8-2019

Algebraic properties of neural codes.

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ALGEBRAIC PROPERTIES OF NEURAL CODES

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A Dissertation
Submitted to the Faculty of the
College of Arts and Sciences of the University of Louisville
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy
in
Applied and Industrial Mathematics

Department of Mathematics
University of Louisville
Louisville, Kentucky

August 2019

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A Dissertation Approved on

May 22, 2019

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DEDICATION

To my husband, Hans.

To my parents, Ron and Laura.

To my siblings, Eliah and Tracy.

ACKNOWLEDGEMENTS

First, I wish to thank my graduate advisor and dissertation director, Dr. Hamid Kulosman, for his immense patience, encouragement, and guidance through this entire process. Without him, I would not have believed myself capable of this degree, nor would I have pursued it. Without him, this dissertation would not be a reality, and I feel truly blessed to have had the privilege of his mentorship.

I wish to thank the members of my dissertation committee, Dr. Csaba Biro, Dr. Jinjia Li, Dr. Thomas Riedel, and Dr. Nicholas Hindy, for their time, interest, and valuable comments, with a special thanks to our department chair, Dr. Thomas Riedel. Without his support, particularly during qualifying exams, I could never have entered candidacy to begin with, and his genuine interest in students' success does not go unnoticed or unappreciated.

I wish to extend my appreciation to the faculties of both the University of Louisville and Indiana University Southeast for providing the preparation and support of my studies. Thanks to April Robinson for initiating my graduate study, to Melissa Neil for inspiring me to teach mathematics, and to Dr. Jessica Taylor for encouraging me to persevere. Thanks to my fellow graduate students for all the understanding and commiserating these last five years. My deepest thanks and gratitude to Dr. Ryann Cartor and Dr. Trevor Leach, for everything.

I wish to thank my friends and family, who always believed in me and cheered me on, especially my husband and my parents, for bearing all the difficulties with patience, understanding, grace, and love. Lastly, and above all else, thanks to the Lord Jesus Christ, without whom none of this matters.

ABSTRACT

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The neural rings and ideals as algebraic tools for analyzing the intrinsic structure of neural codes were introduced by C. Curto, V. Itskov, A. Veliz-Cuba, and N. Youngs in 2013. Since then they have been investigated in several papers, including the 2017 paper by S. Güntürkün, J. Jeffries, and J. Sun, in which the notion of polarization of neural ideals was introduced. We extend their ideas by introducing the polarization of motifs and neural codes, and show that these notions have very nice properties which allow the studying of the intrinsic structure of neural codes of length n via the square-free monomial ideals in $2n$ variables. As a result, we can obtain minimal prime ideals in $2n$ variables which do not come from the polarization of any motifs of length n. For this reason, we introduce the notions for a partial code, including partial motifs and inactive neurons. With these notions, we are able to relate those non-polar primes back to the original neural code. Additionally, we reformulate an existing theorem and provide a shorter, simpler proof. We also give intrinsic characterizations of neural rings and the homomorphisms between them. We characterize monomial code maps as the composition of basic monomial code maps. This work is based on two theorems, introduced by C. Curto and N. Youngs in 2015, and the notions of a trunk and a monomial map between two neural codes, introduced by R. A. Jeffs in 2018.

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CHAPTER 1
INTRODUCTION

The field of neural codes is born from striving to better understand how the brain creates a spatial map of its surroundings. Beginning in 1971, John O'Keefe and his student Jonathan Dostrovsky discovered a specific class of neurons in the hippocampus called place cells, as they were very active in response to changes in the spatial surroundings such as changes in color or shape or the introduction or removal of items [22]. These active regions are aptly named place fields and are roughly analogous to the receptive fields of sensory neurons. Our brain, however, creates its cognitive map only from the information it receives from its neurons, and therein lies the importance of place cells. When a particular group of these cells fire together, it reveals something about the external stimulus space, and this may help us understand how the brain analyzes its neural information [10].

The algebraic study of neural codes began in 2013 with the pioneering paper by Carina Curto, Vladimir Itskov, Alan Veliz-Cuba, and Nora Youngs, where the main algebraic objects of study were introduced: the neural ring, which encodes the underlying stimulus space in its structure, and the related neural ideal, which defines the space itself. These algebraic objects help to yield a minimal description of the receptive field structure called the canonical form of the neural ideal [11]. This area of mathematics has been very active ever since, and some important developments have emerged in the last few years. For example, several important papers have appeared since 2017 studying the convexity and obstructions to convexity of the neural codes, among them [3, 8, 9, 13, 20, 23]. These deal with the visual representation of the receptive fields formed in the brain. Also in 2017, the operation called polarization was introduced for neural ideals by Sema Güntürkün, Jack Jeffries, and Jeffrey Sun in [15]. This operation redefines the neural ideals as square-free monomial ideals, which are very well studied and known for their nice behavior in commutative algebra. Polarization of the neural ideal preserves the intrinsic structure of the neural code while taking advantage of the existing results we have from square-free monomial ideal theory. Explained more in chapter 3, there are some cases where polarization of the neural code might reveal a hidden structure in the neurons, and it is hopeful that this may tell us even more about how the brain analyzes its neural information.

In some cases, there is missing neural information, and discussed more in chapter 4, this led us to introduce the notions for a partial code, including the concept of an inactive neuron, which is a neuron that participates in brain activity, but its state is unknown. We have recently found out that this is related to firing rules in neural computing. As described in [1], a subset of firing or non-firing neurons may depend on already active neurons (that is, having a state of 0 or 1). Even more complicated is that some of these neurons have an undefined output. In fact, it is the variability of the synapses that give the neuron its adaptability. "...such complexity needs eventually to be introduced into some models," [1]. While neurophysiologists still have to confirm if such things exists in a real brain, rather than simply in a neural network, it seems quite natural that they will.

Another important development in the algebraic study of neural codes is the introduction of various maps between neural codes and neural rings. In 2015, Carina Curto and Nora Youngs related the maps between neural codes to the homomorphisms between corresponding neural rings [12]; in 2016, R. Amzi Jeffs, Mohamed Omar, and Nora Youngs focused on neural ring homomorphisms that preserve the neural ideal [19]; in 2018, R. Amzi Jeffs introduced morphisms of neural codes, which preserve the trunk of a neural code [18]. In each of these papers, the structural properties of neural codes and neural rings are studied, and morphisms are the natural mathematical tools for that purpose. In fact, the image of a neural code under a monomial morphism may be the "mirror neural code" on the set of mirror neurons. Mirror neurons are a special class of brain cells discovered in the late 1980s which help may explain why humans are so unique as a species. As described in the abstract of the 2009 paper by Lindsay Oberman and V. S. Ramachandran, "We suggest that mirror neurons are endowed with the precise properties allowing for complex remapping from one domain into another, which may lead to behaviors which arguably distinguish humans from all the other animals, namely our abilities to interact socially, understand others thoughts and emotions, communicate using complex language, and the ability to reflect on ourselves," [21]. The authors continue to suggest that "...perhaps the mirror neuron system serves to connect our own representations with those of others across multiple domains and more generally mapping one dimension onto another in order to abstract what is common to them," [21]. This may be the "neuroscientific" definition of the mathematical notion of morphisms between neural codes.

The organization of this dissertation is as follows. In Chapter 2, we recount some useful results from some of the papers mentioned above, including [11], as well as some results from standard references, including [2, 6, 14]. We introduce the basic notations and definitions of working within the class of neural codes as well as some of the properties we encounter. In Chapter 3, we recount some useful results from [15], and we introduce some new results, including the polarization of the neural code. We analyze in detail the difference between the polarization of the neural code and the formal polarization of the neural code, along with an illustrative example of the difference between them. In Chapter 4, we introduce the notions and properties of a partial code. The notions in this chapter may give us a way to study the neural code when we are missing some neural information. Additionally with these notions, we give a new, simpler proof of an existing result from [15]. In Chapter 5, we give the intrinsic characterizations of neural rings and the homomorphisms between them. As an extension of the work done in [12], [18], and [19], we also give the characterization of monomial code maps as the composition of basic monomial code maps. In Chapter 6, we give our conclusions and recommendations for future work. Note that much of Chapters 3 and 4 appear in our recent paper [4], while much of Chapter 5 appears in our recent paper [5], although we have extended several notions and included several additional proofs.

CHAPTER 2
PRELIMINARIES

As we use techniques from Algebraic Geometry and Commutative Algebra to analyze the properties of neural codes, we first briefly review some notation and existing results needed throughout this dissertation. In particular, since the state of a neuron can be considered as binary, we use elements in the field $\mathbb{F}_2 = \{0, 1\}$, where 0 is considered as "off" or not firing, and 1 is considered as "on" or firing. We denote the set of n neurons as $[n] = \{1, 2, \ldots, n\}$.

2.1 The Neural Ring and Neural Ideal

Definition 2.1. ([11]) We define the element $\mathbf{w} = w_1 \cdots w_n \in \mathbb{F}_2^n$ to be a *codeword* (shortly *word*) of length n, which tracks the state of n neurons. We define a nonempty set of words $\mathcal{C} \subseteq \mathbb{F}_2^n$ to be a *neural code* (shortly *code*) of length n.

Let $\mathbb{F}_2[X_1, \ldots, X_n]$ to be the polynomial ring in n variables $X_1, \ldots, X_n$ over $\mathbb{F}_2$. For any word $\mathbf{w} \in \mathbb{F}_2^n$, there is a natural evaluation map $ev_{\mathbf{w}} : \mathbb{F}_2[X_1, \ldots, X_n] \to \mathbb{F}_2$ by setting $X_i = w_i$. The following definition, relating ideals of this polynomial ring to the varieties of those ideals, is illustrated by Figure 2.1.

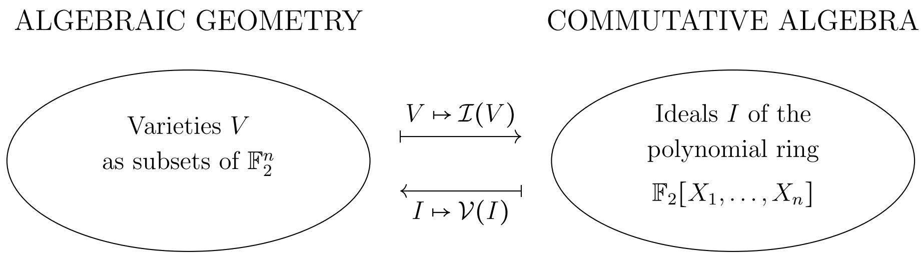
Definition 2.2. ([6], [11]) For an ideal $I \in \mathbb{F}_2[X_1, \ldots, X_n]$, the *variety of* I is

$$\mathcal{V}(I) = \{\mathbf{w} \in \mathbb{F}_2^n : f(\mathbf{w}) = 0 \text{ for all } f \in I\}.$$

For a variety $V \subseteq \mathbb{F}_2^n$, the *ideal of* V is

$$\mathcal{I}(V) = \{f \in \mathbb{F}_2[X_1, \ldots, X_n] : f(\mathbf{w}) = 0 \text{ for all } \mathbf{w} \in V\}.$$

Figure 2.1: Relationship Between Varieties and Ideals.



Note that for any variety $V \subseteq \mathbb{F}_2^n$, we have $\mathcal{I}(V) \supseteq \mathcal{B}$, where $\mathcal{B} = (X_1^2 - X_1, \dots, X_n^2 - X_n)$ is the *Boolean ideal* of $\mathbb{F}_2[X_1, \dots, X_n]$. Moreover, we have $\mathcal{I}(V) = \mathcal{B}$ if and only if $V = \mathbb{F}_2^n$ [11]. Thus we have the following result, in which the second part is called *Hilbert's Nullstellensatz for* $\mathbb{F}_2$ [14].

Theorem 2.3. ([11], [14]) For every variety $V \subseteq \mathbb{F}_2^n$, we have

$$\mathcal{V}(\mathcal{I}(V)) = V.$$

For every ideal $I \subseteq \mathbb{F}_2[X_1, \dots, X_n]$, we have

$$\mathcal{I}(\mathcal{V}(I)) = I + \mathcal{B}.$$

Because we can treat neural codes as varieties in $\mathbb{F}_2^n$, we have the following definition of the neural ring.

Definition 2.4. ([11]) Let $\mathcal{C} \subseteq \mathbb{F}_2^n$ be a neural code. We define the ideal $\mathcal{I}(\mathcal{C})$ by

$$\mathcal{I}(\mathcal{C}) = \{f \in \mathbb{F}_2[X_1, \dots, X_n] : f(\mathbf{w}) = 0 \text{ for all } \mathbf{w} \in \mathcal{C}\}.$$

The *neural ring* $R_\mathcal{C}$ is defined to be the quotient ring

$$R_\mathcal{C} = \frac{\mathbb{F}_2[X_1, \dots, X_n]}{\mathcal{I}(\mathcal{C})} = \mathbb{F}_2[x_1, \dots, x_n],$$

where $x_i = X_i + \mathcal{I}(\mathcal{C})$ for $i \in [n]$. We say the elements of $\mathbb{F}_2[x_1, \dots, x_n]$ are *polynomial expressions*, which behave like polynomials but can be simplified according to the relations given in $\mathcal{I}(\mathcal{C})$.

In particular, notice that $\mathcal{V}(\mathcal{I}(\mathcal{C})) = \mathcal{C}$ and $\mathcal{I}(\mathcal{V}(\mathcal{I}(\mathcal{C}))) = \mathcal{I}(\mathcal{C})$, which follow by Theorem 2.3. As previously stated, $\mathcal{I}(\mathcal{C}) \supseteq \mathcal{B}$ irrespective of $\mathcal{C}$, however, we would like to analyze the ideal generated by only non-Boolean relations in $\mathcal{I}(\mathcal{C})$. This ideal, denoted by $J_{\mathcal{C}}$, is a more convenient object to study for various purposes and why the term *neural ideal* is used to refer to $J_{\mathcal{C}}$ rather than $\mathcal{I}(\mathcal{C})$ [11]. The ideal $J_{\mathcal{C}}$ is generated by the polynomials in the following definition.

Definition 2.5. ([11], [14]) For a word $\mathbf{w} \in \mathbb{F}_2^n$, we define the *Lagrange polynomial* of $\mathbf{w}$, denoted $L_{\mathbf{w}} \in \mathbb{F}_2[X_1, \ldots, X_n]$, in the following way:

$$L_{\mathbf{w}} = \prod_{w_i=1} X_i \prod_{w_j=0} (1 - X_j).$$

Notice that this polynomial is similar to an indicator function as $L_{\mathbf{w}}(\mathbf{w}) = 1$, but for any other word $\mathbf{u} \in \mathbb{F}_2^n$, $L_{\mathbf{w}}(\mathbf{u}) = 0$. Notice also that, since the indices are always disjoint, we avoid capturing the Boolean relations of $\mathbb{F}_2[X_1, \ldots, X_n]$.

Definition 2.6. ([11, page 1582]) For a neural code $\mathcal{C} \subseteq \mathbb{F}_2^n$, we define the *neural ideal* of $\mathcal{C}$, denoted $J_{\mathcal{C}} \in \mathbb{F}_2[X_1, \ldots, X_n]$, in the following way:

$$J_{\mathcal{C}} = (\{L_{\mathbf{w}} : \mathbf{w} \notin \mathcal{C}\}).$$

It is important to note that, since $J_{\mathcal{C}}$ is generated by the Lagrange polynomials which are *not* in the code, $J_{\mathcal{C}}$ consists of polynomials that will vanish for all words in the code $\mathcal{C}$, and hence $J_{\mathcal{C}}$ is the ideal of the variety $\mathcal{C}$. Hence we have the following proposition, also following from Theorem 2.3.

Proposition 2.7. ([11, Lemma 3.2]) The neural ideal of a code $\mathcal{C}$ has the following properties:

$$\begin{aligned} \mathcal{V}(J_{\mathcal{C}}) &= \mathcal{C} \\ \mathcal{I}(\mathcal{C}) &= J_{\mathcal{C}} + \mathcal{B} \end{aligned}$$

2.2 The Receptive Field Structure of the Neural Code

The canonical form of the neural ideal is an important notion as it provides the minimal description of the receptive field (RF) structure of the stimulus space. The RF structure represents where the place cells are firing in the brain [11].

Definition 2.8. ([7], [11]) For a *stimulus space* $X \subseteq \mathbb{R}^d$ $(d \geq 1)$ on n neurons, we denote by $U_i \subseteq X$ the *receptive field* where the neuron i fires, for $i \in [n]$. If $\mathcal{U} = U_1, \ldots, U_n$, we say $\mathcal{U}$ is an *RF cover* of X. We define a *visual realization* of the code $\mathcal{C}$ to be an ordered pair $\mathrm{VR}(\mathcal{C}) = (X,\mathcal{U})$, such that the code $\mathcal{C} = \mathrm{Code}(X,\mathcal{U})$.

Note that the RF cover $\mathcal{U} = U_1, \ldots, U_n$ of X covers all the receptive fields $U_i \subseteq X$, although it may not cover X completely. Additionally, we follow the convention that each receptive field is convex, although the space X need not be.

Definition 2.9. ([7]) Let $X \subseteq \mathbb{R}^d$ and $\mathcal{U} = U_1, \ldots, U_n$ be an RF cover of X. The *atom* of the pair $(X,\mathcal{U})$, corresponding to the set of neurons $\alpha \subseteq [n]$, is the set

$$A_\alpha^{X,\mathcal{U}} = \Big(\bigcap_{i \in \alpha} U_i\Big) \smallsetminus \Big(\bigcup_{j \notin \alpha} U_j\Big).$$

We will write shortly $A_\alpha^{\mathcal{U}}$ instead of $A_\alpha^{X,\mathcal{U}}$ when there is no confusion what X is. Also note that

$$A_\varnothing^{X,\mathcal{U}} = \Big(\bigcap_{i \in \varnothing} U_i\Big) \smallsetminus \Big(\bigcup_{j \notin \varnothing} U_j\Big) = X \smallsetminus \Big(\bigcup_{i=1}^{n} U_i\Big).$$

Proposition 2.10. If $\alpha, \beta \subseteq [n]$ are distinct, and at least one of the atoms $A_\alpha^{\mathcal{U}}$ and $A_\beta^{\mathcal{U}}$ is nonempty, then $A_\alpha^{\mathcal{U}} \neq A_\beta^{\mathcal{U}}$.

Proof. Without loss of generality, suppose $i_o \in \alpha$ and $i_o \notin \beta$. Such an i_o exists since $\alpha \neq \beta$. Then

$$A_\alpha^{\mathcal{U}} = \Big(\bigcap_{i \in \alpha \smallsetminus \{i_o\}} U_i \cap U_{i_o}\Big) \smallsetminus \Big(\bigcup_{j \in^c \alpha} U_j\Big),$$

$$A_\beta^{\mathcal{U}} = \Big(\bigcap_{i \in \beta} U_i\Big) \smallsetminus \Big(\bigcup_{j \in^c \beta \smallsetminus \{i_o\}} U_j \cup U_{i_o}\Big).$$

Here ^{c} denotes the complement in $[n]$. From these two formulas, we can see that every element of $A^{\mathcal{U}}_{\alpha}$ (if there is one) belongs to U_{i_o}, while no element of $A^{\mathcal{U}}_{\beta}$ (if there is one) belongs to U_{i_o}. Hence if at least one of the atoms $A^{\mathcal{U}}_{\alpha}$ and $A^{\mathcal{U}}_{\beta}$ is nonempty, these two atoms are different. $\square$

Definition 2.11. Let $X \subseteq \mathbb{R}^d$, $\mathcal{U} = U_1, \ldots, U_n$ be an RF cover of X, and $\alpha \subseteq [n]$. For each nonempty atom $A^{\mathcal{U}}_{\alpha}$ of the pair $(X, \mathcal{U})$, we define the *word of this atom* as $\mathbf{w} = w_1 \cdots w_n$, where $w_i = 1$ if $i \in \alpha$ and $w_i = 0$ otherwise. The *code* of the pair $(X, \mathcal{U})$ is the set $\mathrm{Code}(X, \mathcal{U})$ of words from the nonempty atoms of $(X, \mathcal{U})$.

In other words, the atoms of the stimulus space X are each of the distinct regions of X, as illustrated by examples in Figure 2.2, and the words of those atoms represent all the neurons firing in that region, as illustrated in Figure 2.3.

For any stimulus space, an element of the canonical form is a general Lagrange polynomial, called a *pseudo-monomial*. In general, we can identify each neuron i with an indeterminate X_i for each $i \in [n]$.

Definition 2.12. ([11, page 1585]) A polynomial $f \in \mathbb{F}_2[X_1, \ldots, X_n]$ is called a *pseudo-monomial* if it has the form

$$f = \prod_{i \in \sigma} X_i \prod_{j \in \tau} (1 - X_j)$$

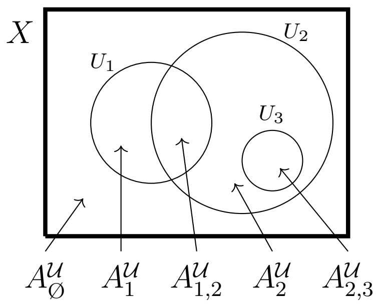
for some disjoint $\sigma, \tau \subseteq [n] = \{1, 2, \ldots, n\}$.

Definition 2.13. ([11, page 1585]) Let $I \subseteq \mathbb{F}_2[X_1, \ldots, X_n]$ be an ideal and $f \in I$ a pseudo-monomial. We say that f is a *minimal pseudo-monomial* of I if there does not exist another pseudo-monomial $g \in I$ such that $\deg(g) < \deg(f)$ and $g | f$ in $\mathbb{F}_2[X_1, \ldots, X_n]$. We say that I is a *pseudo-monomial ideal* if it can be generated by a finite set of pseudo-monomials.

Notice that if f is a pseudo-monomial, and if $g \in \mathbb{F}_2[X_1, \ldots, X_n]$ divides f, then g is *necessarily* a pseudo-monomial and has the form $g = \prod_{i \in \sigma'} X_i \prod_{j \in \tau'} (1 - X_j)$, where $\sigma' \subseteq \sigma$ and $\tau' \subseteq \tau$.

Figure 2.2: Examples of RF structures and Atoms.

Example 1:

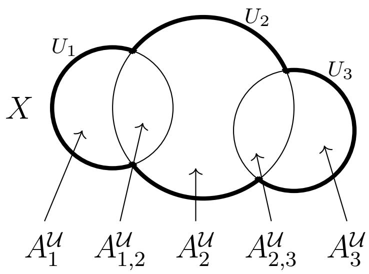


$\mathcal{U} = U_1, U_2, U_3$

Here:

$A^{\mathcal{U}}_{3} = A^{\mathcal{U}}_{1,3} = A^{\mathcal{U}}_{1,2,3} = \emptyset$

Example 2:



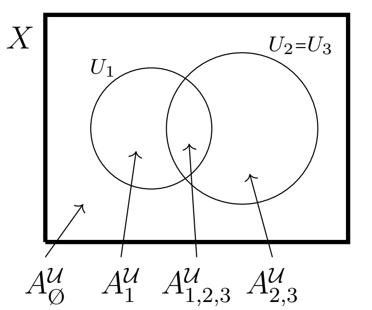
$X = U_1 \cup U_2 \cup U_3$

$\mathcal{U} = U_1, U_2, U_3$

Here:

$A^{\mathcal{U}}_{\emptyset} = A^{\mathcal{U}}_{1,3} = A^{\mathcal{U}}_{1,2,3} = \emptyset$

Example 3:

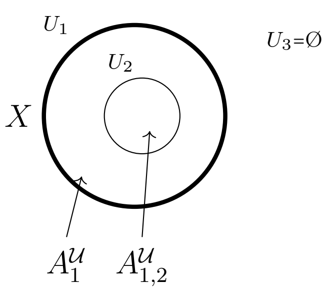


$\mathcal{U} = U_1, U_2, U_3$

Here:

$A^{\mathcal{U}}_{2} = A^{\mathcal{U}}_{3} = \emptyset$

Example 4:



$X = U_1 \cup U_2 \cup U_3$

$\mathcal{U} = U_1, U_2, U_3$

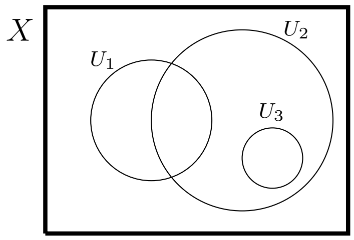
Here:

$A^{\mathcal{U}}_{\emptyset} = A^{\mathcal{U}}_{2} = A^{\mathcal{U}}_{3} = \emptyset$

$A^{\mathcal{U}}_{1,3} = A^{\mathcal{U}}_{2,3} = A^{\mathcal{U}}_{1,2,3} = \emptyset$

Figure 2.3: Examples of Atoms and RF structures.

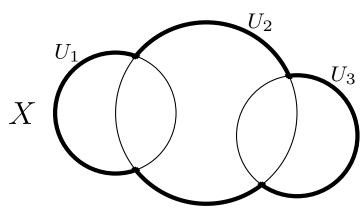
Example 1:



$\mathcal{U} = U_1, U_2, U_3$

$\mathcal{C}(\mathcal{U}) = \{000, 100, 110, 010, 011\}$

Example 2:

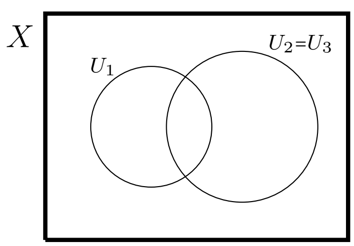


$X = U_1 \cup U_2 \cup U_3$

$\mathcal{U} = U_1, U_2, U_3$

$\mathcal{C}(\mathcal{U}) = \{100, 110, 010, 011, 001\}$

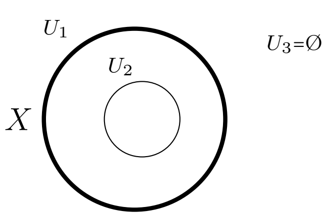
Example 3:



$\mathcal{U} = U_1, U_2, U_3$

$\mathcal{C}(\mathcal{U}) = \{000, 100, 111, 011\}$

Example 4:



$X = U_1 \cup U_2 \cup U_3$

$\mathcal{U} = U_1, U_2, U_3$

$\mathcal{C}(\mathcal{U}) = \{100, 110\}$

Definition 2.14. ([11, page 1585]) Let $I \in \mathbb{F}_2[X_1, \ldots, X_n]$ be a pseudo-monomial ideal. We define the *canonical form* of I, denoted $CF(I)$, to be the (finite) set consisting of all minimal pseudo-monomials of I.

2.3 Properties of the Neural Code

In Algebraic Geometry, a *motif* (or *motive*) can be used to group together similarly behaved cohomology theories. Since we know that we can infer something about the surrounding environment when particular neurons fire at the same time [10], we group together similarly behaved neurons and also call them motifs. These motifs serve as the essence of the neural code, much like the motif in algebraic geometry serves as the essence of a variety, and we use them as the building blocks in the definitions and properties that follow.

Definition 2.15. ([11]) We define the set $\mathbb{M} = \{0, 1, *\}$. We say the sequence $\mathbf{a} = a_1 \cdots a_n \in \mathbb{M}^n$ is a *motif* of length n. For any motif $\mathbf{a} \in \mathbb{M}^n$, we define the *variety of* $\mathbf{a}$, denoted $V_{\mathbf{a}}$, to be the set of all words obtained by replacing the stars of $\mathbf{a}$ by zeros and ones. We say that $\mathbf{a}$ is a *motif of the code* $\mathcal{C}$, and write $\mathbf{a} \in \mathrm{Mot}(\mathcal{C})$, if $V_{\mathbf{a}} \subseteq \mathcal{C}$.

Definition 2.16. ([11]) We define a partial order on $\mathbb{M}$ such that $0 < *$ and $1 < *$. For two motifs $\mathbf{a}, \mathbf{b} \in \mathbb{M}^n$, we say that $\mathbf{a} \le \mathbf{b}$ if $a_i \le b_i$ for each i. We say that a motif $\mathbf{a}$ is a *maximal motif of the code* $\mathcal{C}$, and write $\mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})$, if for any other motif $\mathbf{b} \in \mathrm{Mot}(\mathcal{C})$, $\mathbf{a} \le \mathbf{b}$ implies $\mathbf{a} = \mathbf{b}$.

Remark 2.17. ([11])

(i) Notice that for $\mathbf{a}, \mathbf{b} \in \mathbb{M}^n$, we have $\mathbf{a} \le \mathbf{b} \Leftrightarrow V_{\mathbf{a}} \subseteq V_{\mathbf{b}}$.

(ii) For any $\mathbf{a} \in \mathrm{Mot}(\mathcal{C})$, there exists some $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{C})$ such that $\mathbf{a} \le \mathbf{b}$.

(iii) We have $\mathcal{C}_1 = \mathcal{C}_2$ if and only if $\text{MaxMot}(\mathcal{C}_1) = \text{MaxMot}(\mathcal{C}_2)$ for any two codes $\mathcal{C}_1$ and $\mathcal{C}_2$.

(iv) $\mathcal{C} = \emptyset$ if and only if $\text{MaxMot}(\mathcal{C}) = \emptyset$.

(v) We define the complement of the code $\mathcal{C}$ to be the code $\mathcal{D} = \mathbb{F}_2^n \setminus \mathcal{C}$, denoted $\mathcal{D} = {}^c\mathcal{C}$. We can now write the neural code $\mathcal{C}$ and its neural ideal $J_{\mathcal{C}}$ in the following way:

$$\mathcal{C} = \cup\{V_{\mathbf{a}} : \mathbf{a} \in \text{MaxMot}(\mathcal{C})\}, \tag{2.1}$$

$$J_{\mathcal{C}} = (\{L_{\mathbf{a}} : \mathbf{a} \in \text{MaxMot}(\mathcal{D})\}). \tag{2.2}$$

However, for proper subsets $M \subset \text{MaxMot}(\mathcal{C})$ and $N \subset \text{MaxMot}(\mathcal{D})$, it can happen that we still have $\mathcal{C} = \cup\{V_{\mathbf{a}} : \mathbf{a} \in M\}$ and $J_{\mathcal{C}} = (\{L_{\mathbf{a}} : \mathbf{a} \in N\})$, as we will see in the next example. Either way, since any subset of $\mathcal{C}$ is a *subvariety* in $\mathbb{F}_2^n$, we say the code is the union of its *maximal motivic subvarieties*.

Proposition 2.18. ([11, Lemma 5.7]) Let $\mathcal{C} \subseteq \mathbb{F}_2^n$ be a neural code with complement $\mathcal{D} = {}^c\mathcal{C}$ and neural ideal $J_{\mathcal{C}}$. Then we have

$$CF(J_{\mathcal{C}}) = \{L_{\mathbf{a}} : \mathbf{a} \in \text{MaxMot}(\mathcal{D})\}.$$

Example 2.19. For the code $\mathcal{C} = \{000, 001, 011, 111\} \subseteq \mathbb{F}_2^3$, we have $\text{MaxMot}(\mathcal{C}) = \{00*, 0*1, *11\}$. Clearly Equation 2.1 holds, but we also have $\mathcal{C} = V_{00*} \cup V_{*11}$.

Similarly, since $\mathcal{D} = \{100, 101, 110, 010\}$, then we have $\text{MaxMot}(\mathcal{D}) = \{10*, 1*0, *10\}$. By Proposition 2.18, we have

$$\begin{aligned} CF(J_{\mathcal{C}}) &= \{L_{10*}, L_{1*0}, L_{*10}\} \\ &= \{X_1(1 - X_2), X_1(1 - X_2), X_2(1 - X_3)\}. \end{aligned}$$

The canonical form is unique and is clearly a generating set of the neural ideal, i.e., Equation 2.2 holds; however, $CF(J_{\mathcal{C}})$ is not necessarily a *unique* minimal generating set for $J_{\mathcal{C}}$, and in this case, it is not even a minimal generating set, since each pseudo-monomial in $CF(J_{\mathcal{C}})$ can be generated by the other two. For example, $J_{\mathcal{C}} = (L_{10*}, L_{*10})$ since

$$X_1(1 - X_3) = (1 - X_3) \cdot [X_1(1 - X_2)] + X_1 \cdot [X_2(1 - X_3)].$$

Since the previous example is on three neurons, it is simple to calculate the maximal motifs of the code and the maximal motifs of its complement. In general, however, it is much more tedious to do that. We will see more details in Chapter 3 which concern this process for large n.

Naturally, the next important property we examine is that of the prime ideal. In Algebraic Geometry, it is important to investigate the prime ideals $\mathfrak{p}$ that contain a polynomial ideal I, as the varieties of $\mathfrak{p}$ are the "irreducible subvarieties" of the variety $\mathcal{V}(I)$. More importantly are the *minimal* prime ideals of I, since then, the varieties of $\mathfrak{p}$ correspond to the *maximal* irreducible subvarieties of $\mathcal{V}(I)$.

It turns out that all this is closely related to the *motifs* of a neural code. Considering that we want to investigate the prime ideals and minimal prime ideals of a neural code, we define those in terms of the motifs and maximal motifs of a code. Let $\mathrm{Min}(J_{\mathcal{C}})$ denote the set of *all minimal primes* of the neural ideal.

Definition 2.20. ([11, page 1594]) For a code $\mathcal{C}$ and a motif $\mathbf{a} \in \mathrm{Mot}(\mathcal{C})$, we define a (*motivic*) *prime ideal of* $\mathbf{a}$, denoted $\mathfrak{p}_{\mathbf{a}} \subseteq \mathbb{F}_2[X_1, \ldots, X_n]$, in the following way:

$$\mathfrak{p}_{\mathbf{a}} = (\{X_i : a_i = 0\} \cup \{1 - X_j : a_j = 1\}). \tag{2.3}$$

Note that $V_{\mathbf{a}} \subseteq V_{\mathbf{b}} \Leftrightarrow \mathfrak{p}_{\mathbf{b}} \subseteq \mathfrak{p}_{\mathbf{a}}$ [11, Lemma 5.2]. The next proposition follows from Theorem 2.3.

Proposition 2.21. ([11, page 1594]) Let $\mathbf{a} \in \mathbb{M}^n$. We have

$$\mathcal{V}(\mathfrak{p}_{\mathbf{a}}) = V_{\mathbf{a}},$$

$$\mathcal{I}(V_{\mathbf{a}}) = \mathfrak{p}_{\mathbf{a}} + \mathcal{B}.$$

Proposition 2.22. ([11, Lemma 5.1, Lemma 5.3, and Corollary 5.5]) Let $\mathcal{C} \in \mathbb{F}_2^n$ be a neural code, $\mathbf{a} \in \mathbb{M}^n$ a motif, and $\mathrm{Min}(J_{\mathcal{C}})$ the set of all minimal primes of $J_{\mathcal{C}}$. Then we have

$$\mathbf{a} \in \mathrm{Mot}(\mathcal{C}) \iff \mathfrak{p}_{\mathbf{a}} \supseteq J_{\mathcal{C}} \tag{2.4}$$

$$\mathbf{a} \in \mathrm{MaxMot}(\mathcal{C}) \iff \mathfrak{p}_{\mathbf{a}} \in \mathrm{Min}(J_{\mathcal{C}}). \tag{2.5}$$

Moreover,

$$\mathrm{Min}(J_{\mathcal{C}}) = \{\mathfrak{p}_{\mathbf{a}} : \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})\}. \tag{2.6}$$

One well-known theorem of Emmy Noether (1882-1935) says that every ideal in a Noetherian commutative ring has a unique "irredundant primary decomposition", i.e., it can be written (in a certain unique way) as an intersection of primary ideals containing it [2]. Recall that a commutative ring is Noetherian if every prime ideal of the ring is finitely generated, and an ideal I in a commutative ring R is called *primary* if for every $a, b \in R$, we have $ab \in I \Rightarrow a \in I$ or $b^n \in I$ for some $n \geq 1$ [2]. Since $\mathbb{F}_2[X_1, \ldots, X_n]$ is a Noetherian ring, the neural ideal $J_{\mathcal{C}}$ has a unique irredundant primary decomposition.

Proposition 2.23. ([11, Corollary 5.5]) Let $\mathcal{C} \subseteq \mathbb{F}_2^n$ be a nonempty neural code. Then

$$J_{\mathcal{C}} = \cap\{\mathfrak{p}_{\mathbf{a}} : \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})\}$$

is the unique irredundant primary decomposition of $J_{\mathcal{C}}$.

Remark 2.24. Proposition 2.23 implies that the ideals in the decomposition of $J_{\mathcal{C}}$ are not only primary but actually prime. It also confirms that considering the code as the union of its maximal motivic subvarieties is indeed the most natural approach (from the point of view of Commutative Algebra), and hence Commutative Algebra is the best suited tool for the analysis of $J_{\mathcal{C}}$. From the real-life point of view, this means that we are considering maximal subcodes, where some neurons have fixed states and the remaining neurons have variability.

CHAPTER 3
POLARIZATION OF THE NEURAL CODE

3.1 Motivation

Since the neural ideal of a code is generated by Lagrange polynomials, and since pseudo-monomials are precisely the Lagrange polynomials of motifs, we have that the neural ideal is a pseudo-monomial ideal. However, the analysis of these ideals is very complicated; for example, as we saw in Example 2.19, the set of minimal pseudo-monomials was not a minimal generating set of the neural ideal.

Monomial ideals, on the other hand, are well studied in Commutative Algebra and are known for their nice behavior. Recall that a polynomial of the form $f = X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}$ is called a *monomial* (resp. *square-free monomial*) if $a_i \in \mathbb{N}_0$ (resp. $a_i \in \{0,1\}$) for each i, and an ideal is called a *monomial ideal* (resp. *square-free monomial ideal*) if it can be generated by a finite set of monomials (resp. square-free monomials). Below are just a few nice properties, as found in standard references, including [2, 6].

- **Theorem 1.** Let $I \subseteq \mathbb{F}_2[X_1, \dots, X_n]$ be an ideal. The following are equivalent:

  (i) I is a monomial ideal;

  (ii) for any $f \in I$, all the terms of f are in I.

- **Theorem 2.** Let $I \subseteq \mathbb{F}_2[X_1, \dots, X_n]$ be a monomial ideal, and let $\mathcal{M}$ be a set of monomials in I. Then $\mathcal{M}$ is a set of generators of I if and only if for each monomial $f \in I$, there exists $g \in \mathcal{M}$ such that $g|f$.

- **Theorem 3.** Let I be a monomial ideal. Then there exists a unique minimal set of monomial generators of I.

- **Theorem 4.** Let I be a square-free monomial ideal. Then I is a finite intersection of prime monomial ideals.

This behavior found in monomial ideals and square-free monomial ideals would make the analysis of the neural ideal very nice; however, in most cases, these properties do not hold in pseudo-monomial ideals. The operation introduced in [15] called *polarization of pseudo-monomials* gives us a way to redefine pseudo-monomials in terms of square-free monomials, and thus we can take advantage of the results we have from square-free monomial ideal theory for the analysis of the neural code.

Definition 3.1. ([15, page 6]) For a pseudo-monomial of the form

$$f = \prod_{i \in \sigma} X_i \prod_{j \in \tau} (1 - X_j) \in \mathbb{F}_2[X_1, \dots, X_n],$$

where σ, τ are two disjoint subsets of $[n]$, we define its *polarization* f^p to be the square-free monomial

$$f^p = \prod_{i \in \sigma} X_i \prod_{j \in \tau} Y_j \in \mathbb{F}_2[X_1, \dots, X_n, Y_1, \dots, Y_n].$$

Proposition 3.2. ([15, Lemma 3.1]) Let $f, g \in \mathbb{F}_2[X_1, \dots, X_n]$ be two pseudo-monomials. Then

$$f \mid g \;\Leftrightarrow\; f^p \mid g^p. \tag{3.1}$$

Definition 3.3. ([15, Definition 3.3]) Let J be a pseudo-monomial ideal in $F_2[X_1, \dots, X_n]$ and let $CF(J) = \{f_1, \dots, f_l\}$ be its canonical form. We define the *polarization of J* to be the ideal

$$J^p = (f_1^p, \dots, f_l^p) \subseteq \mathbb{F}_2[X_1, \dots, X_n, Y_1, \dots, Y_n].$$

Remark 3.4. The two previous definitions state that, instead of pseudo-monomial ideals in n variables $X_1, \ldots, X_n$, we may consider related square-free monomial ideals in $2n$ variables which are denoted by $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ (rather than by $X_1, \ldots, X_{2n}$). Because of this difference in the notation for variables, we should be aware that, for example, a pseudo-monomial $f \in \mathbb{F}_2[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ has the form

$$f = \prod_{i \in \sigma} X_i \prod_{j \in \tau} (1 - X_j) \prod_{k \in \mu} Y_k \prod_{l \in \nu} (1 - Y_l),$$

where $\sigma, \tau, \mu, \nu \subseteq [n]$ with $\sigma \cap \tau = \emptyset$ and $\mu \cap \nu = \emptyset$. Similarly, we have that a motif $\mathbf{a} = b_1 \cdots b_n c_1 \cdots c_n \in \mathbb{M}^{2n}$ will have the Lagrange polynomial $L_{\mathbf{a}}$ and the prime ideal $\mathfrak{p}_{\mathbf{a}}$ as follows, respectively:

$$L_{\mathbf{a}} = \prod_{b_i = 1} X_i \prod_{b_j = 0} (1 - X_j) \prod_{c_k = 1} Y_k \prod_{c_l = 0} (1 - Y_l), \tag{3.2}$$

$$\mathfrak{p}_{\mathbf{a}} = \Big(\{X_1 : b_i = 0\} \cup \{1 - X_j : b_j = 1\} \cup \{Y_k : c_k = 0\} \cup \{1 - Y_l : c_l = 1\} \Big).$$

The definitions of these notions with respect to $\mathbb{F}_2^{2n}$ are the same as the ones with respect to $\mathbb{F}_2^n$, so we only need to take into account the notation for the variables. This works for other notions as well (such as the neural ideal, minimal pseudo-monomials, etc.), while some notations, such as minimal primes, can be given in a form that does not depend on the notation for the variables.

In this regard, we will have the following convention: if the length of motifs and codes is denoted by n, then the associated rings and ideals will always be in n variables $X_1, \ldots, X_n$. If the length is denoted by $2n$, then the associated rings and ideals will always be in $2n$ variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$. For the lengths given by concrete numbers, it will always be clear from the context if it is n or $2n$.

3.2 Polarization of the Neural Code

For a code $\mathcal{C}$, we would next like to define the *polarization of the neural code*, $\mathcal{C}^p$. In particular, we would like to define it so that its neural ideal $J_{\mathcal{C}^p}$ consists of monomials only. That is, every maximal motif in the complement of $\mathcal{C}^p$ should have coordinates equal to 1 or $*$ only. In addition, we would like to have monomials related to $\mathcal{C}^p$ be polarizations of the pseudo-monomials related to $\mathcal{C}$, as described in Remark 3.4, so it is natural to try to define the code $\mathcal{C}^p$ with length $2n$. The most important property we would like to hold is the preservation of maximal motifs over polarization, i.e.,

$$\mathrm{MaxMot}(\mathcal{C}^p) = \mathrm{MaxMot}^p(\mathcal{C}),$$

since that would naturally imply that all we must do to obtain $\mathcal{C}^p$ is polarize the maximal motifs of $\mathcal{C}$, i.e., $\mathcal{C}^p = \cup\{V_{\mathbf{a}^p} : \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})\}$. We now face the question of how to polarize the motifs of a neural code.

For a motif $\mathbf{a} = a_1 \cdots a_n \in \mathbb{M}^n$, we want to define its polarization $\mathbf{a}^p$ such that $\mathbf{a}^p \in \mathbb{M}^{2n}$, just as we want to define the polarized code with length $2n$. We will see in the next example what conclusions we can make about the polarization of a motif by assuming only that we have the preservation of maximal motifs of the code over polarization. Moreover, it follows that we have the preservation of minimal prime ideals, proved later in Theorem 3.31.

$$\mathrm{Min}(J_{\mathcal{C}^p}) = \mathrm{Min}^p(J_{\mathcal{C}}). \tag{P0}$$

Example 3.5. For $\mathcal{C} = \{10\} \subseteq \mathbb{F}_2^2$, we have $\mathrm{MaxMot}(\mathcal{C}) = \{10\}$ and hence $\mathrm{Min}(J_{\mathcal{C}}) = \{\mathfrak{p}_{10}\} = \{(1 - X_1, X_2)\}$. Then by (P0), we have

$$\mathrm{Min}^p(J_{\mathcal{C}}) = \{(X_2, 1 - X_1)^p\} = \{(X_2, Y_1)\} = \{\mathfrak{p}_{*00*}\}.$$

Thus, the motif 10 is associated to the polarized motif $*00*$, and we have

$$10^p = *00*\,. \tag{P1}$$

Now since $\mathcal{D} = {}^c\mathcal{C} = \{00, 01, 11\}$, we have $\mathrm{MaxMot}(\mathcal{D}) = \{0*, *1\}$ and $\mathrm{Min}(J_{\mathcal{D}}) = \{\mathfrak{p}_{0*}, \mathfrak{p}_{*1}\} = \{(X_1), (1 - X_2)\}$. Then similarly by (P0), we have

$$\mathrm{Min}^p(J_{\mathcal{D}}) = \{(X_1)^p, (1 - X_2)^p\} = \{(X_1), (Y_2)\} = \{\mathfrak{p}_{0***}, \mathfrak{p}_{***0}\}.$$

Thus, the motifs $0*$ and $*1$ are associated to the polarized motifs $0***$ and $***0$, respectively, and hence,

$$0*^p = 0*** \quad \text{and} \quad *1^p = ***0. \tag{P2}$$

Note that, for a motif $\mathbf{a} \in \mathbb{M}^2$, its polarization $\mathbf{a}^p \in \mathbb{M}^4$ should have the form:

$$\mathbf{a}^p = a_1 a_2{}^p = b_1 b_2 \mid c_1 c_2,$$

where each $a_i = b_i \mid c_i$ is polarized coordinate-wise. Thus, (P1) and (P2) give that

$$10^p = *0 \mid 0* \overset{(P1)}{\Longrightarrow} 1^p = * \mid 0 \;\text{ and }\; 0^p = 0 \mid *,$$
$$*1^p = ** | *0 \;\text{ and }\; 0*^p = 0* | ** \overset{(P2)}{\Longrightarrow} *^p = * \mid *.$$

As we will see in Theorem 3.14, this reasoning can be generalized to motifs for any length n, and hence, the definitions of the polarization of a motif and of the neural code are as follows.

Definition 3.6. Let $\mathbf{a} = a_1 \cdots a_n \in \mathbb{M}^n$. We define its ***polarization*** $\mathbf{a}^p \in \mathbb{M}^{2n}$, where

$$\mathbf{a}^p = a_1 \cdots a_n{}^p = b_1 \cdots b_n \mid c_1 \cdots c_n,$$

in the following way:

$$\text{if } a_i = 0, \text{ then } b_i = 0,\; c_i = *;$$
$$\text{if } a_i = 1, \text{ then } b_i = *,\; c_i = 0;$$
$$\text{if } a_i = *, \text{ then } b_i = *,\; c_i = *.$$

Schematically:

$$\cdots 0 \cdots \;\mapsto\; \cdots 0 \cdots \mid \cdots * \cdots$$

$$\cdots 1 \cdots \;\mapsto\; \cdots * \cdots \mid \cdots 0 \cdots$$

$$\cdots * \cdots \;\mapsto\; \cdots * \cdots \mid \cdots * \cdots$$

Definition 3.7. For a code $\mathcal{C} \subseteq \mathbb{F}_2^n$, we define its *polarization*, denoted $\mathcal{C}^p \subseteq \mathbb{F}_2^{2n}$, by

$$\mathcal{C}^p = \cup\{V_{\mathbf{a}^p} : \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})\}. \tag{3.3}$$

Example 3.8. Continuing with Example 3.5, we can now determine $\mathcal{C}^p$ and $\mathcal{D}^p$. We have the following polarizations:

$$\begin{aligned}
\mathcal{C}^p &= \{10\}^p = V_{10^p} = V_{*00*} \\
&= \{0000, 1000, 0001, 1001\}, \\
\mathcal{D}^p &= \{00, 01, 11\}^p = V_{0*^p} \cup V_{*1^p} = V_{0***} \cup V_{***0} \\
&= \{0000, 0100, 0010, 0001, 0110, 0101, \\
&\qquad 0011, 0111, 1000, 1100, 1010, 1110\}.
\end{aligned}$$

Notice in particular that $\mathcal{C}^p \cap \mathcal{D}^p = \{0000\}$ and $\mathcal{C}^p \cup \mathcal{D}^p = \mathbb{F}_2^n \setminus \{1111\}$. In general, even though $\mathcal{C}$ and $\mathcal{D}$ are complements, $\mathcal{C}^p$ and $\mathcal{D}^p$ are not; in fact, $\mathcal{C}^p \cap \mathcal{D}^p$, as well as the complement of $\mathcal{C}^p \cup \mathcal{D}^p$, can contain several words.

3.3 Properties of the Polarized Code

3.3.1 Preservation of Maximal Motifs

We have many nice properties that hold from our definition of the polarization of the neural code, the most important of which preserves the maximal motifs of a code over polarization for any length n, which will be proved in Theorem 3.14. To prove this, we first introduce a few additional notions.

Definition 3.9. We say that a motif $\mathbf{b} \in \mathbb{M}^{2n}$ is a *polar motif* if there is a motif $\mathbf{a} \in \mathbb{M}^n$ such that $\mathbf{b} = \mathbf{a}^p$. If such a motif $\mathbf{a}$ exists, then it is unique, and we denote $\mathbf{a} = \mathbf{b}^d$ as the *depolarization of* $\mathbf{b}$.

Note that for any two motifs $\mathbf{a}, \mathbf{b} \in \mathbb{M}^n$, we have

$$\mathbf{a}^{pd} = \mathbf{a} \text{ for every } \mathbf{a} \in \mathbb{M}^n;$$

$$\mathbf{b}^{dp} = \mathbf{b} \text{ for every polar motif } \mathbf{b} \in \mathbb{M}^{2n}.$$

Proposition 3.10. Let $\mathbf{a}, \mathbf{b} \in \mathbb{M}^n$. Then

$$\mathbf{a} \le \mathbf{b} \;\Leftrightarrow\; \mathbf{a}^p \le \mathbf{b}^p.$$

Proof. Let $\mathbf{a} = a_1 \cdots a_n, \mathbf{b} = b_1 \cdots b_n \in \mathbb{M}^n$, and suppose $\mathbf{a} \le \mathbf{b}$. For any $i \in [n]$, if we have $(\mathbf{b}^p)_i = 0$, then $\mathbf{b}^p_{n+i} = *$ and $b_i = 0$. Hence $a_i = 0$ and thus $(\mathbf{a}^p)_i = 0$. If $(\mathbf{b}^p)_{n+i} = 0$, then $\mathbf{b}^p_i = *$ and $b_i = 1$. Hence $a_i = 1$ and $(\mathbf{a}^p)_{n+i} = 0$. Thus $\mathbf{a}^p \le \mathbf{b}^p$.

On the other hand, suppose $\mathbf{a}^p \le \mathbf{b}^p$. Then if, for $i \in [n]$, $b_i = 0$, then $(\mathbf{b}^p)_i = 0$, and hence $(\mathbf{a}^p)_i = 0$, hence $a_i = 0$. If $b_i = 1$, then $(\mathbf{b}^p)_{n+i} = 0$, and hence $(\mathbf{a}^p)_{n+i} = 0$, hence $a_i = 1$. Thus $\mathbf{a} \le \mathbf{b}$. $\square$

Corollary 3.11. For any code $\mathcal{C} \subseteq \mathbb{F}_2^n$, we have

$$\mathrm{Mot}^p(\mathcal{C}) \subseteq \mathrm{Mot}(\mathcal{C}^p).$$

Proof. Let $\mathbf{a} \in \mathrm{Mot}(\mathcal{C})$ and let $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{C})$ such that $\mathbf{a} \le \mathbf{b}$. By the previous proposition, we have $\mathbf{a}^p \le \mathbf{b}^p$. Since (by the definition of $\mathcal{C}^p$) we have that $\mathbf{b}^p \in \mathrm{Mot}(\mathcal{C}^p)$, we have $\mathbf{a}^p \in \mathrm{Mot}(\mathcal{C}^p)$. $\square$

Proposition 3.12. For any code $\mathcal{C} \subseteq \mathbb{F}_2^n$,

$$\mathrm{MaxMot}(\mathcal{C}^p) \subseteq \{0, *\}^{2n}.$$

Proof. Suppose that there exists some motif $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{C}^p)$ such that $b_\alpha = 1$ for some $\alpha \in [2n]$. Then for any word $\mathbf{w} \in V_{\mathbf{b}}$, we have $w_\alpha = 1$. Hence $\mathbf{w} \in V_{\mathbf{a}^p}$ for some $\mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})$, and by definition, $(\mathbf{a}^p)_\alpha = *$. But then the word $\mathbf{w}'$, obtained by replacing w_α in $\mathbf{w}$ by 0, is also in $V_{\mathbf{a}^p}$, and hence in $\mathcal{C}^p$. Hence the motif $\mathbf{b}'$, obtained by replacing b_α by $*$ would be a motif of $\mathcal{C}^p$, contradicting the maximality of $\mathbf{b}$. □

Proposition 3.13. For any motif $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{C}^p)$, there is no $i \in [n]$ such that $b_i = b_{i+n} = 0$.

Proof. Suppose to the contrary, and define the following sets:

$$A = \{j \in [n] : b_j = b_{j+n} = *\};$$

$$B = \{j \in [n] : b_j = 0, b_{j+n} = *\};$$

$$C = \{j \in [n] : b_j = *, b_{j+n} = 0\};$$

$$D = \{j \in [n] : b_j = b_{j+n} = 0\};$$

where $i \in D$ by assumption, and the sets A, B, C, D form a partition of $[n]$. Let $\mathbf{w} \in V_{\mathbf{b}}$ be defined in the following way:

$$(\forall j \in A) w_j = w_{j+n} = 1;$$

$$(\forall j \in B) w_j = 0, w_{j+n} = 1;$$

$$(\forall j \in C) w_j = 1, w_{j+n} = 0;$$

$$(\forall j \in D) w_j = w_{j+n} = 0;$$

Since $\mathbf{w} \in \mathcal{C}^p$, there is a motif $\mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})$ such that $\mathbf{w} \in V_{\mathbf{a}^p}$. Since $\mathbf{a}^p$ is a polar motif, we have

$$
\begin{aligned}
&(\forall j \in A)(\mathbf{a}^p)_j = (\mathbf{a}^p)_{j+n} = *;\\
&(\forall j \in B)(\mathbf{a}^p)_j = 0 \text{ or } *, (\mathbf{a}^p)_{j+n} = *;\\
&(\forall j \in C)(\mathbf{a}^p)_j = *, (\mathbf{a}^p)_{j+n} = 0 \text{ or } *;\\
&(\forall j \in D) \text{ at least one of } (\mathbf{a}^p)_j, (\mathbf{a}^p)_{j+n} \text{ is } *.
\end{aligned}
$$

Since D contains at least one element, these relationships imply that $\mathbf{a}^p > \mathbf{b}$, contradicting the maximality of $\mathbf{b}$. □

Theorem 3.14. For any code $\mathcal{C} \subseteq \mathbb{F}_2^n$, we have

$$\mathrm{MaxMot}^p(\mathcal{C}) = \mathrm{MaxMot}(\mathcal{C}^p). \tag{3.4}$$

Proof. Let $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{C}^p)$. By Propositions 3.12 and 3.13, for each $i \in [n]$, we have one of the following three cases:

$$(1)\ b_i = b_{i+n} = *;\quad (2)\ b_i = 0, b_{i+n} = *;\quad (3)\ b_i = *, b_{i+n} = 0.$$

Let $\mathbf{w} \in V_{\mathbf{b}}$ be a word defined in the following way for each case, respectively:

$$(1)\ w_i = w_{i+n} = 1;\quad (2)\ w_i = 0, w_{i+n} = 1;\quad (3)\ w_i = 1, w_{i+n} = 0.$$

Note that this word $\mathbf{w} \in V_{\mathbf{a}^p}$ for some motif $\mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})$. Since $\mathbf{a}^p$ is a polar motif, it follows for each case, respectively, that:

$$(1)\ (\mathbf{a}^p)_i = (\mathbf{a}^p)_{i+n} = *;\quad (2)\ (\mathbf{a}^p)_i = 0, (\mathbf{a}^p)_{i+n} = *;\quad (3)\ (\mathbf{a}^p)_i = *, (\mathbf{a}^p)_{i+n} = 0,$$

and hence $\mathbf{a}^p \geq \mathbf{b}$. Since $\mathbf{a}^p \in \mathrm{Mot}(\mathcal{C}^p)$ and $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{C})$, we have $\mathbf{b} = \mathbf{a}^p$, and hence $\mathrm{MaxMot}^p(\mathcal{C}) \supseteq \mathrm{MaxMot}(\mathcal{C}^p)$.

On the other hand, suppose $\mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})$ such that $\mathbf{a}^p \notin \mathrm{MaxMot}(\mathcal{C}^p)$. By the definition of $\mathcal{C}^p$, we have $\mathbf{a}^p \in \mathrm{Mot}(\mathcal{C}^p)$, and hence there exists some $\mathbf{b} \in$

$\mathrm{MaxMot}(\mathcal{C}^p)$ such that $\mathbf{a}^p < \mathbf{b}$. But, by the first part of the proof, there must also exist some $\mathbf{c} \in \mathrm{MaxMot}(\mathcal{C})$ such that $\mathbf{b} = \mathbf{c}^p$, so by Proposition 3.10, since $\mathbf{a}^p < \mathbf{c}^p$, we would have $\mathbf{a} < \mathbf{c}$, contradicting the maximality of both $\mathbf{a}$ and $\mathbf{c}$. Hence $\mathrm{MaxMot}^p(\mathcal{C}) \subseteq \mathrm{MaxMot}(\mathcal{C}^p)$. □

3.3.2 Formal Polarization of the Neural Code

As we saw in Example 3.5, although the codes $\mathcal{C}$ and $\mathcal{D}$ are complements, their polarizations $\mathcal{C}^p$ and $\mathcal{D}^p$ need not be. This presents an obstacle when considering the polarization of the canonical form, which is defined in terms of the complement of the code. If $\mathcal{C}^p$ and $\mathcal{D}^p$ are *not* complements, then what is the complement of $\mathcal{C}^p$? How do we find it? To answer these questions, we must first analyze the relationship of disjoint motifs.

Definition 3.15. For a motif $\mathbf{a} = a_1 \cdots a_n \in \mathbb{M}^n$, we define $\overline{\mathbf{a}}$ (read as "a-bar") to be the motif $\mathbf{b} = b_1 \cdots b_n \in \mathbb{M}^n$ which satisfies the following condition:

$$\text{for } i \in [n], \text{ if } a_i \neq \star, \text{then } b_i = \overline{a_i} = 1 - a_i.$$

Then for any motifs $\mathbf{a}, \mathbf{b} \in \mathbb{M}^n$, we have:

$$\mathbf{b} = \overline{\overline{\mathbf{a}}^p} \;\Leftrightarrow\; \mathbf{a} = \overline{\overline{\mathbf{b}}}^d. \tag{3.5}$$

Moreover, for any code $\mathcal{C} \subseteq \mathbb{F}_2^n$ and $M \subseteq \mathrm{Mot}(\mathcal{C})$, if we denote $\overline{M} = \{\overline{\mathbf{a}} : \mathbf{a} \in M\}$, then

$$\mathrm{Mot}(\overline{\mathcal{C}}) = \overline{\mathrm{Mot}(\mathcal{C})}, \tag{3.6}$$

$$\mathrm{MaxMot}(\overline{\mathcal{C}}) = \overline{\mathrm{MaxMot}(\mathcal{C})}. \tag{3.7}$$

Proposition 3.16. For any motif $\mathbf{a} \in \mathbb{M}^n$, we have

$$L_{\mathbf{a}}^p = L_{\overline{\overline{\mathbf{a}}^p}}.$$

Proof. This follows from Definitions 3.3, 3.6, 3.15, and Equation 3.2. □

Example 3.17. Let $n = 4$ and let $\mathbf{a} = 11{*}0$. Then by definition, $L_{\mathbf{a}} = X_1X_2(1-X_4)$, and hence $L_{\mathbf{a}}^p = X_1X_2Y_4$. On the other side, we have

$$\overline{\overline{\mathbf{a}}^p} = \overline{\overline{11{*}0}^p} = \overline{00{*}1^p} = \overline{00{*}{*}|\,{*}{*}{*}0} = 11{*}{*}|\,{*}{*}{*}1,$$

corresponding to the Lagrange polynomial $X_1X_2Y_4$. Hence, $L_{\mathbf{a}}^p = L_{\overline{\overline{\mathbf{a}}^p}}$.

Definition 3.18. We say that two motifs $\mathbf{a}, \mathbf{b} \in \mathbb{M}^n$ are *disjoint* if there is an $i \in [n]$ such that $a_i = \overline{b_i}$.

Definition 3.19. One the set $\mathbb{M} = \{0, 1, *\}$, we introduce a commutative operation of *addition* in the following way:

$0 + 0 = 0$	$0 + * = *$
$0 + 1 = 1$	$1 + * = *$
$1 + 1 = 0$	$* + * = *$

We can see the left column represents arithmetic in $\mathbb{F}_2$, while the right column represents max-arithmetic. We then define the addition in $\mathbb{M}^n$ by adding motifs coordinate-wise.

Remark 3.20. It is easy to verify that, with the above operation and the partial order we defined before, $\mathbb{M}^n$ is a *partially ordered monoid.* The importance of this operation lies in the fact that, for two motifs $\mathbf{a}, \mathbf{b} \in \mathbb{M}^n$, the sum $\mathbf{a} + \mathbf{b}$ has at least one coordinate equal to one (called a *1-component*) if and only if the motifs $\mathbf{a}$ and $\mathbf{b}$ are disjoint. Thus, we can *recognize the disjointness* of two motifs algebraically by considering their sum.

Proposition 3.21. (The Disjointness Proposition) For a code $\mathcal{C} \subseteq \mathbb{F}_2^n$ and its complement $\mathcal{D}$, let $\mathbf{a} \in \mathrm{Mot}(\mathcal{C})$ and $\mathbf{b} \in \mathbb{M}^n$. Then $\mathbf{b} \in \mathrm{Mot}(\mathcal{D})$ if and only if $\mathbf{b}$ is disjoint with $\mathbf{a}$.

Moreover, the maximal motifs of $\mathcal{D}$ are the motifs $\mathbf{b}$ that are maximal among the motifs from $\mathbb{M}^n$ which are disjoint from all the maximal motifs of $\mathcal{C}$.

Proof. Easy to see. □

Proposition 3.22. Let $\mathbf{a}, \mathbf{b} \in \mathbb{M}^n$. If $\mathbf{a} + \mathbf{b}$ has a 1-component, then $\mathbf{a} + \mathbf{b}'$ has a 1-component for any $\mathbf{b}' \leq \mathbf{b}$. In particular, for a code $\mathcal{C} \subseteq \mathbb{F}_2^n$ and its complement $\mathcal{D}$, the maximal motifs of $\mathcal{D}$ are the maximal elements $\mathbf{b} \in \mathbb{M}^n$ such that each $\mathbf{a} + \mathbf{b}$ has a 1-component for each $\mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})$.

Proof. Easy to see. □

Corollary 3.23. Let $\mathcal{C} \subseteq \mathbb{F}_2^n$. If $\mathbf{b} \in \mathrm{MaxMot}({}^c(\mathcal{C}^p))$, then every b_i different from $*$ is equal to 1.

Proof. The statement follows from the previous proposition as each 0 can be replaced with $*$, which results in a strictly bigger motif which is disjoint from all maximal motifs of $\mathcal{C}^p$. □

Proposition 3.24. The motifs $\mathbf{a}$ and $\mathbf{b}$ from $\mathbb{M}^n$ are disjoint if and only if the motifs $\mathbf{a}^p$ and $\overline{\overline{\mathbf{b}}^p}$ from $\mathbb{M}^{2n}$ are disjoint.

Proof. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are disjoint. We first consider the case $a_i = 1, b_i = 0$ for some $i \in [n]$. Then $(\mathbf{a}^p)_i = *$ and $(\mathbf{a}^p)_{n+i} = 0$, while $(\overline{\overline{\mathbf{b}}^p})_i = *$ and $(\overline{\overline{\mathbf{b}}^p})_{n+i} = 1$. Hence $\mathbf{a}^p$ and $\overline{\overline{\mathbf{b}}^p}$ are disjoint. The case $\mathbf{a}_i = 0, \mathbf{b}_i = 1$ for some $i \in [n]$ is similar.

On the other hand, suppose that $\mathbf{a}^p$ and $\overline{\overline{\mathbf{b}}^p}$ are disjoint. We first consider the case $(\mathbf{a}^p)_i = 0$, and $(\overline{\overline{\mathbf{b}}^p})_i = 1$ for some $i \in [n]$. Then $\mathbf{a}_i = 0$ and $(\overline{\mathbf{b}}^p)_i = 0$, hence $(\overline{\mathbf{b}})_i = 0$. Hence $b_i = 1$, so that $\mathbf{a}$ and $\mathbf{b}$ are disjoint. The case $(\mathbf{a}^p)_i = 1, (\overline{\overline{\mathbf{b}}^p})_i = 0$ for some $i \in [n]$ is similar. □

Proposition 3.25. For any two codes $\mathcal{C}, \mathcal{D} \subseteq \mathbb{F}_2^n$, we have

$$\mathcal{D} \subseteq {}^c\mathcal{C} \quad \Leftrightarrow \quad \overline{\overline{\mathcal{D}}^p} \subseteq {}^c(\mathcal{C}^p). \tag{3.8}$$

Proof. The following equivalences follow from Proposition 3.10, Proposition 3.24, Theorem 3.14, and Proposition 3.21, respectively.

$$\begin{aligned}
\mathcal{D} \subseteq {}^c\mathcal{C} &\Leftrightarrow (\forall \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C}))(\forall \mathbf{b} \in \mathrm{MaxMot}(\mathcal{D}))\ \mathbf{a} \text{ and } \mathbf{b} \text{ are disjoint} \\
&\Leftrightarrow (\forall \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C}))(\forall \mathbf{b} \in \mathrm{MaxMot}(\mathcal{D}))\ \mathbf{a}^p \text{ and } \overline{\overline{\mathbf{b}}}^p \text{ are disjoint} \\
&\Leftrightarrow (\forall \mathbf{c} \in \mathrm{MaxMot}(\mathcal{C}^p))(\forall \mathbf{d} \in \mathrm{MaxMot}(\overline{\overline{\mathcal{D}}}^p))\ \mathbf{c} \text{ and } d \text{ are disjoint} \\
&\Leftrightarrow \overline{\overline{\mathcal{D}}}^p \subseteq {}^c(\mathcal{C}^p).
\end{aligned}$$

□

Note that, in the previous proposition, equality on the left hand side is not equivalent with the equality on the right hand side, as we are going to see in Example 3.33.

Corollary 3.26. For a code $\mathcal{C} \subseteq \mathbb{F}_2^n$ and its complement $\mathcal{D}$, we have

$$\mathcal{C}^p \subseteq {}^c\overline{\overline{\mathcal{D}}}^p.$$

Proof. Follows immediately from the previous proposition. □

We are now faced with several questions. What set is exactly equal to the complement of $\overline{\overline{\mathcal{D}}}^p$? What is the difference between the sets $\mathcal{C}^p$ and the complement of $\overline{\overline{\mathcal{D}}}^p$? We begin to answer these questions by rewriting its notation in terms of the initial code.

Definition 3.27. For a code $\mathcal{C} \subseteq \mathbb{F}_2^n$ and its complement $\mathcal{D}$, we define the *formal polarization of the code* $\mathcal{C}$, denoted $\mathcal{C}^{[p]}$, by

$$\mathcal{C}^{[p]} = {}^c\overline{\overline{\mathcal{D}}}^p.$$

Proposition 3.28. For a code $\mathcal{C} \subseteq \mathbb{F}_2^n$ and its complement $\mathcal{D}$, we have

$$\mathrm{MaxMot}(\mathcal{C}^p) \subseteq \mathrm{MaxMot}(\mathcal{C}^{[p]}), \tag{3.9}$$

$$\mathrm{MaxMot}({}^c(\mathcal{C}^{[p]})) = \overline{\overline{\mathrm{MaxMot}(\mathcal{D})}}^p \subseteq \mathrm{MaxMot}({}^c(\mathcal{C}^p)). \tag{3.10}$$

Proof. By Theorem 3.14 and the previous definition, if is equivalent to show that $(\forall \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C}))(\forall \mathbf{b} \in \mathrm{MaxMot}(\mathcal{D}))$ $\mathbf{a}^p$ and $\overline{\overline{\mathbf{b}}^p}$ are disjoint, which follows from Proposition 3.24.

For the second statement, let $\mathbf{d} \in \mathrm{MaxMot}({}^c(\mathcal{C}^{[p]})) = \mathrm{MaxMot}(\overline{\overline{\mathcal{D}}^p})$. Then $\mathbf{d} = \overline{\overline{\mathbf{b}}^p}$ for some $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{D})$, and suppose $\mathbf{e} \in \mathrm{MaxMot}({}^c(\mathcal{C}^p))$ such that $\mathbf{d} \leq \mathbf{e}$, i.e., $\overline{\overline{\mathbf{b}}^p} \leq \mathbf{e}$ and hence $\overline{\mathbf{b}}^p \leq \overline{\mathbf{e}}$. Since $\overline{\mathbf{e}}$ is larger than or equal to a polar motif, $\overline{\mathbf{e}}$ must also be a polar motif. That is, for some $\mathbf{f} \in \mathbb{M}^n$, we have $\overline{\mathbf{e}} = \overline{\mathbf{f}}^p$. Hence $\overline{\mathbf{b}}^p \leq \overline{\mathbf{f}}^p$ so that $\overline{\overline{\mathbf{b}}^p} \leq \overline{\overline{\mathbf{f}}^p} = \mathbf{e}$. Then, since $\mathbf{e}$ is disjoint with all the maximal motifs of $\mathcal{C}^p$, we have by Proposition 3.24 that $\mathbf{f}$ is disjoint with all the maximal motifs of $\mathcal{C}$ and $\mathbf{f} \geq \mathbf{b}$. Now since $\mathbf{b}$ is one of the maximal motifs among those which are disjoint from all the maximal motifs of $\mathcal{C}$, we also have that $\mathbf{f} \leq \mathbf{b}$. Thus, $\mathbf{f} = \mathbf{b}$ and $\mathbf{d} = \mathbf{e}$. □

Thus we have the relationship between the maximal motifs for the polarization $\mathcal{C}^p$ and the formal polarization $\mathcal{C}^{[p]}$. In particular, in Equation 3.10, we have the relationship with respect to the complements, which help us characterize the relationship between the canonical forms of $\mathcal{C}^p$ and $\mathcal{C}^{[p]}$.

Theorem 3.29. For a code $\mathcal{C} \subseteq \mathbb{F}_2^n$, we have

$$CF(J_{\mathcal{C}}^p) = CF^p(J_{\mathcal{C}}) = CF(J_{\mathcal{C}^{[p]}}) \subseteq CF(J_{\mathcal{C}^p}).$$

Proof. Let $CF(J_{\mathcal{C}}) = \{f_1, \ldots, f_k\}$. By definition, $J_{\mathcal{C}}^p = (f_1^p, \ldots, f_k^p)$, and recall that $f_1^p, \ldots, f_k^p$ are square-free monomials. By [17], Corollary 1.10, the set $\{f_1^p, \ldots, f_k^p\}$ contains a minimal subset S (with respect to inclusion) which generates $J_{\mathcal{C}}^p$. By [17], Corollary 1.8, if $f_i^p \notin S$, then $f_i^p \mid f_j^p$ for some $f_j^p \in S$. Then by [15], Lemma 3.1, $f_i \mid f_j$, contradicting minimality of the elements in the canonical form. Thus $S = \{f_1^p, \ldots, f_k^p\}$, and hence $CF(J_{\mathcal{C}}^p) = CF^p(J_{\mathcal{C}})$.

For the center equality, let $\mathcal{D} = {}^c\mathcal{C}$. Then we have the following equalities, which follow by Proposition 2.18, Proposition 3.16, and Equation 3.5:

$$\begin{aligned}
CF^p(J_{\mathcal{C}}) &= \{L^p_{\mathbf{a}} : \mathbf{a} \in \mathrm{MaxMot}(\mathcal{D})\} \\
&= \{L_{\overline{\mathbf{a}^p}} : \mathbf{a} \in \mathrm{MaxMot}(\mathcal{D})\} \\
&= \{L_{\mathbf{b}} : \overline{\overline{\mathbf{b}}}^d \in \mathrm{MaxMot}(\mathcal{D})\} \\
&= \{L_{\mathbf{b}} : \mathbf{b} \in \mathrm{MaxMot}(\overline{\overline{\mathcal{D}}}^p)\} \\
&= CF(J_{\mathcal{C}^{[p]}}).
\end{aligned}$$

Finally, the inclusion in the statement follows from the previous proposition and Proposition 2.18. □

We also characterize the relationship between the minimal prime ideals of $\mathcal{C}^p$ and $\mathcal{C}^{[p]}$, but first we need an additional definition.

Definition 3.30. The prime ideals $\mathfrak{p} \subseteq \mathbb{F}_2[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ such that $\mathfrak{p} = \mathfrak{p}_{\mathbf{a}^p}$ for some $\mathbf{a} \in \mathbb{M}^n$ are called *polar motivic primes*.

Note that $\mathfrak{p}_{\mathbf{a}^p} = \mathfrak{p}^p_{\mathbf{a}}$ since for any $\mathbf{a} = a_1 \cdots a_n \in \mathrm{MaxMot}(\mathcal{C})$, we have $\mathbf{a}^p = b_1 \cdots b_n c_1 \cdots c_n$ where

$$\begin{aligned}
\mathfrak{p}_{\mathbf{a}^p} &= (\{X_i : b_i = 0\} \cup \{Y_j : c_j = 0\}) \\
&= (\{X_i : a_i = 0\} \cup \{Y_j : a_j = 1\})^p \\
&= \mathfrak{p}^p_{\mathbf{a}}.
\end{aligned}$$

Theorem 3.31. For a code $\mathcal{C} \subseteq \mathbb{F}_2^n$, we have

$$\mathrm{Min}(J_{\mathcal{C}^p}) \;=\; \mathrm{Min}^p(J_{\mathcal{C}}) \;\subseteq\; \mathrm{Min}(J_{\mathcal{C}^{[p]}}).$$

Proof. The inclusion in the statement follows from Proposition 3.28 and Equation 2.6 from Proposition 2.22. For the equality, by Proposition 2.22 and Theorem 3.14, we have

$$\begin{aligned}
\mathrm{Min}(J_{\mathcal{C}^p}) &= \{\mathfrak{p}_{\mathbf{d}} : \mathbf{d} \in \mathrm{MaxMot}(\mathcal{C}^p)\} \\
&= \{\mathfrak{p}_{\mathbf{a}^p} : \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})\} \\
&= \{\mathfrak{p}_{\mathbf{a}^p} : \mathfrak{p}_{\mathbf{a}} \in \mathrm{Min}(J_{\mathcal{C}})\} \\
&= \{\mathfrak{p}_{\mathbf{a}}^p : \mathfrak{p}_{\mathbf{a}} \in \mathrm{Min}(J_{\mathcal{C}})\}.
\end{aligned}$$

Hence

$$\mathrm{Min}(J_{\mathcal{C}^p}) = \mathrm{Min}^p(J_{\mathcal{C}}).$$

□

Theorem 3.32. For any code $\mathcal{C} \subseteq \mathbb{F}_2^n$, the ideal $J_{\mathcal{C}^p}$ has the unique irredundant primary decomposition, and it is obtained by polarizing the prime ideals from the unique irredundant primary decomposition of $J_{\mathcal{C}}$.

Proof. By Proposition 2.23, the ideals $J_{\mathcal{C}}$ and $J_{\mathcal{C}^p}$ have the following unique irredundant primary decompositions, respectively,

$$\begin{aligned}
J_{\mathcal{C}} &= \cap\{\mathfrak{p}_{\mathbf{a}} : \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})\}, \\
J_{\mathcal{C}^p} &= \cap\{\mathfrak{p}_{\mathbf{b}} : \mathbf{b} \in \mathrm{MaxMot}(\mathcal{C}^p)\},
\end{aligned}$$

and hence, the statement follows from Theorem 3.14. □

3.4 An Illustrative Example

Example 3.33. Consider the neural code $\mathcal{C}$ and its complement $\mathcal{D}$ in $\mathbb{F}_2^3$:

$$\mathcal{C} = \{000, 100, 110, 001\} \text{ and } \mathcal{D} = \{001, 010, 101, 111\}.$$

Then $\mathrm{MaxMot}(\mathcal{C}) = \{*00, 1*0, 011\}$ and $\mathrm{MaxMot}(\mathcal{D}) = \{*01, 1*1, 010\}$, and by Theorem 3.14, we have

$$\mathrm{MaxMot}(\mathcal{C}^p) = \{*00***, **00**, 0***00\}, \tag{3.11}$$

$$\mathrm{MaxMot}(\overline{\overline{\mathcal{D}}}^p) = \{**1*1*, 1*1***, *1*1*1\}. \tag{3.12}$$

By Proposition 2.18, we have $CF(J_{\mathcal{C}^{[p]}}) = \{X_3Y_2, X_1X_3, X_2Y_1Y_3\} = CF^p(J_{\mathcal{C}})$.

By Proposition 3.25, however, we know that $\overline{\overline{\mathcal{D}}}^p \subsetneq {}^c(\mathcal{C}^p)$, and we use Proposition 3.21 (The Disjointness Proposition) to find $\mathrm{MaxMot}({}^c(\mathcal{C}^p))$. Consider the maximal motifs $\mathbf{a}^1 = *00***$, $\mathbf{a}^2 = **00**$, and $\mathbf{a}^3 = 0***00$ of $\mathcal{C}^p$, and define the sets $A_1 = \{2,3\}, A_2 = \{3,4\}, A_3 = \{1,5,6\}$, which represent the coordinates of those motifs that are *zeros*, respectively. Choosing one element from each A_i, we form the set of coordinates of some motif of ${}^c(\mathcal{C}^p)$ that are *ones*, and then from all such sets, we select those which are minimal with respect to inclusion. In that way, we get the following sets, corresponding to the *ones* in the maximal motifs of ${}^c(\mathcal{C}^p)$:

$$B_1 = \{3,5\}, \qquad B_2 = \{1,3\}, \qquad B_3 = \{2,4,6\},$$
$$B_4 = \{3,6\}, \qquad B_5 = \{2,4,5\}, \qquad B_6 = \{1,2,4\},$$

and hence,

$$\mathrm{MaxMot}({}^c(\mathcal{C}^p)) = \{**1*1*, 1*1***, *1*1*1, **1**1, *1*11*, 11*1**\}.$$

Thus by Proposition 2.18

$$CF(J_{\mathcal{C}^p}) = \{X_3Y_2,\ X_1X_3,\ X_2Y_1Y_3,\ X_3Y_3,\ X_2Y_1Y_2,\ X_1X_2Y_1\},$$

illustrating that strict inclusion in Theorem 3.29 is possible: $CF^p(J_{\mathcal{C}}) \subset CF(J_{\mathcal{C}^p})$, and in particular $\mathcal{C}^p \subset \mathcal{C}^{[p]}$. Indeed, $\mathcal{C}^p$ has 29 words while $\mathcal{C}^{[p]}$ has 35 words.

Since Proposition 3.28 also gives us $\mathrm{MaxMot}(\overline{\overline{\mathcal{D}}}^p) = \mathrm{MaxMot}({}^c(\mathcal{C}^{[p]}))$, we can use the same technique to find the maximal motifs of $\mathcal{C}^{[p]}$:

$$\mathrm{MaxMot}(\mathcal{C}^{[p]}) = \{*00***, **00***, **0**0, 00**0*, 0**00*, 0***00\}.$$

Hence, by Proposition 2.22,

$$\operatorname{Min}(J_{\mathcal{C}^p}) = \{(X_2, X_3), (X_3, Y_1), (X_1, Y_2, Y_3)\} = \operatorname{Min}^p(J_{\mathcal{C}}),$$

$$\operatorname{Min}(J_{\mathcal{C}^{[p]}}) = \{(X_2, X_3), (X_3, Y_1), (X_1, Y_2, Y_3), (X_3, Y_3), (X_1, X_2, Y_2), (X_1, Y_1, Y_2)\},$$

and hence, $\operatorname{Min}^p(J_{\mathcal{C}}) \subset \operatorname{Min}(J_{\mathcal{C}^{[p]}})$.

Remark 3.34. Notice that each of the additional monomials in $CF(J_{\mathcal{C}^p})$ all share some index for the X and Y variables. That is, they are not coming from the polarization of any pseudo-monomial.

Also notice that the additional motivic prime ideals in $\operatorname{Min}(J_{\mathcal{C}^{[p]}})$ are coming from maximal motifs of $\mathcal{C}^{[p]}$ which are not the polarization of any motifs from $\mathcal{C}$. That is, as a result of formal polarization, we have obtained some *non-polar* minimal primes.

CHAPTER 4
PARTIAL NEURAL CODES

As we saw in the illustrative example in the previous section, among the minimal primes of $J_{\mathcal{C}[p]}$, in addition to all the minimal primes of $J_{\mathcal{C}^p}$, we also have three *non-polar* minimal primes, namely, $\mathfrak{p}_{**0**0} = (X_3, Y_3)$, $\mathfrak{p}_{00**0*} = (X_1, X_2, Y_2)$, and $\mathfrak{p}_{0**00*} = (X_1, Y_1, Y_2)$, as the motifs in $**0**0, 00**0*, 0**00* \in \mathbb{M}^6$ are *not* the polarization of any motif in $\mathbb{M}^3$. This begs the question: how are these non-polar primes related to $\mathcal{C}$? In particular, if we have some $\mathfrak{p}_{\mathbf{a}} \in \mathrm{Min}(J_{\mathcal{C}[p]})$, then how is the motif $\mathbf{a} \in \mathbb{M}^{2n}$ related to $\mathcal{C}$?

In this section, while trying to answer this question, we introduce the notions of *partial words*, *partial motifs*, *partial codes*, and *inactive neurons*. We can think of a partial word as a word where the state of some neurons is unknown. For example, if a word of length 8 is given by $\mathbf{w}=$ _01_00_1, we say neurons 2, 3, 5, 6, and 8 are *active* (firing or not firing), and the neurons 1, 4, and 7 are *inactive*.

Definition 4.1. For the set $\mathbb{PW} = \{0, 1, \_\}$, we say $\mathbf{w} \in \mathbb{PW}^n$ is a *partial word* of length n, and a set of partial words $\mathcal{C} \subseteq \mathbb{PW}^n$ is a *partial code* of length n. For the set $\mathbb{PM} = \{0, 1, *, \_\}$, we say $\mathbf{a} \in \mathbb{PM}^n$ is a *partial motif* of length n. A neuron i is said to be *inactive* if $a_i = \_$ for $\mathbf{a} \in \mathbb{PM}^n$. We define a partial order on $\mathbb{PM}$ by declaring that $0 < *$ and $1 < *$ (similar to the partial order on $\mathbb{M}$ with the addition that $\_$ is only comparable with itself). For two partial motifs $\mathbf{a}, \mathbf{b} \in \mathbb{PM}^n$, we say that $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for each i. For a partial code $\mathcal{C} \subseteq \mathbb{PW}^n$, the sets of all partial motifs and maximal partial motifs are denoted $\mathrm{ParMot}(\mathcal{C})$ and $\mathrm{MaxParMot}(\mathcal{C})$, respectively.

Remark 4.2. (i) The set of all partial words is denoted by $\mathbb{PW}^n_K$, where $K = \{i_1, \ldots, i_k\} \subseteq [n]$is the set of inactive neurons. Similarly, the set of all partial motifs is denoted by $\mathbb{PM}^n_K$. The sets $\mathbb{PW}^n_K$ and $\mathbb{PM}^n_K$ are naturally in a bijective correspondence with the sets $\mathbb{F}_2^{n-k}$ and $\mathbb{M}^{n-k}$, respectively.

(ii) In some cases, it is useful to deactivate a neuron or set of neurons. A neuron is said to be *deactivated* if it becomes inactive. For a word (or partial word) $\mathbf{w}$ or a motif (or partial motif) $\mathbf{a}$, we denote respectively $\mathbf{w}_K^-$ and $\mathbf{a}_K^-$ as the partial word and partial motif obtained by deactivating the neurons in K. If we deactivate a set of neurons for *every* word in a code $\mathcal{C}$, then we denote $\mathcal{C}_K^-$ as the partial code obtained by deactivating the neurons in K. The partial code $\mathcal{C}_K^-$ is naturally in a bijective correspondence with the code $\mathcal{C} \subseteq \mathbb{F}_2^{n-k}$, obtained by deleting the neurons from K.

Proposition 4.3. Let $K = \{i_1, \ldots, i_k\} \subseteq [n]$. For a code $\mathcal{C} \subseteq \mathbb{F}_2^n$, let $\mathbf{w} \in \mathbb{PW}^n_K$ and suppose $\mathbf{w} \notin \mathcal{C}_K^-$. If the motif $\mathbf{a}$ is obtained from $\mathbf{w}$ by replacing each $\_$ by $*$, then $\mathbf{a} \in \mathrm{Mot}({}^c\mathcal{C})$.

Proof. Easy to see. □

Because we want to better understand the non-polar motifs of $\mathcal{C}^{[p]}$, recall Definition 3.6 which defines the polarization of a motif in $\mathbb{M}^n$.

Definition 4.4. For a partial motif $\mathbf{a} = a_1 \cdots a_n \in \mathbb{PM}^n$, its polarization, denoted $\mathbf{a}^p = b_1 \cdots b_n c_1 \cdots c_n \in \mathbb{PM}^{2n}$, is defined in the following way:

$$\text{if } a_i = 0, \text{ then } b_i = 0,\ c_i = *;$$

$$\text{if } a_i = 1, \text{ then } b_i = *,\ c_i = 0;$$

$$\text{if } a_i = *, \text{ then } b_i = *,\ c_i = *;$$

$$\text{if } a_i = \_, \text{ then } b_i = \_,\ c_i = \_.$$

We say that a partial motif $\mathbf{b} \in \mathbb{PM}^{2n}$ is a *partial polar motif* if there is a partial motif $\mathbf{a} \in \mathbb{PM}^n$ such that $\mathbf{b} = \mathbf{a}^p$. If such a partial motif $\mathbf{a}$ exists, it is unique and we denote $\mathbf{a} = \mathbf{b}^d$ as the *depolarization of* $\mathbf{b}$.

Note that for any two partial motifs $\mathbf{a}, \mathbf{b} \in \mathbb{PM}^n$, we have

$$\mathbf{a}^{pd} = \mathbf{a} \text{ for every } \mathbf{a} \in \mathbb{PM}^n;$$

$$\mathbf{b}^{dp} = \mathbf{b} \text{ for every polar partial motif } \mathbf{b} \in \mathbb{PM}^{2n}.$$

The next theorem is a reformulation of Theorem 5.1 from [15], and we give a different, simpler proof.

Theorem 4.5. ([15, Theorem 5.1]) Let $\mathcal{C} \subseteq \mathbb{F}_2^n$ be a neural code and $\mathbf{c} \in \mathbb{M}^{2n}$ be a motif. Define the motif $\mathbf{a} = a_1 \cdots a_n a_{n+1} \cdots a_{2n}$ by replacing all the ones in $\mathbf{c}$ by $*$. Let K be the set of all such $i \in [n]$ such that $a_i = a_{i+n} = 0$ and $H = K \cup \{i + n : i \in K\}$. Then

$$\mathfrak{p}_{\mathbf{c}} \supseteq J_{\mathcal{C}^{[p]}} \quad \Leftrightarrow \quad \mathbf{a}_{\bar{H}}{}^d \in \mathrm{ParMot}(\mathcal{C}_{\bar{K}}).$$

Proof. Let $\mathcal{D} = {}^c\mathcal{C}$. Recall that ${}^c(\mathcal{C}^{[p]}) = \overline{\overline{\mathcal{D}}}^p$, and hence $\mathrm{MaxMot}({}^c(\mathcal{C}^{[p]})) = \{\overline{\overline{\mathbf{b}}}^p : \mathbf{b} \in \mathrm{MaxMot}(\mathcal{D})\}$. We have:

$$\begin{aligned}
\mathfrak{p}_{\mathbf{c}} \supseteq J_{\mathcal{C}^{[p]}} &\Leftrightarrow \mathbf{c} \in \mathrm{Mot}(\mathcal{C}^{[p]}) && \text{(Prop 2.22)}\\
&\Leftrightarrow \mathbf{c} \text{ and } \overline{\overline{\mathbf{b}}}^p \text{ are disjoint} && \text{(Prop 3.21)}\\
&\Leftrightarrow \mathbf{c} + \overline{\overline{\mathbf{b}}}^p \text{ has a 1-component.} && \text{(Remark 3.20)}
\end{aligned}$$

Since $\mathbf{a} \in \{0, *\}^{2n}$ and $\overline{\overline{\mathbf{b}}}^p \in \{1, *\}^{2n}$, that is equivalent to showing $\mathbf{a} + \overline{\overline{\mathbf{b}}}^p$ has a one 1-component, i.e.,

$$\mathbf{a} + \overline{\overline{\mathbf{b}}}^p < * \cdots * . \tag{4.1}$$

The statement of the theorem follows if we justify the claim that $\mathbf{a}$ satisfies Equation (4.1) if and only if $\mathbf{a}_{\bar{H}}{}^d \in \mathrm{ParMot}(\mathcal{C}_{\bar{K}})$. Clearly, if $\mathbf{a}$ does not satisfy Equation (4.1), then $\mathbf{a}$ and $\overline{\overline{\mathbf{b}}}^p$ are not disjoint, and thus $\mathbf{a}_{\bar{K}}{}^d \notin \mathrm{ParMot}(\mathcal{C}_{\bar{K}})$. On the other hand, suppose Equation (4.1) holds but $\mathbf{a}_{\overline{H}}{}^{d} \notin \mathrm{ParMot}(\mathcal{C}_{K}^{-})$. Then $\mathbf{a} \notin \mathrm{Mot}(\mathcal{C}^{[p]})$, i.e., $\mathbf{a} \in \mathrm{Mot}(\overline{\overline{\mathcal{D}}^{p}}) \subseteq \{1,*\}^{2n}$. But $\mathbf{a} \in \{0,*\}^{2n}$, thus $\mathbf{a} = *\cdots*$, contradicting that $\mathbf{a}$ satisfies Equation (4.1). $\square$

Example 4.6. In the context of Example 3.33, recall that $\mathcal{C} = \{000, 100, 110, 011\}$. To illustrate the previous theorem, consider the following for a motif $\mathbf{c} \in \mathbb{M}^{6}$.

(i) Consider $\mathbf{c} = 00{*}{*}0{*}$. Then $\mathbf{a} = \mathbf{c}$, $K = \{2\}$, and $H = \{2,5\}$ since $a_2 = a_5 = 0$. Then

$$\mathbf{a}_{\overline{H}}{}^{d} = 0\,\_{*}{*}\_\,{*}^{d} = 0\_{*}\,.$$

Thus $\mathcal{C}_{K}^{-} = \{0\_0, 1\_0, 0\_1\} = V_{*\_0} \cup V_{0\_*}$, and hence $\mathbf{a}_{\overline{H}}{}^{d} \in \mathrm{ParMot}(\mathcal{C}_{K}^{-})$ and $\mathfrak{p}_{\mathbf{c}} \supseteq J_{\mathcal{C}^{[p]}}$. In fact, we have that $\mathbf{a}_{\overline{H}}{}^{d} \in \mathrm{MaxParMot}(\mathcal{C}_{K}^{-})$ and $\mathfrak{p}_{\mathbf{c}} \in \mathrm{Min}(J_{\mathcal{C}^{[p]}})$.

(ii) Now consider $\mathbf{c} = 0{*}0{*}{*}0$. Then $\mathbf{a} = \mathbf{c}$, $K = \{3\}$, and $H = \{3,6\}$ since $a_3 = a_6 = 0$. Then

$$\mathbf{a}_{\overline{H}}{}^{d} = 0{*}\_{*}{*}\_^{\,d} = 0{*}\_\,.$$

Thus $\mathcal{C}_{K}^{-} = \{00\_, 10\_, 11\_, 01\_\} = V_{**\_}$, and hence $\mathbf{a}_{\overline{H}}{}^{d} \in \mathrm{ParMot}(\mathcal{C}_{K}^{-})$ and $\mathfrak{p}_{\mathbf{c}} \supseteq J_{\mathcal{C}^{[p]}}$. Notice that $\mathbf{a}_{\overline{H}}{}^{d} \notin \mathrm{MaxParMot}(\mathcal{C}_{K}^{-})$ and $\mathfrak{p}_{\mathbf{c}} \notin \mathrm{Min}(J_{\mathcal{C}^{[p]}})$.

(iii) Lastly, consider $\mathbf{c} = 100{*}0{*}$. Then $\mathbf{a} = {*}00{*}0{*}$, $K = \{2\}$, and $H = \{2,5\}$ since $a_2 = a_5 = 0$. Then, as before,

$$\mathbf{a}_{\overline{H}}{}^{d} = 0\_{*}{*}\_\,{*}^{d} = 0\_{*}\,.$$

Thus $\mathcal{C}_{K}^{-} = \{0\_0, 1\_0, 0\_1\} = V_{*\_0} \cup V_{0\_*}$, and hence $\mathbf{a}_{\overline{H}}{}^{d} \in \mathrm{ParMot}(\mathcal{C}_{K}^{-})$ and $\mathfrak{p}_{\mathbf{c}} \supseteq J_{\mathcal{C}^{[p]}}$. However, in this case, $\mathfrak{p}_{\mathbf{c}} = (1 - X_1, X_2, X_3, Y_2) \notin \mathrm{Min}(J_{\mathcal{C}^{[p]}})$ even though $\mathbf{a}_{\overline{H}}{}^{d} \in \mathrm{MaxParMot}(\mathcal{C}_{K}^{-})$. Indeed, it was shown in Example 3.33 that (X_2, X_3) is a minimal prime of $J_{\mathcal{C}^{[p]}}$.

Thus we have similar conclusions about the relationship between partial motivic primes containing the neural ideal, as related to Equation 2.4, however, we cannot say anything about the *minimal* motivic primes such as Equation 2.5 for *maximal* partial motifs.

CHAPTER 5
MORPHISMS OF NEURAL CODES

As much of this chapter focuses on the category of Neural Codes, we briefly introduce some category theory. The concept of a category is basically a class of objects (usually sets) and a class of morphisms between them (usually maps between those sets) [16]. A category is called *small* if the class of objects is a set. However, when there are too many objects in a class, we can assume that those objects form their own class. In that way, we avoid certain paradoxes of set theory.

Definition 5.1. ([16]) A *category* $\mathbf{C}$ consists of:

1. A class ob $\mathbf{C}$ of *objects* (usually denoted A, B, C, etc.).

2. For each ordered pair of objects (A, B), a set $\mathrm{Hom}_{\mathbf{C}}(A, B)$ (or simply $\mathrm{Hom}(A, B)$ if $\mathbf{C}$ is clear) whose elements are called *morphisms* with *domain* A and *codomain* B (or *from* A *to* B).

3. For each ordered triple of objects (A, B, C), a map $(f, g) \rightsquigarrow g \circ f$ of the product set $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C)$ into $\mathrm{Hom}(A, C)$.

It is assumed that the objects and morphisms satisfy the following conditions:

C1. If $(A, B) \neq (C, D)$, then $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(C, D)$ are disjoint.

C2. (Associativity). If $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, and $h \in \mathrm{Hom}(C, D)$, then $(h \circ g) \circ f = h \circ (g \circ f) = h \circ g \circ f$.

C3. (Unit). For every object A we have a unique element $\mathrm{Id}_A \in \mathrm{Hom}(A, A)$ such that $f \circ \mathrm{Id}_A = f$ for every $f \in \mathrm{Hom}(A, B)$ and $\mathrm{Id}_A \circ g = g$ for every $g \in \mathrm{Hom}(B, A)$.

Definition 5.2. ([16]) An element $f \in \mathrm{Hom}(A, B)$ is called an *isomorphism* if there exists a $g \in \mathrm{Hom}(B, A)$ such that $f \circ g = \mathrm{Id}_B$ and $g \circ f = \mathrm{Id}_A$. It is clear that g is uniquely determined by f, so we can denote it as f^{-1}. This is also an isomorphism and $(f^{-1})^{-1} = f$. If f and h are isomorphisms and $f \circ h$ is defined, then $f \circ h$ is an isomorphism and $(f \circ h)^{-1} = h^{-1} \circ f^{-1}$.

Definition 5.3. ([16]) If $\mathbf{C}$ and $\mathbf{D}$ are categories, a *contravariant functor* F from $\mathbf{C}$ to $\mathbf{D}$ consists of

1. A map $A \rightsquigarrow FA$ of *ob* $\mathbf{C}$ into *ob* $\mathbf{D}$.
2. For every pair of objects (A, B) of $\mathbf{C}$, a map $f \rightsquigarrow F(f)$ of $\mathit{Hom}_{\mathbf{C}}(A, B)$ into $\mathit{Hom}_{\mathbf{D}}(FB, FA)$.

We require that these satisfy the following conditions:

F1. If $g \circ f$ is defined in $\mathbf{C}$, then $F(g \circ f) = F(f) \circ F(g)$.

F2. $F(\mathrm{Id}_A) = \mathrm{Id}_{FA}$.

Definition 5.4. ([16]) A contravariant functor is called *faithful* (*full*) if for every pair of objects A, B in $\mathbf{C}$ the map $f \rightsquigarrow F(f)$ of $\mathrm{Hom}_{\mathbf{C}}(A, B)$ into $\mathrm{Hom}_{\mathbf{D}}(FB, FA)$ is injective (surjective).

Definition 5.5. ([16]) We say that the categories $\mathbf{C}$ and $\mathbf{D}$ are *isomorphic* (or *equivalent*) if there exist functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that $GF = \mathrm{Id}_{\mathbf{C}}$ and $FG = \mathrm{Id}_{\mathbf{D}}$.

5.1 Morphisms Between Neural Rings

Recall from Chapter 2 that the neural ring of $\mathcal{C} \subseteq \mathbb{F}_2^n$ is defined to be the ring

$$R_{\mathcal{C}} = \frac{\mathbb{F}_2[X_1, \dots, X_n]}{\mathcal{I}(\mathcal{C})} = \mathbb{F}_2[x_1, \dots, x_n],$$

where $x_i = X_i + \mathcal{I}(\mathcal{C})$ for $i \in [n]$. We denote the image of $f \in \mathbb{F}_2[X_1, \dots, X_n]$ under the canonical map $\mathbb{F}_2[X_1, \dots, X_n] \to R_{\mathcal{C}}$ by $\overline{f}$ or $\overline{f}(x_1, \dots, x_n)$. In particular, the image of the Lagrange polynomial $L_{\mathbf{w}}$ is denoted by $\overline{L_{\mathbf{w}}}$ or $\overline{L_{\mathbf{w}}}(x_1, \dots, x_n)$. For $A \subseteq \mathcal{C}$ we denote by L_A the polynomial $\sum_{\mathbf{w} \in A} L_{\mathbf{w}}$. It turns out that $R_{\mathcal{C}}$ consists of all $\overline{L_A}$, $A \subseteq \mathcal{C}$, and that they are all distinct. Moreover, if we denote by $\mathcal{P}(\mathcal{C})$ the power set of $\mathcal{C}$, then the bijection $R_{\mathcal{C}} \to (\mathcal{P}(\mathcal{C}), \triangle, \cap)$, given by

$$\overline{L_A} \mapsto A,$$

is a ring isomorphism. For the purpose of this chapter we call the ring $(\mathcal{P}(\mathcal{C}), \triangle, \cap)$ the *neural ring* of $\mathcal{C}$.

Definition 5.6. ([18]) Let $\mathcal{C} \subseteq \mathbb{F}_2^n$ be a code of length n and $\alpha \subseteq [n]$. Then the subset of $\mathcal{C}$,

$$\mathrm{Tk}_{\alpha}^{\mathcal{C}} = \{\mathbf{w} = w_1 \cdots w_n \in \mathcal{C} \mid w_i = 1 \text{ for all } i \in \alpha\},$$

is called the *trunk* of $\mathcal{C}$ determined by α. In particular, $\mathrm{Tk}_{\emptyset}^{\mathcal{C}} = \mathcal{C}$. If $|\alpha| = 1$, $\mathrm{Tk}_{\alpha}^{\mathcal{C}}$ is called a *simple trunk* of $\mathcal{C}$. We will write $\mathrm{Tk}_i^{\mathcal{C}}$ instead of $\mathrm{Tk}_{\{i\}}^{\mathcal{C}}$.

Definition 5.7. The *trunk of the RF cover* $\mathcal{U} = U_1, \dots, U_n$ of X, corresponding to $\alpha \subseteq [n]$, is the set

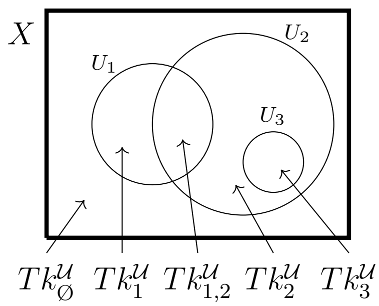
$$Tk_{\alpha}^{\mathcal{U}} = \bigcap_{i \in \alpha} U_i.$$

Examples of the trunks of RF covers are given in Figure 5.1. Note that $Tk_{\emptyset}^{\mathcal{U}} = \bigcap_{i \in \emptyset} U_i = X$.

In the next theorem we give an intrinsic characterization of neural rings. The inspiration for this theorem is coming from [12, Theorem 1.2], where neural rings

Figure 5.1: Examples of Trunks of RF Covers.

Example 1:



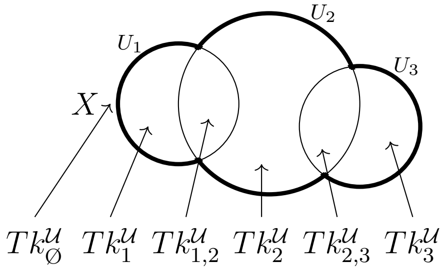
$\mathcal{U} = U_1, U_2, U_3$

Here:

$Tk^{\mathcal{U}}_{1,3} = Tk^{\mathcal{U}}_{1,2,3} = \emptyset$

$Tk^{\mathcal{U}}_{3} = Tk^{\mathcal{U}}_{2,3}$

Example 2:



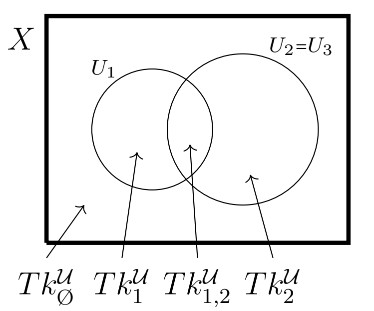
$X = U_1 \cup U_2 \cup U_3$

$\mathcal{U} = U_1, U_2, U_3$

Here:

$Tk^{\mathcal{U}}_{1,3} = Tk^{\mathcal{U}}_{1,2,3} = \emptyset$

Example 3:



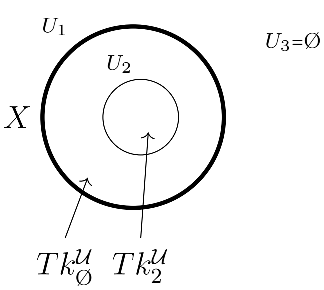
$\mathcal{U} = U_1, U_2, U_3$

Here:

$Tk^{\mathcal{U}}_{2} = Tk^{\mathcal{U}}_{3} = Tk^{\mathcal{U}}_{2,3}$

$Tk^{\mathcal{U}}_{1,2} = Tk^{\mathcal{U}}_{1,3} = Tk^{\mathcal{U}}_{1,2,3}$

Example 4:



$X = U_1 \cup U_2 \cup U_3$

$\mathcal{U} = U_1, U_2, U_3$

Here:

$Tk^{\mathcal{U}}_{3} = Tk^{\mathcal{U}}_{1,3} = Tk^{\mathcal{U}}_{2,3} = Tk^{\mathcal{U}}_{1,2,3} = \emptyset$

$Tk^{\mathcal{U}}_{\emptyset} = Tk^{\mathcal{U}}_{1}$

$Tk^{\mathcal{U}}_{2} = Tk^{\mathcal{U}}_{1,2}$

on n neurons (as modules) were characterized in terms of the actions of the neural ring of the full code. The part of our proof in which we construct the code $\mathcal{C}$ follows the proof of Theorem 1.2 from [12].

Theorem 5.8. A non-zero commutative ring R is isomorphic to the neural ring of some neural code $\mathcal{C}$ if and only if there is a nonempty subset $S = \{s_1, \dots, s_r\}$ of R and a sequence $T = t_1, \dots, t_n$ $(n \geq 1)$ of elements of R such that the following conditions hold:

(N1) Every element $x \in R$ can be uniquely written as a sum $x = s_{j_1} + \cdots + s_{j_p}$ $(p \geq 0)$ of distinct elements of S.

(N2) For any t_i from T and any $s_j \in S$, we have that $t_i s_j \in \{0, s_j\}$.

(N3) For any two distinct elements $s_j, s_k \in S$ there is at least one element t_i from T such that exactly one of the elements $t_i s_j$, $t_i s_k$ is equal to 0.

Moreover, given a non-zero commutative ring R with the properties (N1), (N2), (N3) satisfied by its subset S and a sequence of its elements T, the code $\mathcal{C}$ and the isomorphism $\phi : R \to \mathcal{P}(\mathcal{C})$ can be selected in such a way that the elements of S correspond to the words of $\mathcal{C}$ (as singletons) and the elements of T to the simple trunks of $\mathcal{C}$.

Proof. Let $\mathcal{C}$ be a neural code on n neurons, consisting of r codewords $\mathbf{w}^1, \dots, \mathbf{w}^r$, and let $(\mathcal{P}(\mathcal{C}), \triangle, \cap)$ be its neural ring. Let $s_j = \{\mathbf{w}^j\}$ $(j \in [r])$, $S = \{s_1, \dots, s_r\}$, $t_i = \mathrm{Tk}_i^{\mathcal{C}}$ $(i \in [n])$, $T = t_1, \dots, t_n$. Then for each $X = \{\mathbf{w}^{j_1}, \dots, \mathbf{w}^{j_p}\} \in \mathcal{P}(\mathcal{C})$ the unique way to write X as a "sum" (i.e., symmetric difference) of elements s_j is $X = \{\mathbf{w}^{j_1}\} \triangle \cdots \triangle \{\mathbf{w}^{j_p}\}$. Thus the condition (N1) holds for $\mathcal{P}(\mathcal{C})$. Also for each $i \in [n]$ and $j \in [r]$ we have $\mathrm{Tk}_i^{\mathcal{C}} \cap \{\mathbf{w}^j\} \in \{\varnothing, \{\mathbf{w}^j\}\}$, so that the condition (N2) holds for $\mathcal{P}(\mathcal{C})$. Finally, let $\{\mathbf{w}^j\}$, $\{\mathbf{w}^k\}$ be two distinct elements of S. Let i be a coordinate on which one of $\mathbf{w}^j$, $\mathbf{w}^k$ has 0 and the other one 1. Then exactly one of $\mathrm{Tk}_i^{\mathcal{C}} \cap \{\mathbf{w}^j\}$, $\mathrm{Tk}_i^{\mathcal{C}} \cap \{\mathbf{w}^k\}$ is $\varnothing$. Thus the condition (N3) holds for $\mathcal{P}(\mathcal{C})$.

Conversely, suppose that we have a non-zero commutative ring R which has a subset S and a sequence of its elements T satisfying the conditions (N1), (N2), and (N3).

Claim 1. No element of S is equal to 0.

Proof. Suppose $0 \in S$. If $S = \{0\}$, then, by (N1), $R = \{0\}$, a contradiction. Suppose $S \neq \{0\}$ and let $s \neq 0$ be a non-zero element of S. Then s and $s + 0$ are two different ways to write an element of R as a sum of distinct elements of S, a contradiction. Claim 1 is proved.

Claim 2. If $1 \in S$, then $S = \{1\}$.

Proof. Suppose $1 \in S$ and $S \neq \{1\}$. Let $s \in S$, $s \neq 1$. Then, by (N3), there is a t from the sequence T such that exactly one $t1$, ts is equal to 0. If $t1 = 0$, then, by (N2) and (N3), $ts = s$. However, $t1 = 0$ implies $t = 0$, hence $ts = 0$. Hence $s = 0$, contradicting Claim 1. The other option is that $ts = 0$. Then, by (N2) and (N3), $t1 = 1$, hence $t = 1$, hence $0 = ts = s$, again contradicting Claim 1. Claim 2 is proved.

Proof for the case $S = \{1\}$. Suppose $S = \{1\}$. Then $R = \{0, 1\}$. Hence each t_i is either 0 or 1. We form a codeword $\mathbf{w} = w_1 \cdots w_n \in \mathbb{F}_2^n$ in the following way: if $t_i = 0$, we put $w_i = 0$, and if $t_i = 1$, we put $w_i = 1$. Let $\mathcal{C} = \{\mathbf{w}\}$. Then $\mathcal{P}(\mathcal{C}) = \{\varnothing, \{\mathbf{w}\} = \mathcal{C}\}$. The map $\phi : R \to \mathcal{P}(\mathcal{C})$, defined by $\phi(0) = \varnothing$, $\phi(1) = \mathcal{C}$, is a ring isomorphism. We also have $\phi(t_i) = \varnothing = \mathrm{Tk}_i^{\mathcal{C}}$ if $t_i = 0$, and $\phi(t_i) = \mathcal{C} = \mathrm{Tk}_i^{\mathcal{C}}$ if $t_i = 1$. The proof for the case $S = \{1\}$ is finished.

From now on we assume that $1 \notin S$. Equivalently, $|S| \geq 2$ (due to Claim 2 and the fact that 1 is representable as a sum of distinct elements of S).

Claim 3. For any two distinct elements $s_j, s_k \in S$, $s_j s_k = 0$.

Proof. Let s_j, s_k be two distinct elements of S. By (N3) there is an element t_i from

T such that exactly one of the elements $t_i s_j$, $t_i s_k$ is 0. Say $t_i s_j = 0$. Then, by (N2) and (N3), $t_i s_k = s_k$. Now $t_i s_j s_k = (t_i s_j) s_k = 0 s_k = 0$, and $t_i s_j s_k = s_j (t_i s_k) = s_j s_k$. Hence $s_j s_k = 0$. Claim 3 is proved.

Claim 4. For any element $s_j \in S$, $s_j s_j = s_j$.

Proof. Let $1 = s_{j_1} + \cdots + s_{j_p}$ $(p \geq 2)$ be the unique representation of 1 as a sum of distinct elements of S. If $p < |S|$, then there is an $s_j \in S$ not participating in the representation of 1. Multiplying the representation of 1 by s_j and using Claim 3, we get $s_j = 0$, contradicting to Claim 1. Hence $p = |S|$, i.e., $1 = s_1 + \cdots + s_r$. Now for any $j \in [r]$, when we multiply this representation of 1 by s_j, we get (using Claim 3) that $s_j = s_j s_j$. Claim 4 is proved.

Claim 5. For any element $s_j \in S$, $s_j + s_j = 0$.

Proof. Note that $s_j + s_j \neq s_j$, otherwise, by cancellation, $s_j = 0$, contradicting Claim 1. Suppose that $s_j + s_j = s_j + s_{j_1} + \cdots + s_{j_p}$ with $p \geq 1$ and all s_{j_μ} $(\mu \in [p])$ different than s_j. Cancelling s_j we get $s_j = s_{j_1} + \cdots + s_{j_p}$, contradicting to (N1). Suppose now that $s_j + s_j = s_{j_1} + \cdots + s_{j_p}$ with $p \geq 1$ and all s_{j_μ} $(\mu \in [p])$ different than s_j. If we multiply this equality by s_{j_1} and use the claims 3 and 4, we get $s_{j_1} = 0$, contradicting Claim 1. The only remaining option is $s_j + s_j = 0$. Claim 5 is proved.

Proof for the case $S \neq \{1\}$ (i.e., $|S| \geq 2$). For every element $s \in S$ we construct a word $\mathbf{w} = w_1 \cdots w_n \in \mathbb{F}_2^n$ in the following way: for $i \in [n]$, if $t_i s = 0$ we put $w_i = 0$, otherwise (if $t_i s = s$) we put $w_i = 1$. In that way we get r words $\mathbf{w}^1, \ldots, \mathbf{w}^r$ from $\mathbb{F}_2^n$, corresponding, respectively, to $s_1, \ldots, s_r$. Let $\mathcal{C} = \{\mathbf{w}^1, \ldots, \mathbf{w}^r\}$. For every $x \in R$, if $x = s_{j_1} + \cdots + s_{j_p}$ is the unique representation of x as a sum of distinct elements of S, we define

$$S(x) = \{s_{j_1}, \ldots, s_{j_p}\} \subseteq R,$$

$$W(x) = \{\mathbf{w}^{j_1}, \ldots, \mathbf{w}^{j_p}\} \subseteq \mathcal{C}.$$

Note that for any $x, y \in R$ we have

$$S(x+y) = S(x) \triangle S(y)$$

due to Claim 5, and

$$S(xy) = S(x) \cap S(y)$$

due to the claims 3 and 4. Hence

$$W(x+y) = W(x) \triangle W(y), \tag{5.1}$$

$$W(xy) = W(x) \cap W(y). \tag{5.2}$$

Note also that if $x = 0$, $S(x) = \emptyset$, hence

$$W(0) = \emptyset,$$

and if $x = 1$, $S(x) = S$ by the proof of Claim 4, hence

$$W(1) = \mathcal{C}. \tag{5.3}$$

Now we define a map $\phi : R \to \mathcal{C}$ as $\phi(x) = W(x)$ for any $x \in R$. The relations (5.1), (5.2), and (5.3) show that ϕ is a ring homomorphism. Also

$$\phi(s_j) = \{\mathbf{w}^j\} \text{ for every } j \in [n].$$

It remains to find $\phi(t_i)$ for each $i \in [n]$. Fix an $i \in [n]$. Let $t_i = s_{j_1} + \cdots + s_{j_p}$ $(p \geq 0)$ be the unique representation of the element t_i as a sum of distinct element of S. Multiplying this representation by s_{j_μ} $(\mu \in [p])$ and using the claims 3 and 4 we conclude that

$$t_i s_{j_\mu} = s_{j_\mu} \ (\mu \in [p]). \tag{5.4}$$

We claim that

$$t_i s_j = 0 \text{ for any } s_j \in S \setminus \{s_{j_1}, \ldots, s_{j_p}\}. \tag{5.5}$$

Suppose to the contrary, i.e., $t_i s_j = s_j$ for some $s_j \in S \smallsetminus \{s_{j_1}, \dots, s_{j_p}\}$. Then, by Claim 3, $s_j = t_i s_j = (s_{j_1} + \dots + s_{j_p}) s_j = 0$, a contradiction. Thus

$$\phi(t_i) = \{\mathbf{w}^{j_1}, \dots, \mathbf{w}^{j_p}\},$$

which is precisely the set of all the words from $\mathcal{C}$ that have the i-th coordinate equal to 1 (due to (5.4), (5.5), and the way the code $\mathcal{C}$ is constructed). Thus

$$\phi(t_i) = \mathrm{Tk}_i^{\mathcal{C}} \text{ for all } i \in [n]. \qquad \square$$

Next we give an intrinsic characterization of homomorphisms between neural rings.

Theorem 5.9. Let $\mathcal{C}, \mathcal{D}$ be two codes. A map $\phi : (\mathcal{P}(\mathcal{D}), \triangle, \cap) \to (\mathcal{P}(\mathcal{C}), \triangle, \cap)$ is a ring homomorphism if and only if the following three conditions hold:

(H1) $\phi(\{\mathbf{v}^1\}) \cap \phi(\{\mathbf{v}^2\}) = \emptyset$ for any $\mathbf{v}^1, \mathbf{v}^2 \in \mathcal{D}$.

(H2) $(\forall B \subseteq \mathcal{D})\ \phi(B) = \bigcup_{\mathbf{v} \in B} \phi(\{\mathbf{v}\})$.

(H3) $\phi(\mathcal{D}) = \mathcal{C}$.

Proof. Suppose that $\phi : (\mathcal{P}(\mathcal{D}), \triangle, \cap) \to (\mathcal{P}(\mathcal{C}), \triangle, \cap)$ is a ring homomorphism. Then for two distinct elements $\mathbf{v}^1, \mathbf{v}^2$ of $\mathcal{D}$ we have:

$$\begin{aligned} \emptyset &= \phi(\emptyset) \\ &= \phi(\{\mathbf{v}^1\} \cap \{\mathbf{v}^2\}) \\ &= \phi(\{\mathbf{v}^1\}) \cap \phi(\{\mathbf{v}^2\}). \end{aligned}$$

Thus (H1) holds.

We show (H2) by induction on $|B|$. For $|B| = 1$ the statement is true. Suppose that (H2) holds when $|B| = k$ and suppose that $|B| = k+1$. Let $B = B' \cup \{\mathbf{w}\}$, where

$|B'| = k$. Then

$$\begin{aligned}
\phi(B) &= \phi(B' \cup \{\mathbf{w}\}) \\
&= \phi(B' \triangle \{\mathbf{w}\}) \\
&= \phi(B') \triangle \phi(\{\mathbf{w}\}) \\
&= \Big(\bigcup_{\mathbf{v}' \in B'} \phi(\{\mathbf{v}'\}) \Big) \triangle \phi(\{\mathbf{w}\}) \\
&= \Big(\bigcup_{\mathbf{v}' \in B'} \phi(\{\mathbf{v}'\}) \Big) \cup \phi(\{\mathbf{w}\}) \\
&= \bigcup_{\mathbf{v} \in B} \phi(\{\mathbf{v}\}).
\end{aligned}$$

Thus (H2) holds.

Finally $\phi(\mathcal{D}) = \mathcal{C}$ as the identity element has to be mapped to the identity element. Thus (H3) holds.

Conversely, suppose that $\phi : (\mathcal{P}(\mathcal{D}), \triangle, \cap) \to (\mathcal{P}(\mathcal{C}), \triangle, \cap)$ is a map satisfying the conditions (H1), (H2), and (H3). Let $B_1, B_2 \in \mathcal{P}(\mathcal{D})$. We have:

$$\begin{aligned}
\phi(B_1 \triangle B_2) &= \bigcup_{\mathbf{v} \in B_1 \triangle B_2} \phi(\{\mathbf{v}\}) \\
&= \bigcup_{\mathbf{v} \in B_1} \phi(\{\mathbf{v}\}) \triangle \bigcup_{\mathbf{v} \in B_2} \phi(\{\mathbf{v}\}) \\
&= \phi(B_1) \triangle \phi(B_2).
\end{aligned}$$

We used here the conditions (H1) nd (H2). In the same way we get $h(B_1 \cap B_2) = \phi(B_1) \cap \phi(B_2)$. Finally the condition $\phi(\mathcal{D}) = \mathcal{C}$ is postulated. Thus ϕ is a ring homomorphism. □

Proposition 5.10. ([12, Theorem 1.1]) Let $\mathcal{C}, \mathcal{D}$ be two codes. There is a bijective correspondence between the set of code maps $q : \mathcal{C} \to \mathcal{D}$ and the set of ring homomorphisms $\mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C})$. It is given by associating to each code map $q : \mathcal{C} \to \mathcal{D}$ the homomorphism $q^{-1} : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C})$ and, conversely, by associating to each ring homomorphism $\phi : \mathcal{P}(\mathcal{D} \to \mathcal{P}(\mathcal{C})$ the unique code map $q = \phi_* : \mathcal{C} \to \mathcal{D}$ such that $\phi = q^{-1}$.

We say that the code map q and the ring homomorphism q^{-1}, and the ring homomorphism ϕ and the code map ϕ_*, are *associated* to each other.

5.2 Monomial Morphisms Between Neural Rings

Definition 5.11. Let Codes be the set of all neural codes $\mathcal{C} \subseteq \mathbb{F}_2^n$ of all lengths $n \geq 2$. We call any map $q : \mathcal{C} \to \mathcal{C}'$, where $\mathcal{C}, \mathcal{C}' \in$ Codes, a *code map*. The set Codes, together with code maps as morphisms, forms a small category, which we denote by **Code**.

Definition 5.12. ([12, Section 1.5]) The following maps between the objects of the category **Code** are called *basic linear monomial maps*. Let $\mathcal{C}, \mathcal{C}' \in$ Codes where $\mathcal{C}'$ is the image of $\mathcal{C}$ under each map, and $i \in [n]$.

(1) $\mathrm{acz}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}'$, "adding constant zero", defined by $\mathbf{w} \mapsto \mathbf{w}0$ for all $\mathbf{w} \in \mathcal{C}$;

(2) $\mathrm{aco}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}'$, "adding constant one", defined by $\mathbf{w} \mapsto \mathbf{w}1$ for all $\mathbf{w} \in \mathcal{C}$;

(3) $\mathrm{del}_{\mathcal{C},i} : \mathcal{C} \to \mathcal{C}'$, "deleting the i-th neuron", defined by $\mathbf{w} \mapsto w_1 \cdots \widehat{w_i} \cdots w_n$ for all $\mathbf{w} \in \mathcal{C}$ (here the notation $\widehat{w_i}$ means that the i-th component of $\mathbf{w}$ is omitted);

(4) $\mathrm{rep}_{\mathcal{C},i} : \mathcal{C} \to \mathcal{C}'$, "repeating the i-th neuron", defined by $\mathbf{w} \mapsto \mathbf{w}w_i$ for all $\mathbf{w} \in \mathcal{C}$;

(5) $\mathrm{per}_{\mathcal{C},\sigma} : \mathcal{C} \to \mathcal{C}'$, "permuting the indices", defined by $\mathbf{w} \mapsto w_{\sigma(1)} \cdots w_{\sigma(n)}$ for all $\mathbf{w} \in \mathcal{C}$, where $\sigma \in S_n$;

(6) $\mathrm{inj}_{\mathcal{C}',\mathcal{C}} : \mathcal{C} \to \mathcal{C}'$, "injecting the code into a bigger code", defined by $\mathbf{w} \mapsto \mathbf{w}$ for all $\mathbf{w} \in \mathcal{C}$, where $\mathcal{C} \subseteq \mathcal{C}'$.

We extend the previous definition and introduce the notion of *basic monomial maps* by including all the basic linear monomial maps and adding one new map that we call *adding trunk neuron*.

Definition 5.13. The maps (1) - (6) and the following map (7) are called *basic monomial maps*:

(7) $\mathrm{atn}_{\mathcal{C},\alpha} : \mathcal{C} \to \mathcal{C}'$, "adding trunk neuron", defined by

$$\mathrm{atn}_{\mathcal{C},\alpha}(\mathbf{w}) = \begin{cases} \mathbf{w}1, & \text{if } \mathbf{w} \in \mathrm{Tk}_\alpha^{\mathcal{C}} \\ \mathbf{w}0, & \text{if } \mathbf{w} \notin \mathrm{Tk}_\alpha^{\mathcal{C}}, \end{cases}$$

for all $\mathbf{w} \in \mathcal{C}$, where $\mathcal{C} \in \mathrm{Codes}$ and $\alpha \subseteq [n]$.

Remark 5.14. We will write $\mathrm{atn}_{\mathcal{C},i}$ instead of $\mathrm{atn}_{\mathcal{C},\{i\}}$. Note that

$$\mathrm{atn}_{\mathcal{C},i} = \mathrm{rep}_{\mathcal{C},i}$$

and

$$\mathrm{atn}_{\mathcal{C},\varnothing} = \mathrm{aco}_{\mathcal{C}}.$$

Also, if $\mathbf{1} = 11\cdots1 \notin \mathcal{C}$, then

$$\mathrm{atn}_{\mathcal{C},[n]} = \mathrm{acz}_{\mathcal{C}}.$$

Proposition 5.15. Let $\mathcal{C}$ be a code on the neurons $[n] = 1, \ldots, n$, and $\mathrm{VR}(\mathcal{C}) = (X, \mathcal{U})$ its visual realization. Let $\mathcal{C}, \mathcal{C}' \in \mathrm{Codes}$ where $\mathcal{C}'$ is the image of $\mathcal{C}$ under each map, and $i \in [n]$.

(i) For $\mathrm{acz}_{\mathcal{C}}$, then one of its visual realizations is the pair $\mathrm{VR}(\mathcal{C}') = (X', \mathcal{U}')$ defined by: $U_i' = U_i$, $U_{n+1}' = \varnothing$, $\mathcal{U}' = \mathcal{U} \cup U_{n+1}'$, and $X' = X$. In particular, $\mathrm{odim}(\mathcal{C}') = \mathrm{odim}(\mathcal{C})$.

(ii) For $\mathrm{del}_{\mathcal{C},i}$, then one of its visual realizations is the pair $\mathrm{VR}(\mathcal{C}') = (X', \mathcal{U}')$ defined by: $U_j' = U_j$ for $j \neq i$, $\mathcal{U}' = \mathcal{U} \setminus U_i$, and $X' = X$. In particular, $\mathrm{odim}(\mathcal{C}') \leq \mathrm{odim}(\mathcal{C})$.

(iii) For $\mathrm{per}_{\mathcal{C},\sigma}$, then one of its visual realizations is the pair $\mathrm{VR}(\mathcal{C}') = (X', \mathcal{U}')$ defined by: $U_i' = U_{\sigma(i)}$, $\mathcal{U}' = \mathcal{U}$, and $X' = X$. In particular, $\mathrm{odim}(\mathcal{C}') = \mathrm{odim}(\mathcal{C})$.

(iv) For $\mathrm{atn}_{\mathcal{C},\alpha}$, then one of its visual realizations is the pair $\mathrm{VR}(\mathcal{C}') = (X',\mathcal{U}')$ defined by: $U_i' = U_i$, $U_{n+1}' = \mathrm{Tk}_{\alpha}^{\mathcal{C}}$, $\mathcal{U}' = \mathcal{U} \cup U_{n+1}'$, and $X' = X$. In particular, $\mathrm{odim}(\mathcal{C}') = \mathrm{odim}(\mathcal{C})$.

If $\mathcal{C}$ is open-convexly realizable, then in each of the above four cases, $\mathcal{C}'$ is open-convexly realizable in the Euclidean space $\mathbb{R}^d$ of equal or smaller dimension.

Proof. The cases (i), (iii), and (iv) are clear. In case (ii), if $U_i \subseteq \bigcup_{j\neq i} U_j$, then $U_j' = U_j$ for $j \neq i$ and $X' = X$ in each of the cases $X = \cup U$ and $X \supset \cup U$. If $U_i \setminus \left(\bigcup_{j\neq i} U_j\right) \neq \emptyset$, we can clearly take $X' = X$ if $X \supset \cup U$. However, we can take $X' = X$ in the case $X = \cup U$ as well since we have $\mathbf{0} \in \mathcal{C}'$, and the atom $A_i^{(X,\mathcal{U})}$ becomes $A_{\emptyset}^{(X',\mathcal{U}')}$ (corresponding to $\mathbf{0} \in \mathcal{C}'$). □

Definition 5.16. Let $\mathcal{C} \in \mathrm{Codes}$. We say that the commutative ring $(\mathcal{P}(\mathcal{C}), \triangle, \cap)$ is the *neural ring* of $\mathcal{C}$. We denote

$$\mathrm{NRings} = \{(\mathcal{P}(\mathcal{C}), \triangle, \cap) \mid \mathcal{C} \in \mathrm{Codes}\}$$

and call this set the *set of all neural rings*.

Definition 5.17. The set NRings, together with ring homomorphisms as morphisms, forms a small category, which we denote by **NRing**.

Proposition 5.18. Consider the categories **Code** and **NRing**. If to each code $\mathcal{C} \in \mathrm{Codes}$ we associate its neural ring $F(\mathcal{C}) = (\mathcal{P}(\mathcal{C}), \triangle, \cap)$ and to each code map $q : \mathcal{C} \to \mathcal{D}$ the homomorphism of neural rings $F(q) = q^{-1} : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C})$, then in this way we obtain a functor $F : \mathbf{Code} \to \mathbf{NRing}$, which is an isomorphism of these categories.

Proof. It is easy to verify that F is a functor between these categories. The fact that F is an isomorphism follows easily from Proposition 5.10. □

Definition 5.19. ([12, Section 1.5]) A map $q : \mathcal{C} \to \mathcal{C}'$, where $\mathcal{C}, \mathcal{C}' \in$ Codes, is called a *linear monomial map* if the inverse image under q of every simple trunk of $\mathcal{D}$ is either a simple trunk of $\mathcal{C}$, or the empty set, or $\mathcal{C}$.

Theorem 5.20. ([12, Theorem 1.4]) A map $q : \mathcal{C} \to \mathcal{C}'$, where $\mathcal{C}, \mathcal{C}' \in$ Codes, is a *linear monomial map* if and only if it is the a composition of finitely many basic linear monomial maps.

Definition 5.21. ([18, Definition 2.6]) A map $q : \mathcal{C} \to \mathcal{C}'$, where $\mathcal{C}, \mathcal{C}' \in$ Codes, is called a *monomial map* if the inverse image under q of every simple trunk of $\mathcal{D}$ is either a trunk of $\mathcal{C}$, or the empty set, or $\mathcal{C}$.

Proposition 5.22. (a) Every linear monomial map is a monomial map.

(b) Every basic monomial map is a monomial map.

(c) A composition of two monomial maps is a monomial map.

(d) For any code $\mathcal{C}$ the identity map $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a monomial map.

Proof. (a) Follows from the definitions.

(b) Basic linear monomial maps are linear monomial maps by [12], hence monomial maps. Consider the map $f = \mathrm{atn}_{\mathcal{C},\alpha} : \mathcal{C} \to \mathcal{C}' = \mathrm{atn}_{\mathcal{C},\alpha}(\mathcal{C})$, where $\mathcal{C}$ is a code on n neurons. We have $f^{-1}(\mathrm{Tk}_i^{\mathcal{C}'}) = \mathrm{Tk}_i^{\mathcal{C}}$. Also $f^{-1}(\mathrm{Tk}_{n+1}^{\mathcal{C}'}) = \mathrm{Tk}_\alpha^{\mathcal{C}}$.

(c) and (d): easy to see. □

We now extend Theorem 5.20 (which is [12, Theorem 1.4]) to the case of monomial maps. Our proof follows the proof of Theorem 1.4 from [12].

Theorem 5.23. A map $q : \mathcal{C} \to \mathcal{C}'$, where $\mathcal{C}, \mathcal{C}' \in$ Codes, is a *monomial map* if and only if it is the a composition of finitely many basic monomial maps.

Proof. The forward direction follows from Proposition 5.22. On the other side, let $\mathcal{C}$ be a code of length m, $\mathcal{C}'$ a code of length n, and let $q : \mathcal{C} \to \mathcal{C}'$ be a monomial map. We introduce the codes $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n$ in the following way: $\mathcal{C}_0 = \mathcal{C}$ and, for $i = 1, \ldots, n$, we have

$$\mathcal{C}_i = \{\mathbf{u}v_1 \cdots v_i \mid \mathbf{u} \in \mathcal{C}, \mathbf{v} = v_1 \cdots v_n = q(\mathbf{u})\}.$$

We also introduce the code maps $q_i : \mathcal{C}_{i-1} \to \mathcal{C}_i$ $(i \in [n])$ in the following way:

$$q_i(\mathbf{u}v_1 \cdots v_{i-1}) = \mathbf{u}v_1 v_2 \cdots v_{i-1} v_i,$$

where $\mathbf{u} \in \mathcal{C}$ and $\mathbf{v} = q(\mathbf{u})$. Since $q^{-1}(\mathrm{Tk}_i^{\mathcal{C}'})$ is either $\mathrm{Tk}_\alpha^{\mathcal{C}}$, or $\varnothing$, or $\mathcal{C}$, we have that q_i is, respectively, either $\mathrm{atn}_{\mathcal{C}_{i-1},\alpha}$, or $\mathrm{acz}_{\mathcal{C}_{i-1}}$, or $\mathrm{aco}_{\mathcal{C}_{i-1}} = \mathrm{atn}_{\mathcal{C}_{i-1},\varnothing}$. We also introduce the code $\mathcal{C}_{n+1}$ in the following way:

$$\mathcal{C}_{n+1} = \{\mathbf{v}\mathbf{u} \mid \mathbf{u} \in \mathcal{C}, \mathbf{v} = q(\mathbf{u})\}.$$

Let $\sigma \in S_{m+n}$ be the permutation defined by $\sigma(i) = i + n$ for $i = 1, \ldots, m$, and $\sigma(i) = i - m$ for $i = m+1, m+2, \ldots, m+n$. Let $q_{n+1} : \mathcal{C}_n \to \mathcal{C}_{n+1}$ be defined as

$$q_{n+1} = \mathrm{per}_{\mathcal{C}_n,\sigma}.$$

Now for $i \in [m]$ we introduce the codes $\mathcal{C}_{n+1+i}$ in the following way:

$$\mathcal{C}_{n+1+i} = \{\mathbf{v}u_1 \cdots u_{m-i} \mid \mathbf{u} = u_1 \cdots u_m \in \mathcal{C}, \mathbf{v} = q(\mathbf{u})\}.$$

We also introduce the code maps $q_{n+1+i} : \mathcal{C}_{n+i} \to \mathcal{C}_{n+1+i}$ $(i \in [m])$ in the following way:

$$q_{n+1+i} = \mathrm{del}_{\mathcal{C}_{n+i},n+m+1-i}.$$

Finally, we denote $\mathcal{C}_{n+m+2} = \mathcal{D}$ and introduce the code map $q_{n+m+2} : \mathcal{C}_{n+m+1} \to \mathcal{C}_{n+m+2}$ defined by

$$q_{n+m+2} = \mathrm{inj}_{\mathcal{C}_{n+m+2},\mathcal{C}_{n+m+1}}.$$

We have that

$$q = q_{n+m+2} \circ q_{n+m+1} \circ \cdots \circ q_1$$

and each of the maps $q_1, q_2, \ldots, q_{n+m+2}$ is a basic monomial map. $\square$

Corollary 5.24. Every *surjective* monomial morphism $q : \mathcal{C} \to \mathcal{D}$ is a composition of finitely many basic monomial maps of the following four types: acz, per, del, atn.

Proof. By the Proof of Theorem 5.23, either $q = q_r \circ q_{r-1} \circ \cdots \circ q_1$, or $q = q_{r+1} \circ q_r \circ \cdots \circ q_1$ for some r, where $q_1, \dots, q_r$ are of the types acz, per, del, atn, and q_{r+1} is of the type inj. Since q is surjective, only the former holds. $\square$

The following theorem is a major result from Jeffs paper [18, Theorem 1.4], and it becomes a simple corollary with our characterization of monomial maps in Theorem 5.23.

Corollary 5.25. ([18, Theorem 1.4]) If $\mathcal{C}$ is a convexly realizable code and $\mathcal{D}$ is the image of $\mathcal{C}$ under a monomial map, then $\mathcal{D}$ is convexly realizable, and $\operatorname{odim}(\mathcal{D}) = \operatorname{odim}(\mathcal{C})$.

Proof. Follows from Proposition 5.15 and Corollary 5.24. $\square$

We also give a different proof of Proposition 2.11 from [18].

Corollary 5.26. Consider the partial order on neural codes defined by: $\mathbf{u} = u_1 \cdots u_n \le \mathbf{v} = v_1 \cdots v_n$ if $u_i = 1$ implies $v_i = 1$. Let $q : \mathcal{D} \to \mathcal{D}$ be a monomial map. Then for any $\mathbf{u}, \mathbf{v} \in \mathcal{C}$ if $\mathbf{u} \le \mathbf{v}$, then $q(\mathbf{u}) \le q(\mathbf{v})$.

Proof. This is clear if q is one of the maps acz, per, del. If $q = \operatorname{atn}_\alpha^{\mathcal{C}}$, then for $\mathbf{u} \le \mathbf{v}$ in $\mathcal{C}$, we have $\mathbf{u} \in Tk_\alpha^{\mathcal{C}}$ implies $\mathbf{v} \in Tk_\alpha^{\mathcal{C}}$. Hence $\operatorname{atn}_\alpha^{\mathcal{C}}(\mathbf{u}) \le \operatorname{atn}_\alpha^{\mathcal{C}}(\mathbf{v})$. Now the statement holds if $q = \operatorname{atn}_\alpha^{\mathcal{C}}$, and the corollary follows from Theorem 5.23. $\square$

Proposition 5.27. (a) The set Codes, together with linear monomial maps as morphisms, forms a small category (which we denote **Code_lm**).

(b) ([18]) The set Codes, together with linear monomial maps as morphisms, forms a small category (which we denote **Code_m**).

Proof. The proof of (b) given in [18] works for (a) in a similar way. $\square$

Definition 5.28. Let $\mathcal{C}, \mathcal{D}$ be two neural codes. A map $\phi : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C})$ is called a *linear monomial homomorphism* (resp. *monomial homomorphism*) if the associated code map $q = \phi_* : \mathcal{C} \to \mathcal{D}$ is a linear monomial map (resp. monomial map).

Proposition 5.29. (a) The set NRings, together with linear monomial homomorphisms as morphisms, forms a small category (which we denote **NRing_lm**).

(b) ([18]) The set NRings, together with monomial homomorphisms as morphisms, forms a small category (which we denote **NRing_m**).

Proof. The proof of (b) given in [18] works for (a) in a similar way. □

Proposition 5.30. (a) Consider the categories **Code_lm** and **NRing_lm**. If to each code $\mathcal{C} \in$ Codes we associate its neural ring $F(\mathcal{C}) = (\mathcal{P}(\mathcal{C}), \triangle, \cap)$ and to each linear monomial map $q : \mathcal{C} \to \mathcal{D}$ the linear monomial homomorphism of neural rings $F(q) = q^{-1} : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C})$, then in that way we obtain a functor F : **Code_lm** $\to$ **NRing_lm**, which is an isomorphism of these categories.

(b) ([18]) Consider the categories **Code_m** and **NRing_m**. If to each code $\mathcal{C} \in$ Codes we associate its neural ring $F(\mathcal{C}) = (\mathcal{P}(\mathcal{C}), \triangle, \cap)$ and to each monomial map $q : \mathcal{C} \to \mathcal{D}$ the monomial homomorphism of neural rings $F(q) = q^{-1} : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C})$, then in that way we obtain a functor F : **Code_m** $\to$ **NRing_m**, which is an isomorphism of these categories.

Proof. The proof of (b) given in [18] works for (a) in a similar way. □

CHAPTER 6
CONCLUSIONS

The algebraic study of neural codes began in 2013 when the notions of the neural ring and neural ideal were introduced in [11] as algebraic tools for analyzing the intrinsic structure of neural codes. Neural codes consist of neural words, which are the brain's reaction to external stimuli, represented by sequences of zeros and ones corresponding to the state of an active neuron. The goal of the algebraic theory of neural codes is to understand how the brain deals with them, and in particular, how it stores them, processes them, infers from them properties of the stimulus space and receptive field structure, etc. In this dissertation, we introduced several new notions and investigated the properties of them, advancing in that way the algebraic theory of neural codes as well as reproving some already known statements in a more efficient way.

In Chapter 3, the first new notion that we introduced is the polarization of neural codes. The polarization of monomial ideals is well known in Commutative Algebra as a way to linearize an ideal, and the polarization of pseudo-monomial ideals was introduced in [15] to deal in an easier way with the neural ideals of neural codes. Our idea was to polarize the neural code itself (which was the first instance where some object which is not an ideal was polarized), and then found and analyzed the neural ideal of the polarized code. We found that the comparison of these ideals revealed that we can polarize the neural code in two ways, called polarization and formal polarization of neural codes, both having very nice properties. Each of them allows more efficient procedures for dealing with the neural code since we established straight forward ways for going from code objects to polarized code objects, and vice-versa. Since the polarized code objects are in terms of square-free monomial ideals, they are very easy to handle.

In Chapter 4, we introduced several new notions: partial word, partial motif, partial code, and inactive neuron. Initially, we weren't sure if they corresponded to some real life notions related to brain functioning, or if they were just a convenient and intuitive terminology that made our proofs clearer. After recently finding out that these things do indeed exists in the theory of neural networks (imitations of the brain), and although neurophysiologists still have to confirm it, it seems quite natural that they will.

In Chapter 5, we dealt with the monomial morphisms of neural codes. We introduced the basic monomial morphism called "adding trunk neuron" and proved that any monomial morphism can be decomposed into a sequence of basic monomial morphisms. We found that there is a similarity between the images of neural codes under a monomial morphism and the codes on mirror neurons, which have real-life applications including imitation, action understanding, language, empathy, self-representation, autism, etc [21]. We also found that similar maps to "adding trunk neuron" are used in the theory of neural networks [1]. Additionally, we formulated and proved a simple intrinsic characterization of neural rings.

Although we have made the above advances in the algebraic theory of neural codes, there is still much to be studied and discovered. The following are just a few of the questions that have arisen from our work thus far that need further collaboration and research.

Question 1. For a code $\mathcal{C}$ on n neurons, we would like to better understand the difference between the polarization $\mathcal{C}^p$ and the formal polarization $\mathcal{C}^{[p]}$. As $\mathcal{C}^p \subseteq \mathcal{C}^{[p]}$, what can we say about the words form $\mathbb{F}_2^{2n}$ that are in $\mathcal{C}^{[p]}$ but not in $\mathcal{C}^p$?

Question 2. As a particular case of Question 1, if a motif $\mathbf{a} \in \mathbb{M}^{2n}$ is a non-polar motif with $\mathfrak{p}_{\mathbf{a}} \in \mathrm{Min}(J_{\mathcal{C}[p]})$, how is this motif $\mathbf{a}$ related to $\mathcal{C}$?

Question 3. Since the notions for a partial code appear to have connection to neural networks and the "artificial brain" as suggested in [1], we are motivated to investigate some reactions to the states of other neurons in its network. Which of these reactions lead to monomial morphisms? Do we need to introduce a new notion of morphism which would encompass more of these reactions?

Question 4. As a precondition for Question 3, we would like to thoroughly describe the connection between monomial morphisms of neural codes and the behavior of neurons in real life (in particular, for basic monomial morphisms).

Question 5. What other statements can be proved using the notions of partial words, partial motifs, partial codes, and inactive neurons?

Question 6. What is the best way to visually realize the receptive field of neurons: by convex sets exclusively? by open, closed, or neither? by connected, but not necessarily convex nor open or closed? What is the real life justification for any of those choices?

Question 7. Prove Conjecture 2 from [13]: if $\mathcal{C}$ is open-convexly realizable and $\mathrm{odim}(\mathcal{C}) = 2$, then the minimal convex embedding dimension of $\mathcal{C}$ is 2.

Question 8. Can we find an algebraic feature (called an "algebraic signature" in the literature) of a neural code that can tell us if the code is open-convexly or closed-convexly realizable?

The so-called max-intersection-complete codes are open-convexly realizable [9], and it was indeed shown that codes of this type have an algebraic signature [23]. However, that algebraic signature is quite sophisticated, but the polarization of neural ideals was used in the proof. We hope that the polarization of neural codes will play a role in our attempts to answer the more general above question.

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New Albany, IN 47150
(502)727-1137

EDUCATION:

University of Louisville, Louisville, KY
Ph.D. in Applied and Industrial Mathematics — August 2019
Master of Arts in Mathematics — May 2016

Indiana University Southeast, New Albany, IN
Bachelor of Science in Mathematics — May 2010
Associate of Arts in Mathematics — August 2009

TEACHING EXPERIENCE:

University of Louisville, Louisville, KY
Graduate Teaching Assistant — August 2014 - July 2019

- Independently taught the following courses, averaging 25 students per term:
  - Calculus I, Fall 2018 and Spring 2017
  - Pre-calculus, Fall 2017 and Summer 2017
  - College Algebra, Summer 2016
  - Contemporary Mathematics, Summer 2015
- Instructed two to three recitation sections, averaging a total of 20-75 students per term, Fall 2014 - Spring 2016.
- Available for office hours every term, often accommodating students outside set hours.
- Tutored students from other classes including Calculus II & III, Elements of Calculus, Trigonometry, Elementary Statistics, Finite Mathematics, and Quantitative Reasoning.

Ivy Tech Community College, Sellersburg, IN
Math Tutor — August 2014 - June 2014
Adjunct Instructor — August 2011 - June 2014

- Subjects tutored include Calculus I, Trigonometry, Finite Mathematics, College Algebra, Introductory & Intermediate Algebra, and Quantitative Reasoning.

- Aided campus development of the placement test, IvyPrep.
- Computerized the tutoring center's existing sign-in system with Microsoft Access.
- Implemented an online tutoring environment using Blackboard Collaborate & IM.
- Independently taught 20 credit hours of Introductory & Intermediate Algebra, averaging 100-120 term term.

Simmons College of Kentucky, Louisville, KY
Math Instructor June 2013 - May 2014

- Constructed the summer boot camp in June 2013 for Development Math.
- Incorporated guest speakers from the banking industry.
- Independently taught Developmental Math, averaging 10 students per term.

Mathnasium, New Albany, IN
Teacher May 2010 - June 2012

- Taught K-12 Mathematics with an emphasis on number sense.
- Worked collaboratively with other teachers to increase retention.

Sylvan Learning Center, New Albany, IN
Teacher January 2007 - April 2010

- Taught subjects including SAT/ACT prep, Mathematics, English, Beginning Reading, Study Skills.
- Utilized the diagnostic results and taught prescriptive lessons accordingly.

RESEARCH EXPERIENCE:

University of Louisville, Louisville, KY
Doctoral Research August 2017 - May 2019
Faculty Advisor: Dr. Hamid Kulosman

- Dissertation Title: Defended May 22, 2019
  Algebraic Properties of Neural Codes
- Abstract:
  The neural rings and ideals as algebraic tools for analyzing the intrinsic structure of neural codes were introduced by C. Curto, V. Itskov, A. Veliz-Cuba, and N. Youngs in 2013. Since then they have been investigated in several papers, including the 2017 paper by S. Güntürkün, J. Jeffries, and J. Sun, in which the notion of polarization of neural ideals was introduced. We extend their ideas by introducing the polarization of motifs and neural codes, and show that these notions have very nice properties which allow the studying of the intrinsic structure of neural codes of length n via the square-free monomial ideals in $2n$ variables. As a result, we can obtain minimal prime ideals in $2n$ variables which do not come from the polarization of any motifs of length n. For this reason, we introduce the notions for a partial code, including

partial motifs and inactive neurons. With these notions, we are able to relate those non-polar primes back to the original neural code. Additionally, we reformulate an existing theorem and provide a shorter, simpler proof. We also give intrinsic characterizations of neural rings and the homomorphisms between them. We characterize monomial code maps as the composition of basic monomial code maps. This work is based on two theorems, introduced by C. Curto and N. Youngs in 2015, and the notions of a trunk and a monomial map between two neural codes, introduced by R. A. Jeffs in 2018.

Graduate Research August 2015 - May 2018
Faculty Advisor: Dr. Hamid Kulosman

- Reading courses include "Arithmetic in Integral Domains" and "Gröbner Bases and Algorithms".

Indiana University Southeast, New Albany, IN
Undergraduate Research January 2010 - May 2010
Faculty Advisor: Dr. Crump Baker

- Senior Thesis Title: Non-Euclidean Geometries

Publications:

- Christensen, K., Kulosman, H., "Polarization of Neural Codes," submitted; Cornell University Library, arXiv:1802.01251v2 (April2018).
- Christensen, K., Gipson, R., Kulosman, H., "Irreducibility of certain binomials in semigroup rings for nonnegative rational monoids." International Electronic Journal of Algebra, **24**(2018), 50-61.
- Christensen, K., Gipson, R., Kulosman, H., "A new characterization of principal ideal domains," extended version submitted; Cornell University Library, arXiv:1805.10375v1 (May 2018).

CONFERENCES AND PRESENTATIONS

- 38th Annual Mathematics Symposium, Western Kentucky University (November 2018)
  - Christensen, K., "Algebraic Properties of Neural Codes."
- Algebra and Combinatorics Seminar, University of Louisville (Fall 2018)
- GRADtalks Brown Bag Series, University of Louisville (Fall 2018, Spring 2017)
  - Christensen, K., "Polarizating the Neural Ideal." (April 2019)
  - Christensen, K., "Introduction to Neural Codes." (November 2018)
- MAA Indiana Section Annual Meeting, Hanover College (October 2018)
  - Christensen, K., "Algebraic Properties of Neural Codes."
- AMS: Graduate Student Chapter, University of Louisville (Biweekly 2017-2018)

  - Christensen, K., "Introduction to Neural Rings and Ideals." (October 2018)
  - Christensen, K., "An Introduction to Gröbner Bases." (March 2017)
- John H. Barrett Memorial Lectures, University of Tennessee Knoxville (May 2016)
- ALGECOM-14, Purdue University (October 2016)
- MAA Kentucky Section Annual Meeting, Northern Kentucky University (April 2016)
  - Christensen, K., "New Characterizations of Principal Ideal Domains."
- Graduate Student Seminar, University of Louisville (Biweekly 2015-2016)
  - Christensen, K., "New Characterizations of Principal Ideal Domains." (March 2016)
- William M. Bullitt Memorial Lectures, University of Louisville (March 2015)
- Ohio River Analysis Meeting, University of Cincinnati (February 2015)
- Ivy Tech Adjunct Faculty Conference, Ivy Tech Community College: Indianapolis Campus
  - Christensen, K., "Blackboard IM." (February 2014)

UNIVERSITY INVOLVEMENT

University of Louisville, Louisville, KY
Graduate Network of Arts and Sciences

- President — April 2018 - April 2019
  - Chair monthly meetings with representatives from 20 departments in Arts and Sciences.
  - Collaborate with the Associate Dean of Arts and Sciences and other committees for funding.
  - Include my contact information on promotional materials and communicate accordingly.
  - Prepare monthly agendas, record attendance, plan and set up refreshments.
- Social Event Committee Chair — August 2017 - April 2019
  - Plan and host Fall and Spring events reaching an average of 70-100 attendees.
  - Organize and delegate tasks to approximately five other representatives.
  - Communicate with vendors, request event space, organize set-up and clean-up, etc.
- Secretary — May 2017 - April 2018
  - Record the minutes of each monthly meeting with accuracy.

- Notify any absent department representative(s) to notify potential loss of funding.

- Grant Review Committee Member — March 2017
- Department Representative — August 2016 - April 2017

University of Louisville, Louisville, KY
Graduate Student Council

- Department Proxy Representative — August 2016 - April 2019

Professional Memberships

- Association for Women in Mathematics (2018 - present)
- Mathematical Association of America (2014 - present)
- American Mathematical Society (2014 - present)
- AMS Graduate Student Chapter, University of Louisville (2017 - 2019)

PROFESSIONAL DEVELOPMENT

- Emotional Wellbeing, University of Louisville (May 2018)
- Dealing with difficult people, University of Louisville (April 2018)
- Be Searchable: Developing an Online Portfolio, University of Louisville (June 2016)
- Career Colloquium, University of Louisville (August 2016)
- Women in Alternative Academic Careers Panel, University of Louisville (April 2015)
- Teaching Toolbox, University of Louisville (August 2014)
- Graduate Teaching Assistant Academy, University of Louisville (Fall 2014 Spring 2015)
- Ivy Tech Adjunct Faculty Conference, Ivy Tech Community College (February 2014)