}
\pounds_u \gamma_{\mu\nu} &= [u_{\nu;\mu} + u_{\mu;\nu} + u_\nu a_\mu + u_\mu a_\nu + u_\kappa u^\mu g_{\mu\nu}u^{\kappa}{}_{;\mu} + u_\kappa u^\nu g_{\nu\mu}u^{\kappa}{}_{;\nu}] \\
&= [u_{\nu;\mu} + u_{\mu;\nu} + u_\nu a_\mu + u_\mu a_\nu]. \qquad \text{(B.9)}
\end{aligned}
$$

With this information, rederive the extrinsic curvature tensor as the Lie derivative of the spatial metric via comparison of the coordinate-components of before with those from the derivative:

$$
\begin{aligned}
-\frac{1}{2}\pounds_u \gamma_{\mu\nu} &= -\frac{1}{2}[u_{\nu;\mu} + u_{\mu;\nu} + u_\nu a_\mu + u_\mu a_\nu] \\
&= -\frac{1}{2}[-K_{\nu\mu} - K_{\mu\nu} + u_\nu a^\nu g_{\nu\mu} + u_\mu a^\mu g_{\mu\nu}].
\end{aligned} \tag{B.10}
$$

From the symmetry of the extrinsic curvature tensor:

$$
\begin{aligned}
-K_\mu{}^\nu e_\nu &= \nabla_\mu u^\nu e_\nu \\
\Rightarrow e_\kappa \nabla_\mu u^\nu e_\nu &= e_\kappa(-K_\mu{}^\nu e_\nu) = -K_\mu{}^\nu g_{\kappa\nu} = -K_{\mu\kappa}; \\
e_m u^0 = 0 \Rightarrow \nabla_\mu(e_m u^0) &= (\nabla_\mu e_m)u^0 + e_m(\nabla_\mu u^0) = 0 \\
\Rightarrow (\nabla_\mu e_m)u^0 &= -e_m(\nabla_\mu u^0) \\
\Rightarrow K_{\mu\kappa} &= (\nabla_\mu e_\kappa)u^0 \\
= (\Gamma^\alpha{}_{\mu\kappa} e_\alpha)u^0 &= u^0 e_0 \Gamma^0{}_{\mu\kappa} + u^m e_m \Gamma^m{}_{\mu\kappa} \\
= u^0 e_0 \Gamma^0{}_{\mu\kappa} &= u^0 \nabla_\kappa e_\mu = K_{\kappa\mu}.
\end{aligned} \tag{B.11}
$$

Using again the identity $u_\mu a^\mu = 0$ and exploiting the symmetry of the extrinsic curvature tensor:

$$
-\frac{1}{2}\pounds_u \gamma_{\mu\nu} = -\frac{1}{2}[-2K_{\mu\nu}] = K_{\mu\nu}. \tag{B.12}
$$

Define the unit projection tensor into the hypersurface as $\perp^\mu_\nu \equiv \delta^\mu_\nu + u^\mu u_\nu$; this equates to the mixed-index spatial metric tensor. Also, define $K_{\alpha\beta} = - \perp^\mu_\alpha \perp^\nu_\beta u_{(\mu;\nu)}$ and $\omega_{\alpha\beta} = - \perp^\mu_\alpha \perp^\nu_\beta u_{[\mu;\nu]}$. Then,

$$
u_{\alpha;\beta} \equiv -K_{\alpha\beta} - \omega_{\alpha\beta} - a_\alpha u_\beta
$$

$$
\begin{aligned}
&= \perp_\alpha^\mu \perp_\beta^\nu u_{(\mu;\nu)} + \perp_\alpha^\mu \perp_\beta^\nu u_{[\mu;\nu]} - a_\alpha u_\beta \\
&= \perp_\alpha^\mu \perp_\beta^\nu u_{\mu;\nu} - a_\alpha u_\beta = (\delta_\alpha^\mu + u^\mu u_\alpha)(\delta_\beta^\nu + u^\nu u_\beta) u_{\mu;\nu} - a_\alpha u_\beta \\
&= u_{\alpha;\beta} + u^\nu u_\beta u_{\alpha;\nu} + u^\mu u_\alpha u_{\mu;\beta} + u^\mu u_\alpha u^\nu u_\beta u_{\mu;\nu} - a_\alpha u_\beta \\
&= u_{\alpha;\beta} + u_\beta u^\nu u_{\alpha;\nu} + u_\alpha u^\mu u_{\mu;\beta} + u^\mu u_\alpha u_\beta u^\nu u_{\mu;\nu} - a_\alpha u_\beta \\
&= u_{\alpha;\beta} + u_\beta a_\alpha + u_\alpha u^\mu u_{\mu;\beta} + u^\mu u_\alpha u_\beta a_\mu - a_\alpha u_\beta \\
&= u_{\alpha;\beta} + u_\beta a_\alpha - a_\alpha u_\beta + u_\alpha u^\mu u_{\mu;\beta} + u_\alpha u_\beta u^\mu a_\mu \\
&= u_{\alpha;\beta}. \qquad \text{(B.13)}
\end{aligned}
$$

This relation yields the covariant derivative of this tangent field in terms of extrinsic curvature and the acceleration/curvature vector of the timelike normal curves. Upon further investigation,

$$
\begin{aligned}
\omega_{\alpha\beta} &= - \perp_\alpha^\mu \perp_\beta^\nu u_{[\mu;\nu]} \\
&= -(\delta_\alpha^\mu + u^\mu u_\alpha)(\delta_\beta^\nu + u^\nu u_\beta)\frac{1}{2}(u_{\mu;\nu} - u_{\nu;\mu}) \\
&= -\frac{1}{2}(u_{\alpha;\beta} + u^\nu u_\beta u_{\alpha;\nu} + u^\mu u_\alpha u_{\mu;\beta} + u^\mu u_\alpha u^\nu u_\beta u_{\mu;\nu} - u_{\beta;\alpha} \\
&\quad - u^\mu u_\alpha u_{\beta;\mu} - u^\nu u_\beta u_{\nu;\alpha} - u^\mu u_\alpha u^\nu u_\beta u_{\nu;\mu}) \\
&= -\frac{1}{2}(u_{\alpha;\beta} + u_\beta a_\alpha + u_\alpha u^\mu u_{\mu;\beta} + u_\alpha u_\beta u^\mu a_\mu - u_{\beta;\alpha} \\
&\quad - u_\alpha a_\beta - u_\beta u^\nu u_{\nu;\alpha} - u_\alpha u_\beta u^\nu a_\nu) \\
&= -\frac{1}{2}(u_{\alpha;\beta} + u_\beta a_\alpha - u_{\beta;\alpha} - u_\alpha a_\beta) \\
&= -\frac{1}{2}(u_{\alpha;\beta} + u_\beta u^\beta \nabla_\beta u_\alpha - u_{\beta;\alpha} - u_\alpha u^\alpha \nabla_\alpha a_\beta). \qquad \text{(B.14)}
\end{aligned}
$$

Using the fact that $u_\kappa u^\kappa = -1$:

$$
\begin{aligned}
\omega_{\alpha\beta} &= -\frac{1}{2}(u_{\alpha;\beta} + u_\beta u^\beta \nabla_\beta u_\alpha - u_{\beta;\alpha} - u_\alpha u^\alpha \nabla_\alpha a_\beta) \\
&= -\frac{1}{2}(u_{\alpha;\beta} - \nabla_\beta u_\alpha - u_{\beta;\alpha} + \nabla_\alpha u_\beta)
\end{aligned}
$$

$$= -\frac{1}{2}(u_{\alpha;\beta} - u_{\alpha;\beta} - u_{\beta;\alpha} + u_{\beta;\alpha}) = 0. \tag{B.15}$$

This follows from u^μ being the unit normal field for a family of spacelike hypersurfaces. So,

$$\begin{aligned} u_{\alpha;\beta} &= -K_{\alpha\beta} - a_\alpha u_\beta \\ &= -K_{\alpha\beta} - g_{\alpha\beta} a^\beta u_\beta = -K_{\alpha\beta}. \end{aligned} \tag{B.16}$$

Use this information to redress the Lie derivative of the extrinsic curvature:

$$\pounds_u K_{\mu\nu} = (K_{\mu\nu;\kappa} u^\kappa + K_{\kappa\nu} u^\kappa{}_{;\mu} + K_{\mu\kappa} u^\kappa{}_{;\nu})$$

$$= -\frac{1}{2}([u_{\mu;\nu\kappa} + u_{\nu;\mu\kappa} + u_{\mu;\kappa} a_\nu + u_\mu a_{\nu;\kappa} + a_{\mu;\kappa} u_\nu + a_\mu u_{\nu;\kappa}]u^\kappa$$

$$= (K_{\mu\nu;\kappa} u^\kappa - K_{\kappa\nu} K^\kappa{}_\mu - K_{\mu\kappa} K^\kappa{}_\nu)$$

$$= (-u^\kappa \nabla_\kappa \nabla_\nu u_\mu - K_{\kappa\nu} K^\kappa{}_\mu - K_{\mu\kappa} K^\kappa{}_\nu). \tag{B.17}$$

Consider the Ricci identity:

$$u^\sigma \nabla_\sigma \nabla_\lambda u_\tau = u^\sigma \nabla_\lambda \nabla_\sigma u_\tau +^{(4)} R_{\rho\tau\lambda\sigma} u^\sigma u^\rho. \tag{B.18}$$

Use this in the definition of the Lie derivative of **K**:

$$\pounds_u K_{\mu\nu} = -u^\kappa \nabla_\kappa \nabla_\nu u_\mu - K_{\kappa\nu} K^\kappa{}_\mu - K_{\mu\kappa} K^\kappa{}_\nu$$

$$= -u^\kappa \nabla_\nu \nabla_\kappa u_\mu -^{(4)} R_{\rho\mu\nu\kappa} u^\kappa u^\rho - K_{\kappa\nu} K^\kappa{}_\mu - K_{\mu\kappa} K^\kappa{}_\nu$$

$$\Rightarrow^{(4)} R_{\rho\mu\nu\kappa} u^\kappa u^\rho = -u^\kappa \nabla_\nu \nabla_\kappa u_\mu - K_{\kappa\nu} K^\kappa{}_\mu - K_{\mu\kappa} K^\kappa{}_\nu - \pounds_u K_{\mu\nu}$$

$$= -u^\kappa \nabla_\nu \nabla_\kappa u_\mu - 2K_{\mu\kappa} K_{\kappa\nu} g^{\kappa\kappa} - \pounds_u K_{\mu\nu}. \tag{B.19}$$

Permuting indices:

$$^{(4)}R_{\mu\rho\nu\kappa} u^\kappa u^\rho = u^\kappa \nabla_\nu \nabla_\kappa u_\mu + K_{\kappa\nu} K^\kappa{}_\mu + K_{\mu\kappa} K^\kappa{}_\nu + \pounds_u K_{\mu\nu} \tag{B.20}$$

Project the remaining free indices:

$$\begin{aligned} \perp^\mu{}_\alpha \perp^{\nu(4)}_\beta R_{\mu\rho\nu\kappa} u^\kappa u^\rho &= \frac{1}{2} \perp^\mu{}_\alpha \perp^\nu{}_\beta [^{(3)}\nabla_\mu a_\nu +^{(3)} \nabla_\nu a_\mu] + \perp^\mu_\alpha \perp^\nu_\beta \, a_\mu a_\nu \\ &\quad + K_{\alpha\kappa} K^\kappa{}_\beta + \pounds_{\mathbf{u}} K_{\alpha\beta}. \end{aligned} \tag{B.21}$$

Using this information, write the projection of $^{(4)}R^n{}_{inj}$:

$$\perp^{(4)} R_{ninj} = \frac{1}{N} \pounds_{\mathbf{Nu}} K_{ij} + K_{ik} K^k{}_j + \frac{1}{N} N_{|ji}. \tag{B.22}$$

To determine the evolution equations, rewrite the Lie derivative of $\gamma_{\mu\nu}$ using Eq. (3.9) and the fact that the natural orthogonal vector field is Nu^μ [15]:

$$\begin{aligned} \pounds_{\mathbf{Nu}} \gamma_{\mu\nu} &= \pounds_{\mathbf{t}} \gamma_{\mu\nu} - \pounds_{\mathbf{N}} \gamma_{\mu\nu} \\ &= -2N K_{\mu\nu}. \end{aligned} \tag{B.23}$$

This gives the dynamic equations for the spatial metric. Now, using the projection of

R_{ij}, consider the rewritten Lie derivative of K_{ij}:

$$\begin{aligned} \pounds_{\mathbf{Nu}} K_{ij} &= \pounds_{\mathbf{t}} K_{ij} - \pounds_{\mathbf{N}} K_{ij} \\ &= -N_{|ij} + N[{}^{(3)}R_{ij} - 2K_{ik}K^{k}{}_{j} + K_{ij} Tr\mathbf{K} - 8\pi \gamma_i{}^{\nu} \gamma_j{}^{\mu} T_{\mu\nu} \\ &\quad -4\pi \gamma_{ij}(n^{\mu} n^{\nu} T_{\mu\nu} - \gamma^{ab} \gamma_a{}^{\nu} \gamma_b{}^{\mu} T_{\mu\nu})]. \end{aligned} \tag{B.24}$$

This gives the dynamics of the extrinsic curvature.

Appendix C

Evolution Of Spacetime With Presence Of The C-Field

C.1 The Direct Particle Field Action

This appendix details an example of the ADM formulation applied to a scalar field coupled to matter (non-zero mass scalar field). To attain this formulation, construction of the field theory is required.

Consider an action inspired by Hoyle and Narlikar [16], [17], [23]:

$$S = \frac{1}{16\pi}\int R\sqrt{-g}d^4x + \int L_{C-Field}\sqrt{-g}d^4x. \tag{C.1}$$

The first term corresponds to the geometric action, and the second term represents the contribution from the C-field. From the definitions in Chapter 5, this takes the form [20]

$$\begin{aligned} S[g_{\mu\nu}, C^{(a)}(x^\mu), X^\mu] &= \frac{1}{16\pi}\int R\sqrt{-g}d^4x - \sum_a m_a \int da \\ &+\frac{1}{f}\sum_a\sum_b\int\int G_{;\mu\nu}dX^\mu(a)dX^\nu(b) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{16\pi}\int R\sqrt{-g}d^4x - \sum_a m_a\sqrt{g_{\mu\nu}dX^\mu(a)dX^\nu(a)} \\
&\quad +\frac{1}{f}\sum_a\sum_b\int\int G_{;\mu\nu}dX^\mu(a)dX^\nu(b) \\
&= \frac{1}{16\pi}\int R\sqrt{-g}d^4x - \sum_a m_a\sqrt{g_{\mu\nu}dX^{\mu'}(a)g_{\mu'}{}^{\mu}dX^{\nu'}(a)g_{\nu'}{}^{\nu}} \\
&\quad +\frac{1}{f}\sum_a\sum_b\int\int G_{;\mu\nu}dX^\mu(a)dX^\nu(b) \\
&= \frac{1}{16\pi}\int R\sqrt{-g}d^4x \\
&\quad -\sum_a m_a\int\sqrt{g_{\mu\nu}\frac{dX^{\mu'}(a)}{da}g_{\mu'}{}^{\mu}\frac{dX^{\nu'}(a)}{da}g_{\nu'}{}^{\nu}}da \\
&\quad +\frac{1}{f}\sum_a\sum_b\int\int G_{;\mu\nu}dX^\mu(a)dX^\nu(b) \\
&= \frac{1}{16\pi}\int R\sqrt{-g}d^4x \\
&\quad -\sum_a m_a\int\sqrt{g_{\mu\nu}\frac{dX^{\mu'}(a)}{da}\frac{dX^{\nu'}(a)}{da}g_{\mu'}{}^{\mu}g_{\nu'}{}^{\nu}}da\sqrt{-g}d^4x \\
&\quad +\frac{1}{f}\sum_a\sum_b\int\int G_{;\mu\nu}dX^\mu(a)dX^\nu(b). \qquad \text{(C.2)}
\end{aligned}
$$

This satisfies the condition that $a < b$, with the sum notations a and b representing worldlines of created particles. $X^\mu(a)$ and $X^\mu(b)$ represent the coordinates for given points on the worldlines a and b, respectively. a is the proper time at a given point on the worldline a. $X^\mu(a)$ is a point on the worldline a (parametrized by the worldline) that represents the events (endpoints of worldlines) where particle creation/destruction occurs. This action resembles that of the Fokker action principle, though this work does not address the EM field tensor. Vary this action with respect to the metric tensor:

$$
\delta S = \frac{1}{16\pi}\int \delta[R\sqrt{-g}]d^4x - \sum_a m_a\int\delta[da] + \frac{1}{f}\sum_a\sum_b\int\int\delta[G_{;\mu\nu}dX^\mu(a)dX^\nu(b)] \quad \text{(C.3)}
$$

For the first term, the variation was worked out in Appendix A, and so this yields the term $[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R]\delta g^{\mu\nu}$. Consider variation of the second term [11]:

$$
\begin{aligned}
&\sum_a m_a \int \delta[da] = \sum_a m_a \int \delta[\sqrt{-g_{\mu\nu} dX^\mu(a) dX^\nu(a)}] \\
= \ & \sum_a m_a \int \delta[\sqrt{-g_{\mu\nu}\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}} d\lambda] \\
= \ & -\frac{1}{2}\sum_a m_a \int d\lambda [-g_{\mu\nu}\frac{dX^{\mu'}}{d\lambda}\frac{dX^{\nu'}}{d\lambda} g_{\mu'}{}^{\mu} g_{\nu'}{}^{\nu}]^{-\frac{1}{2}} \frac{dX^{\mu'}}{d\lambda}\frac{dX^{\nu'}}{d\lambda} g_{\mu'}{}^{\mu} g_{\nu'}{}^{\nu}\sqrt{-g} d^4x \delta g_{\mu\nu} \\
= \ & -\frac{1}{2}\sum_a m_a \int d\lambda [-g_{\mu'\nu'}\frac{dX^{\mu'}}{d\lambda}\frac{dX^{\nu'}}{d\lambda}]^{-\frac{1}{2}} \frac{dX^{\mu}}{d\lambda}\frac{dX^{\nu}}{d\lambda}\sqrt{-g} d^4x \delta g_{\mu\nu} \\
= \ & -\frac{1}{2}\sum_a m_a \int da \frac{dX^{\mu}}{da}\frac{dX^{\nu}}{da}\sqrt{-g} d^4x \delta g_{\mu\nu} \\
= \ & \frac{1}{2}\sum_a m_a \int da \frac{\delta^{(4)}(x^\mu - X^\mu(a))}{\sqrt{-g}} \frac{dX^{\mu}(a)}{da}\frac{dX^{\nu}(a)}{da}\sqrt{-g} d^4x \delta g_{\mu\nu},
\end{aligned} \tag{C.4}
$$

where λ is an arbitrary parameter. Vary the last term:

$$
\begin{aligned}
\frac{1}{f}\sum_a\sum_b \int\int \delta[G_{;\mu\nu} dX^\mu(a) dX^\nu(b)] = \ & \frac{1}{f}\sum_a\sum_b \int\int \delta[G_{;\mu\nu}] dX^\mu(a) dX^\nu(b) \\
& + \frac{1}{f}\sum_a\sum_b \int\int G_{;\mu\nu}\delta[dX^\mu(a) dX^\nu(b)].
\end{aligned} \tag{C.5}
$$

Before proceeding, consider variations of the function G, with x a point in spacetime and A a point on the worldline a:

$$
\delta G(x - A(a)) = -\int [\delta(\sqrt{-g} g^{\mu\nu})[G(x' - A(a))]_{,\nu}]_{,\mu} G(x - x') d^4x'. \tag{C.6}
$$

Integrate this term by parts, and use the relation $G(x - x') = G(x' - x)$:

$$
\begin{aligned}
\delta G(x - A(a)) &= \delta(\sqrt{-g} g^{\mu\nu})[G(x' - A(a))] G(x - x') \\
&\quad - \int [\delta(\sqrt{-g} g^{\mu\nu})[G(x' - A(a))]_{,\nu}[G(x - x')]_{,\mu}] d^4x' \\
&= -\int [\delta(\sqrt{-g} g^{\mu\nu})[G(x' - A(a))]_{,\nu}[G(x - x')]_{,\mu}] d^4x'.
\end{aligned} \tag{C.7}
$$

Consider point A on worldline a and point B on worldline b. The latter point replaces the more generic point x. Then,

$$\delta[G(A(a)-B(b))_{;\mu\nu}] = -\int \delta(\sqrt{-g}g^{\mu\nu})[G(A(a)-x')]_{;\nu\mu'}[G(B(b)-x')]_{;\mu\nu'}d^4x'. \quad \text{(C.8)}$$

This means, for the final term,

$$\begin{aligned}
&\frac{1}{f}\sum_a\sum_b\int\int\delta[G_{;\mu\nu}da^\mu db^\nu] \\
= &-\frac{1}{f}\int d^4x'\sum_a\sum_b\int\int\delta(\sqrt{-g}g^{\mu\nu})[G(A(a)-x')]_{;\nu\mu'}[G(B(b)-x')]_{;\mu\nu'}da^{\mu'}db^{\nu'} \\
= &-f\int d^4x'\sum_a\sum_b\delta(\sqrt{-g}g^{\mu\nu})C^{(a)}{}_{;\nu}C^{(b)}{}_{;\mu} \\
= &-f\int d^4x'\sum_a\sum_b[C^{(a)}{}_{;\nu}C^{(b)}{}_{;\mu}g^{\mu\nu}\delta(\sqrt{-g})+C^{(a)}{}_{;\nu}C^{(b)}{}_{;\mu}\sqrt{-g}\delta g^{\mu\nu}] \\
= &-f\int d^4x'\sum_a\sum_b[\frac{1}{2}C^{(a);\kappa}C^{(b)}{}_{;\kappa}g^{\mu\nu}\sqrt{-g}\delta g_{\mu\nu}+C^{(a)}{}_{;\nu}C^{(b)}{}_{;\mu}\sqrt{-g}\delta g^{\mu\nu}] \\
= &-f\int d^4x'\sum_a\sum_b[-\frac{1}{2}C^{(a);\kappa}C^{(b)}{}_{;\kappa}g_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}+C^{(a)}{}_{;\nu}C^{(b)}{}_{;\mu}\sqrt{-g}\delta g^{\mu\nu}] \\
= &-f\int d^4x'\sum_a\sum_b[-\frac{1}{2}C^{(a);\kappa}C^{(b)}{}_{;\kappa}g_{\mu\nu}+C^{(a)}{}_{;\nu}C^{(b)}{}_{;\mu}]\sqrt{-g}\delta g^{\mu\nu} \\
= &\ f\int d^4x'\sum_a\sum_b[-\frac{1}{2}C^{(a);\kappa}C^{(b)}{}_{;\kappa}g^{\mu\nu}+C^{(a);\nu}C^{(b);\mu}]\sqrt{-g}\delta g_{\mu\nu} \qquad \text{(C.9)}
\end{aligned}$$

For Einstein's equations in this representation:

$$\begin{aligned}
R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R = 8\pi[\sum_a m_a\int[&\frac{\delta^{(4)}(x^\mu-X^\mu(a))}{\sqrt{-g}}\frac{dX^{\mu'}(a)}{da}\frac{dX^{\nu'}(a)}{da}g_{\mu'}{}^{\mu}g_{\nu'}{}^{\nu}]da \\
&-2f\sum_a\sum_b[-\frac{1}{2}C^{(a);\kappa}C^{(b)}{}_{;\kappa}g^{\mu\nu}+C^{(a);\nu}C^{(b);\mu}]] \\
= T^{\mu\nu}-f\sum_a\sum_b[C^{(a);\nu}C^{(b);\mu}&+C^{(a);\mu}C^{(b);\nu}-C^{(a);\kappa}C^{(b)}{}_{;\kappa}g^{\mu\nu}]. \quad \text{(C.10)}
\end{aligned}$$

The energy-momentum tensor here contains the mass terms from the created particles.

For the smooth-fluid approximation, $C = \sum_a m_a C^{(a)}$, and

$$\begin{aligned}
R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R &= 8\pi \sum_a m_a \int [\frac{\delta^{(4)}(x^\mu - X^\mu(a))}{\sqrt{-g}} \frac{dX^{\mu'}(a)}{da} \frac{dX^{\nu'}(a)}{da} g_{\mu'}{}^{\mu} g_{\nu'}{}^{\nu}]da \\
&\quad -f[-\frac{1}{2}C^{;\kappa}C_{;\kappa}g^{\mu\nu} + C^{;\nu}C^{;\mu}] \\
&= 8\pi \sum_a m_a \int [\frac{\delta^{(4)}(x^\mu - X^\mu(a))}{\sqrt{-g}} \frac{dX^{\mu}(a)}{da} \frac{dX^{\nu}(a)}{da} da] \\
&\quad -f \cdot [-\frac{1}{2}C^{;\kappa}C_{;\kappa}g^{\mu\nu} + C^{;\nu}C^{;\mu}] \\
&= T^{\mu\nu} - f \cdot [C^{;\nu}C^{;\mu} - \frac{1}{2}C^{;\kappa}C_{;\kappa}g^{\mu\nu}]. \qquad \text{(C.11)}
\end{aligned}$$

These match the form Einstein's equations written by Hoyle and Narlikar [17]. Consider the definition of the C-field:

$$C^{(a)}(x^\mu) = \frac{1}{f} \int G(x^\mu, X^\mu(a))_{;\mu} dX^\mu. \qquad \text{(C.12)}$$

This is the contribution of the worldline a to the total C-field at the point given by coordinates x^μ [19]. For a worldline with endpoints A_1 and A_2,

$$C^{(a)}(x^\mu) = \frac{1}{f}[G(x^\mu, A_2(a)) - G(x^\mu, A_1(a))]. \qquad \text{(C.13)}$$

This implies that the C-field arises only from the ends of worldlines. [21] If the worldline is created at A_1 ($A_2 \to \infty$), then $C^{(a)}(x) = -\frac{1}{f}G(x, A_1(a))$. If it is destroyed at A_2 ($A_1 \to -\infty$), then $C^{(a)}(x) = \frac{1}{f}G(x, A_2(a))$. The result is

$$C^{(a)}(x) = \mp \frac{1}{f} G(x, A(a)), \qquad \text{(C.14)}$$

for creation/destruction at point A. For the Green's function as before,

$$g^{\mu\nu} G(A(a), B(b))_{;\mu\nu} = -\frac{\delta^{(4)}(A(a), B(b))}{\sqrt{-g}}. \qquad \text{(C.15)}$$

These relations imply, for the C-field, that

$$C^{(a),\mu}{}_{;\mu} = \frac{1}{f}[-\frac{\delta^{(4)}(x^\mu - X_1^\mu(a))}{\sqrt{-g}} - -\frac{\delta^{(4)}(x^\mu - X_2^\mu(a))}{\sqrt{-g}}]. \tag{C.16}$$

Making use of Einstein's equations and Weyl's postulate (particles created on congruence of geodesics) for the original C-field action [21],

$$\begin{aligned}
T^{\mu\nu}{}_{;\nu} &= [\sum_a m_a \int [\frac{\delta^{(4)}(x^\mu - X^\mu(a))}{\sqrt{-g}} \frac{dX^\mu(a)}{da} \frac{dX^\nu(a)}{da} da]]_{;\nu} \\
&= \sum_a m_a [\int [\frac{\delta^{(4)}(x^\mu - X^\mu(a))}{\sqrt{-g}} \frac{dX^\mu(a)}{da} \frac{dX^\nu(a)}{da} da]]_{;\nu} \\
&= \sum_{X^\mu(a)} m_a [\frac{\delta^{(4)}(x^\mu - X^\mu(a))}{\sqrt{-g}} \frac{dX^\mu(a)}{da} dX^\nu(a)]_{;\nu} \\
&= \sum_{X^\mu(a)} m_a \frac{\delta^{(4)}(x^\mu - X^\mu(a))}{\sqrt{-g}} \frac{dX^\mu(a)}{da}; \qquad \text{(C.17)}
\end{aligned}$$

$$\begin{aligned}
-f \sum_a \sum_b [C^{(a);\nu} C^{(b);\mu} + C^{(a);\mu} C^{(b);\nu} - C^{(a);\kappa} C^{(b)}{}_{;\kappa} g^{\mu\nu}]_{;\nu} \\
= -\frac{1}{2} f \sum_a \sum_b [C^{(a);\nu}{}_{;\nu} C^{(b);\mu} + C^{(a);\mu} C^{(b);\nu}{}_{;\nu}] \\
= -f \sum_a C^{(a),\nu}{}_{;\nu} \sum_{b\neq a} C^{(b),\mu} = -\sum_{X^\mu(a)} \frac{\delta^{(4)}(x^\mu - X^\mu(a))}{\sqrt{-g}} \sum_{b\neq a} [C^{(b)}(X^\mu(a))]^{,\mu}. \qquad \text{(C.18)}
\end{aligned}$$

The points $X^\mu(a)$ give endpoints where particle creation/destruction occurs. Summation over $X^\mu(a)$ includes both endpoints of the worldline for a finite worldline. The term under the summation with $b \neq a$ yields the contributions to the C-field from the worldlines b at the points with coordinates $X^\mu(a)$. Summing over these points yields all contributions. Since the Einstein tensor has zero divergence, this requires

$$\begin{aligned}
T^{\mu\nu}{}_{;\nu} &= -T_{C-Field}{}^{\mu\nu}{}_{;\nu} \\
\Rightarrow \sum_{X^\mu(a)} m_a \frac{\delta^{(4)}(x^\mu - X^\mu(a))}{\sqrt{-g}} \frac{dX^\mu(a)}{da} &= \sum_{X^\mu(a)} \frac{\delta^{(4)}(x^\mu - X^\mu(a))}{\sqrt{-g}} \sum_{b\neq a} C^{(b)}(X^\mu(a))^{,\mu}
\end{aligned}$$

$$\Rightarrow m_a \frac{dX^\mu(a)}{da} = \sum_{b\neq a}[C^{(b)}(X^\mu(a))]^{,\mu}. \tag{C.19}$$

This condition must be satisfied at the worldline's endpoints. In the smooth-fluid approximation,

$$T^{\mu\nu}{}_{;\nu} = fC^{,\mu}C^{,\nu}{}_{;\nu}. \tag{C.20}$$

As a final consideration, look at the divergence of the gradient of the C-field. For ease of calculation, introduce the vector Green's function:

$$g^{\mu\nu}G_{\kappa\lambda';\mu\nu} + R_\lambda{}^\nu G_{\nu\lambda'} = -\bar{g}_{\kappa\lambda'}\frac{\delta^{(4)}(X^\kappa(a) - X^{\lambda'}(b))}{\sqrt{-\bar{g}}}; \tag{C.21}$$

$\bar{g}_{\kappa\lambda'}$ is Synge's parallel propagator [26]. The parallel propagator gives, for a geodesic connecting two points, a means to describe the transformation of a vector parallel transported to itself along the geodesic. For the divergence of the C-field gradient [19],

$$\begin{aligned}
fC^{(a),\mu}{}_{;\mu} &= \int G_{;\mu'}{}^{;\mu}{}_{;\mu}dX^{\mu'}(a) = -\int G^\nu{}_{\mu';\nu}{}^\mu{}_\mu dX^{\mu'}(a)\\
&= -\int\{G^\nu{}_{\mu'}{}^{;\mu}{}_{;\nu\mu} + (R^\mu{}_\kappa G^\kappa{}_{\mu'})_{;\mu}\}dX^{\mu'}(a)\\
&= -\int\{G^\nu{}_{\mu'}{}^{;\mu}{}_{;\mu\nu} + R^\nu{}_{\kappa\nu\mu}G^\kappa{}_{\mu'}{}^{;\mu} + R^\mu{}_{\kappa\nu\mu}G^\nu{}_{\mu'}{}^{;\kappa} + (R^\mu{}_\kappa G^\kappa{}_{\mu'})_{;\mu}\}dX^{\mu'}(a)\\
&= -[\int\{G^\nu{}_{\mu'}{}^{;\mu}{}_{;\mu} + R_\kappa{}^\nu G^\kappa{}_{\mu'}\}dX^{\mu'}(a)]_{;\nu}.
\end{aligned} \tag{C.22}$$

Consider the current:

$$\begin{aligned}
J^{(a)\mu} &= \int \frac{\delta^{(4)}(x^\mu - X^\mu(a))}{\sqrt{-\bar{g}}}\bar{g}^\mu_{\mu'}dX^{\mu'}(a)\\
&= -\int\{g^{\kappa\tau}G^\mu{}_{\mu';\kappa\tau} + R^{\mu\tau}G_{\tau\mu'}\}dX^{\mu'}(a)\\
&= -\int\{G^\mu{}_{\mu'}{}^{;\tau}{}_{;\tau} + R^{\tau\mu}g_{\tau\tau}G^\tau{}_{\mu'}\}dX^{\mu'}(a)
\end{aligned}$$

$$= \quad -\int \{G^{\mu}{}_{\mu'}{}^{;\tau}{}_{;\tau} + R_{\tau}{}^{\mu} G^{\tau}{}_{\mu'}\} dX^{\mu'}(a). \tag{C.23}$$

For its divergence,

$$\begin{aligned} J^{(a)\mu}{}_{;\mu} &= -[\int \{G^{\mu}{}_{\mu'}{}^{;\tau}{}_{;\tau} + R_{\tau}{}^{\mu} G^{\tau}{}_{\mu'}\} dX^{\mu'}(a)]_{;\mu} \\ &= -[\int \{G^{\nu}{}_{\mu'}{}^{;\mu}{}_{;\mu} + R_{\kappa}{}^{\nu} G^{\kappa}{}_{\mu'}\} dX^{\mu'}(a)]_{;\nu} \\ \Rightarrow f C^{(a),\mu}{}_{;\mu} &= J^{(a)\mu}{}_{;\mu}. \end{aligned} \tag{C.24}$$

For the smooth-fluid approximation, let $C(x^\mu) = \sum_a m_a C^{(a)}(x^\mu)$ be the total C-field at a point x^μ as produced via contributions from all worldlines, and let $J^\mu(x) = \sum_a m_a J^{(a)\mu}(x)$ be the smooth-fluid mass-current. Then,

$$C^{,\mu}{}_{;\mu} = \frac{1}{f} J^{\mu}{}_{;\mu}. \tag{C.25}$$

To obtain the equations of motion for a particle, vary the trajectories [19]:

$$\begin{aligned} \delta S &= \frac{1}{16\pi}\int \delta[R\sqrt{-g}] d^4x - \sum_a m_a \int \delta[da] + \frac{1}{f}\sum_a \sum_b \int\int \delta[G_{;\mu\nu} dX^\mu(a) dX^\nu(b)] \\ &= \frac{1}{16\pi}\int \delta[R\sqrt{-g}] d^4x - \sum_a m_a \int \delta[\sqrt{-g_{\mu\nu}\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}} d\lambda] \\ &\quad + \sum_{b\neq a} \int \delta[C^{(b)}{}_{,\mu}(x) dX^\mu(a)]. \end{aligned} \tag{C.26}$$

For a fixed parametrization, the metric $g_{\mu\nu}[X^\alpha(\lambda)]$ differs between curves by [1]

$$\begin{aligned} \delta g_{\mu\nu} \equiv g_{\mu\nu}[A^\alpha(\lambda) + \delta A^\alpha(\lambda)] - g_{\mu\nu}[A^\alpha(\lambda)] \\ = \frac{\partial g_{\mu\nu}}{\partial X^\sigma}\delta A^\sigma(\lambda). \end{aligned} \tag{C.27}$$

The tangent vector components $\frac{dX^\nu}{d\lambda}$ differ by

$$
\begin{aligned}
\delta(\frac{dX^\nu}{d\lambda}) \equiv \frac{d(A^\nu+\delta A^\nu)}{d\lambda} &- \frac{dA^\nu}{d\lambda} \\
&= \frac{d}{d\lambda}(\delta A^\nu).
\end{aligned} \tag{C.28}
$$

Then, rewrite these terms:

$$
\begin{aligned}
\int_{\lambda_I}^{\lambda_F} \sqrt{-g_{\mu\nu}\frac{dA^\mu}{d\lambda}\frac{dA^\nu}{d\lambda}}d\lambda &= \int_{\lambda_I}^{\lambda_F} \frac{-g_{\mu\nu}\frac{dA^\mu}{d\lambda}\frac{dA^\nu}{d\lambda}}{\sqrt{-g_{\kappa\rho}\frac{dA^\kappa}{d\lambda}\frac{dA^\rho}{d\lambda}}}d\lambda \\
\Rightarrow \delta\tau \quad &= \quad \delta\{\int_{\lambda_I}^{\lambda_F} \frac{-g_{\mu\nu}\frac{dA^\mu}{d\lambda}\frac{dA^\nu}{d\lambda}}{\sqrt{-g_{\kappa\rho}\frac{dA^\kappa}{d\lambda}\frac{dA^\rho}{d\lambda}}}d\lambda\} \\
&= \int_{\lambda_I}^{\lambda_F} \frac{-g_{\mu\nu}\frac{dA^\mu}{d\lambda}\frac{d(\delta A^\nu)}{d\lambda} - \frac{1}{2}(g_{\mu\nu,\sigma}\delta A^\sigma)\frac{dA^\mu}{d\lambda}\frac{dA^\nu}{d\lambda}}{\sqrt{-g_{\kappa\rho}\frac{dA^\kappa}{d\lambda}\frac{dA^\rho}{d\lambda}}}d\lambda.
\end{aligned} \tag{C.29}
$$

Consider the first term, and integrate by parts:

$$
\begin{aligned}
\int_{\lambda_I}^{\lambda_F}(-g_{\mu\nu}\frac{dA^\mu}{d\lambda}\frac{d(\delta A^\nu)}{d\lambda}) &= -g_{\mu\nu}\frac{dA^\mu}{d\lambda}\delta A^\nu|_{\lambda_I}^{\lambda_F} + \int_{\lambda_I}^{\lambda_F}\frac{dg_{\mu\nu}}{d\lambda}\frac{dA^\mu}{d\lambda}\delta A^\nu d\lambda \\
&\quad + \int_{\lambda_I}^{\lambda_F} g_{\mu\nu}\frac{d^2A^\mu}{d\lambda^2}\delta A^\nu d\lambda \\
&= \int_{\lambda_I}^{\lambda_F}\{\frac{d}{d\lambda}[g_{\mu\nu}\frac{dA^\mu}{d\lambda}]\delta A^\nu\}d\lambda \\
&= \int_{\lambda_I}^{\lambda_F}\{\frac{d}{d\lambda}[g_{\mu\nu}\frac{dA^\nu}{d\lambda}]\delta A^\mu\}d\lambda \\
&= \int_{\lambda_I}^{\lambda_F}\{\frac{d}{d\lambda}[g_{\sigma\nu}\frac{dA^\nu}{d\lambda}]\delta A^\sigma\}d\lambda.
\end{aligned} \tag{C.30}
$$

For the full variation,

$$
\begin{aligned}
\delta a = \int_{\lambda_I}^{\lambda_F} &\frac{\{\frac{d}{d\lambda}[g_{\sigma\nu}\frac{dA^\nu}{d\lambda}] - \frac{1}{2}g_{\mu\nu,\sigma}\frac{dA^\mu}{d\lambda}\frac{dA^\nu}{d\lambda}\}}{\sqrt{-g_{\kappa\rho}\frac{dA^\kappa}{d\lambda}\frac{dA^\rho}{d\lambda}}}\delta A^\sigma d\lambda = 0 \\
&\Rightarrow \frac{d}{d\lambda}[g_{\sigma\nu}\frac{dA^\nu}{d\lambda}] - \frac{1}{2}g_{\mu\nu,\sigma}\frac{dA^\mu}{d\lambda}\frac{dA^\nu}{d\lambda} = 0.
\end{aligned} \tag{C.31}
$$

Write $\delta A^\sigma = f(\lambda)\frac{dA^\sigma}{d\lambda}$. Then,

$$\begin{aligned}\int_{\lambda_I}^{\lambda_F} \frac{\{\frac{d}{d\lambda}[g_{\sigma\nu}\frac{dA^\nu}{d\lambda}] - \frac{1}{2}g_{\mu\nu,\sigma}\frac{dA^\mu}{d\lambda}\frac{dA^\nu}{d\lambda}\}}{\sqrt{-g_{\kappa\rho}\frac{dA^\kappa}{d\lambda}\frac{dA^\rho}{d\lambda}}} f(\lambda)\frac{dA^\sigma}{d\lambda}d\lambda = 0 \\ \Rightarrow \{\frac{d}{d\lambda}[g_{\sigma\nu}\frac{dA^\nu}{d\lambda}] - \frac{1}{2}g_{\mu\nu,\sigma}\frac{dA^\mu}{d\lambda}\frac{dA^\nu}{d\lambda}\}\frac{dA^\sigma}{d\lambda} = 0.\end{aligned} \tag{C.32}$$

Note that this holds for the proper time a as well.

Expand the first term:

$$\frac{d}{d\lambda}[g_{\sigma\nu}\frac{dA^\nu}{d\lambda}] = \frac{dg_{\sigma\nu}}{d\lambda}\frac{dA^\nu}{d\lambda} + g_{\sigma\nu}\frac{d^2A^\nu}{d\lambda^2}. \tag{C.33}$$

The contraction over indices in the first part of the expansion can be dealt with by introducing a dummy index:

$$\begin{aligned}\frac{dg_{\sigma\nu}}{d\lambda}\frac{dA^\nu}{d\lambda} &= \frac{1}{2}(\frac{dg_{\sigma\mu}}{d\lambda}\frac{dA^\mu}{d\lambda} + \frac{dg_{\sigma\nu}}{d\lambda}\frac{dA^\nu}{d\lambda}) \\ &= \frac{1}{2}(\frac{dg_{\sigma\mu}}{dA^\nu}\frac{dA^\nu}{d\lambda}\frac{dA^\mu}{d\lambda} + \frac{dg_{\sigma\nu}}{dA^\mu}\frac{dA^\mu}{d\lambda}\frac{dA^\nu}{d\lambda}).\end{aligned} \tag{C.34}$$

Inserting this into the variation:

$$\frac{1}{2}(\frac{dg_{\sigma\mu}}{dA^\nu}\frac{dA^\nu}{d\lambda}\frac{dA^\mu}{d\lambda} + \frac{dg_{\sigma\nu}}{dA^\mu}\frac{dA^\mu}{d\lambda}\frac{dA^\nu}{d\lambda}) + g_{\sigma\nu}\frac{d^2A^\nu}{d\lambda^2} - \frac{1}{2}g_{\mu\nu,\sigma}\frac{dA^\mu}{d\lambda}\frac{dA^\nu}{d\lambda} = 0. \tag{C.35}$$

Specify these computations to the worldline a. Then, $A^\mu \to X^\mu$. So,

$$\begin{aligned}&\frac{1}{2}(\frac{dg_{\sigma\mu}}{dX^\nu}\frac{dX^\nu}{da}\frac{dX^\mu}{da} + \frac{dg_{\sigma\nu}}{dX^\mu}\frac{dX^\mu}{da}\frac{dX^\nu}{da}) + g_{\sigma\nu}\frac{d^2X^\nu}{da^2} - \frac{1}{2}g_{\mu\nu,\sigma}\frac{dX^\mu}{da}\frac{dX^\nu}{da} \\ &= \frac{1}{2}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})\frac{dX^\mu}{da}\frac{dX^\nu}{da} + g_{\sigma\nu}\frac{d^2X^\nu}{da^2} = 0.\end{aligned} \tag{C.36}$$

Raising indices,

$$
\begin{aligned}
& \frac{1}{2}g^{\alpha\sigma}(g_{\sigma\mu,\nu}+g_{\sigma\nu,\mu}-g_{\mu\nu,\sigma})\frac{dX^\mu}{da}\frac{dX^\nu}{da}+g^{\alpha\sigma}g_{\sigma\nu}\frac{d^2X^\nu}{da} \\
= & \frac{1}{2}g^{\alpha\sigma}(g_{\sigma\mu,\nu}+g_{\sigma\nu,\mu}-g_{\mu\nu,\sigma})\frac{dX^\mu}{da}\frac{dX^\nu}{da}+\frac{d^2X^\alpha}{da^2} \\
= & \Gamma^\alpha{}_{\mu\nu}\frac{dX^\mu}{da}\frac{dX^\nu}{da}+\frac{d^2X^\alpha}{da^2} \\
= & 0.
\end{aligned}
\tag{C.37}
$$

These are the geodesic equations, so the C-field does not affect the created particles along their worldlines.

From Eq. (C.30), the surface term yields with Eq. (C.26)

$$
\begin{aligned}
& \sum_a m_a[-g_{\mu\nu}\frac{dX^\mu}{da}\delta X^\nu+\sum_{b\neq a}C^{(b)}{}_{,\mu}\delta X^\mu]|_{\lambda_I}^{\lambda_F}=0 \\
= & \sum_a m_a[-\frac{dX_\nu}{da}\delta X^\nu+\sum_{b\neq a}C^{(b)}{}_{,\mu}\delta X^\mu]|_{\lambda_I}^{\lambda_F} \\
= & \sum_a m_a[-\frac{dX_\mu}{da}\delta X^\mu+\sum_{b\neq a}C^{(b)}{}_{,\mu}\delta X^\mu]|_{\lambda_I}^{\lambda_F}=0 \\
& \Rightarrow \sum_a m_a\frac{dX^\mu}{da}=\sum_{b\neq a}C^{(b),\mu}.
\end{aligned}
\tag{C.38}
$$

This corresponds to conservation of energy concerning created particles and matches Eq. (C.19).

C.2 The Smooth Fluid Action

C.2.1 Einstein's Equations

Now, consider a slightly modified action that includes another Lagrangian for an already-present matter distribution, L_{Matter}. Determine Einstein's equations for the action $S[g_{\mu\nu}, C, X^\mu]$ in Eq. (5.3):

$$
\begin{aligned}
\delta S &= \int \delta((\frac{1}{16\pi}R + L_{Matter} + \frac{1}{2}f \cdot (C_{,\mu}C^{,\mu}))\sqrt{-g} \\
&\quad - \sum_a m_a(1 + C_{,\mu}\frac{dX^{\mu}(a)}{da})da\sqrt{-g})d^4x. \qquad \text{(C.39)}
\end{aligned}
$$

Consider variations of each term with respect to the metric tensor. From Appendix A,

$$\frac{1}{16\pi}\delta(R\sqrt{-g}) = \frac{1}{16\pi}[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R]\delta g^{\mu\nu};$$

$$\delta(L_{Matter}\sqrt{-g}) = \sqrt{-g}\delta L_{Matter} + (\delta\sqrt{-g})L_{Matter} = (\frac{\partial L_{Matter}}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}L_{Matter})\sqrt{-g}\delta g^{\mu\nu};$$

$$\delta(\frac{1}{2}f \cdot (C_{,\mu}C^{,\mu})\sqrt{-g}) = \delta(\frac{1}{2}g^{\mu\nu}f \cdot (C_{,\mu}C_{,\nu})\sqrt{-g})$$

$$= [\frac{1}{2}f \cdot (C_{,\mu}C_{,\nu})\sqrt{-g}]\delta g^{\mu\nu} - [\frac{1}{2}fg_{\mu\kappa}C^{,\kappa}C_{,\kappa}g^{\kappa\mu}]\delta\sqrt{-g}$$

$$= [\frac{1}{2}f \cdot (C_{,\nu}C_{,\mu} - \frac{1}{2}g_{\mu\nu}C_{,\kappa}C^{,\kappa})]\sqrt{-g}\delta g^{\mu\nu};$$

$$\delta[\sum_a m_a \int (1 + C_{,\mu})\frac{dX^{\mu}(a)}{da}da\sqrt{-g}] = \sum_a m_a \int \delta[(1 + C_{,\mu}\frac{dX^{\mu}(a)}{da})da\sqrt{-g}]$$

$$= \sum_a m_a \delta[\int da \frac{\delta^{(4)}(x^{\mu} - X^{\mu}(a))}{\sqrt{-g}}(1 + C_{,\mu}\frac{dX^{\mu}(a)}{da})\sqrt{-g}]$$

$$= \sum_a m_a [\frac{\delta^{(3)}(x^\mu - X^\mu(a))}{\sqrt{-g}}(1 + C_{,\mu}\frac{dX^\mu(a)}{da})\delta\sqrt{-g}]$$

$$= \rho_{Proper}(1 + C_{,\mu}\frac{dX^\mu(a)}{da})\delta\sqrt{-g}$$

$$= -\frac{1}{2}g_{\mu\nu}\sqrt{-g}\rho_{Proper}(1 + C_{,\mu}\frac{dX^\mu(a)}{da})\delta g^{\mu\nu};$$

$$\Rightarrow R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$$

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi[T^{\mu\nu} - f \cdot (C^{,\nu}C^{,\mu} - \frac{1}{2}g^{\mu\nu}C_{,\kappa}C^{,\kappa})]; \tag{C.40}$$

$$\begin{aligned} T^{\mu\nu} &= (g^{\mu\nu}\rho_{Proper}(1 + C_{,\mu}\frac{dX^\mu(a)}{da}) \\ &\quad -2\frac{\partial L_{Matter}}{\partial g_{\mu\nu}} + g^{\mu\nu}L_{Matter}). \end{aligned} \tag{C.41}$$

$\rho_{Proper} = \sum_a m_a \frac{\delta^{(3)}(x^\mu - X^\mu(a))}{\sqrt{-g}}$ is the proper density of the created particles.

Now, consider a variation of the action w.r.t. the C-field (this obtains the wave equations for the C-field) [8]:

$$\frac{\delta S}{\delta C} = \int \frac{\delta}{\delta C}((\frac{1}{16\pi}R + L_{Matter} + \frac{1}{2}f \cdot (C_{,\mu}C^{,\mu}))\sqrt{-g} - \sum_a m_a(1 + C_{,\mu}\frac{dX^\mu(a)}{da})da\sqrt{-g})d^4x. \tag{C.42}$$

C does not appear explicitly in this representation. However, it obtains via integration by parts:

$$\delta S = \delta[\int ((\frac{1}{16\pi}R + L_{Matter} + \frac{1}{2}f \cdot (C_{,\mu}C^{,\mu}))\sqrt{-g} - \sum_a m_a(1 + C_{,\mu}\frac{dX^\mu(a)}{da})da\sqrt{-g})d^4x]$$

$$= \int [[\frac{1}{16\pi}\frac{\partial}{\partial C}R + \frac{\partial}{\partial C}L_{Matter} + \frac{1}{2}f \cdot [[-\frac{\partial}{\partial C_{,\mu}}(C_{,\mu}C_{,\nu}g^{\nu\mu})]_{;\mu} + [-\frac{\partial}{\partial C_{,\nu}}(C_{,\mu}C_{,\nu}g^{\nu\mu})]_{;\nu}]]\sqrt{-g}$$

$$-[\sum_a m_a \int \frac{\delta^{(4)}(x^\mu - X^\mu(a))}{\sqrt{-g}}(-\frac{\partial}{\partial C_{,\mu}}C_{,\mu}\frac{dX^\mu(a)}{da})da]_{;\mu}]\sqrt{-g}\delta C d^4x$$

$$= \int [\frac{1}{2}f[-C^{,\mu}{}_{;\mu} - C^{,\nu}{}_{;\nu}]\sqrt{-g} - [\rho_{Proper}\frac{dX^\mu(a)}{da}]_{;\mu}]\sqrt{-g}\delta C d^4x$$

$$= \int [\frac{1}{2}f[-2\triangle C]\sqrt{-g} - J^\mu{}_{;\mu}\sqrt{-g}]\delta C d^4x$$

$$= \int [f\triangle C - J^\mu{}_{;\mu}]\delta C d^4x; \qquad (C.43)$$

$$\Rightarrow f\triangle C = J^\mu{}_{;\mu}$$

$$\Rightarrow C^{,\mu}{}_{;\mu} = \frac{1}{f}J^\mu{}_{;\mu}. \qquad (C.44)$$

On the boundaries, the variation equals zero. This eliminates any terms not under integrals. $\triangle$ is the d'Alembertian. The term $\rho_{Proper} \equiv \sum_a m_a \frac{\delta^{(3)}(x - A(a))}{\sqrt{-g}}$ yields the proper density of the particles. $J^\mu = \rho_{Proper}\frac{dX^\mu(a)}{da}$ is the mass current.

Variation with respect to the trajectories follows as in the previous section. Leave the action in its original form with integrals over the worldlines;

To study the conservation properties of this action, recall that the divergence of the Einstein tensor (the lefthand side of Einstein's equations) is zero. Manifestly, the righthand side must follow this, so

$$T^{\mu\nu}{}_{;\nu} = [f \cdot (C^{,\nu} C^{,\mu} - \frac{1}{2} g^{\mu\nu} C_{,\kappa} C^{,\kappa})]_{;\nu}$$

$$= f C^{,\nu}{}_{;\nu} C^{,\mu} + f C^{,\nu} C^{,\mu}{}_{;\nu} - \frac{1}{2} g^{\mu\nu} C_{,\kappa;\nu} C^{,\kappa} - \frac{1}{2} g^{\mu\nu} C_{,\kappa} C^{,\kappa}{}_{;\nu}. \quad \text{(C.45)}$$

Matter is created normal to a hypersurface with constant C via Weyl's postulate [16]. Also, this work considers a variable number of particles produced. All non-normal terms vanish, leaving

$$T^{\mu\nu}{}_{;\nu} = f C^{,\mu} C^{,\nu}{}_{;\nu}. \quad \text{(C.46)}$$

If C is not constant, then the particle mass varies along the worldline. For the variable mass scenario, the other terms may remain, but such a case is beyond the scope of this work.

C.2.2 3 + 1 Dynamic Evolution

Using the ADM decomposition on the action, obtain

$$\begin{aligned} S \;=\; & \int (\frac{1}{16\pi}(-g_{ij}\frac{\partial \pi^{ij}}{\partial t} - N\mathcal{H} - N_i \mathcal{H}^i - 2(\pi^{ij} N_j - \frac{1}{2} N^i Tr\Pi + N^{|i}\sqrt{g})_{,i}) \\ & +(L_{Matter} + \frac{1}{2} f \cdot (g_{00} C^{,0} C^{,0} + g_{i0} C^{,0} C^{,i} + g_{0j} C^{,j} C^{,0} + g_{ij} C^{,j} C^{,i}) \\ & - \sum_a m_a \frac{\delta^{(3)}(x^\mu - X^\mu(a))}{N\sqrt{g}}[1 + g_{00} C^{,0} \frac{dX^0(a)}{da} + g_{i0} C^0 \frac{dX^i(a)}{da} + g_{0j} C^{,j} \frac{dX^0(a)}{da} \\ & + g_{ij} C^{,j} \frac{dX^i(a)}{da}]))N\sqrt{g} d^4x. \end{aligned} \quad \text{(C.47)}$$

Recall from the ADM three-plus-one decomposition that $\sqrt{-g} = N\sqrt{g}$ where $g = det|g_{ij}| = g_{ij}A^{ij}$. Rewrite the decomposed covariant metric using ADM. The reason for using the covariant metric is because covariant spatial components of 4-tensors form 3-tensors dependent only on the surface. (Quantities on such a surface are defined via 3-D operations on 3-tensors.) [13] Hoyle's treatment of the gradients of the C-field as equivalent to 4-momenta make this choice more logical. Also, the term under summation corresponds in total to the proper density of the particles, so variations occur only over terms not included in this density. Vary the action w.r.t. the metric, conjugate momentum, lapse, and shift, and obtain Hamilton's equations and the constraint equations:

$$\begin{aligned}
\frac{\delta S}{\delta g_{ij}} &= \int (\frac{1}{16\pi}(-\frac{\partial \pi^{ij}}{\partial t} - N\sqrt{g}(R^{ij} - \frac{1}{2}g^{ij}R) + \frac{N}{2\sqrt{g}}g^{ij}(Tr\Pi^2 - \frac{1}{2}(Tr\Pi)^2) \\
&\quad -\frac{2N}{\sqrt{g}}(\pi^{im}\pi_m{}^j - \frac{1}{2}\pi^{ij}Tr\Pi) + \sqrt{g}(N^{|ij} - g^{ij}N^{|m}{}_{|m}) \\
&\quad +(\pi^{ij}N^m)_{|m} - N^i{}_{|m}\pi^{mj} - N^j{}_{|m}\pi^{mi}) \\
&\quad +\frac{N\sqrt{g}g^{ij}}{2}(L_{Matter} + \frac{1}{2}f((N_kN^k - N^2)C^{,0}C^{,0} + N_iC^{,0}C^{,i} \\
&\quad +N_jC^{,j}C^{,0} + g_{ij}C^{,j}C^{,i})) \\
&\quad +\frac{N\sqrt{g}}{2}fC^{,j}C^{,i} + N\sqrt{g}\frac{\delta L_{Matter}}{\delta g_{ij}} \\
&\quad -N\frac{\sqrt{g}}{2}g^{ij}\rho_{Proper}[1 + (N_kN^k - N^2)C^{,0}\frac{dX^0(a)}{da} + N_iC^0\frac{dX^i(a)}{da} \\
&\quad +N_jC^{,j}\frac{dX^0(a)}{da} + g_{ij}C^{,j}\frac{dX^i(a)}{da}] \\
&\quad -N\sqrt{g}\rho_{Proper}C^{,j}\frac{dx^i}{da})d^4x \qquad \text{(C.48)} \\
\Rightarrow \frac{\partial \pi^{ij}}{\partial t} &= -N\sqrt{g}(R^{ij} - \frac{1}{2}g^{ij}R) + \frac{N}{2\sqrt{g}}g^{ij}(Tr\Pi^2 - \frac{1}{2}(Tr\Pi)^2) \\
&\quad -\frac{2N}{\sqrt{g}}(\pi^{im}\pi_m{}^j - \frac{1}{2}\pi^{ij}Tr\Pi) + \sqrt{g}(N^{|ij} - g^{ij}N^{|m}{}_{|m}) \\
&\quad +(\pi^{ij}N^m)_{|m} - N^i{}_{|m}\pi^{mj} - N^j{}_{|m}\pi^{mi} \\
&\quad +8\pi N\sqrt{g}g^{ij}(L_{Matter} + \frac{1}{2}f((N_kN^k - N^2)C^{,0}C^{,0} + N_iC^{,0}C^{,i} \\
&\quad +N_jC^{,j}C^{,0} + g_{ij}C^{,j}C^{,i})
\end{aligned}$$

$$
\begin{aligned}
&+f g_{ij} C^{,j} C^{,i}) + 16\pi N \sqrt{g} \frac{\delta L_{Matter}}{\delta g_{ij}} \\
&-8\pi N \sqrt{g} g^{ij} \rho_{Proper} [1 + (N_k N^k - N^2) C^{,0} \frac{dX^0(a)}{da} + N_i C^0 \frac{dX^i(a)}{da} \\
&+N_j C^{,j} \frac{dX^0(a)}{da} + g_{ij} C^{,j} \frac{dX^i(a)}{da}] \\
&-16\pi N \sqrt{g} \rho_{Proper} C^{,j} \frac{dX^i(a)}{da}. \qquad \text{(C.49)}
\end{aligned}
$$

Since the particles are created at rest, the 3-momentum of the created particles equals zero ($C^{,i} = 0$), so

$$
\begin{aligned}
\frac{\partial \pi^{ij}}{\partial t} &= -N\sqrt{g}(R^{ij} - \frac{1}{2} g^{ij} R) + \frac{N}{2\sqrt{g}} g^{ij} (Tr\Pi^2 - \frac{1}{2}(Tr\Pi)^2) \\
&-\frac{2N}{\sqrt{g}} (\pi^{im} \pi_m{}^j - \frac{1}{2} \pi^{ij} Tr\Pi) + \sqrt{g}(N^{|ij} - g^{ij} N^{|m}{}_{|m}) \\
&+(\pi^{ij} N^m)_{|m} - N^i{}_{|m} \pi^{mj} - N^j{}_{|m} \pi^{mi} \\
&+8\pi N \sqrt{g} g^{ij} (L_{Matter} + \frac{1}{2} f((N_k N^k - N^2) C^{,0} C^{,0}) + 16\pi N \sqrt{g} \frac{\delta L_{Matter}}{\delta g_{ij}} \\
&-8\pi N \sqrt{g} g^{ij} \rho_{Proper} [1 + (N_k N^k - N^2) C^{,0} \frac{dX^0(a)}{da}]. \qquad \text{(C.50)}
\end{aligned}
$$

For the variation w.r.t. the conjugate momentum:

$$
\begin{aligned}
\frac{\delta S}{\delta \pi^{ij}} &= \int [\frac{1}{16\pi} (-\frac{\partial g_{ij}}{\partial t} + \frac{2N}{\sqrt{g}} (\pi_{ij} - \frac{1}{2} g_{ij} Tr\Pi) + N_{i|j} + N_{j|i})] d^4x \qquad \text{(C.51)} \\
\Rightarrow \frac{\partial g_{ij}}{\partial t} &= \frac{2N}{\sqrt{g}} (\pi_{ij} - \frac{1}{2} g_{ij} Tr\Pi) + N_{i|j} + N_{j|i}. \qquad \text{(C.52)}
\end{aligned}
$$

For the superhamiltonian:

$$
\begin{aligned}
\frac{\delta S}{\delta N} &= \int [-\frac{\mathcal{H}}{16\pi} + \sqrt{g}(L_{Matter} + \frac{1}{2} f((N_k N^k - N^2) C^{,0} C^{,0} \\
&+N_i C^{,0} C^{,i} + N_j C^{,j} C^{,0} + g_{ij} C^{,j} C^{,i})) + \frac{N\sqrt{g}}{2} f(-2N C^{,0} C^{,0})
\end{aligned}
$$

$$
\begin{aligned}
& -\sqrt{g}\rho_{Proper}[1+(N_kN^k-N^2)C^{,0}\frac{dX^0(a)}{da}+N_iC^0\frac{dX^i(a)}{da} \\
& +N_jC^{,j}\frac{dX^0(a)}{da}+g_{ij}C^{,j}\frac{dX^i(a)}{da}] \\
& -N\sqrt{g}\rho_{Proper}[-2NC^{,0}\frac{dX^0(a)}{da}]]d^4x \qquad (C.53)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \mathcal{H} &= \frac{1}{\sqrt{g}}(Tr\Pi^2-\frac{1}{2}(Tr\Pi)^2)-\sqrt{g}R \\
&= 16\pi\sqrt{g}(L_{Matter}+\frac{1}{2}f((N_kN^k-N^2)C^{,0}C^{,0}+N_iC^{,0}C^{,i} \\
& +N_jC^{,j}C^{,0}+g_{ij}C^{,j}C^{,i})+\frac{N}{2}f(-2NC^{,0}C^{,0}) \\
& -\rho_{Proper}[1+(N_kN^k-N^2)C^{,0}\frac{dX^0(a)}{da}+N_iC^0\frac{dX^i(a)}{da} \\
& +N_jC^{,j}\frac{dX^0(a)}{da}]-N\rho[-2NC^{,0}\frac{dX^0(a)}{da}]) \\
&= 16\pi\sqrt{g}(L_{Matter}+\frac{1}{2}f((N_kN^k-N^2)C^{,0}C^{,0}) \\
& +\frac{N}{2}f(-2NC^{,0}C^{,0})-\rho_{Proper}[1+(N_kN^k-N^2)C^{,0}\frac{dX^0(a)}{da}] \\
& -N\rho_{Proper}[-2NC^{,0}\frac{dX^0(a)}{da}]). \qquad (C.54)
\end{aligned}
$$

And for the supermomentum:

$$
\frac{\delta S}{\delta N_i} = \int[-\mathcal{H}^i+N\sqrt{g}(\frac{1}{2}fC^{,0}C^{,i}-\rho_{Proper}C^0\frac{dX^i(a)}{da})]d^4x \qquad (C.55)
$$

$$
\Rightarrow \mathcal{H}^i = -2\pi^{ij}{}_j = N\sqrt{g}[(\frac{1}{2}fC^{,0}C^{,i}-\rho_{Proper}C^0\frac{dX^i(a)}{da})]=0. \qquad (C.56)
$$

C.2.3 Thermodynamics of the C-Field

This section details the thermodynamic computations of this work using the conventions and work of Synge [25]. Introduce terms to describe the system statistically. Let v be the population number, $\mathcal{N}^\mu$ the numerical flux vector, and $\mathcal{N}$ the distribution function. At a point/event on a hypersurface, draw a nullcone with positive (future-forward) orientation. Let n^μ be a timelike unit vector. Consider a slicing of the nullcone such that a target dV (3-dimensional infinitesimal volume element) is obtained. p^μ is the 4-momentum of particles in the gas. Considering the part of the nullcone between the point on the hypersurface and the target, project this region onto the hypersurface. This projection yields a 3-volume, $d\Omega$. Further, this 3-volume can be written as $d\Omega = d\omega|p_\mu n^\mu|$. Synge calls $d\omega$ the 'absolute 2-content' of the 3-volume at the point of intersection of the 4-momentum with the nullcone.

Mathematically,

$$\begin{aligned} v = \mathcal{N}_\mu n^\mu dV &= dV \int \mathcal{N} d\Omega; && \text{(C.57)} \\ \Rightarrow -T_{\mu\nu} n^\nu dV &= dV \int \mathcal{N} p_\mu d\Omega. && \text{(C.58)} \end{aligned}$$

$-T_{\mu\nu}n^\nu$ is the 4-momentum flux across the infinitesimal target. These quantities maximize the entropy integral (as follows from the third law of thermodynamics):

$$F = -dV \int \mathcal{N} \log \mathcal{N} d\Omega \tag{C.59}$$

Treat the equations pertaining to population number and 4-momentum flux as constraints; from the method of Lagrange multipliers,

$$\delta F - \delta[\alpha v - \xi^\mu T_{\mu\nu} n^\nu] = 0, \tag{C.60}$$

where α and ξ^μ are Lagrange multipliers (generally, functions of the coordinates x^μ on the hypersurface but not the 4-momentum). Vary this integral with respect to the distribution function:

$$\begin{aligned} \delta[\mathcal{N} log \mathcal{N}] - \delta[\alpha\mathcal{N} + \xi^\mu \mathcal{N} p_\mu] &= 0 \\ &= \delta\mathcal{N}[\log\mathcal{N}] + \mathcal{N}\delta\log\mathcal{N} - \alpha\delta\mathcal{N} - \xi^\mu p_\mu \delta\mathcal{N} \\ &= [\log\mathcal{N} + \frac{\partial \log\mathcal{N}}{\partial\mathcal{N}}]\delta\mathcal{N} - [\alpha + \xi^\mu p_\mu]\delta\mathcal{N} \\ &= [\log\mathcal{N} + 1]\delta\mathcal{N} - [\alpha + \xi^\mu p_\mu]\delta\mathcal{N} \end{aligned}$$

$$\Rightarrow (\log \mathcal{N} + 1)\delta\mathcal{N} \;=\; \alpha\delta\mathcal{N} + \xi^\mu p_\mu \delta\mathcal{N}. \tag{C.61}$$

This yields the most probable distribution function,

$$\mathcal{N}(x^\mu, p^\mu) = Ae^{\xi_\mu p^\mu}. \tag{C.62}$$

Note that A and ξ^μ are functions of x^μ. Write them as

$$\begin{aligned} A\int e^{\xi_\kappa p^\kappa} d\Omega &= -A\int p_\mu n^\mu e^{\xi_\kappa p^\kappa} d\omega = -\mathcal{N}_\mu n^\mu, \\ A\int p_\mu e^{\xi_\kappa p^\kappa} d\Omega &= -A\int p_\mu p_\nu n^\nu e^{\xi_\kappa p^\kappa} d\omega = -T_{\mu\nu} n^\nu. \end{aligned} \tag{C.63}$$

These equations can be used to solve for A and ξ^μ.

The distribution function is an absolute property of this gas. Arbitrary selection of the timelike unit normal vector yields a sample population. Remove the dependence of $\mathcal{N}$, A, and ξ^μ on this vector, and rewrite the above equations:

$$\begin{aligned} A\int p_\mu e^{\xi_\kappa p^\kappa} d\omega &= \mathcal{N}_\mu, \\ A\int p_\mu p_\nu e^{\xi_\kappa p^\kappa} d\omega &= T_{\mu\nu}. \end{aligned} \tag{C.64}$$

This rewriting gives 14 equations instead of the previous 5. Now, write the 5 conservation equations:

$$\begin{aligned} \mathcal{N}^\mu{}_{;\mu} &= 0; \\ T^{\mu\nu}{}_{;\nu} &= 0. \end{aligned} \tag{C.65}$$

Define Φ as follows:

$$\Phi = \int e^{\xi_\kappa p^\kappa}. \tag{C.66}$$

Rewriting equations (C.27) and (C.28) with this yields

$$\begin{aligned} N_\mu &= A\frac{\partial \Phi}{\partial \xi_\mu}; \\ T_{\mu\nu} &= A\frac{\partial^2 \Phi}{\partial \xi_\mu \partial \xi_\nu}. \end{aligned} \tag{C.67}$$

Let the Lagrange multiplier ξ_μ be timelike normal, and also let it satisfy the relationship

$$\xi = \sqrt{-\xi_\mu \xi^\mu} \tag{C.68}$$

$$\begin{aligned} \Rightarrow \xi^2 = -\xi_\mu \xi^\mu = -Cn_\mu n^\mu = -C(-1) = C \\ \Rightarrow C = \xi^2 \Rightarrow \xi_\mu = \xi n_\mu, \end{aligned} \tag{C.69}$$

where n_μ is the normal to the hypersurface. The quantity ξ is the reciprocal temperature.

At these points on the hypersurface, where a locally Lorentz frame can be specified, so can a 4-momentum space. Then,

$$\begin{aligned} {p_0}^2 &= -m^2 \cosh^2 \chi, \\ {p_1}^2 &= m^2 \sinh^2 \chi \sin^2 \theta \cos^2 \phi, \\ {p_2}^2 &= m^2 \sinh^2 \chi \sin^2 \theta \sin^2 \phi, \\ {p_3}^2 &= m^2 \sinh^2 \chi \cos^2 \theta, \end{aligned} \tag{C.70}$$

for the components of the 4-momentum. The parameters satisfy

$$0 < m < \infty;\ 0 \leq \chi < \infty;\ 0 \leq \theta \leq \pi;\ 0 \leq \phi < 2\pi. \tag{C.71}$$

With fixed m, the 4-momentum vector traces a hypersphere of radius m, and this contains all of the possible 4-momenta of the particles of proper mass m. For a mixture of n gases, n hyperspheres must be considered. This must be modified for photons, but that is beyond the scope of this work.

Using this coordinate system along with the absolute 2-content of a 3-cell of the hypersphere, $d\omega = m^2 \sinh^2 \chi \sin\theta d\chi d\theta d\phi$, rewrite Φ:

$$\Phi = 4\pi m^2 \int_0^\infty e^{-m\xi \cosh\chi} \sinh^2 \chi d\chi. \tag{C.72}$$

This resembles the Bessel function,

$$\begin{aligned} K_n(x) &= \int_0^\infty e^{-x\cosh\chi} \cosh n\chi d\chi \\ &= \frac{x^n}{(2n-1)!} \int_0^\infty e^{-x\cosh\chi} \sinh^{2n} \chi d\chi, \end{aligned} \tag{C.73}$$

where the functions satisfy the following relations:

$$\begin{aligned} xK_n'(x) - nK_n(x) &= -xK_{n+1}(x); \\ xK_n'(x) + nK_n(x) &= -xK_{n-1}(x); \\ K_{n+1}(x) - K_{n-1}(x) &= 2n\frac{K_n(x)}{x}. \end{aligned} \tag{C.74}$$

This implies that $\Phi = 4\pi \frac{mK_1(m\xi)}{\xi}$.

The number density (the number of particles per unit 3-volume) is defined as such:

$$\mathcal{N}_0 = -\mathcal{N}_\mu \bar{u}^\mu \tag{C.75}$$

$$\Rightarrow \mathcal{N}^\mu = \mathcal{N}_0 \bar{u}^\mu. \tag{C.76}$$

where $\bar{u}^\mu$ is the mean 4-velocity of the particles with the standard relation $\bar{u}_\mu \bar{u}^\mu = -1$, and ξ is the reciprocal temperature. Using (C.30) with (C.37), obtain

$$\begin{aligned} N_\mu &= \frac{4\pi A m^2 \xi_\mu K_2(m\xi)}{\xi^2}; \\ T_{\mu\nu} &= \frac{4\pi A m^3 \xi_\mu \xi_\nu K_3(m\xi)}{\xi^3} + \frac{4\pi A m^2 g_{\mu\nu} K_2(m\xi)}{\xi^2}. \end{aligned} \tag{C.77}$$

Define the mean 4-velocity as

$$\bar{u}^\mu = -\frac{\partial \xi}{\partial \xi^\mu} = \frac{\xi^\mu}{\xi}, \tag{C.78}$$

It satisfies the invariant relationship $\bar{u}_\mu \bar{u}^\mu = -1$. In the momentarily comoving reference frame, ξ^μ is purely timelike (orthogonal to the hypersurface as shown in Eq. C.69), and it follows from the definition of a perfect fluid, which is

$$T_{\mu\nu} = (\rho + p)\bar{u}_\mu \bar{u}_\nu + p g_{\mu\nu}. \tag{C.79}$$

For the C-field, the conservation equations fail. So, consider a mixture that conserves the original number of particles. Doing so introduces a modified entropy integral, modified distribution equations, and modified equations of state. The new particles will be taken in the simplest form, a relativistic dust (i.e., no pressure). Also, the particles are created at rest ($\bar{u}^i = 0$). Then,

$$F_{Modified} = -dV \int [\mathcal{N} log \mathcal{N} \;+\; \mathcal{N}_{C-Field} log \mathcal{N}_{C-Field}] d\Omega; \tag{C.80}$$

$$\begin{aligned} \int \mathcal{N}_0 d\Omega &= -\mathcal{N}_\mu n^\mu; \\ \int \mathcal{N}_{0C-Field} d\Omega &= -\mathcal{N}_{\mu C-Field} n^\mu; \end{aligned} \tag{C.81}$$

$$\begin{aligned}
\int (\mathcal{N} + \mathcal{N}_{C-Field}) p_\mu d\Omega &= -T_{\mu\nu} n^\nu; && \text{(C.82)} \\
log\mathcal{N} + 1 &= \alpha + \xi_\mu p^\mu; \\
log\mathcal{N}_{C-Field} + 1 &= \alpha_{C-Field} + \xi_\mu p^\mu; && \text{(C.83)} \\
\Rightarrow \mathcal{N} = Ae^{\xi_\mu p^\mu}, &\qquad \mathcal{N}_{C-Field} = A_{C-Field} e^{\xi_\mu p^\mu}. && \text{(C.84)}
\end{aligned}$$

Write the number flux vectors and the new energy-momentum tensor for the initial perfect fluid and the created particles:

$$A \int p_\mu e^{\xi_\kappa p^\kappa} d\omega = \mathcal{N}_\mu,$$

$$A_{C-Field} \int p_\mu e^{\xi_\kappa p^\kappa} d\omega_{C-Field} = \mathcal{N}_{\mu C-Field}, \tag{C.85}$$

$$A \int p_\mu p_\nu e^{\xi_\kappa p^\kappa} d\omega + A_{C-Field} \int p_\mu p_\nu e^{\xi_\kappa p^\kappa} d\omega_{C-Field} = T_{\mu\nu}. \tag{C.86}$$

From this and the number densities, write expressions for computing the section of the nullcone between the starting point on the hypersurface and the target:

$$A = \frac{\mathcal{N}_0 \xi}{4\pi m^2 K_2(m\xi)}; \tag{C.87}$$

$$A_{C-Field} = \frac{\mathcal{N}_{0C-Field}\xi}{4\pi {m_a}^2 K_2({m_a}^2 \xi)}. \tag{C.88}$$

Using this with (C.30) and (C.37), expand the energy-momentum tensor:

$$\begin{aligned}
\Rightarrow \rho + p &= \frac{4\pi}{\xi}[Am^3 K_3(m\xi) + A_{C-Field} {m_a}^3 K_3(m_a \xi)] \\
&= m\mathcal{N}_0 G(m\xi) + m_a \mathcal{N}_{0C-Field} G(m_a \xi); && \text{(C.89)} \\
p &= \frac{4\pi}{\xi^2} A m^2 K_2(m\xi) \\
&= \frac{\mathcal{N}_0}{\xi}. && \text{(C.90)}
\end{aligned}$$

The function $G(x) = \frac{K_3(x)}{K_2(x)} = \frac{2}{x} - \frac{K_2'(x)}{K_2(x)}$.

The action implies a non-constant number density $\mathcal{N}_{0C-Field}$ for the C-field particles. In the case of inflation/production, this increases. Use the rewritten energy-momentum tensor and the number flux vector:

$$\begin{aligned}
T_{\mu\nu} = (m\mathcal{N}_0 G(m\xi) &+& m_a \mathcal{N}_{0C-Field} G(m_a\xi))\bar{u}_\mu \bar{u}_\nu + g_{\mu\nu}\frac{\mathcal{N}_0}{\xi}, \\
\mathcal{N}^\mu &=& \mathcal{N}_0 \bar{u}^\mu, \\
\mathcal{N}_{C-Field}{}^\mu &=& \mathcal{N}_{0C-Field}\bar{u}^\mu, \\
\Rightarrow [m\mathcal{N}_0 G(m\xi) &+& m_a \mathcal{N}_{0C-Field} G(m_a\xi))\bar{u}_\mu \bar{u}_\nu]_{;\nu} + (\frac{\mathcal{N}_0}{\xi})_{;\mu} \\
&=& (m_a \mathcal{N}_{0C-Field} G(m_a\xi)\bar{u}_\mu \bar{u}_\nu)_{;\nu} \\
&=& f C^{,\mu} C^{,\nu}{}_{;\nu};
\end{aligned} \tag{C.91}$$

$$\begin{aligned}
(\mathcal{N}_0 \bar{u}^\mu)_{;\mu} &=& \mathcal{N}_{0;\mu}\bar{u}^\mu + \mathcal{N}_0 \bar{u}^\mu_{;\mu} = 0; \\
(\mathcal{N}_{0C-Field}\bar{u}^\mu)_{;\mu} &=& \mathcal{N}_{0C-Field;\mu}\bar{u}^\mu + \mathcal{N}_{0C-Field}\bar{u}^\mu_{;\mu} \\
&=& \mathcal{N}_{0C-Field;\mu}\bar{u}^\mu = \frac{\xi^\mu}{\xi}\mathcal{N}_{0C-Field;\mu}.
\end{aligned} \tag{C.92}$$

This follows since the mean 4-velocity is constant and the number density for the C-particles is increasing. These relations imply that

$$f C^{,\mu} C^{,\nu}{}_{;\nu} = \frac{m_a \xi^\mu \xi^\nu}{\xi^2}[\mathcal{N}_{0C-Field} G(m_a\xi)]_{;\nu}. \tag{C.93}$$

Finally, write this expression in terms of split spacetime:

$$\begin{aligned}
f C^{,\mu} C^{,\nu}{}_{;\nu} &=& f \cdot ((g_{00} C^{,0}{}_{;0} C^{,0} + g_{i0} C^{,0}{}_{;0} C^{,i} + g_{0j} C^{,j}{}_{;j} C^{,0} \\
&+& g_{ij} C^{,j}{}_{;j} C^{,i})) \\
&=& f((N_k N^k - N^2) C^{,0}{}_{;0} C^{,0} + N_i C^{,0}{}_{;0} C^{,i} \\
&+& N_j C^{,j}{}_{;j} C^{,0} + g_{ij} C^{,j}{}_{;j} C^{,i}))
\end{aligned}$$

$$\Rightarrow f(N_k N^k - N^2)C^{,0}{}_{;0}C^{,0} = \frac{m_a \xi^0 \xi^0}{\xi^2}[\mathcal{N}_{0C-Field}G(m_a\xi)]_{;0}; \tag{C.94}$$

$$fN_i C^{,0}{}_{;0}C^{,i} = \frac{m_a \xi^i \xi^0}{\xi^2}[\mathcal{N}_{0C-Field}G(m_a\xi)]_{;0} = 0; \tag{C.95}$$

$$fN_j C^{,j}{}_{;j}C^{,0} = \frac{m_a \xi^0 \xi^j}{\xi^2}[\mathcal{N}_{0C-Field}G(m_a\xi)]_{;j} = 0; \tag{C.96}$$

$$fg_{ij} C^{,j}{}_{;j}C^{,i} = \frac{m_a \xi^i \xi^j}{\xi^2}[\mathcal{N}_{0C-Field}G(m_a\xi)]_{;j} = 0. \tag{C.97}$$

These equations give a few more relations by which we can determine properties of the C-field, specifically in the case of such a field producing a dust.

Using the information about 3-momentum (particles are created at rest) and that spatial components of the covariant normal vector equal zero:

$$\begin{aligned}
\mathcal{H} &= \frac{\sqrt{g}}{2}({}^{(3)}R - Tr(\mathbf{K}^2) + (Tr\mathbf{K})^2) \\
&= 8\pi n^\mu n^\nu [T_{\mu\nu} - f\cdot(C_{,\nu}C_{,\mu} - \frac{1}{2}g_{\mu\nu}C_{,\kappa}C^{,\kappa})] \\
&= 8\pi [T_{\mu\nu}n^\mu n^\nu - f\cdot(C_{,\nu}C_{,\mu}n^\mu n^\nu - \frac{1}{2}g_{\mu\nu}n^\mu n^\nu C_{,\kappa}C^{,\kappa})] \\
&= 8\pi [(m\mathcal{N}_0 G(m\xi) + m_a\mathcal{N}_{0C-Field}G(m_a\xi))\bar{u}_\mu\bar{u}_\nu n^\mu n^\nu + g_{\mu\nu}\frac{\mathcal{N}_0}{\xi}n^\mu n^\nu \\
&\quad -f\cdot(C_{,\nu}C_{,\mu}n^\mu n^\nu - \frac{1}{2}g_{\mu\nu}n^\mu n^\nu C_{,\kappa}C^{,\kappa})] \\
&= 8\pi [(m\mathcal{N}_0 G(m\xi) + m_a\mathcal{N}_{0C-Field}G(m_a\xi)) - \frac{\mathcal{N}_0}{\xi} \\
&\quad +fC^{,\mu}C_{,\mu} - \frac{f}{2}C^{,\kappa}C_{,\kappa}] \\
&= 8\pi [(m\mathcal{N}_0 G(m\xi) + m_a\mathcal{N}_{0C-Field}G(m_a\xi)) - \frac{\mathcal{N}_0}{\xi} \\
&\quad +\frac{f}{2}C^{,0}C_{,0}];
\end{aligned} \tag{C.98}$$

$$\begin{aligned}
\mathcal{H}_i &= K_i{}^a{}_{|a} - (Tr\mathbf{K})_{|i} \\
&= -8\pi n^\mu \gamma_i{}^\nu [T_{\mu\nu} - f\cdot(C_{,\nu}C_{,\mu} - \frac{1}{2}g_{\mu\nu}C_{,\kappa}C^{,\kappa})] \\
&= -8\pi [n^\mu \gamma_i{}^\nu (m\mathcal{N}_0 G(m\xi) + m_a\mathcal{N}_{0C-Field}G(m_a\xi))\bar{u}_\mu\bar{u}_\nu + n^\mu\gamma_i{}^\nu g_{\mu\nu}\frac{\mathcal{N}_0}{\xi}
\end{aligned}$$

$$
\begin{aligned}
& -n^\mu \gamma_i{}^\nu f \cdot (C_{,\nu} C_{,\mu} - \frac{1}{2} g_{\mu\nu} C_{,\kappa} C^{,\kappa}) \\
= & -8\pi[n^\mu(\delta^\nu{}_i + n^\nu n_i)(m\mathcal{N}_0 G(m\xi) + m_a \mathcal{N}_{0C-Field} G(m_a \xi))\bar{u}_\mu \bar{u}_\nu \\
& + n^\mu(\delta^\nu{}_i + n^\nu n_i) g_{\mu\nu} \frac{\mathcal{N}_0}{\xi} \\
& - n^\mu(\delta^\nu{}_i + n^\nu n_i) f \cdot (C_{,\nu} C_{,\mu} - \frac{1}{2} g_{\mu\nu} C_{,\kappa} C^{,\kappa})] \\
= & -8\pi[(m\mathcal{N}_0 G(m\xi) + m_a \mathcal{N}_{0C-Field} G(m_a \xi)) n^\mu \bar{u}_\mu (\bar{u}_i + n^\nu n_i \bar{u}_\nu) \\
& + (n_i + n^\mu n_\mu n_i) \frac{\mathcal{N}_0}{\xi} \\
& - f \cdot (n^\mu (C_{,i} C_{,\mu} + n^\nu n_i C_{,\nu} C_\mu) - \frac{1}{2}(n_i + n^\mu n_i n_\mu) C_{,\kappa} C^{,\kappa})] \\
= & \ 0; \qquad \text{(C.99)}
\end{aligned}
$$

$$
\begin{aligned}
(\partial_t - \pounds_{N_i}) K_{ij} = & -N_{|ij} + N R_{ij} - 2N K_{ia} K_j{}^a + N(Tr\mathbf{K}) K_{ij} \\
& -8\pi(N\gamma_i{}^\nu \gamma_j{}^\mu + \frac{1}{2} N \gamma_{ij} n^\mu n^\nu - \frac{1}{2} \gamma_{ij} \gamma^{ab} \gamma_a{}^\nu \gamma_b{}^\mu)[T_{\mu\nu} \\
& - f(C_{,\nu} C_{,\mu} - \frac{1}{2} g_{\mu\nu} C_{,\kappa} C^{,\kappa})] \\
= & -N_{|ij} + N R_{ij} - 2N K_{ia} K_j{}^a + N(Tr\mathbf{K}) K_{ij} \\
& -8\pi(N\gamma_i{}^\nu \gamma_j{}^\mu + \frac{1}{2} N \gamma_{ij} n^\mu n^\nu - \frac{1}{2} N \gamma_{ij} \gamma^{ab} \gamma_a{}^\nu \gamma_b{}^\mu) \\
& [(m\mathcal{N}_0 G(m\xi) + m_a \mathcal{N}_{0C-Field} G(m_a \xi))\bar{u}_\mu \bar{u}_\nu + g_{\mu\nu} \frac{\mathcal{N}_0}{\xi} \\
& - f \cdot (C_{,\nu} C_{,\mu} - \frac{1}{2} g_{\mu\nu} C_{,\kappa} C^{,\kappa})] \\
= & -N_{|ij} + N R_{ij} - 2N K_{ia} K_j{}^a + N(Tr\mathbf{K}) K_{ij} \\
& -8\pi(N(\delta^\nu{}_i + n^\nu n_i)(\delta^\mu{}_j + n^\mu n_j) + \frac{1}{2} N(g_{ij} + n_i n_j) n^\mu n^\nu \\
& - \frac{1}{2}(g_{ij} + n_i n_j)(g^{ab} + n^a n^b)(\delta^\nu{}_a + n^\nu n_a)(\delta^\mu{}_b + n^\mu n_b)) \\
& [(m\mathcal{N}_0 G(m\xi) + m_a \mathcal{N}_{0C-Field} G(m_a \xi))\bar{u}_\mu \bar{u}_\nu + g_{\mu\nu} \frac{\mathcal{N}_0}{\xi} \\
& - f \cdot (C_{,\nu} C_{,\mu} - \frac{1}{2} g_{\mu\nu} C_{,\kappa} C^{,\kappa})] \\
= & -N_{|ij} + N R_{ij} - 2N K_{ia} K_j{}^a + N(Tr\mathbf{K}) K_{ij} \\
& -8\pi(N(\delta^\nu{}_i \delta^\mu{}_j + n^\nu n_i \delta^\mu{}_j + \delta^\nu{}_i n^\mu n_j + n^\nu n_i n^\mu n_j)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2}N(g_{ij}n^\mu n^\nu + n_i n_j n^\mu n^\nu) \\
& -\frac{1}{2}Ng_{ij}(g^{ab}\delta^\nu{}_a\delta^\mu{}_b + g^{ab}\delta^\nu{}_a n^\mu n_b + g^{ab}n^\nu n_a\delta^\mu{}_b + g^{ab}n^\nu n_a n^\mu n_b \\
& +n^a n^b\delta^\nu{}_a\delta^\mu{}_b + n^a n^b\delta^\nu{}_a n^\mu n_b + n^a n^b n^\nu n_a\delta^\mu{}_b + n^a n^b n^\nu n_a n^\mu n_b)) \\
& -f\cdot(C_{,\nu}C_{,\mu} - \frac{1}{2}g_{\mu\nu}C_{,\kappa}C^{,\kappa})] \\
= \quad & -N_{|ij} + NR_{ij} - 2NK_{ia}K_j{}^a + N(Tr\mathbf{K})K_{ij} \\
& -8\pi(N(\delta^\nu{}_i\delta^\mu{}_j + n^\nu n_i\delta^\mu{}_j + \delta^\nu{}_i n^\mu n_j + n^\nu n_i n^\mu n_j) \\
& +\frac{1}{2}N(g_{ij}n^\mu n^\nu + n_i n_j n^\mu n^\nu) \\
& -\frac{1}{2}Ng_{ij}(g^{\nu\mu} + n^\mu n^\nu + n^\nu n^\mu + n^\nu n^b n^\mu n_b \\
& +n^\nu n^\mu + n^\nu n^b n^\mu n_b + n^a n^\mu n^\nu n_a + n^a n^b n^\nu n_a n^\mu n_b)) \\
& [(m\mathcal{N}_0 G(m\xi) + m_a\mathcal{N}_{0C-Field}G(m_a\xi))\bar{u}_\mu\bar{u}_\nu + g_{\mu\nu}\frac{\mathcal{N}_0}{\xi} \\
& -f\cdot(C_{,\nu}C_{,\mu} - \frac{1}{2}g_{\mu\nu}C_{,\kappa}C^{,\kappa})] \\
= \quad & -N_{|ij} + NR_{ij} - 2NK_{ia}K_j{}^a + N(Tr\mathbf{K})K_{ij} \\
& -8\pi(N\delta^\nu{}_i\delta^\mu{}_j + \frac{1}{2}Ng_{ij}n^\mu n^\nu - \frac{1}{2}g_{ij}(g^{\nu\mu} + 3n^\mu n^\nu)) \\
& [(m\mathcal{N}_0 G(m\xi) + m_a\mathcal{N}_{0C-Field}G(m_a\xi))\bar{u}_\mu\bar{u}_\nu + g_{\mu\nu}\frac{\mathcal{N}_0}{\xi} \\
& -f\cdot(C_{,\nu}C_{,\mu} - \frac{1}{2}g_{\mu\nu}C_{,\kappa}C^{,\kappa})] \\
= \quad & -N_{|ij} + NR_{ij} - 2NK_{ia}K_j{}^a + N(Tr\mathbf{K})K_{ij} \\
& -8\pi[(m\mathcal{N}_0 G(m\xi) + m_a\mathcal{N}_{0C-Field}G(m_a\xi))(N\bar{u}_j\bar{u}_i \\
& +\frac{1}{2}Ng_{ij}n^\mu n^\nu\bar{u}_\mu\bar{u}_\nu - \frac{1}{2}g_{ij}(\bar{u}^\nu\bar{u}_\nu + 3n^\mu n^\nu\bar{u}_\mu\bar{u}_\nu)) \\
& +(Ng_{ji} - \frac{1}{2}Ng_{ij} - \frac{1}{2}Ng_{ij}(g^{\nu\mu}g_{\mu\nu} + 3n^\mu n_\nu))\frac{\mathcal{N}_0}{\xi} \\
& -f\cdot(NC_{,i}C_{,j} + \frac{1}{2}Ng_{ij}n^\mu n^\nu C_{,\nu}C_{,\mu} \\
& -\frac{1}{2}g_{ij}(C^{,\mu}C_{,\mu} + 3n^\mu n^\nu C_{,\nu}C_{,\mu}) \\
& -\frac{1}{2}(g_{ji} - \frac{1}{2}g_{ij} - \frac{1}{2}g_{ij}(g^{\nu\mu}g_{\mu\nu} + 3n^\mu n^\nu g_{\mu\nu}))C_{,\kappa}C^{,\kappa})] \\
= \quad & -N_{|ij} + NR_{ij} - 2NK_{ia}K_j{}^a + N(Tr\mathbf{K})K_{ij} \\
& -8\pi[(m\mathcal{N}_0 G(m\xi) + m_a\mathcal{N}_{0C-Field}G(m_a\xi))(\frac{1}{2}Ng_{ij}n^\mu n^\nu\bar{u}_\mu\bar{u}_\nu
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{2}(g_{ij}+n_in_j)(2\bar{u}^{\nu}\bar{u}_{\nu}))+(\frac{1}{2}Ng_{ji}-\frac{1}{2}Ng_{ij})\frac{\mathcal{N}_0}{\xi} \\
& -f\cdot(\frac{1}{2}Ng_{ij}n^{\mu}n^{\nu}C_{,\nu}C_{,\mu}-\frac{1}{2}Ng_{ij}(C^{,\mu}C_{,\mu}) \\
& -\frac{1}{2}(\frac{1}{2}Ng_{ij}-\frac{1}{2}N(g_{ij}))C_{,\kappa}C^{,\kappa})] \\
= \; & -N_{|ij}+NR_{ij}-2NK_{ia}K_j{}^a+N(Tr\mathbf{K})K_{ij} \\
& +8\pi Ng_{ij}[\frac{1}{2}(m\mathcal{N}_0G(m\xi)+m_a\mathcal{N}_{0C-Field}G(m_a\xi)) \\
& -f\cdot(C^{,\mu}C_{,\mu})] \\
= \; & -N_{|ij}+NR_{ij}-2NK_{ia}K_j{}^a+N(Tr\mathbf{K})K_{ij} \\
& +8\pi Ng_{ij}[\frac{1}{2}(m\mathcal{N}_0G(m\xi)+m_a\mathcal{N}_{0C-Field}G(m_a\xi)) \\
& -f\cdot(C^{,0}C_{,0})];
\end{aligned}
\tag{C.100}
$$

$$
(\partial_t-\pounds_{N_i})\gamma_{ij} = -2NK_{ij}. \tag{C.101}
$$

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