ABSTRACT

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The purpose of this literature review is two fold. The first is to examine the types of geometric proof and their purpose within the classroom setting and secondly to examine the factors that influence students' understanding of proofs and their proof construction. The research focuses on three main types of proof 1) Proof for the development of proficiency, 2) proof for understanding, and 3) proof for exploration. In addition to these proof types the research focuses on three components that influence students' development and understanding of proof: 1) the role of technology in students' mastery of proof, 2) the role of curricula and teachers in students' mastery of proof, and 3) the role of the student in their own mastery of proof. Through an investigation into the types of proof and the beliefs and misconceptions of individuals who teacher, learn and write proofs, the author attempts to highlight the reasons for teaching geometric proof and the responsibilities of both teachers and students in the developing understanding in and through geometric proof.

THE EDUCATIONAL PURPOSES OF GEOMETRIC PROOF IN THE HIGH SCHOOL CURRICULUM

by

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BIOGRAPHY

I was born and raised in Hickory, North Carolina a small city at the base of the Appalachian Mountains. I have an older sister who has recently moved back into our old home, with her family, after living several years in Indiana. My father now resides in Hudson, North Carolina which is approximately fifteen miles west of Hickory. My mother passed away during the summer of 2005 from a heart attack in her sleep. Neither of my parents attended college and I was the first individual in my immediate family to earn a college degree and I will also be the first to earn a Master's degree. Most of my childhood years were spent harassing my older sister, in an attempt to be just like her. Growing up, I loved the outdoors and many of our family vacations were spent hiking in the mountains or swimming at the beach.

At an early age I was fascinated with mathematics. It was the only subject in school that ever made sense to me. When I entered college as an undergraduate at North Carolina State University, I enrolled in the computer engineering program but after a few classes discovered it was not for me. Spending days on end in front of a computer screen was anything but enjoyable. So now after two years of college, I had to decide on a new career.

In high school, I had always been everyone's math tutor and really enjoyed helping out my classmates. With this in mind, I changed my major to Mathematics Education and transferred to Appalachian State University where in I earned a Bachelor's degree in middle grades education with concentrations in mathematics and the social sciences and met my husband. In 1999 we both graduated from the same degree program, although his concentrations were mathematics and science. After graduation I applied for and received a teaching position in Johnston County, North Carolina at Clayton Middle School. I taught there for two years during which time I completed a Product of Learning for the state of North Carolina and received a passing score. In the fall of 2001 Johnston County opened a new middle school, Riverwood Middle, and teachers from Clayton Middle School were divided between the two schools. I was sent to Riverwood. I taught at Riverwood middle for two years, and then transferred to Broughton High School in Wake County, North Carolina. This move had two purposes. To be closer to North Carolina State University so that I could pursue a Masters degree in Mathematics Education and to move into the high school curriculum. I discovered that while teaching in middle schools I was unable to teach the amount of geometry that I enjoyed; therefore, I decided to move to a level of education that would allow me to teach all the geometry I wanted. After one year at Broughton, I transferred schools again, hopefully for the last time. Wake County opened a new high school minutes from my home. I could not pass up the opportunity to be close to home and not have to deal with Raleigh traffic every morning. So in the fall of 2004, I started at Knightdale High School, home of the Knights. At Knightdale, I teach Algebra and Geometry and enjoy almost every day. My geometry classes are what inspired my thesis. I notice every year that students struggle more and more with geometric proof, and I wanted to help them learn and understand the usefulness of proof. Thus my thesis is an effort to understand my students' problems and help eliminate any misconceptions and/or difficulties in their path to understanding.

I have been in Raleigh now for 6 years. I have a wonderful husband of three years that has been extremely supportive an understanding and I owe him most of the credit for my success. He and the rest of my family have taken time out of their schedules to help and support me and when necessary to tell me I'm acting a little crazy.

This experience has changed how and teach and how I view my students and I am glad that I have had the opportunity to continue my education and in the process better myself and my teaching skills.

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CHAPTER 1
INTRODUCTION

In the *Principles and Standards*, the National Council of Teachers of Mathematics, NCTM, (2000) considers reasoning and proof to be an integral part of students' mathematical experiences. Mathematical proof is a concept that should help "unify the mathematical experiences" (Vavilis, 2003, p. 2) of students by providing them with the logic skills necessary to construct and deconstruct mathematical arguments. Proof aids in the integrating of seemingly unrelated theories, provides structure, and helps students break down information in all subjects (de Villiers, 1999). While proof is often a subject dreaded by high school geometry students, these students, if they pursue proof as a tool for understanding and exploration will be provided with a skill that is beneficial; the ability to use convincing arguments in an effort to persuade others.

A Perspective on Reasons for Teaching Proof

There are many reasons why teachers teach proof. These reasons range from curriculum requirements, teacher preference, and educational opportunities. In geometry courses across the globe one of the main components of the curriculum is deductive proof, especially for students who are college bound (Senk, 1989). Pacing guides, textbooks, and supplemental materials all reflect the focus that curricula have placed on reasoning and proof. Until recently, many states in America required students in geometry classes to take a proof test. This test consisted of several proof problems for which pupils constructed formal proofs generally in a two-column format. Drawing from my own experience in taking the North Carolina proof test, the proofs were very structured and many included four to five steps. The proofs dealt mainly with figures that were provided and conditional statements were separated into the given and prove statements for us. Students completing this test and similar tests did not necessarily have to understand the theorems, postulates, and definitions they were using, although it would have helped, but instead they only needed to show that they could construct a formal proof using appropriate reasons. Because of testing scenarios like this one, educators often teach the skills involved in constructing a proof, but not the logic needed to understand the proof. While changes in curricula have forced more teachers to incorporate proof into their classroom, it often does not explain how to organize the atmosphere of the classroom so that students are encouraged to produce logical arguments and proofs (Herbst, 2002a). Thus, the addition of proof into the geometry curriculum does not automatically mean students taking a geometry course will learn to utilize proof as a tool to deepen their understanding nor that they will be able to produce a coherent logical argument once they leave the classroom. Teachers who teach proof only because it is in the curriculum will most likely only teach students to mimic the teachers' methods in the structure and rigor of proofs. In my endeavors to teach proof in my own geometry classes, I was interested in learning why geometry curricula have just recently become so focused on students' ability to write and understand proofs.

Several teachers, myself included, teach proof because they enjoy the topic and believe that students should be able to use logic skills to get from point *A* to point *B*. While the reference of getting from point *A* to point *B* may over simplify the argument of proof, it is the foundation of proof, to get from a certain hypothesis to a concluding statement. Educators, who teach proof by choice, not just because it is in their curriculum, may be more open to allowing students the choice of formats for their proofs. Proofs can be written in a variety of formats: two-column, mathematical induction, flow proof, paragraph, and indirect. And within each format there is further room for personal adjustment of arguments which allows students to feel that the proof is their own creation instead of a replica of an instructor's proof. While I enjoy teaching geometry, it is often frustrating to teach proof to students who see it as a waste of time. As part of my research, I was interested in understanding the difficulties that students face in their understanding of the topic of proof. Their difficulties are areas that educators need to focus on, and in order to focus on and hopefully avoid these difficulties; we must know what they are. This way, proof exercises presented in classroom settings can help students to understand that when things happen in mathematics, they happen for a reason (Herbst, 2002a).

One of the most important reasons, in my opinion, for teaching proofs is the opportunity to educate students on the foundations of mathematical principles. So often we tell students in the lower levels of mathematics that they will prove why the sum of the interior angles of a triangle is 180 degrees later in their mathematical career and in doing so we stifle their natural curiosity (Jones, 1994). When students at any age ask why something is valid/true they are asking for proof. It is important that these educational opportunities are not wasted. Students at all levels crave to understand why things happen the way that they do. In geometry classes some of that yearning is lost because students are forced to produce formal proofs that they find uninteresting and unimportant to their daily lives. In an effort to regain their child-like curiosity, geometry teachers must find ways to make proof applicable to the student. Yet, how do we do this? Is it through the use of technology, inductive reasoning instead of formal proof, or is their a middle ground where formal proof, technology, and inductive reasoning can rekindle a students desire to explore new topics and/or develop a better understanding of the geometric world around them?

Purpose of This Literature Review

Since I am an individual that thrives on organization and structure, it is no surprise that I am drawn to the logic and sophistication of geometric proof. On the other hand it may come as a surprise to know that I do not believe that all geometric proofs need to be formal in their structure. Proof is a concept that is largely misunderstood by the students who are learning to construct geometric proofs. They often do not understand the terminology or the premises presented; therefore they struggle with not only the structure of proofs, but also the logic of proofs. This literature review seeks to determine the following:

1. What types of proofs are beneficial to students' geometric understanding?
2. What role does technology have in students' development and understanding of proof?
3. What is the role of curricula and teachers on students' development and understanding of proof?
4. What role do students have in their own development and understanding of proof?

In each of the following four chapters these questions will be discussed and explored using relevant research findings. While each chapter can be taken for its' own merit, the chapters are meant to be used in conjunction with one another. The research findings bridge many of the chapters since students' development and understanding of geometric proof does not rely on any one issue. Conclusion and implications of the research finding are recorded in the final chapter (Chapter 6).

CHAPTER 2

WHAT TYPES OF PROOFS SHOULD STUDENTS BE EXPOSED TO IN HELPING THEM PREPARE FOR ADVANCED MATHEMATICS?

At every stage of development students need to be exposed to reasoning and proof (NCTM 2000). In recent years, the geometry curriculum has changed and evolved so that geometric proof is again a key component. While many educators prefer to develop investigative and problem-solving skills instead of proof construction, the focus must now change so that the heuristic skills that can be developed through problem solving are also used in the process of proof construction (Hanna, 2000). This shift gives educators in this field an important task: expose students to geometric reasoning, so that they can "establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others" (NCTM, 2000, p. 308)

The types of proofs that educators use in order to achieve this goal, may play an important role in the students' understanding of geometric theories, their ability to communicate mathematically and their preparedness for more advanced mathematics classes (Yackel & Hanna, 2003). The classifications for the types of proofs in this document will refer to the purpose of writing proofs. Therefore, the types discussed will focus on the desired understanding and outcome that an instructor is requiring of his/her students. Many researchers use the verbiage 'type of proof' to describe various aspects of proof. One description of proof type is the format in which a proof is presented, these types are two-column, paragraph, flow, indirect, or coordinate proofs; however, the format of a proof is not the focus of this research. The concept of proof types for this study is based on several studies conducted on proof. The first is a study conducted by McCrone, Martin, Dindyal, and Wallace (2002) where two main types of proof are discussed: (a) application proof where students are introduced to a new concept and then shown a proof using the concept, and (b) introduction proof where students are introduced to new concepts using a proof. In most cases students who were shown application proofs, thought they were only relevant with the example given, while introduction proof seemed to encourage students to use them as a building block (McCrone et al., 2002). Similarly, Hanna (1995) discusses proofs that prove and proofs that explain. To prove a theorem using axioms, postulates, definitions, and theorems does fulfill the requirement of proving a statement; however, for students to comprehend the effect of a proof, the proof also needs to explain why these axioms, postulates, definitions, and theorems work and are useful in the context of the proof (Hanna, 1995).

The question now becomes, what type of proofs should teachers use in the instruction of students and in turn expect students to write? Should instructors work with students in the classroom on proofs that are only regurgitations of specific steps for given theorems, or should they practice exercises that encourage students to explore new conjectures? Or does a "good" proof contain aspects of all proof types? In this section three distinct types of proof will be discussed and explored: development of proficiency, understanding, and exploration.

Type I: Proof for the Development of Proficiency

In many geometry classes, students are taught to prove specific theorems, with a specific number of statements and reasons, in a very logical and well planned manner. They are used to establish the skills necessary to properly and logically organize statements and reasons in a proof. In creating these proofs, students are often only becoming proficient in the skills to reproduce the proof a teacher has given them. No new conjectures are introduced and steps cannot be added, deleted or rearranged, even if mathematically sound. Proof for proficiency is simply a regurgitation of steps given to students by the instructor.

Proficiency proofs, do have their place in mathematics education, students do need to know how to logically arrange statements and reasons. "A proof would not succeed with students who had never learned to follow an argument…" (Hanna, 1995, p. 48). The cornerstone of proof is the ability to follow and understand an argument. If students are not well equipped in the art of making statements and providing reasons all other types of proof are nonexistent. According to the NCTM (2000), students "should be able to produce logical arguments and present formal proofs that effectively explain their reasoning." A student who cannot produce a proof using the proper order of the steps is said to be lacking in reasoning skills, due to the fact that they are unable to justify their thought process (Herbst1, 2002). In order to aid these students in their endeavors to create a proof, teachers often have them verify proven theorems or corollaries. Thus, the development and use of predetermined steps is employed. Consider the following proof (Bell, 1976, p. 25):

Theorem: The angle at the center of a circle is twice the angle at the circumference subtended by the same arc.

Given: Circle, center O, points A, P, & B on circumference.

Construction: Join PO and produce to any point X

Let angles a, b, a_1, b_1, x, & y be as marked.

Prove: m∠AOB =2 m∠APB

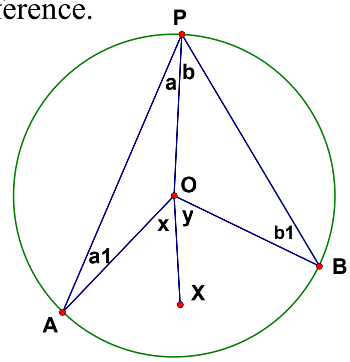


Figure 1

Proof:	OA = OP	$\rightarrow$	radii of a circle are equivalent
	$\therefore a = a_1$	$\rightarrow$	base angles of an isosceles Δ are congruent
	$x = a + a_1$	$\rightarrow$	exterior angel of a triangle
	$\therefore x = 2a$	$\rightarrow$	substitution
	Similarly $y = 2b$		
	$x + y = 2a + 2b$		
	$x + y = 2(a + b)$	$\rightarrow$	distributive property
	m∠AOB =2 m∠APB	$\rightarrow$	substitution

As presented, this proof is a simple conglomeration of steps that an instructor has given to the pupil. The student knows exactly how to begin the proof, which parts of the drawing are relevant, and how to end the proof. In this exercise they are given the object of the proof (the final statement), the necessary construction needed to complete the proof, and labels for all the angles required to complete the exercise. Students who are given this type of proof know that there are required steps that must be followed. There is little room for interpretation. Most students will assume that the only angles needed are those that are labeled, the only needed construction is segment PO and the final result cannot be anything other than m∠AOB = 2m∠APB. This proof simply verifies a fact already in use and allows students a chance to become proficient in their skills of organizing statements and reasons for a proof. Proofs used in developing the proficiency skills of constructing a proof are generally easy with the desirable result only a few short steps away. This causes students to develop the idea that all proofs follow this structure and when they are faced with more complicated topics to prove they are often discouraged (McCrone, et al., 2002)

The idea of proof for proficiency may stem from a desire on the educators part to "preserve the precision and beauty of mathematics" (Alibert & Thomas, 1991). Teachers who are instructing students in what a correct proof looks like may determine that any proof that deviates from the template is an ineffective argument. Many instructors base the correctness of a proof on how it looks and may overlook logical errors if students have written their proof in a valid form with the desired beginning and ending statements (Knuth, 2002). The idea that there is a precision to how a proof is written may for some educators overshadow the logic of the arguments. These types of proofs are also easier to grade. If it has been predetermined that a proof will take six specific steps, then deviation from the template would suggest a student's lack of knowledge making the assessment of the proof an objective activity instead of a subjective one.

Teachers must decide what an acceptable proof is in order to aptly educate students in their endeavors to write proofs; however, teachers need to be cautious in their decision so that they do not create a tunnel vision approach to proof. "Formal proof is sometimes thought of only as chains of logical argument that follow agreed-on rules of deduction and is often characterized by the use of formal notation, syntax, and rules of manipulation" (Yackel & Hanna, 2003, p. 228) Engaging students in these types of exercises can be problematic. Students may not see the need to prove items that already have solutions; they may also feel deterred from expressing any type of difference of opinion for fear of looking stupid in front of their classmates (Balacheff, 1991). In most instances where students are informed of the proper procedure for constructing the argument and structure of a formal proof, the message received by the student is ‘we do this because the instructor/book told us to’ and it has no explicit value (Almeida, 2001, McCrone et al., 2002).

Type II: Proof for Understanding

Often mathematics texts ask students to justify solutions as a process to show that they understand how they arrived at an answer (Almeida, 2001). This is a practice that is central to the process of writing proofs and bridges the process of writing a proof with the systematic solving of a problem. It is important in this application that teachers use explanations that are meaningful for the student and students use explanations that have meaning for his/herself. This use of meaningful explanations internalizes the concepts and allows the individual to express solutions that they understand (Almeida, 2001). If the justification does not have a significant meaning for the pupil, then the entire exercise is procedural (Bell, 1976). According to de Villiers (1990), when students are constructing a proof they need to not only know the theorems, postulates and definitions, but also how they interact with each other and can be integrated with one another to provide a cohesive statement. Writing a proof is like writing a story or telling a joke. There is a beginning, middle and a conclusion. When telling a joke, one does not start with the punch line; the joke would not make any sense. Therefore, when writing a proof, one does not use the information they are trying to prove, it belongs at the end. deVilliers (1990) also stated that proof is a form of communication between mathematical colleagues; therefore, if students wish to communicate effectively, they must have an understanding of the concepts they are trying to express. They cannot fully explain something they, themselves do not understand.

Proving the truth value of a statement is not the sole purpose of proof (Herbst, 2002a). It should be used to show necessary connections to other propositions and their interdependence (Herbst, 2002a). Understanding also plays a role in the eagerness of a student to learn. "Students are more eager to learn what they understand than what they must memorize" (Izen, 1998). To better serve students in this endeavor of understanding, proofs can be used to explain topics so that comprehension is better and topics that students were undecided about or had questions regarding become clearer (Hanna, 1995; deVilliers, 1990). In many cases, students need proof to help them grasp the intricacies of a problem or statement. It is easy to tell students that the sum of the interior angles of any polygon is $(n - 2)180°$, but it may be more helpful for the student if they are shown why this true, using an informal proof. A teacher may start with a triangle, and use the sum of the interior angles of a triangle theorem to introduce the concept that a polygon with three sides has an interior angle sum of 180°. Continuing from there the teacher would draw a quadrilateral and have the students determine the number of triangles formed with the diagonals of one vertex and continue this with several polygons, until students begin to see a pattern. This type of inductive proof, although more correctly referred to as inductive reasoning, where students rely on examples to form opinions, helps students explain why the theorem is true and helps them internalize the information. Proofs used in the explanation of new concepts can help to promote understanding and foster a deeper understanding of the mathematical concepts that are represented in the proof (McCrone, et al., 2002). Using proof as a tool to explain new and difficult concepts increases not only the usefulness of proof but also students' opinion of proof (McCrone et al., 2002). For most educators the problem in teaching proofs is not in students' acceptance of a theorem as a true statement, but the students' views that the concept is unnecessary. Students willingly accept theorems are true and struggle with the need to prove them again (Hersh, 1993). Teacher often here the question, why do we have to prove this? Most of the time, this question stems from the fact that the statement has already been proven and the instructor has told students the statement is true. Students do not struggle with whether a theorem is true or false, but the comprehension of why the theorem is true (Hanna, 1995). Proof for understanding helps put theorems in context so that students develop an appreciation for why the theorem is true (Izen, 1998).

The increase of mathematical understanding is the component of proof that legitimizes it as a convincing argument to a mathematician (Hanna, 1995). "A proof is a *complete* explanation" and in that explanation students begin to understand why particular theorems are used and they are able understand the applications for the theorems (Hersh, 2003, p. 397). "But proof can make its greatest contribution in the classroom only when the teacher is able to use proofs that convey understanding" (Hanna, 2000, p. 7).

Type III: Proof for Exploration

The basis of every mathematical subject was derived by exploration of concepts. “There are numerous examples in the history of mathematics where new results were discovered/ invented in a purely deductive manner…” (deVilliers, 1990). This deductive manner essentially involves proof which by its very definition is exploratory. Proving is a problem solving activity that moves an individual from a known statement to an unknown by the means of finding connections in the data presented, and in order to find those connections one has to explore the possibilities (Hanna, 2000). Students who use proof as an exploration tool are more likely to view proof as help instead of an obstacle. It is generally the case that the lack of logical reasoning does not always stem from a person’s lack of knowledge, but may in fact stem from a lack of interest. Young children are capable of reasoning in situations that are meaningful to them, but they are uninterested in working on topics that contain no value in their daily lives (deVilliers, 1990).

Most mathematics students are instructed in proof by modeling standard logical and deductive reasoning skills. This transforms proof into an ineffective tool. Students are not using proofs to validate their own conjectures; instead, they are proving what has already been proven as a way to refine their proof construction skills. To ‘reinvent’ proof as an effective tool and in order to make it instructive and enlightening as a method for teaching new material, proof needs to be for exploration. “… For mathematicians, proof is much more than a sequence of logical steps; it is also a sequence of ideas and insights (Yackel & Hanna, 2003, p. 228). Students who use proof for exploration investigate their own conjectures and develop insights about the subject through the process of writing a proof. To encourage students to investigate their own conjectures with exploratory proofs, the concept of inductive reasoning as proof may take a more central role than deductive proof (Izen, 1998). Inductive reasoning as proof of a conjecture should not be confused with proof by mathematical induction.

Mathematical induction proves a base case and uses an induction rule to prove a series of other cases. While it is similar to inductive reasoning in the fact that it generalizes from a sample, in most cases a sample of one, proof by induction is deductive because it uses a rule to derive a pattern of true values. This rule is used to examine a number of cases up to the n^{th} case to determine information about every member of the class. In proof by induction the first case is proved true, the second case is derived from the first and if the pattern continues the final result must be true. Inductive reasoning does not prove any case, but rather uses a examples and generalizations in place of a proof.

In generating a proof using inductive reasoning, students investigate relationships in a class of figures and develop a sound conjecture of their own. This conjecture is derived by noticing the repeated appearance of an attribute in several cases of a figure. For example, after drawing several triangles and measuring the interior angle measures, students may conjecture that all triangles have an interior sum of 180°. This may often take place in a dynamic geometry environment (DGE), where students can drag vertices and segments on or within figures to in turn establish patterns that are inductive in nature. While inductive reasoning as proof relies heavily on several series of observations, they are not definitive proof (Jones, 2000). Consider the exterior angle sum theorem (Hadas, Hershkowitz, & Schwarz, 2000). Using a DGE, students can create several examples of polygons, measure exterior angles, and calculate the sum. This will generate a pattern so that students will begin to inductively develop a conjecture about the sum of the exterior angles. However, even in a DGE, students cannot create all possible cases and therefore eventually a formal proof will need to be developed to encompass all cases. Therefore, the exploration of a topic which leads to inductive reasoning, must eventually lead to a deductive proof (Hadas, Hershkowitz, & Schwarz, 2000; Jones, 2000). This does not mean that deductive proof will eventually be unnecessary; logically it will follow after an inductive proof to reinforce the validity of the conjecture (Hanna, 2000). Also proofs of this nature may include having students determine if all possibilities of their conjecture have been accounted for, the process of showing that there are no other possible solutions encourages a higher level of reasoning and may lead students in understanding why a deductive proof is necessary (Bell, 1976).

In order for students to fully understand a topic they must explore it on their own. It is difficult for students to understand topics that are explained to them at a level of reasoning higher than their own (Senk, 1989). Students' who are allowed the opportunity to examine and explore a concept on their own, at their own level of mathematical understanding are more likely to understand the topic. Also, perceptions about concepts and conjectures can be altered and defined by their own explorations of a subject providing more meaning for the concept (Arzarello, Micheletti, Olivero, Paola, & Robutti, 1998). The arguments produced by a student who is exploring a subject for the first time are often more convincing because the student is trying to establish the accuracy of their own ideas (McCrone et al. 2002). This makes proof more meaningful because it proves what is not obvious and helps to dissuade students' doubts about a topic (de Villiers, 1990).

CHAPTER 3

WHAT ROLE DOES TECHNOLOGY HAVE IN STUDENTS' DEVELOPMENT AND UNDERSTANDING OF PROOF?

As the world grows and becomes more advanced with respect to technology, the world of education, must keep up. High school graduates are increasingly required to have computer skills in post-education endeavors. Keeping this in mind, it is time that mathematics teachers begin to employ technology in their classrooms. "Electronic technologies – calculators **and computers** [bold added]– are essential tools for teaching, learning, and doing mathematics" (NCTM, 2000, p. 24). This means more than allowing students to use calculators to compute answers to problems. It also means using software programs that allow students to explore topics and facilitate a deeper understanding of those topics (NCTM , 2000). Dynamic geometry software environments (DGE) can play a key role in a student's development, understanding and writing of proofs, especially as a tool for verification and exploration.

A Tool for Verification

According to Webster's dictionary verification is "an act of verifying" and verify is "to prove the truth by presenting evidence or testimony" (Soukhanov, 1988, p. 1282). Thus using a DGE as a tool for verification would allow students the opportunity to provide a substantial amount of evidence in order to prove or disprove a conjecture. In using a DGE as a tool for verification, students begin with a given conjecture and manipulate figures to determine the validity of the conjecture.

Students may 'drag' parts of a figure, create duplicate or classes of figures and manipulate parts or all of a figure in a DGE. This allows students to explore the relationships that are maintained or changed depending on how the figure is manipulated. For example students are often asked to create geometric constructions of parallel lines, congruent triangles and other objects. When using a DGE, the dragging of objects shows the difference between objects that have been drawn (shapes) verses those that have been constructed (figures). In a DGE each object has a part or parts that are the foundation of the object, such as a point or an initial segment, these parts are referred to as parents. If the 'parent' of an object is deleted then the entire figure disappears, or if the parent is manipulated, the entire object is changed according to that manipulation. For example: in a drawing of a triangle if one side is dragged it becomes disjointed from the rest of the segments (see Table1: Figure 2); however, in a construction of a triangle if a side is dragged the shape of the triangle changes (see Table 1: Figure 3). Therefore, it is easier for students to see the dependencies that are present in geometric figures (Jones, 2000).

When constructing congruent figures, students are also able to see that the order in which congruent elements are copied is an important procedure in the construction process (Hadas, Hershkowitz, & Schwarz, 2000). Constructions of polygons that do not preserve the order of congruence, which is making sure included angles are constructed between the same segments and similarly included segments between the correct angles, will not create congruent figures and in some cases will not create closed figures. These types of investigations with drawing and constructions in a DGE promotes the differences between drawing a shape and constructing a figure, and requires that students focus on the attributes that generate a specific figure (Galindo, 1998; Jones, 2000).

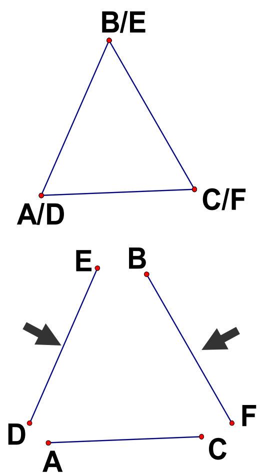
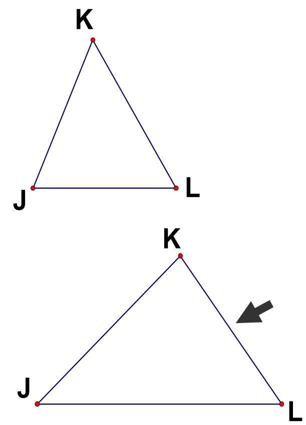
Drawing	Construction
 Figure 2	 Figure 3

Table 1

While dynamic geometry software environments do provide an infinite number of examples, these examples do not provide proof. In using a DGE to promote the understanding of topics, students may come to the conclusion that a formal proof is unnecessary since they can produce an unlimited number of examples validating their conjecture (Galindo, 1998; Hadas, Hershkowitz, & Schwarz, 2000; Hanna, 2000, Jones, 2000). For many students, the idea that 'seeing is believing' is still a major role in how they solve problems and relate to the world around them. Software programs that allow these students to create objects, manipulate shapes, measure lengths and angle measures, and compare attribute provides a solid basis for future conjectures based on physical evidence (Galindo, 1998). Unfortunately, students tend to use this physical evidence as proof (NCTM, 2003). It is the job of the educator to guide students into using proof to show why a conjecture is true, once they have established through the DGE that it appears to be true (Hadas, Hershkowitz, & Schwarz, 2000).

In the process of creating figures and dragging parts of those figures, students are able to select pieces of data or statements that are meaningful to their understanding which allows them to better establish items that do and do not apply to the process of proving (Arzarello, Gallino, et al. 1998; deVillers, 1999). This elimination process comes from the verification of attributes using the DGE.

The use of dynamic software environments leads to the meaningful justification of conjectures that have been verified (Galindo, 1998). Students using a DGE use the relationships of parent elements and properties of a figure to provide justification for conjectures that they notice in the dragging of elements in a construction (Mariotti, 2000). These relationships establish the verification of facts that students need before they begin to create a proof. There needs to be a conviction for the individual that a conjecture is true before they begin to prove it, otherwise the effort used in proving a conjecture would be trivial (deVillers, 1999). Proof using a DGE ties inductive reasoning and deductive reasoning together in a complementary process (Hanna, 2000; Izen, 1998). Students working in a DGE experiment can easily see that a theory is plausible using examples; however, the examples are not proof and eventually students will want/need to know why all the examples work or why a theorem is correct. While a picture may be worth a thousand words, a deductive proof is worth a thousand pictures. "Logically, we require some form of deductive proof, but psychologically it seems we need some experimental exploration or intuitive understanding as well" (deVillers, 1999, p.5).

The manipulation of figures within the DGE "reverses the stream of thought" for students allowing them to state a hypothesis from the action of dragging (Arzarello, Gallino, et al. 1998, p 2-35). This reverse comes from using the outcome of the manipulation to make a conjecture about the original object, thus allowing students to move from a model to a theory. "Dynamic geometry software can help make these connections [between computer and pencil-and-paper and ideal geometric objects] if appropriate exploration tasks are designed and if students are encouraged to see and reflect on the relevant connections" (Galindo, 1998, p. 77). While manipulating figures in a DGE, students may be more inclined to switch between the empirical to the theoretical level (Arzarello, Gallino, et al. 1998). This shift between the levels may be prevalent because of the ability of students to obtain conviction in the ability to create counterexamples or the inability to create a counterexample for a conjecture (de Villers, 1999). The construction of figures and geometric relationships, such as the construction of a perpendicular line through a point on a given line advances the theoretical meaning of the definition of perpendicular lines (Mariotti, 2000). Using a DGE students can construct the perpendicular of a line and use the angle bisector command to verify the correctness of the construction. They can then drag their construction to see that the construction holds for a multitude of cases (NCTM, 2003). The use of a DGE and the multitude of examples that students can create using the software provides bridges between the empirical level of experiments to the theoretical level of deduction by helping them "form a mental image" (Hanna, 2000, p. 13). Examples created by students often help in identifying inconsistencies in conjectures and therefore they are better able to breakdown a theory into its relevant parts that they are then able to explain and use in justifications (de Villers, 1999). According to Harel and Sowder (1998) psychologist have found that it is a natural human instinct to form ideas about concepts based on examples. With this in mind, allowing students to create several examples and form ideas about those examples using a DGE will increase their understanding of geometric topics (NCTM, 2003). When students are not "recipients of formal proofs, but were engaged in an activity of construction and evaluation of arguments in which certainty and understanding were at stake, and they had to use their geometrical knowledge to explain contradictions and overcome uncertainty" (Hadas, Hershkowitz, & Schwarz, 2000, p. 149).

A tool for exploration

According to Webster's dictionary exploration is "to make a careful search or examination" (Soukhanov, 1988, p. 455). In using a DGE as a tool for exploration, students begin with their own conjecture and manipulate figures to determine the validity of their conjecture. The goal in using a DGE is not to give students free-time, let them draw pretty pictures, and then make up something that looks right just to say that technology was used in the classroom. While in the early stages of using a DGE, teachers may and should provide time for students to test out the features and become familiar with the software, this is not how all the time should be spent. It is understandable that teachers may see a new technology as more of a distraction than a helpful and effective teaching tool; however, educators who provide the correct structure and activities in which to use the technology should notice that students begin to develop new strategies for solving problems and increase their interest in the subject (Galindo, 1998; Jones, 2000).

Students using a DGE are not confined to verifying given theorems. They also have the freedom to create a variety of shapes that they can manipulate in order to develop there own ideas and conjectures about geometric relationships enabling them to become more proficient in evaluating the geometric structure of a figure (Jones, 2000). These types of exercises help students to internalize and validate their conjectures based on concrete evidence (Arzarello, Gallino, Micheletti, Olivero, Paola, & Robutti, 1998). The truth value of this evidence can be easily verified or nullified using technology, by creating a variety of figures and using the quantity of examples to convince individuals that a conjecture has merit; however, it does not provide an explanation as to why the conjecture is true or untrue (de Villiers, 1999; Galindo, 1998).

Teachers that tend to employ technology as a means to keep students interested in a topic (which is not a bad idea) or to show students a shortcut in a mathematical process, are not using it as the educational tool it was intended to be and they are definitely not using it as a tool for exploration. According to Jones (2000), most teaching methods tend to devalue or omit the role of exploration in teaching proof. Where as, introducing and using a DGE in the creation of geometric proofs promotes the role of exploration in proof. A key to the use of DGE in proof exploration is good tasks (Galindo, 1998). The task may begin by leading students to certain conclusions and then allow them to discover other conjectures on their own, or they may be completely open ended allowing students to create conjectures about a class of figures. This type of exploration enables students to be involved in their own learning of a geometric topic and thus they are better able to comprehend extensive topics (Hadas, Hershkowitz, & Schwarz, 2000).

The ability to comprehend new and difficult topics transitions a student from following directions to being able to critique, improve, or contribute to a topic (Izen, 1998). Students who understand the topics they are working with can correctly use those topics and explain them to other individuals, they are also able to build on those topics using previous examples and experiences. Dynamic geometry environments allow for the extension of theorems because they make the theorems come alive for the student (Izen, 1998). For example, it is generally accepted that if the midpoints of a kite are connected consecutively, then the resulting figure is a rectangle. However, using a DGE students can investigate the specific attributes necessary to create a rectangle from the midpoints of a quadrilateral (see Figure 4). In such an investigation, students discover that the necessary relationship is that of perpendicular diagonals, not congruent adjacent sides of a kite (deVillers, 1999).

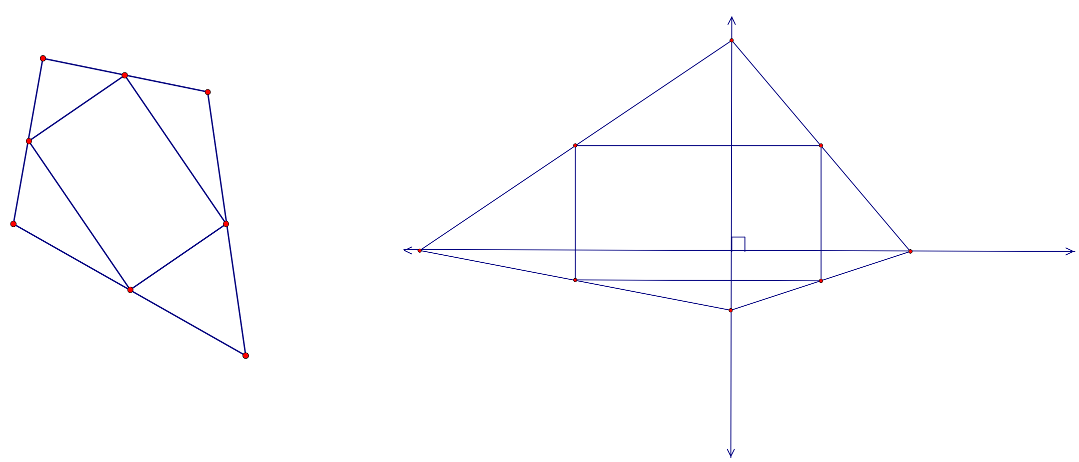


Figure 4

Students using pencil and paper would not generally think of trying to draw other pictures to determine if other quadrilaterals share this attribute and would not attempt a proof of this statement if a teacher asked if the statement was true. Students, like mathematicians, want to begin with some evidence that the statement is true before they waste their time in trying to prove a new conjecture. Working through a proof on a conjecture that has no conclusive evidence that it may be true makes the proof unimportant and seemingly unnecessary, especially if the conjecture is false. The conviction needed to make a proof necessary to a student may be obtained through activities where they begin with a given figure and manipulate only part of the figure in order to create a class of shapes that fit a particular conjecture (Hadas, Hershkowitz, & Schwarz, 2000). This type of exploration for the verification of student ideas is made possible using a DGE because of students' ability to create an infinite number of shapes with particular attributes in a short period of time, where using pencil and paper become tedious and inexact. Students who are able to use exploration and inductive reasoning in order to convince themselves, may better understand the need to use a deductive method to convince other individuals (Sriraman, 2005).

Since most educators explain proof as a method that is used to convince skeptics that a statement is true, they miss the opportunity to use proof as an intellectual challenge (de Villers, 1999). It should also be noted that convincing a skeptic does not always correspond with the need to provide a deductive argument for a theoretical statement, and students will question this reason for developing a proof (Mariotti, 2000). In the case of convincing a skeptic, most skeptics, even some geometry teachers, are willing to accept evidence as proof of a statement. In a study conducted by Knuth (2002), several teachers used drawings to prove to themselves that theorems were in fact true, even after deductive proofs had been presented. Convincing students that a deductive proof is necessary may be done more easily be asking them why their evidence proves their conjecture. Presenting such a question will force students to explain/prove how the explorations of various cases, figures and the attributes of those figures show the truth value of their conjecture.

CHAPTER 4

WHAT IS THE ROLE OF CURRICULA AND TEACHERS ON STUDENTS' DEVELOPMENT AND UNDERSTANDING OF PROOF?

For several years, organizations and researchers in the field of mathematics have been conducting research on mathematical proof. From year to year the demands placed on teachers, students and curricula change and often times these demands conflict (Herbst, 2002a). These changes require everyone employed in the teaching of proofs to be flexible in the structure of their classrooms (DeGroot, 2001). Discussed in this section are ideas presented by the National Council of Teachers of Mathematics (NCTM) and the Committee of Ten, as interpreted by Patricio Herbst, and research from several other authors and their beliefs on how and why proof should be taught.

Curricula

In all aspects of education, one of the most important questions is 'How is the subject being taught relevant?' Most mathematics curricula assume that when students utilize proof, the experience of creating a deductive argument will enhance their analytical abilities so that they are able to experience rigorous mathematical thinking (Herbst, 2002a). However, this is not the answer that most or perhaps any geometry student is looking for when they ask the question 'When am I ever going to use this?' When trying to answer this "simple" question, in a geometry class educators need to focus more on application with explanation rather than just explanation. When students are writing proofs about theorems and using givens to find a predetermined solution, they need to know when and where proofs are applicable in the world they live in. In an effort to make proof accessible to all students who intend to pursue a college education, the form of proof has become more important than the substance of the proof (Fitzgerald, 1996; Knuth, 2002). In order to promote the concept: that the ability to prove a conjecture is a necessary and useful skill, curricula must connect logic with geometry tasks that provide relevance for the subject matter (Epp, 2003). For example, working through a proof that proves the midsegment of a triangle is half the length of the third side does not appear to have an application in the minds of most geometry students. However, if students are asked to find difficult distances, such as the distance across a lake or fissure or to find the height of a tree using the midsegment theorem (see Figures 5 and 6) they will see it is useful and may become interested in knowing why the theorem works. This leads to a formalized proof for a general case and students begin to see how to apply the theorem in various distance problems. The ability to apply a theorem to a variety of problems is one aspect that makes proof so useful. Once a conjecture has been proven, it will always be true and since proof is a general case then the concept is applicable for more than one example.

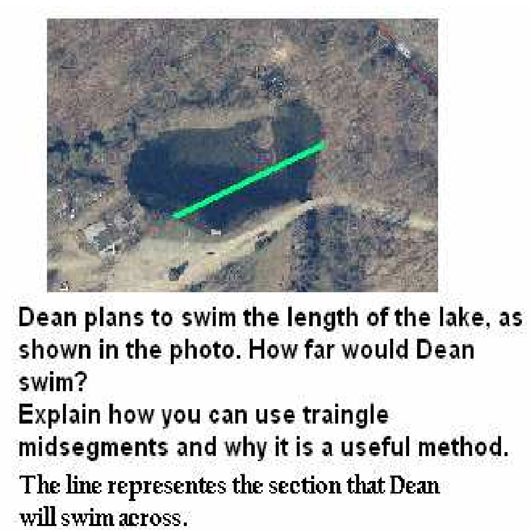


Figure 5
(Bass, Charles, Johnson, & Kennedy, 2004)

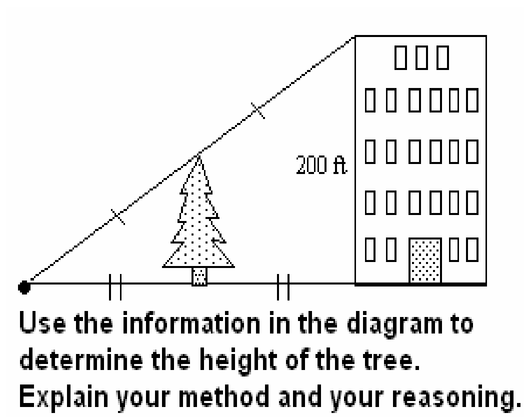


Figure 6
(Bass, Charles, Johnson, & Kennedy, 2004)

Even though curricula tend to separate proof from mathematics and application, these concepts are not independent of each other (Knuth, 2002). The use of proof is what makes mathematics, especially geometry, applicable to life. If geometric concepts and ideas were not true in a generalized state, then the concepts could not be transferred from problem to the next and each new problem would have to have a new conjecture. Being able to prove a conjecture once and then apply the result of that conjecture is why proof is essential in mathematics. Proofs are used to both convince an audience of the validity of a statement and promote a dialogue about the applications of the statement (Knuth, 2002).

In current textbooks, authors have created problems that require students to explain their thinking and logic and produce reasons for their mathematical steps. As textbooks continue to promote these ideas curricula has begun to reflect these concepts (Herbst, 2002a). Geometry, like most of mathematics, is treated like an organic body that can be manipulated to reflect the current trends in education and so texts are constantly changing and revising techniques for teaching proofs, writing proofs, and understanding proofs (Herbst, 2002b).

Proofs have transformed over the years due to changes in curricula in colleges and high school. In the 1890's and early 1900's proof was a subject that was left to higher level mathematics teachers in colleges and universities. These proofs were often lengthy and difficult and therefore they were considered more valuable than less complicated proofs (Herbst, 2002b). For example, a proof of the sum of the interior angles of a triangle using the parallel line postulate is rather short and simplistic, while a proof of the same statement becomes rather difficult and tedious when the postulate is not referenced within the proof (Herbst, 2002b). Therefore for many educators in this time period the latter proof was more valuable. As the years progressed and high schools began to teach geometric proof the textbooks began to change to accommodate a variety of learning styles; however, they still presumed "that students would learn to reason by reasoning" (Herbst, 2002b). Both authors of textbooks and teachers adopted the philosophy that students learn by doing and that the only way to produce students that are capable of logical reasoning is to allow them to reason. While this may seem like a circular argument, this is how most subjects are taught. Students cannot learn a concept if they do not practice it and yet they cannot practice the concept if they have not learned the steps necessary for completion. To this end, texts began to provide exercises in writing proofs where earlier texts had not included methods for writing proofs or descriptions of proofs (Herbst, 2002b).

Recommendations from Mathematical and Educational Studies

According to the NCTM *Principle and Standards for School Mathematics* (2000) "proof should be a significant part of high school students' mathematical experience, as well as an accepted method of communication" (p. 349). Educators at all levels of instruction are responsible for teaching students the reasoning and logic skills necessary to justify and explain their solutions. Thus when students enter a geometry classroom they should have the skills and background necessary for the production of a well organized geometric proof. While reasoning and proof are not special concepts used only in geometry, they are a main focus on geometry. Students need to have the ability to string together a number of logical deductions to help them solve problems and establish knowledge that will help them deduce information about other situations (NCTM, 2000). "The Committee of Ten identified the high school geometry course as a vehicle for students to acquire the *act of demonstration* (or proving)" (Herbst, 2002b, p. 287). In order to acquire the ability to prove, students must be comfortable enough in their reasoning abilities to question the mathematical arguments of themselves and others including the instructor (NCTM, 2000). If students maintain a blind acceptance of any and every statement and theorem that they are presented with, they are relying on the authority of teachers and textbooks. They are not reasoning for themselves. In order to assist students in understanding the need for mathematically proof(s), educators need to understand that students will always accept what they know. This may seem obvious; however, teachers in classrooms across the world are having students prove conjectures they know to be true. This is an inefficient use of proof. Proofs of this nature will not convince students that proof is necessary; instead, conjectures need to be developed that are less obvious to students therefore there is a need to prove the conjecture (Van Dormolen, 1977).

In mathematics classes, especially when teaching proof, all plausible guesses for reasons and justifications should be discussed (NCTM, 2000). This will allow students to become more comfortable in the process of eliminating unnecessary information and retaining pertinent information. Classrooms need to have an atmosphere where students are able to discuss, question, and listen to alternative ideas for proving conjectures. Students should also be expected to "seek, formulate, and critique explanations" (NCTM, 2000, p. 346)

Responsibilities for Educators in Teaching Proof.

It the responsibility of the instructor to help students develop the mathematical language and skills necessary to justify results and present those results in a manner that will correctly communicate their finding (Almeida, 2001, Herbst, 2002a). In order to better aid students in these pursuits, teachers must first determine the level of the students' mathematical knowledge (DeGroot, 2001). It is not possible to move a student forward in their thinking if you do not know where they are starting. This adds to the complexities of teaching proof since students in most geometry classes do not start with the same knowledge base (Almeida, 2001). The difficulty of teaching students proof grows as teachers try to use proofs as a means to advance the students' knowledge and there is no previous knowledge to advance (Almeida, 2001; Van Dormolen, 1977). Therefore it is the role of the teacher to organize and facilitate proof activities that can promote the development of meaning and the deepening of meaning, so that students can use proofs to build a knowledge base or to strengthen a knowledge base (Mariotti, 2000). In building upon a student's knowledge base, teachers are also encouraged to promote a student's acceptance of new rules that may or may not be proven (Almeida, 2001). The idea of promoting acceptance of new rules and proving old ideas may appear to be a contradiction; however, students must understand that certain geometric facts are the building blocks of the ideas they come into the classroom with. Postulates and definitions are the foundation of the Euclidean Geometry that is taught to students. Without the parallel line postulate the entire subject would fall apart, therefore it is necessary for students to accept some facts at face value, while other statements, which rely on these facts, must be proven.

Teachers need to encourage students to write logical and well defined explanations when creating a proof (Almeida, 2001). To do this, teachers need to be flexible and allow students to share a variety of ideas and discuss their findings before they begin to write a formal proof (DeGroot, 2001). The process of discussion allows for the possibility of discourse with others' opinions and will stimulate deeper understanding of the subject matter (DeGroot, 2001). This deeper understanding will help students write proofs that are more deductive than inductive. It is also necessary that teachers refrain from giving hints or suggestions when students are creating their own proofs (Herbst, 2002a). Input from the instructor often diminishes the student's input. They see help from the teacher as a sign they have done something incorrect and they may dispose of their entire proof believing that it is not repairable. Teacher must understand that students "live in a different logical and linguistic world" and that the words they intend may not be the words that students hear (Epp, 2003, p. 886). Asking a student why they included a particular statement or reason is often regarded by students as an attack on their answer. They believe that a teacher would only ask for an explanation if the answer was wrong, students assume that correct answers are never questioned; only accepted.

Teachers must remember that they need to give students' the opportunity to create a proof and they need to learn and determine for themselves what information is necessary in their proofs (Herbst, 2002a). Instructors can facilitate the delineation of necessary information by asking students leading questions and helping them to breakdown problems before students begin to compose their proofs (Almeida, 2001). The experimental process of trying different strategies and justifications will aid students in their endeavors to create a deductive proof and while teachers need to monitor the progress of the students they should not interfere with their methodology (Vavilis, 2003).

Teachers are truly faced with a dilemma due to the pressures of preparing students for standardized tests. Omitting proofs of theorems and using examples because of time constraints has become prevalent in many geometry classes. Teachers often fear that students will not and do not appreciate the use of deductive proof and therefore going through the difference between measuring specific figures and proving is generally considered a waste of time (Chazan, 1993). This gives students the impression that modeling a theorem is a sufficient means of establishing the truth of the theorem (Epp, 2003; Elliott & Knuth, 1998). It is important that educators not allow themselves to become jaded by standardized testing. Deductive reasoning and proof is an essential part of the mathematics curriculum and teachers must present it as such if they expect students to see deductive reasoning as essential in their lives.

Teachers' Understanding of the Concept of Proof

Geometry teachers at every level have different concepts of what a correct proof looks like, the amount of rigor necessary to make a proof relevant and the overall purpose of proofs. Is a correct proof a formal two column proof with every detail explained? Does a paragraph proof that may contain less detail, but exhibit more understanding qualify as a correct proof? Or is an informal proof that is less structured but describes the students' mathematical thinking a correct proof? Educators are torn between the ideas of students' understanding of how to create and correctly write proofs, proof for the development of proficiency, and students' understanding of the actual concept that the proof is explaining, proof for understanding. While teachers are responsible for instructing students on how to reach a conclusion from a set of premises using logical deductions, it is also their responsibility to teach students how to use the conclusions (Herbst, 2002a). Students must be taught how to prove rather than being taught about proof (Jones, 1994).

> It appears that if teachers focus on the overall structure and need for proofs in understanding the underlying mathematical concepts, students will also develop a better sense of the need for proof. On the other hand, if teachers expect students to learn to do proof in a more mechanistic way, the students are likely to see proofs as just another exercise or application and will not develop a more complete understanding of proofs and how to construct proofs (McCrone, et al., 2002, p. 1711).

Thus the viewpoint of the teacher is the key component in how students will view proofs. The atmosphere of leaning that the teacher provides in his or her classroom is a deciding factor in a student's ability to see a need for deductive proof (Martin & McCrone, 2001; Simon & Blume, 1996).

Teachers' Beliefs about the Purpose of Proof

Instructors who see proof as something to be taught instead of learned generally supply students with several examples, explain what a proof looks like in its various forms, provide the correct steps for writing a proof and provide the 'standard' reason for completing proofs as an exercise in determining if a statement is true, but they do not teach students how to prove. Conversely, teachers who instruct students on methods for solving problems and ask for justifications of the thought process behind their solutions teach students how to verify and prove their solutions and in turn discover proof as they work through problems.

While there is little research on in-service teachers conception of proof, most teachers view the primary role of proof as a means to establish the truth of a statement by using a logical step-by-step convincing argument (Knuth, 2002). For some educators the final outcome of a proof overrides the reasoning within the proof, causing teachers to grade proofs as all wrong or all right (Senk, 1985). This concept of proof develops the idea of proof for proficiency and not for the understanding of a statement. Teachers expect students to understand that there is a beginning and an ending statement that will not be altered and that they need to fill in the middle. Students are encouraged to use previous proofs as templates to help in their proof construction so that they will produce proofs that are correct in form if not in function. For many educators, the form of a proof has become more convincing than the argument and therefore, students focus more on the number of steps in completing a proof rather than the substance of the proof (Knuth, 2002).

In some cases, instructors may feel that if students are unable to complete a proof in a classroom setting by providing the instructor with the “correct” statements and reason, then it is time to change the dynamics of the classroom so that the teacher not the students is completing the proof (Herbst, 2002a). In an effort to complete a proof during the instructional time teachers may begin to use rhetorical questions, where students essentially agree with the correct answer(s) begin provided, to speed up the process (Martin & McCrone, 2001). While working on a proof as a class activity, an instructor may elicit recommendations for future statements and reasons from his or her students; however, if students do not offer the ‘correct’ input the instructor may switch roles. Instead of being a facilitator of student ideas, he or she may take over and begin giving students the correct statements and reasons in order to get the proof done (Herbst, 2002a). This authoritative approach to teaching proof reinforces the idea that there are specific statements and reasons for each proof (Martin & McCrone, 2001). While this may not have been the intent of the instructor, they have turned an exercise of proof for exploration or understanding into a proof for proficiency. The formal teaching of proof does not automatically result in a students connected knowledge of the mathematical concepts being proven (Simon & Blume, 1996). Students who are taught proof without having any previous knowledge of the subject matter do not have a frame of reference for the justifications used (Simon & Blume, 1996). Proofs should increase the knowledge of a subject instead of confusing students by asking them to provide reasons for a topic they have not studied.

It is the understanding of most instructors that a proof is not correct unless it is a formal proof, and a proof is deemed formal if it is of a particular form (Knuth, 2002). In a study conducted by Martin and McCrone (2001) they found that “Format for writing formal proofs was over-emphasized in class (understanding the need for proof was under-emphasized)” (p. 592). The format of a proof whether it be two-column, proof by induction, indirect proof, paragraph proof, or a flow proof, may take center stage in the writing of a proof instead of the true substance of the proof, deductive reasoning. According to Knuth (2002b), teachers consider proofs that follow these formats to be acceptable and correct methods for writing a proof as long as they provide sufficient details which are mathematically sound and they are easy to follow.

Teachers' Understanding of the Role of Proof in Mathematics

For many teachers the ability to treat proof as anything other than a formal process is impossible, because they do not completely understand the concept of proof and the mathematical premises on which it is built. Teachers that teach proof must rely heavily on their own mathematical knowledge base and their own beliefs of proof's role in the mathematical world (Knuth, 2002; Martin & McCrone, 2001). "To help students develop productive habits of thinking and reasoning, teachers themselves need to understand mathematics well" (NCTM, 2000, p. 345). Unfortunately, not all teachers have a strong proof knowledge base. In a study by Knuth (2002) many mathematics teachers were uncertain about the validity of certain proofs. They often wanted more evidence than a mathematical proof and felt that a proof may not be true for all cases and may in time become fallible as new ideas and axioms are introduced into the subject of geometry (Knuth, 2002). Teachers who are unsure about the validity of proofs as a general statement transfer that sentiment over to their students when they teach. This may cause conflict for the students since the teacher is explaining that proofs are used to show a statement is true and yet the instructor is relying more on an example than the proof itself.

In his study, Knuth (2002) also found that many educators are unable to determine if a proof is faulty. Several teachers accepted inductive reasoning as a proof, since several examples were presented that displayed several cases. They were unconcerned with the lack of generality provided with the examples and stated that since it works for one case it will work for others and “it was clear from that example the statement was true” (Knuth, 2002, p. 394). While using examples is an excellent way to determine if a conjecture is plausible, it is not proof. In many instances, teachers’ concepts of proof are lacking and incomplete. They like their students do not understand that proof is a definitive set of statements and reasons about the truth of a theorem. Once it is proven, it cannot be unproven and the only possibility for using examples as a proof would require the individual to prove every possible case, an impossible task.

CHAPTER 5

WHAT ROLE DO STUDENTS HAVE IN THEIR OWN DEVELOPMENT AND UNDERSTANDING OF PROOF?

At some point in time everyone must take responsibility for their own learning. Many students realize this very early in their educational studies and far too many others discover this fact too late in life. Proof is no exception to the rule. If students do not take responsibility for their own understanding, then a teacher's guidance and help will offer little reward. A teacher cannot reason for his or her students, they must think for themselves.

Misconceptions of Geometry Students about the Role of Proof

In geometry classrooms across the world, students constantly struggle with concept of formal proof and every teacher has heard their students say that proofs are too hard, too difficult, and too long (Knuth, 2002). Many students are unmotivated to complete proofs because they believe they are forced to complete proofs and provide justification only because their instructor has come to the conclusion that the evidence presented by the student is not valuable without a convincing argument (Almeida, 2002; Mariotti, 2000; Simon & Blume, 1996). Therefore, students are merely satisfying the demands of the teacher by attempting to create a formal proof instead of using proof to better understand and learn a concept (Almeida, 2001). In most cases where proofs are used to prove theorems, students were more than "willing to trust whatever brilliant mathematician thought up the theorem" (Harel & Sowder, 1998).

In constructing a geometric proof for the first time, students are unsure where to begin and are often unable to write original proofs without guidance (Martin &

McCrone, 2001; Weber, 2001). In previous mathematics classes they were able to use some type of trial and error to determine if a solution was reasonable; however, with proofs they do not have an understanding of the strategies necessary to produce a logical argument (Van Dormolen, 1977; Weber, 2001). Students, even those that understand the meaning and usefulness of proofs, may be unable to produce a deductive proof because they lack the strategic knowledge necessary to connect or link different observations (Mitchelmore, 2002; Weber, 2001). With proof they are at a loss because while they are often given a conjecture, they do not understand where a proof is supposed to begin or lead (McCrone, et al., 2002). In creating proofs for theorems, pupils often cite the theorem as proof of the statement (Senk, 1985). For these individuals there has not been a delineation of what is acceptable as a reason and look to a theorem for help and discover that the theorem "proves" what they are trying to explain. They do not see this as circular reasoning, because (a) often they do not understand the idea of circular reasoning and (b) they are looking for an easy answer since they are unsure of the next step because they do not know the real purpose and meaning of a proof. While they know that a formal proof has a beginning and an end, they do not see the importance of the steps that connect the two statements (McCrone, et al., 2002). Other students, however, may create logical, well written, deductive proofs for a conjecture and then erase it or throw it away when another student or the instructor begins to offer their own proof. Students assume that if their proof does not look like the "correct" or final proof then theirs must be incorrect. They are "limited in their sense making with respect to the argument by their understandings" (Simon & Blume, 1996, p. 29). When students are limited in this manner it is difficult for them to conceive that there is more than one way to construct a proof for any given conjecture. When teaching proof, instructors must "remind students that there can be various solution strategies to specific problems" (Vavilis, 2003, p. 17) and that no two proofs must contain the exact same steps.

In using diagrams that are presented with proofs, students' misconceptions tend to increase because students are unsure about the pieces of information they are allowed to assume from a picture or diagram (Herbst, 2002a). Items such as the type of figure (triangle/quadrilateral/etc.), straight lines and vertical angles are items that can be determined by looking at a diagram; however, items such as parallel lines, bisectors and congruence can only be determined from given facts. Students often see lines that appear to be parallel and assume that they are without any hesitation. This often comes from the instructors always using diagrams that 'look the part.' Students focus a great deal on how objects are drawn when they create their proofs and often assume information that is not given (McCrone, et al., 2002). If they cannot match their internal visualizations to the model presented they are unsure about how to proceed in their proof (Degroot, 2001). For example, using figure 7 students may be able to see and in turn prove that the line are parallel to each other more easily, however using figure 8, they are not able to see any evidence that the lines are parallel and may have a more difficult time proving the lines are parallel. This lack of evidence may dissuade them form trying to prove the statement, even though the reasons from figure 7 are applicable to figure 8.

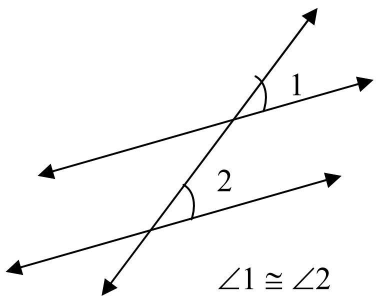


Figure 7

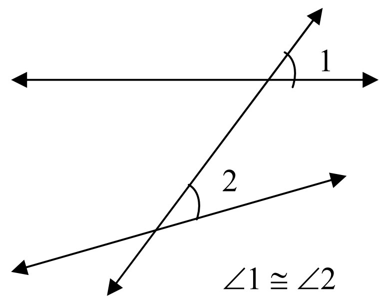


Figure 8

With diagrams, comes another student and sometimes teacher misconception. If enough examples are provided then a proof is unnecessary, because empirical evidence is tangible, where as a generic proof requires abstract thinking (Almeida, 2001; Chazan, 1993). Students have a difficult time understanding that three or four or even twenty or thirty examples cannot prove an infinite set (DeGroot, 2001; Weber, 2001). And yet many other students do not feel that one counter-example is sufficient to disprove a conjecture (Galbraith, 1981). It is a strange twist of fate that students assume that one or two examples will suffice in the proving of a statement and yet they are unconvinced that one example can disprove a statement. Because of their constant dependence on examples for proof, it is apparent that many students lack a true appreciation for the formalization that proof brings to mathematics and may see deductive proof as another form of evidence (Chazan, 1993). They believe that evidence, or examples, should speak for themselves and that further explanations are unnecessary and time consuming (Almeida, 2001; DeGroot, 2001). Students also tend not to trust deductive proof and need further verification such as a diagram after a formal proof has been presented (Chazan, 1993; Epp, 2003). A deductive proof, while generic in respect to the figure(s) represented, is not viewed as generic by students (Chazan, 1993). In most cases students view the diagram presented with the proof as a specific case. They are unable to think abstractly and therefore they do not realize that the diagram, since it does not contain measurements, can be used as a representation for a class of objects that satisfy the givens (Chazan, 1993).

In creating opportunities for students to create proofs and become more comfortable with proof, educators often allow students an extended period of time to work out proofs or to develop conjectures about geometric concepts. For many students this extra time has a direct correlation with the difficulty of the problem. If students are given a longer time frame to work on a problem, then the students assume the problem is harder than previous exercises (Herbst, 2002a). At the same time if a teacher asks students to justify their answer, students will automatically assume that their answer or solution is incorrect (Fitzgerald, 1996). Students who have been in traditional mathematics classes usually only have to supply answers to problems and do not have to provide reasons for their thinking. In most classrooms teachers provide students with proofs that are easily developed and offer very little challenge to students. As stated by McCrone, Martin, Dindyal, and Wallace (2002):

> Teachers contributed to the perception that mathematical problems can be quickly solved by providing examples that were always provable usually in a few steps. As a result, students developed very little perseverance, in terms of reasoning ability, and gave up quite quickly on challenging tasks (p. 1708).

Difficulties of Geometry Students

Proof is a difficult subject with many intricacies and a subtle language. Proofs are characterized by many students as their least favorite activity in any math class and they often do not understand why proofs have to be so difficult. Several aspects of proof play a role in the difficulties that students face in writing a logical geometric proof.

Geometric proofs use a language that is subtle and often confusing and often theorems are given to students in their final form so that students do not gain any insights into how the theorem was derived or its usefulness (Mitchelmore, 2002). For example the Side-Splitter Theorem, as it is called in several high school textbooks, states: If a line is parallel to one side of a triangle and intersects the other two sides, then it divides those sides proportional. In Figure 9, line *BE* is parallel to segment *AD*, therefore $\frac{AB}{BC} = \frac{DE}{EC}$. While this theorem can be proven using similar triangles, educators often show students the theorem and expect them to understand and apply it with very little or no justification being provided. Because of this students may be able to recite the theorem but are generally unable to use the theorem correctly in a proof.

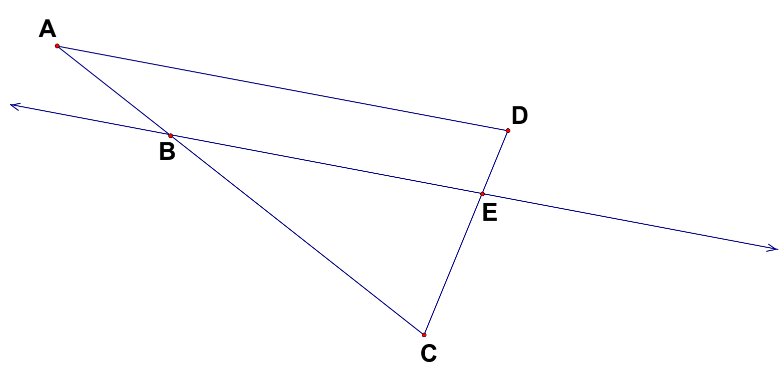


Figure 9

Students generally have difficulties discerning what is meant by statements because they do not understand the precise meaning of symbols or words that are used in a premise because instructors do not spend the time educating them in these areas (McCrone, et al., 2002). Because of their lack of language skills, students also have difficulties communicating their justifications in terms that are mathematically structured (Jones, 1994). It is often the case that an instructor has to help students in structuring a proof and guide them in using the correct verbiage and notation with their proof. Therefore students are always looking to an outside authority for mathematical justification and they may develop the ability to use more abstract techniques when proving a statement (Simon & Blume, 1996). The types of notation and language used in mathematical proofs can appear to students like a foreign language. Shorthand in proof statements can further confuse students because they are trying to process two new concepts at one time. They struggle with the idea of proving facts they know while they are also being asked to use symbols they have never seen to stand for words. For example, how many nonmathematical individuals would understand the following sentence? ABC is an isoceles triangle iff $\overline{BA} \cong \overline{BC}$. Very few individuals would understand that this sentence is a biconditional statement that means that a triangle is an isosceles triangle if and only if two of the sides are congruent. Many of these same individuals may not remember or know what isosceles means and the symbols would only add to the confusion that occurs when terms and notation are unfamiliar to a reader. Thus anyone reading a similar statement may choose to ignore or pay little attention to the meanings of the symbols and words that are used as mathematical language (McCrone, et. al., 2002)

For many students the acceptance of items such as postulates and definitions also seems to work against the concept of proof. In a study conducted by Chazan (1993) as student argued "that deductive proof didn't really prove since you had to 'assume all these things' as part of a deductive proof "(p. 374). It may seem hypocritical in the minds of geometry students to ask them to prove what they can see and 'know' to be true and yet ask them to accept facts that they do not understand and cannot visualize.

When trying to generate a proof, students often alternate between restating the given(s) and jumping to conclusions (Herbst, 2002a). This movement back and forth between givens and hasty conclusions does not promote proof as an activity that can be used for understanding; instead, it promotes proof as a guess and check activity. Students start with a given, guess a conclusion, ask the teacher if they are correct and continue in this pattern until they form, to themselves, what is a logical argument that is devoid of any logic. Students are unable to determine the information necessary for completing a proof and what information is superfluous (Herbst, 2002a; Mitchelmore, 2002). They also struggle with detecting mistakes in proofs that are completed because they are unable to recognize flaws in logical arguments (McCrone, et al., 2002). Students are unable to determine if theorems, axioms, corollaries and definitions are used correctly within a proof because they do not understand the concepts and in turn they unable to correctly apply them (Weber, 2001).

CHAPTER 6

CONCLUSIONS AND IMPLICATIONS

Geometric proof is not an easy subject to teach, learn, or understand. It requires a desire to completely understand a topic which requires a great deal of time and effort on the proof writer's part. According to de Villiers (1999) there is no statistically significant difference in proof performance or appreciation for students who are taught logic before they are introduced to the topic of proof. However, the link between geometric proofs and students' levels of geometric thought cannot be ignored. In a study conducted by Senk (1989), she found that a student's van Hiele level has a direct influence on their ability to write and understand proofs. Students who are at level zero (0) are generally unable to write proofs by the end of the course and those at level one (1) will only be able to write simple standard proofs at the end of the same course. Considering these findings, educators may which to look at the curricula and set prerequisites for students entering geometry classes. These prerequisites may be level two (2) or higher scores on test measuring van Hiele levels, or a course structured in increasing van Hiele levels. In any case, if we want students to be successful in constructing proofs we must give them the tools to do so.

Enhancing the role of proof will require that instructors put more effort into their lessons and planned activities (Knuth, 2002). Proof exercises that are rote memory procedures will not enhance students' logical reasoning skills their understanding of geometric proof. Students need to be presented with a variety of proof types. Every proof that a student writes should not be a proof of a theorem. Proofs can and should be exploratory so that students are able to prove theorems, teacher selected conjectures and student derived conjectures. Proofs that require students to develop and prove their on conjectures have more meaning and purpose for the student. Classroom settings that require students to convince others of the truth or falsehood of their own conjectures transform proof from into a tool which contains personal value for the student and therefore they are more willing to use it in future applications (Alibert & Thomas, 1991). One of the values of mathematics is that it helps train the mind to conceive, judge and reason (Herbst, 2002a). The ability to imagine an endless number of possibilities, judge their relevance and to reason through their applications is useful to any individual and is the essence of proof. These abilities can easily be transferred to other subjects and to life applications and therefore they having meaning to the individual. When the mathematics that is taught in classrooms becomes useful in the world outside the classroom, students are more likely to pay attention and are eager to learn more. The concept of proofs is not different. The concepts presented in geometric proof must be applicable to the student's life outside the walls of the school building. As educators, it is our responsibility to find and incorporate methods of teaching proof so that students see the necessity in constructing them. This will, of course, mean more work for educators and may require that the geometry curriculum be altered to include items other than strict Euclidean Geometry such as: transformational, coordinate, vector and non-Euclidean Geometry as proposed by the College Entrance Exam Board's Commission on Mathematics (Fitzgerald, 1996). Teaching students to think logical in a variety of setting promotes the transferability of proofs.

Perhaps an underlying problem to our students' inability to generate and understand proofs is the instructor's lack of knowledge and interest in the subject area. I know from experience that if you do not enjoy teaching a subject that students can and do sense your lack of enthusiasm. Students, for the most part, will only accept what we encourage them to believe. If the instructor does not like to complete proofs and therefore does not use proofs as an exercise to help students understand concepts, students may view proofs as unnecessary. When students begin to write proofs that base their justifications on an authoritative source, generally the teacher, if this source does not provide sufficient explanations, then students do not have premises upon which to build. Therefore, it is important that teachers enjoy the subject they are teaching so that they will themselves do research on problems the students are completing to better assist students in their endeavors and increase students thirst for justifications.

For educators, students' inability to assess information and the struggles in proof composition prove to be difficult barriers to break through in an effort to help students produce meaningful and logical justifications. Proof should be a process of deductive and logical steps that are gleaned from an individuals insights and understanding of a topic and if students do not have insights or understanding the proof is irrelevant.

Generating proofs in the classroom setting should stir up a debate among students. Proof is intended to be a type of communication so that students can express their reasoning about a problem. Proof should have a social aspect so that students understand its role in the mathematics community (Simon & Blume, 1996). Proofs are not generated so that they can be hidden away; they are created and examined in an effort to promote the understanding of a topic. The student process of constructing proofs begins with inductive reasoning in an effort to determine whether a conjecture is worth the effort of a formal proof (Simon & Blume, 1996). The inductive reasoning process requires students to compare and contrast many examples and in their comparison students working on the same conjecture may disagree. These conflicts help students to understand characteristics of classes of figures that will assist them in generating a valid proof. Students who have their beliefs questioned by peers and authority figures often rethink their beliefs. This either strengthens their resolve by providing justification for their logic or they begin to discover fallacies in their thought processes. This understanding is only possible if proofs are discussed, questioned, and at times rewritten and reexamined. Comparisons of proof strategies and construction can result in a meaningful discourse that will further the students' understanding of the topic (Vavilis, 2003).

To enhance the understanding and construction of proofs by both students and teachers technology may play a crucial role. Technology allows students to compare and contrast a significantly large number of examples in the effort to determine if there is enough evidence to warrant a deductive proof. The use of dynamic geometry software environments brings theorems to life for students so that they can visualize the individual components of a theorem and how the manipulation of a premise changes the validity of a statement. Even though dynamic geometry software environments may initially promote the fallacy that enough examples prove a statement true, the overall potential educational experiences gleaned from using the software far outweigh the negative. Instructors who use technology in their classrooms as a tool for exploration will notice that while students initially believe that examples can prove a conjecture, will eventually want to know why all the examples work. However, not all students will see the need for proof and therefore it is the responsibility of the instructor to guide students toward the

While geometric proof is a difficult subject, it is an important part of the geometry curriculum. It has many roles in the mathematics world and all are equally important. As a tool to develop the logical skills and proficiency necessary for higher level mathematics, proof cannot be surpassed. It provides students with the opportunity to develop skills that are useful in justifying their reasoning and in structuring or ordering their thought processes. A development of proficiency skills prepares students for the rigors ahead of them on both standardized tests in and in future mathematics classes. As a tool for understanding, proof allows students to breakdown theorems and conjectures to discover why they work and how they can be used with different classes of figures. This understanding will assist them when they struggle with new concepts because they will be able to rely on previous knowledge instead of what they have previously heard. As a tool for exploration, proof can provide students with the opportunity to study a concept in-depth that they are interested in, develop a conjecture of their own and use their understanding to justify their conjectures. Using proof to explore new topics and conjectures is the cornerstone of mathematics. Current theorems and corollaries were at some point in time new conjectures that an individual decided to prove. We have moved away from this idea because of time constraints within the classroom and the idea that there are no new theorems to be discovered. It is my opinion that there are hundreds of theorems waiting to be discovered and we need to encourage our students to join the search.

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